THE UNIVERSITY OF
WARWICK

Original citation:
Alphonse, Amal and Elliott, Charles M.. (2015) A Stefan problem on an evolving surface. Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences, 373 (2050).

Permanent WRAP url:
http://wrap.warwick.ac.uk/76379

Copyright and reuse:
The Warwick Research Archive Portal (WRAP) makes this work by researchers of the University of Warwick available open access under the following conditions. Copyright © and all moral rights to the version of the paper presented here belong to the individual author(s) and/or other copyright owners. To the extent reasonable and practicable the material made available in WRAP has been checked for eligibility before being made available.

Copies of full items can be used for personal research or study, educational, or not-for-profit purposes without prior permission or charge. Provided that the authors, title and full bibliographic details are credited, a hyperlink and/or URL is given for the original metadata page and the content is not changed in any way.

Publisher's statement:
Published version: http://dx.doi.org/10.1098/rsta.2014.0279

A note on versions:
The version presented here may differ from the published version or, version of record, if you wish to cite this item you are advised to consult the publisher's version. Please see the 'permanent WRAP url' above for details on accessing the published version and note that access may require a subscription. For more information, please contact the WRAP Team at: publications@warwick.ac.uk

warwickpublicationswrap

highlight your research

http://wrap.warwick.ac.uk

A Stefan problem on an evolving surface

Amal Alphonse and Charles M. Elliott

Mathematics Institute
University of Warwick
Coventry CV4 7AL
United Kingdom

July 10, 2015

Abstract

We formulate a Stefan problem on an evolving hypersurface and study the well-posedness of weak solutions given L^1 data. To do this, we first develop function spaces and results to handle equations on evolving surfaces in order to give a natural treatment of the problem. Then we consider the existence of solutions for L^∞ data; this is done by regularisation of the nonlinearity. The regularised problem is solved by a fixed point theorem and then uniform estimates are obtained in order to pass to the limit. By using a duality method we show continuous dependence which allows us to extend the results to L^1 data.

1 Introduction

The Stefan problem is the prototypical time-dependent free boundary problem. It arises in various forms in many models in the physical and biological sciences [1, 2, 3, 4]. In this paper we present the theory of weak solutions associated with the so-called *enthalpy* approach [1] to the Stefan problem on an evolving curved hypersurface.

Our interest is in the existence, uniqueness and continuous dependence of weak solutions to the Stefan problem

$$\partial^\bullet e(t) - \Delta_{\Omega(t)} u(t) + e(t)\nabla_{\Omega(t)} \cdot \mathbf{w}(t) = f(t) \quad \text{in } \Omega(t)$$
$$e(0) = e_0 \quad \text{on } \Omega(0) \tag{1}$$
$$e \in \mathcal{E}(u)$$

posed on a moving compact hypersurface $\Omega(t) \subset \mathbb{R}^{n+1}$ evolving with (given) velocity field $\mathbf{w}$, where the energy $\mathcal{E} : \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is defined by

$$\mathcal{E}(r) = \begin{cases} r & \text{for } r < 0 \\ [0,1] & \text{for } r = 0 \\ r+1 & \text{for } r > 0. \end{cases}$$

Note that $\mathcal{E}$ is a maximal monotone graph in the sense of Brézis [5]. In (1), $\partial^{\bullet}e$ means the material derivative of e (which we shall also write as $\dot{e}$) and $\nabla_{\Omega(t)}$ and $\Delta_{\Omega(t)}$ are respectively the surface gradient and Laplace–Beltrami operators on $\Omega(t)$. The novelty of this work is that the Stefan problem itself is formulated on a moving hypersurface and our chosen method to treat this problem, which we believe is naturally suited to equations on moving domains, requires the use of some new function spaces and results that we shall introduce, building upon the spaces and concepts presented in [6, 7]. There is, as alluded to above, a rich literature associated to Stefan-type problems [8, 9, 10, 11, 12, 13]. We will show that arguments similar to those used in the standard setting are also amenable to our problem on a moving hypersurface, thanks in part to the function spaces we decide to use. Let us remark that the techniques and functional analysis we develop here can be directly applied to study many other nonlinear PDE problems posed on moving domains.

Let us work out a possible pointwise formulation of (1). Start by supposing $\Omega(t) = \Omega_l(t) \cup \Omega_s(t) \cup \Gamma(t)$ where $\Omega_l(t)$ and $\Omega_s(t)$ divide $\Omega(t)$ into a liquid and a solid phase (respectively) with an *a priori* unknown interface $\Gamma(t)$. The quantity of interest is the temperature $u(t) \colon \Omega(t) \to \mathbb{R}$, which we suppose satisfies

$$
\begin{cases}
u(t) > 0 & \text{in } \Omega_l(t) \\
u(t) = 0 & \text{in } \Gamma(t) \\
u(t) < 0 & \text{in } \Omega_s(t),
\end{cases}
$$

and thus $u = 0$ is the critical temperature where the change of phase occurs. Define

$$
Q_l = \bigcup_{t \in (0,T)} \Omega_l(t) \times \{t\}, \qquad S = \bigcup_{t \in (0,T)} \Gamma(t) \times \{t\},
$$

and Q_s similarly. Given f and u_0, we formally elucidate in Remark 2.12 the relationship between (1) and the following model describing the temperature u:

$$
\begin{aligned}
\partial^{\bullet}u - \Delta_\Omega u + (u+1)\nabla_\Omega \cdot \mathbf{w} &= f & &\text{in } Q_l \\
\partial^{\bullet}u - \Delta_\Omega u + u\nabla_\Omega \cdot \mathbf{w} &= f & &\text{in } Q_s \\
-(\nabla_\Omega u_l - \nabla_\Omega u_s) \cdot \mu &= V & &\text{on } S \\
u &= 0 & &\text{on } S \\
u(0) &= u_0 & &\text{on } \Omega(0),
\end{aligned}
\tag{2}
$$

where u_s denotes the trace of the restriction $u|_{\Omega_s}$ to the interface Γ (likewise with u_l), $V(t)$ is the conormal velocity of $\Gamma(t)$ and $\mu(t)$ is the unit conormal vector pointing into $\Omega_l(t)$ (this vector is tangential to $\Omega(t)$ and normal to $\partial\Omega_l(t)$).

We now introduce some notions of a weak solution, similar to [10]. The function spaces L_X^p below will be made precise in §2 but for now can be thought of as generalisations of Bochner spaces $L^p(0,T;X_0)$ where now $u \in L_X^p$ implies $u(t) \in X(t)$ for almost all t (for a suitable family $\{X(t)\}_{t \in [0,T]}$).

1.1 Definition (Weak solution). *Given $f \in L_{L^1}^1$ and $e_0 \in L^1(\Omega_0)$, a weak solution of (1) is a pair $(u,e) \in L_{L^1}^1 \times L_{L^1}^1$ such that $e \in \mathcal{E}(u)$ and there holds*

$$
-\int_0^T \int_{\Omega(t)} \dot{\eta}(t)e(t) - \int_0^T \int_{\Omega(t)} u(t)\Delta_\Omega \eta(t) = \int_0^T \int_{\Omega(t)} f(t)\eta(t) + \int_{\Omega_0} e_0\eta(0)
$$

2

for all $\eta \in W(L^\infty \cap H^2, L^\infty)$ with $\Delta_\Omega \eta \in L^\infty_{L^\infty}$ and $\eta(T) = 0$.

1.2 Definition (Bounded weak solution). *Given $f \in L^\infty_{L^\infty}$ and $e_0 \in L^\infty(\Omega_0)$, a bounded weak solution of (1) is a pair $(u, e) \in L^2_{H^1} \times L^\infty_{L^\infty}$ such that (u, e) is a weak solution of (1) satisfying*

$$-\int_0^T \int_{\Omega(t)} \dot{\eta}(t) e(t) + \int_0^T \int_{\Omega(t)} \nabla_\Omega u(t) \nabla_\Omega \eta(t) = \int_0^T \int_{\Omega(t)} f(t) \eta(t) + \int_{\Omega_0} e_0 \eta(0) \qquad (3)$$

for all $\eta \in W(H^1, L^2)$ with $\eta(T) = 0$.

We prove the following results.

Theorem 1.3 (Existence of bounded weak solutions). *If $f \in L^\infty_{L^\infty}$, $e_0 \in L^\infty(\Omega_0)$ and $|\Omega| := \sup_{s \in [0,T]} |\Omega(s)| < \infty$, then there exists a bounded weak solution to (1).*

Theorem 1.4 (Uniqueness and continuous dependence of bounded weak solutions). *If for $i = 1, 2$, (u^i, e^i) are two bounded weak solutions of (1) with data $(f^i, e_0^i) \in L^\infty_{L^\infty} \times L^\infty(\Omega_0)$, then*

$$\|e^1(t) - e^2(t)\|_{L^1(\Omega(t))} \leq \int_0^t \|f^1(\tau) - f^2(\tau)\|_{L^1(\Omega(\tau))} + \|e_0^1 - e_0^2\|_{L^1(\Omega_0)}$$

for almost all t.

Theorem 1.5 (Well-posedness of weak solutions). *If $f \in L^1_{L^1}$, $e_0 \in L^1(\Omega_0)$ and $|\Omega| := \sup_{s \in [0,T]} |\Omega(s)| < \infty$, then there exists a unique weak solution to (1). Furthermore, if for $i = 1, 2$, $(u^i, e^i) \in L^1_{L^1} \times L^1_{L^1}$ are two weak solutions of (1) with data $(f^i, e_0^i) \in L^1_{L^1} \times L^1(\Omega_0)$, then*

$$\|e^1 - e^2\|_{L^1_{L^1}} \leq C_T \left(\|f^1 - f^2\|_{L^1_{L^1}} + \|e_0^1 - e_0^2\|_{L^1(\Omega_0)} \right).$$

Below, we shall use the notation $\hookrightarrow$ and $\overset{c}{\hookrightarrow}$ to denote (respectively) a continuous embedding and a compact embedding.

2 Preliminaries

2.1 Abstract evolving function spaces

In [6], we generalised some concepts from [14] and defined the Hilbert space L^2_H given a sufficiently smooth parametrised family of Hilbert spaces $\{H(t)\}_{t \in [0,T]}$. We need a generalisation of this theory to Banach spaces.

For each $t \in [0, T]$, let $X(t)$ be a real Banach space with $X_0 := X(0)$. We informally identify the family $\{X(t)\}_{t \in [0,T]}$ with the symbol X. Let there be a linear homeomorphism $\phi_t \colon X_0 \to X(t)$ for each $t \in [0, T]$ (with the inverse $\phi_{-t} \colon X(t) \to X_0$) such that ϕ_0 is the identity. We assume that there exists a constant C_X independent of $t \in [0, T]$ such that

$$\begin{aligned} \|\phi_t u\|_{X(t)} &\leq C_X \|u\|_{X_0} \qquad \forall u \in X_0 \\ \|\phi_{-t} u\|_{X_0} &\leq C_X \|u\|_{X(t)} \qquad \forall u \in X(t). \end{aligned} \qquad (4)$$

We assume for all $u \in X_0$ that the map $t \mapsto \|\phi_t u\|_{X(t)}$ is measurable.

