ON PEANO SPACES, WITH SPECIAL REFERENCE TO UNICOHERENCE AND NON-CONTINUOUS FUNCTIONS

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To the memory of Dr. G. T. Whyburn

PREFACE

The results presented in this thesis have arisen from my study of Peano spaces over the period of my being enrolled for the doctoral degree at the University of Warwick. Each chapter is conceived of as an independent paper, except for chapter 3. This chapter is primarily concerned with the question of a connectivity function being peripherally continuous, but a number of other aspects of the theory of non-continuous functions are discussed in it. The title of the thesis arises from the preponderance of chapter 3 on non-continuous functions, and the recurrence of the notion of unicoherence in chapters 2, 3, 4 and 8.

On the personal side, I wish to express my appreciation to Prof. K. O. Househam and Mr. P. Baxandall for arousing my interest in topology through their courses at the University of Cape Town.

I am indebted to Prof. E. C. Zeeman for supervising my thesis. Besides showing me the satisfaction deriving from research in topology, his encouragement and guidance have contributed decisively to bringing my thesis to its completion.

A glance at the following pages will show the influence of Dr. G. T. Whyburn on this work. Practically every chapter arises from his published work or his teaching. It is out of gratitude for the invaluable debt that I owe to him that I am dedicating my thesis to his memory.

I wish to thank Dr. G. H. M. Thomas for giving me of his time to discuss questions in this thesis. In several cases conversations with him have contributed substantially towards their solution.

Finally, I acknowledge the assistance that my wife has rendered. She has been a source of constant encouragement to me and I owe the preparation of the manuscript to her.

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CONTENTS

ABSTRACT

We have mentioned that each chapter in this thesis is conceived of as an independent paper, except for chapter 3, which is a collection of results on non-continuous functions. Consequently each chapter contains a clearly marked introductory section, in which its background and content are explained. In this abstract we shall summarize the remarks in these introductory sections.

In chapter 1 we present an n-arc theorem for Peano spaces which is an extension of the theorem in §2 of [32], which Menger called the <u>second</u> n-<u>arc theorem</u> in [17]. Whereas in the second n-arc theorem n disjoint arcs are constructed joining two disjoint closed sets A and B, in chapter 1 we split the closed set A into n disjoint closed subsets $A_1, A_2, \ldots, A_n$ and give necessary and sufficient conditions for there to be n disjoint arcs joining A and B, one meeting each A_i. At the end of chapter 1 we present a conjecture, which we have been able to verify in special cases.

In [35] Whyburn proved a theorem concerning the weak separation of two non-degenerate connected closed sets A and B by a quasi-closed set L in a locally cohesive space X. In chapter 2 we show that A and B can in fact be taken as arbitrary closed sets in this theorem; that is,

we remove the restriction of non-degeneracy and connectedness on A and B.

In chapter 3 we study the circumstances under which a connectivity function is peripherally continuous.

The study of the abstract relations between non-continuous functions was initiated by Stallings in [23]. In this paper he introduced the lpc polyhedron and showed that a connectivity function was peripherally continuous on an lpc polyhedron. Whyburn took up the study of non-continuous functions in [33], [34] and [35]. He introduced the locally cohesive space, which is more general than the lpc polyhedron, and proved that a connectivity function was peripherally continuous on a locally cohesive Peano space.

For technical reasons, the locally cohesive space is not permitted to have local cut points. It is obvious, however, that on many Peano spaces having local cut points a connectivity function remains peripherally continuous. In §2,3 of chapter 3 we formulate a sequence of properties $P_n(X)$, which permit the space X to have local cut points, and we prove in each case that a connectivity function $f : X \to Y$ is peripherally continuous when X has property $P_n(X)$. Each of these properties is an improvement on the last, and the final one, the U-space, satisfactorily incorporates the class of Peano spaces with local cut points on which we are able to prove that a connectivity function is peripherally continuous.

An interesting feature of §3 of chapter 3 is provided by two "weak separation theorems," and more will be found about these in the introduction to chapter 3.

In §4 of chapter 3 we show that a connectivity function is peripherally continuous on a locally compact ANR. This affirmatively answers a question that Stallings raised in [23].

The U-space that we have introduced in §3 of chapter 3 imposes a "unicoherence condition" in the space X (as do all the properties $P_n(X)$ considered in §3, chapter 3). In §5 of chapter 3 we generalize the U-space to the S-space. This imposes a "multicoherence condition" on the space X, and we prove that a connectivity function is peripherally continuous on a cyclic S-space.

We close chapter 3 by considering the question of placing weaker conditions than connectivity on the function $f : X \to Y$ which will still ensure that f is peripherally continuous.

It is well known that if X is a unicoherent Peano continuum and $A_1, A_2, \ldots$ is a sequence of disjoint closed subsets of X no one of which separates X, then $\bigcup_{n=1}^{\infty} A_n$ does not separate X. In [28] van Est proved this theorem for the case where X is a Euclidean space of n dimensions. In chapter 4 we give an example which shows that this theorem does not hold if X is an arbitrary Peano space.

In chapter 5 we provide a new angle to Lebesgue's covering lemma. We show that if the Lebesgue number δ of an open covering $U_1, U_2, \ldots, U_n$ of a compact metric space X, ρ is finite, then it can be defined by the formula $\delta = \min \rho(E, F)$, where E and F are any compartments contained in no common U_i.

In chapter 6 we show that an involution on a cyclic Peano space leaves some simple closed curve setwise invariant.

Whyburn has given a proof of R. L. Moore's decomposition theorem for the 2-sphere in [31] (a refinement of this proof is presented in [36]). His proof is accomplished by showing that the decomposition space satisfies Zippin's characterization theorem for the 2-sphere. In chapter 6 we present an alternative way of showing that the decomposition space satisfies Zippin's characterization theorem. Our proof closely follows Alexander's proof of the Jordan curve theorem as given by Newman in [21], and so consists of arguments that are well-known in another context.

In [30] Whyburn gave a proof of the cyclic connectivity theorem, and in all subsequent appearances of this theorem in the literature Whyburn's proof has been used. Whyburn divided the proof of the theorem into three parts: lemma 1, lemma 2, and the deduction of the theorem from lemmas 1 and 2. In chapter 8 we give an alternative proof of lemma 1. Our proof is based on the fact that a cyclic Peano space has a base of regions whose closures do not separate the space, and it proceeds by an induction on a simple chain of these regions.

CHAPTER 1

AN N-ARC THEOREM FOR PEANO SPACES

1. INTRODUCTION. In this chapter we present a theorem and a conjecture that arise from [32].

We first recall some definitions from [32]. Let A, B and X be closed subsets of a topological space S. We say that X *broadly separates* A *and* B *in* S if $S - X$ is the union of two disjoint open sets (possibly empty) one of which contains $A - X$ and the other of which contains $B - X$. The space S is *n-point strongly connected between* A *and* B provided no set of less than n points broadly separates A and B in S. An arc ab *joins* A *and* B if $ab \cap A = \{a\}$ and $ab \cap B = \{b\}$.

The following theorem, in which we have replaced "completeness" by "local compactness," appears in [32]. It is called the *second n-arc theorem* by Menger in [17].

THE SECOND N-ARC THEOREM. *Let* A *and* B *be disjoint closed subsets of a locally connected, locally compact metric space* S. *A necessary and sufficient condition that there be* n *disjoint arcs in* S *joining* A *and* B *is that* S *be n-point strongly connected between* A *and* B.

In §2 we split the closed set A into n disjoint closed subsets $A_1, A_2, \ldots, A_n$. The theorem then gives a necessary and sufficient condition for there to be n disjoint arcs joining A and B, one meeting each A_i.

In §3 we split A and B into disjoint closed subsets $A_1, A_2, \ldots, A_n$ and $B_1, B_2, \ldots, B_n$. The conjecture then gives a necessary and sufficient condition for there to be n disjoint arcs joining A and B, one meeting each A_i and one meeting each B_i. (I have given a proof of this conjecture for the case $n = 4$, which is the first case that offers difficulties, but it is not included here.) ?

It will be noticed that the space S in the theorem and in the conjecture is not actually a Peano space, as the title of the chapter states, but it becomes one when the property of connectedness is placed on it.

2. Let $A_1, A_2, \ldots, A_n$ and B be disjoint closed subsets of a topological space S. We shall say that a subset X of S is a <u>large point of</u> S (with respect to $A_1, A_2, \ldots, A_n$) if it is a one-point set or one of the sets A_i. We shall say that S is <u>n-point strongly connected between</u> $A_1, A_2, \ldots, A_n$ <u>and</u> B provided the union of less than n large points does not broadly separate $A_1 \cup A_2 \cup \ldots \cup A_n$ and B in S.

We shall say that a system of n disjoint arcs in S <u>joins</u> A_1 $A_2, \ldots, A_n$ <u>and</u> B if each arc joins $A_1 \cup A_2 \cup \ldots \cup A_n$ and B and each A_i is joined to B by exactly one of the arcs.

THEOREM. <u>Let</u> $A_1, A_2, \ldots, A_n$ <u>and</u> B <u>be disjoint closed subsets of a locally connected, locally compact metric space</u> S. <u>A necessary and sufficient condition that there be</u> n <u>disjoint arcs in</u> S <u>joining</u> $A_1, A_2, \ldots, A_n$ <u>to</u> B <u>is that</u> S <u>be n-point strongly connected between</u> $A_1, A_2, \ldots, A_n$ <u>and</u> B.

We need two more definitions for the proof of the theorem. Let $A_1, A_2, \ldots, A_n$ be disjoint closed sets in a topological space S, and let $\beta_1, \beta_2, \ldots, \beta_m$ be disjoint arcs in S. We shall say that A_i is <u>a zero</u>, <u>a single</u> or <u>a multiple</u> with respect to $\beta_1, \beta_2, \ldots, \beta_m$ according as to whether it meets zero, one or more than one of the arcs $\beta_1, \beta_2, \ldots, \beta_m$. A subarc β of some β_i is said to be <u>a bridge of</u> $\beta_1, \beta_2, \ldots, \beta_m$ <u>spanning</u> $A_1, A_2, \ldots, A_n$ if β joins some A_j to some A_k, for $j \neq k$. Clearly there are only a finite number of bridges in $\beta_1, \beta_2, \ldots, \beta_m$ spanning $A_1, A_2, \ldots, A_n$.

PROOF. Using the terminology and notation of the theorem, it is clear that the condition is necessary for the existence of n disjoint arcs joining A_1, A_2, ..., A_n to B in S. So we turn to proving that it is sufficient.

By the arcwise connectivity theorem, the condition is sufficient for $n = 1$. So we assume its sufficiency for each positive integer $< n$ and prove its sufficiency for n by induction.

By the second n-arc theorem there are n disjoint arcs β_1, β_2, ..., β_n in S joining $A_1 \cup A_2 \cup \ldots \cup A_n$ and B. Let p be the number of singles of A_1, A_2, ..., A_n with respect to β_1, β_2, ..., β_n. We shall suppose that $p < n$ and show how to construct a second system of n disjoint arcs joining $A_1 \cup A_2 \cup \ldots \cup A_n$ and B with respect to which the number of singles is $p + 1$. The process can be repeated $n - p$ times to obtain the desired system of arcs joining A_1, A_2, ..., A_n and B.

Let A_1, A_2, ..., A_p be the singles, A_{p+1}, A_{p+2}, ..., A_q the zeros and A_{q+1}, A_{q+2}, ..., A_n the multiples of A_1, A_2, ..., A_n with respect to β_1, β_2, ..., β_n. Since $p < n$ there is at least one zero and at least one multiple here. We shall construct a system of n disjoint arcs joining $A_1 \cup A_2 \cup \ldots \cup A_n$

and B with respect to which $A_1, A_2, \ldots, A_{p+1}$ are singles. To this end we consider the locally connected, locally compact space $S - A_{p+2} \cup A_{p+3} \cup \ldots \cup A_n$. Since it is $(p + 1)$-point strongly connected between $A_1, A_2, \ldots, A_{p+1}$ and B and $p + 1 < q \leq n$, it follows from the inductive hypothesis that it contains $p + 1$ disjoint arcs $\alpha_1, \alpha_2, \ldots, \alpha_{p+1}$ joining $A_1, A_2, \ldots, A_{p+1}$ and B. We suppose, further, that α_r meets A_r for $r \leq p + 1$.

We now use an inductive technique that is familiar from [32]. We relabel $\beta_1, \beta_2, \ldots, \beta_n$ so that β_r meets A_r for $r \leq p$, and we start by defining $\alpha_r^0 = \alpha_r \cap A_r$ for $r \leq p + 1$ and $\beta_r^0 = \beta_r$ for $r \leq p$. Now we suppose that we have defined systems of arcs $\alpha_1^m, \alpha_2^m, \ldots, \alpha_{p+1}^m$ (possibly degenerate) and $\beta_1^m, \beta_2^m, \ldots, \beta_p^m$ such that (a) $\alpha_r \cap A_r \subset \alpha_r^m \subset \alpha_r$ and α_r^m does not meet $B \cup \beta_{p+1} \cup \beta_{p+2} \cup \ldots \cup \beta_n$, (b) $\beta_s \cap B \subset \beta_s^m \subset \beta_s$, (c) if A_r, β_s^m meet then α_r^m is degenerate, (d) if α_r^m, β_s^m meet then they meet in a common end point, (e) exactly one of the sets $\alpha_1^m \cup A_1, \alpha_2^m \cup A_2, \ldots, \alpha_{p+1}^m \cup A_{p+1}$ fails to meet $\beta_1^m \cup \beta_2^m \cup \ldots \cup \beta_p^m$, (f) if b_m is the number of bridges of $\beta_1^m, \beta_2^m, \ldots, \beta_p^m$ that span $\alpha_1 \cup A_1, \alpha_2 \cup A_2, \ldots, \alpha_{p+1} \cup A_{p+1}$, then $b_m < b_{m-1}$ for $m \geq 1$. We now show how the induction may be continued to the next stage and how it leads, after at most a finite number of stages, to the construction of n disjoint arcs joining $A_1 \cup A_2 \cup \ldots \cup A_n$ to B with respect to which $A_1, A_2, \ldots, A_{p+1}$ are singles.

We proceed by denoting by $\alpha_t \cup A_t$ the set, given in (d), which does not meet $\beta_1^m \cup \beta_2^m \cup \ldots \cup \beta_p^m$. We let x be the first point of α_t in the direction $\alpha_t \cap A_t$, $\alpha_t \cap B$ that belongs to the union of the three sets $\beta_1^m \cup \beta_2^m \cup \ldots \cup \beta_p^m$, $\beta_{p+1} \cup \beta_{p+2} \cup \ldots \cup \beta_n$ and $B - \beta_1 \cup \beta_2 \cup \ldots \cup \beta_n$. We consider separately the three mutually exclusive cases (1) $x \in \beta_1^m \cup \beta_2^m \cup \ldots \cup \beta_p^m$, (2) $x \in \beta_{p+1} \cup \beta_{p+2} \cup \ldots \cup \beta_n$ and (3) $x \in B - \beta_1 \cup \beta_2 \cup \ldots \cup$

We first consider case (1) and let $x \in \beta_u^m$. We define $\alpha_r^{m+1} = \alpha_r^m$ for $r \neq t$, $r \leq p + 1$, and α_t^{m+1} as the subarc of α_t whose endpoints are $\alpha_t \cap A_t$, x. We define $\beta_s^{m+1} = \beta_s^m$ for $s \neq u$, $s \leq p$, and β_u^{m+1} as the subarc of β_u^m whose endpoints are $\beta_u \cap B$, x. It is easily seen that (a) - (d) of the inductive hypotheses are preserved. In order to verify that (e) is preserved, we notice that it follows from (a) - (d) that each β_s^m meets at most one $\alpha_r^m \cup A_r$. Thus it follows from (e) that the relation $(\alpha_r^m \cup A_r) \cap \beta_s^m \neq \emptyset$ establishes a 1 - 1 correspondence between the collections $\beta_1^m, \beta_2^m, \ldots, \beta_p^m$ and $\alpha_1^m \cup A_1, \alpha_2^m \cup A_2, \ldots, \alpha_{t-1}^m \cup A_{t-1}, \alpha_{t+1}^m \cup A_{t+1}, \ldots, \alpha_{p+1}^m \cup A_{p+1}$. If we now let $\alpha_v^m \cup A_v$

be the set that corresponds to β_u^m under this relation, it is clear that by (d) $\alpha_v^{m+1} \cup A_v$ does not meet $\beta_1^{m+1} \cup \beta_2^{m+1} \cup \dots \cup \beta_p^{m+1}$, and that it is the only set among $\alpha_1^{m+1} \cup A_1$, $\alpha_2^{m+1} \cup A_2$, ..., $\alpha_{p+1}^{m+1} \cup A_{p+1}$ with this property. It is clear that (f) is also preserved, since $(\beta_u^m - \beta_u^{m+1}) \cup \{x\}$ is an arc that joins $\alpha_v^m \cup A_v$ and $\alpha_t^m \cup A_t$, and so it contains at least one bridge of β_1^m, β_2^m, ..., β_p^m spanning $\alpha_1 \cup A_1$, $\alpha_2 \cup A_2$, ..., $\alpha_{p+1} \cup A_{p+1}$ that is not contained in $\beta_1^{m+1} \cup \beta_2^{m+1} \cup \dots \cup \beta_p^{m+1}$; i.e., $b_{m+1} < b_m$.

Thus in case (1) the inductive hypotheses are preserved. We notice that it follows from (f) that case (1) can occur for only a finite number of values of m, since b_0 is finite. Thus case (2) or case (3) must eventually occur. We complete the proof of the theorem by showing that in either of these cases we can readily obtain a system of n disjoint arcs joining $A_1 \cup A_2 \cup \dots \cup A_n$ and B with respect to which A_1, A_2, ..., A_{p+1} are singles.

We shall only deal with case (2), as case (3) is practically identical to it. Thus we let $x \in \beta_w$, $p + 1 \leq w \leq n$. We define α as the subarc of α_t whose endpoints are $\alpha_t \cap A_t$, x and β as the subarc of β_w whose endpoints are $\beta_w \cap B$, x. We first notice that it follows from (a) - (d) that if $\alpha_r^m \cup A_r$, β_s^m meet, then $\alpha_r^m \cup \beta_s^m$ is an arc joining A_r, B. Since a 1 - 1 correspondence is established between the collections $\alpha_1^m \cup A_1$, $\alpha_2^m \cup A_2$, ..., $\alpha_{t-1}^m \cup A_{t-1}$, $\alpha_{t+1}^m \cup A_{t+1}$, ..., $\alpha_{p+1}^m \cup A_{p+1}$ and β_1^m, β_2^m, ..., β_p^m by the relation $(\alpha_r^m \cup A_r) \cap \beta_s^m \neq \emptyset$ it follows that the union of α_1^m, α_2^m, ..., α_{t-1}^m, α_{t+1}^m, ..., α_{p+1}^m, β_1^m, β_2^m, ..., β_p^m may be expressed as a union of p disjoint arcs joining A_1, A_2, ..., A_{t-1}, A_{t+1}, ..., A_{p+1} and B. Furthermore, by (a), (b) these arcs are disjoint from the arcs β_{p+1}, β_{p+2}, ..., β_{w-1}, β_{w+1}, ..., β_n, α, β. Thus the union of α_1^m, α_2^m, ..., α_{t-1}^m, α_{t+1}^m, ..., α_{p+1}^m, β_1^m, β_2^m, ..., β_p^m, β_{p+1}, β_{p+2}, ..., β_{w-1}, β_{w+1}, ..., β_n, α, β may be expressed as a union of n disjoint arcs joining $A_1 \cup A_2 \cup \ldots \cup A_n$ and B with respect to which A_1, A_2, ..., A_{p+1} are singles. This completes the proof of the theorem.

3. Let A_1, A_2, ..., A_n and B_1, B_2, ..., B_n be disjoint closed subsets of a topological space S. We shall say that a subset X of S is a <u>large point of</u> S (with respect to A_1, A_2, ..., A_n and B_1, B_2, ..., B_n) if it is a one-point set, a set A_i, or a set B_i. We shall say that S is <u>n-point strongly connected between</u> A_1, A_2, ..., A_n <u>and</u> B_1, B_2, ..., B_n

provided the union of less than n large points does not broadly separate $A_1 \cup A_2 \cup \ldots \cup A_n$ and $B_1 \cup B_2 \cup \ldots \cup B_n$ in S.

We shall say that a system of n disjoint arcs in S _joins_ $A_1, A_2, \ldots, A_n$ _and_ $B_1, B_2, \ldots, B_n$ if each arc joins $A_1 \cup A_2 \cup \ldots \cup A_n$ and $B_1 \cup B_2 \cup \ldots \cup B_n$, and each A_i meets just one arc, and each B_i meets just one arc,

CONJECTURE. _Let_ $A_1, A_2, \ldots, A_n$ _and_ $B_1, B_2, \ldots, B_n$ _be disjoint closed subsets of a locally connected, locally compact metric space_ S. _A necessary and sufficient condition that there be_ n _disjoint arcs in_ S _joining_ $A_1, A_2, \ldots, A_n$ _and_ $B_1, B_2, \ldots, B_n$ _is that_ S _be n-point strongly connected between_ $A_1, A_2, \ldots, A_n$ _and_ $B_1, B_2, \ldots, B_n$.

The necessity of the condition is again trivial, so it is the sufficiency of the condition that is interesting.

The conjecture is clearly true if the sets $A_1, A_2, \ldots, A_n$ and $B_1, B_2, \ldots, B_n$ are compact. For in this case the quotient space Q obtained by identifying a pair of points if they belong to a common A_i or a common B_j is locally compact, locally connected and metrizable. If π is the natural projection from S onto Q, it is clear that Q is n-point strongly connected between $\pi(A_1) \cup \pi(A_2) \cup \ldots \cup \pi(A_n)$ and $\pi(B_1) \cup \pi(B_2) \cup \ldots \cup \pi(B_n)$. Consequently it follows from the second n-arc theorem that there are n disjoint arcs in Q joining $\pi(A_1) \cup \pi(A_2) \cup \ldots \cup \pi(A_n)$ and $\pi(B_1) \cup \pi(B_2) \cup \ldots \cup \pi(B_n)$. The π-inverse of each of these arcs contains a connected closed set which meets both $A_1 \cup A_2 \cup \ldots \cup A_n$ and $B_1 \cup B_2 \cup \ldots \cup B_n$, from which it easily follows that there are n-disjoint arcs in S joining $A_1, A_2, \ldots, A_n$ and $B_1, B_2, \ldots, B_n$.

When some of the sets $A_1, A_2, \ldots, A_n$ or $B_1, B_2, \ldots, B_n$ fail to be compact, the above argument does not suffice as the quotient space Q is not in general metrizable.

There ought to be a combinatorial proof of this conjecture along the lines of the proof in §2, which would work equally well whether some of the sets $A_1, A_2, \ldots, A_n$ or $B_1, B_2, \ldots, B_n$ fail to be compact or not. Such a proof has been given for the case $n = 4$, as was remarked earlier.

CHAPTER 2

THE SEPARATION THEOREM FOR QUASI-CLOSED SETS

1. INTRODUCTION. In this chapter we complete a sequence of arguments concerning quasi-closed sets that appear in [35].

In [35] Whyburn proves the following theorem.

THEOREM. _Let_ A _and_ B _be disjoint non-degenerate closed and connected sets in a locally cohesive_ T_1-_space_ X. _Any quasi-closed set_ L _which weakly separates_ A _and_ B _in_ X _contains a closed set_ K _which separates_ $A - K$ _and_ $B - K$ _in_ X.

In the "Concluding Remarks" of [35] Whyburn shows that the requirement that A and B be non-degenerate can be deleted. He also mentions that the condition that A and B be connected can be replaced by the requirement that each of them be of dimension > 0 at each point.

In this chapter we show that A and B can in fact be arbitrary closed sets. The theorem to this effect appears in § 2. We call it _the separation theorem for quasi-closed sets_.

I owe it to Dr. Whyburn for pointing out, in Appendix I of [37], that the results in the "Concluding

Remarks" of [35] are partial versions of the theorem in §2.

2. We first define the necessary terms. We follow the definitions in [35], except for two changes. Firstly, our definition of "unicoherence between two subsets" is weaker than the definition in §5 of [34], on which the definition of "local cohesiveness" in [35] is based. Secondly, we define "local cohesiveness" for arbitrary spaces. This means that we usually have to include certain separation properties in the statements of our results.

In all the definitions that follow X is an arbitrary topological space unless otherwise stated.

A set E in a space X is <u>quasi-closed</u> in X if each point in $X - E$ has a base of neighbourhoods whose frontiers do not meet E.

Let E and F be two disjoint subsets of a connected space X. We say that X is <u>unicoherent between</u> E <u>and</u> F if however X is expressed as the union of two connected closed sets M and N such that $M - N$ and $N - M$ contain E and F, respectively, $M \cap N$ is always connected. If p is a point of a space X, we say that R is <u>a canonical region about</u> p <u>in</u> X if R is a connected neighbourhood of p, the frontier $\mathrm{Fr}\, R$ of R is connected, and $\overline{R}$ is unicoherent between $\{p\}$ and $\mathrm{Fr}\, R$ (or, equivalently, in case X is connected, X is unicoherent between $\{p\}$ and $X - R$).

A space X is <u>locally cohesive</u> if each of its points has a base of canonical regions. Notice that a locally cohesive space is locally connected.

Let E, F and L be subsets of a space X. We say that L <u>separates</u> E <u>and</u> F <u>in</u> X if $X - L$ is the union of two sets M and N which contain E and F, respectively, and which are separated in X (M and N are separated in X if $\overline{M} \cap N = \emptyset = M \cap \overline{N}$). We say that L <u>weakly separates</u> E <u>and</u> F <u>in</u> X if no component of $X - L$ meets both E and F. Notice that we may have $E \cap F \cap L \neq \emptyset$ in this last definition.

Before giving the theorem, we state two simple lemmas. These can be found as statements in §1 of [35]. They are, in any event, easily proved on the basis of our definitions.

2.1. LEMMA. <u>If</u> R <u>is a canonical region about a point</u> p <u>in a locally cohesive space</u> X, <u>and</u> K <u>is a closed set in</u> $\overline{R}$ <u>that separates</u> p <u>and</u> $\mathrm{Fr}\, R$ <u>in</u> $\overline{R}$, <u>then there is a canonical region</u> S <u>about</u> p <u>such that</u> $\overline{S} \subset R$ <u>and</u> $\mathrm{Fr}\, S \subset K$.

2.2. LEMMA. <u>If</u> L <u>is a quasi-closed set in a locally cohesive regular space</u> X, <u>then each point of</u> $X - L$ <u>has a base of canonical regions whose frontiers do not meet</u> L.

2.3. THEOREM. Let A and B be closed sets in a locally cohesive regular T_1-space X. Any quasi-closed set L which weakly separates A and B in X contains a closed set K which separates $A - K$ and $B - K$ in X.

PROOF. We notice that we may suppose without loss of generality that X is connected, for on the one hand the restriction of L to a component of X is quasi-closed, and on the other hand the union of a collection of closed sets, each contained in a component of X, is closed. ?
Thus we shall suppose that X is connected.

We first consider the case of a point $p \in A - L$ which lies in a non-degenerate component H_p of $X - L$, and we show that there is a region G_p about p which does not meet B and for which $\mathrm{Fr}\, G_p \subset \overline{H}_p \cap L$.

First notice that $\overline{H}_p - H_p \subset L$; for if $x \in \overline{H}_p - L$, then $H_p \cup \{x\}$ is a connected set in $X - L$ and so is contained in H_p. Now let V be the union of all the components of $X - \overline{H}_p$ that meet B. Then $\mathrm{Fr}\, V \cap H_p = \emptyset$. For let $x \in H_p$ and let R be a canonical region about x which neither meets B nor contains H_p (H_p is non-degenerate and X is a T_1-space) and whose boundary $\mathrm{Fr}\, R$ does not meet L. Then H_p meets both R and its complement and so contains $\mathrm{Fr}\, R$. However, each component of V meets $X - R$ but not $\mathrm{Fr}\, R$, and so does not meet R.

Consequently V does not meet R and $x \notin \operatorname{Fr} V$. Thus $\operatorname{Fr} V \subset \overline{H}_p - H_p$. Thus $X - (\overline{V} \cup (B \cap \overline{H}_p))$ is a neighbourhood of p which does not meet B and whose frontier is contained in $\overline{H}_p \cap L$. If we let G_p be the component of $X - (\overline{V} \cup (B \cap \overline{H}_p))$ that contains p, then G_p is a region about p which does not meet B and for which $\operatorname{Fr} G_p \subset \overline{H}_p \cap L$.

Now we consider the case of a point $p \in A - L$ which lies in a degenerate component of $X - L$, and we show that there is a region G_p about p whose closure does not meet B and whose boundary is a connected subset of L.

