The Grayson spectral sequence for hermitian K-theory

by

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Contents

"But we are self aware for sure
That doesn't mean there's nothing more
We both have dreams we need to find
To expand our pretty minds"

Ashbury Heights

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Declarations

This thesis is submitted to the University of Warwick in support of my application for the degree of Doctor of Philosophy. It has been composed by myself and has not been submitted in any previous application for any degree. I declare that to the best of my knowledge, the material contained in this thesis is original and my own work except where otherwise indicated.

Abstract

Let R be a regular ring such that 2 is invertible. We construct a spectral sequence converging to the hermitian K-theory, alias the Grothendieck-Witt theory, of R. In particular, we construct a tower for the hermitian K-groups in even shifts, whose terms are given by the hermitian K-theory of automorphisms. The spectral sequence arises as the homotopy spectral sequence of this tower and is analogous to Grayson's version of the motivic spectral sequence [Gra95].

Further, we construct similar towers for the hermitian K-theory in odd shifts if R is a field of characteristic different from 2. We show by a counter example that the arising spectral sequence does not behave as desired. We proceed by proposing an alternative version for the tower and verify its correctness in weight 1. Finally we give a geometric representation of the (hermitian) K-theory of automorphisms in terms of the general linear group, the orthogonal group, or in terms of ϵ-symmetric matrices, respectively.

The K-theory of automorphisms can be identified with motivic cohomology if R is local and of finite type over a field. Therefore the hermitian K-theory of automorphisms as presented in this thesis is a candidate for the analogue of motivic cohomology in the hermitian world.

Chapter 1

Introduction

1.1 Synopsis

Since the turn of the last century a lot of attention was given to the spectral sequence relating motivic cohomology to algebraic K-theory. This spectral sequence is built to model the Atiyah-Hirzebruch spectral sequence for topological K-theory and it takes on the form

$$E_2^{p,q} = H^{p-q}(X, \mathbb{Z}(-q)) \Rightarrow K_{-p-q}(R) \tag{1.1}$$

where R is a regular ring, $X = \mathrm{Spec}(R)$, H^* denotes motivic cohomology and K_* is algebraic K-theory.

A good overview of the search for such a spectral sequence and also some of the motivation for this endeavour is given in [Gra05]. There are several ways to obtain the spectral sequence, one of which goes back to an idea of Goodwillie and Lichtenbaum and is investigated by Grayson in [Gra95]. There, he uses the K-theory of commuting automorphisms to construct a filtration of the K-theory space of a ring, which ultimately yields the spectral sequence. Explicitly, he constructs a tower

$$K(R) \simeq W^0 \leftarrow W^1 \leftarrow \cdots \tag{1.2}$$

with $W^t = \Omega^{-t} K(R\Delta^{\cdot}, \mathbb{G}_m^{\wedge t})$, the t-fold delooping of the (direct sum) K-theory of automorphisms, stabilised in a suitable way. The spaces W^t fit into fibration sequences of the form

$$\Omega^{-t-1} K(R\Delta^{\cdot}, \mathbb{G}_m^{\wedge t+1}) \to \Omega^{-t} K(R\Delta^{\cdot}, \mathbb{G}_m^{\wedge t}) \to \Omega^{-t} K_0(R\Delta^{\cdot}, \mathbb{G}_m^{\wedge t}) \tag{1.3}$$

so that the E_2-term is given by homotopy groups of the base-space. Later, Suslin [Sus03] identified these non-standard motivic cohomology groups with the standard ones, for smooth semi-local varieties over a field, so that the spectral sequence takes on the form as presented in (1.1).

This paper studies an analogous spectral sequence for hermitian K-theory, alias higher Grothendieck-Witt theory. For any exact category with duality there are essentially four different kinds of such groups, indicated by a certain "shift". The Grothendieck-Witt groups in even shifts $GW^{[2n]}$, also denoted by ${}_\epsilon GW$ for $\epsilon = (-1)^n$, are essentially just the K-theory of the symmetric monoidal category of ϵ-symmetric inner product spaces, and therefore it is possible to generalise Grayson's work to this setting. In particular we construct filtrations

$$ {}_{\epsilon}GW(R) \simeq W^0 \leftarrow W^1 \leftarrow \cdots \tag{1.4} $$

for any regular ring R with involution and $2 \in R^{\times}$, with $W^t = \Omega^{-t}{}_{\epsilon}GW(R\Delta^{\cdot}, \mathbb{G}_m^{\wedge t})$. The spaces W^t fit into fibration sequences

$$ \Omega^{-t-1}{}_{\epsilon}GW(R\Delta^{\cdot}, \mathbb{G}_m^{\wedge t+1}) \to \Omega^{-t}{}_{\epsilon}GW(R\Delta^{\cdot}, \mathbb{G}_m^{\wedge t}) \to \Omega^{-t}{}_{\epsilon}GW_0(R\Delta^{\cdot}, \mathbb{G}_m^{\wedge t}) \tag{1.5} $$

The Grothendieck-Witt groups of an exact (symmetric monoidal) category with duality $\mathcal{E}$ in odd shifts were initially defined by Karoubi [Kar73] as the fibre of the hyperbolic functor

$$ GW^{[2n-1]}(\mathcal{E}) \to K(\mathcal{E}) \to GW^{[2n]}(\mathcal{E}) \tag{1.6} $$

Later, Schlichting defined all four Grothendieck-Witt theories simultaneously in the framework of exact categories with weak equivalences and duality. In particular, given any exact category with duality we may consider chain complexes in this category with quasi-isomorphisms as weak equivalences. We have an additional degree of freedom for the duality as we may shift a chain complex to either side. It turns out that the process of shifting results in four-periodic groups and we recover the spaces ${}_{\epsilon}GW$ in even shifts. The defining sequence (1.6) then becomes a theorem (cf. [Sch12, Theorem 6.1]).

Because of their different nature the construction for Grothendieck-Witt groups in odd shifts does not work exactly as in even shifts. In fact, we provide a counterexample for the naive analogue of (1.3) in these cases.
We proceed by showing the existence of fibration sequences

$$ \Omega^{-1}{}_{(-1)^n}S(R\Delta^{\cdot}) \to GW^{[2n+1]}(R\Delta^{\cdot}) \to GW_0^{[2n+1]}(R\Delta^{\cdot}) \tag{1.7} $$

where ${}_{\epsilon}S(R)$ is the set of ϵ-symmetric matrices with entries in R. Moreover we construct a homotopy equivalence

$$ {}_{\epsilon}S(R\Delta^{\cdot}) \xrightarrow{\sim} {}_{\epsilon}GW(R\Delta^{\cdot}, \widetilde{\mathbb{G}}_m^{\wedge 1}). \tag{1.8} $$

Together these two results give the weight 1 part of the tower for hermitian K-theory in odd shifts. The generalisation to higher weighs remains conjectural. A supporting argument for the correctness of the proposed filtration would be if certain groups in the E_2-term for a field k, here indexed as ${}_{[p]}E_2^{0,-p}$, would satisfy

$$K_p^{MW}(k) \cong {}_{[p]}E_2^{0,-p} \tag{1.9}$$

in concordance with the situation in K-theory, where we have

$$K_p^M(k) \cong E_2^{0,-p} \tag{1.10}$$

We verify this for $p = 0$ and $p = 1$, and construct a map for $p = 2$.

One of the central ideas in the construction of motivic cohomology is the category of finite correspondences Cor_k over a field k. Presheaves on Cor_k are also called presheaves with transfer, and motivic cohomology is the homology of a certain complex in the category of those sheaves. However many interesting presheaves on the category of smooth schemes do not have transfers and there is a promising preprint by Calmés and Fasel who define a category of finite Chow-Witt correspondences and use this to define generalised motivic cohomology groups. We hope that the E_2-term of the spectral sequence we construct here, indeed consists of such groups.

1.2 Main results

This thesis contains three main theorems. In particular we have:

Theorem 6.1.1. *Let R be a ring. Then the sequence*

$$\Omega^{-1}K(R\Delta^{\cdot}, \mathbb{G}_m^{\wedge t+1}) \to K(R\Delta^{\cdot}, \mathbb{G}_m^{\wedge t}) \to K_0(R\Delta^{\cdot}, \mathbb{G}_m^{\wedge t}) \tag{1.11}$$

is a homotopy fibration. If R is a ring with involution and 2 is invertible, then also

$$\Omega^{-1}{}_{\epsilon}GW(R\Delta^{\cdot}, \mathbb{G}_m^{\wedge t+1}) \to {}_{\epsilon}GW(R\Delta^{\cdot}, \mathbb{G}_m^{\wedge t}) \to {}_{\epsilon}GW_0(R\Delta^{\cdot}, \mathbb{G}_m^{\wedge t}) \tag{1.12}$$

is a homotopy fibration.

together with

Proposition 6.2.2. *The maps*

$$K(R\Delta^{\cdot}, \mathbb{G}_m^{\wedge t+1}) \to \Omega K(R\Delta^{\cdot}, \mathbb{G}_m^{\wedge t}) \tag{1.13}$$

and

$${}_{\epsilon}GW(R\Delta^{\cdot}, \mathbb{G}_m^{\wedge t+1}) \to \Omega_{\epsilon}GW(R\Delta^{\cdot}, \mathbb{G}_m^{\wedge t}) \tag{1.14}$$

in Theorem 6.1.1 above are induced by multiplication with elements $U \in K_1(\mathbb{G}_m)$ and $U \in GW_1(\mathbb{G}_m)$, respectively.

The fibration sequences can be assembled to provide a tower filtering the (hermitian) K-theory space of R. In Chapter 9 we see how such a tower gives rise to a spectral sequence converging to the (hermitian) K-groups of R. The explicit description of the maps in the tower as multiplication by an element can be used to gain valuable insights on the differentials in these spectral sequences.

Theorem 7.2.2. *Let $t \geq 0$ be an integer. Further let $\mathcal{GL}^t(R\Delta^\cdot)$ and ${}_\epsilon\mathcal{O}^t(R\Delta^\cdot)$ be the simplicial sets of t-tuples of pairwise commuting matrices in $Gl(R\Delta^\cdot)$ and ${}_\epsilon O(R\Delta^\cdot)$, respectively. Then there exist homotopy equivalences*

$$\mathcal{GL}^t(R\Delta^\cdot)^+ \xrightarrow{\simeq} K(R\Delta^\cdot, \mathbb{G}_m^{\wedge t}) \tag{1.15}$$

and

$${}_\epsilon\mathcal{O}^t(R\Delta^\cdot)^+ \xrightarrow{\simeq} {}_\epsilon GW(R\Delta^\cdot, \mathbb{G}_m^{\wedge t}) \tag{1.16}$$

This rather technical looking result can be used to describe (hermitian) K-groups explicitly in terms of generators and relations. In particular it describes the (hermitian) K-theory space as investigated in Section 4.5 in terms of simplicial sets of commuting automorphsism.

Theorem 8.5.9. *Let R be a regular, local ring with involution such that 2 is invertible. Then there are homotopy fibration sequences of the form*

$$\Omega^{-1}GW^{[2n]}(R\Delta^\cdot, \widetilde{\mathbb{G}}_m^{\wedge 1}) \xrightarrow{U} GW^{[2n+1]}(R\Delta^\cdot) \to GW_0^{[2n+1]}(R\Delta^\cdot), \tag{1.17}$$

where U is the element of $GW_1^{[1]}(\widetilde{\mathbb{G}}_m)$ represented by the symmetric matrix $U \in Gl_1(\widetilde{\mathbb{G}}_m) = Gl_1(k[U, U^{-1}])$ (cf. Proposition 8.4.4).

in conjunction with

Theorem 8.4.1. *Let R be a regular, local ring such that 2 is invertible. Then the natural map*

$${}_{(-1)^n}S(R\Delta^\cdot) \to \Omega GW^{[2n+1]}(R\Delta^\cdot) \tag{1.18}$$

is a homotopy equivalence.

The proof of Theorem 8.5.9 differs from that of Theorem 6.1.1 and the generalisation to higher terms in the tower remains conjectural (cf. Chapter 9). Theorem 8.4.1 then allows us to describe the hermitian K-theory space in odd shifts explicitly via ${}_\epsilon S(R\Delta^\cdot)$ the simplicial set of invertible ϵ-symmetric matrices, much in the spirit of Theorem 7.2.2.

1.3 Structure of this thesis

This paper is organised as follows:

- Chapter 2 introduces some notation that we use throughout the paper. Notably we describe how we write simplicial objects in a given category. We also address some subtleties of the geometric realisation of simplicial spaces and of simplicial categories. We recall the construction of the classifying space of a category via its nerve and introduce the simplicial ring $R\Delta^{\cdot}$. A technical problem arising from the introduction of the simplicial ring $R\Delta^{\cdot}$ is that a given sequence of three simplicial spaces, which is a homotopy fibration sequence in each degree, is not necessarily a homotopy fibration sequence after geometric realisation. We recall (and slightly modify) a result by Grayson [Gra95], which proves that the sequence
$$|d \mapsto I_* X_d| \to |X| \to |d \mapsto \pi_0 X_d| \tag{1.19}$$
is a fibration sequence for a group-complete simplicial H-space $X = ([d] \mapsto X_d)$ (cf. Lemma 3.2.3). This fibration sequence can be seen as the prototype of what will later become one of the sequences (1.3) and (1.5).
Finally we define various categories with certain additional structures, namely symmetric monoidal categories, exact categories and categories with duality and/or weak equivalences.

- Chapter 3 deals with the multifarious different constructions of algebraic and hermitian K-theory. We begin by describing the group-completion of a symmetric monoidal category in some detail, because this construction will be the most important one for the remainder of the article. Further, we introduce inner product spaces in a category with duality and define hermitian K-theory as the group-completion thereof. The first section is concluded by the cofinality theorem. Here, a functor $F\colon \mathcal{S} \to \mathcal{T}$ is cofinal if we can restrict diagrams on $\mathcal{T}$ along F to diagrams on $\mathcal{S}$ without changing their colimit. Cofinal functors are important to us because they induce homotopy equivalences on the base-point components of the K-theory spaces and thereby allow us to study the group-completion of a simpler subcategory without losing much information.
We proceed by briefly explaining Quillen's Q-construction that constructs a K-theory space from an exact category $\mathcal{E}$, the hermitian Q-construction for exact categories with duality $(\mathcal{E}, *, \eta)$ and Schlichting's hermitian S-construction for exact categories with weak equivalences and duality $(\mathcal{E}, \omega, *, \eta)$. Many categories we will consider as symmetric monoidal categories do in fact carry an

exact structure and our main focus will be on various comparison results that allow us to apply statements valid for exact categories to symmetric monoidal categories.
We pay special attention to the theory of shifted Grothendieck-Witt groups. Here, we consider the category $\mathrm{Ch}^b(\mathcal{E})$ of bounded chain complexes in a given exact category with duality $(\mathcal{E}, *, \eta)$ and take quasi-isomorphisms to be the weak equivalences. For a given chain complex, the 'naive' dual complex can be shifted to the left or to the right, resulting in countably many dualities $(*^n, \eta^n)$ in this category. Consequently there are countably many hermitian K-theories, defined as $GW^{[n]}(\mathcal{E}) := GW(\mathrm{Ch}^b(\mathcal{E}), quis, *^n, \eta^n)$.
The Grothendieck-Witt groups are in fact four-periodic with respect to this shift $GW^{[n]}(\mathcal{E}) \cong GW^{[n+4]}(\mathcal{E})$, and the Grothendieck-Witt groups ${}_\epsilon GW(\mathcal{E})$ of an exact (symmetric monoidal) category can naturally be identified with the Grothendieck-Witt groups in even shifts ${}_+GW(\mathcal{E}) \cong GW^{[0]}(\mathcal{E})$ and ${}_-GW(\mathcal{E}) \cong GW^{[2]}(\mathcal{E})$, respectively. We conclude with Schlichting's version of Karoubi's fundamental theorem.
There is an abundance of different sign conventions for the category of chain complexes and dualities upon it. This section also fixes the notation for the remainder of this paper. They are chosen to be consistent with [Sch12]. For that reason many of the results cited from other sources differ somewhat from their original formulation.
Section3.6 in this chapter computes the 0-th homotopy group of the hermitian K-theory of an exact category with duality $(\mathcal{E}, *, \eta)$ in terms of explicit generators and relations. Because of the isomorphism $GW_0^{[2n]}(\mathcal{E}) \cong {}_{(-1)^n}GW_0(\mathcal{E})$ we already know that the groups in even shifts are generated by symmetric and skew-symmetric inner product spaces in $\mathcal{E}$, respectively. We show how these generators can be expressed as inner product spaces in $\mathrm{Ch}^b(\mathcal{E})$, always having sign conventions and notations in mind. By a theorem of Walter [Wal03], the groups in odd shifts $GW_0^{[2n+1]}(\mathcal{E})$ are generated by so-called (skew-)symmetric short complexes, i.e. forms on complexes with (at most) two non-trivial entries.

- Chapter 4 gives an exposition of the objects we study in this paper. The example that stays with us is the symmetric monoidal category of automorphisms of projective modules over a given ring $\mathcal{P}(R, \mathbb{G}_m^t)$, whose K-theory provides the building blocks for the tower (1.2). Explicitly, an object of this category is a tuple $(P, \theta_1, \ldots, \theta_t) = (P, \{\theta_j\})$ consisting of a projective R-module P and t commuting automorphisms $\theta_j \in Aut(P)$. The K-theory space of this category is denoted by $K(R, \mathbb{G}_m^t)$.

The category of ϵ-symmetric inner product spaces in $\mathcal{P}(R, \mathbb{G}_m^t)$ is denoted by ${}_\epsilon\mathcal{P}_h(R, \mathbb{G}_m^t)$. Here $\epsilon = \pm 1$, the duality on $\mathcal{P}(R, \mathbb{G}_m^t)$ is defined by $(P, \{\theta_j\})^* = (P^*, \{(\theta_j^*)^{-1}\})$, and an ϵ-symmetric inner product space is an isomorphism $\phi\colon (P, \{\theta_j\}) \xrightarrow{\sim} (P, \{\theta_j\})^*$. The tower (1.4) is built of the K-theory spaces ${}_\epsilon GW(R, \mathbb{G}_m^t) := K({}_\epsilon\mathcal{P}_h(R, \mathbb{G}_m^t))$. The duality is chosen such that an inner product space in the category of automorphisms $\mathcal{P}(R, \mathbb{G}_m^t)$ is the same as t commuting automorphism of an inner product space in the category $\mathcal{P}(R)$.
We conclude this section by defining cofinal subcategories of both, $\mathcal{P}(R, \mathbb{G}_m^t)$ and ${}_\epsilon\mathcal{P}_h(R, \mathbb{G}_m^t)$. In particular the category of free modules with automorphisms $\mathcal{F}(R, \mathbb{G}_m^t)$ is cofinal in $\mathcal{P}(R, \mathbb{G}_m^t)$ and it suffices to study sets of commuting invertible matrices over R rather than abstract automorphisms of projective modules.
For ${}_\epsilon\mathcal{P}_h(R, \mathbb{G}_m^t)$ we define two different cofinal subcategories. Both constructions make use of the hyperbolic functor ${}_\epsilon H\colon \mathcal{P}(R, \mathbb{G}_m^t) \to {}_\epsilon\mathcal{P}_h(R, \mathbb{G}_m^t)$ as defined in 3.2.5. The hyperbolic functor is not cofinal itself but can be extended to a cofinal one by changing the sets of morphism in the source, provided that 2 is a unit in R. While we have seen that an automorphism of an inner product space is the same as an inner product space in the category of automorphisms, this clearly changes when we start changing the morphisms and as a result we obtain two distinct cofinal subcategories of ${}_\epsilon\mathcal{P}_h(R, \mathbb{G}_m^t)$. Either we extend the functor ${}_\epsilon H\colon \mathcal{P}(R) \to {}_\epsilon\mathcal{P}_h(R)$ to a cofinal one and then consider automorphisms in these categories or we extend the functor ${}_\epsilon H\colon \mathcal{P}(R, \mathbb{G}_m^t) \to {}_\epsilon\mathcal{P}_h(R, \mathbb{G}_m^t)$ directly. In one case we will end up with sets of commuting matrices in ${}_\epsilon O(R)$, i.e. orthogonal or symplectic matrices, while the other case is more of technical interest and slightly harder to describe in succinct form.
Let $\mathbb{G}_m = k[U, U^{-1}]$ be a ring, where we choose k to be either $\mathbb{Z}$, $\mathbb{Z}[\frac{1}{2}]$ or a field with characteristic different from two, depending on the circumstances. Then objects in $\mathcal{P}(R, \mathbb{G}_m^t)$ can then be interpreted as modules over the ring $R \times \mathbb{G}_m^{\times t}$ that are finitely generated and projective over R. Similarly an object in ${}_\epsilon\mathcal{P}_h(R, \mathbb{G}_m^t)$ can be understood as an inner product space of $R \times \mathbb{G}_m^{\times t}$-modules. There are t distinguished inclusions

$$\mathbb{G}_m = k[U_j, U_j^{-1}] \subset k[U_1, \ldots U_t, U_1^{-1}, \ldots, U_t^{-1}] = \mathbb{G}_m^t \tag{1.20}$$

given by $U_j \mapsto U_j$ for any positive integer $j \leq t$. In particular, any object of $\mathcal{P}(R, \mathbb{G}_m^t)$ can be given the structure of a $\mathbb{G}_m$-module in t different ways, and for each of them the tensor product of modules (inner product spaces) over $\mathbb{G}_m$

induces a product on the K-theory $K(R, \mathbb{G}_m^t) \wedge K(G_m) \to K(R, \mathbb{G}_m^{t-1})$ and on the GW-theory ${}_\epsilon GW(R, \mathbb{G}_m^t) \wedge GW(G_m) \to {}_\epsilon GW(R, \mathbb{G}_m^{t-1})$. Moreover there are distinct elements $U_j \in K_1(\mathbb{G}_m) = \mathbb{G}_m^\times$ and $U_j \in {}_+GW_1(\mathbb{G}_m)$, multiplication with which yields maps $K(R, \mathbb{G}_m^t) \to \Omega K(R, \mathbb{G}_m^{t-1})$ and ${}_\epsilon GW(R, \mathbb{G}_m^t) \to \Omega_\epsilon GW(R, \mathbb{G}_m^{t-1})$. Of course, the notation is suggestive and there is no actual difference between U_i and U_j, $i \neq j$, as elements of $K_1(\mathbb{G}_m)$.
There are also various functors

$$\mathbb{G}_m^t = k[U_1^\pm, \ldots, U_t^\pm] \to k[U_1^\pm, \ldots, \widehat{U_j}^\pm, \ldots, U_t^\pm] = \mathbb{G}_m^{t-1} \tag{1.21}$$

which map $U_j \mapsto 1$ for a fixed index j. These functors induce inclusions $\mathcal{P}(R, \mathbb{G}_m^{t-1}) \subset \mathcal{P}(R, \mathbb{G}_m^t)$ and ${}_\epsilon\mathcal{P}_h(R, \mathbb{G}_m^{t-1}) \subset {}_\epsilon\mathcal{P}_h(R, \mathbb{G}_m^t)$. Explicitly, these functors equip a module with an additional identity-morphism at the j-th position. It is evident from the definition that multiplication by U_j kills the j-th subcategory of this form. Further, because these functors are split by a forgetful functor, the resulting subspaces are direct factors. The remainder of this section describes how we can define the complement of this factor in a functorial way. It does so by employing the language of cubes in a given category. We will refer to this complement as the stabilised or reduced K-theory (GW-theory) of automorphisms and write $K(R, \mathbb{G}_m^{\wedge t})$ and ${}_\epsilon GW(R, \mathbb{G}_m^{\wedge t})$, respectively.
Of particular importance is a result that states that the sequence

$$\mathcal{S} \to \mathcal{S}(\mathbb{G}_m) \to \langle \mathcal{S}, \mathcal{S}(\mathbb{G}_m) \rangle \tag{1.22}$$

is a homotopy fibration sequence after group-completion (and under certain conditions on the symmetric monoidal category $\mathcal{S}$). On the one hand this readily provides us with a categorical model for the complementary factor of $\mathcal{S}^+ \subset \mathcal{S}(\mathbb{G}_m)^+$ and on the other hand we show that any cofinal functor $\mathcal{S} \to \mathcal{T}$ induces a homotopy equivalence $\langle \mathcal{S}, \mathcal{S}(\mathbb{G}_m) \rangle^+ \xrightarrow{\sim} \langle \mathcal{T}, \mathcal{T}(\mathbb{G}_m) \rangle^+$. Therefore we may restrict ourselves to a suitable cofinal subcategory as introduced in the previous section.
An important application is Lemma 4.6.2 which shows that the map

$$K_0(R, \mathbb{G}_m^{\wedge t}) \to GW_0^{[2n+1]}(R, \mathbb{G}_m^{\wedge t}) \tag{1.23}$$

is surjective, where R is either a field of characteristic unequal to 2 or, if $t = 0$, a local ring with 2 invertible. In light of the discussion in Section 5 on homotopy fibration sequences of simplicial spaces, this lemma is sufficient to apply (one version of) the fundamental theorem to the simplicial category of automorphisms (with duality).

The main theorems of this paper only hold *modulo simplicial homotopy.* Roughly speaking this means that, given a functor F from rings to groups, say, and a fixed ring R, we will be interested in the colimit of the diagram $F(R[T]) \rightrightarrows F(R)$, where the two morphisms are induced by the evaluations $T = 0$ and $T = 1$ respectively. We introduced the simplicial ring $R\Delta^{\cdot} = ([d] \mapsto R\Delta^d)$ with $R\Delta^d = R[T_0, \cdots, T_d]/_{(\sum T_i = 1)}$ to formally deal with arising higher homotopies.

The symmetric monoidal categories introduced in the previous sections, such as the category of projective modules over a ring R, depend functorially on said ring. If $\mathcal{C}(R)$ is such a category, we may therefore study the simplicial category $\mathcal{C}(R\Delta^{\cdot}) = ([d] \mapsto \mathcal{C}(R\Delta^d))$. This way we define the simplicial K-theory (GW-theory) of automorphisms $K(R\Delta^{\cdot}, \mathbb{G}_m^{\wedge t}) = |[d] \mapsto K(R\Delta^d, \mathbb{G}_m^{\wedge t})|$ and ${}_{\epsilon}GW(R\Delta^{\cdot}, \mathbb{G}_m^{\wedge t}) = |[d] \mapsto {}_{\epsilon}GW(R\Delta^d, \mathbb{G}_m^{\wedge t})|$, respectively.

As a first application, we show that certain categories, notably some whose group-completion are, essentially by definition, the stabilised K-theory (GW-theory) of automorphisms, are already group complete when evaluated at a simplicial ring. The objects in these categories are tuples of pairwise commuting, invertible matrices, which form an abelian monoid under block-wise addition. The key argument, in a nutshell, is therefore a generalised Whitehead-lemma: Given an invertible matrix A with entries in R, we want to find a matrix $B(T)$ with entries in the polynomial ring $R[T]$, such that $B(0) = A \oplus A^{-1}$ and $B(1)$ is the identity. Of course there are multifarious additional conditions to satisfy, making the proof more intricate. This argument will have its second coming in the proof of Theorem 6.1.1.

- Chapter 5 is a stand-alone discussion of the simplicial version of Quillen's Theorem B. We recall the original (non-simplicial) version of the theorem that allows us to express the homotopy fibre of a given functor as the classifying space of yet another category. There are two versions of this statement one in terms of so-called right-fibres, the other in terms of left-fibres.

  The idea is now, to express a simplicial category by a homotopy equivalent (non-simplicial) category, apply the ordinary Theorem B and finally translate the left- or right-fibres back to simplicial categories. Explicitly, there are two different ways to do this, due to Thomason [Tho79] and Segal [Seg74], resulting

in a total of four different versions of the simplicial Theorem B.
One version of the result goes back to Waldhausen [Wal82], another one appears in [Gra95]. We believe, however, that this is the first exhaustive treatment in the literature.

- Chapter 6 contains with Theorem 6.1.1 the first major result. The version for K-theory is due to Grayson, but the statement for GW-theory as well as the proofs given for both of them are new. Moreover, we think that our presentation is significantly more explicit.
  We begin by considering the sequence (1.20) for $X = K(R\Delta^{\cdot}, \mathbb{G}_m^{\wedge t})$ and $X = {}_{\epsilon}GW(R\Delta^{\cdot}, \mathbb{G}_m^{\wedge t})$, respectively. Then, the heavy weight is pulled by Lemma 6.1.5 which uses the simplicial Theorem B to show that the fibre in this sequence is a delooping of $X = K(R\Delta^{\cdot}, \mathbb{G}_m^{\wedge t+1})$ or of $X = {}_{\epsilon}GW(R\Delta^{\cdot}, \mathbb{G}_m^{\wedge t+1})$, respectively.
  In either case, the space X is defined as the group-completion $\mathcal{S}^{-1}\mathcal{S}$ of some simplicial symmetric monoidal category $\mathcal{S}$. Thus, we are interested in the loop-space of $|d \mapsto I_*\mathcal{S}^{-1}\mathcal{S}_d|$ or equivalently, the homotopy fibre of the map $\langle \mathcal{S}, \mathcal{S}\rangle \to |d \mapsto I_*\mathcal{S}^{-1}\mathcal{S}_d|$. It is not hard to see that an object of the right-fibre at the object $(0, 0)$ is uniquely determined by an object $S \in \mathcal{S}$ together with an automorphism θ of S, i.e. an object in $\mathcal{S}(\mathbb{G}_m)$.
  The remainder of the proof deals with the technically intricate endeavour to verify that all transition maps are homotopy equivalences. Here, we need an argument akin to the generalised Whitehead-lemma to construct explicit homotopies. It also turns out that we have to restrict ourselves to suitable cofinal subcategories for this to work. Fortunately this doesn't change the base-point component $I_*\mathcal{S}^{-1}\mathcal{S}$.
  We conclude this chapter by showing that the first map in (1.3) and (1.5) that is induced by Theorem B, is indeed multiplication by U, as defined in Section 4.4.

- The simplest case of the Theorem proven in the previous section is the fibration sequence
  $$\Omega^{-1}K(R\Delta^{\cdot}, \mathbb{G}_m^{\wedge 1}) \to K(R\Delta^{\cdot}) \to K_0(R\Delta^{\cdot}) \tag{1.24}$$
  However, it is well known from the classical point of view that the fibre of the sequence above is also given by $BGl(R\Delta^{\cdot})^+ \simeq BGl(R\Delta^{\cdot}) \simeq \Omega^{-1}|Gl(R\Delta^{\cdot})|$, where the last term is the geometric realisation of the simplicial set $([d] \mapsto Gl(R\Delta^d))$. Any element of $Gl(R\Delta^{\cdot})$ may naturally be identified with an object

of $\mathcal{P}(R, \mathbb{G}_m)$ and the evident inclusion map

$$Gl(R\Delta^{\cdot}) \to K(R\Delta^{\cdot}, \mathbb{G}_m^{\wedge 1}) \tag{1.25}$$

is a homotopy equivalence. In Chapter 7 we generalise this insight through Theorem 7.2.2. The idea of the prove is to consider both sides, a symmetric monoidal category, say $\mathcal{S}$, on the right, and its set of objects $\mathrm{Ob}(\mathcal{S})$ on the left, which is understood to be a discrete category, as subcategories of one bigger category. Truncating these new bigger categories in a suitable way allows us to show that both inclusion functors are homotopy equivalences.

- Chapter 8.1 investigates the analogue of 6.1.1 for the Grothendieck-Witt-theory of rings in odd shifts. Unfortunately the naive generealisation doesn't work as demonstrated in Lemma 8.1.4. We proceed with an alternative approach, which proposes that the set of invertible ϵ-symmetric matrices should play an important role in the tower for odd Grothendieck-Witt groups, very much like orthogonal matrices do so for even shifts, and invertible matrices for K-theory. In particular, we prove the Theorems 8.4.1 and 8.5.9.

- Finally, Chapter 9 recalls how to construct the homotopy spectral sequence of a tower of topological spaces. In particular we see how the Grayson spectral sequence for K-theory arises in such a way and how this generalises to hermitian K-theory in even shifts. Further we conjecture how the corresponding spectral sequence for odd shifts should look like. We end with a short discussion of the relation of these spectral sequences to Milnor-Witt K-theory.

Chapter 2

Preliminaries

2.1 Simplicial objects

This section will set up some of the notation we use in this paper. Most terms will be defined in the body of the paper, but some, notably simplicial objects, are so ubiquitous that it seems prudent to describe them beforehand. We assume most of the constructions to be common knowledge and refrain from giving any proofs. The standard reference is [GJ09].

Definition 2.1.1. Denote by $[n]$ the ordered set with $n+1$ elements, i.e.

$$[n] = \{0 < 1 < \ldots < n\} \tag{2.1}$$

Sometimes we will interpret $[n]$ as a category where an object is an integer a with $0 \leq a \leq n$. For any two objects a and b there is exactly one morphism if and only if $a \leq b$ and none otherwise. The composition law is clear.

Definition 2.1.2. Denote by Δ the category which has objects $[n]$ for $n \in \mathbb{N}_0$. A morphism, also called simplicial map, $\lambda\colon [n] \to [m]$ is a monotone map of ordered sets. There are two types of distinguished maps, the so-called face- and degeneracy-maps, that generate the set of morphisms by composition. Given any n, there are face-maps $\delta_j\colon [n{-}1] \to [n]$ and degeneracy-maps $\sigma_j\colon [n{+}1] \to [n]$ for each $0 \leq j \leq n$. The map δ_j “misses” the element j in the target, while the map σ_j hits the element j in the target twice.

Definition 2.1.3. Denote by Δ^+ the subcategory of Δ, which has the same objects and only strictly increasing maps.

Definition 2.1.4. Let $\mathcal{C}, \mathcal{D}$ be categories. Then denote by $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ the functor category, which has functors $F\colon \mathcal{C} \to \mathcal{D}$ as objects and where a morphism is a natural transformation of functors.

Definition 2.1.5. Let $\mathcal{C}$ be a category. A simplicial object in $\mathcal{C}$ is a functor

$$X\colon \Delta^{op} \to \mathcal{C} \tag{2.2}$$

A cosimplicial object in $\mathcal{C}$ is a functor

$$Y\colon \Delta \to \mathcal{C} \tag{2.3}$$

We sometimes just write $([n] \mapsto X_n)$ when we define a simplicial object and can assume that the simplicial maps are understood. Further we use $s\mathcal{C}$ as a short-hand for the category $\mathrm{Fun}(\Delta^{op}, \mathcal{C})$ of simplicial objects in $\mathcal{C}$.

Notable examples of categories, in which we will study simplicial objects, are the category of sets, the category of rings, the category of small categories and the category of simplicial sets itself.

Definition 2.1.6. The topological n-simplex is given as

$$\Delta^n_{top} := \{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} | \sum_i x_i = 1, x_i \geq 0\} \tag{2.4}$$

The functor

$$\Delta \to \mathrm{Top} \tag{2.5}$$
$$[n] \mapsto \Delta^n_{top} \tag{2.6}$$

is a cosimplicial topological space. Here a map $\lambda\colon [n] \to [m]$ induces a map $\Delta^n_{top} \to \Delta^m_{top}\colon x \mapsto y$ via $y_i = \sum_{\lambda(j)=i} x_j$.

Definition 2.1.7. Let $X\colon \Delta^{op} \to \mathrm{Top}$ (Sets) be a simplicial topological space (or a simplicial set). Then its topological realisation is the quotient space

$$|X| = \coprod_{n \geq 0} X_n \times \Delta^n_{top}/_\sim \tag{2.7}$$

where $(\lambda^* x, t) \sim (x, \lambda_* t)$ for all $x \in X_m$, $t \in \Delta^n_{top}$ and $\lambda\colon [n] \to [m] \in \Delta$.
The *fat* realisation of X is the topological space

$$||X|| = \coprod_{n \geq 0} X_n \times \Delta^n_{top}/_{\sim_+} \tag{2.8}$$

where $(\lambda^* x, t) \sim_+ (x, \lambda_* t)$ for all $x \in X_m$, $t \in \Delta^n_{top}$ and $\lambda\colon [n] \to [m] \in \Delta^+$.

Definition 2.1.8. A simplicial topological space X is called *good* if all degeneracy-maps $X_{n-1} \to X_n$ are closed (Hurewicz) cofibrations.

Proposition 2.1.9. *If X is a good simplicial space, then the natural map*

$$||X|| \xrightarrow{\simeq} |X| \tag{2.9}$$

is a homotopy equivalence.

Proposition 2.1.10. *Let $X = ([n] \mapsto X_n \in \mathrm{Top})$ be a simplicial space. Then there is a homotopy equivalence*

$$||X|| \simeq \operatorname{hocolim}_{\Delta^{op}} X_n \tag{2.10}$$

Definition 2.1.11. Let $\mathcal{C}$ be a category. A bisimplicial object in $\mathcal{C}$ is a functor

$$\Delta^{op} \times \Delta^{op} \to \mathcal{C} \tag{2.11}$$

Equivalently a bisimplicial object is a simplicial object in the category of simplicial objects

$$\Delta^{op} \to \mathrm{Fun}(\Delta^{op}, \mathcal{C}) \tag{2.12}$$

Definition 2.1.12. Let $X = (([n],[m]) \mapsto X_{n,m})$ be a bisimplicial set. Define simplicial sets as follows:

$$\begin{aligned} \mathrm{diag}(X) \colon \Delta^{op} &\to \mathrm{Sets} & X_{n,*} \colon \Delta^{op} &\to \mathrm{Sets} \\ [n] &\mapsto X_{n,n} & [m] &\mapsto X_{n,m} \end{aligned} \tag{2.13}$$

$$\begin{aligned} X_{*,m} \colon \Delta^{op} &\to \mathrm{Sets} & \pi_i^v(X) \colon \Delta^{op} &\to \mathrm{Sets} \\ [n] &\mapsto X_{n,m} & [n] &\mapsto \pi_i |X_{n,*}| \end{aligned} \tag{2.14}$$

Definition/Proposition 2.1.13. Let $X = (([n],[m]) \mapsto X_{n,m})$ be a bi-simplicial set. There are three functorially homeomorphic spaces that can be used to define the geometric realisation of X:

$$|X| := |\,\mathrm{diag}(X)| \cong |[n] \mapsto |X_{n,*}|| \cong |[m] \mapsto |X_{*,m}|| \tag{2.15}$$

Definition 2.1.14. Let $\mathcal{C}$ be a small category. Then

$$\Delta^{op} \to \mathrm{Sets} \tag{2.16}$$

$$n \mapsto \mathrm{Fun}([n], \mathcal{C}) =: \mathcal{N}_n(\mathcal{C}) \tag{2.17}$$

is a simplicial set, called the nerve of $\mathcal{C}$. Explicitly we may visualise a functor $A \colon [n] \to \mathcal{C}$ as a string of maps $A_0 \xrightarrow{f_0} \cdots \xrightarrow{f_{n-1}} A_n$. To illustrate what happens to morphisms consider the face-map δ_i and compute

$$\delta_i^*(A) = A \circ \delta_i = A_0 \xrightarrow{f_0} \cdots \xrightarrow{f_{i-1}} A_{i-1} \xrightarrow{f_i \circ f_{i-1}} A_{i+1} \cdots \xrightarrow{f_{n-1}} A_n \tag{2.18}$$

Similarly, for a degeneracy-map σ_i we observe

$$\sigma_i^*(A) = A \circ \sigma_i = A_0 \xrightarrow{f_0} \cdots \xrightarrow{f_{i-1}} A_i \xrightarrow{1} A_i \xrightarrow{f_i} \cdots \xrightarrow{f_{n-1}} A_n \tag{2.19}$$

Definition 2.1.15. Let $\mathcal{C}$ be a small category. Define $B\mathcal{C}$, the classifying space of

$\mathcal{C}$, to be the geometric realisation of the nerve $\mathcal{NC}$ of $\mathcal{C}$.

$$B\mathcal{C} = |\mathcal{NC}| \tag{2.20}$$

Let $\mathcal{C} = ([d] \mapsto C_d)$ be a simplicial category. Define the classifying space $B\mathcal{C}$ of $\mathcal{C}$ to be the geometric realisation of the bisimplicial set $(([d],[e]) \mapsto \mathcal{N}_d\mathcal{C}_e)$.

Proposition 2.1.16. *A monomorphism of simplicial sets induces a closed cofibration after geometric realisation.*

Remark 2.1.17. Essentially by definition we see that the degeneracy-maps in the nerve of a category are all injective. Therefore the simplicial space $([d] \mapsto B\mathcal{C}_d)$ is good in the sense of Definition 2.1.8, and the ordinary geometric realisation coincides with the fat one.

Following the exposition in [GJ09] we see that the geometric realisation functor takes values in the category of compactly generated Hausdorff spaces. We consider it as such for the following reason:

Proposition 2.1.18. *The functor*

$$|\cdot|\colon \text{sSet} \to \text{CGHaus} \tag{2.21}$$

preserves finite limits.

