A Fundamental Study into the Theory and Application of the Partial Metric Spaces

by

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Thesis

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Contents

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Declarations

During the course of my research I have had one paper published [ONe97]. None of the material in this paper appears directly in the thesis, although it has been the starting point for much research. The conjugate pmetric of section 3.2.4 first appeared in this paper, as did an initial definition of the weighted spaces of section 4.3. My initial attempts at relating partial metric spaces to their T_0-topological properties, section 4.1.1, can also be found here.

Abstract

Our aim is to establish the partial metric spaces within the context of Theoretical Computer Science. We present a thesis in which the big "idea" is to develop a more (classically) analytic approach to problems in Computer Science. The partial metric spaces are the means by which we discuss our ideas. We build directly on the initial work of Matthews and Wadge in this area. Wadge introduced the notion of healthy programs corresponding to complete elements in a semantic domain, and of size being the extent to which a point is complete. To extend these concepts to a wider context, Matthews placed this work in a generalised metric framework. The resulting partial metric axioms are the starting point for our own research.

In an original presentation, we show that T_0-*metrics* are either quasi-metrics, if we discard symmetry, or partial metrics, if we allow non-zero self-distances. These self-distances are how we capture Wadge's notion of size (or weight) in an abstract setting, and Edalat's computational models of metric spaces are examples of partial metric spaces. Our contributions to the theory of partial metric spaces include abstracting their essential topological characteristics to develop the *hierarchical spaces*, investigating their T_0-topological properties, and developing metric notions such as completions. We identify a *quantitative domain* to be a continuous domain with a T_0-metric inducing the Scott topology, and introduce the *weighted spaces* as a special class of partial metric spaces derived from an auxiliary weight function.

Developing a new area of application, we model deterministic Petri nets as dynamical systems, which we analyse to prove liveness properties of the nets. Generalising to the framework of weighted spaces, we can develop model-independent analytic techniques. To develop a framework in which we can perform the more difficult analysis required for non-deterministic Petri nets, we identify the measure-theoretic aspects of partial metric spaces as fundamental, and use valuations as the link between weight functions and *information measures*. We are led to develop a notion of *local sobriety*, which itself appears to be of interest.

Notation

$\downarrow x$, $\uparrow x$	Set of elements below (above) x.
$\mathrm{Max}(X)$	Set of maximal elements of a poset X.
$\bigsqcup^{\uparrow} A$	Supremum of the directed set A.
$x \ll y$	x approximates y.
$\Downarrow x$, $\Uparrow x$	Set of elements approximating (approximated by) x.
$x[n]$	n-truncation of $x \in S^{\infty}$.
$\mathcal{I}(B)$	Ideal completion of B.
$\mathrm{Int}(A)$	Topological interior of A.
$\mathrm{cl}(A)$	Topological closure of A.
$\pi^{\uparrow}$, $\pi^{\downarrow}$	Collection of open upper (lower) sets of a partially ordered space.
κ	Collection of compact upper sets.
τ^k	Cocompact topology of a T_0-space (X, τ).
$\leq_\tau$	Specialisation order of a T_0-space (X, τ).
$\overline{X}$	Completion of metric space X.
$\mathbf{I}\Re$	Domain of intervals.
$\mathbf{B}X$	Space of formal balls of a metric space X.
$\tau_{[d]}$, $\tau_{[d^*]}$	Induced pmetric (metric) topology of a partial metric space.
ϕ_d	Weight function of a partial metric space.
$\leq_d$	Specialisation order of a partial metric space.
$\tau_{[\phi]}$, $\tau_{[\phi^*]}$	Induced pmetric (metric) topology of a weighted space.
$\tau_{[\nu]}$, $\tau_{[\nu^*]}$	Induced pmetric (metric) topology of a valuation space.

Chapter 1

Introduction

Our aim is to establish the partial metric spaces within the context of Theoretical Computer Science. We present a thesis in which the big "idea" is to develop a more (classically) analytic approach to problems in Computer Science. The partial metric spaces are the means by which we discuss our ideas. Taking the initial work of Matthews [Mat94, Mat95] and Wadge [Wad81] in this area, we develop some original mathematics that provides a platform from which to build the theories that we hint at therein.

Our work falls naturally into three parts. In the first part, chapter 2, we give some established material from Theoretical Computer Science that will be the reference points for placing our work into context. It is worth remarking that our work doesn't necessarily fit into any one of the areas that we cover, which means that it can be difficult to characterise, but that it doesn't exist independently of them either. For the second part, chapters 3 and 4, we build directly on the initial work of Matthews [Mat94]. In these chapters, we give a firm foundation for the subject, filling in the obvious gaps in the theory, and establishing the work within the context of the material in chapter 2. It is in the third part of the thesis, chapters 5 and 6, that we take the subject in some new directions.

In chapter 2 we begin with Scott's theory of domains [AJ94], which was introduced to give a rigorous mathematical theory for the semantics of programming languages. A less familiar approach, but one which has the same objectives, is that of de Bakker [dBZ82], in which metrics are used in place of partial orders. We cover the necessary details on partial orders in the chapter while metric spaces, and the more abstract topological spaces, are in the appendices. The essential features of both these approaches for solving domain equations can be abstracted to a unifying framework. The most significant way of doing this, for our purposes, uses the quasi-metrics [Smy87], which are a generalised notion of a metric without symmetry.

The fundamental concepts of continuous domains have alternative topological descriptions in the form of the Scott topology. This leads us to a study of the more obscure T_0-spaces, that are appropriate for the study of partial orders, rather than the familiar Hausdorff spaces. Aside from denotational semantics, continuous domains have also found application as computational models for classical spaces from mathematics, with the classical space embedded as the set of maximal elements. This has had some interesting consequences. For example, by approximating measures on a space by simpler valuations, Edalat [Eda95a] has been able to give a generalisation of Riemann integration that is arguably more faithful than the Lebesgue generalisation. Of more direct relevance to this thesis, the computational models of metric spaces provide a useful source of motivating examples and intuition for the partial metric spaces.

In chapter 3 we turn to the partial metric spaces themselves, and begin with Wadge's cycle sum test [Wad81], an elegant non-operational test for showing that many of Kahn's data flow networks are free from deadlock. Wadge uses the test to motivate the notion of healthy programs corresponding to complete elements in a semantic domain, where complete elements are defined as those elements that can not be further completed. Wadge remarks that these are not the same as the maximal elements, but goes no further in characterising them. The size of an element is introduced as the extent to which it is complete, and Wadge remarks that both of these concepts should extend to a wider context. This motivated Matthews [Mat94] to place Wadge's work in a generalised metric framework, and the resulting partial metric axioms are the starting point for our own research.

With the benefit of hindsight, we present Matthews' axioms from a different perspective, helping us to place them within the context of some of the material from chapter 2. We define a T_0*-metric* as a "metric" appropriate for T_0-spaces, and find that we must either work without symmetry, in which case we have quasi-metrics, or work with non-zero self-distances, in which case we have the partial metrics. It is by introducing these non-zero self-distances that we can capture Wadge's notion of size (or weight) in an abstract setting, from which a notion of a complete element can be derived. The computational models of metric spaces from chapter 2 are partial metric spaces, and help shape our intuition. It is familiar that elements in the model approximate points in the metric space, and if a quasi-metric captures the Scott topology, then the quasi-metric distance approximates the metric distance. What partial metrics can give us in addition, is a measure of how vague an element in the model is; that is we can quantify the degree of approximation of the element, or how "deep" it is in the domain.

Although our presentation of the partial metric axioms is original, the material is basically taken from [Mat94]. For our first original results, we are motivated to abstract the essential topological characteristics of partial metric spaces. If we think of a partial metric space as consisting of layers of metric spaces, each of which consists of points with the same size, then we see how to place a total order on the collection of open sets, which in turn leads us to what we consider to be the essential additional structure for the open sets. We call spaces with this structure *hierarchical spaces*, and show that they lie between T_0-spaces and Hausdorff spaces. As well as helping us to understand the essential features of partial metric spaces, these spaces give us a framework in which to discuss the notion of a boundedly observable property, which Smyth [Smy92] uses to motivate the introduction of quasi-metrics on a domain.

We now turn to the basic theory of partial metric spaces, for which Matthews gives the initial results in [Mat94]. In chapter 4, we significantly extend these results, investigating the partial metric spaces within the context of T_0-topological properties. Our main results include characterising order-consistency and sobriety of the pmetric topology in terms of the pmetric, and hence showing precisely how sobriety is a notion of completeness. In domain theory, the Scott topology naturally captures the basic (qualitative) notions of a domain. We identify a *quantitative domain* to be a continuous domain with some additional structure, such as a T_0-metric, that captures the Scott topology as well as adding some quantitative information to the domain. We then give some conditions for the pmetric topology to be the Scott topology. We also develop metric notions such as completions, and see that for the computational models of metric spaces, our partial metric completion agrees with the ideal completion of domain theory and induces the metric completion on the metric space.

Partial metrics on a domain induce a weight function. Turning this around, if we have a notion of weight inherent in a domain, then we can consider when we induce a partial metric. In this way we again motivate the introduction of metric ideas to a domain. To capture this situation, we introduce the *weighted spaces*, which are derived from an auxiliary weight function over a structured poset, in much the same way that normed spaces are a special class of metric spaces. These spaces are a particularly useful class of partial metric spaces, in which the distance between any two points is a measure of their common information.

Although we have many original results in chapters 3 and 4, it is only in chapter 5 that we take this work in a new direction, but one that is directly motivated by Wadge's work on the cycle sum test [Wad81]. What we propose is to model a deterministic Petri net as a dynamical system, and then to perform some quantitative and qualitative analysis to prove liveness properties of the net. Working initially in a domain-theoretic framework, our quantitative analysis is to establish the existence of unique fixed points, and our qualitative analysis is to find information on this fixed point, without explicitly finding it. Wadge's cycle sum test is the motivation for this analysis, since a function associated with a data flow network passes the test only if its unique fixed point is complete, and the network is free from deadlock. This is essentially the qualitative analysis of some function with a particularly simple dynamics. When we develop a domain suitable for the analysis of liveness in deterministic Petri nets, then we find that we have incomplete maximal elements, which further helps us understand the intuition behind Wadge's work.

We extend these basic ideas to develop more sophisticated analytic techniques, and then generalise to the framework of weighted spaces, abstracting the essential details. We are therefore developing new areas of application for the partial metric spaces. Furthermore, we demonstrate some ideas on developing model-independent analytic techniques within this framework, which can be re-applied to the models for data flow networks and Petri nets. Such a possibility was seen by Matthews in [Mat95], although we go much further in our scope. The challenge is to extend these ideas to non-deterministic Petri nets, and it is clear that this will present significant difficulties. Rather than attempt the details, we discuss some ideas on how this could be achieved. Notably, we discuss the possibility of a suitable dynamical system with *many* fixed points for modeling a non-deterministic Petri net. This would clearly require a far more sophisticated analysis than anything we have so far considered, and developing a framework in which to be able to work is the motivation for our final chapter.

We begin chapter 6 by observing that an initial step in developing such a framework would be to develop our understanding of how a partial metric can be put on a function space, which is a difficult problem in its own right. In domain theory, problems regarding function spaces inevitably lead to a discussion on cartesian closed categories. The material in chapter 5, however, seems to be leading us towards an analytic theory analogous to some of real analysis, where metrics on function spaces are derived from their constituent metrics. We are therefore led to seek some combination of the two approaches for partial metric spaces. We take the first step towards this objective, and identify the measure-theoretic aspects of partial metric spaces as fundamental, by observing that "well-behaved" partial metrics seem to be closely related to measures. We cover the necessary details of measure theory in the appendices.

We observe that valuations provide a link between the weight function of a weighted space and measures, and so we develop a special class of weighted spaces, the *valuation spaces*, whose weight functions are derived from valuations. Taking this to its logical conclusion, we consider when weight functions for weighted spaces can be derived from suitable measures, which we call *information measures.* However, building cartesian closed categories of such spaces would pose significant problems, so we consider the problem of inducing information measures from a valuation space. Building cartesian closed categories of these spaces should be easier.

We find that we can use existing results from the literature on extending valuations to measures, provided that we have a suitable notion of "local compactness" for our spaces. Somewhat surprisingly, this leads to us weakening the notion of sobriety, to something that we call *local sobriety*, where locally sober spaces lie between locally compact sober spaces and order-consistent topologies. We investigate such spaces in some detail, and further develop a notion of *local coherence*, with the property that locally coherent spaces induce locally compact ordered spaces. Borel measures on locally coherent spaces are closely related to valuations, and this provides us with the connection that we seek. Although this material still requires a great deal more work, we feel that the spaces under consideration are of interest in their own right, as well as providing a suitable platform for investigating the problem of partial metrics for function spaces.

Some final comments are appropriate. Our driving motivation is to develop a more (classically) analytic approach to problems in Computer Science. Clearly this is way beyond the scope of one thesis, but the work that we present is aimed towards this goal. The material in chapter 5 is really the key, as it motivates our ideas on the modeling of systems and the analysis of their properties. We present the chapter informally so that we can discuss our ideas to the full. Clearly formalisation will have to follow, but it is the basic ideas that are essential.

With regards to the modeling of systems, one feature of what we propose is that the models themselves should be developed with regard to the problem at hand. This is one way that we can simplify the analysis, but seems to be at odds with much of semantics where the objective is to seek a full semantic model. However, it is the analysis aspect about which we have the most to say. In common with many areas in Computer Science and Mathematics, what we would like is to develop general model-independent analytic techniques that can be applied to suitable models. In this way the more difficult mathematics involved in developing the techniques can remain hidden in their application. We demonstrate our ideas within the framework of partial metric spaces, and in doing so establish the work begun by Matthews within the context of Theoretical Computer Science.

Chapter 2

Domains of Computation

We begin with a review of some of the more established areas in Theoretical Computer Science relative to which we wish to position our work on the partial metric spaces. We consider the partial order and metric approaches to solving domain equations (section 2.1), the ordered topological structures such as quasi-metrics and compact ordered spaces (section 2.2) and Edalat's work on computational models for classical spaces from mathematics (section 2.3). In each case we dwell on the details that will enable us to place our work in context, and paint the more general picture with broader strokes.

2.1 Solving domain equations

To support the correct description and implementation of a programming language, it is important that we understand the *semantics* of the language constructs. Two of the established methodologies for giving a rigorous mathematical theory for the semantics of programming languages (explained succinctly in [dBZ82]) are the *operational semantics* [Hen90], in which operations described by the language constructs are modelled by computational steps on an abstract machine, and the *denotational semantics* [Sto77, Ten94], in which expressions in a programming language denote values in a mathematical domain with an appropriate structure. We consider the denotational semantics, but only in the context of (1) How to construct a suitable semantic domain, and (2) How to assign meaning to recursive constructs. We consider the partial order and metric approaches to tackling these problems, and see how they can be placed in a unifying framework.

2.1.1 Elements of domain theory

Domain theory was first introduced by Dana Scott in 1970 [Sco70, Sco72] when he gave a mathematical model for the type-free λ-calculus [Bar84]. The constructs that Scott used have led to a general mathematical foundation for the semantics of programming languages [Sto77, Ten94]. In this section we consider the basic domain theoretic ideas of convergence and approximation, and in the next we consider categories of domains. Our exposition draws heavily on the presentation given in the excellent [AJ94] (but see also [SLG94]), and we refer to this for details and further references.

The basic structures in domain theory are the *partially ordered sets* P which have a binary relation $\sqsubseteq$ satisfying,

1. (Reflexive) $x \sqsubseteq x$.

2. (Antisymmetric) If $x \sqsubseteq y$ and $y \sqsubseteq x$ then $x = y$.

3. (Transitive) If $x \sqsubseteq y$ and $y \sqsubseteq z$ then $x \sqsubseteq z$.

A subset A of a partially ordered set P is *directed* if it is non-empty, and each pair of elements of A has an upper bound in A. An ω-chain $x_1 \sqsubseteq x_2 \sqsubseteq \cdots$ in P is a simple example of a directed set. If a directed set A has a supremum (least upper bound), then this is denoted by $\bigsqcup^{\uparrow} A$. The suprema of directed sets are the limits in which we are interested, and Mislove [Mis97] calls this *convergence in order.* A partially ordered set D in which every directed subset has a supremum is called a *directed complete partial order* (*dcpo*), and is *pointed* if D has a least element, denoted by $\bot \in D$. As an example, the collection of finite and infinite sequences over some set S, denoted by S^∞, is called the *domain of streams*, and is a dcpo with the subsequence ordering.

If D and E are dcpos, then a function $f : D \to E$ is said to be *continuous* if, whenever A is a directed set in D, then $f(\bigsqcup^\uparrow A) = \bigsqcup^\uparrow f(A)$. It is Scott's thesis that computable functions (those that produce a finite output in a finite time) over domains are continuous. For this to make sense we obviously need, as well as a notion of convergence, a notion of approximation in our domains, so that "infinite" objects can be seen as the coherent limit of their finite approximations.

Suppose x and y are elements of a dcpo D, then we say that x *approximates* y ($x \ll y$) if, for all directed subsets A of D, $y \sqsubseteq \bigsqcup^\uparrow A$ implies that $x \sqsubseteq a$ for some $a \in A$. This was originally called the "way-below" relation [Sto77], but we follow [AJ94] and call it the *order of approximation*. The crucial property of the order of approximation is that of *interpolation*,

$$x \ll y \quad \Longrightarrow \quad \exists z \in D \quad \text{such that} \quad x \ll z \ll y.$$

If a point approximates itself, then that point is said to be *compact*. The idea is that these correspond to finite pieces of information, and are the computable elements in a domain. For example, in S^∞, the compact elements are the finite sequences, and $x \ll y$ if, and only if, x is a finite subsequence of y.

A subset B of a dcpo D is a *basis* for D if, for every $x \in D$, the set $B_x = \Downarrow x \cap B$ (where $\Downarrow x$ is the set of elements approximating x) is directed with supremum x. Every basis must contain the set of compact elements, $K(D)$, of D. A dcpo D is called a *continuous domain* if it has a basis, and an *algebraic domain* if it has a basis of compact elements. We use the prefix ω- if a basis is countable. An equivalent characterisation of a continuous domain is that $x = \bigsqcup^\uparrow \Downarrow x$, for all $x \in D$. For an algebraic domain $K(D)_x = \Downarrow x \cap K(D)$ and we have, $x = \bigsqcup^\uparrow K(D)_x$, for all $x \in D$.

It is easy to see that S^∞ is an ω-algebraic domain, and that any algebraic domain is a continuous domain. An example of a non-algebraic but continuous domain is the unit interval $[0, 1]$ with the usual ordering, where $x \ll y$ precisely when $x < y$ or $x = 0$. A domain in which each pair of elements that are bounded above have a supremum is called a *bounded-complete domain* (or *bc-domain*). The ωbc-algebraic domains are usually sufficient for the purposes of semantics, and are called the *Scott domains.*

In the continuous or algebraic domains, infinite (or ideal) elements are given in a coherent way as *limits* of their finite *approximations.* A basis approximates the order relation since $x \sqsubseteq y$ precisely when $B_x \subseteq B_y$, and a continuous function is completely determined by its action on a basis. So a domain can really be considered as the "completion" of a basis.

To make this clearer, we define an *abstract basis* to be a set B with a transitive relation $\prec$ such that

$$M \prec x \quad \Longrightarrow \quad \exists y \in B \quad \text{such that} \quad M \prec y \prec x,$$

for all $x \in B$ and M finite subsets of B. Examples are, of course, the actual bases of continuous domains and also posets. We let $\mathcal{I}(B)$ be the set of ideals (directed lower sets) of B ordered by inclusion. This is the *ideal completion* of B and is a continuous domain with basis given by $i(B)$ (where $i : B \to \mathcal{I}(B)$ maps $x \in B$ to $\downarrow x$, the set of elements below x). If D is a continuous domain with basis B and we take $(B, \ll)$ as an abstract basis, then $\mathcal{I}(B)$ is isomorphic[1] to D. Posets are precisely the bases of compact elements of algebraic domains.

2.1.2 Categories of domains

To find a mathematical model for the type-free λ-calculus, Scott found a continuous domain D such that the terms of the λ-calculus could be interpreted as elements in D, and application in the λ-calculus could be interpreted as function application, effectively solving the "domain equation"

$$D \cong [D \to D].$$

[1]An *isomorphism* between two posets is a monotone bijection with monotone inverse.

The following simple result, originally given in [Tar55], lies at the heart of the domain theoretic approach to solving such equations.

Theorem 2.1.1 *Suppose D is a pointed dcpo, and f is a continuous function on D, then f has a least fixed point given by $\bigsqcup^{\uparrow}{}_{n \in \mathbb{N}} f^n(\perp)$.*

The least fixed point operator, which assigns to a continuous function its least fixed point, and is itself continuous, is the means by which we give meaning to recursive constructs. The above result gives us a canonical construction of the least fixed point.

In general, we work in a category of domains in which we can construct appropriate domain equations, and associated "continuous" functors, and apply the theorem at the functorial level to solve these equations. As an example, *cartesian closed categories*, which are closed under finite products and function space, are suitable for modelling the typed λ-calculus. We briefly consider the problem of cartesian closed categories of continuous domains, keeping the category theory to a minimum and refering to [Mac71, Poi94] for details.

We denote the category of dcpos and continuous functions by **DCPO**. When we consider a subcategory of pointed objects, then we use the subscript $\perp$. The cartesian product of two dcpos D and E is the usual product of sets with the coordinatewise ordering, denoted by $D \times E$. This is once again a dcpo with directed suprema calculated coordinatewise, and is the categorical product.

The function space of two dcpos D and E consists of all continuous functions ordered pointwise, and is denoted by $[D \to E]$. This is also a dcpo with directed suprema calculated pointwise. To see that the function space is the exponential in **DCPO**, we use the *apply* morphism which sends a function $f \in [D \to E]$ and a point $x \in D$ to $f(x) \in E$. The category **DCPO** is therefore cartesian closed.

The order of approximation naturally carries over from dcpos to their finite product, but not to their function space. The following result then tells us that neither the full (containing all continuous functions) subcategories **CONT** of continuous domains, or **ALG** of algebraic domains, of **DCPO** are cartesian closed.

Theorem 2.1.2 *Suppose* **C** *is a cartesian closed full subcategory of* **DCPO**, *and* D, E *are objects of* **C**, *then the product of* D *and* E *is isomorphic to the cartesian product* $D \times E$, *and the exponential of* D *and* E *is isomorphic to the function space* $[D \to E]$.

We finish by identifying the maximal full subcategories of $\mathbf{CONT}_\perp$ which are closed under cartesian product and function space, and which must therefore be cartesian closed since the necessary universal properties are inherited from **DCPO**. A pointed continuous domain E is an *L-domain* if each pair of elements in E, bounded above by $e \in E$, has a supremum in $\downarrow e$. For comparison, a lattice has a supremum for each pair of elements. As an example, if two sequences in S^∞ have a common upper bound then one is a subsequence of the other (they are cofinal), and so S^∞ is an L-domain. A pointed continuous domain D is *coherent* (or stably locally compact) if the intersection of compact upper sets (with respect to the Scott topology, which we consider in the next section) are compact. If D and E are pointed continuous domains and $[D \to E]$ is continuous, then either D is a coherent domain or E is an L-domain.

If f is a function over D, then f is *finitely separated* from the identity on D, id_D, if there exists a finite set M such that for any $x \in D$, there is $m \in M$ with $f(x) \sqsubseteq m \sqsubseteq x$. We say that D is an *FS-domain* if we have a directed family $\{f_i\}_{i \in I}$ of continuous functions, each finitely separated from id_D, with supremum id_D. A pointed continuous domain D is an FS-domain if, and only if, both D *and* $[D \to D]$ are coherent. The categories **L** of L-domains and continuous functions, and **FS** of FS-domains and continuous functions, are cartesian closed full subcategories of $\mathbf{CONT}_\perp$, and we have the following result.

Theorem 2.1.3 *Every cartesian closed full subcategory of* $\mathbf{CONT}_\perp$ *is contained in* $\mathbf{L}$ *or in* $\mathbf{FS}$.

2.1.3 Metric domains for denotational semantics

We now consider, at a more elementary level, the metric approach for solving domain equations. We assume familiarity with metric spaces, but give a summary in the appendices. In the last section we saw that semantic domains can be constructed as solutions of suitable domain equations, and that the meaning of a recursive construct can be defined as the least fixed point of an associated operator over a continuous domain. One alternative to using dcpos, especially in the presence of concurrency, is to use complete (bounded) metric spaces. Domain equations were first solved in a metric setting by de Bakker and Zucker in [dBZ82]. Consider for example, the domain equation (taken from [dBZ82], but see also [AR89, Wag94])

$$P \cong \{\surd\} + A \times P,$$

which models deterministic processes that can perform a sequence of events from A, possibly ending in a termination symbol $\surd$ not in A.

To solve this equation, we begin with the singleton set $A^{(0)} = \{\surd\}$ together with d_0, the trivial metric. We recursively enrich our space, so that $A^{(n+1)} = \{\surd\} + A \times A^{(n)}$. The metric for a disjoint sum, $+$, is such that elements in different components are a distance 1 apart, and elements in the same component inherit their distance from that component. For the product, $\times$, the distance between two elements is the distance between their first co-ordinate, if they differ, and half the distance between their second co-ordinate otherwise. In more detail we have,

$$\begin{aligned} d_{n+1}(\surd, q) &= 1, \qquad \text{if } q \in A^{(n+1)} \text{ and } q \neq \surd, \\ d_{n+1}(\langle a_1, p_1\rangle, \langle a_2, p_2\rangle) &= \begin{cases} 1, & \text{if } a_1 \neq a_2, \\ d_n(p_1, p_2)/2, & \text{if } a_1 = a_2. \end{cases} \end{aligned}$$

The union, $A^* = \bigcup_{n \in \mathbf{N}} A^{(n)}$, is isometric to the set of finite sequences over A, and the metric completion, $A^\infty = \overline{A^*}$, is isometric to the set of finite and infinite sequences over A, with the metric,

$$d(x, y) = 2^{-[x,y]}, \qquad \forall x, y \in A^\infty,$$

where $[x, y]$ is the first place at which x and y differ, and we take $2^{-\infty} = 0$. The solution to our domain equation is (A^∞, d), and the intuition is that the distance between two processes is inversely proportional to the time needed to distinguish them.

A second example, also taken from [dBZ82] but see [Wag94], is the domain equation

$$P \cong \{\surd\} + \mathcal{P}_c(A \times P),$$

which models non-deterministic processes that either terminate or act as one of a set of possible continuations, each of which performs an action from A and then behaves as a process again. The power set operator $\mathcal{P}_c(\cdot)$ takes closed sets together with the Hausdorff metric, and the domain equation is solved as before.

The emphasis in [dBZ82] is on developing a canonical way in which to solve domain equations over metric spaces. The basic ideas are subsequently reformulated and extended to the (cartesian closed) category of complete bounded metric spaces in [AR89]. Domain equations over this category are solved as the (unique) fixed point of a suitable (contracting) functor. The techniques used can be suitably described as a lifting of Banach's contraction mapping theorem to a categorical setting.

2.1.4 Unifying partial orders and metric spaces

Both partial orders and metric spaces can therefore be used to solve domain equations, and in each case there is a convenient fixed point theorem for interpreting recursive definitions. In his thesis [Wag94], Wagner seeks to establish a common framework unifying the partial order and metric approaches to semantics so as to identify the pre-requisites for solving domain equations. He shows how the choice of a notion of approximation in a domain, leads to a category of domains that supports that notion and is suitable for solving domain equations.

For example, the notion of approximation for partial orders is a binary one; an element either approximates another, or it does not. This is not the order of approximation that we met in section 2.1.1, but rather the information ordering on a domain. In the metric approach, one element can be thought of as approximating another to the extent of their mutual distance, and so values are in $[0, \infty)$ rather than $\mathbf{2} = \{\top, \bot\}$.

Wagner uses the enriched categories in which the hom functor can map into categories other than **SET** (which is the case for traditional categories), and so include partial orders and metric spaces. For example, categories enriched over **2** are precisely the pre-orders (partial orders without anti-symmetry), and categories enriched over $[0, \infty]$ are metric spaces in which we allow infinite distances. Two elements in a pre-order are classified by **2**, and in a metric space by their distance. Rules of composition reduce to transitivity (for pre-orders) and the triangle inequality (for metric spaces). Within this framework Wagner generalises the partial order and metric approaches for solving domain equations.

A related, but topological rather than categorical, approach is with the continuity spaces of Flagg and Kopperman [FK98]. A value quantale $\mathcal{V}$ generalises the order and additive properties of $[0, \infty]$, and a $\mathcal{V}$-continuity space is a generalised metric space with the metric taking values in $\mathcal{V}$. To recover partial orders we can take $\mathcal{V} = \mathbf{2}$.

These two approaches are called *quantitative domain theories* in [FSW98]. They are essentially "big" unification theories (another example is Rutten's generalised ultrametric domain theory given in [Rut95]) and seek to show the unity of the partial order and metric approaches to semantics. We will meet another such approach in section 2.2.3 when we look at quasi-metrics. However, we will also see a different approach emerge, in which we seek a common generalisation of partial orders and metric spaces, securing the advantages of both approaches. That is retaining the partial order, while being able to make quantitative distinctions between elements in a domain.

2.2 Topology and order

We now consider the topological aspects of domain theory, and see that the fundamental concepts of continuous domains have alternative topological descriptions. Our topologies are not Hausdorff, as in classical mathematics (we also give a brief survey of topology in the appendices), but are appropriate for the study of partial orders. We look at the many guises in which partially ordered topological spaces arise, and consider how they are related. We refer to [Law87, Law91, Smy92, AJ94, JS96] as our main references, and to many other papers which we acknowledge in the subsequent sections.

2.2.1 The Scott topology

For a dcpo D we define the *Scott topology* to have as closed sets, the lower sets that are closed under the suprema of directed sets. The Scott topology is not Hausdorff, but does satisfy the weaker T_0-separation axiom. These T_0-spaces are really the focus of our attention in this section. The key to understanding a T_0-space, (X, τ), is the *specialisation order* given by $x \leq_\tau y$ if, and only if, every open set containing x also contains y or, equivalently,

$$x \leq_\tau y \iff x \in \mathrm{cl}\,\{y\}.$$

For a Hausdorff space this order would be discrete, but for the Scott topology on a dcpo it is the original partial order.

In general, for a given partial order on a set, there is a complete lattice of topologies with that order as the specialisation order. The finest of these is the *Alexandroff topology* consisting of all the upper sets. The coarsest is the *weak topology* with the sets $\downarrow x$ as a subbasis for the closed sets. We work with the Scott topology on a dcpo, since it is the finest topology that captures our convergence in order. To make this precise, we say that a topology on a dcpo is *order-consistent* if the specialisation order agrees with the original order, and every closed set is closed under the suprema of directed sets. The Scott topology is the finest order-consistent topology, and the weak topology is the coarsest order-consistent topology on a dcpo.

A function between dcpos is continuous, as defined in section 2.1.1, if, and only if, it is continuous with respect to the Scott topology on each dcpo. For a continuous domain, the sets $\Uparrow x$ (of elements approximated by x) form a basis for the Scott topology. It follows that the Scott topology captures the order of approximation, since $x \ll y$ if, and only if, $y \in \mathrm{Int}(\uparrow x)$. The importance of the Scott topology in domain theory has motivated a more general investigation into T_0-spaces, some of which we consider in this section.

2.2.2 Partially ordered spaces

There is a natural duality to a T_0-space (X, τ), since every open set is an upper set, and every closed set a lower set, with respect to the specialisation order. We say that two T_0-topologies τ and τ' on X are *complementary* whenever $\leq_\tau = \geq_{\tau'}$. We define the *cocompact topology* τ^k to have as a subbasis for the closed sets, the compact upper sets in (X, τ). This is a complementary topology for τ. We call $\pi = \tau \vee \tau^k$ the *patch topology* on X and $(X, \pi, \leq_\tau)$ the *patch space.* For a continuous domain, D, with the Scott topology, σ, the cocompact topology σ^k is the weak topology for $\geq_\sigma$ and the patch topology is called the *Lawson topology*, denoted by λ.

For a topology π and a partial order $\leq$ on X, the order is said to be *closed* if its graph $\mathrm{Gr}(\leq) = \{(x, y) \mid x \leq y\}$ is closed as a subset of $X \times X$. The topology π is then Hausdorff, and $(X, \pi, \leq)$ is called a *partially ordered space.* We let $\pi^\uparrow$ denote the collection of open upper sets, and $\pi^\downarrow$ the collection of open lower sets, so that $\pi^{\uparrow}$ and $\pi^{\downarrow}$ are complementary topologies. A *compact ordered space* [Nac65] is a partially ordered space, $(X, \pi, \leq)$, with π compact. We will see that the compact ordered spaces can be characterised in terms of T_0-spaces.

We say that a T_0-space (X, τ) is *locally compact* if every open neighbourhood of a point contains a compact neighbourhood, in which case the patch space is a partially ordered space. In a T_0-space, distinct points have distinct neighbourhood filters[2], and every neighbourhood filter of a point is completely prime[3], so every completely prime filter over τ is the neighbourhood filter of *at most* one point. A T_0-space is *sober* if every completely prime filter over τ is the neighbourhood filter of precisely one point. Alternatively, every irreducible closed set[4] is the closure of a unique point. Every sober space is, with respect to the specialisation order, a dcpo with an order-consistent topology.

Theorem 2.2.1 (Hofmann-Mislove Theorem [KP94]) *For every sober space (X, τ), there is a bijection between the set of Scott-open filters[5] of τ, and the collection, κ, of compact upper subsets of X, ordered by reverse inclusion.*

For a sober space, (X, τ), κ is therefore a dcpo. Furthermore, if τ is locally compact then τ is a continuous lattice, κ is a continuous domain and we have:

$$\begin{aligned} K' \ll K \text{ in } \kappa &\iff \exists O \in \tau \text{ with } K \subseteq O \subseteq K', \\ O' \ll O \text{ in } \tau &\iff \exists K \in \kappa \text{ with } O' \subseteq K \subseteq O. \end{aligned}$$

So there is a natural duality between the open sets and the compact upper sets of a locally compact sober space. We define (X, τ) to be *coherent* if it is sober, compact and locally compact with the intersection of compact upper sets again compact. In which case the duality is particularly strong, since κ is a continuous *lattice*.

[2] A *filter* $\mathcal{U}$ is a non-empty upwardly-closed subset such that $U, U' \in \mathcal{U}$ implies $U \cap U' \in \mathcal{U}$.
[3] A filter $\mathcal{F}$ of τ is *completely prime* if a subset $\mathcal{A}$ of open sets with $\bigcup \mathcal{A} \in \mathcal{F}$ implies $\mathcal{A} \cap \mathcal{F} \neq \emptyset$.
[4] A closed set is *irreducible* if it is non-empty and is not the union of two closed proper subsets.
[5] A filter $\mathcal{F}$ of τ is *Scott-open* if a directed family $\mathcal{A}$ of open sets with $\bigcup \mathcal{A} \in \mathcal{F}$ implies $\mathcal{A} \cap \mathcal{F} \neq \emptyset$.

Theorem 2.2.2 *If $(X, \pi, \leq)$ is a compact ordered space, then $(X, \pi^{\uparrow})$ is a coherent space. Conversely, if (X, τ) is a coherent space, then the patch space $(X, \pi, \leq_{\tau})$, where $\pi = \tau \vee \tau^k$, is a compact ordered space, $\tau = \pi^{\uparrow}$, $\tau^k = \pi^{\downarrow}$, and these constructions are mutual inverses.*

For a continuous domain, the Scott-open sets are precisely the Lawson-open sets, so a pointed continuous domain is coherent (section 2.1.2) precisely when the Lawson topology is compact. We see that the locally compact sober spaces, generalise the Scott topology for continuous domains and the coherent spaces generalise the Scott topology for coherent domains.

2.2.3 The logical approach to quasi-metric spaces

We now consider Smyth's work on quasi-metrics [Smy87, Smy91, Smy92] which is another of the quantitative domain theories (see section 2.1.4) that seeks to unify the partial order and metric approaches to semantics. A *quasi-metric* on a set X is a map $d : X \times X \to [0, \infty)$ satisfying

Q1. $d(x, x) = 0$.

Q2. $d(x, y) = d(y, x) = 0 \quad \Longrightarrow \quad x = y$.

Q3. $d(x, y) \leq d(x, z) + d(z, y)$.

A quasi-metric space is clearly a generalisation of a metric space, but we can also regard it as a generalisation of a poset if we think of $d(x, y)$ as the "truth value" of the assertion $x \sqsubseteq y$, with zero corresponding to true [FSW98]. We recover a poset P with the *discrete* quasi-metric, defined by

$$d(x, y) \quad = \quad \begin{cases} 0, & \text{if } x \sqsubseteq y, \\ 1, & \text{otherwise.} \end{cases}$$

In the logical approach to quasi-metrics, definitions (for example of Cauchy sequences or completeness) should, in the case of the discrete quasi-metric over a poset, have natural analogues in domain theory. Of course, we also require that in the metric case, they agree with the usual definitions.