Let R be a canonical region about p whose complement is non-degenerate and contains B and such that $\operatorname{Fr} R \cap L = \emptyset$. Then $L \cap R$ is a quasi-closed set, and we assert that it weakly separates the closed sets $\{p\}$ and $X - R$. For let H be the component of $X - L \cap R$ that contains the connected closed set $X - R$. Then $H \cap \overline{R}$ is connected, because it is a closed subset of H which contains the connected set $\operatorname{Fr} R$. It follows that p cannot belong to H, because if it did $\{p\} \cup (H \cap \overline{R})$ would be a non-degenerate connected subset of $X - L$, contradicting the assumption that p lies in a degenerate component of $X - L$. Since H is non-degenerate, there is by the second paragraph of this proof a region G which contains H and does not meet the closed set $\{p\}$, and whose boundary lies in $L \cap R$. Let G_p be the component of $X - \overline{G}$ that contains p. Then $\operatorname{Fr} G_p$ is connected. For $X = \overline{G}_p \cup (X - G_p)$, where $\overline{G}_p$ and $X - G_p$ are two connected closed subsets of X such that $\{p\} \cap (X - G_p) = \emptyset$ and $(X - R) \cap \overline{G}_p = \emptyset$. Therefore, since R is a canonical region about p, $\operatorname{Fr} G_p = \overline{G}_p \cap (X - G_p) \cap \overline{R}$ is connected. That is, G_p is a region about p whose closure does not meet B and whose boundary is a connected subset of L.

We shall suppose hereafter that B is non-degenerate, for if B is degenerate we can prove the theorem by interchanging the letters "A" and "B" in the second and fourth paragraphs of the proof when $B \subset X - L$, and by removing the set B from X when $B \subset L$.

Now we show that $\operatorname{Fr} \bigcup G_p \subset L$, the union being taken over all points $p \in A - L$. Suppose that $x \in (\operatorname{Fr} \bigcup G_p) - L$. Then $x \notin A$, so there is a canonical region R about x such that $R \cap A = \emptyset$, $R \not\supset B$ and $\operatorname{Fr} R \cap L = \emptyset$. Then G_p meets R for some $p \in A - L$, and consequently $\operatorname{Fr} G_p$ meets R, because R is connected and not contained in G_p. It follows that p cannot lie in a degenerate component of $X - L$, for in this case $\operatorname{Fr} G_p$ is a connected subset of L and so is contained in R. Thus, since $X - R$ is a connected set in the complement of $\operatorname{Fr} G_p$ which meets B, $\operatorname{Fr} G_p$

separates not only p and B but also p and $X - R$. That is, $p \in R$, which is false because $R \cap A = \emptyset$. So p lies in a non-degenerate component H_p of $X - L$. Further, since $\mathrm{Fr}\, G_p \subset \overline{H}_p$, H_p meets both R and its complement, and so meets and contains $\mathrm{Fr}\, R$. However $x \notin \overline{G}_p$, so there is a point $q \in A - L$ such that G_q meets $R - \overline{G}_p$, and as before q lies in a non-degenerate component H_q of $X - L$ which contains $\mathrm{Fr}\, R$. But this implies that $H_p \cap H_q \neq \emptyset$, and so $H_p = H_q$. Consequently $G_p = G_q$, by construction, which is a contradiction.

Let $K = (\mathrm{Fr} \bigcup G_p) \cup (A - \bigcup G_p)$, the union again being taken over all $p \in A - L$. Then K is a subset of L which is closed in X, and it separates $A - K$ and $B - K$ in X.

3. In conclusion we wish to point out the relation between certain results in [8] and [37] and the theorem given above.

We consider the following three results, which are proved in [8]:

(a) Theorem, p.54 [8].

(b) Corollary 1, p.57 [8],

(c) Separation Theorem, p.59 [8]

Referring to [8], we see that (a) implies (b) and (b) easily implies (c). There is also an easy implication from (c) to (a). Thus (a), (b) and (c) are all equivalent. Again referring to [9], we see that the conclusion in each of (a), (b) and (c) is the same, namely that a closed set E can be found in a set L which separates two sets $A - L$ and $B - L$ in a space X. Let us replace this conclusion in (a), (b) and (c) by, "a closed set E can be found in L which separates $A - E$ and $B - E$ in X." Then we get three propositions (a)', (b)' and (c)'. It is clear that (a)' and (c)' are untrue, and (b)' is simply our theorem above. It will be noticed that (b)' is a better result than (b), because (b) can be immediately deduced from (b)', but not conversely.

In Appendix 1 of [37] the proof of our theorem is broken into three steps:

(d) Theorem 1, p.58 [37],

(e) Theorem 2, p.59 [37],

(f) Separation Theorem, p.61 [37].

It is shown in [37] that (d) and (e) imply (f), which is our theorem of §2. It will be noticed that it also follows immediately that (f) implies (d) and (e).

CHAPTER 3

ON CONNECTIVITY FUNCTIONS BEING PERIPHERALLY CONTINUOUS

1. INTRODUCTION. We first explain how the study in this chapter arose.

BACKGROUND TO CHAPTER. In 1957 in [12] O.H. Hamilton showed that a connectivity function $f : I^n \to I^n$, where I^n was the closed Euclidean n-cell, had the fixed point property. The principal part of his argument involved showing that a connectivity function was peripherally continuous. He was then able to prove that a peripherally continuous function had the fixed point property.

In 1959 in [23] Stallings initiated the study of the relations between different kinds of non-continuous functions. He introduced the notion of an almost continuous function, and studied the relations between connectivity functions, peripherally continuous functions and almost continuous functions defined on polyhedral spaces. He also considered local connectivity functions and polyhedrally almost continuous functions. Stallings was aware of the limitation involved in using polyhedral objects, rather than purely topological objects, and questioned to what extent this limitation could be removed (see §2 of [23]). In §6 of [23], Stallings listed a number of interesting questions, several of which have subsequently been answered (see [27], [6], [10] and [33]). In §4 of this chapter another of these questions is answered.

What concerns us here, however, is that Stallings noticed that Hamilton's proof that a connectivity function $f : I^n \to I^n$ was peripherally continuous contained a gap. In filling in this gap, Stallings placed the theorem in a wider setting. He showed that a (local) connectivity function $f : X \to Y$ was peripherally continuous, where X was an lpc polyhedron and Y was a regular T_1-space. The lpc polyhedron then retains the pertinent properties of the n-cell that Hamilton used: namely, it has a base of regions $\{U_\alpha\}_\alpha$ whose closures are unicoherent and whose boundaries are connected. Notice that an lpc polyhedron is simply a polyhedron with no local cut points.

In 1966 and 1967 Dr. G.T. Whyburn published a series of three papers on non-continuous functions, namely [33], [34] and [35]. In the last two of these he introduced the notion of a locally cohesive space, and he used this to prove a number of interesting theorems about peripherally continuous functions (see also [37]).

However, what particularly interests us is that Whyburn proved that a connectivity function $f : X \to Y$

was peripherally continuous, where X was a locally cohesive Peano space and Y was a regular T_1-space.

Now the locally cohesive Peano space, like the lpc polyhedron, has a base of regions $\{U_\alpha\}_\alpha$ whose boundaries are connected. But the "unicoherence condition" that is imposed on it is more subtle than the requirement that each $\overline{U}_\alpha$ be unicoherent. It is only required that each $\overline{U}_\alpha$ be unicoherent modulo $\mathrm{Fr}\, U_\alpha$ (see theorem (3.1) of this chapter). Thus the locally cohesive Peano space is a considerable improvement over the lpc polyhedron. Besides being a purely topological notion, it also includes some infinitely multicoherent spaces within the terms of its definition. For example, the space in figure (1.1) which is the closure of the set of all points (x, y, z)

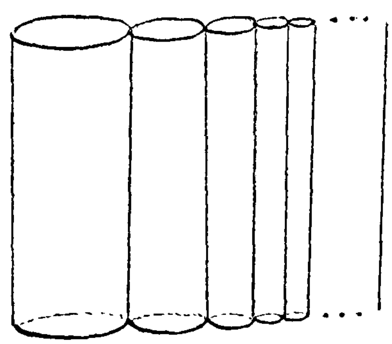

fig. (1.1)

in Euclidean 3-space such that $(x - (1 - 3/2^{n+1}))^2 + y^2 = 1/2^{n+1}$ and $0 \le z \le 1$ for some positive integer n, is a locally cohesive Peano continuum.

It will be noticed that the locally cohesive Peano space has no local cut points. In 1967 I wrote out a proof, which closely followed the arguments of Hamilton and Stallings, of a theorem concerning a connectivity function being peripherally continuous in which the domain space was permitted to have local cut points. In fact, the domain space was a Peano space which had a covering by unicoherent regions. This theorem and its original proof appear in §2 of this chapter.

When Dr. Whyburn was shown this theorem, he deduced it as a consequence of his theorem concerning a connectivity function being peripherally continuous on a locally cohesive Peano space, and it was in this form that it appeared in Appendix II of [37]. However, it is presented here with its original proof, because this proof contains the beginnings of many techniques that are used in subsequent sections of the chapter.

CONTENT OF CHAPTER. We have explained above how the work of this chapter arose. The purpose of the chapter is to further the study of the circumstances under which a connectivity function is peripherally continuous.

We let $P(X)$ stand for a statement which asserts that the topological space X has certain properties, and we let $Th(P(X))$ stand for this statement:

$Th(P(X)) \equiv$ <u>if</u> $f : X \to Y$ <u>is a connectivity function</u> <u>and</u> $P(X)$ <u>and</u> Y <u>is a regular</u> T_1-<u>space, then</u> f <u>is</u> <u>peripherally continuous</u>.

If we now put

$P_1(X) \equiv X$ is an lpc polyhedron,

$P_2(X) \equiv X$ is a locally cohesive Peano space,

$P_3(X) \equiv X$ is a Peano space with a covering by unicoherent regions,

then $Th(P_n(X))$ is a theorem for $n = 1,2,3$, as we have seen.

It will be noticed that neither $P_2(X)$ nor $P_3(X)$ is contained in the other. In §3 we first combine the better features of each, thus obtaining

$P_4(X) \equiv X$ is a Peano space with a covering (or base) of regions $\{U_\alpha\}_\alpha$ such that each $\bar{U}_\alpha$ is unicoherent modulo $\operatorname{Fr} U_\alpha$.

$P_4(X)$ then contains both $P_2(X)$ and $P_3(X)$, and $Th(P_4(X))$ appears as theorem (3.2).

After this, we improve the statement $P_4(X)$ with respect to cut points. We point out in example (3.2) and the paragraphs which immediately follow it, that while $Th(P_4(X))$ adequately deals with cyclic Peano spaces, the statement $P_4(X)$ does not adequately cover the class of Peano spaces with cut points on which we are able to prove that a connectivity function is peripherally continuous.

This consideration leads us to formulate

$$P_5(X) \equiv X \text{ is a U-space,}$$

the significance of the "U" being that we still have a "unicoherence condition" as a part of $P_5(X)$. In theorem (3.4) we show that $P_5(X)$ contains $P_4(X)$. The remainder of §3 is principally concerned with proving $Th(P_5(X))$, which appears as theorem (3.6). The U-space, then, provides a satisfactory solution to this problem of cut points.

That there are significant U-spaces which are not covered by $P_4(X)$ is shown by the example in figure (1.2) This space is the closure of the set of all points

(x, y) in the Euclidean plane such that $1/2^{n+2} \leq (x - (1 - 3/2^{n+1}))^2 + y^2 \leq 1/2^{n+1}$ for some positive integer n.

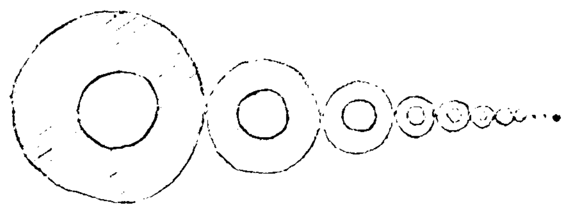

fig. (1.2)

The obstacle in proving $Th(P_5(X))$ is the necessity of knowing that the quasi-components and components of a semi-open set (i.e., the complement of a semi-closed set) are identical in a cyclic U-space. This can be deduced from either theorem (3.5) or theorem (3.5a) (from theorem (3.5) in the text).

Theorems (3.5) and (3.5a) constitute an interesting feature of the chapter for, besides being the most difficult part of §3 (notice that, as conceived here, the proofs of lemmas (3.12), (3.13) and (3.14) are parts of the proof of theorem (3.5)), they contain a good deal more than is required to prove $Th(P_5(X))$. They are "weak separation theorems" of a type that have already occurred elsewhere in the literature, In this category we mention the following:

(a) theorem 4, [12],
(b) theorem 2.1, [35],
(c) "Separation theorem," chap.IV, [8],
(d) theorem of §2, chap.II, this thesis,
(e) lemma I, [7],
(f) theorems (3.5), (3.5a), this chapter.

(a) - (d) all concern the weak separation of two closed sets A and B by a quasi-closed set, and they are all subsumed under (d). Their principal use has been to prove that the n-cell has the fixed point property under peripherally continuous functions. If we rephrase (e)(1), we see that it concerns the weak separation of two degenerate closed sets A and B by a totally disconnected semi-closed set. It is the key to proving the principal theorem of [7]. (f) concerns the weak separation of two closed sets A and B by a semi-closed set, and its proof offers considerably more difficulties than that of

(1) Let L be a subset of a space X. The following two statements are then equivalent: (i) the quasi-components and components of X - L are identical, (ii) if L weakly separates two points p, q in X, then L broadly separates p, q in X.

(e) (because A and B are not necessarily degenerate).
(f), with A and B degenerate, is used to prove $Th(P_5(X))$.

It would seem that these weak separation theorems are a principal feature of the study of non-continuous functions, a fact which does not seem to be properly appreciated yet. For example, with A and B non-degenerate, (f) can be used to prove that a connectivity function $f : I^n \to I^n$ has a fixed point, without first showing that f is peripherally continuous (c.f., the proof of the fixed point property for peripherally continuous functions in [35]). Also lemma (3.12) has applications in its own right. Using it, we can show that a pseudo-continuous(2) function on a cyclic U-space (a) preserves connectedness and (b) is peripherally continuous (these results will be published separately).

In §6 of [23] Stallings raised the question as to what extent the theorems of his paper were valid for ANR's. In §4 we answer this question affirmatively for the theorem concerning a connectivity function being peripherally continuous. In fact, theorems (4.2) and (4.3) are the propositions $Th(P_6(X))$ and $Th(P_7(X))$, where

(2) We shall call a function $f : X \to Y$ pseudo-continuous if $f^{-1}(F)$ is a semi-closed subset of X whenever F is a closed subset of Y.

$P_6(X) \equiv X$ is a locally contractible Peano space,

$P_7(X) \equiv X$ is a locally compact ANR($\mathfrak{M}$).

It will be noticed that each proposition $P_n(X)$, $n = 1, 2, \ldots, 5$, imposes some sort of "unicoherence condition" on X, and it is by virtue of this that $Th(P_n(X))$ is proved. The "unicoherence condition" is used in this way: a certain set L is found which separates X, and the "unicoherence condition" is used to deduce that a component of L separates X. In all cases, however, it would be sufficient to know just that a finite number of components of L separates X. In all cases, however, it would be sufficient to know just that a finite number of components of L separates X. This consideration leads us to formulate the definition of an S-space. Putting

$P_8(X) \equiv X$ is a cyclic S-space,

the statement $Th(P_8(X))$[3] appears as theorem (5.2). The case of the S-space with cut points is not dealt with in this chapter, as we have not yet attempted to prove the weak separation theorem for cyclic S-spaces that corresponds

(3) In theorem (5.2) the space Y is only assumed to be regular, not regular and T_1. This is possible because of lemmas (5.4) and (5.5). As these lemmas can be applied in the same way to the proofs of each of the preceding theorems $Th(P_n(X))$, it is only necessary to demand that Y be regular in these theorems.

to theorems (3.5) and (3.5a).

We remark that the S-space is the natural setting for the theorem concerning a connectivity function being peripherally continuous. Not only is it the most general space to which the argument applies, but also in it we no longer have to concern ourselves with local cut points, as these are dealt with implicitly. The S-space is so-called in this chapter because A. H. Stone was the first to investigate weakly finitely multicoherent spaces in [26], and a considerable amount of inspiration has been obtained from this paper.

We close §5 and the chapter by considering the possibility of ascribing weaker properties than connectivity to the non-continuous function $f : X \to Y$ which will ensure that f is peripherally continuous. In theorem (5.3) we show that if $f : X \to Y$ is pseudo-continuous and connectedness preserving, where X is a cyclic S-space and Y is a regular space, then f is peripherally continuous. Finally, we remark that there is reason to believe that the hypothesis that f is connectedness preserving is redundant in this theorem.

2. We first present the necessary definitions and state a number of lemmas that we shall need in the proof of theorem (2.1).

A <u>Peano space</u> is a locally compact, connected and locally connected metric space. It was shown in [1] that a locally separable connected metric space was separable. A proof of this may also be found on p.75 of [21]. From this it follows that <u>a Peano space has a countable open base</u>.

If $f : X \to Y$ is a function, <u>the graph of</u> f, written $\Gamma(f)$, is defined to be $\{(x, y) : x \in X \ \& \ y = f(x)\}$. It is a subset of the Cartesian product $X \times Y$.

Let X and Y be arbitrary topological spaces. A function $f : X \to Y$ is called <u>a connectivity function</u> if for each connected set C in X the graph $\Gamma(f|C)$ of the restricted function $f|C : C \to Y$ is a connected subset of the topological product space $X \times Y$.

Again let X and Y be any spaces. A function $f : X \to Y$ is said to be <u>peripherally continuous at a point</u> $x \in X$ if for each pair of neighbourhoods U and V of x and $f(x)$, respectively, there is a neighbourhood W of x such that $W \subset U$ and $f(\mathrm{Fr}\, W) \subset V$ $(\mathrm{Fr}\, W = \overline{W} - W)$. A function $f : X \to Y$ is <u>peripherally continuous</u> if it is peripherally continuous at each point of X.

Let X be a space with a countable open base. A subset S of X is <u>semi-closed in</u> X if for each sequence $K_1, K_2, \ldots$ of components of S which converges in X, $\lim K_i$ is contained in S or is a single point. (The definition of the convergence of a sequence of sets may be found in [31].) We remark that a semi-closed set can be defined in an arbitrary topological space by replacing the convergent sequence of components in the above definition by a convergent net of components, as is done in [22]. However, for our purposes the above definition will suffice.

We make the following observation, which is an immediate consequence of the definition: if S is a semi-closed set in a T_1-space X which has a countable open base, then the components of S are closed in X.

Lastly, we say that a connected space X is <u>unicoherent</u> if for each representation of X as the union of two connected closed sets M and N, $M \cap N$ is always connected.

We need the following lemmas in order to prove theorem (2.1).

LEMMA (2.1). <u>If</u> X <u>is a Peano space and</u> p <u>is a point</u>

of X, then each region U about p contains a region V about p such that $\overline{V}$ is compact and contained in U and no component of $U - p$ contains more than one component of $X - \overline{V}$.

LEMMA (2.2). Let $f : X \to Y$ be a connectivity function, where X and Y are arbitrary T_1-spaces. Then, for each non-degenerate connected subset C of X, the graph $\Gamma(f|C)$ has no isolated points.

LEMMA (2.3). Let X be a locally connected Hausdorff space with a countable open base and Y a T_1-space. If $f : X \to Y$ is a connectivity function, then for each closed set F in Y, $f^{-1}(F)$ is semi-closed in X.

LEMMA (2.4). Let X be a connected and locally connected space. Then the following are equivalent:

(i) X is unicoherent,

(ii) if a closed set F separates two points p, q in X, then so does some component of F,

(iii) if a closed set F separates X, then so does some component of F.

LEMMA (2.5). Let X be a connected, locally connected completely normal space. Then X is unicoherent if and only if every set that separates X has a component that separates X.

Lemma (1.1) is a special case of theorem 1, p.188 of [14]. It may also be easily proved along the lines of the theorem of Whyburn quoted in §2 of chapter 8 of this thesis. Lemma (2.2) is proved in [12], although in [12] the spaces X and Y are required to be Hausdorff spaces. Lemma (2.3) is proved for compact spaces in [23] and [9]. It is proved as stated here in [37]. It is given in its most extended form in [22]. In [29], [38] and [24] a number of properties are proved to be equivalent to unicoherence, but (iii) of lemma (2.4) is not among them. For this reason, and because the proofs of lemmas (3.2) and (3.10) will be patterned on the proof of lemma (2.4), we prove lemma (2.4) here. Lemma (2.5) follows directly from lemma (2.4).

PROOF OF LEMMA (2.4). Although it is shown in [24] that (i) implies (ii), we shall prove it here for convenience.

Suppose that X is unicoherent. Let C be the component of X - F that contains p, and let D be the component of $X - \overline{C}$ that contains q. Then

$X = (X - D) \cup \overline{D}$ is a representation of X as the union of two connected closed sets. Thus $(X - D) \cap \overline{D} = \mathrm{Fr}\, D$ is connected, and it separates p, q. Thus, since $\mathrm{Fr}\, D \subset \mathrm{Fr}\, C \subset F$, it follows that a component of F separates p, q.

That (ii) implies (iii) is trivial.

In order to prove that (iii) implies (i) we suppose that X is not unicoherent. Then there are two connected closed sets M and N such that $X = M \cup N$ and $M \cap N = P \cup Q$, where P and Q are disjoint non-empty closed sets. By the local connectedness of X, there is a component C of $X - N$ such that $\overline{C}$ meets both P and Q. Let $A = P \cap \mathrm{Fr}\, C$ and $B = Q \cap \mathrm{Fr}\, C$. Let A' be the union of A and all the components of $X - \overline{C}$ whose closures do not meet B, and let B' be the union of B and all the components of $X - \overline{C}$ whose closures do not meet A. Then it follows from the local connectedness of X that A' and B' are disjoint closed sets, and neither of them separates X. There is, however, a component D of $X - \overline{C}$ that has closures points in both A and B, for if not N would be contained in $A' \cup B'$ and so would not be connected. Thus D is not in $A' \cup B'$, and so $F = A' \cup B'$ is a closed set which separates X, but no component of F separates X. The contradiction shows that (iii) implies (i).

THEOREM (2.1). *Let X be a Peano space that has a covering by unicoherent regions, and Y a regular T_1-space. If $f : X \to Y$ is a connectivity function, then f is peripherally continuous.*

PROOF. Let p be an arbitrary point of X. The space X has a covering by unicoherent regions $X_1, X_2, \ldots$, and we shall let $p \in X_i$.

We wish to prove that f is peripherally continuous at p. Since this is clearly so if $X_i = \{p\}$, we may suppose that $X_i \neq \{p\}$. Let U and V be any neighbourhoods of p and $f(p)$ such that $U \subset X_i$ and $X_i - U \neq \phi$. We shall show that there is a neighbourhood W of p such that $W \subset U$ and $f(\mathrm{Fr}\, W) \subset V$.

By lemma (2.1) there is a neighbourhood U_1 of p such that $\overline{U}_1$ is compact and contained in U, and no component of $U - \{p\}$ contains more than one component of $U - \overline{U}_1$. Since Y is regular, there is a neighbourhood V_1 of $f(p)$ such that $\overline{V}_1 \subset V$.

Consider the sets which are expressible as the union of $\{p\}$ and a component of $U - \{p\}$ which is not separated from $X_i - U$. There are only a finite number of them, and we shall denote them by $Q_1, Q_2, \ldots, Q_n$.

Now let Q be a typical set from the sequence

Q_1, Q_2, ..., Q_n. Then Q is a unicoherent Peano space and $f|Q : Q \to Y$ is a connectivity function. From now on until the beginning of the final paragraph we shall work in the space Q, and in this period all topological terms and operations will refer to the space Q.

By lemma (2.3), $(f|Q)^{-1}(\overline{V}_1)$ is a semi-closed set in Q. Thus it is easily shown that $\overline{U}_1 \cap (f|Q)^{-1}(\overline{V}_1)$ is semi-closed in Q. Let $\{F_\alpha\}_\alpha$ be the collection of components of $\overline{U}_1 \cap (f|Q)^{-1}(\overline{V}_1)$. Then the sets F_α are closed. Now notice that $Q - \overline{U}_1$ is a non-empty connected set. For each F_α, let E_α be the union of F_α and all the components of $Q - F_\alpha$ except the one that contains $Q - \overline{U}_1$. Then each E_α is closed and by definition does not disconnect Q.

The main part of the proof rests on showing that p belongs to the interior of some E_α.

We first establish some relations among the sets E_α. For any pair α, β such that $\alpha \neq \beta$ we have just one of the following three relations holding:

$$\left.\begin{aligned} &E_\alpha \cap E_\beta = \emptyset, \\ &E_\alpha \subset E_\beta - F_\beta, \\ &E_\beta \subset E_\alpha - F_\alpha. \end{aligned}\right\} \cdots\cdots \text{(I)}$$

To see this, consider the components of $Q - (F_\alpha \cup F_\beta)$, which we will simply call "the components" in this paragraph. If $Q - \overline{U}_1$ lies in a component whose closure meets both F_α and F_β, then E_α consists of F_α and the components whose closures meet F_α alone (that is, whose closures do not meet F_β), while E_β consists of F_β and the components whose closures meet F_β alone, and so $E_\alpha \cap E_\beta = \phi$. If, on the other hand, $Q - \overline{U}_1$ lies in a component whose closure meets F_β alone, then E_α consists of F_α and the components whose closures meet F_α alone, while E_β consists of F_α, F_β and all the components except the one containing $Q - \overline{U}_1$, and so $E_\alpha \subset E_\beta - F_\beta$. Similarly we get the third relation when $Q - \overline{U}_1$ lies in a component whose closure meets F_α alone.

Now we set up an equivalence relation on $\{E_\alpha\}_\alpha$. We write $E_\alpha \sim E_\beta$ whenever there is a γ such that $E_\gamma \supset E_\alpha, E_\beta$. This relation is reflexive and symmetric. It is also transitive, for if $E_\alpha \sim E_\beta$ and $E_\beta \sim E_\gamma$ then we can find δ, ϵ such that $E_\alpha, E_\beta \subset E_\delta$ and $E_\beta, E_\gamma \subset E_\epsilon$. This means that $E_\delta \cap E_\epsilon \neq \phi$, and so by (I) either $E_\delta \subset E_\epsilon$ or $E_\epsilon \subset E_\delta$. Hence the relation is transitive.

We now turn our attention to the properties of an equivalence class that contains no maximal element; that is, an equivalence class with no element that contains every other element in the equivalence class. Let $\mathcal{E}$ be such an equivalence class, and let G be the union of all the elements in $\mathcal{E}$. We shall prove that G is open and has just one boundary point, which of course does not belong to $\overline{U}_1 \cap (f|Q)^{-1}(\overline{V}_1)$.

G is open. For let $E_\alpha \in \mathcal{E}$. Since $\mathcal{E}$ contains no maximal element, there is an element E_β in $\mathcal{E}$ such that $E_\beta \not\subset E_\alpha$. By the equivalence relation there is an element E_γ in $\mathcal{E}$ such that $E_\gamma \supset E_\alpha, E_\beta$. By (I), $E_\alpha \subset E_\gamma - F_\gamma$, which is an open set. That is,

$$G = \bigcup \{E_\alpha - F_\alpha : E_\alpha \in \mathcal{E}\}, \qquad \ldots\ldots \text{(II)}$$

which is an open set.

