SOME THEOREMS IN COMBINATORIAL TOPOLOGY

by

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CONTENT

This thesis consists of four papers:

(1) ON THE FUNDAMENTAL GROUP OF AN ORBIT SPACE

(2) TRANSVERSALITY FOR PIECEWISE LINEAR MANIFOLDS

(3) TRANSVERSALITY FOR POLYHEDRA

(4) EXTENDING TRIANGULATIONS

(1) was accepted for the degree of Master of Science at the University of Birmingham, December 1964.

(2) is the result of joint work with Professor E.C. Zeeman. It must be emphasised that the notion of a "transimplicial map", which has borne fruit in both (2) and (3), is due entirely to Zeeman.

At the beginning of each paper there is an introduction which contains a survey of the problems to be tackled, and statements of the main results obtained.

Each paper is presented in a form suitable for publication.

The material in this thesis is original, exept where otherwise indicated by reference to the work of others.

M. A. Armstrong

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On the fundamental group of an orbit space

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0. *Introduction.* Let K be a connected simplicial complex, finite or infinite, its polyhedron ((2), page 45) being the space X. Then X is connected. Suppose further that X is simply connected. For any group G of simplicial transformations of X, H will denote the normal subgroup generated by elements which have a non-empty fixed-point set.

The aim of this paper is to show that the fundamental group of the orbit space X/G is isomorphic to the factor group G/H. Path-connectedness of the orbit space is of course ensured by the path-connectedness of X, as a connected polyhedron, and the continuity of the natural mapping from X to X/G.

As a particular example of this situation, consider a Fuchsian group Γ acting on the upper half-plane U in such a way that U/Γ is compact, as in (3). A fundamental region for Γ may be obtained in the form of a convex non-Euclidean polygon with a finite number of sides and all its vertices in U, consequently U may be triangulated in such a way that Γ acts simplicially. Further, if Γ has orbit genus g then it is defined by generators

$$x_1, x_2, \ldots, x_r,$$

$$a_1, b_1, \ldots, a_g, b_g,$$

and the relations

$$x_i^{m_i} = 1 \quad (i = 1, \ldots, r),$$

$$x_1 x_2 \ldots x_r \prod_{i=1}^{g} (a_i b_i a_i^{-1} b_i^{-1}) = 1.$$

The generators $x_1, \ldots, x_r$ are elliptic transformations and each leaves fixed exactly one point of U, whereas the a_j, b_j $(j = 1, \ldots, g)$ as hyperbolic elements cannot have any fixed points in U. Let Γ_1 be the normal subgroup of Γ generated by the elements $x_1, \ldots, x_r$; then viewing U/Γ as a closed surface S exhibits the classical result $\pi_1(S) \cong \Gamma/\Gamma_1$.

In section 1 we derive necessary and sufficient conditions for the action of G on X to 'induce' a triangulation of X/G, and show further that these conditions are always satisfied after at most two barycentric subdivisions of K. Throughout sections 2 and 3 K is assumed subdivided in accordance with the requirements of section 1, allowing the use of an edge-path lifting procedure to establish an isomorphism between G/H and the 'edge group' of the induced triangulation.

Related problems have been considered by Fox in (1).

I am grateful to Prof. A. M. Macbeath for providing the motivation for this work, and for many helpful suggestions.

Notation. The natural mapping or projection from $X = |K|$ to X/G will be denoted by p, for $x \in X$ the point $p(x)$ of X/G being written $\hat{x}$. Letters a, b, c, d, v will be reserved for vertices (or 0-simplexes), the occurrence of the letter e always denoting a 1-simplex. The stabilizer of $x \in X$, i.e. $\{g \in G: g(x) = x\}$, will be written stab (x).

1. *Induced triangulation of the orbit space.* Ability to triangulate the orbit space X/G in such a way that p becomes a dimension-preserving simplicial mapping will lead to great simplification in the proof of our main result. In this section we investigate the conditions which must be imposed on the action of G on X in order to make possible such a triangulation.

By means of the projection $p: X \to X/G$ we may define an abstract complex K/G as follows:

(i) The vertices of K/G are the orbits (projections) of the vertices of K.

(ii) The orbits $\hat{a}_0, \ldots, \hat{a}_k$ span a simplex of K/G if and only if there exist vertices $a_0, \ldots, a_k$ of K, with $p(a_i) = \hat{a}_i$ for $0 \leqslant i \leqslant k$, spanning a simplex of K.

There is a natural mapping $p_1: X \to |K/G|$; suppose $x \in X$ with carrier $(a_0 \ldots a_k)$; represent x as $\lambda_0 a_0 + \ldots + \lambda_k a_k$ where $0 < \lambda_i \leqslant 1$ for $0 \leqslant i \leqslant k$, and $\sum_{i=0}^{k} \lambda_i = 1$; then p_1 maps x to the point $\lambda_0 p(a_0) + \ldots + \lambda_k p(a_k)$ of $|K/G|$. Moreover, for any $g \in G$, $x \in X$, we have $p_1(g(x)) = p_1(x)$, so that p_1 induces a map

$$\psi: X/G \to |K/G|$$

for which, given $\hat{x} \in X/G$ and any point $x \in p^{-1}(\hat{x})$, $\psi(\hat{x}) = p_1(x)$.

The situation is best represented diagrammatically

$$\begin{tikzcd} X = |K| \arrow[d, "p"'] \arrow[dr, "p_1"] & \\ X/G \arrow[r, "\psi"'] & |K/G| \end{tikzcd}$$

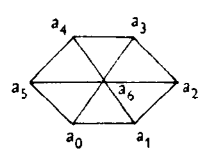
This map ψ is obviously onto, and since our projection map p is open will be continuous if and only if p_1 is continuous. However, p_1 was constructed as a simplicial map from K to K/G, ensuring its continuity.

If ψ was a homeomorphism we could regard the spaces X/G, $|K/G|$ and the maps p, p_1 as essentially the same. In this case the action of G on X will be said to *induce*, via the triangulation K of X, a triangulation K/G of X/G.

In general ψ need not be 1–1.

Example 1. Take K to be a 2-simplex, vertices a_0, a_1, a_2, together with all its faces. Let G be the cyclic group Z_3 generated by the permutation $(a_0 a_1 a_2)$ of the vertices of K, then the elements of G determine simplicial maps $|K| \to |K|$ by linearity inside each simplex. The space X/G is homeomorphic to a disc, whereas $|K/G|$ consists only of a single point and therefore ψ is not 1–1.

Example 2. Suppose K is the 2-dimensional complex illustrated and G the cyclic group Z_3 generated by the permutation $(a_0 a_2 a_4)(a_1 a_3 a_5)$ viewed as a rotation of K which fixes a_6.



Now to say that ψ is 1–1 means, for $\hat{x}, \hat{y} \in X/G$, that $\psi(\hat{x}) = \psi(\hat{y})$ implies $\hat{x} = \hat{y}$, or equivalently that for $x, y \in X p_1(x) = p_1(y)$ shall imply $p(x) = p(y)$. In this example ψ cannot possibly be 1–1, for consider the points

$$\left.\begin{aligned} x &= \lambda a_0 + (1-\lambda) a_1 \\ y &= (1-\lambda) a_1 + \lambda a_2 \end{aligned}\right\} \quad (0 < \lambda < 1).$$

Then x and y cannot be in the same orbit, but

$$p_1(x) = \lambda p(a_0) + (1-\lambda) p(a_1) = (1-\lambda) p(a_1) + \lambda p(a_2) = p_1(y).$$

These two simple examples point respectively to the following two conditions, which will be shown to be separately necessary and together sufficient for ψ to be a homeomorphism.

CONDITION 1. *Given any* 1-*simplex* $(a_i a_j)$ *of* K *no element of* G *is allowed to map* a_i *on* a_j.

Necessity. Suppose $g \in G$ with $g(a_i) = a_j$ then $p(a_i) = p(a_j)$. Consider any point $x = \lambda a_i + (1-\lambda) a_j$ where $0 < \lambda < 1$, then

$$p_1(x) = \lambda p(a_i) + (1-\lambda) p(a_j) = p(a_i) = p_1(a_i).$$

However, since g is a simplicial transformation, we cannot have a_i, a vertex, and x, a non-vertex, in the same orbit. Thus $p(x) \neq p(a_i)$, and ψ cannot be 1–1.

CONDITION 2. *Given two simplexes* $(a_0 \dots a_k b)$ *and* $(a_0 \dots a_k c)$ *of* K *with* b *and* c *in the same orbit, then there exists an element* $g^* \in G$ *such that*

$$\begin{aligned} g^*(a_i) &= a_i \quad (0 \leqslant i \leqslant k), \\ g^*(b) &= c. \end{aligned}$$

Necessity. Suppose ψ is a homeomorphism and $g(b) = c$. Consider points

$$x = \sum_{i=0}^{k} \lambda_i a_i + \mu b, \quad y = \sum_{i=0}^{k} \lambda_i a_i + \mu c,$$

where $0 < \lambda_i$, $\mu < 1$ and $\Sigma \lambda_i + \mu = 1$. Then

$$p_1(x) = \Sigma \lambda_i p(a_i) + \mu p(b) = \Sigma \lambda_i p(a_i) + \mu p(c) = p_1(y).$$

But ψ is 1–1, therefore we must have $p(x) = p(y)$, showing the existence of $g^* \in G$ with $g^*(x) = y$. Remembering that, since ψ is assumed to be a homeomorphism, Condition 1 is satisfied, then clearly g^* does all that is required.

THEOREM 1. *If Conditions* 1 *and* 2 *are satisfied then* ψ *is a homeomorphism.*

Proof. (*a*) ψ is 1–1.

Consider any two points $x, y \in X$, say

$$x = \lambda_0 a_0 + \dots + \lambda_k a_k, \quad y = \mu_0 b_0 + \dots + \mu_l b_l,$$

where $0 < \lambda_i, \mu_i \leqslant 1$ and $\displaystyle\sum_{i=0}^{k} \lambda_i = \sum_{j=0}^{t} \mu_j = 1.$

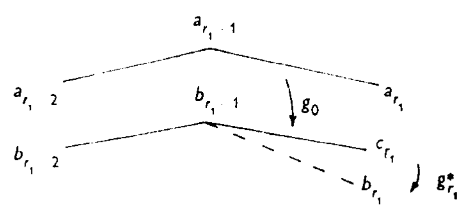
Suppose $p_1(x) = p_1(y)$; then $\Sigma\lambda_i p(a_i) = \Sigma\mu_j p(b_j)$. Condition 1 ensures that

$$\left.\begin{matrix} p(a_i) \neq p(a_j) \\ p(b_i) \neq p(b_j) \end{matrix}\right\} \quad (i \neq j);$$

therefore $t = k$, and we may re-order, if necessary, so that $\mu_i = \lambda_i$, $p(a_i) = p(b_i)$ for $0 \leqslant i \leqslant k$. In view of this there exist in G elements g_i satisfying $g_i(a_i) = b_i$ for $0 \leqslant i \leqslant k$. Consider the sequences of vertices $a_0, \ldots, a_k$; $b_0, \ldots, b_k$. If $g_0(a_i) = b_i$ for $0 \leqslant i \leqslant k$ there is nothing to prove since then $g_0(x) = y$. Otherwise there exists a first integer r_1 such that $g_0(a_{r_1}) \neq b_{r_1}$; let $g_0(a_{r_1}) = c_{r_1}$. Now $g_{r_1}(a_{r_1}) = b_{r_1}$; therefore a_{r_1}, b_{r_1} and c_{r_1} are all in the same G-orbit so that there exists $g \in G$ with $g(c_{r_1}) = b_{r_1}$. Applying Condition 2 to the simplexes $(b_0 \ldots b_{r_1-1} b_{r_1})$, $(b_0 \ldots b_{r_1-1} c_{r_1})$, we have the existence of an element $g^*_{r_1} \in G$ which satisfies

$$g^*_{r_1}(b_i) = b_i \quad (0 \leqslant i \leqslant r_1 - 1),$$
$$g^*_{r_1}(c_{r_1}) = b_{r_1}.$$

Therefore $g^*_{r_1} g_0(a_i) = b_i$ for $0 \leqslant i \leqslant r_2 - 1$, $r_1 < r_2$.



If now $r_2 - 1 = k$ the proof is complete, otherwise we repeat the above argument. Clearly the process terminates after at most k steps, and by successively applying Condition 2 at each step we provide an element $g \in G$ such that $g(a_i) = b_i$ for $0 \leqslant i \leqslant k$, i.e. $g(x) = y$, giving $p(x) = p(y)$ as required.

(*b*) ψ is an open mapping.

We have already seen that ψ is onto, continuous and 1–1, giving $p_1\colon |K| \to |K/G|$ to be a dimension-preserving simplicial mapping. Hence the restriction of p_1 to any simplex, indeed to the closure of any simplex, of K is a homeomorphism. This means that ψ^{-1} restricted to the closure of any simplex of K/G is continuous, and reference to the two well-known Propositions 1·3·3, 1·10·4 of (2) for the finite and infinite cases respectively shows ψ^{-1} to be continuous and hence completes the argument.

The group G is equally well a group of simplicial transformations of $K^{(r)}$, the rth derived complex of K.

THEOREM 2. *The action of G on $K^{(2)}$ will always satisfy Conditions* 1 *and* 2.

Proof. (*a*) We show, first, that the action of G on $K^{(1)}$ always satisfies Condition 1 (when it must certainly satisfy this condition 'on' $K^{(2)}$). For suppose (ab) is a 1-simplex of $K^{(1)}$; then a, b are the barycentres of simplexes σ_a, σ_b of K and we order so that σ_b is a face of σ_a, written $\sigma_b < \sigma_a$. Then given $g \in G$, since it is simplicial and preserves dimension, it cannot possibly map a to b.

(*b*) It remains only to deal with Condition 2. In view of the above we know that Condition 1 is satisfied on $K^{(1)}$, ensuring that no element of G permutes the vertices of a simplex of $K^{(1)}$. Given two simplexes $(a_0 \ldots a_k b)$, $(a_0 \ldots a_k c)$ of $K^{(2)}$ regard $a_0, \ldots, a_k, b, c$ as the barycentres of simplexes $\sigma_0, \ldots, \sigma_k, \sigma_b, \sigma_c$ of $K^{(1)}$. Then we may re-order, if necessary, so that

$$\sigma_0 < \sigma_1 < \ldots < \sigma_i < \sigma_b < \sigma_{i+1} < \ldots < \sigma_k,$$

$$\sigma_0 < \sigma_1 < \ldots < \sigma_j < \sigma_c < \sigma_{j+1} < \ldots < \sigma_k.$$

Suppose then that b and c are in the same orbit, with $g(b) = c$, when certainly $i = j$. If now b and c are not the leading vertices of the two given simplexes, σ_b and σ_c are both faces of σ_k; but since g is simplicial on $K^{(1)}$, $g(\sigma_b) = \sigma_c$ implying a permutation of the vertices of σ_k and contradicting our previous remark. In the case where b and c are the leading vertices we have

$$\sigma_0 < \ldots < \sigma_k < \sigma_b,$$

$$\sigma_0 < \ldots < \sigma_k < \sigma_c.$$

But $g(\sigma_b) = \sigma_c$ and $\sigma_k < \sigma_b$ together imply $g(\sigma_k) < \sigma_c$, then $\sigma_k < \sigma_c$ and Condition 1 on $K^{(1)}$ must give g acting as the identity on σ_k, and hence as the identity on σ_i for $0 \leqslant i \leqslant k$. Therefore $g(a_i) = a_i$ for $0 \leqslant i \leqslant k$ and g itself has the properties of the required g^*.

Theorems 1 and 2 allow us to assume from now on an induced triangulation K/G of X/G.

2. *Edge-paths and their lifting properties.* An edge-path in a complex L is defined as a sequence of vertices $a_{i_0}, a_{i_1}, \ldots, a_{i_r}$, with $r \geqslant 0$, in which each successive pair span a 1-simplex of L. Given subsets S, $\hat{S}$ of X, X/G with $p(S) = \hat{S}$ we say that S lifts $\hat{S}$.

For reasons which will become apparent in the next section we are interested in lifting edge-paths in K/G to edge-paths in K; all results derived here will be given as Lemmas.

LEMMA 1. *Any edge-path in K/G with initial vertex $\hat{a}_0$ may be lifted to an edge-path in K which begins at any point of $p^{-1}(\hat{a}_0)$.*

Proof. Consider in K/G any 1-simplex $\hat{e}$ with vertices $\hat{a}$ and $\hat{b}$; we first show that this can be lifted to a 1-simplex e in K having as a vertex any specified vertex $a \in p^{-1}(\hat{a})$. Certainly $\hat{e}$ may be lifted to some 1-simplex e_0 in K which has as a vertex some point $a_0 \in p^{-1}(\hat{a})$. Then there exists $g \in G$ with $g(a_0) = a$; but g is a dimension-preserving simplicial mapping and therefore maps e_0 to a 1-simplex e which has a as a vertex; then clearly $p(e) = \hat{e}$.

Suppose now the given edge-path in K/G is $\hat{a}_0 \hat{a}_1 \ldots \hat{a}_k$, where $\hat{a}_{i-1} \hat{a}_i$ span the 1-simplex $\hat{e}_i$ $(1 \leqslant i \leqslant k)$. If $a_0 \in p^{-1}(\hat{a}_0)$ is to be the initial point of the lifted path, lift $\hat{e}_1$ to e_1 having a vertex at a_0; this determines the second vertex a_1 of e_1 and $a_1 \in p^{-1}(\hat{a}_1)$. Now lift $\hat{e}_2$ to e_2 'on' a_1; clearly this process may be continued and terminates after precisely k steps. The lifted path is seen to have the same number of vertices as the original.

This path-lifting procedure is not unique.

By an *admissible operation* on an edge-path $a_{i_0}a_{i_1}\dots a_{i_r}$ we mean its replacement by the edge-path:
$$a_{i_0}a_{i_1}\dots a_{i_n}h(a_{i_n+1})h(a_{i_n+2})\dots h(a_{i_r})$$
for some $0 \leqslant n \leqslant r$ and $h \in \operatorname{stab}(a_{i_n})$.

LEMMA 2. *If two edge-paths in K have common first vertex and project to the same edge-path in K/G, then one may be obtained from the other by a finite number of admissible operations.*

Proof. Let the two paths in K be
$$\left.\begin{aligned} E_1 &= a_0a_1^1\dots a_k^1 \\ E_2 &= a_0a_1^2\dots a_k^2\end{aligned}\right\} \quad (a_0 = a_0^1 = a_0^2), \quad \left.\begin{aligned} a_i^1a_{i+1}^1 &\quad \text{span} \quad e_i^1 \\ a_i^2a_{i+1}^2 &\quad \text{span} \quad e_i^2\end{aligned}\right\} \quad (0 \leqslant i \leqslant k-1),$$
with common projection
$$\hat{E} = \hat{a}_0\hat{a}_1\dots\hat{a}_k, \quad \text{where} \quad \hat{a}_i\hat{a}_{i+1} \quad \text{span} \quad \hat{e}_i \quad \text{for} \quad 0 \leqslant i \leqslant k-1.$$
Assuming E_1 and E_2 to be distinct there will be a first integer r_1 such that $a_{r_1+1}^1 \neq a_{r_1+1}^2$. Then $\hat{e}_{r_1}$ lifts to two distinct 1-simplexes $e_{r_1}^1$ and $e_{r_1}^2$ which have a common vertex; reference to Conditions 1 and 2 provides an element $h_{r_1} \in \operatorname{stab}(a_{r_1}^1)$ satisfying
$$h_{r_1}(a_{r_1+1}^1) = a_{r_1+1}^2.$$
It is admissible to replace E_1 by
$$E_{r_1} = a_0a_1^1\dots a_{r_1}^1h_{r_1}(a_{r_1+1}^1)\dots h_{r_1}(a_k^1).$$
If now $E_{r_1} = E_2$ the proof is complete; otherwise there will be a first integer $r_2 > r_1$ for which $h_{r_1}(a_{r_2+1}^1) \neq a_{r_2+1}^2$ when we repeat the above argument. In this way we obtain a finite sequence of paths
$$E_1, E_{r_1}, E_{r_2}, \dots, E_{r_n} = E_2 \quad (n < k),$$
each one being obtained from its predecessor by an admissible operation.

It is clear that if E_1 and E_2 are edge-paths in K which are obtainable from one another by a finite number of admissible operations then:

(*a*) $p(E_1) = p(E_2)$;

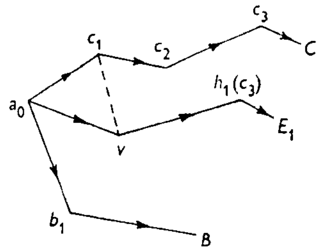
(*b*) there exists $h \in H$ which maps the end-point of E_1 to that of E_2.

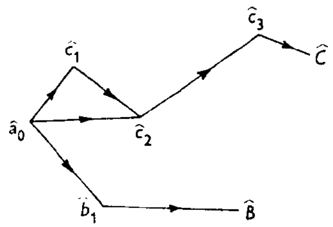
An *allowable operation* on edge-paths is defined as follows. If three consecutive vertices $a_ia_ja_k$ span a 2-simplex of a complex L, the triple may be replaced in an edge-path by the pair a_ia_k, or conversely the pair may be replaced by the triple. It is also allowable to replace a triple $a_ia_ja_i$ by the single vertex a_i, or conversely to alter a_i to a_iba_i providing a_i, b span a 1-simplex of L.

LEMMA 3. *Let $\hat{B} = \hat{a}_0\hat{b}_1\dots\hat{b}_{k-1}\hat{a}_k$, $\hat{C} = \hat{a}_0\hat{c}_1\dots\hat{c}_{l-1}\hat{a}_k$ be homotopic edge-paths in K/G which lift to paths $B = a_0b_1\dots b_k$, $C = a_0c_1\dots c_l$ in K. Then there exists an edge-path D in K which projects to $\hat{B}$, and an element $h \in H$ which maps c_l to the end-point of D.*

Proof. Since $\hat{B}$ is homotopic to $\hat{C}$, it may be obtained from $\hat{C}$ by a finite number N of allowable operations. Consider the first allowable operation on $\hat{C}$; we cannot always lift this to a corresponding allowable operation on C, for example, $\hat{c}_{i-1}\hat{c}_i\hat{c}_{i+1}$ may well span a 2-simplex in K/G without $c_{i-1}c_ic_{i+1}$ spanning one in K. However, we can at least lift our operation to an allowable operation on a path which is obtained from C by an admissible operation. There is no loss of generality in assuming that this first allowable operation involves $\hat{a}_0$, since if $\hat{B}$ and $\hat{C}$ coincide as far as $\hat{b}_r = \hat{c}_r$ we merely refer to Lemma 2 and define D to coincide with C as far as c_r.

(1) If the operation replaces $\hat{a}_0\hat{c}_1\hat{c}_2$ by $\hat{a}_0\hat{c}_2$, lift the 2-simplex $(\hat{a}_0\hat{c}_1\hat{c}_2)$ to a 2-simplex in K which has (a_0c_1) as a side, say (a_0c_1v). Clearly this is possible (refer to the corresponding argument for 1-simplexes given in proof of Lemma 1). Then Conditions 1 and 2 provide $h_1 \in \operatorname{stab}(c_1)$ which maps c_2 to v; an admissible operation on C gives the path $a_0c_1h_1(c_2)\ldots h_1(c_l)$, on which we operate allowably to obtain $a_0h_1(c_2)\ldots h_1(c_l)$.





Conversely if $\hat{a}_0\hat{c}_1$ is replaced by $\hat{a}_0\hat{v}\hat{c}_1$, lift the 2-simplex $(\hat{a}_0\hat{v}\hat{c}_1)$ to a 2-simplex in K which has (a_0c_1) as a side, say (a_0vc_1). Here we may directly operate allowably on C to give $a_0vc_1c_2\ldots c_l$.

(2) Suppose $\hat{c}_2 = \hat{a}_0$ and the operation replaces $\hat{a}_0\hat{c}_1\hat{c}_2$ by $\hat{a}_0$; then if $c_2 = a_0$ we may directly operate allowably on C to give $a_0c_3c_4\ldots c_l$. If $c_2 \neq a_0$ then Conditions 1 and 2 imply the existence of $h_1 \in \operatorname{stab}(c_1)$ with $h_1(c_2) = a_0$. An admissible operation on C gives $a_0c_1a_0h_1(c_3)\ldots h_1(c_l)$, on which we operate allowably to obtain $a_0h_1(c_3)\ldots h_1(c_l)$.

Conversely, and finally, if $\hat{a}_0$ is altered to $\hat{a}_0\hat{v}\hat{a}_0$, lift the 1-simplex $(\hat{a}_0\hat{v})$ to a 1-simplex in K which has a_0 as a vertex, say (a_0v). Operating allowably on C we 'lift' the initial operation to give $a_0va_0c_1c_2\ldots c_l$.

The allowable operations in K/G give us a sequence of paths $\hat{C}, \hat{E}_1, \ldots, \hat{E}_N = \hat{B}$. By repetition of the above process we obtain in K a lifted sequence

$$C = E_0, E_1, \ldots, E_N = D$$

together with group elements $h_1, \ldots, h_N$ of H (some of which may be the identity) where h_i maps the end-point of E_{i-1} to that of E_i for $1 \leqslant i \leqslant N$. Our construction ensures $p(D) = \hat{B}$, and the required group element $h \in H$ is simply $h_N \ldots h_2h_1$.

As a direct corollary of Lemmas 2 and 3 we have

Lemma 4. *With the same hypotheses as for Lemma* 3 *there exists* $h \in H$ *satisfying* $h(c_l) = b_k$.

3. *Main result.* We are now in a position to prove

THEOREM 3. $$\pi_1(X/G) \cong G/H.$$

Proof. As in the previous section we assume 'compatible' triangulations $K, K/G$ of $X, X/G$.

Take as base-point in X/G any vertex $\hat{a}$ of the induced triangulation; since X/G is path-connected this choice is arbitrary. Let $\pi_1(K/G, \hat{a})$ denote the edge-group of homotopy classes of edge loops on $\hat{a}$ and $\pi_1(X/G, \hat{a})$ denote the fundamental group of $X/G = |K/G|$ based at $\hat{a}$; then $\pi_1(K/G, \hat{a}) \cong \pi_1(X/G, \hat{a})$ ((2), page 237). In view of this we are able to restrict ourselves to looking at edge loops on $\hat{a}$. Choose a vertex $a \in p^{-1}(\hat{a})$ as base point in X; again the choice is arbitrary.

We set up a mapping $\phi: G \to \pi_1(K/G, \hat{a})$ as follows: given $g \in G$, join a to $g(a)$ by an edge-path E in K; then $p(E)$ is an edge loop on $\hat{a}$ in K/G. Define $\phi(g) = \{p(E)\}$, where $\{\ \}$ denotes the homotopy class of the edge loop under consideration. Since $X = |K|$ is simply connected any two edge-paths joining a to $g(a)$ must be homotopic and therefore, by the continuity of p, project to homotopic edge loops in X/G. Consequently the above definition is independent of the choice of E and ϕ is well defined.