We begin with Cauchy sequences. The natural definition for a sequence $\{x_n\}$ to be *Cauchy* in a quasi-metric space (X, d) seems to be that,

$$\forall \varepsilon > 0, \quad \exists \ell \geq 1 \quad \text{such that} \quad \forall n, m \geq \ell, \quad d(x_m, x_n) < \varepsilon. \tag{2.1}$$

However, for the discrete quasi-metric over a poset, Cauchy sequences then correspond to eventually constant sequences, whereas we would like "Cauchy" sequences to roughly correspond to ω-chains and "limits" to correspond to supremum. For this reason we define a sequence $\{x_n\}$ to be *forward Cauchy* if

$$\forall \varepsilon > 0, \quad \exists \ell \geq 1 \quad \text{such that} \quad \forall n \geq m \geq \ell, \quad d(x_m, x_n) < \varepsilon.$$

For the discrete quasi-metric over a poset, forward Cauchy sequences then correspond to eventually increasing sequences.

We are led to define an element $x \in X$, to be the *upper limit* of a forward Cauchy sequence, $\{x_n\}$, if

$$d(x, y) \quad = \quad \lim_{n \to \infty} d(x_n, y), \qquad \forall y \in X.$$

This last limit exists since $\{x_n\}$ is forward Cauchy although, in the general theory [Smy91], it is convenient to admit infinite distances. For the discrete quasi-metric over a poset, upper limits correspond to supremum, and so *complete* quasi-metric spaces, in which every forward Cauchy sequence has an upper limit, correspond to dcpos, and completion by Cauchy sequences corresponds to the ideal completion of a poset to an algebraic domain (see section 2.1.1).

2.2.4 Topologies induced by quasi-metrics

For a quasi-metric d on a set X, the (standard) topology has basis the ε-balls, $B_\varepsilon(x) = \{y \in X \mid d(x, y) < \varepsilon\}$. The *conjugate* quasi-metric d^{-1} is defined by

$d^{-1}(x,y) = d(y,x)$, and the *associated* metric d^* is defined by

$$d^*(x,y) = \max\{d(x,y), d^{-1}(x,y)\}.$$

The definition of a Cauchy sequence, originally given in (2.1), is then lifted from the metric space (X, d^*). Other properties for the quasi-metric space (X, d), such as completeness and total boundedness, can similarly be lifted from (X, d^*).

In the logical approach to quasi-metrics, the limit of a forward Cauchy sequence need not be a topological limit (with respect to the standard topology) and, for the discrete quasi-metric over a dcpo, the standard topology is the Alexandroff topology. In section 2.2.1, we saw that we require the topology for a dcpo to be at least order-consistent ("computationally satisfactory" [Smy87]), so Smyth defines alternative topologies for a quasi-metric which (1) Reduce to order-consistent topologies for the discrete quasi-metric over a dcpo, (2) Reduce to the usual topology for metric spaces, and (3) For which upper limits are topological limits.

The problem however, recognised by Smyth in [Smy91], is that we lose the simplicity of metric spaces, which was one of the advantages we had hoped to secure. To regain this simplicity, Smyth proposes that we restrict our attention to the totally bounded quasi-metric spaces. We can then avoid the difficulties (and controversies) of the more general theory, since the forward Cauchy and Cauchy sequences agree [Smy91], and the upper limit of a (forward) Cauchy sequence is the greatest topological limit with respect to the standard topology (and specialisation order). It follows that for the totally bounded quasi-metric spaces, we can avoid the alternative topologies of [Smy87] altogether.

2.2.5 The computational significance of topology and metric

We now consider a justification of topologies for domains of computation, and the introduction of metrical ideas, taken from [Smy92]. Consider the domain of streams, S^∞, and suppose, as in section 2.1.3, that S^∞ represents a class of simple processes that can perform a sequence of actions. If we observe a process as it proceeds then, at any given time, we can only see a finite segment of its output. Some properties of a sequence (that is a process) are finitely observable, if they are present, and others are infinite in nature. For the condition of *finite observability*, we require that if the property holds of some $x \in S^\infty$, then knowledge of some initial finite segment (approximation) of x suffices to establish this.

If we understand a property extensionally as a subset of S^∞, then the class of finitely observable properties form a topology. In the context of domain theory, appropriate for the notions of finiteness and approximation, we require that a finitely observable property satisfies (1) If a certain amount of information establishes the success of a test, then so does any further information, and (2) If the limit of a directed set of better approximations passes a test, then some approximant passes the test. The finitely observable properties are therefore Scott-open. If $x[n]$ is the n-truncation of $x \in S^\infty$, then the condition of finite observability for $A \subseteq S^\infty$ becomes, for all $x \in A$, there exists $n \geq 1$ such that $\uparrow x[n] \subseteq A$, and the topology of observable properties is the Scott topology.

With this intuition, the specialisation order of a topology, τ, clearly becomes an information ordering, since $x \leq_\tau y$ implies that any finitely observable property of x holds of y. Identifying points with the same information content is therefore the T_0-separation axiom. Continuity can naturally be seen as a sufficient condition for computability since, for a function f to be computable, to obtain finite information about $f(x)$ it suffices to have finite information about x.

We can refine the notion of a finitely observable property to a *boundedly* observable property, whose instances can be verified within a number of steps fixed in advance. Since every finitely observable property is a disjunction of boundedly observable properties, then the latter form a basis for our topology. We call a boundedly observable property whose instances can be verified within k steps, a *depth k property*, and can define a family of relations of closeness,

$$x \leq_\tau^k y \iff \text{Every depth } k \text{ property of } x \text{ holds for } y,$$

which clearly refine the specialisation order. The depth k properties in S^∞ are the $\uparrow x$ for which the length of the sequence is $|x| \leq k$. It follows that $x[k] \sqsubseteq y[k]$ if, and only if, $x \leq_\sigma^k y$.

It now seems natural to define a quasi-metric on S^∞, by

$$d(x,y) = \inf\{2^{-n} \mid x \leq_\sigma^n y\},$$

which is small if, and only if, a "deep" property is required to distinguish x from y. If $x \in S^\infty$ is a finite sequence and we let $0 < \varepsilon < 2^{-|x|}$, then $\uparrow x = B_\varepsilon(x)$. Conversely, each $B_\varepsilon(x)$ is the join of the $\uparrow y$ where $y \in B_\varepsilon(x)$ and y is a finite sequence. So the (standard) topology induced by d is the Scott topology (or topology of observable properties) on S^∞. Furthermore, the associated metric,

$$d^*(x,y) = \inf\{2^{-n} \mid x[n] = y[n]\},$$

is the metric from section 2.1.3.

We have therefore motivated from first principles, a unification of the two approaches to semantics on S^∞ by a quasi-metric. We can further motivate the Lawson topology if we first observe that the condition for finite observability is based on *positive information*. Suppose $a, b \in S$ are distinct, and consider the property $A = \uparrow\langle a\rangle \setminus \uparrow\langle a, b\rangle$, which is not finitely observable. The problem is that if $x \in S^\infty$, and at some given time we have observed $\langle a\rangle$, then is $x \in A$? Smyth suggests, in [Smy92], that if in some finite time we can know whether the next output will *not* be some given element (that is we introduce some negative information), then properties such as A become finitely observable. The topology so induced is the Lawson topology.

2.2.6 Quasi-uniformities

One objection to the approach of the last section, is the arbitrary way in which we assign distances from the relations of closeness $\{\leq_\tau^k\}$. We briefly consider quasi-uniformities [FL82, Law91], which lie between T_0-topologies and quasi-metrics, and give us a global notion of nearness, allowing us to define Cauchy sequences and completeness, without the arbitrary distances of quasi-metrics. Furthermore, quasi-uniformities are central to the study of the interdependence between topologies and orders.

A *quasi-uniformity* on a set X is a filter $\mathcal{U}$ of $X \times X$ such that:

1. Each member of $\mathcal{U}$ contains the diagonal $\{(x,x) \mid x \in X\}$.

2. If $U \in \mathcal{U}$ then $V \circ V = \{(x,z) \mid (x,y),(y,z) \in V \text{ for some } y \in X\} \subseteq U$ for some $V \in \mathcal{U}$.

A quasi-metric d determines a quasi-uniformity $\mathcal{U}_d$ by taking as a base (of the filter $\mathcal{U}_d$) the sets $U_\varepsilon = \{(x,y) \mid d(x,y) < \varepsilon\}$. For a quasi-uniform space $(X,\mathcal{U})$, we have the induced pre-order, $\bigcap \mathcal{U}$, which we denote by $\leq_{\mathcal{U}}$, and the (standard) topology, which we denote by $\mathcal{T}(\mathcal{U})$, the natural analogue to the quasi-metric topology. We will assume that $\mathcal{T}(\mathcal{U})$ is T_0 so that $\leq_{\mathcal{U}}$, the specialisation order for $\mathcal{T}(\mathcal{U})$, is a partial order.

For a quasi-uniformity $\mathcal{U}$ on X, the *conjugate* quasi-uniformity $\mathcal{U}^{-1}$ and the *associated* uniformity $\mathcal{U}^*$, are the natural analogues of the quasi-metric definitions. As an example, the sets of the form $\{(x,y) \mid x-y < \varepsilon\}$, for any $\varepsilon > 0$, are a basis for the *lower quasi-uniformity*, $\mathcal{U}$, on $\Re$. The induced (standard) topology $\mathcal{T}(\mathcal{U})$ is the lower topology on $\Re$, with basic sets of the form (x,∞), and the topology induced by the associated uniformity $\mathcal{U}^*$ is the Euclidean topology on $\Re$.

The topologies $\mathcal{T}(\mathcal{U})$ and $\mathcal{T}(\mathcal{U}^{-1})$ are complementary and $(X,\mathcal{T}(\mathcal{U}^*),\leq_{\mathcal{U}})$ is a partially ordered space. A quasi-uniformity $\mathcal{U}$ is *compatible* with a topology τ on X if $\mathcal{T}(\mathcal{U}) = \tau$. A complemented bitopological space (X,τ,τ') is *determined* by a quasi-uniformity $\mathcal{U}$ if $\mathcal{T}(\mathcal{U}) = \tau$ and $\mathcal{T}(\mathcal{U}^{-1}) = \tau'$, and a partially ordered space $(X,\pi,\leq)$ is *determined* by $\mathcal{U}$ if $\mathcal{T}(\mathcal{U}^*) = \pi$ and $\leq_{\mathcal{U}}=\leq$.

A quasi-uniform space $(X,\mathcal{U})$ is totally bounded (complete) if the uniform space $(X,\mathcal{U}^*)$ is totally bounded (complete). We have seen, in sections 2.2.3 and

2.2.4, that Smyth advocates the complete totally bounded quasi-uniform (or quasi-metric) spaces as domains of computation [Smy91]. Totally bounded spaces can also be justified in the context of section 2.2.5 since, for any given depth of testing, there is a bound on the number of points that we can distinguish by observation, and distance is a measure of the difficulty of distinguishing points. An important connection with section 2.2.2 is that these spaces can alternatively be characterised [Smy91, Law91] as compact ordered spaces.

Theorem 2.2.3 *A compact ordered space $(X, \pi, \leq)$ is uniquely determined by the complete totally bounded quasi-uniformity consisting of open neighbourhoods of the graph $Gr(\leq)$ in $X \times X$. Conversely, if $(X,\mathcal{U})$ is a complete totally bounded quasi-uniform space, then $(X, \mathcal{T}(\mathcal{U}^*), \leq_\mathcal{U})$ is a compact ordered space.*

2.3 Computational models

Aside from denotational semantics, continuous domains have found application as computational models for classical spaces from mathematics, with the classical space embedding as the set of maximal elements. We met our first example of such a model in section 2.1.1, where the Cantor space with the product topology is homeomorphic to the maximal elements of the domain of streams, S^∞, for a finite set S, together with the subspace Scott topology. Most of the material in this section is based on Edalat's recent survey [Eda98], as well as the original papers [Eda95a, Eda95b, EH98].

2.3.1 The upper space as a computational model

Consider an example from [Law97], where we seek numerical approximations of a zero for a continuous function $f : [0,1] \to [0,1]$ with $f(0) < 0$ and $f(1) > 0$. Using the bisection method, we divide our interval in two subintervals of equal length, and choose a subinterval for which f is positive at the right-hand endpoint, and negative

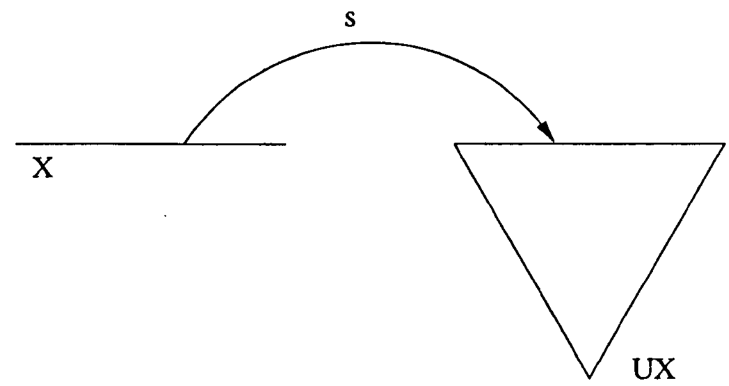


Figure 2.1: Embedding X into its upper space.

at the left-hand endpoint. In this way we construct a sequence of nested intervals with a single point intersection which, by continuity, is a zero of our function. Each interval is a better approximation of the zero. We can use domain theory to identify the essential mathematical details of such a computational example.

We let $\mathbf{I}\Re$ denote the collection of closed bounded (and hence compact) intervals in $\Re$ ordered by reverse inclusion. Intuitively, an interval approximates each of the numbers that it contains. We use reverse inclusion as our information ordering since, as we saw in our example, smaller intervals give a more precise approximation. Matthews calls these intervals vague real numbers in [Mat94], and Lawson calls them approximate reals in [Law97]. This *domain of intervals* is an ω-continuous domain in which suprema of directed sets are given by the intersection of the intervals and $a \ll b$ in $\mathbf{I}\Re$ if, and only if, b is in the interior of a. The map $s : \Re \to \mathbf{I}\Re$ defined by $s(x) = \{x\}$, for each $x \in \Re$, is a homeomorphism from $\Re$ onto the set of maximal elements of $\mathbf{I}\Re$ with the subspace Scott topology. So $\mathbf{I}\Re$ is a computational model of $\Re$, and we can study continuous and computable functions on $\Re$ domain-theoretically, as originally suggested by Scott in [Sco70].

This domain of intervals is itself an example of a more general construction. Suppose X is a second countable locally compact Hausdorff space. We define the *upper space*, $\mathbf{U}X$, to be the collection of non-empty compact subsets of X ordered by reverse inclusion. This is an ω-continuous domain with suprema of a directed set of elements given by the intersection of the elements and $a \ll b$ in $\mathbf{U}X$ if, and only if, b is in the interior of a. Maximal elements of $\mathbf{U}X$ are the singleton subsets of X, and $s : X \to \mathbf{U}X$ given by $s(x) = \{x\}$, for all $x \in X$, is a homeomorphism from X onto the set of maximal elements of its upper space with the subspace Scott topology, as illustrated in figure 2.1. So the upper space is a computational model for a second countable locally compact Hausdorff space.

2.3.2 The space of formal balls

Another important class of spaces in mathematics are the *Polish spaces*, which are topologically complete separable metrisable spaces, and include the Banach spaces. We cannot in general construct the upper space computational model for these spaces, since they need not have enough compact subsets. However, by choosing a separable complete metric, we can construct an alternative computational model.

For a metric space (X, d), we define a *formal ball* to be a pair (x, r), with $x \in X$ and $r \in [0, \infty)$. We define the *space of formal balls*, $\mathbf{B}X$, to be the collection of formal balls, ordered by

$$(x, r) \sqsubseteq (y, s) \iff d(x, y) \leq r - s.$$

If we let $C_\varepsilon(x) = \{y \in X \mid d(x, y) \leq \varepsilon\}$, with $\varepsilon > 0$ and $x \in X$, denote the *closed balls* in X, then $(x, r) \sqsubseteq (y, s)$ implies that $C_r(x) \supseteq C_s(y)$. In general, the converse does not hold, but for the normed vector spaces (which include the Banach spaces) $\mathbf{B}X$ is isomorphic to the collection of closed balls ordered by reverse inclusion.

One of the pleasing aspects in this construction is that the order-theoretic properties of $\mathbf{B}X$ are closely related to the metric properties of X. For example, directed sets in $\mathbf{B}X$ have an ω-chain with the same upper bounds, and ω-chains in $\mathbf{B}X$ correspond to Cauchy sequences in X. Similarly, suprema in $\mathbf{B}X$ correspond to limits in X, so we have that X is complete if, and only if, $\mathbf{B}X$ is a dcpo.

We can extend the "order of approximation" given in section 2.1.1 to arbitrary posets, by replacing "for every directed set" in the definition by "for every directed set with a supremum". The space of formal balls is then a continuous poset, with order of approximation given by

$$(x,r) \ll (y,s) \iff d(x,y) < r-s.$$

Bases of $\mathbf{B}X$ correspond to dense subsets of X, so we see that $\mathbf{B}X$ is ω-continuous if, and only if, X is separable. Maximal elements of $\mathbf{B}X$ are of the form $(x,0)$, and $i: X \to \mathbf{B}X$ given by $i(x) = (x,0)$, for all $x \in X$, is a homeomorphism from X onto the set of maximal elements of $\mathbf{B}X$ with the subspace Scott topology. It follows that if X is a separable complete metric space then our computational model $\mathbf{B}X$ is an ω-continuous domain. In this framework, we can give a domain-theoretic proof of Banach's contraction mapping theorem.

Finally, we consider the metric and domain-theoretic notions of completion. Suppose $(\overline{X},\overline{d})$ is the metric completion of (X,d), then $\mathbf{B}\overline{X}$ is a continuous domain. On the other hand, if we consider $(\mathbf{B}X,\ll)$ as an abstract basis (see section 2.1.1) then its ideal completion $\mathcal{I}(\mathbf{B}X)$ is also a continuous domain. Since X is dense in $\overline{X}$, then $\mathbf{B}X = X \times [0,\infty)$ is a basis for $\mathbf{B}\overline{X}$. The order of approximation of $\mathbf{B}\overline{X}$ restricts to $\mathbf{B}X$, so $\mathbf{B}\overline{X}$ is isomorphic to the ideal completion of its basis $\mathbf{B}X$, which is $\mathcal{I}(\mathbf{B}X)$. In this way the metric completion can be constructed domain-theoretically, since $\overline{X}$ can be identified with $\mathrm{Max}(\mathcal{I}(\mathbf{B}X))$. In chapter 4 we will consider a natural way to recover the complete metric $\overline{d}$, from the ideal completion.

2.3.3 Edalat's generalisation of Riemann integration

We now turn to Edalat's generalisation of Riemann integration [Eda95a]. For a topological space X, we let $\mathbf{M}^1X$ denote the collection of probability distributions, or normalised Borel measures (we give a brief survey of measure theory in the appendices). We construct a computational model for this space so that we can approximate a measure in $\mathbf{M}^1X$ by some "simpler measures" in our model. To find the integral of a function with respect to a measure we then integrate with respect to these simpler measures, and take the limit. This is essentially the dual to the Lebesgue approach, where we take the limit of the integrals of approximating simple functions. The advantage of Edalat's generalisation is that we retain the computational features of the original and gain new techniques for computing the integral.

We begin by defining these "simpler measures". Suppose (X, π) is a topological space, then a *valuation* [Bir67, Law82, Jon89] $\nu : \pi \to [0, \infty)$ satisfies

1. $\nu(a) + \nu(b) = \nu(a \cap b) + \nu(a \cup b)$.
2. $\nu(\emptyset) = 0$.
3. $a \subseteq b$ implies that $\nu(a) \leq \nu(b)$.

Clearly a valuation is a measure-like function, but one that is only defined on open sets, and any measure restricted to the open sets gives a valuation. A *continuous valuation* further satisfies

4. If $\mathcal{A} \subseteq \pi$ is a directed set, then $\nu\left(\bigcup_{U \in \mathcal{A}} U\right) = \sup_{U \in \mathcal{A}} \nu(U)$.

For any $x \in X$, we define the *point valuation*, δ_x, by

$$\delta_x(U) = \begin{cases} 1, & \text{if } x \in U, \\ 0, & \text{if } x \notin U. \end{cases}$$

We then define a *simple valuation* to be a linear combination of point valuations, $\sum_{i=1}^{n} r_i \delta_{x_i}$, with $x_i \in X$, $r_i \in [0, \infty)$.

We define the *probabilistic power domain*, $\mathbf{P}X$, to be the collection of continuous valuations on X bounded by 1, with the pointwise ordering. The probabilistic power domain is a dcpo, and suprema of directed sets are computed pointwise. If X is now a second countable locally compact Hausdorff space, then the upper space, $\mathbf{U}X$, is an ω-continuous domain. An important result from [Jon89] is that $\mathbf{PU}X$ is then an ω-continuous domain with a basis consisting of simple valuations.

As we shall discuss further in the next section, we can extend $s : X \to \mathbf{U}X$ to $\sigma : \mathbf{M}^1X \to \mathbf{PU}X$ by $\sigma(\mu) = \mu \circ s^{-1}$, for all $\mu \in \mathbf{M}^1X$, and this is a bijection between the probability distributions on X and the set of maximal elements of $\mathbf{PU}X$. If we give $\mathbf{M}^1X$ the weak topology, then σ is a homeomorphism, and $\mathbf{PU}X$ is a computational model for $\mathbf{M}^1X$.

For a probability distribution in $\mathbf{M}^1X$, we can use this model to find an approximating ω-chain of simple valuations in $\mathbf{PU}X$. For example, suppose λ is the Lebesgue measure on $[0,1]$ and P is the partition $0 = x_0 < x_1 < \cdots < x_k = 1$ with $\|P\| = \max_{1 \leq i \leq k} x_i - x_{i-1}$. We then have

$$\nu_P = \sum_{i=1}^{k} (x_i - x_{i-1}) \delta_{[x_{i-1}, x_i]} \in \mathbf{PU}[0,1],$$

and $\nu_p \sqsubseteq \lambda$. Furthermore, if P' refines P, then $\nu_P \sqsubseteq \nu_{P'}$ and if $\{P_n\}$ is a refining sequence of partitions with $\|P_n\| \to 0$, then the ω-chain $\{\nu_{P_n}\}$ approximates the Lebesgue measure.

We are now in a position to give a brief account of Edalat's generalisation of Riemann integration. Suppose X is a compact metric space, $f : X \to \Re$ is bounded and $\mu \in \mathbf{M}^1X$ is a probability measure. We let $\mathbf{P}^1\mathbf{U}X$ denote the subdcpo of the normalised valuations on $\mathbf{U}X$, which is also an ω-continuous domain with a basis consisting of normalised simple valuations.

For a simple valuation $\nu = \sum_{b \in B} r_b \delta_b \in \mathbf{P}^1\mathbf{U}X$, with $B \subseteq \mathbf{U}X$ finite, we define the *lower sum*, and *upper sum*, of f with respect to ν, to be

$$S^{\ell}(f, \nu) = \sum_{b \in B} r_b \inf f(b), \quad \text{and} \quad S^u(f, \nu) = \sum_{b \in B} r_b \sup f(b),$$

respectively. If we take $\nu \sqsubseteq \nu'$, then the lower sum increases and the upper sum decreases, and provided $\nu, \nu' \ll \mu$, then $S^{\ell}(f, \nu) \leq S^u(f, \nu')$. So we can define the *lower R-integral*, and *upper R-integral*, of f with respect to μ, to be

$$R\underline{\int} f d\mu = \sup_{\nu \ll \mu} S^{\ell}(f, \nu), \quad \text{and} \quad R\overline{\int} f d\mu = \inf_{\nu \ll \mu} S^u(f, \nu),$$

respectively. We say that f is *R-integrable* with respect to μ, if these are equal, and denote this value by $R\int f d\mu$.

A function f is R-integrable with respect to μ if, and only if, for any $\varepsilon > 0$ we can find a simple valuation $\nu \in \mathbf{P}^1\mathbf{U}X$ with $\nu \ll \mu$ such that $S^u(f,\nu) - S^\ell(f,\nu) < \varepsilon$. If f is R-integrable, then for any ω-chain $\{\nu_n\}$ in $\mathbf{P}^1\mathbf{U}X$ approximating μ, we have

$$R\int f d\mu = \lim_{n\to\infty} S^\ell(f,\nu_n) = \lim_{n\to\infty} S^u(f,\nu_n).$$

It is these simple approximating sequences for the integral that are the essential feature of R-integration.

2.3.4 Extending valuations to measures

Returning to the key result of the last section, the embedding of $\mathbf{M}^1X$ onto the maximal elements of $\mathbf{PU}X$, Edalat [Eda95b] uses some existing results from the literature, notably from [Pet51, Law82], on extending continuous valuations to Borel measures. We briefly consider some of these extension results, as they will prove useful in placing our work in chapter 6 into context.

Our starting point will be the work of Pettis [Pet51] on extending valuations to measures. In the last section a valuation was defined over the open sets of a topology. In [Pet51] the definition is more general, and a valuation $\nu : \mathcal{L} \to [0,\infty)$ can be defined over any lattice $\mathcal{L}$, of subsets of some set X, provided $\nu(\emptyset) = 0$ if $\emptyset \in \mathcal{L}$. If we let $H(\mathcal{L}) = \{A \setminus B \mid A, B \in \mathcal{L}, B \subseteq A\}$, then our valuation induces a map $\psi : H(\mathcal{L}) \to [0,\infty)$, given by $\psi(A \setminus B) = \nu(A) - \nu(B)$, which Pettis shows is well-defined and additive. In general we will be extending ψ to a measure over $S(H(\mathcal{L}))$, the σ-ring generated by $H(\mathcal{L})$, rather than extending ν directly. We now give Pettis' extension result, which we remark can be given in even more general form.

Theorem 2.3.1 ([Pet51], p.192) *Suppose (X,π) is a Hausdorff space, $\mathcal{K}$ a family of closed sets and $\mathcal{W}$ a lattice of open sets, such that $K \setminus W$ is compact, for each*

$K \in \mathcal{K}$ and $W \in \mathcal{W}$. If $\nu : \mathcal{W} \to \Re$ is a valuation such that, for each $W \in \mathcal{W}$ and $\varepsilon > 0$, there exists $K \in \mathcal{K}$ and $W' \in \mathcal{W}$ with

$$W' \subseteq K \subseteq W \quad \textit{and} \quad \nu(W) \leq \nu(W') + \varepsilon,$$

then there exists a unique σ-finite measure ψ^* on $S(H(\mathcal{W}))$ extending ψ on $H(\mathcal{W})$. If $\emptyset \in \mathcal{W}$ then ψ^* is the unique measure on $S(\mathcal{W})$ extending ν on $\mathcal{W}$.

Pettis remarks that his work can be viewed as an abstraction of the method for deriving the Lebesgue measure on $\Re$. To see this we take $\Re$ with the usual topology, $\mathcal{W} = \{(a,b) \mid a \leq b \text{ in } \Re\}$, $\mathcal{K} = \{[a,b] \mid a \leq b \text{ in } \Re\}$ and $\nu((a,b)) = b - a$. Since $\emptyset \in \mathcal{W}$ then ψ^* uniquely extends ν on $\mathcal{W}$ and is the Lebesgue measure. In chapter 6 we will consider another way in which we can use this result to derive the Lebesgue measure. Returning to Edalat's work [Eda95b], Pettis' theorem can be used to deduce that any continuous valuation on a locally compact Hausdorff space has a unique extension to a measure, and hence that, for a second-countable locally compact Hausdorff space, the finite measures and continuous valuations are in one-to-one correspondence.

Lawson also uses Pettis' theorem in [Law82] to prove a number of extension results, from which we can deduce that any continuous valuation over an ωbc-domain extends uniquely to a finite regular Borel measure (with respect to the Lawson topology). This has recently been generalised, in [AME98], where the authors show that if a bounded valuation over a dcpo is the directed supremum of a family of simple valuations, then it has a unique extension to a Borel measure (with respect to the Scott topology). It follows that every bounded continuous valuation over a continuous domain can be extended uniquely to a Borel measure.

2.4 Concluding remarks

We finished section 2.1 with some comments on using partial orders and metrics together so as to be able to make quantitative distinctions between elements in a domain. This is something that we will be working towards, and we will seek to place our work within the context of the material on T_0-spaces from section 2.2, particularly the quasi-metrics. The space of formal balls from section 2.3 will also be useful in establishing context, and the material on extending valuations to measures will re-surface in chapter 6.

Chapter 3

Partial Metric Axioms

Having set out the general area in which we wish to work, we now embark on our study of the partial metric spaces. We begin with Wadge's work on the cycle sum test (section 3.1), from which we identify the essential motivating ideas that set the partial metrics apart from the material in chapter 2. We give an original presentation of the partial metric axioms (section 3.2) with the emphasis on the context in which to place them, and give some original material in which we consider the essential topological characteristics of the partial metric spaces (section 3.3).

3.1 Wadge's cycle sum test

The work on partial metrics can be traced directly back to a paper by Bill Wadge in 1981 [Wad81], in which Wadge discusses Kahn's data flow networks, and gives an elegant non-operational test for proving that many of them are free from deadlock. We summarise the details of this paper.

3.1.1 The extensional semantics of data flow

Consider the Kahn data flow network of figure 3.1 (taken from [Wad81], but see also [Kah74]) which generates the sequence $\langle 1, 2, 3, 5, \ldots \rangle$ of Fibonnacci numbers

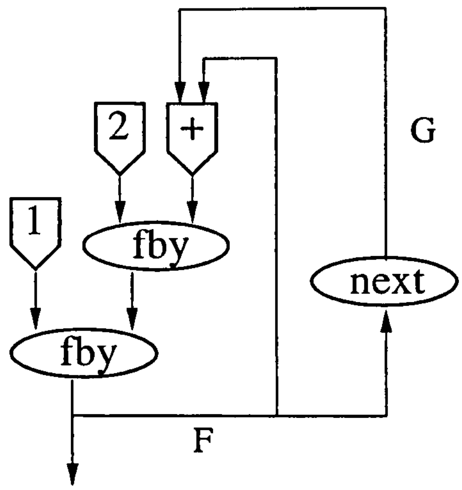


Figure 3.1: Generating the Fibonnacci Numbers

in the following way. The nodes **1** and **2** produce endless streams of 1's and 2's respectively. The + node waits for a value on each input arc then returns their sum down the output arc. The **fby** (followed by) nodes give as output the first value on the left input arc, followed by the values on the right input arc. The **next** node discards the first value on its input arc, and then passes the rest onto its output arc.

As an extensional (denotational) semantics we assign to each arc a sequence, recording the tokens which travel along that arc. In a healthy network, this flow of tokens proceeds indefinitely, and our sequences will be infinite. It is possible however that this flow might cease at some point (that is deadlock), so we will also require finite sequences. For a network with k arcs, we take as our semantic domain S^k, the product of k copies of the domain of streams, S^∞, from section 2.1.1, with $S = \mathbf{N}$, and denote a typical element by $\vec{x} = (x_1, \ldots, x_k) \in S^k$.

The outputs of our nodes are completely determined by their inputs (the nodes are functional), which means that there is a function describing the correspondence between the inputs and outputs of that node. It follows that to a data flow network, there corresponds a set of recursive equations, the right-hand side of which defines a function over our semantic domain. For example, the equations for our Fibonnacci network can be written as,

$$\begin{aligned} F &= 1 \,\mathrm{fby}\, (2 \,\mathrm{fby}\, F + G), \\ G &= \mathrm{next}\, F, \end{aligned} \tag{3.1}$$

which defines the continuous function, $f : S^2 \to S^2$,

$$\begin{pmatrix} f(\vec{x})_1 \\ f(\vec{x})_2 \end{pmatrix} = \begin{pmatrix} 1 \,\mathrm{fby}\, (2 \,\mathrm{fby}\, x_1 + x_2) \\ \mathrm{next}\, x_1 \end{pmatrix}, \qquad \forall \vec{x} \in S^2.$$

Kahn conjectured in [Kah74], and Faustini proved in [Fau82], that the operational behaviour of a network is described by the least fixed point of its associated function.

3.1.2 Circularity and deadlock

Deadlock will occur in a network if some node is directly or indirectly consuming its own output (the network has a *cycle*), and starves itself of tokens. Consider the following simple examples. The present value of the variable I in

$$I = 1 \,\mathrm{fby}\, I + 1, \tag{3.2}$$

depends only on the previously computed values of I. The least fixed point of the associated function is the sequence $\langle 1, 2, 3, \ldots \rangle$, and the network corresponding to this equation is free from deadlock. On the other hand, the present values of J and K in

$$J = 1 + J \quad \text{and} \quad K = 2 * \mathrm{next}\, K, \tag{3.3}$$

depend on the present and future values of J and K respectively. The least fixed point of the associated functions is the empty sequence $\emptyset$, and the networks corresponding to them deadlock.

To formulate the cycle sum test we associate, to each of the arguments of an operation, a number which measures the extent to which the output leads the argument:

- 0 is associated with each argument of $+$.
- 0 and 1 are associated respectively to the arguments of **fby**.
- -1 is associated to the argument of **next**.

The *cycle sum* of a cycle is the minimum extent to which a variable in that cycle depends on itself, and is found by summing these numbers as we compose operations. For example, I has cycle sum $+1$ in (3.2), and we see that to compute the first n values of I we require the first $n-1$ values of I. On the other hand, J and K have cycle sums 0 and -1 respectively in (3.3). It seems clear that a positive cycle sum indicates a healthy dependency. This is the *cycle sum test* and our claim is that if every cycle in a network has a positive cycle sum, then that network is free from deadlock.

3.1.3 Justification of the cycle sum test

A network cannot deadlock if the sequences associated to each of its arcs are infinite. We say that the *size* of a sequence $x \in S^\infty$, is its length, $|x|$, and define the size of an element $\vec{x} \in S^k$ to be the size of its least component, and denote this by $|\vec{x}|$. The *complete* elements in a domain are defined to be those with infinite size. An element is therefore complete in S^k if, and only if, each of its components is complete in S^∞. We prove that a network is free from deadlock by showing that the least fixed point of an associated function is complete. To justify the cycle sum test, we must therefore show that whenever a set of equations corresponding to a network passes the test, then the least fixed point of the associated function is complete.

Consider the continuous function, $f : S^\infty \to S^\infty$, associated with (3.2) whose least fixed point is, by Theorem 2.1.1, the supremum of the chain

$$\emptyset, \quad f(\emptyset) = \langle 1 \rangle, \quad f^2(\emptyset) = \langle 1, 2 \rangle, \quad \ldots$$

which is complete since the size of the terms in the chain increase by one at each step;

$$|f(x)| = |x| + 1, \qquad \forall x \in S^{\infty}.$$

More generally, consider a data flow network with k equations defining k variables over S^{∞}. We form the $k \times k$ matrix M with entries in $\mathbf{Z} \cup \{\infty\}$ so that M_{ij} is the extent to which the ith equation depends on the jth variable, with ∞ signifying no dependency. For example, the Fibonnacci network has, from the equations in (3.1), the associated matrix

$$M = \begin{pmatrix} 2 & 2 \\ -1 & \infty \end{pmatrix}.$$

If we use the min/sum product of matrices, then a network passes the cycle sum test precisely when the diagonal elements of

$$\min_{1 \leq s \leq k} M^s$$

are positive. For example, the Fibonnacci network has two cycles in (3.1), with cycle sums +1 and +2, and these are the diagonal elements of M and M^2. It follows that a network passes the cycle sum test precisely when some power of M has positive entries. In our example M^2 has positive entries.

If we let $f : S^k \to S^k$ be the function associated with our equations, and define $\sigma : S^k \to [0, \infty]^k$ by

$$\sigma(\vec{x}) = \begin{pmatrix} |x_1| \\ \vdots \\ |x_k| \end{pmatrix}, \qquad \forall \vec{x} \in S^k,$$

then M is such that

$$\sigma(f(\vec{x})) \;=\; M * \sigma(\vec{x}), \qquad \forall \vec{x} \in S^k,$$

where $*$ is the min/sum product. Furthermore, for any $s \geq 1$, we have

$$\sigma(f^s(\vec{x})) \;=\; M^s * \sigma(\vec{x}), \qquad \forall \vec{x} \in S^k.$$

It is clear that if M^s has positive entries then the size of the least component of $f^s(\vec{x})$ is greater than the size of the least component of $\vec{x}$, and hence the least fixed point of f is complete in S^k. This finishes the proof that every network which passes the cycle sum test is free from deadlock.

3.1.4 Size and completeness

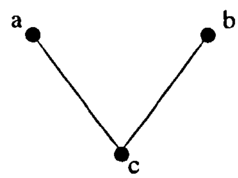
The concepts that Wadge's paper introduces are those of size and completeness, which Wadge notes "*should extend to a much wider context*". Complete elements in a domain are understood, in the vaguest sense, as those elements which cannot be further completed (which is not the same as maximal), and size is a measure of the extent to which they are complete. Wadge indicated that to understand this more fully would require an extended notion of a domain that included some quantitative measure of convergence. To achieve this, within a generalised metric framework, has been the motivation behind the work of Matthews [Mat94, Mat95] on the partial metric spaces, which in turn has been the direct precursor to this thesis.