Fr G is a single point. To prove this let $R_0, R_1, \ldots$ be a countable covering of G by open sets whose closures are compact and lie in G. We shall define a sequence of elements $E_{\alpha_0}, E_{\alpha_1}, \ldots$ in $\mathcal{E}$ such that the sequence $F_{\alpha_0}, F_{\alpha_1}, \ldots$ converges to Fr G. Select E_{α_0} as any element in $\mathcal{E}$. Suppose now that for $k = 0$ we have selected E_{α_0} in this way, and for $k > 0$ we have selected E_{α_k} as an element of $\mathcal{E}$ such that $E_{\alpha_k} - F_{\alpha_k} \supset E_{\alpha_{k-1}} - F_{\alpha_{k-1}}, \overline{R}_{k-1}$. In order to select $E_{\alpha_{k+1}}$ we consider $\overline{R}_k$. By (II), $\{E_\beta - F_\beta : E_\beta \in \mathcal{E}\}$ is an opening covering of the compact set $\overline{R}_k$ and as such contains a finite subcovering $E_{\beta_1} - F_{\beta_1}$, $E_{\beta_2} - F_{\beta_2}$, ..., $E_{\beta_n} - F_{\beta_n}$ of $\overline{R}_k$. Since E_{β_1}, E_{β_2}, ..., E_{β_n}, E_{α_k} are all equivalent to each other, there is an element in $\mathcal{E}$ which contains all of them. We shall denote this element by $E_{\alpha_{k+1}}$. It is then evident from (I) that the inductive hypothesis is preserved. Now we show that F_{α_0}, F_{α_1}, ... converges to Fr G. Let $x \in$ Fr G and let R be a region about x. Then $R \cap R_k \neq \emptyset$ for some k, and so $R \cap (E_{\alpha_i} - F_{\alpha_i}) \neq \emptyset$ for each $i > k$. But R is a connected set which meets the complement of E_{α_i} for all i. Thus, for $i > k$, R meets E_{α_i} - int E_{α_i}, which is contained in F_{α_i}. That is, for $i > k$, $R \cap F_{\alpha_i} \neq \emptyset$, and so Fr G $\subset$ lim inf F_{α_i}. But if $y \in G$ then y lies in some R_k, which is contained in $E_{\alpha_i} - F_{\alpha_i}$ for $i > k$. Thus lim sup $F_{\alpha_i} \subset$ Fr G. That is, F_{α_0}, F_{α_1}, ... converges to Fr G. We now show that Fr G is a single point. Since $\overline{U}_1 \cap (f|Q)^{-1}(\overline{V}_1)$ is a semi-closed set there are two possibilities: Fr G is contained in $\overline{U}_1 \cap (f|Q)^{-1}(\overline{V}_1)$ or is a single point. However, the first cannot occur because Fr G is a continuum, as the limit of a sequence of continua in the compact set $\overline{U}_1$, and so it would lie in some component F_α of $\overline{U}_1 \cap (f|Q)^{-1}(\overline{V}_1)$. Then $G - F_\alpha$ (or indeed G, for one can easily see that $G \cap F_\alpha = \phi$) would be separated from $Q - \bar{U}_1$ by F_α, and so G would be contained in E_α. But then E_α would belong to ε, and this contradicts the assumption that the equivalence class ε has no maximal element. Thus $\mathrm{Fr}\ G$ is a single point.

It follows that G does not disconnect Q, since G is open and $\mathrm{Fr}\ G$ is a single point.

We shall denote by $\{G_\gamma\}_\gamma$ the collection of all sets such as G which are the union of an equivalence class with no maximal element.

Consider now an equivalence class which does have a maximal element. We shall discard all of its elements except the maximal element. The collection of all maximal elements that we get in this way we shall denote by $\{E_{\alpha_\beta}\}_\beta$.

We now let L be the union of the collection $\{E_{\alpha_\beta}\}_\beta \cup \{G_\gamma\}_\gamma$ of disjoint sets. Then the components of L are the sets E_{α_β} and the sets G_γ. For from the fact that G_γ is open and $\mathrm{Fr}\ G_\gamma \cap L = \phi$, it follows that G_γ is a component of L. Thus we have only to show that a non-degenerate subcollection of $\{E_{\alpha_\beta}\}_\beta$ does not have a connected union. Let K be the union of the collection $\{E_{\alpha_\beta}\}_\beta$, which we shall suppose is non-degenerate. It suffices to show that K is not connected, as will be apparent from the argument that follows. Since the union of $\{F_{\alpha_\beta}\}_\beta$ is not connected, it has a separation $F_1 \cup F_2$. For $i = 1, 2$, let K_i be the union of the elements E_{α_β} for which $F_{\alpha_\beta} \subset F_i$. Then K_1 and K_2 are disjoint and, furthermore, they are separated. For if this is not so we may suppose without loss of generality that there is a point $x \in K_1 \cap \overline{K}_2$. Then $x \in F_1$, for $K_1 - F_1$ is an open set which is disjoint from K_2. Let R be a region about x which does not meet F_2. Then R must meet some open set $E_{\alpha_\beta} - F_{\alpha_\beta}$ for which $F_{\alpha_\beta} \subset F_2$. But this is impossible because R does not meet F_{α_β}, which separates x and $E_{\alpha_\beta} - F_{\alpha_\beta}$. Thus $K_1 \cup K_2$ is a separation of K. This completes the argument, and shows that the components of L are the sets E_{α_β} and the sets G_γ.

It now follows that $p \in \operatorname{int} E_\alpha$, for some α. To see this we consider the set L. Since no component of L separates Q, it follows from lemma (2.4) that L itself does not separate Q, which is unicoherent. Thus by lemma (2.2) the connected set $Q - L$ does not contain p in its closure, for there is no point $q \in (Q - L) \cap U_1$ such that $f(q) \in V_1$. Thus $p \in \operatorname{int} L$. Let R be a connected region about p which is contained in L. Then R must be contained in a component of L. That is, $p \in \operatorname{int} E_{\alpha_\beta}$ for some α_β, or $p \in G_\gamma$ for some γ. In the latter case it follows from (II) that $p \in E_\alpha - F_\alpha$ for some $E_\alpha - F_\alpha \subset G_\gamma$. Thus in either case there is an α such that $p \in \text{int } E_\alpha$.

So we have shown that there is a set E_α such that $p \in \text{int } E_\alpha$. The closure of E_α is of course compact, and $(f|Q)(\text{Fr int } E_\alpha) \subset \overline{V}_1$. All this has been in the subspace Q.

To complete the proof we return to the space X. We have shown that in each of the subspaces $Q_1, Q_2, \ldots, Q_n$ there is a relatively open subset W_i containing p such that $\text{Fr}_{Q_i}(W_i)$ (the frontier of W_i in the space Q_i) is compact and $(f|Q_i)(\text{Fr}_{Q_i}(W_i)) \subset \overline{V}_1$. Let $W = (\bigcup_i W_i) \cup (U - \bigcup_i Q_i)$. Then W is a neighbourhood of p which is contained in U, and $\text{Fr } W = \bigcup_i \text{Fr}_{Q_i} W_i$. Thus $f(\text{Fr } W) = \bigcup_i (f|Q_i)\,\text{Fr}_{Q_i}(W_i) \subset V$. This completes the proof.

3. We start off this section, the purpose of which has been described in §1, by commenting on §5 and §6 of [34].

At the beginning of §5 of [34] the following definition is given. "A connected space or set M is said to be unicoherent, or cohesive, between disjoint connected subsets (or points) A and B of M provided $H_a \cdot H_b$ is connected for every representation $M = H_a + H_b$, where H_a and H_b are closed and connected and contain A and B, respectively, <u>in their interiors relative to</u> M." (My italics.) The definitions of canonical region and locally cohesive space in [34] are of course based on this definition, and so are the subsequent proofs in §5 and §6.

There is, however, something puzzling in this definition of unicoherence between a pair of points or subsets. Somewhat later in §5 the following statement is made, "<u>Remarkably enough</u>, a cyclic, locally connected continuum M is necessarily unicoherent if it is unicoherent between one pair of distinct points" (my italics). However, it may easily be shown that if M is any connected, locally connected regular T_1-space, and M is unicoherent between some pair of distinct points in the sense of the above definition, then M is unicoherent (see Appendix).

We suggest that the definition of unicoherence between a pair of points should be altered to the following, and remark that all the proofs of §5 and §6 of [34] go through without change under this revised definition.

We shall say that a space X is unicoherent between two subsets (or points) A and B if for each representation $X = M \cup N$, where M and N are connected closed sets such that $A \subset M - N$ and $B \subset N - M$, $M \cap N$ is connected. Notice that this definition imposes a lesser degree of unicoherence on the space than the definition of [34].

In particular, we still have theorem (5.2) of [34]: a Peano continuum X is unicoherent between a pair of its points a and b if and only if the cyclic chain C(a,b) is unicoherent.

Our definition of a canonical region and a locally cohesive space are verbally the same as those in [34], except they are based on our revision of the definition of "unicoherence between two subsets." If p is a point of a space X, we say that R is a canonical region about $\{p\}$ in X if R is a connected neighbourhood of p, the frontier Fr R of R is connected, and $\overline{R}$ is unicoherent between $\{p\}$ and Fr R. A space X is locally cohesive if each of its points has a base of canonical regions.

We now make the following definition. Let A be a subset of a connected space X. We say that X is

<u>unicoherent modulo</u> A if for each representation $X = M \cup N$ in which M and N are connected closed subsets of X such that $A \subset M - N$, $M \cap N$ is connected.

We then have the following theorem.

THEOREM (3.1). <u>Let</u> X <u>be a Peano space</u>. <u>Then</u> X <u>is locally cohesive if and only if it has a base of regions</u> $\{R_i\}$ <u>such that</u> $\operatorname{Fr} R_i$ <u>is connected and</u> $\overline{R}_i$ <u>is unicoherent modulo</u> $\operatorname{Fr} R_i$ <u>for each</u> i.

PROOF. The proof of this theorem is nothing but the relevant portion of the proof of theorem (6.2) of [34].

Let R be a canonical region about a point p in X such that $\overline{R}$ is compact. We show that $\overline{R}$ is unicoherent modulo $\operatorname{Fr} R$.

We form the quotient space $\overline{R} / \operatorname{Fr} R$ on $\overline{R}$ by identifying the points in $\operatorname{Fr} R$. Let $\pi : \overline{R} \to \overline{R} / \operatorname{Fr} R$ be the natural projection from $\overline{R}$ onto $\overline{R} / \operatorname{Fr} R$. It follows that $\overline{R} / \operatorname{Fr} R$ is a Peano continuum, since it cannot fail to be locally connected at just the single point $\pi(\operatorname{Fr} R)$. Since $\overline{R} / \operatorname{Fr} R$ is unicoherent between $\pi(p)$ and $\pi(\operatorname{Fr} R)$, it follows that the cyclic chain $C(\pi(p), \pi(\operatorname{Fr} R))$ is unicoherent, by theorem (5.2) of [34], which, as we remarked, still holds. But since the locally cohesive space X has no local cut points, it follows that R, $\overline{R}$ and $\pi(\overline{R})$ are cyclic. Hence in fact $\pi(\overline{R})$ is unicoherent. Thus suppose that $\overline{R} = M \cup N$ is a representation of $\overline{R}$ as the union of two closed connected sets M and N such that $\mathrm{Fr}\, R \subset M - N$. By the unicoherence of $\pi(\overline{R})$ it follows that $\pi(M) \cap \pi(N)$ is connected. Thus $M \cap N$ is connected, which proves that $\overline{R}$ is unicoherent modulo $\mathrm{Fr}\, R$.

Since X has a base of canonical regions $\{R_i\}$ with compact closures, it follows that X has a base of regions $\{R_i\}$ such that $\mathrm{Fr}\, R_i$ is connected and $\overline{R}_i$ is unicoherent modulo $\mathrm{Fr}\, R_i$ for each i. As the converse is trivial, the theorem is proved.

We remark in passing that if R is a region in a space X such that $\overline{R}$ is unicoherent modulo $\mathrm{Fr}\, R$, then it does <u>not</u> follow that $\overline{R}$ is unicoherent modulo $\overline{R}$ - int $\overline{R}$. This is shown by the following example.

EXAMPLE (3.1). Let X be the subset of the Euclidean plane consisting of the points (x, y) such that $|y| \leq 1$ and either $|x| \geq 1/2$ or $|y| \geq 1/2$, and let R be the set of points (x, y) in X such that $|x| < 1$ and $|y| < 1$. Then $\overline{R}$ is unicoherent modulo $\mathrm{Fr}\, R = \{(x, y) : |x| = 1 \text{ or } |y| = 1\}$, but $\overline{R}$ is not unicoherent modulo $\overline{R}$ - int $\overline{R} = \{(x, y) : |x| = 1$ and $|y| \leq 1\}$.

However, we can ask the following question: if X has a covering by regions $\{R_i\}$ such that each $\overline{R}_i$ is unicoherent modulo $\mathrm{Fr}\, R_i$, does X also have a covering by regions $\{S_j\}$ such that each $\overline{S}_j$ is unicoherent modulo $\overline{S}_j$ - int S_j ? We shall not, however, attempt to answer this here.

LEMMA (3.1). <u>If</u> X <u>is a Peano space and</u> p <u>is a non-cut point of</u> X, <u>then</u> p <u>has a base of regions whose complements are connected</u>.

PROOF. Let U be a neighbourhood of p with a compact closure. Then $\mathrm{Fr}\, U$ has a covering by regions $U_1, U_2, \ldots, U_m$, the closures of which do not contain p. Since $X - \{p\}$ is connected, there is a simple chain of regions

$$U_1 = V_{i,1}, V_{i,2}, \ldots, V_{i,n_i} = U_i$$

such that $\overline{V}_{i,j}$ does not contain p. Let V be the complementary component of $(X - U) \cup \bigcup_{i,j} \overline{V}_{i,j}$ that contains p. Then $V \subset U$ and $X - V$ is connected.

LEMMA (3.2). *Let* X *be a connected, locally connected normal space and let* Y *be a connected subset of* X. *Then the following properties are equivalent*:

(i) X *is unicoherent modulo* Y,

(ii) *if a closed set* F *which is disjoint from* Y *separates two points* p, q *in* X, *then a component of* F *does*,

(iii) *if a closed set* F *which is disjoint from* Y *separates* X, *then a component of* F *does*.

LEMMA (3.3). *Let* X *be a connected, locally connected completely normal space, and* A *a connected subset of* X. *Then* X *is unicoherent modulo* A *if and only if each set in* $X - A$ *that separates* X *has a component that separates* X.

Lemma (3.3) follows immediately from lemma (3.2). So it is only necessary to prove lemma (3.2), the proof of which is very similar to that of lemma (2.4).

PROOF OF LEMMA (3.2). The proof that (i) implies (ii) is identical to the proof that (i) implies (ii) in lemma (2.4), since the connected set Y takes care of itself. That (ii) implies (iii) is trivial. So we prove that (iii) implies (i).

We suppose that X is not unicoherent modulo Y. Then there are two connected closed sets M and N such that $X = M \cup N$, $Y \subset M - N$ and $M \cap N = P \cup Q$, where P and Q are disjoint non-empty closed sets. Since X is normal, there are neighbourhoods U and V of P and Q such that $\overline{U} \cap \overline{V} = \emptyset$. Further, we may suppose that each component of U meets P and each component of V meets Q. There is now a component C of $X - M \cup \overline{U} \cup \overline{V}$ which has closure points in both $\overline{U}$ and $\overline{V}$. Then $M \cup U \cup V$ lies in a component D of $X - \overline{C}$. Let $A = \overline{U} \cap \overline{C}$ and $B = \overline{V} \cap \overline{C}$. Let A' be the union of A and all the components of $X - \overline{C}$ whose closures do not meet B, and let B' be the union of B and all the components of $X - \overline{C}$ whose closures do not meet A. Then A' and B' are disjoint closed sets neither of which separates X. However $F = A' \cup B'$ is a closed set that has both C and D among its complementary components. Thus F is a closed set in $X - Y$ which separates X, but no component of F separates X. This contradiction proves the lemma.

LEMMA (3.4). <u>Let</u> X <u>be a Peano space and</u> R <u>a region in</u> X <u>such that</u> $\overline{R}$ <u>is unicoherent modulo</u> $\operatorname{Fr} R$. <u>Let</u>

p <u>be a point in</u> R <u>and</u> Q <u>a set expressible as the union of</u> $\{p\}$ <u>and a component of</u> $R - \{p\}$. <u>Then</u> $\overline{Q}$ <u>is unicoherent modulo</u> $\overline{Q} \cap \mathrm{Fr}\, R$.

PROOF. If $\overline{Q} \cap \mathrm{Fr}\, R = \emptyset$, then it follows immediately that $\overline{Q}$ is unicoherent; that is, $\overline{Q}$ is unicoherent modulo $\overline{Q} \cap \mathrm{Fr}\, R = \emptyset$.

Thus let $\overline{Q} \cap \mathrm{Fr}\, R \neq \emptyset$, and suppose $\overline{Q}$ is not unicoherent modulo $\overline{Q} \cap \mathrm{Fr}\, R$. Then there is a representation $\overline{Q} = M \cup N$, where M and N are connected closed sets such that $\overline{Q} \cap \mathrm{Fr}\, R \subset M - N$ and $M \cap N = A \cup B$, where A and B are non-empty disjoint closed sets. Since $\overline{Q} \cap \mathrm{Fr}\, R = \overline{Q} - Q$, it follows that $A \cup B \subset Q$. Let α be an arc, possibly degenerate, from p to $A \cup B$ in the Peano space Q. We may suppose without loss of generality that α meets A and is disjoint from B. Then $\overline{R} = (M \cup \alpha \cup \overline{R - Q}) \cup N$ is a representation of $\overline{R}$ as the union of the two connected closed sets $M \cup \alpha \cup \overline{R - Q}$ and N, and Fr R does not meet N. But the intersection of $M \cup \alpha \cup \overline{R - Q}$ and N is not connected, for it contains $A \cup B$ and is contained in $A \cup B \cup \alpha$. Thus $\overline{R}$ is not unicoherent modulo Fr R, which is a contradiction.

We have the following converse to lemma (3.4).

LEMMA (3.5). *Let* X *be a Peano space,* p *a point in* X *and* S *a region about* p *with this property: if* Q *is a set expressible as the union of* $\{p\}$ *and a component of* $S - \{p\}$, *then* $\overline{Q}$ *is unicoherent modulo* $\overline{Q} \cap \mathrm{Fr}\, S$. *Then there is a region* R *such that* $p \in R \subset S$ *and* $\overline{R}$ *is unicoherent modulo* $\mathrm{Fr}\, R$.

Further, if the closures of the components of $S - \{p\}$ *meet only in the point* p, *then we may take* $R = S$.

PROOF. There are only a finite number of sets $Q_1, Q_2, \ldots, Q_n$ which are expressible as the union of $\{p\}$ and a component of $S - \{p\}$ which is not separated from $\mathrm{Fr}\, S$. By lemma (2.1) there is a region U_i about p in the subspace Q_i such that $\overline{U}_i$ is compact and $Q_i - U_i$ is connected. Notice that $\overline{U}_i$ is unicoherent modulo $\overline{U}_i \cap (Q_i - U_i)$, and that $\overline{U}_i - \{p\}$, $\overline{U}_j - \{p\}$ are separated for $i \neq j$. Let $R = S - \bigcup_i (Q_i - U_i)$. Then $\mathrm{Fr}\, R = \bigcup_i \overline{U}_i \cap (Q_i - U_i)$ and it easily follows that $\overline{R}$ is unicoherent modulo $\mathrm{Fr}\, R$.

If the closures of the sets $Q_1, Q_2, \ldots, Q_n$ meet only in $\{p\}$ then it is immediately clear that we may take $R = S$.

THEOREM (3.2). *Let* X *be a Peano space which has a* covering by regions $\{R_i\}$ such that each $\bar{R}_i$ is unicoherent modulo Fr R_i, and Y a regular T_1-space. If $f: X \to Y$ is a connectivity function, then f is peripherally continuous.

PROOF. The proof is the same as the proof of theorem (2.1) except for the changes that must be made where the unicoherence property is used. Thus we refer to the proof of theorem (2.1) and indicate the changes that must be made.

Instead of being a unicoherent region about p, X_i becomes a region about p such that $\bar{X}_i$ is unicoherent modulo Fr X_i. From this it follows that $\bar{U}$ is unicoherent modulo Fr U (see lemma (3.6)). Thus from lemma (3.4) it follows that $\bar{Q}$ is unicoherent modulo $\bar{Q} \cap$ Fr U. Thus Q is unicoherent modulo the connected set $Q - \bar{U}_1$, since $\bar{U}_1 \cap \mathrm{Fr}\, U = \emptyset$.

The only other change that is necessary is in the third last paragraph, which begins, "It now follows that ..." The reasoning which enables us to conclude in this paragraph that L does not separate Q must be altered slightly. No component of L separates Q and $L \subset Q - \bar{U}_1$. Since Q is unicoherent modulo $Q - \bar{U}_1$, it now follows from lemma (3.3) that L does not separate Q.

No other part of the proof of theorem (2.1) needs to be altered.

We notice from theorem (3.1), that theorem (3.2) contains theorem (6.2) of [34] as a special case. However, we also notice that the way in which the unicoherence property is used in the proof of theorem (3.2), namely by means of lemma (3.3), is exactly the same as the way in which the unicoherence of the locally cohesive space is used in the proof of theorem (6.2) of [34].

We also point out that although theorem (3.2) is stated for a Peano space having a covering of regions $\{R_i\}$ such that each $\overline{R}_i$ is unicoherent modulo Fr R_i, it could equally well be stated for a Peano space having a base of such regions. This is because of the following simple lemma.

LEMMA (3.6). _If_ R _is a region in a locally connected space_ X _which is unicoherent modulo_ Fr R, _and_ S _is a subregion of_ R, _then_ S _is unicoherent modulo_ Fr S.

PROOF. Suppose that $\overline{S}$ is not unicoherent modulo Fr S. Then there are two connected closed sets M and N such that $\overline{S} = M \cup N$, $N \cap \mathrm{Fr}\, S = \emptyset$ and $M \cap N$ is not connected. Since $N \cap \mathrm{Fr}\, R = \emptyset$, it follows that $\mathrm{Fr}\, R \subset \overline{R - N}$. But $R - N \subset (R - \overline{S}) \cup M$, which is a connected set. Therefore $\mathrm{Fr}\, R \cup (R - \overline{S}) \cup M$ is a connected set. But this latter set is equal to $(\overline{R} - S) \cup M$. Therefore $(\overline{R} - S) \cup M$ and N are two connected closed subsets of $\overline{R}$ whose union is $\overline{R}$ and whose intersection is $M \cap N$, which is not connected. But $N \cap \mathrm{Fr}\, R = \emptyset$, which is a contradiction.

EXAMPLE (3.2). Let X be the subset of the plane which is the union of $I = \{(x, y) : 0 \le x \le 1$ and $-1/2 \le y \le 0\}$ and $C_n = \{(x, y) : (x - 1/2^n)^2 + (y - 1/2^{n+2})^2 = 1/2^{n+2}\}$, for $n = 0, 1, 2, \ldots$ (see figure (3.1).

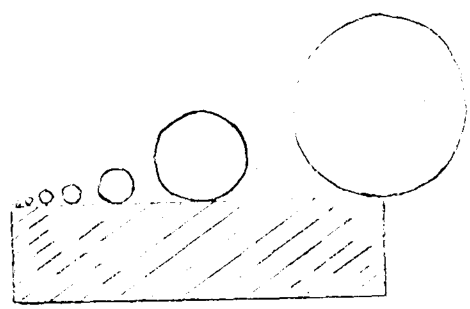

fig. (3.1)

Also let $f : X \to Y$ be a connectivity function, where Y is an arbitrary regular T_1-space. It then follows from theorem (3.2) that f is peripherally continuous on $X - \{(0, 0)\}$. However, since by theorem (3.2), $f|I : I \to Y$ is peripherally continuous at $(0, 0)$, it follows that f is also peripherally continuous at $(0, 0)$. However, X does not have a covering by regions $\{R_i\}$ such that each $\overline{R}_i$ is unicoherent modulo Fr R_i.

Example (3.2) shows that when X is not a cyclic space, the property ascribed to X in the hypothesis of theorem (3.2) does not adequately define the class of spaces on which we are able to prove that a connectivity function is peripherally continuous. It is to adequately deal with the cut points of X that the remainder of this section will be concerned. We shall show that if each true cyclic element of X has the property ascribed to X in theorem (3.2), then each connectivity function $f : X \to Y$ is peripherally continuous, Y being as usual a regular T_1-space.

In the meantime we remark that the statement of theorem (3.2) adequately covers the case in which X is a cyclic Peano space. The only way in which it can be improved for such spaces is by altering the "unicoherence condition" to a "multicoherence condition." Thus we state this case as a separate theorem:

THEOREM (3.3). _If_ X _is a cyclic Peano space which has a covering by regions_ $\{R_i\}$ _such that each_ $\overline{R}_i$ _is unicoherent modulo_ $\mathrm{Fr}\ R_i$, _and_ Y _is a regular_ T_1-_space, then every connectivity function_ $f : X \to Y$ _is peripherally continuous_.

For the definition of the terms _conjugate element_, _cyclic element_, and _true cyclic element_, _cut point_ and _end point_ we refer to chap. IV of [31]. In the sequel we shall use only the most elementary properties that arise from these concepts, and these too can be found in chap. IV of [31].

We shall say that X is _a U-space_ if X is a Peano space and if for each true cyclic element C of X there is a collection of regions $\{R_i\}$ in the subspace C which cover C and such that each $\mathrm{cl}_C\ R_i$ is unicoherent modulo $\mathrm{Fr}_C\ R_i$ (see notation (3.1)).

NOTATION (3.1) When we are working in a subspace A of X, as we shall often be doing, we shall denote the closure, frontier and interior of a set relative to the subspace A by cl_A ( ), Fr_A ( ) and int_A ( ). When no confusion is likely to arise, we shall omit the subscript "A".

The remainder of this section will be concerned with establishing theorem (3.6), in which we prove that if $f : X \to Y$ is a connectivity function, where X is a U-space and Y is a regular T_1-space, then f is peripherally continuous. However, it will first be in order to show that theorem (3.2) is subsumed under theorem (3.6). This is done in the next theorem.

THEOREM (3.4). Let X be a Peano space which has a covering by regions R_i such that each $\overline{R}_i$ is unicoherent modulo $\mathrm{Fr}\, R_i$. Then X is a U-space.

PROOF. We first remark that whenever we use the closure operator "$^-$" in this proof, it will stand for the closure in the space X.

Let C be a true cyclic element of X and p a point of C. Let R be a region about p in X such that $\overline{R}$ is unicoherent modulo $\mathrm{Fr}_X\, R$. Denote the sets that are expressible as the union of $\{p\}$ and a component of $R - \{p\}$ that meets C by $Q_1, Q_2, \ldots$, and suppose that $\overline{Q}_i \cap \mathrm{Fr}_X\, R \neq \emptyset$ if and only if $i \leq n$.

Suppose first that there is some $i > n$. By lemma (3.4), Q_i is unicoherent. Since C is a true cyclic element of Q_i, it easily follows that C is unicoherent.

That is, C is unicoherent modulo $Fr_C\ C = \phi$, and the theorem is proved in such a case.

So we may suppose that each $i \le n$.

By lemma (3.1), there is a region U_i about p in the subspace Q_i such that $cl_{Q_i}\ U_i$ is compact and $Q_i - U_i$ is connected. Notice firstly that $cl_{Q_i}\ U_i = \overline{U}_i$, and secondly that $U_i - \{p\}$ is an open subset of X such that $Fr_X\ (U_i - \{p\}) = (\overline{U}_i - U_i) \cup \{p\}$. Now notice that $\overline{U}_i$ and $\overline{Q}_i - U_i$ are two connected closed subsets of $\overline{Q}_i$ such that $\overline{U}_i \cap (\overline{Q}_i \cap Fr_X\ R) = \phi$. Since, by lemma (3.4), $\overline{Q}_i$ is unicoherent modulo $\overline{Q}_i \cap Fr_X\ R$, it follows that $\overline{U}_i \cap (\overline{Q}_i - U_i) = \overline{U}_i - U_i$ is connected. Similarly we deduce from lemma (3.4) that $\overline{U}_i$ is unicoherent modulo $\overline{U}_i - U_i$.

We first show that the connected set $\overline{U}_i \cap C$ is unicoherent modulo $(\overline{U}_i - U_i) \cap C$. Let $\overline{U}_i \cap C = M \cup N$, where M and N are connected closed subsets of $\overline{U}_i \cap C$ such that $(\overline{U}_i - U_i) \cap C \subset M - N$. Now let $\{A_k\}_k$ be the collection of components of $X - C$ which meet $\overline{U}_i$. Then $\overline{A}_k \cap \overline{U}_i$ is connected. Further, supposing that $(\overline{U}_i - U_i) \cap C \neq \phi$, it follows that $\overline{A}_k \cap (\overline{U}_i - U_i)$ is a connected set which meets $M - N$. If $(\overline{U}_i - U_i) \cap C = \phi$, then the connected set $\overline{U}_i - U_i$ lies in just one of the components A_k, which we may suppose without loss of generality has its boundary point in $M - N$. Define M' as the union of M and all the $\overline{A}_k \cap \overline{U}_i$'s which meet M, and N' as the union of N and all the $\overline{A}_k \cap \overline{U}_i$'s which do not meet M. Then M' and N' are connected closed subsets of $\overline{U}_i$ such that $\overline{U}_i = M' \cup N'$ and $\overline{U}_i - U_i \subset M' - N'$. Thus $M' \cap N'$ is connected, and so $M \cap N$ is connected. That is, $\overline{U}_i \cap C$ is unicoherent modulo $(\overline{U}_i - U_i) \cap C$.

Secondly, we notice that $\overline{U}_i - U_i \subset C$ or $(\overline{U}_i - U_i) \cap C = \emptyset$. For suppose that $\overline{U}_i - U_i$ meets a component A of $X - C$. This implies that U_i meets A. Since U_i is a connected set which also meets $X - A$ (in the point p), it follows that the single point in $\overline{A} - A$ is in U_i. Thus $\overline{U}_i - U_i$ is contained in A, since it is a connected set which meets A but does not meet $\overline{A} - A$. That is, $\overline{U}_i - U_i \cap C = \emptyset$.