In particular, given two small categories $\mathcal{C}$ and $\mathcal{D}$ we have a homeomorphism $B(\mathcal{C} \times \mathcal{D}) \xrightarrow{\cong} B\mathcal{C} \times B\mathcal{D}$.
We conclude this section with the definition of the simplicial ring $R\Delta^{\cdot}$ which is familiar from the Karoubi-Villamayor K-theory:

Definition 2.1.19. Let R be a ring. Define the simplicial ring $R\Delta^{\cdot}$ by

$$R\Delta^d = R[T_0, \cdots, T_d]/_{(\sum T_i = 1)} \tag{2.22}$$

as a quotient of the polynomial ring in variables T_i for $0 \leq i \leq d$ and set

$$\phi^*(T_j) = \sum_{\phi(i)=j} T_i \tag{2.23}$$

for any map $[d] \xrightarrow{\phi} [e]$. If $(R, \bar{\cdot})$ is a ring with involution, then $(R\Delta^d, \bar{\cdot})$ is also a ring with involution, where $\bar{T_i} = T_i$ for all i.
Note that $R\Delta^d \cong R[T_1, \cdots, T_d]$, the polynomial ring in d variables, but that the maps ϕ^* are harder to describe succinctly in this notation.

2.2 Symmetric monoidal and exact categories, dualities and weak equivalences

In this section we describe a number of different additional structures a category can have, namely a symmetric monoidal or exact structure, possibly with duality and weak equivalences.

We begin by recalling the definition of symmetric monoidal categories. These are categories that come with a suitable notion of a direct sum of objects. Later on we will give a number of different examples relevant to this article. For the time-being, the reader is invited to think of the category of projective, finitely generated modules with the usual direct sum as a motivating example.

Definition 2.2.1. A (unital) symmetric monoidal category is a category $\mathcal{S}$ equipped with a functor $\oplus\colon \mathcal{S}\times\mathcal{S}\to\mathcal{S}$, a unit object 0 and natural isomorphisms $\alpha\colon S\oplus(T\oplus U)\xrightarrow{\sim}(S\oplus T)\oplus U$, $\sigma\colon S\oplus T\xrightarrow{\sim}T\oplus S$ and $\eta\colon 0\oplus S\xrightarrow{\sim}S$ for all objects $S,T,U\in\mathcal{S}$ such that certain diagrams commute (cf. [Gra76]). A symmetric monoidal category is strict if α and η are the identity morphism.
A functor in the category of small symmetric monoidal categories is a functor $F\colon\mathcal{S}\to\mathcal{S}'$ together with natural transformations $f\colon F(S)\oplus F(T)\xrightarrow{\sim}F(S\oplus T)$ and $\epsilon\colon 0\xrightarrow{\sim}F(0)$ making certain diagrams commute (cf. [Gra76]). A morphism in the category of small strict monoidal categories is a morphism of small monoidal categories such that f and ϵ are the identity transformation. Every symmetric monoidal category is naturally equivalent to a strict monoidal category (cf. [May74, Proposition 4.2]). Hence we may assume that all symmetric monoidal categories are strict.
A monoidal category $\mathcal{S}$ acts on the category $\mathcal{C}$ from the left, if there is a functor $\oplus\colon\mathcal{S}\times\mathcal{C}\to\mathcal{C}$ with natural isomorphisms $S\oplus(T\oplus C)\cong(S\oplus T)\oplus C$ and $0\oplus C\cong C$ for $S,T\in\mathcal{S}$ and $C\in\mathcal{C}$. Again, certain diagrams have to commute (cf. [Gra76]). Right actions are defined analogously.

The category of projective modules has, a fortiori, the structure of an exact category. In fact this will be the case for most categories we will study.

Definition 2.2.2. An exact category $\mathcal{E}$ is an additive category, together with a family of sequences

$$A\rightarrowtail B\twoheadrightarrow C \tag{2.24}$$

called conflations, admissible exact sequences or just exact sequences, satisfying a number of axioms (see e.g. [Qui73]). Equivalently it can be characterised as follows:

A small additive category $\mathcal{E}$ is an exact category if and only if there exists a full and faithful embedding $\mathcal{E} \subset \mathcal{A}$ into an abelian category $\mathcal{A}$ such that $\mathcal{E}$ is closed under extensions in $\mathcal{A}$ and such that a sequence in $\mathcal{E}$ is admissible exact if and only if its image in $\mathcal{A}$ is exact (in the sense of abelian categories).

Definition 2.2.3. A category with duality is a triple $(\mathcal{C}, *, \eta)$ where $\mathcal{C}$ is a category, $*\colon \mathcal{C}^{op} \to \mathcal{C}$ an functor and $\eta\colon 1 \xrightarrow{\cong} **$ a natural isomorphism such that for all objects $A \in \mathcal{C}$ we have $\eta_A^* \circ \eta_{A^*} = 1_{A^*}$.
A duality preserving functor $F\colon (\mathcal{C}, *, \eta) \to (\mathcal{D}, *, \eta)$, is a functor $F\colon \mathcal{C} \to \mathcal{D}$, such that $F(C^*) = F(C)^*$ and $F(f^*) = F(f)^*$ for objects C and maps f in $\mathcal{C}$.
If $f\colon C \to D^*$ is a morphism in $\mathcal{C}$ for some objects C and D, we sometimes use f^t as a shorthand for the composition $f^t = f^*\eta_D\colon D \to C^*$.

Definition 2.2.4. An exact category with duality is a category with duality $(\mathcal{E}, *, \eta)$ such that $\mathcal{E}$ is an exact category and such that the duality $*\colon \mathcal{C}^{op} \to \mathcal{C}$ is an exact functor.
A functor $F\colon (\mathcal{C}, *, \eta) \to (\mathcal{D}, *, \eta)$ of exact categories with duality, is a functor $F\colon \mathcal{C} \to \mathcal{D}$ that respects both, the exact structure and the duality.

Definition 2.2.5. Let $(\mathcal{C}, *, \eta)$ be an exact category with duality. Denote by $\mathcal{C}_h$ the associated category of inner product spaces on $(\mathcal{C}, *, \eta)$. An object in this category is a pair (C, ϕ), consisting of an object $C \in \mathcal{C}$ and a morphism $\phi\colon C \to C^*$ such that $\phi^*\eta = \phi$. A morphism $(C, \phi) \to (C', \phi')$ in $\mathcal{C}_h$ is a morphism $f\colon C \to C'$ in $\mathcal{C}$ such that $\phi = f^*\phi' f$. Note that a functor of exact categories with duality naturally induces a functor on the associated categories of inner product spaces. The category of inner product spaces is a symmetric monoidal category under orthogonal sum.

Remark 2.2.6. The index in $\mathcal{C}_h$ anticipates the fact that it will often be enough to study *hyperbolic* inner product spaces (cf. Definition 3.2.5).

Definition 2.2.7. A small exact category with weak equivalences and duality is a tuple $(\mathcal{E}, \omega, *, \eta)$, where

- $\mathcal{E}$ is a (small) exact category,
- $\omega \subset Mor(\mathcal{E})$ a set of morphisms called weak equivalences, which
  - is closed under composition, isomorphism and retracts,
  - contains all identities, and
  - satisfies the "two out of three" property, i.e. if two out of the three maps f, g, fg are weak equivalences then so is the third.

- $*\colon (\mathcal{E}^{op}, \omega) \to (\mathcal{E}, \omega)$ is an exact functor which respects weak equivalences $*(\omega) \subset \omega$ and

- $\eta\colon 1 \to **$ is a natural transformation such that for all $V \in \mathcal{E}$

  - $\eta_V : V \to V^{**}$ is a weak equivalence and
  - $\eta_V^* \circ \eta_{V^*} = 1_{V^*}$.

Example 2.2.8. Let $(\mathcal{E}, *, \eta)$ be an exact category with duality.

- Then $(\mathcal{E}, iso, *, \eta)$ is an exact category with duality and weak equivalences. Here iso is the set of isomorphisms in $\mathcal{E}$.

- Let further $n \in \mathbb{Z}$ be an integer. Then we can turn the category of bounded chain complexes $\mathrm{Ch}^b(\mathcal{E})$ into an exact category with weak equivalences and duality $(\mathrm{Ch}^b(\mathcal{E}), quis, *^n, \eta^n)$. Here, an object is a bounded chain complex $(E_i, d_i) = (\ldots \to E_i \xrightarrow{d_i} E_{i+1} \to \ldots)$ of objects in $\mathcal{E}$, the duality is given by $(E_i, d_i)^{*^n} = (E^*_{-i-n}, (-1)^{i+1} d^*_{-i-1-n})$, the natural transformation is given degree-wise as $\eta^n_{E_i} = (-1)^{i(i+n)} \eta_{E_i}$ and the weak equivalences are precisely the quasi-isomorphisms of chain complexes.

Remark 2.2.9. There exist different sign conventions for the duality on the category of chain complexes $\mathrm{Ch}^b(\mathcal{E})$ but the one chosen here seems to be the most natural as it is induced by the internal hom.

Chapter 3

Constructions of K-theory and hermitian K-theory

3.1 The K-theory of a symmetric monoidal category

Throughout this paper, when we say K-theory, we will usually mean the K-theory of symmetric monoidal categories. In this section we will recall the necessary definitions and constructions. In a later section we will see a different method that assigns a K-theory space to an exact category.

3.1.1 Group-completion

The K-theory of a symmetric monoidal category is defined via the so-called group-completion, a construction that generalises the Grothendieck-construction that associates an abelian group $G = M^+$ with a given abelian monoid M in a universal way. Here, the starting point is the classifying space $B\mathcal{S}$ of a symmetric monoidal category $\mathcal{S}$, which is an H-space. The universal object $B\mathcal{S}^+$ is then a *group-like* H-space. The set of connected components of $B\mathcal{S}$ is an abelian monoid and its Grothendieck group is precisely given by the connected components of $B\mathcal{S}^+$.

Definition 3.1.1. An H-space is a topological space X, together with a "multiplication" map $\mu\colon X \times X \to X$ with an identity element, i.e. an element $e \in X$ such that $\mu(e, -)$ and $\mu(-, e)$ are (homotopic to) the identity. Further, we assume H-spaces to be associative, i.e. that there exists a homotopy $\mu \circ (\mu \times 1) \simeq \mu \circ (1 \times \mu)$. An H-space is called group-like if the multiplication map has a homotopy inverse, i.e. if there exists a map $i\colon X \to X$ such that the maps $x \mapsto \mu(i(x), x)$ and $x \mapsto \mu(x, i(x))$ are homotopic to the constant map $x \mapsto e$.

The set $\pi_0(X)$ of connected components of an H-space X is a monoid and automatically a group if X is group-like. If X is a CW-complex, the converse holds (cf. [Whi78, X.2]):

Proposition 3.1.2. *Let X be a CW-complex that is also an H-space. Then X is group-like if and only if $\pi_0(X)$ is a group.*

Proposition 3.1.3. *Let $\mathcal{S}$ be a symmetric monoidal category. Then the classifying space $B\mathcal{S}$ is an H-space.*

Proof. The direct sum functor $\oplus\colon \mathcal{S} \times \mathcal{S} \to \mathcal{S}$ induces the multiplication map

$$B\mathcal{S} \times B\mathcal{S} \cong B(\mathcal{S} \times \mathcal{S}) \to B\mathcal{S} \tag{3.1}$$

The identity element e is given by the object $0 \in \mathcal{S}$. □

We are now in the position that we can define the group-completion of an H-space:

Definition 3.1.4. Let X be a homotopy commutative and homotopy associative H-space. A group-completion of X is a group-like H-space Y together with a map of H-spaces $X \to Y$ such that the natural map on (localised) homology groups

$$\pi_0(X)^{-1}H_*(X,R) \to H_*(Y,R) \tag{3.2}$$

is an isomorphism for all commutative rings R and such that the natural map

$$\pi_0(X) \to \pi_0(Y) \tag{3.3}$$

is the classical group-completion after Grothendieck.

Definition 3.1.5. Two maps $f, g\colon X \to Y$ of topological spaces are called weakly homotopic if they induce the same map on homotopy classes $[A,X] \to [A,Y]$ for every finite CW-complex A. A weak H-map is a map of H-spaces compatible with the multiplication maps up to weak homotopy.

The group-completion is universal in the following sense:

Proposition 3.1.6 ([CCMT84, Proposition 1.2])**.** *Let X be an H-space such that $\pi_0(X)$ is either countable or contains a countable cofinal submonoid. Moreover let $g\colon X \to Y$ be a group-completion. Then for any group-like H-space Z and weak H-map $f\colon X \to Z$, there exists a weak H-map $\tilde{f}\colon Y \to Z$, unique up to weak homotopy such that*

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[dr, "f"'] & Y \arrow[d, "\tilde{f}"] \\ & Z \end{tikzcd} \tag{3.4}$$

is weakly homotopy-commutative.

In particular group-completions are unique as follows:

Corollary 3.1.7. *Let X be an H-space such that $\pi_0(X)$ is either countable or contains a countable cofinal submonoid. If $f'\colon X \to X'$ and $f''\colon X \to X''$ are two group-completions, then there is a homotopy equivalence $g\colon X' \to X''$, unique up to weak homotopy, such that gf' and f'' are weakly homotopic.*

In regard of the theorem above we speak of *the* group-completion of an H-space X and denote it by $X \to X^+$.

Definition 3.1.8. Let $\mathcal{S}$ be a symmetric monoidal category. We define the K-theory space of $\mathcal{S}$ to be the group-completion of the classifying space $B\mathcal{S}$:

$$K(\mathcal{S}) := B\mathcal{S}^+ \tag{3.5}$$

The K-groups of $\mathcal{S}$ are the homotopy groups of $K(\mathcal{S})$, in symbols:

$$K_n(\mathcal{S}) := \pi_n K(\mathcal{S}) \tag{3.6}$$

Sometimes it is more convenient to think of K-theory in terms of spectra rather than topological spaces. This will not influence the K-groups:

Proposition 3.1.9 ([Tho82, Appendix])**.** *There exists a functor K from the category of symmetric monoidal categories to the category of -1-connected Ω-spectra which naturally exhibits the group-completion of (the classifying space of) a symmetric monoidal as the zeroth space in the spectrum. Given a symmetric monoidal category $\mathcal{S}$ we call the spectrum $K(\mathcal{S})$ the connective K-theory spectrum of $\mathcal{S}$. Its homotopy groups are the K-groups of $\mathcal{S}$.*

3.2 Simplicial fibrations and group-complete H-spaces

When we work over the simplicial ring $R\Delta^{\cdot}$ we are often faced with the following technical obstacle: We are given spaces $E(R)$, $B(R)$ and $F(R)$ that depend functorially on the ring R and that fit as total space, base space and fibre into the fibration sequence

$$F(R) \to E(R) \to B(R) \tag{3.7}$$

The question is now, whether the sequence

$$F(R\Delta^{\cdot}) \to E(R\Delta^{\cdot}) \to B(R\Delta^{\cdot}) \tag{3.8}$$

of simplicial spaces is still a fibration sequence after geometric realisation. We solve this problem by showing that a certain map satisfies the so-called π_*-Kan condition. For an exact formulation of the π_*-Kan condition we refer the reader to Section B3 of [BF78]. In the same reference there is also a sufficient condition for it to be satisfied:

Lemma 3.2.1 ([BF78, B3.1])**.** *Let X be a bisimplicial set with $X_{m,*}$ simple for $m \geq 0$. Then X satisfies the π_*-Kan condition if and only if the simplicial map*

$$\pi_n^v(X)_{free} \to \pi_0^v(X) \tag{3.9}$$

is a fibration for all $n \geq 1$.

Here, $\pi_n^v(X)_{\mathrm{free}}$ is the group of unpointed homotopy classes of X and the map is the obvious surjection. Moreover, we call a simplicial set X simple, if every connected

component of $|X|$ is a simple space.
We have defined K-theory spaces as the geometric realisation of a certain simplicial set, similarly we may consider simplicial K-theory as the geometric realisation of a bisimplicial set. If we denote this bisimplicial set by X, then $X_{m,*}$ is a group-complete H-space. In particular it is simple with homotopy equivalent connected components, and $\pi_0^v(X)$ is already an abelian group. Moreover, a surjective map of simplicial abelian groups is automatically a fibration. So the π_*-Kan condition will be satisfied trivially. This allows us to apply the following result to our obstruction problem:

Proposition 3.2.2 ([BF78, B4]). *Let*

$$\begin{array}{ccc} V & \longrightarrow & X \\ \downarrow & & \downarrow \\ W & \longrightarrow & Y \end{array} \tag{3.10}$$

be a commutative square of bisimplicial sets, such that the terms $V_{m,*}, X_{m,*}, Y_{m,*}$ and $Z_{m,*}$ form a homotopy fibre square for each $m \geq 0$. If X and Y satisfy the π_*-Kan condition and if $\pi_0^v X \to \pi_0^v Y$ is a fibration, then

$$\begin{array}{ccc} \operatorname{diag} V & \longrightarrow & \operatorname{diag} X \\ \downarrow & & \downarrow \\ \operatorname{diag} W & \longrightarrow & \operatorname{diag} Y \end{array} \tag{3.11}$$

is a homotopy fibre square.

As a first application of this proposition we quote a lemma that appears in a slightly modified version as Theorem 7.1 in [Gra95].

Lemma 3.2.3. *Let X be a simplicial group-like H-space. Then the sequence*

$$|d \mapsto I_* X_d| \to |X| \to |d \mapsto \pi_0 X_d| \tag{3.12}$$

is a fibration sequence, where $I_* X_d$ denotes the connected component of the basepoint in X_d.

Proof. All connected components of a group-like H-space are homotopy equivalent, so it is clear that the sequence is level-wise a fibration. Because X is group-like, the simplicial spaces X and $(d \mapsto \pi_0 X_d)$ satisfy the π_*-Kan condition, and the identity map of $|d \mapsto \pi_0 X_d|$ is clearly a fibration. The result follows. $\square$

3.2.1 Hermitian K-theory via inner product spaces

We can define the Grothendieck-Witt theory of an exact category with duality using the machinery of the previous Section: From an exact category with duality we pass to inner product spaces and consider the group-completion of the resulting symmetric monoidal category. In fact, this is the point of view we will take for the bigger part of this article.

Definition 3.2.4. Let $(\mathcal{C}, *, \eta)$ be an exact category with duality. Define the Grothendieck-Witt space of $\mathcal{C}$ as the group-completion of $B\mathcal{C}_h$

$$GW(\mathcal{C}) := B\mathcal{C}_h^+ \tag{3.13}$$

The Grothendieck-Witt groups are the homotopy groups

$$GW_n(\mathcal{C}) := \pi_n GW(\mathcal{C}) \tag{3.14}$$

There are two important functors linking the category $\mathcal{C}_h$ to the category $\mathcal{C}$:

Definition 3.2.5. Let $(\mathcal{C}, *, \eta)$ be an exact category with duality. The hyperbolic functor $H\colon i\mathcal{C} \to i\mathcal{C}_h$ is defined via $H(C) = C \oplus C^*$ equipped with the form

$$\begin{aligned} C \oplus C^* &\to C^* \oplus C^{**} \\ (x, f) &\mapsto (f, \eta_C(x)) \end{aligned} \tag{3.15}$$

An isomorphism $g\colon C \to C'$ in $\mathcal{C}$ is mapped to the isometry $\begin{pmatrix} g & \\ & (g^*)^{-1} \end{pmatrix}$. An object (C, ϕ) of $\mathcal{C}_h$ is called hyperbolic if it is in the essential image of the hyperbolic functor. The forgetful functor $i\mathcal{C}_h \to i\mathcal{C}$ simply maps $(C, \phi) \to C$ and for morphisms $g \mapsto g$.

Any symmetric inner product space can be completed to an object in the essential image of the hyperbolic functor:

Proposition 3.2.6. *Let $(\mathcal{C}, *, \eta)$ be an additive category with duality such that 2 is invertible. Further, let (C, ϕ) be a symmetric inner product space in $\mathcal{C}_h$, i.e. an isomorphism $\phi\colon V \to V^*$ with $\phi = \phi^*\eta$. Then $(V, \phi) \oplus (V^*, -\eta\phi^{-1})$ is hyperbolic.*

Proof. We proceed in two steps. First, we consider the automorphism $\begin{pmatrix} 1 & 0 \\ \phi & -1 \end{pmatrix}$ of

$V \oplus V^*$ and readily compute

$$\begin{pmatrix} 1 & \phi\eta^{-1} \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \phi & 0 \\ 0 & -\eta\phi^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \phi & -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \eta & -\eta\phi^{-1} \end{pmatrix} \tag{3.16}$$

Subsequently, consider the automorphism $\begin{pmatrix} 1 & \frac{\phi^{-1}}{2} \\ 0 & 1 \end{pmatrix}$ and compute

$$\begin{pmatrix} 1 & 0 \\ \frac{\eta\phi^{-1}}{2} & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ \eta & -\eta\phi^{-1} \end{pmatrix} \begin{pmatrix} 1 & \frac{\phi^{-1}}{2} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \eta & 0 \end{pmatrix} \tag{3.17}$$

We conclude that $(V, \phi) \oplus (V^*, -\eta\phi^{-1})$ is isometric to an object in the image of the hyperbolic functor via the composition of these two automorphisms. □

3.2.2 A model for the group-completion

In the previous sections we abstractly defined the K-theory space of a symmetric monoidal category as the group-completion of the classifying space. In this section we construct an explicit model which we will use for the remainder of this article.

Definition 3.2.7. A monoidal category $\mathcal{S}$ acts on a category $\mathcal{C}$ if there is a functor

$$\oplus : \mathcal{S} \times \mathcal{C} \to \mathcal{C} \tag{3.18}$$

with natural isomorphisms $(S \oplus T) \oplus C \cong S \oplus (T \oplus C)$ and $e \oplus C \cong C$, for $S, T \in \mathcal{S}$, $C \in \mathcal{C}$ satisfying further coherence conditions analogous to those defining $\mathcal{S}$.

Definition 3.2.8. Let $\mathcal{S}$ be a symmetric monoidal category acting on the category $\mathcal{C}$ from the right. Let $\langle \mathcal{S}, \mathcal{C} \rangle$ denote the category which has the same objects as $\mathcal{C}$, and where a morphism $C \to C'$ is given by equivalence classes of pairs $[S, \alpha]$ with $S \in \mathcal{S}$ and $\alpha\colon C \oplus S \to C'$ being a morphism in $\mathcal{C}$. Here, two classes $[S, \alpha]$ and $[S', \alpha']$ coincide if there is an isomorphism $u\colon S \to S'$ such that the diagram

$$\begin{tikzcd} C \oplus S \arrow[rr, "1+u"] \arrow[dr, "\alpha"'] & & C \oplus S' \arrow[dl, "\alpha'"] \\ & C' & \end{tikzcd} \tag{3.19}$$

commutes.

If $[S, \alpha]\colon C \to C'$ and $[S', \alpha']\colon C' \to C''$ are two morphisms in $\langle \mathcal{S}, \mathcal{C} \rangle$, then their composition is defined to be the morphism $[S \oplus S', \alpha' \circ (\alpha + 1_{S'})]\colon C \to C''$.

If $\mathcal{S}$ acts diagonally on $\mathcal{S} \times \mathcal{C}$, we write

$$\mathcal{S}^{-1}\mathcal{C} := \langle \mathcal{S}, \mathcal{S} \times \mathcal{C} \rangle . \tag{3.20}$$

Proposition 3.2.9 ([Gra76, p. 221])**.** *Let $\mathcal{S}$ be a symmetric monoidal category. If translations in $\mathcal{S}$ are faithful, we write $\mathcal{S}^+ = (i\mathcal{S})^{-1}\mathcal{S}$. The inclusion $\mathcal{S} \to \mathcal{S}^+$ is a group-completion.*

3.2.3 Cofinality

In this section we will see that there are often simpler subcategories which carry all the essential K-theoretical information. The first major ingredient is the so-called cofinality theorem which states that cofinal subcategories result in the same connected component on K-theory. In Section 4.5 we will see that for the stabilised (hermitian) K-theory of automorphisms also the number of connected components will coincide.

Definition 3.2.10. Let $\mathcal{S}, \mathcal{T}$ be symmetric monoidal categories. A symmetric monoidal functor $F : \mathcal{S} \to \mathcal{T}$ is called cofinal, if

(a) for every $T \in \mathcal{T}$ there is a $T' \in \mathcal{T}$ and a $S \in \mathcal{S}$ such that $T \oplus T' \cong F(S)$, and

(b) $Aut_{\mathcal{S}}(S) \cong Aut_{\mathcal{T}}(F(S))$ holds for all $S \in \mathcal{S}$.

Theorem 3.2.11 (Cofinality Theorem, [Wei13, 4.11])**.** *Let $F : \mathcal{S} \to \mathcal{T}$ be a cofinal functor of symmetric monoidal categories. Then the base point components of $K(\mathcal{S})$ and $K(\mathcal{T})$ are homotopy equivalent. In particular $K_n(\mathcal{S}) \cong K_n(\mathcal{T})$ for all $n \geq 1$.*

3.3 The K-theory space of an exact category

Most symmetric monoidal categories we will work with are actually isomorphism categories of some exact category. The K-theory space of an exact category is homotopy equivalent to that of the underlying symmetric monoidal category, if the exact structure is split. Although this condition is hardly ever satisfied for our examples, we can force any exact category to be split by changing the exact structure. Throughout this paper we will regularly use results on the K-theory of exact categories and apply them on the K-theory of symmetric monoidal categories.

Definition 3.3.1 (Quillen)**.** Given an exact category $\mathcal{E}$ we define a category $Q\mathcal{E}$ with the same objects as $\mathcal{E}$ and where a morphism $A \to B$ is an equivalence class of triples (U, p, i). Here $U \in \mathcal{E}$ and p and i are morphisms in $\mathcal{E}$ as follows

$$A \overset{p}{\twoheadleftarrow} U \overset{i}{\rightarrowtail} B \tag{3.21}$$

The triples $(U, p, i) \sim (U', p', i')$ are equivalent, if there exists an isomorphism $g\colon U \xrightarrow{\cong} U'$ such that $p = p'g$ and $i = i'g$. The composition of $A \xrightarrow{(U,p,i)} B \xrightarrow{(W,q,j)} C$ is defined as $(U \times_B W, pq', ji')$, where $U \times_B W$, q' and i' are defined as a pull-back:

$$\begin{tikzcd}
 & & C \\
 & B & W \arrow[l, two heads, "q"'] \arrow[u, tail, "j"] \\
A & U \arrow[l, two heads, "p"'] \arrow[u, tail, "i"] & U \times_B W \arrow[l, two heads, "q'"'] \arrow[u, tail, "i'"]
\end{tikzcd} \tag{3.22}$$

Definition 3.3.2. Let $\mathcal{E}$ be an exact category. Define the K-theory space of $\mathcal{E}$ as the loop-space of the category $Q\mathcal{E}$:

$$K^Q(\mathcal{E}) := \Omega B Q \mathcal{E} \tag{3.23}$$

Now, Quillen's famous "Q=+"-theorem provides a connection to the K-theory of a symmetric monoidal category:

Definition 3.3.3. Let $\mathcal{C}$ be a category. We denote by $i\mathcal{C}$ the isomorphism category, i.e. the category with the same objects as $\mathcal{C}$ and isomorphisms in $\mathcal{C}$ as morphisms.

Theorem 3.3.4 ([Gra76, p. 228])**.** *Let $\mathcal{C}$ be an exact category such that every exact sequence splits and let $\mathcal{S} = i\mathcal{C}$ be the isomorphism category of $\mathcal{C}$. Then $\mathcal{S}$ is a symmetric monoidal category that inherits the functor $\oplus$ from the additive structure of $\mathcal{C}$. Finally, there exists a homotopy equivalence*

$$K^Q(\mathcal{C}) \simeq K(\mathcal{S}) \tag{3.24}$$

of Quillen's K-theory to the K-theory of symmetric monoidal categories.

3.4 Hermitian K-theory of an exact category with duality

Definition 3.4.1 ([Uri90], [CL86])**.** Let $(\mathcal{C}, *, \eta)$ be an exact category with duality. Define $Q^h\mathcal{C}$ to be the category with inner product spaces (A, ϕ) (in $\mathcal{C}$) as objects. A morphisms $(A, \phi) \to (B, \psi)$ is given by an equivalence class of triples (U, p, i)

$$A \overset{p}{\twoheadleftarrow} U \overset{i}{\rightarrowtail} B \tag{3.25}$$

as in Quillen's Q-construction such that $\phi|_U = \psi|_U$, i.e. $p^*\phi p = i^*\psi i$ and such that $Ker\ i^* \circ \psi =: U^\perp$ can be identified with $Ker\ p$ via i. The composition is defined as in Quillen's Q-construction $Q\mathcal{E}$.

Definition 3.4.2. Let $(\mathcal{C}, *, \eta)$ be an exact category with duality. The Grothendieck-Witt space is defined as the homotopy fibre

$$GW^Q(\mathcal{C}) = \mathrm{hofib}(BQ^h\mathcal{C} \to BQ\mathcal{C}) \tag{3.26}$$

$$(V, \phi) \mapsto V \tag{3.27}$$

Then the Grothendieck-Witt groups are given by

$$GW_i^Q(\mathcal{C}) = \pi_i GW^Q(\mathcal{C}). \tag{3.28}$$

Proposition 3.4.3 ([Sch04])**.** *Let $(\mathcal{C}, *, \eta)$ be an exact category with duality such that every exact sequence splits. Then the Grothendieck-Witt spaces according to the hermitian Q-construction and as the group-completion of the category of inner product spaces coincide:*

$$GW(\mathcal{C}) \simeq GW^Q(\mathcal{C}) \tag{3.29}$$

3.5 Hermitian K-theory of exact categories with weak equivalences and duality

This section gives an exposition of the theory of shifted Grothendieck-Witt groups. Then main references are [Sch10] and [Sch12]. We have already seen two different ways to construct a Grothendieck-Witt space from an exact category with duality, once as the group-completion of the symmetric monoidal category of inner product spaces and once via the hermitian Q-construction. We have also seen that the resulting Grothendieck-Witt spaces coincide if the exact category is split. In this section we will learn how to construct the Grothendieck-Witt space of an exact category with duality and weak equivalences via the hermitian $S_\bullet$ construction. As the central example we will study the category of chain complexes in an exact category with duality and quasi-isomorphisms as weak equivalences. In this scenario there are infinitely many possible dualities, where one is the "naive" choice, and countably many arise essentially from shifting a chain complex to either side. It turns out that the resulting theories are four-periodic while the Grothendieck-Witt spaces of symmetric and anti-symmetric forms of Section 4.2 can be naturally identified with the Grothendieck-Witt spaces of even index in the new sense. The highlight of this section will be Theorem 3.5.20, which can be seen as a generalisation of Karoubi's fundamental Theorem and displays a strong relation between the various shifted theories amongst each other and with ordinary K-theory. This implies that the Grothendieck-Witt groups in odd shifts, introduced in this section, can be identified with Karoubi's U- and V-theory.

3.5.1 The hermitian $S_\bullet$-construction

In this section we will present the hermitian $S_\bullet$-construction that allows us to associate a Grothendieck-Witt space to an exact category with weak equivalences and duality. We will also see how the resulting space relates to the Grothendieck-Witt spaces we have encountered before. First, recall the classical $S_\bullet$-construction, according to Waldhausen [Wal85]:

Definition 3.5.1. Let $n \in \mathbb{N}$ be a positive integer. Define the arrow category by $\mathcal{A}r[d] := \mathrm{Fun}([1],[d])$. Explicitly an object is of the form $a \leq b$ for $0 \leq a, b \leq d$ and there is exactly one morphism $(a \leq b) \to (a' \leq b')$ if and only if $a \leq a'$ and $b \leq b'$, i.e. if we have a diagram in $[d]$ of the form

$$\begin{tikzcd} a \arrow[r] \arrow[d] & b \arrow[d] \\ a' \arrow[r] & b' \end{tikzcd} \tag{3.30}$$

The functor

$$\Delta \to \mathrm{Cat} \tag{3.31}$$

$$[d] \mapsto \mathcal{A}r[d] \tag{3.32}$$

is a cosimplicial category. If $A\colon \mathcal{A}r[d] \to \mathcal{C}$ is a functor for some category $\mathcal{C}$, we write $A_{p,q} := A(p \leq q)$.

Definition 3.5.2. Let $(\mathcal{E}, \omega)$ be an exact category with weak equivalences. We have a simplicial exact category with weak equivalences

$$\Delta^{op} \to \mathrm{Cat} \tag{3.33}$$
$$[d] \mapsto \mathrm{Fun}(\mathcal{A}r[d], \mathcal{E}) \tag{3.34}$$

Here a sequence $A \to B \to C$ is exact if for all $p \leq q$ the sequence $A_{p,q} \to B_{p,q} \to C_{p,q}$ is exact in $\mathcal{E}$, and a morphism $A \to B$ is a weak equivalence if for all $p \leq q$ the morphism $A_{p,q} \to B_{p,q}$ is a weak equivalence.

Definition 3.5.3. Let $S_d\mathcal{E} \subset \mathrm{Fun}(\mathcal{A}r[d], \mathcal{E})$ be the full subcategory of those functors $A\colon \mathcal{A}r[d] \to \mathcal{E}$ such that for all $0 \leq p \leq q \leq r \leq d$ the sequence

$$A_{p,q} \rightarrowtail A_{p,r} \twoheadrightarrow A_{q,r} \tag{3.35}$$

is an admissible short exact sequence in $\mathcal{E}$ and $A_{p,p} = 0$.

Definition 3.5.4. Let $(\mathcal{C}, \omega)$ be an exact category with weak equivalences. Write $\omega\mathcal{C}$ for the category with the same objects as $\mathcal{C}$ and weak equivalences as morphisms.

Definition 3.5.5. Let $(\mathcal{E}, \omega)$ be an exact category with weak equivalences. Then

$$\omega S_\bullet\mathcal{E}\colon \Delta^{op} \to \mathrm{Cat} \tag{3.36}$$
$$[d] \mapsto \omega S_d\mathcal{E} \tag{3.37}$$

is a simplicial exact category. Define $K(\mathcal{E}, \omega) = \Omega|\omega S_\bullet\mathcal{E}|$.

Proposition 3.5.6 ([Wal85, 1.9 Appendix])**.** *Let $(\mathcal{E}, iso)$ be an exact category with weak equivalences, where iso is the class of isomorphisms in $\mathcal{E}$. Then*

$$|Q\mathcal{E}| \simeq |iS_\bullet\mathcal{E}|. \tag{3.38}$$

The hermitian $S_\bullet$-construction, due to Schlichting [Sch10], generalises these ideas:

Definition 3.5.7. Let A, B be ordered sets. Then we write AB for the concatenation, i.e. the ordered set $A \sqcup B$ with $a < b$ for all $a \in A$ and $b \in B$. In particular write $[d]^{op}[d] = [2d+1] = \{d' < \ldots < 1' < 0' < 0 < 1 < \ldots < d\}$. If we interpret $[d]^{op}[d]$ as a category, then it has the duality $p \mapsto p'$ and $p' \mapsto p$.

Definition 3.5.8. Let $(\mathcal{E}, \omega, *, \eta)$ be an exact category with weak equivalences and duality. Define the simplicial category with duality

$$\Delta^{op} \to \mathrm{Cat} \tag{3.39}$$
$$[d] \mapsto \mathcal{R}_d\mathcal{E} \tag{3.40}$$

where $\mathcal{R}_d\mathcal{E} = S_{[d]^{op}[d]}\mathcal{E}$ and $(A^*)_{p,q} = (A_{p',q'})^*$ for functors $A\colon Ar([d]^{op}[d]) \to \mathcal{E}$. We refer to this process as edge-wise subdivision.

Definition 3.5.9 ([Sch10])**.** Let $(\mathcal{E}, \omega, *, \eta)$ be an exact category with weak equivalences and duality. Then we define the Grothendieck-Witt space of $\mathcal{E}$ as

$$GW(\mathcal{E}) = GW(\mathcal{E}, \omega, *, \eta) = \mathrm{hofib}(|(\omega R_\bullet\mathcal{E})_h| \to |\omega S_\bullet\mathcal{E}|) \tag{3.41}$$
$$(A, \phi) \mapsto A \circ i \tag{3.42}$$

where $i\colon [d] \to [d]^{op}[d]$ is the map with $p \mapsto p$.

Proposition 3.5.10 ([Sch10, Proposition 2])**.** *Let $(\mathcal{E}, *, \eta)$ be an exact category with duality, then $(\mathcal{E}, iso, *, \eta)$ is an exact category with duality and weak equivalences. There exists a homotopy equivalence*

$$|(iR_\bullet\mathcal{E})_h| \simeq |Q^h\mathcal{E}|. \tag{3.43}$$

Corollary 3.5.11. *Recall that for any exact category with duality $(\mathcal{E}, *, \eta)$ there is the exact category with duality and weak equivalences $(\mathcal{E}, iso, *, \eta)$. The Grothendieck-Witt space of $(\mathcal{E}, *, \eta)$ according to Definition 3.4.2 and of $(\mathcal{E}, iso, *, \eta)$ in the sense of the definition above are homotopy equivalent.*

Proof. This follows directly from the definitions, Proposition 3.5.10 and Proposition 3.5.6. □

Definition 3.5.12. Let $(\mathcal{E}, \omega)$ be an exact category with weak equivalences. Define $\mathcal{HE} := \mathcal{E} \times \mathcal{E}^{op}$ to be the exact category with weak equivalences and duality given by $(E, F)^* = (F, E)$.
Now assume $(\mathcal{E}, *, \eta, \omega)$ is an exact category with weak equivalences and duality. We define the forgetful functor

$$\begin{aligned} F\colon \mathcal{E} &\to \mathcal{HE} \\ X &\mapsto (X, X^*) \\ f &\mapsto (f, f^*) \end{aligned} \tag{3.44}$$

and the hyperbolic functor

$$\begin{aligned} H\colon \mathcal{HE} &\to \mathcal{E} \\ (X,Y) &\mapsto X \oplus Y^* \\ (f,g) &\mapsto f + g^* \end{aligned} \tag{3.45}$$

Proposition 3.5.13 ([Sch10, Proposition 1])**.** *Let $(\mathcal{E},\omega)$ be an exact category with weak equivalences. Then there is a natural weak equivalence*

$$GW(\mathcal{HE},\omega) \simeq K(\mathcal{E},\omega) \tag{3.46}$$

Remark 3.5.14. Together with the proposition above, the forgetful and hyperbolic functor induce maps $F\colon GW(\mathcal{E},*,\eta,\omega) \to K(\mathcal{E},\omega)$ and $H\colon K(\mathcal{E},\omega) \to GW(\mathcal{E},*,\eta,\omega)$, respectively.

3.5.2 Shifted Grothendieck-Witt spaces

In the previous section we have defined a Grothendieck-Witt space of an exact category with weak equivalences and duality. In this section we specialise on the category of chain complexes in an exact category $\mathcal{E}$ with the various dualities as defined in Example 2.2.8.

Definition 3.5.15. Let $(\mathcal{E},*,\eta)$ be an exact category with duality, let $\epsilon \in \{\pm 1\}$, and let $n \in \mathbb{Z}$. We define the shifted Grothendieck-Witt spaces of $\mathcal{E}$ via

$${}_{\epsilon}GW^{[n]}(\mathcal{E}) := GW(\mathrm{Ch}^b(\mathcal{E}), quis, *^n, \epsilon\eta^n) \tag{3.47}$$

Let $T\colon \mathrm{Ch}^b(\mathcal{E}) \to \mathrm{Ch}^b(\mathcal{E})$ be the shifting functor, explicitly given by $T(E_i) = E_{i+1}, T(f)_i = f_{i+1}$ and $(d_{TE})_i = d_{i+1}$. Then T together with the duality compatibility isomorphism $(-1)^i id_{T(E^{*^n})_i} = (-1)^i id_{T(E)_i^{*^{n+2}}}$ induces an isomorphism of exact categories with duality and weak equivalences

$$T\colon (\mathrm{Ch}^b(\mathcal{E}), quis, *^n, \eta^n) \to (\mathrm{Ch}^b(\mathcal{E}), quis, *^{n+2}, -\eta^{n+2}) \tag{3.48}$$

Therefore we conclude:

Proposition 3.5.16 (cf. [Sch10, p.45f])**.** *Let $(\mathcal{E},*,\eta)$ be an exact category with duality, let $\epsilon \in \{\pm 1\}$, and let $n \in \mathbb{Z}$. The shifting functor $T\colon \mathrm{Ch}^b(\mathcal{E}) \to \mathrm{Ch}^b(\mathcal{E})$ induces homotopy equivalences*

$${}_{\epsilon}GW^{[n]}(\mathcal{E}) \simeq {}_{-\epsilon}GW^{[n+2]}(\mathcal{E}) \tag{3.49}$$

and in particular

$$ {}_{\epsilon}GW^{[n]}(\mathcal{E}) \simeq {}_{\epsilon}GW^{[n+4]}(\mathcal{E}) \tag{3.50}$$

Remark 3.5.17.

- There is an alternative, naive sign convention for the shifted dualities, such that the duality compatibility isomorphism for the shift T is just the identity. Since we won't use this convention anywhere else, we do not go into detail.