2.1 Definition. *Define the Banach spaces*

$$L_X^p = \{u : [0,T] \to \bigcup_{t \in [0,T]} X(t) \times \{t\}, \quad t \mapsto (\hat{u}(t),t) \quad | \quad \phi_{-(\cdot)}\hat{u}(\cdot) \in L^p(0,T;X_0)\} \quad \text{for } p \in [1,\infty)$$

$$L_X^\infty = \{u \in L_X^2 \mid \operatorname{ess\,sup}_{t \in [0,T]} \|u(t)\|_{X(t)} < \infty\}$$

endowed with the norm

$$\|u\|_{L_X^p} = \begin{cases} \left(\int_0^T \|u(t)\|_{X(t)}^p\right)^{\frac{1}{p}} & \text{for } p \in [1,\infty) \\ \operatorname{ess\,sup}_{t \in [0,T]} \|u(t)\|_{X(t)} & \text{for } p = \infty. \end{cases} \tag{5}$$

Note that we made an abuse of notation after the definition of the first space and identified $u(t) = (\hat{u}(t),t)$ with $\hat{u}(t)$. That (5) defines a norm is easy to see once one checks that the integrals are well-defined (the case $p = \infty$ is easy), which can be shown by a straightforward adaptation of the proof of Theorem 2.8 in [6] for the case when each $X(t)$ is separable (see the appendix) and the proof of Lemma 3.5 in [14] for the non-separable case. The fact that L_X^p is a Banach space follows from Lemma 2.3 below.

2.2 Important Notation. *Given a function $u \in L_X^p$, the notation $\tilde{u}$ will be used to mean the pullback $\tilde{u}(\cdot) := \phi_{-(\cdot)}u(\cdot) \in L^p(0,T;X_0)$, and vice-versa.*

2.3 Lemma. *The spaces $L^p(0,T;X_0)$ and L_X^p are isomorphic via $\phi_{(\cdot)}$ with an equivalence of norms:*

$$\frac{1}{C_X}\|u\|_{L_X^p} \leq \|\phi_{-(\cdot)}u(\cdot)\|_{L^p(0,T;X_0)} \leq C_X \|u\|_{L_X^p} \quad \text{for all } u \in L_X^p.$$

Proof. We show the case $p = \infty$ here; an adaptation of the $p = 2$ case done in [6] easily proves the lemma for $p \in [1,\infty)$ (see the appendix). Let $u \in L_X^\infty$. Measurability of $\tilde{u}$ follows since $u \in L_X^2$. Now, by definition, we have that for all $t \in [0,T]\backslash N$, $\|u(t)\|_{X(t)} \leq A$ where N is a null set and $A = \|u\|_{L_X^\infty}$. This means that for all $t \in [0,T]\backslash N$, $C_X^{-1}\|\tilde{u}(t)\|_{X_0} \leq \|u(t)\|_{X(t)} \leq A$ by the assumption (4), i.e.,

$$\|\tilde{u}\|_{L^\infty(0,T;X_0)} = \operatorname{ess\,sup}_{t \in [0,T]} \|\tilde{u}(t)\|_{X_0} \leq C_X A = C_X \|u\|_{L_X^\infty},$$

so $\tilde{u} \in L^\infty(0,T;X_0)$. Similarly, we conclude that if $\tilde{u} \in L^\infty(0,T;X_0)$ then $u \in L_X^\infty$. $\qquad\square$

2.4 Remark. *The dual operator $\phi_{-t}^* : X_0^* \to X^*(t)$ is also a linear homeomorphism with $\|\phi_{-t}^*\| = \|\phi_{-t}\|$ and $(\phi_{-t}^*)^{-1} = \phi_t^*$ [15, Theorem 4.5-2 and §4.5], and if X_0 is separable, $t \mapsto \|\phi_{-t}^* f\|_{X^*(t)}$ is measurable for $f \in X_0^*$; thus, in the separable setting, the dual operator also satisfies the same boundedness properties as ϕ_t. This means that the spaces $L_{X^*}^p$ are also well-defined Banach spaces given separable $\{X(t)\}_{t \in [0,T]}$ (the map $\phi_{-(\cdot)}^*$ plays the same role as $\phi_{(\cdot)}$ did for the spaces L_X^p).*

The following subspaces will be of use later:

$$C_X^k = \{\xi \in L_X^2 \mid \phi_{-(\cdot)}\xi(\cdot) \in C^k([0,T];X_0)\} \quad \text{for } k \in \{0,1,...\}$$
$$\mathcal{D}_X = \{\eta \in L_X^2 \mid \phi_{-(\cdot)}\eta(\cdot) \in \mathcal{D}((0,T);X_0)\}.$$

2.1.1 Dual spaces

In this subsection, we assume that $\{X(t)\}_{t\in[0,T]}$ is reflexive. In order to retrieve weakly convergent subsequences from sequences that are bounded in L_X^p, we need L_X^p to be reflexive. This leads us to consider a characterisation of the dual spaces. We let $p \in [1,\infty)$ and (p,q) be a conjugate pair in this section.

Theorem 2.5. *The space $(L_X^p)^*$ is isometrically isomorphic to $L_{X^*}^q$, and hence we may identify $(L_X^p)^* \equiv L_{X^*}^q$ and the duality pairing of $f \in L_{X^*}^q$ with $u \in L_X^p$ is given by*

$$\langle f, u \rangle_{L_{X^*}^q, L_X^p} = \int_0^T \langle f(t), u(t) \rangle_{X^*(t), X(t)}.$$

To prove this theorem, although we can exploit the fact that the pullback is in a Bochner space, showing that the natural duality map is isometric is not so straightforward because $\phi_{(\cdot)}$ is not assumed to be an isometry. In fact, we have to go back to the foundations and emulate the proof for the dual space identification for Bochner spaces; see [16, §IV].

2.6 Lemma. *For every $g \in L_{X^*}^q$, the expression*

$$l(f) = \int_0^T \langle g(t), f(t) \rangle_{X^*(t), X(t)} \qquad \text{for all } f \in L_X^p \tag{6}$$

defines a functional $l \in (L_X^p)^*$ such that $\|l\| = \|g\|_{L_{X^*}^q}$.

Proof. Let $g \in L_{X^*}^q$ and define $l \colon L_X^p \to \mathbb{R}$ by (6); the integral is well-defined by similar reasoning as before (see Lemma 2.13 in [6]). By Hölder's inequality, we have $|l(f)| \leq \|g\|_{L_{X^*}^q} \|f\|_{L_X^p}$, so $l \in (L_X^p)^*$ and $\|l\| \leq \|g\|_{L_{X^*}^q}$. We now show the reverse inequality. First suppose g has the form $g(t) = \sum x_{i,t}^* \chi_{E_i}(t)$ where the $x_{i,t}^* \in X^*(t)$ and the E_i are measurable, pairwise disjoint and partition $[0,T]$. It is clear that $\|g(t)\|_{X^*(t)} = \sum \|x_{i,t}^*\|_{X^*(t)} \chi_{E_i}(t)$. Let $h(t) = \|g(t)\|_{X^*(t)}^{q/p} / \|g\|_{L_{X^*}^q}^{q/p}$ which satisfies $\|h\|_{L^p(0,T)}^p = 1$ and $\int_0^T \|g(t)\|_{X^*(t)} h(t) = \|g\|_{L_{X^*}^q}$, hence for any $\epsilon > 0$ we have

$$\int_0^T \|g(t)\|_{X^*(t)} h(t) \geq \|g\|_{L_{X^*}^q} - \frac{\epsilon}{2}. \tag{7}$$

Now choose $x_{i,t} \in X(t)$, $\|x_{i,t}\|_{X(t)} = 1$ such that

$$\|x_{i,t}^*\|_{X^*(t)} - \langle x_{i,t}^*, x_{i,t} \rangle_{X^*(t), X(t)} \leq \frac{\epsilon}{2 \|h\|_{L^1(0,T)}}. \tag{8}$$

Define $f \in L_X^p$ by $f(t) = \sum x_{i,t} h(t) \chi_{E_i}(t)$ and note that $\|f\|_{L_X^p}^p = \|h\|_{L^p(0,T)}^p$. We obtain using (8) and (7) that $l(f) \geq \|g\|_{L_{X^*}^q} - \epsilon$. This proves that $\|l\| = \|g\|_{L_{X^*}^q}$ whenever $g(t) = \sum x_{i,t}^* \chi_{E_i}(t)$ is of the stated form. Now suppose $g \in L_{X^*}^q$ is arbitrary. Then there exist $\tilde{g}_n(t) = \sum \tilde{g}_{i,n} \chi_{E_i}(t)$ with $\tilde{g}_{i,n} \in X_0^*$ such that $\tilde{g}_n \to \tilde{g}$ in $L^q(0,T;X_0^*)$ and so the sequence $g_n(t) := \phi_{-t}^* \tilde{g}_n(t) = \sum \phi_{-t}^* \tilde{g}_{i,n} \chi_{E_i}(t)$ satisfies $g_n \to g$ in $L_{X^*}^q$. Because the $\phi_{-t}^* \tilde{g}_{i,n} \in X^*(t)$, we know by our efforts above that $l_n \colon L_X^p \to \mathbb{R}$ defined $l_n(f) = \int_0^T \langle g_n(t), f(t) \rangle_{X^*(t), X(t)}$ has norm $\|l_n\| = \|g_n\|_{L_{X^*}^q}$. We also have

$$\|l_n - l\| \leq \|g_n - g\|_{L_{X^*}^q} \to 0$$

which implies $\lim_{n\to\infty} \|l_n\| = \|l\|$ and also $\lim_{n\to\infty} \|l_n\| = \lim_{n\to\infty} \|g_n\|_{L_{X^*}^q} = \|g\|_{L_{X^*}^q}$. $\qquad\square$

We have shown that $\mathcal{J}\colon L^q_{X^*} \to (L^p_X)^*$ defined by $\mathcal{J}(g) := l(\cdot) = \int_0^T \langle g(t), (\cdot)(t)\rangle_{X^*(t),X(t)}$ is isometric: $\|\mathcal{J}g\|_{(L^q_X)^*} = \|l\| = \|g\|_{L^q_{X^*}}$. We now show that $\mathcal{J}$ is onto. Given $l \in (L^p_X)^*$, define $\tilde{L}\colon L^p(0,T;X_0) \to \mathbb{R}$ by $\tilde{L}(\tilde{v}) = l(\phi_{(\cdot)}\tilde{v}(\cdot)) = l(v)$ for all $\tilde{v} \in L^p(0,T;X_0)$. It is obvious that $\tilde{L} \in L^p(0,T;X_0)^*$, and by the dual space identification for Bochner spaces, there exists an $\tilde{L}^* \in L^q(0,T;X_0^*)$ such that

$$\langle l, v\rangle_{(L^p_X)^*,L^p_X} = \langle \tilde{L}, \tilde{v}\rangle_{L^p(0,T;X_0)^*,L^p(0,T;X_0)} = \int_0^T \langle \phi^*_{-t}\tilde{L}^*(t), v(t)\rangle_{X^*(t),X(t)},$$

so $\mathcal{J}(\phi^*_{-(\cdot)}\tilde{L}^*(\cdot)) = l$ where $\phi^*_{-(\cdot)}\tilde{L}^*(\cdot) \in L^q_{X^*}$. Hence $\mathcal{J}$ is onto, and we have proved Theorem 2.5.