Given an element $\alpha \in \pi_1(K/G, \hat{a})$, choose any representative edge loop; then by Lemma 1 this may be lifted to an edge-path in K beginning at a. The final point of this path must belong to the G-orbit of a and therefore there exists $g_0 \in G$ which maps this final point to a. Then $\phi(g_0) = \alpha$, showing ϕ to be *onto*.

The map ϕ is a *homomorphism*; let g_1, g_2 be any two elements of G and consider $\phi(g_2 g_1)$. Join a to $g_1(a)$, $g_2(a)$ by edge-paths E_1, E_2 respectively; then E_2 followed by $g_2(E_1)$ is an edge-path joining a to $g_2 g_1(a)$. By definition

$$\phi(g_2 g_1) = \{p(E_2 g_2(E_1))\} = \{p(E_2)\}.\{p(E_1)\} = \phi(g_2).\phi(g_1),$$

as required.

Finally, we show that the *kernel* of this homomorphism is H.

(a) $H \subseteq \ker\phi$. Any generator of H must fix a vertex of K, Condition 1 being satisfied for the action of G on K. If b is a vertex of K and $g(b) = b$, join a to b by an edge-path C, then the path consisting of C followed by $g(C)$ in reverse joins a to $g(a)$ and projects to a null-homotopic loop in K/G. Thus since the stabilizer of any vertex of K is contained in the kernel of ϕ, by our earlier remark we must have $H \subseteq \ker\phi$.

(b) $\ker\phi \subseteq H$. Suppose $\phi(k) = 1$, where $k \in G$ and 1 here denotes the unit of $\pi_1(K/G, \hat{a})$; join a to $k(a)$ by an edge-path C in K; then $\hat{C} = p(C)$ is a loop on $\hat{a}$ homotopic to the constant path $\hat{B}$ at $\hat{a}$. Applying Lemma 4 with B as the constant path at a, there exists $h \in H$ such that $h(k(a)) = a$; thus $hk \in \operatorname{stab}(a)$ and therefore $k \in H$. This completes the proof of Theorem 3.

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TRANSVERSALITY FOR PIECEWISE LINEAR MANIFOLDS

by M.A. ARMSTRONG and E.C. ZEEMAN

We prove three transversality theorems in the piecewise linear category. For the standard definitions and properties of this category see [12]. All maps considered will be piecewise linear, all manifolds compact, and all submanifolds locally flat (which is always the case for codimension ≥ 3 by [11]). We say M is a <u>proper</u> submanifold of Q if the boundary $\dot{M} \subset \dot{Q}$ and the interior $\mathring{M} \subset \mathring{Q}$.

The main result of this paper (Theorem 1) says that if M, P are proper submanifolds of Q then we can ambient isotop M until it is transversal to P.

Perhaps we should straightway point out some inherent difficulties. We do not assume that P has a normal bundle in Q (or, equivalently, a normal microbundle). As yet the existence of normal bundles in the piecewise linear category is an open question. Haefliger and Wall [5] have proved that normal bundles exist in the stable range, but Hirsch [6] has shown that normal disk bundles do not always exist in the unstable range, and this gives weight to the conjecture that normal bundles also may not always exist.

If P did have a normal bundle in Q, then one could slide M along the fibres until it was transversal. This essentially is the geometrical idea behind Thom's original transversality theorem [8] for smooth maps, and behind Williamson's extension [10] to piecewise linear maps.

However, we are interested in the case where P may not have a normal bundle, and therefore we do not assume anything about normal bundles. Also we are primarily interested in ambient isotoping embeddings to be transversal, rather than homotoping maps, although in Theorem 2 we do deduce a result about maps.

Given $M, P \subset Q$, if we want to isotop M transversal to P, then the following method of attack at once suggests itself. Choose a triangulation K of Q in which M and P appear as subcomplexes. Let K^* denote the dual cell complex of K, and attempt to isotop M into the m-skeleton of K^*. But this is not always possible, because if it were one could infer that M always had a normal disc bundle in Q contradicting Hirsch's result [6].

Therefore we cannot isotop M into the m-skeleton of K^*. Instead we have to isotop M step by step so as to be transversal to each simplex of K. In other words our proof is by bare hands - the subtlety lying in the interplay between the linear and the piecewise linear. If one uses only the piecewise linear structure, then one runs into a difficulty illustrated by the following example.

The folded disc : let D be a folded disc crossing an interval I in Euclidean 3-space (E^3) as shown in Figure 1.

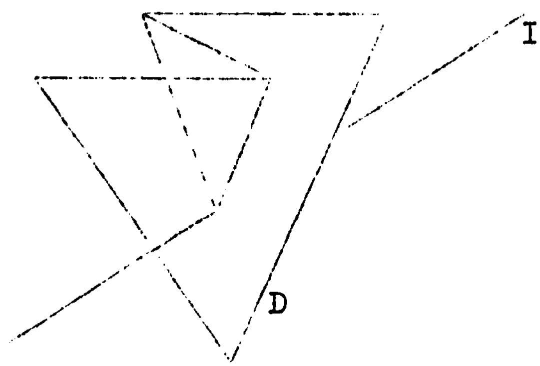


Figure 1

This picture is piecewise linearly homeomorphic to a standard linear disc in E^3 together with a perpendicular line through its centre, consequently D and I are transversal in E^3. If we now multiply by an extra dimension, we obtain $D \times I$ crossing $I \times I$ transversally in E^4. However, on tilting $I \times I$ upwards a little keeping $I \times 0$ fixed the transversality is destroyed, since the intersection of $D \times I$ with $I \times I$ becomes three concurrent lines and is no longer a manifold. With this example in mind it is easy to manufacture the following more disheartening situation. Let Δ^q be a q-simplex and S^{m-1}, S^{p-1} spheres crossing transversally in its boundary. Let D^m, D^p be discs formed by joining the spheres to two points in general position in the interior of Δ^q. Then D^m and D^p may cross transversally at all interior points, yet fail to be transversal at their boundaries.

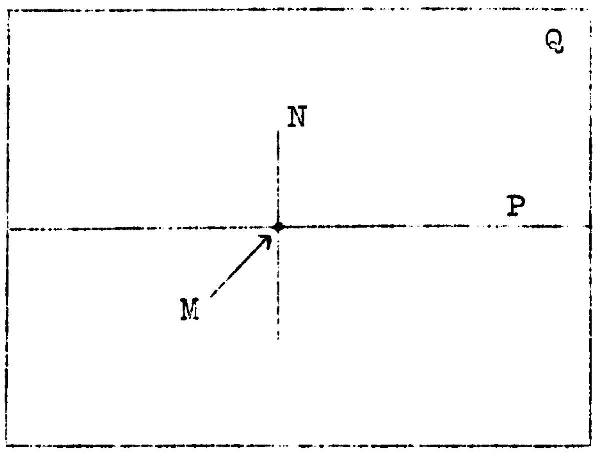
So as not to meet with this kind of difficulty in the inductive step of our proof, we shall introduce the notion of M being <u>transimplicial</u> to the triangulation K of the ambient manifold Q. Being transimplicial is roughly the opposite of being a subcomplex. It is not a piecewise linear invariant, but rather is a technical device introduced for the purposes of proof; it uses not only the piecewise linear structure but also the local linear structure of K, and consequently is a stronger property than transversality. With this extra structure we are able to produce (transimplicial) Theorems 4 and 5 that have our main (transversality) result, Theorem 1, as a corollary.

The same techniques are used in Theorem 2 to extend the result from embeddings to maps : any map $f:M \to Q$ is homotopic to a map g transversal to the submanifold P of Q, and the cobordism class of $g^{-1}P$ depends only on the homotopy class of f. It should be noted that in the analogous differential setting [8], the set of all transversal maps is open in the function space, whereas this is not true in piecewise linear theory (we have no derivatives to "control" local movement). This defect accounts for our more directly geometrical approach.

We should point out that although Theorem 5 is a relative transimplicial theorem, we have no corresponding relative transversality theorem. This omission is discussed at the end of the paper.

Our third main result, Theorem 3, can be thought of as an existence theorem for <u>quotient regular</u>

<u>neighbourhoods</u> (analogous to quotient vector bundles) - the inherent difficulty here being that in a regular neighbourhood there are no convenient fibres to play with. More precisely, given manifolds $M \subset P \subset Q$, we produce a fourth manifold N in Q that cuts P transversally along M.



<u>Figure 2</u>

At the end of the paper we show how this result can be used to construct induced regular neighbourhoods, and Whitney sums. However, we are unable to prove any uniqueness theorems for these constructions.

We should like to acknowledge an unpublished paper by V. Poenaru and one of us, which contained incomplete proofs of some of the results below.

Contents.

THE MAIN THEOREMS

Firstly we give a precise definition of what we mean by transversality. Let M, P be two proper submanifolds of the manifold Q. Denote by E^n n-dimensional Euclidean space and by E^n_+ the closed half space obtained by restricting the first coordinate to be non-negative.

Definition 1. The submanifolds M, P are transversal at the point $x \in \mathring{M} \cap \mathring{P}$ (respectively $\dot{M} \cap \dot{P}$) if there is a coordinate neighbourhood $h:E^q \to Q$ ($h:E^q_+ \to Q$) of x in Q such that $h^{-1}M$, $h^{-1}P$ are two linear subspaces of E^q (E^q_+) in general position.

M and P are transversal if they are transversal at all points of $M \cap P$.

It follows immediately that if M, P are transversal in Q, then $M \cap P$ is a proper submanifold of dimension $m + p - q$, which is locally flat in both M and P.

Theorem 1. If Q is a manifold with proper submanifolds M and P, then M can be ambient isotoped transversal to P by an arbitrarily small ambient isotopy of Q.

We want an analogous definition and theorem for maps. For simplicity we confine ourselves to closed manifolds, although there are similar results for bounded manifolds.

Definition 2. (i) Let M, P, Q be closed manifolds, with P a submanifold of Q. Let $f:M \to Q$ be an embedding; we say that the embedding f is transversal to P if fM and P are transversal as submanifolds.

(ii) Now suppose $f:M \to Q$ is an arbitrary piecewise linear map. We say that the map f is graph-transversal to P if its graph

$$\Gamma f:M \to M \times Q$$

is transversal to $M \times P$ as an embedding. Two properties follow at once.

A) If $f:M \to Q$ is an embedding that is transversal to P as an embedding, then it is graph-transversal to P as a map. In other words graph-transversality is a generalisation.

B) If $f:M \to Q$ is a map that is graph-transversal to P then $f^{-1}P$ is a locally-flat submanifold of M of codimension $q - p$. This is because the homeomorphism $\Gamma f:M \to (\Gamma f)M$ maps $f^{-1}P$ onto $(\Gamma f)M \cap (M \times P)$, which is a locally flat submanifold of dimension $m + (m + p) - (m + q)$ by the remark above.

<u>Theorem 2.</u> <u>Given closed manifolds M, P, Q with $P \subset Q$, and given a map $f: M \to Q$, then there exists an arbitrarily close homotopic map g that is graph-transversal to P. The inverse image $g^{-1}P$ is a locally flat submanifold of M of codimension $q - p$, and the cobordism class $\{g^{-1}P\}$ depends only on the homotopy class $[f]$.</u>

<u>Remark.</u> All our results in this paper concern manifolds; a subsequent paper by one of us will deal with polyhedra [2]. In particular a stronger definition of transversality for maps will be given in [2], and a strengthened version of Theorem 2 proved.

<u>Theorem 3.</u> <u>Given manifolds $M \subset P \subset Q$, both inclusions being proper, then there exists a fourth manifold N, contained in Q, that intersects P transversally in M.</u>

<u>Remark.</u> N will not be a proper submanifold of Q, because in general the boundary $\dot{N} \not\subset \dot{Q}$. However it will be proper in the neighbourhood of M, and so the definition of transversality of N and P makes sense.

We proceed now with the business of setting up sufficient machinery to prove Theorems 1, 2 and 3.

(p, q)-DISC FIBERINGS

The ideas introduced in this section will be of fundamental importance throughout the rest of the paper. Let X, Y, Z be polyhedra, and let D^n denote a standard n-dimensional disc with centre 0.

<u>Definition 3.</u> A map $g:Y \to Z$ will be said to be locally a q-disc fibering at $y \in Y$, or more briefly <u>F(q) at y</u>, if there exists a neighbourhood N of gy in Z and an embedding $\psi:N \times D^q \to Y$ onto a neighbourhood of y, such that the diagram

$$\begin{tikzcd} N \times D^q \arrow[r, "p_1"] \arrow[d, "\psi"'] & N \arrow[d, dashed, "i"] \\ Y \arrow[r, "g"'] & Z \end{tikzcd}$$

is commutative. Here p_1 denotes projection onto the first factor, and i the inclusion of N in Z.

<u>Definition 4.</u> The pair of maps $X \xrightarrow{f} Y \xrightarrow{g} Z$ is said to be locally a (p, q)-disc fibering at $x \in X$, abbreviated to <u>F(p, q) at x</u>, if there exists a neighbourhood N of gfx in Z, embeddings $\varphi:N \times D^p \to X$, $\psi:N \times D^q \to Y$ onto neighbourhoods of x, fx respectively, and a map $k:D^p, 0 \to D^q, 0$ such that

$$\begin{tikzcd} N \times D^p \arrow[r, "1\times k"] \arrow[d, "\varphi"'] & N \times D^q \arrow[r, "p_1"] \arrow[d, "\psi"'] & N \arrow[d, "i"] \\ X \arrow[r, "f"'] & Y \arrow[r, "g"'] & Z \end{tikzcd}$$

commutes.

Note : (i) We can choose φ so that $\varphi(gfx, 0) = x$.

(ii) There is a natural generalisation to sequences of maps of greater length.

(iii) If the pair $X \xrightarrow{f} Y \xrightarrow{g} Z$ is $F(p, q)$ at $x \in X$, then the composition $X \xrightarrow{gf} Z$ is $F(p)$ at x.

(iv) The same diagram shows that the pair f, g is also $F(p, q)$ at all points in some neighbourhood of x.

We prove three basic lemmas.

Lemma 1. (Restriction).

Suppose $X \to Y \to Z$ is $F(p, q)$ at $x \in X$, where $gfx \in Z_0$, a subpolyhedron of Z. Let $Y_0 = g^{-1}Z_0$, $X_0 = f^{-1}Y_0$. Then $X_0 \xrightarrow{f|X_0} Y_0 \xrightarrow{g|Y_0} Z_0$ is also $F(p, q)$ at x.

Proof. By restriction.

Lemma 2. (Glueing).

Given $X \xrightarrow{f} Y \xrightarrow{g} Z$, let Z_i $i = 1, \ldots, t$ be subpolyhedra of Z, and suppose $\bigcup_{i=1}^{t} Z_i$ is a neighbourhood of gfx in Z. Let $Y_i = g^{-1}Z_i$, $g_i = g|Y_i$, $X_i = f^{-1}Y_i$ and $f_i = f|X_i$. Then $X \xrightarrow{f} Y \xrightarrow{g} Z$ is $F(p, q)$ at x if and only if each $X_i \xrightarrow{f_i} Y_i \xrightarrow{g_i} Z_i$ is $F(p, q)$ at x.

<u>Proof.</u> Given that $X \xrightarrow{f} Y \xrightarrow{g} Z$ is $F(p, q)$ at x, restriction shows each $X_i \xrightarrow{f_i} Y_i \xrightarrow{g_i} Z_i$ to be $F(p, q)$ at x.

Conversely, suppose we are given for each i a neighbourhood N_i of gfx in Z_i, embeddings $\varphi_i : N_i \times D^p \to X_i$, $\psi_i : N_i \times D^q \to Y_i$ and a map $k_i : D^p, 0 \to D^q, 0$ such that

$$\begin{tikzcd}
N_i \times D^p \arrow[r, "1 \times k_i"] \arrow[d, "\varphi_i"'] & N_i \times D^q \arrow[r, "p_1"] \arrow[d, "\psi_i"'] & N_i \arrow[d, "\subset"] \\
X_i \arrow[r, "f_i"'] & Y_i \arrow[r, "g_i"'] & Z_i
\end{tikzcd}$$

commutes.

Triangulate Z so that gfx is a vertex and each N_i is a subcomplex. §Let $K = \overline{st}(gfx, Z)$, then each simplex $A \in K$ is contained in some N_i. Consider a conewise expansion

$$gfx = K_0 \nearrow K_1 \nearrow \ldots \nearrow K_m = K$$

each K_i being a cone, vertex gfx.

Let $K_{i,\varepsilon}$ denote the cone K_i shrunk by ε, and D^p_ε, D^q_ε the discs D^p, D^q shrunk by ε.

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§ Let v be a vertex of a complex K; we denote the open, closed star of v in K by $st(v, K)$, $\overline{st}(v, K)$ respectively.

We shall define, inductively on j, a number $\varepsilon_j > 0$, embeddings $\Phi_j : K_{j,\varepsilon_j} \times D^p_{\varepsilon_j} \to X$, $\Psi_j : K_{j,\varepsilon_j} \times D^q_{\varepsilon_j} \to Y$ and a map $k : D^p_{\varepsilon_j}, 0 \to D^q_{\varepsilon_j}, 0$ such that

$$\begin{tikzcd}
K_{j,\varepsilon_j} \times D^p_{\varepsilon_j} \arrow[r, "1\times k"] \arrow[d, "\Phi_j"'] & K_{j,\varepsilon_j} \times D^q_{\varepsilon_j} \arrow[r, "p_1"] \arrow[d, "\Psi_j"'] & K_{j,\varepsilon_j} \arrow[d, "\subset"] \\
X \arrow[r, "f"'] & Y \arrow[r, "g"'] & Z
\end{tikzcd}$$

commutes.

<u>Begin</u>, for $j = 0$, with $\varepsilon_0 = 1$ and $\Phi_0 = \varphi_i | gfx \times D^p$, $\Psi_0 = \psi_i | gfx \times D^q$, $k = k_i$, for some chosen i.

(Without loss of generality we may assume $k(D^p_\varepsilon) \subset D^q_\varepsilon$ for all ε such that $0 \leqslant \varepsilon \leqslant 1$, for if not proceed as follows. Choose λ, $0 < \lambda \leqslant 1$, such that D^p_λ is contained in the star of the origin in some triangulation of D^p with respect to which k is simplicial. Then $k(D^p_{\lambda\varepsilon}) \subset D^q_\varepsilon$ for all $\varepsilon \in [0, 1]$. Let $\Lambda : D^p \to D^p_\lambda$ be the shrinking map, and replace k, Φ_0 by $k\Lambda$ and $\Phi_0(1 \times \Lambda)$ respectively.)

<u>Inductive step</u>, $j \to j + 1$.

Suppose $K_{j+1} = K_j \cup A$, let $L = K_j \cap A$ and $\rho : A \to L$ be a retraction. Choose r such that $A \subset N_r$. Given $a \in A$, $u \in D^p$, $v \in D^q$, define $\varphi_{r,a} : D^p \to X$ and $\psi_{r,a} : D^q \to Y$

by

$$\varphi_{r,a}(u) = \varphi_r(a, u)$$
$$\psi_{r,a}(v) = \psi_r(a, v).$$

Now $\varphi_r(L \times D^p)$ is a neighbourhood of x in $f^{-1}g^{-1}L$, and moreover Φ_j maps $\begin{cases} L_{\varepsilon_j} \times D^p_{\varepsilon_j} \text{ into } f^{-1}g^{-1}L \\ gfx \times 0 \text{ to } x. \end{cases}$

Also $\psi_r(L \times D^q)$ is a neighbourhood of fx in $g^{-1}L$, and Ψ_j maps $\begin{cases} L_{\varepsilon_j} \times D^q_{\varepsilon_j} \text{ into } g^{-1}L \\ gfx \times 0 \text{ to } fx. \end{cases}$

Therefore there is a positive ε, $\varepsilon \leqslant \varepsilon_j$, such that

$$\Phi_j(L_\varepsilon \times D^p_\varepsilon) \subset \varphi_r(L \times D^p)$$
$$\Psi_j(L_\varepsilon \times D^q_\varepsilon) \subset \psi_r(L \times D^q).$$

Choose then $\varepsilon_{j+1} = \varepsilon$ and define

$$\Phi_{j+1}(z, u) = \begin{cases} \Phi_j(z, u) \text{ on } K_{j,\varepsilon} \times D^p_\varepsilon \\ \varphi_{r,z}\varphi^{-1}_{r,\rho z}\,\Phi_j(\rho z, u) \text{ on } A_\varepsilon \times D^p_\varepsilon. \end{cases}$$

$$\Psi_{j+1}(z, v) = \begin{cases} \Psi_j(z, v) \text{ on } K_{j,\varepsilon} \times D^q_\varepsilon \\ \psi_{r,z}\psi^{-1}_{r,\rho z}\Psi_j(\rho z, v) \text{ on } A_\varepsilon \times D^q_\varepsilon. \end{cases}$$

In both cases we have agreement on the overlap, because here $\rho z = z$. Our map Φ_{j+1} is piecewise linear on $A_\varepsilon \times D^p_\varepsilon$ because it is the composition

$$A_\varepsilon\times D^p_\varepsilon \xrightarrow{\Gamma\rho\times 1} A_\varepsilon\times L_\varepsilon\times D^p_\varepsilon \xrightarrow{1\times\Phi_j} A_\varepsilon\times X \xrightarrow{1\times\varphi_r^{-1}} A_\varepsilon\times L\times D^p \xrightarrow{\text{proj.}} A_\varepsilon\times D^p \xrightarrow{\varphi_r} X.$$

Similarly for Ψ_{j+1}.

We are left to show the commutativity of

$$\begin{tikzcd}
A_\varepsilon \times D^p_\varepsilon \arrow[r, "1\times k"] \arrow[d, "\Phi_{j+1}"'] & A_\varepsilon \times D^q_\varepsilon \arrow[r, "p_1"] \arrow[d, dashed, "\Psi_{j+1}"'] & A_\varepsilon \arrow[d, dashed, "\subset"] \\
X \arrow[r, "f"'] & Y \arrow[r, "g"'] & Z.
\end{tikzcd}$$

For the right hand square, if $a \in A_\varepsilon$, $v \in D^q_\varepsilon$, then

$$\begin{aligned}
g\Psi_{j+1}(a,\ v) &= g\psi_{r,a}\psi^{-1}_{r,\rho a}\Psi_j(\rho a,\ v) \\
&\in g\psi_{r,a}(D^q) \\
&= a \\
&= p_1(a,\ v).
\end{aligned}$$

In the left hand square, for $a \in A_\varepsilon$, $u \in D^p_\varepsilon$, we have

$$\begin{aligned}
\Psi_{j+1}(1\times k)(a,\ u) = \Psi_{j+1}(a,\ ku) &= \psi_{r,a}\psi^{-1}_{r,\rho a}\Psi_j(\rho a,\ ku) \\
&= \psi_{r,a}\psi^{-1}_{r,\rho a}\Psi_j(1\times k)(\rho a,\ u) \\
&= \psi_{r,a}\psi^{-1}_{r,\rho a}\, f\,\Phi_j(\rho a,\ u) \text{ by inductive hypothesi} \\
&= \psi_{r,a}\underbrace{\psi^{-1}_{r,\rho a}\, f\,\varphi_{r,\rho a}}\varphi^{-1}_{r,a}\Phi_{j+1}\ (a,\ u)
\end{aligned}$$

$= \psi_{r,a}k_r\ \varphi^{-1}_{r,a}\Phi_{j+1}(a,\ u)$ since

$$\begin{tikzcd}
D^p \arrow[r, "k_r"] \arrow[d, dashed, "\varphi_{r,\rho a}"'] & D^q \arrow[d, dashed, "\psi_{r,\rho a}"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

commutes,

$= f\Phi_{j+1}(a,\ u)$ since

$$\begin{tikzcd}
D^p \arrow[r, dashed, "k_r"] \arrow[d, dashed, "\varphi_{r,a}"'] & D^q \arrow[d, dashed, "\psi_{r,a}"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

commutes.

This completes the inductive step $j \to j + 1$.

Eventually, at the end of the induction, we obtain a commutative diagram

$$\begin{tikzcd} K_\varepsilon \times D^p_\varepsilon \arrow[r, "1\times k"] \arrow[d, "\Phi_m"'] & K_\varepsilon \times D^q_\varepsilon \arrow[r, "p_1"] \arrow[d, "\Psi_m"'] & K_\varepsilon \arrow[d, "\subset"] \\ X \arrow[r, "f"'] & Y \arrow[r, "g"'] & Z \end{tikzcd},$$

where $\varepsilon = \varepsilon_m$. Since K_ε is a neighbourhood of gfx in Z, this shows that $X \xrightarrow{f} Y \xrightarrow{g} Z$ is $F(p, q)$ at x, and so completes the proof of Lemma 2.

<u>Lemma 3.</u> (Composition).

<u>If $X \xrightarrow{f} Y \xrightarrow{g} Z$ is $F(p, q)$ at $x \in X$ and $Z \xrightarrow{h} W$ is $F(n)$ at gfx, then $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$ is $F(n+p, n+q, n)$ at x.</u>

<u>Proof.</u> We have a neighbourhood N' of gfx in Z, embeddings φ', ψ' and a map k which give rise to a commutative diagram -

$$\begin{tikzcd} N' \times D^p \arrow[r, "1\times k"] \arrow[d, "\varphi'"'] & N' \times D^q \arrow[r, "p_1"] \arrow[d, "\psi'"'] & N' \arrow[d, "\subset"] \\ X \arrow[r, "f"'] & Y \arrow[r, "g"'] & Z \end{tikzcd}.$$

Choose a neighbourhood N of hgfx in W and an embedding $e: N \times D^n \to Z$ onto a neighbourhood of gfx <u>in N'</u> such that

$$\begin{tikzcd} N \times D^n \arrow[r, "p_1"] \arrow[d, "e"'] & N \arrow[d, "\subset"] \\ Z \arrow[r, "h"'] & W \end{tikzcd}$$

commutes.

Define $\quad \psi : N \times D^n \times D^q \to Y$ by

$$\psi(t, u, v) = \psi'(e(t, u), v)$$

and $\quad \varphi : N \times D^n \times D^p \to X$ by

$$\varphi(t, u, v) = \varphi'(e(t, u), v).$$

Then

$$\begin{tikzcd}
N \times D^n \times D^p \arrow[r, "1\times 1\times k"] \arrow[d, dashed, "\varphi"'] & N \times D^n \times D^q \arrow[r, dashed, "proj^n"] \arrow[d, "\psi"'] & N \times D^n \arrow[r, "p_1"] \arrow[d, "e"'] & N \arrow[d, dashed, "\subset"] \\
X \arrow[r, dashed, "f"'] & Y \arrow[r, dashed, "g"'] & Z \arrow[r, dashed, "h"'] & W
\end{tikzcd}$$

commutes as required.

<u>Corollary.</u> <u>With the same hypotheses, $X \xrightarrow{f} Y \xrightarrow{hg} W$ is $F(n + p, n + q)$ at x.</u>

TRANSIMPLICIAL MAPS

Let Q be a manifold, and K a triangulation of Q. If A is an a-dimensional simplex of K, let

$$L^A = \mathrm{lk}(A, K)$$

denote the link of A in K. Then § $AL^A = \overline{\mathrm{st}}(A, K)$.