3.2 The partial metric axioms

We are now ready to give the partial metric axioms of Matthews [Mat94]. Our presentation is original and has been influenced by [BS97, Hec98], with some of the material already appearing in [ONe97]. We will find that in seeking "metrics" for T_0-spaces in general, we are naturally lead to the axioms of Matthews, which allow us to discuss Wadge's ideas in an abstract setting.

3.2.1 Weakening the metric axioms

Let us consider how the metric axioms can be weakened to a T_0-topological framework. We define a *T_0-metric* to be a "distance function" $d : X \times X \to [0, \infty)$, on a set X, that "induces" a T_0-topology on X. More precisely, we require that the



d(a,a) = 0 = d(b,b)
d(c,c) = 1
d(a,c) = d(c,a) = 1
d(b,c) = d(c,b) = 1
d(a,b) = d(b,a) = 2

Figure 3.2: A distance function which is not a T_0-metric.

ε-balls, $B_\varepsilon(x) = \{y \in X \mid d(x,y) < \varepsilon\}$ (for $x \in X$, $\varepsilon > 0$), are the basis for a T_0-topology, $\tau_{[d]}$, on X. We let $\leq_d$ denote the specialisation order of $\tau_{[d]}$. Suppose this is a non-trivial partial order, so that $a <_d b$ for some $a, b \in X$. We can then find $\varepsilon > 0$ such that $b \in B_\varepsilon(b)$ but $a \notin B_\varepsilon(b)$, and it follows that $\varepsilon > d(b,b)$ and $d(b,a) \geq \varepsilon$. We have two cases to consider:

$$\begin{array}{llcl} \text{either} \quad a \in B_\varepsilon(a) & \Longrightarrow \quad b \in B_\varepsilon(a) & \Longrightarrow & d(a,b) < \varepsilon \leq d(b,a), \\ \text{or} \qquad\qquad & a \notin B_\varepsilon(a) & \Longrightarrow & d(a,a) \geq \varepsilon > d(b,b). \end{array}$$

In the first instance d cannot be symmetric, and in the second d must have non-zero self-distances. In weakening the metric axioms to induce non-trivial specialisation orders, we must therefore choose between symmetry and zero self-distances, we cannot have both.

If we insist on zero self-distances, then the quasi-metric axioms, from section 2.2.3, are appropriate. If we require symmetry then Matthews proposes, in [Mat94], that the appropriate axioms are,

P1. $d(x,y) \geq d(x,x)$.

P2. $d(x,x) = d(x,y) = d(y,y) \implies x = y$.

P3. $d(x,y) = d(y,x)$.

P4. $d(x,y) \leq d(x,z) + d(z,y) - d(z,z)$.

To see why we must strengthen the metric triangle inequality, consider the example space in figure 3.2, in which the distance function satisfies P1-3 together with the metric triangle inequality. Since $c \in B_\varepsilon(c)$ implies $b \in B_\varepsilon(c)$, then we cannot have

$$c \in B_\varepsilon(c) \quad \subseteq \quad B_2(a) = \{a, c\},$$

and so the distance function is not a T_0-metric.

The combination P1-4 are a minimal generalisation of the metric axioms, retaining symmetry, and are called the *partial metric* axioms. We will call distance functions that satisfy the partial metric axioms either partial metrics or *pmetrics.* A minimal set of axioms, for which we retain symmetry, are the combination P2-4, and these are called the *weak partial metric* axioms. This weaker position is argued for by Heckmann in [Hec98], who shows that we can deduce a weak P1,

$$2d(x, y) \geq d(x, x) + d(y, y),$$

from P4, and that $d' : X \times X \to [0, \infty)$ defined by

$$d'(x, y) = \max\{d(x, y), d(x, x), d(y, y)\},$$

is a partial metric with $\tau_{[d']} = \tau_{[d]}$. However, we feel the intuition to be that much simpler with the P1 axiom, since we retain that idea that a point y is at least as far from x as x is itself.

In the presence of non-zero self-distances we find it convenient to change our ε-ball definition so that

$$B_\varepsilon(x) = \{y \in X \mid d(x, y) < d(x, x) + \varepsilon\}, \qquad \forall x \in X, \quad \forall \varepsilon > 0.$$

This induces the same topology, $\tau_{[d]}$, on X and is more intuitive, since $x \in B_\varepsilon(x)$ for all $\varepsilon > 0$.

To conclude our discussion on axioms for distance functions, we remark that the minimal T_0-metric axioms are the combination P2 and P4, where we have neither symmetry nor zero self-distances, and that the P4 axiom alone is sufficient for the ε-balls to induce a topology, although this need not be T_0.

3.2.2 Understanding partial metrics

In seeking distance functions for T_0-topologies, the choice is essentially between the quasi-metrics and the partial metrics. As we have already seen in section 2.2, the quasi-metrics have been around for significantly longer than the partial metrics, and have been extensively studied in their own right. One reason for the emergence of the quasi-metrics earlier than the partial metrics is the relatively simple intuition in the presence of non-symmetry, as opposed to non-zero self-distances; colloquially, suppose metric measures "amount of effort" and think of walking in hilly terrain [Smy92]. It has proved more difficult to find an entirely satisfactory intuition for the partial metrics.

We consider examples of partial metric spaces and try to establish some of the intuition behind them. We begin with the domain of intervals, $\mathbf{I}\Re$, from section 2.3.1, for which Matthews [Mat94] defines a pmetric by

$$\begin{aligned} d([a,b],[c,d]) &= \sup\{|x-y| \mid x,y \in [a,b] \cup [c,d]\} \\ &= \max\{b,d\} - \min\{a,c\}. \end{aligned}$$

This pmetric agrees with the usual metric on the set of maximal elements of $\mathbf{I}\Re$, and induces the Scott topology on $\mathbf{I}\Re$. It follows that $(\mathbf{I}\Re, d)$ is a computational model for $\Re$ together with the usual metric. In this model, we can not only approximate points in $\Re$ by elements in $\mathbf{I}\Re$, but we can also approximate distances, since the distance between any two elements in $\mathbf{I}\Re$ is the extent to which the points they approximate can differ in $\Re$.

A natural step is to now extend this to the space of formal balls, $\mathbf{B}X$, for a metric space (X,d). We define a pmetric $\delta : \mathbf{B}X \times \mathbf{B}X \to \Re$ by

$$\delta((x,r),(y,s)) = \max\{2r, r + d(x,y) + s, 2s\} \geq 0.$$

This is an original definition which was also found independently by Heckmann in [Hec98]. Intuitively we think of (x,r) as the "solid" closed ball $C_r(x)$ in X, and define δ to measure the "diameter" of the formal union of (x, r) and (y, s). Our pmetric clearly extends d on X and, once we prove that δ induces the Scott topology on $\mathbf{B}X$, then $(\mathbf{B}X, \delta)$ will be a computational model for (X, d).

Lemma 3.2.1 *The induced partial metric topology on* $\mathbf{B}X$ *is the Scott topology.*

Proof. If $(y, s) \in B_\varepsilon((x, r))$, then

$$r + d(x, y) + s \leq \delta((x, r), (y, s)) < 2r + \varepsilon,$$

so that $d(x, y) < r + \varepsilon - s$ and $(x, r + \varepsilon) \ll (y, s)$. Conversely, if $(x, r + \varepsilon/2) \ll (y, s)$ then $(y, s) \in B_\varepsilon((x, r))$ so we see that

$$\Uparrow(x, r + \varepsilon/2) \subseteq B_\varepsilon((x, r)) \subseteq \Uparrow(x, r + \varepsilon).$$

If $(y, s) \ll (x, r)$ and we take $0 < \varepsilon < s - r - d(x, y)$, then $(y, s) \ll (x, r + \varepsilon) \ll (x, r)$. It easily follows that the partial metric topology and the Scott topology on $\mathbf{B}X$ agree.

QED

We will see in the next section that such computational models can also be given using quasi-metrics. The partial metrics however have one crucial feature that makes them more suitable for our purposes; they capture the notion of size that Wadge was searching for in [Wad81]. Consider the domain of intervals once more, where $d([a, b], [a, b]) = b - a$, for any $[a, b] \in \mathbf{I}\Re$. We clearly see that self-distance is a measure of how vague a point is. We call this self-distance, the *size* or *weight* of a point, and identify the least vague points, those with size zero, as *complete* points. It is interesting that we have been lead to an abstract setting in which to discuss Wadge's ideas simply by trying to develop T_0-metrics.

For any partial metric space (X, d), we define the *weight function* $\phi_d : X \to [0, \infty)$, by $\phi_d(x) = d(x, x)$. This allows us to quantify how "deep" a point is in a domain, or how much information it contains, and is the crucial distinction between the partial metrics and the quasi-metrics. The specialisation order of the induced pmetric topology is naturally captured by the pmetric as follows,

$$x \leq_d y \iff d(x,y) = d(x,x),$$

and we see that this is an information ordering in a precise sense, since $x \leq_d y$ implies that $\phi_d(x) \geq \phi_d(y)$. Furthermore, if one element is strictly below another, then it must be strictly more vague, and we have

$$x \leq_d y \text{ and } x \neq y \implies \phi_d(x) > \phi_d(y).$$

It is easy to see that complete points must be maximal, although the converse need not hold. This distinction between complete and maximal points will become clearer in chapter 5 when we give an example of a partial metric space with incomplete maximal elements.

3.2.3 Weighted quasi-metrics and metrics

We hinted in the last section that partial metrics and quasi-metrics are related, and that what distinguishes them is the induced weight function of the partial metrics. We clarify this with a result of Matthews [Mat94].

Lemma 3.2.2 ([Mat94]) *Suppose (X,d) is a partial metric space. Then*

$$q(x,y) = d(x,y) - d(x,x), \qquad \forall x,y \in X,$$

defines a quasi-metric q on X such that $\tau_{[d]} = \tau_{[q]}$ and

$$q(x,y) + \phi_d(x) = q(y,x) + \phi_d(y), \qquad \forall x,y \in X. \tag{3.4}$$

Conversely, if q is a quasi-metric on X with a function $\phi : X \to [0,\infty)$ that satisfies (3.4) with ϕ in place of ϕ_d, then

$$d(x,y) = q(x,y) + \phi(x), \qquad \forall x,y \in X,$$

defines a pmetric on X such that $\tau_{[q]} = \tau_{[d]}$, and ϕ is the induced weight function.

The partial metrics are therefore equivalent to the class of quasi-metrics that can be symmetrised by a weight function. We call these the *weightable* quasi-metrics, and say that a pair $<q, \phi>$ is a *weighted* quasi-metric. As a corollary we see that every pmetric has an associated metric, d^*, given by

$$d^*(x,y) = d(x,y) - \min\{d(x,x), d(y,y)\}.$$

This is not the metric that Matthews gives in [Mat94], which comes from symmetrising the induced quasi-metric, q, with $q + q^{-1}$ rather than $\max\{q, q^{-1}\}$, but gives the same topology.

Seeing that the partial metrics are a special class of quasi-metrics, Matthews then seeks a comparable result for metrics in [Mat94]. By defining a metric $d^* : X \times X \to [0, \infty)$ to be *weightable* [Mat94], if there exists a function $\phi : X \to [0, \infty)$ such that

$$d^*(x,y) \geq \phi(x) - \phi(y), \qquad \forall x, y \in X,$$

and calling the pair $<d^*, \phi>$ a *weighted* metric, Matthews gives the following result.

Lemma 3.2.3 ([Mat94]) *Suppose (X, d) is a partial metric space, then*

$$d^*(x,y) = 2d(x,y) - d(x,x) - d(y,y), \qquad \forall x, y \in X,$$

defines a weighted metric $<d^*, \phi_d>$ on X. Conversely, if $<d^*, \phi>$ is a weighted metric on X, then

$$d(x,y) = (d^*(x,y) + \phi(x) + \phi(y))/2, \qquad \forall x, y \in X,$$

defines a pmetric on X.

However, this is much weaker than lemma 3.2.2, since distinct partial metric spaces can induce the same weighted metric space. For example, suppose $X = \{a, b_1, b_2, \ldots\}$ and we have two pmetrics d_1 and d_2 on X such that

$$d_1(a,a) = 1 = d_2(a,a),$$

$$d_1(b_n, b_n) = 0 = d_2(b_n, b_n), \qquad \forall n \geq 1,$$

$$d_1(b_n, b_m) = 1 = d_2(b_n, b_m), \qquad \forall n, m \geq 1, n \neq m,$$

$$\text{and} \quad d_1(a, b_n) = 1 + 1/2^n, \quad d_2(a, b_n) = 2 + 1/2^n, \quad \forall n \geq 1.$$

The partial metric spaces (X, d_1) and (X, d_2) are distinct, since $\{a\} \in \tau_{[d_2]} \setminus \tau_{[d_1]}$. However, the induced metrics (from the lemma) are such that

$$d_1^*(b_n, b_m) = 2 = d_2^*(b_n, b_m), \qquad \forall n, m \geq 1, n \neq m,$$

$$\text{and} \quad d_1^*(a, b_n) = 1 + 2/2^n, \quad d_2^*(a, b_n) = 3 + 2/2^n, \quad \forall n \geq 1,$$

which both induce the discrete topology. Since the induced weight functions are the same, then the induced weighted metric spaces are topologically equivalent.

3.2.4 Conjugate partial metrics

We now consider a slight anomaly in the definition of the partial metrics. If we observe that for a quasi-metric, the associated metric comes from symmetrising the quasi-metric, then we are left to wonder where the associated metric for a partial metric comes from. In more detail, the immediate consequence of non-symmetry for a quasi-metric q, on a set X, is that we have a natural conjugate quasi-metric q^{-1} which is used to symmetrise q. Now, as Künzi discusses in [KV94], if both q and q^{-1} are weightable by ϕ and ϕ' respectively, then $\phi + \phi'$ is a constant function on X. It follows that if a pmetric d induces a weighted quasi-metric $<q, \phi_d>$, and ϕ_d is unbounded, then q^{-1} cannot be weightable. It is still the case however, that we have an inherent duality from our T_0-topology, and so we should be able to "symmetrise" the pmetric to recover the associated metric.

An original observation, that we first made in [ONe97], is that we can recover a natural duality for partial metrics if we admit negative distances. We argue that *positivity* in the axioms is superfluous, since $d(x, y) \geq 0$ has been effectively replaced by the P1 axiom, $d(x, y) \geq d(x, x)$. An immediate difficulty that this introduces is that it now makes little sense to define a complete element as one with zero size.

Our response is to call a point *complete* if, and only if, it has size $\inf\{\phi_d(x) \mid x \in X\}$, when this exists. In fact, this helps to clarify that the basic concept is really size, from which a notion of completeness can be derived.

Another difficulty in adopting negative distances is our intuition, since we consider $\phi_d(x)$ as measuring the vagueness of a point. The problem seems to be more with the labels "positive", "zero" and "negative" information, than anything fundamental. If we think in terms of allowing the vagueness of points in a space to be measured without a lower bound, then the situation becomes more acceptable.

Once we allow $d : X \times X \to \Re$, then the *conjugate* partial metric, d^{-1}, can be defined by $d^{-1}(x,y) = d(x,y) - d(x,x) - d(y,y)$. We immediately see that if q is the induced quasi-metric of d, then q^{-1} is the induced quasi-metric of d^{-1}, and $-\phi_d$ is a weight function for q^{-1}. So there is a precise sense in which d^{-1} is conjugate to d. Furthermore, the metric topology is the join of the pmetric and conjugate pmetric topologies. To see this, if we let $B_\varepsilon(x; d^{-1})$ and $B_\varepsilon(x; d^*)$ denote the ε-balls in (X, d^{-1}) and (X, d^*) respectively, then we show that $B_\varepsilon(x) \cap B_\varepsilon(x; d^{-1}) = B_\varepsilon(x; d^*)$. Suppose $x, y \in X$ and $\varepsilon > 0$, then the result immediately follows from,

$$\left.\begin{array}{r} d(x,y) < d(x,x) + \varepsilon \\ d^{-1}(x,y) < d^{-1}(x,x) + \varepsilon \end{array}\right\} \iff d(x,y) < \min\{d(x,x), d(y,y)\} + \varepsilon.$$

In this thesis, our partial metrics will take positive and negative values, unless specifically stated otherwise.

3.2.5 Upper semicontinuity of T_0-metrics

We finish our presentation of the partial metric axioms with one other potentially useful property that need not hold for quasi-metrics. For any metric space (X, d), the metric $d : X \times X \to [0, \infty)$ is continuous with respect to the product metric topology on $X \times X$, and the usual topology on $[0, \infty)$. If (X, d) is a T_0-metric space, then the equivalent result is that $d : X \times X \to \Re$ is *upper semicontinuous* with respect to the product T_0-metric topology, $\tau_{[d]}$, on $X \times X$ (that is $d^{-1}(-\infty, r)$ is open for any $r \in \Re$). Intuitively, if two points are close together then any two points sufficiently close to them will also be close together.

Suppose d is a T_0-metric with $a <_d b$ $(a, b \in X)$. For all $r > d(a,a)$, we have $(a,a) \in d^{-1}(-\infty, r)$. However, if d were upper semicontinuous, then $(b,a) \in d^{-1}(-\infty, r)$, which implies that $d(b,a) = d(a,a) = d(a,b)$. So in the presence of a non-trivial specialisation ordering, which is precisely when we are interested in the T_0-metrics, non-zero self-distances are a necessary condition for upper semicontinuity. That the quasi-metrics need not be upper semicontinuous is surely well-known, although we have not seen it in the literature. However, both Bukatin [BS97] and Heckmann [Hec98] point out, in different guises, that the partial metrics are upper semicontinuous.

Lemma 3.2.4 *For any partial metric space (X, d), the function $d : X \times X \to \Re$ is upper semicontinuous with respect to the product partial metric topology on $X \times X$.*

Proof. Suppose $r \in \Re$ and $d(x,y) < r$. We let $\varepsilon = r - d(x,y) > 0$, then $x \in B_{\varepsilon/2}(x)$, $y \in B_{\varepsilon/2}(y)$, and if $a \in B_{\varepsilon/2}(x)$ and $b \in B_{\varepsilon/2}(y)$, then we have

$$d(a,b) \quad \leq \quad d(a,x) - d(x,x) + d(x,y) + d(y,b) - d(y,y) \quad < \quad r.$$

QED

This seems to be a further justification for the partial metrics over the quasi-metrics (or for symmetry over zero self-distances). Our case is supported by the following argument of Bukatin [BS97]. Within the context of denotational semantics, Bukatin considers the space of distances as a datatype, which should be represented by a continuous domain, and requires that the distance function $d : X \times X \to D$ is computable. In domain theory, continuity can be seen as a sufficient condition for computability, and if we take $D = ([0, \infty), \geq)$, then continuity with respect to the Scott topology on D and the product T_0-topology on $X \times X$, is precisely upper semicontinuity. Bukatin argues that we can therefore define computationally meaningful distances between programs.

3.3 Hierarchical spaces

We now give our first section of original material, in which we abstract the essential topological characteristics of partial metric spaces. As well as helping to develop our understanding of the partial metric spaces, these spaces will also give us a framework in which to discuss the notion of a boundedly observable property from section 2.2.5.

3.3.1 Hierarchies of open sets

We begin by motivating and introducing some of our ideas, and delay until the next section the precise definitions. Our starting point is the simple observation that, for a partial metric space (X, d) and $r \in \Re$, the set

$$X_r = \{x \in X \mid \phi_d(x) = r\},$$

together with the induced metric

$$d_r^*(x, y) = d(x, y) - r,$$

is a metric space. We can therefore think of a partial metric space as being layers of (possible empty) metric spaces, indexed by the weight function taking values in the total order $(\Re, \leq)$. These layers are a measure of the vagueness of the points in X, and can be used to compare the vagueness of points unrelated in the information ordering. This is the essential feature of a partial metric space that we wish to capture topologically. To do this we must consider the open sets as fundamental, rather than the points themselves, and see how we can deduce the vagueness of a point.

Consider the generalised situation given in figure 3.3, where r, s and t are values in some totally ordered set, which indexes layers of points in a set X, with t less than r less than s. We again think of these layers as a measure of vagueness, and observe that points in the open set U are less vague than points in the layer indexed by s, but that we cannot say anything relative to points in the layer indexed by r.

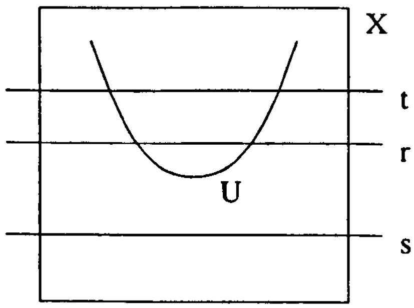


Figure 3.3: Motivating hierarchies of open sets

We will say that U is *of depth* (of vagueness) r but not of depth s. We immediately notice, that once U is of depth r then it must be of depth t. We can now divide our topology into classes of open sets of the same depth, and we call this a *hierarchy of open sets.*

We remark that this is similar to the situation in section 2.2.5, in which open sets in the domain of streams, S^{∞}, were identified as finitely observable properties, and boundedly observable properties were those whose instances could be verified within a number of steps given in advance. Such open sets were said to be of depth indexed by the total order $(\mathbf{N}, \geq)$.

3.3.2 Defining a hierarchical space

We now make our ideas precise, and define a class of topological spaces which, we claim, capture the essential topological characteristics of partial metric spaces. For a T_0-space, (X, τ), with basis $\mathcal{B}$, we let $\mathcal{B}_x \subseteq \tau$ denote the collection of basic open sets that contain $x \in X$. We define an *index set* to be a conditionally complete[1] totally ordered set, $(R, \leq)$, which satisfies an interpolation property; for $r < s$ in R we can find $t \in R$ such that $r < t < s$.

Definition 3.3.1 *For a T_0-space (X, τ), a* **hierarchy of open sets** *is a collection*

[1]A poset is *conditionally complete* if every set bounded above has a supremum.

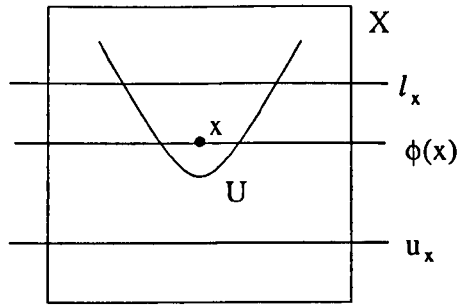


Figure 3.4: Induced weight function

of subsets of τ, $\{\mathcal{B}_r\}_{r\in R}$, *with index set* $(R, \leq)$, *such that* $\bigcup_{r\in R}\mathcal{B}_r$ *is a basis for* τ, $r \leq s$ *in* R *implies that* $\mathcal{B}_s \subseteq \mathcal{B}_r$, *and for each* $x \in X$ *we can find* $\ell_x, u_x \in R$ *such that* $\mathcal{B}_x \subseteq \mathcal{B}_{\ell_x}$ *but* $\mathcal{B}_x \not\subseteq \mathcal{B}_{u_x}$ *. Each open set in* $\mathcal{B}_r$ *is said to be* **of depth** r, *and the collection* $\mathcal{B}_r$ *is said to be the* **class of depth** r.

For a T_0-space, (X, τ), with a hierarchy of open sets, $\{\mathcal{B}_r\}_{r\in R}$, we can deduce the vagueness of a point, and hence find layers of points indexed by $(R, \leq)$. We define a *weight function* $\phi : X \to R$ by

$$\phi(x) = \sup\{r \in R \mid \mathcal{B}_x \subseteq \mathcal{B}_r\} \in R,$$

which exists since $\mathcal{B}_x \subseteq \mathcal{B}_{\ell_x}$ and if $\mathcal{B}_x \subseteq \mathcal{B}_r$ then $r < u_x$. It is immediate that $\ell_x \leq \phi(x) \leq u_x$, and we illustrate this in figure 3.4. Furthermore, for any $r < \phi(x) < s$ we have $\mathcal{B}_x \subseteq \mathcal{B}_r$ and $\mathcal{B}_x \not\subseteq \mathcal{B}_s$. Now, $x \leq_\tau y$ implies that $\phi(y) \leq \phi(x)$, but we do not yet have $x <_\tau y$ implies that $\phi(y) < \phi(x)$, which we seek since this is an essential feature of the partial metric spaces. This will require an additional condition on our open sets.

We recall that in a T_0-space, (X, τ), if $x \not\leq_\tau y$, then we can find an open set containing x but not y, and in a Hausdorff space we can separate any two points using disjoint open sets. We suggest that our extra condition should ensure that the hierarchy of open sets plays an essential role in separating points in our space.

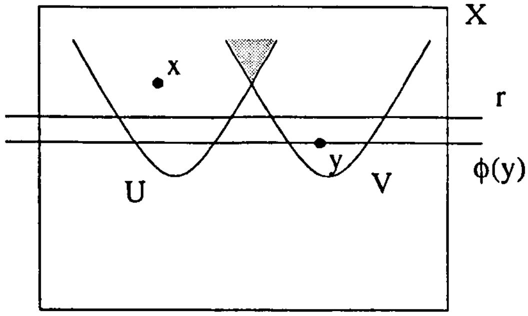


Figure 3.5: Separating points

This will position our spaces between the T_0-spaces and the Hausdorff spaces. If we refer to figure 3.5 for intuition, then our condition is that whenever $x \not\leq_\tau y$, then we can find open sets containing x and y, and some level of vagueness below $\phi(y)$, such that any points in the intersection are below this level of vagueness. The precise definition follows.

Definition 3.3.2 *A* **hierarchical space** $(X, \{\mathcal{B}_r\}_{r \in R})$ *is a T_0-space (X, τ), with a hierarchy of open sets, $\{\mathcal{B}_r\}_{r \in R}$, such that whenever $x \not\leq_\tau y$ then we can find $U \in \mathcal{B}_x$, $V \in \mathcal{B}_y$ and $r < \phi(y)$ in R such that for any $z \in U \cap V$, $\mathcal{B}_z \not\subseteq \mathcal{B}_r$.*

Suppose $(X, \{\mathcal{B}_r\}_{r \in R})$ is a hierarchical space and $x <_\tau y$, then $y \not\leq_\tau x$ implies that we can find $U \in \mathcal{B}_y$, $V \in \mathcal{B}_x$ and $r < \phi(x)$ in R such that for any $z \in U \cap V$, $\mathcal{B}_z \not\subseteq \mathcal{B}_r$. Since $y \in U \cap V$, then $\phi(y) \leq r$ and we have that $\phi(y) < \phi(x)$. Obviously hierarchical spaces are T_0-spaces, and it is clear that any Hausdorff space, (X, π), is trivially a hierarchical space $(X, \{\pi\})$. In the next section we see that partial metric spaces are naturally hierarchical spaces.

3.3.3 Partial metric spaces as hierarchical spaces

Suppose (X, d) is a partial metric space, then for each $r \in \Re$, we define

$$\mathcal{B}_r = \{U \in \tau_{[d]} \mid \exists x \in U \text{ with } \phi_d(x) \geq r\}.$$

It is clear that every non-empty open set is in some $\mathcal{B}_r$. If $r \le s$ and $U \in \mathcal{B}_s$, then we can find $x \in U$ such that $\phi_d(x) \ge s \ge r$ and so $U \in \mathcal{B}_r$. For each $x \in X$, we let $\ell_x, u_x \in \Re$ be such that $\ell_x \le \phi_d(x) < u_x$. It is clear that $\mathcal{B}_x \subseteq \mathcal{B}_{\ell_x}$, and if we let $\varepsilon = u_x - \phi_d(x) > 0$, then $y \in B_\varepsilon(x)$ implies that

$$\phi_d(y) \le d(x,y) < \phi_d(x) + \varepsilon = u_x,$$

and hence $B_\varepsilon(x) \notin \mathcal{B}_{u_x}$. So we have a natural hierarchy of open sets, $\{\mathcal{B}_r\}_{r\in\Re}$, indexed by $(\Re, \le)$. Now suppose that $x \not\le_d y$, then $d(x,y) - \phi_d(x) > 0$, and we can find $r \in \Re$ such that

$$\phi_d(y) > r > \phi_d(y) + \phi_d(x) - d(x,y).$$

We let

$$\varepsilon = d(x,y) + r - \phi_d(x) - \phi_d(y) > 0,$$

and take $U = B_{\varepsilon/2}(x)$ and $V = B_{\varepsilon/2}(y)$. For any $z \in U \cap V$, we have

$$\begin{aligned} d(x,y) &\le d(x,z) + d(z,y) - \phi_d(z) \\ &< \phi_d(x) + \varepsilon + \phi_d(y) - \phi_d(z) \\ &= d(x,y) + r - \phi_d(z), \end{aligned}$$

and so $\phi_d(z) < r$ and hence $\mathcal{B}_z \not\subseteq \mathcal{B}_r$. It follows that $(X, \{\mathcal{B}_r\}_{r\in\Re})$ is a hierarchical space.

We now show that this hierarchical space has the same weight function as the partial metric space. If we fix $x \in X$, then we immediately see that $\mathcal{B}_x \subseteq \mathcal{B}_{\phi_d(x)}$. Suppose $\mathcal{B}_x \subseteq \mathcal{B}_r$, for some $r \in R$. Then, for any $\varepsilon > 0$, we have $B_\varepsilon(x) \in \mathcal{B}_r$, which implies that, for any $y \in B_\varepsilon(x)$

$$r \le \phi_d(y) \le d(x,y) < \phi_d(x) + \varepsilon,$$

and hence $r \le \phi_d(x)$. So we see that $\phi_d(x) = \phi(x)$, for all $x \in X$, and our induced weight functions agree. We have therefore proved the following result.

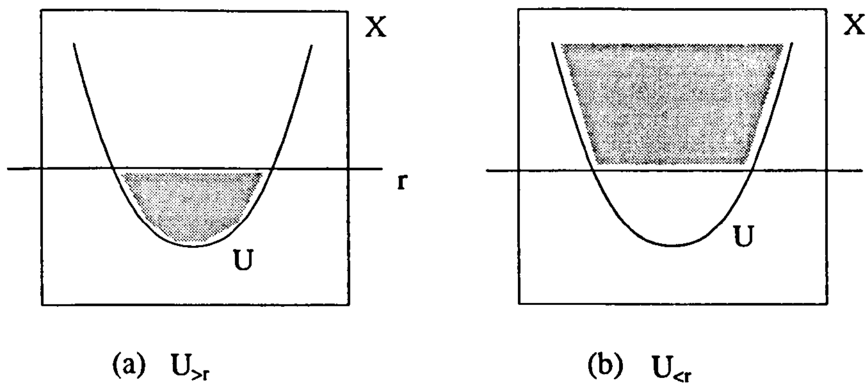


Figure 3.6: Upper and lower components of an open set

Theorem 3.3.3 *Suppose (X, d) is a partial metric space, then $(X, \{\mathcal{B}_r\}_{r\in\Re})$ is a hierarchical space with the same weight function.*

3.3.4 Hierarchical spaces as partially ordered spaces

To help place the hierarchical spaces in context, we now show that every hierarchical space is naturally a partially ordered space. Suppose $(X, \{\mathcal{B}_r\}_{r\in R})$ is a hierarchical space with $\mathcal{B} = \bigcup_{r\in R} \mathcal{B}_r$. For $U \in \mathcal{B}$ and $r \in R$, we define

$$U_{>r} = \{x \in U \mid r < \phi(x)\} \quad \text{and} \quad U_{<r} = \{x \in U \mid \phi(x) < r\},$$

and illustrate the situation in figure 3.6.

Lemma 3.3.4 *Suppose $(X, \{\mathcal{B}_r\}_{r\in R})$ is a hierarchical space with $\mathcal{B} = \bigcup_{r\in R} \mathcal{B}_r$, then the collection $\{U_{<r} \mid U \in \mathcal{B}, r \in R\}$ is a basis for τ.*

Proof. Suppose $x \in X$ and $U \in \mathcal{B}_x$, then $\phi(x) < u_x$ and $x \in U_{<u_x} \subseteq U$. Since $\mathcal{B}$ is a basis for τ, then we are left to show that the $U_{<r}$ are open. Suppose $x \in U_{<r}$ and $s \in R$ is such that $\phi(x) < s < r$, then we can find $V \in \mathcal{B}_x$ such that $V \notin \mathcal{B}_s$. For any $y \in V$ it follows that $\mathcal{B}_y \not\subseteq \mathcal{B}_s$ and hence $\phi(y) \leq s < r$, so we see that $x \in U \cap V \subseteq U_{<r}$.

QED

It is clear that $\{U_{>r} \mid U \in \mathcal{B}, r \in R\}$ and $\{U_{<r} \mid U \in \tau, r \in R\}$ together form the basis for a topology, π, which will be our induced Hausdorff topology.

Lemma 3.3.5 *Suppose $(X, \{\mathcal{B}_r\}_{r \in R})$ is a hierarchical space, then $(X, \pi, \leq_\tau)$ is a partially ordered space.*

Proof. If $x \not\leq_\tau y$, then we can find $U \in \mathcal{B}_x$, $V \in \mathcal{B}_y$ and $r < \phi(y)$ such that, for any $z \in U \cap V$, $\mathcal{B}_z \not\subseteq \mathcal{B}_r$. It is clear that $(x, y) \in U \times V_{>r}$, and if $a \leq_\tau b$ with $(a, b) \in U \times V_{>r}$ then $b \in U \cap V$ and $r < \phi(b)$ which implies that $\mathcal{B}_b \subseteq \mathcal{B}_r$, which is a contradiction.

QED

We assume that a partial metric space, (X, d), induces the natural hierarchy of open sets, $\{\mathcal{B}_r\}_{r \in \Re}$, from the last section. We show that the induced Hausdorff topology, π, agrees with the induced metric topology, $\tau_{[d^*]}$.

Lemma 3.3.6 *Suppose (X, d) is a partial metric space, inducing the hierarchical space $(X, \{\mathcal{B}_r\}_{r \in \Re})$, then $\pi = \tau_{[d^*]}$.*

Proof. Suppose $\varepsilon > 0$, $x \in X$ and we let $U = B_\varepsilon(x)$ and $r = \phi_d(x) - \varepsilon$. Then $y \in B_\varepsilon(x; d^*)$ implies that

$$d(x, y) - \phi_d(y) \leq d^*(x, y) < \varepsilon,$$

and hence $r < \phi_d(y)$. So we have $y \in U_{>r}$. Now suppose that $y \in U_{>r}$, then

$$d(x, y) < d(x, x) + \varepsilon = r + 2\varepsilon < \phi_d(y) + 2\varepsilon,$$

and so $d^*(x, y) < 2\varepsilon$. It follows that

$$B_\varepsilon(x; d^*) \subseteq U_{>r} \subseteq B_{2\varepsilon}(x; d^*),$$

from which the result follows.

QED

3.3.5 A framework for boundedly observable properties

We return to the domain of streams, S^∞, and observe that $x \in S^\infty$ has length $|x| = k$ if, and only if, x satisfies some depth k property, but no depth $k+1$ properties. It follows that in this example, the ideas of Smyth [Smy92], on boundedly observable properties, and Wadge [Wad81], on the size of elements in a domain, are closely related. We investigate this to see how the hierarchical spaces can be a common framework for both approaches.

Matthews gives a pmetric on S^∞ [Mat94], by defining

$$d(x, y) = \inf\{2^{-n} \mid x[n] = y[n],\, n \leq |x|, |y|\},$$

which is small if, and only if, we require a "deep" property to distinguish x and y. This is a symmetrised version of the motivation given for the quasi-metric in section 2.2.5, and $\tau_{[d]}$ is similarly the Scott topology on S^∞. Furthermore, we also capture the notion of size since $\phi_d(x) = 2^{-|x|}$. We can immediately deduce that S^∞ with the Scott topology is a hierarchical space.

For the boundedly observable properties however, an alternative hierarchical space is appropriate. Suppose we take $(\mathbf{N}^*, \geq)$ as our index set, where $\mathbf{N}^* = \mathbf{N} \cup \{\infty\}$. We recall that for any $x \in S^\infty$ with $|x| < \infty$, then $\uparrow x$ is open in the Scott topology. We define, for each $r \in \mathbf{N}$,

$$\mathcal{B}_r = \{\uparrow x \mid |x| \leq r\}.$$

Furthermore, we define $\mathcal{B}_\infty = \{\uparrow x \mid |x| < \infty\}$, so that the collection of $\mathcal{B}_r$, $r \in \mathbf{N}^*$, is a basis for the Scott topology on S^∞.

We show that $\{\mathcal{B}_r\}_{r \in \mathbf{N}^*}$ is a hierarchy of open sets. It is clear that if $m \geq n$ in $\Re$ and $\uparrow x \in \mathcal{B}_n$, then $|x| \leq n \leq m$ and hence $\uparrow x \in \mathcal{B}_m$, so that $\mathcal{B}_m \subseteq \mathcal{B}_n$, and for any $n \in \mathbf{N}$, $\mathcal{B}_n \subseteq \mathcal{B}_\infty$. For $x \in S^\infty$, we let $\ell_x = |x|$ and $u_x = |x| - 1$ or $u_x < \infty$ if $|x| = \infty$. If $y \in S^\infty$ with $|y| < \infty$ and $x \in \uparrow y$, then $|y| \leq |x| = \ell_x$, so that $\uparrow y \in \mathcal{B}_{\ell_x}$ and $\mathcal{B}_x \subseteq \mathcal{B}_{\ell_x}$. Also, since $\uparrow x \notin \mathcal{B}_{u_x}$, then we have $\mathcal{B}_x \not\subseteq \mathcal{B}_{u_x}$. To see that $(S^\infty, \{\mathcal{B}_r\}_{r\in\mathbf{N}^*})$ is a hierarchical space, we suppose that $x \not\sqsubseteq y$ in S^∞, and let $r = |y| + 1/2$. We either have $\uparrow x \cap \uparrow y = \emptyset$, in which case the result is trivial, or $y \sqsubseteq x$ but with $x \neq y$ so that $|y| < \infty$, in which case $|x| \geq |y| + 1 > r$, and $\uparrow x \notin \mathcal{B}_r$. Finally, it is clear that $\mathcal{B}_x \subseteq \mathcal{B}_{|x|}$, and if $\mathcal{B}_x \subseteq \mathcal{B}_r$ then $|x| = \infty$ implies that $r = \infty$, otherwise $|x| \leq r$, and we see that $\phi(x) = |x|$.