2.2 Function spaces on evolving surfaces

We now make precise the assumptions on the evolving surface $\Omega(t)$ our Stefan problem is posed on and we discuss function spaces in the context of the previous subsections. For each $t \in [0,T]$, let $\Omega(t) \subset \mathbb{R}^{n+1}$ be an orientable compact (i.e., no boundary) n-dimensional hypersurface of class C^3, and assume the existence of a flow $\Phi\colon [0,T] \times \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ such that for all $t \in [0,T]$, with $\Omega_0 := \Omega(0)$, the map $\Phi^0_t(\cdot) := \Phi(t,\cdot)\colon \Omega_0 \to \Omega(t)$ is a C^3-diffeomorphism that satisfies $\frac{d}{dt}\Phi^0_t(\cdot) = \mathbf{w}(t,\Phi^0_t(\cdot))$ and $\Phi^0_0(\cdot) = \mathrm{Id}(\cdot)$ for a given C^2 velocity field $\mathbf{w}\colon [0,T]\times\mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$, which we assume satisfies the uniform bound $|\nabla_{\Omega(t)}\cdot\mathbf{w}(t)| \le C$ for all $t \in [0,T]$. A C^2 normal vector field on the hypersurfaces is denoted by $\nu\colon [0,T]\times\mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$. It follows that the Jacobian $J^0_t := \det \mathbf{D}\Phi^0_t$ is C^2 and is uniformly bounded away from zero and infinity.

For $u\colon \Omega_0 \to \mathbb{R}$ and $v\colon \Omega(t) \to \mathbb{R}$, define the pushforward $\phi_t u = u \circ \Phi^t_0$ and pullback $\phi_{-t}v = v \circ \Phi^0_t$, where $\Phi^t_0 := (\Phi^0_t)^{-1}$. We showed in [7] that $\phi_t\colon L^2(\Omega_0) \to L^2(\Omega(t))$ and $\phi_t\colon H^1(\Omega_0) \to H^1(\Omega(t))$ are linear homeomorphisms (with uniform bounds) and (thus) with $L^2 \equiv \{L^2(\Omega(t))\}_{t\in[0,T]}$, $H^1 \equiv \{H^1(\Omega(t))\}_{t\in[0,T]}$ and $H^{-1} \equiv \{H^{-1}(\Omega(t))\}_{t\in[0,T]}$, the spaces $L^2_{L^2}$, $L^2_{H^1}$ and $L^2_{H^{-1}}$ are well-defined (see [7, 17] for an overview of Lebesgue and Sobolev spaces on hypersurfaces) and we let $L^2_{H^1} \subset L^2_{L^2} \subset L^2_{H^{-1}}$ be a Gelfand triple.

A function $u \in C^1_{L^2}$ has a *strong material derivative* defined by $\dot{u}(t) = \phi_t\left(\frac{d}{dt}(\phi_{-t}u(t))\right)$. Given a function $u \in L^2_{H^1}$, we say that it has a *weak material derivative* $g \in L^2_{H^{-1}}$ if

$$(u, \dot{\eta})_{L^2_{L^2}} = -\langle g, \eta\rangle_{L^2_{H^{-1}},L^2_{H^1}} - (u, \eta\nabla_\Omega\cdot\mathbf{w})_{L^2_{L^2}} \qquad \forall \eta \in \mathcal{D}_{H^1}$$

holds, and we write $\dot{u}$ or $\partial^\bullet u$ instead of g. Define the Hilbert spaces (see [6, 7] for more details)

$$\mathcal{W}(H^1(\Omega_0), H^{-1}(\Omega_0)) = \{u \in L^2(0,T;H^1(\Omega_0)) \mid u' \in L^2(0,T;H^{-1}(\Omega_0))\}$$
$$W(H^1, H^{-1}) = \{u \in L^2_{H^1} \mid \dot{u} \in L^2_{H^{-1}}\}$$

endowed with the natural inner products. For subspaces $X \hookrightarrow H^1$ and $Y \hookrightarrow H^{-1}$, we also define the subset $W(X,Y) \subset W(H^1,H^{-1})$ in the natural manner.

2.7 Lemma (See [6, 7]). *Let either $X = W(H^1, H^{-1})$ and $X_0 = \mathcal{W}(H^1(\Omega_0), H^{-1}(\Omega_0))$, or $X = W(H^1, L^2)$ and $X_0 = \mathcal{W}(H^1(\Omega_0), L^2(\Omega_0))$. For such pairs, the space X is isomorphic to X_0 via $\phi_{-(\cdot)}$ with an equivalence of norms:*

$$C_1 \left\|\phi_{-(\cdot)}v(\cdot)\right\|_{X_0} \le \|v\|_X \le C_2 \left\|\phi_{-(\cdot)}v(\cdot)\right\|_{X_0}.$$

We showed in [6, 7] that for $u, v \in W(H^1, H^{-1})$, the map $t \mapsto (u(t), v(t))_{L^2(\Omega(t))}$ is absolutely continuous, and

$$\frac{d}{dt} \int_{\Omega(t)} u(t)v(t) = \langle \dot{u}(t), v(t) \rangle + \langle \dot{v}(t), u(t) \rangle + \int_{\Omega(t)} u(t)v(t) \nabla_\Omega \cdot \mathbf{w}(t)$$

holds for almost all t, where the duality pairing is between $H^{-1}(\Omega(t))$ and $H^1(\Omega(t))$.

2.2.1 Some useful results

In this subsection, p and q are not necessarily conjugate. The first part of the following lemma is a particular realisation of Lemma 2.3.

2.8 Lemma. *For $p, q \in [1, \infty]$, the spaces $L^p_{L^q}$ and $L^p(0, T; L^q(\Omega_0))$ are isomorphic via the map $\phi_{(.)}$ with an equivalence of norms. If $q = \infty$ the spaces are isometrically isomorphic. The embedding $L^\infty_{L^\infty} \subset L^p_{L^q}$ is continuous.*

2.9 Lemma. *The space $W(H^1, H^{-1})$ is compactly embedded in $L^2_{L^2}$.*

Theorem 2.10 (Dominated convergence theorem for $L^p_{L^q}$). *Let $p, q \in [1, \infty)$. Let $\{w_n\}$ and w be functions such that $\{\tilde{w}_n\}$ and $\tilde{w}$ are measurable (eg. membership of $L^1_{L^1}$ will suffice). If for almost all $t \in [0, T]$,*

$$w_n(t) \to w(t) \quad \text{almost everywhere in } \Omega(t)$$
$$\exists g \in L^p_{L^q} : |w_n(t)| \leq g(t) \quad \text{almost everywhere in } \Omega(t) \text{ and for all } n,$$

then $w_n \to w$ in $L^p_{L^q}$.

2.11 Lemma. *If $u \in W(H^1, H^{-1})$, then*

$$2 \int_0^T \langle \dot{u}(t), u^+(t) \rangle_{H^{-1}(\Omega(t)), H^1(\Omega(t))} = \int_{\Omega(T)} u^+(T)^2 - \int_{\Omega_0} u^+(0)^2 - \int_0^T \int_{\Omega(t)} u^+(t)^2 \nabla_\Omega \cdot \mathbf{w}. \quad (9)$$

Proof. By density, we can find $\{u_n\} \subset W(H^1, L^2)$ with $u_n \to u$ in $W(H^1, H^{-1})$. It follows that $\partial^\bullet(u_n^+) = \dot{u}_n \chi_{u_n \geq 0} \in L^2_{L^2}$ (this is sensible because $w \in H^1(\Omega)$ implies $w^+ \in H^1(\Omega)$) and therefore (9) holds for u_n. Since $W(H^1, H^{-1}) \hookrightarrow C^0_{L^2}$, it follows that $u_n^+(t) \to u^+(t)$ in $L^2(\Omega(t))$ (for example see [18, Lemma 2.88] or [19, Lemma 1.22]). So we can pass to the limit in the first two terms on the right hand side.

Now we just need to show that $u_n^+ \to u^+$ in $L^2_{H^1}$. It is easy to show the convergence in $L^2_{L^2}$, so we need only to check the convergence of the gradient. Let $g(r) = \chi_{\{r>0\}}$. Then, using $g \leq 1$,

$$|\nabla_\Omega u_n^+(t, x) - \nabla_\Omega u^+(t, x)| \leq |\nabla_\Omega u_n(t, x) - \nabla_\Omega u(t, x)| + |g(u_n(t, x)) - g(u(t, x))||\nabla_\Omega u(t, x)|.$$

For the second term, let us note that since $u_n \to u$ in $L^2_{H^1}$, for almost all t, $u_n(t, x) \to u(t, x)$ almost everywhere in $\Omega(t)$ for a subsequence (which we have not relabelled). Let us fix t. Then for almost every $x \in \Omega(t)$, it follows that $g(u_n(t, x))\nabla_\Omega u(t, x) \to g(u(t, x))\nabla_\Omega u(t, x)$ pointwise. Because $g \leq 1$, the dominated convergence theorem gives overall $\nabla_\Omega u_n^+ \to \nabla_\Omega u^+$ in $L^2_{L^2}$. $\square$

2.3 Preliminary results

2.12 Remark. *It is well-known in the standard setting that a mushy region (the interior of the set where the temperature is zero) can arise in the presence of heat sources [20, 1]; with no heat sources, the initial data may give rise to mushy regions. We will content ourselves with the following heuristic calculations under the assumption that there is no mushy region.*

Let the bounded weak solution of (1) *(in the sense of Definition* 1.2*) have the additional regularity* $u \in W(H^1, L^2)$ *and* $\Delta_\Omega u \in L^2_{L^2}$, *and suppose that the sets* $\Omega_l(t) = \{u > 0\}$ *and* $\Omega_s(t) = \{u < 0\}$ *divide* $\Omega(t)$ *with a common interface* $\Gamma(t)$, *which we assume is a sufficiently smooth* $n-$*dimensional hypersurface (of measure zero with respect to the surface measure on* $\Omega(t)$*). Then the bounded weak solution is also a classical solution in the sense of* (2)*. To see this, suppose that* (u, e) *is a weak solution satisfying the equality in* (3)*. The integration by parts formula on each subdomain of* Ω *implies*

$$\int_0^T \int_{\Omega(t)} \nabla_\Omega u(t) \nabla_\Omega \eta(t) = - \int_0^T \int_{\Omega(t)} \eta(t) \Delta_\Omega u(t) + \int_0^T \int_{\Gamma(t)} \eta(t) (\nabla_\Omega u_s(t) - \nabla_\Omega u_l(t)) \cdot \mu.$$
(10)

With $e(t)\eta(t)\nabla_\Omega \cdot \mathbf{w} = \nabla_\Omega \cdot (e(t)\eta(t)\mathbf{w}) - \mathbf{w} \cdot \nabla_\Omega(e(t)\eta(t))$ *and the divergence theorem [17, §2.2],*

$$\int_0^T \int_{\Omega_s(t)} e(t)\eta(t)\nabla_\Omega \cdot \mathbf{w} = \int_0^T \int_{\Gamma(t)} e(t)\eta(t)\mathbf{w} \cdot \mu + \int_0^T \int_{\Omega_s(t)} \mathbf{w} \cdot (e(t)\eta(t)\nu H - \nabla_\Omega(e(t)\eta(t))).$$

We use this result in the formula for integration by parts over time over Ω_s:

$$\int_0^T \int_{\Omega_s(t)} \dot{\eta}(t)e(t) = \int_0^T \frac{d}{dt} \int_{\Omega_s(t)} e(t)\eta(t) - \int_0^T \int_{\Omega_s(t)} \dot{e}(t)\eta(t) - \int_0^T \int_{\Gamma(t)} e(t)\eta(t)\mathbf{w} \cdot \mu$$
$$- \int_0^T \int_{\Omega_s(t)} e(t)\eta(t)\mathbf{w} \cdot \nu H + \int_0^T \int_{\Omega_s(t)} \mathbf{w} \cdot \nabla_\Omega(e(t)\eta(t)).$$