Let v be a vertex of A, and

$$s^A : AL^A \to vL^A$$

denote the simplicial map defined as the join of $A \to v$ to the identity on L^A.

Let M be another manifold, and $f:M \to Q$ be a map. Given a point x of M, let A be the unique simplex of K such that $fx \in \mathring{A}$.

Definition 5. We say that the map f is transimplicial to K at x if the pair

$$f^{-1}AL^A \xrightarrow{f} AL^A \xrightarrow{s^A} vL^A$$

is $F(m + a - q, a)$ at x. If this is the case for all $x \in M$, we say f is transimplicial to K.

Note 1. Our definition is independent of the choice of v (by an application of the composition lemma).

---

§ We denote the join of two complexes K and L by KL.

<u>Note 2.</u> The restriction and glueing lemmas of the previous section show that equivalent to Definition 5 is : for every principal simplex $AB \in K$, the pair $f^{-1}AB \xrightarrow{f} AB \xrightarrow{s^A} vB$ is $F(m + a - q, a)$ at x.

<u>Note 3.</u> Often it will be convenient to use the idea of a submanifold (i.e. the image of an embedding rather than the embedding itself) being transimplicial to a triangulation. The definition is the obvious one. Given a manifold Q, submanifold M, and triangulation K of Q, we say M is transimplicial to K at $x \in M$ if the pair

$$M \cap AL^A \subset AL^A \xrightarrow{s^A} vL^A$$

is $F(m + a - q, a)$ at x, where $x \in \mathring{A}$, $A \in K$, and we use the above notation.

<u>Note 4.</u> The concept is designed to cut out the folding phenomenon described in our introduction. We illustrate below a non-transimplicial embedding of a 2-disc in 3-dimensions. The disc lies in the star of a 1-simplex, and has a fold running down to a point in the 1-simplex.

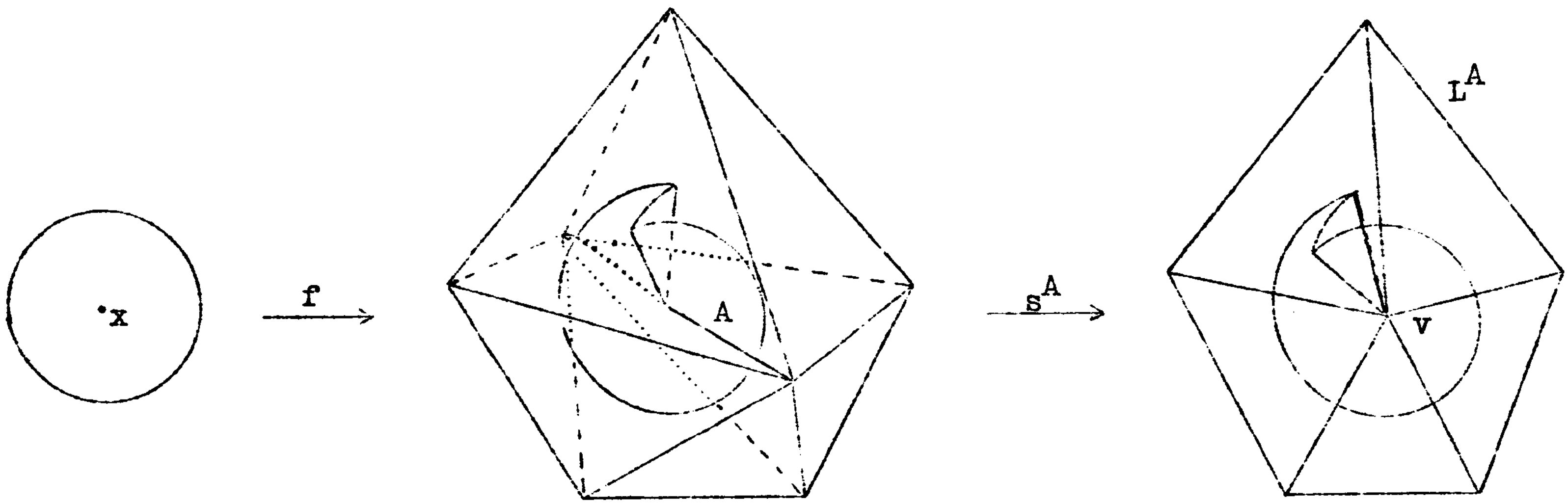

Figure 3

The embedding f fails to be transimplicial at x, because if it were, then the composition $s^A f$ would be F(0), i.e. would be an embedding; but it is not an embedding because it is three-to-one where the fold gets flattened down.

Notice that if we move the fold point into the interior of a 3-simplex, then the embedding does become transimplicial. In fact this is the geometric idea behind our main proof. Given an embedding $M \to Q$ and a triangulation K of Q, we cannot isotop M into the m-skeleton of K^* (by Hirsch's result [6]), but nevertheless we shall show that we can push the worst fold and kink points into top dimensional simplexes, and so make M transimplicial to K.

Note 5. To prove the theorems in this paper we need only consider transimplicial embeddings rather than transimplicial maps. However, maps are just as easy to handle as embeddings at this stage, and several of the more general results that we prove for maps will be useful in [2].

Lemma 4. (Openness)

If f is transimplicial to K at $x \in M$, then f is transimplicial to K at each point in some neighbourhood of x.

Proof. Using the previous notation, the pair

$$f^{-1}AL^A \xrightarrow{f} AL^A \xrightarrow{s^A} vL^A$$

is $F(m + a - q, a)$ at x. By the openness of disc fiberings, there is a neighbourhood U of x in M such that this pair is $F(m + a - q, a)$ at all points of U. Let $y \in U$ and suppose $fy \in \mathring{B}$, $B \in K$; then A is a face of B and consequently $BL^B \subset AL^A$; let $B = AC$. By restriction the pair $f^{-1}BL^B \xrightarrow{f} BL^B \xrightarrow{s^A} vCL^B$ is $F(m + a - q, a)$ at y. But $s^{vC}:vCL^B \to vL^B$ is $F(b - a)$ at $s^A fy$, and $s^{vC}s^A = s^B:BL^B \to vL^B$. Therefore by the corollary to Lemma 3

$$f^{-1}BL^B \xrightarrow{f} BL^B \xrightarrow{s^B} vL^B$$

is $F(m + b - q, b)$ at y, completing the proof.

Lemma 5.

For any subdivision K' of K, f transimplicial to K' implies f transimplicial to K.

Proof. Given $x \in M$, suppose $fx \in \mathring{A}'$, where $A' \in K'$ and $\mathring{A}' \subset \mathring{A}$, $A \in K$. Let v' be a vertex of A', v a vertex of A, $L' = \mathrm{lk}(A', K')$ and $L = \mathrm{lk}(A, K)$. Then $s^A:AL \to vL$ induces a linear (i.e. each simplex is mapped linearly) map $\lambda:v'L' \to vL$ which makes the following diagram commute

$$\begin{array}{ccccc} f^{-1}A'L' & \xrightarrow{f} & A'L' & \xrightarrow{s^{A'}} & v'L' \\ \cap & & \cap & & \downarrow\lambda \\ f^{-1}AL & \xrightarrow{f} & AL & \xrightarrow{s^A} & vL \end{array} .$$

Since f is transimplicial to K' the pair $f^{-1}A'L' \to A'L' \to v'L'$ is $F(m + a' - q, a')$ at x.

If we show that λ is $F(a - a')$ at v', then $f^{-1}AL \to AL \to vL$ is $F(m + a - q, a)$ by composition, and so the lemma follows. Therefore it remains to show that λ is $F(a - a')$ at v'.

K is contained in some Euclidean space E. Let F be the decomposition space of E consisting of all a-planes parallel to A, and let $g:E \to F$ be the natural map. Then g embeds vL in F because A is joinable to L. Similarly g' embeds $v'L'$ in F', where $g':E \to F'$ is the natural map onto the decomposition space of all a'-planes parallel to A'. We have a commutative diagram

$$\begin{tikzcd} v'L' \arrow[r, dashed, "\lambda"] \arrow[d, dashed, "g'"'] & vL \arrow[d, "g"] \\ F' \arrow[r, "\mu"] & F \end{tikzcd}$$

where μ is the natural map. Since μ is linear it is $F(a - a')$ everywhere.

Let $N = g(vL)$, $N' = g'(v'L')$. Then N' is a neighbourhood of $g'v'$ in $\mu^{-1}N$ because $A'L'$ is a neighbourhood of x in AL. Therefore $\mu:N' \to N$ is $F(a - a')$ at $g'v'$ by restriction. Therefore

$\lambda : v'L' \to vL$ is $F(a - a')$ at v', and the proof of Lemma 5 is complete.

Let P be a proper submanifold of Q, and let K be a triangulation of the pair Q, P; in other words K is a triangulation of Q in which P appears as a subcomplex K_1.

Lemma 6. (Consistency)

If M is a proper submanifold of Q that is transimplicial to K, then M is transversal to P.

Proof. Given $x \in M \cap P$, suppose $x \in \mathring{A}$, $A \in K_1$. Let $L = \mathrm{lk}(A, K)$, $L_1 = \mathrm{lk}(A, K_1)$ and v be a vertex of A. Since M is transimplicial to K we have, with the usual notation, a commutative diagram:

$$\begin{tikzcd} N \times D \arrow[r, "1\times k"] \arrow[d, "\varphi"'] & N \times D_* \arrow[rr, "\text{projection}"] \arrow[d, "\psi"'] & & N \arrow[d, "\subset"] \\ M \cap AL \arrow[r, "\subset"'] & AL \arrow[rr, "s^A"'] & & vL \end{tikzcd},$$

where $D = D^{m+a-q}$ and $D_* = D^a$. Let $N_1 = N \cap vL_1$. Since Q, P is a locally flat manifold pair, we can choose N such that N, N_1 is an unknotted ball pair. The above left hand square can be rewritten:

$$\begin{tikzcd} N \times D \arrow[r, "1\times k"] \arrow[d, "\varphi"'] & N \times D_* \arrow[d, "\psi"] \\ M \arrow[r, "\subset"'] & Q \end{tikzcd}.$$

Since M is locally flat in Q, we know that $N \times kD$ is locally flat at $(v, 0)$ in $N \times D_*$, and therefore that kD is locally flat at 0 in D_*. Meanwhile N_1 is locally flat at v in N. Therefore $N \times kD$ and $N_1 \times D_*$ are transversal at $(v, 0)$ in $N \times D_*$. Taking the image under ψ we deduce that M and P are transversal at x in Q. This is true for all $x \in M \cap P$, and so M, P are transversal.

We shall require triangulations of our manifolds that possess a certain local linearity property.

<u>Definition 6.</u> A combinatorial manifold K, of dimension q, is called <u>Brouwer</u> if:

(i) For each $A \in \overset{\circ}{K}$ there is a linear embedding

$$\overline{st}(A, K) \to E^q.$$

(ii) For each $A \in \dot{K}$ there is a linear embedding

$$\overline{st}(A, K), \overline{st}(A, \dot{K}) \to E^q_+, E^{q-1}.$$

Notes: 1. If only (ii) holds we say K is <u>Brouwer at the boundary.</u>

2. Not every combinatorial manifold is Brouwer, see Cairns [4].

3. Any subdivision of a Brouwer manifold is Brouwer.

The following lemma is due, in a sharpened form, to Whitehead [9].

Lemma 7.

(a) *Any combinatorial manifold K has a Brouwer subdivision K'.*

(b) *If K is already Brouwer at the boundary, we can choose K' such that $\dot{K}' = \dot{K}$.*

Proof. (a) Choose an atlas of q-simplexes $f_i : \Delta \to K$, $1 \leq i \leq r$, that cover K in the sense that every point has some $f_i\Delta$ as a closed neighbourhood. Now produce K' by subdividing so that all the f_i are simultaneously simplicial (using [12] Theorem 1).

(b) If K is already Brouwer at the boundary, we can confine our attention to a subatlas not meeting $\dot{K}$ that covers every simplex not meeting $\dot{K}$. In order to make the subatlas simplicial, it is not necessary to subdivide any simplex on the boundary.

The main burden of this paper will be to prove the following two theorems.

Theorem 4. *If $f : M \to Q$ is an embedding between closed manifolds, and K any triangulation of Q, then f can be ambient isotoped, by an arbitrarily small ambient isotopy, to an embedding g that is transimplicial to K.*

This theorem is in fact true for maps (see [2]).

We now give a relative version.

<u>Theorem 5. Let P be a proper submanifold of Q, and J a Brouwer triangulation of the boundary $\dot{Q}$, $\dot{P}$. Let $f:M \to Q$ be a proper embedding such that $f|\dot{M}$ is transimplicial to J. Then there exists an extension of J to a Brouwer triangulation K of Q, P, and an arbitrarily small ambient isotopy keeping $\dot{Q}$ fixed carrying f into an embedding g that is transimplicial to K.</u>

<u>Remark.</u> Let K be an arbitrary extension of J to a Brouwer triangulation of Q, P. Then although $f|\dot{M}$ is transimplicial to J, it may well happen that f is not transimplicial to K at points of $\dot{M}$. For example, let D be a disc properly embedded in a tetrahedron T as shown in Figure 4. Then $\dot{D}$ is transimplicial to $\dot{T}$, but the fold ensures that D is not transimplicial to T at the boundary point x.

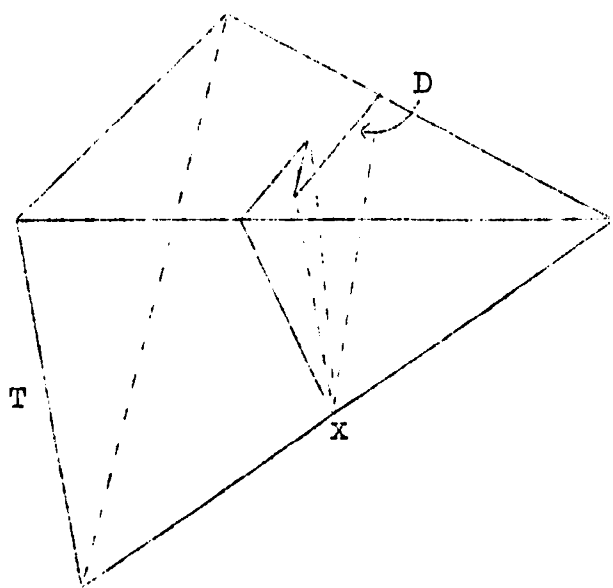

Figure 4

In our proof of Theorem 5, we get round this difficulty by using a collaring technique to construct a particular extension K relative to which such folds are straightened out.

Before proving these transimplicial results, we give applications in the form of proofs of our transversality theorems.

PROOF OF THEOREM 1.

We are given proper submanifolds M, P of Q, and have to ambient isotop M transversal to P.

By Lemma 7, it is possible to choose a Brouwer triangulation of the pair $\dot{Q}$, $\dot{P}$. Apply Theorem 4 to ambient isotop $\dot{M}$ transimplicial to J, and extend this ambient isotopy from $\dot{Q}$ to the whole of Q by [7] Addendum (2.2). Suppose the effect of this isotopy has been to move M to $M_1 \subset Q$, then $\dot{M}_1$ is transimplicial to J. We are now in a position to apply Theorem 5. This provides:

(a) an extension of J to a Brouwer triangulation K of the pair Q, P.

(b) an arbitrarily small ambient isotopy which moves M_1 transimplicial to K whilst keeping $\dot{Q}$ fixed.

Reference to Lemma 6 shows that the composition of our two isotopies produces the required result.

PROOF OF THEOREM 2.

We are given closed manifolds M, and $P \subset Q$, together with a map $f:M \to Q$ which we want to homotop graph-transversal to P. The graph $\Gamma f:M \to M \times Q$ is an embedding. Choose Brouwer triangulations K_1 of M and K_2 of Q, P, and let K_3 be a subdivision of the cell complex $K_1 \times K_2$ triangulating $M \times Q$, $M \times P$. Using Theorem 4, ambient isotop Γf into an embedding F that is transimplicial to K_3.

Lemma 8.

We can choose F so that the composition

$$M \xrightarrow{F} M \times Q \xrightarrow{p_1} M$$

is a homeomorphism, where p_1 is the projection.

The proof of this lemma is postponed, it can be found directly following the proof of Theorem 4.

Meanwhile, let $e = (p_1F)^{-1}$, the inverse homeomorphism. Define $G = (e \times 1)F:M \to M \times Q$, and let g denote the composition

$$M \xrightarrow{G} M \times Q \xrightarrow{p_2} Q.$$

Then g is homotopic to f and $G = \Gamma g$, the graph of g.

The triangulation K_3 of $M \times Q$ is really a homeomorphism $t:K_3 \to M \times Q$. Let K denote the triangulation

$$(e \times 1)t:K_3 \to M \times Q.$$

Then since $e \times 1$ maps $M \times P$ to itself, K is also a triangulation of $M \times Q$, $M \times P$. Now F is transimplicial to the triangulation K_3, and since we have applied the homeomorphism $e \times 1$ to both embedding and triangulation, we deduce that G is transimplicial to K. Therefore by Lemma 6 we know G is transversal to $M \times P$. Hence g is graph-transversal to P, because $\Gamma g = G$, and consequently $g^{-1}P$ is a locally flat submanifold of M of codimension $q - p$.

It remains to show the invariance of the cobordism class $\{g^{-1}P\}$. There were two choices involved in the above construction namely those of triangulation and isotopy. Let K_*, g_* arise from different choices. Then g, g_* are connected by a homotopy $h:M \times I \to Q$ say.

The graph

$$\Gamma h:M \times I \to M \times I \times Q$$

is a proper embedding, whose restriction to the boundary

$$\Gamma g \cup \Gamma g_*:\partial(M \times I) \to \partial(M \times I \times Q)$$

is transimplicial to the Brouwer triangulation $K \cup K_*$ of $\partial(M \times I \times Q)$. By Theorem 5 extend $K \cup K_*$ to a triangulation of $M \times I \times Q$, $M \times I \times P$ and ambient isotop

Γh, keeping the boundary fixed, to a transimplicial embedding H, say.

By Lemma 6 H is transversal to $M \times I \times P$, and so $H^{-1}(M \times I \times P)$ is an $(m + 1 + p - q)$-dimensional submanifold of $M \times I$ with boundary $g^{-1}P \cup (- g_*^{-1}P)$, the minus sign referring to orientation. In other words $g^{-1}P$ and $g_*^{-1}P$ are cobordant. If f_* is homotopic to f then the same g will do for both, and so the cobordism class $\{g^{-1}P\}$ depends only upon the homotopy class $[f]$.

Remark.

There is a small but subtle point here. If f happened to be already graph-transversal to P we could not infer that $f^{-1}P \in \{g^{-1}P\}$, because f might not be transimplicial to any triangulation, and so we could not use the relative transimplicial Theorem 5, as in the proof above. Nor do we have a relative transversal theorem to use instead (see the end of the paper).

PROOF OF THEOREM 3.

We are given manifolds $M \subset P \subset Q$, with both inclusions proper, and need to construct a "perpendicular" manifold N. Begin as for Theorem 1, combining the results of Theorems 4 and 5 to obtain a triangulation J of P and an ambient isotopy of P moving M to M_1, where M_1 is transimplicial to J. By [7] Corollary (2.3) extend the ambient isotopy of P to give an ambient isotopy of the whole of Q. Extend J to a triangulation K of Q, this is possible since P is proper and locally flat in Q (see [3]). Let K' denote a first derived of K mod J.

For each simplex $A \in J$, let
$L_A = \{\text{simplexes } B \in K' : AB \in K' \text{ and } B \cap J = \emptyset\}$.
Define

$$X = \bigcup_{A \in J} (A \cap M_1) L_A ,$$

the joins being made linearly inside the simplexes of K. Note firstly that the dimension of X is $m + q - p$. X need not be a manifold, however we shall show that it is a manifold "near" M_1.

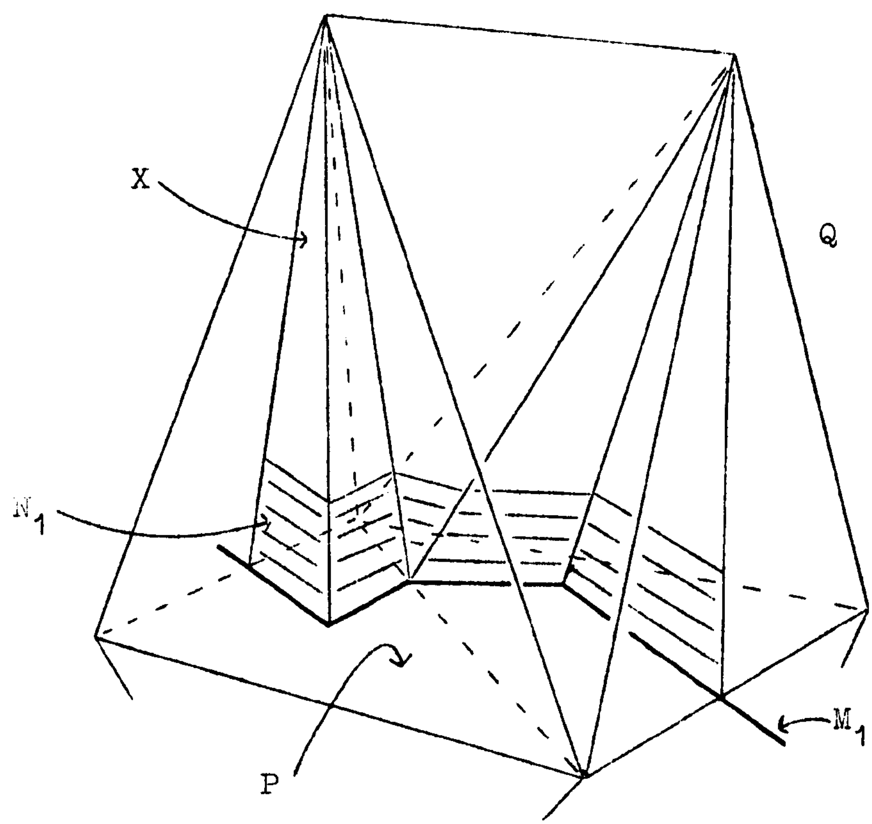

Figure 5

For $x \in M_1$, suppose $x \in \mathring{A}$, $A \in J$, and write $L^P = \mathrm{lk}(A, J)$, $L^Q = \mathrm{lk}(A, K')$. Let v be a vertex of A. Since M_1 is transimplicial to J, the pair

$$M_1 \cap AL^P \subset AL^P \xrightarrow{s^A} vL^P$$

is $F(m + a - p, a)$ at x. This implies that

$$X \cap AL^Q \subset AL^Q \xrightarrow{s^A} vL^Q$$

is also $F(m + a - p, a)$ at x. So as not to interrupt the main line of argument, we ask the reader to temporarily accept this implication; a proof will be given following Lemma 12. We have therefore a neighbourhood D^{q-a} of v in vL^Q and an embedding of $D^{q-a} \times D^{m+a-p} = D^{m+q-p}$ onto a neighbourhood of x in X. Consequently there is a neighbourhood N_1 of M_1 in X (for example take a second derived neighbourhood) which is an $(m + q - p)$ - manifold and transimplicial to K'. By Lemma 6 N_1 is transversal to P in Q. By construction $N_1 \cap P = M_1$. Now reverse the original ambient isotopy of Q to obtain the required manifold N.

THE t-SHIFT OF AN EMBEDDING.

For the proof of Theorem 4 we shall use a sequence of special local moves (first introduced in [12] Chapter 6) called t-shifts. The parameter t concerns dimension, and the construction involves choice of local coordinate systems (i.e. replacing the piecewise linear structure by local linear structures) and choices of points in general position.

Suppose $f:M \to Q$ is a proper embedding between manifolds. By Lemma 7, we can find triangulations K_1, K_2 of M, Q such that $f:K_1 \to K_2$ is simplicial and K_2 Brouwer. If $K_1^{(2)}$, $K_2^{(2)}$ denote the barycentric second derived complexes of K_1, K_2, then $f:K_1^{(2)} \to K_2^{(2)}$ remains simplicial.

Let T_1 be a t-simplex of K_1 such that $\mathring{T}_1 \subset \mathring{M}$, and let $T_2 = fT_1$. Take a simplicial neighbourhood of T_2 modulo its boundary in $K_2^{(2)}$ (i.e. this consists of all closed simplexes of $K_2^{(2)}$ which meet the interior of T_2) and call the resulting q-ball B_2. Let $B_1 = f^{-1}B_2$, this is an m-ball (it is in fact the corresponding simplicial neighbourhood of T_1 mod $\dot{T}_1$ in $K_1^{(2)}$). For $i = 1, 2$ let $\hat{T}_i$ denote the barycentre of T_i, and let $S_i = \dot{B}_i$. Then the polyhedron $|B_i| = \hat{T}_i S_i$, although of course as a complex B_i is a subdivision of $\hat{T}_i S_i$.

Denote by $f_T : B_1 \to B_2$ the restriction of f, Thus f_T is the join of the two maps $\hat{T}_1 \to \hat{T}_2$, $S_1 \to S_2$. The idea is to construct another embedding $g_T : B_1 \to B_2$ that agrees with f_T on the boundary $\dot{B}_1$, and is ambient isotopic to f_T keeping the boundary $\dot{B}_2$ fixed. We shall give the explicit construction below; it will be apparent that g_T can be chosen arbitrarily close to f_T, and the ambient isotopy made arbitrarily small.

Define a new embedding of M in Q by

$$g = \begin{cases} f \text{ on } M - B_1 \\ g_T \text{ on } B_1. \end{cases}$$

Then g is ambient isotopic to f. We call the move $f \to g$ a <u>local t-shift</u> with respect to the triangulation K_2.

<u>Construction of the local shift :</u> Choose a linear embedding h of $\overline{st}(T_2, K_2)$ in E^q (this is possible since K_2 is Brouwer), then h embeds B_2 linearly in E^q.

Let X denote the combinatorial q-ball hB_2 $Y = \dot{X}$, and $v = h\hat{T}_2$. Choose a point $w \in E^q$ near v which satisfies:

(i) $w \in st(v, X)$

(ii) w and Y are joinable

(iii) w is in general position with respect to the vertices of X.

Define a homeomorphism $j: X \to X$ as the join of the identity on Y to the map $v \to w$. Thus $h^{-1}jh$ is a homeomorphism of the ball B_2 which keeps its boundary fixed. Define $\underline{g_T = h^{-1}jhf_T}$. Then g_T is ambient isotopic to f_T keeping $\dot{B}_2$ fixed in view of:

<u>Alexander's Lemma.</u> <u>Any homeomorphism of a ball keeping the boundary fixed is isotopic to the identity keeping the boundary fixed.</u>

Suppose we now let T_1 run over a sequence of "interior" t-simplexes of K_1, then the corresponding balls $\{B_1\}$ overlap only in their boundaries, on which the $\{g_T\}$ agree with f, and therefore with each other. Consequently the resulting embeddings, and ambient isotopies, may be combined to give an embedding g ambient isotopic to f. We call $f \to g$ a <u>global t-shift</u> or, more briefly, a t-shift.