- If $\epsilon = 1$, it is often dropped from the notation.

The following Theorem relates the Grothendieck-Witt space of an exact category with duality as in Section 3.4 to the Grothendieck-Witt space as defined in this section:

Theorem 3.5.18 ([Sch10, Proposition 6]). *Let $(\mathcal{E}, *, \eta)$ be an exact category with duality. Then the functor $(\mathcal{E}, iso, *, \eta) \to (\mathrm{Ch}^b(\mathcal{E}), quis, *^0, \eta^0)$, that maps an object to the corresponding chain complex concentrated in degree 0, induces a homotopy equivalence*

$$GW(\mathcal{E}, *, \eta) \xrightarrow{\simeq} GW(\mathrm{Ch}^b(\mathcal{E}), quis, *^0, \eta^0) \tag{3.51}$$

Remark 3.5.19. The analogous statement for K-theory holds and is known as the Gillet-Waldhausen Theorem [Wei13, Theorem V.2.2]. The forgetful functor and the hyperbolic functor according to Section 3.5.1 and to Definition 3.2.5 are compatible.

We conclude this exposition with a generalisation of Karoubi's fundamental theorem:

Theorem 3.5.20 ([Sch12, Theorem 6.1]). *Let $(\mathcal{E}, *, \eta)$ be an exact category with duality such that $\frac{1}{2} \in \mathcal{E}$ and $\epsilon \in \{\pm 1\}$. Then there is a fibration sequence*

$$ {}_{\epsilon}GW^{[n]}(\mathcal{E}) \xrightarrow{F} K(\mathcal{E}) \xrightarrow{H} {}_{\epsilon}GW^{[n+1]}(\mathcal{E}) \tag{3.52}$$

where the maps are induced by the forgetful and the hyperbolic functor, respectively.

Remark 3.5.21. In fact, Schlichting proves a stronger result, for he constructs positive Ω-spectra representing Grothendieck-Witt theory and demonstrates that there is an exact triangle in the homotopy category of spectra. Since the Ω^∞-functor takes exact triangles in this category to fibration sequences in the category of spaces, the weaker result follows.

3.6 Grothendieck-Witt groups

Apart from the various Grothendieck-Witt *spaces* themselves, a very central role is played by their set of connected components, i.e. the 0-th Grothendieck-Witt *groups*. In this section we study these groups for the shifted Grothendieck-Witt theories of automorphisms. Methodically, we study chain complexes of free modules with automorphisms as an example of exact categories with duality and weak equivalences. However we will use some result that stem from the work on Grothendieck-Witt groups of triangulated categories. The main references for this section are [Wal03] and, to a certain extent, also [Sch10].

3.6.1 Ad-hoc definitions of Grothendieck-Witt groups

The idea of the Grothendieck-Witt group of an exact category with duality goes back to Knebusch [Kne77], it was generalised to exact categories with duality and weak equivalences in [Sch10]:

Definition 3.6.1. The Grothendieck-Witt group of an exact category with weak equivalences and duality $(\mathcal{E}, \omega, *, \eta)$ is the abelian group $GW_0(\mathcal{E}) = GW_0(\mathcal{E}, \omega, *, \eta)$ generated by inner product spaces $[X, \phi]$ in $(\mathcal{E}, \omega, *, \eta)$, i.e. objects $X \in \mathcal{E}$ together with weak equivalences $\phi\colon X \to X^*$ such that $\phi^*\eta_X = \phi$, subject to the relations

(a) $[V, \phi] = [W, \psi]$ if there is a weak equivalence $f\colon V \xrightarrow{\sim} W$ such that $\phi = f^*\psi f$

(b) $[V \oplus W, \phi \oplus \psi] = [V, \phi] + [W, \psi]$

(c) $[V, \phi_V] = [U \oplus W, \begin{pmatrix} & \phi_W \\ \phi_U & \end{pmatrix}]$ for any given inner product space in the category of exact sequences in $\mathcal{E}$ of the form

$$\begin{tikzcd} U \arrow[r, tail] \arrow[d, "\sim"', "\phi_U"] & V \arrow[r, two heads] \arrow[d, "\sim"', "\phi_V"] & W \arrow[d, "\sim"', "\phi_W"] \\ W^* \arrow[r, tail] & V^* \arrow[r, two heads] & U^* \end{tikzcd} \tag{3.53}$$

such that $\phi^*\eta = \phi$ for the weak equivalence $\phi = (\phi_U, \phi_V, \phi_W)$.

Naturally, this ad-hoc definition of the Grothendieck-Witt groups through generators and relations yields the same result as applying π_0 to the Grothendieck-Witt space

Proposition 3.6.2 ([Sch10, Proposition 3])**.** *Let $(\mathcal{E}, \omega, *, \eta)$ be an exact category with weak equivalences and duality. Then there exists a natural isomorphism:*

$$GW_0(\mathcal{E}, \omega, *, \eta) \cong \pi_0 GW(\mathcal{E}, \omega, *, \eta) \tag{3.54}$$

Walter takes a different approach in [Wal03] and studies the Grothendieck-Witt groups of *complicial* categories with weak equivalences and of triangulated categories. Here, a complicial category is a full sub-category of the category of chain complexes over some given additive category, satisfying some additional conditions. The category of chain complexes itself satisfies these conditions trivially. The localisation $\mathcal{C}[\omega^{-1}]$ of a complicial category with weak equivalences $(\mathcal{C}, \omega)$ at said weak equivalences is a triangulated category [Wal03, Theorem 4.1].

Definition 3.6.3. Let $(\mathcal{C}, \omega)$ be a small complicial category with weak equivalences and a simplicially graded duality containing $\frac{1}{2}$. Then the Grothendieck-Witt groups of $\mathcal{C}$ is the free abelian group on symmetric objects $[V, \phi]$ modulo the relations (a) and (b) as above and a third relation:

(c') If $h\colon W[-1] \to W^*$ is a morphism which is symmetric for the -1-st shifted duality, then $[Cone(W, h)] = [H(W)]$.

Here, $[Cone(W, h)]$ is the class of an inner product space $(V, \phi\colon V \to V^*)$ that fits into an exact triangle of the form $W[-1] \xrightarrow{h} W^* \xrightarrow{\phi^{-1}v^*} V \xrightarrow{v} W$. By Theorem 2.6 of [Bal00] the pair (V, ϕ) exists and is uniquely determined up to isomorphism. Now this sequence of maps fits into a diagram

$$\begin{tikzcd} W^* \arrow[r, tail] \arrow[d, "="] & V \arrow[r, two heads] \arrow[d, "\phi_V", "\sim"'] & W \arrow[d, "\eta_W", "\sim"'] \\ W^* \arrow[r, tail] & V^* \arrow[r, two heads] & W^{**} \end{tikzcd} \tag{3.55}$$

which is of the form 3.53 via the equivalence $(\phi_U, id, id)\colon (U, V, W) \to (W^*, V, W)$. We conclude that condition (c') implies condition (c). Conversely, assume we are given a diagram of the form 3.53 with the sequence $U \rightarrowtail V \twoheadrightarrow W$ being split. Then V is (equivalent to) the cone of some morphism $W[-1] \to U$ and condition (c) implies (c') in this special case. For the general case it suffices to realise that a short exact sequence $U \overset{f}{\rightarrowtail} V \twoheadrightarrow W$ is equivalent to the split exact sequence $U \rightarrowtail Cyl(f) \twoheadrightarrow Cone(f)$.

3.6.2 Explicit generators for the Grothendieck-Witt group of chain complexes

Let $(\mathcal{E}, *, \eta)$ be an exact category with duality and let $(\mathrm{Ch}^b(\mathcal{E}), quis, *^n, {}_\epsilon\eta^n)$ be the associated exact category of chain complexes with weak equivalences and duality as in Example 2.2.8. In this section we will explicitly describe the generators of the Grothendieck-Witt groups in these special cases. In light of periodicity (Proposition 3.5.16) it suffices to consider one fixed ϵ, say $\epsilon = 1$, and the cases $n = -1, 0, 1, 2$.

Even shifts

A generator of $GW^{[0]}(\mathcal{E})$ is given by a quasi-isomorphism of a chain complex to its dual

$$\begin{tikzcd} \cdots \arrow[r] & E_{-1} \arrow[r, "d_{-1}"] \arrow[d, "\phi_{-1}"] & E_0 \arrow[r, "d_0"] \arrow[d, "\phi_0"] & E_1 \arrow[r] \arrow[d, "\phi_1"] & \cdots \\ \cdots \arrow[r] & E_1^* \arrow[r, "d_0^*"] & E_0^* \arrow[r, "-d_{-1}^*"] & E_{-1}^* \arrow[r] & \cdots \end{tikzcd} \tag{3.56}$$

which is symmetric in the sense that $\phi_i = (-1)^i \phi_{-i}^* \eta_{E_i}$. By Theorem 3.5.18 we have

$$GW^{[0]}(\mathcal{E}) = GW(\mathrm{Ch}^b(\mathcal{E}), quis, *^0, \eta^0) \xleftarrow{\cong} GW(\mathcal{E}, *, \eta) \tag{3.57}$$

where the isomorphism sends a symmetric inner product space $\phi\colon E \to E^*$ to the complex concentrated in degree 0. This is consistent with the sign conventions as $\phi = \phi^* \eta_E$.

A generator of $GW^{[2]}(\mathcal{E})$ is given by a quasi-isomorphism

$$\begin{tikzcd} \cdots \arrow[r] & E_{-1} \arrow[r, "d_{-1}"] \arrow[d, "\phi_{-1}"] & E_0 \arrow[r, "d_0"] \arrow[d, "\phi_0"] & E_1 \arrow[r] \arrow[d, "\phi_1"] & \cdots \\ \cdots \arrow[r] & E_{-1}^* \arrow[r, "d_{-2}^*"] & E_{-2}^* \arrow[r, "-d_{-3}^*"] & E_{-3}^* \arrow[r] & \cdots \end{tikzcd} \tag{3.58}$$

symmetric in the sense that $\phi_i = (-1)^i \phi_{-i-2}^* \eta_{E_i}$. By Theorem 3.5.18 and Proposition 3.5.16 we have

$$GW^{[2]}(\mathcal{E}) \cong {}_{-}GW^{[0]}(\mathcal{E}) \xleftarrow{\cong} GW(\mathcal{E}, *, -\eta) \tag{3.59}$$

where the isomorphism on the right sends an anti-symmetric form $\phi\colon E \to E^*$ to the complex concentrated in degree 0 and the isomorphism on the right follows this up by a shift to the left. Hence, $\phi\colon E \to E^*$ is included in degree -1. Again, the sign conventions are consistent, since $\phi = -\phi^*\eta$.

Odd shifts

Similar to the two cases with even shifts, it is also possible to find a "smaller" set of generators for the cases with odd shifts. These generators are quasi-isomorphisms of *short complexes*:

Definition 3.6.4 ([Wal03, §7])**.** Let $(\mathcal{E}, *, \eta)$ be an exact category with duality. An ϵ-symmetric short complex (A, ϕ) in $\mathcal{E}$ is a quasi-isomorphism of the form

$$\begin{tikzcd} 0 \arrow[r] & A_0 \arrow[r, "d"] \arrow[d, "\phi^t"] & A_1 \arrow[r] \arrow[d, "\phi"] & 0 \\ 0 \arrow[r] & A_1^* \arrow[r, "\epsilon d^*"] & A_0^* \arrow[r] & 0 \end{tikzcd} \tag{3.60}$$

In particular we have $\epsilon d^* \phi^t = \phi d$.

Remark 3.6.5.

- Although the definition is essentially taken from [Wal03], the notation differs, due to different sign conventions.
- Note that for $d = id$ we have that ϕ is ϵ-symmetric, which explains the name.

Definition 3.6.6 ([Wal03, §7])**.** Let $(\mathcal{E}, *, \eta)$ be an exact category with duality. The Grothendieck-Witt group of ϵ-symmetric short complexes, denoted by $GW^{\epsilon}_{short}(\mathcal{E})$, is the free abelian group generated by ϵ-symmetric short complexes in $\mathcal{E}$ modulo the following relations

- $[A, \phi] + [A', \phi'] = [(A, \phi) \perp (A', \phi')]$
- If $f\colon A \to B$ is a quasi-isomorphism, then we have $[B, \phi] = [A, f^*\phi f]$
- Chain homotopic complexes are in the same class, i.e. for any ϵ-symmetric morphism $h\colon A_1 \to A_1^*$ we have that $[A, \phi]$ is equivalent to the class of

$$\begin{tikzcd} 0 \arrow[r] & A_0 \arrow[r, "d"] \arrow[d, "\phi^t - hd"'] & A_1 \arrow[r] \arrow[d, "\phi - \epsilon d^* h"] & 0 \\ 0 \arrow[r] & A_1^* \arrow[r, "\epsilon d^*"] & A_0^* \arrow[r] & 0 \end{tikzcd} \tag{3.61}$$

- A graph complex $\Gamma_{(A,d)}$, i.e. a short complex of the form

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "d"] \arrow[d, "\eta"'] & A^* \arrow[r] \arrow[d, "="] & 0 \\ 0 \arrow[r] & A^{**} \arrow[r, "\epsilon d^*"] & A^* \arrow[r] & 0 \end{tikzcd} \tag{3.62}$$

is equivalent to the trivial graph complex $[H(A)] := [\Gamma_{(A,0)}]$.

Remark 3.6.7. The trivial graph complex $[H(A)]$ is indeed the image of the class $[A] \in K(\mathcal{E})$ under the hyperbolic functor.

We can naturally understand ϵ-symmetric short complexes as generators of the Grothendieck-Witt groups in odd shifts:

A generator of $GW^{[1]}(\mathcal{E})$ is given by a quasi-isomorphism

$$\begin{tikzcd} \cdots \arrow[r] & E_{-1} \arrow[r, "d_{-1}"] \arrow[d, "\phi_{-1}"] & E_0 \arrow[r, "d_0"] \arrow[d, "\phi_0"] & E_1 \arrow[r] \arrow[d, "\phi_1"] & \cdots \\ \cdots \arrow[r] & E_0^* \arrow[r, "d_{-1}^*"] & E_{-1}^* \arrow[r, "-d_{-2}^*"] & E_{-2}^* \arrow[r] & \cdots \end{tikzcd} \tag{3.63}$$

symmetric in the sense that $\phi_i = \phi_{-i-1}^* \eta_{E_i}$. Observe that we may consider a symmetric short complex as a quasi-isomorphism of complexes concentrated in degrees -1 and 0. This induces a well-defined map

$$GW^+_{short}(\mathcal{E}) \to GW^{[1]}(\mathcal{E}) \tag{3.64}$$

Indeed, the first two conditions are clearly satisfied. The third condition follows from the surgery formula [Wal03, p. 18] and the final condition follows from condition c) of Definition 3.6.1 with $B = (A_{-1} \to A_0)$, $A = (A_{-1} \to 0)$, $C = (0 \to A^*)$, $\phi_B = (\phi_0^t, \phi_0)$, and $\phi_A = \phi_C^* = (id, 0)$. Similarly, a generator of $GW^{[-1]}(\mathcal{E})$ is given by a quasi-isomorphism

$$\begin{tikzcd} \cdots \arrow[r] & E_{-1} \arrow[r, "d_{-1}"] \arrow[d, "\phi_{-1}"] & E_0 \arrow[r, "d_0"] \arrow[d, "\phi_0"] & E_1 \arrow[r] \arrow[d, "\phi_1"] & \cdots \\ \cdots \arrow[r] & E_2^* \arrow[r, "d_1^*"] & E_1^* \arrow[r, "-d_{-0}^*"] & E_0^* \arrow[r] & \cdots \end{tikzcd} \tag{3.65}$$

and an anti-symmetric short complex may be considered as such a quasi-isomorphism concentrated in degrees 0 and 1, inducing a map

$$GW^-_{short}(\mathcal{E}) \to GW^{[-1]}(\mathcal{E}) \tag{3.66}$$

It turns out that these maps from the Grothendieck-Witt groups of ϵ-symmetric short complexes to the Grothendieck-Witt groups in odd shifts are already isomorphisms:

Theorem 3.6.8 ([Wal03, Theorem 7.1]). *If $\mathcal{E}$ is a small exact category with duality containing $\frac{1}{2}$, then the natural maps from Grothendieck-Witt groups of short complexes to shifted Grothendieck-Witt groups are isomorphisms*

$$GW^{\epsilon}_{short}(\mathcal{E}) \xrightarrow{\simeq} GW^{[\epsilon]}(\mathcal{E}) \tag{3.67}$$

Remark 3.6.9. To be precise, Walter compares the Grothendieck-Witt groups of short complexes to the odd triangulated Grothendieck-Witt groups of the derived category $D^b(\mathcal{E})$. However, by Theorem 5.1 of the same reference we know that the localisation map from complicial Grothendieck-Witt groups to triangulated Grothendieck-Witt groups is also an isomorphism.

Theorem 3.6.10 ([Wal03, Theorem 10.1]). *Let R be a commutative local ring such that 2 is invertible. Then the Grothendieck-Witt groups are given as*

$$GW^{[0]}(R) = GW(R), \quad GW^{[1]}(R) = 0, \quad GW^{[2]}(R) = \mathbb{Z}, \quad GW^{[3]}(R) = \mathbb{Z}/2 \tag{3.68}$$

3.7 Milnor and Milnor-Witt K-theory

An earlier attempt to define higher K-theory is Milnor K-theory for fields, which remains an important invariant. The generalisation to the hermitian world, the Milnor-Witt K-theory, was introduced by Morel.

Definition 3.7.1. Let k be a field. The Milnor K-theory of k is the graded associative ring $K^M_*(k)$ generated by symbols $[a]$, one for each unit $a \in k^\times = k - \{0\}$, in degree 1, subject to the relations

- Steinberg relation: $[a][1-a] = 0$ for all $a \in k^\times - \{1\}$
- Multiplicativity: $[a] + [b] = [ab]$ for all $a, b \in k^\times$

Definition 3.7.2. Let k be a field. The Milnor-Witt K-theory of k is the graded associative ring $K^{MW}_*(k)$ generated by symbols $[a]$, one for each unit $a \in k^\times = k - \{0\}$, in degree 1, and one generator η in degree -1, subject to the relations

- Steinberg relation: $[a][1-a] = 0$ for all $a \in k^\times - \{1\}$
- $[a] + [b] + \eta[a][b] = [ab]$ for all $a, b \in k^\times$

- $\eta[a] = [a]\eta$ for all $a \in k^\times$
- Set $h = \eta[-1] + 2$, then $\eta h = 0$.

Lemma 3.7.3. *Let k be a field. There exists a natural epimorphism*

$$\begin{aligned} K_*^{MW}(k) &\to K_*^M(k) \\ \eta &\mapsto 0 \end{aligned} \tag{3.69}$$

Chapter 4

K-theory and hermitian K-theory of automorphisms

4.1 K-theory of automorphisms

Definition 4.1.1. Let R be a (unital, associative) ring. Denote by $\mathcal{P}(R)$ the category of finitely generated projective (right-)modules over R. $\mathcal{P}(R)$ naturally has the structure of an exact category. We denote the K-theory of $i\mathcal{P}(R)$ by $K(R)$. Contained in $\mathcal{P}(R)$ is the full subcategory $\mathcal{F}(R)$ of free modules.

Definition 4.1.2 ([Gra95, p.3])**.** Let R be a ring and $t \in \mathbb{N}$. Denote by $\mathcal{P}(R, \mathbb{G}_m^t)$ the category of t commuting automorphism in $\mathcal{P}(R)$. Explicitly, an object is a tuple $(P, \theta_1, \ldots, \theta_t)$ consisting of a projective module P together with pairwise commuting automorphisms $\theta_j \in Aut(P)$, one for every $1 \leq j \leq t$. A morphism $f\colon (P, \theta_1, \ldots, \theta_t) \to (P', \theta_1', \ldots, \theta_t')$ in $\mathcal{P}(R, \mathbb{G}_m^t)$ is a morphism $f : P \to P'$ that is compatible with the automorphisms, i.e. a morphism that satisfies $f\theta_j = \theta_j' f$ for all j. By convention we set $\mathcal{P}(R, \mathbb{G}_m^0) = \mathcal{P}(R)$. The category $\mathcal{P}(R, \mathbb{G}_m^t)$ is symmetric monoidal under direct sum. We denote the K-theory of $i\mathcal{P}(R, \mathbb{G}_m^t)$ by $K(R, \mathbb{G}_m^t)$. The category $\mathcal{P}(R, \mathbb{G}_m^t)$ contains the category $\mathcal{F}(R, \mathbb{G}_m^t)$ of free modules with automorphisms as a full subcategory.

Definition 4.1.3. Let R be a ring and $t \in \mathbb{N}$. Denote by $\mathcal{GL}^t(R)$ the category, where an object is a tuple $(n, \{\theta_j\}) = (n, \theta_1, \ldots, \theta_t)$, consisting of a natural number $n \in \mathbb{N}_0$, together with a string of pairwise commuting invertible matrices θ_j in $Gl_n(R)$, one for every $1 \leq j \leq t$. A morphism $(n, \{\theta_j\}) \to (n', \{\theta_j'\})$ is then another element η of $Gl_n(R)$ such that $\theta_j = \eta^{-1}\theta_j'\eta$ for all j. In particular there are no morphisms between objects of different rank.

Remark 4.1.4. The category $\mathcal{GL}^t(R)$ can be thought of as a skeleton of $i\mathcal{F}(R, \mathbb{G}_m^t)$, although this is only true if R has the invariant basis number property, i.e. if $R^n \cong R^m$ implies $n = m$. But even if it doesn't, the inclusion $\mathcal{GL}^t(R) \subset i\mathcal{F}(R, \mathbb{G}_m^t)$ is still cofinal.

Definition 4.1.2 generalises easily to arbitrary categories:

Definition 4.1.5. Let $\mathcal{C}$ be a category. Denote by $\mathcal{C}(\mathbb{G}_m)$ the category of automorphisms in $\mathcal{C}$. An object is a pair $(C \in \mathcal{C}, \theta \in Aut(C))$, and a morphism $f\colon (C, \theta) \to (C', \theta')$ is a morphism $f\colon C \to C'$ in $\mathcal{C}$ that respects the automorphism, i.e. $f\theta = \theta' f$. We then inductively define $\mathcal{C}(\mathbb{G}_m^t) := \mathcal{C}(\mathbb{G}_m^{t-1})(\mathbb{G}_m)$. If $\mathcal{C}$ is monoidal, then so is $\mathcal{C}(\mathbb{G}_m)$.

4.2 Grothendieck-Witt theory of automorphisms

In this section we will revisit the categories discussed in Section 4.1 and will see how we can naturally equip them with a duality. Given a right R-module P we define its dual by $P^* = Hom(P, R)$. If R is not commutative, P^* is in general only a left module and we need an involution on R to turn it into a right module. This motivates the notation of the following definition.

Definition 4.2.1. Let $(R, \bar{\cdot})$ be a ring with involution and let ϵ be a central element with $\bar{\epsilon}\epsilon = 1$, then $(\mathcal{P}(R), *, {}_\epsilon\eta)$ is an exact category with duality, where $* = \mathrm{Hom}_R(-, R)$ and ${}_\epsilon\eta_M \colon M \to M^{**}$ is given by $m \mapsto (f \mapsto \bar{\epsilon}\overline{f(m)})$. Given a right R-module M, $\mathrm{Hom}_R(M, R)$ is made into a right module via $(f \cdot r)(m) := \bar{r}f(m)$. The corresponding category of inner product spaces is denoted by ${}_\epsilon\mathcal{P}_h(R)$.

Remark 4.2.2. In this paper we usually consider $\epsilon = \pm 1$.

Now, there are essentially two different ways to define categories ${}_\epsilon\mathcal{P}_h(R, \mathbb{G}_m^t)$. Either, one considers automorphisms in the category ${}_\epsilon\mathcal{P}_h(R)$ or one defines a duality on $\mathcal{P}(R, \mathbb{G}_m^t)$ and passes to inner product spaces. We will define a duality such that the resulting categories are isomorphic:

Definition 4.2.3. Let $(R, \bar{\cdot})$ be a ring with involution and let ϵ be a central element with $\bar{\epsilon}\epsilon = 1$, then $(\mathcal{P}(R, \mathbb{G}_m^t), *, {}_\epsilon\eta)$ is an exact category with duality, where $(P, \theta_1, \ldots, \theta_t)^* = (P^*, (\theta_1^*)^{-1}, \ldots, (\theta_t^*)^{-1})$. Denote the category of inner product spaces by ${}_\epsilon\mathcal{P}_h(R, \mathbb{G}_m^t)$.

Explicitly an object of ${}_\epsilon\mathcal{P}_h(R, \mathbb{G}_m^t)$ is a tuple $(P, \theta_1, \ldots, \theta_t)$ together with an isomorphism to its dual $\phi\colon (P, \theta_1, \ldots, \theta_t) \to (P^*, (\theta_1^*)^{-1}, \ldots, (\theta_t^*)^{-1})$, i.e. an isomorphism $\phi\colon P \to P^*$ such that $\phi\theta_j = (\theta_j^*)^{-1}\phi$. This is the same as an object of ${}_\epsilon\mathcal{P}_h(R)$ together with a string of t commuting automorphisms, i.e. an isomorphism $\phi\colon P \to P^*$ together with automorphisms θ_j such that $\theta_j^*\phi\theta_j = \phi$. A morphism $f\colon (P, \theta_1, \ldots, \theta_t, \phi) \to (P', \theta_1', \ldots, \theta_t', \phi')$ is a morphism $f\colon P \to P'$ that satisfies $f^*\phi' f = \phi$ and $f\theta_j = \theta_j' f$ for all j.

Remark 4.2.4. The duality on $\mathcal{GL}^t(R)$ that is consistent with the one above, is given by $(n, \{\theta_j\})^* = (n, \{(\theta_j^*)^{-1}\})$. Because an automorphism θ_j is the same as an invertible matrix with entries in R, we can explicitly write $\theta_j^t = \theta_j^*$ for the transpose of θ_j and will do so on and off. The duality defined like this is strict.

Definition 4.2.5. Let ${}_\epsilon H\colon i\mathcal{P}(R) \to i_\epsilon\mathcal{P}_h(R)$ be the hyperbolic functor. Then denote by ${}_\epsilon O_n(R)$ the group of isometries of ${}_\epsilon H(R^n)$. Explicitly, we can identify

${}_\epsilon O_n(R)$ with the group of invertible $2n \times 2n$ matrices θ such that

$$\theta^t \begin{pmatrix} 0 & 1 \\ \epsilon & 0 \end{pmatrix} \theta = \begin{pmatrix} 0 & 1 \\ \epsilon & 0 \end{pmatrix} \tag{4.1}$$

Definition 4.2.6. Denote by ${}_\epsilon\mathcal{O}^t(R)$ the category where objects are tuples $(n, \{\theta_j\}) = (n, \theta_1, \ldots, \theta_t)$, with $n \in \mathbb{N}$ a non-negative integer and $\theta_j \in {}_\epsilon O_n(R)$ a family of pair-wise commuting matrices. A morphism $(n, \{\theta_j\}) \to (n', \{\theta'_j\})$ is another matrix $\eta \in {}_\epsilon O_n(R)$ such that $\theta_j = \eta^{-1}\theta'_j\eta$ for all j. In particular, there are no morphisms between objects of different rank.

4.3 Cofinal subcategories

In the proceeding sections we defined categories $i\mathcal{P}(R, \mathbb{G}_m^t)$ and $i_\epsilon\mathcal{P}_h(R, \mathbb{G}_m^t)$ with subcategories $\mathcal{GL}^t(R)$ and ${}_\epsilon\mathcal{O}^t(R)$, respectively. The two latter categories are significantly easier to study, since objects are based on explicit invertible matrices, rather than abstract automorphisms of projective modules (with forms). In the case of inner product spaces we will introduce the additional condition that $2 \in R$ will be invertible, which will stick with us for the remainder of this paper.

Lemma 4.3.1. *Let R be a ring. Then the natural functor*

$$\mathcal{GL}^t(R) \to \mathcal{P}(R, \mathbb{G}_m^t) \tag{4.2}$$

is cofinal in the sense of Definition 3.2.10.

Proof. Let $(P, \theta_1, \ldots, \theta_t)$ be an object of $\mathcal{P}(R, \mathbb{G}_m^t)$. Then there exists a projective module Q such that $P \oplus Q \cong R^n$ for some natural number n. Clearly the object $(P, \theta_1, \ldots, \theta_t) \oplus (Q, 1_Q, \ldots, 1_Q)$ is in the essential image of the inclusion. The assertion on automorphisms is obvious. □

Lemma 4.3.2. *Let $\mathcal{S}, \mathcal{T}$ be two symmetric monoidal categories and let $F \colon \mathcal{S} \to \mathcal{T}$ be a symmetric monoidal functor. If F is cofinal , then so is the induced functor $\mathcal{S}(\mathbb{G}_m) \to \mathcal{T}(\mathbb{G}_m)$.*

Proof. If $f \colon T \oplus T' \to F(S)$ is an isomorphism, then so is $f \colon (T, \theta) \oplus (T', 1_{T'}) \to (S, f(\theta + 1_{T'})f^{-1})$, which verifies condition (a) of the cofinality theorem 3.2.11. Condition (b) of 3.2.10 is trivially satisfied. □

There are now essentially two ways to formulate a hermitian analogue to Lemma 4.3.1. Either, one can find a cofinal subcategory of $i_\epsilon\mathcal{P}_h(R)$ using Proposition 3.2.6, and let Lemma 4.3.2 above induce cofinal subcategories for $i_\epsilon\mathcal{P}_h(R,\mathbb{G}_m^t)$. Alternatively one can apply Proposition 3.2.6 directly to the category $i_\epsilon\mathcal{P}_h(R,\mathbb{G}_m^t)$, resulting in a different subcategory. In detail:

Lemma 4.3.3. *Let R be a ring with involution such that $2 \in R$ is invertible and $\epsilon \in \{\pm 1\}$. Then the natural functor*

$$ {}_\epsilon\mathcal{O}^t(R) \to i_\epsilon\mathcal{P}_h(R,\mathbb{G}_m^t) \tag{4.3}$$

is cofinal.

Proof. We proceed in several steps:

- Because we have $i_\epsilon\mathcal{P}_h(R,\mathbb{G}_m^t)(\mathbb{G}_m) = i_\epsilon\mathcal{P}_h(R,\mathbb{G}_m^{t+1})$ and ${}_\epsilon\mathcal{O}^t(R)(\mathbb{G}_m) = {}_\epsilon\mathcal{O}^{t+1}(R)$ we can reduce the proof to the case $t = 0$ by Lemma 4.3.2.
- Condition (b) is true, essentially by definition.
- The category ${}_\epsilon\mathcal{P}(R)$ is additive. Therefore any inner product space $(P,\phi) \in {}_\epsilon\mathcal{P}(R)$ can be completed to a hyperbolic one by Proposition 3.2.6.
- Finally if P is projective with Q such that $P \oplus Q \cong R^n$ we have an isometry $H(P) \oplus H(Q) \cong H(R^n)$.

$\square$

Definition 4.3.4. Denote by ${}_\epsilon\mathcal{HGL}^t(R)$ the full subcategory of ${}_\epsilon\mathcal{O}^t(R)$ consisting of those objects that are in the image of the hyperbolic functor

$$ {}_\epsilon H\colon \mathcal{GL}^t(R) \to {}_\epsilon\mathcal{O}^t(R) \tag{4.4}$$

Explicitly, an object is a tuple $(n, \{\theta_j\})$, with t pairwise commuting $\theta_j \in Gl_n(R)$. A morphism in this category is a matrix in ${}_\epsilon O_n(R)$ that works as a base-change.

Remark 4.3.5. Note that we have

$$ {}_\epsilon\mathcal{HGL}^{t+1}(R) \subsetneq {}_\epsilon\mathcal{HGL}^t(R)(\mathbb{G}_m) \tag{4.5}$$

because objects and morphisms are recruited from a different set of matrices.

Lemma 4.3.6. *Let R be a ring with involution such that $2 \in R$ is invertible and $\epsilon \in \{\pm 1\}$. Then the composition functor*

$$ F\colon {}_\epsilon\mathcal{HGL}^t(R) \to {}_\epsilon\mathcal{O}^t(R) \to i_\epsilon\mathcal{P}_h(R,\mathbb{G}_m^t) \tag{4.6}$$

is cofinal.

Proof. Let $(P, \{\theta_j\}, \phi)$ be an object of $i_\epsilon\mathcal{P}_h(R, \mathbb{G}_m^t)$. Because the category ${}_\epsilon\mathcal{P}_h(R, \mathbb{G}_m^t)$ is additive we have

$$(P, \{\theta_j\}, \phi) \oplus (P, \{\theta_j^{-*}\}, \eta\phi^{-1}) \cong {}_\epsilon H(P, \{\theta_j\}) \tag{4.7}$$

and there exists a projective module Q such that $P \oplus Q$ is free. Finally the object

$${}_\epsilon H(P, \{\theta_j\}) \oplus {}_\epsilon H(Q, \{1\}) \tag{4.8}$$

lies in the image of the F. Condition (b) of the cofinality theorem 3.2.11 is satisfied by definition. □

4.4 Multiplication and Stabilisation

4.4.1 Products

There is a different interpretation of the category $\mathcal{S}(\mathbb{G}_m)$. To this end we assume that we are working over a fixed commutative "base-ring" k, i.e. we consider the category of rings R which come with a fixed homomorphism $k \to R$ (and where ring-maps respect these morphisms). When we talk about (ordinary) K-theory, then we usually just set $k = \mathbb{Z}$ and any ring comes with a unique homomorphism $\mathbb{Z} \to R$. In the setting of hermitian K-theory we regularly require 2 to be invertible and set $k = \mathbb{Z}[\frac{1}{2}]$. Another option would be to let k be any fixed field, with characteristic different from 2. In either case we then set $\mathbb{G}_m := k[U, U^{-1}]$, the ring of Laurent-polynomials in the variable U .

Definition 4.4.1 (cf. [Gra95])**.** Given an additive category $\mathcal{C}$ and a ring R, let $\mathcal{C}(R)$ denote the category of pairs (C, ρ), where $C \in \mathcal{C}$ and $\rho\colon R \to \mathrm{End}_{\mathcal{C}}(C)$ is a ring-homomorphism. Morphisms $(C, \rho) \to (C', \rho')$ in this category are morphisms $f\colon C \to C'$ such that the following diagram commutes

$$\begin{tikzcd} R \arrow[r, "\rho"] \arrow[d, "\rho'"] & \mathrm{End}_{\mathcal{C}}(C) \arrow[d, "f\circ -"] \\ \mathrm{End}_{\mathcal{C}}(C') \arrow[r, "-\circ f"] & \mathrm{Hom}_{\mathcal{C}}(C, C') \end{tikzcd} \tag{4.9}$$

If $\mathcal{C}$ is an exact category, then $\mathcal{C}(R)$ is an exact category, where a sequence is exact in $\mathcal{C}(R)$ if it is exact in $\mathcal{C}$ after forgetting the R-module structure ρ.
If $(\mathcal{C}, *, \eta)$ is an exact category with duality and R a ring with involution, then $(\mathcal{C}(R), *, \eta)$ is an exact category with duality, where $(C, \rho)^* := (C^*, \rho^*)$ with $\rho^*(r) := \rho(\bar{r})^*$ and $\eta_{(C,\rho)}\colon (C, \rho) \to (C, \rho)^{**}$ is simply given by $\eta_C\colon C \to C^{**}$.

If we now plug the ring $\mathbb{G}_m = \mathbb{Z}[U, U^{-1}]$ into the definition above, then the category $\mathcal{C}(\mathbb{G}_m)$ is isomorphic to the category of the same name defined in Section 4.1. Indeed, a homomorphism $\rho\colon \mathbb{G}_m \to \mathrm{End}_{\mathcal{C}}(C)$ is uniquely determined by the image θ of U and θ is invertible with $\theta^{-1} = \rho(U)^{-1} = \rho(U^{-1})$.
For a given R-module P, a homomorphism $\mathbb{G}_m \to \mathrm{End}_R(P)$ is the same as an $\mathbb{G}_m$-module structure on P and we may consider the category $\mathcal{P}(R, \mathbb{G}_m) := \mathcal{P}(R)(\mathbb{G}_m)$ as the category of finitely generated R-$\mathbb{G}_m$-bimodules, projective over R. Similarly, and object of $\mathcal{P}(R, \mathbb{G}_m^t)$ can be interpreted as a module over the ring $R \otimes \mathbb{G}_m^{\otimes t}$, finitely generated and projective over R.

Any inclusion $\mathbb{G}_m = k[U_j, U_j^{-1}] \subset k[U_1, \dots U_t, U_1^{-1}, \dots, U_t^{-1}] = \mathbb{G}_m^t$ given by $U_j \mapsto U_j$ for any fixed positive integer $j \leq t$ allows us to consider an object of $\mathcal{P}(R \otimes \mathbb{G}_m^{\otimes t})$ as a $\mathbb{G}_m$-module. Thus, there are bi-exact functors

$$\mathcal{P}(R, \mathbb{G}_m^{t+1}) \otimes_{\mathbb{G}_m} \mathcal{P}(\mathbb{G}_m) \to \mathcal{P}(R, \mathbb{G}_m^t) \tag{4.10}$$

given by $(P, Q) \mapsto P \otimes_{\mathbb{G}_m} Q$, one for each $1 \leq j \leq t+1$. These functors clearly induce morphisms

$$K(R, \mathbb{G}_m^{t+1}) \wedge K(\mathbb{G}_m) \to K(R, \mathbb{G}_m^t) \tag{4.11}$$

Now, the class $U_j \in k[U_j^{\pm 1}]^\times = K_1(\mathbb{G}_m) = [S^1, K(\mathbb{G}_m)]$ induces a map

$$K(R, \mathbb{G}_m) \wedge S^1 \xrightarrow{U_j} K(R) \tag{4.12}$$

and by adjunction a map

$$K(R, \mathbb{G}_m) \xrightarrow{U_j} \Omega K(R) \tag{4.13}$$

Remark 4.4.2. The notation is chosen suggestively: In general multiplication by U_j will depend on j, because the $\mathbb{G}_m$-module structure of an object in $\mathcal{P}(R, \mathbb{G}_m^{t+1})$ varies. However the U_j are virtually the same element of $\mathbb{G}_m$ for all j.

In order to construct a similar map for hermitian K-theory equip $\mathbb{G}_m = \mathbb{Z}[\frac{1}{2}, U, U^{-1}]$ with the involution $\bar{U} = U^{-1}$. If $(\mathcal{C}, *, \eta)$ is a category with duality and $(C, \rho \colon \mathbb{G}_m \to \mathrm{End}(C)) \in \mathcal{C}(\mathbb{G}_m)$ is an object, then ρ is uniquely determined by $\theta := \rho(U)$. By definition we have $\rho^*(U) = \rho(\bar{U})^* = \rho(U^{-1})^* = (\theta^{-1})^*$, which is consistent with the duality defined in Section 4.2. Hence we may consider the category ${}_\epsilon\mathcal{P}_h(R, \mathbb{G}_m^t)$ as the category of ϵ-symmetric inner product spaces of finitely generated modules over the ring $R \otimes \mathbb{G}_m^{\otimes t}$ with involution, projective over R. Hence, we are able to define bi-exact functors

$${}_\epsilon\mathcal{P}_h(R, \mathbb{G}_m^{t+1}) \otimes \mathcal{P}_h(\mathbb{G}_m) \to {}_\epsilon\mathcal{P}_h(R, \mathbb{G}_m^t) \tag{4.14}$$

More precisely, let $(P, \phi) \in {}_\epsilon\mathcal{P}_h(R, \mathbb{G}_m^{t+1})$ and $(Q, \psi) \in \mathcal{P}_h(\mathbb{G}_m)$ be inner product spaces that are ϵ-symmetric and symmetric, respectively. Then, $(P \otimes Q, \phi \otimes \psi) \in {}_\epsilon\mathcal{P}_h(R, \mathbb{G}_m^t)$ will be also ϵ-symmetric:

$$
\begin{tikzcd}
P \otimes Q \arrow[r, "\phi\otimes\psi"] \arrow[d, "\epsilon\eta_P\otimes\eta_Q"'] & P^* \otimes Q^* \\
P^{**} \otimes Q^{**} \arrow[ur, "\phi^*\otimes\psi^*"'] &
\end{tikzcd}
\tag{4.15}
$$

If moreover both, (P, ϕ) and (Q, ψ) are hyperbolic, then so is $(P \otimes Q, \phi \otimes \psi)$. The tensor products induce maps

$$
{}_{\epsilon}GW(R, \mathbb{G}_m^{t+1}) \wedge GW(\mathbb{G}_m) \to {}_{\epsilon}GW(R, \mathbb{G}_m^t) \tag{4.16}
$$

It remains to construct elements $U_j \in GW_1(\mathbb{G}_m)$ to obtain multiplication maps

$$
{}_{\epsilon}GW(R, \mathbb{G}_m^{t+1}) \xrightarrow{U_j} \Omega_{\epsilon}GW(R, \mathbb{G}_m^t) \tag{4.17}
$$

To this end recall that $\mathbb{G}_m^*$ is a right $\mathbb{G}_m$-module via $(f \cdot a)(x) = \bar{a}f(x)$, for $f \in \mathbb{G}_m^*$ and $a, x \in \mathbb{G}_m$. Therefore there is a canonical isomorphism

$$
\begin{aligned}
\phi\colon \mathbb{G}_m &\to \mathbb{G}_m^* \\
a &\mapsto \bar{a} \cdot (-)
\end{aligned}
\tag{4.18}
$$

and multiplication by U defines an automorphism of the symmetric inner product space $(\mathbb{G}_m, \phi)$.
The canonical isomorphism $\eta\colon \mathbb{G}_m \to \mathbb{G}_m^{**}$ is given by $x \mapsto (f \mapsto \overline{f(x)})$. To verify the equation $\phi = \phi^*\eta$ we compute

$$
\phi^*(\eta(a))(x) = \eta(a)\phi(x) = \overline{\phi(x)(a)} = \overline{\bar{x}a} = x\bar{a} = \phi(a)(x) \tag{4.19}
$$

Further, the diagram

$$
\begin{tikzcd}
\mathbb{G}_m \arrow[r, "\phi"] \arrow[d, "U"'] & \mathbb{G}_m^* \\
\mathbb{G}_m \arrow[r, "\phi"] & \mathbb{G}_m^* \arrow[u, "U^*"']
\end{tikzcd}
\tag{4.20}
$$

commutes.
The inner product space $(\mathbb{G}_m, \phi)$ is a direct summand of the hyperbolic space $(\mathbb{G}_m \oplus \mathbb{G}_m^*, (\phi, -\phi^{-1}))$. Indeed, the automorphism $f = \begin{pmatrix} \frac{1}{2} & \frac{\phi^{-1}}{2} \\ \phi & -1 \end{pmatrix}$ of $\mathbb{G}_m \oplus \mathbb{G}_m^*$ satisfies

$$
f^t \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} f = \begin{pmatrix} \phi & \\ & -\phi^{-1} \end{pmatrix} \tag{4.21}
$$

We conclude that $(\mathbb{G}_m, \phi)$ is represented by $f \begin{pmatrix} U & \\ & 1 \end{pmatrix} f^{-1} = \begin{pmatrix} \frac{U+1}{2} & \frac{(U-1)\phi^{-1}}{4} \\ \phi(U-1) & \frac{U+1}{2} \end{pmatrix}$ in $O_2(\mathbb{G}_m) \subseteq O(\mathbb{G}_m)$. Therefore it can be considered as an element of $GW_1(\mathbb{G}_m) = O(\mathbb{G}_m)/[O(\mathbb{G}_m), O(\mathbb{G}_m)]$.