A similar expression over Ω_l *can also be derived this way, the difference being that the term with* μ *has the opposite sign. Then, using* $\dot{e} = \partial^\bullet(\mathcal{E}(u)) = \dot{u}$, $e_s(t)|_{\Gamma(t)} = 0$, *and* $e_l(t)|_{\Gamma(t)} = 1$, *we get*

$$\int_0^T \int_{\Omega(t)} \dot{\eta}(t)e(t) = \int_0^T \frac{d}{dt} \int_{\Omega(t)} e(t)\eta(t) - \int_0^T \int_{\Omega(t)} \dot{u}(t)\eta(t) + \int_0^T \int_{\Gamma(t)} \eta(t)\mathbf{w} \cdot \mu$$
$$- \int_0^T \int_{\Omega(t)} e(t)\eta(t)\mathbf{w} \cdot \nu H + \int_0^T \int_{\Omega(t)} \mathbf{w} \cdot \nabla_\Omega(e(t)\eta(t)).$$
(11)

Since by the partial integration formula $\int_{\Omega(t)} \underline{D}_i(g) = \int_{\Omega(t)} gH\nu_i$, *we have (with* $g = \mathbf{w}_i e(t)\eta(t)$*) that the fourth term in the right hand side of* (11) *is*

$$\int_{\Omega(t)} e(t)\eta(t)\mathbf{w} \cdot \nu H = \sum_i \int_{\Omega(t)} e(t)\eta(t)\mathbf{w}_i\nu_i H = \int_{\Omega(t)} \nabla_\Omega(e(t)\eta(t)) \cdot \mathbf{w} + \int_{\Omega(t)} \eta(t)e(t)\nabla_\Omega \cdot \mathbf{w}.$$

So the calculation (11) *becomes*

$$\int_0^T \int_{\Omega(t)} \dot{\eta}(t)e(t) = \int_0^T \left(\frac{d}{dt} \int_{\Omega(t)} e(t)\eta(t) - \int_{\Omega(t)} (\dot{u}(t)\eta(t) + \eta(t)e(t)\nabla_\Omega \cdot \mathbf{w}) + \int_{\Gamma(t)} \eta(t)\mathbf{w} \cdot \mu \right).$$

(12)

Now, taking the weak formulation (3) *and substituting* (12) *together with the expression for the spatial term* (10)*, we get for η with $\eta(T) = \eta(0) = 0$*

$$\int_0^T \int_{\Omega(t)} f(t)\eta(t) = -\int_0^T \int_{\Omega(t)} \dot{\eta}(t)e(t) + \int_0^T \int_{\Omega(t)} \nabla_\Omega u(t)\nabla_\Omega \eta(t)$$

$$= \int_0^T \int_{\Omega(t)} (\dot{u}(t) + e(t)\nabla_\Omega \cdot \mathbf{w} - \Delta_\Omega u(t))\eta(t) + \int_{\Gamma(t)} \eta(t) \left((\nabla_\Omega u_s(t) - \nabla_\Omega u_l(t)) \cdot \mu - (\mathbf{w} \cdot \mu) \right).$$

Taking η to be compactly supported in Q_s, and afterwards taking η compactly supported in Q_l, we recover exactly the first two equations in (2)*. So we may drop the first integral on the left and the right hand side. Then with a careful choice of η, we will obtain precisely the interface condition in* (2)*.*

2.13 Lemma. *Given $\xi \in C^1(\Omega_0)$ and $\tilde{\alpha} \in C^2([0,T] \times \Omega_0)$ satisfying $0 < \epsilon \le \alpha \le \alpha_0$ a.e., there exists a unique solution $\varphi \in W(H^1, L^2)$ with $\Delta_\Omega \varphi \in L^2_{L^2}$ to*

$$\dot{\varphi} - \alpha(x,t)\Delta_\Omega \varphi = 0 \tag{13}$$
$$\varphi(x,0) = \xi(x)$$

satisfying $\|\varphi\|_{L^\infty_{L^\infty}} \le \|\xi\|_{L^\infty(\Omega_0)}$ and (cf. [21, Chapter V, §9])

$$\int_0^t \int_{\Omega(\tau)} (\dot{\varphi}(\tau))^2 + \int_0^t \int_{\Omega(\tau)} \alpha |\Delta_\Omega \varphi|^2 + \int_{\Omega(t)} |\nabla_\Omega \varphi(t)|^2 \le (1+\alpha_0)(1+e^{2C_\mathbf{w}(1+\alpha_0)t}) \int_{\Omega_0} |\nabla_\Omega \xi|^2.$$

(14)

Proof. Define the bilinear form $a(t;\varphi,\eta) = \int_{\Omega(t)} \alpha(x,t)\nabla_\Omega \varphi \nabla_\Omega \eta + \int_{\Omega(t)} \nabla_\Omega \alpha(x,t)\nabla_\Omega \varphi \eta$ which is clearly bounded and coercive on $H^1(\Omega(t))$. Split $a(t;\cdot,\cdot)$ into the forms $a_s(t;\varphi,\eta) := \int_{\Omega(t)} \alpha(x,t)\nabla_\Omega \varphi \nabla_\Omega \eta$ and $a_n(t;\varphi,\eta) := \int_{\Omega(t)} \nabla_\Omega \alpha(x,t)\nabla_\Omega \varphi \eta$. One sees that $a_s(t;\eta,\eta) \ge 0$ and that both $a_n(t;\cdot,\cdot) \colon H^1(\Omega(t)) \times L^2(\Omega(t)) \to \mathbb{R}$ and $a_s(t;\cdot,\cdot) \colon H^1(\Omega(t)) \times H^1(\Omega(t)) \to \mathbb{R}$ are bounded. Also, letting $\chi_j^t := \phi_t \chi_j^0$ where χ_j^0 are the normalised eigenfunctions of $-\Delta_{\Omega_0}$, we have for $\eta \in \tilde{C}^1_{H^1} := \{u \mid u(t) = \sum_{j=1}^m \alpha_j(t)\chi_j^t, \ m \in \mathbb{N}, \ \alpha_j \in AC([0,T]) \text{ and } \alpha_j' \in L^2(0,T)\}$,

$$\frac{d}{dt} a_s(t;\eta(t),\eta(t)) = 2a_s(t;\dot{\eta}(t),\eta(t)) + r(t;\eta(t))$$

where r is such that $|r(t;\eta(t))| \le C \|\eta(t)\|^2_{H^1(\Omega(t))}$ (see [17, Lemma 2.1], note that $\tilde{\alpha} \in C^1([0,T];C^1(\Omega_0))$ and thus $\alpha \in C^1_{H^1}$). Hence by [6, Theorem 3.13] we have the unique existence of $\varphi \in W(H^1, L^2)$. Rearranging the equation (13) shows that $\alpha \Delta_\Omega \varphi \in L^2_{L^2}$. Since α is uniformly bounded by positive constants, it follows that $\Delta_\Omega \varphi \in L^2_{L^2}$.

9

The L^∞ bound Let $K := \|\xi\|_{L^\infty(\Omega_0)}$. Test the equation with $(\varphi - K)^+$:

$$\frac{1}{2}\frac{d}{dt}\left\|(\varphi(t) - K)^+\right\|^2_{L^2(\Omega(t))} + \int_{\Omega(t)} \alpha(t)\nabla_\Omega((\varphi(t) - K)^+)\nabla_\Omega\varphi(t)$$

$$= \frac{1}{2}\int_{\Omega(t)} ((\varphi(t) - K)^+)^2\nabla_\Omega \cdot \mathbf{w} - \int_{\Omega(t)} \nabla_\Omega\alpha(t)\nabla_\Omega\varphi(t)(\varphi(t) - K)^+$$

which becomes, through the use of Young's inequality with δ,

$$\frac{1}{2}\frac{d}{dt}\left\|(\varphi(t) - K)^+\right\|^2_{L^2(\Omega(t))} \leq \left(\frac{C_\mathbf{w}}{2} + \|\nabla_\Omega\alpha\|_{L^\infty} C_\delta\right)\left\|(\varphi(t) - K)^+\right\|^2_{L^2(\Omega(t))}.$$

An application of Gronwall's inequality and noticing $(\varphi(0) - K)^+ = (\xi - \|\xi\|_{L^\infty})^+ = 0$ yields $\varphi(t) \leq \|\xi\|_{L^\infty(\Omega_0)}$. Repeating this process with $(-\varphi(t) - K)^+$ allows us to conclude.

The inequality (14) Multiplying the equation (13) by $\Delta_\Omega\varphi$ and integrate: formally,

$$\int_0^t \int_{\Omega(\tau)} \alpha|\Delta_\Omega\varphi|^2 = -\int_0^t \int_{\Omega(\tau)} \nabla_\Omega\dot\varphi\nabla_\Omega\varphi$$

$$= -\int_0^t \frac{1}{2}\frac{d}{d\tau}\int_{\Omega(\tau)} |\nabla_\Omega\varphi|^2 + \frac{1}{2}\int_0^t \int_{\Omega(\tau)} |\nabla_\Omega\varphi|^2\nabla_\Omega \cdot \mathbf{w} - \int_0^t \int_{\Omega(\tau)} D(\mathbf{w})\nabla_\Omega\varphi\nabla_\Omega\varphi$$

$$\leq \frac{1}{2}\int_{\Omega_0} |\nabla_\Omega\xi|^2 - \frac{1}{2}\int_{\Omega(t)} |\nabla_\Omega\varphi(t)|^2 + C_\mathbf{w}\int_0^t \int_{\Omega(\tau)} |\nabla_\Omega\varphi|^2. \tag{15}$$

See [17, Lemma 2.1] or [7] for the definition of the matrix $D(\mathbf{w})$. This calculation is merely formal because we have not shown that $\dot\varphi(t) \in H^1(\Omega(t))$; however the end result of the calculation is still valid by Lemma 2.14. We also have by squaring (13), integrating and using (15):

$$\int_0^t \int_{\Omega(\tau)} (\dot\varphi(\tau))^2 \leq \alpha_0\int_0^t \int_{\Omega(\tau)} \alpha(\Delta_\Omega\varphi)^2 \leq \frac{\alpha_0}{2}\int_{\Omega_0} |\nabla_\Omega\xi|^2 + \alpha_0 C_\mathbf{w}\int_0^t \int_{\Omega(\tau)} |\nabla_\Omega\varphi|^2.$$

Adding the last two inequalities then we obtain

$$\int_0^t \int_{\Omega(\tau)} (\dot\varphi(\tau))^2 + \int_0^t \int_{\Omega(\tau)} \alpha|\Delta_\Omega\varphi|^2 + \frac{1}{2}\int_{\Omega(t)} |\nabla_\Omega\varphi(t)|^2 \leq \frac{1 + \alpha_0}{2}\int_{\Omega_0} |\nabla_\Omega\xi|^2$$

$$+ C_\mathbf{w}(1 + \alpha_0)\int_0^t \int_{\Omega(\tau)} |\nabla_\Omega\varphi|^2.$$

Gronwall's inequality can be used to deal with the last term on the right hand side. $\qquad\square$

2.14 Lemma. *With $\varphi \in W(H^1, L^2)$ from the previous lemma, the following inequality holds:*

$$\int_0^t \int_{\Omega(\tau)} \alpha|\Delta_\Omega\varphi|^2 \leq \frac{1}{2}\int_{\Omega_0} |\nabla_\Omega\xi|^2 - \frac{1}{2}\int_{\Omega(t)} |\nabla_\Omega\varphi(t)|^2 + C_\mathbf{w}\int_0^t \int_{\Omega(\tau)} |\nabla_\Omega\varphi|^2. \tag{16}$$

10

Proof. Let $C_{H^2}^\infty := \{\eta \mid \phi_{-(\cdot)}\eta(\cdot) \in C^\infty([0,T];H^2(\Omega_0))\}$. We start with a few preliminary results. Let us show $C_{H^2}^\infty \subset W(H^2, H^1)$. Take $\eta \in C_{H^2}^\infty$ so that $\tilde\eta \in C^\infty([0,T];H^2(\Omega_0)) \subset \mathcal{W}(H^2, H^1)$. By smoothness of $\Phi_0^{(\cdot)}$, it follows that $\eta = \phi_{(\cdot)}\tilde\eta \in L_{H^2}^2$, and $\dot\eta = \partial^\bullet(\phi_{(\cdot)}\tilde\eta) = \phi_{(\cdot)}(\tilde\eta') \in L_{H^1}^2$ because $\tilde\eta' \in C^\infty([0,T];H^2(\Omega_0)) \subset L^2(0,T;H^1(\Omega_0))$. So $\eta \in W(H^2, H^1)$.