In case $(\overline{U}_i - U_i) \cap C = \emptyset$, it is now easily seen that C is unicoherent. For by hypothesis $C \cap Q_i$ contains some other point besides p, and so is a non-degenerate connected set. Thus the neighbourhood $C \cap U_i$ of p in the space $C \cap Q_i$ contains some point other than p; that is, $(C - \{p\}) \cap (U_i - \{p\}) \neq \emptyset$. Therefore $C - \{p\}$ is a connected set which meets the open subset $U_i - \{p\}$ of X but does not meet $Fr_X(U_i - \{p\}) = (\overline{U}_i - U_i) \cup \{p\}$.

Thus $C - \{p\} \subset U_i - \{p\}$ and $C \subset U_i$. The previously proved statement "$\overline{U}_i \cap C$ is unicoherent modulo $(\overline{U}_i - U_i) \cap C$" now implies that C is unicoherent.

Thus we may suppose without loss of generality that $\overline{U}_i - U_i \subset C$ for each $i \leq n$. We first show that $\overline{U}_i \cap C \subset \overline{U_i \cap C}$, in order to do which it will suffice to show that $\overline{U}_i - U_i \subset \overline{U_i \cap C}$. Thus let $x \in \overline{U}_i - U_i$. Then $\{x\} \cup U_i$ is a connected set and so $\{x\} \cup (U_i \cap C)$ is a connected set which is, furthermore, non-degenerate. Thus each neighbourhood of x meets $U_i \cap C$. That is, $x \in \overline{U_i \cap C}$. Now, since $\overline{U}_i \cap C \subset \overline{U}_i \cap C$ and C is closed in X, we have the following:

$$\begin{aligned}
Fr_C\left(\bigcup_j U_j \cap C\right) &= \overline{\bigcup_j U_j \cap C} - \bigcup_j U_j \cap C \quad , \\
&= \bigcup_j \overline{U_j \cap C} - \bigcup_j U_j \cap C \quad , \\
&= \bigcup_j \overline{U}_j \cap C - \bigcup_j U_j \cap C \quad , \\
&= \bigcup_j (\overline{U}_j - U_j),
\end{aligned}$$

where the index j always runs from 1 to n. Now we observe that $\bigcup_j U_j \cap C$ is a region about p in the subspace C. The sets which are expressible as the union of $\{p\}$ and a component of $(\bigcup_j U_j \cap C) - \{p\}$ are the sets $U_i \cap C$, $i \leq n$. In view of the identities $cl_C(U_i \cap C) = \overline{U_i \cap C} = \overline{U}_i \cap C$, we have

$$\begin{aligned} &\mathrm{cl}_C\,(U_i \cap C) \cap \mathrm{Fr}_C\,(\bigcup_j U_j \cap C) \\ &= \overline{U}_i \cap C \cap \bigcup_j (\overline{U}_j - U_j)\,, \\ &= \overline{U}_i - U_i\,. \end{aligned}$$

The previously proved statement "$\overline{U}_i \cap C$ is unicoherent modulo $(\overline{U}_i - U_i) \cap C$" now implies that $\mathrm{cl}_C\,(U_i \cap C)$ is unicoherent modulo $\mathrm{cl}_C\,(U_i \cap C) \cap \mathrm{Fr}_C\,(\bigcup_j U_j \cap C)$. Thus by lemma (3.5) there is a region S about p in the subspace C such that $S \subset \bigcup_j U_j \cap C$ and $\mathrm{cl}_C\, S$ is unicoherent modulo $\mathrm{Fr}_C\, S$ (a glance at the proof of lemma (3.5) will show that S can actually be taken to be equal to $\bigcup_j U_j \cap C$). This completes the proof of the theorem.

Before proving theorems (3.5) and (3.6) we shall need some definitions and lemmas.

Let X be an arbitrary topological space. A decomposition $\mathcal{D}$ of X is a collection of non-empty disjoint closed subsets of X which cover X.

NOTATION (3.2) If A is an arbitrary subset of X and $\mathcal{D}$ is a decomposition of X, we define

$$\begin{aligned} A^+_{\mathcal{D}} &= \bigcup \{D \in \mathcal{D} : D \cap A \neq \emptyset\}, \\ A^-_{\mathcal{D}} &= \bigcup \{D \in \mathcal{D} : D \subset A\}. \end{aligned}$$

(A^- is not to be mistaken for $\overline{A}$, which is the closure of A.) Where no confusion arises, we simply write A^+ and A^-, instead of $A_{\mathcal{D}}^+$ and $A_{\mathcal{D}}^-$.

Let X be an arbitrary space and $\mathcal{D}$ a decomposition of X. We say that $\mathcal{D}$ is <u>an upper semi-continuous (usc) decomposition of</u> X if for each closed subset F of X, $F_{\mathcal{D}}^+$ is closed; or, alternatively, if for each open subset G of X, $G_{\mathcal{D}}^-$ is open.

The following lemma appears as proposition (5.2) on p. 132 of [31].

LEMMA (3.7). *If X is a Peano continuum and L is a semi-closed set in X, then the decomposition of X into the components of L and the points of $X - L$ is usc.*

If L is a set with closed components in a T_1-space X, and $\mathcal{D}$ is the decomposition of X consisting of the components of L and the points of $X - L$, then we shall call $\mathcal{D}$ <u>the decomposition of</u> X <u>associated with</u> L.

NOTATION (3.3). Under the circumstances of the previous paragraph, instead of writing $A_{\mathcal{D}}^+$ and $A_{\mathcal{D}}^-$ for an arbitrary subset A of X, we write A_L^+ and A_L^-. If there is no confusion, we still of course just write A^+ and A^-.

If X is a Peano space and L is a semi-closed subset of X, then it is not in general true that the decomposition of X associated with L is usc. But we can say something useful about this decomposition, which we do in lemma (3.9).

LEMMA (3.8). (A variation of Janiszewski's border theorem). *Let* K *be a connected closed set in a metric space* X *and* G *an open subset of* X *with a compact closure. If* K *meets both* G *and* $X - G$, *then each component of* $K \cap \overline{G}$ *meets* $\operatorname{Fr} G$.

Lemma (3.8) can be proved by making only the smallest alteration to the proof of Janiszewski's border theorem, which can be found on p. 184 of [13].

LEMMA (3.9). *Let* L *be a semi-closed set in a Peano space* X, *and form the decomposition of* X *associated with* L. *If* G *is an open subset of* X *with a compact closure, then* G^- *is open*.

PROOF. Except for the use of lemma (3.8), the proof of lemma (3.9) is very similar to the proof of lemma (3.7).

Let $F = X - G$. We show that F^+ is closed. If on the contrary F^+ is not closed, then there is a point $p \in G - F^+$ and a base of neighbourhoods $U_1, U_2, \ldots$ about p such that $U_i \subset G$ and $U_i \cap K_i \neq \emptyset$, where K_i is a component of L which meets F. Thus a component C_i of $K_i \cap \overline{G}$ meets U_i. By Janiszewski's border theorem, $C_i \cap \mathrm{Fr}\, G \neq \emptyset$.

From $\{C_i\}_i$ we can now pick a convergent subsequence $\{C_{M_i}\}_i$. Let $C_{M_i} \to C$. Then C is a continuum which meets $\mathrm{Fr}\, G$ and contains p. Now $C_{M_i} \subset K_{M_i}$, and from $\{K_{M_i}\}$ we can choose a convergent sequence $\{K_{M_{N_i}}\}_i$. Let $K_{M_{N_i}} \to H$. Then $H \supset C$. Thus H is non-degenerate and so is contained in L. Thus C is contained in L and so is contained in a component K of L. Thus $K \subset F^+$ and so $p \in F^+$. This contradiction establishes the lemma.

LEMMA (3.10). _Let_ X _be a connected, locally connected normal space and_ A _a closed connected subset of_ X _which does not separate_ X. _Then the following properties are equivalent:_

(i) X _is unicoherent modulo_ A,

(ii) _if_ F _is a closed subset of_ X _which is disjoint from_ A, _and_ F _separates_ p, q _in_ $X - A$, _then_ a component of F separates p, q in X,

(iii) if F is a closed subset of X which is disjoint from A and which separates $X - A$, then a component of F separates X.

LEMMA (3.11). If in the statement of lemma (3.10) we suppose that X is in addition completely normal and T_1, and $\overline{X - A}$ is compact, and if in (ii) and (iii) we suppose that F is merely a set with closed components, then properties (i), (ii) and (iii) are equivalent.

We shall turn to proving lemma (3.10), and we shall omit the proof of lemma (3.11).

PROOF OF LEMMA (3.10). We first notice the trivial fact that for any set $Y \subset X - A$, $cl_{X - A}(Y) = \bar{Y} \cap (X - A)$.

In order to prove that (i) implies (ii), let p belong to a component C of $(X - A) - F$, and let q belong to a component D of $(X - A) - \bar{C}$. Then $\bar{D} \cap (X - A)$ and $(X - A) - D$ are two relatively closed connected subsets of the subspace $X - A$ whose union is $X - A$ and whose intersection is a non-empty subset of F. We assert that one of the two sets $\bar{D} \cap (X - A)$, $(X - A) - D$ is separated from A. For suppose that this is not the case. Let R be a region about A such that $\overline{R} \cap F = \phi$. Then $\overline{R} \cap (X - A)$ can be expressed as the union of the two non-empty disjoint sets $\overline{R} \cap \overline{D} \cap (X - A)$, $\overline{R} \cap ((X - A) - D)$, which are closed subsets of the space $X - A$. Since $X - A$ is connected, there is a component E of $(X - A) - \overline{R}$ which has closure points in both $\overline{R} \cap \overline{D} \cap (X - A)$ and $\overline{R} \cap ((X - A) - D)$. But now $X = \overline{E} \cup (X - E)$, where $\overline{E}$ and $X - E$ are two connected closed subsets of X such that $A \cap \overline{E} = \phi$ and $\overline{E} \cap (X - E)$ is not connected. This contradicts the fact that X is unicoherent modulo A. Thus either $\overline{D} \cap (X - A)$ or $(X - A) - D$ is separated from A, and we may without loss of generality suppose it is the latter. Thus $(X - A) - D$ is a closed subset of X. Thus $A \cup \overline{D}$ and $(X - A) - D$ are two closed connected subsets of X whose union is X, and the second of them does not meet A. Thus the intersection H of these two sets is connected. But $H = (\overline{D} \cap (X - A)) \cap ((X - A) - D)$, and so is contained in a component of F. Further, since $p \notin A \cup \overline{D}$ and $q \notin (X - A) - D$, it follows that H separates p and q in X. This shows that (i) implies (ii). That (ii) implies (iii) is obvious, and with the aid of lemma (3.2) we easily show that (iii) implies (i). For suppose that X is not unicoherent modulo A. Then by lemma (3.2) there is a closed set F in X such that $F \cap A = \emptyset$, F separates X and no component of F separates X. It follows now that F must also separate $X - A$, for A is a closed set. But this contradicts (iii).

NOTATION (3.4). Let $\mathbf{A}$ stand for one of the upper-case Latin letters A, B, C, ..., Z. In subsequent lemmas when we consider a point x in a space X, we shall often use the symbol "$\mathbf{A}(x)$" to stand for some subset of X that contains x, and in this case we shall denote the closure $\overline{\mathbf{A}(x)}$ of $\mathbf{A}(x)$ in X by the abbreviated symbol $\overline{\mathbf{A}}(x)$.

Let $S(x)$ be a region about a point x in a locally connected T_1-space X. If $Q(x)$ is a set which is expressible as the union of $\{x\}$ and a component of $S(x) - \{x\}$, then we shall call $Q(x)$ <u>an arm of</u> $S(x)$.

The following three lemmas may all be looked upon as a part of the proof of theorem (3.5). We have isolated them in order to make the proof of theorem (3.5) more manageable.

LEMMA (3.12). *Let* X *be a cyclic U-space,* L *a semi-closed subset of* X *no component of which separates* X, *and* x *a point in* $X - L$. *Then there is an arbitrarily* *small region* $S(x)$ *about* x *such that for each arm* $Q(x)$ *of* $S(x)$

(i) $Q(x) - L$ *is connected or* $\overline{Q}(x) - Q(x)$ *lies in a component of* L,

(ii) $Q(x) = (Q(x))^-$,

(iii) $\overline{Q}(x) - Q(x)$ *is connected*,

(iv) $\overline{Q}(x)$ *is unicoherent modulo* $\overline{Q}(x) - Q(x)$.

PROOF. By hypothesis there is a region R about x in X such that $\overline{R}$ is unicoherent modulo $\operatorname{Fr} R$. By virtue of lemmas (3.1) and (3.6), R may be chosen as an arbitrarily small region about x for which $\overline{R}$ is compact.

Now form the decomposition of X associated with L. By lemma (3.9), R^- is an open set about x. Let R' be the component of R^- which contains x. By lemma (3.6), the closure of R' is unicoherent modulo $\operatorname{Fr} R'$. Also, of course, $R' = (R')^-$.

Since X is a cyclic space, each component of $R' - \{x\}$ has closeure points in $\operatorname{Fr} R'$ (which we may obviously suppose to be non-empty). Let $Q_1, Q_2, \ldots, Q_n$ be the sets that are expressible as the union of $\{x\}$ and a component of $R' - \{x\}$. By lemma (3.4), each $\overline{Q}_i$ is unicoherent modulo $\overline{Q}_i - Q_i$.

By lemma (3.1), there is a region U_i about x in the subspace Q_i such that $\overline{U}_i$ is compact and contained in Q_i, and $Q_i - U_i$ is connected. Now $Q_i = Q_i^-$ and $L \cap Q_i$ is a semi-closed set in the subspace Q_i. Thus U_i^- is a neighbourhood of x in the subspace Q_i. Further, $Q_i - U_i^- = (Q_i - U_i)^+$ is a closed connected set in the subspace Q_i, and each component of U_i^- has its frontier (with respect to the subspace Q_i) in $(Q_i - U_i)^+$. Let V_i be the component of U_i^- that contains x. Then $Q_i - V_i$ is a connected closed subset of the subspace Q_i. Further, $\overline{V}_i \cap (\overline{Q}_i - Q_i) = \phi$. From this and the fact that $\overline{Q}_i$ is unicoherent modulo $\overline{Q}_i - Q_i$, it easily follows that Q_i is unicoherent modulo $Q_i - V_i$. Also $V_i = V_i^-$.

Suppose in the first case that $V_i - L \cap V_i$ is connected. Then we define $Q_i(x) = V_i$.

In the second case we suppose that $V_i - L \cap V_i$ is not connected. Then by lemma (3.11), some component of $L \cap V_i$ separates Q_i. But if a component F of $L \cap V_i$ separates Q_i, then $Q_i - F$ must have just two components, the one containing x and the other containing the connected set $Q_i - V_i$. For if this were not the case then F, which is a component of L, would separate X, which is contrary to hypothesis.. Thus let F be some component of $L \cap V_i$ which separates Q_i, and let $Q_i(x)$ be the component of $Q_i - F$ that contains x.

We now define $S(x) = \bigcup_{i=1}^{n} Q_i(x)$. Then the arms of $S(x)$ are just the sets $Q_i(x)$, $i \leq n$, and it is clear that each $Q_i(x)$ has the properties (i) - (iv).

Let X be a cyclic U-space, L a semi-closed subset of X, and x a point of $X - L$. If $S(x)$ is a region about x and each arm $Q(x)$ of $S(x)$ has the properties (i) - (iv) listed in lemma (3.12), then we shall call $S(x)$ _a special region about_ x (_in_ X _with respect to_ L). If $Q(x)$ is an arm of a special region $S(x)$ about x and $Q(x) - L$ is connected, then we shall say that $Q(x)$ is _an arm of_ $S(x)$ _of type (a)_; if $Q(x) - L$ is not connected, then we shall say that $Q(x)$ is _an arm of_ $S(x)$ _of type (b)_. (It should be noticed that the properties "$Q(x) - L$ is connected" and "$\overline{Q}(x) - Q(x)$ lies in a component of L" do not in general divide the arms of $S(x)$ into two mutually exclusive classes, although they can be made to do so by suitably "cutting back" the arms of $S(x)$ of type (a).)

NOTATION (3.5). With the notation of the previous paragraph, we shall always denote the component of $X - L$ to which x

belongs by $H(x)$.

REMARK (3.1). The special region $S(x)$ about x in lemma (3.12) can also be chosen so that $X - S(x)$ is connected. This can be done by taking the region R at the beginning of the proof of lemma (3.12) to have the property that $X - R$ is connected (which is possible because X is cyclic). With the remainder of the proof of lemma (3.12) unchanged, it will follow that $X - S(x)$ is connected.

LEMMA(3.13). _Let_ X _be a cyclic U-space_, L _a semi-closed subset of_ X _no component of which separates_ X, _and_ x _a point of_ $X - L$. _If_ $S(x)$ _is a special region about_ x _and_ $H(x)$ _is the component of_ $X - L$ _in which_ x _lies_, _then_ $\mathrm{Fr}\, S(x) \subset H(x) \cup L$.

PROOF. In order to prove this, it is only necessary to show that for each arm $Q(x)$ of $S(x)$ of type (a), $\overline{Q}(x) - L$ is connected.

Let $y \in (\overline{Q}(x) - Q(x)) - L$. Then in an arbitrarily small neighbourhood of y there is an arc uv such that $uv - \{v\} \subset Q(x)$ and $v \in \overline{Q}(x) - Q(x)$. Since each component of L that meets $Q(x)$ is contained in $Q(x)$, it follows that $uv - \{v\} \not\subset L$. Thus $uv - \{v\}$ contains some point of $Q(x) - L$. Thus $y \in \overline{Q(x) - L}$. That is, $(\overline{Q}(x) - Q(x)) - L \subset \overline{Q(x) - L}$. Since $Q(x) - L$ is connected, we therefore deduce that $\overline{Q}(x) - L$ is connected.

LEMMA (3.14). Let X be a cyclic U-space and L a semi-closed subset of X no component of which separates X. Let H be the union of a collection of components of $X - L$ such that $\overline{H} - H \subset L$. Let x and y be points of H and $X - H \cup L$, respectively, and let $S(x)$ and $R(y)$ be special regions about x and y such that $\overline{S}(x)$ is compact, $\overline{R}(y) \cap \overline{H} = \emptyset$ and $R(y) \not\subset S(x)$. Then there is a special region $T(x)$ about x such that $T(x) \subset S(x)$ and

$$\overline{T}(x) \cap R(y) = \emptyset \quad \ldots\ldots\ldots \quad (I)$$

$$S(x) \cap H \subset T(x) \quad \ldots\ldots\ldots \quad (II)$$

PROOF. In order to prove this lemma we examine the arms $Q(x)$ of $S(x)$.

In the first case let $Q(x)$ be an arm of $S(x)$ of type (a), so that $Q(x) - L$ is connected. We show that $\overline{Q}(x) \cap R(y) = \emptyset$. For suppose that this is not the case, so that $Q(x) \cap R(y) \neq \emptyset$. Since $Q(x) \not\supset R(y)$, there is an arc uv lying in $R(y)$ such that $uv - \{v\} \subset Q(x)$ and $v \in \overline{Q}(x) - Q(x)$. Since $Q(x) = (Q(x))^-$, it follows that $uv - \{v\}$ does not lie in L; that is, it meets $H(x)$ (see notation (3.5)). But this implies that $R(y) \cap H(x) \neq \emptyset$, which is false because $H(x) \subset H$.

For each arm $Q(x)$ of $S(x)$ of type (a) we define $Q_T(x) = Q(x)$.

Now let $Q(x)$ be an arm of $S(x)$ of type (b), so that $\overline{Q}(x) - Q(x)$ lies in some component of L, and suppose that $\overline{Q}(x) \cap R(y) \neq \emptyset$. We first show that $\overline{Q}(x) - Q(x) \subset R(y)$.

We have $R(y) \cap Q(x) \neq \emptyset$ and $R(y) \not\subset Q(x)$. Thus, since $R(y)$ is connected, it follows that $\overline{Q}(x) - Q(x)$ meets $R(y)$. But $\overline{Q}(x) - Q(x)$ lies in a component of L and $R(y) = (R(y))^-$. Thus $\overline{Q}(x) - Q(x) \subset R(y)$.

Now we show that $\overline{Q}(x) \cap R(y)$ is connected. Suppose on the contrary that $\overline{Q}(x) \cap R(y) = M \cup N$, where M and N are two disjoint, non-empty relatively closed subsets of the space $\overline{Q}(x) \cap R(y)$. Then M and N are a relatively closed subsets of $R(y)$, for $\overline{Q}(x) \cap R(y)$ is a relatively closed subset of $R(y)$. On the other hand, $Q(x) - \overline{M} \cup \{x\}$ is an open subset of the space X, and $(Q(x) - \overline{M} \cup \{x\}) \cap R(y) = N$, because $\overline{M} \cap N = \emptyset$. So N is a non-empty relatively open and closed subset of

$R(y)$ and $R(y) - N \neq \phi$. This contradicts the connectedness of $R(y)$. Thus $\overline{Q}(x) \cap R(y)$ is connected.

Since $\mathrm{Fr}\, R(y) \cap (\overline{Q}(x) - \mathrm{int}\, Q(x)) = \phi$, it follows that $\overline{Q}(x) \cap \overline{R}(y) = \overline{\overline{Q}(x) \cap R(y)}$. Thus from the fact that $\overline{Q}(x) \cap R(y)$ is connected, we may deduce that $\overline{Q}(x) \cap \overline{R}(y)$ is also connected.

Now we notice that $\overline{Q}(x) - R(y)$ is a closed set in X, and that $\overline{Q}(x) - R(y) = (\overline{Q}(x) - R(y))^+$. Let $\{F_\alpha\}_\alpha$ be the collection of all components of L that lie in $\overline{Q}(x) - R(y)$ and separate $\overline{Q}(x)$. Since F_α does not separate X and $\overline{Q}(x) - Q(x)$ is connected, it follows that $\overline{Q}(x) - F_\alpha$ has precisely two components, one of which contains x and the other of which contains $\overline{Q}(x) - Q(x)$. We denote by $\lceil F_\alpha \rceil$ the union of F_α and the component of $\overline{Q}(x) - F_\alpha$ that contains x. Since $\overline{Q}(x) \cap R(y)$ is a connected set which contains $\overline{Q}(x) - Q(x)$, it follows that $\lceil F_\alpha \rceil \subset \overline{Q}(x) - R(y)$. We obtain a total ordering on $\{F_\alpha\}_\alpha$ by defining

$$F_\alpha \leq F_\beta \qquad \text{if and only if} \qquad \lceil F_\alpha \rceil \subset \lceil F_\beta \rceil.$$

Under this ordering $\{F_\alpha\}_\alpha$ may or may not have a maximal element, and we treat the two cases differently.

In the first case we suppose that $\{F_\alpha\}_\alpha$ has a maximal element, which we denote by F_ω. We assert that

$H \cap Q(x) \subset \ulcorner F_\omega \urcorner - F_\omega$.

In order to prove this we let K be the union of $\ulcorner F_\omega \urcorner$ and all the components of L that lie in $(\overline{Q}(x) - R(y)) - \ulcorner F_\omega \urcorner$. Then the sets whose union has just been given are the components of K. Thus no component of K separates $\overline{Q}(x)$. Also K is a semi-closed subset of the compact space $\overline{Q}(x)$, and the proof of lemma (3.7) shows that the decomposition of $\overline{Q}(x)$ associated with K is usc (lemma (3.7) cannot be applied directly to show this because $\overline{Q}(x)$ may not be locally connected at some points of $\overline{Q}(x) - Q(x)$; this, however, presents no difficulty because $\overline{K} \cap (\overline{Q}(x) - Q(x) = \emptyset)$.

In the remaining paragraphs concerned with proving that $H \cap Q(x) \subset \ulcorner F_\omega \urcorner - F_\omega$, we work in the subspace $\overline{Q}(x)$, and the index operations "$(\ )^+_K$" and "$(\ )^-_K$" are taken with respect to the usc decomposition of $\overline{Q}(x)$ associated with K (see notation (3.3)).

We have seen that $\overline{Q}(x) \cap \overline{R}(y)$ is a connected set. Thus $(\overline{Q}(x) \cap \overline{R}(y))^+_K$ is a connected closed subset of $\overline{Q}(x)$. Let C be a component of $\overline{Q}(x) - (\overline{Q}(x) \cap \overline{R}(y))^+_K$. We show that $C \cap H = \emptyset$.

In order to do this we first show that $C - K$ is connected. $\overline{Q}(x) - C$ is a connected closed subset of $\overline{Q}(x)$, and $\overline{Q}(x)$ is unicoherent modulo $\overline{Q}(x) - C$. We notice that $C = C^-_K$, so that the components of $K \cap C$

are all components of K. Since no component of $K \cap C$ separates $\overline{Q}(x)$, we may apply lemma (3.11), which tells us that $K \cap C$ does not separate C; i.e., $C - K$ is connected.

Now we let M be the union of all the components of L that lie in $(\overline{Q}(x) - R(y)) - ([F_\omega] - F_\omega)$ and meet $\operatorname{Fr} R(y)$. From the three inclusions $\operatorname{Fr} C \subset (\overline{Q}(x) \cap \operatorname{Fr} R(y))^+_K$, $\operatorname{Fr} C \cap [F_\omega] \subset F_\omega$ and $\operatorname{Fr} R(y) \subset L \cup H(y)$ (see lemma (3.13)), we deduce that $\operatorname{Fr} C \subset M \cup H(y)$. We wish to show that $\operatorname{Fr} C \cap H(y) \neq \emptyset$.

Suppose on the contrary that $\operatorname{Fr} C \cap H(y) = \emptyset$, so that $\operatorname{Fr} C \subset M$. Now $\overline{C} \cap (\overline{Q}(x) - Q(x)) = \emptyset$, $\overline{Q}(x) - C$ is connected and $\overline{Q}(x)$ is unicoherent modulo $\overline{Q}(x) - Q(x)$. This implies that $\operatorname{Fr} C$ is connected, and so lies in a component F of M. Since C is a component of $\overline{Q}(x) - F$ and $C \cap (\overline{Q}(x) - Q(x)) = \emptyset$, it follows that F separates $\overline{Q}(x)$. But a component of M separates $\overline{Q}(x)$ if and only if $F_\omega \subset M$. Thus we must have $F = F_\omega$. But now C meets neither $[F_\omega] - F_\omega$ nor $\overline{Q}(x) - Q(x)$, both of which are contained in $(\overline{Q}(x) \cap \overline{R}(y))^+_K$. Thus C is a component of $\overline{Q}(x) - F_\omega$ which contains neither x nor $\overline{Q}(x) - Q(x)$. This implies that F_ω disconnects X, which is contrary to hypothesis. The contradiction shows that $\operatorname{Fr} C \cap H(y) \neq \emptyset$.

We now show that $C - K \subset H(y)$. Let $z \in H(y) \cap \operatorname{Fr} C$. In an arbitrarily small neighbourhood of z there is then an arc uv such that $uv - \{v\} \subset C$ and $v \in \mathrm{Fr}\, C$. Since $C = C_K^-$, it follows that $uv - \{v\}$ contains points of $C - K$. This shows that $z \in \overline{C - K}$. Now $C - K \subset X - L$, and so $(C - K) \cup \{z\}$ is a connected subset of $X - L$. Since $z \in H(y)$ it follows that $C - K \subset H(y)$.

Now we can show that $H \cap Q(x) \subset [F_\omega]$. For if $p \in H \cap Q(x) - [F_\omega]$ then in fact $p \notin \overline{R}(y) \cup K$, because $\overline{H} \cap \overline{R}(y) = \phi$. Thus p belongs to some component C of $\overline{Q}(x) - (\overline{Q}(x) \cap \overline{R}(y))_K^+$, and in fact $p \in C - K$. But we have seen that $C - K \subset H(y)$, which is disjoint from H.. This contradiction shows that $H \cap Q(x)$ is contained in $[F_\omega]$, and consequently in $[F_\omega] - F$.

In this case in which $Q(x)$ is an arm of $S(x)$ of type (b) and $\{F_\alpha\}_\alpha$ has as a maximal element F_ω, we define $Q_T(x) = [F_\omega] - F_\omega$.