4.4.2 Stabilisation

There are various ring homomorphisms of the form

$$\mathbb{G}_m^{t+1} = k[U_1^{\pm}, \dots, U_{t+1}^{\pm}] \to k[U_1^{\pm}, \dots, \widehat{U_j^{\pm}}, \dots, U_{t+1}^{\pm}] = \mathbb{G}_m^t \tag{4.22}$$

which map $U_j \mapsto 1$ for a fixed index j and these morphisms induce functors

$$\mathcal{P}(R, \mathbb{G}_m^t) \to \mathcal{P}(R, \mathbb{G}_m^{t+1}). \tag{4.23}$$

Explicitly an object $(P\{\theta_j\}) \in \mathcal{P}(R, \mathbb{G}_m^t)$ is equipped with an additional identity-morphism at the j-th position. Therefore we obtain maps

$$i_j \colon K(R, \mathbb{G}_m^t) \to K(R, \mathbb{G}_m^{t+1}). \tag{4.24}$$

and similarly

$$i_j \colon {}_{\epsilon}GW(R, \mathbb{G}_m^t) \to {}_{\epsilon}GW(R, \mathbb{G}_m^{t+1}). \tag{4.25}$$

Both maps are split by the corresponding forgetful functor and therefore are inclusions on a direct factor:

Lemma 4.4.3. *Let $i \colon X \to Y$ be an inclusion of group-like CW complexes, split by a map $p \colon Y \to X$ (i.e. $p \circ i = 1$). Then X is (up to homotopy) a direct factor of Y. In other words, there exists a space Z such that $Y \simeq X \times Z$.*

Proof. Let Z be the homotopy fibre of p. Then there exists a long exact sequence of homotopy groups

$$\dots \to \pi_{n+1}X \to \pi_n Z \to \pi_n Y \xrightarrow{p_*} \pi_n X \to \pi_{n-1}Z \to \dots \tag{4.26}$$

split into short exact sequences by i_*:

$$\begin{tikzcd} 0 \arrow[r] & \pi_n Z \arrow[r] & \pi_n Y \arrow[r, "p_*"] & \pi_n X \arrow[r] \arrow[l, bend left, "i_*"] & 0 \end{tikzcd} \tag{4.27}$$

It follows that the natural map $X \times Z \to Y$ is a homotopy equivalence. ∎

Now, multiplication by U_j vanishes on the image of ι_j. More precisely the compositions

$$K(R, \mathbb{G}_m^t) \xrightarrow{i_j} K(R, \mathbb{G}_m^{t+1}) \xrightarrow{U_j} \Omega K(R, \mathbb{G}_m^t) \tag{4.28}$$

and

$$_\epsilon GW(R, \mathbb{G}_m^t) \xrightarrow{i_j} {}_\epsilon GW(R, \mathbb{G}_m^{t+1}) \xrightarrow{U_j} \Omega_\epsilon GW(R, \mathbb{G}_m^t) \tag{4.29}$$

are trivial.
This leads us to the following (naive) definition, which defines the stabilised (hermitian) K-theory as the complementary factor of the respective inclusion:

Definition 4.4.4 (cf. [Gra95])**.** Let $\mathcal{S}$ be a symmetric monoidal category. The stabilised K-theory of automorphisms $K(\mathcal{S}(\mathbb{G}_m^{\wedge 1}))$ is defined as the fibre

$$K(\mathcal{S}(\mathbb{G}_m^{\wedge 1})) \to K(\mathcal{S}(\mathbb{G}_m)) \to K(\mathcal{S}) \tag{4.30}$$

of the map induced by the forgetful functor.

To generalise this construction to the K-theory of a category with multiple commuting automorphism $\mathcal{S}(\mathbb{G}_m^t)$ we use the language of cubes as in [Gra92, §4]:

Definition 4.4.5. Let [1] denote the ordered set $\{0 < 1\}$ considered as a category and $t \in \mathbb{N}_0$ a non-negative integer. A t-dimensional cube in a category $\mathcal{C}$ is a functor $F\colon [1]^t \to \mathcal{C}$. In particular, a 0-dimensional cube is, by convention, just an object, a 1-dimensional cube is a morphism, a 2-dimensional cube a commuting square of morphisms, etc. A t-dimensional opposite cube in a category $\mathcal{C}$ is a functor $F\colon ([1]^t)^{op} \to \mathcal{C}$.
We will denote an element of $[1]^t$ by a string $i = (\epsilon_1, \ldots, \epsilon_t)$ of numbers $\epsilon_j \in \{0, 1\}$ and write $|i|$ for the number of entries of i that are equal to 1. Write $\widehat{[1]}^t$ for the full subcategory of all objects different from $(1, \ldots, 1)$.

Example 4.4.6.

- By convention, we consider the morphism $K(\mathcal{S}) \to K(\mathcal{S}(\mathbb{G}_m))$ as a one-dimensional cube in the category of spaces and the morphism $K(\mathcal{S}(\mathbb{G}_m)) \to K(\mathcal{S})$ as an opposite one-dimensional cube.
- More generally we can consider the t-dimensional cube with $i \mapsto K(\mathcal{S}(\mathbb{G}_m^{|i|}))$ for any $i \in [1]^t$. A morphism $i = (\epsilon_1, \ldots, \epsilon_t) \to (\epsilon'_1, \ldots, \epsilon'_t) = i'$ in $[1]^t$ is sent to the morphism induced by the natural inclusion functor $S(\mathbb{G}_m^{|i|}) \to S(\mathbb{G}_m^{|i'|})$

that adds an identity morphism whenever $0 = \epsilon_j < \epsilon'_j = 1$. The t-dimensional opposite cube $i \mapsto K(\mathcal{S}(\mathbb{G}_m^{|i|}))$ has morphisms induced by various forgetful functors.

We are now able to generalise the (naive) definition of the stabilised K-theory space to categories with multiple automorphisms.

Definition 4.4.7. Let $\mathcal{S}$ be a symmetric monoidal category and $i = (\epsilon_1, \ldots, \epsilon_t) \mapsto K(\mathcal{S}(\mathbb{G}_m^{|i|}))$ the t-dimensional opposite cube defined in Example 4.4.6. Define the stabilised K-theory $\mathcal{S}(\mathbb{G}_m^t)$ as the homotopy fibre

$$K(\mathcal{S}(\mathbb{G}_m^{\wedge t})) = \mathrm{hofib}(K(\mathcal{S}(\mathbb{G}_m^t)) \to \operatorname*{holim}_{i \in \widehat{[1]}^n} K(\mathcal{S}(\mathbb{G}_m^{|i|}))) \tag{4.31}$$

4.4.3 Alternative definition of stabilised K-theory

For technical reasons it is more convenient to consider $K(\mathcal{S}(\mathbb{G}_m^{\wedge 1}))$ as the base space of a fibration sequence

$$K(\mathcal{S}) \to K(\mathcal{S}(\mathbb{G}_m)) \to K(\mathcal{S}(\mathbb{G}_m^{\wedge 1})) \tag{4.32}$$

which exists by Lemma 4.4.3. To avoid various choices of splittings, we follow Grayson ([Gra95, Lemma 4.3], [Gra92, Lemma 4.1]), who generalises the relative K-theory space of a functor to cubes. For an n-dimensional cube $\mathcal{C}$ he naturally constructs an $n+1$-simplicial set $S^{\oplus}\mathcal{C}$ such that the following properties are satisfied:

- If $n = 0$, then $\Omega|S^{\oplus}C| \simeq K(\mathcal{C})$.
- Suppose we are given an additive map $\mathcal{C} \to \mathcal{C}'$ of n-dimensional cubes of additive categories and let $[\mathcal{C} \to \mathcal{C}']$ denote the corresponding $n+1$-dimensional cube of additive categories. Then there is a fibration sequence

$$S^{\oplus}\mathcal{C} \to S^{\oplus}\mathcal{C}' \to S^{\oplus}[\mathcal{C} \to \mathcal{C}']. \tag{4.33}$$

- The geometric realisation of $S^{\oplus}\mathcal{C}$ is connected, so that the induced fibration

$$\Omega|S^{\oplus}\mathcal{C}| \to \Omega|S^{\oplus}\mathcal{C}'| \to \Omega|S^{\oplus}[\mathcal{C} \to \mathcal{C}']| \tag{4.34}$$

is surjective on π_0.

Definition 4.4.8. Let $\mathcal{S}$ be a symmetric monoidal category. Write $\mathcal{S}(\mathbb{G}_m^{\wedge t}) := (i \mapsto \mathcal{S}(\mathbb{G}_m^{|i|}))$ for the t-dimensional cube of Example 4.4.6.

Proposition 4.4.9. *Let $\mathcal{S}$ be a symmetric monoidal category. Then there is a homotopy equivalence*

$$\Omega|S^{\oplus}\mathcal{S}(\mathbb{G}_m^{\wedge t})| \simeq \mathrm{hofib}(K(\mathcal{S}(\mathbb{G}_m^t)) \to \operatorname*{holim}_{i\in\widehat{[1]}^n} K(\mathcal{S}(\mathbb{G}_m^{|i|}))) \tag{4.35}$$

We will need the following lemma:

Lemma 4.4.10 (Four fibration lemma [Cor94])**.** *Given a homotopy commutative diagram of the form*

$$\begin{tikzcd}
& A \arrow[rr] \arrow[dl] \arrow[dd] & & C \arrow[dl] \arrow[dd] \\
B \arrow[rr] \arrow[dd] & & D \arrow[dd] & \\
& A' \arrow[rr] \arrow[dl] \arrow[dd] & & C' \arrow[dl] \arrow[dd] \\
B' \arrow[rr] \arrow[dd] & & D' \arrow[dd] & \\
& A'' \arrow[rr] \arrow[dl] & & C'' \arrow[dl] \\
B'' \arrow[rr] & & D'' &
\end{tikzcd} \tag{4.36}$$

where the vertical sequences $B \to B' \to B''$, $C \to C' \to C''$ and $D \to D' \to D''$ are homotopy fibration sequences and where the middle and bottom square are homotopy pull-backs, then the top square is a homotopy pull-back if and only if the sequence $A \to A' \to A''$ is a homotopy fibration.

Prof of Proposition 4.4.9. We proceed by induction on t. For the base of the induction consider the sequence

$$\Omega|S^{\oplus}\mathcal{S}| \to \Omega|S^{\oplus}\mathcal{S}(\mathbb{G}_m)| \to \Omega|S^{\oplus}\mathcal{S}(\mathbb{G}_m^{\wedge 1})| \tag{4.37}$$

By construction the second map is a fibration that is surjective on connected components. The first space can be identified with $K(\mathcal{S})$, the second space with $K(\mathcal{S}(\mathbb{G}_m))$ and the map is the natural map induced by the inclusion of categories, split by the forgetful functor. By Lemma 4.4.3 there is a homotopy fibration sequence

$$K(\mathcal{S}) \leftarrow K(\mathcal{S}(\mathbb{G}_m)) \leftarrow \Omega|S^{\oplus}\mathcal{S}(\mathbb{G}_m^{\wedge 1})| \tag{4.38}$$

For the induction step, consider the commutative diagram

$$
\begin{tikzcd}
& \Omega|S^{\oplus}\mathcal{S}(\mathbb{G}_m^{\wedge t})| \arrow[rr] \arrow[dl] \arrow[dd] & & \Omega|S^{\oplus}\mathcal{S}(\mathbb{G}_m)(\mathbb{G}_m^{\wedge t-1})| \arrow[dl] \arrow[dd] \\
* \arrow[rr] \arrow[dd] & & \Omega|S^{\oplus}\mathcal{S}(\mathbb{G}_m^{\wedge t-1})| \arrow[dd] & \\
& K(\mathcal{S}(\mathbb{G}_m^{t})) \arrow[rr] \arrow[dl] \arrow[dd] & & K(\mathcal{S}(\mathbb{G}_m^{t})) \arrow[dl] \arrow[dd] \\
K(\mathcal{S}(\mathbb{G}_m^{t-1})) \arrow[rr] \arrow[dd] & & K(\mathcal{S}(\mathbb{G}_m^{t-1})) \arrow[dd] & \\
& \operatorname{holim}_{i\in\widehat{[1]}^{t}} K(\mathcal{S}(\mathbb{G}_m^{|i|})) \arrow[rr] \arrow[dl] & & \operatorname{holim}_{i\in\{1\}\times\widehat{[1]}^{t-1}} K(\mathcal{S}(\mathbb{G}_m^{|i|})) \arrow[dl] \\
K(\mathcal{S}(\mathbb{G}_m^{t-1})) \arrow[rr] & & \operatorname{holim}_{i\in\{0\}\times\widehat{[1]}^{t-1}} K(\mathcal{S}(\mathbb{G}_m^{|i|})) &
\end{tikzcd}
\tag{4.39}
$$

The bottom square is a homotopy pull-back, because $\{0\} \times \widehat{[1]}^{t-1} \cup \{1\} \times \widehat{[1]}^{t-1} \cup \{(0,1,\dots,1)\} = \widehat{[1]}^{t}$, the middle square is trivially a homotopy pull-back and the top square is a homotopy pull-back by construction. The front left vertical sequence is trivially a fibration and the two vertical sequences on the right are fibrations by the induction hypothesis. By the four-fibration lemma we are done. $\square$

This allows us to formulate the alternative definition (cf. Definition 4.4.7 and [Gra95, Definition 4.2]):

Definition 4.4.11. Let $\mathcal{S}$ be a symmetric monoidal category and $\mathcal{S}(\mathbb{G}_m^{\wedge t}) = (i \mapsto \mathcal{S}(\mathbb{G}_m^{|i|}))$ the t-dimensional cube of symmetric monoidal categories. Define the stabilised K-theory of $\mathcal{S}(\mathbb{G}_m^{\wedge t})$ by

$$K(\mathcal{S}(\mathbb{G}_m^{\wedge t})) := \Omega|S^{\oplus}\mathcal{S}(\mathbb{G}_m^{\wedge t})| \tag{4.40}$$

4.5 A different model for the stabilised K-theory

In the previous section we have seen that for any symmetric monoidal category $\mathcal{S}$ the space $K(\mathcal{S}(\mathbb{G}_m^{\wedge 1}))$ fits into a fibration sequence of the form

$$K(\mathcal{S}) \to K(\mathcal{S}(\mathbb{G}_m)) \to K(\mathcal{S}(\mathbb{G}_m^{\wedge 1})) \tag{4.41}$$

where the first map is induced by the functor $\mathcal{S} \to \mathcal{S}(\mathbb{G}_m)$ that on objects maps $S \mapsto (S, 1_S)$. Moreover the second map is surjective on π_0, which characterises the homotopy type of the base completely. The following lemma enables us to express $K(\mathcal{S}(\mathbb{G}_m^{\wedge 1}))$ as the group-completion of yet another symmetric monoidal category.

Lemma 4.5.1 ([HS04, Lemma 3.9])**.** *Let $\mathcal{T}$ be a symmetric monoidal category, such that any morphism in $\mathcal{T}^{-1}\mathcal{T}$ is monic and that $\mathcal{S} \subset \mathcal{T}$ is a full monoidal subcategory. Then the diagram*

$$\begin{tikzcd} \mathcal{S} \arrow[r] \arrow[d] & \mathcal{T} \arrow[d] \\ \langle \mathcal{S}, \mathcal{S} \rangle \arrow[r] & \langle \mathcal{S}, \mathcal{T} \rangle \end{tikzcd} \tag{4.42}$$

is homotopy cartesian after group-completion, and $\langle \mathcal{S}, \mathcal{S} \rangle$ is contractible.

We will apply this Lemma to the category $\mathcal{T} := \mathcal{S}(\mathbb{G}_m)$ with the full subcategory $\mathcal{S}$, where $\mathcal{S}$ is either $\mathcal{GL}^t(R)$ or ${}_\epsilon\mathcal{O}(R)$. It remains to check that any morphism in $\mathcal{T}^{-1}\mathcal{T} = \langle \mathcal{T}, \mathcal{T} \times \mathcal{T} \rangle$ is monic. Recall that an object of the category $\langle \mathcal{T}, \mathcal{C} \rangle$ is simply an object $C \in \mathcal{C}$ and a morphism $C \to C'$ is given by an equivalence class of data $[T \in \mathcal{T}, \alpha \colon C \oplus T \to C']$.
To simplify the notation we will only show that any morphism in $\langle \mathcal{T}, \mathcal{T} \rangle$ is monic. Because objects and morphisms in $\mathcal{T}^{-1}\mathcal{T}$ are pairs of objects and morphisms $\langle \mathcal{T}, \mathcal{T} \rangle$ (with some additional conditions), the result for $\mathcal{T}^{-1}\mathcal{T}$ will follow.
Now, let $f = [U, \alpha] \colon T' \to T''$ and $g_i = [U_j, \alpha_i] \colon T \to T'$ for $i = 1, 2$ be morphisms in $\langle \mathcal{T}, \mathcal{T} \rangle$ such that $f \circ g_1 = f \circ g_2$. Then we want to show that $g_1 = g_2$. Explicitly the condition $f \circ g_1 = f \circ g_2$ means, that there exists an isomorphism $u \colon U \oplus U_1 \to U \oplus U_2$ such that the following diagram commutes:

$$\begin{tikzcd} T \oplus U_1 \oplus U \arrow[r, "\alpha_1 + 1_U"] \arrow[d, "1_T + u"] & T' \oplus U \arrow[r, "\alpha"] & T'' \arrow[d, "="] \\ T \oplus U_2 \oplus U \arrow[r, "\alpha_2 + 1_U"] & T' \oplus U \arrow[r, "\alpha"] & T'' \end{tikzcd} \tag{4.43}$$

This is equivalent to the condition $1_T + u = \alpha_2^{-1}\alpha_1 + 1_U \colon T \oplus U_1 \oplus U \to T \oplus U_2 \oplus U$. It follows that we can write $u = \begin{pmatrix} u' & 0 \\ 0 & 1_U \end{pmatrix}$ and $\alpha_2^{-1}\alpha_1 = \begin{pmatrix} 1_T & 0 \\ 0 & u' \end{pmatrix}$ for some isomorphism $u' \colon U_1 \to U_2$. In other words, the diagram

$$\begin{tikzcd} T \oplus U_1 \arrow[r, "\alpha_1"] \arrow[d, "1_T + u'"] & T' \arrow[d, "="] \\ T \oplus U_2 \arrow[r, "\alpha_2"] & T' \end{tikzcd} \tag{4.44}$$

commutes and $g_1 = g_2$.

Because ${}_\epsilon\mathcal{HGL}^t(R) \subset {}_\epsilon\mathcal{O}^t(R)$ and ${}_\epsilon\mathcal{HGL}^t(R)(\mathbb{G}_m) \subset {}_\epsilon\mathcal{O}^{t+1}(R)$ are subcategories, the conditions of Lemma 4.5.1 are equally satisfied for ${}_\epsilon\mathcal{HGL}^t(R)$.

The stabilised K-theory space $K(\mathcal{S}(\mathbb{G}_m^{\wedge 1}))$ has one further advantage, namely that cofinal functor $\mathcal{S} \to \mathcal{T}$ now also induce isomorphisms on K_0:

Proposition 4.5.2. *Let $f\colon \mathcal{S} \to \mathcal{T}$ be a cofinal functor of symmetric monoidal categories. Moreover assume that $\mathcal{S} \subset \mathcal{S}(\mathbb{G}_m)$ and $\mathcal{T} \subset \mathcal{T}(\mathbb{G}_m)$ satisfy the conditions of Lemma 4.5.1. Then f induces a homotopy equivalence*

$$\langle \mathcal{S}, \mathcal{S}(\mathbb{G}_m)\rangle^+ \xrightarrow{\simeq} \langle \mathcal{T}, \mathcal{T}(\mathbb{G}_m)\rangle^+ \tag{4.45}$$

Proof. Consider the commutative diagram

$$\begin{tikzcd} \mathcal{S}^+ \arrow[r] \arrow[d] & \mathcal{T}^+ \arrow[d] \\ \mathcal{S}(\mathbb{G}_m)^+ \arrow[r] & \mathcal{T}(\mathbb{G}_m)^+ \end{tikzcd} \tag{4.46}$$

We want the induced map on vertical homotopy cofibres to be an equivalence. This is the case if and only if the induced map on horizontal homotopy cofibres is an equivalence. By Theorem 3.2.11 this map is just

$$K_0(\mathcal{T})/K_0(\mathcal{S}) \to K_0(\mathcal{T}(\mathbb{G}_m))/K_0(\mathcal{S}(\mathbb{G}_m)). \tag{4.47}$$

It remains to show that this group-homomorphism is an isomorphism. Injectivity is clear, since the forgetful map in the other direction is a left-inverse. To show surjectivity consider a generator $[(T,\theta)] \in K_0(\mathcal{T}(\mathbb{G}_m))$. Since $f\colon \mathcal{S} \to \mathcal{T}$ is cofinal there exists a $T' \in \mathcal{T}$ and $S \in \mathcal{S}$ such that $T \oplus T' \cong S$ and consequently $(T,\theta) \oplus (T',1) \cong (S, \theta \oplus 1)$. Thus $[(T,\theta)] = -[(T',1)]$ in $K_0(\mathcal{T}(\mathbb{G}_m))/K_0(\mathcal{S}(\mathbb{G}_m))$. We are done.

□

Corollary 4.5.3. *There is a commutative triangle of homotopy equivalences*

$$\begin{tikzcd} \left\langle {}_\epsilon\mathcal{HGL}^t(R), {}_\epsilon\mathcal{HGL}^t(R)(\mathbb{G}_m)\right\rangle^+ \arrow[r, "\simeq"] \arrow[d, "\simeq"] & \left\langle {}_\epsilon\mathcal{O}^t(R), {}_\epsilon\mathcal{O}^{t+1}(R)\right\rangle^+ \arrow[dl, "\simeq"] \\ \left\langle i_\epsilon\mathcal{P}_h(R, \mathbb{G}_m^t), i_\epsilon\mathcal{P}_h(R, \mathbb{G}_m^{t+1})\right\rangle^+ & \end{tikzcd} \tag{4.48}$$

Proof. The two vertical maps are homotopy equivalences by Proposition 4.5.2, the remaining horizontal map is a homotopy equivalence by the two-out-of-three property. $\square$

From the expression of $K(\mathcal{S}(\mathbb{G}_m^{\wedge t}))$ as the iterated base of a fibration that is surjective on π_0, the following is evident.

Lemma 4.5.4. *Let $\mathcal{S}$ be a symmetric monoidal category. Then the group of connected components $\pi_0 K(\mathcal{S}(\mathbb{G}_m^{\wedge t}))$ is given by the group $\pi_0 K(\mathcal{S}(\mathbb{G}_m^t))$ modulo the subgroup generated by all those $(S, \{\theta_j\})$ with $\theta_j = 1$ for some j.*

4.6 Shifted Grothendieck-Witt groups of automorphisms

In this section we will prove that the sequence

$$GW_0^{[2n]}(k, \mathbb{G}_m^{\wedge t}) \to K_0(k, \mathbb{G}_m^{\wedge t}) \to GW_0^{[2n+1]}(k, \mathbb{G}_m^{\wedge t}) \to 0 \tag{4.49}$$

is exact for any field k, such that 2 is invertible. This is, at least partially, already known by Theorem 3.5.20. However, the theorem does not imply that the map $K_0(k, \mathbb{G}_m^{\wedge t}) \to GW_0^{[2n+1]}(k, \mathbb{G}_m^{\wedge t})$ is surjective. Moreover, we will see more explicitly, in terms of generators, that the composition $GW_0^{[2n]}(R, \mathbb{G}_m^{\wedge t}) \to GW_0^{[2n+1]}(R, \mathbb{G}_m^{\wedge t})$ is trivial for any ring R such that 2 is invertible. Essentially, these results follow from applying the formulas discussed in Paragraph 7.2 of [Wal03].

Definition 4.6.1. A sublagrangian of an ϵ-symmetric complex (A, ϕ) is a chain complex L together with a morphism $i\colon L \to A$ of chain complexes, such that i and ϕi are essential monomorphisms and such that $i^*\phi i = 0$. Given such a sublagrangian, we may define $L^\perp := \mathrm{Ker}(i^*\phi)$ and obtain an induced ϵ-symmetric complex $(L^\perp/L, \phi')$.

For a sublagrangian L of (A, ϕ) thatis concentrated in degree -1, Walter proves the formula [Wal03, Formulas 7.2]

$$[A, \phi] = [L^\perp/L, \phi'] + [H(L)] \tag{4.50}$$

Lemma 4.6.2. *Let $R = k$ be a field, or let R be local and $t = 0$. Assume further that 2 is invertible. Then the homomorphism*

$$K_0(R, \mathbb{G}_m^{\wedge t}) \to GW_0^{[2n+1]}(R, \mathbb{G}_m^{\wedge t}) \tag{4.51}$$

is surjective.

Proof. The case R local, $t = 0$ is [Bal01, Theorem 5.6]. We treat the case $R = k$. Recall from Section 4.2 that $\mathcal{F}(k, \mathbb{G}_m^t)$ is the exact category with duality of k-vector spaces, equipped with t-tuples of commuting automorphisms and that the duality is defined as $(A, \{\theta_j\})^* = (A^*, \{(\theta_j^*)^{-1}\})$. Consider an ϵ-symmetric short complex in $\mathcal{F}(k, \mathbb{G}_m^t)$:

$$\begin{tikzcd}
0 \arrow[r] & (A_0, \{\theta_{0j}\}) \arrow[r, "d"] \arrow[d, "\phi^t"] & (A_1, \{\theta_{1j}\}) \arrow[r] \arrow[d, "\phi"] & 0 \\
0 \arrow[r] & (A_1^*, \{(\theta_{1j}^*)^{-1}\}) \arrow[r, "\epsilon d^*"] & (A_0^*, \{(\theta_{0j}^*)^{-1}\}) \arrow[r] & 0
\end{tikzcd} \tag{4.52}$$

Let $L := ker(d) \subset A_0$ and observe that all θ_{0j} map L onto itself. Indeed, if $d(x) = 0$, then $d\theta_{0j}(x) = \theta_{1j}d(x) = 0$ as well. It follows that there is a map

$$\begin{tikzcd} 0 \arrow[r] & (L, \theta_{0j}|_L) \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & (A_0, \{\theta_{0j}\}) \arrow[r] & (A_1, \{\theta_{1j}\}) \arrow[r] & 0 \end{tikzcd} \tag{4.53}$$

of short complexes, and it can be easily seen that this morphism defines a sublagrangian. Further, we observe that $L^{\perp}/L$ is the short complex defined by the isomorphism $(A_0, \{\theta'_{0j}\})/ker(d) \xrightarrow{\cong} Im(d) \subseteq (A_1, \{\theta_{1j}\})$ and is therefore quasi-isomorphic to the trivial complex. $\square$

Lemma 4.6.3. *The composition $GW_0^{[2n]}(R, \mathbb{G}_m^{\wedge t}) \to GW_0^{[2n+1]}(R, \mathbb{G}_m^{\wedge t})$ vanishes.*

Proof. An inner product space (X, ϕ) representing a generator in $GW^{[2n]}(\mathcal{C})$ for some exact category with duality $(\mathcal{C}, *, \eta)$ is given by an ϵ-symmetric isomorphism $\phi\colon X \to X^*$. The composition of the forgetful functor followed by the hyperbolic functor maps this generator to the ϵ-symmetric short complex $H(X)$:

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, "0"] \arrow[d, "\eta"'] & X^* \arrow[r] \arrow[d, "="] & 0 \\ 0 \arrow[r] & X^{**} \arrow[r, "0"] & X^* \arrow[r] & 0 \end{tikzcd} \tag{4.54}$$

The class $[H(X)]$ is now equivalent to the class of the graph complex

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, "\phi"] \arrow[d, "\eta"'] & X^* \arrow[r] \arrow[d, "="] & 0 \\ 0 \arrow[r] & X^{**} \arrow[r, "\epsilon\phi^*"] & X^* \arrow[r] & 0 \end{tikzcd} \tag{4.55}$$

where the short complex $0 \to X \xrightarrow{\phi} X^* \to 0$ is clearly quasi-isomorphic to the trivial complex. We conclude that the composition $GW_0^{[2n]}(R, \mathbb{G}_m^{\wedge t}) \to GW_0^{[2n+1]}(R, \mathbb{G}_m^{\wedge t})$ is the zero map.

$\square$

4.7 K-theory and hermitian K-theory of simplicial rings and automorphisms

The central results of this work will be true *modulo polynomial homotopy.* More concretely, if we are presented with a functor F from rings to groups, say, we will be interested in the colimit

$$\operatorname{colim}(F(R[T]) \rightrightarrows F(R)) \tag{4.56}$$

where the two morphisms are induced by the evaluations $T = 0$ and $T = 1$ respectively. To take care of higher homotopies this is formalised using the simplicial ring $R\Delta^{\cdot}$ (Definition 2.1.19). In particular we have $R\Delta^0 = R[T_0]/(T_0 = 1) \cong R$ and $R\Delta^1 = R[T_0, T_1]/(T_1 = 1 - T_0) \cong R[T_0]$. The face-maps now satisfy $d_0(T_0) = 0$ and $d_1(T_0) = T_0 = 1$, i.e. they are precisely the two evaluations mentioned above.
The constructions in Sections 4.1 and 4.2, which first assign a symmetric monoidal category to a ring R (with involution) and then take the group-completion, depend naturally on the ring and therefore extend to functors from rings (with involution) to simplicial spaces

$$R \mapsto K(R\Delta^{\cdot}, \mathbb{G}_m^{\wedge t}) \tag{4.57}$$

and

$$R \mapsto {}_{\epsilon}GW(R\Delta^{\cdot}, \mathbb{G}_m^{\wedge t}) \tag{4.58}$$

respectively. We will use the notations $K(R\Delta^{\cdot}, \mathbb{G}_m^{\wedge t})$ and ${}_{\epsilon}GW(R\Delta^{\cdot}, \mathbb{G}_m^{\wedge t})$ interchangeably for both, the simplicial space and its geometric realisation. The notation $K_0(R\Delta^{\cdot}, \mathbb{G}_m^{\wedge t})$ is used for the simplicial abelian group $(d \mapsto K_0(R\Delta^d, \mathbb{G}_m^{\wedge t}))$ and its realisation, respectively, and similarly for ${}_{\epsilon}GW$. It is important to note that this is not the same as $\pi_0 K(R\Delta^{\cdot}, \mathbb{G}_m^{\wedge t})$ which denotes the monoid of connected components of the simplicial space $K(R\Delta^{\cdot}, \mathbb{G}_m^{\wedge t})$:

Proposition 4.7.1. *Let F be a functor from rings to spaces (sets). Then there are natural isomorphisms*

$$\pi_0 F(R\Delta^{\cdot}) \cong \pi_0 |[d] \mapsto \pi_0 F(R\Delta^d)| \tag{4.59}$$

Proof. Both sides are given as

$$\pi_0 F(R\Delta^{\cdot}) = \operatorname{colim}(\pi_0 F(R[T]) \rightrightarrows \pi_0(F(R))). \tag{4.60}$$

$\square$

The next two propositions show that the spaces $\langle \mathcal{S}, \mathcal{S}(\mathbb{G}_m)\rangle$ are group complete for $\mathcal{S} = \mathcal{GL}^t(R\Delta^{\cdot})$ and $\mathcal{S} = {}_\epsilon\mathcal{HGL}^t(R\Delta^{\cdot})$.

Proposition 4.7.2. *The classifying spaces of $\left\langle \mathcal{GL}^t(R\Delta^{\cdot}), \mathcal{GL}^{t+1}(R\Delta^{\cdot})\right\rangle$ for $t \in \mathbb{N}_0$ are group complete.*

Proof. According to Proposition 3.1.2 it suffices to show that

$$\pi_0 \left\langle \mathcal{GL}^t(R\Delta^{\cdot}), \mathcal{GL}^{t+1}(R\Delta^{\cdot})\right\rangle \tag{4.61}$$

is already a group. The case $t = 0$ is treated by the well-known Whitehead-lemma (see e.g. [Sri96, Lemma 1.4]) and it is not hard to generalise it to positive t. We treat the cases for all $t \in \mathbb{N}_0$ simultaneously.
We have

$$\pi_0(\langle \mathcal{S}, \mathcal{S}(\mathbb{G}_m)\rangle) = \pi_0(\langle \mathcal{GL}(R, \mathbb{G}_m^t), \mathcal{GL}(R, \mathbb{G}_m^{t+1})\rangle)/\sim \tag{4.62}$$

where an element of π_0 represented by an object $(n, {}_0\theta_1, \ldots, {}_0\theta_{t+1})$ of $\mathcal{GL}(R, \mathbb{G}_m^{t+1})$ is equivalent to the element represented by $(n, {}_1\theta_1, \ldots, {}_1\theta_{t+1})$, if there exists an object $(n, \theta_1(T), \ldots, \theta_{t+1}(T))$ in $\mathcal{GL}(R[T], \mathbb{G}_m^{t+1})$ such that $\theta_j(0) = {}_0\theta_j$ and $\theta_j(1) = {}_1\theta_j$ for all j.
We will construct an inverse of the fixed object $(n, \theta_1, \ldots, \theta_{t+1})$ in $\mathcal{GL}(R, \mathbb{G}_m^{t+1})$. To this end let

$$M(T) = \begin{pmatrix} 1 & 0 \\ T & 1 \end{pmatrix} \begin{pmatrix} 1 & -T \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ T & 1 \end{pmatrix} \tag{4.63}$$

be a block-matrix in $Gl_{2n}(R[T])$ where each $n \times n$ block consists of a scalar multiple of the identity matrix. Consider the object

$$X(T) = \left(2n, \begin{pmatrix} \theta_1 & 0 \\ 0 & \theta_1 \end{pmatrix}, \ldots, \begin{pmatrix} \theta_t & 0 \\ 0 & \theta_t \end{pmatrix}, M(T)^{-1} \begin{pmatrix} \theta_{t+1} & 0 \\ 0 & 1 \end{pmatrix} M(T) \begin{pmatrix} 1 & 0 \\ 0 & \theta_{t+1}^{-1} \end{pmatrix}\right) \tag{4.64}$$

in $\mathcal{GL}(R[T], \mathbb{G}_m^{t+1})$, where the θ_j are understood to be extended to $R[T]$. Note that the automorphisms commute pairwise, because the blocks of M are scalar multiples of the identity matrix and therefore in the centre of $Gl_n(R[T])$.
We observe that

$$X(0) = (\theta_1, \ldots, \theta_t, \theta_{t+1}) \oplus (\theta_1, \ldots, \theta_t, \theta_{t+1}^{-1}) \tag{4.65}$$

and

$$X(1) = 2(\theta_1, \dots, \theta_t, 1) \tag{4.66}$$

but the latter is in the image of $\mathcal{GL}(R, \mathbb{G}_m^t) \to \mathcal{GL}(R, \mathbb{G}_m^{t+1})$ and therefore trivial in $\pi_0\langle \mathcal{GL}(R, \mathbb{G}_m^t), \mathcal{GL}(R, \mathbb{G}_m^{t+1})\rangle$. We conclude that $(\theta_1, \dots, \theta_t, \theta_{t+1}^{-1})$ is the inverse of $(\theta_1, \dots, \theta_t, \theta_{t+1})$. □

There is also a Whitehead-lemma for ${}_\epsilon\mathcal{O}$ [KM70, Lemma 1] that show that

$$\left\langle {}_\epsilon\mathcal{O}^0(R\Delta^{\cdot}), {}_\epsilon\mathcal{O}^1(R\Delta^{\cdot})\right\rangle \tag{4.67}$$

is group-complete. However, it is not clear at all if and how this constructive approach generalises to higher t, because there are more compatibility conditions to satisfy. In short, we were lucky that the matrix

$$M(T)^{-1}\begin{pmatrix}\theta_{t+1} & 0\\ 0 & 1\end{pmatrix} M(T)\begin{pmatrix}1 & 0\\ 0 & \theta_{t+1}^{-1}\end{pmatrix} \tag{4.68}$$

commutes with all the other automorphisms $\theta_i \oplus \theta_i$ in $Gl(R)$, while for ${}_\epsilon O(R)$ this is not the case. This is the reason we introduced the categories ${}_\epsilon\mathcal{HGL}^t(R)$, for we have:

Proposition 4.7.3. *The classifying spaces of* $\left\langle {}_\epsilon\mathcal{HGL}^t(R\Delta^{\cdot}), {}_\epsilon\mathcal{HGL}^t(R\Delta^{\cdot})(\mathbb{G}_m)\right\rangle$ *for* $t \in \mathbb{N}_0$ *are group complete.*

Proof. An object S of ${}_\epsilon\mathcal{HGL}^t(R)(\mathbb{G}_m)$ consists of pairwise commuting, invertible matrices $\theta_j \in Gl_n(R)$ for $1 \le j \le t$ of rank $n \in \mathbb{N}_0$ and one matrix $\theta \in {}_\epsilon O_n(R)$ that commutes with all the ${}_\epsilon H(\theta_j)$. If θ has the block-structure $\begin{pmatrix}A & B\\ C & D\end{pmatrix}$ then $\theta^{-1} = {}_\epsilon H(R^n)^{-1}\theta^t {}_\epsilon H(R^n) = \begin{pmatrix}D^t & \epsilon B^t\\ \epsilon C^t & A^t\end{pmatrix}$.
Define the string

$$X(T) = \left(2n, \begin{pmatrix}\theta_1 & 0\\ 0 & \theta_1\end{pmatrix}, \dots, \begin{pmatrix}\theta_t & 0\\ 0 & \theta_t\end{pmatrix}, Y(T)\right) \tag{4.69}$$

for

$$Y(T) = {}_\epsilon H(M(T))^{-1}\begin{pmatrix}A & 0 & B & 0\\ 0 & 1 & 0 & 0\\ C & 0 & D & 0\\ 0 & 0 & 0 & 1\end{pmatrix} {}_\epsilon H(M(T))\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & D^t & 0 & \epsilon B^t\\ 0 & 0 & 1 & 0\\ 0 & \epsilon C^t & 0 & A^t\end{pmatrix} \tag{4.70}$$

consisting of commuting matrices $\begin{pmatrix} \theta_j & 0 \\ 0 & \theta_j \end{pmatrix} \in Gl_{2n}(R)$, $1 \leq j \leq t$, and a matrix $Y(T)$ in ${}_\epsilon O_{2n}(R)$ that commutes with all the ${}_\epsilon H \begin{pmatrix} \theta_j & 0 \\ 0 & \theta_j \end{pmatrix}$. Evaluating $X(1)$ yields an object in ${}_\epsilon\mathcal{HGL}^t(R) \subset {}_\epsilon\mathcal{HGL}^t(R)(\mathbb{G}_m)$, and S is a direct summand of $X(0)$. We are done.