We shall want to perform a succession of t-shifts, one for each value of t, $\dim K_1 \geq t \geq 0$. But after the first shift the resulting embedding will no longer be simplicial with respect to K_1, K_2. However, in the construction of a shift, our initial assumption that f be simplicial was a luxury rather than a necessity, and the construction can be adapted as follows. Suppose $r > t$, $e: K_1 \to K_2$ simplicial, and that we perform an r-shift $e \to f$. Then given a t-simplex $T_1 \in K_1$:

(a) f maps T_1 linearly onto a t-simplex $T_2 \in K_2$.

(b) If B_2 is as above, and if $B_1 = f^{-1}B_2$, then B_1 is an m-ball and $f^{-1}\dot{B}_2 = \dot{B}_1$.

(c) $f_T: B_1 \to B_2$ is the join of $\dot{B}_1 \to \dot{B}_2$ to $\hat{T}_1 \to \hat{T}_2$.

Property (a) is satisfied because the r-shift does not move the $(r-1)$-skeleton, and properties (b) and (c) follow from property (i) of w in each local r-shift.

With the amount of structure contained in (a), (b) and (c) we can construct a local t-shift $f \to g$ exactly as before. Only one minor modification is needed, and that is in property (iii) for the point w: for this choose subdivisions such that $B_1' \to B_2'$ is simplicial, let X' be the corresponding subdivision of X, and choose w in general position with respect to the vertices of X'. The remainder of the construction is unaltered.

In this way we can construct t-shifts for all t, $m = \dim K_1 \geqslant t \geqslant 0$, in descending order, because for each t-simplex, the preceeding higher dimensional shifts preserve the structure (a), (b) and (c).

PROOF OF THEOREM 4.

Let X be a combinatorial q-ball, with boundary Y, linearly embedded in E^q, and S^{m-1} an $(m-1)$-sphere in Y. Suppose that Y is joinable to the interior point w of X; in other words X and wY have the same underlying polyhedron. We have the following two lemmas.

Lemma 9.

If S^{m-1} is transimplicial to Y at y, then wS^{m-1} is transimplicial to X at y.

Lemma 10.

If S^{m-1} is a subcomplex of Y, and if w is in general position with respect to the vertices of X, then wS^{m-1} is transimplicial to X at all interior points of X.

Proof of 9. (See Figure 6) Suppose $y \in \mathring{A}$, $A \in Y$. Let v be a vertex of A, $L = \mathrm{lk}(A, X)$, $L_1 = \mathrm{lk}(A, Y)$ and s the simplicial map $AL \to vL$. We know that $S^{m-1} \cap AL_1 \subset AL_1 \xrightarrow{s} vL_1$ is $F(m + a - q, a)$ at y: i.e. there is a neighbourhood N_1 of v in vL_1 and a commutative diagram

$$\begin{array}{ccccc}
N_1 \times D_1^{m+a-q} & \xrightarrow{1\times\subset} & N_1 \times D_1^a & \xrightarrow{\text{proj}^n} & N_1 \\
\downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle \subset} \\
S^{m-1} \cap AL_1 & \xrightarrow{\subset} & AL_1 & \xrightarrow{s} & vL_1
\end{array}$$

where φ_1 embeds $N_1 \times D_1^a$, $N_1 \times D_1^{m+a-q}$ as neighbourhoods of y in AL_1, $S^{m-1} \cap AL_1$ respectively. Since w is joinable to Y, every ray radiating from w meets Y in a unique point. The same is true for points near w. Thus any ray near wy and parallel to wy also meets Y in a unique point. Therefore given a neighbourhood V of y in Y, there exists a neighbourhood U of y in X such that projection parallel to wy gives a map $r:U \to V$. Now choose V, U sufficiently small so that $V \subset \varphi_1(N_1 \times D_1^a)$ and $U \subset AL$. Define $\theta:U \to D_1^a$ as the composition

$$U \xrightarrow{r} \varphi_1(N_1 \times D_1^a) \xleftarrow{\varphi_1} N_1 \times D_1^a \xrightarrow{\text{proj}^n} D_1^a.$$

Then $s \times \theta:U \to vL \times D_1^a$ is piecewise linear and onto a neighbourhood of $v \times 0$ in $vL \times D_1^a$. Moreover, $s \times \theta$ is an embedding, for suppose u_1, u_2 have the same image under $s \times \theta$. Since $su_1 = su_2$ the interval u_1u_2 is parallel to A. Therefore the interval $(ru_1)(ru_2)$ is also parallel to A and of the same length, consequently the points $\varphi_1^{-1}ru_1$, $\varphi_1^{-1}ru_2$ have the same first coordinate in $N_1 \times D_1^a$. Since $\theta u_1 = \theta u_2$, they also have the same last coordinate. Therefore they are equal, giving $ru_1 = ru_2$, and so $u_1 = u_2$. Thus $s \times \theta$ is an embedding as required.

Choose neighbourhoods N of v in vL, D^a of 0 in D_1^a, D^{m+a-q} of 0 in D_1^{m+a-q} such that

$$N \times D^a \subset (s \times \theta)U, \text{ and}$$
$$D^{m+a-q} \subset D^a .$$

Define $\varphi : N \times D^a \to AL$ by $\varphi = (s \times \theta)^{-1} | N \times D^a$. By construction

$$\begin{tikzcd} N \times D^{m+a-q} \arrow[r, "1\times\subset"] \arrow[d, "\varphi"] & N \times D^a \arrow[r, "\mathrm{proj}^n"] \arrow[d, "\varphi"] & N \arrow[d, "\subset"] \\ wS^{m-1} \cap AL \arrow[r, "\subset"] & AL \arrow[r, "s"] & vL \end{tikzcd}$$

commutes, showing wS^{m-1} transimplicial to X at y.

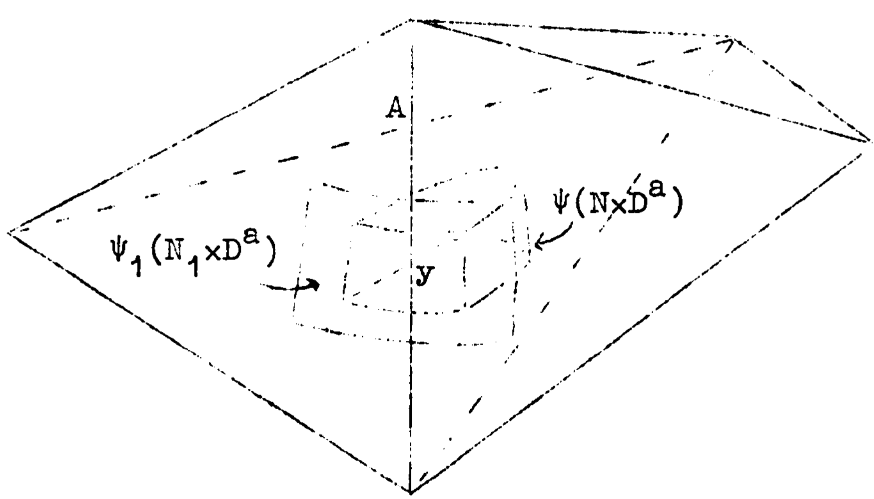

Figure 6

Proof of 10. (See Figure 7) Since w is in general position it must lie in the interior of a principal simplex of X, hence trivially wS^{m-1} is transimplicial to X at w. Given an interior point x of wS^{m-1}, $x \neq w$, suppose that $x \in \mathring{A}$ where A is a simplex of X (we may assume $\dim A < q$, otherwise the lemma is again trivial). Let $L = \mathrm{lk}(A, X)$. We need to show that

$$wS^{m-1} \cap AL \subset AL \xrightarrow{s} vL$$

is $F(m+a-q, a)$ at x. Denote by [A] the linear subspace of E^q spanned by A. Then $w \notin [A]$, by the general position of w. Let [wx] meet Y in y, where $y \in \mathring{C}$, $C \in Y$. Again using the general position of w, we infer that [A] and [C] together span E^q. Therefore $[wA] \cap C$ is a convex linear cell, containing y in its interior, of dimension $(a+1+c-q)$. Call this cell E.

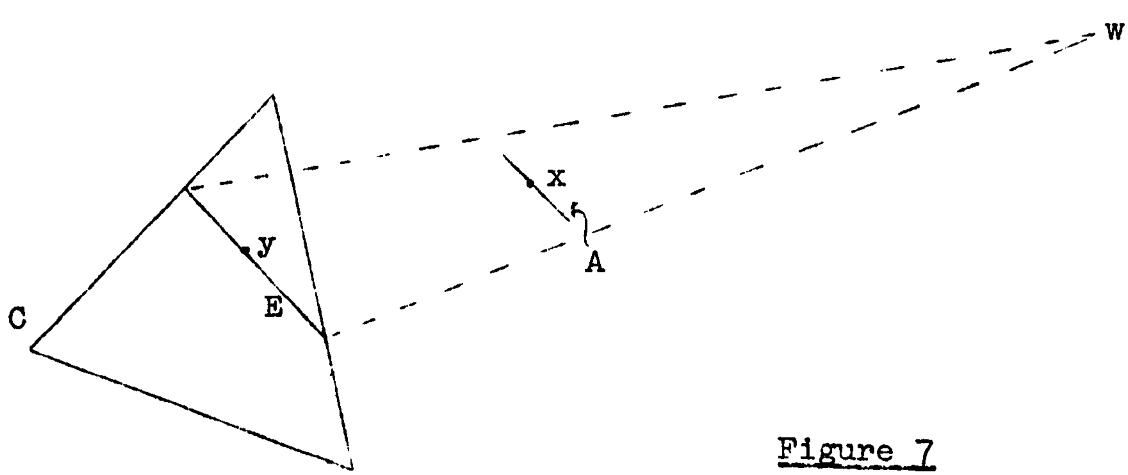

Figure 7

Let $L_1 = \mathrm{lk}(C, S^{m-1})$, $L_2 = \mathrm{lk}(C, Y)$. Then EL_1, EL_2 are respectively $m + a - q$, a - balls.

Let $\rho: C \to [E]$ denote orthogonal projection, and V be the neighbourhood $(\rho^{-1}E)L_2$ of y in Y. Let $\bar{\rho}: V \to EL_2$ be the join of ρ to the identity on L_2. As in the proof of the previous lemma any ray parallel and sufficiently close to wx meets Y in a unique point, and therefore there exists a neighbourhood U of x in X such that projection parallel to wx gives a map $r: U \to V$. We can choose U sufficiently small so that $U \subset AL$. Let θ be the composition

$$U \xrightarrow{r} V \xrightarrow{\bar{\rho}} EL_2.$$

Then θ is a projection in a direction complementary to the projection

$$U \xrightarrow{\subset} AL \xrightarrow{s} vL.$$

Therefore the product

$$s \times \theta: U \to vL \times EL_2$$

is a piecewise linear embedding onto a neighbourhood of $v \times y$ in $vL \times EL_2$. Choose neighbourhoods N of v in vL, D^{m+a-q} of y in EL_1, D^a of y in EL_2, such that $D^{m+a-q} \subset D^a$ and $N \times D^a \subset (s \times \theta)U$. Define $\psi: N \times D^a \to AL$ by $\psi = (s \times \theta)^{-1} | N \times D^a$. By construction we have a commutative diagram

$$\begin{tikzcd}
N \times D^{m+a-q} \arrow[r, hook, dashed, "\subset"] \arrow[d, dashed, "\psi"'] & N \times D^{a} \arrow[r, "\mathrm{proj}^{n}"] \arrow[d, dashed, "\psi"'] & N \arrow[d, "\subset"] \\
wS^{m-1} \cap AL \arrow[r, dashed, "\subset"] & AL \arrow[r, dashed, "s"] & vL
\end{tikzcd}$$

and therefore the proof of Lemma 10 is complete.

We shall also need:

Lemma 11. *Let M, Q be closed manifolds, and $f:M \to Q$ an embedding. Suppose B_2 is a q-ball contained in Q such that $(B_2, B_2 \cap fM)$ is a (q, m)-ball pair. Let $B_1 = f^{-1}(B_2 \cap fM)$, and let K be a triangulation of Q, B_2. Then if x is a point of $\dot{B}_1$ such that both*

$$f|B_1 : B_1 \to B_2, \text{ and}$$
$$f|M - \mathring{B}_1 : M - \mathring{B}_1 \to Q - \mathring{B}_2$$

are transimplicial to K at x, then f is transimplicial to K at x.

Proof. A straightforward application of the glueing lemma. (Of course in saying $f|B_1 : B_1 \to B_2$ is transimplicial to K, we mean that it is transimplicial to the subcomplex of K triangulating B_2; similarly for the statement about $f|M - \mathring{B}_1$. Where no confusion can arise this abbreviation will be constantly used.)

Inductive proof of Theorem 4.

Recall the statement of Theorem 4. We are given an embedding $f:M \to Q$ between closed manifolds, together with a triangulation K of Q, and we have to ambient isotop f to g such that g is transimplicial to K.

Choose a triangulation K_1 of M and a subdivision K_2 of K so that $f:K_1 \to K_2$ is simplicial and K_2 is Brouwer. Let K_1^t denote the t-skeleton of K_1, and $K_2^{(2)}$ the barycentric second derived of K_2. We shall produce inductively a sequence of embeddings of M in Q

$$f = g_{m+1},\ g_m,\ \dots,\ g_0 = g$$

such that

(i) g_t is transimplicial to $K_2^{(2)}$ at points of $K_1 - K_1^{t-1}$, and

(ii) g_t is ambient isotopic to g_{t+1} by an arbitrarily small ambient isotopy.

Application of Lemma 5 shows that the final embedding g is transimplicial to K.

<u>Beginning of induction</u> : Apply a local m-shift to f, with respect to K_2, for each m-simplex of K_1. Define g_m to be the embedding which results from the global m-shift. Then (ii) is satisfied. Let A_1 be an m-simplex of K_1 and $A_2 = fA_1$. It is sufficient to show that g_m is transimplicial to $K_2^{(2)}$ at points of $\mathring{A}_1$. Recall the local m-shift process. Using the notation of the previous section, we have

$$g_m = h^{-1}jhf:\hat{A}_1\mathring{A}_1 \to \hat{A}_2S_2.$$

By Lemma 10, $jhfA_1$ is transimplicial to X at all interior points. Therefore, since the property of being transimplicial is preserved under an isomorphism, g_mA_1 is transimplicial to $K_2^{(2)}$ at points of $g_m\mathring{A}_1$ as required.

Inductive step: Assume that, as a result of r-shifts for $m \geqslant r > t$, we have

$$g_m, \ldots, g_{t+1}$$

satisfying (i) and (ii).

Apply a local t-shift to g_{t+1}, with respect to K_2, for each t-simplex of K_1, and define g_t as the embedding resulting from the global t-shift. Again (ii) is immediately satisfied, and in proving (i) it is sufficient to examine the effect of a local shift, say that associated with $T_1 \in K_1$. We again use the notation of the previous section. Then:

$$g_t = g_{t+1} \text{ on } M - \mathring{B}_1, \text{ and}$$
$$g_t = h^{-1}jhg_{t+1}:B_1 \to B_2.$$

We claim that g_t is transimplicial to $K_2^{(2)}$ at points of

(a) $K_1 - K_1^t$, and

(b) $\mathring{T}_1$.

By the inductive hypothesis and restriction,

$$g_t:M - \mathring{B}_1 \to Q - \mathring{B}_2$$

is transimplicial to $K_2^{(2)}$ at points of $K_1 - K_1^t$. It remains to show

$$g_t : B_1 \to B_2$$

transimplicial to $K_2^{(2)}$ at all points except those of $\dot{T}_1$.

For then (b) is automatically taken care of, and (a) follows at once by application of Lemma 11. Our aim is accomplished using Lemmas 9 and 10. By Lemma 10, $jhg_{t+1}B_1$ is transimplicial to X', and therefore to X, at all interior points. Consequently $h^{-1}jhg_{t+1}B_1 = g_tB_1$ is transimplicial to $K_2^{(2)}$ at all points in its interior. Before the move we see by restriction that $hg_{t+1}\dot{B}_1$ is transimplicial to Y except at points of $hg_{t+1}\dot{T}_1$. Therefore, since j keeps Y fixed, Lemma 9 shows $jhg_{t+1}B_1$ transimplicial to X at all points of $jhg_{t+1}(\dot{B}_1 - \dot{T}_1)$. Consequently $g_tB_1 \subset B_2$ is transimplicial to $K_2^{(2)}$ at points of $g_t(\dot{B}_1 - \dot{T}_1)$, and the induction is complete.

Proof of Lemma 8.

Let us recall and simplify the statement of Lemma 8. We are given two closed manifolds M, Q. Let $\mathcal{E}$ denote the set of embeddings $e : M \to M \times Q$ with the property that the composition

$$M \xrightarrow{e} M \times Q \xrightarrow{proj^n} M$$

is a homeomorphism. In particular if $f:M \to Q$ is an arbitrary map, then its graph $\Gamma f \in \mathcal{E}$. Let K_1, K_2 be Brouwer triangulations of M, Q and let K_3 be a simplicial subdivision of the convex linear cell complex $K_1 \times K_2$. Then Lemma 8 follows from:

<u>Lemma 8*.</u> <u>Given $e \in \mathcal{E}$, there exists $e' \in \mathcal{E}$ transimplicial to K_3 and ambient isotopic to e.</u>

<u>Proof.</u> By Theorem 4 we can ambient isotop e to e' transimplicial to K_3. The only thing left is to make sure $e' \in \mathcal{E}$, and this is achieved by taking care over the t-shifts. The ambient isotopy e to e' consists of a finite sequence of local shifts

$$e \to e_1 \to e_2 \to \dots \to e_r = e'.$$

We proceed by induction on the number of local shifts. This begins trivially since $e \in \mathcal{E}$. Suppose we have managed to ensure $e_i \in \mathcal{E}$, and consider the local shift $e_i \to e_{i+1}$. It takes place inside a ball AL, where $A \in K_3'$, K_3' some subdivision of K_3, and $L = \mathrm{lk}(A, K_3')$. Since K_3' is a subdivision of $K_1 \times K_2$, there exist simplexes $A_1 \in K_1$, $A_2 \in K_2$ such that

$$AL \subset \mathrm{st}(A_1, K_1) \times \mathrm{st}(A_2, K_2).$$

Also, since K_1, K_2 are both Brouwer, we can choose linear embeddings $h_1:\mathrm{st}(A_1, K_1) \to E^m$, $h_2:\mathrm{st}(A_2, K_2) \to E^q$. We shall use the linear embedding

$$h = h_1 \times h_2 : AL \to E^m \times E^q$$

in order to construct the shift.

In detail, if $X = h(AL)$ and $v = h\hat{A}$, then $X = v\dot{X}$. Choose w in general position in $\mathring{X}$ sufficiently near v such that $X = w\dot{X}$. Define $j: X \to X$ by mapping $v \to w$, keeping $\dot{X}$ fixed, and joining linearly. Use $h^{-1}jh: AL \to AL$ to define the shift $e_i \to e_{i+1}$.

Now let $M_0 = e_i^{-1}(AL) \subset M$, and let $Z = he_iM_0$. Then Z is an m-cell, and $Z \subset X$, $\dot{Z} \subset \dot{X}$, $Z = v\dot{Z}$. Let $\pi: E^m \times E^q \to E^m$ denote the projection. Then since $e_i \in \mathcal{E}$, π embeds Z as an m-cell in E^m, and

$$\pi Z = (\pi v)(\pi \dot{Z}).$$

We now choose w sufficiently close to v such that

$$\pi Z = (\pi w)(\pi \dot{Z}).$$

As a consequence, although $e_iM_0 \neq e_{i+1}M_0$, nevertheless the projection $M \times Q \to M$ will map both e_iM_0 and $e_{i+1}M_0$ homeomorphically onto the same m-cell in M. Then $e_{i+1} \in \mathcal{E}$, and the inductive step is complete.

We end this section by filling the gap left in the proof of Theorem 3. For this we need:

Lemma 12. *Let E be a simplex, F a principal face of E, v the vertex opposite F, and W a submanifold of F. If W is transimplicial to F at a point x, then vW is transimplicial to E at x.*

Proof. By exactly the same technique as was used for Lemma 9.

Corollary. *Let F, C be simplexes, and W a submanifold of F. If W is transimplicial to F, then CW is transimplicial to CF at points of W.*

Proof. Join successively to the vertices of C, applying the lemma at each step.

Recall the proof of Theorem 3. With the previous notation, we needed to show that for any point $x \in M_1$,

$$X \cap AL^Q \subset AL^Q \xrightarrow{s} vL^Q$$

is $F(m + a - p, a)$ at x.

Given $B \in L^Q$, write $AB = CF$ where $F = AB \cap J$ and C is the face of AB opposite F. Since M_1 is transimplicial to J, we have by restriction $M_1 \cap F$ transimplicial to F. But $X \cap AB = C(M_1 \cap F)$ and so by the Corollary above $X \cap AB$ is transimplicial to AB at x. In other words

$$X \cap AB \subset AB \xrightarrow{s} vB$$

is $F(m + a - p, a)$ at x. Therefore by glueing (Lemma 2) over all $B \in L^Q$, we have the desired result. This completes the proof of Theorem 3.

PROOF OF THEOREM 5.

It is necessary to do a considerable amount of preparatory work.

Collars.

Let Q be a manifold with boundary. A collar c_Q of Q is an embedding

$$c_Q : \dot{Q} \times I \to Q$$

such that $c(x, 0) = x$ for all $x \in \dot{Q}$. Any compact manifold has a collar, and any two collars are ambient isotopic keeping the boundary fixed ([12], Theorem 13).

Given a proper embedding $f : M \to Q$ then by [12], Lemma 24 we can choose collars c_M, c_Q of M, Q that are compatible with f, that is to say the following diagram commutes

$$\begin{tikzcd} \dot{M} \times I \arrow[r, "c_M"] \arrow[d, "(f|\dot{M})\times 1"'] & M \arrow[d, "f"] \\ \dot{Q} \times I \arrow[r, "c_Q"] & Q \end{tikzcd} .$$

In particular if P is a proper submanifold of Q, then we can choose compatible collars, that is to say $c_P = c_Q | \dot{P} \times I$.

Suppose we are now given a collar c_Q of Q and a triangulation J of the boundary $\dot{Q}$. If Q_1 denotes the image of c_Q, then we can extend J to a triangulation of the collar Q_1, in a canonical way, as follows. $J \times I$ is a convex linear cell complex, which has a canonical simplicial subdivision, $(J \times I)'$ say, obtained by starring in order of decreasing dimension all simplexes $A \times 1$, $A \in J$. The resulting triangulation

$$(J \times I)' \to \dot{Q} \times I \xrightarrow{c_Q} Q_1$$

is called the <u>canonical extension</u> of J to the collar. The canonical extension is functorial in the following sense. Let P be a proper submanifold of Q, and suppose we are given compatible collars c_Q, c_P and a triangulation J of $\dot{Q}$, $\dot{P}$. If Q_1, P_1 denote the images of c_Q, c_P, then the canonical extension of J to Q_1 is a triangulation of the pair Q_1, P_1 and the restriction to P_1 is the canonical extension of the restriction of J to $\dot{P}$.

<u>Lemma 13.</u> <u>Let P be a proper submanifold of Q. Given a triangulation J of $\dot{Q}$, $\dot{P}$ then there exists an extension of J to a triangulation K of Q, P. Further, if J is Brouwer then K can be chosen to be Brouwer.</u>

<u>Proof.</u> Choose compatible collars c_Q, c_P, let Q_1, P_1 denote their images, and let $Q_2 = \overline{Q - Q_1}$, $P_2 = \overline{P - P_1}$. Let $(J \times I)'$ be the canonical extension of J to Q_1 and let J' denote the subcomplex triangulating the inside of the collar, $\dot{Q}_2$.

Choose any triangulation L of Q_2, P_2. Then both J' and $\dot{L}$ triangulate $\dot{Q}_2$, and so they have a common subdivision, say $J'' = \dot{L}'$ (see [12] Lemma 4). These subdivisions extend uniquely to subdivisions $(J \times I)''$, L' of $(J \times I)'$, L without introducing any more vertices. Identifying $J'' = \dot{L}'$, the union $K = (J \times I)'' \cup L'$ gives a triangulation of Q, P and provides the required extension of J.

Finally, if J is Brouwer then so is the canonical extension to the collar. Therefore K is Brouwer at the boundary, and so by Lemma 7(b) we can choose a Brouwer subdivision K' that also extends J.

Relative t-shifts.

In proving Theorem 5 we shall need to be more precise in our t-shift process; recall the considerable choice available for the position of the point w. The necessary accuracy is expressed in the following lemma.

Let M, Q be manifolds and K a triangulation of Q. Given a map $f:M \to Q$ let

$$T_K^f = \{x \in M : f \text{ is transimplicial to } K \text{ at } x\}.$$

<u>Lemma 14. Suppose $f: M \to Q$ is a proper embedding, K a Brouwer triangulation of Q, and $K^{(2)}$ a second derived of K. Let K_1 be a triangulation of M, and K_2 a subdivision of $K^{(2)}$ such that $f: K_1 \to K_2$ is simplicial. Let T be a t-simplex of K_1 such that $T \subset \mathring{M}$, and $f \to g$ the associated local t-shift made in the local linear structure of K. If the shift is sufficiently small then $T_K^f \subset T_K^g$.</u>

<u>Remark.</u> The proof of Lemma 14 is long, and more complicated than our corresponding work in the proof of Theorem 4. The difficulty is that we are in a situation where the glueing lemma is no longer applicable.

<u>Proof of Lemma 14.</u>

Since f is a proper embedding we know $fT \subset \mathring{Q}$. Define, as before, B_2 to be a simplicial neighbourhood of fT modulo its boundary in $K_2^{(2)}$, and $B_1 = f^{-1}B_2$. Now

$$\begin{aligned} fB_1 \subset B_2 &\subset \overline{st}(fT, K_2) \\ &\subset \overline{st}(u'', K^{(2)}) \quad \text{for some vertex } u'' \in \mathring{K}^{(2)} \\ &\subset st(u, K) \qquad \text{for some vertex } u \in \mathring{K}. \end{aligned}$$

Therefore the problem is localised both with respect to K and K_2. Using the Brouwer property of K choose a linear embedding

$$h: \overline{st}(u, K) \to E^q.$$

Then h automatically embeds B_2 linearly in E^q. The local shift may now be defined as before; in particular we write

$$f_0 = hf : f^{-1}\overline{st}(u, K) \to E^q, \text{ and}$$
$$g_0 = jhf : f^{-1}\overline{st}(u, K) \to E^q.$$

Remark. The above construction explains our reason for calling this section "relative t-shifts". We are t-shifting f with respect to the triangulation K_2, but with the reservation that we do so relative to the local linear structure of K.

Suppose f is transimplicial to K at $x \in M$, we want to ensure that g is also. If $x \notin B_1$, the result is trivial because a neighbourhood of x is not moved by the shift. Also if $x \in \overset{\circ}{B}_1$, application of Lemma 10 as in the proof of Theorem 4 shows g transimplicial to $K_2^{(2)}$, and therefore to K, at x.

