The Topology of the Higher Projective Planes

by

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Contents

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Declarations

This thesis is submitted to the University of Warwick in support of my application for the degree of Doctor of Philosophy. It has been composed by myself and has not been submitted in any previous application for any degree.

Abstract

In this thesis we study two of the exceptional projetive planes $\mathbb{P}^2(\mathbb{C}\otimes\mathbb{O})$ and $\mathbb{P}^2(\mathbb{H}\otimes\mathbb{O})$. These are the homogenous spaces $E_6/S^1 \times_{C_4} \mathrm{Spin}(10)$ and $E_7/S^3 \times_{C_2} \mathrm{Spin}(12)$. These spaces both have natural actions by the compact Lie groups F_4 and $S^1 \times E_6$ respectively. The method that we will use to study these spaces is via the decompositions associated to these actions. In particular we will describe the homotopy type of $\mathbb{P}^2(\mathbb{C}\otimes\mathbb{O})$ in terms of the octonionic projective plane $\mathbb{P}^2(\mathbb{O})$ and spaces associated to $\mathbb{P}^2(\mathbb{O})$. We use this to compute the cohomology of $\mathbb{P}^2(\mathbb{C}\otimes\mathbb{O})$. Finally, we give a description of certain orbits of the action on $\mathbb{P}^2(\mathbb{H}\otimes\mathbb{O})$.

Chapter 1

Introduction

In this thesis we study two of the exceptional projetive planes $\mathbb{P}^2(\mathbb{C}\otimes\mathbb{O})$ and $\mathbb{P}^2(\mathbb{H}\otimes\mathbb{O})$. These are the homogenous spaces $E_6/S^1 \times_{C_4} \mathrm{Spin}(10)$ and $E_7/S^3 \times_{C_2} \mathrm{Spin}(12)$. These spaces both have natural actions by the compact Lie groups F_4 and $S^1 \times E_6$ respectively. The method that we will use to study these spaces is via the decompositions associated to these actions. In particular we will describe the homotopy type of $\mathbb{P}^2(\mathbb{C}\otimes\mathbb{O})$ in terms of the octonionic projective plane $\mathbb{P}^2(\mathbb{O})$ and spaces associated to $\mathbb{P}^2(\mathbb{O})$. We use this to compute the cohomology of $\mathbb{P}^2(\mathbb{C}\otimes\mathbb{O})$. Finally, we give an description of certain orbits of the action on $\mathbb{P}^2(\mathbb{H}\otimes\mathbb{O})$.

Chapter 2 covers the preliminaries which we will use later in this thesis. It is principly based on the work of Adams [1996] and Baez [2002] and covers the basics of division algebras and triality. These provide a basis for a definition of the Fruedenthal magic square coming from Rosenfeld [1997].

This allows us to define the exceptional projective spaces, the study of which, will be the focus of this thesis. We then give concrete definitions and some basic facts for the exceptional Lie groups G_2, F_4, E_6, E_7 and E_8.

In Chapters 3 and 4 we turn our attention to oriented Grassmannian spaces. In Chapter 3 we give the statement of Mostert's theorem and then apply this theorem to obtain 2 different decompositions of oriented Grassmannian manifolds. We then use these decompositons in Chapter 4 to compute the cohomology of the Grassmannian mainfold of oriented 2 planes.

These two chapters serve as the simplest non-trivial example and give an illustration of the basic techniques which we will use later in this thesis.

Chapter 5 gives concrete descriptions of the Lie group E_6 and its Lie algebra along with certain subalgebras. This alows us to compute various decompositions for homogenous spaces with fundamental group E_6. In this chapter we focus on 2

particular examples.

We will use one of the decompositions which has been obtained in Chapter 5 to compute the cohomology of the space $E_6/S^1 \times_{C_4} \mathrm{Spin}(10)$ in Chapter 6.

In Chapter 7 we study the orthogonal action of a compact Lie group on a sphere via a representation. In particular we are interested in conditions for it to act transitively or with codimension one. Also in this chapter we give some results on the particular decompositions associated to some codimension one actions. These will primarily be used in Chapter 9 to help reduce the possible Cartan types of stabilizers of orbits.

We study, in Chapter 8, the Weyl group of E_8 with reference to the Weyl subgroup associated to $\mathrm{Spin}(16) \subset E_8$. This is performed to obtain some reults on the double quotient of Weyl groups which we will use in the next chapter to index certain orbits of a decomposition.