We are now in a position to capture the intuition from section 2.2.5 on finitely observable and boundedly observable properties in terms of this hierarchy of open sets. We say that $U \subseteq S^\infty$ is *finitely observable* if

$$\forall x \in U \quad \exists n \in \mathbf{N} \quad \exists V \in \mathcal{B}_n \quad \text{such that} \quad x \in V \subseteq U,$$

and that $U \subseteq S^\infty$ is *boundedly observable* if

$$\exists n \in \mathbf{N} \quad \forall x \in U \quad \exists V \in \mathcal{B}_n \quad \text{such that} \quad x \in V \subseteq U.$$

In this case we also say that U is a *depth n property*. What is important of course, is the finitary nature of the $\mathcal{B}_n$. It is to be hoped that this will be useful in seeing how the material from section 2.2.5 can be generalised.

3.4 Concluding remarks

In section 3.1, we gave the last of our background material and motivated the essential characteristics of a partial metric. Although most of the material in section 3.2 was from [Mat94], the presentation was original, and we established the axioms in context. Our original material in section 3.3, on the hierarchical spaces, completes the foundations and we are now in a position to build the general theory.

Chapter 4

Partial Metric Space Theory

The theory of partial metric spaces begins with the work of Matthews, who gives some initial results in [Mat94], to which Heckmann [Hec98] contributes some further results. We significantly extend the existing work, by considering the partial metric spaces as T_0-topological spaces, and seeing how additional conditions on the pmetric can lead to stronger topological properties (section 4.1). We extend the metric notions of isometries and completions to partial metric spaces (section 4.2), and introduce a particularly useful class of partial metric spaces derived from an auxiliary weight function over a more structured poset (section 4.3).

4.1 Topological properties

We investigate the partial metric spaces within the context of the T_0-topological properties from sections 2.2.1 and 2.2.2. We will find natural conditions on the pmetric for the induced topology to be order-consistent or sober, and consider completeness, compactness and coherence. We then investigate connections with the Scott topology, and develop a notion of a quantitative domain. Unless stated otherwise, the material in this section is original.

4.1.1 Order-consistency

From section 3.2.3 we recall that a partial metric space, (X, d), induces a natural quasi-metric on X and has an associated metric space (X, d^*). As with the quasi-metrics (sections 2.2.3 and 2.2.4), we can lift a notion of Cauchy sequences, and hence completeness, from (X, d^*). In terms of the pmetric, we see that a sequence $\{x_n\}$ in X is Cauchy [Mat94] if, and only if,

$$\lim_{n,m\to\infty} d(x_n, x_m) \quad \text{exists.}$$

Any ω-chain $\{x_n\}$ in X which has $\{\phi_d(x_n) \mid n \geq 1\}$ bounded below in $\Re$, is therefore a Cauchy sequence. For example, if $\{x_n\}$ has a supremum in X, then $\{x_n\}$ is a Cauchy sequence.

Suppose $\{x_n\}$ is a sequence in X and $a \in X$. Convergence with respect to both $\tau_{[d]}$ and the metric d^* are naturally captured by the pmetric [Mat94];

$$x_n \to a \text{ in } (X, \tau_{[d]}) \iff \lim_{n\to\infty} d(x_n, a) = d(a, a),$$

$$\text{and} \quad x_n \to a \text{ in } (X, d^*) \iff \lim_{n\to\infty} d(x_n, a) = \lim_{n\to\infty} d(x_n, x_n) = d(a, a).$$

We call the latter *proper convergence* in [ONe97]. A partial metric space is therefore complete if, and only if, every Cauchy sequence has a proper limit. Matthews states, but does not prove, the following result in [Mat94].

Lemma 4.1.1 *Suppose (X, d) is a partial metric space, then an ω-chain $\{x_n\}$ in X, with proper limit $a \in X$, has $a = \bigsqcup^{\uparrow} x_n$.*

Proof. Suppose $n \geq 1$, then for any $m \geq n$ we have

$$\begin{aligned} d(a, x_n) &\leq d(a, x_m) + d(x_m, x_n) - d(x_m, x_m) \\ &= d(a, x_m) + d(x_n, x_n) - d(x_m, x_m), \end{aligned}$$

and this has limit $d(x_n, x_n)$ as $m \to \infty$, so that $x_n \leq_d a$. If $b \in X$ is such that $x_n \leq_d b$ for all $n \geq 1$, then

$$d(a, b) \leq d(a, x_n) + d(x_n, b) - d(x_n, x_n) = d(x_n, x_n),$$

which has limit $d(a,a)$ as $n \to \infty$, so that $a \leq_d b$.

QED

The following results are all original, although the first was found independently by Heckmann [Hec98] for the (positive) weak partial metrics.

Lemma 4.1.2 *Suppose (X,d) is a partial metric space and $A \subseteq X$ is directed, then there exists an ω-chain in A with the same upper bounds.*

Proof. Suppose first that $S = \{\phi_d(a) \mid a \in \Re\}$ is bounded below in $\Re$, then for each $n \geq 1$, we can find $a_n \in A$ such that $d(a_n, a_n) < \inf S + 1/n$. We inductively construct our ω-chain $\{x_n\}$ so that $x_1 = a_1$ and $x_{n+1} \in A$ is above $x_n, a_{n+1} \in A$. Now suppose that $y \in X$ is an upper bound of $\{x_n\}$ and let $a \in A$. For $n \geq 1$ we let $y_n \in A$ be above $x_n, a \in A$ so that $a_n \leq_d x_n \leq_d y, y_n$ and $d(y_n, y_n) \geq \inf S$. From axioms P1 and P4 we have

$$\begin{aligned} d(a,a) \leq d(a,y) &\leq d(a,y_n) + d(y_n,a_n) + d(a_n,y) - d(y_n,y_n) - d(a_n,a_n) \\ &= d(a,a) + d(a_n,a_n) - d(y_n,y_n) \\ &< d(a,a) + 1/n. \end{aligned}$$

Since this holds for all $n \geq 1$, then $a \leq_d y$ as required.
If S is not bounded below, then we can find $a_n \in A$ such that $d(a_n, a_n) < -n$, and inductively construct an ω-chain $\{x_n\}$ as before. Since S is not bounded below then A has no upper bounds and the result is trivial.

QED

We will say that the pmetric topology is order-consistent if it is order-consistent with respect to its specialisation order. We can characterise order-consistency in terms of the pmetric.

Lemma 4.1.3 *Suppose (X,d) is a partial metric space, then the pmetric topology is order-consistent if, and only if, every ω-chain with a supremum has a proper limit.*

Proof. We first suppose that the pmetric topology is order-consistent and $\{x_n\}$ is an ω-chain in X with $a = \bigsqcup^{\uparrow} x_n \in X$. Clearly $\{x_n\}$ converges to a, and

$$d(a,a) \le d(x_n, x_n) \le d(x_n, a).$$

Since the right-hand side converges to $d(a,a)$, then $\{x_n\}$ converges to a properly. Conversely, by lemma 4.1.2, we can consider ω-chains in place of directed sets, when their suprema exist, and the proper limit of an ω-chain is its supremum.

QED

Such a result cannot hold for quasi-metrics since, for example, for the discrete quasi-metric on a poset, an ω-chain converges to its supremum if, and only if, it attains its supremum. We will say that $\phi_d : X \to \Re$ is *continuous* if, for every ω-chain $\{x_n\}$ in X with supremum $a \in X$, then $\phi_d(a) = \inf\{\phi_d(x_n) \mid n \ge 1\}$.

Corollary 4.1.4 *Suppose (X, d) is a partial metric space, then the pmetric topology is order-consistent if, and only if, ϕ_d is continuous.*

Since ω-chains with supremum are Cauchy sequences, then order-consistency can be thought of as a weaker notion of completeness for a partial metric space. We will say that a partial metric space (X, d) is *bounded* if $\{d(x,y) \mid x, y \in X\}$ is bounded below in $\Re$.

Corollary 4.1.5 *If (X, d) is a complete partial metric space, then $\tau_{[d]}$ is order-consistent. Furthermore, if (X, d) is also bounded then $(X, \le_d)$ is a dcpo.*

We can also use lemma 4.1.3 to further justify our use of negative, or more precisely unbounded, distances. We consider $\Re$, but with the partial order, $\ge$, rather than, $\le$, as this will simplify some further work in section 4.3

Corollary 4.1.6 *Any partial metric d on $\Re$ for which $\le_d = \ge$ and d^* is the usual metric, must be unbounded.*

It follows that we can only pmetrise the partially ordered space $(\Re, \geq)$, with the usual topology, if we admit negative distances. Such a pmetric is given by $d(x, y) = \max\{x, y\}$, and we will have much more to say about such partial metrics in section 4.3.

4.1.2 Sobriety and completeness

Sobriety for a T_0-space (section 2.2.2) is a notion of completeness which is quite difficult to work with, since we must use either completely prime filters of open sets or irreducible closed sets. Sünderhauf presents a more intuitive approach in [Sün95] by defining the notion of an *observative net* which, together with an appropriate notion of convergence, characterises sobriety. Before we became aware of this work, we found our own intuitive understanding of sobriety as a notion of completeness for partial metric spaces, which we now present.

Sober spaces are, with respect to the specialisation order, dcpos with an order-consistent topology. We begin by strengthening lemma 4.1.3 so that $(X, \leq_d)$ is also a dcpo.

Lemma 4.1.7 *Suppose (X, d) is a partial metric space such that every ω-chain has a proper limit, then $(X, \leq_d)$ is a dcpo and the pmetric topology is order-consistent.*

Proof. Any directed set in X has an ω-chain with the same upper bounds, whose proper limit is the supremum of the directed set. Order-consistency is immediate from lemma 4.1.3.

QED

Since arbitrary ω-chains need not be Cauchy sequences, then we have started to move away from our usual notion of completeness. The following generalised notion of an ω-chain is an instance of Sünderhauf's observative nets, but notice that, as with metric spaces, partial metric spaces allow us to work with sequences rather than nets.

Definition 4.1.8 *Suppose (X, d) is a partial metric space, then a sequence $\{x_n\}$ is* **self-convergent** *if, for each $k \geq 1$, $\{x_n\}$ converges to x_k.*

The following result is our own, but we now realise that it could be deduced from Sünderhauf's work.

Theorem 4.1.9 *Suppose (X, d) is a partial metric space, then the pmetric topology is sober if, and only if, every self-convergent sequence has a proper limit.*

Proof. We first suppose that the pmetric topology is sober, and let $\{x_n\}$ be a self-convergent sequence. We let A be the non-empty collection of limit points of $\{x_n\}$. If $\{y_n\}$ is a sequence in A converging to $a \in X$, then it follows from

$$d(x_n, a) \leq d(x_n, x_m) + d(x_m, y_n) + d(y_n, a) - d(x_m, x_m) - d(y_n, y_n),$$

for $m \geq n$, that $\{x_n\}$ converges to a. So $a \in A$, and A is closed. To see that A is an irreducible closed set, suppose B and C are proper closed subsets of A with $y \in B \setminus C$ and $z \in C \setminus B$. Since $\{x_n\}$ converges to both y and z, then $\{x_n\}$ is eventually in $X \setminus B$ and $X \setminus C$, but this is a contradiction since each $x_n \in A$. We let $a \in X$ be such that A is the closure of a, so that $A = \downarrow a$. It follows that $\{x_n\}$ converges to a, each $x_k \leq_d a$ and a is the proper limit of $\{x_n\}$.
Now suppose that $A \subseteq X$ is an irreducible closed set. If U, V are open sets such that $U \cap A$ and $V \cap A$ are non-empty, then $U \cap V \cap A$ is non-empty since otherwise $A \setminus U$ and $A \setminus V$ are proper closed subsets of A with union A. Suppose $a, b \in A$, then a simple induction allows us to find, for each $n \geq 1$,

$$x_n \in A \cap B_{1/2^n}(a) \cap B_{1/2^n}(b) \cap B_{1/2^n}(x_1) \cap \cdots \cap B_{1/2^n}(x_{n-1}).$$

Clearly $\{x_n\}$ is self-convergent, and converges to a and b. We let $c \in X$ be the proper limit of $\{x_n\}$, so that $a, b \leq_d c$. Since A is closed, then $c \in A$, and A is a directed set with $\bigsqcup^{\uparrow} A \in X$. Order-consistency implies that $\bigsqcup^{\uparrow} A \in A$, and the closure of $\bigsqcup^{\uparrow} A$ is $\downarrow \bigsqcup^{\uparrow} A$ which is A.

QED

We will say that a self-convergent sequence $\{x_n\}$ is *bounded* if $\{\phi_d(x_n) \mid n \geq 1\}$ is bounded below in $\Re$. In the next lemma we see that these are precisely the Cauchy self-convergent sequences.

Lemma 4.1.10 *Suppose (X, d) is a partial metric space, then a self-convergent sequence is bounded if, and only if, it is Cauchy.*

Proof. Suppose first that $\{x_n\}$ is a bounded self-convergent sequence, and let

$$\ell = \inf\{\phi_d(x_n) \mid n \geq 1\} \in \Re.$$

If we fix $\varepsilon > 0$ then we can find $k \geq 1$ such that $\ell \leq \phi_d(x_k) < \ell + \varepsilon/3$. Since $\{x_n\}$ is self-convergent, then we can find $N \geq 1$ such that for all $n \geq N$,

$$d(x_n, x_k) < \phi_d(x_k) + \varepsilon/3.$$

So, for all $n, m \geq N$, we have

$$\ell \leq d(x_n, x_m) \quad \leq \quad d(x_n, x_k) + d(x_k, x_m) - \phi_d(x_k) < \phi_d(x_k) + 2\varepsilon/3 < \ell + \varepsilon.$$

So we see that $\lim_{n,m\to\infty} d(x_n, x_m) = \ell$ and $\{x_n\}$ is Cauchy as required.
Now suppose that $\{x_n\}$ is a Cauchy self-convergent sequence, and let

$$\ell = \lim_{n,m\to\infty} d(x_n, x_m) \in \Re.$$

It is clear that $\lim_{n\to\infty} \phi_d(x_n) = \ell$, so we are left to show that each $\phi_d(x_k) \geq \ell$ for $\{x_n\}$ to be bounded. If we suppose that $\phi_d(x_k) < \ell$ for some $k \geq 1$, then we can find $\varepsilon > 0$ such that $\phi_d(x_k) < \ell - \varepsilon$. We can find $N \geq 1$ such that, for all $n \geq N$,

$$|\phi_d(x_n) - \ell| < \varepsilon/2 \quad \text{and} \quad \phi_d(x_n) - \phi_d(x_k) \leq d(x_n, x_k) - \phi_d(x_k) < \varepsilon/2.$$

We can immediately deduce that $\ell - \phi_d(x_k) < \varepsilon$, which is a contradiction.

QED

We are now in a position to improve corollary 4.1.5.

Corollary 4.1.11 *Suppose (X, d) is a complete bounded partial metric space, then the pmetric topology is sober.*

We conclude this section by comparing the completeness of sober partial metric spaces, with the complete partial metric spaces in general. We first introduce the notion of a self-convergent sequence approximating a Cauchy sequence.

Definition 4.1.12 *Suppose (X, d) is a partial metric space, then a Cauchy sequence $\{x_n\}$ is* **approximated** *by a self-convergent sequence $\{y_n\}$, if*

$$\lim_{n\to\infty} d(x_n, x_n) = \lim_{n\to\infty} d(x_n, y_n) = \lim_{n\to\infty} d(y_n, y_n).$$

Lemma 4.1.13 *Suppose (X, d) is a partial metric space, then the pmetric topology is sober and every Cauchy sequence is approximated by a self-convergent sequence if, and only if, (X, d) is complete and every self-convergent sequence is a Cauchy sequence.*

Proof. It is clear that if the pmetric topology is sober, then every self-convergent sequence is Cauchy, and if every Cauchy sequence is approximated by a self-convergent sequence, which has a proper limit, then this is the proper limit of the Cauchy sequence, and so (X, d) is complete. Conversely, if every self-convergent sequence is a Cauchy sequence, then completeness implies that the pmetric topology is sober, and for every Cauchy sequence $\{x_n\}$ with proper limit $a \in X$, then $\{a\}$ is a self-convergent sequence approximating $\{x_n\}$.

QED

4.1.3 Compactness and the patch topology

We say that a partial metric space (X, d) is *compact* if the induced metric space (X, d^*) is compact. From section 3.2.3 we know that a pmetric induces a quasi-metric on X with the same topology. From section 2.2.6 we see that $(X, \tau_{[d^\bullet]}, \leq_d)$

is a partially ordered space, and hence a compact ordered space. It is clear that a compact partial metric space is complete, and we will see from the next lemma that it is also sober, so that both notions of completeness from the last section are satisfied by compact partial metric spaces.

Lemma 4.1.14 *Suppose (X, d) is a compact partial metric space, then $(X, \tau_{[d]})$ is a coherent space.*

Proof. By theorem 2.2.2, and the above comments, we need only show that $\tau_{[d]} = {\tau_{[d^*]}}^{\uparrow}$. It is immediate that $\tau_{[d]} \subseteq {\tau_{[d^*]}}^{\uparrow}$, so we suppose, for a contradiction, that $U \in {\tau_{[d^*]}}^{\uparrow} \setminus \tau_{[d]}$. Then there exists $a \in U$ such that, for all $n \geq 1$, we can find some $x_n \in B_{1/2^n}(a) \setminus U$. Compactness implies that the sequence $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ with proper limit $b \in X$, and it follows from

$$d(a, b) \leq d(a, x_{n_k}) + d(x_{n_k}, b) - d(x_{n_k}, x_{n_k}),$$

that $a \leq_d b$, and so $b \in U$. But $\{x_{n_k}\}$ converges properly to b, and so is eventually in U, which is our contradiction.

QED

To see that the converse need not hold, consider the domain of streams, S^∞, with S an infinite set, and the pmetric, d, from section 3.3.5. We know that $\tau_{[d]}$ is the Scott topology, and it is clear that the patch topology $\pi = \tau_{[d]} \vee {\tau_{[d]}}^k$ is compact. It follows that $(S^\infty, \tau_{[d]})$ is a coherent space. However, (X, d) is not a compact partial metric space. In particular we see that $\tau_{[d^*]}$ is not the patch topology of $\tau_{[d]}$. We know from section 3.2.4, that $\tau_{[d^*]} = \tau_{[d]} \vee \tau_{[d^{-1}]}$, so we are lead to compare the cocompact topology, ${\tau_{[d]}}^k$, with the conjugate pmetric topology, $\tau_{[d^{-1}]}$, for partial metric spaces in general. We will then see that for compact partial metric spaces, $\tau_{[d^*]}$ is the patch topology of $\tau_{[d]}$.

Lemma 4.1.15 *Suppose (X, d) is a partial metric space, then ${\tau_{[d]}}^k \subseteq \tau_{[d^{-1}]}$.*

Proof. Suppose $V \subseteq X$ is a compact upper set in $(X, \tau_{[d]})$. Once we show that $X \setminus V \in \tau_{[d^{-1}]}$ then we are done. Suppose $y \in X \setminus V$, then for all $x \in V$ we must have $x \not\leq_d y$, and we can define

$$\varepsilon_x = [d(x,y) - d(x,x)]/2 > 0.$$

Since $\{B_{\varepsilon_x}(x) \mid x \in V\}$ is an open cover of V, then there exists a finite subcover $\{B_{\varepsilon_{x_n}}(x_n) \mid 1 \leq n \leq N, x_n \in V\}$. We define $\varepsilon = \min\{\varepsilon_{x_n} \mid 1 \leq n \leq N\} > 0$, and show that $B_\varepsilon(y; d^{-1}) \subseteq X \setminus V$. Suppose $z \in V$, then $z \in B_{\varepsilon_{x_n}}(x_n)$, for some $1 \leq n \leq N$, and we have

$$\begin{aligned}
\varepsilon \leq \varepsilon_{x_n} &= d(x_n, y) - d(x_n, x_n) - \varepsilon_{x_n} \\
&\leq d(x_n, z) + d(z, y) - d(z, z) - d(x_n, x_n) - \varepsilon_{x_n} \\
&< d(z, y) - d(z, z) \\
&= d^{-1}(z, y) - d^{-1}(y, y).
\end{aligned}$$

So $z \notin B_\varepsilon(y; d^{-1})$ which implies that $B_\varepsilon(y; d^{-1}) \subseteq X \setminus V$.

QED

Theorem 4.1.16 *Suppose (X, d) is a partial metric space, then (X, d) is compact if, and only if, $(X, \tau_{[d]})$ is a coherent space, and $\tau_{[d^*]}$ is the patch topology of $\tau_{[d]}$.*

Proof. We need only prove that when (X, d) is a compact partial metric space, then ${\tau_{[d]}}^k = \tau_{[d^{-1}]}$. Suppose $U \in \tau_{[d^{-1}]}$, then $U \in \tau_{[d^*]}$ and so $X \setminus U$ is $\tau_{[d^*]}$-compact. Any $\tau_{[d]}$-open cover of $X \setminus U$ is a $\tau_{[d^*]}$-open cover and so there exists a finite subcover. So $X \setminus U$ is $\tau_{[d]}$-compact, and since $X \setminus U$ is clearly an upper set with respect to $\leq_d$, then $U \in {\tau_{[d]}}^k$ and so ${\tau_{[d]}}^k = \tau_{[d^{-1}]}$.

QED

4.1.4 The Scott topology

We now investigate connections between the pmetric topology of a partial metric space, (X, d), and the Scott topology on $(X, \leq_d)$. We begin by returning to domain theory, and giving some thoughts on what we understand by the term *quantitative domain*, as opposed to the quantitative domain theories of sections 2.1.4 and 2.2.3. We have seen that the Scott topology naturally captures the basic (qualitative) notions of domain theory, namely limits and approximation. It therefore seems reasonable to define a *quantitative domain* to be a continuous domain together with some additional structure, such as a quasi-metric or partial metric, that captures the Scott topology, and adds some quantitative information to the domain. This is at odds with the definition in [FSW98], since the discrete quasi-metric on a continuous domain need not induce the Scott topology. We will further develop our ideas in section 4.3, but for now return to the more general setting, and seek a condition for the pmetric and Scott topologies to agree.

Definition 4.1.17 *A sequence $\{x_n\}$* **surpasses** *a sequence $\{y_n\}$ in X if, for every $n \geq 1$, there exists an $m \geq 1$ such that $y_n \leq_d x_m$.*

Definition 4.1.18 *A partial metric space has* **convergence in order** *if every sequence $\{x_n\}$ converging to $a \in X$, surpasses an ω-chain $\{y_n\}$ with supremum a.*

Lemma 4.1.19 *Suppose (X, d) is a partial metric space with order-consistent pmetric topology, then it has convergence in order if, and only if, $\tau_{[d]}$ is the Scott topology.*

Proof. Since the pmetric topology is order-consistent, then we know that $\tau_{[d]} \subseteq \sigma$. Suppose $\tau_{[d]} = \sigma$, and let $\{x_n\}$ be a sequence converging to $a \in X$. We let $A = \bigcup_{n=1}^{\infty} \downarrow x_n$, and $\bar{A}$ be the Scott-closure of A. Then each $x_n \in \bar{A}$, so we must have $a \in \bar{A}$. If $a \in A$, then $a \leq_d x_n$ for some $n \geq 1$, and we take $\{a\}$ as our ω-chain. Otherwise, there must exist some directed set, and hence some ω-chain $\{y_n\}$, in A with supremum a, and $\{y_n\} \subseteq A$ is precisely the condition for $\{x_n\}$ to surpass $\{y_n\}$.

Conversely, we suppose, for a contradiction, that $U \in \sigma \setminus \tau_{[d]}$. So there exists some $a \in U$ such that, for all $n \geq 1$, we can find $x_n \in B_{1/2^n}(a) \setminus U$, and $\{x_n\}$ converges to a. We let $\{x_n\}$ surpass the ω-chain $\{y_n\}$ with supremum a, then some $y_n \in U$ and hence some $x_m \in U$, which is our contradiction.

QED

A problem with the above result is that convergence in order is not an easy property to achieve, since it requires the existence of points that even complete partial metrics need not have. In the section 4.3 we will meet a special class of partial metric spaces whose additional structure ensures that completeness does imply convergence in order.

4.2 Isometries and completions

Our next step in developing the general theory, is to extend the basic metric notions of isometry and completion to the partial metric spaces. We can then apply our notion of completion to the space of formal balls to see that, in this instance, ideal completion and partial metric completion agree, and we induce the metric completion on the metric space. The material in this section is original.

4.2.1 Isometries

Given two partial metric spaces (X, d) and (X', d'), Heckmann [Hec98] takes what appears to be the natural definition for an isometry $f : X \to X'$, and requires that $d'(fx, fy) = d(x, y)$. We argue that this is a little too simplistic. Consider the subspaces $[0, 1]$ and $[1, 2]$ of $\Re$, with the pmetric $d(x, y) = max\{x, y\}$ from section 4.1.1. These should intuitively be isometric, with the map $f : [0, 1] \to [1, 2]$, given by $f(x) = x + 1$, being an isometry. However, for any $x, y \in [0, 1]$, we have

$$d(fx, fy) = \max\{x + 1, y + 1\} = d(x, y) + 1.$$

The problem is that if we uniformly adjust the weight function for a partial metric space, then the underlying metric structure remains unaffected, but Heckmann's definition does not reflect this. We therefore define an *isometry* to be a bijection $f : X \to X'$, for which there exists $k \in \Re$, such that

$$d'(fx, fy) = d(x, y) + k, \qquad \forall x, y \in X. \tag{4.1}$$

We say that (X, d) and (X', d') are *isometric*, and use the term *isometry into* for any map which satisfies (4.1) but need not be a bijection. With this definition, any bounded partial metric space is isometric to a positive partial metric space, and this reinforces our interpretation of a complete element in section 3.2.4.

4.2.2 Partial metric completions

We define the *completion* of a partial metric space (X, d) to be a complete partial metric space $(\overline{X}, \overline{d})$ and a map $i : X \to \overline{X}$ such that i is an isometry into $\overline{X}$ and $i(X)$ is dense in $(\overline{X}, \overline{d}^*)$. Clearly, the metric space $(\overline{X}, \overline{d}^*)$, together with i, is then a completion of the metric space (X, d^*). We show that every partial metric space has a unique completion (up to isometry), by generalising the metric case (see [Sut75] for example).

We begin with the following inequality, which is easily derived from the P4 axiom;

$$|d(x, y) - d(z, w)| \leq d^*(x, z) + d^*(y, w), \qquad \forall x, y, z, w \in X. \tag{4.2}$$

We let $\overline{X}$ be the set of equivalence classes of Cauchy sequences, where $\{x_n\} \sim \{y_n\}$ if, and only if, $\lim_{n\to\infty} d^*(x_n, y_n) = 0$. For any $\overline{x}, \overline{y} \in \overline{X}$, represented by the Cauchy sequences $\{x_n\}$ and $\{y_n\}$ respectively, we define

$$\overline{d}(\overline{x}, \overline{y}) = \lim_{n\to\infty} d(x_n, y_n).$$

To see that this exists, for any $n, m \geq 1$, we use (4.2) to get

$$0 \leq |d(x_n, y_n) - d(x_m, y_m)| \leq d^*(x_n, x_m) + d^*(y_n, y_m).$$

Since the right-hand side tends to 0 as $n, m \to \infty$, then $\{d(x_n, y_n)\}$ is a Cauchy sequence in $\Re$, and $\lim_{n\to\infty} d(x_n, y_n)$ exists. To see that $\overline{d}$ is well-defined, suppose the Cauchy sequence $\{x'_n\}$ also represents $\overline{x}$, and we again use (4.2) to get

$$0 \leq |d(x'_n, y_n) - d(x_n, y_n)| \leq d^*(x'_n, x_n).$$

The right-hand side tends to 0 as $n \to \infty$, so we must have

$$\lim_{n\to\infty} d(x'_n, y_n) = \lim_{n\to\infty} d(x_n, y_n),$$

and $\overline{d}$ is well-defined.

Most of the partial metric axioms for $\overline{d}$ are immediate from those for d and taking limits. For P2 we see that $\overline{d}(\overline{x}, \overline{x}) = \overline{d}(\overline{x}, \overline{y}) = \overline{d}(\overline{y}, \overline{y})$ if, and only if,

$$\lim_{n\to\infty} d(x_n, x_n) = \lim_{n\to\infty} d(x_n, y_n) = \lim_{n\to\infty} d(y_n, y_n).$$

But this is precisely when $\lim_{n\to\infty} d^*(x_n, y_n) = 0$, so $\{x_n\} \sim \{y_n\}$ and $\overline{x} = \overline{y}$, and vice versa. Since

$$\overline{d}^*(\overline{x}, \overline{y}) = \lim_{n\to\infty} d^*(x_n, y_n),$$

is the usual metric given in constructing the completion of (X, d^*) in the literature (see [Sut75] for example), then $(\overline{X}, \overline{d})$ is complete. We define $i : X \to \overline{X}$ by $i(x) = \{x\}$, so that

$$\overline{d}(i(x), i(y)) = \lim_{n\to\infty} d(x, y) = d(x, y), \qquad \forall x, y \in X,$$

and i is an isometry into $\overline{X}$. Since $(\overline{X}, \overline{d}^*)$ is the metric completion of (X, d^*), it follows that $i(X)$ is dense in $(\overline{X}, \overline{d}^*)$ and so $(\overline{X}, \overline{d})$, together with i, is a partial metric completion of (X, d).

Now suppose that $(Y, \hat{d})$ is another partial metric completion of (X, d), with $\hat{\imath} : X \to Y$ an isometry into Y. For any $x \in Y$, there exists $\{x_n\}$ Cauchy in (X, d) such that $x = \lim_{n\to\infty} \hat{\imath}(x_n)$ in $(Y, \hat{d}^*)$, so we can define $f : Y \to \overline{X}$ by $f(x) = \{x_n\}$. If $\{y_n\}$ is another such sequence, then

$$\lim_{n\to\infty} d^*(x_n, y_n) = \lim_{n\to\infty} \hat{d}^*(\hat{\imath}(x_n), \hat{\imath}(y_n)) = 0,$$

so that $\{x_n\} \sim \{y_n\}$ and f is well-defined.

If $x, y \in Y$ and $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in (X, d) such that $x = \lim_{n\to\infty} \hat{\imath}(x_n)$ and $y = \lim_{n\to\infty} \hat{\imath}(y_n)$ in $(Y, \hat{d}^*)$ then, for some $k \in \Re$, we can use (4.2) to see that

$$\overline{d}(f(x), f(y)) = \lim_{n\to\infty} d(x_n, y_n) = \lim_{n\to\infty} \hat{d}(\hat{\imath}(x_n), \hat{\imath}(y_n)) + k = \hat{d}(x, y) + k.$$

So f is an isometry into $\overline{X}$. To see that f is an isometry, suppose $\{x_n\} \in \overline{X}$, then $\{\hat{\imath}(x_n)\}$ is Cauchy in $(Y, \hat{d}^*)$ and so has a limit $x \in Y$. We then have $f(x) \sim \{x_n\}$, and we have proved the following result.

Theorem 4.2.1 *Every partial metric space (X, d) has a unique completion (up to isometry).*

4.2.3 Completing the space of formal balls

Suppose (X, d) is a metric space, $\mathbf{B}X$ is the space of formal balls from section 2.3.2 and δ is the partial metric on $\mathbf{B}X$ from section 3.2.2. We know that $i : X \to \mathbf{B}X$ given by $i(x) = (x, 0)$, for all $x \in X$, is a homeomorphism from X onto the set of maximal elements of $\mathbf{B}X$ with the subspace Scott topology, and that $\tau_{[\delta]}$ is the Scott topology on $\mathbf{B}X$. It is now clear that i is an isometry into $\mathbf{B}X$, and that (X, d) is isometric to $(\mathrm{Max}(\mathbf{B}X), \delta)$. Similarly, $(\overline{X}, \overline{d})$ is isometric to $(\mathrm{Max}(\mathbf{B}\overline{X}), \hat{\delta})$, where $\hat{\delta}$ is the pmetric on $\mathbf{B}\overline{X}$ derived from $\overline{d}$. We now give an original result, in which we show that the partial metric completion $(\overline{\mathbf{B}X}, \overline{\delta})$ is isometric to $(\mathbf{B}\overline{X}, \hat{\delta})$, from which it follows that $(\mathrm{Max}(\overline{\mathbf{B}X}), \overline{\delta})$ is the metric completion of (X, d), and that we can either take formal balls first, and then complete, or vica versa.

Lemma 4.2.2 *$(\overline{\mathbf{B}X}, \overline{\delta})$ is isometric to $(\mathbf{B}\overline{X}, \hat{\delta})$.*

Proof. We first observe that,

$$\delta^*((x, r), (y, s)) = \max\{d(x, y) + |r - s|, 2|r - s|\},$$

and that $\{(x_n, r_n)\}$ is a Cauchy sequence in $(\mathrm{B}X, \delta)$ if, and only if,

$$\lim_{n,m\to\infty} \max\{d(x_n, x_m) + |r_n - r_m|, 2|r_n - r_m|\} = 0,$$

which is precisely when $\{x_n\}$ is Cauchy in (X, d) and $\lim_{n\to\infty} r_n$ exists.
So, if $\{x_n\}$ is a Cauchy sequence in (X, d), and $r \geq 0$, then $\{(x_n, r)\}$ is a Cauchy sequence in $(\mathrm{B}X, \delta)$, and we can define $f : \mathrm{B}\overline{X} \to \overline{\mathrm{B}X}$ by

$$f(\{x_n\}, r) = \{(x_n, r)\}.$$

Once we show that f is an isometry, then we are done. For any $(\{x_n\}, r), (\{y_n\}, s) \in \mathrm{B}\overline{X}$, we have

$$\begin{aligned}
\hat{\delta}((\{x_n\}, r), (\{y_n\}, s)) &= \max\{2r, r + \overline{d}(\{x_n\}, \{y_n\}) + s, 2s\} \\
&= \max\{2r, r + \lim_{n\to\infty} d(x_n, y_n) + s, 2s\} \\
&= \lim_{n\to\infty} \delta((x_n, r), (y_n, s)) \\
&= \overline{\delta}(f(\{x_n\}, r), f(\{y_n\}, s)).
\end{aligned}$$

If we now suppose that $\{(x_n, r_n)\} \in \overline{\mathrm{B}X}$, then $\lim_{n\to\infty} r_n = r$ (say) and $\{x_n\}$ Cauchy in (X, d). So $(\{x_n\}, r) \in \mathrm{B}\overline{X}$, $f(\{x_n\}, r) = \{(x_n, r)\}$ and $\{(x_n, r_n)\} \sim \{(x_n, r)\}$ as required.

QED

In section 2.3.2 we saw that $\mathrm{B}\overline{X}$ is isomorphic to the ideal completion, $\mathcal{I}(\mathrm{B}X)$, of the abstract basis $(\mathrm{B}X, \ll)$. It follows that $\overline{\mathrm{B}X}$ is isomorphic to $\mathcal{I}(\mathrm{B}X)$ and that our two processes of completion on $\mathrm{B}X$ agree. In a rather ad hoc manner, Heckmann seeks the same result in [Hec98], by defining a (weak) pmetric, $\tilde{\delta}$, on $\mathcal{I}(\mathrm{B}X)$, for which $(\mathrm{Max}(\mathcal{I}(\mathrm{B}X)), \tilde{\delta})$ is the metric completion of (X, d). Our approach however is the more natural, and further develops the unity of the metric and ideal completion processes.

4.3 Weighted spaces

In section 4.1.4 we suggested that a quantitative domain could be a continuous domain together with a quasi-metric or partial metric capturing the Scott topology, and investigated when the pmetric and Scott topologies agree. We saw in chapter 3, that the important additional feature partial metrics bring to a domain is a weight function. However, we observe that many of the examples we have considered, such as $\mathbf{I}\Re$, S^∞ and even $\Re$, already have an inherent notion of weight. It therefore seems reasonable to consider when we can use this to build a pmetric that captures the Scott topology. The material in this section is original and will lead us to a class of partial metric spaces that will be particularly useful in chapters 5 and 6.

4.3.1 Defining weighted spaces

A partial metric is a distance function over a set. We now develop the situation where we have a set with some basic structure, and an auxiliary weight function over this set, and deduce a natural pmetric. We begin by identifying the basic structure that we require of our sets. These will be posets for which, using some domain theoretic intuition, any two points have a point with their common information, and if two points are not contradictory, then they have a point with their combined information. We first introduced such structures in [ONe97].