Therefore there remains the case $x \in \dot{B}_1$; here $fx = gx$. Let A be the simplex of K such that $fx \in \overset{\circ}{A}$, and let $L^A = \mathrm{lk}(A, K)$. Then $AL^A \subset \overline{st}(u, K)$. Define $E^a = [hA]$, the linear subspace of E^q spanned by hA, and $E^{q-a} = E^q//E^a$, the decomposition space whose points are a-dimensional linear subspaces of E^q parallel to E^a. Let $\pi : E^q \to E^{q-a}$ be the natural projection and $\pi^* : E^q \to E^a$ the orthogonal projection (see Figure 8).

Since f is transimplicial to K at x, the pair

$$f^{-1}AL^A \xrightarrow{f_0} E^q \xrightarrow{\pi} E^{q-a}$$

is $F(m + a - q, a)$ at x. Therefore if $y = f_0x$, $z = \pi y$, there is a neighbourhood N of z in E^{q-a} (which we may take to be a simplex), and embeddings φ, ψ onto neighbourhoods of x, y in $f^{-1}AL^A$, E^q respectively, such that the following diagram commutes

$$\begin{tikzcd} N \times D^{m+a-q} \arrow[r, "1\times k"] \arrow[d, "\varphi"'] & N \times D^a \arrow[r, "\mathrm{proj}^n"] \arrow[d, "\psi"'] & N \arrow[d, "\subset"] \\ f^{-1}AL^A \arrow[r, "f_0"'] & E^q \arrow[r, "\pi"'] & E^{q-a} \end{tikzcd} .$$

Call E^a "the <u>vertical</u>". Given two points $y_1, y_2 \in E^q$, let $\alpha(y_1, y_2)$ denote the angle that the vector y_1y_2 makes with the vertical. More precisely

$$\alpha(y_1,y_2) = \tan^{-1}\left(\frac{d(\pi y_1, \pi y_2)}{d(\pi^* y_1, \pi^* y_2)}\right)$$

$$0 \leqslant \alpha \leqslant \pi/2$$

where d denotes Euclidean distance.

<u>Sublemma 1.</u> <u>There exists $\alpha_0 > 0$ such that given any two distinct points $x_1, x_2 \in N$ and any $y \in D^a$, then</u>

$$\underline{\alpha(\psi(x_1, y), \psi(x_2, y)) \geqslant \alpha_0}.$$

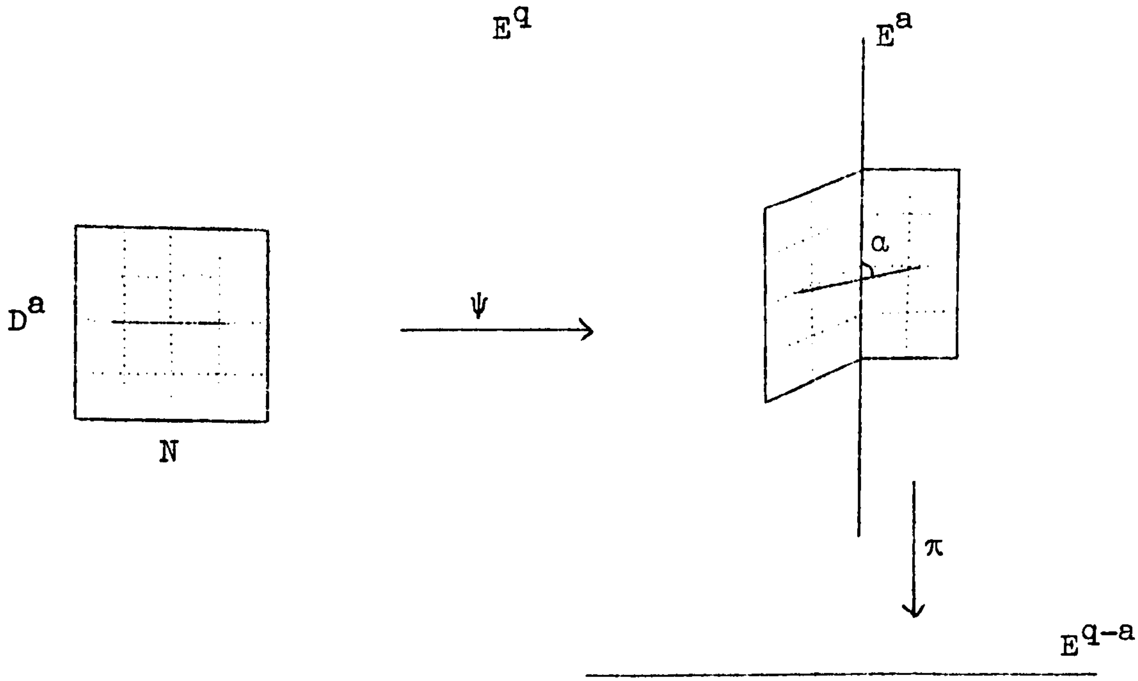

Figure 8

Proof. We chose N to be a simplex and we can regard D^a as a simplex, therefore $N \times D^a$ is a convex linear cell. Let J be a simplicial subdivision of $N \times D^a$ such that $\psi : J \to E^q$ is linear.

Case (i): Suppose (x_1, y), (x_2, y) both lie in a simplex $S \in J$. Then their images $\psi(x_1, y)$, $\psi(x_2, y)$ lie in $\psi(S \cap (N \times y))$, which is a convex linear cell in E^q. This cell is embedded in E^{q-a} by π (because $\pi\psi : N \times D^a \to N$ is the projection), and therefore it makes an angle $\alpha_S > 0$ (independent of y since $\psi|S$ is linear) with the vertical. Let $\alpha_0 = \min(\alpha_S : S \in J)$. Then $\alpha(\psi(x_1, y), \psi(x_2, y)) \geq \alpha_S \geq \alpha_0$.

<u>Case (ii)</u>: (x_1, y) and (x_2, y) do not lie in the same simplex of J. Since $\psi(N \times y) \xrightarrow{\pi} N$ is a homeomorphism, the vector $x_1x_2 \subset N$ lifts under π^{-1} to an arc, I say, in $\psi(N \times y)$ which joins $\psi(x_1, y)$ to $\psi(x_2, y)$. Then I consists of a finite number of linear segments each one of which makes an angle greater than or equal to α_0 with the vertical. Therefore the vector joining the ends of I also makes an angle $\geq \alpha_0$ with the vertical. This completes Sublemma 1, and we now continue with the proof of Lemma 14.

As before we denote the combinatorial ball hB_2 by X, and its boundary by Y. Recall the homeomorphism $j: X \to X$, defined by moving $f_0\hat{T} = v$ to a suitable point $w = g_0\hat{T}$ in general position with respect to the vertices of X, and joining linearly to Y. Extend j by the identity to the whole of E^q.

<u>Sublemma 2.</u> <u>Given $\alpha_0 > 0$, there exists $\varepsilon > 0$ such that if $d(f_0\hat{T}, g_0\hat{T}) < \varepsilon$ then for all $y_1, y_2 \in E^q$</u>

$$\underline{\alpha(y_1, y_2) \geq \alpha_0 \Longrightarrow \alpha(jy_1, jy_2) > 0.}$$

<u>Proof.</u> Let S be a simplex of X. Since $j|S$ is linear, there exists $\varepsilon_S > 0$ such that if j moves $f_0\hat{T}$ less than ε_S, then any vector in S changes direction by less than α_0.

Let $\varepsilon = \min(\varepsilon_S : S \in X)$. Suppose now that j moves $f_0\hat{T}$ by less than ε. Given y_1, y_2 in E^q, the vector y_1y_2 consists of a finite number of segments, each one lying either in some simplex of X or in $E^q - X$. Therefore $j(y_1y_2)$ is an arc, consisting of a finite number of linear segments each making an angle less than α_0 with y_1y_2. Therefore the vector $(jy_1)(jy_2)$ joining the ends of this arc also makes an angle less than α_0 with y_1y_2. But y_1y_2 makes an angle $\geqslant \alpha_0$ with the vertical, and therefore $(jy_1)(jy_2)$ makes an angle > 0 with the vertical. This completes Sublemma 2.

We now make our local shift within the ε given by Sublemma 2; it remains to show this ensures g transimplicial to K at x. To do this it is sufficient to construct a commutative diagram

$$\begin{tikzcd}
N_* \times D_*^{m+a-q} \arrow[r, "1\times k_*"] \arrow[d, dashed, "\varphi_*"'] & N_* \times D_*^a \arrow[r, dashed, "\mathrm{proj}^n"] \arrow[d, dashed, "\psi_*"'] & N_* \arrow[d, dashed, "\subset"] \\
g^{-1}{}_{AL}{}^{A} \arrow[r, dashed, "g_0"'] & E^q \arrow[r, "\pi"'] & E^{q-a}
\end{tikzcd}$$

which we now proceed to do. Let $U = j\psi(N \times D^a)$; since $jy = y$, U is a neighbourhood of y in E^q. Define $\theta: U \to D^a$ as the composition

$$U \xleftarrow{j} \psi(N \times D^a) \xleftarrow{\psi} N \times D^a \xleftarrow{\mathrm{proj}^n} D^a.$$

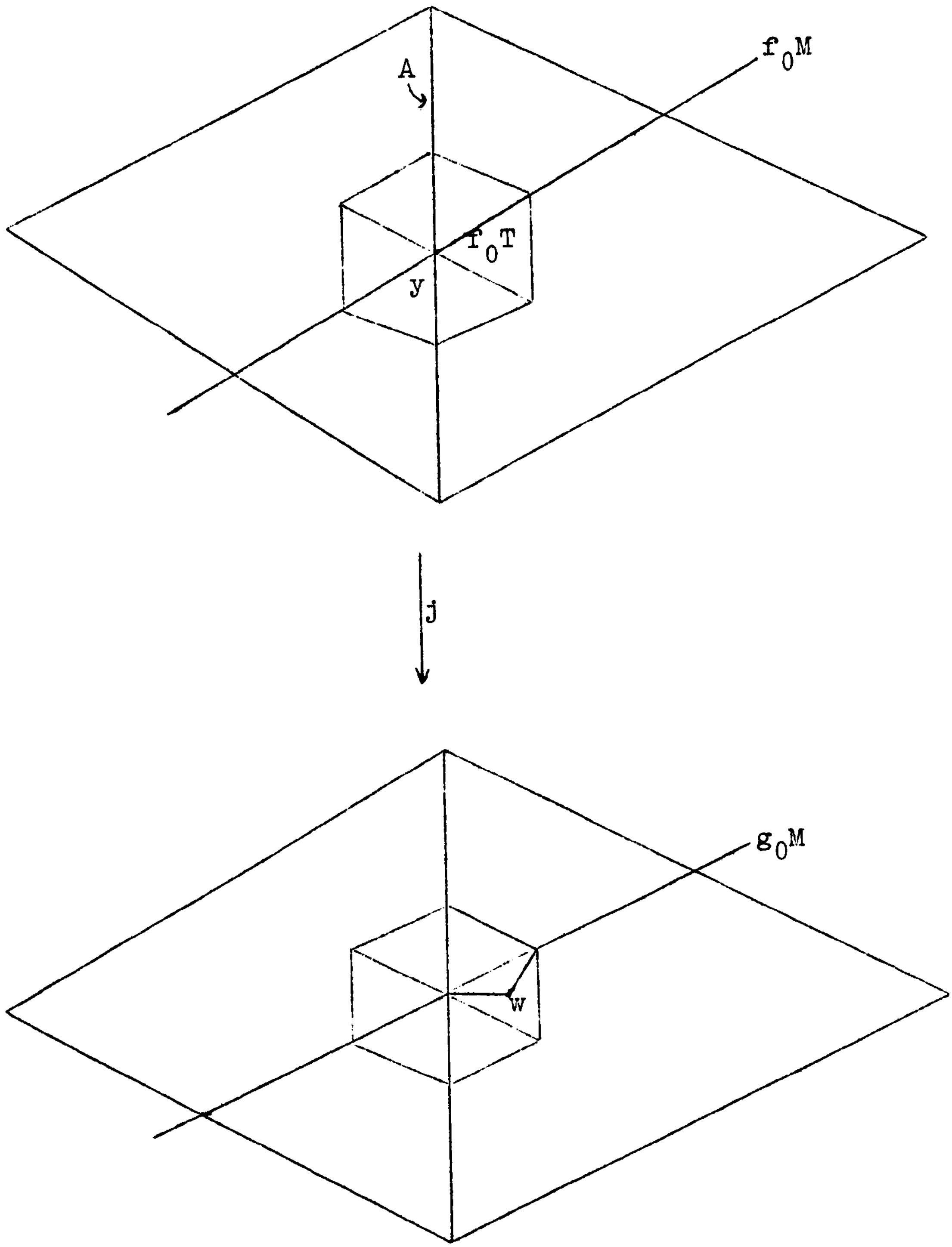

Figure 9

Then the product $\pi \times \theta : U \to E^{q-a} \times D^a$ is piecewise linear and onto a neighbourhood of $(z, 0)$. We claim that it is an embedding; for given $y_1 \neq y_2 \in U$ with $\theta y_1 = \theta y_2$, then $\alpha(y_1, y_2) > 0$ by Sublemmas 1 and 2, thus $\pi y_1 \neq \pi y_2$. Choose a neighbourhood N_* of z in E^{q-a}, and a disc neighbourhood D^a_* of 0 in D^a such that $N_* \times D^a_* \subset (\pi \times \theta)U$. Define $\psi_* = (\pi \times \theta)^{-1} : N_* \times D^a_* \to E^q$. We have therefore produced the right hand half of our diagram. Since $k : D^{m+a-q}, 0 \to D^a, 0$ is an embedding, choose D^{m+a-q}_* as a disc neighbourhood of 0 in $k^{-1}D^a_*$ and define $k_* = k|D^{m+a-q}_* : D^{m+a-q}_* \to D^a_*$. Finally we need to define φ_*. It is elementary to check that

$$\psi_*(1 \times k_*)(N_* \times D^{m+a-q}_*) \subset g_0 AL^A \ ,$$

therefore since g_0 is an embedding we can define

$$\varphi_* = g_0^{-1}\psi_*(1 \times k_*) : N_* \times D^{m+a-q}_* \to g_0^{-1} AL^A \ .$$

We have not finished the proof of Lemma 14 yet : so far we have shown that, given $x \in \dot{B}_1 \cap T^f_K$, then there exists $\varepsilon > 0$, such that if $d(f_0\hat{T}, g_0\hat{T}) < \varepsilon$ then $x \in \dot{B}_1 \cap T^g_K$. Notice that ε depends upon x. Suppose that $x' \in \dot{B}_1 \cap T^f_K$ and x, x' lie in the interior of the same simplex $S \in K_1$.

<u>Sublemma 3.</u> <u>The same ε will do for x'.</u>

<u>Proof.</u> Choose neighbourhoods V, V' of x, x' in $\mathrm{st}(S, K_1)$, such that linear translation by the vector xx'

maps V into V'. Let $\lambda : V, x \to V', x'$ denote this linear translation. Since f_0 maps $st(S, K_1)$ linearly into E^q, there are linear translations λ', λ'' of E^q, E^{q-a} respectively such that the diagram

$$\begin{tikzcd}
V, x \arrow[r, "f_0"] \arrow[d, dashed, "\lambda"] & E^q \arrow[r, "\pi"] \arrow[d, dashed, "\lambda'"] & E^{q-a} \arrow[d, "\lambda''"] \\
V', x' \arrow[r, dashed, "f_0"] & E^q \arrow[r, "\pi"] & E^{q-a}
\end{tikzcd}$$

is commutative. Recall the commutative diagram

$$\begin{tikzcd}
N \times D^{m+a-q} \arrow[r, "1 \times k"] \arrow[d, "\varphi"] & N \times D^a \arrow[r, dashed, "proj^n"] \arrow[d, "\psi"] & N \arrow[d, dashed, "\subset"] \\
f^{-1}(AL^A) \arrow[r, "f_0"] & E^q \arrow[r, dashed, "\pi"] & E^{q-a}
\end{tikzcd}$$

expressing the fact that f is transimplicial to K at x. We can choose N, D^{m+a-q} such that $\operatorname{im}\varphi \subset V$ (replacing them by subballs if necessary); note that this replacement does not alter the angle α_0 of Sublemma 1. Now replace the three vertical arrows by $\lambda\varphi$, $\lambda'\psi$, λ'' respectively, and we have an expression of the transimpliciality of f to K at x'. Again α_0 is unaltered. Therefore the ε of Sublemma 2 is unaltered. This completes the proof of Sublemma 3, and we now conclude the lemma.

$\dot{B}_1$ is covered by the interiors of a finite number of simplexes of K_1; for each of these choose an ε by Sublemmas 2 and 3, and select the minimum such ε.

Therefore if $d(f_0\hat{T}, g_0\hat{T}) < \varepsilon$ then $\dot{B}_1 \cap T_K^f \subset \dot{B}_1 \cap T_K^g$. In other words if the shift is sufficiently small $T_K^f \subset T_K^g$. This completes the proof of Lemma 14.

Proof of Theorem 5.

Recall the statement of Theorem 5. We are given a manifold-pair Q, P, a Brouwer triangulation J of the boundary $\dot{Q}$, $\dot{P}$ and a proper embedding $f:M \to Q$ such that $f|\dot{M}$ is transimplicial to J. We have to extend J to a Brouwer triangulation K of Q, and ambient isotop f to g keeping $\dot{Q}$ fixed, so that g is transimplicial to K.

First choose compatible collars c_Q, c_P of Q, P. Then choose collars c_M, c_Q^* of M, Q compatible with $f:M \to Q$. By [12] Theorem 13 ambient isotop c_Q^* to c_Q keeping $\dot{Q}$ fixed, and suppose that this ambient isotopy carries f to g. The result is that c_M, c_Q are now compatible with g.

Intuitively what we have done so far is unfold M near the boundary, and get rid of the sort of kinks that are illustrated in the diagram of the Remark after Theorem 5. More precisely, we shall describe this unfolding in transimplicial terms, as follows.

Extend the triangulation J to the collars by the canonical extension, which is Brouwer, and then extend further over the rest of the manifolds by Lemma 13 to give a Brouwer triangulation K of Q, P. We claim that g is transimplicial to K at points of $\dot{M}$ (notice that before the unfolding we only knew that $f|\dot{M}$ was transimplicial to J at points of $\dot{M}$). To prove this claim we use the compatibility of the collars c_M, c_Q with g, because it then suffices to show that

$$(g|\dot{M}) \times 1 : \dot{M} \times I \to \dot{Q} \times I$$

is transimplicial at points of $\dot{M} \times 0$ to the canonical triangulation $(J \times I)'$ of $\dot{Q} \times I$. Now we can use the fact that $g|\dot{M} = f|\dot{M}$, which is transimplicial to J. Given $x \in \dot{M} = \dot{M} \times 0$, suppose $fx \in \mathring{A}$, $A \in J = J \times 0$. Let v be a vertex of A, $L = \mathrm{lk}(A, K)$, $L_1 = \mathrm{lk}(A, J)$. By the transimpliciality of $f|\dot{M}$ we have a commutative diagram

$$\begin{array}{ccccc}
N \times D^{m+a-q} & \xrightarrow{1 \times k} & N \times D^a & \xrightarrow{\mathrm{proj}^n} & N \\
\downarrow \varphi & & \downarrow & & \downarrow \subset \\
f^{-1} A L_1 & \xrightarrow{f} & A L_1 & \xrightarrow{s^A} & v L_1
\end{array}$$

Let $U = [\psi(N \times D^a) \times I] \cap AL$, and let $r : \dot{Q} \times I \to \dot{Q}$ be the projection. Define $\theta : U \to D^a$ as the composition

$$U \xrightarrow{r} \psi(N \times D^a) \xleftarrow{\psi} N \times D^a \xrightarrow{\mathrm{proj}^n} D^a .$$

Then $s^A \times \theta : U \to vL \times D^a$ is a piecewise linear map onto a neighbourhood of $(v, 0)$. Moreover it is an embedding because given $u_1 \neq u_2$ such that $s^A u_1 = s^A u_2$, then $u_1 u_2$ is parallel to A, and so is $(vu_1)(vu_2)$, implying that $\theta u_1 \neq \theta u_2$. Therefore, choosing discs $N_* \subset vL$, $D_*^a \subset D^a$ such that $N_* \times D_*^a \subset (s^A \times \theta)U$, we can define

$$\psi_* = (s^A \times \theta)^{-1} : N_* \times D_*^a \to AL.$$

The required diagram for the transimpliciality of $(f|\dot{M}) \times 1$ at x can now be built up in the usual fashion. Therefore g is transimplicial to K at points of $\dot{M}$.

There remains to isotop g transimplicial on the interior (keeping $\dot{Q}$ fixed) as follows. By Lemma 4 g is transimplicial to K at all points in some open neighbourhood U of $\dot{M}$. Let $K^{(2)}$ be the second barycentric derived of K. Choose a triangulation K_1 of M and a subdivision K_2 of $K^{(2)}$ such that

(a) $g : K_1 \to K_2$ is simplicial, and

(b) if V is the closed simplicial neighbourhood of $\dot{K}_1$ in K_1, then $V \subset U$.

Now perform the t-shifts of Lemma 14 in order of decreasing dimension for all simplexes $T \in K_1$ such that $\mathring{T} \subset M - V$. Then, as in the proof of Theorem 4, we see that g becomes transimplicial to K_2, and therefore to K, at all points of $M - V$. By Lemma 14 g remains transimplicial to K at points of V.

The proof of Theorem 5 is complete.

Remark. The significance of Lemma 14 in the above proof should now be apparent. At the last stage we had an embedding g transimplicial to K at points of $\dot{M}$. If we had just haphazardly made interior shifts of g with respect to some subdivision of K, then we may well have introduced new folds at boundary points, and so lost the transimplicial property there.

RELATIVE TRANSVERSALITY?

We were able to prove relative transimpliciality (in Theorem 5) but not relative transversality. We tried the procedure

transversal on the boundary	$\Longrightarrow$	transimplicial on the boundary	$\Longrightarrow$	isotop transimplicial on the interior	$\Longrightarrow$	transversal on the interior,

and although the second two steps are given by Theorem 5 and Lemma 6, we failed to achieve the first step. Essentially it is a passage from local to global, because transversality is local but transimpliciality is global, in the sense that an atlas is local while a triangulation is global. It is true that given manifolds $M \subset Q$, it is possible to triangulate Q so that M is transimplicial as follows: triangulate Q anyhow, ambient isotop M transimplicial, and then apply the inverse isotopy to move both M and the triangulation back. But the question is whether it is possible to have another manifold as a subcomplex at the same time.

<u>Conjecture 1</u> <u>Given two transversal submanifolds of Q, then it is possible to triangulate Q so that one is a subcomplex and the other transimplicial.</u>

Conjecture 1 would supply the missing step to prove:

<u>Conjecture 2</u> (Relative Transversality) <u>If M, P are proper submanifolds of Q such that $\dot{M}$, $\dot{P}$ are transversal in $\dot{Q}$, then M can be ambient isotoped transversal to P keeping $\dot{Q}$ fixed.</u>

A special case of Conjecture 2, which in fact turns out to be equivalent to Conjecture 2 is:

<u>Conjecture 3</u> <u>Transversal spheres S^{m-1}, $S^{p-1} \subset S^{q-1}$ can be spanned by transversal discs D^m, $D^p \subset D^q$.</u>

Joining linearly to interior points is no good, because if we join them to the same point the discs fail to be transversal at that point, and if we join them to separate points, they fail to be transversal at the boundary (by the folded disc phenomenon). Conjecture 2 would imply:

<u>Conjecture 4</u> <u>If M, Q are closed and f, g:M $\to$ Q are homotopic maps transversal to P, then $f^{-1}P$, $g^{-1}P$ are cobordant.</u>

Summarising:

Conjecture 1 $\Longrightarrow$ Conjecture 2 $\Longleftrightarrow$ Conjecture 3 $\Longrightarrow$ Conjecture 4.

TUBES

<u>Definition.</u> We use the word <u>tube</u> as an abbreviation for the term "abstract regular neighbourhood", which is rather a mouthful. Let M^m be closed. Define a <u>t-tube on M</u> to be a manifold T^{m+t} together with a proper (locally flat) embedding $e:M \to T$ such that $T \searrow eM$. In other words T is a regular neighbourhood of a homeomorphic copy of M. We call t the dimension of the tube.

Two tubes are <u>homeomorphic</u> if there exists a homeomorphism h making a commutative diagram

$$\begin{tikzcd} & T_1 \arrow[d, dashed, "h"] \\ M \arrow[ur, "e"] \arrow[r, "h"'] & T_2 \end{tikzcd}$$

Let $\mathcal{T}^t(M)$ denote the set of homeomorphy classes of t-tubes on M, and let $\mathcal{T}(M) = \sum_0^\infty \mathcal{T}^t(M)$.

<u>Remarks.</u>

1. Tubes are the natural analogue in piecewise linear theory of vector bundles in differential theory. The existence and uniqueness of regular neighbourhoods show that any proper embedding $M \subset Q$ determines a unique

element of $\mathfrak{J}^{q-m}(M)$, which we call the <u>normal tube</u>.

2. The important thing about tubes is that, like tubes in ordinary life, they are not fibered. In fact Hirsch's example is a 3-tube on S^4 that cannot be fibered. In some sense the lack of fibering is more "geometrical" because the tube is more homogeneous.

3. In the stable range, $t \geqslant m + 2$ Haefliger and Wall [5] have shown that any tube can be fibered with t-discs, and so $\mathfrak{J}^t(M)$ coincides with $K^t_{top}(M)$ of piecewise linear microbundle theory.

4. The collapse $T \searrow cM$ determines a homotopy equivalence $\pi : T \to M$ such that $\pi c = 1$. However π is not natural, not unique, and not in general a fibering. The non-naturality of π reveals itself, when it turns out to be no good for defining induced tubes.

5. There is a trivial tube $0 \in \mathfrak{J}^t(M)$ containing $M \times D^t$, and a suspension $\mathfrak{J}^t(M) \to \mathfrak{J}^{t+1}(M)$ given by product with I, which stabilises in the stable range. To examine the structure of $\mathfrak{J}(M)$ more thoroughly we define below subtubes, quotient tubes, induced tubes and Whitney sums.

6. The concept of tube generalises to polyhedra other than manifolds, to give a theory totally different from vector bundle theory, even in the stable range.

<u>Subtubes</u>

Call $e_1 : M \to T_1$ a subtube of $e : M \to T$ if T_1 is a proper (locally flat) submanifold of T such that $T \searrow T_1$ and the diagram

$$\begin{tikzcd} & T_1 \arrow[d, "\subset"] \\ M \arrow[ur, "e_1"] \arrow[r, "e"'] & T \end{tikzcd}$$

is commutative. Call two subtubes $T_1, T_2 \subset T$ <u>transversal</u> if T_1, T_2 intersect transversally in eM. Notice that in this case $t = t_1 + t_2$. We call the class of T_2 the <u>quotient tube</u> T/T_1.