In Chapter 9 we apply the results of Chapter 8 to study the orbit structure of an action of $S^1 \times E_6$ on the space $E_7/S^3 \times_{C_2} \mathrm{Spin}(12)$. We study particular orbits, the generic orbit and make some notes on other features of the decomposition.

Chapter 2

Preliminaries

2.1 Algebras

We first establish some basic definitions. A *(real) algebra* is a real vector space A, along with a bilinear multiplication operation $\circ : A \otimes A \to A$.

An algebra morphism between algebras $A, \circ_A$ and $B, \circ_B$ is a linear map $T : A \to B$ such that the following diagram commutes:

$$\begin{tikzcd} A \otimes A \arrow[r, "T \otimes T"] \arrow[d, "\circ_A"'] & B \otimes B \arrow[d, "\circ_B"'] \\ A \arrow[r, "T"] & B \end{tikzcd}$$

If the algebra bilinear multiplication operation $\circ_A$ is obvious from the context we will denote $a \circ_A b$ just by ab. An algebra is *unital* if it has a multiplicative unit and it is a *division algebra*, if every non-zero element has a multiplicative inverse.

A normed division algebra is a real division algebra A along with a norm on the underlying vector space $|\cdot| : A \to \mathbb{R}$, such that $|ab| = |a||b|$ for all $a, b \in A$.

It was been shown by Hurwitz [1898] that, up to isomorphism, the only normed division algebras are the real numbers $\mathbb{R}$, the complex numbers $\mathbb{C}$, the quaternions $\mathbb{H}$, and the octonions (or Cayley numbers) $\mathbb{O}$. For a more modern proof see [Conway and Smith, 2003, Theorem 1, pg. 72]. In particular given a normed division algebra A there exists a unique involution $a \mapsto \bar{a}$ such that $|a| = a\bar{a}$.

An algebra A is:

Associative If for all $a, b, c \in A$ we have $a(bc) = (ab)c$

Alternative If for any $a, b \in A$ the algebra generated by a and b is associative.

Power Associative If for any $a \in A$ the algebra generated by a is associative. Thus for any $a \in A$ the expression a^n is well defined.

While the algebras $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$ are all associative $\mathbb{O}$ is not. As is shown in [Conway and Smith, 2003, Theorem 2, pg. 76] the algebra generated by any 2 octonions is one of $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, thus $\mathbb{O}$ is alternative.

The classical Cayley-Dickson construction originally described in Dickson [1919] is described in detail in [Conway and Smith, 2003, Chapter 6]). Applied to $\mathbb{H}$ this construction produces the algebra $\mathbb{O}$. Applied to $\mathbb{O}$ it produces an algebra known as the sedonion algebra $\mathbb{S}$. This algebra has zero divisors and so it is not a division algebra. It is not alternative but it is power associative. In general if the Cayley-Dickinson construction is repeatedly applied to the octonions the resulting algebra is not a division algebra, it is not alternative but it is power associative.

The tensor product $\mathbb{C} \otimes \mathbb{O}$ also has zero divisors and so it is not division algebra. It is of course associative. In contrast the algebras $\mathbb{H} \otimes \mathbb{O}$ and $\mathbb{O} \otimes \mathbb{O}$ are not division algebras, and they are not alternative or even power associative.

A linear map $\partial : A \to A$ is a derivation for an algebra A if it satisfies the Leibnitz identity: for all $a, b \in A$

$$\partial(ab) = \partial(a)b + a\partial(b)$$

The space of deriviations of A forms a Lie algebra $\mathfrak{der}(A)$ with bracket given by

$$[\partial_a, \partial_b] = \partial_a \circ \partial_b - \partial_b \circ \partial_a$$

Here $\partial_a, \partial_b \in \mathfrak{der}(A)$ and $\circ$ is the composition of linear maps.

Let $\mathrm{Aut}(A)$ be the group of algebra automorphisms of a real n-dimensional algebra A then the natural embedding $Aut(A) \subset GL(\mathbb{R}^n)$ gives A the structure of a topological group. The following classical theorem is proved in [Rosenfeld, 1997, Chapter 1]

Theorem 2.1. *Let A be an algebra then $Aut(A)$ is a Lie group with Lie algebra $\mathfrak{der}(A)$*

2.2 Division algebras and triality

Here we give an account of the notion of a triality which was first described in [Adams, 1996, pg. 111]. A *triality* is given by three real vector spaces V_1, V_2, V_3 and

a linear map

$$t : V_1 \otimes V_2 \otimes V_3 \to \mathbb{R}$$

such that given any $(v_1, v_2) \in V_1 \times V_2$ with both v_1 and v_2 nonzero, there exists $v_3 \in V_3$ such that $t(v_1, v_2, v_3) \neq 0$, and of course the symmetric condition with (v_1, v_2) replaced by (v_1, v_3) and also by (v_2, v_3).