Definition 4.3.1 *A* **consistent semilattice** *is a poset* $(P, \sqsubseteq)$ *such that*

1. $\forall x, y \in P$, $x \sqcap y \in P$.
2. *If* $\{x, y\} \subseteq P$ *is consistent (bounded above), then* $x \sqcup y \in P$.

We recall from section 3.2.2, that for a partial metric space (X, d), the weight function $\phi_d : X \to \Re$ is strictly monotonic decreasing; $x \leq_d y$ implies $\phi_d(x) \geq \phi_d(y)$ and $x <_d y$ implies that $\phi_d(x) > \phi_d(y)$. We say that a weight function is *semi-modular* if

$$\phi_d(x \sqcap y) + \phi_d(x \sqcup y) \leq \phi_d(x) + \phi_d(y), \tag{4.3}$$

whenever $x \sqcap y$ and $x \sqcup y$ exist in $(X, \leq_d)$.

Definition 4.3.2 *A* **weighted space** $(X, \sqsubseteq, \phi)$ *is a consistent semilattice* $(X, \sqsubseteq)$ *together with a strictly monotonic decreasing semi-modular function* $\phi : X \to \Re$, *which we call the* **weight function**.

In [ONe97] we defined a "valuation space" to be a weighted space with a modular weight function, which requires equality in (4.3). In this thesis however, we will reserve the term for chapter 6 when we further develop our ideas by using the valuations from section 2.3.3. In the next lemma we see that the weighted spaces are a special class of partial metric spaces in much the same way that normed spaces are a special class of metric spaces.

Lemma 4.3.3 *Suppose* $(X, \sqsubseteq, \phi)$ *is a weighted space, then* $d(x, y) = \phi(x \sqcap y)$ *defines a pmetric for which* $\leq_d = \sqsubseteq$ *and* $\phi_d = \phi$.

Proof. Axioms P1 and P3 are immediate, and P2 follows since $d(x, y) = d(x, x)$ if, and only if, $x \sqsubseteq y$. For the P4 axiom we have

$$\begin{aligned} d(x,z) + d(y,y) &= \phi(x \sqcap z) + \phi(y) \\ &\leq \phi(x \sqcap y \sqcap z) + \phi((x \sqcap y) \sqcup (y \sqcap z)) \\ &\leq \phi(x \sqcap y) + \phi(y \sqcap z) \\ &= d(x,y) + d(y,z). \end{aligned}$$

It is immediate that $\phi_d = \phi$ and $\leq_d = \sqsubseteq$.

QED

The distance between two points in a weighted space is therefore a measure of their common information. Whenever we consider a pmetric for a weighted space, then this will be the induced pmetric from the lemma, and we will write $\tau_{[\phi]}$ for the induced pmetric topology and $\tau_{[\phi^*]}$ for the induced metric topology. Many of the partial metric spaces that we have already met are weighted spaces. For example, the pmetric on $\mathbf{I}\Re$, from section 3.2.2, is induced by the weight function $\phi([a,b]) = b-a$, the pmetric on S^∞, from section 3.3.5, is induced by $\phi(x) = 2^{-|x|}$, and the pmetric on $(\Re, \geq)$, from section 4.1.1, is induced by $\phi(x) = x$. We will meet more examples in section 4.3.3.

4.3.2 Topological properties of weighted spaces

We say that a weighted space is complete (or compact) if its induced pmetric space is complete (or compact). We return to the material from section 4.1.4, and show that complete weighted spaces have convergence in order. We can then emphasis some of the more desirable T_0-topological properties of complete and compact weighted spaces.

Lemma 4.3.4 *Every complete weighted space $(X, \sqsubseteq, \phi)$ has convergence in order.*

Proof. Suppose $\{w_n\}$ is a sequence converging to $a \in X$, and $\{x_n\}$ is a subsequence satisfying $x_n \in B_{1/2^n}(a)$, for each $n \geq 1$. We fix $n \geq 1$ and, for each $k \geq 1$, we let

$$z_k = a \sqcap x_n \sqcap \cdots \sqcap x_{n+k} \in X.$$

So $z_{k+1} = (a \sqcap x_{n+k+1}) \sqcap z_k$, and it follows that

$$\phi(a \sqcap x_{n+k+1}) + \phi(z_k) \geq \phi(z_{k+1}) + \phi(a).$$

We therefore see that

$$0 \leq \phi(z_{k+1}) - \phi(z_k) \leq d(a, x_{n+k+1}) - d(a,a) < 1/2^{n+k+1},$$

and $\{z_k\}$ is a Cauchy sequence. We let $y_n \in X$ be the proper limit, so that

$$y_n = a \sqcap x_n \sqcap x_{n+1} \sqcap \cdots.$$

Clearly $\{w_n\}$ surpasses the ω-chain $\{y_n\}$, and for each $n \geq 1$, we have $y_n = (a \sqcap x_n) \sqcap y_{n+1}$. It follows that

$$\phi(a \sqcap x_n) + \phi(y_{n+1}) \geq \phi(y_n) + \phi(a),$$

and we see that

$$\phi(y_n) - \phi(a) \leq \sum_{i=0}^{\infty} \phi(a \sqcap x_{n+i}) - \phi(a) \leq \sum_{i=0}^{\infty} 1/2^{n+i} = 1/2^{n-1}.$$

So $\phi(a) = \inf_{n\geq 1} \phi(y_n)$, and since each $y_n \sqsubseteq a$, then $a = \bigsqcup^{\uparrow} y_n$, and we are done.

QED

For a complete weighted space $(X, \sqsubseteq, \phi)$, the consistent semilattice need not be a dcpo, which was something we insisted on in [ONe97]. However, any set in X bounded above has a directed set in X with the same upper bounds, and so $(X, \sqsubseteq)$ is conditionally complete. For a pmetric space (X, d), we will say that $(X, \leq_d)$ is a *conditional dcpo* if every directed set $A \subseteq X$ which has $\{\phi_d(a) \mid a \in A\}$ bounded below in $\Re$, has a supremum in X.

Theorem 4.3.5 *Suppose $(X, \sqsubseteq, \phi)$ is a weighted space, then it is complete if, and only if, $(X, \sqsubseteq)$ is a conditional dcpo, $\tau_{[\phi]}$ is the Scott topology and every Cauchy sequence surpasses an approximating ω-chain.*

Proof. We first suppose that $(X, \sqsubseteq, \phi)$ is complete. That $(X, \sqsubseteq)$ is a conditional dcpo and $\tau_{[\phi]}$ is order-consistent is immediate, and by lemma 4.3.4 and 4.1.19, $\tau_{[\phi]}$ is the Scott topology. Suppose $\{x_n\}$ is a Cauchy sequence, then it has a proper limit $a \in X$, and lemma 4.3.4 implies that $\{x_n\}$ surpasses an approximating ω-chain $\{y_n\}$ with supremum a. Order-consistency implies that $\{y_n\}$ converges properly to a, and the result follows.

Conversely, suppose $\{x_n\}$ is a Cauchy sequence, and let $\{y_n\}$ be the appropriate approximating ω-chain. Since $(X, \sqsubseteq)$ is a conditional dcpo, we can let $a = \bigsqcup^{\uparrow} y_n$, and order-consistency implies that $\lim_{n\to\infty} d(y_n, y_n) = d(a, a)$. It easily follows that $\{x_n\}$ converges to a properly.

QED

We immediately see that since the weight function $\phi(x) = x$ on $(\Re, \geq)$ induces the Euclidean metric, which is complete, then for this example, $\tau_{[\phi]}$ is the Scott topology.

Corollary 4.3.6 *Suppose $(X, \sqsubseteq, \phi)$ is a compact weighted space, then $(X, \tau_{[\phi]})$ is a coherent space, $\tau_{[\phi]}$ is the Scott topology and $\tau_{[\phi^*]}$ is its patch topology.*

Proof. Immediate from lemma 4.3.5 and 4.1.16.

QED

4.3.3 Scott-domains as weighted spaces

Künzi has shown, in [KV94], that every second-countable T_0-topology is pmetrisable, but that there are quasi-metrisable topologies that are not pmetrisable. These results may be of more interest to topologists, than useful for our purposes, since we have been more interested in investigating the implications of the weight function. However, we now give an example of Künzi's construction that fits neatly into our framework of weighted spaces. We recall from section 2.1.1 that Scott-domains are ωbc-algebraic domains, and we will further assume that they have a bottom element. It is then clear that a Scott-domain is a consistent semilattice. The following results were found independently of Künzi's work.

Lemma 4.3.7 *Suppose D is a Scott-domain, $K = \{k_1, k_2, \ldots\}$ is the collection of compact elements and we define $\phi : D \to [0,1]$, by*

$$\phi(x) = 1 - \sum_{k_n \in K_x} 1/2^n,$$

then $(D, \sqsubseteq, \phi)$ is a weighted space.

Proof. If $x \sqsubseteq y$ but $x \neq y$, then $K_x \subset K_y$, and clearly $\phi(x) > \phi(y)$. Now suppose that $\{x, y\} \subseteq D$ is consistent, then

$$\phi(x) + \phi(y) \quad = \quad 1 - \sum_{k_n \in K_x} 1/2^n + 1 - \sum_{k_n \in K_y} 1/2^n$$

$$
\begin{aligned}
&= 1 - \sum_{k_n \in K_x \cup K_y} 1/2^n + 1 - \sum_{k_n \in K_x \cap K_y} 1/2^n \\
&\geq 1 - \sum_{k_n \in K_{x \sqcup y}} 1/2^n + 1 - \sum_{k_n \in K_{x \sqcap y}} 1/2^n \\
&= \phi(x \sqcup y) + \phi(x \sqcap y).
\end{aligned}
$$

QED

In [ONe97] we gave a similar result for the weighted spaces with modular weight function, but had to work with the prime-algebraic domains. In chapter 6 we consider how we can give a similar result for ω-continuous domains. To finish this section, we observe that the metric induced on D from the lemma, is given by

$$
d^*(x,y) = \max\left\{ \sum_{k_n \in K_x \setminus K_y} 1/2^n, \sum_{k_n \in K_y \setminus K_x} 1/2^n \right\}.
$$

We use this to show that our weighted space is compact.

Lemma 4.3.8 *Suppose D is a Scott-domain, then the weighted space $(D, \sqsubseteq, \phi)$ is compact.*

Proof. We first show that it is complete. Suppose $\{x_n\}$ is a Cauchy sequence D. we let $y_n = \sqcap\{x_m \mid m > n\}$, for all $n \geq 1$, and $x = \bigsqcup^{\uparrow}\{y_n \mid n \geq 1\}$. For any $\varepsilon > 0$, we let $J \geq 1$ be such that $\delta = 1/2^J < \varepsilon$. We can find $N \geq 1$ such that $d^*(x_n, x_m) < \delta$ for $n, m \geq N$. Clearly, if $k_j \in K_{x_n} \setminus K_{x_m}$ or $k_j \in K_{x_m} \setminus K_{x_n}$, then we must have $j > J$. So we see that, for all $n \geq N$,

$$
d^*(x_n, x) \leq \sum_{j=J+1}^{\infty} 1/2^j = \delta < \varepsilon,
$$

and so $\{x_n\}$ converges properly to x.

We now show that the metric space (D, d^*) is totally bounded, from which it follows that $(D, \sqsubseteq, \phi)$ is compact. For any $\varepsilon > 0$, there exists $J \geq 1$ such that $1/2^J < \varepsilon$. We define

$$
A = \{\bigsqcup Q \mid Q \subseteq \{k_j \mid 1 \leq j \leq J\},\, Q \text{ consistent}\} \bigcup \{\bot\}.
$$

Then for any $x \in D$, there must exist $a \in A$ such that for all $1 \leq j \leq J$, $k_j \sqsubseteq a$ if, and only if $k_j \sqsubseteq x$, so we have

$$d^*(a, x) \leq \sum_{j=J+1}^{\infty} 1/2^j = 1/2^J < \varepsilon.$$

So A is a finite ε-net and (D, d^*) is totally bounded.

QED

Corollary 4.3.9 *If D is a Scott-domain with weighted space $(D, \sqsubseteq, \phi)$, then the Scott and pmetric topologies agree as do the Lawson and metric topologies.*

Proof. Immediate from corollary 4.3.6 and lemma 4.3.8.

QED

4.3.4 The completion of a weighted space

We finally consider the completion of a weighted space to see that this is again a weighted space, which will prove useful in the next chapter. Suppose $(X, \sqsubseteq, \phi)$ is a weighted space, with induced pmetric $d(x, y) = \phi(x \sqcap y)$, and let $(\overline{X}, \overline{d})$ be the partial metric completion from section 4.2.2. As in section 4.2.2, we will suppose that our $\overline{x} \in \overline{X}$ are represented by Cauchy sequences $\{x_n\}$ in X, and vice versa. For simplicity, we will denote the specialisation order of $\overline{d}$ by $\sqsubseteq$. We will show that $(\overline{X}, \sqsubseteq)$ is a consistent semilattice, and that $\overline{\phi} : \overline{X} \to \Re$, given by

$$\overline{\phi}(\overline{x}) = \lim_{n \to \infty} \phi(x_n), \qquad \forall \overline{x} \in \overline{X},$$

defines a weight function that induces $\overline{d}$.

We begin with some preliminary results. Suppose $\overline{x}, \overline{y} \in \overline{X}$, and define $z_n = x_n \sqcap y_n$, for each $n \geq 1$. From section 4.2.2, we know that $\lim_{n \to \infty} \phi(x_n \sqcap y_n)$ exists. We will show that $\lim_{n,m \to \infty} d(z_n, z_m) = \lim_{n \to \infty} \phi(x_n \sqcap y_n)$, and hence that

$\{z_n\}$ is a Cauchy sequence in X. For any $n, m \geq 1$, we have

$$\begin{aligned}
\phi(x_n \sqcap y_n) \leq d(z_n, z_m) &= d(x_n \sqcap x_m, y_n \sqcap y_m) \\
&\leq d(x_n \sqcap x_m, x_m) + d(x_m, y_m) + d(y_m, y_n \sqcap y_m) \\
&\quad - d(x_m, x_m) - d(y_m, y_m) \\
&= d(x_n, x_m) + \phi(x_m \sqcap y_m) + d(y_n, y_m) - \phi(x_m) - \phi(y_m).
\end{aligned}$$

The result follows since $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in X.

Now suppose that $\overline{x}, \overline{y} \in \overline{X}$ are such that each $\{x_n, y_n\}$ is consistent in $(X, \sqsubseteq)$, and define $z_n = x_n \sqcup y_n$, for each $n \geq 1$. It is clear that $\lim_{n\to\infty} \phi(x_n \sqcup y_n)$ exists. We will show that $\lim_{n,m\to\infty} d(z_n, z_m) = \lim_{n\to\infty} \phi(x_n \sqcup y_n)$, and hence that $\{z_n\}$ is a Cauchy sequence in X. For any $n, m \geq 1$, we have

$$\begin{aligned}
\phi(x_n \sqcup y_n) \leq d(z_n, z_m) &= \phi((x_n \sqcup y_n) \sqcap (x_m \sqcup y_m)) \\
&\leq \phi((x_n \sqcap x_m) \sqcup (y_n \sqcap y_m)) \\
&= \phi(x_n \sqcap x_m) + \phi(y_n \sqcap y_m) - \phi(x_n \sqcap x_m \sqcap y_n \sqcap y_m) \\
&= d(x_n, x_m) + d(y_n, y_m) - d(x_n \sqcap y_n, x_m \sqcap y_m),
\end{aligned}$$

from which the result follows.

We now show that $(\overline{X}, \sqsubseteq)$ is a consistent semilattice. Suppose $\overline{x}, \overline{y} \in \overline{X}$, and we define $z_n = x_n \sqcap y_n$, for each $n \geq 1$. We know that $\{z_n\}$ is a Cauchy sequence in X, and we have

$$\overline{d}(\overline{x}, \overline{z}) = \lim_{n\to\infty} d(x_n, z_n) = \lim_{n\to\infty} d(z_n, z_n) = \overline{d}(\overline{z}, \overline{z}).$$

It follows that $\overline{z} \sqsubseteq \overline{x}$ and similarly for $\overline{y}$. Now suppose that $\overline{w} \in \overline{X}$ is below $\overline{x}$ and $\overline{y}$, and we define $w'_n = w_n \sqcap z_n$, for each $n \geq 1$. It is clear that, for each $n \geq 1$,

$$\begin{aligned}
d(w'_n, w'_n) = d(w_n \sqcap x_n, w_n \sqcap y_n) &\leq d(w_n \sqcap x_n, w_n) + d(w_n, w_n \sqcap y_n) - d(w_n, w_n) \\
&= d(w_n, x_n) + d(w_n, y_n) - d(w_n, w_n),
\end{aligned}$$

so we see that

$$\overline{d}(\overline{w}, \overline{w}') = \overline{d}(\overline{w}', \overline{w}') \leq \overline{d}(\overline{w}, \overline{x}) + \overline{d}(\overline{w}, \overline{y}) - \overline{d}(\overline{w}, \overline{w}) = \overline{d}(\overline{w}, \overline{w}),$$

and so $\overline{w} = \overline{w}'$. Since $\overline{w}' \sqsubseteq \overline{z}$, then it follows that $\overline{z} = \overline{x} \sqcap \overline{y}$.

Now suppose that $\{\overline{x}, \overline{y}\}$ is consistent in $(\overline{X}, \sqsubseteq)$. Notice that we cannot assume that the $\{x_n, y_n\}$ are consistent in X. Suppose $\overline{w} \in \overline{X}$ is an upper bound for $\overline{x}$ and $\overline{y}$, and define

$$z_n = (x_n \sqcap w_n) \sqcup (y_n \sqcap w_n) \sqsubseteq w_n, \qquad \forall n \geq 1.$$

We know that $\{x_n \sqcap w_n\}$ and $\{y_n \sqcap w_n\}$ are Cauchy sequences in X, so it follows that $\{z_n\}$ is a Cauchy sequence in X, and $\overline{z} \sqsubseteq \overline{w}$. We show that $\overline{z}$ is an upper bound for $\overline{x}$ and $\overline{y}$. For each $n \geq 1$, we have that

$$\phi(x_n) \leq d(x_n, z_n) \leq d(x_n, x_n \sqcap w_n) + d(x_n \sqcap w_n, z_n) - d(x_n, w_n) = d(x_n, w_n).$$

It follows that $\overline{d}(\overline{x}, \overline{z}) = \overline{d}(\overline{x}, \overline{x})$ and hence that $\overline{x} \sqsubseteq \overline{z}$, and similarly for $\overline{y}$. We observe that

$$\begin{aligned}
\overline{d}(\overline{z}, \overline{z}) &= \lim_{n\to\infty} \phi(x_n \sqcap w_n) + \phi(y_n \sqcap w_n) - \phi(x_n \sqcap y_n \sqcap w_n) \\
&= \overline{d}(\overline{x}, \overline{w}) + \overline{d}(\overline{y}, \overline{w}) - \overline{d}(\overline{x} \sqcap \overline{y}, \overline{w}) \\
&= \overline{d}(\overline{x}, \overline{x}) + \overline{d}(\overline{y}, \overline{y}) - \overline{d}(\overline{x} \sqcap \overline{y}, \overline{x} \sqcap \overline{y}),
\end{aligned}$$

which is independent of $\overline{w}$. Now suppose that $\overline{w}' \in \overline{X}$ is another upper bound for $\overline{x}$ and $\overline{y}$, then so is $\overline{w} \sqcap \overline{w}'$, and we define,

$$z'_n = (x_n \sqcap w_n \sqcap w'_n) \sqcup (y_n \sqcap w_n \sqcap w'_n) \sqsubseteq w_n \sqcap w'_n, \qquad \forall n \geq 1.$$

It is clear that $\overline{d}(\overline{z}', \overline{z}') = \overline{d}(\overline{z}, \overline{z})$ and so $\overline{z} = \overline{z}'$. It follows that $\overline{z} \sqsubseteq \overline{w}'$, and we have that $\overline{z} = \overline{x} \sqcup \overline{y}$, and $(\overline{X}, \sqsubseteq)$ is a consistent semilattice.

If we define $\overline{\phi} : \overline{X} \to \Re$ by $\overline{\phi}(\overline{x}) = \lim_{n\to\infty} \phi(x_n)$, then

$$\overline{\phi}(\overline{x} \sqcap \overline{y}) = \lim_{n\to\infty} \phi(x_n \sqcap y_n) = \lim_{n\to\infty} d(x_n, y_n) = \overline{d}(\overline{x}, \overline{y}), \qquad \forall \overline{x}, \overline{y} \in \overline{X}.$$

Once we show that $\overline{\phi}$ is a weight function for $(\overline{X}, \sqsubseteq)$, then we are done. It is immediate that $\overline{\phi}$ is a strictly monotonic decreasing function, and if $\{\overline{x}, \overline{y}\}$ is consistent in $\overline{X}$, then $\overline{\phi}$ is semi-modular since

$$\overline{d}(\overline{x} \sqcup \overline{y}, \overline{x} \sqcup \overline{y}) = \overline{d}(\overline{x}, \overline{x}) + \overline{d}(\overline{y}, \overline{y}) - \overline{d}(\overline{x} \sqcap \overline{y}, \overline{x} \sqcap \overline{y}).$$

Theorem 4.3.10 *The partial metric completion of a weighted space is again a weighted space.*

4.4 Concluding remarks

This completes our exposition of a basic theory for partial metric spaces. In section 4.1 we established the partial metric spaces within the context of T_0-properties, and investigated the notion of partial metric completion in section 4.2. We have discussed our ideas on quantitative domains, and used these to motivate a particular class of partial metric spaces, the weighted spaces, in section 4.3. We will use much of this material in developing a potential area of applications in the next chapter, before returning to more theoretic developments in chapter 6.

Chapter 5

Applications in the Modeling and Analysis of Systems

We consider some new ideas on the modeling of systems and techniques suitable for the analysis of their properties, motivated by Wadge's work on the data flow networks from section 3.1. More specifically, we present a new approach to the modeling and analysis of liveness in deterministic Petri nets (sections 5.1 and 5.2), and consider the weighted spaces as a framework in which to work (section 5.3). We will see that to extend our methods to include non-determinism will require more sophisticated techniques than we have available (section 5.4), and developing a framework in which to be able to work is the motivation for the next chapter. Aside from the background material on Petri nets, the material in this chapter is original.

5.1 A model for the analysis of liveness in Petri nets

Given a system N and a property P, then our general approach can be described as follows. We seek to model N as the fixed points of some function in such a way that N satisfies P if, and only if, the fixed points satisfy some property P'. We take Petri nets as our systems, and consider the property of liveness. Notice that we are setting up the model specific to the problem in which we are interested. As we are trying to get across our general approach, we will proceed with a degree of informality, refering to specific examples throughout, rather than attempting full generality in our arguments. Most of our background material on Petri nets is taken from [BCO92], although [Mur89] has a fuller account. The rest of the material is original.

5.1.1 A primer on Petri nets and liveness

Petri nets are a graphical and mathematical modeling tool particularly well suited to modeling systems which involve, for example, concurrency or synchronisation. Consider the Petri net in figure 5.1(a) (taken from [BCO92]). The *places* p_i are drawn as circles and can be considered to represent conditions. These are satisfied when a place contains a token (drawn as a dot) such as p_1 and p_5 in our example. The *transitions* q_i are drawn as bars and represent events, which are said to occur whenever a transition fires. A transition is *enabled* when all its input (upstream) places have at least one token. In figure 5.1(a), q_2 is enabled but q_1 and q_3 are not. The *firing* of an enabled transition removes a token from each input place, and adds one to each output (downstream) place. More general firing and enabling rules exist, for example with integer weights on the arcs, which require multiple tokens in a place to enable a transition, and adding multiple tokens to a downstream place after a transition fires. We refer to [Mur89, BCO92] for details.

The net in figure 5.1(a) is of a particularly simple nature in that each place has precisely one upstream and one downstream transition. These are called *event graphs* in [BCO92] and model synchronisation. Slightly more generally we could allow multiple upstream transitions for a place, such as p_3 in figure 5.1(b). A significant extension is to allow non-determinism, as in figure 5.1(c), where a token at p_1 enables both downstream transitions q_0 and q_1, and there are no general rules

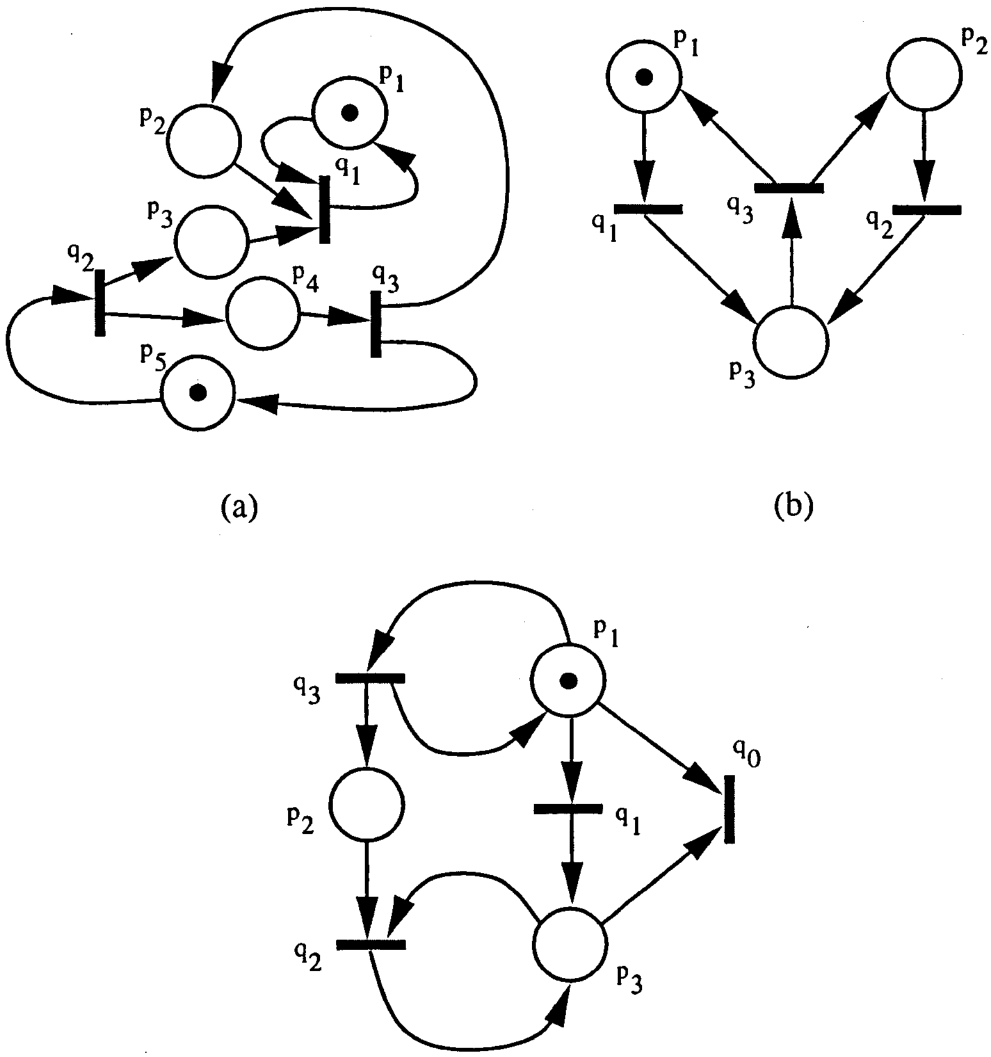


Figure 5.1: Example Petri nets

as to which should fire. Such nets can be used to model decisions or choices.

A further extension to the basic model is to introduce a timing aspect, either as the duration of a firing for transitions, or as how long it takes before a new token at a place can contribute to enabling downstream transitions. These timings can be variable or constant. Timed Petri nets are used in analysing performance evaluation and scheduling problems.

The initial placement of tokens in a net is called the *initial marking* of the net. The marking of a net is interpreted as its state, which evolves as its transitions fire. This is the dynamic behaviour of a Petri net, and we may now analyse behavioural properties such as which states it is possible to reach in a sequence of firings from a given state. The particular property in which we are interested is liveness, where a Petri net is said to be *live* if, after any finite initial sequence of firings, any transition can be fired infinitely often. The nets in figures 5.1(a) and 5.1(b) are live, while the net in figure 5.1(c) is not. A live Petri net guarantees deadlock-free operation regardless of the firing sequence of its transitions.

In general liveness is an ideal property which is impractical to consider. We therefore relax the liveness condition, and define the following *levels of liveness* (taken from [Mur89]). For a given Petri net, a transition is:

L0-live (dead) if it never fires.

L1-live (potentially firable) if it can be fired at least once.

L2-live if, for any $n \geq 1$, the transition can be fired at least n times.

L3-live if it can be fired infinitely often.

L4-live (live) if it can always be fired infinitely often.

For example, consider the Petri net in figure 5.1(c), where the transitions q_0, q_1, q_2 and q_3 are dead, L1-live, L2-live and L3-live, respectively.

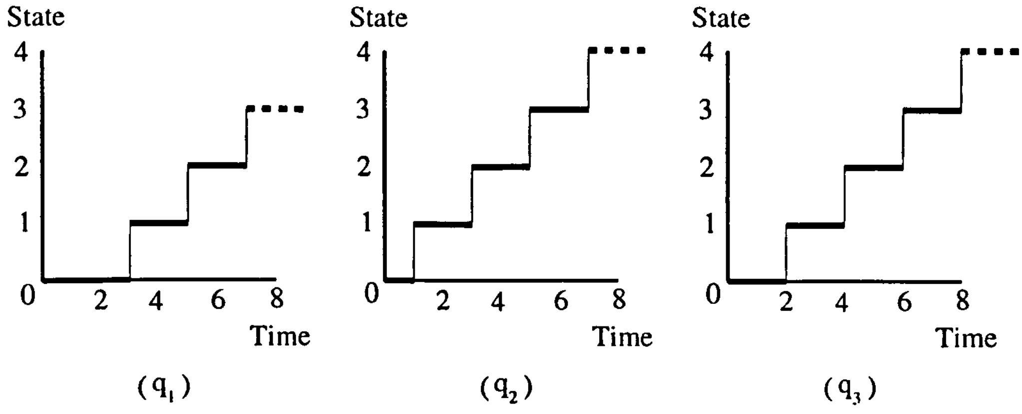


Figure 5.2: Example of a Petri net trajectory

5.1.2 A notion of state suitable for analysing liveness

The classical approach to analysing liveness is to look for necessary and sufficient conditions for the existence of a live marking for a given Petri net structure. We propose a quite different approach based on the ideas behind Wadge's cycle sum test. In this approach we consider our Petri nets to have a timed aspect in which transitions have instantaneous firing times, and places have a holding time of one. We begin with a new interpretation of the state of a Petri net, suitable for the analysis of liveness.

At a given time $t \in \Re^+$, we define the *state* of a Petri net to be the number of times that each transition has fired. The *trajectory* or *motion* of a Petri net is then the evolution of state over time. For example, the trajectory of the net in figure 5.1(a) is given by the three graphs in figure 5.2. This is an example of a discrete event dynamical system [Ho89], since the state of the system takes values in a discrete set, and changes at discrete intervals of time, rather than continuously, so the trajectory is a sequence of piecewise constant segments. The trajectory of a (deterministic) Petri net with k-transitions is given by a *state function* $f : \Re^+ \to \mathbf{N}^k$ where, for each $t \in \Re^+$, $f(t)_i$ is the number of times that transition q_i ($1 \leq i \leq k$) has fired at time t. A non-deterministic Petri nets may have many possible trajectories and therefore many possible state functions.

Suppose $\mathcal{S}$ is the collection of state functions for a given Petri net with k-transitions, then for $f \in \mathcal{S}$, we define

$$|f_i| = \sup\{f(t)_i \mid t \in \Re^+\} \in \mathrm{N} \cup \{\infty\},$$

for each $1 \leq i \leq k$, and

$$|f| = \min\{|f_1|, \ldots, |f_k|\}.$$

We can see that our net is live if, and only if, $|f| = \infty$ for each $f \in \mathcal{S}$. Furthermore, the transition q_i is

L0-live (dead) if $|f_i| = 0$ for all $f \in \mathcal{S}$.

L1-live (potentially firable) if there exists $f \in \mathcal{S}$ such that $|f_i| \geq 1$.

L2-live if, for any $n \geq 1$, there exists $f \in \mathcal{S}$ such that $|f_i| \geq n$.

L3-live if there exists $f \in \mathcal{S}$ such that $|f_i| = \infty$.

L4-live (live) if $|f_i| = \infty$ for all $f \in \mathcal{S}$.

5.1.3 Network functions for modeling deterministic Petri nets

We seek to model a deterministic Petri net by a function, over a suitably defined domain, whose fixed point is precisely the state function of the net. Before we can consider how to define such a function, we must specify our basic domain. We fix $k \geq 1$, and let $X^{(k)}$ denote the collection of partial and total upper semicontinuous[1] functions, $f : \Re^+ \to \mathrm{N}^k$, for which there exists $T_f \in [0, \infty]$ such that f is defined on precisely $[0, T_f)$. State functions of Petri nets with k-transitions are clearly total functions in $X^{(k)}$.

It is clear that there are many functions over $X^{(k)}$ whose fixed point is precisely the state function of a given deterministic Petri net. For example, suppose

[1]A function $f : \Re^+ \to \mathrm{N}^k$ is *upper semicontinuous* if, for each $n \in \mathrm{N}$ and $1 \leq i \leq k$, there exists $T \in \Re^+$ such that $\{t \in \Re^+ \mid f(t)_i < n\} = [0, T)$.

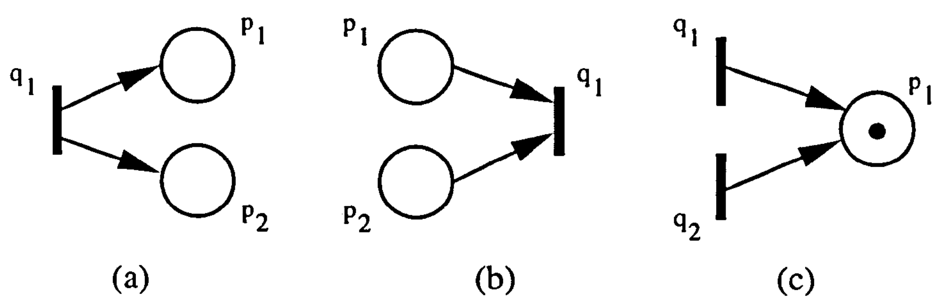


Figure 5.3: Basic Petri net structures

$f \in X^{(k)}$ is such a state function, then the constant function that sends everything to f, would satisfy this criteria. However, such a function pre-supposes knowledge of the state function, which is exactly the situation that we wish to avoid. Our aim is to use the structure of the net to describe a suitable function over $X^{(k)}$, which will then allow us to get information on the fixed point, that is the state function of the net, without necessarily working it out.

So how can we find such a function? In [HW90], Hubbard and West propose that whenever you wish to model a system, you should describe the forces involved, and derive the motions from them, rather than describing the motions directly. In our situation, we regard the motion of a Petri net to be the evolution of state over time, in other words, the firing of its transitions. It seems reasonable therefore to consider the places as being the "forces" from which this motion is derived.

Consider the basic Petri net structures in figure 5.3, together with the following observations:

(a) The number of tokens to have arrived at p_1 and p_2 at time $t \in \Re^+$, is the number of firings of q_1 at time t.

(b) The number of firings of q_1 at time $t \geq 1$ is the *minimum* of the number of tokens to have arrived at p_1 and p_2 at time $t - 1$, and is 0 at time $t \in [0, 1)$.

(c) The number of tokens to have arrived at p_1 at time $t \in \Re^+$ is one plus the number of firings of q_1 and q_2 at time t.

We take the net in figure 5.1(a) as a worked example, and suppose that it has state function $f \in X^{(3)}$. From the above observations we see that $f(t)_1$, the number of firings of q_1 at time $t \geq 1$, is the minimum of the number of tokens to have arrived at p_1, p_2, and p_3 at time $t-1$, which are; one plus the number of firings of q_1 at time $t-1$, $f(t-1)_1+1$, the number of firings of q_3 at time $t-1$, $f(t-1)_3$, and the number of firings of q_2 at time $t-1$, $f(t-1)_2$, respectively. We can easily deduce the following equations, analogous to the equations we had for a data flow network in section 3.1,

$$\begin{aligned} f(t)_1 &= \min\{f(t-1)_1+1, f(t-1)_3, f(t-1)_2\}, \\ f(t)_2 &= f(t-1)_3+1, \\ f(t)_3 &= f(t-1)_2, \end{aligned}$$

for each $t \geq 1$, and $f(t) = (0,0,0)$ at $t \in [0,1)$.

We consider the right-hand side of these equations as defining a (deterministic) *network function* $H : X^{(3)} \to X^{(3)}$ so that, for each $g \in X^{(3)}$,

$$H(g)(t) = \begin{pmatrix} \min\{g(t-1)_1+1, g(t-1)_3, g(t-1)_2\} \\ g(t-1)_3+1 \\ g(t-1)_2 \end{pmatrix}, \qquad \forall t \in [1, T_g+1),$$

and $H(g)(t) = (0,0,0)$ for all $t \in [0,1)$. The state function of our net is clearly a fixed point of H, and we will show in section 5.2.1 that it is the unique fixed point.