□

4.8 Homotopy invariance

Proposition 4.8.1 (Homotopy invariance of K-theory [Qui73, Corollary to Theorem 8])**.** *Let R be a regular ring. Then the natural map*

$$K(R[T]) \xrightarrow{\simeq} K(R) \tag{4.71}$$

is a homotopy equivalence.

Corollary 4.8.2. *Let R be a regular ring. Then the simplicial K-theory coincides with the ordinary K-theory*

$$K(R\Delta^{\cdot}) \xrightarrow{\simeq} K(R) \tag{4.72}$$

Proof. $K(R\Delta^{\cdot})$ is equivalent to the constant simplicial space and good in the sense of Definition 2.1.8. □

Proposition 4.8.3 (Homotopy invariance for GW-theory [Sch12, Theorem 9.8])**.** *Let R be a regular noetherian ring with 2 invertible. Further let $n \in \mathbb{Z}$ be any integer. Then the natural map*

$$GW^{[n]}(R[T]) \xrightarrow{\simeq} GW^{[n]}(R) \tag{4.73}$$

is a homotopy equivalence.

Corollary 4.8.4. *Let R be a regular noetherian ring with 2 invertible. Further let $n \in \mathbb{Z}$ be any integer. Then the simplicial GW-theory coincide with the ordinary GW-theory*

$$GW^{[n]}(R\Delta^{\cdot}) \xrightarrow{\simeq} GW^{[n]}(R) \tag{4.74}$$

Chapter 5

Simplicial Theorem B'

In this chapter we recall Quillen's famous Theorem B that allows us to describe the homotopy fibre of a functor as the classifying space of a certain comma category [Qui73] and Waldhausen's Theorem B' that generalises the result to simplicial categories [Wal82]. The original Theorem B comes in essentially two different versions, one using left-fibres and one using right-fibres. The generalised version B' for simplicial categories can be reduced to the classical case, because any simplicial category is homotopy equivalent to a (non-simplicial) category. Waldhausen explains this idea and refers to Thomason's (unpublished) thesis for a full proof. Moreover there is more than one equivalent non-simplicial category that can be associated with a simplicial one. Notably there exists a construction by Segal [Seg74] that associates a homotopy equivalent category $simp(A)$ to a given simplicial space A. We show how this can be applied if A is actually a simplicial category. Using both versions of a non-simplicial equivalent category and both versions of Theorem B, we ultimately end up with four distinct versions of Theorem B'.

5.1 Left-fibres, right-fibres and Theorem B

Definition 5.1.1. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor of small categories and $D \in \mathcal{D}$ an object. The comma category or right-fibre $D \downarrow F$ has pairs $(C \in \mathcal{C}, f\colon D \to F(C))$ as objects and a morphism $(C, f) \to (C', f')$ is given by a morphism $g\colon C \to C'$ in $\mathcal{C}$ such that the following diagram commutes:

$$\begin{tikzcd} D \arrow[r, "f"] \arrow[dr, "f'"'] & F(C) \arrow[d, "F(g)"] \\ & F(C') \end{tikzcd} \tag{5.1}$$

There is a forgetful functor $D \downarrow F \to \mathcal{C}$ and for any morphism $D \to D'$ in $\mathcal{D}$ there is an induced functor $D' \downarrow F \to D \downarrow F$ in the opposite direction on right-fibres. The induced maps on the geometric realisation are called transition maps.

Similarly the left-fibre $F \downarrow D$ has pairs $(C \in \mathcal{C}, f\colon F(C) \to D)$ as objects and morphisms $(C, f) \to (C', f')$ are given by a morphism $g\colon C \to C'$ in $\mathcal{C}$ such that the diagram

$$\begin{tikzcd} F(C) \arrow[r, "f"] \arrow[d, "F(g)"'] & D \\ F(C') \arrow[ur, "f'"'] & \end{tikzcd} \tag{5.2}$$

commutes. There exists a forgetful functor $F \downarrow D \to \mathcal{C}$ and transition maps $F \downarrow D \to F \downarrow D'$ for any morphism $D \to D'$.

Theorem 5.1.2 (Theorem B, [Qui73, p. 97]). *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor of small categories such that for any morphism $D \to D'$ in $\mathcal{D}$ the transition map induced by $D' \downarrow F \to D \downarrow F$ is a homotopy equivalence. Then the sequence*

$$D \downarrow F \to \mathcal{C} \to \mathcal{D} \tag{5.3}$$

induces a homotopy fibration on classifying spaces for any $D \in \mathcal{D}$. Similarly, if all transition maps of left-fibres $F \downarrow D \to F \downarrow D'$ are homotopy equivalences, then the sequence

$$F \downarrow D \to \mathcal{C} \to \mathcal{D} \tag{5.4}$$

induces a homotopy fibration on classifying spaces for any $D \in \mathcal{D}$.

5.2 Categories associated with simplicial categories

In this section we will show how any simplicial category is homotopy equivalent to a non-simplicial one. We will consider two different versions of this idea. The first one is based on Thomason's homotopy colimit theorem [Tho79]:

Theorem 5.2.1. *Let $F\colon K \to \mathrm{Cat}$ be a functor from a small category K to the category of small categories. Denote by $K \int F$ the category with pairs $(k \in K, x \in F(k))$ as objects and where a morphism $(k, x) \to (k', x')$ is given by a pair (f, g) with $f\colon k \to k'$ and $g\colon F(f)(x) \to x'$. Then there is a natural homotopy equivalence*

$$\operatorname{hocolim}_K F \to K \int F \tag{5.5}$$

A simplicial category $\mathcal{C} = (d \mapsto \mathcal{C}_d)$ is the same as a functor $\mathcal{C}\colon \Delta^{op} \to \mathrm{Cat}$ and we have

$$|\mathcal{C}| \simeq \operatorname{hocolim}_{\Delta^{op}} \mathcal{C}_d \simeq |\Delta^{op} \int \mathcal{C}| \tag{5.6}$$

Explicitly an object of $\Delta^{op} \int \mathcal{C}$ is a pair $(d, C \in \mathcal{C}_d)$ and a morphism $(d, C) \to (d', C')$ is given by a pair $\phi\colon d' \to d$ and $f\colon \phi^* C \to C'$. A functor $F\colon \mathcal{C} \to \mathcal{D}$ of simplicial categories is nothing else than a natural transformation of functors $\mathcal{C}, \mathcal{D}\colon \Delta^{op} \to \mathrm{Cat}$ and therefore induces a functor $\Delta^{op} \int F\colon \Delta^{op} \int \mathcal{C} \to \Delta^{op} \int \mathcal{D}$ that maps $(d, C) \mapsto (d, F(C))$ and $(\phi, f) \mapsto (\phi, F(f))$.

We will now describe Segal's version ([Seg74]).

Definition 5.2.2. Let $\mathcal{C} = (d \mapsto \mathcal{C}_d)$ be a simplicial category. Denote by $\mathrm{simp}(\mathcal{C})$ the category which has pairs $(d, C \in \mathcal{C}_d)$ as objects and where a morphism $(d, C) \to (d', C')$ is given by a pair $\phi\colon d \to d'$ and $f\colon C \to \phi^* C'$. A functor $F\colon \mathcal{C} \to \mathcal{D}$ of simplicial categories induces a functor $\mathrm{simp}(F)\colon \mathrm{simp}(\mathcal{C}) \to \mathrm{simp}(\mathcal{D})$, with $\mathrm{simp}(F)(d, C) = (d, F(C))$ and $\mathrm{simp}(F)(\phi, f) = (\phi, F(f))$.

Remark 5.2.3. The definitions of $\mathrm{simp}(\mathcal{C})$ and of $\Delta^{op} \int \mathcal{C}$ look very similar. Note that in one category morphisms in the first entry behave covariantly, while they behave contravariantly in the other.

The goal is now to show the following

Proposition 5.2.4. *Let $\mathcal{C} = (d \mapsto \mathcal{C}_d)$ be a simplicial category. Then there exists a homotopy equivalence*

$$\chi\colon |\mathcal{C}| \xrightarrow{\simeq} |\,\mathrm{simp}(\mathcal{C})| \tag{5.7}$$

We prove this by a sequence of Lemmas:

Definition 5.2.5. Let $F = (d \mapsto F_d)$ be a co-simplicial space and $C = (d \mapsto C_d)$ a simplicial space. Write $F(C) := \coprod_d (F_d \times C_d)/\sim$ with the relations $(\xi, \theta^* c) \sim (\theta_* \xi, c)$.

Lemma 5.2.6 ([Seg74, Lemma A.5])**.** *Let $F, F'\colon \Delta \to \mathrm{sSet}$ be two co-simplicial spaces and $T\colon F \to F'$ natural transformation. Then $F(C) \xrightarrow{\simeq} F'(C)$ for all good simplicial spaces C if*

- $F_d \xrightarrow{\simeq} F'_d$
- $F(\partial\Delta^d) \to F_d$ *and* $F'(\partial\Delta^d) \to F'_d$ *are cofibrations.*

Proof. We have $F(C) = \mathrm{colim}\, F_d(C)$ where $F_d(C)$ is defined as the push-out

$$\begin{tikzcd} (F_d \times \cup_i C_{d-1,i}) \cap (F(\partial\Delta_d) \times C_d) \arrow[r, "i"] \arrow[d] & F_d \times C_d \arrow[d] \\ F_{d-1}(C) \arrow[r] & F_d(C) \end{tikzcd} \tag{5.8}$$

for $d > 0$ and $F_0(C) := F_0 \times C_0$. By induction we may conclude that $F_d(C) \simeq F'_d(C)$ for all d ($F_0(C)$ is just $F_0 \times C_0$). The map i is a cofibration by the hypotheses and therefore $F_{d-1}(C) \to F_d(C)$ is a cofibration as well. Taking a colimit along cofibrations is the same as taking the homotopy colimit and the result follows. $\square$

Recall that for each d we denote by Δ^d the simplicial set with $\Delta^d_k = \mathrm{Hom}_\Delta(k,d)$ and $\lambda^*\phi := \phi \circ \lambda$ for any $\lambda\colon k \to k'$ and $\phi \in \Delta^d_k$. It is well known that upon geometric realisation we obtain $|\Delta^d| = \Delta^d_{top}$ and the assignment

$$F\colon [d] \mapsto |\Delta^d| \tag{5.9}$$

forms a co-simplicial space. We may consider Δ^d as a simplicial (discrete) category and therefore we can make sense of the category $\mathrm{simp}(\Delta^d)$. Explicitly an object is a pair $(k, \phi \in \mathrm{Hom}_\Delta(k,d))$ and a morphism $(k,\phi) \to (k',\phi')$ is an element $\lambda \in \mathrm{Hom}_\Delta(k,k')$ with $\phi = \phi'\lambda$. The assignment

$$F'\colon [d] \mapsto |\,\mathrm{simp}(\Delta^d)| \tag{5.10}$$

is a co-simplicial space. We may understand $\mathrm{simp}(\Delta^d)$ as a barycentric subdivision of Δ^d. Indeed $\mathrm{simp}(\Delta^d)$ contains an object (and therefore a 0-simplex) for any simplex of Δ^d, the higher elements in the nerve of $\mathrm{simp}(\Delta^d)$ fill in the additional simplices as illustrated in the picture below, that show all non-degenerate simplices for $d = 1$

$$\bullet\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\bullet \qquad\qquad \bullet\!\!-\!\!\!-\!\!\bullet\!\!-\!\!\!-\!\!\bullet \tag{5.11}$$

On the left the two 0-simplices are given by the two possible maps $\delta_0, \delta_1\colon [0] \to [1]$ and the connecting 1-simplex corresponds to the identity $id\colon [1] \to [1]$. On the right the 0-simplices are given by the objects $(0,\delta_0)$, $(1,id)$ and $(0,\delta_1)$ and they are connected by the strings $(0,\delta_i) \xrightarrow{\delta_i} (1,id)$. We observe that there is a natural homotopy equivalence

$$|\Delta^d| \to |\,\mathrm{simp}(\Delta^d)| \tag{5.12}$$

inducing a transformation of co-simplicial spaces. Therefore we may apply Lemma 5.2.6, with F and F' as above, and with the simplicial space $d \mapsto |\mathcal{C}_d|$ where $\mathcal{C} = (d \mapsto \mathcal{C}_d)$ is a simplicial category.

Definition 5.2.7. There is a simplicial map

$$\chi'\colon \coprod_d (|\,\mathrm{simp}(\Delta^d)| \times |\mathcal{C}_d|)/_\sim \to |\,\mathrm{simp}(\mathcal{C})| \tag{5.13}$$

that maps the simplex given by

$$(d_0 \xrightarrow{\nu_0} \cdots \xrightarrow{\nu_{k-1}} d_k \xrightarrow{\nu_k} d), (C_0 \xrightarrow{f_0} \cdots \xrightarrow{f_{k-1}} C_k) \tag{5.14}$$

to the simplex

$$(d_0, \nu_0^* \cdots \nu_k^* C_0) \xrightarrow{(\nu_0, \nu_0^* \cdots \nu_k^* f_0)} \cdots \xrightarrow{(\nu_{k-1}, \nu_{k-1}^* \nu_k^* f_{k-1})} (d_k, \nu_k^* C_k) \tag{5.15}$$

Define the map χ in Proposition 5.2.4 to be the composition of χ' with the map of Lemma 5.2.6.

Before we prove Proposition 5.2.4, we introduce the following auxiliary categories: Given the simplicial category $\mathcal{C} = (d \mapsto \mathcal{C}_d)$ denote by $\mathcal{C}'_d$ the category with objects $(\lambda\colon [d] \to [k], C \in \mathcal{C}_k)$. A morphism $(\lambda, C) \to (\lambda', C')$ in this category is given by a pair $(\mu\colon k \to k', f\colon C \to \mu^* C')$ that satisfies $\lambda' = \mu\lambda$. There is an adjoint pair of functors

$$\begin{aligned} \mathcal{C}_d &\to \mathcal{C}'_d \\ C &\mapsto (1, C) \\ f &\mapsto (1, f) \end{aligned} \tag{5.16}$$

and

$$\begin{aligned} \mathcal{C}'_d &\to \mathcal{C}_d \\ (\lambda, C) &\mapsto \lambda^* C \\ (\mu, f) &\mapsto \lambda^* f \end{aligned} \tag{5.17}$$

We conclude:

Lemma 5.2.8. *The pair of functors above induces a homotopy equivalence of simplicial categories:*

$$([d] \mapsto \mathcal{C}_d) \to ([d] \mapsto \mathcal{C}'_d). \tag{5.18}$$

Proof of Proposition 5.2.4. Consider the two maps

$$\coprod_d (|\operatorname{simp}(\Delta^d)| \times |\mathcal{C}'_d|)/_\sim \to |\operatorname{simp}(\mathcal{C})| \tag{5.19}$$

that map the simplex

$$(d_0 \xrightarrow{\nu_0} \cdots \xrightarrow{\nu_{k-1}} d_k \xrightarrow{\nu_k} d), d \xrightarrow{\lambda_{-1}} l_0, ((l_0, C_0) \xrightarrow{(\lambda_0, f_0)} \cdots \xrightarrow{(\lambda_{k-1}, f_{k-1})} (l_k, C_k)) \tag{5.20}$$

to

$$(l_0, C_0) \xrightarrow{(\lambda_0, f_0)} \cdots \xrightarrow{(\lambda_{k-1}, f_{k-1})} (l_k, C_k) \tag{5.21}$$

or to

$$
\begin{aligned}
(d_0, \nu_0^* \cdots \nu_k^* \lambda_{-1}^* C_0) &\xrightarrow{(\nu_0, \nu_0^* \cdots \nu_k^* \lambda_{-1}^* f_0)} \cdots \\
&\cdots \xrightarrow{(\nu_{k-1}, \nu_{k-1}^* \nu_k^* \lambda_{-1}^* \cdots \lambda_{k-2}^* f_{k-1})} (d_k, \nu_k^* \lambda_{-1}^* \cdots \lambda_{k-1}^* C_k),
\end{aligned} \tag{5.22}
$$

respectively. The former map can easily be seen to be a homotopy equivalence, by considering it as the diagonal of a map of bisimplicial sets. The latter map induces the map χ' when composed with the equivalence

$$
\coprod_d (|\operatorname{simp}(\Delta^d)| \times |\mathcal{C}_d|)/\sim \xrightarrow{\sim} \coprod_d (|\operatorname{simp}(\Delta^d)| \times |\mathcal{C}'_d|)/\sim \tag{5.23}
$$

Finally the two maps are homotopic by an argument very similar to that of [Tho79, Lemma 1.2.5]. This concludes the proof. $\square$

5.3 Simplicial Theorem B' four ways

Let $\mathcal{C}, \mathcal{D}$ be small simplicial categories and let $F \colon \mathcal{C} \to \mathcal{D}$ be a functor. We have seen that we can turn F into a functor between non-simplicial categories in two ways:

$$
\operatorname{simp}(F) \colon \operatorname{simp}(\mathcal{C}) \to \operatorname{simp}(\mathcal{D}) \tag{5.24}
$$

or

$$
\Delta^{op} \int F \colon \Delta^{op} \int \mathcal{C} \to \Delta^{op} \int D \tag{5.25}
$$

We may apply Theorem B to both functors and obtain two different left- and right-fibres each. We will call them left- and right-fibres of type S (for Segal or simp) and of type T (for Thomason), respectively. A priori the fibres are just categories, but as it turns out we may apply the constructions of Section 5.2 once more, this time in the other direction to obtain four distinct simplicial categories. Waldhausen [Wal82] sketches the proof for left-fibres of type S, and Grayson [Gra95] quotes the result for right-fibres of type T. To our knowledge there exists no comprehensive treatment in the literature. We demonstrate the case of right-fibres of type T explicitly, because that is the case we will use first. The other three cases work similarly:

We apply the right-fibre formulation of Theorem B to the functor $\Delta^{op} \int F \colon \Delta^{op} \int \mathcal{C} \to \Delta^{op} \int D$. Concretely, we fixate a pair $(l, D \in \mathcal{D}_l)$ and consider the category

$$
(l, D) \downarrow (\Delta^{op} \int F) \tag{5.26}
$$

An object is a tuple (d, C, ϕ, f), where $(d, C \in \mathcal{C}_d)$ is an object in $\Delta^{op} \int \mathcal{C}$ and $(\phi\colon d \to l, f\colon \phi^* D \to F(C))$ is a morphism $(l, D) \to (d, \Delta^{op} \int F(C))$ in this category. A morphism $(d, C, \phi, f) \to (d', C', \phi', f')$ consists of a morphism $\lambda\colon d' \to d$ with $\phi\lambda = \phi'$ and a morphism $g\colon \lambda^* C \to C'$ such that the following diagram commutes:

$$\begin{tikzcd} & F(\lambda^* C) \arrow[d, "F(g)"] \\ (\phi')^* D \arrow[ur, "f"] \arrow[r, "f'"] & F(C') \end{tikzcd} \tag{5.27}$$

Definition 5.3.1. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor of simplicial categories. Fix some $l \geq 0$ and an object $D \in \mathcal{D}_l$. The right-fibre ${}_T(D \downarrow F)$ of type T is defined as the simplicial category $d \mapsto {}_T(D \downarrow F)_d$ with

$${}_T(D \downarrow F)_d = \coprod_{\phi\colon [d] \to [l]} \phi^* D \downarrow F_d \tag{5.28}$$

where the disjoint union runs over all simplicial maps $\phi\colon [d] \to [l]$.

From the definitions it is clear that

$$\Delta^{op} \int {}_T(D \downarrow F) = (l, D) \downarrow (\Delta^{op} \int F) \tag{5.29}$$

Transition maps are induced by morphisms in $(\Delta^{op} \int \mathcal{D})$. Now, any morphism $(\phi, f)\colon (d, D) \to (d', D')$ factors as

$$\begin{tikzcd} (d, D) \arrow[r, "(\phi, f)"] \arrow[d, "(\phi, id)"'] & (d', D') \\ (d', \phi^* D) \arrow[ur, "(id, f)"'] & \end{tikzcd} \tag{5.30}$$

Thus, we may consider two different types of transition maps between right-fibres. On the one hand, given a morphism $g\colon D \to D' \in \mathcal{D}_l$, there is a functor $u(g)\colon {}_T(D' \downarrow F) \to {}_T(D \downarrow F)$ and on the other hand, there is a functor $v(\delta)\colon {}_T((\delta^* D) \downarrow F)_d \to {}_T(D \downarrow F)_e$, for any simplicial map $\delta\colon [e] \to [d]$. If we write an object of ${}_T(D \downarrow F)_d$ as a triple (ϕ, C, f) with $\phi\colon [d] \to [l]$, $C \in \mathcal{C}_d$ and $f\colon \phi^* D \to F_d(C)$, then the transition maps are explicitly given by $u(g)(\phi, f, C) = (\phi, f \circ \phi^* g, C)$ and $v(\delta)(\phi, f, C) = (\delta \circ \phi, f, C)$. We conclude:

Theorem 5.3.2 (cf. Theorem B' [Wal82, p. 166])**.** *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor of simplicial categories. If all transition maps between right-fibres are homotopy* *equivalences, then*

$$_T(D \downarrow F) \to \mathcal{C} \to \mathcal{D} \tag{5.31}$$

is a fibration for all $l \geq 0$ and $D \in \mathcal{D}_l$.

Remark 5.3.3. There exist analogous statements for the three other kinds of fibres.

Definition 5.3.4. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor of simplicial categories. Fix some $l \geq 0$ and an object $D \in \mathcal{D}_l$. The right-fibre of type S is the simplicial category

$$_S(D \downarrow F)_d = \coprod_{\phi\colon [l]\to[d]} D \downarrow F_d \circ \phi^* \tag{5.32}$$

The left-fibre of type S has the form

$$_S(F \downarrow D)_d = \coprod_{\phi\colon [d]\to[l]} F \downarrow \phi^* D \tag{5.33}$$

Finally, the left-fibre of type T is given by

$$_T(F \downarrow D)_d = \coprod_{\phi\colon [l]\to[d]} F \circ \phi^* \downarrow D \tag{5.34}$$

The addendum to Theorem B' ([Wal82, p. 26]) holds for right-fibres of type T and left-fibres of type S:

Lemma 5.3.5 (Addendum to Theorem B')**.** *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor of simplicial categories. Assume further that objects in $\mathcal{D}$ have simplicial dimension 0 up to isomorphism, i.e. that for all $D \in \mathcal{D}_l$ there exists a $D_0 \in \mathcal{D}$ with $\phi_l^* D_0 = D$ for the unique map $\phi_l\colon [l] \to [0]$. Finally suppose the the conditions of Theorem B' hold for transition maps coming from morphisms in $\mathcal{D}_0$. Then the conclusion of Theorem B' holds for right-fibres of type T and left-fibres of type S.*

Proof. Fix $D \in \mathcal{D}_l$ and a $D_0 \in \mathcal{D}$ with $\phi_l^* D_0 = D$. Let $\phi_d\colon [d] \to [0]$ be the unique map. Then we readily observe

$$\begin{aligned} _T(D \downarrow F) &= \coprod_{\phi\colon [d]\to[l]} \phi^* D \downarrow F_d \\ &= \coprod_{\phi\colon [d]\to[l]} \phi_d^* D_0 \downarrow F_d \simeq \\ &= \coprod_{\phi\colon [d]\to[l]} * \times {}_T(D_0 \downarrow F) \simeq {}_T(D_0 \downarrow F) \end{aligned} \tag{5.35}$$

and similarly

$$ {}_S(F \downarrow D)_d = \coprod_{\phi\colon [d]\to[l]} F \downarrow \phi^* D \simeq {}_S(F \downarrow D_0)_d \tag{5.36} $$

The result then follows. $\square$

A related version was proven by Grayson for right-fibres of type T. The proof works identically for left-fibres of type S.

Theorem 5.3.6 ([Gra95, Theorem 5.3])**.** *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor of simplicial categories, let each $\mathcal{D}_l$ be connected and let all transition maps arising from morphisms $f\colon D \to D'$ be homotopy equivalences. Then all transition maps are already homotopy equivalences.*

Chapter 6

Fibrations via commuting automorphisms

6.1 The homotopy fibration sequence

The goal of this Chapter is to prove the following

Theorem 6.1.1. *Let R be a ring. Then the sequence*

$$\Omega^{-1}K(R\Delta^{\cdot}, \mathbb{G}_m^{\wedge t+1}) \xrightarrow{\sigma} K(R\Delta^{\cdot}, \mathbb{G}_m^{\wedge t}) \to K_0(R\Delta^{\cdot}, \mathbb{G}_m^{\wedge t}) \tag{6.1}$$

is a homotopy fibration. If R is a ring with involution and 2 is invertible, then also

$$\Omega^{-1}{}_{\epsilon}GW(R\Delta^{\cdot}, \mathbb{G}_m^{\wedge t+1}) \xrightarrow{\sigma} {}_{\epsilon}GW(R\Delta^{\cdot}, \mathbb{G}_m^{\wedge t}) \to {}_{\epsilon}GW_0(R\Delta^{\cdot}, \mathbb{G}_m^{\wedge t}) \tag{6.2}$$

is a homotopy fibration.

Remark 6.1.2. We explicitly realise $K(R\Delta^{\cdot}, \mathbb{G}_m^{\wedge t+1})$ and ${}_{\epsilon}GW(R\Delta^{\cdot}, \mathbb{G}_m^{\wedge t+1})$ as loop spaces in Lemma 6.1.5 below. Here, it doesn't matter whether the delooping or the simplicial dimnsion has higher precedence, since for any simplicial loop space $X = (d \mapsto X_d)$ we have

$$\Omega^{-1}|d \mapsto X_d| \simeq |d \mapsto \Omega^{-1}X_d|. \tag{6.3}$$

Indeed, the delooping is connected and we compute

$$\Omega|d \mapsto \Omega^{-1}X_d| \simeq |d \mapsto \Omega\Omega^{-1}X_d| = |d \mapsto X_d|. \tag{6.4}$$

Remark 6.1.3. The maps σ above are abstractly induced by an application of Theorem B' in Lemma 6.1.5. We will recall the construction in Section 6.2.2 and eventually identify the map with something more explicit.

Remark 6.1.4. The first part of the above theorem appears as Corollary 9.6 in [Gra95]. However the proof and the result for Grothendieck-Witt groups are new.

If R is regular, then the fibrations for various t in this theorem fit together to form a tower for the K-theory space and Grothendieck-Witt theory space, respectively:

$$K(R) \simeq W^0 \leftarrow W^1 \leftarrow \dots \tag{6.5}$$

with $W^t = \Omega^{-t}(K(R\Delta^{\cdot}, \mathbb{G}_m^{\wedge t}))$ or

$${}_{\epsilon}GW(R) \simeq W^0 \leftarrow W^1 \leftarrow \dots \tag{6.6}$$

with $W^t = \Omega^{-t}({}_{\epsilon}GW(R\Delta^{\cdot}, \mathbb{G}_m^{\wedge t}))$. The homotopy spectral sequences of these towers will then converge to the K-groups and GW-groups, respectively. The homotopy groups of the base spaces $K_0(R\Delta^{\cdot}, \mathbb{G}_m^{\wedge t})$ and ${}_\epsilon GW_0(R\Delta^{\cdot}, \mathbb{G}_m^{\wedge t})$ will be the building blocks of the E_2 term. For more details on the homotopy spectral sequence of a tower, the reader is referred to Chapter 9.

We have laid the foundation for the proof of Theorem 6.1.1 in the previous section, the only piece missing is the following highly technical result:

Lemma 6.1.5. *Let $\mathcal{S}$ be either $\mathcal{GL}(R\Delta^{\cdot}, \mathbb{G}_m^t)$ or ${}_\epsilon\mathcal{HGL}(R\Delta^{\cdot}, \mathbb{G}_m^t)$. Then*

$$|\,\langle \mathcal{S}, \mathcal{S}(\mathbb{G}_m)\rangle\,| \simeq \Omega|d \mapsto I_*\mathcal{S}^{-1}\mathcal{S}_d| \tag{6.7}$$

are homotopy equivalent, where $I_*\mathcal{S}^{-1}\mathcal{S}_d$ is the connected component of the base-point in $\mathcal{S}^{-1}\mathcal{S}_d$.

Before we come to the proof of this lemma , we discuss how the results patch together to prove Theorem 6.1.1.

Proof of Theorem 6.1.1. Let $\mathcal{T}$ be either $i\mathcal{P}(R\Delta^{\cdot}, \mathbb{G}_m^t)$ or $i_\epsilon\mathcal{P}_h(R\Delta^{\cdot}, \mathbb{G}_m^t)$ and let $\mathcal{S}$ be either $\mathcal{GL}(R\Delta^{\cdot}, \mathbb{G}_m^t)$ or ${}_\epsilon\mathcal{HGL}(R\Delta^{\cdot}, \mathbb{G}_m^t)$.

By Lemma 3.2.3 there is a fibration sequence

$$|d \mapsto I_*\mathcal{T}^{-1}\mathcal{T}_d| \to |\mathcal{T}^{-1}\mathcal{T}| \to |d \mapsto \pi_0\mathcal{T}^{-1}\mathcal{T}_d| \tag{6.8}$$

Now, $I_*\mathcal{S}^{-1}\mathcal{S}_d \xrightarrow{\simeq} I_*\mathcal{T}^{-1}\mathcal{T}_d$ by the cofinality theorem 3.2.11 and

$$|d \mapsto I_*\mathcal{S}^{-1}\mathcal{S}_d| \simeq \Omega^{-1}|\langle \mathcal{S}, \mathcal{S}(\mathbb{G}_m)\rangle| \tag{6.9}$$

by Lemma 6.1.5. By Proposition 4.5.2 we have

$$\Omega^{-1}|\langle \mathcal{S}, \mathcal{S}(\mathbb{G}_m)\rangle|^+ \xrightarrow{\simeq} \Omega^{-1}|\langle \mathcal{T}, \mathcal{T}(\mathbb{G}_m)\rangle|^+ \tag{6.10}$$

Moreover

$$\langle \mathcal{S}, \mathcal{S}(\mathbb{G}_m)\rangle \xrightarrow{\simeq} \langle \mathcal{S}, \mathcal{S}(\mathbb{G}_m)\rangle^+ \tag{6.11}$$

since $\pi_0(\langle \mathcal{S}, \mathcal{S}(\mathbb{G}_m)\rangle)$ is already a group.

Finally, $|\mathcal{T}^{-1}\mathcal{T}| = K(\mathcal{T})$, $|d \mapsto \pi_0\mathcal{T}^{-1}\mathcal{T}_d| = K_0(\mathcal{T})$ and, by Lemma 4.5.1,

$$\Omega^{-1}|\langle \mathcal{T}, \mathcal{T}(\mathbb{G}_m)\rangle^+| = \Omega^{-1}|K(\mathcal{T}(\mathbb{G}_m^{\wedge 1}))| \tag{6.12}$$

Thus,

$$\Omega^{-1}|K(\mathcal{T}(\mathbb{G}_m^{\wedge 1}))| \to K(\mathcal{T}) \to K_0(\mathcal{T}) \tag{6.13}$$

is a fibration sequence. Repeating the argument for $\mathcal{T}(\mathbb{G}_m)$ in place of $\mathcal{T}$ we also see that

$$\Omega^{-1}|K(\mathcal{T}(\mathbb{G}_m)(\mathbb{G}_m^{\wedge 1}))| \to K(\mathcal{T}(\mathbb{G}_m)) \to K_0(\mathcal{T}(\mathbb{G}_m)) \tag{6.14}$$

is a fibration sequence. We proceed by induction on t. To this end consider the diagram

$$\begin{tikzcd}
\Omega^{-1}K(\mathcal{T}(\mathbb{G}_m^{\wedge t})) \arrow[r] \arrow[d] & K(\mathcal{T}(\mathbb{G}_m^{\wedge t-1})) \arrow[r] \arrow[d] & K_0(\mathcal{T}(\mathbb{G}_m^{\wedge t-1})) \arrow[d] \\
\Omega^{-1}K(\mathcal{T}(\mathbb{G}_m)(\mathbb{G}_m^{\wedge t})) \arrow[r] \arrow[d] & K(\mathcal{T}(\mathbb{G}_m)(\mathbb{G}_m^{\wedge t-1})) \arrow[r] \arrow[d] & K_0(\mathcal{T}(\mathbb{G}_m)(\mathbb{G}_m^{\wedge t-1})) \arrow[d] \\
\Omega^{-1}K(\mathcal{T}(\mathbb{G}_m^{\wedge t+1})) \arrow[r] & K(\mathcal{T}(\mathbb{G}_m^{\wedge t})) \arrow[r] & K_0(\mathcal{T}(\mathbb{G}_m^{\wedge t}))
\end{tikzcd} \tag{6.15}$$

The first two vertical sequences are fibrations by construction (because the map $K(\mathcal{S}(\mathbb{G}_m)(\mathbb{G}_m^{\wedge t})) \to K(\mathcal{S}(\mathbb{G}_m^{\wedge t+1}))$ is surjective on π_0, the connected delooping is still a fibration). The third vertical sequence is a fibration, because $K_0(\mathcal{S}(\mathbb{G}_m^{\wedge t-1})) \to K_0(\mathcal{S}(\mathbb{G}_m)(\mathbb{G}_m^{\wedge t-1}))$ is an injective map of simplicial abelian groups with cokernel $K_0(\mathcal{S}(\mathbb{G}_m^{\wedge t}))$. The first two horizontal sequences are fibrations by the induction hypothesis. We are done.

□

Before we finally prove Lemma 6.1.5, we describe a modified version of the central argument we used to prove Propositions 4.7.2 and 4.7.3. Let T_0 be an object in $\mathcal{S}_l$ for $\mathcal{S} = \mathcal{GL}(R\Delta^{\cdot}, \mathbb{G}_m^t)$ or ${}_\epsilon\mathcal{HGL}(R\Delta^{\cdot}, \mathbb{G}_m^t)$, respectively. In both cases T_0 is explicitly given by a string of t commuting automorphisms in $Gl_n(R\Delta^l)$ for some n. Further let $\lambda\colon [d] \to [1]$ and $\phi\colon [d] \to [l]$ be two maps.
We define the following matrix of rank $2n$ with entries in $R\Delta^1 = R[X_0, X_1]/(X_0 + X_1 = 1)$:

$$M(X_0) = \begin{pmatrix} 1 & 0 \\ X_0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -X_0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ X_0 & 1 \end{pmatrix} \tag{6.16}$$

Because $M(X_0)$ consist block-wise of scalar multiples of the identity matrix, we may conclude that $\lambda^* M(X_0)$, respectively ${}_\epsilon H(\lambda^* M(X_0))$ is an automorphism of $\phi^*(T_0 \oplus T_0)$. We will use this automorphism in the proof below.

Note that if λ factors through one of the inclusions $i_j\colon [0] \to [1]$, for $j = 0, 1$, then

$$\lambda^* M(X_0) = \begin{cases} M(1) = \begin{pmatrix} 0 & -1_{\phi^* T_0} \\ 1_{\phi^* T_0} & 0 \end{pmatrix} & \text{if } j = 0 \\ M(0) = \begin{pmatrix} 1_{\phi^* T_0} & 0 \\ 0 & 1_{\phi^* T_0} \end{pmatrix} & \text{if } j = 1 \end{cases} \tag{6.17}$$

Proof of Lemma 6.1.5. The main idea is to use Waldhausen's Theorem B' to show that the sequence

$$\langle \mathcal{S}, \mathcal{S}(\mathbb{G}_m) \rangle \to \langle \mathcal{S}, \mathcal{S} \rangle \to |d \mapsto I_* \mathcal{S}^{-1} \mathcal{S}_d| \tag{6.18}$$

is a homotopy fibration. Since the total space is contractible, the assertion follows. We will proceed in several steps:

- Notation in this proof

  We begin by setting up some notation. This information is essentially redundant, but we hope it will increase readability.

  - An object of the category $\mathcal{S}$ will usually be denoted by S. If there there is the need for a second distinct object in $\mathcal{S}$, we will denote it by S'. If we consider a pair of emancipated objects in $\mathcal{S} \times \mathcal{S}$ we will use (S_0, S_1). Once the transition map $u^*(T_0)$, as defined below, is established, we may consider the pair (S_0, S_1) as fixed. There will still be pairs in $\mathcal{S} \times \mathcal{S}$ that will vary, but these pairs will be of the special form (S, S), or (S', S'), respectively.
  - Morphisms in the bracket categories $\langle \mathcal{S}, \mathcal{S} \rangle$ and $\langle \mathcal{S}, \mathcal{S} \times \mathcal{S} \rangle$ will also depend on objects of $\mathcal{S}$. To make the distinction easier we will denote these objects by T and T'. The transition map in the proof will depend on an object T_0, which can be considered as fixed. Moreover, a morphism in these bracket categories contains one or two morphisms in $\mathcal{S}$. These will be denoted by α and α' for single maps and by (α_0, α_1) or (α_0', α_1') for pairs. As it happens, the maps in a pair might coincide. Then we simply use (α, α) as a notation.
  - In the right-fibres, an object will consist of a tuple that will in particular contain objects *and* morphisms of $\mathcal{S}$ (and therefore objects denoted by S

and T). A morphism in these fibres will depend on yet another object in $\mathcal{S}$, which we call U. The corresponding maps will be called γ.

– Finally we will use canonical switching isomorphisms: The isomorphisms $\sigma_i \colon S_i \oplus T_0 \to T_0 \oplus S_i$ depend on S_i, which is indicated by an index. The switching isomorphisms $\sigma \colon S \oplus T_0 \to T_0 \oplus S$ change with varying S and have no subscript.

- Construction of the functor $D \colon \langle \mathcal{S}, \mathcal{S} \rangle \to |d \mapsto I_* \mathcal{S}^{-1} \mathcal{S}_d|$

Recall that an object of $\mathcal{S}^{-1}\mathcal{S}_d$ is a pair (S_0, S_1) of objects of $\mathcal{S}_d$, and that a morphism $(S_0, S_1) \to (S_0', S_1')$ is an equivalence class of data, consisting of an object $T \in \mathcal{S}_d$ and morphisms $\alpha_i \colon S_i \oplus T \to S_i'$. For an object (S_0, S_1) to be in the connected component of the base-point means that there is a zig-zag of morphisms, connecting (S_0, S_1) to the zero object $(0, 0)$. For the categories $\mathcal{GL}(R\Delta^{\cdot}, \mathbb{G}_m^t)$ and ${}_{\epsilon}\mathcal{O}(R\Delta^{\cdot}, \mathbb{G}_m^t)$ this is actually equivalent to S_0 and S_1 being stably isomorphic, i.e. to the existence of an object $T \in \mathcal{S}_d$, such that $S_0 \oplus T \cong S_1 \oplus T$.
Define D to be the diagonal functor, which maps $S \mapsto (S, S)$ and $[T, \alpha] \mapsto [T, \alpha, \alpha]$.

- The right-fibres are non-empty

Recall that the right-fibre for the object $(S_0, S_1) \in I_* \mathcal{S}^{-1} \mathcal{S}_l$ is the simplicial category which in degree d takes on the form

$$(S_0, S_1 \downarrow D)_d = \coprod_{\phi \colon [d] \to [l]} (\phi^*(S_0, S_1) \downarrow D_d) \tag{6.19}$$

where the disjoint union runs over all simplicial maps $\phi \colon [d] \to [l]$. An object $(S_0, S_1 \downarrow D)_d$ is a tuple $(\phi, S, [T, \alpha_0, \alpha_1])$ where $\phi \colon [d] \to [l]$, $S \in \mathcal{S}_d$ and $[T, \alpha_0, \alpha_1] \colon \phi^*(S_0, S_1) \to (S, S)$. A morphism

$$(\phi, S, [T, \alpha_0, \alpha_1]) \to (\phi', S', [T', \alpha_0', \alpha_1'])$$

can only exist if $\phi = \phi'$ and is then given by a morphism $[U, \gamma] \colon S \to S' \in \langle \mathcal{S}, \mathcal{S} \rangle_d$ such that $[U, \gamma, \gamma] \circ [T, \alpha_0, \alpha_1] = [T', \alpha_0', \alpha_1'] \in I_* \mathcal{S}^{-1} \mathcal{S}_d$. This category is non-empty, since S_0 and S_1 are stably isomorphic.