If each V_i has an inner product then t is a *normed triality* if

$$|t(v_1, v_2, v_3)) \leq \|v_1\|\|v_2\|\|v_3\|$$

and for all $v_1, v_2 \neq 0$ there is a v_3 such that this bound is attained, and of course the symmetric condition with v_1, v_2 replaced by v_1, v_3 and also replaced by v_2, v_3.

Examples of normed trialities are given by $V_1 = V_2 = V_3 = \mathbb{R}, \mathbb{C}$ or $\mathbb{O}$ and

$$t(x, y, z) = Re(xy)z.$$

Normed trialities are the same as normed division algebras. Given a normed triality we set $A = V_1$ and choose unit vectors in V_2 and V_3. Then using the triality we can identify V_2^*, and V_3^* with A and using the inner products on V_2, V_3 we can now identify V_2 and V_3 with A. Transposing the triality appropriately we get a product

$$A \otimes A \to A.$$

It can be checked that this makes A into a normed division algebra.

Associated to a normed triality t is a Lie group $\mathrm{Tri}(t)$. This is the subgroup of $O(V_1) \times O(V_2) \times O(V_3)$ consisting of those triples (f_1, f_2, f_3) such that

$$t(v_1, v_2, v_3) = t(f_1(v_1), f_2(v_2), f_3(v_3))$$

This is a closed subgroup of $O(V_1) \times O(V_2) \times O(V_3)$. It turns out that

$$1 = \mathrm{Aut}(\mathbb{R}) \subseteq \mathrm{Tri}(\mathbb{R}) = \{(g_1, g_2, g_3) \in O(1)^3 : g_1 g_2 g_3 = 1\}$$

$$C_2 = \mathrm{Aut}(\mathbb{C}) \subseteq \mathrm{Tri}(\mathbb{C}) = \{(g_1, g_2, g_3) \in U(1)^3 : g_1 g_2 g_3 = 1\} \times C_2$$

$$\mathrm{SO}(3) = \mathrm{Aut}(\mathbb{H}) \subseteq \mathrm{Tri}(\mathbb{H}) = \mathrm{Sp}(1)^3/\pm(1, 1, 1)$$

$$G_2 = \mathrm{Aut}(\mathbb{O}) \subseteq \mathrm{Tri}(\mathbb{O}) = \mathrm{Spin}(8)$$

The Lie algebras of the three triality groups are given by:

$$\mathfrak{tri}(\mathbb{R}) = 0$$

$$\mathfrak{tri}(\mathbb{C}) = \mathfrak{u}(1)^2$$

$$\mathfrak{tri}(\mathbb{H}) = \mathfrak{sp}(1)^3$$

$$\mathfrak{tri}(\mathbb{O}) = \mathrm{spin}(8).$$

2.3 Triality and Freudenthal's magic square

In Tits [1966] a method of associating to a pair of division algebras A, B a Lie algebra $L_{A,B}$ is developed. This construction has been refined by Barton and Sudbery [2003] and it is this formulation we use here.

As a vector space $L_{A,B}$ is given by

$$\mathfrak{tri}(A) \oplus \mathfrak{tri}(B) \oplus (A \otimes B)^3.$$

The formula for the bracket is given in Barton and Sudbury's paper. We will not repeat it here as we do not really need it. We will however make some comments on the shape of these formulas.

- If $x, y \in \mathfrak{tri}(A) \subset L_{A,B}$ then $[x, y]$ is just their Lie backet in $\mathfrak{tri}(A)$. The analogous statement applies if $x, y \in \mathfrak{tri}(B) \subset L_{A,B}$.

- The Lie algebra $\mathfrak{tri}(A)$ is sub-Lie algebra of $\mathrm{Hom}_{\mathbb{R}}(A, A)^3$. Therefore if $x \in \mathfrak{tri}(A)$ then x gives three linear maps $x_1, x_2, x_3 : A \to A$ and three linear maps $x_1 \otimes 1, x_2 \otimes 1, x_3 \otimes 1 : A \otimes B \to A \otimes B$. This defines a linear map which we shall denote by $x \otimes 1 : (A \otimes B)^3 \to (A \otimes B)^3$. Now if $y \in (A \otimes B)^3)$ then $[x, y] = x \otimes 1(y)$. There is a completely analogous description for the bracket of elements in $\mathfrak{tri}(B)$ and elements of $(A \otimes B)^3$.