<u>Corollary to Theorem 3.</u> <u>Quotient tubes exist.</u>

<u>Question.</u> <u>Are they unique?</u>

We can question not only whether two such T_2's are unique up to homeomorphism, but whether they are unique up to ambient isotopy, keeping T_1 fixed.

<u>Proof of Corollary.</u> Given $eM \subset T_1 \subset T$, Theorem 3 furnishes a manifold P intersecting T_1 transversally in eM. So far P is not proper. Triangulate everything and let N be a second derived neighbourhood of T_1 in T. Then N is a tube, and T_1 a subtube because $N \searrow T_1$. Also $N \cap P$ is a subtube because $N \searrow (N \cap P) \cup T_1 \searrow N \cap P$, and $N \cap P$ cuts T_1 transversally. By uniqueness of

regular neighbourhoods, there is a homeomorphism $N \to T$ keeping T_1 fixed, and throwing $N \cap P$ onto T_2, say. We have shown T_2 exists.

<u>Quotient normal tubes.</u> Suppose we are given proper embeddings $M^m \subset P^p \subset Q^q$, where M is closed. Define the quotient normal tube on M to be the quotient tube T_Q/T_P where T_P, T_Q are regular neighbourhoods of M in P, Q such that T_P is a subtube of T_Q. Notice that $\dim(T_Q/T_P) = q - p$.

<u>Induced tubes.</u>

Given a map $f:M_1 \to M_2$ between closed manifolds and a tube $e_2:M_2 \to T_2$ on the target, define the <u>induced tube</u> on M_1 to be the quotient normal tube of

$$M_1 \xrightarrow{\Gamma f} M_1 \times M_2 \xrightarrow{1 \times e_2} M_1 \times T_2.$$

Notice that the induced tube has the same dimension as the given tube. By the above, induced tubes exist, but we do not know if they are unique.

<u>Remark.</u> Normally induced objects are defined categorically. For example if $\Pi:V_2 \to M_2$ is a vector bundle then the induced vector bundle is the pull-back of

$$\begin{tikzcd} & V_2 \arrow[d, "\pi"] \\ M_1 \arrow[r, "f"] & M_2 \end{tikzcd} .$$

However in the case of tubes π is non-natural, and consequently the pull-back is not in general a manifold. What is natural is the embedding $e_1 : M_1 \to T_1$ of a tube on the source of f, but the push-out of

$$\begin{array}{ccc} T_1 & & \\ \uparrow{\scriptstyle e_1} & & \\ M_1 & \longrightarrow & M_2 \end{array}$$

is again not in general a manifold. Therefore neither pull-backs nor push-outs give induced tubes, and we have to work for them.

Whitney sums.

Given tubes $e_1 : M \to T_1$ and $e_2 : M \to T_2$ on the same manifold M, define the Whitney sum $T = T_1 \oplus T_2$ to be the quotient normal tube of

$$M \xrightarrow{\text{diagonal}} M \times M \xrightarrow{e_1 \times e_2} T_1 \times T_2.$$

Notice that $t = t_1 + t_2$, and so the Whitney sum gives a product $\mathfrak{J}^{t_1} \times \mathfrak{J}^{t_2} \to \mathfrak{J}^{t_1+t_2}$. Again we have existence, but uniqueness is unsolved.

Questions. (i) Can T_1, T_2 be embedded transversally in $T_1 \oplus T_2$?

(ii) Is the Whitney sum homeomorphic to the tube induced from $e_1 : M \to T_1$ by $\pi_2 : T_2 \to M$, and vice versa?

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TRANSVERSALITY FOR POLYHEDRA

by M.A. ARMSTRONG

Some transversality results for piecewise linear manifolds were announced and proved in [1], [2]. In this paper the notion of transversality will firstly be extended so as to be applicable to subpolyhedra of a piecewise linear manifold, and then a transversality theorem for polyhedra will be proved using the techniques developed in [2]. Transversality can be considered as a refinement of general position, and in this respect the result given below is an improvement on one of Zeeman's general position theorems ([7] Chapter 6). At each point in a polyhedron there is a natural local product structure, and the transversality of two subpolyhedra requires not only minimality of the dimension of their intersection, but also that their local product structures tie together nicely in the ambient manifold.

It will be assumed, without further mention, that all spaces have a piecewise linear structure, and that all maps are piecewise linear. The standard reference is [7]. Any polyhedra considered will always be compact. However, the ambient piecewise linear manifold is allowed to be compact or non-compact, and bounded or unbounded.

The main theorem may be stated as follows. Let Q be a piecewise linear manifold and X, Y compact subpolyhedra, both of codimension $\geqslant 3$ in Q. If the intersection $X \cap Y$ is contained in the interior $\mathring{Q}$, then X can be ambient isotoped transversal to Y by an arbitrarily small ambient isotopy of Q that keeps the boundary $\dot{Q}$ fixed. The codimension restrictions ensure, by Lickorish's theorem on unknotting cones [5], that the embeddings $X \subset Q$, $Y \subset Q$ are locally quite respectable at interior points of Q.

The approach will be to avoid the inhomogeneity of X and Y by deducing the theorem from a stronger result about maps between manifolds. To simplify the present discussion, assume that Q is closed (i.e. compact and without boundary). Let K be a triangulation of Q, and let M denote a compact manifold. It will be shown that any map $f:M \to Q$ can be ambient isotoped "transimplicial" to K by an arbitrarily small ambient isotopy of Q. Notice that this procedure does not alter the image of the map, but merely changes its position in Q. Consider now the situation $X, Y \subset Q$. Coat X in a regular neighbourhood M, and collapse M to X in some way to produce a map $M \to Q$ which has X as its image. Then ambient isotoping this map transimplicial to some triangulation of Q in which Y is a subcomplex, has the effect of moving X transversal to Y.

In the above sketch use has been made of rather more than just the piecewise linear structure of Q - namely of the local linearity of a particular triangulation. There is an obstruction to a direct proof using only piecewise linearity, and the difficulty may be indicated as follows. Let B^q be a q-ball and S^{q-1} its boundary. Suppose $X, Y \subset S^{q-1}$ are transversal polyhedra of codimension ≥ 3. Then it is natural to ask if X and Y can be spanned in B^q by transversal cones. If this were possible, then by use of Lickorish's cone theorem, one could proceed immediately to a proof of the previous transversality theorem. However, if X and Y are joined to the same point in the interior of B^q the result is transversality everywhere except at this point, and if they are joined to different points then transversality may well be lost at the boundary. This last phenomenon is exhibited in detail, for X and Y spheres, in the introduction to [2].

A transversality theorem for maps, which is stronger than that given in [2], will also be deduced. More precisely, let M, Q be closed manifolds and P a closed locally flat submanifold of Q. Then given a map $f:M \to Q$, there is an arbitrarily close ambient isotopic map that is transversal to P. Notice it is not assumed that P has a normal bundle (equivalently a normal microbundle) in Q.

Familiarity with [2] is recommended, though principal definitions and results will be recalled where necessary.

It is a pleasure to express my gratitude to E.C. Zeeman for the help and encouragement he has given me during this research.

THE MAIN THEOREMS

Our first job is to give a sensible definition of transversality for two polyhedra in a manifold. We shall keep a couple of restrictions in mind - that our definition be free from particular triangulations of the manifold and polyhedra, and that it agree with the following standard version if we are dealing with closed submanifolds of a closed manifold. We shall use D^t to denote a t-disc.

Definition 1. If Q^q is a closed manifold and M^m, P^p closed submanifolds, then M and P are transversal at the point $z \in M \cap P$ if there is an embedding

$$h: D^{m+p-q} \times D^{q-m} \times D^{q-p} \to Q$$

onto a neighbourhood of z in Q, such that

$$h^{-1}M = D^{m+p-q} \times v \times D^{q-p}$$

$$h^{-1}P = D^{m+p-q} \times D^{q-m} \times w$$

where v, w are determined by $h^{-1}z \in D^{m+p-q} \times v \times w$.

M and P are transversal in Q if they are transversal at each point of their intersection.

Let X be a polyhedron. We shall associate with each point $x \in X$ an integer $I(X, x)$, called the intrinsic dimension of X at x, as follows.

<u>Definition 2.</u> $I(X, x)$ is the largest integer t for which there is a polyhedron V, and an embedding

$$f : D^t \times V \to X$$

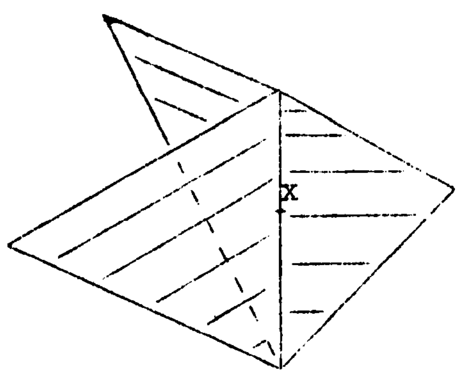
that embeds $\mathring{D}^t \times V$ onto a neighbourhood of x in X.

<u>Examples</u>

1. If X is a manifold of dimension n, then

$$I(X, x) = \begin{cases} n \text{ if } x \text{ lies in the interior } \mathring{X}, \\ n-1 \text{ if } x \text{ lies in the boundary } \dot{X}. \end{cases}$$

2. Let X, x be as illustrated, then $I(X, x) = 1$.
Here V is the cone on three points.



<u>Remarks</u>

1. The set of points of X with intrinsic dimension $\leqslant t$ is what E.C. Zeeman has called the intrinsic t-skeleton of X [8].
2. The set of points of X with intrinsic dimension t forms an open t-manifold.

So as not to interrupt progress towards the main theorems, any further discussion of intrinsic dimension is postponed until the next section. In particular Definition 2 will be reformulated more elegantly there.

Suppose now we have $X, Y \subset Q$. The manifold Q may be compact or non-compact, and with or without boundary. The subpolyhedra X, Y will always be compact. The <u>codimension</u> of $X \subset Q$ is the dimension of Q minus that of X. We shall restrict ourselves to the case $\underline{X \cap Y \subset \mathring{Q}}$. If boundary intersection points are allowed the situation is more complicated, and we have no corresponding result to Theorem 1 below. Therefore we leave discussion of this case until the end of the paper. Let z be a point of $X \cap Y$ and suppose

$$I(X, z) = t,$$
$$I(Y, z) = s.$$

<u>Definition 3.</u> The polyhedra X, Y are <u>transversal at z</u> if there is an embedding

$$h: D^{t+s-q} \times D^{q-t} \times D^{q-s} \to Q$$

onto a neighbourhood of z in $\mathring{Q}$, and subpolyhedra $V \subset D^{q-t}$, $W \subset D^{q-s}$ such that

$$h^{-1}X = D^{t+s-q} \times V \times D^{q-s},$$
$$h^{-1}Y = D^{t+s-q} \times D^{q-t} \times W.$$

X and Y are <u>transversal</u> in Q if they are transversal at each point of their intersection.

In the case where X and Y are closed manifolds, t, s become their respective dimensions, V, W each reduce to a single point, and so the definition agrees with that given earlier. Figures 1a and 1b illustrate transversality and non-transversality situations respectively.

We are now in a position to state our main result.

<u>Theorem 1.</u> <u>Let Q be a piecewise linear manifold and X, Y compact subpolyhedra, both of codimension ≥ 3 in Q. If $X \cap Y \subset \mathring{Q}$, then X can be ambient isotoped transversal to Y by an arbitrarily small ambient isotopy of Q that keeps $\dot{Q}$ fixed.</u>

Our second theorem concerns maps. Let M^m, $P^p \subset Q^q$ be closed manifolds. Given a map $f:M \to Q$ let $\Gamma f:M \to M \times Q$ denote its graph. In [2] the notion of graph transversality was introduced - f is graph transversal to P if ΓfM and $M \times P$ are transversal as submanifolds of $M \times Q$ - and it was shown that arbitrarily close to any map there is a homotopic graph transversal map, provided P is locally flat in Q. We now give a stronger definition and result.

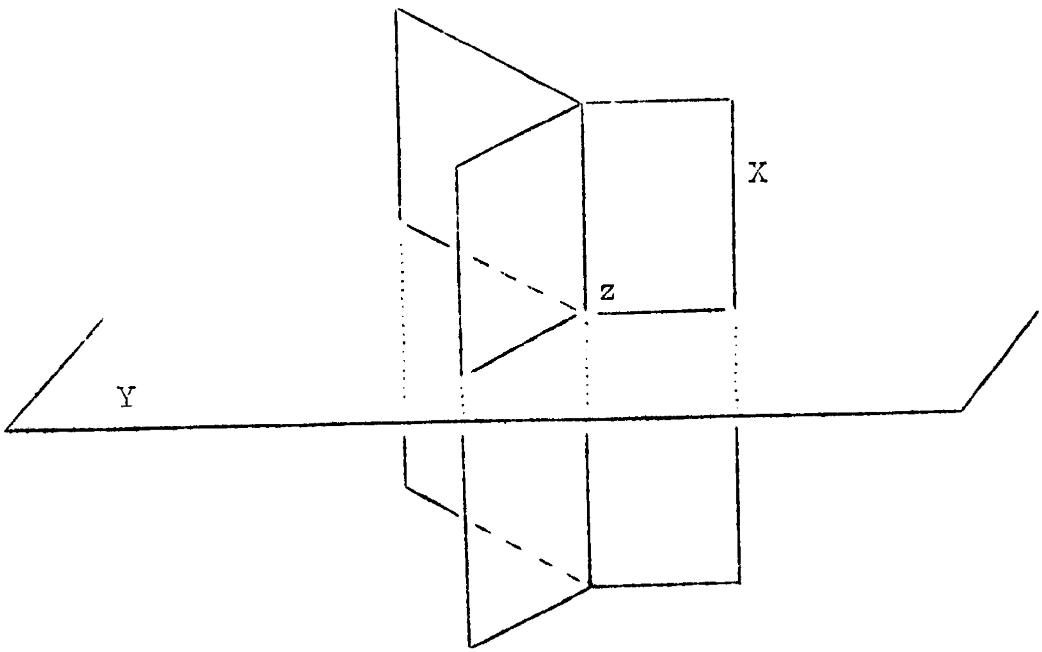


Figure 1a

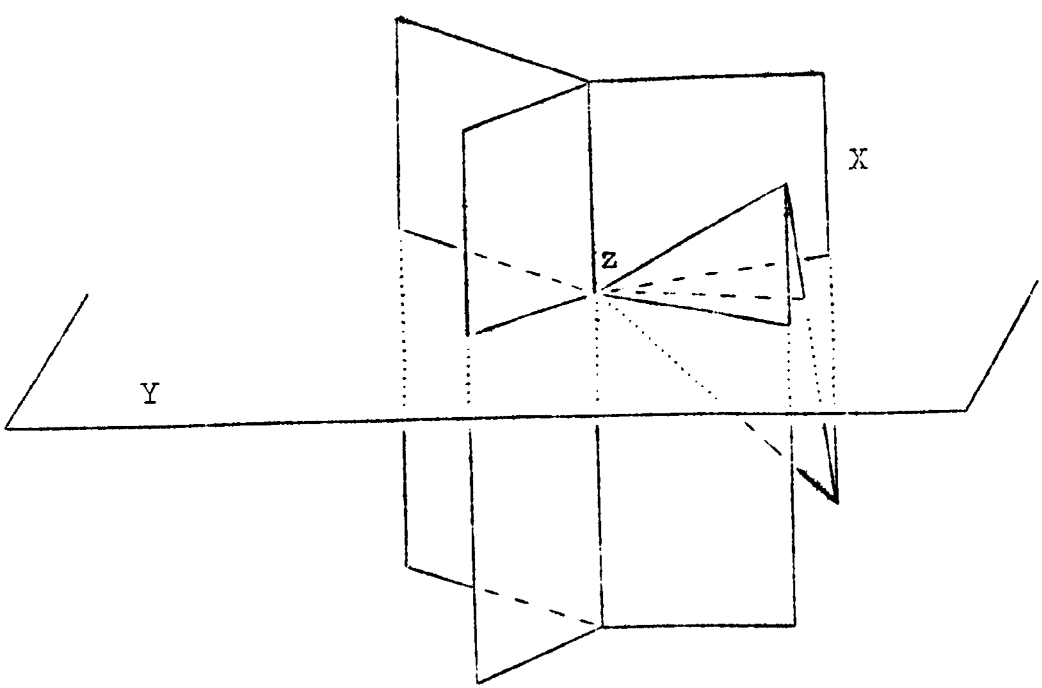


Figure 1b

<u>Definition 4.</u> Let x be a point of M such that $fx \in P$. The map f is <u>transversal to P at x</u> if there is a commutative diagram

$$\begin{array}{ccc} D^{q-p} \times D^{m+p-q} & \xrightarrow{1 \times k} & D^{q-p} \times D^{p} \\ \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\varphi} \\ M & \xrightarrow[f]{} & Q \end{array}$$

where ψ, φ are embeddings onto neighbourhoods of x, fx such that

$$\varphi^{-1}P = \pi\psi^{-1}x \times D^{p}$$

(π being projection $D^{q-p} \times D^{m+p-q} \to D^{q-p}$).

f is <u>transversal</u> to P if it is transversal at all such points x.

We see straightway from the definitions that, if f is either graph transversal or transversal to P, then $f^{-1}P$ is a closed locally flat submanifold of M of codimension $q - p$.

<u>Theorem 2.</u> <u>Let M, Q be closed manifolds, and P a locally flat closed submanifold of Q. Given a map $f:M \to Q$, it can be ambient isotoped to a map that is transversal to P by an arbitrarily small ambient isotopy of Q.</u>

We shall not be able to verify:

<u>Conjecture.</u> If f, $g:M \to Q$ are homotopic maps, both transversal to P, then $f^{-1}P$ and $g^{-1}P$ are cobordant manifolds.

INTRINSIC DIMENSION

We now investigate more fully the notion of intrinsic dimension introduced earlier, and prove some useful lemmas. Firstly we give two new definitions that are equivalent to Definition 2.

Given a polyhedron X and a point x of X, let K be a triangulation of X in which x is a vertex and define the <u>link of x in X</u> by

$$\text{link}(x, X) = \text{link}(x, K).$$

Up to piecewise linear homeomorphism, this definition is independent of the choice of K. For, since any two triangulations have a common subdivision, it is enough to consider an arbitrary subdivision K' of K and prove $\text{link}(x, K)$, $\text{link}(x, K')$ homeomorphic. This last is easily accomplished using the standard technique of pseudo radial projection. That is to say one can obtain a piecewise linear homeomorphism

$$\text{link}(x, K') \to \text{link}(x, K)$$

as the linear extension of the radial projection from x (itself not piecewise linear) on the vertices.

<u>Definition 2a</u> $I(X, x)$ is the largest integer t such that $\text{link}(x, X)$ is a t-fold suspension. To say $\text{link}(x, X)$

is a t-fold suspension means there is a polyhedron W and a homeomorphism

$$\mathrm{link}(x, X) \to S^{t-1} * W,$$

where * denotes linear join and S^{t-1} a (t-1)-sphere. Notice that S^{t-1} is itself a t-fold suspension - for W take the empty polyhedron.

Alternatively, let $\mathfrak{T}$ be the piecewise linear structure of X - i.e. $\mathfrak{T}$ is a maximal family of piecewise linearly related triangulations - and for each $K \in \mathfrak{T}$ let d(K, x) be the dimension of the simplex of K that has x in its interior.

<u>Definition 2b</u> $\quad I(X, x) = \max\limits_{\mathfrak{T}} d(K, x).$

Consider now the equivalence of our three definitions. It is evident that:

(1) If there is a triangulation of X in which x lies in the interior of a t-simplex, then link(x, X) is a t-fold suspension.

(2) If link(x, X) is a t-fold suspension, then $I(X, x) \geq t$ in the sense of Definition 2.

Therefore to complete the equivalence it is enough to prove -

Lemma 1   *If there is a polyhedron V and an embedding*

$$f : D^t \times V \to X$$

that embeds $\mathring{D}^t \times V$ onto a neighbourhood of x, then there is a triangulation of X such that x lies in the interior of a t-simplex.

Proof. We can assume that D^t is a t-simplex Δ^t. Choose triangulations of V, X - for brevity we denote them by the same letters. Let $(\Delta^t \times V)'$ be a simplicial subdivision of $\Delta^t \times V$, and X' a subdivision of X, such that

(i) x is a vertex of X',

(ii) if $v \in V$ is the projection of $f^{-1}x$, then $\Delta^t \times v$ is a subcomplex $(\Delta^t \times v)'$ of $(\Delta^t \times V)'$,

(iii) f is simplicial.

Choose a point y of $\overline{\text{star}}[f^{-1}x, (\Delta^t \times v)']$ in general position with respect to the vertices of $(\Delta^t \times v)'$. Then y is joinable to $\text{link}[f^{-1}x, (\Delta^t \times V)']$ in the linear structure of $(\Delta^t \times V)'$, and in

$$y * \text{link}[f^{-1}x, (\Delta^t \times V)']$$

$f^{-1}x$ lies in the interior of a t-simplex. Therefore, using f, we may replace $\overline{\text{star}}(x, X')$ by $y * \text{link}[f^{-1}x, (\Delta^t \times V)']$ and so obtain a new triangulation of X with the required property.

For the remainder of this section it will be most convenient to work with Definition 2a.

Let $\mathrm{Iso}(W \subset S^n)$ denote the set of ambient isotopy classes of embeddings of a polyhedron W in the n-sphere S^n. Then suspension induces a map

$$\Sigma:\mathrm{Iso}(W \subset S^n) \to \mathrm{Iso}(\Sigma W, S^{n+1}).$$

Using the relative regular neighbourhoods of Hudson and Zeeman [4] it is not hard to show that

(a) Σ is injective, and

(b) by Lickorish's result [5] on unknotting cones Σ is bijective if the codimension n-dim W is ≥ 3.

(In [5] Theorem 5 it is shown that if W unknots in S^n, then ΣW unknots in S^{n+1}. Using the argument given there as a model the reader will have little difficulty in verifying (a) and (b).)

Consequently one has by induction:

Theorem The map

$$\Sigma^t:\mathrm{Iso}(W \subset S^n) \to \mathrm{Iso}(\Sigma^t W \subset S^{n+t})$$

induced by t-fold suspension, is bijective if n-dim $W \geq 3$.

We now use this to prove:

Lemma 2 Let Q be a manifold, X a subpolyhedron of codimension ≥ 3, and x a point of X satisfying $x \in \mathring{Q}$ and $I(X, x) = t$. Then there is a subpolyhedron $V \subset D^{q-t}$, and an embedding

$$f:D^t \times D^{q-t} \to Q$$

onto a neighbourhood of x in Q, such that $f^{-1}X = D^t \times V$.

Proof. Consider the pair

$$\mathrm{link}(x, X) \subset \mathrm{link}(x, Q)$$

- defined, up to homeomorphism, from some triangulation of Q in which X is a subcomplex and x a vertex. Now $\mathrm{link}(x, Q)$ is a $(q-1)$-sphere since $x \in \mathring{Q}$, and $\mathrm{link}(x, X)$ is a t-fold suspension since $I(X, x) = t$. Therefore by the above theorem it is possible to find a homeomorphism

$$S^{t-1} * S^{q-t-1},\ S^{t-1} * W \to \mathrm{link}(x, Q),\ \mathrm{link}(x, X)$$

where $W \subset S^{q-t-1}$. This extends conewise to an embedding

$$D^t * S^{q-t-1},\ D^t * W \to Q,\ X$$

onto neighbourhoods of x. It is now routine to produce the required product structure from the join already obtained.

The next lemma and its corollary are due jointly to H.R. Morton and the author. Denote piecewise linear homeomorphism by $\cong$.

Lemma 3 *Let X, Y be polyhedra and suppose*

$$\Sigma^r X \cong \Sigma^n Y \quad \text{for } r < n.$$

Then X is a suspension.

Proof. If $r = 0$ then for any $n > 0$ and any Y the result is certainly true. Proceed by induction on r. Suppose $r > 0$ and assume that for any $n > r - 1$ and any Y

$$\Sigma^{r-1} X \cong \Sigma^n Y \Rightarrow X \text{ a suspension.}$$

Now $\Sigma^r X \cong \Sigma^n Y$ means we have a homeomorphism

$$h: S^{r-1} * X \to S^{n-1} * Y.$$

Choose a point $z \in S^{r-1}$, then

$$\text{link}(z, \Sigma^r X) \cong \text{link}(z, S^{r-1}) * X \cong \Sigma^{r-1} X.$$

Consider now $z' = hz \in \Sigma^n Y$. Our definition of link was arranged so as to be invariant under piecewise linear homeomorphism, and so

$$\text{link}(z, \Sigma^r X) \cong \text{link}(z', \Sigma^n Y).$$

There arise three cases:

(i) $z' \in S^{n-1}$ when $\text{link}(z', \Sigma^n Y) \cong \text{link}(z', S^{n-1}) * Y$
$$\cong \Sigma^{n-1} Y.$$

(ii) $z' \in Y$ when $\text{link}(z', \Sigma^n Y) \cong S^{n-1} * \text{link}(z', Y)$
$$= \Sigma^n \text{ link}(z', Y).$$

(iii) Finally, if z' is neither in the suspension ring, nor in Y, it must lie on a unique ray joining say $x \in S^{n-1}$ to $y \in Y$. Thus

$$\text{link}(z', \Sigma^n Y) \cong S^0 * \text{link}(x, S^{n-1}) * \text{link}(y, Y)$$
$$\cong \Sigma^n \text{ link}(y, Y).$$

Therefore by induction X is seen to be a suspension in each case.

<u>Corollary</u> <u>$\Sigma X \cong \Sigma Y \Longrightarrow X \cong Y$.</u>

<u>Proof.</u> Desuspend X and Y as far as possible to give

$X = \Sigma^r X'$, and

$Y = \Sigma^n Y'$

where X', Y' are not suspensions. By the lemma we must have $r = n$, and therefore $\Sigma^r X' \cong \Sigma^r Y'$. It is enough to show $X' \cong Y'$. Again induct on r, the induction beginning trivially for $r = 0$. Suppose $r > 0$; as above choose a point z on the suspension ring of $\Sigma^r X'$ and consider its image in $\Sigma^r Y'$. Since X' is not a suspension we see, again by use of the lemma, that only case (i) can occur. Therefore $\Sigma^{r-1} X' \cong \Sigma^{r-1} Y'$, which implies $X' \cong Y'$ by induction. This completes the proof. The corollary itself will not be used here, but it does not appear to be well known and so seems worth mention.

Lemma 4 *If $I(X, x) = t$ and $y \in \mathring{D}^n$, then*

$$I(X \times D^n, x \times y) = t + n.$$

Proof. $\operatorname{link}(x \times y, X \times D^n)$ is homeomorphic to $\Sigma^n \operatorname{link}(x, X)$. Also $\operatorname{link}(x, X) \cong \Sigma^t W$, where W is not a suspension, since $I(X, x) = t$. Therefore $I(X \times D^n, x \times y) \geq t + n$, and application of Lemma 3 shows we must have equality.