- <u>The right-fibre of the zero object $(0, 0 \downarrow D)$ can be identified with $\langle \mathcal{S}, \mathcal{S}(\mathbb{G}_m)\rangle$</u>

Before we show that all transition maps induce homotopy equivalences on right-fibres, we observe that we can identify the fibre $(0, 0 \downarrow D)$ with $\langle \mathcal{S}, \mathcal{S}(\mathbb{G}_m)\rangle$. Indeed, an object of $(0, 0 \downarrow D)_d$ is given by a morphism $(0, 0) \to (S, S)$ for some $S \in \mathcal{S}_d$. Such a morphism can be uniquely represented by a pair of morphisms $(S \xrightarrow{=} S, S \xrightarrow{\theta} S)$ for some automorphism θ of S. The pair (S, θ) can be considered as an object of $\mathcal{S}(\mathbb{G}_m)$. A morphism $(S, \theta) \to (S', \theta')$ in this category is given by a morphism $[T, v \colon S \oplus T \to S']$ in $\langle \mathcal{S}, \mathcal{S}\rangle$ such that the diagram

$$\begin{tikzcd} (S \oplus T, S \oplus T) \arrow[r, "(1,\theta\oplus 1)"] \arrow[d, "(u,u)"] & (S \oplus T, S \oplus T) \arrow[d, "(v,v)"] \\ (S', S') \arrow[r, "(1,\theta')"] & (S', S') \end{tikzcd} \tag{6.20}$$

commutes for some isomorphism $u \colon S \oplus T \to S'$. It is easy to see that $u = v$ and therefore $v(\theta \oplus 1)v^{-1} = \theta'$.

- <u>The transition maps $u^*(T_0)$</u>

The categories $I_*\mathcal{S}^{-1}\mathcal{S}_d$ are connected by definition. Hence, we may apply Theorem 5.3.6 to restrict ourselves to transition maps induced by morphisms $[T_0, \alpha_0, \alpha_1] \colon (S_0, S_1) \to (S_0', S_1') \in I_*\mathcal{S}^{-1}\mathcal{S}_d$. For further simplification choose a representative $(T_0, \alpha_0, \alpha_1)$ and consider the commutative diagram in $I_*\mathcal{S}^{-1}\mathcal{S}_d$:

$$\begin{tikzcd} (S_0, S_1) \arrow[r, "(T_0,\alpha_0,\alpha_1)"] \arrow[dr, "(T_0,1,1)"'] & (S_0', S_1') \arrow[d, "(0,\alpha_0^{-1},\alpha_1^{-1})"] \\ & (S_0 \oplus T_0, S_1 \oplus T_0) \end{tikzcd} \tag{6.21}$$

The vertical map is an isomorphism and therefore induces a homotopy equivalence. Thus, it suffices to verify that morphisms of the form $[T_0, 1, 1] \colon (S_0, S_1) \to (S_0 \oplus T_0, S_1 \oplus T_0)$ induce homotopy equivalences. We will denote this functor by $u^*(T_0)$. Explicitly it is given by

$$\begin{aligned} u^*(T_0) \colon (S_0 \oplus T_0, S_1 \oplus T_0 \downarrow D) &\to (S_0, S_1 \downarrow D) \\ (\phi, S, [T, \alpha_0, \alpha_1]) &\mapsto (\phi, S, [\phi^* T_0 \oplus T, \alpha_0, \alpha_1]) \\ [U, \gamma] &\mapsto [U, \gamma] \end{aligned} \tag{6.22}$$

where the α_i are morphisms $\phi^*(S_i \oplus T_0) \oplus T \to S$ and γ is a morphism of the form $\gamma\colon S \oplus U \to S'$ for some S'.

- <u>The homotopy inverse functor of $u^*(T_0)$</u>

We claim that the following functor is a homotopy inverse of $u^*(T_0)$:

$$\begin{aligned} v\colon (S_0, S_1 \downarrow D) &\to (S_0 \oplus T_0, S_1 \oplus T_0 \downarrow D) \\ (\phi, S, [T, \{\alpha_i\}]) &\mapsto (\phi, \phi^* T_0 \oplus S, [T, \{(1_{\phi^* T_0} + \alpha_i) \circ (\sigma_i + 1_T)\}]) \\ [U, \gamma] &\mapsto [U, 1 + \gamma] \end{aligned} \tag{6.23}$$

Here, $\sigma_i\colon \phi^*(S_i \oplus T_0) \to \phi^*(T_0 \oplus S_i)$ is the canonical switching isomorphism. Explicitly, a morphism $[U, \gamma]\colon (\phi, S, [T, \alpha_0, \alpha_1]) \to (\phi, S', [T', \alpha'_0, \alpha'_1])$ can be represented by diagrams of the form

$$\begin{tikzcd} \phi^*(S_i) \oplus T \oplus U \arrow[r, "\alpha_i \oplus 1_U"] \arrow[d, "1_{\phi^*(S_0)} \oplus u"] & S \oplus U \arrow[d, "\gamma"] \\ \phi^*(S_i) \oplus T' \arrow[r, "\alpha'_i"] & S' \end{tikzcd} \tag{6.24}$$

for $i = 0, 1$ and an isomorphism $u\colon T \oplus U \to T'$. Then the morphism $[U, 1 + \gamma]$ is represented by the outer squares in the diagrams

$$\begin{tikzcd} \phi^*(S_i \oplus T_0) \oplus T \oplus U \arrow[r, bend left, "\sigma_i + 1_T + 1_U"] \arrow[d, "1_{\phi^* S_i} + 1_{\phi^*(T_0)} + u"] & \phi^*(T_0 \oplus S_i) \oplus T \oplus U \arrow[r, bend left, "1_{\phi^*(T_0)} + \alpha_i + 1_U"] \arrow[d, "1_{\phi^*(T_0)} + 1_{\phi^*(S_i)} + u"] & \phi^*(T_0) \oplus S \oplus U \arrow[d, "1_{\phi^* T_0} + \gamma"] \\ \phi^*(S_i \oplus T_0) \oplus T' \arrow[r, bend right, "\sigma_i + 1_{T'}"'] & \phi^*(T_0 \oplus S_i) \oplus T' \arrow[r, bend right, "1_{\phi^*(T_0)} + \alpha'_i"'] & \phi^*(T_0) \oplus S' \end{tikzcd} \tag{6.25}$$

- <u>The composition $u^*(T_0) \circ v$</u>

The composition $u^*(T_0) \circ v$ is given by

$$\begin{aligned} u^*(T_0) \circ v\colon (S_0, S_1 \downarrow D) &\to (S_0, S_1 \downarrow D) \\ (\phi, S, [T, \alpha_0, \alpha_1]) &\mapsto (\phi, \phi^* T_0 \oplus S, [\phi^* T_0 \oplus T, g_0, g_1]) \\ [U, \gamma] &\mapsto [U, 1 + \gamma] \end{aligned} \tag{6.26}$$

with $g_i = (1_{\phi^*T_0} + \alpha_i) \circ (\sigma_i + 1_T)$. In order to construct a natural transformation from the identity functor to $u^*(T_0) \circ v$, consider the morphism $[\phi^*T_0, \sigma]\colon (\phi, S, [T, \alpha_0, \alpha_1]) \to (\phi, [\phi^*T_0 \oplus T, g_0, g_1])$, where $\sigma\colon S \oplus \phi^*T_0 \to \phi^*T_0 \oplus S$ is the natural switching isomorphism. This is indeed a well defined morphism, represented by the diagrams

$$\begin{tikzcd}
\phi^*S_i \oplus T \oplus \phi^*T_0 \arrow[r, "\alpha_i + 1_{\phi^*T_0}"] \arrow[d, "1_{\phi^*S_i} + \sigma"] & S \oplus \phi^*T_0 \arrow[d, "\sigma"] \\
\phi^*(S_i \oplus T_0) \oplus T \arrow[r, "(1_{\phi^*T_0} + \alpha_i)\circ(\sigma_i + 1_T)"] & \phi^*T_0 \oplus S
\end{tikzcd} \tag{6.27}$$

To verify that the morphisms $[\phi^*T_0, \sigma]$ define a natural transformation, it remains to check that $[U, 1+\gamma] \circ [\phi^*T_0, \sigma] = [\phi^*T_0, \sigma] \circ [U, \gamma]$ holds for any morphism $[U, \gamma]\colon (\phi, S, [T, \alpha_0, \alpha_1]) \to (\phi, S', [T', \alpha'_0, \alpha'_1])$. Explicitly this means that the following diagram commutes:

$$\begin{tikzcd}
S \oplus \phi^*(T_0) \oplus U \arrow[r, "\sigma + 1_U"] \arrow[d, "1_S + \sigma"] & \phi^*(T_0) \oplus S \oplus U \arrow[r, "1_{\phi^*T_0} + \gamma"] & \phi^*(T_0) \oplus S' \arrow[d, "="] \\
S \oplus U \oplus \phi^*(T_0) \arrow[r, "\gamma + 1_{\phi^*T_0}"] & S' \oplus \phi^*(T_0) \arrow[r, "\sigma"] & \phi^*(T_0) \oplus S'
\end{tikzcd} \tag{6.28}$$

This can be checked manually.

- <ins>The composition $v \circ u^*(T_0)$</ins>

The composition $v \circ u^*(T_0)$ is given by

$$\begin{aligned}
v \circ u^*(T_0)\colon (S_0 \oplus T_0, S_1 \oplus T_0 \downarrow D) &\to ((S_0 \oplus T_0, S_1 \oplus T_0 \downarrow D) \\
(\phi, S, [T, \alpha_0, \alpha_1]) &\mapsto (\phi, \phi^*T_0 \oplus S, [\phi^*T_0 \oplus T, g_0, g_1]) \\
[U, \gamma] &\mapsto [U, 1+\gamma]
\end{aligned} \tag{6.29}$$

With $g_i = (1_{\phi^*T_0} + \alpha_i) \circ (\sigma_i + 1_{\phi^*T_0 + 1_T})$. Unfortunately, there is no immediate natural transformation from the identity functor to $v \circ u^*(T_0)$, since the naive choice $[\phi^*T_0, \sigma]\colon (\phi, S, [T, \alpha_0, \alpha_1]) \to (\phi, \phi^*(T_0 \oplus S), [\phi^*T_0 \oplus T, g_0, g_1])$ is *not* a well-defined morphism, because the following diagram does *not* commute:

$$
\begin{tikzcd}
\phi^*(S_i \oplus T_0) \oplus T \oplus \phi^* T_0 \arrow[r, bend left, "\alpha_i + 1_T + 1_U"] \arrow[d, "1_{\phi^*(S_i)} + 1_{\phi^*(T_0)} + \sigma"] & S \oplus \phi^* T_0 \arrow[r, bend left, "\sigma"] & \phi^* T_0 \oplus S \arrow[d, "="] \\
\phi^*(S_i \oplus T_0 \oplus T_0) \oplus T \arrow[r, bend right, "\sigma_i + 1_{\phi * T_0} + 1_T"'] & \phi^*(T_0 \oplus S_i \oplus T_0) \oplus T \arrow[r, bend right, "1_{\phi^*(T_0)} + \alpha_i"'] & \phi^* T_0 \oplus S
\end{tikzcd}
\tag{6.30}
$$

In fact, this is the point where we need the simplicial dimension to perform an additional switch of $\phi^*(T_0 \oplus T_0)$.

Given any simplicial category $\mathcal{C} = ([d] \mapsto \mathcal{C}_d)$ we may consider the product $\mathcal{C} \times \Delta^1$ where we consider the simplicial set $[d] \mapsto \Delta^1_d = \mathrm{Hom}_\Delta(d, 1)$ as a discrete simplicial category. Thus, an object of $(\mathcal{C} \times \Delta^1)_d$ is a pair $(C \in \mathcal{C}_d, \lambda \colon [d] \to [1])$, a morphism $(C, \lambda) \to (C', \lambda')$ only exists if $\lambda = \lambda'$ and is then given by a morphism $C \to C'$. For $\phi \colon [d'] \to [d]$ we have $\phi^*(C, \lambda) = (\phi^* C, \lambda \circ \phi)$. Because geometric realisation respects fibre products in the category of CW-complexes we have $|\mathcal{C} \times \Delta^1| \simeq |\mathcal{C}| \times I$. Therefore we can define a homotopy by giving a functor:

$$
\begin{aligned}
H \colon (S_0 \oplus T_0, S_1 \oplus T_0 \downarrow D) \times \Delta^1 &\to (S_0 \oplus T_0, S_1 \oplus T_0 \downarrow D) \\
((\phi, S, [T, \alpha_0, \alpha_1]), \lambda) &\mapsto (\phi, \phi^* T_0 \oplus S, [\phi^* T_0 \oplus T, h_0, h_1]) \\
[U, \gamma] &\mapsto [U, 1 + \gamma]
\end{aligned}
\tag{6.31}
$$

with $h_i = g_i \circ (1_{\phi^* S_i} + \lambda^* M(X_0) + 1_T)$. Write $d^i = \delta_i^* \colon \Delta^0 \to \Delta^1$. Then clearly $H \circ (1 \times d^1) = v \circ u^*(T_0)$ and, after a short calculation, $H \circ (id \times d^0)$ is given by

$$
\begin{aligned}
H \circ d^0 \colon (S_0 \oplus T_0, S_1 \oplus T_0 \downarrow D) &\to (S_0 \oplus T_0, S_1 \oplus T_0 \downarrow D) \\
(\phi, S, [T, \alpha_0, \alpha_1]) &\mapsto (\phi, \phi^* T_0 \oplus S, [T \oplus \phi^* T_0, f_0, f_1]) \\
[U, \gamma] &\mapsto [U, 1 + \gamma]
\end{aligned}
\tag{6.32}
$$

with $f_i = \sigma \circ (\alpha_i + (-1)_{\phi^* T_0})$. Now $[\phi^* T_0, \sigma] \colon (\phi, S, [T, \alpha_0, \alpha_1]) \to (\phi, \phi^* T_0 \oplus S, [T \oplus \phi^* T_0, f_0, f_1])$ is a well-defined morphism, and a diagram, similar to diagram (6.28), shows that they form a natural transformation from the identity functor to $H \circ d^0$.

$\square$

6.2 Agreement

In Section 4.4.1 we have seen how to define maps

$$K(R, \mathbb{G}_m^{\wedge t+1}) \xrightarrow{\cdot U} \Omega K(R, \mathbb{G}_m^{\wedge t}) \tag{6.33}$$

and

$${}_\epsilon GW(R, \mathbb{G}_m^{\wedge t+1}) \xrightarrow{\cdot U} \Omega_\epsilon GW(R, \mathbb{G}_m^{\wedge t}) \tag{6.34}$$

To treat both cases simultaneously set $\mathcal{S}(R) = i\mathcal{P}(R, \mathbb{G}_m^t)$ or $\mathcal{S}(R) = i_\epsilon \mathcal{P}_h(R, \mathbb{G}_m^t)$, respectively (and similarly $\mathcal{S}(R, \mathbb{G}_m) = i\mathcal{P}(R, \mathbb{G}_m^{t+1})$ or $\mathcal{S}(R, \mathbb{G}_m) = i_\epsilon \mathcal{P}_h(R, \mathbb{G}_m^{t+1})$). Moreover set $\mathcal{T} = i\mathcal{P}(\mathbb{G}_m)$ or $\mathcal{T} = i_\epsilon \mathcal{P}_h(\mathbb{G}_m)$, respectively. Then the maps above are induced by the bi-exact tensor product functor

$$\otimes \colon \mathcal{S}(R, \mathbb{G}_m) \times \mathcal{T} \to \mathcal{S}(R) \tag{6.35}$$

For future reference write τ for the induced map

$$\tau \colon K(\mathcal{S}(R, \mathbb{G}_m)) \wedge K(\mathcal{T}) \to K(\mathcal{S}(R)) \tag{6.36}$$

Multiplication by U as above is then the adjoint map of the composition

$$K(\mathcal{S}(R, \mathbb{G}_m)) \wedge S^1 \xrightarrow{1 \wedge f} K(\mathcal{S}(R, \mathbb{G}_m)) \wedge K(\mathcal{T}) \xrightarrow{\tau} K(\mathcal{S}(R)) \tag{6.37}$$

where $f \colon S^1 \to K(\mathcal{T})$ identifies a unique element $U \in K_1(\mathcal{T})$.
On the other hand Lemma 6.1.5 states that there are induced maps

$$\sigma_R \colon K(\mathcal{S}(R, \mathbb{G}_m)) \to \Omega K(\mathcal{S}(R)) \tag{6.38}$$

that arise from the application of Theorem B. The goal of this section is to demonstrate that the σ and multiplication by $U \in K_1(\mathcal{T})$ coincide.

Remark 6.2.1. Of course we are interested in these maps when studied for the simplicial ring $R\Delta^\cdot$. However, unravelling the definitions shows that in both cases the map $K(\mathcal{S}(R\Delta^\cdot, \mathbb{G}_m)) \to \Omega K(\mathcal{S}(R\Delta^\cdot))$ is in fact induced by the corresponding maps for (non simplicial) rings. To decide if these maps agree it hence suffices to study the non simplicial case.

6.2.1 The natural map from the right-fibre to the homotopy fibre

Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor of small categories and $D \in \mathcal{D}$ a fixed object. In this section we recall from [Qui73, §1] how to define a map from the right-fibre $D \downarrow F$ to the homotopy fibre of F at the base point D. Theorem B gives a criterion for when this map is an equivalence, but it exists irrespective.

We have seen that there is a forgetful functor $V\colon D \downarrow F \to \mathcal{C}$ from the right-fibre to the source of F that maps $(C, f\colon D \to F(C)) \mapsto C$. Now consider the composition

$$\begin{aligned} F \circ V\colon D \downarrow F &\to \mathcal{D} \\ (C, f) &\mapsto F(C) \end{aligned} \tag{6.39}$$

Note that there is a natural transformation from the constant functor $(C, f) \mapsto D$ to $F \circ V$ given by $f\colon D \to F(C) = F \circ V(C, f)$. By definition the map $V\colon D \downarrow F \to \mathcal{C}$ together with a homotopy from $F \circ V$ to the constant map induced a map

$$\sigma\colon D \downarrow F \to \mathrm{hofib}(F, D) \tag{6.40}$$

In our situation we are presented with a symmetric monoidal category $\mathcal{S}$ and consider the functor

$$\begin{aligned} F\colon \langle \mathcal{S}, \mathcal{S} \rangle &\to I_*\mathcal{S}^{-1}\mathcal{S} \\ S &\mapsto (S, S) \end{aligned} \tag{6.41}$$

Moreover we fix the object $(0,0) \in I_*\mathcal{S}^{-1}\mathcal{S}$. We have identified the right-fibre $(0,0) \downarrow F$ with the category $\langle \mathcal{S}, \mathcal{S}(\mathbb{G}_m) \rangle$, the forgetful functor $\langle \mathcal{S}, \mathcal{S}(\mathbb{G}_m) \rangle \to \langle \mathcal{S}, \mathcal{S} \rangle$ maps $(S, \theta) \mapsto S$ and the natural transformation is given by $[S, 1, \theta]\colon (0,0) \to (S, S)$. The category $\langle \mathcal{S}, \mathcal{S} \rangle$ is contractible and $\mathrm{hofib}(F, (0,0)) = \Omega K(\mathcal{S})$.

The induced map is hence of the form $\langle \mathcal{S}, \mathcal{S}(\mathbb{G}_m) \rangle \to \Omega K(\mathcal{S})$. By the universal property of group-completion we obtain the map

$$\sigma\colon K(\mathcal{S}(\mathbb{G}_m)) = \langle \mathcal{S}, \mathcal{S}(\mathbb{G}_m) \rangle^+ \to \Omega K(\mathcal{S}) \tag{6.42}$$

6.2.2 Proof of agreement

Proposition 6.2.2. *Let $\mathcal{S}(R) = \mathcal{P}(R, \mathbb{G}_m^t)$ and let $\mathcal{T} = i\mathcal{P}(\mathbb{G}_m)$ or let $\mathcal{S}(R) = {}_\epsilon\mathcal{P}_h(R, \mathbb{G}_m^t)$ and $\mathcal{T} = i_\epsilon\mathcal{P}_h(\mathbb{G}_m)$. Then the maps*

$$K(\mathcal{S}(R, \mathbb{G}_m)) \xrightarrow{\cdot U} \Omega K(\mathcal{S}(R)) \tag{6.43}$$

and

$$K(\mathcal{S}(R, \mathbb{G}_m)) \xrightarrow{\sigma_R} \Omega K(\mathcal{S}(R)) \tag{6.44}$$

coincide up to natural homotopy.

Proof. Consider the diagram

$$\begin{tikzcd}
S^1 \wedge S^0 \wedge K(\mathcal{S}(R, \mathbb{G}_m)) \arrow[r, "="] \arrow[d, "1\wedge e\wedge 1"] & S^1 \wedge K(\mathcal{S}(R, \mathbb{G}_m)) \arrow[d, "f\wedge 1"] \\
S^1 \wedge K(\mathcal{T}(\mathbb{G}_m)) \wedge K(\mathcal{S}(R, \mathbb{G}_m)) \arrow[r, "\sigma_{\mathbb{G}_m}\wedge 1"] \arrow[d, "1\wedge\tau"] & K(\mathcal{T}) \wedge K(\mathcal{S}(R, \mathbb{G}_m)) \arrow[d, "\tau"] \\
S^1 \wedge K(\mathcal{S}(R, \mathbb{G}_m)) \arrow[r, "\sigma_R"] & K(\mathcal{S}(R))
\end{tikzcd} \tag{6.45}$$

where τ and σ are as discussed, $f\colon S^1 \to K(\mathcal{T})$ identifies the unique element $U \in K_1(\mathcal{T})$ and $e\colon S^0 \to K(\mathcal{T}(\mathbb{G}_m))$ hits either the element $[\mathbb{G}_m, U] \in K_0(\mathbb{G}_m, \mathbb{G}_m))$ or the element $[\mathbb{G}_m, U, (\bar{\cdot})] \in GW_0(\mathbb{G}_m, \mathbb{G}_m))$ as discussed in Section 4.4.1.
The upper square commutes, essentially by definition. Indeed, the class of the object (T, θ) in $K_0(\mathcal{T}(\mathbb{G}_m))$, is mapped to the loop in $K_1(\mathcal{T})$ that is described by θ and based at T. In particular we have $\sigma_{\mathbb{G}_m} \circ (1 \wedge e) = f$.
Consider the tensor product

$$\otimes\colon \mathcal{S}(\mathbb{G}_m) \times \mathcal{T}(\mathbb{G}_m) \to \mathcal{S}(\mathbb{G}_m) \tag{6.46}$$

Clearly we have $(S, \theta) \otimes (\mathbb{G}_m, U) \cong (S, \theta)$. It follows that the left vertical map is the identity. Moreover the right vertical map is, by definition, multiplication by U. It remains to show that the lower square is homotopy commutative, because then commutativity of the outer square yields the result.

- <u>The composition $\sigma_R \circ (1 \wedge \tau)\colon S^1 \wedge K(\mathcal{T}(\mathbb{G}_m)) \wedge K(\mathcal{S}(R, \mathbb{G}_m)) \to K(\mathcal{S}(R))$</u>

  In fact we consider the adjoint map $K(\mathcal{T}(\mathbb{G}_m)) \wedge K(\mathcal{S}(R, \mathbb{G}_m)) \to \Omega K(\mathcal{S}(R))$.

It is given by the outer dashed map in the following diagram:

$$\begin{tikzcd}
 & \Omega\mathcal{S}^{-1}\mathcal{S} \arrow[r] & \langle \mathcal{S},\mathcal{S}\rangle \arrow[r] & I_*\mathcal{S}^+ \\
 & \langle \mathcal{S},\mathcal{S}(\mathbb{G}_m)\rangle \arrow[u, dashed, "\sigma"] \arrow[ur, "\text{forget}"] & & \\
\langle \mathcal{T},\mathcal{T}(\mathbb{G}_m)\rangle \wedge \langle \mathcal{S},\mathcal{S}(\mathbb{G}_m)\rangle \arrow[ur, "\tau"] \arrow[uur, dashed] & & &
\end{tikzcd} \tag{6.47}$$

The diagonal composition $\langle \mathcal{T},\mathcal{T}(\mathbb{G}_m)\rangle \wedge \langle \mathcal{S},\mathcal{S}(\mathbb{G}_m)\rangle \to \langle \mathcal{S},\mathcal{S}\rangle$ maps objects $((T,\eta),(S,\theta)) \mapsto T\otimes(S,\theta)$, and the natural transformation from the constant functor to the composition $\langle \mathcal{T},\mathcal{T}(\mathbb{G}_m)\rangle \wedge \langle \mathcal{S},\mathcal{S}(\mathbb{G}_m)\rangle \to I_*\mathcal{S}^+$ is given by $[T\otimes(S,\theta),1,\eta\otimes 1]\colon (0,0) \to (T\otimes(S,\theta), T\otimes(S,\theta))$.

- <u>The composition $\tau\circ\sigma\wedge 1\colon S^1 \wedge K(\mathcal{T}(\mathbb{G}_m)) \wedge K(\mathcal{S}(R,\mathbb{G}_m)) \to K(\mathcal{S}(R))$</u>

Again we consider the adjoint map $K(\mathcal{T}(\mathbb{G}_m)) \wedge K(\mathcal{S}(R,\mathbb{G}_m)) \to \Omega K(\mathcal{S}(R))$. It is the outer dashed map in the following diagram

$$\begin{tikzcd}
\langle \mathcal{T},\mathcal{T}(\mathbb{G}_m)\rangle \wedge \langle \mathcal{S},\mathcal{S}(\mathbb{G}_m)\rangle \arrow[r, dashed, "\sigma\wedge 1"] \arrow[dr, "\text{forget}\wedge 1"'] \arrow[rr, dashed, bend left] & \Omega\mathcal{T}^+ \wedge \langle \mathcal{S},\mathcal{S}(\mathbb{G}_m)\rangle \arrow[r, "\tau"] \arrow[d] & \Omega\mathcal{S}^+ \arrow[d] \\
 & \langle \mathcal{T},\mathcal{T}\rangle \wedge \langle \mathcal{S},\mathcal{S}(\mathbb{G}_m)\rangle \arrow[r, "\tau"] \arrow[d] & \langle \mathcal{S},\mathcal{S}\rangle \arrow[d] \\
 & I_*\mathcal{T}^+ \wedge \langle \mathcal{S}(\mathbb{G}_m),\mathcal{S}(\mathbb{G}_m)\rangle \arrow[r, "\tau"] & I_*\mathcal{S}^+
\end{tikzcd} \tag{6.48}$$

The two squares on the right commute and the vertical sequences are homotopy fibration sequences. A short diagram chase reveals that the composition $\tau\circ(\text{forget}\wedge 1)$ is also given on objects by $((T,\eta),(S,\theta)) \mapsto T\otimes(S,\theta)$, and the natural transformation from the constant functor to the composition $\langle \mathcal{T},\mathcal{T}(\mathbb{G}_m)\rangle \wedge \langle \mathcal{S},\mathcal{S}(\mathbb{G}_m)\rangle \to I_*\mathcal{S}^+$ is $[T,1,\eta]\wedge[(S,\theta),1] = [T\otimes(S,\theta),1,\eta\otimes 1]\colon (0,0) \to (T\otimes(S,\theta), T\otimes(S,\theta))$. This finishes the proof.

□

Chapter 7

A simpler model for the K-theory and hermitian K-theory of automorphisms

In the previous chapter a central role was played by the simplicial category $\langle \mathcal{S}, \mathcal{S}(\mathbb{G}_m)\rangle$. In this section we will see how the classifying space of this simplicial category is homotopy equivalent to the underlying simplicial set of objects and thereby find a significantly simpler model for the (hermitian) K-theory tower.

7.1 Motivation

Consider the fibre sequence of Theorem 6.1.1 for K-theory and $t = 0$, namely the sequence

$$\left\langle \mathcal{GL}^0(R), \mathcal{GL}^1(R)\right\rangle \to K(R\Delta^{\cdot}) \to K_0(R\Delta^{\cdot}) \tag{7.1}$$

Classically we know that the fibre of $K(R\Delta^d) \to K_0(R\Delta^d)$ in each level is given by $BGl(R\Delta^d)^+$, so that we have a fibration sequence

$$BGl(R\Delta^{\cdot})^+ \to K(R\Delta^{\cdot}) \to K_0(R\Delta^{\cdot}) \tag{7.2}$$

If we now follow the slogan that simplicially we are already group complete, we can identify the fibre with $BGl(R\Delta^{\cdot})$. This space is connected in every degree and we conclude that $\Omega|BGl(R\Delta^{\cdot})| \simeq |d \mapsto \Omega BGl(R\Delta^d)| \simeq |Gl(R\Delta^{\cdot})|$, where the last space is the realisation of the simplicial abelian group $Gl(R\Delta^{\cdot})$ as used in Karoubi-Villamayor K-theory.

We notice that, because an object of $\mathcal{S}$ is uniquely determined by its rank, the set $\coprod_n Gl_n(R\Delta^d)$ is precisely the set of objects of $\mathcal{S}(\mathbb{G}_m)$. Every set can be considered as a discrete category and thus this discussion implies that

$$Gl(R\Delta^{\cdot}) \xrightarrow{\simeq} \left\langle \mathbb{N}, \coprod_{n\in\mathbb{N}} Gl_n(R\Delta^{\cdot})\right\rangle \xrightarrow{\simeq} \langle \mathcal{S}, \mathcal{S}(\mathbb{G}_m)\rangle \tag{7.3}$$

is a homotopy equivalence. In other words the homotopy type of the simplicial category $\langle \mathcal{S}, \mathcal{S}(\mathbb{G}_m)\rangle$ is completely determined by its underlying simplicial set of objects. In this section we will generalise this observation to higher terms in the (hermitian) K-theory tower.

7.2 Notation and formulation of the result

We first recall and introduce some of the notation used in this section. Throughout this section $\mathcal{S}$ will either denote $\mathcal{GL}^t(R\Delta^{\cdot})$ or ${}_\epsilon\mathcal{O}^t(R\Delta^{\cdot})$ for an unspecified non-negative integer t. An object of simplicial dimension d is determined by a natural

number n and t commuting matrices in $Gl_n(R\Delta^d)$ or ${}_\epsilon O_n(R\Delta^d)$, respectively. A morphism is a base-change in the same class of matrices. Observe that $\mathcal{S}(\mathbb{G}_m)$ can be identified with $\mathcal{GL}^{t+1}(R\Delta^\cdot)$ or ${}_\epsilon\mathcal{O}^{t+1}(R\Delta^\cdot)$.
We give a special name to $\mathcal{S}$ for $t = 0$ and let $\mathcal{S}^0$ denote the simplicial category $\mathcal{GL}^0(R\Delta^\cdot)$ or ${}_\epsilon\mathcal{O}^0(R\Delta^\cdot)$, respectively. An object of $\mathcal{S}^0$ is simply a non-negative integer n, independently of the simplicial dimension, and a morphism in dimension d is a matrix either in $Gl_n(R\Delta^d)$ or ${}_\epsilon O_n(R\Delta^d)$. Observe that we can reconstruct $\mathcal{S}$ from $\mathcal{S}^0$, since $\mathcal{S}^0(\mathbb{G}_m^t) = \mathcal{S}$. In the other direction we have forgetful functors $\mathcal{S} \to \mathcal{S}^0$ that strip an object of all attached automorphisms.
The underlying set of objects of a category $\mathcal{C}$ can be considered as a category itself:

Definition 7.2.1. Let $\mathcal{C}$ be a category. Then $\mathrm{Ob}(\mathcal{C})$ denotes the category which has the same objects as $\mathcal{C}$ and only identity morphisms. We will call a category with $\mathcal{C} = \mathrm{Ob}(\mathcal{C})$ *discrete.*

The classifying space of a discrete category is just the set of objects, considered as a discrete topological space. The classifying space of a simplicial discrete category is similarly just (the geometric realisation of) a simplicial set.

Theorem 7.2.2. *Let $t \geq 0$ be an integer and let $\mathcal{S}$ be either $\mathcal{GL}^t(R\Delta^\cdot)$ or ${}_\epsilon\mathcal{O}^t(R\Delta^\cdot)$. Then the natural inclusion of categories*

$$\langle \mathrm{Ob}(\mathcal{S}), \mathrm{Ob}(\mathcal{S}(\mathbb{G}_m))\rangle \xrightarrow{\simeq} \langle \mathcal{S}, \mathcal{S}(\mathbb{G}_m)\rangle \tag{7.4}$$

induces a homotopy equivalence after group-completion.

Remark 7.2.3. For $t = 0$ we regain the homotopy equivalence (7.3).

7.3 Organisation of the proof

The proof of Theorem 7.2.2 is quite intricate and the weight is pulled by a series of lemmas: In the next section we will use Lemma 7.4.1 to show how the theorem follows from an easier statement (Lemma 7.4.2). The subsequent section is then devoted to proving this assertion: Lemma 7.5.3 will imply Lemmas 7.5.4 and 7.5.5. Together they will finally yield Lemma 7.4.2.

7.4 Technical prelude

The morphisms in the category $\langle \mathcal{S}, \mathcal{S}\rangle$ for $t > 0$ are significantly harder to deal with than in the case $t = 0$. For example, the category $\left\langle \mathcal{S}^0, \mathcal{S}^0 \right\rangle$ (the case $t = 0$) is

filtered, whereas $\langle \mathcal{S}, \mathcal{S} \rangle$ is quite far from it in general. A similar situation arises for $\langle \mathrm{Ob}(\mathcal{S}), \mathrm{Ob}(\mathcal{S}) \rangle$. Another advantage of the simplicial category $\left\langle \mathcal{S}^0, \mathcal{S}^0 \right\rangle$ is that any object is (up to isomorphism) of simplicial dimension 0, i.e. for any object $S \in \mathcal{S}^0_d$ exists an object $S_0 \in \mathcal{S}^0_0$ such that $S = \phi^*_d(S_0)$. In other words $\left\langle \mathrm{Ob}(\mathcal{S}^0), \mathrm{Ob}(\mathcal{S}^0) \right\rangle$ is a constant simplicial set.
It turns out that the categories $\left\langle \mathcal{S}^0, \mathcal{S} \right\rangle$ and $\left\langle \mathrm{Ob}(\mathcal{S}^0), \mathrm{Ob}(\mathcal{S}) \right\rangle$ are a good compromise, since these categories have simpler morphisms than $\langle \mathcal{S}, \mathcal{S} \rangle$ or $\langle \mathrm{Ob}(\mathcal{S}), \mathrm{Ob}(\mathcal{S}) \rangle$ without losing too much information. Indeed, the morphisms of $\left\langle \mathcal{S}^0, \mathcal{S} \right\rangle$ are a subset of the morphisms of $\left\langle \mathcal{S}^0, \mathcal{S}^0 \right\rangle$ and we have the following lemma:

Lemma 7.4.1. *Let $t \geq 0$ be an integer and let $\mathcal{S}$ be either $\mathcal{GL}^t(R\Delta^{\cdot})$ or ${}_{\epsilon}\mathcal{O}^t(R\Delta^{\cdot})$. Then the following sequences are fibration sequences after group-completion*

$$\left\langle \mathrm{Ob}(\mathcal{S}^0), \mathrm{Ob}(\mathcal{S}) \right\rangle \to \left\langle \mathrm{Ob}(\mathcal{S}^0), \mathrm{Ob}(\mathcal{S}(\mathbb{G}_m)) \right\rangle \to \langle \mathrm{Ob}(\mathcal{S}), \mathrm{Ob}(\mathcal{S}(\mathbb{G}_m)) \rangle \tag{7.5}$$

and

$$\left\langle \mathcal{S}^0, \mathcal{S} \right\rangle \to \left\langle \mathcal{S}^0, \mathcal{S}(\mathbb{G}_m) \right\rangle \to \langle \mathcal{S}, \mathcal{S}(\mathbb{G}_m) \rangle \tag{7.6}$$

Proof. We prove the second fibration, the proof for the first one is identical. Consider the diagram

$$\begin{tikzcd}
\mathcal{S}^0 \arrow[r] \arrow[d] & \mathcal{S}^0 \arrow[r] \arrow[d] & * \arrow[d] \\
\mathcal{S} \arrow[r] \arrow[d] & \mathcal{S}(\mathbb{G}_m) \arrow[r] \arrow[d] & \langle \mathcal{S}, \mathcal{S}(\mathbb{G}_m) \rangle \arrow[d] \\
\left\langle \mathcal{S}^0, \mathcal{S} \right\rangle \arrow[r] & \left\langle \mathcal{S}^0, \mathcal{S}(\mathbb{G}_m) \right\rangle \arrow[r] & \langle \mathcal{S}, \mathcal{S}(\mathbb{G}_m) \rangle
\end{tikzcd} \tag{7.7}$$

All sequences, apart from possibly the bottom horizontal one, are fibrations after group-completion. This is either true for trivial reasons (if the one-point space $*$ is involved) or by Lemma 4.5.1. Because the morphism $\pi_0 \left\langle \mathcal{S}^0, \mathcal{S}(\mathbb{G}_m) \right\rangle \to \pi_0 \langle \mathcal{S}, \mathcal{S}(\mathbb{G}_m) \rangle$ is surjective with kernel $\pi_0(\left\langle \mathcal{S}^0, \mathcal{S} \right\rangle)$, we are done. $\square$

Therefore we are now able to deduce Theorem 7.2.2 from the following lemma:

Lemma 7.4.2. *Let $t \geq 0$ be an integer and let $\mathcal{S}$ be either $\mathcal{GL}^t(R\Delta^{\cdot})$ or ${}_{\epsilon}\mathcal{O}^t(R\Delta^{\cdot})$. Then the natural inclusion of categories*

$$\left\langle \mathrm{Ob}(\mathcal{S}^0), \mathrm{Ob}(\mathcal{S}) \right\rangle \xrightarrow{\simeq} \left\langle \mathcal{S}^0, \mathcal{S} \right\rangle \tag{7.8}$$

induces a homotopy equivalence.

Proof of Theorem 7.2.2. By Lemma 7.4.1 there is a commutative diagram

$$\begin{tikzcd}
\left\langle \mathrm{Ob}(\mathcal{S}^0), \mathrm{Ob}(\mathcal{S}) \right\rangle \arrow[r] \arrow[d, "f"] & \left\langle \mathrm{Ob}(\mathcal{S}^0), \mathrm{Ob}(\mathcal{S}(\mathbb{G}_m)) \right\rangle \arrow[r] \arrow[d, "e"] & \langle \mathrm{Ob}(\mathcal{S}), \mathrm{Ob}(\mathcal{S}(\mathbb{G}_m)) \rangle \arrow[d, "b"] \\
\left\langle \mathcal{S}^0, \mathcal{S} \right\rangle \arrow[r] & \left\langle \mathcal{S}^0, \mathcal{S}(\mathbb{G}_m) \right\rangle \arrow[r] & \langle \mathcal{S}, \mathcal{S}(\mathbb{G}_m) \rangle
\end{tikzcd} \tag{7.9}$$

where both rows are fibration sequences. By the Lemma 7.4.2 we know that f and e are homotopy equivalences. By the 5-Lemma we conclude that b is a homotopy equivalence on base-point components. Because both fibrations in the diagram above are surjective on π_0, the result follows. $\square$

7.5 One category to contain them both

In this section we will prove Lemma 7.4.2. The main idea is to consider a certain category $\mathcal{U}$, which will contain both, $\left\langle \mathrm{Ob}(\mathcal{S}^0), \mathrm{Ob}(\mathcal{S}) \right\rangle$ and $\left\langle \mathcal{S}^0, \mathcal{S} \right\rangle$, as full subcategories. We will then show that both of these inclusions are homotopy equivalences.

Definition 7.5.1. Let $\mathcal{S}$ be either $\mathcal{GL}^t(R\Delta^\cdot)$ or ${}_\epsilon\mathcal{O}^t(R\Delta^\cdot)$. Denote by $\mathcal{U}$ the category which has objects $(d, X, S, \{\theta_j\})$, with $d \geq 0$, $X, S \in \mathcal{S}_d^0$ and a family $\theta_j \in Aut(X \oplus S)$, $1 \leq j \leq t$ of pairwise commuting automorphisms. A morphism $(d, X, S, \{\theta_j\}) \to (d', X', S', \{\theta'_j\})$ is given by a tuple $(\phi, Y, [T, \alpha])$ with $\phi\colon [d] \to [d']$, $Y \in \mathcal{S}_d^0$ such that $Y \oplus X = \phi^* X'$ and a morphism $[T, \alpha]\colon S \to \phi^* S' \in \left\langle \mathcal{S}_d^0, \mathcal{S}_d^0 \right\rangle$ such that the following diagram commutes for all $1 \leq j \leq t$:

$$\begin{tikzcd}
Y \oplus X \oplus S \oplus T \arrow[r, "1+\alpha"] \arrow[d, "1+\theta_j+1"] & \phi^* X' \oplus \phi^* S' \arrow[d, "\theta'_j"] \\
Y \oplus X \oplus S \oplus T \arrow[r, "1+\alpha"] & \phi^* X' \oplus \phi^* S'
\end{tikzcd} \tag{7.10}$$

The full subcategory of $\mathcal{U}$ consisting of all objects of the form $(d, X, 0, \{\theta_j\})$ can be naturally identified with $\left\langle \mathrm{Ob}(\mathcal{S}^0), \mathrm{Ob}(\mathcal{S}) \right\rangle$ (where $\mathrm{Ob}(\mathcal{S}^0)$ acts on $\mathrm{Ob}(\mathcal{S})$ from the left) and the subcategory of objects $(d, 0, S, \theta)$ coincides with $\left\langle \mathcal{S}^0, \mathcal{S} \right\rangle$ ($\mathcal{S}^0$ acts on $\mathcal{S}$ from the right). In a way, $\mathcal{U}$ extends to the left to encompass $\left\langle \mathrm{Ob}(\mathcal{S}^0), \mathrm{Ob}(\mathcal{S}) \right\rangle$ and to the right to contain $\left\langle \mathcal{S}^0, \mathcal{S} \right\rangle$. We define some auxiliary categories that can be seen as truncations of the category $\mathcal{U}$:

Definition 7.5.2. Let $A \in \mathcal{S}_0^0$ be fixed and for all $d \geq 0$ let $\phi_d\colon [d] \to [0]$ be the unique map in the simplex category Δ.