- If $x \in A$ and $y \in B$ write $u_1(x \otimes y) = [x \otimes y, 0, 0] \in (A \otimes B)^3$, $u_2(x \otimes y) = [0, x \otimes y, 0] \in (A \otimes B)^3$, $u_3(x \otimes y) = [0, 0, x \otimes y] \in (A \otimes B)^3$. Then

  $$[u_1(x \otimes y), u_2(x' \otimes y')] = u_3(\overline{x'x} \otimes \overline{y'y})$$

  There are of course the obvious analogues of these formulas with $(1, 2)$ replaced by $(1, 3)$ and $(2, 3)$.

Finally it remains to give the formula for $[u_1(x \otimes y), u_1(x' \otimes y')]$, $[u_2(x \otimes y), u_2(x' \otimes y')]$, and $[u_3(x \otimes y), u_3(x' \otimes y')]$. These formulas are given explicitly in [Barton and Sudbery, 2003, Theorem 4.4]

This gives rise to the following table of $L_{A,B}$:

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}$	$\mathfrak{so}(3)$	$\mathfrak{su}(3)$	$\mathfrak{sp}(3)$	$\mathfrak{f}(4)$
$\mathbb{C}$	$\mathfrak{su}(3)$	$\mathfrak{su}(3) \oplus \mathfrak{su}(3)$	$\mathfrak{su}(6)$	$\mathfrak{e}(6)$
$\mathbb{H}$	$\mathfrak{sp}(3)$	$\mathfrak{su}(6)$	$\mathfrak{so}(12)$	$\mathfrak{e}(7)$
$\mathbb{O}$	$\mathfrak{f}(4)$	$\mathfrak{e}(6)$	$\mathfrak{e}(7)$	$\mathfrak{e}(8)$

This is Freudenthal's magic square of Lie algebras as it appears in the works of [Freudenthal, 1964, 6.14,pg. 172] although he derives the table via a different construction.

The construction above define real forms for $L_{\mathbb{A},\mathbb{B}}$. These Lie algebras are all non-compact under this construction as they are non-compact they have a well defined maximal compact subalgebra these are given by the folloing table:

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}$	$\mathfrak{so}(1) \oplus \mathfrak{so}(2)$	$\mathfrak{su}(1) \oplus \mathfrak{su}(2)$	$\mathfrak{sp}(1) \oplus \mathfrak{sp}(2)$	$\mathfrak{spin}(9)$
$\mathbb{C}$	$\mathfrak{su}(1)\mathfrak{su}(2)$	$\mathfrak{su}(1) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(1) \oplus \mathfrak{su}(2)$	$\mathfrak{su}(2) \oplus \mathfrak{su}(4)$	$\mathbb{R} \oplus \mathfrak{spin}(10)$
$\mathbb{H}$	$\mathfrak{sp}(1) \oplus \mathfrak{sp}(2)$	$\mathfrak{su}(2) \oplus \mathfrak{su}(4)$	$\mathfrak{so}(4) \oplus \mathfrak{so}(8)$	$\mathfrak{su}(2) \oplus \mathfrak{spin}(12)$
$\mathbb{O}$	$\mathfrak{spin}(9)$	$\mathbb{R} \oplus \mathfrak{spin}(10)$	$\mathfrak{su}(2) \oplus \mathfrak{spin}(12)$	$\mathfrak{spin}(16)$

Taking the simple lie groups assosicated to the compact forms of these pairs of groups and sub groups we get the following table of symmetric space:

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}$	$\frac{\mathrm{SO}(3)}{S(O(1)\times O(2))}$	$\frac{\mathrm{SU}(3)}{S(U(1)\times U(2))}$	$\frac{Sp3}{Sp1\times Sp2}$	$\frac{F_4}{Spin9}$
$\mathbb{C}$	$\frac{\mathrm{SU}(3)}{S(U(1)\times U(2))}$	$\frac{\mathrm{SU}(3)}{S(U(1)\times U(2))} \times \frac{\mathrm{SU}(3)}{S(U(1)\times U(2))}$	$\frac{\mathrm{SU}(6)}{S(U(2)\times U(4))}$	$\frac{E_6}{S^1\times_{C_2} Spin10}$
$\mathbb{H}$	$\frac{Sp3}{Sp1\times Sp2}$	$\frac{\mathrm{SU}(6)}{S(U(2)\times U(4))}$	$\frac{\mathrm{SO}(12)}{S(O(4)\times O(8))}$	$\frac{E_7}{S^3\times_{C_4} Spin12}$
$\mathbb{O}$	$\frac{F_4}{Spin9}$	$\frac{E_6}{S^1\times_{C_2} Spin10}$	$\frac{E_7}{S^3\times_{C_4} Spin12}$	$\frac{E_8}{Spin16}$