5.1.4 Comparable models and extensions

In [BCO92], Baccelli *et al* have a similar approach to modeling Petri nets, although their aim is for performance evaluation techniques, rather than analysing liveness. For the transitions q_i in figure 5.1(a) for example, they let $x_i(j)$ $(j \geq 1)$ be the time of the jth firing of q_i, and derive the evolution equations:

$$\begin{aligned} x_1(j+1) &= \max\{x_1(j)+1, x_2(j+1)+1, x_3(j+1)+1\}, \\ x_2(j+1) &= x_3(j)+1, \\ x_3(j+1) &= x_2(j+1)+1, \end{aligned}$$

with an initial condition $x_2(1) = 1$. This is clearly the dual to our approach, and we can deduce one from the other.

Now suppose that a place has multiple upstream transitions, such as p_3 in the net in figure 5.1(b), then we cannot use evolution equations to find the firing times, since we cannot tell if the jth firing of q_3 is due to a firing of q_1 or a firing of q_2. However, we can avoid this problem with our approach, and the deterministic network function $H : X^{(3)} \to X^{(3)}$ is such that, for all $g \in X^{(3)}$,

$$H(g)(t) = \begin{pmatrix} g(t-1)_3 + 1 \\ g(t-1)_3 \\ g(t-1)_1 + g(t-1)_2 \end{pmatrix}, \qquad \forall t \in [1, T_g + 1),$$

and $H(g)(t) = (0,0,0)$ for all $t \in [0,1)$.

One notable difference from our presentation however, is that the presentation given in [BCO92] is in complete generality. We conclude this section with the comment, or perhaps hypothesis, that we can use our approach to model *any* deterministic Petri net. Furthermore, we claim that we can extend our methods to Petri nets with more general timing constraints or with multiple tokens in a place. Rather than attempt to prove these claims however, we would like to see where our approach can take us and then, if this work should turn out to be sufficiently useful, return to the beginning and set up a formal framework.

5.2 An analysis of the deterministic network functions

Our basic approach is to consider a (deterministic) network function as describing a dynamical system, and then to give some qualitative and quantitative analysis of this system. Our qualitative analysis will show that deterministic network functions have unique fixed points, which must therefore be the state function of the net. Our quantitative analysis will initially be based on Wadge's cycle sum test for the data flow networks, and will give us a test for liveness. We will further develop this analysis so as to gain information on the levels of liveness of the transitions of the net. What is important is that we get this information from the fixed point of a deterministic network function without having to find it.

5.2.1 Qualitative analysis: uniqueness of fixed points

Taking a domain theoretic approach, we will easily be able to give a complete description of the dynamics of a deterministic network function. Suppose $k \geq 1$ and $X^{(k)}$ is the space of functions given in section 5.1.3. We begin by showing that $X^{(k)}$ is a continuous domain.

Lemma 5.2.1 *$(X^{(k)}, \sqsubseteq)$ is a continuous domain, with*

$$f \sqsubseteq g \iff T_f \leq T_g \text{ and } f = g \text{ on } [0, T_f).$$

Proof. It is clear that $(X^{(k)}, \sqsubseteq)$ is a poset, and that $f, g \sqsubseteq h$ in $X^{(k)}$ implies either $f \sqsubseteq g$ or $g \sqsubseteq f$. So every directed set in $X^{(k)}$ is a chain. Suppose $\{f_\alpha\}_{\alpha \in A}$ is a chain in $X^{(k)}$, and let

$$T = \sup\{T_{f_\alpha} \mid \alpha \in A\} \in [0, \infty].$$

Then $f : [0, T) \to \mathrm{N}^k$, where $f = f_\alpha$ on $[0, T_{f_\alpha})$, is a well-defined partial (or total) function from $\Re^+$ to N^k with $T_f = T$. For any $n \in \mathrm{N}$ and $1 \leq i \leq k$, we have

$$\{t \in \Re^+ \mid f(t)_i < n\} = \bigcup_{\alpha \in A} \{t \in \Re^+ \mid f_\alpha(t)_i < n\},$$

so f is upper semicontinuous and is the supremum of the chain in $X^{(k)}$.

Suppose $f \sqsubseteq g$ in $X^{(k)}$, with $T_f < T_g$, and $\{h_\alpha\}_{\alpha \in A}$ is a chain in $X^{(k)}$ with supremum h above g. Then $T_f \leq T_{h_\alpha}$, for some $\alpha \in A$, and we have $f = g = h = h_\alpha$ on $[0, T_f)$.

So $f \sqsubseteq h_\alpha$ and it follows that $f \ll g$. Now let $f \in X^{(k)}$ and define $h_n = f$ on $[0, n)$, if f is total, or on $[0, T_f - 1/2^n)$ otherwise. So each $h_n \ll f$ and f is the supremum of the chain $h_1 \sqsubseteq h_2 \sqsubseteq \cdots$. Since the collection of functions well-below f must be a chain, then it follows that $X^{(k)}$ is a continuous domain.

QED

It is immediate from the definition of meets that, for any $f, g \in X^{(k)}$, $f = g$ on $[0, T_{f \sqcap g})$. Now, if $H : X^{(k)} \to X^{(k)}$ is one of the deterministic network functions from the last section, then we can observe that $Hf = Hg$ on $[0, T_{f \sqcap g} + 1)$, for all $f, g \in X^{(k)}$. We will assume this property to be true of all deterministic network functions. We are then in a position to prove that a deterministic network function has a unique fixed point, which must be the state function of the net, and that it is an *attracting fixed point*, to which each sequence, $\{H^n g\}$, in the domain converges.

Lemma 5.2.2 *Every deterministic network function has a unique attracting fixed point.*

Proof. Suppose $H : X^{(k)} \to X^{(k)}$ is a deterministic network function, then from our above assumption, we see that for any $f, g \in X^{(k)}$ we have $T_{Hf \sqcap Hg} = T_{f \sqcap g} + 1$. A simple induction shows us that $H^n f = H^n g$ on $[0, n)$, for any $n \geq 1$. Furthermore, if $n \geq m \geq 1$, then $H^m(H^{n-m}(g)) = H^m(g)$ on $[0, m)$ and hence $H^n(g) = H^m(g)$ on $[0, m)$. Now suppose that $g \in X^{(k)}$ is arbitrary, and we define $f \in X^{(k)}$ by

$$f = H^n(g) \qquad \text{on } [0, n) \qquad \forall n \geq 1,$$

which is well-defined by the above discussion. It also follows that since $H^{n+1}(g) = H^n(g)$ on $[0, n)$, for any $n \geq 1$, then we have $H(f) = f$, which is clearly a unique fixed point. Finally, if $g' \in X^{(k)}$, then for any $n \geq 1$, we have $H^n(g') = f$ on $[0, n)$, and it follows that $H^n(g') \to f$.

QED

5.2.2 Quantitative analysis: generalising the cycle sum test

It is clear that Wadge's cycle sum test, from section 3.1, is essentially a quantitative analysis of the function associated with a data flow network, since success with the test proves that the fixed point of the function is complete. We generalise this so that we can perform some quantitative analysis on a deterministic network function, and get information on its fixed point appropriate to the study of liveness.

We follow Wadge's intuition and identify the complete elements in $X^{(k)}$, as the state functions associated with a live Petri net. In fact, this example throws into focus Wadge's thoughts on maximality and completeness in a domain, since although the state functions are *always* maximal, we only consider the live state functions as complete. To make this precise, we recall from section 5.1.2, the definitions of $|f|$ and $|f_i|$ for state functions, and extend these to any $f \in X^{(k)}$, so that

$$|f_i| = \sup\{f(t)_i \mid t \in [0, T_f)\} \in \mathbb{N} \cup \{\infty\},$$

for each $1 \leq i \leq k$, and

$$|f| = \min\{|f_1|, \dots, |f_k|\}.$$

We refer to $|f|$ as the *size* of f, and define the *complete* elements in $X^{(k)}$ to be those with $|f| = \infty$. We see that a deterministic Petri net is live if, and only if, its state function is complete.

We make some observations regarding the basic functions used in building deterministic network functions. For any $f, g \in X^{(k)}$, we have that

$$|\min\{f, g\}| = \min\{|f|, |g|\},$$

$$|f + g| = |f| + |g|,$$

$$\text{and if} \quad g(t) = \begin{cases} 0, & t \in [0, 1), \\ f(t-1), & t \in [1, T_f + 1), \end{cases} \quad \text{then } |g| = |f|.$$

We now make explicit a simple result that was used in proving the cycle sum test in section 3.1.3. We let $\Re_{\min}$ denote the set $\Re \cup \{\infty\}$ together with the operations min and $+$, and let $*$ denote the matrix multiplication in this algebra.

Lemma 5.2.3 *Suppose M is a $k \times k$ matrix such that M^s has strictly positive entries, for some $s \geq 1$, and $\vec{v}$ is a k-vector with $\vec{v} \geq M * \vec{v}$, then $\vec{v} = (\infty, \ldots, \infty)$.*

Proof. The linear function, $\vec{v} \mapsto M * \vec{v}$, defined by M is monotonic, so that $\vec{v} \geq M^s * \vec{v}$. If $v_i \neq \infty$, for some $1 \leq i \leq k$, then we have

$$v_i \geq \min_{1 \leq j \leq k} ((M^s)_{ij} + v_j) > \min_{1 \leq j \leq k} v_j.$$

So we must have $\vec{v} = (\infty, \ldots, \infty)$ for otherwise, $\min_{1 \leq i \leq k} v_i > \min_{1 \leq j \leq k} v_j$ which is a contradiction.

QED

In section 3.1.3, the existence of such a matrix M and $s \geq 1$, was a direct consequence of the equations corresponding to a network passing the cycle sum test, and we took $\vec{v} = \sigma(\vec{x})$, for the least fixed point $\vec{x} \in S^k$ of the associated function. It followed that $\vec{x}$ was complete, and the network was free from deadlock.

Adopting the notation from section 3.1.3, we define $\sigma : X^{(k)} \to [0, \infty]^k$ by

$$\sigma(f) = \begin{pmatrix} |f_1| \\ \vdots \\ |f_k| \end{pmatrix}, \qquad \forall f \in X^{(k)}.$$

We say that $\sigma(f) \in [0, \infty]^k$ is the **vector of sizes** for f. We work once more with the example net given in figure 5.1(a), which has the network function $H : X^{(3)} \to X^{(3)}$ defined in section 5.1.3. If we let

$$M = \begin{pmatrix} 1 & 0 & 0 \\ \infty & \infty & 1 \\ \infty & 0 & \infty \end{pmatrix},$$

then, from our observations on the basic functions, it is clear that $\sigma(H(g)) = M * \sigma(g)$, for all $g \in X^{(3)}$. If $f \in X^{(3)}$ is the state function of our net (the fixed point of H), then $\sigma(f) = M * \sigma(f)$, and since M^4 has strictly positive entries then we can apply the lemma to see that f must be complete, and hence that the net is live.

If we now take our second worked example, from figure 5.1(b), which has network function $H : X^{(3)} \to X^{(3)}$ defined in section 5.1.4, then this is slightly more complicated since the map is not linear. However, we do have

$$\begin{pmatrix} |H(g)_1| \\ |H(g)_2| \\ |H(g)_3| \end{pmatrix} = \begin{pmatrix} |g_3| + 1 \\ |g_3| \\ |g_1| + |g_2| \end{pmatrix}, \qquad \forall g \in X^{(3)},$$

and if we take

$$M = \begin{pmatrix} \infty & \infty & 1 \\ \infty & \infty & 0 \\ 0 & \infty & \infty \end{pmatrix},$$

then it is clear that $\sigma(H(g)) \geq M * \sigma(g)$, for all $g \in X^{(3)}$. If $f \in X^{(3)}$ is the state function of our net, then $\sigma(f) \geq M * \sigma(f)$ still satisfies the conditions of the lemma, and since M^4 has strictly positive entries, then we can again deduce that our net is live.

5.2.3 A more sophisticated quantitative analysis

The analysis from the last section is useful as a test for liveness, but clearly goes no further than that. We now extend this so that we actually find the vector of sizes for the fixed point of a deterministic network function. Although this requires more work than before, it is still easier than finding the fixed point itself, and will give us the levels of liveness of the transitions of the Petri net. We begin with a lemma, which is simple to prove once we assume that deterministic network functions are only built from the basic functions that we considered in the last section.

Lemma 5.2.4 *Suppose $H : X^{(k)} \to X^{(k)}$ is a deterministic network function, then for any $f, g \in X^{(k)}$, if $\sigma(f) = \sigma(g)$ then $\sigma(H(f)) = \sigma(H(g))$.*

For such functions we can work with the vector of sizes for functions in $X^{(k)}$, rather than the functions themselves. We define,

$$\mathcal{R}(X^{(k)}) = \{\vec{u} \in [0, \infty]^k \mid \vec{u} = \sigma(f) \text{ for some } f \in X^{(k)}\},$$

then for any deterministic network function, $H : X^{(k)} \to X^{(k)}$, the **derived function**, $H' : \mathcal{R}(X^{(k)}) \to \mathcal{R}(X^{(k)})$, can be defined as follows. Suppose $\vec{u} \in \mathcal{R}(X^{(k)})$, then we can find $f \in X^{(k)}$ such that $\vec{u} = \sigma(f)$. We define

$$H'(\vec{u}) = \sigma(H(f)).$$

The next result shows that the least fixed point of H' in $[0,\infty]^k$ is the vector of sizes for the fixed point of H, which will give us the levels of liveness for the transitions of the Petri net.

Theorem 5.2.5 *Suppose $H : X^{(k)} \to X^{(k)}$ is a deterministic network function with unique fixed point $f \in X^{(k)}$, then the derived function, $H' : \mathcal{R}(X^{(k)}) \to \mathcal{R}(X^{(k)})$, has least fixed point $\sigma(f) \in \mathcal{R}(X^{(k)})$.*

Proof. If we let $L \subseteq \mathcal{R}(X^{(k)})$ be the collection of fixed points of H', then $L \neq \emptyset$ since $\sigma(f) \in L$. We let $\vec{\ell} = \min L$ (pointwise minimum in $[0,\infty]^k$). Suppose, for a contradiction, that for some $1 \leq i \leq k$, $\ell_i < |f_i|$, then there exists $\vec{u} \in L$ such that $u_i < |f_i|$. We let $g \in X^{(k)}$ be such that $\vec{u} = \sigma(g)$. It follows that $|g_i| < |f_i|$ so we can find $t \in [0, T_f)$ such that $f(t)_i > |g_i|$. We let $n > 0$ be such that $t \in [0, n)$, then we see that

$$|g_i| = u_i = (H')^n(\vec{u})_i = |H^n(g)_i| \geq H^n(g)(t)_i = f(t)_i \quad > \quad |g_i|,$$

which is a contradiction, and the result follows.

QED

We consider some worked examples. For the nets in figures 5.1(a) and 5.1(b) respectively, we have the derived functions,

$$H'(\vec{u}) = \begin{pmatrix} \min\{u_1+1, u_2, u_3\} \\ u_3+1 \\ u_2 \end{pmatrix}, \quad \text{and} \quad H'(\vec{u}) = \begin{pmatrix} u_3+1 \\ u_3 \\ u_1+u_2 \end{pmatrix},$$

for all $\vec{u} \in \mathcal{R}(X^{(3)})$. Both of these have least fixed point (∞, ∞, ∞), and we again deduce that our nets are live. Notice that these functions could be derived directly from the Petri nets themselves, and that we are still essentially performing a counting argument.

To consider a non-live example, suppose we take the Petri net in figure 5.1(a) once more, but remove the initial token from p_1. In this case our network function is such that, for each $g \in X^{(3)}$,

$$H(g)(t) = \begin{pmatrix} \min\{g(t-1)_1, g(t-1)_3, g(t-1)_2\} \\ g(t-1)_3 + 1 \\ g(t-1)_2 \end{pmatrix}, \qquad \forall t \in [1, T_g + 1),$$

and $H(g)(t) = (0,0,0)$ for all $t \in [0,1)$. It follows that the derived function becomes

$$H'(\vec{u}) = \begin{pmatrix} \min\{u_1, u_2, u_3\} \\ u_3 + 1 \\ u_2 \end{pmatrix}, \qquad \forall \vec{u} \in \mathcal{R}(X^{(3)}),$$

which has least fixed point $(0, \infty, \infty)$. We deduce that transitions q_2 and q_3 are live, whereas transition q_1 is dead.

5.3 A partial metric framework for the analysis

Having seen how to extend Wadge's ideas behind the cycle sum test for data flow networks to an analysis of liveness for (deterministic) Petri nets, we now consider how we can abstract the essential details. We will see that both our models, S^k and $X^{(k)}$, are weighted spaces, and that the weighted spaces can be used to develop the techniques from our analysis in a model-independent framework.

5.3.1 The data flow model

In section 3.1.3 our data flow model was S^k, the product of k copies of S^∞. If we seek an appropriate partial metric, then this should capture the domain theoretic aspects of the space, that is the Scott topology, as well as the complete elements which were crucial to our analysis. We will also find that the analytic techniques we develop will require that a suitable partial metric is complete. In section 3.3.5 we saw that S^∞ was naturally a partial metric space, and in section 4.3.1, that it was a weighted space. We begin by considering product spaces in general, from which we will deduce that S^k is a complete weighted space in which the complete elements are naturally captured.

Suppose (X_1, d_1) and (X_2, d_2) are partial metric spaces. A **product pmetric** on $X_1 \times X_2$ is any partial metric whose topology is the product of the pmetric topologies $\mathcal{T}_{[d_1]}$ and $\mathcal{T}_{[d_2]}$. For an element $\vec{x} \in X_1 \times X_2$ we write $\vec{x} = (x_1, x_2)$ with $x_1 \in X_1$ and $x_2 \in X_2$.

Lemma 5.3.1 *Suppose (X_1, d_1) and (X_2, d_2) are partial metric spaces, then*

$$d(\vec{x}, \vec{y}) = d_1(x_1, y_1) + d_2(x_2, y_2), \qquad \forall \vec{x}, \vec{y} \in X_1 \times X_2,$$

is a product pmetric on $X_1 \times X_2$ whose associated metric, d^*, is Lipschitz equivalent to the product metric $d_1^* + d_2^*$ on $X_1 \times X_2$.

Proof. The partial metric axioms are straightforward to verify. Now suppose that $\delta, \varepsilon_1, \varepsilon_2 > 0$, then

$$B_\varepsilon(\vec{x}; d) \subseteq B_{\varepsilon_1}(x_1; d_1) \times B_{\varepsilon_2}(x_2; d_2), \quad \text{where } \varepsilon = \min\{\varepsilon_1, \varepsilon_2\},$$

$$\text{and} \qquad B_{\delta/2}(x_1; d_1) \times B_{\delta/2}(x_2; d_2) \subseteq B_\delta(\vec{x}; d).$$

It follows that $\mathcal{T}_{[d]} = \mathcal{T}_{[d_1]} \times \mathcal{T}_{[d_2]}$, and d is a product pmetric. We now observe that

$$\begin{aligned}
[d_1^*(x_1, y_1) + d_2^*(x_2, y_2)]/2 &\leq \max\{d_1^*(x_1, y_1), d_2^*(x_2, y_2)\} \\
&= \max\{d_1(x_1, y_1) - d_1(x_1, x_1), d_1(x_1, y_1) - d_1(y_1, y_1), \\
&\qquad d_2(x_2, y_2) - d_2(x_2, x_2), d_2(x_2, y_2) - d_2(y_2, y_2)\} \\
&\leq \max\{d(\vec{x}, \vec{y}) - d(\vec{x}, \vec{x}), d(\vec{x}, \vec{y}) - d(\vec{y}, \vec{y})\} \\
&= d^*(\vec{x}, \vec{y}).
\end{aligned}$$

Since we also have $d^*(\vec{x},\vec{y}) \leq d_1^*(x_1,y_1) + d_2^*(x_2,y_2)$, then d^* is Lipschitz equivalent to the product metric $d_1^* + d_2^*$ on $X_1 \times X_2$.

QED

Lemma 5.3.2 *Suppose $(X_1, \sqsubseteq_1, \phi_1)$ and $(X_2, \sqsubseteq_2, \phi_2)$ are weighted spaces, then the product $(X_1 \times X_2, \sqsubseteq)$ is a consistent semilattice, and $\phi : X_1 \times X_2 \to \Re$ defined by*

$$\phi(\vec{x}) = \phi_1(x_1) + \phi_2(x_2), \qquad \forall \vec{x} \in X_1 \times X_2,$$

is a weight function on $(X_1 \times X_2, \sqsubseteq)$ that induces the above product pmetric.

Proof. Straightforward.

QED

From section 4.3.1, we recall that $\phi(x) = 2^{-|x|}$ is a weight function for S^∞, and it follows from the lemma that S^k is a weighted space. We show that S^∞ is complete, and then it follows that S^k is a complete weighted space.

Lemma 5.3.3 *The weighted space $(S^\infty, \sqsubseteq, \phi)$ is complete.*

Proof. Suppose $\{x_n\}$ is a Cauchy sequence in S^∞, then $\lim_{n,m\to\infty} d(x_n, x_m) = 2^{-t}$ for some $t \in \mathrm{N} \cup \{\infty\}$. If $t < \infty$, then we can find $N \geq 1$ such that for any $n, m \geq N$, $d(x_n, x_m) = 2^{-t}$. It follows that $|x_N| = t$, and for each $n \geq N$, $x_N = x_n$, so we see that x_N is the proper limit of the sequence. If $t = \infty$, then we let $N_0 = 1$ and inductively find, for each $k \geq 1$, $N_k \geq N_{k-1}$ such that for all $n, m \geq N_k$,

$$d(x_n, x_m) < 2^{-(k+1)} + 2^{-t} \leq 2^{-k}.$$

It follows that $|x_{N_k}| > k$ and for each $n \geq N_k$, $x_{N_k}[k] = x_n[k]$. We define

$$a_k = x_{N_k}[k] \in S^\infty, \quad \text{and} \quad a = \bigsqcup_{k=1}^{\infty} a_k \in S^\infty,$$

and see that $|a| = \infty$. Since ϕ is continuous, and for any $k \geq 1$ and $n \geq N_k$, we have $a_k \sqsubseteq a \sqcap x_n$, then

$$\phi(a) \leq d(a, x_n) = \phi(a \sqcap x_n) \leq \phi(a_k),$$

and a is the proper limit of our sequence.

QED

We already know, but it also follows from the lemma and theorem 4.3.5, that the induced pmetric topology on S^∞, and hence S^k, is the Scott topology. We are left to show that our pmetric captures the complete elements that we identified in section 3.1.3. We see that a point $\vec{x} \in S^k$ is complete if, and only if, $\phi(\vec{x}) = 0$, which is precisely when $|x_i| = \infty$ for each $1 \leq i \leq k$. So we see that our data flow model fits neatly into a complete weighted space framework. Once we show that the same is true of the Petri net model, then we will develop some analytic techniques in this more general setting.

5.3.2 The Petri net model

We therefore seek a complete partial metric on the Petri net model, $X^{(k)}$, from section 5.1.3, which captures the Scott topology and the appropriate complete elements. We let X be the collection of partial and total upper semicontinuous functions, $f : \Re^+ \to \mathbf{N}$, for which there exists $T_f \in [0, \infty]$ such that f is defined on precisely $[0, T_f)$. For $k \geq 1$, we let X^k be the product of k copies of X, and for $f \in X^k$, we let $f_i \in X$, $1 \leq i \leq k$, be defined by

$$f_i(t) = f(t)_i, \qquad \forall t \in [0, \infty].$$

It is clear that $X^{(k)}$ is not the product space X^k, but that it embeds in X^k as those functions for which the T_{f_i} are constant for all $1 \leq i \leq k$. We choose to work with the larger space X^k since it is easier to show that X^k is a weighted space, by giving the result for X, than it is to show directly that $X^{(k)}$ is a weighted space. We begin by showing that $(X, \sqsubseteq)$ is a consistent semilattice.

Lemma 5.3.4 *$(X, \sqsubseteq)$ is a consistent semilattice.*

Proof. For any $f, g \in X$ we define

$$T_{f \sqcap g} = \sup\{t \in [0, \min\{T_f, T_g\}) \mid f(t) = g(t)\},$$

and $(f \sqcap g)(t) = f(t) = g(t)$ on $[0, T_{f\sqcap g})$, so that $f \sqcap g$ is the meet of f and g in X. Also, if f, g are below h in X then either $f \sqsubseteq g$ or $g \sqsubseteq f$.

QED

One problem we have in defining a weight function on $(X, \sqsubseteq)$ is that functions in X map from an unbounded domain, $\Re_+$, to an unbounded range $\mathbf{N}$. We will therefore make use of $\varphi : [0, \infty] \to [0, 1]$, which we define as the bijection $\varphi(t) = 1/2^t$.

Lemma 5.3.5 *The function $\phi : X \to [0, 2]$, defined by,*

$$\phi(f) = \varphi(|f|)[\varphi(T_f) + 1], \qquad \forall f \in X,$$

is a weight function on the consistent semilattice $(X, \sqsubseteq)$, whose induced pmetric topology is the Scott topology.

Proof. It is clear that if $f \sqsubseteq g$, then $T_f \le T_g$ and $|f| \le |g|$, so that $\phi(f) \ge \phi(g)$. Suppose further that $\phi(f) = \phi(g) > 0$, then $T_f = T_g$ and $f = g$. If $\phi(f) = \phi(g) = 0$, then $|f| = |g| = \infty$, which implies that $T_f = T_g$ and $f = g$ once more. Since $f, g \in X$ below h in X implies that either $f \sqsubseteq g$ or $g \sqsubseteq f$, then ϕ is a weight function.

We now show that the induced pmetric topology is the Scott topology. We can deduce from lemma 5.2.1 that $f \ll g$ in $(X, \sqsubseteq)$ if, and only if $f \sqsubseteq g$ and $T_f < T_g$. Suppose $g \in \Uparrow f$ and we let $f \ll g' \ll g$, then g' is strictly below g which implies that $|g'| < \infty$. We let

$$\varepsilon = \varphi(|g'|)[\varphi(T_f) - \varphi(T_{g'})] > 0.$$

If $h \in B_\varepsilon(g')$, then $\phi(g' \sqcap h) < \phi(g') + \varepsilon$, which implies that

$$\varphi(T_{g'\sqcap h}) + 1 \le \frac{\varphi(|g' \sqcap h|)}{\varphi(|g'|)}[\varphi(T_{g'\sqcap h}) + 1] \quad < \quad \varphi(T_{g'}) + 1 + \varphi(T_f) - \varphi(T_{g'}).$$

It follows that $T_f < T_{g' \sqcap h} \leq T_h$ and hence $f \ll h$ and we see that $g \in B_\varepsilon(g') \subseteq \Uparrow f$. So the Scott open sets are open in the induced pmetric topology.

Now suppose that $g \in X$, $\varepsilon > 0$ and $h \in B_\varepsilon(g)$. We find $f \sqsubseteq g$ such that

$$0 < \varphi(T_f) - \varphi(T_g) < \varepsilon/2 \quad \text{and} \quad \varphi(|f|) - \varphi(|g|) < \frac{\varepsilon}{\varepsilon + 4}.$$

We see that $T_f < T_g$ and $f \ll g$. Furthermore, we have

$$\begin{aligned}
\phi(f \sqcap g) = \phi(f) &< \left[\varphi(|g|) + \frac{\varepsilon}{\varepsilon+4}\right]\left[\varphi(T_g) + \frac{\varepsilon}{2} + 1\right] \\
&= \phi(g) + \frac{\varepsilon\varphi(|g|)}{2} + \frac{\varepsilon}{\varepsilon+4}\left[\varphi(T_g) + \frac{\varepsilon}{2} + 1\right] \\
&\leq \phi(g) + \frac{\varepsilon}{2} + \frac{\varepsilon}{\varepsilon+4}\left[\frac{\varepsilon+4}{2}\right] \\
&= \phi(g) + \varepsilon.
\end{aligned}$$

It follows that $f \in B_\varepsilon(g)$ and hence that $g \in \Uparrow f \subseteq B_\varepsilon(g)$, and the result follows.

QED

Having captured the Scott topology, we now turn to the complete elements in $X^{(k)}$, which we identified in section 5.2.2, and see that $\phi(f) = 0$ if, and only if, $|f_i| = \infty$ for each $1 \leq i \leq k$ which implies that $|f| = \infty$. So it appears that our partial metric is appropriate for our purposes. The problem however, is that it is not complete. To see this, consider the decreasing chain $\{f_n\}$ in X, defined by

$$f_n(t) = \begin{cases} 0, & t \in [0, 1), \\ 1, & t \in [1, 1 + 1/n), \end{cases}$$

which is a Cauchy sequence since $\sup\{\phi(f_n) \mid n \geq 1\} = 3/4$. The infimum of this chain is given by $f(t) = 0$ on $[0, 1)$, which is not the proper limit since $\phi(f) = 3/2$. The difficulty is that the "proper limit" should be

$$g(t) = \begin{cases} 0, & t \in [0, 1), \\ 1, & t = 1, \end{cases}$$

which is not in X. One way around this problem is to work with the completion, $\overline{X}^k$, of X^k, which we know to be a complete weighted space from section 4.3.4. In this way the Petri net model also fits into the framework of the weighted spaces, although we must be careful when we seek to interpret the results of the more abstract analysis.

5.3.3 The cycle sum test revisited

Having established the framework of complete weighted spaces as suitable, we now place the analysis of section 5.2.2 in this more general framework. Matthews attempts something similar in [Mat95], but his results are sketchy, and the details more intricate than necessary. However, his use of contraction maps is still relevant and we pursue this to begin with. We remark that by restricting ourselves to the weighted spaces, we can simplify the presentation, while leaving it clear how to extend the results to partial metrics.

We begin with Matthews' definition [Mat94] of a contraction map over a (positive) partial metric space, and the contraction mapping theorem that he generalises directly from the metric case. Suppose (X, d) is a complete positive partial metric space, and $f : X \to X$ is such that there exists $\alpha < 1$ with

$$d(fx, fy) \leq \alpha d(x, y), \qquad \forall x, y \in X, \tag{5.1}$$

then we say that f is a *contraction* on X.

Theorem 5.3.6 (Contraction Mapping Theorem [Mat94]) *Suppose (X, d) is a complete positive partial metric space, and $f : X \to X$ is a contraction, then f has a unique attracting fixed point $a \in X$ and $\phi_d(a) = 0$.*

For the weighted spaces, we can simplify the contraction condition (5.1) so that we need only consider points, rather than pairs of points, but require that our functions are monotonic.

Lemma 5.3.7 *Suppose $(X, \sqsubseteq, \phi)$ is a complete positive weighted space, and $f : X \to X$ is a monotonic function for which there exists a positive $\alpha < 1$ such that*

$$\phi(fx) \leq \alpha\phi(x), \qquad \forall x \in X,$$

then f is a contraction on X.

Proof. Monotonicity implies that $f(x \sqcap y) \sqsubseteq fx \sqcap fy$, for any $x, y \in X$, so we immediately have

$$d(fx, fy) = \phi(fx \sqcap fy) \leq \phi(f(x \sqcap y)) \quad \leq \quad \alpha\phi(x \sqcap y) = \alpha d(x, y).$$

QED

We recall that in section 5.2.2, we had a deterministic network function, $H : X^{(k)} \to X^{(k)}$, and a matrix, M, for which some power had strictly positive entries, satisfying $\sigma(H(g)) \geq M * \sigma(g)$, for all $g \in X^{(k)}$. To place this within the framework of complete weighted spaces, we first fix $g \in X^{(k)}$, so that for any $1 \leq i \leq k$, we have

$$|H(g)_i| \geq \min_{1 \leq j \leq k} M_{ij} + |g_j|.$$

Taking the definition of φ from the last section, we see that

$$\varphi(|H(g)_i|) \leq \max_{1 \leq j \leq k} \varphi(M_{ij})\varphi(|g_j|).$$

An important property of deterministic network functions that we make use of is that $\varphi(T_{H(g)}) \leq \varphi(T_g)$, and so

$$\phi(H(g)_i) \leq \max_{1 \leq j \leq k} \varphi(M_{ij})\phi(g_j).$$

If we adapt the notation of section 5.2.2, so that $\sigma_\phi : X^{(k)} \to \Re^k$ is given by

$$\sigma_\phi(\vec{x}) = \begin{pmatrix} \phi(x_1) \\ \vdots \\ \phi(x_k) \end{pmatrix}, \qquad \forall \vec{x} \in X^{(k)},$$

then our inequality becomes,

$$\sigma_\phi(H(g)) \leq \varphi(M) * \sigma_\phi(g), \qquad \forall g \in X^{(k)},$$

where $*$ now denotes matrix multiplication in the algebra $\Re_{\max}$, which has operations max and product, and $\varphi(M)$ is the $k \times k$ matrix with each $\varphi(M)_{ij} = \varphi(M_{ij})$. Abstracting to the product of k copies of an arbitrary complete positive weighted space, we get a result that combines the uniqueness result of section 5.2.1 together with the generalised cycle sum test of section 5.2.2.

Theorem 5.3.8 *Suppose $(X, \sqsubseteq, \phi)$ is a complete weighted space with $k \geq 1$, and we have $f : X^k \to X^k$ monotonic, a $k \times k$ matrix, M, such that, M^s has entries strictly less than some $\alpha < 1$, for some $s \geq 1$, and*

$$\sigma_\phi(f(\vec{x})) \leq M * \sigma_\phi(\vec{x}), \qquad \forall \vec{x} \in X^k,$$

then f has a unique attracting fixed point, $\vec{a} \in X^k$, and $\phi(\vec{a}) = 0$.

Proof. We first find $t \geq 1$ such that $\alpha^t < 1/k$, and show that f^{st} is a contraction on X^k, from which the result follows. We immediately see that

$$\phi(f^{st}\vec{x}) = \sum_{i=1}^{k} \phi((f^{st}\vec{x})_i) \quad \leq \quad \sum_{i=1}^{k} \max_{1 \leq j \leq k} M_{ij}^{st} \phi(x_j) \leq \sum_{i=1}^{k} \alpha^t \phi(\vec{x}) = k\alpha^t \phi(\vec{x}).$$

It follows that f^{st} is a contraction on X^k since $k\alpha^t < 1$.

QED

We remark that to re-apply this general result to the Petri net model, $\overline{X}^k$, we would still need to know that the state function of a deterministic Petri net is a fixed point of its network function to ensure that the unique fixed point of the theorem is in $X^{(k)} \subseteq \overline{X}^k$.

5.3.4 Uniform and derived functions

Continuing this process of abstracting to the framework of weighted spaces, we now look at section 5.2.3, and abstract an essential property of deterministic network functions. Suppose $(X, \sqsubseteq, \phi)$ is a weighted space and $k \geq 1$, then we define

$$\mathcal{R}(\phi)^k = \{\vec{u} \in \Re^k \mid \vec{u} = \sigma_\phi(\vec{x}) \text{ for some } \vec{x} \in X^k\}.$$

Definition 5.3.9 *We say that a function $f : X^k \to X^k$ is* **uniform** *if, for any $\vec{x}, \vec{y} \in X^k$, then $\sigma_\phi(\vec{x}) = \sigma_\phi(\vec{y})$ implies that $\sigma_\phi(f\vec{x}) = \sigma_\phi(f\vec{y})$.*

As with the deterministic network functions, for any uniform function we can define the **derived function,** $f' : \mathcal{R}(\phi)^k \to \mathcal{R}(\phi)^k$ as follows. Suppose $\vec{u} \in \mathcal{R}(\phi)^k$, then we can find $\vec{x} \in X^k$ such that $\vec{u} = \sigma_\phi(\vec{x})$. We define

$$f'(\vec{u}) = \sigma_\phi(f\vec{x}).$$

To generalise theorem 5.2.5, we also require that our uniform functions have an attracting fixed point. If we let the vector of sizes now refer to $\sigma_\phi(\vec{x}) \in \mathcal{R}(\phi)^k$, then for uniform functions with an attracting fixed point, we can find the vector of sizes of that fixed point, without finding the point itself. However, to do this we further require that our weighted space, $(X, \sqsubseteq, \phi)$, is *bounded above*, so that $\{\phi(x) \mid x \in X\}$ has an upper bound in $\Re$.