- Denote by $\langle \mathrm{Ob}(\mathcal{S}^0), (\mathrm{Ob}(\mathcal{S}^0) \oplus A)(\mathbb{G}_m^t) \rangle$ the category where an object is a tuple $(d, X, \{\theta_j\})$ with $X \in \mathcal{S}_d^0$ and $\theta_j \in Aut(X \oplus A)$. A morphism $(d, X, \{\theta_j\}) \to (d', X', \{\theta'_j\})$ consists of a map $\phi\colon d \to d'$ and an object $Y \in \mathcal{S}_d^0$ such that $Y \oplus X = \phi^* X'$ and $1 + \theta_j = \phi^* \theta'_j$ for all j.

- Denote by $\langle \mathcal{S}^0, (A \oplus \mathcal{S}^0)(\mathbb{G}_m^t) \rangle$ the category where objects are tuples $(d, S, \{\theta_j\})$ with $S \in \mathcal{S}_d^0$ and $\theta_j \in Aut(A \oplus S)$. A morphism $(d, S, \{\theta_j\}) \to (d', S', \{\theta'_j\})$ consists of a map $\phi\colon d \to d'$ and a morphism $[T, \alpha]\colon S \to \phi^* S'$ in $\langle \mathcal{S}_d^0, \mathcal{S}_d^0 \rangle$ such that the diagrams

$$\begin{tikzcd} A \oplus S \oplus T \arrow[r, "1+\alpha"] \arrow[d, "\theta_j + 1"] & A \oplus S' \arrow[d, "\theta'_j"] \\ A \oplus S \oplus T \arrow[r, "1+\alpha"] & A \oplus S' \end{tikzcd} \tag{7.11}$$

commute for all j.

For $A = R^n$, the category $\langle \mathrm{Ob}(\mathcal{S}^0), (\mathrm{Ob}(\mathcal{S}^0) \oplus A)(\mathbb{G}_m^t) \rangle$ can be identified with $\langle \mathrm{Ob}(\mathcal{S}^0), \mathrm{Ob}(\mathcal{S}) \rangle_{\geq n}$, the full subcategory of $\langle \mathrm{Ob}(\mathcal{S}^0), \mathrm{Ob}(\mathcal{S}) \rangle$ consisting of modules with rank $\geq n$. Indeed, there is a natural embedding:

$$\begin{aligned} \langle \mathrm{Ob}(\mathcal{S}^0), (\mathrm{Ob}(\mathcal{S}^0) \oplus A)(\mathbb{G}_m^t) \rangle &\to \langle \mathrm{Ob}(\mathcal{S}^0), \mathrm{Ob}(\mathcal{S}) \rangle \\ (d, X, \{\theta_j\}) &\mapsto (d, X \oplus A, \{\theta_j\}) \\ (\phi, Y) &\mapsto (\phi, Y) \end{aligned} \tag{7.12}$$

Moreover it is not hard to see that the inclusion

$$\langle \mathrm{Ob}(\mathcal{S}^0), \mathrm{Ob}(\mathcal{S}) \rangle_{\geq n} \subseteq \langle \mathrm{Ob}(\mathcal{S}^0), \mathrm{Ob}(\mathcal{S}) \rangle \tag{7.13}$$

induces a homotopy equivalence on classifying spaces, since any module may be stably identified with a module of higher rank. The situation is significantly more complicated when we truncate on the other side. There exist a functor

$$\begin{aligned} \langle \mathcal{S}^0, (A \oplus \mathcal{S}^0)(\mathbb{G}_m^t) \rangle &\to \langle \mathcal{S}^0, \mathcal{S}^0(\mathbb{G}_m^t) \rangle \\ (d, S, \{\theta_j\}) &\mapsto (d, \phi_d^* A \oplus S, \{\theta_j\}) \\ (\phi, [T, \alpha]) &\mapsto (\phi, [T, 1 + \alpha]) \end{aligned} \tag{7.14}$$

that we can use to identify $\langle \mathcal{S}^0, (A \oplus \mathcal{S}^0)(\mathbb{G}_m^t) \rangle$ as a subcategory of $\langle \mathcal{S}^0, \mathcal{S}^0(\mathbb{G}_m^t) \rangle$, albeit not a full one. However we are able to show that this functor induces homotopy equivalences:

Lemma 7.5.3. *Let $A = R^n$. Then the functor $\langle \mathcal{S}^0, (A \oplus \mathcal{S}^0)(\mathbb{G}_m^t) \rangle \to \langle \mathcal{S}^0, \mathcal{S}^0(\mathbb{G}_m^t) \rangle$ as in (7.14) induces homotopy equivalences on the classifying spaces.*

The proof is very similar that of Lemma 6.1.5:

Proof. The homotopy inverse can be explicitly given as

$$\begin{aligned}\left\langle \mathcal{S}^0, \mathcal{S}^0(\mathbb{G}_m^t)\right\rangle &\to \left\langle \mathcal{S}^0, (A \oplus \mathcal{S}^0)(\mathbb{G}_m^t)\right\rangle \\ (d, S, \{\theta_j\}) &\mapsto (d, S, \{1+\theta_j\}) \\ (\phi, [T, \alpha]) &\mapsto (\phi, [T, \alpha])\end{aligned} \tag{7.15}$$

One of the compositions is then

$$\begin{aligned}\left\langle \mathcal{S}^0, \mathcal{S}^0(\mathbb{G}_m^t)\right\rangle &\to \left\langle \mathcal{S}^0, \mathcal{S}^0(\mathbb{G}_m^t)\right\rangle \\ (d, S, \{\theta_j\}) &\mapsto (d, \phi^* A \oplus S, \{1+\theta_j\}) \\ (\phi, [T, \alpha]) &\mapsto (\phi, [T, 1+\alpha])\end{aligned} \tag{7.16}$$

Recall that a natural transformation of functors induces a homotopy between the induced maps on classifying spaces. Thus it suffices to find a natural transformation from the identity functor on $\left\langle \mathcal{S}^0, \mathcal{S}^0(\mathbb{G}_m^t)\right\rangle$ to the composition as described above. Such a transformation is given by morphisms

$$(id, [\phi_d^* A, \sigma]) \colon (d, S, \{\theta_j\}) \to (d, \phi_d^* A \oplus S, \{1+\theta_j\}) \tag{7.17}$$

where $\sigma \colon S \oplus \phi^* A \to \phi^* A \oplus S$ is the natural switching isomorphisms. Indeed, these morphisms are well defined because

$$\begin{tikzcd} S \oplus \phi_d^* A \arrow[r, "\sigma"] \arrow[d, "\theta_j+1"] & \phi_d^* A \oplus S \arrow[d, "1+\theta_j"] \\ S \oplus \phi_d^* A \arrow[r, "\sigma"] & \phi_d^* A \oplus S \end{tikzcd} \tag{7.18}$$

commutes, and they define a natural transformation because for any morphism

$$(\phi, [T, \alpha]) \colon (d, S, \{\theta_j\}) \to (d', S', \{\theta'_j\}) \tag{7.19}$$

we have

$$(\phi, [T, 1+\alpha]) \circ (id, [\phi_d^* A, \sigma]) = (id, [\phi_{d'}^* A, \sigma]) \circ (\phi, [T, \alpha]). \tag{7.20}$$

Indeed, the following diagram commutes:

$$\begin{tikzcd} S \oplus \phi_d^* A \oplus T \arrow[r, "\sigma+1"] \arrow[d, "1+\sigma"] & \phi_d^* A \oplus S \oplus T \arrow[r, "1+\alpha"] & \phi_d^* A \oplus \phi^* S' \\ S \oplus T \oplus \phi_d^* A \arrow[r, "\alpha+1"] & \phi^* S \oplus \phi_d^* A \arrow[ur, "\sigma"] & \end{tikzcd} \tag{7.21}$$

The composition in the opposite direction is given by

$$\begin{aligned} F_1\colon \left\langle \mathcal{S}^0, (A \oplus \mathcal{S}^0)(\mathbb{G}_m^t)\right\rangle &\to \left\langle \mathcal{S}^0, (A \oplus \mathcal{S}^0)(\mathbb{G}_m^t)\right\rangle \\ (d, S, \{\theta_j\}) &\mapsto (d, \phi^* A \oplus S, \{1+\theta_j\}) \\ (\phi, [T, \alpha]) &\mapsto (\phi, [T, 1+\alpha]) \end{aligned} \tag{7.22}$$

Similar to the situation in diagram (6.30) the naive choice for a natural transformation from the identity functor to F_1, given by morphisms $(id, [\phi^* A, \sigma])\colon (d, S, \{\theta_j\}) \to (d, \phi^* A \oplus S, \{1+\theta_j\})$ does not work. These maps rather define a natural transformation from the identity functor to F_2, given as

$$\begin{aligned} F_2\colon \left\langle \mathcal{S}^0, (A \oplus \mathcal{S}^0)(\mathbb{G}_m^t)\right\rangle &\to \left\langle \mathcal{S}^0, (A \oplus \mathcal{S}^0)(\mathbb{G}_m^t)\right\rangle \\ (d, S, \{\theta_j\}) &\mapsto (d, \phi^* A \oplus S, \{(\sigma+1)(1+\theta_j)(\sigma+1)\}) \\ (\phi, [T, \alpha]) &\mapsto (\phi, [T, 1+\alpha]) \end{aligned} \tag{7.23}$$

It remains to relate F_1 to F_2. To this end, let $\phi_1\colon [1] \to [0]$ be the unique map and recall from (6.16) how one can defined a polynomial homotopy $M\colon \phi_1^*(A \oplus A) \to \phi_1^*(A \oplus A)$ that switches the two summands around. Recall that we can define a simplicial category $\left\langle \mathcal{S}^0, (A \oplus \mathcal{S}^0)(\mathbb{G}_m^t)\right\rangle \times \Delta^1$ with objects $(d, S, \{\theta_j\}, \lambda\colon [d] \to [1])$ and where a morphisms $(d, S, \{\theta_j\}, \lambda) \to (d', S', \{\theta_j\}, \lambda')$ is given by a morphism

$$(\phi, [T, \alpha])\colon (d, S, \{\theta_j\}) \to (d', S', \{\theta_j'\}) \tag{7.24}$$

that satisfies $\lambda' \phi = \lambda$.
Now a homotopy between F_1 and F_2 can be described as

$$\begin{aligned} H\colon \left\langle \mathcal{S}^0, (A \oplus \mathcal{S}^0)(\mathbb{G}_m^t)\right\rangle \times \Delta^1 &\to \left\langle \mathcal{S}^0, (A \oplus \mathcal{S}^0)(\mathbb{G}_m^t)\right\rangle \\ (d, S, \{\theta_j\}, \lambda) &\mapsto (d, \phi^* A \oplus S, \{(\lambda^* M + 1)^{-1}(1+\theta_j)(\lambda^* M + 1)\}) \\ (\phi, [T, \alpha]) &\mapsto (\phi, [T, 1+\alpha]) \end{aligned} \tag{7.25}$$

This finishes the proof of the lemma.

□

As a final ingredient we demonstrate that the inclusions of the truncated categories induce homotopy equivalences:

Lemma 7.5.4. *The sequence*

$$\begin{aligned} \big\langle \mathrm{Ob}(\mathcal{S}^0), \mathrm{Ob}(\mathcal{S})\big\rangle \to \mathcal{U} &\to \big\langle \mathcal{S}^0, \mathcal{S}^0\big\rangle \\ (d, X, S, \{\theta_j\}) &\mapsto (d, S) \\ (\phi, Y, [T, \alpha]) &\mapsto (\phi, [T, \alpha]) \end{aligned} \tag{7.26}$$

induces a fibration sequence on classifying spaces. In particular the inclusion

$$\big\langle \mathrm{Ob}(\mathcal{S}^0), \mathrm{Ob}(\mathcal{S})\big\rangle \to \mathcal{U} \tag{7.27}$$

induces a homotopy equivalence.

Proof. We aim to identify the fibre of the functor

$$\begin{aligned} F\colon \mathcal{U} &\to \big\langle \mathcal{S}^0, \mathcal{S}^0\big\rangle \\ (d, X, S, \{\theta_j\}) &\mapsto (d, S) \\ (\phi, Y, [T, \alpha]) &\mapsto (\phi, [T, \alpha]) \end{aligned} \tag{7.28}$$

with $\big\langle \mathrm{Ob}(\mathcal{S}^0), \mathrm{Ob}(\mathcal{S})\big\rangle$. Because every object in the target category is of simplicial dimension 0, it suffices to verify the condition for Theorem B' on left-fibres of the form $F \downarrow (0, A)$, where A is a based free R-module of rank n. An object of $F \downarrow (0, A)$ is a tuple $(d, X, S, \{\theta_j\}, [U, \beta])$ with $S, X \in \mathcal{S}^0_d$, $\theta_j \in Aut(X \oplus S)$ and $[U, \beta]\colon S \to \phi_d^*(A) \in \big\langle \mathcal{S}^0_d, \mathcal{S}^0_d\big\rangle$. A morphism $(d, X, S, \{\theta_j\}, [U, \beta]) \to (d', X', S', \{\theta'_j\}, [U', \beta'])$ is given by a morphism $(\phi, Y, [T, \alpha])$ in $\mathcal{U}$ such that

$$\begin{tikzcd} S \arrow[r, "{[U,\beta]}"] \arrow[d, "{[T,\alpha]}"'] & \phi_d^* A \\ S' \arrow[ur, "{[U',\beta']}"'] & \end{tikzcd} \tag{7.29}$$

commutes. The full subcategory of $F \downarrow (0, A)$ with objects $(d, X, \phi_d^* A, \{\theta_j\}, [0, id])$ can be identified with the category

$$\big\langle \mathrm{Ob}(\mathcal{S}^0), (\mathrm{Ob}(\mathcal{S}^0) \oplus A)(\mathbb{G}^t_m)\big\rangle \cong \big\langle \mathrm{Ob}(\mathcal{S}^0), \mathrm{Ob}(\mathcal{S})\big\rangle_{\geq n}. \tag{7.30}$$

The transfer map induced on left-fibres by $[A, id]\colon 0 \to A \in \big\langle \mathcal{S}^0_0, \mathcal{S}^0_0\big\rangle$ restricts to the inclusion

$$\big\langle \mathrm{Ob}(\mathcal{S}^0), \mathrm{Ob}(\mathcal{S})\big\rangle_{\geq n} \subseteq \big\langle \mathrm{Ob}(\mathcal{S}^0), \mathrm{Ob}(\mathcal{S})\big\rangle \tag{7.31}$$

which we know to be a homotopy equivalence.
It remains to show that the inclusion $G\colon \left\langle \mathrm{Ob}(\mathcal{S}^0), (\mathrm{Ob}(\mathcal{S}^0) \oplus A)(\mathbb{G}_m^t)\right\rangle \subseteq F \downarrow (0, A)$ induces a homotopy equivalence. We construct a functor in the opposite direction that will function as a homotopy inverse:

$$\begin{aligned} H\colon F \downarrow (0, A) &\to \left\langle \mathrm{Ob}(\mathcal{S}^0), (\mathrm{Ob}(\mathcal{S}^0) \oplus A)(\mathbb{G}_m^t)\right\rangle \\ (d, X, S, \{\theta_j\}, [U, \beta]) &\mapsto (d, X, \{(1+\beta)(\theta_j+1)(1+\beta^{-1})\}) \\ (\phi, Y, [T, \alpha]) &\mapsto (\phi, Y) \end{aligned} \tag{7.32}$$

This functor is well-defined because the following diagrams commute (for some isomorphism $u\colon U \to T \oplus S'$):

$$\begin{tikzcd}
Y \oplus X \oplus S \oplus U \arrow[rr, "1+\beta"] \arrow[dr, "1+u"] \arrow[ddd, "1+\theta_j+1"] & & X' \oplus \phi^* A \\
& Y \oplus X \oplus S \oplus T \oplus U' \arrow[r, "1+\alpha+1"] \arrow[d, "1+\theta_j+1"] & X' \oplus S' \oplus U' \arrow[u, "1+\beta'"] \arrow[d, "\theta'_j+1"] \\
& Y \oplus X \oplus S \oplus T \oplus U' \arrow[r, "1+\alpha+1"] & X' \oplus S' \oplus U' \arrow[d, "1+\beta'"] \\
Y \oplus X \oplus S \oplus U \arrow[rr, "1+\beta"] \arrow[ur, "1+u"] & & X' \oplus \phi^* A
\end{tikzcd} \tag{7.33}$$

The composition $H \circ G$ is the identity and the composition $G \circ H$ is given by

$$\begin{aligned} F \downarrow (0, A) &\to F \downarrow (0, A) \\ (d, X, S, \{\theta_j\}, [U, \beta]) &\mapsto (d, X, \phi_d^* A, \{(1+\beta)(\theta_j+1)(1+\beta^{-1})\}, [0, 1]) \\ (\phi, Y, [T, \alpha]) &\mapsto (\phi, Y, [0, 1]) \end{aligned} \tag{7.34}$$

It is easy to see that there are well-defined morphisms

$$(id, 0, [U, \beta])\colon (d, X, S, \{\theta_j\}, [U, \beta]) \to (d, X, \phi_d^* A, \{(1+\beta)(\theta_j+1)(1+\beta^{-1})\}, [0, 1]) \tag{7.35}$$

and it follows essentially from diagram (7.33) that they define a natural transformation from the identity functor to $G \circ H$.

$\square$

Lemma 7.5.5. *The sequence*

$$\begin{aligned} \langle \mathcal{S}^0, \mathcal{S} \rangle &\to \mathcal{U} \to \langle \mathrm{Ob}(\mathcal{S}^0), \mathrm{Ob}(\mathcal{S}^0) \rangle \\ (d, X, S, \{\theta_j\}) &\mapsto (d, X) \\ (\phi, Y, [T, \alpha]) &\mapsto (\phi, Y) \end{aligned} \tag{7.36}$$

induces a fibration sequence on classifying spaces. In particular the inclusion

$$\langle \mathcal{S}^0, \mathcal{S} \rangle \to \mathcal{U} \tag{7.37}$$

induces a homotopy equivalence.

Proof. Let

$$\begin{aligned} F \colon \mathcal{U} &\to \langle \mathrm{Ob}(\mathcal{S}^0), \mathrm{Ob}(\mathcal{S}^0) \rangle \\ (d, X, S, \{\theta_j\}) &\mapsto (d, X) \\ (\phi, Y, [T, \alpha]) &\mapsto (\phi, Y) \end{aligned} \tag{7.38}$$

The proof is essentially identical to the proof of Lemma 7.5.4. Here, we show that the functors

$$\begin{aligned} G \colon F \downarrow (0, A) &\to \langle \mathcal{S}^0, (A \oplus \mathcal{S}^0)(\mathbb{G}_m^t) \rangle \\ (d, X, S, \{\theta_j\}, Z) &\mapsto (d, S, \{1_Z + \theta_j\}) \\ (\phi, Y, [T, \alpha]) &\mapsto (\phi, [T, \alpha]) \end{aligned} \tag{7.39}$$

induce homotopy equivalences, where Z is a morphism $F(d, X, S, \{\theta_j\}) = (d, X) \to (0, A)$, i.e an object $Z \in \mathrm{Ob}(\mathcal{S}^0)$ with $Z \oplus X = \phi_d^* A$ for the unique map $\phi_d \colon [d] \to [0]$. The result follows then from Lemma 7.5.3. A homotopy inverse to G is given by

$$\begin{aligned} H \colon \langle \mathcal{S}^0, (A \oplus \mathcal{S}^0)(\mathbb{G}_m^t) \rangle &\to F \downarrow (0, A) \\ (d, S, \{\theta_j\}) &\mapsto (d, \phi_d^* A, S, \{\theta_j\}, 0) \\ (\phi, [T, \alpha]) &\mapsto (\phi, 0, [T, \alpha]) \end{aligned} \tag{7.40}$$

where 0 is the trivial object in $\mathrm{Ob}(\mathcal{S}^0)$. Then the composition $G \circ H$ is the identity, and the morphisms

$$(id, Z, [0, 1]) \colon (d, X, S, \{\theta_j\}, Z) \to (d, \phi_d^* A, S, \{1 + \theta_j\}, 0) \tag{7.41}$$

provide a natural transformation from the identity functor to $H \circ G$. $\square$

Proof of Lemma 7.4.2. The following diagram commutes

$$\begin{tikzcd}
 & & & \left\langle \mathcal{S}^0, \mathcal{S}^0 \right\rangle \\
 & \left\langle \mathcal{S}^0, \mathcal{S} \right\rangle \arrow[r] & \mathcal{U} \arrow[r] \arrow[ur] & \left\langle \mathrm{Ob}(\mathcal{S}^0), \mathrm{Ob}(\mathcal{S}^0) \right\rangle \arrow[u] \\
 & \left\langle \mathrm{Ob}(\mathcal{S}^0), \mathrm{Ob}(\mathcal{S}) \right\rangle \arrow[u] \arrow[ur] & &
\end{tikzcd} \tag{7.42}$$

The result follows from Lemma 7.5.4 and 7.5.5. $\square$

Chapter 8

Fibrations via commuting automorphisms in odd shifts

8.1 Fibrations with obstructions in the base

In Theorem 6.1.1 we constructed certain fibrations for the Grothendieck-Witt theory of automorphisms in even shifts, namely fibrations of the form

$$\Omega^{-1}GW^{[2n]}(R\Delta^{\cdot}, \mathbb{G}_m^{\wedge t+1}) \to GW^{[2n]}(R\Delta^{\cdot}, \mathbb{G}_m^{\wedge t}) \to GW_0^{[2n]}(R\Delta^{\cdot}, \mathbb{G}_m^{\wedge t}) \tag{8.1}$$

where R is a ring with involution such that 2 is invertible. The goal of this chapter is to construct similar fibrations for hermitian K-theory in odd shifts, i.e. the spaces $GW^{[2n+1]}$.

Consider the diagram

$$\begin{tikzcd}
{}_\epsilon GW^{[n]}(R, \mathbb{G}_m^{\wedge t}) \arrow[r] \arrow[d] & K(R, \mathbb{G}_m^{\wedge t}) \arrow[r] \arrow[d] & {}_\epsilon GW^{[n+1]}(R, \mathbb{G}_m^{\wedge t}) \arrow[d] \\
{}_\epsilon GW^{[n]}(R, \mathbb{G}_m^{t}) \arrow[r] \arrow[d] & K(R, \mathbb{G}_m^{t}) \arrow[r] \arrow[d] & {}_\epsilon GW^{[n+1]}(R, \mathbb{G}_m^{t}) \arrow[d] \\
\operatorname*{holim}_{i\in\widehat{[1]}^n} {}_\epsilon GW^{[n]}(R, \mathbb{G}_m^{|i|}) \arrow[r] & \operatorname*{holim}_{i\in\widehat{[1]}^n} K(R, \mathbb{G}_m^{|i|}) \arrow[r] & \operatorname*{holim}_{i\in\widehat{[1]}^n} {}_\epsilon GW^{[n+1]}(R, \mathbb{G}_m^{|i|})
\end{tikzcd} \tag{8.2}$$

By Proposition 4.4.9 the vertical sequences are homotopy fibrations, and the lower and middle horizontal sequence are homotopy fibration by Theorem 3.5.20 and because homotopy limits commute with each other. It follows that the top sequence, the sequence of stabilised (hermitian) K-theory spaces, is a fibration as well.

Unfortunately, this does not mean that

$$GW^{[2n]}(R\Delta^{\cdot}, \mathbb{G}_m^{\wedge t}) \xrightarrow{F} K(R\Delta^{\cdot}, \mathbb{G}_m^{\wedge t}) \xrightarrow{H} GW^{[2n+1]}(R\Delta^{\cdot}, \mathbb{G}_m^{\wedge t}) \tag{8.3}$$

is also a fibration sequence, even in simple cases (unless $t = 0$ of course).

Lemma 3.2.1 provides a sufficient condition for the sequence to be a fibration after geometric realisation. Explicitly, we need the map

$$K_0(R\Delta^d, \mathbb{G}_m^{\wedge t}) \to GW_0^{[2n+1]}(R\Delta^d, \mathbb{G}_m^{\wedge t}) \tag{8.4}$$

to be surjective, which is satisfied for $d = 0$ by Lemma 4.6.2, but is not so clear for higher d.

Definition 8.1.1. Let R be a ring with involution. Define $F^{[n]}(R, t)$ to be the homotopy fibre in the following diagram

$$F^{[n]}(R,t) \to GW_0^{[2n]}(R\Delta^{\cdot}, \mathbb{G}_m^{\wedge t}) \to K_0(R\Delta^{\cdot}, \mathbb{G}_m^{\wedge t}) \tag{8.5}$$

Further, define $E^{[n]}(R,t)$ to be the homotopy fibre in

$$E^{[n]}(R,t) \to \Omega^{-1}GW^{[2n]}(R\Delta^{\cdot}, \mathbb{G}_m^{\wedge t}) \xrightarrow{F} \Omega^{-1}K(R\Delta^{\cdot}, \mathbb{G}_m^{\wedge t}) \tag{8.6}$$

Corollary 8.1.2. *Assume that* 2 *is invertible. Then there may identify*

$$E^{[n]}(R,0) = GW^{[2n+1]}(R\Delta^{\cdot}) \tag{8.7}$$

Proof. By Lemma 4.6.2 we obtain a homotopy fibration when we deloop the sequence in Proposition **??**. The result follows. $\square$

Lemma 8.1.3. *Let R be a ring with involution such that* 2 *is invertible. Then there exist fibration sequences of the form*

$$E^{[n]}(R,t+1) \to \Omega E^{[n]}(R,t) \to F^{[n]}(R,t) \tag{8.8}$$

Moreover the map $\Omega E^{[n]}(R,t) \to F^{[n]}(R,t)$ induces a surjective map on the sets of connected components.

Proof. Consider the diagram

$$\begin{tikzcd}
F^{[n]}(R,t) \arrow[r] & GW_0^{[2n]}(R\Delta^{\cdot}, \mathbb{G}_m^{\wedge t}) \arrow[r] & K_0(R\Delta^{\cdot}, \mathbb{G}_m^{\wedge t}) \\
\Omega E^{[n]}(R,t) \arrow[u] \arrow[r] & GW^{[2n]}(R\Delta^{\cdot}, \mathbb{G}_m^{\wedge t}) \arrow[u] \arrow[r] & K(R\Delta^{\cdot}, \mathbb{G}_m^{\wedge t}) \arrow[u] \\
E^{[n]}(R,t+1) \arrow[u] \arrow[r] & \Omega^{-1}GW^{[2n]}(R\Delta^{\cdot}, \mathbb{G}_m^{\wedge t+1}) \arrow[u] \arrow[r] & \Omega^{-1}K(R\Delta^{\cdot}, \mathbb{G}_m^{\wedge t+1}) \arrow[u]
\end{tikzcd} \tag{8.9}$$

The right and central vertical sequences are fibrations by Theorem 6.1.1 and the horizontal sequence are fibrations by definition. By the four fibration lemma 4.4.10, the left vertical sequence is a fibration, too. It remains to show that

$$\pi_1(E^{[n]}(R,t)) \to \pi_0(F^{[n]}(R,t)) \tag{8.10}$$

is surjective, which follows from the long exact homotopy sequences and the 4-lemma applied to the diagram

$$
\begin{tikzcd}
\pi_1 GW_0^{[2n]} \arrow[r] & \pi_1 K_0 \arrow[r] & \pi_0 F^{[n]}(R,t) \arrow[r] & \pi_0 GW_0^{[2n]} \arrow[r] & \pi_0 K_0 \\
\pi_1 GW^{[2n]} \arrow[r] \arrow[u, two heads] & \pi_1 K \arrow[r] \arrow[u, two heads] & \pi_1 E^{[n]}(R,t) \arrow[r] \arrow[u] & \pi_0 GW^{[2n]} \arrow[r] \arrow[u, "\cong"] & \pi_0 K \arrow[u, "\cong"]
\end{tikzcd}
\tag{8.11}
$$

where the argument $R\Delta^{\cdot}, \mathbb{G}_m^{\wedge t}$ has been removed from the notation to increase readability. $\square$

Lemma 8.1.4. *The sequence*

$$\Omega^{-1} GW^{[2n+1]}(R\Delta^{\cdot}, \mathbb{G}_m^{\wedge t+1}) \to GW^{[2n+1]}(R\Delta^{\cdot}, \mathbb{G}_m^{\wedge t}) \to GW_0^{[2n+1]}(R\Delta^{\cdot}, \mathbb{G}_m^{\wedge t}) \tag{8.12}$$

is not always a fibration sequence.

Proof. It suffices to give a counter-example. To this end consider the case $t = n = 0$ and $R = k$ a field. By homotopy invariance we know that $GW^{[1]}(k\Delta^{\cdot}) \simeq GW^{[1]}(k)$ and, in particular (cf. [Wal03, Theorem 10.1]),

$$|d \mapsto GW_0^1(k\Delta^d)| \simeq GW_0^1(k) \simeq * \tag{8.13}$$

If the sequence above were a fibration sequence this would imply that the map

$$\pi_0 GW^{[1]}(k\Delta^{\cdot}, \mathbb{G}_m^{\wedge 1}) \to \pi_1 GW^{[1]}(k\Delta^{\cdot}) \tag{8.14}$$

was an isomorphism. By Lemma 4.6.2 we know that

$$\pi_0 K(k\Delta^{\cdot}, \mathbb{G}_m^{\wedge 1}) \cong k^{\times} \tag{8.15}$$

surjects on the left hand side, while the right hand side is Karoubi's V-theory, a contradiction. Details on Karoub's V-theory can be found in the subsequent chapters. $\square$

8.2 The first obstructions

The homotopy fibration sequence in Lemma 8.1.3 is still a homotopy fibration sequence after delooping. The base space is then of the form $\Omega^{-1}F^{[n]}(R,t) \times \pi_0 E^{[n]}(R,t)$. This section is devoted to the study of these base spaces in certain special cases.

Lemma 8.2.1. *Let R be a regular, local ring such that 2 is invertible. The space $F^{[1]}(R,0)$ is contractible. In particular there is a fibration sequence*

$$\Omega^{-1}E^{[1]}(R,1) \to GW^{[3]}(R\Delta^{\cdot}) \to GW_0^{[3]}(R\Delta^{\cdot}) \tag{8.16}$$

Proof. By homotopy invariance we can readily identify the sequence

$$GW_0^{[2]}(R\Delta^{\cdot}) \xrightarrow{F} K_0(R\Delta^{\cdot}) \xrightarrow{H} GW_0^{[3]}(R\Delta^{\cdot}) \tag{8.17}$$

with the sequence of constant simplicial sets

$$\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \to \mathbb{Z}/2 \tag{8.18}$$

which is clearly a fibration. Similarly we see that the fibre of $GW_0^{[n]}(R\Delta^{\cdot}) \to K_0(R\Delta^{\cdot})$ vanishes. The second part follows (again using homotopy invariance) from Theorem 8.1.3.

□

Lemma 8.2.2. *Let R be a regular, local ring such that 2 is invertible. Then we have*

$$F^{[0]}(R,0) \simeq \mathrm{Ker}(\mathrm{rk}\colon GW_0^{[0]}(R) \to \mathbb{Z}) \tag{8.19}$$

where $\mathrm{rk}\colon GW_0^{[0]}(R) \to \mathbb{Z}$ is the rank homomorphism. In particular $\pi_i(F^{[0]}(R,0)) = 0$ for $i \geq 1$. Moreover, we have $F^{[0]}(R,0) \simeq *$ if every unit in R is a square.

Proof. The sequence

$$GW_0^{[0]}(R\Delta^{\cdot}) \xrightarrow{F} K_0(R\Delta^{\cdot}) \xrightarrow{H} GW_0^{[1]}(R\Delta^{\cdot}) \tag{8.20}$$

coincides with the sequence

$$GW_0^{[0]}(R) \to \mathbb{Z} \to 0 \tag{8.21}$$

If every unit in R is a square, then $GW_0^{[0]}(R) = \mathbb{Z}$ and the rank homomorphism is an isomorphism. □

8.3 The spaces $E^{[n]}(R,1)$

Consider the fibration sequence

$$GW^{[2n]}(R\Delta^{\cdot}, \mathbb{G}_m^{\wedge 1}) \to K(R\Delta^{\cdot}, \mathbb{G}_m^{\wedge 1}) \to E^{[n]}(R,1) \tag{8.22}$$

We can identify the first two spaces in the sequence as follows:

$$\left.\begin{array}{l} Gl(R\Delta^{\cdot}) \\ {}_{\epsilon}O(R\Delta^{\cdot}) \end{array}\right\} \simeq \langle \mathrm{Ob}(\mathcal{S}), \mathrm{Ob}(\mathcal{S}(\mathbb{G}_m))\rangle \overset{(a)}{\simeq} \langle \mathcal{S}, \mathcal{S}(\mathbb{G}_m)\rangle$$
$$\overset{(b)}{\simeq} \langle \mathcal{S}, \mathcal{S}(\mathbb{G}_m)\rangle^{+} \overset{(c)}{\simeq} \left\{\begin{array}{l} K(R\Delta^{\cdot}, \mathbb{G}_m^{\wedge 1}) \\ {}_{\epsilon}GW(R\Delta^{\cdot}, \mathbb{G}_m^{\wedge 1}) \end{array}\right. \tag{8.23}$$

for $\mathcal{S} = \mathcal{GL}^0(\Delta^{\cdot})$ or $\mathcal{S} = {}_{\epsilon}\mathcal{O}^0(\Delta^{\cdot})$, respectively. The homotopy equivalences are given by

(a) Theorem 7.2.2,

(b) Proposition 4.7.2 or 4.7.3, and

(c) Proposition 4.5.2.

Now the simplicial sets $Gl(R\Delta^{\cdot})$ and ${}_{\epsilon}O(R\Delta^{\cdot})$ actually have the structure of simplicial groups and the forgetful map ${}_{\epsilon}O(R) \to Gl(R)$ is injective for any ring R. The following is immediate:

Proposition 8.3.1. *Let R be a ring with involution. Then there are fibrations of simplicial sets*

$${}_{\epsilon}O(R\Delta^{\cdot}) \to Gl(R\Delta^{\cdot}) \xrightarrow{F} Gl/_{\epsilon}O(R\Delta^{\cdot}) \tag{8.24}$$

Corollary 8.3.2. *Let R be a ring with involution such that 2 is invertible. Then there are homotopy equivalences*

$$E^{[n]}(R, 1) \simeq Gl/_{(-1)^n}O(R\Delta^{\cdot}) \tag{8.25}$$

Proof. By Proposition 8.3.1 and the discussion above both spaces are the homotopy fibre of the map

$$\Omega^{-1}({}_{\epsilon}O(R\Delta^{\cdot})) \to \Omega^{-1}(Gl(R\Delta^{\cdot})) \tag{8.26}$$

□

Definition 8.3.3. Let R be a ring. Denote by ${}_{\epsilon}S_n(R)$ the set of $n \times n$ invertible symmetric matrices with entries in R. Further let ${}_{\epsilon}H_{2n}$ be the n-fold block sum of matrices of the form ${}_{\epsilon}H_2 = \begin{pmatrix} 0 & 1 \\ \epsilon & 0 \end{pmatrix}$. Taking the block sum with ${}_{\epsilon}H_2$ provides an embedding ${}_{\epsilon}S_n(R) \to {}_{\epsilon}S_{n+2}(R)$ and we write ${}_{\epsilon}S(R) = \operatorname{colim} {}_{\epsilon}S_n(R)$.

Lemma 8.3.4 ([Oja84]). *Let R be a ring and $A \in {}_\epsilon S_n(R[T])$. Then there exists an integer k, an elementary matrix $P \in E_{n+2k}(R[T])$ and matrices $A_0, A_1 \in M_{n+2k}(R)$ such that*

$$P^*(A \oplus {}_\epsilon H_{2k})P = A_0 + A_1 T \tag{8.27}$$

Remark 8.3.5. The sum on the right hand side is element-wise addition, A_0 and A_1 are both ϵ-symmetric, but only A_0 is invertible in general.

Corollary 8.3.6 ([Oja84]). *Let R be a ring such that 2 is invertible and let $A \in {}_\epsilon S(R[T])$. Then there exists an $A_0 \in {}_\epsilon S(R)$ and an element $P \in Gl(R[T])$ with $P(T = 0) = 1$, such that*

$$A = P^* A_0 P \tag{8.28}$$

Proposition 8.3.7. *Let R be a ring such that 2 is invertible. Then there are homotopy equivalences*

$$\Omega(Gl/{}_\epsilon O(R\Delta^\cdot)) \xrightarrow{\simeq} \Omega({}_\epsilon S(R\Delta^\cdot)) \tag{8.29}$$

If moreover R is local, there is even a homotopy equivalence

$$Gl/{}_{-}O(R\Delta^\cdot) \xrightarrow{\simeq} {}_{-}S(R\Delta^\cdot) \tag{8.30}$$

Proof. Consider the map

$$\begin{aligned} Gl(R) &\to {}_\epsilon S(R) \\ A &\mapsto A^t_\epsilon H A \end{aligned} \tag{8.31}$$

Two matrices A and A' have the same image if and only if $AA'^{-1} \in {}_\epsilon O(R)$, i.e. if $[A] = [A'] \in Gl/{}_\epsilon O(R)$.
We claim that the morphism of simplicial sets

$$F\colon Gl/{}_\epsilon O(R\Delta^\cdot) \to {}_\epsilon S(R\Delta^\cdot) A \quad \mapsto A^t_\epsilon H A \tag{8.32}$$

is surjective on the connected component of ${}_\epsilon H$. We proceed by induction on the simplicial dimension:
Assume $X \in {}_\epsilon S(R)$ is in the connected component of ${}_\epsilon H$. This means there are $A_i \in {}_\epsilon S(R[T])$, $1 \le i \le r$ such that $A_1(1) = X$, $A_i(0) = A_{i+1}(1)$ for $i < r$ and $A_r(0) = {}_\epsilon H$. By Corollary 8.3.6 there are invertible matrices $P_i \in Gl_n(R[T])$ with $A_i = P_i^* A_i(0) P_i$. Finally set $P = (\Pi_{i=n}^2 P_i(1))P_1$ and conclude $P^*{}_\epsilon H P = X$.
For the induction step assume that all matrices of simplicial dimension d are in the image of F, i.e. for all $X \in {}_\epsilon S(R\Delta^d)$ that are in the connected component of ${}_\epsilon H$ there exists a matrix $P \in Gl(R\Delta^d)$ such that $X = P^*{}_\epsilon H P$. Now, the following

diagram commutes

$$\begin{array}{ccc} R\Delta^{d+1} = \frac{R[T_0,\dots,T_{d+1}]}{\sum T_i = 1} & \xrightarrow{T_{d+1}=0} & R\Delta^{d} = \frac{R[T_0,\dots,T_d]}{\sum T_i = 1} \\ \cong \uparrow T_i \mapsto T_i & & \cong \uparrow T_i \mapsto T_i \\ R[T_1,\dots,T_{d+1}] & \xrightarrow{T_{d+1}=0} & R[T_1,\dots,T_d] \end{array} \tag{8.33}$$

Let $X \in {}_{\epsilon}S(R\Delta^{d+1})$ be in the connected component of ${}_{\epsilon}H$. Then X can be represented by a matrix with entries in $R[T_1,\dots,T_d][T_{d+1}]$. By Corollary 8.3.6 there exists a matrix $P \in Gl(R[T_1,\dots,T_d][T_{d+1}])$ such that $X = P^* X(T_{d+1} = 0)P$ and by the induction hypothesis there exists a $Q \in Gl(R[T_1,\dots,T_d]$ such that $X(T_{d+1} = 0) = Q^* {}_{\epsilon}HQ$. This finished the first part of the Proposition.
For the second statement it now suffices to show that there is an isomorphism

$$\pi_0(Gl/{}_{-}O(R\Delta^{\cdot})) \xrightarrow{\cong} \pi_0({}_{-}S(R\Delta^{\cdot})) \tag{8.34}$$

But for a local ring R every skew-symmetric inner product space has a symplectic basis, i.e. for each $X \in {}_{-}S(R)$ there is a $P \in Gl(R)$ such that $X = P^* {}_{\epsilon}HP$. We are done. $\square$

Corollary 8.3.8. *Let R be a local ring with involution such that 2 is invertible. Then there are homotopy equivalences*

$$\Omega(E^{[0]}(R,1)) \simeq \Omega(S(R\Delta^{\cdot})) \tag{8.35}$$

and

$$E^{[1]}(R,1) \simeq {}_{-}S(R\Delta^{\cdot}) \tag{8.36}$$

Proof. This is just a simple combination of the results of this section. $\square$

Remark 8.3.9. In [ST14] Schlichting and Tripathi prove results resembling the ones above in the case of regular rings R. See e.g. Theorem 6, op. cit.