The top row is the four projective planes $\mathbb{P}^2(\mathbb{R})$, $\mathbb{P}^2(\mathbb{C})$, and $\mathbb{P}^2(\mathbb{O})$. The last column is often denoted by $\mathbb{P}^2(\mathbb{O})$, $\mathbb{P}^2(\mathbb{C} \otimes \mathbb{O})$, $\mathbb{P}^2(\mathbb{H} \otimes \mathbb{O})$ and $\mathbb{P}^2(\mathbb{O} \otimes \mathbb{O})$.

Leung in his PhD thesis Huang and Leung [2011] derived another table of spaces via a similar construction these hae the same total lie algebras but different maximal compact subspaces in particular we will use the notation $\mathbb{X}^2(\mathbb{A} \otimes \mathbb{B})$ for

his spaces formed this way. In this thesis we will only use 2 of them $\mathbb{X}(\mathbb{O}) \cong F_4/S^3 \times \mathrm{SU}(4)$ and $\mathbb{X}(\mathbb{O} \otimes \mathbb{C}) \cong E_6/S^3 \times_{C_2} \mathrm{SU}(6)$. We unfortunatly lacked the time to give a more detailed account of the decomposition of these spaces in methods simliar to those used on the spaces $\mathbb{P}^2(\mathbb{A} \otimes \mathbb{B})$.

2.4 Lie algebras and Lie groups

Cartan in his thesis Cartan [1894] classified the simple Lie algebras. They fall into four infinite families the classical Lie algebras and five so-called exceptional Lie algebras.

The four infinite classes $\mathfrak{su}(n)$, $\mathfrak{spin}(2n+1)$, $\mathfrak{sp}(n)$ and $\mathfrak{spin}(2n)$ are the Lie algebras of $\mathrm{SU}(n)$, $\mathrm{Spin}(2n+1)$, $\mathrm{Sp}(n)$ and $\mathrm{Spin}(2n)$ respectively.

The five exceptional Lie algebras are denoted by $\mathfrak{g}(2)$, $\mathfrak{f}(4)$, $\mathfrak{e}(6)$, $\mathfrak{e}(7)$ and $\mathfrak{e}(8)$ which are the Lie to simply connected compact Lie groups G_2 , F_4 , E_6 , E_7 and E_8.

2.5 General notation

We briefly state a few general points of notation for representations of compact Lie groups. Given a real representation ρ its complexification will be denoted by $\rho_{\mathbb{C}}$. The underlying representation of a complex representation ν will be the denoted by $\nu_{\mathbb{R}}$.

Given $i : G \to H$ and a representation ρ of H the restriction of this representation to G will be denoted by $i^*\rho$ if the map i is obvious from the context we may also refer to this representation as $\rho|_G$.

A reasonable familiarity with the classical groups is assumed. We will largely follow the notation in Adams [1996] for the representations of the classical groups in particular we define:

n The trivial representation of dimension n.

ξ The fundamental complex representation of S^1.

ξ^k The complex representation $\bigotimes_{\mathbb{C}}^k \xi$ of S^1.

λ_i The vector representation of $\mathrm{Spin}(i)$.

Δ_i The real spin representation of $\mathrm{Spin}(i)$ if i is not divisible by 4.

$\Delta_{4j}^{\pm}$ The two real spin representations of $\mathrm{Spin}(4j)$.

μ_i The complex vector representation of $\mathrm{SU}(i)$.

H_i The complex vector representation of $\mathrm{Sp}(i)$.

$\mathfrak{g}$ The real adjoint representation of G for any Lie group. G.

2.6 The exceptional Lie groups

Adams gives constructions of the exceptional Lie groups in [Adams, 1996, Chapter 8]. We summarise Adams's approach to these constructions and the key facts we will need. First Adams constructs the smallest and the biggest of these exceptional Lie groups, G_2 and E_8. He then constructs F_4, E_6 and E_7 as subgroups of E_8.

2.6.1 G_2

The spin representation of Spin(7) is an 8-dimensional real representation. This gives a transitive action of Spin(7) on S^7 and the stabilizer of a point in S^7 is a closed 14-dimensional subgroup of Spin(7). Adams shows that this is indeed G_2 in [Adams, 1996, Theorem 5.5,pg. 32].