Lemma 5.3.10 *Suppose $(X, \sqsubseteq, \phi)$ is a complete positive weighted space, bounded above, with $k \geq 1$, and $f : X^k \to X^k$ is a monotonic uniform function with an attracting fixed point $\vec{a} \in X^k$, then the derived function $f' : \mathcal{R}(\phi)^k \to \mathcal{R}(\phi)^k$ has a maximal fixed point $\sigma_\phi(\vec{a}) \in \mathcal{R}(\phi)^k$.*

Proof. We first observe that for any $\vec{x} \in X^k$ and $1 \leq i \leq k$, we have $\phi(a_i) \leq \phi((f^n\vec{x} \sqcap \vec{a})_i)$, and since

$$\lim_{n\to\infty} \sum_{i=1}^{k} \phi((f^n\vec{x} \sqcap \vec{a})_i) = \lim_{n\to\infty} \phi(f^n\vec{x} \sqcap \vec{a}) = \phi(\vec{a}) = \sum_{i=1}^{k} \phi(a_i),$$

then for any $1 \leq i \leq k$, we have

$$\lim_{n\to\infty} \phi((f^n\vec{x})_i) \leq \lim_{n\to\infty} \phi((f^n\vec{x} \sqcap \vec{a})_i) = \phi(a_i).$$

If we let $L \subseteq X^k_{\sigma_\phi}$ be the collection of fixed points of f', then $L \neq \emptyset$ since $\sigma_\phi(\vec{a}) \in L$. We let $\vec{\ell} = \sup L$ (pointwise supremum in $\Re^k$). Suppose, for a contradiction, that for some $1 \leq i \leq k$, $\ell_i > \phi(a_i)$, then there exists $\vec{u} \in L$ such that $u_i > \phi(a_i)$. We let $\vec{x} \in X^k$ be such that $\vec{u} = \sigma_\phi(\vec{x})$. However,

$$\phi(a_i) \geq \lim_{n\to\infty} \phi((f^n\vec{x})_i) = \lim_{n\to\infty} (f')^n(\sigma_\phi(\vec{x}))_i = \phi(x_i) = u_i,$$

which is our contradiction, and the result follows.

QED

Our final result allows us to deduce that a function has an attracting fixed point, and will be useful in applying the above lemma. As a shorthand notation, we write $x_n \searrow x$ for a sequence $\{x_n\}$ in $\Re$ converging to $x \in \Re$ from above.

Lemma 5.3.11 *Suppose $(X, \sqsubseteq, \phi)$ is a complete weighted space, $f : X \to X$ is continuous and $\ell \in \Re$ is such that $\phi(x) \geq \ell$, for some $x \in X$, and $\phi(x) \geq \ell$ implies that $\phi(f^n x) \searrow \ell$, then f has an attracting fixed point $a \in X$ with $\phi(a) = \ell$.*

Proof. Suppose $x \in X$ with $\phi(x) \geq \ell$, then for any $m \geq n \geq N$, $\phi(x \sqcap f^{m-n}x) \geq \phi(x) \geq \ell$, and

$$\ell \leq \phi(f^n x) \leq \phi(f^n x \sqcap f^m x) \leq \phi(f^n(x \sqcap f^{m-n}x)),$$

which converges to ℓ from above, and so $\{f^n x\}$ is Cauchy. We let $a \in X$ be the proper limit of this sequence, then $\phi(a) = \ell$ and continuity implies that a is a fixed point of f. Now suppose that $x \in X$ is any point, then for any $n \geq 1$,

$$\ell \leq \phi(a) \leq \phi(a \sqcap f^n x) \leq \phi(f^n(a \sqcap x)),$$

which converges to ℓ from above. It follows that $f^n x \to a$ and we are done.

QED

These results are an indication of the techniques that we can develop within the framework of complete weighted spaces, and are effectively the techniques that we used in our analysis of deadlock in data flow networks and liveness of Petri nets. There is clearly much scope for improvement as well as for developing further techniques.

5.4 The problem of non-determinism

We now consider the problem of extending our methods from section 5.1 to non-deterministic Petri nets. Rather than giving detailed results, this section will be essentially discursive, and we will try to motivate our ideas and some possible further directions for the work.

One way to deal with non-determinism is to introduce an external choice function, or oracle, knowledge of which would allow us to assign tokens from places to transitions. Suppose a Petri net has k-transitions and ℓ-places, then we define an *oracle* to be a map

$$\gamma : \{1, \ldots, k\} \times \{1, \ldots, \ell\} \times \mathrm{N} \rightarrow \mathrm{N} \cup \{\infty\},$$

such that $\gamma(i, j, r)$ is the number of tokens to have become available from place p_i to transition q_j if there have been r tokens available from p_i, and ∞ if there is no arc from p_i to q_j. We use ∞, rather than 0, to distinguish the situation where there is no arc from a place to a transition, from the situation where there are no tokens available from a place. We write $\gamma_{i \rightarrow j}(r)$ for $\gamma(i, j, r)$.

Consider the basic structure for non-determinism, given in figure 5.4(a), where the number of firings of q_1 and q_2 at time $t \geq 1$ is the number of tokens to have arrived at p_1 at time $t - 1$, and is 0 at time $t \in [0, 1)$. If we consider the non-deterministic net in figure 5.4(b) as our worked example, then it is clear that

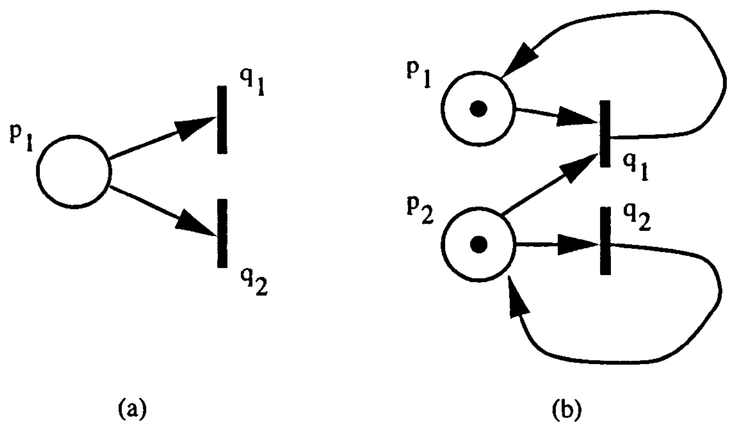


Figure 5.4: Non-determinism in Petri nets

any oracle must have $\gamma_{1\to 1}(r) = r$ and $\gamma_{1\to 2}(r) = \infty$, for any $r \in \mathbb{N}$. The (non-deterministic) network function $H_\gamma : X^{(2)} \to X^{(2)}$, which depends on the oracle γ, is such that, for each $g \in X^{(2)}$,

$$H_\gamma(g)(t) = \begin{pmatrix} \min\{g(t-1)_1 + 1, \gamma_{2\to 1}(g(t-1)_2 + 1)\} \\ \gamma_{2\to 2}(g(t-1)_2 + 1) \end{pmatrix}, \qquad \forall t \in [1, T_g + 1),$$

and $H_\gamma(g)(t) = (0,0)$ for all $t \in [0,1)$.

If we know the oracle *a priori* then our Petri net is essentially deterministic, and the analysis is as before. For example, suppose the oracle γ sends every token from p_2 to q_2, then $\gamma_{2\to 1}(r) = 0$, and $\gamma_{2\to 2}(r) = r$, for any $r \in \mathbb{N}$. Suppose as well that the oracle γ' sends the first token at p_2 to q_1, and subsequent tokens to q_2, so that

$$\gamma'_{2\to 1}(r) = \begin{cases} 0, & r = 0, \\ 1, & r \geq 1, \end{cases} \qquad \gamma'_{2\to 2}(r) = \begin{cases} 0, & r = 0, \\ r - 1, & r \geq 1. \end{cases}$$

The network functions $H_\gamma, H_{\gamma'} : X^{(2)} \to X^{(2)}$ are such that, for each $g \in X^{(2)}$,

$$H_\gamma(g)(t) = \begin{pmatrix} 0 \\ g(t-1)_2 + 1 \end{pmatrix}, \quad H_{\gamma'}(g)(t) = \begin{pmatrix} 1 \\ g(t-1)_2 \end{pmatrix}, \qquad \forall t \in [1, T_g + 1),$$

and $H_\gamma(g)(t) = H_{\gamma'}(g)(t) = (0,0)$ for all $t \in [0,1)$. The derived functions $H'_\gamma, H'_{\gamma'}$: $\mathcal{R}(X^{(2)}) \to \mathcal{R}(X^{(2)})$ are then

$$H'_\gamma(\vec{u}) = \begin{pmatrix} 0 \\ u_2+1 \end{pmatrix}, \quad \text{and} \quad H'_{\gamma'}(\vec{u}) = \begin{pmatrix} 1 \\ u_2 \end{pmatrix}, \qquad \forall \vec{u} \in \mathcal{R}(X^{(2)}),$$

which have least fixed points $(0,\infty)$ and $(1,0)$ respectively. So we see that, for our non-deterministic Petri net, the transition q_1 is L1-live and transition q_2 is L3-live.

The challenge is to perform the analysis when the oracle is unknown. In effect, what we are trying to do, is to analyse many deterministic network functions, indexed by the oracles. We suggest that this is not necessarily the most natural extension of the deterministic position, and rather moves away from our attempts to analyse a dynamical system associated with a Petri net. In a more natural extension, we would expect a (non-deterministic) network function $H : X^{(k)} \to X^{(k)}$ to have as fixed points precisely the possible state functions of the net. The dynamics of such a function will obviously be more complicated, and require a more sophisticated analysis, but this is part of the natural evolution away from the analysis of systems with particularly simple dynamics.

A natural question arises. Suppose we have such a network function, what would it be like? Intuitively we can think of network functions as taking points in $X^{(k)}$ towards a "nearest" state function. Whenever we are "close" to a state function in $X^{(k)}$ our network function then behaves like a deterministic network function. We can think of a non-deterministic network function as being locally like a deterministic network function.

But how could we find such a function? Of course, if we knew the state functions of the net, then we could just send each $g \in X^{(k)}$ to the nearest state function, but we want to avoid finding the state functions explicitly. Consider the following approach. Suppose we let Γ be the collection of oracles for a non-deterministic Petri net, then for any map $\tau : X^{(k)} \to \Gamma$, which we call *selective* *maps*, we can define a network function $H_\tau : X^{(k)} \to X^{(k)}$ by

$$H_\tau(g) = H_{\tau(g)}(g), \qquad \forall g \in X^{(k)}.$$

Although this depends on the selective map, it is independent of any particular oracle. Any fixed point, $g \in X^{(k)}$, of H_τ must be the fixed point of $H_{\tau(g)}$, and therefore a state function of the net. To ensure that we capture all the state functions we restrict our choice of selective maps.

We can use the induced metric on our space $X^{(k)}$ from the last section, to define a selective map to be *minimal* whenever

$$d^*(g, H_{\tau(g)}(g)) = \min\{d^*(g, H_\gamma(g)) \mid \gamma \in \Gamma\}, \qquad \forall g \in X^{(k)},$$

and consider the network functions induced by minimal selective maps. For any state function of our net, $f \in X^{(k)}$, we can find an oracle $\gamma \in \Gamma$, such that $d^*(f, H_\gamma(f)) = 0$. It follows that $d^*(f, H_{\tau(f)}(f)) = 0$, and $H_\tau(f) = f$. The fixed points of network functions induced by minimal selective functions are therefore precisely the state functions of the net.

Admittedly this approach doesn't seem entirely natural, and we haven't said anything about the existence of such minimal selective maps, but it does, at least theoretically, give us a network function whose fixed points are the state functions of a non-deterministic Petri net, without knowing these in advance. An interesting point that we wish to make, and this is where we invoke the further theoretical investigations of the next chapter, is that a distance function on some space of functions over $X^{(k)}$ would allow us to define a network function as the particular H_τ that is closest to the identity, as we range over all minimal selective maps. Such a network function would have the required properties, and does have a certain naturality about it.

Chapter 6

Developing a Partial Metric Framework for the Analysis

What we would ideally like is some framework in which we can develop the analytic techniques that we were building towards in chapter 5. We suggest that an important initial step is to develop our understanding of how we can put a partial metric on a function space. In itself this is a very difficult problem, beyond the scope of our thesis. So we will restrict ourselves to identifying the measure-theoretic aspects of partial metric spaces as fundamental (section 6.1) with the "well-behaved" partial metrics closely related to measures. In developing this link (section 6.2) we will be lead to a detailed investigation of some new properties of T_0-spaces (sections 6.3 and 6.4), which turn out to be of interest in their own right. What we hope is that this material will prove sufficient as a platform from which to tackle the more general problems.

6.1 Motivating a measure-theoretic approach

In a domain-theoretic approach to the problem of function spaces, we would seek cartesian closed categories of partial metric spaces. We have already made some progress in this direction since, in section 4.3.3, we showed that the Scott-domains, which form a cartesian closed category, are weighted spaces. We observe however, that the weight function on a function space is derived from the underlying set, by enumerating the compact elements, rather than from the constituent weight functions. This is at odds with the following metrics on the space, $C[0,1]$, of continuous functions over $[0,1]$;

$$d_1(f,g) = \int_0^1 |f(x) - g(x)|dx,$$

$$\text{and} \quad d_\infty(f,g) = \sup_{x\in[0,1]} |f(x) - g(x)|dx, \quad \forall f,g \in C[0,1],$$

where it is clear that each metric is derived from the Euclidean metric on $[0,1]$. We suggest that since the material from chapter 5 seems to be leading us towards an analytic theory analogous to some of real analysis, then we should want the weight functions on a function space to be derived from their constituent weight functions.

Consider the problem of finding a partial metric for $C[0,1]$. Since we used the sup approach in section 2.1.3 to give a cartesian closed category of metric domains, then it may seem the obvious choice here. However, suppose we could find a partial metric, d, on $C[0,1]$ such that the induced metric, d^*, is the sup-metric, and the specialisation order is such that $f \leq_d g$ if, and only if, $f(x) \geq g(x)$, for all $x \in [0,1]$. Suppose $f, g, h : [0,1] \to [0,1]$ are the functions $f(x) = x^2$, $g(x) = x$ and $h(x) = 1$, for any $x \in [0,1]$. Clearly $h \leq_d g \leq_d f$, and we have

$$\phi_d(h) - \phi_d(f) = d_\infty(h,f) = 1 = d_\infty(h,g) = \phi_d(h) - \phi_d(g),$$

from which we see that $\phi_d(f) = \phi_d(g)$ and hence $f = g$ which is a contradiction. We are therefore naturally lead to the alternative, which is to consider integration and measures.

Suppose μ is the Lebesgue measure on $[0,1]$ and recall from section 4.3.1, the weighted space $([0,1], \geq, \phi)$, where $\phi : [0,1] \to [0,1]$ is given by $\phi(x) = x$. It is clear that $(C[0,1], \geq)$ is a consistent semilattice, and we can define a weight function

$\Phi : C[0,1] \to [0,\infty)$, by

$$\Phi(f) = \int_{[0,1]} \phi \circ f d\mu = \int_0^1 f(x)dx, \qquad \forall f \in C[0,1].$$

The induced pmetric is $d(f,g) = \int_0^1 \max\{f(x), g(x)\}dx$, and the induced metric is d_1. An example of Schellekens [Sch95], in which he demonstrates the applicability of the partial metrics to the complexity analysis of programs, is a second example of how a partial metric on a function space can be derived from a measure.

An abstract *complexity measure* [DW83] is a binary partial function, $C(k,n)$, on $\mathbf{N}^2$ such that $C(k,n)$, the complexity of a program P with coding k on input n is defined if, and only if, P converges on input n, and the predicate $C(k,n) \leq y$ is recursive. An example of a complexity measure would be the running time of a program. If we suppose that $C(k,n)$ has infinite complexity when it is undefined, and that the complexity measure is always non-zero, then a program, P, with coding k has *complexity function* $C_P \in (0,\infty]^{\mathbf{N}}$ given by $C_P(n) = C(k,n)$, for all $n \in \mathbf{N}$. Schellekens [Sch95] defines a partial metric on the function space $(0,\infty]^{\mathbf{N}}$ by

$$d(f,g) = \sum_{n=0}^{\infty} \frac{1}{2^n \min\{f(n), g(n)\}}, \qquad \forall f,g \in (0,\infty]^{\mathbf{N}},$$

and defines the *complexity distance* between two programs P and Q as the distance between C_P and C_Q in $(0,\infty]^{\mathbf{N}}$. Schellekens goes on to give a complexity analysis of mergesort, showing that divide and conquer algorithms naturally induce contraction maps over $(0,\infty]^{\mathbf{N}}$. However, we are more interested in how this fits into our general framework, although we hope that it does hint at some future applications.

We can define a counting measure on $\mathbf{N}$ by taking the power set as our σ-algebra and defining $\mu(A) = \sum_{n \in A} 1/2^n$, for any $A \subseteq \mathbf{N}$. For the consistent semilattice $((0,\infty], \leq)$, we can define a positive weight function $\phi : (0,\infty] \to [0,\infty)$ by $\phi(x) = 1/x$. The function space $((0,\infty]^{\mathbf{N}}, \leq)$ is a consistent semilattice, and we can define a positive weight function $\Phi : (0,\infty]^{\mathbf{N}} \to [0,\infty)$ by

$$\Phi(f) = \int_{\mathbf{N}} \phi \circ f d\mu = \sum_{n=0}^{\infty} \frac{1}{2^n f(n)}, \qquad \forall f \in (0,\infty]^{\mathbf{N}}.$$

The partial metric induced by this weight function is Schellekens' partial metric.

We can abstract the essential details of these examples as follows. Suppose X is a measure space with positive measure μ, and Y is a weighted space with positive weight function ϕ, then we define, for some collection of measurable functions $f : X \to Y$, the weight function

$$\Phi(f) = \int_X \phi \circ f d\mu.$$

We must be careful to restrict our choice of functions so that the above integral is always finite, and take care when two functions agree almost everywhere. So that the weight function for the function space is derived from the weight functions of *both* constituent weighted spaces, we seek to replace the measure space (X, μ) with a weighted space whose weight function is closely related to the measure μ. We are therefore motivated to investigate the connections between measures and the weight functions of weighted spaces.

6.2 Valuation spaces

We recall from section 4.3.1, that for a weighted space, the weight functions are strictly monotonic decreasing functions that satisfy a semi-modular inequality. They are therefore similar to, but weaker than, the valuations from section 2.3.3, which are themselves measure-like functions. Valuations will provide us with our link between weight functions and measures, and we will use these to develop special classes of weighted spaces. The material in this section is original.

6.2.1 Defining valuation spaces

Our first problem is that for a weighted space, $(X, \sqsubseteq, \phi)$, the weight function is defined on the underlying consistent semilattice $(X, \sqsubseteq)$, whereas a valuation is defined on some powerset of X. One way to reconcile this is given by the map $x \mapsto X \setminus \downarrow x$. We take $\mathcal{F}_X = \{X \setminus \downarrow F \mid F \subseteq X \text{ finite}\}$ to be the smallest lattice containing the

$X \setminus \downarrow x$, and define our valuation ν over $\mathcal{F}_X$. The intuition will be that where $\phi(x)$ is a measure of how vague $x \in X$ is, $\nu(X \setminus \downarrow x)$ will be a measure of how much information x has. Our valuations will be able to take negative, but not infinite, values. Since $x \sqsubseteq y$ if, and only if, $X \setminus \downarrow y \subseteq X \setminus \downarrow x$, then we have also resolved the difficulty of a weight function being monotonic decreasing while a valuation is monotonic increasing. This leaves the problem of strictness, for which we use the condition that $\nu(X \setminus \downarrow x \sqcap y) = \nu(X \setminus \downarrow y)$ implies that $x \sqsubseteq y$.

Definition 6.2.1 *A* **valuation space** $(X, \sqsubseteq, \nu)$ *is a consistent semilattice* $(X, \sqsubseteq)$ *together with a valuation* $\nu : \mathcal{F}_X \to \Re$ *such that*

$$\nu(X \setminus \downarrow x \sqcap y) = \nu(X \setminus \downarrow x) \quad \Longrightarrow \quad x \sqsubseteq y.$$

In the next lemma we see that the valuation spaces are a special class of weighted spaces, those that can be derived from a valuation on $\mathcal{F}_X$.

Lemma 6.2.2 *Suppose* $(X, \sqsubseteq, \nu)$ *is a valuation space, then* $\phi(x) = \nu(X \setminus \downarrow x)$ *defines a weight function for the consistent semilattice* $(X, \sqsubseteq)$.

Proof. The result is clear once we show that ϕ is semi-modular. Suppose $\{x, y\} \subseteq X$ is consistent, then

$$\begin{aligned}
\phi(x) + \phi(y) &= \nu(X \setminus \downarrow x) + \nu(X \setminus \downarrow y) \\
&= \nu(X \setminus \downarrow x \cap \downarrow y) + \nu(X \setminus \downarrow x \cup \downarrow y) \\
&\geq \nu(X \setminus \downarrow x \sqcap y) + \nu(X \setminus \downarrow x \sqcup y) \\
&= \phi(x \sqcap y) + \phi(x \sqcup y).
\end{aligned}$$

QED

In a more precise sense than for the weighted spaces, the distance between two points in a valuation space is a measure of their common information. Whenever we consider a pmetric for a valuation space, this will be the pmetric induced by the weight function in the lemma, and we will write $\tau_{[\nu]}$ for the induced pmetric topology and $\tau_{[\nu^*]}$ for the induced metric topology. As an example of a valuation space, we have the consistent semilattice $(\Re, \geq)$ for which $\mathcal{F}_{\Re} = \{(-\infty, a) \mid a \in \Re\}$. The valuation $\nu : \mathcal{F}_{\Re} \to \Re$, given by $\nu(-\infty, a) = a$, satisfies the strictness condition since $\nu(-\infty, \max\{a, b\}) = \nu(-\infty, a)$ implies that $a \geq b$, and induces the weight function $\phi(a) = a$. We will find examples of more generalised valuation spaces in the next section, when we extend the results of section 4.3.3 to ω-continuous domains.

6.2.2 Generalised valuation spaces

It is clear that ω-continuous domains will only fit into our framework of valuation spaces if they are consistent semilattices. However, one of the benefits of moving away from weighted functions to valuations is that we can actually work with posets, rather than consistent semilattices, and still recover a partial metric space. In this case however, we must take $\mathcal{F}_X$ to be the closure of $\{X \setminus \downarrow x \mid x \in X\}$ under finite unions and intersections. We show that ω-continuous domains are examples of these "generalised valuation spaces" which, although these results were found independently, is another example of Künzi's construction [KV94]. We remark that this will give us a larger cartesian closed category of partial metric spaces, but one that has the same deficiences as in section 4.3.3.

Lemma 6.2.3 *Suppose $(X, \sqsubseteq)$ is a poset and $\nu : \mathcal{F}_X \to \Re$ is a valuation satisfying*

$$\nu(X \setminus \downarrow x \cap \downarrow y) = \nu(X \setminus \downarrow x) \quad \Longrightarrow \quad x \sqsubseteq y,$$

then $d(x, y) = \nu(X \setminus \downarrow x \cap \downarrow y)$, defines a pmetric for which $\leq_d = \sqsubseteq$ and $\phi_d(x) = \nu(X \setminus \downarrow x)$. We say that $(X, \sqsubseteq, \nu)$ is a **generalised valuation space**.

Proof. Axiom P3 is immediate, and P1 follows since $X \setminus \downarrow x \subseteq X \setminus \downarrow x \cap \downarrow y$. For P2, we have that $d(x, y) = d(x, x)$ implies $\nu(X \setminus \downarrow x \cap \downarrow y) = \nu(X \setminus \downarrow x)$ and hence by strictness, $x \sqsubseteq y$. For the P4 axiom, we have

$$\begin{aligned} d(x,z) + d(y,y) &= \nu(X \setminus {\downarrow}x \cap {\downarrow}z) + \nu(X \setminus {\downarrow}y) \\ &\leq \nu(X \setminus {\downarrow}x \cap {\downarrow}y \cap {\downarrow}z) + \nu(X \setminus {\downarrow}y \cap ({\downarrow}x \cup {\downarrow}z)) \\ &= \nu(X \setminus {\downarrow}x \cap {\downarrow}y) + \nu(X \setminus {\downarrow}y \cap {\downarrow}z) \\ &= d(x,y) + d(y,z). \end{aligned}$$

It is immediate that $\leq_d = \sqsubseteq$ and $\phi_d(x) = \nu(X \setminus {\downarrow}x)$.

QED

Lemma 6.2.4 *Suppose D is an ω-continuous domain with basis $B = \{b_1, b_2, \ldots\}$, and define $\nu : \mathcal{F}_P \to \Re$ to be the restriction of the measure on $\mathcal{P}D$, given by*

$$\nu(A) = \sum_{b_n \in A} 1/2^n,$$

to $\mathcal{F}_D$, then $(D, \sqsubseteq, \nu)$ is a generalised valuation space, and the Scott and pmetric topologies agree.

Proof. Suppose $\{x_n\}$ is an ω-chain with supremum $a \in D$, then $B_a = \bigcup_{n=1}^{\infty} B_{x_n}$. It clearly follows that ν is a continuous valuation on $\mathcal{F}_D$. Suppose $\nu(D \setminus {\downarrow}x \cap {\downarrow}y) = \nu(D \setminus {\downarrow}x)$, then it follows that $B_x \cap B_y = B_x$, and hence that $x \sqsubseteq y$. So $(D, \sqsubseteq, \nu)$ is a valuation space, and the induced partial metric is

$$d(x,y) = 1 - \nu(B_x \cap B_y).$$

Since ϕ_d is continuous, then $\tau_{[\nu]}$ is order-consistent. To show that $\tau_{[\nu]}$ is the Scott topology, we use lemma 4.1.19 and show that every sequence $\{x_n\}$ converging to $a \in D$ surpasses some ω-chain with supremum a. By lemma 4.1.2 we can find an ω-chain $\{b_{i_n}\}$ in B_a with supremum a. If we fix $n \geq 1$ and let $\varepsilon = 1/2^{i_n} > 0$, then we can find some $m \geq 1$ such that $x_m \in B_\varepsilon(a)$. It follows that $\nu(B_a \setminus B_{x_m}) < \varepsilon$, and hence $b_{i_n} \in B_a$ implies that $b_{i_n} \sqsubseteq x_m$. So $\{x_n\}$ surpasses $\{b_{i_n}\}$ and $\tau_{[\nu]}$ is the Scott topology.

QED

In the case of ω-algebraic domains, the Lawson and induced metric topologies also agree.

Lemma 6.2.5 *If D is an ω-algebraic domain then, for the generalised valuation space $(D, \sqsubseteq, \nu)$, the Scott and pmetric topologies agree as do the Lawson and metric topologies.*

Proof. From lemma 4.1.15, we immediately see that the Lawson topology is contained in the induced metric topology $\tau_{[\nu^*]}$. We show that $B_\varepsilon(x; d^*)$ is Lawson-open, for any $\varepsilon > 0$ and $x \in D$. We let $N \geq 1$ be such that

$$1/2^N = \sum_{n=N+1}^{\infty} 1/2^n < \varepsilon,$$

and define $\delta = 1/2^N$ and

$$B' = \{b_n \notin B_x \mid 1 \leq n \leq N\}.$$

If we let

$$U = B_\delta(x; d) \cap \bigcap_{b_n \in B'} D \setminus \uparrow b_n,$$

then we have $x \in U$ Lawson-open, since $B_\delta(x; d)$ is Scott-open. Now suppose that $y \in U$, and consider

$$d^*(x, y) = \max\{\nu(B_x \setminus B_y), \nu(B_y \setminus B_x)\}.$$

Now $y \in B_\delta(x; d)$ implies that $\nu(B_x \setminus B_y) < \delta < \varepsilon$. Furthermore, if $b_n \in B_y \setminus B_x$, then $b_n \sqsubseteq y$ and $b_n \not\ll x$. So $b_n \not\sqsubseteq x$ (since D is algebraic) and so $n > N$, and

$$\nu(B_y \setminus B_x) \leq \sum_{n=N+1}^{\infty} 1/2^n < \varepsilon.$$

So we see that $d^*(x, y) < \varepsilon$ and so $x \in U \subseteq B_\varepsilon(x; d^*)$.

QED

6.2.3 Information measures

In section 6.2.1, we saw that the valuation spaces are a special class of weighted spaces, namely those that can be derived from a valuation. This was our first step in establishing a connection between the weighted spaces and measures. The next logical step is to consider a class of weighted spaces that can be derived from a measure. For a lattice $\mathcal{L}$ of subsets of some set X, we recall from section 2.3.4, that $H(\mathcal{L}) = \{A \setminus B \mid A, B \in \mathcal{L}, B \subseteq A\}$.

Definition 6.2.6 *Suppose $(X, \sqsubseteq)$ is a consistent semilattice. An* **information measure,** *μ, on X is a Borel measure (with respect to the weak topology) which is finite on $H(\mathcal{F}_X)$ and satisfies*

$$\mu(\downarrow x \setminus \downarrow x \sqcap y) = 0 \quad \Longrightarrow \quad x \sqsubseteq y.$$

Similar to a valuation space, the intuition behind an information measure is that $\mu(X \setminus \downarrow x)$ is a measure of how much information $x \in X$ has. Although, each point may have an infinite amount of information (think of the Lebesgue measure on $(\Re, \geq)$, where $(-\infty, a)$ has infinite measure), we can always consider relative information since our measure is finite on $H(\mathcal{F}_X)$. Our strictness condition tells us that if we have no information on x that we don't have on y, then x must be below y in the information ordering.

Lemma 6.2.7 *Suppose $(X, \sqsubseteq)$ is a consistent semilattice with information measure μ, then we can find a continuous valuation $\nu : \mathcal{F}_X \to \Re$ such that $(X, \sqsubseteq, \nu)$ is a valuation space.*

Proof. We fix $W^* \in \mathcal{F}_X$ and define

$$\nu(W) = \mu(W \setminus W^* \cap W) - \mu(W^* \setminus W^* \cap W) \in \Re, \qquad \forall W \in \mathcal{F}_X.$$

Suppose first that $W_2 \subseteq W_1$ in $\mathcal{F}_X$, then we have

$$\nu(W_1) - \nu(W_2) \;=\; \mu(W_1 \setminus W^* \cap W_1) - \mu(W^* \setminus W^* \cap W_1)$$

$$
\begin{aligned}
&\quad -\mu(W_2 \setminus W^* \cap W_2) + \mu(W^* \setminus W^* \cap W_2) \\
&= \mu(W_1 \setminus W^* \cap W_1) - \mu(W_2 \setminus W^* \cap W_2) \\
&\quad + \mu(W^* \cap W_1 \setminus W^* \cap W_2) \\
&= \mu(W_1 \setminus W^* \cap W_2) - \mu(W_2 \setminus W^* \cap W_2) \\
&= \mu(W_1 \setminus W_2).
\end{aligned}
$$

It is immediate that $\nu(W_1) \geq \nu(W_2)$. Furthermore, for any $W_1, W_2 \in \mathcal{F}_X$, we have

$$\nu(W_1) - \nu(W_1 \cap W_2) = \mu(W_1 \setminus W_1 \cap W_2) = \mu(W_1 \cup W_2 \setminus W_2) = \nu(W_1 \cup W_2) - \nu(W_2),$$

and so ν is a valuation on $\mathcal{F}_X$. The strictness condition is satisfied since, if $\nu(X \setminus {\downarrow}x \sqcap y) = \nu(X \setminus {\downarrow}x)$, then $\mu({\downarrow}x \setminus {\downarrow}x \sqcap y) = 0$ and $x \sqsubseteq y$. To see that ν is continuous, suppose $\{W_n\}$ is an ω-chain in $\mathcal{F}_X$ with supremum $W \in \mathcal{F}_X$, then

$$W \setminus W^* \cap W = \bigcup_{n=1}^{\infty} W_n \setminus W^* \cap W_n, \quad \text{and} \quad W^* \setminus W^* \cap W = \bigcap_{n=1}^{\infty} W^* \setminus W^* \cap W_n,$$

and it follows that $\nu(W) = \sup\{\nu(W_n) \mid n \geq 1\}$.

QED

One drawback of the above lemma is that a different $W^* \in \mathcal{F}_X$ in the proof induces a different continuous valuation $\nu' : \mathcal{F}_X \to \Re$. However, it is clear that $\nu - \nu'$ will be constant and so the induced partial metric spaces will be isometric. A more significant problem, is that to build a cartesian closed category of consistent semilattices with information measures, we must be able to deduce an information measure for a function space from its constituent information measures. It would be simpler if we could identify a cartesian closed category of valuation spaces, for which the valuations induce information measures, which we could then use to build similar valuations on function spaces. Ultimately we seek a cartesian closed category of weighted spaces whose weight functions induce information measures appropriate to build similar weight functions on function spaces.

As an example, consider the valuation space $(\Re, \geq, \nu)$ from section 6.2.1, for which the induced pmetric topology is the Scott topology. We can extend ν to an information measure by using Pettis' theorem from section 2.3.4. We take $\mathcal{W} = \{(-\infty, a) \mid a \in \Re\}$, which are open in the Scott topology on $(\Re, \geq)$, and $\mathcal{K} = \{(-\infty, a] \mid a \in \Re\}$, which are closed. Since $(-\infty, b] \setminus (-\infty, a) = [a, b]$ or $\emptyset$, then the compactness criteria is satisfied. The continuity condition follows by local compactness. The collection $H(\mathcal{W})$ consists of sets of the form $[a, b)$ and $\psi : H(\mathcal{W}) \to [0, \infty)$ is given by $\psi([a, b)) = b - a$. From the theorem we have a unique measure, ψ^*, on $\sigma(H(\mathcal{W}))$ which extends ψ on $H(\mathcal{W})$, and is therefore the Lebesgue measure, which is an information measure on $(\Re, \geq)$.

To extend this to more general valuation spaces requires some condition on our T_0-spaces stronger than local compactness. An example of such a condition is coherence, from section 2.2.2, which is our T_0-characterisation of compact ordered spaces. However, our motivating example, $(\Re, \pi, \geq)$, where π is the usual topology on $\Re$, is only a *locally compact ordered space*, which we define to be a partially ordered space with a locally compact Hausdorff topology. We therefore seek some condition weaker than coherence, which includes our example.

6.3 A local notion of sobriety

We recall, from section 2.2.2, that our first step in giving a T_0-characterisation of a compact ordered space was to consider the locally compact sober spaces, which generalise the Scott topology for continuous domains. If we consider the locally compact ordered space $(\Re, \pi, \geq)$, where π is the usual topology on $\Re$, then it is clear that $\pi^{\uparrow}$, the Scott topology, is locally compact but not sober, since $(\Re, \geq)$ is not a dcpo. Our aim in this section, is to weaken the framework of locally compact sober spaces to include the Scott topology for $(\Re, \geq)$. This material is original as well as being our most technical.

6.3.1 Filtered open sets and local sobriety

We begin with a lemma used in [KP94] to prove the Hofmann-Mislove Theorem (see section 2.2.2), and from which we abstract a crucial property of open sets of a sober space.

Lemma 6.3.1 ([KP94], p.302) *Suppose (X, τ) is a sober space and $\mathcal{F}$ is a Scott-open filter, then every open set U containing $\bigcap \mathcal{F}$ is already in $\mathcal{F}$.*

Definition 6.3.2 *Suppose (X, τ) is a T_0-space, then an open set $W \subseteq X$ is* **filtered** *if, whenever $\mathcal{F}$ is a Scott-open filter and $\bigcap \mathcal{F} \subseteq W$, then $W \in \mathcal{F}$.*

Definition 6.3.3 *A closed subset of a T_0-space (X, τ) is* **sober** *if it is sober as a subspace of X.*

Lemma 6.3.4 *Suppose (X, τ) is a T_0-space, then $X \setminus Z$ is a filtered open set if $Z \subseteq X$ is sober.*

Proof. We show that any $U \in \tau$ with $X \setminus Z \subseteq U$ is a filtered open set. Suppose $\mathcal{F}$ is a Scott-open filter in τ with $\bigcap \mathcal{F} \subseteq U$. The restriction $\mathcal{F}_Z$ of $\mathcal{F}$ to Z is clearly a Scott-open filter with respect to the subspace topology, and $\bigcap \mathcal{F}_Z \subseteq U \cap Z$. By lemma 6.3.1, $U \cap Z \in \mathcal{F}_Z$ and hence $U \in \mathcal{F}$.

QED

We immediately see that the Scott-open sets of $(\Re, \geq)$, with the exception of the empty set, are filtered. Rather than weaken the framework of locally compact sober spaces directly, we strengthen the notion of local compactness for a T_0-space, and then show that this is satisfied by the locally compact sober spaces. We recall from section 2.2.2, that for a T_0-space (X, τ), we denote the collection of compact upper sets by κ.

Definition 6.3.5 *A T_0-space (X,τ) is* **locally sober** *if, for any open neighbourhood N of $x \in X$, there exists $K \in \kappa$ and $Z \subseteq X$ sober, such that*

$$x \in X \setminus Z \subseteq K \subseteq N.$$

We immediately see that the Scott topology for $(\Re, \geq)$ is locally sober, and that locally sober spaces are locally compact. It follows that not every sober space is locally sober. However, a locally compact sober space is locally sober, since every open set in a sober space is filtered. One important property of sober spaces which we preserve is that locally sober spaces are order consistent, with respect to their specialisation order.

Lemma 6.3.6 *Suppose (X,τ) is a locally sober space, then τ is order consistent with respect to the specialisation order.*

Proof. Suppose $A \subseteq X$ is a directed set with $x = \bigsqcup^{\uparrow} A \in X$, and $U \in \tau$ is an open neighbourhood of x. We can find a filtered open set $W \subseteq X$ such that $x \in W \subseteq U$. If we let $\mathcal{F}$ be the Scott-open filter of open neighbourhoods of points in A, then $\bigcap \mathcal{F} = \uparrow x \subseteq W$. So we must have $W \in \mathcal{F}$, and then U is an open neighbourhood of some point in A.