In the second case we suppose that $\{F_\alpha\}_\alpha$ has no maximal element. Then $\bigcup_\alpha [F_\alpha] = \bigcup_\alpha ([F_\alpha] - F_\alpha)$, which is an open subset of $\overline{Q}(x)$. We denote it by G. Let $R_1, R_2, \ldots$ be a sequence of regions in $\overline{Q}(x)$ that cover G and such that $\overline{R}_i$ is a compact subset of G. Then we are easily able to find a sequence $F_{\alpha_1}, F_{\alpha_2}, \ldots$ from $\{F_\alpha\}_\alpha$ such that $[F_{\alpha_k}] - F_{\alpha_k} \supset \overline{R}_k$ and $F_{\alpha_{k-1}} \leq F_{\alpha_k}$. It now easily follows that $F_{\alpha_k} \to \mathrm{Fr}\, G$. Thus $\mathrm{Fr}\, G$ is either a subcontinuum of L or a single point in the complement of L. The former, however, cannot occur, for this would imply that the collection $\{F_\alpha\}_\alpha$ has a maximal element. Thus $\mathrm{Fr}\ G$ is a single point in the complement of L, and we shall denote it by g.

We let $P(x) = Q(x) - G$. Then we show that $\overline{P}(x) \cap H = \emptyset$ in exactly the same way that we showed that $(\overline{Q}(x) - ([F_\omega] - F_\omega)) \cap H = \emptyset$ in the case when $\{F_\alpha\}_\alpha$ had F_ω as a maximal element. The only difference is that we work in the space $\overline{P}(x)$ instead of the space $\overline{Q}(x)$.

Now since $g \notin H \cup L$, it follows that $g \notin \overline{H}$. Thus $\overline{H} \cap G = \overline{H} \cap \overline{G}$ is a compact subset of G. Let U be a region in the subspace $\overline{Q}(x)$ such that $\overline{H} \cap G \subset U \subset \overline{U} \subset G$. Since $\overline{U}$ does not contain g, we can find a set F_{α_k} such that $F_{\alpha_k} \cap \overline{U} = \emptyset$. It then follows that $\overline{U} \subset [F_{\alpha_k}] - F_{\alpha_k}$; i.e., $\overline{H} \cap G \subset [F_{\alpha_k}] - F_{\alpha_k}$.

We define $Q_T(x) = [F_{\alpha_k}] - F_{\alpha_k}$ for this case in which $Q(x)$ is an arm of $S(x)$ of type (b) and $\{F_\alpha\}_\alpha$ has no maximal element.

Now we define $T(x) = \bigcup Q_T(x)$, the union being taken over all the arms $Q(x)$ of $S(x)$. Then we notice that the arms of $T(x)$ are precisely the sets $Q_T(x)$, each one of which satisfies the conditions (i) - (iv) of lemma (3.12). Thus $T(x)$ is a special region about x. Further, from the relations $\overline{Q}_T(x) \cap R(y) = \emptyset$ and $Q(x) \cap H \subset Q_T(x)$, it follows that $T(x)$ satisfies (I) and (II). This completes the proof of lemma (3.14).

Let A, B and L be subsets of a space X. We say that L <u>separates</u> A <u>and</u> B <u>in</u> X if $X - L$ is the union of two separated sets, the one containing A and the other containing B (two sets M and N are <u>separated</u> if $\overline{M} \cap N = \emptyset = M \cap \overline{N}$). We say that L <u>broadly separates</u> A <u>and</u> B <u>in</u> X if L separates $A - L$ and $B - L$ in X. Finally, we say that L <u>weakly separates</u> A <u>and</u> B <u>in</u> X if no component of $X - L$ meets both A and B. The latter two definitions may be found in [32] and [35], respectively.

THEOREM (3.5). *Let X be a cyclic U-space and L a semi-closed subset of X no component of which separates X. Let A and B be two closed subsets of X which are weakly separated by L in X. Then L contains a closed subset K of X which broadly separates A and B in X.*

PROOF. We let $H = \bigcup\{H(x) : x \in A - L\}$, where $H(x)$ is defined in notation (3.5), and we first show that $\overline{H} - H \subset L$.

For let $y \in X - H \cup L$ and let $R(y)$ be a special region about y such that $\overline{R}(y) \cap A = \emptyset$. Then by lemma

(3.13), $\operatorname{Fr} R(y) \subset H(y) \cup L$, and so $H \cap \operatorname{Fr} R(y) = \emptyset$. Thus every component of H meets $A \subset X - \overline{R}(y)$, and no component of H meets $\operatorname{Fr} R(y)$. Thus no component of H meets $R(y)$, and consequently H does not meet $R(y)$; i.e., $y \notin \overline{H}$. This shows that $\overline{H} - H \subset L$.

For each point $y \in X - H \cup L$, let $R(y)$ be a special region about y such that $\overline{R}(y) \cap \overline{H} = \emptyset$. Then the open covering $\{R(y) : y \in X - H \cup L\}$ of $X - H \cup L$. has a countable subcovering $R(y_1), R(y_2), \ldots$ of $X - H \cup L$.

Similarly, for each point $x \in H$ there is a special region $S(x)$ about x such that $\overline{S}(x)$ is compact and disjoint from B. From the covering $\{S(x) : x \in H\}$ of H, we select a countable subcovering $S(x_1), S(x_2), \ldots$ of H.

Now we show that there is a special region $T(x_n)$ about x_n such that

(I) for each $i \leq n$, $T(x_n) \supset R(y_i)$ or $\overline{T}(x_n) \cap R(y_i) = \emptyset$,

(II) $S(x_n) \cap H \subset T(x_n) \subset S(x_n)$

We define $T(x_n)$ as follows. Define $T_0(x_n) = S(x_n)$. If each of the sets $R(y_1), \ldots, R(y_n)$ is contained in $T_0(x_n)$, then let $T(x_n) = T_0(x_n)$. If not select one set from $R(y_1), \ldots, R(y_n)$ which is not contained in $T_0(x_n)$ and call it $R(y_{N_1})$. By lemma (3.14) there is a special region $T_1(x_n)$ about x_n such that $\overline{T}_1(x_n) \cap R(y_{N_1}) = \emptyset$ and $T_0(x_n) \cap H \subset T_1(x_n) \subset T_0(x_n)$. If each of the sets $R(y_1)$, ..., $R_{y_{N_1-1}}$, $R(y_{N_1+1})$, ..., $R(y_n)$ is contained in $T_1(x_n)$, define $T(x_n) = T_1(x_n)$. If not select one of these sets which is not contained in $T_1(x_n)$ and call it $R(y_{N_2})$. By using lemma (3.14) again, we find a special region $T_2(x_n)$ about x_n such that $\overline{T}_2(x_n) \cap R(y_{N_2}) = \emptyset$ and $T_1(x_n) \cap H \subset T_2(x_n) \subset T_1(x_n)$. Continuing in this way, we arrive at the definition $T(x_n) = T_m(x_n)$ for some $m \leq n$, and it is clear that (I) and (II) hold.

Let $G = \bigcup_{n=1}^{\infty} T(x_n)$. By virtue of (II) it follows that $H \subset G$ and $G \cap B = \emptyset$. We assert that $\operatorname{Fr} G \subset L$.

In order to prove that $\operatorname{Fr} G \subset L$, suppose that there is a point $y \in \operatorname{Fr} G - L$. Since $y \in X - H \cup L$, it follows that $y \in R(y_k)$, for some k. Now $R(y_k)$ is not contained in any of the sets $T(x_k)$, $T(x_{k+1})$, ... (because $y \notin G$), and so by (I) each of the sets $\overline{T}(x_k)$, $\overline{T}(x_{k+1})$, ... is disjoint from $R(y_k)$. On the other hand, if $n < k$ then $y \notin T(x_n)$ and $y \notin \operatorname{Fr} T(x_n)$, which is contained in $H(x_n) \cup L$, by lemma (3.13). Therefore $R(y_k) - \bigcup_{n=1}^{k-1} \overline{T}(x_n)$ is a neighbourhood of y which does not meet G, and so $y \notin \operatorname{Fr} G$. This contradiction shows that $\operatorname{Fr} G \subset L$.

Let $K = (A - G) \cup \operatorname{Fr} G$. Then K is a subset of L

which is closed in X, and it broadly separates A and B in X.

Theorem (3.5) is as much as we will need for the proof of theorem (3.6). However, by modifying the proofs of lemmas (3.12) - (3.14) slightly, we can prove the following extension of theorem (3.5).

THEOREM (3.5a). Let X be a cyclic U-space and L a semi-closed subset of X. If A and B are two closed subsets of L which are weakly separated by L in X, then L contains a closed subset K of X which broadly separates A and B in X.

THEOREM (3.6) Let X be a U-space and Y a regular T_1-space. If $f : X \to Y$ is a connectivity function, then f is peripherally continuous.

PROOF. Let U be a neighbourhood of a point p of X such that $\overline{U}$ is compact, and V a neighbourhood of $f(p)$. We shall show that there is a neighbourhood W of p such that $W \subset U$ and $f(\mathrm{Fr}\ W) \subset V$.

Let $A_1, A_2, \ldots, A_n$ be the sets that are expressible as the union of $\{p\}$ and a component of $X - \{p\}$ that meets $\mathrm{Fr}\ U$ (we may naturally suppose that $\mathrm{Fr}\ U \neq \emptyset$), and let A be a typical set from this sequence.

Since p is not a cut point of the Peano space A, it is either an end point of A, or it belongs to a true cyclic element C of A (see (1.1), p.64 of [30]). We deal with the latter case first.

In the first case let p belong to a true cyclic element C of A. Let U_1 be a neighbourhood of p in the subspace C such that $\overline{U}_1 \subset U \cap C$. Then among the components of $C - A$ whose closures meet U_1, only a finite number $B_1, B_2, \ldots, B_n$ meet $\operatorname{Fr} U$. For suppose on the contrary that $C - A$ has an infinite number of such components $B_1, B_2, \ldots$. Select $b_i \in B_i \cap \operatorname{Fr} U$. Then the infinite set $b_1, b_2, \ldots$ has a point of accumulation $b \in \operatorname{Fr} U$. It is now clear that no neighbourhood of b that is contained in $A - \overline{U}_1$ can be connected, which contradicts the local connectedness of A. Since p is not a cut point A, it follows that $U_1 - \bigcup_{i=1}^{n} \overline{B}_i$ is a neighbourhood of p in the subspace C.

Now consider the connectivity function $f|C : C \to Y$. Since C is also a true cyclic element of X, it follows that C is a cyclic U-space. Thus, by theorem (3.3), $f|C : C \to Y$ is peripherally continuous; i.e., there is a neighbourhood W' of p in the subspace C such that $W' \subset U_1 - \bigcup_{i=1}^{n} \overline{B}_i$ and $f(\operatorname{Fr}_C W') \subset V_1$. Let W_A be the union of W' and all the components of $A - C$ whose closures meet W'. Then W_A is a neighbourhood of p in the subspace A such that $W_A \subset U \cap A$ and $f(\mathrm{Fr}_A W_A) \subset V$.

In the second case we suppose that p is an end point of A. Then in the space A we can find a neighbourhood U_1 of p such that $\overline{U}_1 \subset U \cap A$ and $\mathrm{Fr}_A U_1$ is a single point q. We let $E(p,q)$ be the set of all points in U_1 that separate p,q in $\overline{U}_1$.

If there is a point $r \in \{q\} \cup E(p,q)$ such that $f(r) \in V$, then the component W_A of $\overline{U}_1 - \{r\}$ is a neighbourhood of p in A such that $W_A \subset U \cap A$ and $f(\mathrm{Fr}_A W_A) \subset V$. So we may suppose that this does not happen.

Now let C be a true cyclic element of $\overline{U}_1$ which contains exactly two distinct points $r, s \in \{q\} \cup E(p,q)$, and suppose that $f^{-1}(V)$ separates r, s in C. Then it follows that $f^{-1}(V)$ contains some closed subset K of C such that $C - K = M \cup N$, where M and N are disjoint open subsets of C that contain r and s, respectively. Let P be the component of $\overline{U}_1 - C$ that contains p. Then $\overline{P} - P$ is equal to $\{r\}$ or $\{s\}$, and we may suppose it is the former. Let W_A be the union of M and all the components of $\overline{U}_1 - C$ whose closures meet M. Then $q \notin M$, and so W_A is a neighbourhood of p such that $W_A \subset U \cap A$ and $f(\mathrm{Fr}_A W_A) \subset V$. So we shall also suppose that this case does not happen;

that is, the case in the first sentence of this paragraph.

Now let V_1 be a neighbourhood of $f(p)$ in Y such that $\overline{V}_1 \subset V$.

Let C again be a true cyclic element of $\overline{U}_1$ which contains exactly two distinct points $r, s \in \{q\} \cup \Xi(p,q)$. Let $\{F_\alpha\}_\alpha$ be the collection of components of the semi-closed subset $f^{-1}(\overline{V}_1) \cap C$ of C. By supposition, F_α does not contain r or s, and does not separate r and s in C. Let $[F_\alpha]$ be the union of F_α and all the components of $C - [F_\alpha]$ except the one that contains r and s. We now define $[F_\alpha] \sim [F_\beta]$ if and only if some $[F_\gamma] \supset [F_\alpha] \cup [F_\beta]$, and we easily prove that this is an equivalence relation on $\{[F_\alpha]\}_\alpha$. Now let $\mathcal{E}$ be an equivalence class of $[F_\alpha]$'s. The assumption that $\mathcal{E}$ has no maximal element (that is, no element that contains every other element in $\mathcal{E}$) leads us to the conclusion, as in the proof of theorem (2.1), that $\bigcup\{[F_\alpha] : [F_\alpha] \in \mathcal{E}\}$ is an open subset of C whose boundary in the subspace C is a single point. But as a cyclic space, C contains no such open sets whose complements are non-degenerate. This proves that every equivalence class $\mathcal{E}$ contains a maximal element.

We let $\{[F_{\alpha_\beta}]\}_\beta$ be the collection of maximal elements of $\{[F_\alpha]\}_\alpha$, and define $H_\beta = [F_{\alpha_\beta}]$. Let $L = \bigcup_\beta H_\beta$. Then L is a semi-closed subset of C whose components are the sets H_β, and none of these components separates r, s in C. Since C is a cyclic U-space, as a true cyclic element of the U-space X, it now follows from theorem (3.5) that L does not weakly separate r and s in C. But L contains $f^{-1}(\overline{V}_1) \cap C$. Thus r,s lie in the same component D_C of $C - f^{-1}(\overline{V}_1)$.

Now we recall that, since p is an end point of $\overline{U}_1$, the cyclic chain $C(p,q)$ from p to q in the Peano continuum $\overline{U}_1$ is expressible as the union of $\{q\} \cup E(p,q)$ and all the true cyclic elements of $\overline{U}_1$ that contain just two points in $\{q\} \cup E(p,q)$ (see (5.2), p.71 of [31]). Further, the true cyclic elements of $C(p,q)$ are exactly the same as the true cyclic elements of $\overline{U}_1$ that meet $\{q\} \cup E(p,q)$ in just two points.

Let $D = \{p,q\} \cup E(p,q) \cup \bigcup_C D_C$, the union being taken over all true cyclic elements C of $C(p,q)$. We assert that D is connected.

For suppose that D is not connected. Then $D = M \cup N$, where M and N are two non-empty separated subsets of $C(p,q)$. Let M' be the union of M and all the true cyclic elements of $C(p,q)$ such that $C \cap D \subset M$, and let N' be the union of N and all the true cyclic elements of $C(p,q)$ such that $C \cap D \subset N$. Then $C(p,q) = M' \cup N'$. Further, since for each true cyclic element C of $C(p,q)$, $C - \{q\} \cup E(p,q)$ is an open subset of C, it readily follows that M' and N' are separated in $C(p,q)$. This contradicts the connectedness of $C(p,q)$. Thus D is connected.

Now consider the connectivity function $f|A : A \to Y$. There is no point of D other than p which this function maps into V_1. This contradicts lemma (2.2).

Returning to the sequence of sets $A_1, A_2, \ldots, A_n$, we have now shown that for each i there is a neighbourhood W_i of p in the space A_i such that $W_i \subset U \cap A_i$ and $f(\mathrm{Fr}_{A_i} W_i) \subset V$. Let $W = (\bigcup_{i=1}^{n} W_i) \cup (X - \bigcup_{i=1}^{n} A_i)$. Then W is a neighbourhood of p in X such that $W \subset U$ and $f(\mathrm{Fr}\ W) \subset V$. This completes the proof.

4. In this section we answer a question that Stallings reaised in [23].

On p.253 of [23] Stallings showed that _if_ $f : P \to Y$ _is a local connectivity map of the lpc polyhedron_ P _into a regular Hausdorff space_ Y, _then_ f _is peripherally continuous_.

On p.262 of [23] he asks whether this theorem remains true when the lpc polyhedron P is replaced by an ANR. In this section we give an affirmative answer to this question. We shall use theorem (3.2) to show that P may be replaced by any locally compact ANR($\mathfrak{m}$).

We first give the necessary definitions. If X and Y are topological spaces, then a function $f : X \to Y$ _is a local connectivity function_ if there is an open covering $\{U_\alpha\}_\alpha$ of X such that $f|U_\alpha : U_\alpha \to Y$ is a connectivity function.

A glance at the proofs of theorems (2.1), (3.2), (3.3) and (3.6) will show that in each case the connectivity of the function f was only used locally. Thus each of these theorems holds if the connectivity function f is replaced by a local connectivity function f.

Following chap. IV of [5], we shall say that X is _an absolute neighbourhood retract for metrizable spaces_ (or X is _an ANR($\mathfrak{m}$)_) if X is a metrizable space and for each homeomorphism h mapping X onto a closed subset $h(X)$ of a metrizable space Y, $h(X)$ is a neighbourhood retract in Y.

A topological space X is said to be <u>locally contractible</u> provided that for each point $x \in X$ and each neighbourhood U of x, there is a neighbourhood V of x such that $V \subset U$ and V is contractible to a point in U. We notice that a locally contractible space is locally connected.

Let X be an $\mathrm{ANR}(\mathfrak{m})$. We notice from (3.3), chap. IV of [5], that X is locally contractible.

Let X be a locally compact $\mathrm{ANR}(\mathfrak{m})$. Then it follows that, when X is metrized, each component of X is a Peano space.

Let X be an arbitrary topological space and Y a subset of X. We say that a continuous mapping $f : X \to S^1$, where S^1 is the circle of complex numbers of unit modulus, is <u>exponentially equivalent to 1 on</u> Y (written "$f \sim 1$ on Y") if there is a continuous real-valued function φ on Y such that $f(x) = \exp[i\varphi(x)]$ for each $x \in Y$.

The following is a standard lemma on connected spaces which are not unicoherent.

LEMMA (4.1). <u>Let</u> X <u>be a connected normal space which is not unicoherent, and</u> M <u>and</u> N <u>two connected closed sub-</u>

<u>sets of</u> X <u>such that</u> $X = M \cup N$ <u>and</u> $M \cap N$ <u>is not connected. Then there is a continuous function</u> $f : X \to S^1$ <u>such that</u> $f \sim 1$ <u>on</u> M, $f \sim 1$ <u>on</u> N <u>and</u> $f \not\sim 1$ <u>on</u> X.

PROOF. Let $M \cap N = P \cup Q$, where P and Q are disjoint non-empty closed sets. By Urysohn's lemma, there is a continuous function $\varphi_1 : M \to [0, \pi]$ such that $\varphi_1(P) = 0$ and $\varphi_1(Q) = 1$, and a continuous function $\varphi_2 : N \to [\pi, 2\pi]$ such that $\varphi_2(Q) = \pi$ and $\varphi_2(P) = 2\pi$. Let $f_1 : X \to S^1$ be defined by

$$f(x) = \begin{cases} \exp[i\varphi_1(x)] & \text{for } x \in M, \\ \exp[i\varphi_2(x)] & \text{for } x \in N. \end{cases}$$

Then f is well-defined, since $\exp[i\varphi_1(x)] = \exp[i\varphi_2(x)]$ for $x \in M \cap N$, and f is continuous, because the restricted functions $f|M$ and $f|N$ are continuous on the closed subsets M and N of X.

By definition $f \sim 1$ on M and $f \sim 1$ on N. We show that $f \not\sim 1$ on X by supposing on the contrary that $f \sim 1$ on X. Then there is a continuous real-valued function φ on X such that $f(x) = \exp[i\varphi(x)]$ for all $x \in X$. Thus $\varphi(P) \subset \{0, \pm 2\pi, \pm 4\pi, \pm \ldots\}$ and $\varphi(Q) \subset \{\pm\pi, \pm 3\pi, \pm 5\pi, \pm \ldots\}$. Let p and q be points of $\varphi(P)$ and $\varphi(Q)$, respectively, and let r be a point between p and q such that r is not a multiple of π. Since $\varphi(M)$ and $\varphi(N)$ are connected sets, it follows that $r \in \varphi(M) \cap \varphi(N)$. Thus $\exp[r] \in f(M) \cap f(N)$. But $\exp[r]$ is not equal to either of the two complex numbers $+1$ or -1, and $f(M) \cap f(N)$ is precisely the set of these two complex numbers. This contradiction shows that $f \not\sim 1$ on X.

THEOREM (4.1). _If_ X _is a locally contractible Peano space, then_ X _has a covering by regions_ $\{R_i\}_i$ _such that each_ $\overline{R}_i$ _is unicoherent modulo_ $\mathrm{Fr}\, R_i$.

PROOF. Suppose on the contrary that X does not have a covering by such regions. Then there is some point $p \in X$ such that each region R that contains p has the property that $\overline{R}$ is not unicoherent modulo $\mathrm{Fr}\, R$.

Let U and V be regions about p such that $V \subset U$ and V is contractible to a point in U. Since each subregion of V is also contractible to a point in U, we may clearly suppose that $\overline{V}$ is compact and contained in U.

Since by supposition $\overline{V}$ is not unicoherent modulo $\mathrm{Fr}\, V$, there are two connected closed sets M and N such that $\overline{V} = M \cup N$, $(\overline{V} - V) \cap N = \emptyset$ and $M \cap N$ is not connected. By lemma (4.1), there is a continuous function

$f : \overline{V} \to S^1$, the latter being the set of complex numbers of unit modulus, such that $f \sim 1$ on M and $f \not\sim 1$ on $\overline{V}$.

We shall now produce a contradiction. Let φ be a real-valued function on M such that $f(x) = \exp[i\varphi(x)]$ for each $x \in M$. Then $\varphi|\overline{V} - V$ is a real-valued function on the compact set $\overline{V} - V$, and so by Tietze's extension theorem there is a real-valued function ψ on $U - V$ such that $\psi|\overline{V} - V = \varphi|\overline{V} - V$.

Define

$$g(x) = \begin{cases} f(x), \text{ for } x \in \overline{V}, \\ \exp[i\psi(x)], \text{ for } x \in U - V. \end{cases}$$

Then $g : U \to S^1$ is a well-defined continuous function, because $f(x) = \exp[i\psi(x)]$ for $x \in \overline{V} - V$, and $\overline{V}$ and $U - V$ are relatively closed subsets of U.

Since $\overline{V}$ is contractible to a point in U, there is a mapping $h : \overline{V} \times [0,1] \to U$ such that $h(x,0) = x$ and $h(x,1) = q$ for all $x \in \overline{V}$, where q is some point in U. The composition $gh : \overline{V} \times [0,1] \to S^1$ is a homotopy between the mapping $g|\overline{V} : \overline{V} \to S^1$ and the constant mapping from $\overline{V}$ into S^1. By theorem (6.2), chap. XI of [31], this implies that $g|\overline{V} \sim 1$ on $\overline{V}$. However, $g|\overline{V} = f$, and $f \not\sim 1$ on $\overline{V}$. This contradiction proves the theorem.

THEOREM (4.2). <u>Let</u> X <u>be a locally contractible Peano space and</u> Y <u>a regular</u> T_1-<u>space. If</u> $f : X \to Y$ <u>is a connectivity function (or local connectivity function) then</u> f <u>is peripherally continuous</u>.

THEOREM (4.3). <u>Let</u> X <u>be a locally compact</u> ANR($\mathfrak{m}$) <u>and</u> Y <u>a regular</u> T_1-<u>space. If</u> $f : X \to Y$ <u>is a connectivity function (or local connectivity function) then</u> f <u>is peripherally continuous</u>.

Theorem (4.2) follows immediately from theorems (4.1) and (3.2) (the latter also holds for local connectivity functions, as it was remarked earlier in this section). Theorem (4.3) is a corollary of theorem (4.2), since each component of a locally compact ANR($\mathfrak{m}$) is a locally contractible Peano space.

5. The purpose of this section has been outlined in §1.

We shall say that a connected space X is *weakly finitely multicoherent* if for each pair of connected closed subsets M, N of X such that $X = M \cup N$, $M \cap N$ always has a finite number of components. Such spaces were investigated by A.H. Stone in [26].

Let A be a subset of a connected space X. We shall say that X is *weakly finitely multicoherent modulo* A if for each pair of connected closed subsets M, N of X such that $X = M \cup N$ and $A \cap N = \phi$, $M \cap N$ is always connected.

We then have the following result, the proof of which is simple and is omitted.

LEMMA (5.1). *Let* X *be a connected, locally connected and completely normal space*, *and let* A *be a connected subset of* X *such that* X *is weakly finitely multicoherent modulo* A. *If* L *is now a subset of* $X - A$ *which separates* X, *then a finite number of components of* L *separate* X.

The following are also two straightforward lemmas, and their proofs are omitted.

LEMMA (5.2). *Let* $A_1 \supset A_2 \supset \ldots$ *be a sequence of* connected closed sets in a metric space X. Suppose, furthermore, that there is a compact set K such that $X - A_n \subset K$ for all n. Then $\bigcap_{n=1}^{\infty} A_n$ is connected.

LEMMA (5.3). Let $C_1, C_2, \ldots, C_n$ be components of a space X such that $X \neq C_1 \cup C_2 \cup \ldots \cup C_n$. Then there are two non-empty separated subsets M, N of X such that $X = M \cup N$ and $N \supset C_1, C_2, \ldots, C_n$.

The above three lemmas are needed for the proof of theorem (5.2). Because in this theorem the range space is only regular, and not regular and T_1 (as it is in §2,3,4 of this chapter and in [23], [34], [35]) we also need the appropriate modifications of lemmas (2.2) and (2.3), and these are given next.

LEMMA (5.4). Let $f : X \to Y$ be a connectivity function, where X and Y are both regular spaces. Then if C is a connected subset of X, the graph $\Gamma(f|C)$ has no isolated points.

PROOF. Suppose $\Gamma(f|C)$ has an isolated point $(p, f(p))$. Let $U \times V$ be a basic open set about $(p, f(p))$ in $X \times Y$ which does not meet $\Gamma(f|C - \{p\})$. We can find neighbourhoods U_1, V_1 of p, $f(p)$, respectively, such that $\overline{U}_1 \subset U$, $\overline{V}_1 \subset V$. But now the two disjoint open sets $U_1 \times V_1$, $(X \times Y) - (\overline{U}_1 \times \overline{V}_1)$ provide a separation of $\Gamma(f|C)$, which is a contradiction.

LEMMA (5.5). <u>Let</u> $f : X \to Y$ <u>be a connectivity function, where</u> X <u>is a locally connected, Hausdorff space with a countable basis and</u> Y <u>is a regular space. Then for each closed subset</u> F <u>of</u> Y, $f^{-1}(F)$ <u>is a semi-closed subset of</u> X.

PROOF. Let F be a closed subset of Y, and let F_{α_1}, F_{α_2}, ... be a convergent sequence of components of $f^{-1}(F)$ whose limit is L. Let p,q be two distinct points of L such that $p \in L - f^{-1}(F)$. Let U, V be disjoint regions containing p,q, respectively. We may without loss of generality suppose that V meets all the sets F_{α_1}, F_{α_2}, ... Let $Q = V \cup \bigcup_{n=1}^{\infty} F_{\alpha_n}$. Then Q is a connected set, and so $Q \cup \{p\}$ is also a connected set. However, there is an open subset G of Y such that $f(p) \in G \subset \overline{G} \subset Y - F$, and therefore the two disjoint open sets $U \times G$, $(X \times (Y - \overline{G})) \cup (V \times Y)$ provide a separation of $\Gamma(f|Q \cup \{p\})$ in $X \times Y$. This contradiction proves the lemma.

In the proof of theorem (5.2) we also use the fact that a connectivity function preserves connectedness (though it is not necessary to use this fact), and we state this as a lemma.

LEMMA (5.5a). Let $f : X \to Y$ be a connectivity function, where X and Y are arbitrary topological spaces. Then for each connected subset C of X, $f(C)$ is a connected subset of Y.

The following two lemmas are used to prove lemmas (5.8) and (5.9), and theorem (5.2).

LEMMA (5.6). Let A be a connected subset of a connected, locally connected and normal space X which is weakly finitely multicoherent modulo A. Let F be a closed and connected subset of $X - A$. Then all but a finite number of components of $X - F$ have connected frontiers.

This result is stated for the case where F is an arbitrary subset of X and X is a weakly finitely multicoherent space, in the footnote of p.298 of [26]. This argument used to prove this footnote may also be used to prove lemma (5.6). This argument is similar to the argument given in the proof of theorem 5, §4 of [25].