2.6.2 E_8

Adams gives an explicit construction, in terms of spinors, of the exceptional Lie algebra $\mathfrak{e}(8)$ of dimension 248. He then defines G to be the subgroup of $GL(\mathbb{R}^{248})$ consisting of the automorphism of this Lie algebra preserving the Killing form, that is the invariant inner product on the Lie algebra E_8. He goes on to show that this Lie group G is compact and simply connected. It follows that G must be isomorphic to the simply connected compact Lie group E_8 occurring in the classification of Lie groups.

Adams also shows that $\mathfrak{spin}(16)$ is a Lie subalgebra of $\mathfrak{e}(8)$. He then goes on to deduce that there is a homomorphism of Lie groups

$$\pi : \mathrm{Spin}(16) \to E_8$$

with the following properties.

- The derivative of π at the identity is the inclusion of the Lie algebra $\mathfrak{spin}(16)$ in $\mathfrak{e}(8)$.
- The kernel of π is the cyclic group of order 2 generated by the central element of Spin(16) which acts by $+1$ in Δ^+ and -1 in Δ^-. This element does of

course depend on the choice of Δ^+ and Δ^- as does the embedding of $\mathfrak{spin}(16)$ in $\mathfrak{e}(8)$.

Furthermore we have the following isomorphism of representations:

$$\pi^*(\mathfrak{e}(8)) = \mathfrak{spin}(16) \oplus \Delta^+_{16}.$$

2.6.3 E_7

Starting from the homomorphism $\pi : \mathrm{Spin}(16) \to E_8$ described in the previous section we can form the diagram

$$\begin{array}{ccccc} \mathrm{Spin}(12) \times \mathrm{Spin}(4) & \longrightarrow & \mathrm{Spin}(16) & \xrightarrow{\pi} & E_8 \\ \| & & & & \\ (\mathrm{Spin}(12) \times S^3) \times \mathrm{SU}(2) & & & & \end{array}$$

Where S^3 and $\mathrm{SU}(2)$ are embedded into $\mathrm{Spin}(4)$ as $\ker(\Delta^+_4)$ and $\ker(\Delta^-_4)$ respectively.

Theorem 2.2. *E_7 is the stabilizer in E_8 of the image of* $\mathrm{SU}(2)$ *in E_8.*

From this description of E_7 it is easy to deduce the following facts.

- There is a homomorphism

$$h : S^3 \times Spin(12) \to E_7$$

  with kernel C_4. In the above diagram we see that the copy of $S^3 \times Spin(12)$ centralises the copy of S^3 in $S^3 \times S^3 \times Spin(12)$. This defines a homomorphism $h : S^3 \times Spin(12) \to E_7$. A routine argument using the diagram shows that the kernel of this homomorphism is the subgroup of $S^3 \times Spin(12)$ generated by (i, ω_{12}) and evidently this is a cyclic group of order 4. Here ω_{12} is the central element of $\mathrm{Spin}(12)$ which acts as $+1$ in Δ^+ and -1 in Δ^-.

- The representation $h^*\mathfrak{e}(7)$ is uniquely determined by:

$$h^*(\mathfrak{e}(7)_{\mathbb{C}}) = \mathfrak{spin}(3)_{\mathbb{C}} \oplus \mathfrak{spin}(12)_{\mathbb{C}} \oplus (\Delta_3)_{\mathbb{C}} \otimes_{\mathbb{C}} (\Delta^+_{12})_{\mathbb{C}}.$$

- E_7 also has an irreducible complex representation of complex dimension 56, which we will denote by S^+_{56}, such that:

$$h^*(S^+_{56}) = (\Delta_3)_{\mathbb{C}} \otimes (\lambda_{12})_{\mathbb{C}} \oplus (\Delta^+_{12})_{\mathbb{C}}$$

2.6.4 E_6

This time we write down the natural analogue of the previous diagram we get this by replacing $\mathrm{Spin}(4)\times\mathrm{Spin}(12)$ by $\mathrm{Spin}(6)\times\mathrm{Spin}(10)$. This is the following diagram.

$$\begin{tikzcd}
\mathrm{Spin}(10)\times\mathrm{Spin}(6) \arrow[r] & \mathrm{Spin}(16) \arrow[r, "\pi"] & E_8 \\
\mathrm{Spin}(10)\times S^1\times\mathrm{SU}(3) \arrow[u, hook] & &
\end{tikzcd}$$

Where the map from $S^1\times\mathrm{SU}(3)$ into $\mathrm{Spin}(6)$ is given by following the diagram:

$$\begin{tikzcd}
(z,g) \arrow[d, mapsto] & S^1\times\mathrm{SU}(3) \arrow[r] \arrow[d] & \mathrm{Spin}(6) \arrow[d] \\
(z^2,g) & S^1\times\mathrm{SU}(3) \arrow[r] & \mathrm{U}(3) \arrow[r] & \mathrm{SO}(6)
\end{tikzcd}$$

Where the maps here are the natural ones.