Let us also prove that $C_{H^2}^\infty \subset W(H^2, L^2)$ is dense. Let $w \in W(H^2, L^2)$; then $\tilde w \in \mathcal{W}(H^2, L^2)$ since $\tilde w \in L^2(0,T;H^2(\Omega_0))$ by smoothness of $\Phi_0^{(\cdot)}$ and since $\tilde w' = \phi_{-(\cdot)}\dot w \in L^2(0,T;L^2(\Omega_0))$ (because $\dot w \in L_{L^2}^2$). By [22, Lemma II.5.10] there exists $\tilde w_n \in C^\infty([0,T];H^2(\Omega_0))$ with $\tilde w_n \to \tilde w$ in $\mathcal{W}(H^2, L^2)$. Then, $w_n := \phi_{(\cdot)}\tilde w_n \in C_{H^2}^\infty$ (by definition) and

$$\|w_n - w\|_{W(H^2,L^2)} \le C \left(\|\tilde w_n - \tilde w\|_{L^2(0,T;H^2(\Omega_0))} + \|\tilde w_n' - \tilde w'\|_{L^2(0,T;L^2(\Omega_0))} \right) \to 0,$$

where we used the smoothness of $\Phi_0^{(\cdot)}$ and the reasoning behind Assumption 2.37 of [6] (see also [6, Theorem 2.33]).

Given $\varphi \in W(H^2, L^2)$, by the density result, there exists $\varphi_n \in C_{H^2}^\infty \subset W(H^2, H^1)$ such that $\varphi_n \to \varphi$ in $W(H^2, L^2)$ with φ_n satisfying (16):

$$\int_0^t \int_{\Omega(\tau)} \alpha|\Delta_\Omega\varphi_n|^2 \le \frac{1}{2}\int_{\Omega_0} |\nabla_\Omega\varphi_n(0)|^2 - \frac{1}{2}\int_{\Omega(t)} |\nabla_\Omega\varphi_n(t)|^2 + C_{\mathbf{w}}\int_0^t \int_{\Omega(\tau)} |\nabla_\Omega\varphi_n|^2. \quad (17)$$

We know that $\tilde\varphi_n \to \tilde\varphi$ in $\mathcal{W}(H^2, L^2)$ (this is just how we construct the sequence φ_n; see above), and $\mathcal{W}(H^2, L^2) \hookrightarrow C^0([0,T];H^1(\Omega_0))$ [22, Lemma II.5.14] implies $\varphi_n(t) \to \varphi(t)$ in $H^1(\Omega(t))$. Now we can pass to the limit in every term in (17). $\square$

3 Well-posedness

We can approximate $\mathcal{E}$ by C^∞ bi-Lipschitz functions $\mathcal{E}_\epsilon$ such that (for example see [12, 13])

$$\mathcal{E}_\epsilon \to \mathcal{E} \text{ uniformly in the compact subsets of } \mathbb{R}\backslash\{0\}$$
$$\mathcal{E}_\epsilon^{-1} \to \mathcal{E}^{-1} \text{ uniformly in the compact subsets of } \mathbb{R}$$
$$\mathcal{E}_\epsilon(0) = 0 \text{ and } \mathcal{E}_\epsilon = \mathcal{E} \text{ on } (-\infty, 0) \cup (\epsilon, \infty)$$
$$1 \le \mathcal{E}_\epsilon'(r) \le 1 + L_\epsilon \text{ and } (1 + L_\epsilon)^{-1} \le (\mathcal{E}_\epsilon^{-1}(r))' \le 1 \quad \text{for all } r \in \mathbb{R}$$

(where $L_\epsilon = \mathcal{O}(1/\epsilon)$ is the Lipschitz constant of the approximation to the Heaviside function). We write $\mathcal{U} := \mathcal{E}^{-1}$ and $\mathcal{U}_\epsilon := \mathcal{E}_\epsilon^{-1}$. In order to prove Theorem 1.3, that of the well-posedness of L^∞ weak solutions given bounded data, we consider the following approximation of (1).

3.1 Definition. *Find for each $\epsilon > 0$ a function $e_\epsilon \in W(H^1, H^{-1})$ such that*

$$\partial^\bullet e_\epsilon - \Delta_\Omega(\mathcal{U}_\epsilon e_\epsilon) + e_\epsilon \nabla_\Omega \cdot \mathbf{w} = f \quad in \ L_{H^{-1}}^2$$
$$e_\epsilon(0) = e_0. \tag{$\mathbf{P}_\epsilon$}$$

Theorem 3.2. *Given $f \in L_{H^{-1}}^2$ and $e_0 \in L^2(\Omega_0)$, the problem $(\mathbf{P}_\epsilon)$ has a weak solution $e_\epsilon \in W(H^1, H^{-1})$.*

11

Proof. Using the chain rule on the nonlinear term leads us to consider for fixed $w \in W(H^1, H^{-1})$

$$\langle \partial^\bullet (Sw), \eta \rangle_{L^2_{H^{-1}}, L^2_{H^1}} + (\mathcal{U}'_\epsilon(w) \nabla_\Omega(Sw), \nabla_\Omega \eta)_{L^2_{L^2}} + (Sw, \eta \nabla_\Omega \cdot \mathbf{w})_{L^2_{L^2}} = \langle f, \eta \rangle_{L^2_{H^{-1}}, L^2_{H^1}}$$

$$Sw(0) = e_0.$$

$$(\mathbf{P}(w))$$

If S denotes the solution map of $(\mathbf{P}(w))$ that takes $w \mapsto Sw$, then we seek a fixed point of S. First, note that since the bilinear form involving the surface gradients is bounded and coercive, the solution $Sw \in W(H^1, H^{-1})$ of $(\mathbf{P}(w))$ does indeed exist by [6, Theorem 3.6], and moreover, it satisfies the estimate

$$\|Sw\|_{W(H^1, H^{-1})} \leq C \left(\|f\|_{L^2_{H^{-1}}} + \|u_0\|_{L^2(\Omega_0)} \right) =: C_* \tag{18}$$

where the constant C does not depend on w because $\mathcal{U}'_\epsilon(w(t))$ is uniformly bounded from below (in w). Then the set $E := \{w \in W(H^1, H^{-1}) \mid w(0) = e_0, \|w\|_{W(H^1, H^{-1})} \leq C_*\}$, which is a closed, convex, and bounded subset of $X := W(H^1, H^{-1})$, is such that $S(E) \subset E$ by (18). We now show that S is weakly continuous. Let $w_n \rightharpoonup w$ in $W(H^1, H^{-1})$ with $w_n \in E$. From the estimate (18), we know that Sw_n is bounded in $W(H^1, H^{-1})$, so for a subsequence

$$Sw_{n_j} \rightharpoonup \chi \quad \text{in } W(H^1, H^{-1})$$
$$Sw_{n_j} \to \chi \quad \text{in } L^2_{L^2}$$

by the compact embedding of Lemma 2.9. Now we show that $\chi = Sw$. Due to $W(H^1, H^{-1}) \hookrightarrow C^0_{L^2}$, $Sw_{n_j} \rightharpoonup \chi$ in $C^0_{L^2}$. This implies $Sw_{n_j}(0) \rightharpoonup \chi(0)$ in $L^2(\Omega_0)$ (to see this consider for arbitrary $f \in L^2(\Omega_0)$ the functional $G \in (C^0_{L^2})^*$ defined $G(u_n) = \int_{\Omega_0} f u_n(0)$). Since $Sw_{n_j}(0) = e_0$, it follows that

$$\chi(0) = e_0. \tag{19}$$

On the other hand, since w_n are weakly convergent in $W^1(H^1, H^{-1})$, they are bounded in the same space. Now, $W(H^1, H^{-1}) \overset{c}{\hookrightarrow} L^2_{L^2}$, hence $w_n \to w$ in $L^2_{L^2}$. It follows that the subsequence $w_{n_j} \to w$ in $L^2_{L^2}$ too, and so there is a subsequence such that for almost every $t \in [0, T]$, $w_{n_{j_k}}(t) \to w(t)$ a.e. in $\Omega(t)$. By continuity, for a.a. t, $\mathcal{U}'_\epsilon(w_{n_{j_k}}(t)) \nabla_\Omega \eta(t) \to \mathcal{U}'_\epsilon(w(t)) \nabla_\Omega \eta(t)$ a.e., and also we have $|\mathcal{U}'_\epsilon(w_{n_{j_k}}) \nabla_\Omega \eta| \leq |\nabla_\Omega \eta|$ with the right hand side in $L^2_{L^2}$. Thus we can use the dominated convergence theorem (Theorem 2.10) which tells us that $\mathcal{U}'_\epsilon(w_{n_{j_k}}) \nabla_\Omega \eta \to \mathcal{U}'_\epsilon(w) \nabla_\Omega \eta$ in $L^2_{L^2}$. Now we pass to the limit in the equation $(\mathbf{P}(w))$ with w replaced by $w_{n_{j_k}}$ to get

$$\int_0^T \langle \partial^\bullet \chi(t), \eta(t) \rangle + \int_{\Omega(t)} \mathcal{U}'_\epsilon(w(t)) \nabla_\Omega \chi(t) \nabla_\Omega \eta(t) + \int_{\Omega(t)} \chi(t) \eta(t) \nabla_\Omega \cdot \mathbf{w} = \int_0^T \langle f(t), \eta(t) \rangle$$

which, along with (19), shows that $\chi = Sw$, so $Sw_{n_j} \rightharpoonup Sw$. However, we have to show that the whole sequence converges, not just a subsequence. Let $x_n = Sw_n$ and equip the space $X = W(H^1, H^{-1})$ with the weak topology. Let $x_{n_m} = Sw_{n_m}$ be a subsequence. By the bound of S, it follows that x_{n_m} is bounded, hence it has a subsequence such that

$$x_{n_{m_l}} \rightharpoonup x^* \text{ in } X \qquad \text{and} \qquad x_{n_{m_l}} \to x^* \text{ in } L^2_{L^2}.$$

By similar reasoning as before, we identify $x^* = Sw$, and Theorem 3.3 below tells us that indeed $x_n = Sw_n \rightharpoonup Sw$. Then by the Schauder–Tikhonov fixed point theorem [23, Theorem 1.4, p. 118], S has a fixed point. $\qquad \square$

Theorem 3.3. *Let x_n be a sequence in a topological space X such that every subsequence x_{n_j} has a subsequence $x_{n_{j_k}}$ converging to $x \in X$. Then the full sequence x_n converges to x.*

3.1 Uniform estimates

We set $u_\epsilon = \mathcal{U}_\epsilon(e_\epsilon)$. Below we denote by M a constant such that $\|u_0\|_{L^\infty(\Omega_0)} \le M$.