COROLLARY. Let U, V be two conditionally compact[1]

[1] A set A is conditionally compact if $\bar{A}$ is compact.

regions in a Peano space X *such that* $\overline{U} \subset V$, $X - V$ *is connected and* X *is weakly finitely multicoherent modulo* $X - V$. *Let* F *be a connected closed subset of* X *such that* $X - U \subset F$. *Then all but a finite number of components of* $X - F$ *have connected boundaries*.

PROOF. Because X is locally connected and $\overline{V}$ is compact, it follows from Janiszewski's border theorem (p.184, [13]) that each component of $F \cap V$ has closure points in $\operatorname{Fr} V$. Thus only a finite number of components. $F_1, F_2, \ldots, F_m$ of $F \cap V$ meet $\overline{U}$.

Suppose now that $X - F$ has an infinite number of components $C_1, C_2, \ldots$ with disconnected boundaries. Since $\overline{U}$ is compact, the frontiers of only a finite number of the sets $C_1, C_2, \ldots$ can meet more than one of the sets $F_1, F_2, \ldots, F_m$. Thus there is a set F_i and a subsequence $C_{n_1}, C_{n_2}, \ldots, C_{n_k}, \ldots$ such that $\operatorname{Fr} C_{n_k}$ is disconnected and $\operatorname{Fr} C_{n_k} \subset F_i$. But F_i is a connected closed subset of $\overline{V}$ and X is weakly finitely multicoherent modulo $X - \overline{V}$. This contradicts lemma (5.6)

We introduce the following definition. It is equivalent to the definition given in §2.2 of [24] and §3.3 of [25]. Let A and B be subsets of some space.

We say that A is <u>connected relative to</u> B if every non-empty relatively open and closed subset of the subspace $A \cup B$ meets B.

We state the following obvious properties of this relation:

(i) if $B \subset A \subset \overline{B}$, then A is connected relative to B,

(ii) if each A_α is connected relative to B, then so is $\bigcup_\alpha A_\alpha$,

(iii) if A is connected relative to B and B is connected, then $A \cup B$ is connected,

(iv) if A is connected relative to B and B is connected relative to C, then $A \cup B$ is connected relative to C,

(v) if A is connected relative to B, and $B \subset C$, then A is connected relative to C.

Notice that (iii) and (iv) are generalized versions of (1) and (2) in §3.3 of [25], and (v) is stated in §2.2 of [24]. Finally, in connection with this definition, notice that if A is a compact set and B is a closed set, then A is connected relative to B if and only if each component of A meets B.

Let A be a subset of a space X. We say that X

is locally connected modulo A if X is *locally connected modulo* A if X is locally connected at each point in $X - A$.

LEMMA (5.7). *Let* X *be a continuum (metric) and let* A *and* B *be two disjoint closed subsets of* X *such that* X *is locally connected modulo* $A \cup B$. *Then there is an open set* U *such that* $U \supset A$, U *is connected relative to* A, *and* $\overline{U} \cap B = \emptyset$.

PROOF. Let V be an open set such that $V \supset A$ and $\overline{V} \cap B = \emptyset$. We show that all the components of V are open.

Since it is clear that any component of V which does not meet A is open, we consider a component C of V which does meet A. Let x be a point in C such that $x \notin \operatorname{int} C$. Then clearly $x \in A$. Also there is a sequence of components $C_1, C_2, \ldots$ of V, all different from C, such that $x \in \liminf C_m$. Let $C_{m_1}, C_{m_2}, \ldots$ be a convergent subsequence of $C_1, C_2, \ldots$. Then $\lim C_{m_n} = \lim \overline{C}_{m_n} = K$, which is a continuum. Since, by Janiszewski's border theorem (p.184 of [13]; c.f., lemma (3.8)) $\overline{C}_{m_n} \cap \operatorname{Fr} V \neq \emptyset$ for each m_n, we have $K \cap \operatorname{Fr} V \neq \emptyset$. However, we also have $K \cap A \neq \emptyset$ and $K \cap (V - A) = \emptyset$. This shows that K is not connected, which is false. Thus each component of V is open.

To complete the proof, we now define U as the union of the finite number of components of V which meet A.

Following [24], we say that a set E is <u>a simple subset of a space</u> X if both E and $X - E$ are connected.

LEMMA (5.8). <u>Let</u> U, V <u>be conditionally compact regions in a Peano space</u> X <u>such that</u> $\overline{U} \subset V$, $X - V$ <u>is connected and</u> X <u>is weakly finitely multicoherent modulo</u> $X - V$. <u>Let</u> $E_1 \subset E_2 \subset \ldots \subset U$ <u>be a sequence of simple closed subsets of</u> X, <u>and suppose that</u> $\mathrm{Fr}\, E_n = A_1 \cup A_2 \cup \ldots \cup A_n \cup B_n$, $B_n \subset \mathrm{int}\, E_{n+1}$, <u>where</u> $A_1, A_2, \ldots, A_n, B_n$ <u>are disjoint closed sets</u>. Then $(A_1 \cup A_2 \cup \ldots \cup A_{n+1}) \cap \mathrm{Fr}(\mathrm{int}\, E_{n+1} - E_n) \neq \emptyset$ <u>for at most a finite number of different values of</u> n.

PROOF. We suppose that the conclusion of the lemma is false. Then we may suppose, without loss of generality, that $(A_1 \cup A_2 \cup \ldots \cup A_{n+1}) \cap \mathrm{Fr}(\mathrm{int}\, E_{n+1} - E_n) \neq \emptyset$ for every value of n. Again, without loss of generality, we may suppose that $A_1 \neq \emptyset$.

Since X is locally connected, the hypothesis that

$(A_1 \cup A_2 \cup \dots \cup A_{n+1}) \cap \mathrm{Fr}(\mathrm{int}\, E_{n+1} - E_n) \neq \emptyset$ implies that there is a component C_n of $\mathrm{int}\, E_{n+1} - E_n$ such that $(A_1 \cup A_2 \cup \dots \cup A_{n+1}) \cup \overline{C}_n \neq \emptyset$. Let C_n lie in a component D_n of $E_{n+1} - \mathrm{int}\, E_n$.

Notice that $D_n \cap (A_1 \cup A_2 \cup \dots \cup A_n \cup B_n) \neq \emptyset$ and $D_n \cap (A_{n+1} \cup B_{n+1}) = \emptyset$. For if $D_n \cap (A_1 \cup A_2 \cup \dots \cup A_n \cup B_n) = \emptyset$ then $D_n \cap E_n = \emptyset$ and we easily deduce that E_{n+1} is not connected; and if $D_n \cap (A_{n+1} \cup B_{n+1}) = \emptyset$ then $\mathrm{Fr}\, C_n \subset A_1 \cup A_2 \cup \dots \cup A_n \cup B_n$, and so E_n separates X.

Now we use lemma (5.7). Since D_n is locally connected modulo $D_n \cap (A_1 \cup A_2 \cup \dots \cup A_{n+1} \cup B_n \cup B_{n+1})$, it follows from lemma (5.7) that the subspace D_n contains two relatively open subsets U_n and V_n such that U_n contains and is connected relative to $D_n \cap (A_1 \cup A_2 \cup \dots \cup A_n)$, V_n contains and is connected relative to $D_n \cap A_{n+1}$, $\overline{U}_n \cap \overline{V}_n = \emptyset$ and $(\overline{U}_n \cup \overline{V}_n) \cap (B_n \cup B_{n+1}) = \emptyset$.

We notice that $(\overline{U}_n - U_n) \cup (\overline{V}_n - V_n) \neq \emptyset$, for all possible choices of U_n, V_n. For suppose that this is false. Since D_n is connected, it then follows that $D_n = U_n$ or $D_n = V_n$. But if $D_n = U_n$, then $D_n \cap (A_{n+1} \cup B_{n+1}) = \emptyset$, and, if $D_n = V_n$, then $D_n \cap (A_1 \cup A_2 \cup \dots \cup A_n \cup B_n) = \emptyset$, both of which conclusions are false.

Now we define $F = \bigcap_{n=1}^{\infty} (X - \text{int } E_n) \cup \bigcup_{n=1}^{\infty}(\overline{U}_n \cup \overline{V}_n)$. Then F is a closed, connected set which contains $X - U$. F is closed because the complement of F is equal to $\bigcup_{n=1}^{\infty} (\text{int } E_{n+1} - \bigcup_{m=1}^{n} \overline{U}_m \cup \overline{V}_m)$, which is open. To see that F is connected, we observe that $X - \text{int } E_1 \supset X - \text{int } E_2 \supset \ldots$ is a decreasing sequence of connected closed sets the complements of which are contained in $\overline{U}$. Therefore, by lemma (5.2), $\bigcap_{n=1}^{\infty} (X - \text{int } E_n)$ is connected, and it easily follows from this that F is connected.

We notice that the sets $(\overline{U}_n - U_n) \cup (\overline{V}_n - V_n)$, $n = 1,2,\ldots,$ form a sequence of relatively open and closed, disjoint non-empty subsets of the subspace $\text{Fr } F$. Further, X is weakly **finitely multicoherent** modulo $X - V$. Thus, on the one hand, the boundary of a component of $X - F$ can meet at most a finite number of the sets $(\overline{U}_n - V_n) \cup (\overline{V}_n - V_n)$, $n = 1,2,\ldots$. On the other hand, only a finite number of components of $X - F$ can have disconnected boundaries, by the corollary to lemma (5.6). Thus we can find an integer k with this property: if G is a component of $X - F$ and $(\text{Fr } G) \cap ((\overline{U}_k - U_k) \cup (\overline{V}_k - V_k)) \neq \emptyset$, then $\text{Fr } G \subset (\overline{U}_k - U_k) \cup (\overline{V}_k - V_k)$. We shall use this property to produce a contradiction.

Firstly, suppose that $D_k \cap (B_k \cup B_{k+1}) = \emptyset$. Then $\text{Fr } D_k \subset A_1 \cup A_2 \cup \ldots \cup A_{k+1}$. Thus $\overline{U}_k \neq \emptyset$, $\overline{V}_k \neq \emptyset$;

for if $\overline{U}_k$ (or $\overline{V}_k$) were empty, then we could have defined V_k (or U_k) to be equal to D_k in the first place, and for these choices of U_k, V_k we should have had $(\overline{U}_k - U_k) \cup (\overline{V}_k - V_k) = \phi$, which, as we have seen, is false. Since D_k is connected and locally connected modulo $\overline{U}_k \cup \overline{V}_k$, it now follows that we can find a component G of $D_k - \overline{U}_k \cup \overline{V}_k$ such that $(\mathrm{Fr}\ G) \cap (\overline{U}_k - U_k) \neq \phi$, $(\mathrm{Fr}\ G) \cap (\overline{V}_k - V_k) \neq \phi$. But G is also a component of $X - F$, and, since $\mathrm{Fr}\ G$ is not connected, the choice of k is contradicted.

Secondly, suppose that $D_k \cup (B_k \cup B_{k+1}) \neq \phi$. Let H be a component of $D_k - \overline{U}_k \cup \overline{V}_k$ such that $H \cap (B_k \cup B_{k+1}) \neq \phi$. By Janiszewski's border theorem (see p.184 of [13]; c.f., lemma (3.8)), $(\mathrm{Fr}\ H) \cap ((\overline{U}_k - U_k) \cup (\overline{V}_k - V_k)) \neq \phi$. Let G be the component of $X - F$ that contains H. Then $G \cap (B_k \cup B_{k+1}) \neq \phi$ and, because of the choice of k, $\mathrm{Fr}\ G \subset (\overline{U}_k - U_k) \cup (\overline{V}_k - V_k)$. But, if $G \cap B_k \neq \phi$, then $G \cap E_k$ is a relatively open and closed, non-empty proper subset of E_k (proper because $E_k - G \supset A_1$), and, if $G \cap B_{k+1} \neq \phi$, then $G \cap (X - E_{k+1})$ is a relatively open and closed, non-empty proper subset of $X - E_{k+1}$. This is a contradiction, and so the lemma is proved.

In corollaries (1) and (2) to lemma (5.8), the notation and hypotheses of the statement of lemma (5.8) are assumed.

COROLLARY (1). *Let* N *be an integer such that* $(A_1 \cup A_2 \cup \dots \cup A_{n+1}) \cap \mathrm{Fr}(\mathrm{int}\, E_{n+1} - E_n) = \emptyset$ *for* $n \geq N$. *Then* $A_n = \emptyset$ *for* $n > N$.

This follows immediately from the fact that each E_n is connected.

COROLLARY (2). $\lim_{n\to\infty} B_n$ *exists*.

PROOF. We have to prove that $\limsup B_n \subset \liminf B_n$. Thus we suppose that there is a point $x \in \limsup B_n - \liminf B_n$. Then we can select two subsequences B_{m_1}, B_{m_2}, ... and B_{n_1}, B_{n_2}, ... such that $N \leq m_k < n_k < m_{k+1}$ for each k and $x \in \limsup B_{n_k} - \limsup B_{m_k}$, N being the integer in corollary (1).

Now select $x_k \in B_{n_k}$ so that $x_k \to x$, and let C_k be the component of $\mathrm{int}\, E_{m_{k+1}} - E_{m_k}$ which contains x. Then $\overline{C}_k - C_k \subset B_{m_k} \cup B_{m_{k+1}}$.

Let R be a region about x such that $\overline{R} \cap \limsup B_{m_k} = \emptyset$. Select the integer k so large that $C_k \cap R = \emptyset$, and $(B_{m_k} \cup B_{m_{k+1}}) \cap R = \emptyset$. Notice that $x \notin C_k$, because C_k is an open set which meets only B_{n_k} among the sets

B_{n_1}, B_{n_2}, ... Thus $C_k \cap R$ is a relatively open and closed, non-empty proper subset of R, because $(\overline{C}_k - C_k) \cap R = \emptyset$. This contradiction proves corollary (2).

LEMMA (5.9). *Let* U, V *be conditionally compact regions in a Peano space* X *such that* $\overline{U} \subset X$, $X - V$ *is connected and* X *is weakly finitely multicoherent modulo* $X - V$, *and let* $E_1 \subset E_2 \subset \ldots \subset U$ *be a sequence of simple closed subsets of* X. *Let* $\operatorname{Fr} E_n = A_1 \cup A_2 \cup \ldots \cup A_n \cup B_n$, *where* $B_n \subset \operatorname{int} E_{n+1}$ *and* A_1, A_2, ..., A_n, B_n *are disjoint closed sets. Let the components of* $E_{n+1} - \operatorname{int} E_n$, $n = 1,2,\ldots,$ *be* $D_{n,m}$, $m = 1,2,\ldots, n_p$. *Then* $D_{n,m} \cap E_n$ *is connected for all but a finite number of pairs* (n,m).

PROOF. Suppose that the conclusion of the lemma is false. Then we can suppose without loss of generality, that for each value of n there is a component D_{n,m_n} of $E_{n+1} - \operatorname{int} E_n$ such that $D_{n,m_n} \cap E_n$ is not connected. For convenience we shall write $D_{n,m_n} = D_n$.

Let N be the integer given in corollary (1) to lemma (5.8), so that $A_n = \emptyset$ for $n > N$. For $n > N$, let C_n be a component of $D_n - E_n$ such that $B_n \cap \overline{C}_n = P_n \cup Q_n$, where P_n and Q_n are two disjoint non-empty closed sets. Join each component of $B_{n+1} \cap C_n$ to Q_n by an arc which lies in $C_n - P_n$, and denote the union of the finite number of arcs so obtained by α_n. Let $M_n = Q_n \cup \alpha_n \cup (B_{n+1} \cap C_n)$.

Now define $F = X - \bigcup_{n>N} (C_n - M_n)$. Then F is a closed set containing $X - U$, and it is also connected. This can be seen by observing, firstly, that $E_{N+1} \cup \bigcap_{n=1}^{\infty} (X - E_n)$ is connected, by lemma (5.2), and, secondly, that it follows by induction that $F \supset B_n$ for each $n > N$.

Thus F is a connected closed set containing $X - U$. But the components of $C_n - M_n$, $n > N$, are components of $X - F$, and at least one component of $C_n - M_n$ has a disconnected boundary for each $n > N$. This contradicts the corollary to lemma (5.6).

In the following two corollaries to lemma (5,9), the notation and hypotheses in the statement of lemma (5.9) are assumed.

COROLLARY (1). <u>There is an integer</u> b <u>such that</u> B_n <u>has</u> $\leq b$ <u>components for all</u> n.

This is a straightforward consequence of lemma (5.9).

COROLLARY (2). lim B_n <u>has only a finite number of components</u>.

This is an immediate consequence of corollary (2) to lemma (5.8) and corollary (1) to lemma (5.9).

We sum up the pertinent parts of lemmas (5.8) and (5.9) and their corollaries in the following theorem.

THEOREM (5.1). _Let_ U, V _be conditionally compact regions in a Peano space_ X _such that_ $\overline{U} \subset V$, $X - V$ _is connected and_ X _is weakly finitely multicoherent modulo_ $X - V$. _Let_ $E_1 \subset E_2 \subset \ldots \subset U$ _be a sequence of simple closed subsets of_ X. _Let_ $\mathrm{Fr}\, E_n = A_1 \cup A_2 \cup \ldots \cup A_n \cup B_n$, $B_n \subset \mathrm{int}\, E_{n+1}$, _where_ $A_1, A_2, \ldots, A_n, B_n$ _are disjoint closed sets. Then_

(i) $A_n \neq \emptyset$ _for at most a finite number of values of_ n,

(ii) $\lim B_n$ _exists and has only a finite number of components._

In the context in which we shall use this theorem, we shall not in general have $\mathrm{Fr}\, E_n = A_1 \cup A_2 \cup \ldots \cup A_n \cup B_n$; we shall have $\mathrm{Fr}\, E_n \subset A_1 \cup A_2 \cup \ldots \cup A_n \cup B_n$, where $A_1, A_2, \ldots, A_n, B_n$ are disjoint closed subsets of E_n which meet $\mathrm{Fr}\, E_n$. We remark that lemmas (5.8) and (5.9) and their corollaries can all be proved under these circumstances without change, except for the few obvious modifications. Thus we have this theorem.

THEOREM (5.1)$'$. <u>If the hypotheses of theorem</u> (5.1) <u>remain unaltered, except that it is supposed that</u> $\mathrm{Fr}\, E_n \subset A_1 \cup A_2 \cup \ldots \cup A_n \cup B_n$, $B_n \subset \mathrm{int}\, E_{n+1}$, <u>where</u> $A_1, A_2, \ldots, A_n, B_n$ <u>are disjoint closed subsets of</u> E_n <u>which meet</u> $\mathrm{Fr}\, E_n$, <u>then the conclusions of theorem</u> (5.1) <u>remain unaltered</u>.

It is obvious that the first conclusion of theorem (5.1) or theorem (5.1)$'$ may not hold if X fails to be weakly finitely multicoherent modulo $X - V$. The following example shows that the second conclusion of these theorems may not hold if X fails to be weakly finitely multicoherent modulo $X - V$; viz it shows that $\lim B_n$ may not exist.

EXAMPLE (5.1). Let X be the set-theoretic difference between $[0, 1] \times [0, 1]$ and the set of all points (x, y) such that $1/2^{2m+1} < x < 1/2^{2m}$, $1/2 < y < 1/2 + 1/2^{2m+1}$, for some non-negative integer m, and let $V = X$. Let A_{2m} be the set of all points (x, y) such that $1/2^{2m+1} \le x \le 1/2^{2m}$, $0 \le y \le 1/2$. The sets $A_0, A_2, A_4, \ldots$ are shown by the diagonal shading in fig. (5.1). Let $A_{2m+1} = \emptyset$. Let E_n be the union of A_n and all the points $(x, y) \in X$ such that $x \ge 3/2^{n+2}$, and let $B_n = (\mathrm{Fr}\, E_n) - A_n$. The sets $B_0, B_1, B_2, \ldots$ are shown in fig. (5.1) by the thick vertical lines.

Then $B_{2m} \to \{0\} \times [1/2, 1]$, while $\lim B_{2m+1} \to \{0\} \times [0, 1]$, and so $\lim B_n$ does not exist.

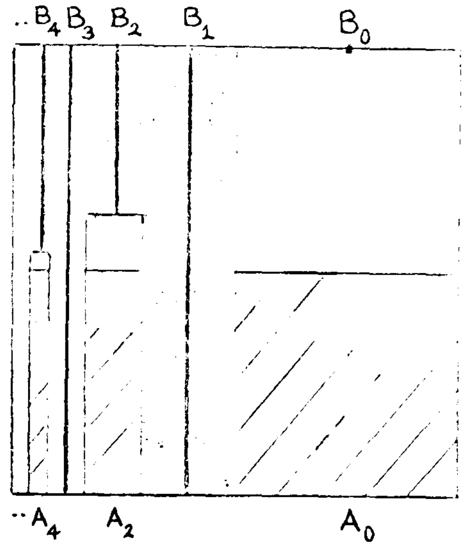

fig. (5.1)

We shall say that X is <u>an S-space</u> if X is a Peano space and if for each true cyclic element C of X, there is a base of regions $\{U_\alpha\}$ for the subspace C such that $cl_C\ U_\alpha$ is weakly finitely multicoherent modulo $Fr_C\ U_\alpha$. It is easily shown that we can without loss of generality take $C - U_\alpha$ to be connected for each U_α in C.

THEOREM (5.2). *If $f : X \to Y$ is a connectivity function (or local connectivity function), where X is a cyclic S-space and Y is a regular space, then f is peripherally continuous.*

PROOF. Let p be a point in X, and let U and V be neighbourhoods of p and $f(p)$, respectively, where

U is a conditionally compact region such that $X - U$ is connected and X is weakly finitely multicoherent modulo $X - U$. (In addition select U so small that $f|\overline{U} : \overline{U} \to Y$ is a connectivity function in case f is only a local connectivity function). The existence of the region U follows from lemma (3.1). It is required to show that there is a neighbourhood W of p such that $W \subset U$ and $f(\mathrm{Fr}\, W) \subset V$.

<u>The sets</u> U_1 <u>and</u> V_1. Let V_1 be a neighbourhood of $f(p)$ such that $\overline{V}_1 \subset V$. Let U' be a region about p such that $\overline{U}' \subset U$ and $X - U'$ is connected. Such a region exists by lemma (3.1). Then $\overline{U}' \cap f^{-1}(\overline{V}_1)$ is a semi-closed set. Denote by $(X - U')^+$ the union of $X - U'$ and all the components of $\overline{U}' \cap f^{-1}(\overline{V}_1)$ that meet $X - U'$. By lemma (3.9), $(X - U')^+$ is a closed and connected set. Let U_1 be the component of $X - (X - U')^+$ to which p belongs. Then U_1 is a region about p such that $X - U_1$ is connected, and a component of $\overline{U}' \cap f^{-1}(\overline{V}_1)$ which meets U_1 is wholly contained in U_1. It is the components of $U_1 \cap f^{-1}(\overline{V}_1)$ that will interest us.

<u>The notation</u> ∂A. We introduce the following notation. For any subset A of U_1 we shall denote by ∂A the

union of all the components of $U_1 \cap f^{-1}(\overline{V}_1)$ which meet $\operatorname{Fr} A = \overline{A} - \operatorname{int} A$.

The notation $\lceil A \rceil$. We also introduce this notation. Let A be any closed subset of X lying in U_1. We shall denote by $\lceil A \rceil$ the union of A and all the components of $X - A$ except the one containing the non-empty connected set $X - U_1$. Thus $\lceil A \rceil$ is a closed subset of X which is contained in U_1 and does not disconnect X. Further, if A and B are any two closed subsets of X which lie in U_1, then we have the following simple relations, the third of which is a consequence of the first two:

(a) $\lceil \lceil A \rceil \rceil = \lceil A \rceil$,

(b) if $A \subset B$ then $\lceil A \rceil \subset \lceil B \rceil$,

(c) $\lceil \lceil A \rceil \cup \lceil B \rceil \rceil = \lceil A \cup B \rceil$.

The enclosures. For each finite number of components $F_1, F_2, \ldots, F_m$ of $U_1 \cap f^{-1}(\overline{V}_1)$, we shall call $\lceil F_1 \cup F_2 \cup \ldots \cup F_m \rceil$ an enclosure if it is connected.

Now let $E = \lceil F_1 \cup F_2 \cup \ldots \cup F_m \rceil$ be an enclosure, where $F_1, F_2, \ldots, F_m$ are components of $U_1 \cap f^{-1}(\overline{V}_1)$. Then $\operatorname{Fr} E \subset \partial E \subset F_1 \cup F_2 \cup \ldots \cup F_m$. Thus $E = \lceil \partial E \rceil$. In future, when we express an enclosure E as $E = \lceil F_1 \cup F_2 \cup \ldots \cup F_m \rceil$, where $F_1, F_2, \ldots, F_m$ are components of $U_1 \cap f^{-1}(\overline{V}_1)$, we shall always assume that $\partial E = F_1 \cup F_2 \cup \dots \cup F_m$.

Equivalence relation on the enclosures. We set up an equivalence relation on the collection of enclosures as follows. We say that two enclosures E_1 and E_2 are equivalent, written $E_1 \sim E_2$, if there is a third enclosure E_3 such that $E_1 \cup E_2 \subset E_3$. That this relation is reflexive and symmetric is clear. In order to prove that it is transitive, let $E_1 \sim E_2$ and $E_2 \sim E_3$. Then there are enclosures E_4 and E_5 such that $E_1 \cup E_2 \subset E_4$ and $E_2 \cup E_3 \subset E_5$. Consider $[E_4 \cup E_5]$. It is a connected set and, by (c),

$$\begin{aligned} [E_4 \cup E_5] &= [[\partial E_4] \cup [\partial E_5] \;, \\ &= [\partial E_4 \cup \partial E_5] . \end{aligned}$$

Thus $[E_4 \cup E_5]$ is an enclosure. But $E_1 \cup E_3 \subset [E_4 \cup E_5]$, by (b). Thus $E_1 \sim E_3$. This completes the proof that the relation on the enclosures is an equivalence relation.

Chains. Let $\mathcal{E}$ be an equivalence class of enclosures. Let

$$\begin{aligned} H(\mathcal{E}) &= \bigcup\{E : E \in \mathcal{E}\} , \\ G(\mathcal{E}) &= \bigcup\{E - \partial E : E \in \mathcal{E}\} . \end{aligned}$$

We shall call $H(\mathcal{E})$ <u>a chain</u>. In the case where $\mathcal{E}$ contains a maximal element with respect to inclusion, $H(\mathcal{E})$ will be an enclosure.

<u>Properties of an equivalence class with no maximal element with respect to inclusion</u>. Let $\mathcal{E}$ be an equivalence class with no maximal element with respect to inclusion. For the purposes of this section, write $H = H(\mathcal{E})$ and $G = G(\mathcal{E})$.

We first show that $H - G$ <u>has only a finite number of components</u>. Suppose, on the contrary, that from the collection of components of $H - G$ we can select an infinite sequence of distinct elements $F_0, F_1, \ldots$. We select a sequence of elements $E_0 \subset E_1 \subset \ldots$ from $\mathcal{E}$ as follows. Let E_0 be any element in $\mathcal{E}$ such that $F_0 \subset \partial E_0$. Suppose now that E_n has been selected. Since $G \cap \partial E_n$ is a union of components of ∂E_n (for if F is a component of ∂E_n and $F \cap (H - G) \neq \phi$, then $F \cap (E - \partial E) = \phi$ for each $E \in \mathcal{E}$; i.e., $F \subset H - G$) we can find a set E_{n+1} in $\mathcal{E}$ such that $E_n, F_n \subset E_{n+1}$, $\partial E_n \cap G \subset E_{n+1} - \partial E_{n+1}$. Now put

$$\begin{aligned} A_0 &= E_0 \cap (H - G), \\ A_{n+1} &= (E_{n+1} - E_n) \cap (H - G), \\ B_n &= \partial E_n - A_1 \cup A_2 \cup \ldots \cup A_n. \end{aligned}$$

Then the hypotheses of theorem (5.1)′ are satisfied, but $A_n \neq \phi$ for an infinite number of values of n, which is false.

Next we construct a sequence $E_0 \subset E_1 \subset \ldots$ of elements from $\mathcal{E}$ such that

$$H - G \subset \partial E_0 ,$$
$$G \cap \partial E_n \subset E_{n+1} - \partial E_n ,$$
$$G = \bigcup_{n=0}^{\infty} (E_n - \partial E_n) .$$

To do this, let $R_0, R_1, \ldots$ be a sequence of regions the union of whose closures is equal to G. Since $H - G$ has only a finite number of components, we can find an element E_0 in $\mathcal{E}$ such that $H - G \subset \partial E_0$. So suppose that E_n has been selected. We select E_{n+1} as follows. $G \cap \partial E_n$ is a union of components of ∂E_n, as was pointed out in the preceding paragraph. Thus $\bar{R}_n \cup (G \cap \partial E_n)$ is a compact subset of G, and so the collection of open sets $\{E - \partial E : E \in \mathcal{E}\}$ contains a finite subcollection $E^1 - \partial E^1, E^2 - \partial E^2, \ldots, E^m - \partial E^m$ whose union covers $\bar{R}_n \cup (G \cap \partial E_n)$. There is now an element E_{n+1} in $\mathcal{E}$ which contains $E^1, E^2, \ldots, E^m, E_n$. It is then clear that the sequence $E_0, E_1, \ldots$ has the desired properties.