Theorem 2.3. *E_6 is the stabilizer in E_8 of the image of* $\mathrm{SU}(3)$ *in E_8.*

Once more we can easily deduce the following facts from this description of E_6.

- There is a homomorphism $h : S^1\times\mathrm{Spin}(10)\to E_6$ with kernel C_2. This follows because $S^1\times\mathrm{Spin}(10)$ centralises $\mathrm{SU}(3)$ in $S^1\times\mathrm{SU}(3)\times\mathrm{Spin}(10)$. The kernel of this homomorphism is the cyclic group of order 2 generated by $(-1,\omega_{10})\in S^1\times Spin(10)$ where ω_{10} is the generator of the centre of $\mathrm{Spin}(10)$ which acts as (+1) on Δ_{10}^+ and (-1) on Δ_{10}^-.

- The representation $h^*(\mathfrak{e}_6)$ is determined by:

$$h^*(\mathfrak{e}(6)_{\mathbb{C}}) = 1 + \mathfrak{spin}10^{\mathbb{C}} + \xi^3\otimes_{\mathbb{C}}\Delta_{10}^+ + \xi^{-3}\otimes_{\mathbb{C}}\Delta_{10}^-$$

- E_6 also has two 27 dimensional complex irreducible representations which we denote by $S_{27}^{\pm}$, such that:

$$h^*(S_{27}^+) = \xi^{-1}\otimes_{\mathbb{C}}\Delta_{10}^+ + \xi^2\otimes_{\mathbb{C}}\lambda_{10} + \xi^{-4}$$

$$h^*(S_{27}^-) = \xi\otimes_{\mathbb{C}}\Delta_{10}^- + \xi^{-2}\otimes_{\mathbb{C}}\lambda_{10} + \xi^4.$$

2.6.5 F_4

Now we replace $Spin(4) \times Spin(12)$ in the diagram of E_7 by $Spin(7) \times Spin(9)$. This gives the following diagram.

$$\begin{tikzcd}
G_2 \arrow[r, hook] & Spin(7) \arrow[r, hook] & Spin(7) \times Spin(9) \arrow[r] & Spin(16) \arrow[r, "\pi"] & E_8 \\
 & & Spin(9) \arrow[u, hook] & &
\end{tikzcd}$$

All the homomorphisms in this diagram are the natural ones.

Theorem 2.4. *F_4 is the stabilizer in E_8 of the image of G_2 in E_8.*

Once more we get a list of elementary deductions from this definition of F_4.

- There is an injective homomorphism

$$h : Spin(9) \to F_4.$$

- $\mathfrak{f}(4)$ pulls back under h as:

$$h^*(\mathfrak{f}(4)) = \mathfrak{spin}(9) + \Delta_9.$$

- F_4 has a 26 dimensional irreducible real representation which we will denote by S_{26} such that

$$h^*(S_{26}) = \Delta_9 + \lambda_9 + 1.$$

2.6.6 Rosenfeld's projective planes

Definition 2.5. *A projective element in an algebra A is an element $p \in A$ such that*

$$p \circ p = p$$

Given $p \in A$, right multiplication by p defines a linear map $R_p : A \to A$. If p is projective then R_p is a linear map representing a projection onto a subspace. We define the rank of an element p to be the rank of R_p so a rank n projective element has R_p the projection onto an n-dimensional subspace.

Given an associative algebra A the algebra $\mathbb{J}_n(A)$ is a Jordan algebra given by 3×3-matrices over A with the commutator as a product. Taking the subspace

of rank 1 projective elements in $\mathbb{J}_n(A)$ defines a space which is shown in [Rosenfeld, 1997, pg. 346] is equal to the classical projective spaces $\mathbb{P}^n(A)$ where

$$\mathbb{P}^n(A) = \frac{N_n(A)}{A^*}$$

where A^* are the units of A and N_n is the subset for non-singular elements of A^{n+1}

$$N_n(A) = \{(x_1, \ldots, x_{n+1}) \ \mid \ qx_1 = \ldots = qx_{n+1} = 0 \iff q = 0\}$$

and the group A^* acts on $N_n(A)$ from the natural action of A on A^{n+1}.