3.4 Lemma. *The following bound holds independent of ϵ:*

$$\|u_\epsilon\|_{L_{L^\infty}^\infty} + \|\mathcal{E}_\epsilon(u_\epsilon)\|_{L_{L^\infty}^\infty} \le 2e^{\|\nabla_\Omega \cdot \mathbf{w}\|_\infty T}\left(T\|f\|_{L_{L^\infty}^\infty} + \|u_0\|_{L^\infty(\Omega_0)} + 1\right) + 1.$$

Proof. We substitute $w(t) = e^{-\lambda t}e_\epsilon(t)$ in $(\mathbf{P}_\epsilon)$ and use $\partial^\bullet(e^{\lambda t}w(t)) = \lambda e^{\lambda t}w(t) + e^{\lambda t}\dot{w}(t)$ to get

$$\dot{w}(t) - e^{-\lambda t}\Delta_\Omega(\mathcal{U}^\epsilon(e^{\lambda t}w(t))) + \lambda w(t) + w(t)\nabla_\Omega \cdot \mathbf{w} = e^{-\lambda t}f(t).$$

Let $\alpha = \|f\|_{L_{L^\infty}^\infty}$ and $\beta = \|e_0\|_{L^\infty(\Omega_0)}$ and define $v(t) = \alpha t + \beta$. Note that $\dot{v}(t) = \alpha$ and $v(0) = \beta$. Subtracting $\dot{v}(t)$ from the above and testing with $(w(t) - v(t))^+$, we get

$$\langle \dot{w}(t) - \dot{v}(t), (w(t) - v(t))^+\rangle_{H^{-1}(\Omega(t)),H^1(\Omega(t))} + \int_{\Omega(t)} e^{-\lambda t}\nabla_\Omega(\mathcal{U}^\epsilon(e^{\lambda t}w(t)))\nabla_\Omega(w(t) - v(t))^+$$

$$+ \int_{\Omega(t)}(\lambda + \nabla_\Omega \cdot \mathbf{w})w(t)(w(t) - v(t))^+ = \int_{\Omega(t)}(e^{-\lambda t}f(t) - \alpha)(w(t) - v(t))^+. \quad (20)$$

Note that $e^{-\lambda t}\nabla_\Omega(\mathcal{U}_\epsilon(e^{\lambda t}w(t)))\nabla_\Omega(w(t) - v(t))^+ = \mathcal{U}_\epsilon'(e^{\lambda t}w(t))|\nabla_\Omega(w(t) - v(t))^+|^2$ because $\nabla_\Omega v(t) = 0$. Set $\lambda := \|\nabla_\Omega \cdot \mathbf{w}\|_{L^\infty}$, then the last term on the LHS of (20) is non-negative because if $w > v$, $w > 0$ since $v \ge 0$. So we can throw away that and the gradient term to find

$$\langle \dot{w}(t) - \dot{v}(t), (w(t) - v(t))^+\rangle_{H^{-1}(\Omega(t)),H^1(\Omega(t))} \le \int_{\Omega(t)}(e^{-\lambda t}f(t) - \alpha)(w(t) - v(t))^+.$$

Integrating this and using Lemma 2.11, we find

$$\frac{1}{2}\int_{\Omega(T)}((w(t) - v(t))^+)^2 \le \frac{1}{2}\|\nabla_\Omega \cdot \mathbf{w}\|\int_0^T\int_{\Omega(t)}((w(t) - v(t))^+)^2$$

since $e^{-\lambda t}f(t) - \alpha = e^{-\lambda t}f(t) - \|f(t)\|_{L^\infty(\Omega(t))} \le 0$ and $w(0) - v(0) = e_0 - \|e_0\|_{L^\infty(\Omega_0)} \le 0$. The use of Gronwall's inequality gives $w(t) \le T\|f\|_{L_{L^\infty}^\infty} + (1 + M)$ almost everywhere on $\Omega(t)$. So we have shown that for all $t \in [0,T]\backslash N_1$, $w(t,x) \le C$ for all $x \in \Omega(t)\backslash M_1^t$, where $\mu(N_1) = \mu(M_1^t) = 0$. A similar argument yields for all $t \in [0,T]\backslash N_2$, $w(t,x) \ge -C$ for all $x \in \Omega(t)\backslash M_2^t$, where $\mu(N_2) = \mu(M_2^t) = 0$. Taking these statements together tells us that for all $t \in [0,T]\backslash N$, $|w(t,x)| \le C$ on $\Omega(t)\backslash M^t$ where $N = N_1 \cup N_2$ and $M^t = M_1^t \cup M_2^t$ have measure zero. This gives $\|w\|_{L_{L^\infty}^\infty} \le T\|f\|_{L_{L^\infty}^\infty} + (1 + M)$. From this and $u_\epsilon = \mathcal{U}_\epsilon(e^{\lambda(\cdot)}w(\cdot)) \le e^{\lambda T}|w|$, we obtain the bound on u_ϵ. The bound on $\mathcal{E}_\epsilon(u_\epsilon)$ follows from $\mathcal{E}_\epsilon(u_\epsilon) \le 1 + |u_\epsilon|$. $\square$

3.5 Lemma. *The following bound holds independent of ϵ:*

$$\|\nabla_\Omega u_\epsilon\|_{L_{L^2}^2} + \|\partial^\bullet(\mathcal{E}_\epsilon u_\epsilon)\|_{L_{H^{-1}}^2} \le C(T, \Omega, M, \mathbf{w}, f). \quad (21)$$

13

Proof. Testing with $\mathcal{E}_\epsilon(u_\epsilon)$ in $(\mathbf{P}_\epsilon)$, using $\nabla_\Omega u_\epsilon \nabla_\Omega(\mathcal{E}_\epsilon(u_\epsilon)) = (\mathcal{E}_\epsilon)'(u_\epsilon)|\nabla_\Omega u_\epsilon|^2 \geq |\nabla_\Omega u_\epsilon|^2$, integrating over time and using the previous estimate, we find

$$\frac{1}{2}\|\mathcal{E}_\epsilon(u_\epsilon(T))\|^2_{L^2(\Omega(T))} + \int_0^T \int_{\Omega(t)} |\nabla_\Omega u_\epsilon(t)|^2 \leq \frac{1}{2}(1+M)^2|\Omega_0| + C_1(T, M, \mathbf{w}, f).$$

The bound on the time derivative follows by taking supremums. $\qquad\square$

3.6 Lemma. *Define $\tilde{u}_\epsilon = \phi_{-(\cdot)} u_\epsilon$. The following limit holds uniformly in ϵ:*

$$\lim_{h\to 0} \int_0^{T-h} \int_{\Omega_0} |\tilde{u}_\epsilon(t+h) - \tilde{u}_\epsilon(t)| = 0.$$

Proof. We follow the proof of Theorem A.1 in [8] here. Fix $h \in (0, T)$ and consider

$$\int_0^{T-h} (\mathcal{E}_\epsilon(\tilde{u}_\epsilon(t+h)) - \mathcal{E}_\epsilon(\tilde{u}_\epsilon(t)), \tilde{u}_\epsilon(t+h) - \tilde{u}_\epsilon(t))_{L^2(\Omega_0)} \, \mathrm{d}t$$

$$= \int_0^{T-h} \int_t^{t+h} \frac{d}{d\tau}(\mathcal{E}_\epsilon(\tilde{u}_\epsilon(\tau)), \tilde{u}_\epsilon(t+h) - \tilde{u}_\epsilon(t))_{L^2(\Omega_0)} \, \mathrm{d}\tau \, \mathrm{d}t$$

$$\leq \sqrt{h}\, \|(\mathcal{E}_\epsilon(\tilde{u}_\epsilon))'\|_{L^2(0,T;H^{-1}(\Omega_0))} \int_0^{T-h} (\|\tilde{u}_\epsilon(t+h)\|_{H^1(\Omega_0)} + \|\tilde{u}_\epsilon(t)\|_{H^1(\Omega_0)}) \, \mathrm{d}t$$

$$\leq C_1(T, \Omega, M, \mathbf{w}, f)\sqrt{h}\, \|(\mathcal{E}_\epsilon(\tilde{u}_\epsilon))'\|_{L^2(0,T;H^{-1}(\Omega_0))} \qquad \text{(by the uniform estimates)}$$

$$\leq C_2(T, \Omega, M, \mathbf{w}, f)\sqrt{h}\, \|\partial^\bullet(\mathcal{E}_\epsilon(u_\epsilon))\|_{L^2_{H^{-1}}} \qquad \text{(see the proof of Theorem 2.33 in [6])}$$

$$\leq C_3(T, \Omega, M, \mathbf{w}, f)\sqrt{h}, \tag{22}$$

with the last inequality by (21). Now, since the $\mathcal{U}_\epsilon'$ are uniformly bounded above, they are uniformly equicontinuous. Therefore, for fixed δ, there is a σ_δ (depending solely on δ) such that

$$\text{if } |y - z| < \sigma_\delta, \text{ then } |\mathcal{U}_\epsilon(y) - \mathcal{U}_\epsilon(z)| < \delta \qquad \text{for } \textit{any } \epsilon. \tag{23}$$

So in the set $\{|\tilde{u}_\epsilon(t+h) - \tilde{u}_\epsilon(t)| > \delta\} = \{|\mathcal{U}_\epsilon(\mathcal{E}_\epsilon(\tilde{u}_\epsilon(t+h))) - \mathcal{U}_\epsilon(\mathcal{E}_\epsilon(\tilde{u}_\epsilon(t)))| > \delta\}$, we must have $|\mathcal{E}_\epsilon(\tilde{u}_\epsilon(t+h)) - \mathcal{E}_\epsilon(\tilde{u}_\epsilon(t))| \geq \sigma_\delta$ (this is the contrapositive of (23)). This implies from (22) that

$$\int_0^{T-h} \int_{\Omega_0} |\tilde{u}_\epsilon(t+h) - \tilde{u}_\epsilon(t)|\chi_{\{|\tilde{u}_\epsilon(t+h)-\tilde{u}_\epsilon(t)|>\delta\}} \leq \frac{C_3\sqrt{h}}{\sigma_\delta}.$$

Writing $\mathrm{Id} = \chi_{\{|\tilde{u}_\epsilon(t+h)-\tilde{u}_\epsilon(t)|>\delta\}} + \chi_{\{|\tilde{u}_\epsilon(t+h)-\tilde{u}_\epsilon(t)|\leq\delta\}}$, notice that

$$\int_0^{T-h} \int_{\Omega_0} |\tilde{u}_\epsilon(t+h) - \tilde{u}_\epsilon(t)| \leq \int_0^{T-h} \int_{\Omega_0} |\tilde{u}_\epsilon(t+h) - \tilde{u}_\epsilon(t)|\chi_{\{|\tilde{u}_\epsilon(t+h)-\tilde{u}_\epsilon(t)|>\delta\}} + \delta|\Omega_0|(T-h)$$

$$\leq \frac{C_3\sqrt{h}}{\sigma_\delta} + \delta|\Omega_0|T.$$

Taking the limit as $h \to 0$, using the arbitrariness of $\delta > 0$ and the fact that the right hand side of the above does not depend on ϵ gives us the result. $\qquad\square$

3.2 Existence of bounded weak solutions

With all the uniform estimates acquired, we can extract (weakly) convergent subsequences. In fact, we find (we have not relabelled subsequences)

$$
\begin{aligned}
u_\epsilon \to u & \quad \text{in } L^p_{L^q} \text{ for any } p, q \in [1, \infty) \\
\nabla_\Omega u_\epsilon \rightharpoonup \nabla_\Omega u & \quad \text{in } L^2_{L^2} \\
\mathcal{E}_\epsilon(u_\epsilon) \rightharpoonup \chi & \quad \text{in } L^2_{L^2}
\end{aligned}
\tag{24}
$$

where only the first strong convergence listed requires an explanation. Indeed, the point is to apply [24, Theorem 5] with $H^1(\Omega_0) \overset{c}{\hookrightarrow} L^1(\Omega_0) \subset L^1(\Omega_0)$, which gives us a subsequence $\tilde{u}_{\epsilon_j} \to \tilde{\rho}$ strongly in $L^1(0, T; L^1(\Omega_0))$. It follows that $u_{\epsilon_j} \to \rho$ in $L^1_{L^1}$, whence for a.a. t, $u_{\epsilon_{j_k}}(t) \to \rho(t)$ a.e. in $\Omega(t)$. We also know that for a.a. t, $|u_{\epsilon_{j_k}}(t)| \le C$ a.e. in $\Omega(t)$ by Lemma 3.4, and so for a.a. t, the limit satisfies $|\rho(t)| \le C$ a.e. in $\Omega(t)$ too. By Theorem 2.10, $u_{\epsilon_{j_k}} \to \rho$ in $L^p_{L^q}$ for all $p, q \in [1, \infty)$. Since $u_{\epsilon_{j_k}} \rightharpoonup u$ (subsequences have the same weak limit), it must be the case that $\rho = u$.