Let us now write

$$A_n = \begin{cases} H - G & \text{for } n = 0\,, \\ \phi & \text{for } n > 0\,, \end{cases}$$

$$B_n = G \cap \partial E_n\,.$$

Then we have the hypotheses of theorem (5.1)$'$ satisfied. Thus, in particular, $\lim B_n$ exists and has a finite number of components. We shall denote these components by $L_1, L_2, \ldots, L_s$.

Now we show that, if n is sufficiently large, then $\overline{G} - (E_n \cap G)$ has precisely s components $M_1, M_2, \ldots, M_s$ (these components depend on n, of course), where $M_i \supset L_i$ and $\overline{M}_i \cap B_n \neq \phi$, for $i = 1,2,\ldots,s$. Such a set E_n will be called a special set. To show this, let $S_1, S_2, \ldots, S_s$ be neighbourhoods of $L_1, L_2, \ldots, L_s$, respectively, with mutually disjoint closures. Then there is an integer m such that $B_n \subset S_1 \cup S_2 \cup \ldots \cup S_s$ for each $n > m$. Now we notice that for only a finite number of values of $n > m$, say $n = m_1, m_2, \ldots, m_r$, is there a component of $(E_{n+1} - B_{n+1}) - E_n$ which meets more than one of the sets $S_1, S_2, \ldots, S_s$. If we now take $n \geq m_1, m_2, \ldots, m_r$, then E_n is easily shown to have the properties required of a special set.

Now we show that L_i is either a single point in the complement of $U_1 \cap f^{-1}(\overline{V}_1)$, or is a subset of $X - U_1$. In order to do this, we first observe that there are at most a finite number of points $x_1, x_2, \ldots, x_v$ in $\lim B_n - \overline{U}' \cap f^{-1}(\overline{V}_1)$. For suppose x is a point in $\lim B_n - \overline{U}' \cap f^{-1}(\overline{V}_1)$. Then it is possible to select a component F_n of B_n so that $x \in \liminf F_n$. But it then follows that $\{x\} = \lim F_n$, for otherwise some convergent subsequence $\{F_{n_k}\}_{k=0,1,\ldots}$ of $\{F_n\}_{n=0,1,\ldots}$ can be chosen which has a non-degenerate limit, and this limit consequently lies in a component of $\overline{U}' \cap f^{-1}(\overline{V}_1)$; this, however, contradicts the fact that $x \notin \overline{U}' \cap f^{-1}(\overline{V}_1)$. That there can now be only a finite number of points $x_1, x_2, \ldots, x_v$ in $\lim B_n - \overline{U}' \cap f^{-1}(\overline{V}_1)$ follows from the fact, given in corollary (1) to lemma (5.9), that the supremum of the number of components of B_n is finite.

Now we show that each component L_i of $\lim B_n$ which meets $U_1 \cap f^{-1}(\overline{V}_1)$ is actually contained in $U_1 \cap f^{-1}(\overline{V}_1)$, and so is contained in a component of $U_1 \cap f^{-1}(\overline{V}_1)$. For let L_i be a component of $\lim B_n$ which meets $U_1 \cap f^{-1}(\overline{V}_1)$ and suppose that $x_j \in L_i - \overline{U}' \cap f^{-1}(\overline{V}_1)$, for some $j \leq v$. Let N be a neighbourhood of x_j such that $L_i \not\subset \overline{N}$ and $x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_v \notin \overline{N}$. Let C be the component of $L_i \cap \overline{N}$ to which x_j belongs. By lemma (3.8), C meets $\mathrm{Fr}\, N$, and so is a non-degenerate connected set. But the only point of C whose image under f lies in the open set $Y - \overline{V}_1$ is x_j, and this contradicts lemma (5.4) (or, alternatively, it contradicts lemma (5.5a), because Y is a regular space, and so $f(C)$ is not connected).

Suppose now that L_i meets a component F of $U_1 \cap f^{-1}(\overline{V}_1)$. By the preceding paragraph, L_i then lies in F. Also $F \subset H - G$. Let E_n be a special set and let L_i lie in a component M_i of $\overline{C} - (E_n \cap G)$. Let C_i be a component of $M_i - L_i$. Since $F \subset \partial E_n$, it now follows that E_n separates X, for $\partial C_i \subset \partial E_n$ and $C \not\supset X - U_1$. This is a contradiction, and it proves the assertion that L_i is either a single point in the complement of $U_1 \cap f^{-1}(V_1)$ or a subset of $X - U_1$.

Finally, since $\overline{H} \cap U_1 = H \cup \bigcup\{L_i : L_i \subset U_1\}$, and since L_i is a single point in the complement of $U_1 \cap f^{-1}(\overline{V}_1)$ whenever $L_i \subset U_1$ (as we have just shown), <u>we notice that</u> $\partial H = H - G$.

<u>The union of a finite number of chains does not separate</u> X. Let $\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_r$ be distinct equivalence classes. We shall assume that $\mathcal{E}_k$ has no maximal element with respect to inclusion if and only if $k \leq q$ $(q \leq r)$. We write $H_k = H(\mathcal{E}_k)$, and it is required to prove that $X - H_1 \cup H_2 \cup \ldots \cup H_r$ is connected.

We suppose that the contrary is the case, so that

$$X - H_1 \cup H_2 \cup \ldots \cup H_r = P \cup Q,$$

where P, Q are disjoint non-empty separated sets. For $k \leq q$, let E_{k,n_k} be a special set, and denote the components of $\overline{G}_k - (E_{k,n_k} \cap G_k)$ by $M_{k,1}, M_{k,2}, \ldots, M_{k,s_k}$ $(G_k = \bigcup\{E - \partial E : E \in \mathcal{E}_k\})$. For $k > q$, let $E_{k,n_k} = H_k$, which is an enclosure. Let

$$P' = P \cup \bigcup\{M_{k,j} : M_{k,j} \cap P \neq \emptyset\},$$
$$Q' = Q \cup \bigcup\{M_{k,j} : M_{k,j} \cap Q \neq \emptyset\}.$$

Then P', Q' are easily shown to be separated sets, but the complement of their union is $\bigcup_{k=1}^{r} E_{k,n_k}$. This implies that $\bigcup_{k=1}^{r} E_{k,n_k}$ separates X, which is false.

$H(\mathcal{E}) - G(\mathcal{E})$ <u>is disconnected for only a finite number of equivalence classes</u> $\mathcal{E}$. Suppose on the contrary that an infinite number of equivalence classes $\mathcal{E}_0, \mathcal{E}_1, \ldots, \mathcal{E}_m, \ldots$ can be found such that $H(\mathcal{E}_m) - G(\mathcal{E}_m)$ is not connected. Write $H_m = H(\mathcal{E}_m)$, $G_m = G(\mathcal{E}_m)$. For a fixed m, let $E_{m,0} \subset E_{m,1} \subset \ldots$ be a sequence of enclosures such that $H_m - G_m \subset \partial E_{m,0}$ and $G_m = \bigcup_{n=0}^{\infty} (E_{m,n} - \partial E_{m,n})$ (such a sequence of enclosures was described previously).
Let

$$A_k = X - \bigcup\{E_{m,n} - \partial E_{m,n} : m+n \leq k\}.$$

Then A_k is connected, because $X - \bigcup\{E_{m,n} : m+n \leq k\}$ is connected. Thus $A_0 \supset A_1 \supset A_2 \supset \ldots$ is a contracting sequence of connected closed subsets of X. So, by lemma (5.2),

$$\bigcap_{k=0}^{\infty} A_k = X - \bigcup_{n=0}^{\infty} G_m$$

is a connected closed set, and it contains $X - U_1$. Now $H_m - G_m$ is disconnected, and so it follows that there is a component C_m of G_m such that Fr C_n is disconnected. But C_m is a component of $X - \bigcap_{k=0}^{\infty} A_k$, and this contradicts the corollary to lemma (5.6).

<u>The set</u> L. Let us now denote the collection of chains by $\{H_\alpha\}_\alpha$, and let $L = \bigcup_\alpha H_\alpha$.

<u>The components of</u> L <u>are the sets</u> H_α. Let $\{H_{\alpha_\beta}\}_\beta$ be a non-degenerate subcollection of $\{H_\alpha\}_\alpha$. We show that $\bigcup_\beta H_{\alpha_\beta}$ is not connected. Let

$$\bigcup_\beta \partial H_{\alpha_\beta} = M \cup N,$$

where M, N are non-empty, disjoint separated sets such that $\partial H_{\alpha_\beta} \subset N$ whenever $\partial H_{\alpha_\beta}$ is disconnected. Such a separation exists by lemma (5.3). Let

$$M' = M \cup \bigcup\{H_{\sigma_\beta} : \partial H_{\sigma_\beta} \subset M\},$$
$$N' = N \cup \bigcup\{H_{\alpha_\beta} : \partial H_{\alpha_\beta} \subset N\}.$$

Then it follows easily from the local connectedness of the space X that M', N' are separated sets, and so $\bigcup_\beta H_{\sigma_\beta}$ is not connected.

$p \in \operatorname{int} L$. Suppose $p \notin \operatorname{int} L$. Since no finite number of components of L separates X, it follows from lemma (5.1) that L does not separate X. The supposition that $p \notin \operatorname{int} L$ implies that $p \in \overline{X - L}$. But then $(X - L) \cup \{p\}$ is a connected set whose graph meets $U_1 \times V_1$ in the isolated point $(p, f(p))$. This contradicts lemma (5.4).

f <u>is peripherally continuous at</u> p. Since $p \in \operatorname{int} L$ and X is locally connected, it follows that $p \in \operatorname{int} H_\alpha$, for some α. Let $H = H_\alpha = H(\varepsilon_\alpha)$, $G = G_\alpha = G(\varepsilon_\alpha)$. Let $E_0 \subset E_1 \subset \ldots$ be a sequence of enclosures (previously described) such that $H - G \subset \partial E_0$ and $G = \bigcup_{n=0}^{\infty} (E_n - \partial E_n)$. Thus, if $p \in H - G$, then $p \in \operatorname{int} E_0$, while if $p \in G$ then $p \in \operatorname{int} E_n$ for some n. Thus, in either case we find an enclosure E_n such that $p \in \operatorname{int} E_n$. Put $W = \operatorname{int} E_n$. Then $W \subset U_1 \subset U$ and $f(\operatorname{Fr} W) \subset \overline{V}_1 \subset V$. Thus f is peripherally continuous at p.

This completes the proof of theorem (5.2).

REMARK (5.1). It will be noticed that in lemmas (5.8) and (5.9) and theorems (5.1) and (5.2) we have a system of two regions in the space X, which for the purposes of this remark we shall call U_1, U_2. These regions have the properties that $U_1 \supset \overline{U}_2$, $X - U_1$ is connected and X is weakly finitely multicoherent modulo $X - U_1$ (in theorem (5.2) $X - U_2$ is also connected, U_2 being the "U_1" of that theorem). The reason for the use of two regions (instead of one) is that in all these lemmas and theorems we have used the corollary to lemma (5.6). If we had used lemma (5.6) itself, which we could have done, only one region U would have been necessary, with the properties that $X - U$ was connected and X was weakly finitely multicoherent modulo $X - U$.

We conclude this section by making some general remarks.

If we examine the proofs of $Th(P_1(X))$ and $Th(P_2(X))$ (see §1 of this chapter) as given in [23] and [34], respectively, and the proofs of $Th(P_3(X))$ - $Th(P_8(X))$ in this chapter, then we see that it is purported to have been proved that a connectivity function $f : X \to Y$ is peripherally continuous. However, something more than this has actually been proved. In all these theorems the only two properties of the connectivity function $f : X \to Y$ that have been used are these:

(a) for each non-degenerate connected subset C of X, the graph $\Gamma(f|C)$ has no isolated points,

(b) for each closed set F in Y, $f^{-1}(F)$ is semi-closed in X.

Since (a) implies (b) (if X and Y have the appropriate properties), it is seen that what has actually been proved in $Th(P_1(X)) - Th(P_8(X))$ is that a function $f : X \to Y$ which has property (a) is peripherally continuous. Since property (a) is hardly interesting in itself, it may be wondered whether we cannot assign some more satisfactory property (or properties) to f and still draw the conclusion that f is peripherally continuous. The following considerations, culminating in theorem (5.3), show that this can indeed be done.

LEMMA (5.10). _Let_ G _be a region in a connected and locally connected space_ X, _and_ E _a component of_ $X - G$. _Then_ $X - E$ _is connected_.

PROOF. Let F be a component of $X - G$ which is different from E. Let R be the component of $X - E$ which contains F. Then $R \neq F$, for otherwise X would not be connected.

Thus $R \not\subset X - G$, for otherwise F would not be a component of $X - G$. Thus $R \cap G \neq \phi$. But this implies that F, G are contained in the same component of $X - E$, and this holds for all F. Thus $X - E$ is connected.

LEMMA (5.11). *Let* U, V *be conditionally compact simple regions in a Peano space* X *such that* $\overline{U} \subset V$ *and* X *is weakly finitely multicoherent modulo* $X - V$. *Let* L *be a subset of* U *such that the components of* L *are closed subsets of* X *and no finite number of components of* L *separates* X. *Then* $U - L$ *has only a finite number of components*.

PROOF. Suppose on the contrary that $U - L$ has an infinite number of components. Then there are two disjoint open subsets G_1, G^2 of U such that $U - L \subset G_1 \cup G^2$, $(U - L) \cap G_1 \neq \phi$, $(U - L) \cap G^2 \neq \phi$. We may suppose without loss of generality that G^2 contains more than one component of $U - L$. Thus there are two disjoint open subsets G_2, G^3 of G^2 such that $(U - L) \cap G^2 \subset G_2 \cup G^3$, $(U - L) \cap G_2 \neq \phi$, $(U - L) \cap G^3 \neq \phi$. Again we may suppose without loss of generality that G^3 contains more than one component of $U - L$. Thus there are two disjoint open subsets G_3, G^4 of G^3 such that

$(U - L) \cap G^3 \subset G_3 \cup G^4$, $(U - L) \cap G_3 \neq \emptyset$, $(U - L) \cap G^4 = \emptyset$. Continuing in this way, we define a sequence $G_1, G_2, \ldots, G_n, \ldots$ of non-empty disjoint open subsets of U, and we notice that $U \cap \operatorname{Fr} G_n \subset L$.

Now let G_n' be a component of G_n. Then $U \cap \operatorname{Fr} G_n' \subset U \cap \operatorname{Fr} G_n \subset L$. Also $\overline{G}_n' \cap (X - U) \neq \emptyset$ for otherwise L would separate X (which is impossible, because no finite number of components of L separate X; see lemma (5.1)). Let E_n be the component of $X - G_n'$ which contains the connected set $X - U$. Let $H_n = X - E_n$.

By lemma (5.10), H_n is a region in U such that $U \cap \operatorname{Fr} H_n \subset U \cap \operatorname{Fr} G_n' \subset L$. Also $\overline{H}_n \cap (X - U) \neq \emptyset$. We show that $H_n \cap H_m = \emptyset$ for $n \neq m$. To see this consider the following. G_m' is disjoint from G_n' and is not separated from $X - U$. Therefore $G_m' \subset E_n$. Now suppose that there is some component E of $X - G_m'$ such that $E \neq E_m$ and $E \cap H_n \neq \emptyset$. Since E is closed and is not separated from G_m', $E \cap E_n \neq \emptyset$. Since H_n is the union of G_n' and the collection of all the components of $X - G_n'$ except E_n, it follows that $E \cap G_n' \neq \emptyset$. Now E is a component of $X - G_m'$ and so $G_n' \subset E$. But E is closed, and this implies that $E \cap (X - U) \neq \emptyset$; that is, $E = E_n$, which is false.

Now we show that $U \cap \operatorname{Fr} H_n$ is a closed subset of X.

Since both $\overline{H}_n$ and $X - H_n$ are connected closed subsets of X, and $\overline{H}_n \subset V$ and X is weakly finitely multi-coherent modulo $X - V$, it follows that $\mathrm{Fr}\, H_n = \overline{H}_n \cap (X - H_n)$ has only a finite number of components, which we shall denote by C_i, $i = 1, 2, \ldots, p$. Recall that $U \cap \mathrm{Fr}\, H_n \subset L$, and let C_i meet U. We prove that $C_i \subset U$. Suppose that this is not the case, and let D_i be a component of $C_i - (X - U)$. Then by Janiszewski's border theorem, $\overline{D}_i \cap (X - U) \neq \phi$. But this means that D_i is not a closed set, and this implies that the component of L in which D_i is contained is not closed, which is false. This shows that $U \cap \mathrm{Fr}\, H_n$ is equal to the union of a finite number of components of $\mathrm{Fr}\, H_n$, and so is a closed subset of X.

Now let σ_n be the union of a finite number of arcs lying in $\overline{H}_n \cap U$ such that each component of $U \cap \mathrm{Fr}\, H_n$ is joined to each other component of $U \cap \mathrm{Fr}\, H_n$ by an arc in α_n. Let R_n be a relatively open subset of the subspace $\overline{H}_n$ such that R_n contains $\overline{H}_n \cap (X - U)$ and contains no point whose distance from $\overline{H}_n \cap (X - U)$ is $\geq 1/2^n$, R_n is connected relative to $\overline{H}_n \cap (X - U)$, and $\overline{R}_n \cap (\alpha_n \cup (U \cap \mathrm{Fr}\, H_n)) = \phi$. Such a set R_n exists by lemma (5.7). Let M_n be the union of $\overline{R}_n$ and all the components of $\overline{H}_n - \overline{R}_n$ that do not meet $\alpha_n \cup (U \cap \mathrm{Fr}\, H_n)$. Let $F = (X - U) \cup \bigcup_{n=1}^{\infty} M_n$. Then it follows that F is a closed and connected set which contains $X - U$. But each of the sets $H_n \cap \operatorname{Fr} R_n$ is a non-empty, relatively open and closed proper subset of $\operatorname{Fr} F$, and each component of $X - F$ whose frontier meets $H_n \cap \operatorname{Fr} R_n$ has a disconnected frontier (since $\overline{H}_n$ does not separate X). This contradicts the corollary to lemma (5.6).

We shall say that a function $f : X \to Y$ is <u>pseudo-continuous</u> if for each closed subset F of Y, $f^{-1}(F)$ is a semi-closed subset of X (we assume that X has a countable open base -- see definition of semi-closed set in §2 of this chapter).

THEOREM (5.3). *Let* X *be a cyclic S-space and* Y *a regular space. Let* $f : X \to Y$ *be a pseudo-continuous and connectedness preserving function. Then* f *is peripherally continuous.*

PROOF. In order to prove this theorem, we indicate the changes that have to be made to the proof of theorem (5.2). Thus we adopt the notation of the proof of theorem (5.2).

In place of the semi-closed set $\overline{U}' \cap f^{-1}(\overline{V}_1)$, we work with the semi-closed set $\overline{U}' \cap f^{-1}(\operatorname{Fr} V_1)$, which does not contain p, and we form the enclosures using the components of this set. Thus, wherever $\overline{U}' \cap f^{-1}(\overline{V}_1)$ and

and its components were used in the proof of theorem (5.2), we now use $\overline{U}' \cap f^{-1}(\mathrm{Fr}\ V_1)$ and its components.

The proof is now identical to the proof of theorem (5.2) down to the second last paragraph. The changes that then have to be made are these. The components of L are not necessarily closed sets. But since X is cyclic, and $\mathrm{Fr}\ H$ is disconnected for only a finite number of chains, it follows that there are only a finite number of chains $H_1, H_2, \ldots, H_r$ which are not closed. We assume that $p \notin \bigcup_{i=1}^{r} G_i$. Let $X' = X - \bigcup_{i=1}^{r} G_i$, $U' = X' \cap U_1$ and $L' = X' \cap L$. Then X' is still weakly finitely multicoherent modulo $X' - X' \cap V = X - V$, and the components of L' are closed subsets of X' no finite number of which disconnect X'. Thus, by lemma (.), $U_1' - L'$ has only a finite number of components $C_1', C_2', \ldots, C_t'$, and we may suppose that $p \in C_1$. Since f preserves connectedness, by lemma (5.5a), it follows that $f(C_1') \subset V_1$. In the space X', let W' be a neighbourhood of p such that $\overline{W}' \cap (C_2 \cup \ldots \cup C_t) = \emptyset$ and $\overline{W}' \subset U_1'$. Let $(X' - W')^+$ be the union of $X' - W'$ and all the components of L' that meet $X' - W'$. Then $(X' - W')^+$ is a closed set in X'. Let $W'' = X' - (X' - W')^+$. Then $\mathrm{Fr}_{X'}\ W'' \subset (U_1' \cap f^{-1}(\mathrm{Fr}\ V_1)) \cup C_1'$, and so $f(\mathrm{Fr}_{X'}\ W'') \subset \overline{V}_1$, but W'' may not be open in X. Thus let E_i be a special set for the chain H_i, for

$i = 1, 2, \ldots, r$. If W'' contains a point in $\overline{H}_i - E_i$, add to W'' the component of $\overline{H}_i - E_i$ containing that point. Denote the set formed in this way by W, which is then an open neighbourhood of p in X. Further, $\mathrm{Fr}\, W \subset (U_1 \cap f^{-1}(\mathrm{Fr}\, V_1)) \cup C_1$, and so $f(\mathrm{Fr}\, W) \subset \overline{V}_1 \subset V$. This proves the theorem.

The results of §5 can be summed up in the diagram of fig. (5.2), where the arrows represent implications.

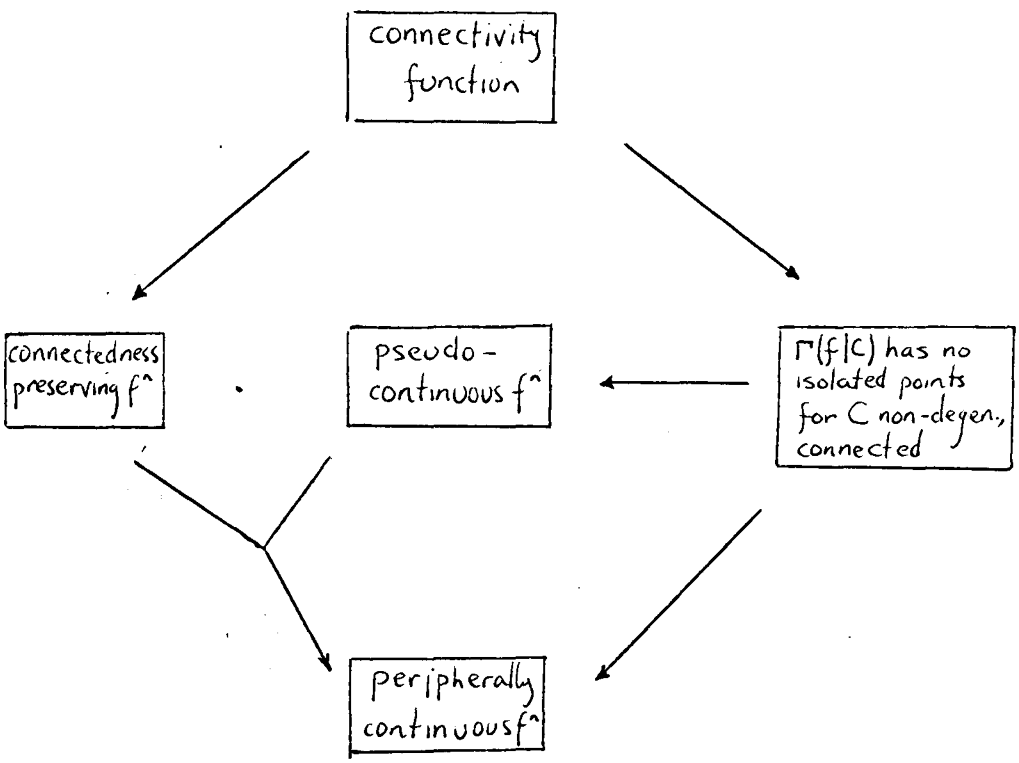

fig. (5.2)

In this diagram, the implications (i) (ii) and (iii) are given by lemmas (5.4), (5.5)[1] and (5.5a), respectively. The implication (iv) is theorem (5.2), and the implication (v) is theorem (5.3). What makes (v) interesting is that there is evidence to believe that the implication (vi), marked by a dotted line, also holds (see the remark on pseudo-continuous functions in §1). If this is the case, then theorem (5.3) is a considerable improvement over theorem (5.2).

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[1] The only property of the connectivity function $f : X \to Y$ that is used in the proof of lemma (5.5) is that for each non-degenerate connected set C, $\Gamma(f|C)$ has no isolated points.

CHAPTER 4

A COUNTER EXAMPLE ON UNICOHERENT PEANO SPACES

1. INTRODUCTION. In this chapter we give an example of a sequence of disjoint closed sets $A_1, A_2, \ldots$ in a unicoherent Peano space X such that $X - A_n$ is connected for each n, and yet $X - \bigcup_{n=1}^{\infty} A_n$ is not connected. This example is described in §3, and in §4 it is proved that it has the stated properties. In §5 we raise a question which arises from this example and the paper of van Est [28]. In §2 we explain the significance of the example.

2. A Peano space is a locally compact, connected and locally connected metric space. A Peano continuum is a compact Peano space. A connected space is said to be unicoherent if however it is expressed as the union of two connected closed subsets A and B, $A \cap B$ is always connected. We then have the following well-known theorem:

If X is a unicoherent Peano continuum and $A_1, A_2, \ldots$ is a sequence of disjoint closed subsets of X no one of which separates X, then $\bigcup_{n=1}^{\infty} A_n$ does not separate X.

This theorem has also been proved for certain non-compact unicoherent Peano spaces. In 1923 Miss mullikin proved it in [20] for the case in which X is the plane (this proof was considerably simplified in 1924 by Mazurkiewicz in [16]), and in 1952 van Est proved it in [28] for the case in which X is a Euclidean space of any (finite) dimension. Our example shows that the theorem does not hold when X is an arbitrary Peano space.

The proof of the theorem that has been quoted was shown to me by Dr. G.T. Whyburn, and runs briefly as follows. If on the contrary $\bigcup_{n=1}^{\infty} A_n$ separates X, then it follows from the unicoherence of X that some subset F of $\bigcup_{n=1}^{\infty} A_n$ which is closed and connected in X also separates X. But now F is a continuum which can be decomposed into the sequence of disjoint closed sets $A_1 \cap F$, $A_2 \cap F$, ..., and this contradicts Sierpenski's theorem on continua (see p.113 of [14] or p.16 of [31]).

So in trying to construct our example, we look for an example of a locally compact connected space which can be decomposed into a sequence of disjoint closed sets. Such space was given by Kuratowski on p.115 of [14]. As it is the essential feature in the construction of our example, we begin §3 by describing this space of Kuratowski.

3. In the Euclidean plane let A_n consist of the points (x, y) which satisfy one of the following conditions:

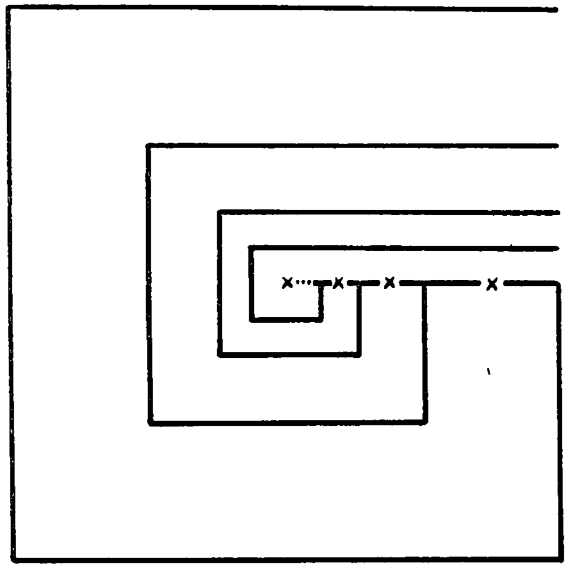

fig. (1)

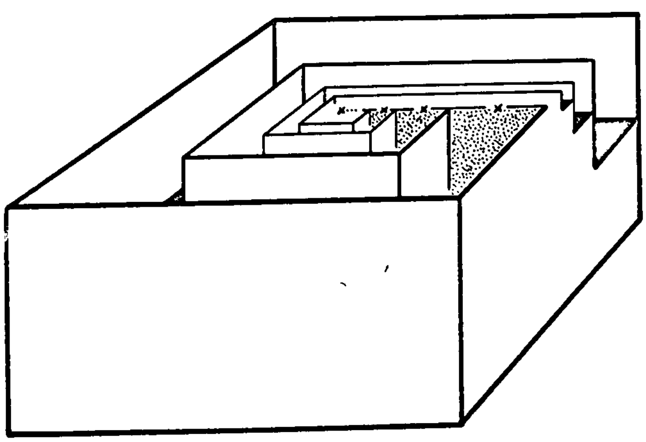

fig. (2)

(i) $-1/2^n \le x \le 1,\ y = 1/2^n$,

(ii) $x = -1/2^n,\ -1/2^n \le y \le 1/2^n$,

(iii) $-1/2^n \le x \le 1/2^n,\ y = -1/2^n$,

(iv) $x = 1/2^n,\ -1/2^n \le y \le 0$,

(v) $3/2^{n+2} < x < 3/2^{n+1},\ x \le 1,\ y = 0$.