8.4 Fibrations via ϵ-symmetric matrices

Theorem 8.1.3 for $t = 0$ states the existence of a fibration sequence of the form

$$\Omega^{-1}E^{[n]}(R,1) \to GW^{[2n+1]}(R\Delta^{\cdot}) \to \Omega^{-1}F^{[n]}(R,0) \times \pi_0(GW^{[2n+1]}(R\Delta^{\cdot})) \tag{8.37}$$

where the obstruction term $F^{[n]}(R,0)$ vanishes for $n=1$ and is at least discrete for $n=0$. In Corollary 8.3.8 at the end of the last section we observed that $E^{[1]}(R,1)$ can be expressed in terms of skew-symmetric matrices and that at least the base-point component of $E^{[0]}(R,1)$ can be described via symmetric matrices. It turns out that the two deviances in the case $n=0$, that is the non-vanishing of certain lower homotopy groups, exist for the very same reason. This leads us to new candidates of fibration sequences that appear to be better than the sequence (8.8), since they don't contain any obstruction term. More precisely we show:

Theorem 8.4.1. *Let R be a regular, local ring such that 2 is invertible. Then the natural map*

$$ {}_{(-1)^n}S(R\Delta^\cdot) \to \Omega GW^{[2n+1]}(R\Delta^\cdot) \tag{8.38}$$

is a homotopy equivalence.

The following is an easy consequence and the notation is more in the flavour of things:

Corollary 8.4.2. *Let R be a regular, local ring such that 2 is invertible. Then there are homotopy fibration sequences*

$$\Omega^{-1}{}_{(-1)^n}S(R\Delta^\cdot) \to GW^{[2n+1]}(R\Delta^\cdot) \to GW_0^{[2n+1]}(R\Delta^\cdot) \tag{8.39}$$

First, we explicitly construct the natural map as used in Theorem 8.4.1. To this end recall from [Kar73] that Karoubi defined ${}_\epsilon\mathcal{V}(R)$-theory as the fibre of the forgetful functor

$$ {}_\epsilon\mathcal{V}(R) \to {}_\epsilon GW(R) \to K(R) \tag{8.40}$$

By the fundamental Theorem 3.5.20 this implies

$$ {}_{(-1)^n}\mathcal{V}(R) \simeq \Omega GW^{[2n+1]}(R) \tag{8.41}$$

Moreover there exists an explicit description of ${}_\epsilon V(R) = \pi_{0\epsilon}\mathcal{V}(R)$ in terms of generators and relations (cf. e.g [Kar78, II]):

Definition 8.4.3. Let R be a ring such that 2 is invertible. Denote by ${}_\epsilon V(R)$ the abelian group generated by symbols of the form $[L,\phi_0,\phi_1]$, where the ϕ_i are ϵ-symmetric bilinear forms on the free R-module L, modulo the relations $[L,\phi_0,\phi_1] = [L',\phi_0\circ a,\phi_1\circ a]$ for any isomorphism $a\colon L'\to L$, $[L,\phi_0,\phi_1]+[L',\phi_0',\phi_1'] = [L\oplus L',\phi_0\oplus\phi_0',\phi_1\oplus\phi_1']$ and $[L,\phi_0,\phi_1]+[L,\phi_1,\phi_2] = [L,\phi_0,\phi_1]$.

Definition/Proposition 8.4.4. There is a natural map ${}_{\epsilon}S(R) \to {}_{\epsilon}V(R)$ which maps a representative $A \in {}_{\epsilon}S_{2n}(R)$ to the class $[R^{2n}, A, {}_{\epsilon}H]$. For regular R this induces a map, well-defined up to homotopy,

$$|S(R\Delta^{\cdot})| \to |{}_{\epsilon}V(R\Delta^{\cdot})| \simeq {}_{\epsilon}V(R) \subset {}_{\epsilon}\mathcal{V}(R) \tag{8.42}$$

This is the map in the formulation of Theorem 8.4.1.

Definition 8.4.5. A ring R is called K_1-rigid, if the evaluation at 0, $R[T] \to R$ induces an isomorphism $K_1(R[T]) \cong K_1(R)$.

Proposition 8.4.6 ([BL08, Section 4.5])**.** *Let R be a K_1-rigid ring with 2 invertible. Then the application*

$$\begin{aligned} {}_{\epsilon}V(R) &\to \pi_0{}_{\epsilon}S(R\Delta^{\cdot}) \\ [L, \phi_0, \phi_1] &\mapsto [\phi_0] - [\phi_1] \end{aligned} \tag{8.43}$$

is an isomorphism.

Proof of Theorem 8.4.1. Consider the fibration sequence

$$\Omega^{-1}E^{[n]}(R,1) \to GW^{[2n+1]}(R\Delta^{\cdot}) \to \Omega^{-1}F^{[n]}(R,0) \times \pi_0 GW^{[2n+1]}(R\Delta^{\cdot}) \tag{8.44}$$

By Lemma 8.2.1 and Lemma 8.2.2, respectively, the homotopy groups of the base-space $\pi_i(\Omega^{-1}F^{[n]}(R,0) \times \pi_0 GW^{[2n+1]}(R\Delta^{\cdot}))$ vanish for $i \geq 1$ (for odd n) and $i \geq 2$ (for even n). This implies that there are homotopy equivalences

$$\Omega GW^{[3]}(R) \simeq E^{[1]}(R,1) \tag{8.45}$$

and

$$\Omega^2 GW^{[1]}(R) \simeq \Omega E^{[0]}(R,1) \tag{8.46}$$

By Proposition 8.4.6 and Corollary 8.3.8 we are done. $\square$

8.5 Symmetric matrices and a new duality

Recall from Definition 4.4.1 how the duality on the category ${}_{\epsilon}\mathcal{P}(R, \mathbb{G}_m)$ is induced by the involution $\bar{\cdot}$ on the ring $\mathbb{G}_m = k[U, U^{-1}]$ with $\bar{U} = U^{-1}$. There, an object of ${}_{\epsilon}\mathcal{P}(R, \mathbb{G}_m)$ is described as a pair $(P, \rho\colon \mathbb{G}_m \to \mathrm{End}(P))$, ρ is uniquely determined by the image $\theta = \rho(U)$ and the duality is by definition $\theta^* = \rho^*(U) := (\rho(\bar{U}))^* = \theta^{-*}$.
There is a natural identification ${}_{\epsilon}\mathcal{P}_h(R, \mathbb{G}_m^t)(\mathbb{G}_m) = {}_{\epsilon}\mathcal{P}_h(R, \mathbb{G}_m^{t+1})$ of the categories of inner product spaces, and a string $(P, \phi, \{\theta_j\}, \theta)$ with $1 \le j \le t$, can either be understood as an object of ${}_{\epsilon}\mathcal{P}_h(R, \mathbb{G}_m^{t+1})$ or as an object of ${}_{\epsilon}\mathcal{P}_h(R, \mathbb{G}_m^t)$ together with an automorphism θ in the same category. This interpretation was a key idea for the results of chapter 6.
The ring $\mathbb{G}_m = k[U, U^{-1}]$ comes with another obvious involution $\tilde{\cdot}$ given by $\tilde{U} = U$. In this section we will study this involution and the dualities induced by it.

Definition 8.5.1. Let $\widetilde{\mathbb{G}}_m$ be the ring $k[U, U^{-1}]$ of Laurent polynomials over a ring k as in section 4.4.1, equipped with the trivial involution. Denote by ${}_{\epsilon}\mathcal{P}(R, \widetilde{\mathbb{G}}_m^t)$ the category with duality with objects $(P, \{\theta_j\})$ for $1 \le j \le t$ and pairwise commuting automorphisms $\theta_j \in Aut(P)$. A morphism $(P, \{\theta_j\}) \to (P', \{\theta'_j\})$ is given by an isomorphism $\eta\colon P \to P'$ that commutes with θ_j for all j. Finally the duality is given by

$$(P, \{\theta_j\})^* = (P^*, \{\theta_j^*\}) \tag{8.47}$$

Definition 8.5.2. Denote by ${}_{\epsilon}\mathcal{P}_h(R, \tilde{\mathbb{G}}_m^t)$ the category of inner product spaces in the category ${}_{\epsilon}\mathcal{P}(R, \tilde{\mathbb{G}}_m^t)$. Explicitly an object is an isomorphism

$$\phi\colon (P, \{\theta_j\}) \to (P^*, \{\theta_j^*\}) \tag{8.48}$$

or, equivalently, an isomorphism $\phi\colon P \to P^*$ that satisfies $\phi\theta_j = \theta_j^*\phi$ for all $1 \le j \le t$. A morphism

$$\eta\colon (P, \phi, \{\theta_j\}) \to (P', \phi', \{\theta'_j\}) \tag{8.49}$$

is an isomorphism $\eta\colon P \to P'$ that satisfies $\eta\theta_j = \theta'_j\eta$ for all $1 \le j \le t$ and $\phi = \eta^*\phi'\eta$.

Warning 8.5.3. We have the identification ${}_{\epsilon}\mathcal{P}_h(R, \widetilde{\mathbb{G}}_m^t)(\widetilde{\mathbb{G}}_m) = {}_{\epsilon}\mathcal{P}_h(R, \widetilde{\mathbb{G}}_m^{t+1})$. This is inconvenient since we no longer can identify a tuple $(P, \phi, \{\theta_j\}, \theta) \in {}_{\epsilon}\mathcal{P}_h(R, \widetilde{\mathbb{G}}_m^t) \times \mathrm{Aut}(P, \phi, \{\theta_j\})$ with an object of ${}_{\epsilon}\mathcal{P}_h(R, \widetilde{\mathbb{G}}_m^{t+1})$, as we did in the proof of Lemma 6.1.5. Indeed θ satisfies $\phi = \theta^*\phi\theta$, whereas the θ_j satisfy $\phi\theta_j = \theta_j^*\phi$.

As before we can describe a category which is cofinal in ${}_{\epsilon}\mathcal{P}_h(R, \widetilde{\mathbb{G}}_m^t)$ and whose objects are given in terms of matrices over R:

Definition 8.5.4. Let ${}_\epsilon\mathcal{S}^t(R)$ be the category where an object is a string $(n, \{A_j\})$, consisting of a non-negative integer n and ϵ-symmetric matrices (not necessarily commutating) $A_j \in {}_\epsilon S_{2n}(R)$ for $1 \le j \le t$ such that all $A_i({}_\epsilon H)A_j$ are ϵ-symmetric as well. A morphism $(n, \{A_j\}) \to (n', \{A'_j\})$ only exists if $n = n'$ and is then given by a matrix $\eta \in {}_\epsilon O(R)$ that satisfies $\eta^* A'_j \eta = A_j$ for all j.

Proposition 8.5.5. *The category ${}_\epsilon\mathcal{S}^t(R)$ may be identified with a subcategory of ${}_\epsilon\mathcal{P}_h(R, \widetilde{\mathbb{G}}^t_m)$. Moreover the inclusion of categories is cofinal.*

Proof. Let $(P, \phi, \{\theta_j\}) \in {}_\epsilon\mathcal{P}_h(R, \widetilde{\mathbb{G}}^t_m)$ be an object and consider $A_j = \phi \circ \theta_j$. Then A_j satisfies

$$A_j^* = (\phi\theta_j)^* = \theta_j^*\phi^* = \epsilon\theta_j^*\phi = \epsilon\phi\theta_j = \epsilon A_j \tag{8.50}$$

If $\eta\colon (P, \phi, \{\theta_j\}) \to (P', \phi', \{\theta'_j\})$ is a morphism, then it satisfies $\theta'_j\eta = \eta\theta_j$ and $\eta^*\phi'\eta = \phi$. If we set $A'_j = \phi' \circ \theta'_j$ we then have

$$\eta^* A'_j\eta = \eta^*\phi'\theta'_j\eta = \eta^*\phi'\eta\theta_j = \phi\theta_j = A_j \tag{8.51}$$

This process can be reversed by setting $\theta_j = \phi^{-1}A_j$ for any string of ϵ-symmetric isomorphisms A_j such that all the $A_i\phi^{-1}A_j$ are ϵ-symmetric. Indeed, we compute

$$\theta_i\theta_j = \phi^{-1}A_i\phi^{-1}A_j = \phi^{-1}(A_j\epsilon\phi^{-1}A_i)^* = \epsilon^2\phi^{-1}A_j\phi^{-1}A_i = \theta_j\theta_i \tag{8.52}$$

In other words, the functor

$$\begin{aligned} {}_\epsilon\mathcal{S}^t(R) &\to {}_\epsilon\mathcal{P}_h(R, \widetilde{\mathbb{G}}^t_m) \\ (n, \{A_j\}) &\mapsto ({}_\epsilon H(R^n), \{{}_\epsilon H^{-1}A_j\}) \\ \eta &\mapsto \eta \end{aligned} \tag{8.53}$$

is an inclusion onto the subcategory of those objects $(P, \phi, \{\theta_j\}) \in {}_\epsilon\mathcal{P}_h(R, \widetilde{\mathbb{G}}^t_m)$ that satisfy $(P, \phi) = {}_\epsilon H(R^n)$ for some n. The verification of cofinality works as before. $\square$

Corollary 8.5.6. *Let R be a ring with involution such that 2 is invertible. Then the inclusion functor above induces homotopy equivalences*

$$\left\langle {}_\epsilon\mathcal{S}^0(R), {}_\epsilon\mathcal{S}^1(R) \right\rangle^+ \to {}_\epsilon GW(R, \widetilde{\mathbb{G}}^{\wedge 1}_m) \tag{8.54}$$

Proof. We first apply Lemma 4.5.1. Moreover Proposition 4.5.2 holds if we replace $\mathbb{G}_m$ by $\widetilde{\mathbb{G}}_m$. $\square$

Lemma 8.5.7. *The category $\left\langle {}_\epsilon\mathcal{S}^0(R\Delta^\cdot), {}_\epsilon\mathcal{S}^1(R\Delta^\cdot) \right\rangle$ is group complete.*

Proof. Let A be any ϵ-symmetric matrix. By [BL08, Lemma 4.5.1.9] there exists an elementary matrix Q such that

$$Q^*(M \oplus -M^{-1})Q = {}_\epsilon H \tag{8.55}$$

Therefore

$$\pi_0 \left\langle {}_\epsilon\mathcal{S}^0(R\Delta^\cdot), {}_\epsilon\mathcal{S}^1(R\Delta^\cdot)\right\rangle \tag{8.56}$$

is a group. □

Lemma 8.5.8. *The inclusion*

$$\epsilon S(R\Delta^\cdot) = \left\langle \mathrm{Ob}({}_\epsilon\mathcal{S}^0(R\Delta^\cdot)), \mathrm{Ob}({}_\epsilon\mathcal{S}^1(R\Delta^\cdot))\right\rangle \to \left\langle {}_\epsilon\mathcal{S}^0(R\Delta^\cdot), {}_\epsilon\mathcal{S}^1(R\Delta^\cdot)\right\rangle \tag{8.57}$$

is a homotopy equivalence.

Proof. The statement follows as in Lemma 7.4.2. The only change is in the proof of the analogue of Lemma 7.5.3. Here, the functor H (cf. (7.25)) is instead defined as

$$\begin{aligned} H\colon \left\langle \mathcal{S}^0, (A \oplus \mathcal{S}^0)(\widetilde{\mathbb{G}}^1_m)\right\rangle \times \Delta^1 &\to \left\langle \mathcal{S}^0, (A \oplus \mathcal{S}^0)(\widetilde{\mathbb{G}}^1_m)\right\rangle \\ (d, S, \{\theta_j\}, \lambda) &\mapsto (d, \phi^* A \oplus S, \{(\lambda^* M + 1)^*(1+\theta_j)(\lambda^* M + 1)\}) \\ (\phi, [T, \alpha]) &\mapsto (\phi, [T, 1+\alpha]) \end{aligned} \tag{8.58}$$

Everything else remains unchanged. □

Theorem 8.5.9. *Let R be a regular, local ring with involution such that 2 is invertible. Then there are homotopy fibration sequences of the form*

$$\Omega^{-1} GW^{[2n]}(R\Delta^\cdot, \widetilde{\mathbb{G}}^{\wedge 1}_m) \xrightarrow{U} GW^{[2n+1]}(R\Delta^\cdot) \to GW^{[2n+1]}_0(R\Delta^\cdot), \tag{8.59}$$

where U is the element of $GW^{[1]}_1(\widetilde{\mathbb{G}}_m)$ represented by the symmetric matrix $U \in Gl_1(\widetilde{\mathbb{G}}_m) = Gl_1(k[U, U^{-1}])$ (cf. Proposition 8.4.4).

Proof. The existence of the homotopy fibration sequence follows directly from Corollary 8.4.2 and the results of this section. It remains to show that the map defined in 8.4.4 coincides with multiplication by $U \in GW^{[1]}_1(\widetilde{\mathbb{G}}_m)$. Consider the following diagram, similar to that in the proof of Proposition 6.2.2:

$$
\begin{tikzcd}
S^1 \wedge S^0 \wedge GW^{[2n]}(R, \widetilde{\mathbb{G}}_m) \arrow[r, "="] \arrow[d, "1\wedge e\wedge 1"] & S^1 \wedge GW^{[2n]}(R, \widetilde{\mathbb{G}}_m) \arrow[d, "f\wedge 1"] \\
S^1 \wedge GW^{[0]}(\widetilde{\mathbb{G}}_m, \widetilde{\mathbb{G}}_m) \wedge GW^{[2n]}(R, \widetilde{\mathbb{G}}_m) \arrow[r] \arrow[d, "1\wedge\tau"] & GW^{[1]}(\widetilde{\mathbb{G}}_m) \wedge GW^{[2n]}(R, \widetilde{\mathbb{G}}_m) \arrow[d, "\tau"] \\
S^1 \wedge GW^{[2n]}(R, \widetilde{\mathbb{G}}_m) \arrow[r] & GW^{[2n+1]}(R)
\end{tikzcd}
\tag{8.60}
$$

where e identifies the element of $GW_0^{[0]}(\widetilde{\mathbb{G}}_m, \widetilde{\mathbb{G}}_m)$ that is represented by the "identity" morphism

$$(\widetilde{\mathbb{G}}_m, U) \to (\widetilde{\mathbb{G}}_m, U)^* = (\widetilde{\mathbb{G}}_m^*, U^*) \tag{8.61}$$

and f detects the element $U \in GW_1^{[1]}(\widetilde{\mathbb{G}}_m)$. The upper square is commutative as before and the vertical compositions are the identity and multiplication by U, respectively.

The vertical maps in the lower square are by definition induced by the tensor product of the underlying categories. By Lemma 8.5.8 the horizontal maps are also determined by their behaviour on objects. Let $[(X,\theta),\phi] \in \mathcal{F}_h(\widetilde{\mathbb{G}}_m, \widetilde{\mathbb{G}}_m)$ and $[(Y,\eta),\psi] \in {}_{(-1)^n}\mathcal{F}_h(R, \widetilde{\mathbb{G}}_m)$. We compute for the composition along the upper right corner:

$$[(X,\theta),\phi] \otimes [(Y,\eta),\psi] \mapsto [X, \phi\theta, \phi] \otimes [(Y,\eta),\psi] \mapsto [X \otimes (Y,\eta), \phi\theta \otimes \psi, \phi \otimes \psi] \tag{8.62}$$

And along the lower left corner:

$$[(X,\theta),\phi] \otimes [(Y,\eta),\psi] \mapsto [(X \otimes (Y,\eta), \theta \otimes 1), \phi \otimes \psi] \mapsto [X \otimes (Y,\eta), (\phi \otimes \psi) \circ (\theta \otimes 1), \phi \otimes \psi] \tag{8.63}$$

This concludes the proof. $\square$

Chapter 9

The spectral sequence and open questions

9.1 From towers to spectral sequences

In this section we quickly recall how a tower of topological spaces (spectra) can give rise to a spectral sequence.

Definition 9.1.1. A spectral sequence is a collection of tri-graded abelian groups $E_r^{p,q}$ together with differentials

$$d_r \colon E_r^{p,q} \to E_r^{p+r,q-r+1} \tag{9.1}$$

and isomorphisms

$$E_{r+1}^{*,*} \cong H(E_r^{*,*}, d_r). \tag{9.2}$$

Here the index r ranges over positive integers and p, q over all integers.

Definition 9.1.2. An exact couple is a pair of bi-graded abelian groups (C, A) together with three maps f, g and h, such that the following diagram is commutative and exact.

$$\begin{tikzcd} A \arrow[rr, "f"] & & A \arrow[dl, "g"] \\ & C \arrow[ul, "h"] & \end{tikzcd} \tag{9.3}$$

Let (C, A) be an exact couple. Define $d = g \circ h \colon C \to C$. The composition $d^2 = g \circ h \circ g \circ h$ is trivial, since the couple is defined to be exact. We can hence form the quotient

$$C' = \mathrm{Ker}(d)/Im(d). \tag{9.4}$$

The map $h \colon C \to A$ induces a map $h' \colon C' \to Im(f) := A'$ by $[x] \mapsto h(x)$. Two representatives x and x' of the class $[x]$ differ by an element $d(y)$ for some $y \in C$. Since we have $Im(d) = Im(gh) \subseteq Im(g) = \mathrm{Ker}(h)$, h' is well defined. Moreover, any representative x is in the kernel of d. In particular, $h(x) \in \mathrm{Ker}(g) = Im(f)$.
Analogously, the map $g \colon A \to C$ induces a map $g' \colon A' \to C'$. This is a bit more complicated. Let $x \in A'$, then we may pick a preimage y with $f(y) = x$. The image $g'(x)$ is then defined to be $[g(y)] \in C'$. If y' is another preimage of x, then $y - y' \in \mathrm{Ker}(f) = Im(h)$ and thus $g(y-y') \in Im(gh) = Im(d)$. Hence, the definition of g' is independent of the choice of y. Finally $f \colon A \to A$ induces a map $f' \colon A' \to A'$. Here, we simply define $f' = f|_{A'}$.

It is straightforward to check that the diagram

$$\begin{tikzcd} A' \arrow[rr, "f'"] & & A' \arrow[dl, "g'"] \\ & C' \arrow[ul, "h'"] & \end{tikzcd} \tag{9.5}$$

is again exact and commutative.

Definition 9.1.3. Let (C, A) be an exact couple. The exact couple $(C', A') =: (C^{(1)}, A^{(1)})$ is called the derived couple of (C, A). Inductively, we define the r-th derived couple by $(C^{(r)}, A^{(r)}) := ((C^{(r-1)})', (A^{(r-1)})')$ for $r \in \mathbb{N}$.

Definition 9.1.4. Let (C, A) be an exact couple. We define a spectral sequence $(E_r^{p,q}, d_r)$ by setting $E_r = C^{(r)}$ and $d_r = g^{(r)} \circ h^{(r)} \colon E_r \to E_r$. The condition $E_{r+1}^{*,*} \cong H(E_r^{*,*}, d_r)$ is then immediate from the construction.

Definition 9.1.5. A tower of topological spaces (spectra) is a sequence of spaces (spectra) W^j for $j \in \mathbb{N}_0$ together with maps $f^j \colon W^{j+1} \to W^j$ for all j. If further there exist spaces B^j such that there are homotopy fibration sequences

$$W^{j+1} \xrightarrow{f^j} W^j \to B^j \tag{9.6}$$

then the B^j are considered part of the data. The tower can then be depicted as

$$\begin{tikzcd} \cdots \arrow[r] & W^2 \arrow[r] \arrow[d] & W^1 \arrow[r] \arrow[d] & W^0 \arrow[d] \\ & B^2 & B^1 & B^0 \end{tikzcd} \tag{9.7}$$

Proposition 9.1.6. *Given a tower of topological spaces as in Definition 9.1.5, then the assignment $A_{p,q} = \pi_p(W^q)$ and $C_{p,q} = \pi_p(B^q)$ yields an exact couple (C, A) and therefore a spectral sequence, the so-called homotopy spectral sequence of the tower. This spectral sequence converges to $\pi_p(W^0)$.*

Proof. For a discussion of convergence of spectral sequences see [Boa99, Theorem 6.1]. The reason for convergence here is that the tower is bounded from above. □

9.2 The spectral sequence for K-theory

By the first part of Theorem 6.1.1 there is a tower as in Definition 9.1.5, with

$$W^t = \Omega^{-t} K(R\Delta^*, \mathbb{G}_m^{\wedge t}) \tag{9.8}$$

and

$$B^t = \Omega^{-t}|d \mapsto K_0(R\Delta^d, \mathbb{G}_m^{\wedge t})|. \tag{9.9}$$

Theorem 9.2.1 ([Sus03, Theorem 6.1])**.** *Let $X = \operatorname{Spec} R$ be a smooth semilocal scheme, essentially of finite type over a field F. Then there exists an isomorphism*

$$\pi_{t-s}|d \mapsto K_0(R\Delta^d, \mathbb{G}_m^{\wedge t})| \cong H^s(X, \mathbb{Z}(t)) \tag{9.10}$$

After a suitable index shift we conclude

Corollary 9.2.2. *Let $X = \operatorname{Spec} R$ be a smooth semilocal scheme, essentially of finite type over a field F. Then there exists a spectral sequence of the form*

$$E_2^{p,q} = H^{p-q}(X, \mathbb{Z}(-q)) \Rightarrow K_{-p-q}(R). \tag{9.11}$$

Theorem 9.2.3 ([MVW06, Theorem 5.1])**.** *Let k be a field. Then there exists an isomorphism*

$$E_2^{0,-p} = H^p(k, \mathbb{Z}(p)) \cong K_p^M(k) \tag{9.12}$$

9.3 The spectral sequence for hermitian K-theory

9.3.1 Even shifts

By the second part of Theorem 6.1.1 there is a tower as in Definition 9.1.5, with

$$W^t = \Omega^{-t} GW^{[2n]}(R\Delta^*, \mathbb{G}_m^{\wedge t}) \tag{9.13}$$

and

$$B^t = \Omega^{-t}|d \mapsto GW_0^{[2n]}(R\Delta^d, \mathbb{G}_m^{\wedge t})|. \tag{9.14}$$

Corollary 9.3.1. *Let R be a regular ring with involution such that 2 is invertible. Then there exists a spectral sequence of the form*

$$_{[2n]}E_2^{p,q} = \pi_{-p}|d \mapsto GW_0^{[2n]}(R\Delta^{\cdot}, \mathbb{G}_m^{\wedge(-q)})| \Rightarrow GW_{-p-q}^{[2n]}(R) \tag{9.15}$$

9.3.2 Odd shifts

Most results in the section remain conjectural:

Question 1. *Does Theorem 8.5.9 generalise to form a full tower for hermitian K-theory in odd shifts and for all local rings with involution R such that 2 is invertible? In other words, are there homotopy fibration sequences of the form*

$$\Omega^{-1}GW^{[2n-k]}(R\Delta^{\cdot}, \widetilde{\mathbb{G}}_m^{\wedge k+1}) \to GW^{[2n+1-k]}(R\Delta^{\cdot}, \widetilde{\mathbb{G}}_m^{\wedge k}) \to GW_0^{[2n+1-k]}(R\Delta^{\cdot}, \widetilde{\mathbb{G}}_m^{\wedge k})? \tag{9.16}$$

As a consequence there would be a tower as in Definition 9.1.5, with

$$W^t = \Omega^{-t}GW^{[2n+1-t]}(R\Delta^*, \widetilde{\mathbb{G}}_m^{\wedge t}) \tag{9.17}$$

and

$$B^t = \Omega^{-t}|d \mapsto GW_0^{[2n-t]}(R\Delta^d, \widetilde{\mathbb{G}}_m^{\wedge t})|. \tag{9.18}$$

and further, the answer to the following question would be affirmative.

Question 2. *Is there a spectral sequence of the form*

$$_{[2n+1]}E_2^{p,q} = \pi_{-p}|d \mapsto GW_0^{[2n+1+q]}(R\Delta^{\cdot}, \widetilde{\mathbb{G}}_m^{\wedge(-q)})| \Rightarrow GW_{-p-q}^{[2n+1]}(R)? \tag{9.19}$$

9.3.3 Connection to Milnor-Witt K-theory

Question 3. *Are there isomorphisms*

$$K_p^{MW}(k) \cong {}_{[p]}E_2^{0,-p} = \begin{cases} \pi_0|d \mapsto GW_0^{[p]}(k\Delta^{\cdot}, \mathbb{G}_m^{\wedge p})| & \textit{if } p \textit{ even} \\ \pi_0|d \mapsto GW_0^{[0]}(k\Delta^{\cdot}, \widetilde{\mathbb{G}}_m^{\wedge p})| & \textit{if } p \textit{ odd} \end{cases}? \tag{9.20}$$

We conclude with a discussion of ${}_{[p]}E_2^{0,-p}$ for small p. To this end recall first that there is a natural homomorphism

$$K_*^{MW}(k) \to GW_*^{[*]}(k) \tag{9.21}$$

that is an isomorphism in degree 0 and 1 by construction, in degree 2 by a result of Suslin [Sus87, §6], and in degree 3 by a result of Asok and Fasel [AF14]. The second reference also contains an explicit construction of this map. Furthermore we know, for example by [Sch12, Corollary A2], that there are isomorphisms

$$GW_*^{[2]}(R) \cong KSp_*(R) \tag{9.22}$$

In degree 2, the latter group can be given explicitly as the universal central extension of the elementary symplectic group

$$0 \to KSp_2(R) \to St^{Sp}(R) \to E^{Sp}(R) \to 0 \tag{9.23}$$

(See [KM70, Theorem 1] for the statement, and the preceding sections in the same source for the notation.)

- $p = 0$: By definition we have

$$_{[0]}E_2^{0,0} = GW_0^{[0]}(k) \cong K_0^{MW}(k) \tag{9.24}$$

- $p = 1$: By the result of Section 8.5 and by Proposition 8.4.6 we observe

$$_{[1]}E_2^{0,-1} = \pi_0|d \mapsto GW_0^{[0]}(k\Delta^{\cdot}, \widetilde{\mathbb{G}}_m^{\wedge 1})| \cong \pi_0 S(R\Delta^{\cdot}) \cong GW_1^{[1]}(k) \cong K_1^{MW}(k) \tag{9.25}$$

- $p = 2$: We will construct a homomorphism

$$_{[2]}E_2^{0,-2} = \pi_0|d \mapsto GW_0^{[2]}(k\Delta^{\cdot}, \mathbb{G}_m^{\wedge 2})| \to KSp_2(k) \cong K_2^{MW}(k) \tag{9.26}$$

First, take any two commuting matrices X and Y in the elementary symplectic group $E^{Sp}(k)$. We may choose lifts x and y of X and Y, respectively, in the symplectic Steinberg group $St^{Sp}(R)$. Now the commutator $[x, y]$ represents an element in $KSp_2(k)$, and because the Steinberg group is central, this element does not depend on the lift. Moreover, it is easy to see that this construction is invariant under base-changes, and that a pair (X, Y) is lifted to the trivial element, whenever one of them is the identity matrix.

Now let (A, B) be two commuting symplectic matrices, representing an element of $\pi_0|d \mapsto GW_0^{[2]}(k\Delta^{\cdot}, \mathbb{G}_m^{\wedge 2})|$. The matrices

$$\begin{pmatrix} A & & \\ & A^{-1} & \\ & & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} B & & \\ & 1 & \\ & & B^{-1} \end{pmatrix} \tag{9.27}$$

are elementary and we can associate an element of $KSp_2(k)$ as above. It remains to show that this construction yields the same element for two pairs (A, B) and (A', B') that are polynomially homotopic, for then we have defined a well-defined group homomorphism:

$$\pi_0|d \mapsto GW_0^{[2]}(k\Delta^{\cdot}, \mathbb{G}_m^{\wedge 2})| \to KSp_2(k) \tag{9.28}$$

Hence assume that there is a pair $(A(T), B(T))$ of commuting matrices in $Sp(k[T])$ that specialises to (A, B) and (A', B') for $T = 0$ and $T = 1$, respectively. Now the pair $(A(T), B(T))$ can be lifted to an element of $KSp(k[T]) \cong GW_2^{[2]}(k[T])$. By homotopy invariance the two induced maps

$$GW_2^{[2]}(k[T]) \rightrightarrows GW_2^{[2]}(k) \tag{9.29}$$

coincide. We are done.

Bibliography

[AF14] Aravind Asok and Jean Fasel, *An explicit KO-degree map and applications*, arXiv:1403.4588[math.KT] (2014), 35.

[Bal00] Paul Balmer, *Triangular Witt groups. I. The 12-term localization exact sequence*, K-Theory **19** (2000), no. 4, 311–363. MR 1763933 (2002h:19002)

[Bal01] ______, *Triangular Witt groups. II. From usual to derived*, Math. Z. **236** (2001), no. 2, 351–382. MR 1815833 (2002h:19003)

[BF78] A. K. Bousfield and E. M. Friedlander, *Homotopy theory of Γ-spaces, spectra, and bisimplicial sets*, Geometric applications of homotopy theory (Proc. Conf., Evanston, Ill., 1977), II, Lecture Notes in Math., vol. 658, Springer, Berlin, 1978, pp. 80–130. MR 513569 (80e:55021)

[BL08] Jean Barge and Jean Lannes, *Suites de Sturm, indice de Maslov et périodicité de Bott*, Progress in Mathematics, vol. 267, Birkhäuser Verlag, Basel, 2008. MR 2431916 (2010d:19004)

[Boa99] J. Michael Boardman, *Conditionally convergent spectral sequences*, Homotopy invariant algebraic structures (Baltimore, MD, 1998), Contemp. Math., vol. 239, Amer. Math. Soc., Providence, RI, 1999, pp. 49–84. MR 1718076 (2000m:55024)

[CCMT84] J. Caruso, F. R. Cohen, J. P. May, and L. R. Taylor, *James maps, Segal maps, and the Kahn-Priddy theorem*, Trans. Amer. Math. Soc. **281** (1984), no. 1, 243–283. MR 719669 (86g:55007)

[CL86] Ruth Charney and Ronnie Lee, *On a theorem of Giffen*, Michigan Math. J. **33** (1986), no. 2, 169–186. MR 837575 (88e:18009)

[Cor94] Octavian Cornea, *Cone-length and Lusternik-Schnirelmann category*, Topology **33** (1994), no. 1, 95–111. MR 1259517 (95d:55008)

[GJ09] Paul G. Goerss and John F. Jardine, *Simplicial homotopy theory*, Modern Birkhäuser Classics, Birkhäuser Verlag, Basel, 2009, Reprint of the 1999 edition [MR1711612]. MR 2840650

[Gra76] Daniel Grayson, *Higher algebraic K-theory. II (after Daniel Quillen)*, Algebraic K-theory (Proc. Conf., Northwestern Univ., Evanston, Ill., 1976), Springer, Berlin, 1976, pp. 217–240. Lecture Notes in Math., Vol. 551. MR 0574096 (58 #28137)

[Gra92] Daniel R. Grayson, *Adams operations on higher K-theory*, K-Theory **6** (1992), no. 2, 97–111. MR 1187703 (94g:19005)

[Gra95] ———, *Weight filtrations via commuting automorphisms*, K-Theory **9** (1995), no. 2, 139–172. MR 1340843 (96h:19001)

[Gra05] ———, *The motivic spectral sequence*, Handbook of K-theory. Vol. 1, 2, Springer, Berlin, 2005, pp. 39–69. MR 2181820 (2006i:19005)

[HS04] Jens Hornbostel and Marco Schlichting, *Localization in Hermitian K-theory of rings*, J. London Math. Soc. (2) **70** (2004), no. 1, 77–124. MR 2064753 (2005b:19007)

[Kar73] Max Karoubi, *Périodicité de la K-théorie hermitienne*, Algebraic K-theory, III: Hermitian K-theory and geometric applications (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972), Springer, Berlin, 1973, pp. 301–411. Lecture Notes in Math., Vol. 343. MR 0382400 (52 #3284)

[Kar78] ———, *K-theory*, Springer-Verlag, Berlin-New York, 1978, An introduction, Grundlehren der Mathematischen Wissenschaften, Band 226. MR 0488029 (58 #7605)

[KM70] I.S. Klein and A.V. Mikhalev, *Steinberg orthogonal group over a ring with involution*, Algebra and Logic **9** (1970), no. 2, 88–103 (English).

[Kne77] Manfred Knebusch, *Symmetric bilinear forms over algebraic varieties*, Conference on Quadratic Forms—1976 (Proc. Conf., Queen's Univ., Kingston, Ont., 1976), Queen's Univ., Kingston, Ont., 1977, pp. 103–283. Queen's Papers in Pure and Appl. Math., No. 46. MR 0498378 (58 #16506)

[May74] J. P. May, *E_∞ spaces, group completions, and permutative categories*, New developments in topology (Proc. Sympos. Algebraic Topology, Oxford, 1972), Cambridge Univ. Press, London, 1974, pp. 61–93. London Math. Soc. Lecture Note Ser., No. 11. MR 0339152 (49 #3915)

[MVW06] Carlo Mazza, Vladimir Voevodsky, and Charles Weibel, *Lecture notes on motivic cohomology*, Clay Mathematics Monographs, vol. 2, American Mathematical Society, Providence, RI, 2006. MR 2242284 (2007e:14035)

[Oja84] Manuel Ojanguren, *On Karoubi's theorem:* $W(A) = W(A[t])$, Arch. Math. (Basel) **43** (1984), no. 4, 328–331. MR 802307 (87i:18010)

[Qui73] Daniel Quillen, *Higher algebraic K-theory. I*, Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972), Springer, Berlin, 1973, pp. 85–147. Lecture Notes in Math., Vol. 341. MR 0338129 (49 #2895)

[Sch04] Marco Schlichting, *Hermitian K-theory on a theorem of Giffen*, K-Theory **32** (2004), no. 3, 253–267. MR 2114168 (2005m:19004)

[Sch10] ______, *The Mayer-Vietoris principle for Grothendieck-Witt groups of schemes*, Invent. Math. **179** (2010), no. 2, 349–433. MR 2570120 (2011m:19005)

[Sch12] ______, *Hermitian K-theory, derived equivalences and Karoubi's fundamental theorem*, arXiv:1209.0848v1 [math.KT.] (2012), 110.

[Seg74] Graeme Segal, *Categories and cohomology theories*, Topology **13** (1974), 293–312. MR 0353298 (50 #5782)

[Sri96] V. Srinivas, *Algebraic K-theory*, second ed., Progress in Mathematics, vol. 90, Birkhäuser Boston Inc., Boston, MA, 1996. MR 1382659 (97c:19001)

[ST14] Marco Schlichting and Girja S. Tripathi, *Geometric models for higher Grothendieck–Witt groups in $\mathbb{A}^1$-homotopy theory*, Mathematische Annalen (2014), 1–25 (English).

[Sus87] A. A. Suslin, *Torsion in K_2 of fields*, K-Theory **1** (1987), no. 1, 5–29. MR 899915 (89a:11123)

[Sus03] Andrei Suslin, *On the Grayson spectral sequence*, Tr. Mat. Inst. Steklova **241** (2003), no. Teor. Chisel, Algebra i Algebr. Geom., 218–253. MR 2024054 (2005g:14043)

[Tho79] Robert W. Thomason, *Homotopy colimits in the category of small categories*, Math. Proc. Cambridge Philos. Soc. **85** (1979), no. 1, 91–109. MR 510404 (80b:18015)

[Tho82] ———, *First quadrant spectral sequences in algebraic K-theory via homotopy colimits*, Comm. Algebra **10** (1982), no. 15, 1589–1668. MR 668580 (83k:18006)

[Uri90] M. Uridia, *U-theory of exact categories*, K-theory and homological algebra (Tbilisi, 1987–88), Lecture Notes in Math., vol. 1437, Springer, Berlin, 1990, pp. 303–313. MR 1079971 (92e:19001)

[Wal82] Friedhelm Waldhausen, *Algebraic K-theory of spaces, a manifold approach*, Current trends in algebraic topology, Part 1 (London, Ont., 1981), CMS Conf. Proc., vol. 2, Amer. Math. Soc., Providence, R.I., 1982, pp. 141–184. MR 686115 (84f:18025)

[Wal85] ———, *Algebraic K-theory of spaces*, Algebraic and geometric topology (New Brunswick, N.J., 1983), Lecture Notes in Math., vol. 1126, Springer, Berlin, 1985, pp. 318–419. MR 802796 (86m:18011)

[Wal03] Charles Walter, *Grothendieck-Witt groups of triangulated categories*, Preprint, July 1, 2003, K-theory Preprint Archives, http://www.math.uiuc.edu/K-theory/0643/, 2003.

[Wei13] Charles A. Weibel, *The K-book*, Graduate Studies in Mathematics, vol. 145, American Mathematical Society, Providence, RI, 2013, An introduction to algebraic K-theory. MR 3076731

[Whi78] George W. Whitehead, *Elements of homotopy theory*, Graduate Texts in Mathematics, vol. 61, Springer-Verlag, New York-Berlin, 1978. MR 516508 (80b:55001)