Proof of Theorem 1.3. In $(\mathbf{P}_\epsilon)$, we can test with a function $\eta \in W(H^1, L^2)$ with $\eta(T) = 0$, integrate by parts and then pass to the limit to obtain

$$
-\int_0^T \int_{\Omega(t)} \dot{\eta}(t)\chi(t) + \int_0^T \int_{\Omega(t)} \nabla_\Omega u(t) \nabla_\Omega \eta(t) = \int_0^T \int_{\Omega(t)} f(t)\eta(t) + \int_{\Omega_0} e_0 \eta(0)
$$

and it remains to be seen that $\chi \in \mathcal{E}(u)$ or equivalently $u = \mathcal{U}(\chi)$. By monotonicity of $\mathcal{E}_\epsilon$, we have for any $w \in L^2_{L^2}$

$$
\int_0^T \int_{\Omega(t)} (\mathcal{E}_\epsilon(u_\epsilon) - w)(u_\epsilon - \mathcal{U}_\epsilon(w)) \ge 0.
$$

Because $\mathcal{U}_\epsilon \to \mathcal{U}$ uniformly, for a.a. t, $\mathcal{U}_\epsilon(w(t)) \to \mathcal{U}(w(t))$ a.e. in $\Omega(t)$, and $|\mathcal{U}_\epsilon(w)| \le |w|$, and the dominated convergence theorem shows that $\mathcal{U}_\epsilon(w) \to \mathcal{U}(w)$ in $L^2_{L^2}$. Using this and (24), we can easily pass to the limit in this inequality and obtain

$$
\int_0^T \int_{\Omega(t)} (\chi - w)(u - \mathcal{U}w) \ge 0 \quad \text{for all } w \in L^2_{L^2}.
$$

By Minty's trick we find $u = \mathcal{U}(\chi)$. To see why $\chi \in L^\infty_{L^\infty}$, we have from the estimate in Lemma 3.4 that for a.a. $t \in [0, T]$, $\|\mathcal{E}_\epsilon(\tilde{u}_\epsilon(t))\|_{L^\infty(\Omega_0)} \le C$, giving $\mathcal{E}_\epsilon(\tilde{u}_\epsilon(t)) \overset{*}{\rightharpoonup} \tilde{\zeta}(t)$ in $L^\infty(\Omega(t))$ and (by weak-* lower semicontinuity) $\|\tilde{\zeta}(t)\|_{L^\infty(\Omega(t))} \le C$ for a.a. t, and we just need to identify $\tilde{\zeta} \in \mathcal{E}(\tilde{u})$. It follows from (24) that $\mathcal{E}_\epsilon(u_\epsilon) \to \chi$ in $L^2_{H^{-1}}$ by Lions–Aubin, and so for a.e. t and for a subsequence (not relabelled), $\mathcal{E}_\epsilon(u_\epsilon(t)) \to \chi(t)$ in $H^{-1}(\Omega(t))$. This allows us to conclude that $\chi = \zeta$ (the weak-* convergence of $\mathcal{E}_\epsilon(\tilde{u}_\epsilon(t))$ to $\tilde{\zeta}(t)$ also gives weak convergence in any $L^p(\Omega(t))$ to the same limit). $\qquad \square$

3.3 Continuous dependence and uniqueness of bounded weak solutions

The next lemma allows us to drop the requirement for our test functions to vanish at time T.

3.7 Lemma. *If (u, e) is a bounded weak solution (satisfying (3)), then (u, e) also satisfies*

$$\int_{\Omega(T)} e(T)\eta(T) - \int_0^T \int_{\Omega(t)} \dot{\eta}(t)e(t) + \int_0^T \int_{\Omega(t)} \nabla_\Omega u(t)\nabla_\Omega \eta(t) = \int_0^T \int_{\Omega(t)} f(t)\eta(t) + \int_{\Omega_0} e_0 \eta(0)$$

for all $\eta \in W(H^1, L^2)$.

Proof. To see this, for $s \in (0, T]$, consider the function $\chi_{\epsilon,s}(t) = \min\left(1, \epsilon^{-1}(s-t)^+\right)$ which has a weak derivative $\chi'_{\epsilon,s}(t) = -\epsilon^{-1}\chi_{(s-\epsilon,s)}(t)$. Take the test function in (3) to be $\chi_{\epsilon,T}\eta$ where $\eta \in W(H^1, L^2)$, send $\epsilon \to 0$ and use the Lebesgue differentiation theorem. $\square$

We can finally prove Theorem 1.4.

Proof of Theorem 1.4. We can prove the continuous dependence like in [21, Chapter V, §9]. As explained in Lemma 3.7, we drop the requirement $\eta(T) = 0$ in our test functions and we now suppose that $\Delta_\Omega \eta \in L_{L^2}^2$. Suppose for $i = 1, 2$ that (u_i, e_i) is the solution to the Stefan problem with data (f_i, u_0^i), so

$$\int_{\Omega(t)} (e_i(t) - e_2(t))\eta(t) - \int_0^t \int_{\Omega(\tau)} \dot{\eta}(\tau)(e_1(\tau) - e_2(\tau)) - \int_0^t \int_{\Omega(\tau)} (u_1(\tau) - u_2(\tau))\Delta_\Omega \eta(\tau)$$

$$= \int_0^t \int_{\Omega(\tau)} (f_1(\tau) - f_2(\tau))\eta(\tau) + \int_{\Omega_0} (e_0^1 - e_0^2)\eta(0). \quad (25)$$

Define $a = (u_1 - u_2)/(e_1 - e_2)$ when $e_1 \neq e_2$ and $a = 0$ otherwise, and note that $0 \leq a(x, t) \leq 1$. Let η_ϵ solve in $\cup_{\tau \in (0,t)}\{\tau\} \times \Omega(\tau)$ the equation

$$\partial_\tau^\bullet \eta_\epsilon(\tau) + (a_\epsilon(x, \tau) + \epsilon)\Delta_\Omega \eta_\epsilon(\tau) = 0$$
$$\eta_\epsilon(t) = \xi \quad \text{on } \Omega_0 \quad (26)$$

with $\xi \in C^1(\Omega_0)$ and where a_ϵ satisfies $\phi_{-(\cdot)}a_\epsilon \in C^2([0,T] \times \Omega_0)$ and $0 \leq a_\epsilon \leq 1$ a.e. and $\|a_\epsilon - a\|_{L^2(Q)} \leq \epsilon$. This is well-posed by Lemma 2.13. Equation (25) can be written in terms of a_ϵ, and if we choose $\eta = \eta_\epsilon$ and use (26), we find

$$\int_{\Omega(t)} (e_1(t) - e_2(t))\xi \leq \|e_1 - e_2\|_{L_{L^\infty}^\infty} \int_0^t \int_{\Omega(\tau)} (|a(x, \tau) - a_\epsilon(x, \tau)| + \epsilon)|\Delta_\Omega \eta_\epsilon(\tau)|$$

$$+ \|\xi\|_{L^\infty(\Omega_0)} \int_0^t \|f_1(\tau) - f_2(\tau)\|_{L^1(\Omega(\tau))} + \|\xi\|_{L^\infty(\Omega_0)} \int_{\Omega_0} |e_0^1 - e_0^2| \quad (27)$$

using the L^∞ bound from Lemma 2.13. We can estimate the first integral on the right hand side:

$$\int_0^t \int_{\Omega(\tau)} |a(x, \tau) - a_\epsilon(x, \tau)||\Delta_\Omega \eta_\epsilon(\tau)| \leq \sqrt{\epsilon}\|a - a_\epsilon\|_{L_{L^2}^2} \sqrt{(2+\epsilon)(1 + e^{2C_{\mathbf{w}}(2+\epsilon)t})} \|\nabla_\Omega \xi\|_{L^2(\Omega_0)}$$

and

$$\int_0^t \int_{\Omega(\tau)} |\epsilon \Delta_\Omega \eta_\epsilon| \leq \sqrt{t|\Omega|\epsilon(2+\epsilon)(1 + e^{2C_{\mathbf{w}}(2+\epsilon)t})} \left(\int_{\Omega_0} |\nabla_\Omega \xi|^2\right)^{\frac{1}{2}}$$

16

by the results in Lemma 2.13. Sending $\epsilon \to 0$ in (27) gives us (recalling $\xi \leq 1$),

$$\int_{\Omega(t)} (e_1(t) - e_2(t))\xi \leq \int_0^t \|f_1(\tau) - f_2(\tau)\|_{L^1(\Omega(\tau))} + \|e_0^1 - e_0^2\|_{L^1(\Omega_0)}.$$

Now pick $\xi = \xi_n$ where $\xi_n(x) \to \text{sign}(e_1(t, x) - e_2(t, x)) \in L^2(\Omega(t))$ a.e. in $\Omega(t)$. □

3.4 Well-posedness of weak solutions

Proof of Theorem 1.5. Suppose $(e_0, f) \in L^1(\Omega_0) \times L^1_{L^1}$ are data and consider functions $e_{0n} \in L^\infty(\Omega_0)$ and $f_n \in L^\infty_{L^\infty}$ satisfying

$$(f_n, e_{0n}) \to (f, e_0) \quad \text{in } L^1_{L^1} \times L^1(\Omega_0).$$

The existence of f_n holds because by density, there exist $\tilde{f}_n \in C^0([0, T] \times \Omega_0)$ such that $\tilde{f}_n \to \tilde{f}$ in $L^1((0, T) \times \Omega_0) \equiv L^1(0, T; L^1(\Omega_0))$. Denote by (u_n, e_n) the respective (bounded weak) solutions to the Stefan problem with the data (e_{0n}, f_n). By virtue of these solutions satisfying the continuous dependence result, it follows that $\{e_n\}_n$ is a Cauchy sequence in $L^1_{L^1}$ and thus $e_n \to \chi$ in $L^1_{L^1}$ for some χ. Recall that $|u_n| = |\mathcal{U}(e_n)| \leq |e_n|$ so by consideration of an appropriate Nemytskii map, we find $u_n = \mathcal{U}(e_n) \to \mathcal{U}(\chi)$. Now we can pass to the limit in

$$-\int_0^T \int_{\Omega(t)} \dot{\eta}(t)e_n(t) - \int_0^T \int_{\Omega(t)} u_n(t)\Delta_\Omega\eta(t) = \int_0^T \int_{\Omega(t)} f_n(t)\eta(t) + \int_{\Omega_0} e_{n0}\eta(0)$$

and doing so gives

$$-\int_0^T \int_{\Omega(t)} \dot{\eta}(t)\chi(t) - \int_0^T \int_{\Omega(t)} \mathcal{U}(\chi(t))\Delta_\Omega\eta(t) = \int_0^T \int_{\Omega(t)} f(t)\eta(t) + \int_{\Omega_0} e_0\eta(0)$$

and overall this shows that there exists a pair $(\chi, \mathcal{E}^{-1}(\chi)) \in L^1_{L^1} \times L^1_{L^1}$ which is a weak solution of the Stefan problem. For these integrals to make sense, we need $\eta \in W^1(L^\infty \cap H^2, L^\infty)$ with $\Delta_\Omega\eta \in L^\infty_{L^\infty}$.