Remark. It is not always true that if $I(X, x) = t$ and $I(Y, y) = s$ then $I(X \times Y, x \times y) = t + s$. For example take $X = Y = D^1$ and let x, y be end points. Then $I(X, x) = 0 = I(Y, y)$ but $I(X \times Y, x \times y) = 1$.

Lemma 5   Let M, P be manifolds and $X \subset M$, $Y \subset P$ subpolyhedra both of codimension $\geqslant 3$. If z is an interior point of $M \times P$ that lies in $(M \times Y) \cap (X \times P)$, then $M \times Y$ and $X \times P$ are transversal at z.

Proof.   Project z into M, P so obtaining points $x \in X$, $y \in Y$; i.e. $z = x \times y \in M \times P$. Suppose $I(X, x) = t$ and $I(Y, y) = s$. By Lemma 2 there are embeddings

$$f : D^t \times D^{m-t},\ D^t \times V \to M,\ X$$
$$g : D^s \times D^{p-s},\ D^s \times W \to P,\ Y$$

onto neighbourhoods of x, y where $V \subset D^{m-t}$ and $W \subset D^{p-s}$. The product of these gives rise to an embedding

$$h : D^t \times D^s \times D^{m-t} \times D^{p-s} \to M \times P$$

onto a neighbourhood of z, and certainly

$$h^{-1}(M \times Y) = D^t \times D^s \times D^{m-t} \times W,$$
$$h^{-1}(X \times P) = D^t \times D^s \times V \times D^{p-s}.$$

The proof is completed by observing that, due to Lemma 4,

$$I(M \times Y, z) = m + s\text{, and}$$
$$I(X \times P, z) = t + p.$$

TRANSIMPLICIAL MAPS

A detailed discussion of, and motivation for, the ideas introduced briefly below can be found in [2].

Let M, Q be compact manifolds, f a map of M into Q, and K a triangulation of Q. Consider a point x of M and suppose A is the simplex of K such that $fx \in \mathring{A}$. Choose a vertex v of A, and let L be the link of A in K, and $s: AL \to vL$ the simplicial map defined as the join of $A \to v$ to the identity on L.

Definition 5 The map $f: M \to Q$ is transimplicial to K at the point $x \in M$ if there exists a neighbourhood N of v in vL, and a commutative diagram

$$\begin{array}{ccccc} N \times D^{m+a-q} & \xrightarrow{1\times k} & N \times D^{a} & \xrightarrow{\text{projection}} & N \\ \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\varphi} & & \cap \\ f^{-1}AL & \xrightarrow[f]{} & AL & \xrightarrow[s]{} & vL \end{array}$$

where a is the dimension of A and ψ, φ are embeddings onto neighbourhoods of x, fx respectively. We say f is transimplicial to K if it is so at all points of M.

Remarks

1. The definition is independent of the choice of v.
2. If f is transimplicial to K at $x \in M$, then f is transimplicial to K at all points in some neighbourhood of x (see [2] Lemma 4).

3. Let K' be a subdivision of K. If f is transimplicial to K' at $x \in M$, then f is also transimplicial to K at x (see [2] Lemma 5).

Our main chore will be to prove:

Theorem 3 Let M, Q be compact manifolds, and K a triangulation of Q. Any map $f: M \to Q$ can be ambient isotoped to a new map g that is transimplicial to K at all points of $g^{-1}\mathring{Q}$. The ambient isotopy can be chosen arbitrarily small, and made to keep $\dot{Q}$ fixed.

Corollary If in addition Q is closed, any map $f: M \to Q$ can be ambient isotoped transimplicial to K by an arbitrarily small ambient isotopy of Q.

For the case where f is an embedding, a proof of the corollary has been given in [2].

Before proving Theorem 3 we shall apply it to obtain proofs of our polyhedral and map transversality theorems. The importance of transimplicial maps for our purposes is contained in the following two lemmas.

Lemma 6 Suppose Q is a compact manifold, X, Y subpolyhedra both of codimension $\geqslant 3$ such that $X \cap Y \subset \mathring{Q}$, and K a triangulation of Q in which Y is a subcomplex. Let M be a regular neighbourhood of X in Q, and $f: M \to Q$ some retraction of M onto X. Then if f is transimplicial to K, the subpolyhedra X and Y are transversal in Q.

<u>Proof.</u> Let $z \in X \cap Y$ and A be the simplex of K such that $\cancel{fx}\ z \in \mathring{A}$. Therefore, with the notation introduced earlier, we have a commutative diagram

$$\begin{tikzcd} N \times D^a \arrow[r, "1\times k"] \arrow[d, dashed, "\psi"] & N \times D^a \arrow[r, "\text{projection}"] \arrow[d, dashed, "\varphi"] & N \arrow[d, phantom, "\cap"] \\ f^{-1}AL \arrow[r, dashed, "f"'] & AL \arrow[r, "s"'] & vL \end{tikzcd}$$

where ψ, φ are embeddings onto neighbourhoods of $\cancel{x}, \cancel{fx}$. $z = fz$

Let K_1 be the subcomplex of K that triangulates Y, and $L_1 = \text{link}(A, K_1)$. The commutativity of the left hand square implies

$$\varphi^{-1} fM = N \times kD^a.$$

That of the right hand square implies

$$\varphi^{-1} Y = (N \cap vL_1) \times D^a.$$

Therefore it is enough to check that $N \times kD^a$ and $(N \cap vL_1) \times D^a$ are transversal at the interior point $\varphi^{-1}\cancel{fx}\,z$ of $N \times D^a$. However, due to the codimension restrictions this is ensured by Lemma 5.

<u>Lemma 7</u> <u>Let $M, P \subset Q$ be closed manifolds, with P locally flat in Q, and let K be a triangulation of Q in which P is a subcomplex. If $f:M \to Q$ is transimplicial to K, then f is transversal to P.</u>

<u>Proof.</u> Let $x \in f^{-1}P$, and A be the simplex of K such that $fx \in \mathring{A}$. Again consider the diagram

$$\begin{tikzcd} N \times D^{m+a-q} \arrow[r, dashed, "1\times k"] \arrow[d, dashed, "\psi"'] & N \times D^{a} \arrow[r, "\text{projection}"] \arrow[d, dashed, "\varphi"] & N \arrow[d, phantom, "\cap"] \\ f^{-1}AL \arrow[r, "f"'] & AL \arrow[r, "s"'] & vL \end{tikzcd}.$$

Suppose P appears as the subcomplex K_1 of K, and let $L_1 = \text{link}(A, K_1)$. Now P is locally flat in Q, therefore we can choose N so that $(N, N \cap vL_1)$ is an unknotted $(q - a, p - a)$ ball pair. Thus there exists an unknotting homeomorphism

$$h: D^{q-p} \times D^{p-a}, \quad z \times D^{p-a} \to N, \ N \cap vL_1$$

where $z \in \mathring{D}^{q-p}$. The diagram

$$\begin{tikzcd} D^{q-p} \times D^{p-a} \times D^{m+a-q} \arrow[r, "1\times 1\times k"] \arrow[d, "\psi(h\times 1)"'] & D^{q-p} \times D^{p-a} \times D^{a} \arrow[d, "\varphi(h\times 1)"] \\ M \arrow[r, "f"'] & Q \end{tikzcd}$$

commutes, and exhibits the transversality of f to P at x.

<u>Proof of Theorem 1</u>

Recall the statement of the theorem. We are given a manifold Q together with subpolyhedra X and Y, both of codimension $\geqslant 3$, such that $X \cap Y \subset \mathring{Q}$. The thesis is that we can ambient isotop X transversal to Y by an arbitrarily small ambient isotopy of Q that keeps $\dot{Q}$ fixed.

First observe that it is enough to consider the case where Q is compact. For otherwise we can work entirely inside a regular neighbourhood of $X \cup Y$ in Q. Suppose then that Q is compact, and choose a triangulation K of Q in which Y is a subcomplex. Let M be a regular neighbourhood of X, then M is a manifold and collapses to X. Collapse M to X in some way, thus defining a map $f:M \to Q$ such that $fM = X$. Apply Theorem 3 to this map, then refer to Lemma 6. The proof of Theorem 1 is complete.

Proof of Theorem 2

Let $M, P \subset Q$ be closed manifolds and suppose P is locally flat in Q. We are given a map $f:M \to Q$ and want to ambient isotop it transversal to P by an arbitrarily small ambient isotopy of Q.

Choose a triangulation K of Q in which P appears as a subcomplex. Apply the Corollary to Theorem 3 to ambient isotop f transimplicial to K, then by Lemma 7 this ambient isotopy produces the required result.

t-SHIFTS

It will be necessary to work with triangulations that have a particular property. Let E^q denote Euclidean q-space, E^q_+ the closed half space of points with non-negative last coordinate, and E^{q-1} the subspace of points with last coordinate zero.

Definition Let K be a combinatorial manifold of dimension q. Then K is called a Brouwer manifold if:

(i) For each $v \in \overset{\circ}{K}$ there is a linear embedding

$$\overline{\text{star}}(v, K) \to E^q.$$

(ii) For each $v \in \dot{K}$ there is a linear embedding

$$\overline{\text{star}}(v, K), \quad \overline{\text{star}}(v, \dot{K}) \to E^q_+, \quad E^{q-1}.$$

Remarks

1. Not all combinatorial manifolds are Brouwer, see Cairns [3].
2. Any subdivision of a Brouwer manifold is Brouwer.

The following lemma is due, in a sharpened form, to Whitehead [6]. An alternative proof, given by Zeeman, can be found in [2].

Lemma 8 Any combinatorial manifold has a Brouwer subdivision.

In proving Theorem 3 we shall ambient isotop our map by means of a sequence of special shifts applied to its image. The shifts constructed below are a variation of those first introduced in [7] Chapter 6.

Let M, Q be compact manifolds, and suppose we are given a map $f:M \to Q$ together with a Brouwer triangulation K of Q. Let K'' denote a second derived of K. Choose a subdivision K_1 of K'' and a triangulation J of M such that $f:J \to K_1$ is simplicial. We call $A \in K_1$ an <u>interior</u> simplex of K_1 if $\mathring{A} \subset \mathring{K}_1$. Let A be an interior simplex of K_1 that lies in fJ, let K_1' be the barycentric first derived of K_1 and $\hat{A}$ the barycentre of A. Let

$$\begin{aligned} W &= \overline{\text{star}}(\hat{A}, K_1') \\ &= \hat{A} * \text{link}(\hat{A}, K_1'). \end{aligned}$$

Since A is an interior simplex, $\text{link}(\hat{A}, K_1')$ is a q-sphere. Choose a vertex z of $\mathring{K}$ such that

$$W - A \cap \dot{K}_1 \subset \text{star}(z, K).$$

We can find such a vertex z since A is an interior simplex of some subdivision of a second derived of K. Using the Brouwer property of K, let

$$\lambda:\overline{\text{star}}(z, K) \to E^q$$

be a linear embedding. Thus λ embeds W linearly in E^q. We denote the complex λW by V. Choose a point v near

$\lambda\hat{A}$ in V such that:

(i) v is joinable to $\dot{V}$ in the linear structure of E^q, and

(ii) v is in general position with respect to the vertices of V.

Let $\mu: V \to V$ be the homeomorphism defined as the join of the identity on $\dot{V}$ to the map $\lambda\hat{A} \to v$. Finally, define a homeomorphism

$$h_A : Q \to Q$$

by

$$h_A(x) = \begin{cases} x \text{ if } x \in Q - W, \\ \lambda^{-1}\mu\lambda(x) \text{ if } x \in W. \end{cases}$$

Then h_A is ambient isotopic to the identity keeping $Q - \mathring{W}$ fixed in view of:

<u>Alexander's Lemma</u> <u>Any homeomorphism of a ball that keeps the boundary fixed is isotopic to the identity keeping the boundary fixed.</u>

We call the move $f \to h_A f$ a <u>local shift</u> of f in the triangulation K. Notice that K entered into the construction when we chose λ, i.e. our shift has been made with respect to the local linear structure of K.

Now let A vary over all interior t-simplexes of fJ, and for each simplex construct a corresponding homeomorphism h_A. The $\{\overline{\text{star}}(\hat{A}, K_1')\}$ overlap only in their boundaries on which the $\{h_A\}$ agree as the identity. Therefore we may combine these homeomorphisms to give a new homeomorphism

$$h_t : Q \to Q$$

that is the identity on $\dot{Q}$. The previous local ambient isotopies can also be combined, showing h_t to be ambient isotopic to the identity keeping $\dot{Q}$ fixed.

We call the move $f \to h_t f$ a <u>t-shift</u> of f in the triangulation K (with respect to J, K_1). It is clear that, by judicious choice of v in each local shift, we can make $h_t f$ arbitrarily close to f, and the ambient isotopy arbitrarily small.

<u>Lemma 9</u> <u>Using the above notation, $h_A f$ is transimplicial to K at all points of $f^{-1}\mathring{W}$.</u>

In fact we shall prove a stronger result, namely that $h_A f$ is transimplicial to K_1' at these points.

<u>Proof.</u> We consider μ as a homeomorphism $E^q \to E^q$, extending it by the identity outside V, and we write g for the map

$$\mu\lambda f : f^{-1}\,\overline{\text{star}}(z, K) \to E^q.$$

Let J' be a first derived of J such that $f : J' \to K_1'$ is simplicial. Then g is simplicial from $f^{-1}W \subset J'$ to $v * \dot{V}$. Suppose x is a point of M such that $fx \in \mathring{W}$, and let B be the simplex of V for which $gx \in \mathring{B}$. Choose a vertex u of B, let $L = \operatorname{link}(B, V)$ and s be the usual simplicial map $BL \to uL$. Now λ is a <u>linear</u> embedding. Therefore, in order to show $h_A f$ transimplicial to K_1' at x, it is enough to produce a commutative diagram

$$\begin{tikzcd}
N \times D^{m+b-q} \arrow[r, "1\times k"] \arrow[d, dashed, "\psi"'] & N \times D^{b} \arrow[r, "\text{projection}"] \arrow[d, dashed, "\varphi"'] & N \arrow[d, phantom, "\cap"] \\
g^{-1}BL \arrow[r, "g"'] & BL \arrow[r, "s"'] & uL
\end{tikzcd}$$

where $b = \dim B$, N is a neighbourhood of u in uL, and ψ, φ are embeddings onto neighbourhoods of x, gx respectively.

In the particular case $gx = v$ there is no problem since, by general position, v lies in the interior of a principal simplex of V.

Suppose now $gx \neq v$. Let $x \in \mathring{C}$, where C is a simplex of J', and let $gC = D$. Then D is the linear join in E^q of v to some simplex of $\dot{V}$. By the general position of v in V, we may infer that $D \cap B$ is a convex linear cell (henceforth abbreviated to "cell") of dimension $(d + b - q)$. Let $E = [g^{-1}(D \cap B)] \cap C$, a cell of dimension $(d + b - q) + (c - d) = (c + b - q)$. Let F be the $(q - b)$-cell through x that is perpendicular to E in C. Consider now a simplex, C^* say, of J' that has C as a face.

Let $gC^* = D^*$, when D will be a face of D^* (of course it may happen that $D^* = D$). Corresponding to E we have a $(c^* + b - q)$-cell $E^* = [g^{-1}(D^* \cap B)] \cap C^*$. Now although E^* is not necessarily perpendicular to F in C^*, it certainly has the property that any $(q - b)$-cell parallel to F in C^*, and sufficiently close to F, meets it in exactly one point. Therefore, for some neighbourhood U of x in star(C, J'), we can obtain a well defined map $\rho_1 : U \to g^{-1}B$ by projecting each $U \cap C^*$ parallel to F onto the corresponding E^*. Return now to E^q. Since we defined F perpendicular to E in C, we know that the linear subspaces $[B]$, $[gF]$ of E^q, spanned by B, gF respectively, are complementary. Let $\rho_2 : E^q \to [B]$ be projection parallel to $[gF]$. Our constructions of ρ_1, ρ_2 ensure that

$$g\rho_1 = \rho_2 g.$$

Finally, define

$$\alpha = sg \times \rho_1 : U \cap g^{-1}\mathrm{star}(B, V) \to uL \times g^{-1}B,$$

$$\beta = s \times \rho_2 : \mathrm{star}(B, V) \to uL \times [B].$$

One can check that α and β are both piecewise linear embeddings onto neighbourhoods of (u, x) and (u, gx) respectively. Choose ball neighbourhoods

N of u in uL

D^{m+b-q} of x in $g^{-1}B$

D^b of gx in B

such that

$$N \times D^{m+b-q} \subset \text{image of } \alpha,$$

$$N \times D^{b} \subset \text{image of } \beta, \text{ and}$$

$$gD^{m+b-q} \subset D^{b}.$$

Hence the following diagram commutes

$$\begin{array}{ccccc} N \times D^{m+b-q} & \xrightarrow{1 \times g} & N \times D^{b} & \xrightarrow{\text{projection}} & N \\ \alpha^{-1} \downarrow & & \beta^{-1} \downarrow & & \cap \\ g^{-1}BL & \xrightarrow[g]{} & BL & \xrightarrow[s]{} & uL . \end{array}$$

This completes the proof of Lemma 9.

Suppose now that our map f is initially transimplicial to K at some points of $f^{-1}\mathring{Q} \subset M$, and let $\underline{T(f)}$ denote the set of such points. We would like to make our local shift so that the new map $h_A f$ is also transimplicial to K at these points.

<u>Lemma 10</u> <u>If our local shift is made small enough</u>

$$\underline{T(f) \subset T(h_A f).}$$

<u>Remark.</u> A corresponding lemma was proved in [2] under the assumption that f was an embedding. The crucial and main part of that proof applies equally well here and we shall not repeat it, but simply give the reference when necessary.

Before proving Lemma 10 we do a little preparatory work. Suppose x is a point of M such that $fx \in \text{star}(z, K)$.

Let Z be the complex

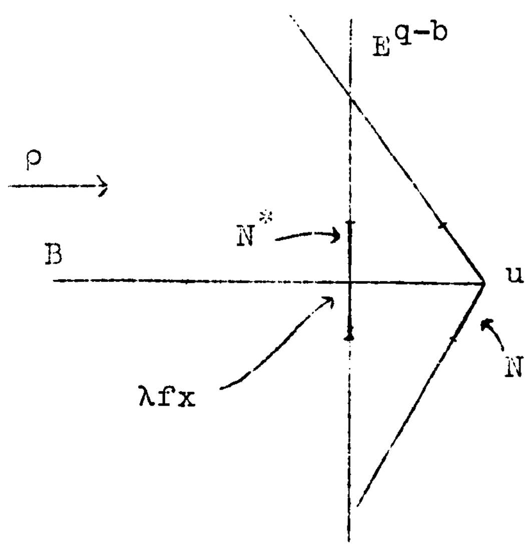
$$\lambda[\overline{\mathrm{star}}(z, K)] \subset E^q,$$

and B the simplex of Z such that $\lambda f x \in \mathring{B}$. We denote by E^b the linear subspace of E^q spanned by B, by E^{q-b} the orthogonal subspace through $\lambda f x$, and by ρ the projection $E^q \to E^{q-b}$ parallel to E^b.

Lemma 11 $x \in T(f)$ if and only if there is a commutative diagram

$$\begin{tikzcd}
N^* \times D^{m+b-q} \arrow[r, "1\times k"] \arrow[d, "\Psi"'] & N^* \times D^b \arrow[r, "\text{projection}"] \arrow[d, "\Phi"'] & N^* \arrow[d, phantom, "\cap"] \\
(\lambda f)^{-1}\overline{\mathrm{star}}(B, Z) \arrow[r, "\lambda f"'] & E^q \arrow[r, "\rho"'] & E^{q-b}
\end{tikzcd}$$

where N^* is a neighbourhood of $\lambda f x$ in E^{q-b}, and Ψ, Φ are embeddings onto neighbourhoods of x, $\lambda f x$.



Proof. Choose a vertex u of B, let $L = \mathrm{link}(B, Z)$ and s be the natural simplicial map $BL \to uL$. Suppose $x \in T(f)$, then since λ is a linear embedding we have a commutative diagram

$$\begin{array}{ccccc}
N \times D^{m+b-q} & \xrightarrow{1\times k} & N \times D^{b} & \xrightarrow{\text{projection}} & N \\
\downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\varphi} & & \cap \\
(\lambda f)^{-1}BL & \xrightarrow[\lambda f]{} & BL & \xrightarrow[s]{} & uL ,
\end{array}$$

where N is a neighbourhood of u in uL, and ψ, φ embeddings onto neighbourhoods of x, λfx. Now $\rho|uL$ is an embedding of uL in E^{q-b}, and so embeds N onto a neighbourhood N^* of λfx in E^{q-b}. Also

$$\rho|BL = (\rho|uL)s : BL \to uL.$$

Therefore the following diagram commutes and completes half our proof:

$$\begin{array}{ccccc}
N^* \times D^{m+b-q} & \xrightarrow{1\times k} & N^* \times D^{b} & \xrightarrow{\text{projection}} & N^* \\
\downarrow{\scriptstyle(\rho|N)^{-1}\times 1} & & \downarrow{\scriptstyle(\rho|N)^{-1}\times 1} & & \\
N \times D^{m+b-q} & \xrightarrow{1\times k} & N \times D^{b} & & \downarrow{\scriptstyle\subset} \\
\downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\varphi} & & \\
(\lambda f)^{-1}BL & \xrightarrow[\lambda f]{} & E^{q} & \xrightarrow[\rho]{} & E^{q-b} .
\end{array}$$

An argument in the opposite direction is equally straightforward.

<u>Proof of Lemma 10.</u> Let x be a point of M that lies in $T(f)$. If $fx \notin W$, the local shift has no effect on f in a neighbourhood of x, and so the lemma is trivial. Secondly, if $fx \in \overset{\circ}{W}$, then by Lemma 9 we know

$$x \in T(h_A f).$$

It remains to consider the case $fx \in \dot{W}$. For this we shall use Lemma 11 and the notation introduced there. Thus we have a commutative diagram

$$\begin{array}{ccccc}
N \times D^{m+b-q} & \xrightarrow{1 \times k} & N \times D^b & \xrightarrow{\text{projection}} & N \\
\downarrow \psi & & \downarrow \varphi & & \cap \\
(\lambda f)^{-1} BL & \xrightarrow[\lambda f]{} & E^q & \xrightarrow[\rho]{} & E^{q-b}.
\end{array}$$

Notice now that in the case under consideration $\mu\lambda fx = \lambda fx$. We again write g for

$$\mu\lambda f : f^{-1}\overline{\mathrm{star}}(z, K) \to E^q.$$

Let

$$I = \varphi(N \times D^b) \subset E^q,$$
$$I^* = \mu I.$$

Define a map $\theta_1 : I^* \to D^b$ as the composition

$$I^* \xleftarrow{\mu} I \xleftarrow{\varphi} N \times D^b \xrightarrow{\text{projection}} D^b.$$

We now make reference to the proof of Lemma 14 in [2] where it is shown that, if the local shift is made small enough (i.e. if v is chosen near enough $\lambda\hat{A}$), then

$$\alpha = \rho \times \theta_1 : I^* \to E^{q-b} \times D^b$$

is an embedding onto a neighbourhood of $(gx,\ \theta_1 gx)$. Moreover, this can be <u>simultaneously</u> ensured for all points of $T(f) \cap f^{-1}\dot{W}$.

<u>Remark.</u> It is in proving these facts that our careful use of the linear structure of K, in making the local shift, plays a vital role.

Define $\theta_2 : \psi(N \times D^{m+b-q}) \to D^{m+b-q}$ as the composition

$$\psi(N \times D^{m+b-q}) \xleftarrow{\psi} N \times D^{m+b-q} \xrightarrow{\text{projection}} D^{m+b-q}.$$

We claim that

$$\beta = \rho g \times \theta_2 : \psi(N \times D^{m+b-q}) \to E^{q-b} \times D^{m+b-q}$$

is an embedding onto a neighbourhood of $(gx,\ \theta_2 x)$. For let $y \neq y'$ be points of $\psi(N \times D^{m+b-q})$ such that

$$\theta_2 y = \theta_2 y',$$

then $\theta_1 gy = \theta_1 gy'$ (by commutativity in the previous diagram) and therefore

$$\rho gy \neq \rho gy' \quad (\text{since } \alpha \text{ is an embedding}).$$

Finally, choose new disc neighbourhoods

N^* of gx in E^{q-b}

D_*^{m+b-q} of $\theta_2 x$ in D^{m+b-q}

D_*^b of $\theta_1 gx$ in D^b

such that

$$N^* \times D_*^{m+b-q} \subset \text{image of } \beta$$
$$N^* \times D_*^{b} \subset \text{image of } \alpha, \text{ and}$$
$$kD_*^{m+b-q} \subset D_*^{b}.$$

Our construction ensures commutativity in:

$$\begin{array}{ccccc}
N^* \times D_*^{m+b-q} & \xrightarrow{1\times k} & N^* \times D_*^{b} & \xrightarrow{\pi} & N^* \\
\beta^{-1}\downarrow & & \alpha^{-1}\downarrow & & \cap \\
g^{-1}\overline{\text{star}}(B,\ Z) & \xrightarrow[g]{} & E^q & \xrightarrow[\rho]{} & E^{q-b}.
\end{array}$$

Remembering that g and $h_A f$ differ only by λ, a second application of Lemma 11 completes the proof.

<u>Proof of Theorem 3</u>

We are given $f:M \to Q$ together with a triangulation K of Q, and want to ambient isotop f, keeping $\dot{Q}$ fixed, to a new map g that is transimplicial to K at all points of $g^{-1}\overset{\circ}{Q}$.

For any triangulation L of Q, let L^r denote its r-skeleton and

$$L(r) = L - L^r - \dot{L}.$$

Suppose the dimension of fM is n. Using Lemma 8, subdivide K in some way to give a Brouwer triangulation L. We shall ambient isotop f to a map g that is transimplicial to L, and therefore to K, at points of $g^{-1}\overset{\circ}{Q}$. Let L'' be a second derived of L. Choose a triangulation J_1 of M and a subdivision L_1 of L'' such that $f:J_1 \to L_1$ is simplicial. Perform an n-shift of f in L with respect to J_1, L_1. Then by Lemma 9 we know

$$f^{-1}L_1(n - 1) \subset T(h_n f).$$

Choose subdivisions J_2, L_2 of J_1, L_1 so that $h_n f$ becomes simplicial, and apply an $(n - 1)$-shift to $h_n f$ in L with respect to J_2, L_2. By Lemmas 9 and 10 we can ensure

$$(h_n f)^{-1}L_2(n - 2) \subset T(h_{n-1}h_n f).$$

Repeat this procedure, working in order of decreasing dimension. After precisely $(n + 1)$-steps we obtain a subdivision L_{n+1} of L and a map

$$g = h_0 h_1 \ldots h_n f : M \to Q$$

such that

(a) $(h_1 \ldots h_n f)^{-1}\mathring{L}_{n+1} \subset T(g)$

(b) g is ambient isotopic to f keeping $\dot{Q}$ fixed.

Therefore the proof of Theorem 3 is complete.