QED

6.3.2 Local sobriety and partial metric spaces

In section 4.1.2, we saw that sobriety for partial metric spaces could be given a simple intuition as a notion of pmetric completeness. We can similarly simplify the intuition for local sobriety, but will see that it is more than a straightforward notion of completeness.

Lemma 6.3.7 *Suppose (X,d) is a partial metric space, then $(X,\tau_{[d]})$ is locally sober if, and only if, $\tau_{[d]}$ is locally compact and every self-convergent sequence either has a proper limit or converges to every point in X.*

Proof. Suppose first that $(X, \tau_{[d]})$ is locally sober, $\{x_n\}$ is a self-convergent sequence and $a \in X$. If $\{x_n\}$ does not converge to a, then we can find $U \in \tau_{[d]}$ such that $a \in U$ and some self-convergent subsequence, $\{x_{n_k}\}$, of $\{x_n\}$ that is not in U. By local sobriety we can find $Z \subseteq X$ sober such that $x \in X \setminus Z \subseteq U$ so that $\{x_{n_k}\} \subseteq Z$. By theorem 4.1.9, $\{x_{n_k}\}$ has a proper limit in Z, and it follows that $\{x_n\}$ has a proper limit in Z.

For the converse, we suppose that $N \subseteq X$ is an open neighbourhood of some $a \in X$. By local compactness, we can find $K \subseteq X$ compact upper, and $U \in \tau_{[d]}$ such that $a \in U \subseteq K \subseteq N$. Suppose $\{x_n\}$ is a self-convergent sequence in $X \setminus U$, then $\{x_n\}$ does not converge to a, so we must have a proper limit for $\{x_n\}$ in $X \setminus U$. It follows that $X \setminus U$ is sober, and $(X, \tau_{[d]})$ is locally sober.

QED

We recall from section 4.1.2, that a self-convergent sequence is bounded if, and only if, it is Cauchy. So for a complete partial metric space, (X, d), we need only check that $\tau_{[d]}$ is locally compact and that every unbounded self-convergent sequence converges to every point in X. For a bounded partial metric space, (X, d), with $\tau_{[d]}$ locally sober, we can "complete" $(X, \tau_{[d]})$ to a locally compact sober space by introducing a top element. This is not possible if, as with $(\Re, \geq)$, the space is unbounded.

6.3.3 Some consequences of local sobriety

From section 2.2.2 we recall that for a locally compact sober space (X, τ), τ is a continuous lattice and κ is a continuous domain. We now show that the locally sober spaces have similar properties. Our first lemma is equivalent to our definition of local sobriety.

Lemma 6.3.8 *Suppose (X, τ) is a locally sober space, then τ is a continuous lattice with the filtered open sets as a basis.*

Proof. Since (X, τ) is locally compact, then τ is a continuous lattice. Suppose $O' \ll O$ in τ. For each $x \in O$ we can find $K_x \in \kappa$ and $W_x \subseteq X$ filtered open, such that

$$x \in W_x \subseteq K_x \subseteq O.$$

So we see that $O' \subseteq O = \bigcup_{x \in O} K_x = \bigcup_{x \in O} W_x$. Since $O' \ll O$, we can find $x_1, \ldots, x_n \in O$ such that

$$O' \subseteq \bigcup_{i=1}^{n} W_{x_i} \subseteq \bigcup_{i=1}^{n} K_{x_i} \subseteq O,$$

and we take $K = \bigcup_{i=1}^{n} K_{x_i} \in \kappa$ and $W = \bigcup_{i=1}^{n} W_{x_i}$ filtered open. So $W \ll O$ and O is the directed supremum of such sets.

QED

We now have a sequence of three lemmata, which allow us to show that the poset of compact upper sets, κ, is continuous.

Lemma 6.3.9 *The collection of sober sets of a T_0-space (X, τ) is a lattice.*

Proof. Suppose $Z, Z' \subseteq X$ are sober and $A \subseteq Z \cup Z'$ is an irreducible closed set, then A is the union of the two closed sets $A \cap Z$ and $A \cap Z'$. If $A \not\subseteq Z$ then $A \cap Z$ is a proper closed subset of A, so that $A \cap Z' = A$ and A is the closure of a unique point in Z' and hence in $Z \cup Z'$. So $Z \cup Z'$ is sober, and it is trivial that $Z \cap Z'$ is sober.

QED

Lemma 6.3.10 *Suppose (X, τ) is a locally sober space, $K \in \kappa$ and $x \in X \setminus K$, then we can find $K' \in \kappa$ and $Z \subseteq X$ sober such that*

$$x \in X \setminus K' \subseteq Z \subseteq X \setminus K.$$

Proof. For any $y \in K$, we know that $y \not\sqsubseteq x$, so we can find $U_y \in \tau$ such that $y \in U_y$ but $x \notin U_y$. Local sobriety implies that we can find $K_y \in \kappa$ and $W_y \subseteq X$ filtered open, such that

$$y \in W_y \subseteq K_y \subseteq U_y.$$

In which case we have

$$K \subseteq \bigcup_{y \in K} W_y \subseteq \bigcup_{y \in K} K_y \subseteq \bigcup_{y \in K} U_y.$$

Since each $W_y \in \tau$, then we can find $y_1, \ldots y_n \in K$ such that

$$K \subseteq W_x \subseteq K_x \subseteq U_x,$$

where $W_x = \bigcup_{i=1}^{n} W_{y_i}$ filtered open, $K_x = \bigcup_{i=1}^{n} K_{y_i} \in \kappa$ and $U_x = \bigcup_{i=1}^{n} U_{y_i}$. Since $x \notin U_x$ then, if we let $Z_x = X \setminus W_x$ sober, we can finally conclude that

$$x \in X \setminus K_x \subseteq Z_x \subseteq X \setminus K.$$

QED

Lemma 6.3.11 *Suppose (X, τ) is a locally sober space, then κ is a continuous poset such that*

$$K' \ll K \text{ in } \kappa \qquad \iff \qquad \exists W \subseteq X \text{ filtered open with } K \subseteq W \subseteq K'.$$

Proof. We first suppose that $W \subseteq X$ is filtered open with $K \subseteq W \subseteq K'$. If we let $\{K_i\}_{i \in I}$ be a filtered family in κ with $\bigcap_{i \in I} K_i \in \kappa$ and contained in K, then we let $\mathcal{F}$ be the union of the collection of open neighbourhoods of each K_i. So $\mathcal{F}$ is a Scott-open filter with $\bigcap \mathcal{F} = \bigcap_{i \in I} K_i \subseteq K \subseteq W$, which is a filtered open set. So $W \in \mathcal{F}$ and hence contains some K_i, which is then contained in K'.

Now suppose that $K' \ll K$ and $K \neq X$ since otherwise the result is trivial as X is a filtered open set. By lemma 6.3.10, for each $x \in X \setminus K$ we can find $K_x \in \kappa$ and $Z_x \subseteq X$ sober such that

$$x \in X \setminus K_x \subseteq Z_x \subseteq X \setminus K.$$

It is clear that

$$X \setminus K = \bigcup_{x \in X \setminus K} X \setminus K_x = \bigcup_{x \in X \setminus K} Z_x,$$

and hence that $K = \bigcap \{K_x \mid x \in X \setminus K\}$. We can therefore find $x_1, \ldots, x_n \in X \setminus K$ such that

$$K \subseteq \bigcap_{i=1}^{n} X \setminus Z_{x_i} \subseteq \bigcap_{i=1}^{n} K_{x_i} \subseteq K'.$$

Since $\bigcup_{i=1}^{n} Z_{x_i}$ is sober, then $X \setminus \bigcup_{i=1}^{n} Z_{x_i}$ is filtered open and we are done.

QED

To conclude our investigation into the locally sober spaces, we give a result that will be useful in the next section, which is that the patch space of a second-countable locally sober space is second-countable.

Lemma 6.3.12 *Suppose (X, τ) is a second-countable locally sober space, then τ^k is second countable, and hence the patch topology $\pi = \tau \vee \tau^k$ is second countable.*

Proof. Suppose we let $\mathcal{B}$ be a countable basis for τ, with $\mathcal{B}^*$ the collection of finite unions of elements of $\mathcal{B}$. For each $B, B' \in \mathcal{B}^*$, if $B \ll B'$, then we can find $K_{B,B'} \in \kappa$ such that $B \subseteq K_{B,B'} \subseteq B'$. We show that

$$\mathcal{B}^k = \{X \setminus K_{B,B'} \in \tau^k \mid B, B' \in \mathcal{B}^*, B \ll B'\}$$

is a basis for τ^k. Suppose $x \in X \setminus K$ with $K \in \kappa$, then by interpolation on the continuous poset κ, we can find $K', K'' \in \kappa$ such that $K' \ll K'' \ll K$ and $x \in X \setminus K'$. So we can find $W, W' \subseteq X$ filtered open, such that

$$K \subseteq W \subseteq K'' \subseteq W' \subseteq K'.$$

Since $W, W' \in \tau$, then we can find $B, B' \in \mathcal{B}^*$ such that

$$K \subseteq B \subseteq W \subseteq K'' \subseteq B' \subseteq W' \subseteq K'.$$

But then $B \ll B'$ in τ, and so we have

$$K \subseteq B \subsetneq K_{B,B'} \subseteq B' \subseteq K',$$

and

$$x \in X \setminus K' \subseteq X \setminus K_{B,B'} \subseteq X \setminus K.$$

Since $\mathcal{B}^k$ is countable, then we are done.

QED

We remark that locally sober spaces seem to be of sufficient interest in their own right for further study. However, we now return to our investigations regarding weight functions, valuations and measures, to see where local sobriety fits in.

6.4 Valuation spaces, local coherence and measures

We recall from section 6.2.3 that we would like to identify some conditions on our valuation spaces so that the valuation naturally induces an information measure, which we can then use in developing function spaces. We have seen how to extend the valuation, ν, for the valuation space $(\Re, \geq, \nu)$ to an information measure, and developed the locally sober spaces as a framework that includes locally compact sober spaces and the Scott topology on $(\Re, \geq)$. We now take the next step, which is to develop a notion of local coherence, and consider how this can be used to achieve our goal. The material in this section is original.

6.4.1 Local coherence

In section 2.2.2, the coherent spaces strengthened the locally compact sober spaces to characterise the compact ordered spaces. We similarly define local coherence, and give some connections with locally compact ordered spaces.

Definition 6.4.1 *A locally sober space (X, τ) is* **locally coherent** *if the collection, κ, of compact upper sets is a lattice.*

It is clear from the definition, that coherent spaces are locally coherent. In the following lemma, we will see that locally coherent spaces have locally compact ordered patch spaces, and so identify a class of T_0-spaces for which the patch space is a locally compact ordered space. We have not given a T_0-characterisation of locally compact ordered spaces, as the example following the lemma will show.

Lemma 6.4.2 *Suppose (X, τ) is a locally coherent space, then the patch space $(X, \pi, \leq_\tau)$, where $\pi = \tau \vee \tau^k$, is a locally compact ordered space.*

Proof. We need only show that the patch topology is locally compact. Suppose $x \in U \setminus K$ with $U \in \tau$ and $K \in \kappa$. We can find $K', K'' \in \kappa$, $W \subseteq X$ filtered open and $Z \subseteq X$ sober, such that

$$x \in W \subseteq K' \subseteq U,$$
$$\text{and} \quad x \in X \setminus K'' \subseteq Z \subseteq X \setminus K.$$

So we have

$$x \in W \setminus K'' \subseteq K' \cap Z \subseteq U \setminus K,$$

and since $W \in \tau$, then $W \setminus K'' \in \pi$ and $K' \cap Z$ is a coherent subspace of X and hence π-compact.

QED

Consider the poset $(X, \sqsubseteq)$ in figure 6.1 together with the Alexandroff topology, τ, of upper sets. Suppose $\mathcal{F}$ is the Scott-open filter of open neighbourhoods of the y_n, then $\bigcap \mathcal{F} = \emptyset \subseteq \uparrow x_1$, but $\uparrow x_1 \notin \mathcal{F}$. So $\uparrow x_1$ is not a filtered open set. Since $x_1 \in \uparrow x_1 \in \tau$, then (X, τ) is not locally sober. If we consider the patch space, then it is clear that every singleton set is open in the patch topology, $\pi = \tau \vee \tau^k$, and so $(X, \pi, \sqsubseteq)$ is a locally compact ordered space with $\tau = \pi^\uparrow$.

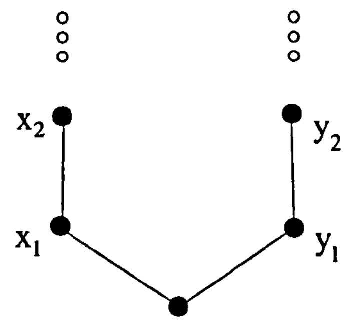


Figure 6.1: A locally compact ordered space

6.4.2 Borel measures on locally coherent spaces

We now consider measures and locally coherent spaces. In the following simple lemma we will see that, for a second-countable locally compact T_0-space, the Borel sets generated by the T_0-topology and the patch topology are the same. This result is surely well-known, although we haven't seen it in the literature. It follows that second-countable locally coherent spaces have patch spaces that have proved particularly useful in measure theory.

Lemma 6.4.3 *Suppose (X, τ) is a second-countable locally compact T_0-space, then $\sigma(\kappa) = \sigma(\tau) = \sigma(\pi)$, where $\pi = \tau \vee \tau^k$.*

Proof. Suppose we let $\mathcal{B}$ be a countable base for τ, with $\mathcal{B}^*$ the collection of finite unions of elements of $\mathcal{B}$. Suppose $K \in \kappa$ is non-empty, otherwise $K \in \tau$, and let $K \subseteq U \in \tau$. We can find $B_U \in \mathcal{B}^*$ such that $K \subseteq B_U \subseteq U$. So we have

$$K = \bigcap \{B_U \in \mathcal{B}^* \mid K \subseteq U \in \tau\},$$

which implies that $K \in \sigma(\tau)$ and $\sigma(\kappa) \subseteq \sigma(\tau)$. Now suppose that $U \in \tau$, and let $B_1 \subseteq B_2 \subseteq \cdots \ll U$ in $\mathcal{B}^*$ and $U = \bigcup_{i=1}^{\infty} B_i$. We can then find $K_i \in \kappa$ $(i = 1, 2, \ldots)$ with $B_i \subseteq K_i \subseteq U$, so that $U = \bigcup_{i=1}^{\infty} K_i \in \sigma(\kappa)$, and $\sigma(\tau) = \sigma(\kappa)$.

Now suppose that $x \in V \in \pi$. We can find $B_x \in \mathcal{B}$ and $K_x \in \kappa$ such that $x \in B_x \setminus K_x \subseteq V$. Using the above notation, we have

$$B_x \setminus K_x = B_x \setminus \bigcap\{B_U \in \mathcal{B}^* \mid K_x \subseteq U \in \tau\} = \bigcup\{B_x \setminus B_U \mid K_x \subseteq U \in \tau\}.$$

Since the set $\{B \setminus B^* \mid B \in \mathcal{B}, B^* \in \mathcal{B}^*\}$ is countable, then

$$V = \bigcup\{B_x \setminus K_x \mid x \in V\} = \bigcup\{B_x \setminus B_U \mid x \in V, K_x \subseteq U \in \tau\} \in \sigma(\tau).$$

So $\sigma(\pi) \subseteq \sigma(\tau)$ and since the converse is trivial, then we have completed the proof.

QED

Corollary 6.4.4 *Suppose (X, τ) is a second-countable locally coherent space, with $\pi = \tau \vee \tau^k$, then (X, π) is a second-countable locally compact Hausdorff space with the same Borel sets.*

We are now in a position to generalise to the locally coherent spaces, the extension result from section 6.2.3 for $(\Re, \geq, \nu)$. We will find it useful to call a basis for a topology, which is also a lattice of open sets, a *lattice-basis*. We recall the general results from section 2.3.4 on extending valuations to measures.

Lemma 6.4.5 *Suppose (X, τ) is a second-countable locally coherent space, $\mathcal{W}$ is a lattice-basis of filtered open sets and $\nu : \mathcal{W} \to \Re$ is a continuous valuation, then ν induces a unique regular Borel measure, ψ^*, on X extending ψ on $H(\mathcal{W})$.*

Proof. We take the collection of compact upper sets, κ, as our family of π-closed sets, where $\pi = \tau \vee \tau^k$. For $K \in \kappa$ and $W \in \mathcal{W}$, $K \setminus W$ is π-compact since $X \setminus W$ is sober. Now suppose that $W \in \mathcal{W}$ and $\varepsilon > 0$. Since W is the union of an ω-chain of filtered open sets in $\mathcal{W}$ and ν is continuous, then we can find $W' \ll W$ such that $\nu(W') \geq \nu(W) - \varepsilon$. Local compactness implies that we can find $K \in \kappa$ such that $W' \subseteq K \subseteq W$. We can therefore apply Pettis' theorem to find a unique σ-finite measure, ψ^*, on $S(H(\mathcal{W}))$ such that

$$\psi^*(W \setminus W') = \nu(W) - \nu(W') < \infty,$$

for any $W' \subseteq W$ in $\mathcal{W}$. Since every open set is the union of an ω-chain of filtered open sets in $\mathcal{W}$, then $\tau \subseteq S(H(\mathcal{W}))$ which is a σ-algebra. So ψ^* is a Borel measure on X, and is regular since (X, π) is a second-countable locally compact Hausdorff space.

QED

Continuing with the assumptions of the theorem, suppose now that $\nu, \nu' : \mathcal{W} \to \Re$ are continuous valuations inducing the same measure, ψ^*, on X. For $W, W' \in \mathcal{W}$, we have

$$\nu'(W \cup W') - \nu'(W) = \psi^*(W \cup W' \setminus W) = \nu(W \cup W') - \nu(W),$$
$$\text{and} \quad \nu'(W) - \nu'(W \cap W') = \psi^*(W \setminus W \cap W') = \nu(W) - \nu(W \cap W').$$

Since ν and ν' are valuations on the lattice $\mathcal{W}$, then we see that

$$\nu'(W') - \nu'(W) = \nu(W') - \nu(W),$$

and it follows that $\nu' - \nu$ is constant on $\mathcal{W}$. Conversely, if $\nu' - \nu$ is constant on $\mathcal{W}$, then by uniqueness ν and ν' must induce the same measure on X. We can use the proof of lemma 6.2.7 from section 6.2.3 to see that a Borel measure, finite on $H(\mathcal{W})$, gives a continuous valuation on $\mathcal{W}$ that induces the measure. We have therefore proved the following result.

Theorem 6.4.6 *Suppose (X, τ) is a second-countable locally coherent space and $\mathcal{W}$ is a lattice-basis of filtered open sets, then there is a correspondence between the continuous valuations on $\mathcal{W}$ that differ by a constant, and the regular Borel measures on X, finite on $H(\mathcal{W})$.*

6.4.3 Information measures on valuation spaces

We must take one more step before we can use local coherence to deduce an information measure for a valuation space $(X, \sqsubseteq, \nu)$. Our valuation, ν, is defined on $\mathcal{F}_X$, and we must extend this to a lattice-basis of filtered open sets. If $U \subseteq X$ is a proper upper set, then

$$U = \bigcap\{X \setminus {\downarrow}F \mid F \subseteq X \setminus U \text{ finite}\}.$$

We therefore define

$$\mathcal{U}_X = \{U \subseteq X \mid U \text{ is a proper upper set and } \nu(U) > -\infty\},$$

and in the following lemma, see that $\mathcal{U}_X$ is a lattice to which the valuation, ν, naturally extends.

Lemma 6.4.7 *Suppose $(X, \sqsubseteq, \nu)$ is a valuation space, then*

$$\nu(U) = \inf\{\nu(X \setminus {\downarrow}F) \mid F \subseteq X \setminus U \textit{ finite}\},$$

defines a valuation on the lattice $\mathcal{U}_X$.

Proof. We first remark that if $U, U' \subseteq X$ are proper upper sets with $x \notin U$ and $x' \notin U'$, then $x \sqcap x' \notin U \cap U'$ and $U \cap U', U \cup U'$ are proper upper sets. Now suppose that $U \subseteq U'$ in $\mathcal{U}_X$, then whenever $F \subseteq X \setminus U'$ is finite we have $F \subseteq X \setminus U$ and we easily see that $\nu(U) \leq \nu(U')$. Now suppose that $U.U' \in \mathcal{U}_X$ and fix $\varepsilon > 0$. We find $F \subseteq X \setminus U$ and $F' \subseteq X \setminus U'$ finite, such that

$$\nu(X \setminus {\downarrow}F) < \nu(U) + \varepsilon, \quad \text{and} \quad \nu(X \setminus {\downarrow}F') < \nu(U') + \varepsilon.$$

Clearly $F \cup F' \subseteq X \setminus U \cap U'$ and we let $G \subseteq X \setminus U \cup U'$ be the finite set for which ${\downarrow}G = {\downarrow}F \cap {\downarrow}F'$. We then have

$$\begin{aligned}
\nu(U \cup U') + \nu(U \cap U') &\leq \nu(X \setminus {\downarrow}G) + \nu(X \setminus {\downarrow}F \cup F') \\
&= \nu(X \setminus {\downarrow}F) + \nu(X \setminus {\downarrow}F') \\
&< \nu(U) + \nu(U') + 2\varepsilon.
\end{aligned}$$

Since this holds for all $\varepsilon > 0$, then we have

$$\nu(U \cup U') + \nu(U \cap U') \leq \nu(U) + \nu(U').$$

We immediately see that if either of $\nu(U \cup U')$ or $\nu(U \cap U')$ is infinite, then at least one of $\nu(U)$ or $\nu(U')$ is infinite, which is a contradiction. So $U \cup U', U \cap U' \in \mathcal{U}_X$ and $\mathcal{U}_X$ is a lattice. Furthermore, for $\varepsilon > 0$ we find $F \subseteq X \setminus U \cup U'$, $G \subseteq X \setminus U \cap U'$ finite such that

$$\nu(X \setminus \downarrow F) < \nu(U \cup U') + \varepsilon \quad \text{and} \quad \nu(X \setminus \downarrow G) < \nu(U \cap U') + \varepsilon.$$

We let $H = G \cap X \setminus U$ and $H' = G \cap X \setminus U'$, then $H \cup F \subseteq X \setminus U$ and $H' \cup F \subseteq X \setminus U'$ are finite and

$$\begin{aligned}
\nu(U) + \nu(U') &\leq \nu(X \setminus \downarrow H \cup F) + \nu(X \setminus \downarrow H' \cup F) \\
&= \nu(X \setminus \downarrow G \cup F) + \nu(X \setminus \downarrow F \cup (\downarrow H \cap \downarrow H')) \\
&\leq \nu(X \setminus \downarrow F \cup G) + \nu(X \setminus \downarrow F \cap \downarrow G) \\
&= \nu(X \setminus \downarrow F) + \nu(X \setminus \downarrow G) \\
&< \nu(U \cup U') + \nu(U \cap U') + 2\varepsilon.
\end{aligned}$$

Since this holds for all $\varepsilon > 0$, then we have modularity and ν is a valuation as required.

QED

We know that the collection of filtered open sets is a lattice-basis for a locally coherent space, and that the filtered open sets are upper sets. The problem however is to ensure that $\mathcal{U}_X$ restricted to the filtered open sets is a lattice-basis for our topology. For our last result, we will have to assume such a property, but we hypothesis that it should be possible to establish some connection with local coherence, and hence simplify the lemma.

Lemma 6.4.8 *Suppose $(X, \sqsubseteq, \nu)$ is a valuation space for which $\nu : X \to \Re$ is a continuous valuation, $(X, \tau_{[\nu]})$ is locally coherent and*

$$\mathcal{W}_X = \{W \subseteq X \mid W \text{ a proper filtered open set with } \nu(W) > -\infty\},$$

is a lattice-basis, then $(X, \sqsubseteq)$ has an information measure.

Proof. It follows by lemma 6.4.7 and theorem 6.4.6, that we can find a unique regular Borel measure ψ^* on X such that $\psi^*(W \setminus W') = \nu(W) - \nu(W')$, for all $W' \subseteq W$ in $\mathcal{W}_X$. Since

$$\psi^*(\downarrow x \setminus \downarrow x \sqcap y) = \psi^*((X \setminus \downarrow x \sqcap y) \setminus (X \setminus \downarrow x)),$$

then $\psi^*(\downarrow x \setminus \downarrow x \sqcap y) = 0$ implies that $\nu(X \setminus \downarrow x \sqcap y) = \nu(X \setminus \downarrow x)$ and hence $x \sqsubseteq y$. So ψ^* is an information measure on X.

QED

We have therefore given some conditions on a valuation space for the valuation to induce an information measure. There is clearly much scope to improve this result. The next step is to then use the induced information measure to build a function space, and finally a cartesian closed category. We hypothesis that these steps are possible, even though we are not in a position to take this work any further. Instead we hope that we have given a platform to build on.

Chapter 7

Conclusions and Further Work

We clarify the position that we have reached. Prior to this thesis, the body of work related to the partial metrics consisted of Wadge's paper on the cycle sum test [Wad81], Matthews' initial paper giving the axioms and basic results [Mat94] and his subsequent paper [Mat95] attempting to generalise the cycle sum test to the framework of partial metric spaces. What we do, is to present the partial metric axioms within the context of the more general T_0-metrics (section 3.2), give a firm foundation for the theory of partial metric spaces (chapters 3 and 4) and discuss the foundational issues by introducing the hierarchical spaces (section 3.3). More specific contributions, are the introduction of the weighted spaces (section 4.3), the valuation spaces (section 6.2) and the subsequent fundamental connections with measures (chapter 6). From our development of the theory of partial metric spaces, we contribute to domain theory (section 4.1.4) by identifying a suitable notion of a quantitative domain, and the more general area of T_0-topologies (section 6.3) by introducing local sobriety. Furthermore, we extend the applications of the partial metric spaces to Petri nets (chapter 5).

We finish by looking at our last two chapters once more, to consider the potential for further work. In chapter 5, we demonstrated our ideas on the modeling and analysis of systems by considering liveness of deterministic Petri nets. As we remarked in the chapter, there is a clear need to formalise these results to see if we really can model any deterministic Petri net in this way. There is also a lot of scope for extending these results. For example, we can look at Petri nets with more general timing constraints or multiple tokens, and we can extend to properties other than liveness, such as performance evaluation techniques relevant to timed Petri nets. Having considered Petri nets and data flow networks, we could see if these results extend to more general discrete event systems. In section 5.4 we raised the issue of extending to non-deterministic Petri nets, and there is clearly much scope for work here. At the same time, we need to further develop our model-independent analytic techniques. In this way, we see a more (classically) analytic approach to problems in Computer Science developing.

Our work in chapter 6 provided a platform to begin developing such techniques, but clearly raises more issues than it settles. We believe the locally compact ordered spaces to be worthy of further study and would like, if possible, to give a T_0-characterisation of such spaces. Local sobriety is our attempt at this, and we believe it to be of sufficient interest for further study, but question whether some other approach may be more fruitful? The connections between well-behaved partial metrics and measures are central to the chapter, and are certainly worthy of further investigation. The information measures seem to be a useful contribution that require more work, and we would hope that this could lead to a satisfactory development of function spaces, which in turn could be the framework in which to develop further analytic techniques.

Appendix A

Mathematical Background

We give the relevant background material on topological spaces, metric spaces and measure theory. We give the definitions and state some key results, and refer to our sources [Kel55, Apo74, Sut75, Coh80, Rud87, Smy92] for the proofs and further explanations.

A.1 Topological spaces

A.1.1 A *topology* on a set X is a family, π, of subsets of X such that $\emptyset, X \in \pi$ and π is closed under arbitrary unions and finite intersections. The pair (X, π) is a *topological space.* The *open sets* are those in π and an *open neighbourhood* of $x \in X$ is an open set containing x. The collection of open neighbourhoods of a point is called its *neighbourhood filter.* The *interior* of a set is the largest open set that it contains.

A.1.2 Suppose π and π' are topologies on a set X and $\pi \subseteq \pi'$, then π is *coarser* (*weaker*) than π', and π' is *finer* than π. The finest topology on X is the *discrete topology* consisting of all subsets of X.

A.1.3 Suppose X is a topological space, then a set is *closed* if its complement is open. The *closure* of a set is the smallest closed set that contains it. A point $x \in X$

is a *limit point* of $A \subseteq X$ if every open neighbourhood of x contains some other point of A. A set is closed if, and only if, it contains all its limit points.

A.1.4 Suppose (X, π) is a topological space and $Y \subseteq X$. If we let π_Y denote the collection of subsets of Y of the form $U \cap Y$ for some $U \in \pi$, then π_Y is the *induced topology* on Y and (Y, π_Y) is the *subspace* of (X, π).

A.1.5 Suppose (X, π) is a topological space, then $\mathcal{B}$ is a *basis* if for each $V \in \pi$ and $x \in V$ we can find $U \in \mathcal{B}$ such that $x \in U \subseteq V$. The *Euclidean topology* on $\Re$ has basic open sets of the form $(a, b) \subseteq \Re$. A topological space is *second countable* if it has a countable basis.

A.1.6 Suppose X and Y are topological spaces. A function $f : X \to Y$ is *continuous* if $f^{-1}(U)$ is open in X whenever U is open in Y. The function f is a *homeomorphism* if it is a bijection and both f and f^{-1} are continuous.

A.1.7 Suppose $\{(X_\alpha, \pi_\alpha)\}$ is an indexed family of topological spaces, and let $\prod_\alpha X_\alpha$ be the product of the indexed family of sets $\{X_\alpha\}$. The *product topology* on $\prod_\alpha X_\alpha$ is the weakest topology that makes each of the projections $\pi_\beta : \prod_\alpha X_\alpha \to X_\beta$ continuous.

A.1.8 Suppose X is a topological space and $A \subseteq X$, then A is *dense* if X is the closure of A. A space is *separable* if it has a countable dense subset.

A.1.9 A topological space X is *Hausdorff* if for each pair of distinct $x, y \in X$, there are disjoint open sets U, V such that $x \in U$ and $y \in V$. A topological space is a *T_0-space* if for each pair of distinct $x, y \in X$, there exists an open set containing one but not the other.

A.1.10 Suppose X is a topological space and $A \subseteq X$. An *open cover* of A is a collection of open sets such that A is in their union. A *subcover* is a subfamily that is itself an open cover of A. The set A is *compact* if every open cover has a finite subcover.

A.1.11 A space X is *locally compact* if for any $x \in X$, every open neighbourhood of x contains a compact neighbourhood of x. A Hausdorff space is locally compact if and only if every point has a compact neighbourhood. Every compact Hausdorff space is locally compact.

A.1.12 From $[0,1]$ we remove the middle third to get $A_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$. We then remove the middle third of $[0, \frac{1}{3}]$ and $[\frac{2}{3}, 1]$ to get $A_2 \subseteq A_1$. Continuing in this way, we define $A_3, A_4, \ldots$, and let $C = \bigcap_{n=1}^{\infty} A_n$. This is called the *Cantor space*, and $x \in C$ if, and only if, $x = \sum_{n=1}^{\infty} a_n/3^n$ where each a_n is either 0 or 2.

A.2 Metric spaces

A.2.1 A *metric* on a set X is a function $d : X \times X \to [0, \infty)$ such that

1. $d(x,y) = 0$ if, and only if, $x = y$.
2. $d(x,y) = d(y,x)$.
3. $d(x,z) \leq d(x,y) + d(y,z)$.

The pair (X,d) is a *metric space*. If $x \in X$ and $\varepsilon > 0$ then

$$B_\varepsilon(x) = \{y \in X \mid d(x,y) < \varepsilon\},$$

is called an *open ball*. The collection of open balls is the basis for a topology on X.

A.2.2 A topological space (X, π) is *metrizable* if there is a metric on X such that π is the induced topology.

A.2.3 Suppose (X,d) and (X',d') are metric spaces, then a *Lipschitz equivalence* $f : X \to X'$ is an injection for which there exists $h, k > 0$ such that

$$hd'(f(x), f(y)) \leq d(x,y) \leq kd'(f(x), f(y)), \qquad \forall x, y \in X.$$

An *isometry* $f : X \to X'$ is a bijection such that $d'(f(x), f(y)) = d(x,y)$, for all $x, y \in X$. If f is not surjective, then we say that it is an *isometry into* X'.

A.2.4 Suppose (X, d) is a metric space, then for $x \in X$ and $B \subseteq X$ non-empty closed, we define $d(x, B) = \inf\{d(x, y) \mid y \in B\}$. The *Hausdorff metric* on the collection of non-empty closed sets is defined by

$$d_H(A, B) = \sup\{d(x, B), d(A, y) \mid x \in A, y \in B\}.$$

A.2.5 Suppose (X, d) is a metric space, $x \in X$ and $\{x_n\}$ is a sequence in X, then $\{x_n\}$ *converges* to x if $\lim_{n\to\infty} d(x_n, x) = 0$. We say that x is the *limit* of $\{x_n\}$.

A.2.6 Suppose (X, d) is a metric space, then a sequence $\{x_n\}$ in X is *Cauchy* if for every $\varepsilon > 0$ there exists $N \geq 1$ such that $d(x_n, x_m) < \varepsilon$ forall $n, m \geq N$. The metric space is *complete* if every Cauchy sequence converges to a point in X.

A.2.7 The completion of a metric space (X, d) is a complete metric space $(\overline{M}, \overline{d})$ and an isometry into $\overline{M}$, $i : M \to \overline{M}$, such that $i(M)$ is dense in $\overline{M}$. Every metric space has a unique completion up to isometry.

A.2.8 Suppose (X, d) is a metric space, then a *contraction* $f : X \to X$ is a map for which there exists $0 \leq k < 1$ such that $d(f(x), f(y)) \leq kd(x, y)$, for all $x, y \in X$. If (X, d) is a complete metric space and $f : X \to X$ is a contraction, then f has a unique fixed point.

A.2.9 A metric space (X, d) is *totally bounded* if for each $\varepsilon > 0$ there is a finite subset $S \subseteq X$ such that

$$X = \bigcup\{B_\varepsilon(x) \mid x \in S\}.$$

The following are equivalent:

1. X is compact.
2. X is complete and totally bounded.
3. Every sequence has a subsequence converging to a point in X.

A.2.10 Given two metric spaces (X, d) and (X', d') we can define many possible *product metrics* on $X \times X'$. For example, if we let $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in $X \times X'$, then we can define

$$d_1(x, y) = d(x_1, y_1) + d'(x_2, y_2),$$
$$d_2(x, y) = [d(x_1, y_1)^2 + d'(x_2, y_2)^2]^{1/2},$$
$$d_\infty(x, y) = \max\{d(x_1, y_1), d(x_2, y_2)\}.$$

A.2.11 A complex vector space X *is a normed space* if for each $x \in X$, there is $\|x\| \in \Re^+$ called the *norm* of x such that

1. $\|x + y\| \leq \|x\| + \|y\|$, for all $x, y \in X$.
2. $\|\alpha x\| = |\alpha|\|x\|$, for all $x \in X$, α a scalar.
3. $\|x\| = 0$ implies that $x = 0$.

The norm induces a metric on X as $d(x, y) = \|x - y\|$. A *Banach space* is a normed space complete in the induced metric.

A.3 Measure theory

A.3.1 If $\mathcal{M}$ is a non-empty class of subsets of some set X, then $\mathcal{M}$ is a *σ-ring* if it is closed under differences and countable unions, and a *σ-algebra* if it contains X and is closed under complements and countable unions and intersections. A σ-ring that contains X is a σ-algebra. For any class of subsets $\mathcal{M}$ of X, there exists a smallest σ-ring $S(\mathcal{M})$, and a smallest σ-algebra $\sigma(\mathcal{M})$ containing $\mathcal{M}$. If $\mathcal{M}$ is a σ-algebra then X is a *measurable space* and the members of $\mathcal{M}$ are the *measurable sets*.

A.3.2 For a set X and σ-algebra $\mathcal{M}$, a (*positive*) *measure* $\mu : \mathcal{M} \to [0, \infty]$ has $\mu(\emptyset) = 0$ and is countably additive, so that if $\{A_i\}$ is a disjoint collection of members of $\mathcal{M}$, then

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i).$$

A *measure space* is a measurable space with a positive measure defined on its measurable sets. A *normalised measure* has $\mu(X) = 1$.

A.3.3 Suppose X is a topological space and let $\mathcal{B}$ be the smallest σ-algebra containing every open set. The members of $\mathcal{B}$ are called the *Borel sets* of X. A *Borel measure* is a measure defined on the Borel sets.

A.3.4 For a measure space X, a *σ-finite measure* is a measure such that X is the countable union of a collection of sets with finite measure.

A.3.5 Suppose μ is a Borel measure on a topological space X, then μ is *regular* if μ is finite on compact sets, for each $A \subseteq X$,

$$\mu(A) = \inf\{\mu(U) \mid A \subseteq U \text{ and } U \text{ open}\},$$

and for each open $U \subseteq X$,

$$\mu(U) = \sup\{\mu(K) \mid K \subseteq U \text{ and } K \text{ compact}\}.$$

A.3.6 Suppose (X, π) is a second-countable locally compact Hausdorff space with a σ-finite Borel measure μ, then μ is regular.

A.3.7 The *Lebesgue measure* on $\Re$ is such that every interval has measure its length, every Borel set is Lebesgue measurable, it is translation invariant and regular.

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