Now suppose that (u^1, e^1) and (u^2, e^2) are two weak solutions of class L^1 to the Stefan problem with data (f^1, e_0^1) and (f^2, e_0^2) in $L^1_{L^1} \times L^1(\Omega_0)$ respectively. We know that there exist approximations $(f_n^1, e_{0n}^1), (f_n^2, e_{0n}^2) \in L^\infty_{L^\infty} \times L^\infty(\Omega_0)$ of the data satisfying

$$(f_n^1, e_{0n}^1) \to (f^1, e_0^1) \quad \text{and} \quad (f_n^2, e_{0n}^2) \to (f^2, e_0^2) \quad \text{in } L^1_{L^1} \times L^1(\Omega_0).$$

These approximate data give rise to the approximate solutions e_n^1 and e_n^2 both of which are elements of $L^\infty_{L^\infty}$. It follows from above that $e_n^1 \to e^1$ and $e_n^2 \to e^2$ in $L^1_{L^1}$. Now consider the continuous dependence result that e_n^1 and e_n^2 satisfy:

$$\|e_n^1 - e_n^2\|_{L^1_{L^1}} \leq T\left(\|f_n^1 - f_n^2\|_{L^1_{L^1}} + \|e_{0n}^1 - e_{0n}^2\|_{L^1(\Omega_0)}\right). \tag{28}$$

Regarding the right hand side, by writing $e_{0n}^1 - e_{0n}^2 = e_{0n}^1 - e_0^1 + e_0^1 - e_0^2 + e_0^2 - e_{0n}^2$, (and similarly for the f_n^i) and using triangle inequality, along with the fact that $e_n^1 - e_n^2 \to e^1 - e^2$ in $L^1_{L^1}$, we can take the limit in (28) as $n \to \infty$ and we are left with what we desired. □

A   Proofs

Proof that L_X^p is a Banach space when X_0 is separable. It is easy to verify that the expressions in (5) define norms if the integrals on the right hand sides are well-defined, which we now check. So let $u \in L_X^p$. Then $\tilde{u} := \phi_{-(\cdot)} u(\cdot) \in L^p(0, T; X_0)$. Define $F \colon [0, T] \times X_0 \to \mathbb{R}$ by $F(t, x) = \|\phi_t x\|_{X(t)}$. By assumption, $t \mapsto F(t, x)$ is measurable for all $x \in X_0$, and if $x_n \to x$ in X_0, then by the reverse triangle inequality,

$$|F(t, x_n) - F(t, x)| \leq \|\phi_t(x_n - x)\|_{X(t)} \leq C_X \|x_n - x\|_{X_0} \to 0,$$

so $x \mapsto F(t, x)$ is continuous. Thus F is a Carathéodory function. Due to the condition $|F(t, x)| \leq C_X \|x\|_{X_0}$, by Remark 3.4.5 of [25], the Nemytskii operator N_F defined by $(N_F x)(t) := F(t, x(t))$ maps $L^p(0, T; X_0) \to L^p(0, T)$, so that

$$\|N_F \tilde{u}\|_{L^p(0,T)}^p = \int_0^T \|u(t)\|_{X(t)}^p < \infty.$$

$\square$

Proof of Lemma 2.3. First we show that if $u \in L^p(0, T; X_0)$, then $\phi_{(\cdot)} u(\cdot) \in L_X^p$.

Let $u \in L^p(0, T; X_0)$ be arbitrary. By density, there exists a sequence of simple functions $u_n \in L^p(0, T; X_0)$ with

$$\|u_n - u\|_{L^p(0,T;X_0)} \to 0$$

and thus for almost every t,

$$\|u_n(t) - u(t)\|_{X_0} \to 0$$

for a subsequence, which we relabelled. We have that $\phi_t u_n(t) \to \phi_t u(t)$ in $X(t)$ by continuity; this implies

$$\|\phi_t u_n(t)\|_{X(t)} \to \|\phi_t u(t)\|_{X(t)} \qquad \text{pointwise a.e.} \tag{29}$$

Write $u_n(t) = \sum_{i=1}^{M_n} u_{n,i} \mathbf{1}_{B_i}(t)$ where the $u_{n,i} \in X_0$ and the B_i are measurable, disjoint and partition $[0, T]$. Then

$$\phi_t u_n(t) = \sum_{i=1}^{M_n} \phi_t(u_{n,i}) \mathbf{1}_{B_i}(t) \in X(t).$$

Taking norms and exponentiating, we get

$$\|\phi_t u_n(t)\|_{X(t)}^p = \sum_{i=1}^{M_n} \|\phi_t u_{n,i}\|_{X(t)}^p \mathbf{1}_{B_i}^p(t),$$

which is measurable (with respect to t) since, by assumption, the $\|\phi_t u_{n,i}\|_{X(t)}$ are continuous and a finite sum of measurable functions is measurable. Thus, by (29), $\|\phi_t u(t)\|_{X(t)}$, is measurable. Finally,

$$\int_0^T \|\phi_t u(t)\|_{X(t)}^p \leq \int_0^T C_X^p \|u(t)\|_{X_0}^p = C_X^p \|u\|_{L^p(0,T;X_0)}^p,$$

so $\phi_{(\cdot)} u(\cdot) \in L_X^p$.

So there is a map from $L^p(0, T; X_0)$ to L_X^p and vice-versa from the definition of L_X^p. The isomorphism between the spaces is $T\colon L^p(0, T; X_0) \to L_X^p$ where

$$Tu = \phi_{(\cdot)}u(\cdot), \quad \text{and} \quad T^{-1}v = \phi_{-(\cdot)}v(\cdot).$$

It is easy to check that T is linear and bijective. The equivalence of norms follows by the bounds on $\phi_{-t}\colon X(t) \to X_0$

$$\frac{1}{C_X} \|u(t)\|_{X(t)} \le \|\phi_{-t}u(t)\|_{X_0} \le C_X \|u(t)\|_{X(t)}.$$

$\square$

Acknowledgments

References

[1] Elliott CM, Ockendon JR. 1982 *Weak and variational methods for moving boundary problems.* Research Notes in Mathematics, 59. Pitman (Advanced Publishing Program), Boston, Mass.-London.

[2] Friedman A. 1982 *Variational principles and free-boundary problems.* A Wiley-Interscience Publication. Pure and Applied Mathematics. John Wiley & Sons, Inc., New York.

[3] Meirmanov AM. 1992 *The Stefan problem.* Translated from the Russian by Marek Niezgódka and Anna Crowley. With an appendix by the author and I. G. Götz. de Gruyter Expositions in Mathematics, 3. Walter de Gruyter & Co., Berlin.

[4] Rodrigues J-F. 1987 *Obstacle problems in mathematical physics.* North-Holland Mathematics Studies, 134. Mathematical Notes, 114. North-Holland Publishing Co., Amsterdam.

[5] Brézis H. 1973 *Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert.* North-Holland Mathematics Studies, 5. Mathematical Notes, 50. North-Holland Publishing Co., Amsterdam.

[6] Alphonse A, Elliott CM, Stinner B. 2015 An abstract framework for parabolic PDEs on evolving spaces. *Port. Math.* **72**, 1–46.

[7] Alphonse A, Elliott CM, Stinner B. 2015 On some linear parabolic PDEs on moving hypersurfaces. *Interfaces Free Bound.* (to appear).

[8] Blanchard D, Porretta A. 2005 Stefan problems with nonlinear diffusion and convection. *J. Differential Equations.* **210**, 383–428.

[9] Friedman A. 1968 The Stefan problem in several space variables. *Trans. Amer. Math. Soc.* **133**, 51–87.

[10] Kamenomostskaja SL. 1961 On Stefan's problem. *Mat. Sb. (N.S.)* **53**, 489–514.

[11] Oleĭnik OA. 1960 A method of solution of the general Stefan problem. *Soviet Math. Dokl.* **1**, 1350–1354.

[12] Rodrigues J-F, Urbano JM. 1999 On a Darcy--Stefan problem arising in freezing and thawing of saturated porous media. *Contin. Mech. Thermodyn.* **11**, 181-191.

[13] Rodrigues J-F. 1994 Variational methods in the Stefan problem. In Augusto Visintin, editor, *Phase Transitions and Hysteresis*, volume 1584 of *Lecture Notes in Mathematics*, pages 147–212. Springer Berlin Heidelberg.

[14] Vierling M. 2014 Parabolic optimal control problems on evolving surfaces subject to point-wise box constraints on the control — theory and numerical realization. *Interfaces Free Bound.* **16**, 137–173.

[15] Kreyszig E. 1978 *Introductory functional analysis with applications.* John Wiley & Sons, New York-London-Sydney.

[16] Diestel J, Uhl JJ. 1977 *Vector Measures.* Mathematical surveys and monographs. American Mathematical Society.

[17] Dziuk G, Elliott CM. 2013 L^2-estimates for the evolving surface finite element method. *Math. Comp.* **82**, 1–24.

[18] Carl S, Le VK, Motreanu D. 2007 *Nonsmooth Variational Problems and Their Inequalities: Comparison Principles and Applications.* Springer Monographs in Mathematics. Springer, 2007.

[19] Heinonen J, Kilpeläinen T, Martio O. 2012 *Nonlinear Potential Theory of Degenerate Elliptic Equations.* Dover Books on Mathematics Series. Dover Publications.

[20] Bertsch M, de Mottoni P, Peletier LA. 1986 The Stefan problem with heating: appearance and disappearance of a mushy region. *Trans. Amer. Math. Soc.* **293** 677–691. (DOI 10.1090/S0002-9947-1986-0816319-3)

[21] Ladyženskaja OA, Solonnikov VA, Ural'ceva, NN. 1986 *Linear and quasilinear equations of parabolic type.* Translations of Mathematical Monographs, Vol. 23 American Mathematical Society, Providence, R.I.

[22] Boyer F, Fabrie P. 2012 *Mathematical Tools for the Study of the Incompressible Navier-Stokes Equations and Related Models.* Applied Mathematical Sciences. Springer, 2012.

[23] Emmrich E, Wittbold P, Technische Universität Berlin. 2009 *Analytical and Numerical Aspects of Partial Differential Equations: Notes of a Lecture Series.* De Gruyter Proceedings in Mathematics Series. Walter De Gruyter.

[24] Simon J. 1986 Compact sets in the space $L^p(0, T; B)$. *Ann. Mat. Pura Appl.* **146**, 65–96.

[25] Gasinski L, Papageorgiou NS. 2006 *Nonlinear analysis.* Series in Mathematical Analysis and Applications, 9. Chapman & Hall/CRC, Boca Raton, FL.