BOUNDARY INTERSECTION POINTS

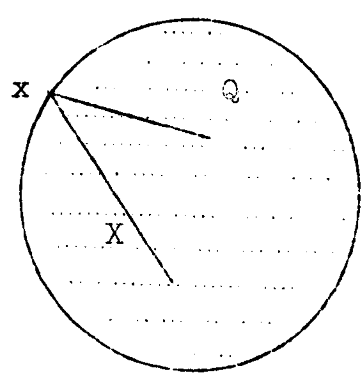
In this final section we shall complete our definition of transversality so as to include the case where X and Y intersect at points of $\dot{Q}$. To do this we follow an idea of Zeeman [8] and make the notion of intrinsic dimension "ambient".

Suppose $X \subset Q$ and consider a point $x \in X$. We define the ambient intrinsic dimension of x, written $I(X \subset Q, x)$, as follows.

Definition 6 $I(X \subset Q, x)$ is the largest integer t for which there is a polyhedron $V \subset D^{q-t}$, and an embedding

$$f: D^t \times D^{q-t} \to Q$$

onto a neighbourhood of x in Q such that $f^{-1}X = D^t \times V$.



Notes:

1. $I(X \subset Q, x) \leqslant I(X, x)$.

2. As for the notion of intrinsic dimension, two equivalent definitions (based on links and triangulations) can be given.

3. In the above illustration $I(X, x) = 1$, $I(X \subset Q, x) = 0$.

4. Suppose q-dim $X \geqslant 3$ and that $x \in \mathring{Q}$. Then Lemma 2 of this paper shows $I(X, x) = I(X \subset Q, x)$.

Consider now the situation $X, Y \subset Q$. Let z be a point of $X \cap Y \cap \dot{Q}$ and suppose

$$I(X \subset Q, z) = t,$$
$$I(Y \subset Q, z) = s.$$

Definition 7 The polyhedra X, Y are transversal at z if there is an embedding

$$h : D^{t+s-(q-1)} \times D^{(q-1)-t} \times D^1 \times D^{(q-1)-s} \to Q$$

onto a neighbourhood of z in Q, and subpolyhedra

$$V \subset D^{(q-1)-t} \times D^1, \quad W \subset D^1 \times D^{(q-1)-s}$$

such that

$$h^{-1}\dot{Q} = D^{t+s-(q-1)} \times D^{(q-1)-t} \times 0 \times D^{(q-1)-s},$$
$$h^{-1}X = D^{t+s-(q-1)} \times V \times D^{(q-1)-s},$$
$$h^{-1}Y = D^{t+s-(q-1)} \times D^{(q-1)-t} \times W.$$

Transversality at points of $X \cap Y \cap \mathring{Q}$ is defined exactly as before, and again we say simply that X and Y are

<u>transversal</u> in Q if they are transversal at all points of their intersection.

<u>Conjecture</u> Let Q be a manifold and X, Y compact subpolyhedra, both of codimension $\geqslant 4$ in Q. Then X can be ambient isotoped transversal to Y by an arbitrarily small ambient isotopy of Q.

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EXTENDING TRIANGULATIONS

by M.A. ARMSTRONG.

The results given below are useful in piecewise linear (PL) topology. They do not seem to be well known, or to have appeared previously in the literature.

Let Q be a compact PL-manifold, and M a <u>proper</u> compact PL-submanifold of Q. To say M is a proper submanifold means that the boundary $\dot{M} = M \cap \dot{Q}$. Recall that a triangulation of M is a combinatorial manifold K together with a homeomorphism $k:K \to M$; where no confusion can arise k is usually omitted. A triangulation L of Q is said to <u>extend</u> K if, in the diagram

$$\begin{tikzcd} K \arrow[r, "k"] \arrow[d, dashed, "s"'] & M \arrow[d, phantom, "\cap"] \\ L \arrow[r, "l"'] & Q \end{tikzcd},$$

the induced map $s:K \to L$ is <u>simplicial</u>. Recall also the notion of local unknottedness. The submanifold M is said to be <u>locally unknotted</u> in Q if, for some triangulation K of M and extension L of K over Q, the closed star ball pair

$$(\overline{\text{star}}\ (sv, L)\ ,\ s[\overline{\text{star}}\ (v, K)])$$

is unknotted for each vertex $v \in K$. The choice of K and L is irrelevant, since if this is true for a particular pair K, L, it is true for any subdivisions K', L' and consequently for any other choice. Of course, by [1], local knotting can only occur in codimension 2, and possibly in codimension 1, depending on the validity or otherwise of the PL Schönflies conjecture.

Theorem. Every triangulation of M can be extended over Q if and only if M is locally unknotted in Q.

Corollary 1. Any triangulation of the boundary of a compact PL-manifold can be extended to a triangulation of the whole manifold.

Corollary 2. If M, instead of being proper in Q, is contained in the interior of Q, and if the codimension is ≥ 3, then any triangulation of M can be extended over Q.

As an example of a non-extendable triangulation in codimension 2, consider the cone on a knotted PL-sphere pair (S^{n+1}, S^{n-1}). This is a ball pair (B^{n+2}, B^n) in which B^n is locally knotted at the cone point. Triangulate B^n as an n-simplex, and suppose this triangulation can be extended to B^{n+2}. Then the ball pair consisting of the closed star of B^n in this extension, and B^n itself, is unknotted, contradicting the local knottedness of the pair (B^{n+2}, B^n).

Proofs of the theorem and its corollaries will follow a couple of elementary lemmas.

Lemma 1. Let V be a compact polyhedron, W a compact subpolyhedron of V, and K a triangulation of W. Then there is a derived $K^{(r)}$ of K that can be extended to a triangulation of V.

Proof. Since the embedding of W in V is PL, some subdivision K' of K can be extended to a triangulation L of V. By [2] Lemma 4, there is an rth derived $K^{(r)}$ of K that is isomorphic to some subdivision K'' of K'. Finally, by the Corollary to Lemma 3 of [2], there is a subdivision L' of L that extends $K^{(r)}$.

Lemma 2. Let (X, Y) and (X_1, Y_1) be two unknotted PL-ball pairs. Then any PL-homeomorphism

$$h: \dot{X}_1 \cup Y_1, Y_1 \to \dot{X} \cup Y, Y$$

can be extended to a PL-homeomorphism

$$\bar{h}: X_1, Y_1 \to X, Y.$$

This result occurs as Lemma 18 in [2].

Proof of the Theorem

(a) Suppose M is locally flat in Q. Given a triangulation K of M there is, by Lemma 1, an rth derived that can be extended over Q. Now any derived of a finite complex is obtained by a finite number of stellar subdivisions - each such being the result of starring some simplex at an interior point. Therefore (by induction on the number of stellar subdivisions) it is sufficient to prove that if σK is obtained from K by a single stellar subdivision, and if σK can be extended over Q, then K can be extended over Q.

Let σK be obtained by starring the simplex $A \in K$ at the interior point $\hat{A}$. It is convenient to divide up the proof into two cases.

Firstly suppose M, Q closed. Let J be an extension of σK over Q, i.e. in

$$\sigma K \xrightarrow{k} M \subset Q \xleftarrow{j} J$$

the induced map $s: \sigma K \to J$ is simplicial. Some further notation is needed; let F be the subcomplex $s(\sigma K)$ of J, and u the vertex $s\hat{A}$. Take a first derived J' of J mod F (the reason for working in J', rather than in J, will appear later) and define

$$X = \overline{\text{star}}\ (u, J'),$$
$$Y = \overline{\text{star}}\ (u, F).$$

Then, since M is a proper locally unknotted submanifold of Q, the pair (X, Y) is an unknotted ball pair. It is in fact a cone pair with vertex u, and the idea behind the remainder of this proof is simply to replace this pair by a suitable new cone pair - this replacement will have the effect of straightening out $s[\overline{\text{star}}\ (A, K)]$ so that s looks linear.

To make this precise, let v be a vertex of $s\dot{A}$ and

$$B = X \cup \overline{\text{star}}\ (v, J').$$

Then B is seen (Figure 1) to be the union of two balls

$$\overline{\text{star}}\ (v, J')$$
$$X - \text{star}\ (v, J')$$

glued along the common face link (v, X), and consequently is a ball. Construct a new complex L as follows. Embed J linearly in some Euclidean space $E^n \subset E^{n+1}$, choose a point $w \in E^{n+1} - E^n$, and define

$$L = (J' - B) \cup (w * \dot{B})$$

where * denotes linear join. It remains to produce a suitable homeomorphism $l: L \to Q$.

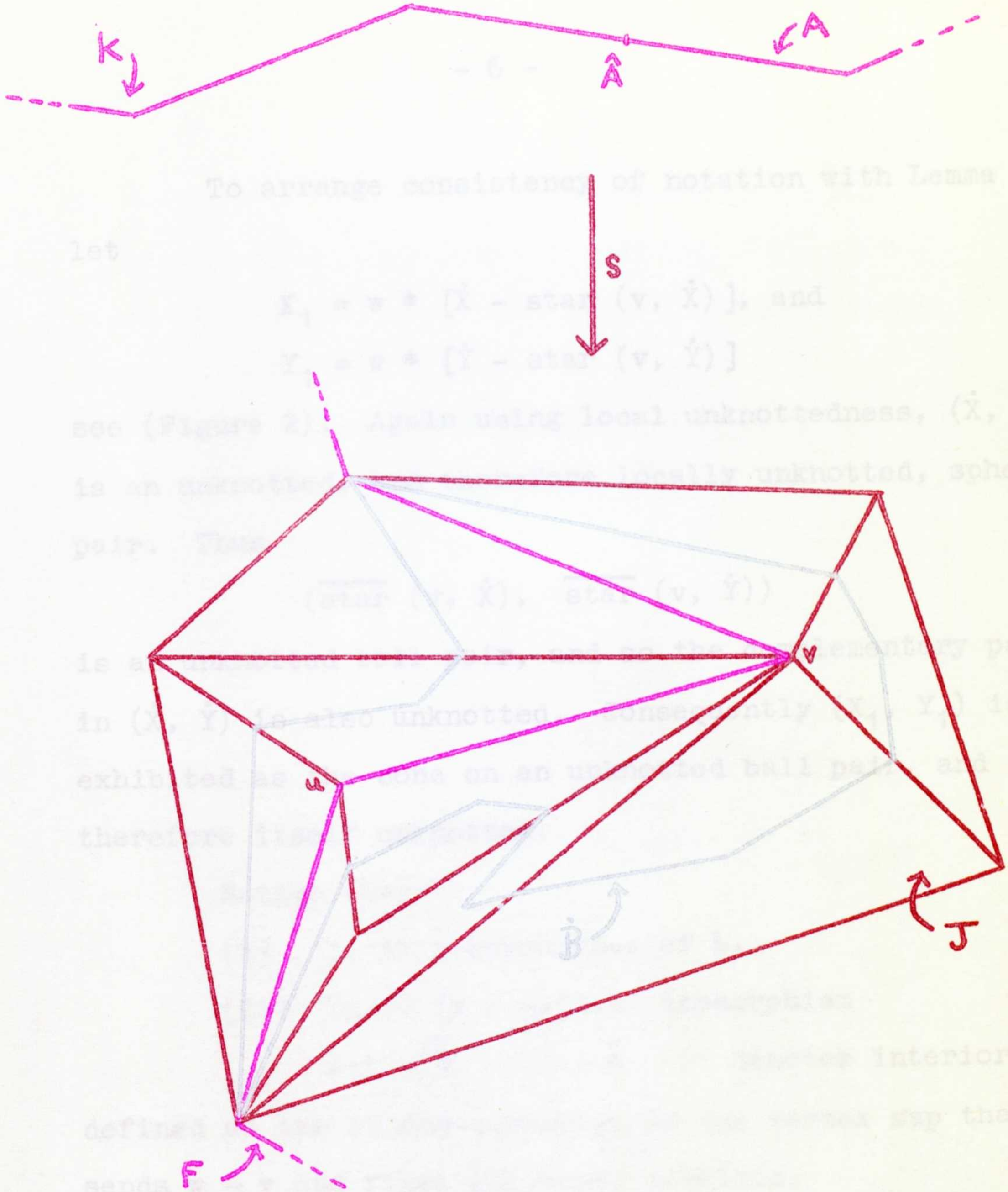

Fig. 1

To arrange consistency of notation with Lemma 2, let

$$X_1 = w * [\dot{X} - \text{star}\ (v, \dot{X})], \text{ and}$$
$$Y_1 = w * [\dot{Y} - \text{star}\ (v, \dot{Y})]$$

see (Figure 2). Again using local unknottedness, $(\dot{X}, \dot{Y})$ is an unknotted, and therefore locally unknotted, sphere pair. Thus

$$(\overline{\text{star}}\ (v, \dot{X}),\ \overline{\text{star}}\ (v, \dot{Y}))$$

is an unknotted ball pair, and so the complementary pair in $(\dot{X}, \dot{Y})$ is also unknotted. Consequently (X_1, Y_1) is exhibited as the cone on an unknotted ball pair, and is therefore itself unknotted.

Notice that:

(i) X_1 is a subcomplex of L.

(ii) There is a natural isomorphism

$$f{:}L - \overset{\circ}{X}_1 \to J' - \overset{\circ}{X} \quad (^\circ \text{ denotes interior})$$

defined as the linear extension of the vertex map that sends $w \to v$ and fixes all other vertices.

<u>Remark</u>. (i) and (ii) follow because

$$\overline{\text{star}}\ (v, J') \cap \overline{\text{star}}\ (u, J') = \overline{\text{star}}\ (v, \overline{\text{star}}\ (u, J')).$$

This equality need not be satisfied in the initial extension J (one only has to draw $\overline{\text{star}}\ (u, J)$ a little concave as in Figure 1). However, since A is a simplex of K, certainly the equality is satisfied with J' replaced by F. Therefore in deriving J so as to ensure the above, it is not necessary to subdivide simplexes of F.

Let $g: Y_1 \to \overline{\text{star}}\,(A, K)$ be the isomorphism defined as the linear extension of $s^{-1}f$ on the vertices. Then f and the composition

$$Y_1 \xrightarrow{g} \overline{\text{star}}\,(A, K) \xrightarrow{s} Y$$

together define a PL-homeomorphism $h: \dot{X}_1 \cup Y_1,\ Y_1 \to \dot{X} \cup Y,\ Y.$ By Lemma 2, this may be extended to $\bar{h}: X_1,\ Y_1 \to X,\ Y.$ Finally, define $l: L \to Q$ by

$$l|L - \overset{\circ}{X}_1 = jf, \text{ and}$$

$$l|X_1 = j\bar{h}.$$

Then, by construction, $l: L \to Q$ extends $k: K \to M$.

Now consider the <u>general case</u> where M, Q are allowed to have boundary. The proof goes through exactly as before, except that the expressions for L, X_1 and Y_1 are rather more complicated since A may well meet the boundary of K. Let:

$$L = (J' - B) \cup (w * [\dot{B} - \text{star}(v, \dot{B}) - \text{star}(u, \dot{B})])$$

$$X_1 = w * [\dot{X} - \text{star}\ (v, \dot{X}) - \text{star}\ (u, \dot{X})]$$

$$Y_1 = w * [\dot{Y} - \text{star}\ (v, \dot{Y}) - \text{star}\ (u, \dot{Y})].$$

Of course if $A \subset \overset{\circ}{K}$ these expressions reduce to those

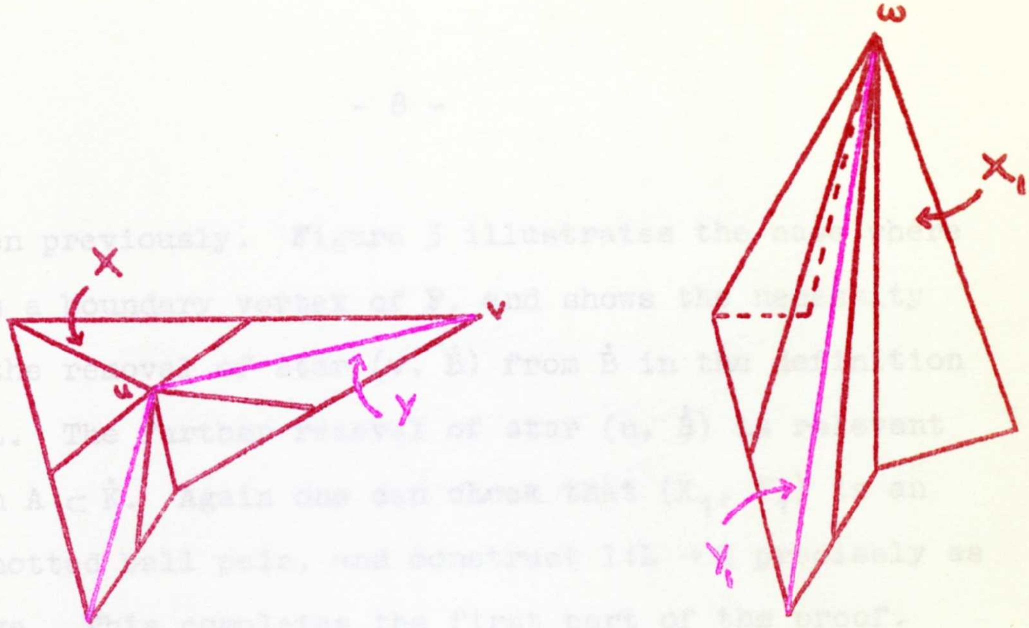

Fig. 2

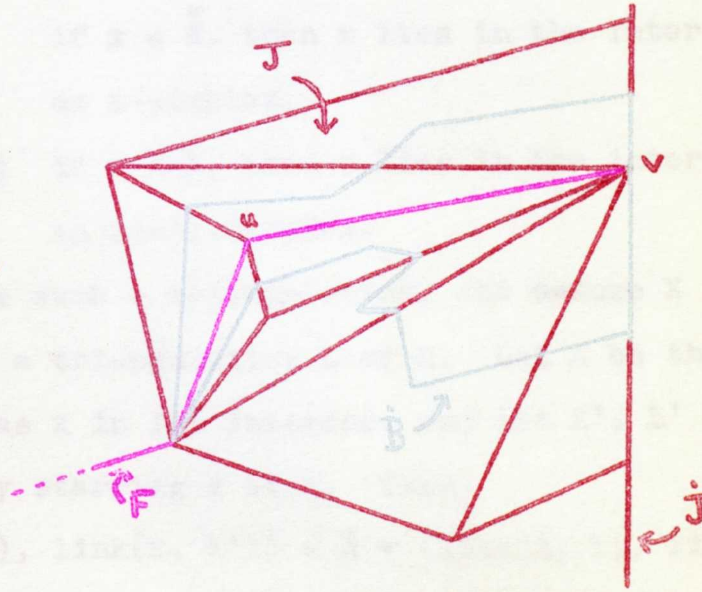

Fig. 3

given previously. Figure 3 illustrates the case where v is a boundary vertex of F, and shows the necessity of the removal of star $(v, \dot{B})$ from $\dot{B}$ in the definition of L. The further removal of star $(u, \dot{B})$ is relevant when $A \subset \dot{K}$. Again one can check that (X_1, Y_1) is an unknotted ball pair, and construct $1:L \to Q$ precisely as above. This completes the first part of the proof.

(b) Conversely, suppose M is locally knotted in Q. To complete the theorem one needs a triangulation of M that cannot be extended over Q.

Let x be a point at which M is locally knotted in Q. Then it is enough to produce a triangulation of M in which

(i) if $x \in \overset{\circ}{M}$, then x lies in the interior of an m-simplex,

or (ii) if $x \in \dot{M}$, then x lies in the interior of an $(m-1)$-simplex.

For let K be such a triangulation, and assume K can be extended to a triangulation L of Q. Let A be the simplex of K that has x in its interior, and let K', L' result from K, L by starring A at x. Then

$(\text{link}(x, L'), \text{link}(x, K')) = \dot{A} * (\text{link}(A, L), \text{link}(A, K))$

and therefore is an unknotted sphere (ball) pair for

$x \in \mathring{M}$ $(\dot{M})$, contradicting the local knotting of M in Q at x.

Triangulations of the required type can be constructed directly as follows. Suppose firstly $x \in \mathring{M}$, and let $f:\Delta^m \to \mathring{M}$ be an embedding of an m-simplex onto a neighbourhood of x. Choose a subdivision Δ' and a triangulation $l:L \to M$ such that $l^{-1}x$ is a vertex of L, and the induced map $s:\Delta' \to L$ is simplicial. Let B denote the subcomplex $s\Delta'$ of L. A new complex K may now be constructed by embedding L in a Euclidean space $E^n \subset E^{n+1}$, choosing a point $z \in E^{n+1} - E^n$, and defining

$$K = (L - B) \cup z * \dot{B}.$$

Choose a point $y \in \mathring{\Delta}$ in general position with respect to the vertices of Δ', and let $g:z * \dot{B} \to \Delta$ be the join of $z \to y$ to s^{-1} on $\dot{B}$. Finally, define $k:K \to M$ by

$$k = l \text{ on } L - B, \text{ and}$$

$$k = fg \text{ on } z * \dot{B}.$$

Then $k^{-1}x$ lies in the interior of an m-simplex of K, as required.

If $x \in \dot{M}$, the construction generalises in the obvious manner. Choose an embedding $f:\Delta^m \to M$ onto a neighbourhood of x such that $f^{-1}\dot{M}$ is a principal face

Δ^{m-1} of Δ^m. Proceed now as above, except that of course $\dot{B}$ is replaced by

$$\dot{B} - s\mathring{\Delta}^{m-1},$$

and y is chosen in $\mathring{\Delta}^{m-1}$ in general position with respect to $(\Delta^{m-1})'$.

The proof of the theorem is now complete.

Proof of Corollary 1

Let M be the manifold in question. Add a collar to M and denote the resulting manifold by Q. Then $\dot{M}$ is a proper locally unknotted submanifold of Q and so the theorem is applicable. Therefore any triangulation of $\dot{M}$ can be extended to a triangulation of Q, and of course M must appear as a subcomplex.

Proof of Corollary 2

Suppose K is a given triangulation of M, and let N be a relative regular neighbourhood of M mod $\dot{M}$ in $\mathring{Q}$. By [1] M is locally unknotted in N. First apply the theorem to extend K over N, then apply Corollary 1 to the manifold $Q - \mathring{N}$ to complete the extension.

Two questions have been neglected in this paper:

(a) If M is locally knotted in Q, which triangulations of M are extendable over Q?

(b) When is it possible to extend triangulations for polyhedra?

Information on both of these will be given in a subsequent paper by E.C. Zeeman [3].

I would like to thank Professor Zeeman for his encouragement during this work, and for pointing out a gap in an earlier version.

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APPENDIX 1

The following remarks were omitted from (1).

1. Suppose X is not simply connected, and let $\tilde{X}$ be the universal cover of X. Then G can be lifted to a group $\tilde{G}$ of homeomorphisms of $\tilde{X}$ in such a way that $\tilde{G}$ is an extension of $\pi_1(X)$ by G, the quotients $\tilde{X}/\tilde{G}$ and X/G are homeomorphic, and $\tilde{G}$ acts simplicially on some triangulation of $\tilde{X}$. Thus, if the geometry of the situation allows one to recognise $\tilde{G}$, then $\pi_1(X/G)$ can be found by applying the theorem of (1) to the pair $\tilde{X}, \tilde{G}$.

2. That G act simplicially on some triangulation of X is of course a severe restriction. One would like to prove the result of (1) under somewhat weaker hypotheses, especially since the only theorem known about triangulating group actions requires X to be a compact polyhedron, and G a finite group of piecewise linear homeomorphisms. However, our theorem is not true in complete generality. For example consider the reals acting on the real line by addition, then X/G is a point and G/H is isomorphic to the reals. This example suggests that a discontinuity

condition is necessary on the pair X, G. Suppose then that X is a simply connected topological space, and G a properly discontinuous group of homeomorphisms of X. Let F be the set of points of X that have non-trivial stabilizer in G. The proof given in (1) depended on a path lifting procedure. It can be modified to deal with this situation if one can verify that given a path in X/G there is a homotopic path that meets F/G in only a finite set of points.

APPENDIX 2

A problem is suggested below. A positive solution to it is enough to provide a proof of Conjecture 1 of (2), and therefore of relative transversality. Such a solution would also show the uniqueness of quotient normal tubes.

The 5 balls problem: Let $B_1 \subset B_2 \subset B_3$ be a triple of PL-balls, both inclusions being proper, and both pairs unknotted. Further, let $B_4, B_5 \subset B_3$ be PL-balls that satisfy:

(a) B_4, B_5 are both transversal to B_2 in B_3

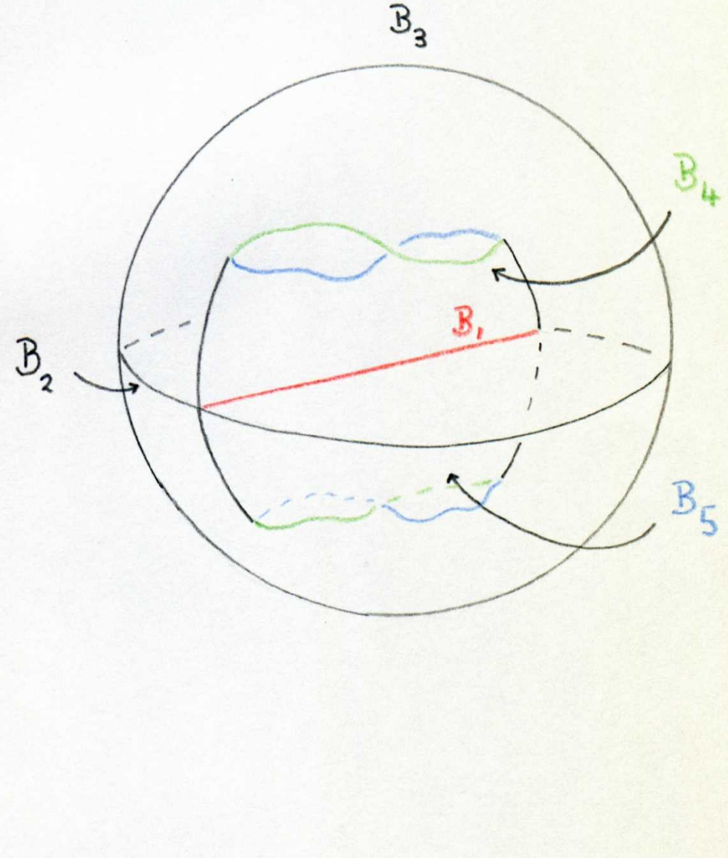
(b) $B_4 \cap B_2 = B_1 = B_5 \cap B_2$

(c) $B_1 \subset B_4$, $B_1 \subset B_5$ are unknotted

(d) $B_4 \cap \dot{B}_3 = \dot{B}_4 \cap \dot{B}_3 = \dot{B}_5 \cap \dot{B}_3 = B_5 \cap \dot{B}_3$ and this common intersection is a tube on $\dot{B}_1$.

Can we ambient isotop B_4 to B_5 keeping $\dot{B}_3 \cup B_2$ fixed?

(A picture is provided overleaf.)



$\mathcal{B}_3$
$\mathcal{B}_4$
$\mathcal{B}_1$
$\mathcal{B}_2$
$\mathcal{B}_5$