Then the set $A = \bigcup_{n=0}^{\infty} A_n$ is the space given by Kuratowski in [14]. It is shown in fig. (1), where the crosses indicate the points on the line segment $0 \le x \le 1,\ y = 0$ which are not in A.

In order to describe the space of our counter example, we identify the point (x, y) in the Euclidean plane with the point $(x, y, 0)$ in Euclidean 3-space, of which the set A therefore becomes a subset.

Let B_n be the component of $[-1, 1] \times [-1, 1] \times \{0\} - \bar{A}$ whose frontier lies in $\bar{A}_{n-1} \cup \bar{A}_n$, for $n = 1, 2, 3, \ldots$. We define a set Y' by subtracting from the cube $[-1, 1] \times [-1, 1] \times [0, -1]$ the two sets $\bigcup_{n=1}^{\infty} B_n \times [0, -1/2^n)$ and $[0, 1] \times \{0\} \times \{0\} - A$. The set Y is shown in fig. (2).

Let Z be the reflection of Y in the plane $z = 0$, and let $X = Y \cup Z$. The space X is our counter example.

4. It is clear that X is a Peano space in which $A_0, A_1, \ldots$ is a sequence of disjoint closed sets such that $X - A_n$ is connected for each n and yet $X - \bigcup_{n=0}^{\infty} A_n$ is not connected.

Thus it remains only to show that X is unicoherent.

In order to do this we shall quote three theorems which can be found with small changes in chap. XI of [31]. We first make two definitions.

We denote by S^1 the circle of complex numbers of unit modulus. We say that a space X is <u>contractible with respect to</u> S^1 if each mapping $f : X \to S^1$ is homotopic to the constant mapping from X into S^1. We say that a space X has <u>property (b)</u> if for each mapping $f : X \to S^1$ there is a real-valued mapping φ on X such that $f(x) = \exp[i\,\varphi(x)]$ for each $x \in X$. The first of these definitions may be found in [14]; the second in [31].

We then have the following three theorems, in which it is assumed for convenience that the spaces in question are separable and metric.

THEOREM 1. *A space* X *is contractible with respect to* S^1 *if and only if it has property (b).*

THEOREM 2. *Let* X_1 *and* X_2 *be closed subsets of their union* $X = X_1 \cup X_2$ *such that* $X_1 \cap X_2$ *is connected. Then if* X_1 *and* X_2 *both have property (b), so does* X.

THEOREM 3. *A connected space* X *which has property (b)* is unicoherent.

Now we show that the space X of §3 is unicoherent. We notice that Y has this property: if (x, y, z) belongs to Y so do all the points on the line segment joining (x, y, z) and $(x, y, -1)$. From this it follows that the square $[-1, 1] \times [-1, 1] \times \{-1\}$ is a deformation retract of Y, and so Y is contractible. Therefore Y is contractible with respect to S^1, and so Z is as well. Thus, by theorem 1, both Y and Z have property (b). Since Y and Z are closed subsets of X and $Y \cap Z = A$, it now follows from theorem 2 that X has property (b). Thus, by theorem 3, X is unicoherent.

5. We have seen that the theorem of §2 does not hold for an arbitrary Peano space, and yet it does hold for some non-compact Peano spaces, as has been shown by Miss Mullikin and van Est in [20] and [28], respectively. These considerations lead us to seek a precise analytical definition of the class of unicoherent Peano spaces for which the theorem of §2 holds.

We notice that the space X of §3 has this property: some of its points (namely those of the form $x \neq 3/2^{n+1}$, for $n = 1, 2, 3, \ldots,$ $y = 0$, $z = 0$) do not lie in unicoherent regions with compact closures. Since the Euclidean spaces (and likewise the locally Euclidean spaces) do not suffer from this deficiency, we are prompted to ask:

QUESTION. Let X be a unicoherent Peano space which has a covering by unicoherent regions with compact closures. If $A_1, A_2 \ldots$ is a sequence of disjoint closed sets no one of which separates X, is $X - \bigcup_{n=1}^{\infty} A_n$ necessarily connected?

If this fails we can try imposing stronger conditions on the unicoherent regions that cover X. We can for example demand that their closures be unicoherent Peano continua.

CHAPTER 5

A NEW PROOF OF LEBESGUE'S COVERING LEMMA

1. INTRODUCTION. Lebesgue's covering lemma states that, given an open covering $U_1, \ldots, U_n$ of a compact metric space X, ρ, there is a positive number δ such that if $\rho(x, y) < \delta$ then both x and y belong to some U_i. The purpose of this short note is to enlarge upon this conclusion and thereby provide a more interesting proof of the lemma than the usual ones.

We first explain how we arrive at the new result. Figure 1 shows a compact metric space covered by two open subsets U and V. If δ is the distance between $U - V$ and $V - U$, then any two points whose distance apart is less than δ both lie in U or V; further, no number greater than δ will ensure this. Figure 2 shows a compact metric space X, ρ covered by a finite number of open subsets $U_1, U_2, \ldots, U_n$ and one may suspect that the same idea holds. The lines of the figure divide the set X up into a number of "compartments" (those white regions crossed by no lines) and by analogy one may suspect that two of these compartments A and B, at a positive distance apart, have the properties

(i) if $\rho(x, y) < \rho(A, B)$ then both x and y belong to some U_i,

(ii) no number greater than $\rho(A, B)$ has this property.

Except in a trivial case this is so, and it is the extension of Lebesgue's lemma that we shall prove.

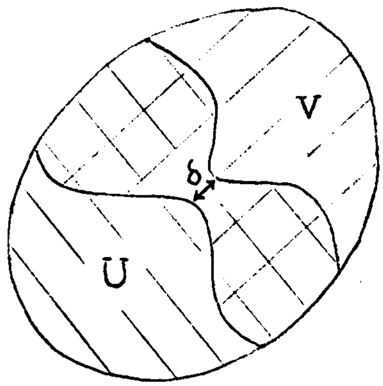

fig. (1)

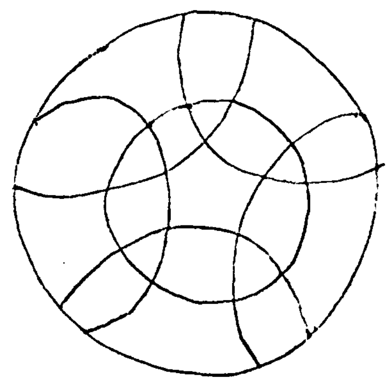

fig. (2)

2. We first notice the trivial exception. When each pair of points is contained in some U_i, no pair of compartments satisfies (ii), because every positive number satisfies (i). In this case, however, Lebesgue's lemma is trivial.

Now we define a "compartment" (in [15] this is called a constituent). Let X be a set covered by a finite number of subsets $X_1, X_2, \ldots, X_n$. A <u>compartment</u> (of the covering $X_1, X_2, \ldots, X_n$) is a non-empty set expressible as the intersection of n distinct sets consisting of X_i's and complements of X_i's.

It follows from the definition that the compartments of a finite covering of X form a finite, disjoint covering of X. Where no confusion arises, we simply speak of compartments, instead of compartments of a particular covering. We do this below.

THEOREM. *If $U_1, \ldots, U_n$ is an open covering of a compact metric space X, ρ, and some pair of points is contained in no U_i, then there are two compartments A and B at a positive distance apart such that*

(i) *if $\rho(x, y) < \rho(A, B)$, then both x and y belong to some U_i,*

(ii) *no number greater than $\rho(A, B)$ has property (i).*

PROOF. The two points contained in no common U_i belong to a pair of compartments contained in no U_i. Thus we may define $\delta = \min \rho(E, F)$, where E and F are any compartments contained in no common U_i. Then δ is attained as the distance between some pair of compartments A and B, and it satisfies the requirements of the theorem.

First, $\delta > 0$. For let E and F be compartments such that $\rho(E, F) = 0$. Then from E and F we can select sequences $\{x_i\}$ and $\{y_i\}$ such that $\rho(x_i, y_i) \to 0$. By compactness, there is a point x and a subsequence $\{x_{N_i}\}$ of $\{x_i\}$ such that $x_{N_i} \to x$. Since $y_{N_i} \to x$ as well, x belongs to both $\bar{E}$ and $\bar{F}$. But x belongs to some open U_k. Thus U_k meets both E and F and so, by the definition of compartment, contains both E and F.

Also δ satisfies (i) and (ii). For let $\rho(x, y) < \delta$. Then x and y belong to compartments E and F. If $E = F$ then both x and y necessarily belong to some common U_i, because each compartment is contained in some U_i. If $E \neq F$ then $\rho(E, F) < \delta$ and some U_i contains both E and F. Thus some U_i contains both x and y. On the other hand, if $\delta' > \delta$ then there are two compartments E and F, contained in no common U_i, such that $\rho(E, F) < \delta'$. In E and F we can select points x and y such that $\rho(x, y) < \delta'$. Then x and y belong to no common U_i since otherwise U_i would contain both E and F.

3. The above theorem has a simple formulation in terms of Lebesgue numbers.

Let $U_1, U_2, \ldots, U_n$ be a finite open covering of a compact metric space X, ρ. We shall call $\delta > 0$ <u>the Lebesgue number</u> of the covering $U_1, U_2, \ldots, U_n$ if

(i) $\rho(x, y) < \delta$ implies both x and y belong to some U_i,

(ii) no number greater than δ satisfies (i).

If every positive number satisfies (i) we shall say that the Lebesgue number of the covering is infinite.

It is then a trivial conclusion that the Lebesgue number is infinite if and only if each pair of points lies in some U_i. Thus our interest is turned to the case where some pair of points lies in no U_i.

THEOREM (alternative form). *If* $U_1, U_2, \ldots, U_n$ *is a finite open covering of a compact metric space* X, ρ *such that some pair of points is contained in no* U_i, *then the Lebesgue number* δ *of the covering* $U_1, U_2, \ldots, U_n$ *is given by* $\delta = \min \rho(E, F)$, *where* E *and* F *are any compartments contained in no common* U_i.

CHAPTER 6

A THEOREM ON INVOLUTIONS ON CYCLIC PEANO SPACES

The purpose of this note is to prove that an involution f on a cyclic Peano space S leaves some simple closed curve in S setwise invariant.

We shall first define the required terms. A _Peano space_ is a locally compact, connected and locally connected metric space. A connected space is called _cyclic_ if it has no cut point. An _involution_ on a space is a periodic mapping whose period is 2; it is necessarily a homeomorphism. A mapping $f : X \to X$ is said to leave a subset E of S _setwise invariant_ if $f(E) = E$. These definitions may be found, for example, in [31].

We shall use the following lemma, which is a variation of lemma 1 of [30].

LEMMA. _If_ U, V _are disjoint non-empty open sets in a cyclic Peano space_ S, _then there are two disjoint arcs_ ab, cd _in_ S _such that_ $a, c \in A$ _and_ $b, d \in B$.

An arc whose end points are a, b will generally be denoted by ab. If A and B are closed sets, we say that ab _is an arc from_ A _to_ B if $ab \cap A = \{a\}$ and $ab \cap B = \{b\}$.

THEOREM. An involution f on a cyclic Peano space S leaves some simple closed curve in S setwise invariant.

PROOF. Let f be an involution on a cyclic Peano space S. Since $f(x) \neq x$ for some point x in S, it follows that there is a non-empty region R in S such that $R \cap f(R) = \emptyset$. By the lemma, there are two disjoint arcs ab and cd in S such that $a, c \in R$ and $b, d \in f(R)$.

In the first case suppose that one of these arcs is disjoint from its image, say $ab \cap f(ab) = \emptyset$. Let pq be an arc in R from ab to $f(ab)$. Then f leaves the simple closed curve $pq \cup qf(p) \cup f(pq) \cup f(q)p$ setwise invariant, where $qf(p) \subset f(ab)$ and $f(q)p \subset ab$.

In the second case suppose that both of the arcs meet their images. First consider ab. Let m be the first point on ab in the order a, b such that $am \cap f(am) \neq \emptyset$, where $am \subset ab$. Then $am \cap f(am)$ contains just the points $m, f(m)$. If $m \neq f(m)$ then the subarcs of am and $f(am)$ from m to $f(m)$ form a simple closed curve which is left setwise invariant under f. So suppose that $m = f(m)$. Also, let n be the first point on cd in the order c, d such that $cn \cap f(cn) \neq \emptyset$, and suppose that $n = f(n)$. Then $am \cup f(am)$ and $cn \cup f(cn)$ are setwise invariant arcs under f. If $am \cap f(cn) = \emptyset$, then $am \cup f(am)$ and $cn \cup f(cn)$ are disjoint, and the construction of an arc pq in R from am to cn shows, as in the first case, that there is a simple closed curve which is setwise invariant under f. So suppose that $am \cap f(cn) \neq \emptyset$. Let r be the first point on am in the order a, m which lies on $f(cn)$. Then $r \neq m, n$ so that $ar \cup rf(c)$ and $f(a)\,f(r) \cup f(r)\,c$ are disjoint arcs, where $ar \subset am$, $rf(c) \subset f(cn)$, $f(a)\,f(r) \subset f(am)$ and $f(r)\,c \subset cn$. Further $ar \cup rf(c)$ and $f(a)\,f(r) \cup f(r)\,c$ are images of each other under f, and ar and $f(r)\,c$ both meet R. Thus the construction of an arc pq in R from ar to $f(r)\,c$ again shows that there is a simple closed curve which is left setwise invariant by f.

REMARK. The well-known cyclic connectivity theorem of [30] can be used to prove this theorem, in which case the region R is replaced by a point and the construction of the arc pq in each case becomes unnecessary. But use of the cyclic connectivity theorem does not change the ideas of the proof, and eliminates only the trivial constructions of the arc pq. On the other hand the proof of the cyclic connectivity theorem is based upon the theory of cyclic elements, none of which is required in the above proof. Thus in our proof we have avoided the cyclic connectivity theorem and used only the lemma and in so doing have kept the proof at its most elementary level.

CHAPTER 7

ON R. L. MOORE'S DECOMPOSITION THEOREM

1. INTRODUCTION. The decomposition theorem that R.L. Moore proved in [19] states that _if_ $\mathfrak{D}$ _is a non-degenerate monotone upper semi-continuous decomposition of a 2-sphere_ S _and no element of_ $\mathfrak{D}$ _separates_ S, _then the decomposition space_ $S/\mathfrak{D}$ _is also a 2-sphere_. The proofs of this theorem that appear in the literature all show that the space $S/\mathfrak{D}$ has some properties which it is well-known characterize the 2-sphere. Thus in R.L. Moore's paper [19] it is shown that $S/\mathfrak{D}$ satisfies the eight axioms of [18], which characterize the plane (S is a plane in [19]). In chap. IX of [14] Kuratowski shows that $S/\mathfrak{D}$ is a Janiszewski space, which it is known is homeomorphic to the 2-sphere. In chap. IX of [31] and chap. XVII of [36] Whyburn shows that $S/\mathfrak{D}$ satisfies the hypotheses of Zippin's characterization theorem of the 2-sphere (the argument in [31] has been refined in [36]; in the former it is shown that no arc separates the decomposition space; in the latter it is merely shown that no arc that lies on a simple closed curve separates the decomposition space). (Zippin's theorem on the characterization of the 2-sphere may be found as theorem (5.1) in chap. VI of [31] or as theorem (4.2) in chap. III of [38].)

In this note we give a proof that the decomposition space $S/\mathfrak{D}$ satisfies the hypotheses of Zippin's theorem which is different from those given in [31] and [36]. Our proof follows Alexander's proof of the Jordan curve theorem as given by Newman in [21] very closely, and thus consists of arguments that are already familiar.

2. We first quote the two results from [21] that we shall need. We shall suppose throughout that S is the 2-sphere.

THEOREM 1. *If the common part of two closed subsets* A *and* B *of* S *is connected, then two points which are separated by neither* A *nor* B *in* S *are not separated by* $A \cup B$ *in* S.

THEOREM 2. *If the common part of two connected closed subsets* A *and* B *of* S ~~is connected~~ *has two components, and neither* A *nor* B *separates* S, *then* $S - A \cup B$ has *just two components*.

Theorem 1 is given as theorem (9.2), p.112 of [21], and is an immediate consequence of Alexander's lemma. Theorem 2 is proved for the case where A and B are arcs in the proof of the Jordan curve theorem in [21]. But in this proof the only property of the arc that is used is that it is a continuum which does not separate the plane (or sphere).

The definition of an upper semi-continuous (usc) decomposition is given in chap. VII of [31]. A usc decomposition is *monotone* if each of its elements is a continuum.

THEOREM. *Let* $\mathcal{D}$ *be a non-degenerate monotone usc decomposition of* S *no element of which separates* S. *Then the decomposition space* $S/\mathcal{D}$ *is a Peano continuum which satisfies the hypotheses of Zippin's theorem on the characterization of the 2-sphere; i.e.,* $S/\mathcal{D}$ *satisfies these three properties:*

(a) $S/\mathcal{D}$ *contains at least one simple closed curve*,

(b) *no arc in* $S/\mathcal{D}$ *separates* $S/\mathcal{D}$,

(c) *every simple closed curve in* $S/\mathcal{D}$ *separates* $S/\mathcal{D}$.

PROOF. Since the decomposition space is a Peano continuum with no cut points, it is clear that (a) is satisfied.

In order to prove (b) we suppose that there is an arc α in $S/\mathcal{D}$ which does separate $S/\mathcal{D}$. We denote by $\pi : S \to S/\mathcal{D}$ the natural projection. Then $\pi^{-1}(\alpha)$ is a closed subset of S which separates S. Let x, y be two points in S that are separated by $\pi^{-1}(\sigma)$.

We let $\varphi : [0, 1] \to \alpha$ be a homeomorphism and we use Alexander's "pinching process." Let $\alpha' = \varphi([0, 1/2])$

and $\alpha'' = \varphi([1/2, 1])$. Then $\pi^{-1}(\alpha')$ and $\pi^{-1}(\alpha'')$ are closed sets whose union is $\pi^{-1}(\alpha)$ and whose intersection is the connected set $\pi^{-1}(\alpha' \cap \alpha'')$. Thus, by theorem 1, one of the two sets $\pi^{-1}(\alpha')$, $\pi^{-1}(\alpha'')$ separates x, y in S. We may suppose that it is the former, and we define $\alpha_1 = \alpha'$. Applying the same argument to α_1 as we have applied to α, we get a subarc α_2 of α_1 which separates x, y in S. Continuing in this manner, we get a sequence of arcs $\alpha_1 \supset \alpha_2 \supset \alpha_3 \supset \ldots$ such that $\pi^{-1}(\alpha_n)$ separates x, y in S for each n, and $\delta(\alpha_n) \to 0$ by construction.

Let $\{p\} = \bigcap_{n=1}^{\infty} \alpha_n$. Then $\pi^{-1}(p)$ is a closed set which does not separate x, y in S. From this we obtain a contradiction as follows. Let γ be an arc in $S - \pi^{-1}(p)$ whose end points are x, y. Let U be the union of all the elements of $\mathfrak{D}$ which do not meet γ. It follows from the upper semi-continuity of $\mathfrak{D}$ that U is a neighbourhood of $\pi^{-1}(p)$ in S. Thus $\pi(U)$ is a neighbourhood of p in $S/\mathfrak{D}$, and so we can find an n such that $\alpha_n \subset \pi(U)$. Therefore $\pi^{-1}(\alpha_n) \subset U$, and so $\pi^{-1}(\alpha_n)$ does not separate x, y in S. This contradiction shows that there is no arc α which separates $S/\mathfrak{D}$.

In order to prove (c), let $J = \alpha \cup \beta$ be a simple closed curve in $S/\mathfrak{D}$, where α, β are two arcs such that $\alpha \cap \beta = \{a, b\}$, where a, b are two points. Then

$\pi^{-1}(J) = \pi^{-1}(\alpha) \cup \pi^{-1}(\beta)$ and $\pi^{-1}(\alpha) \cap \pi^{-1}(\beta) = \pi^{-1}(a) \cup \pi^{-1}(b)$, which has exactly two components. We have just shown that no arc separates $S/\mathcal{D}$. From this and the fact that π is monotone, it follows that $\pi^{-1}(\alpha)$ and $\pi^{-1}(\beta)$ are continua which do not separate S (see (2.2), p.138 of [31]). Thus by theorem 2, $S - \pi^{-1}(J)$ has exactly two components U, V. Since the decomposition is monotone, $\pi(U) \cap \pi(V) = \phi$. Thus $(S/\mathcal{D}) - J = \pi(U) \cup \pi(V)$ is a separation of $(S/\mathcal{D}) - J$. This shows that the decomposition space satisfies (c).

CHAPTER 8

A NOTE ON THE CYCLIC CONNECTIVITY THEOREM

1. INTRODUCTION. A locally compact, connected and locally connected metric space is called a _Peano space_. A _cut point_ of a Peano space is a point whose complement is not connected. In [2] Ayres proved the well-known _cyclic connectivity theorem_, which states that _every two points of a Peano space_ X _having no cut points lie together on a simple closed curve in_ X. Whyburn simplified the proof of this theorem in [30], using some elementary properties of cyclic elements. In this simplification he first proved these two lemmas.

LEMMA 1. _If_ A _and_ B _are non-degenerate, closed and disjoint subsets of_ X, _then there are two disjoint arcs in_ X _joining_ A _and_ B.

LEMMA 2. _Every point_ x _in_ X _is an interior point of some arc_ a x b _in_ X.

The proofs of these two lemmas constitute the main part of the proof in [30], the fact that each two points lie together on a simple closed curve being a simple consequence of the two lemmas.

Since its first appearance in 1931, the proof of the cyclic connectivity theorem that Whyburn gave in [30] has appeared in several places in the literature, namely in [31], [38] and [11]. In this note we shall show that lemma 1 can be proved differently from [30].

(A second proof of the cyclic connectivity theorem has been given by Ayres in [3]. In this paper the organization and proof of the theorem are different from those in [30]. Our proof and the proof of the corresponding part of [3] have in common the use of a "finiteness" argument, but our techniques are different.)

2. We base the proof of lemma 1 on a theorem[1] of Whyburn that appeared in 1933, two years after the appearance of Whyburn's proof of the cyclic connectivity theorem in [30]. This theorem states that

Each non-cut point of a Peano space S lies in an arbitrarily small region U such that U has "property S" and $S - \overline{U}$ is connected.

(A second method of proving this theorem has been given by Bing in the proof of theorem 1' of [4]). It follows as a corollary of the above theorem that the region U given there has a locally connected closure (see p.20 of [31]).

(1) theorem, §4 of G.T. Whyburn, Concerning S-regions in locally connected continua, Fund. Math. 20 (1933) pp.131-139.

When we speak of an arc in the sequel we permit it to be degenerate. We shall say that an arc ab <u>joins a closed set</u> A <u>and a closed set</u> B if $ab \cap A = \{a\}$ and $ab \cap B = \{b\}$.

NEW PROOF OF LEMMA 1. Let X be a Peano space with no cut points and let A and B be non-degenerate, closed and disjoint subsets of X. It follows from the theorem and its corollary that have been quoted, that for each point $x \in X$, there is a region U_x about x such that $A \not\subset \overline{U}_x$, $B \not\subset \overline{U}_x$ and $\overline{U}_x$ is a Peano space which does not separate X. We shall in addition suppose that $U_x = \operatorname{int} \overline{U}_x$.

By the simple chain theorem, the covering $\{U_x\}_x$ of X contains a simple chain from A to B, which we shall denote by $U_1, U_2, \ldots, U_n$.

Supposing that $n > 1$, we see that $\overline{U}_2 \not\subset \overline{U}_1$. For if $n > 2$ this follows from the relations $U_2 \cap U_3 \neq \emptyset$ and $U_1 \cap U_3 = \emptyset$, and if $n = 2$ it follows from the relations $U_2 \cap B \neq \emptyset$, $U_1 \cap B = \emptyset$ and $U_1 = \operatorname{int} \overline{U}_1$. Thus in the Peano space $X - \overline{U}_1$, there is an arc α_1 that joins A and U_2. Also, in the Peano space $\overline{U}_1$, there is an arc β_1 that joins A and $\overline{U}_2$.

Supposing that $n > 2$, it follows by the same reasoning as before that $\overline{U}_3 \not\subset \overline{U}_2$. Let γ be an arc in the Peano space $X - \overline{U}_2$ that joins $A \cup \alpha_1 \cup \beta_1$ and $\overline{U}_3$. If γ meets neither α_1 nor β_1, we define $\alpha_2 = \gamma$. If γ

meets either α_1 or β_1, we may suppose without loss of generality that it is the former, and we define α_2 as the union of γ and the subarc of α_1 that joins A and γ. We let δ be an arc in the Peano space $\overline{U}_2$ that joins β_1 and $\overline{U}_3$, and we define $\beta_2 = \beta_1 \cup \delta$.

We continue inductively in this manner. If we put $B = U_{n+1}$, then we finish up with two disjoint arcs joining A and B.

We remark that in the above proof we can get by demanding only that the sets $\overline{U}_1, \overline{U}_2 \ldots, \overline{U}_n$ do not separate X; we do not need the closures of these sets to be locally connected. For in this case, after having selected the arc α_k in $X - \overline{U}_k$ in the proof, we can select β_k as an arc joining A and $\overline{U}_{k+1}$ in the component of $X - \alpha_k$ that contains the connected set $\beta_{k-1} \cup \overline{U}_k$.

Finally we remark that it does not seem that the method that we have used to prove lemma 1 can be used to prove the second n-arc theorem (see §1, chap. 1 of this thesis). This is because if X is a Peano space which is not separated by any pair of points, it does not necessarily follow that each point $x \in X$ lies in an arbitrarily small region U of X such that $X - \overline{U}$ is a Peano space with no cut points. An example of such a space X can easily be given.

APPENDIX

At the beginning of §3, chap. 3 of this thesis, we said that, using the definition of unicoherence between two subsets as given in §5 of [34] (see also p.44 of this thesis) we can easily show that if M is any connected, locally connected regular T_1-space, and M is unicoherent between some pair of distinct points, then M is unicoherent. We demonstrate this below.

Suppose M is unicoherent between a pair of distinct points p_1, p_2, but that M is not unicoherent. Thus there are two connected closed subsets A_1, A_2 such that $M = A_1 \cup A_2$ and $A_1 \cap A_2 = B_1 \cup B_2$, where B_1 and B_2 are disjoint non-empty closed sets. We may suppose without loss of generality that p_1 and p_2 both belong to A_2. For each point $x \in M$ there is a region U_x about x such that $\overline{U}_x$ contains p_i if and only if $x = p_i$, and $\overline{U}_x \cap B_i \neq \phi$ if and only if $x \in B_i$. From the covering $\{U_x\colon x \in M\}$ of M we can select a simple chain $U_1, U_2, \ldots, U_n$ from the set $\{p_1, p_2\}$ to the set $B_1 \cup B_2$. We may without loss of generality suppose that $p_1 \in U_1$ and $U_n \cap B_1 \neq \phi$. Let V be a region about p_2 such that $\overline{V} \cap \bigcup_{i=1}^{n} \overline{U}_i = \phi$ and $\overline{V} \cap B_i \neq \phi$ if and only if $p_2 \in B_i$.

Define

$$A_1' = A_1 \cup \bigcup_{i=1}^{n} \overline{U}_i,$$
$$A_2' = A_2 \cup \overline{V}.$$

Then $A_1' \cap A_2'$ is the union of the three non-empty closed sets $B_1 \cup \bigcup_{i=1}^{n} (\overline{U}_i \cap A_2)$, B_2 and $A_1 \cap \overline{V}$. As the first two of these sets are disjoint, and the third does not meet the first two, it follows that $A_1' \cap A_2'$ is not connected. But for $i = 1, 2$, A_i' is a connected closed set and $p_i \in \operatorname{int} A_i'$. This is a contradiction.

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