In particular for associative normed division algebras $\mathbb{A}$ and $\mathbb{B}$ we have that $\mathbb{P}^2(\mathbb{A} \otimes \mathbb{B})$ is the symmetric space in the $\mathbb{A}$, $\mathbb{B}$ position of Freudenthal's table.

Further it is shown in [Rosenfeld, 1997, 7.2, pg 332] that the space of rank 1 projective elements in the exceptional Jordan algebras $J_3(\mathbb{O})$ and $J_3(\mathbb{C} \otimes \mathbb{O})$ are the symmetric spaces in positions $\mathbb{O}$ and $\mathbb{C} \otimes \mathbb{O}$ of the table. Alas as $\mathbb{J}_3(\mathbb{H} \otimes \mathbb{O})$ and $\mathbb{J}_3(\mathbb{O} \otimes \mathbb{O})$ are not Jordon algebras we cannot continue in this manner to define all the symmetric spaces in Fredenthal's table.

In [Rosenfeld, 1997, Chapter 7] Rosenfeld attempts to relate the geometry of elements of Freudenthal's table $\mathbb{P}^2(\mathbb{A} \otimes \mathbb{B})$ to the algebras $\mathbb{A} \otimes \mathbb{B}$.

Throughout the rest of this thesis we will use the definition of these spaces as homogeneous spaces $\mathbb{P}^2(\mathbb{A} \otimes \mathbb{O})$ when we refer to the exceptional projective spaces (sometimes referred to as the Rosenfeld projective spaces).

Chapter 3

Mostert's theorem and decompositions of oriented Grassmannians

Much of this thesis is based on decomposing a compact smooth manifold M by using the orbit structure of an action of a compact Lie group G on M. The simplest example is given by a theorem of Mostert in [Mostert, 1957a, Theorem 4]. The purpose of this section is to describe Mostert's theorem and work out in detail how this can be used to give a decomposition of the manifold $\mathrm{Gr}_2^+(\mathbb{R}^m)$, the Grassmannian of oriented two planes in $\mathbb{R}^m$. There are two main reasons for doing this in some detail. The first is that it provides very nice illustrative examples of the decompositions of homogeneous spaces we can obtain by using Mostert's theorem and its generalisations. The second is that we need some of these ideas in subsequent sections. We note that an Errata, Mostert [1957b], was published for this paper but it has no effect on the result we use.

3.1 Mostert's theorem

Let G be a compact Lie group and let M be a closed compact manifold equipped with a smooth action of G. Then the the action of G has *cohomogeneity* 1 if the generic orbit has codimension 1. Then a general theorem in Mostert [1957a] tells us that one of the following two possibilities must hold.

1. The orbit space is a circle and there is a subgroup $H \subset G$ such that $M = G/H \times S^1$ as a G space.

2. The orbit space is a closed interval. In this case there are three closed subgroups A, B, C of G which fit into the following diagram of inclusions:

$$\begin{array}{ccc} C & \longrightarrow & A \\ \downarrow & & \downarrow \\ B & \longrightarrow & G \end{array}$$

with the following properties.

(a) The generic orbits, corresponding to the points in the interior of the orbit space, are diffeomorphic as G spaces to G/C.

(b) There are two exceptional orbits, corresponding to the end points of the orbit space, and as G spaces these are G/A and G/B.

(c) The projections

$$\pi_A : G/A \to G/C, \quad \pi_B : G/B \to G/C$$

are both sphere bundles.

(d) The manifold M is homeomorphic to the double mapping cylinder of the maps

$$G/A \xleftarrow{\pi_A} G/C \xrightarrow{\pi_B} G/B$$

Recall that the double mapping cylinder of the maps

$$Y \xleftarrow{f} X \xrightarrow{g} Z$$

is the space

$$Y \cup_f X \times I \cup_g Z$$

where $(x, 0) \in X \times I$ is identified with $f(x) \in Y$ and $(x, 1) \in X \times I$ is identified with $g(x) \in Z$. Another way to describe the mapping cylinder is as the homotopy colimit of the diagram

$$Y \xleftarrow{f} X \xrightarrow{g} Z.$$

There is an alternative way to describe this decomposition: there is a Morse–Bott function on M with two critical values.

3.2 Preliminaries on Grassmannians

The oriented Grassmannian $\mathrm{Gr}_k^+(\mathbb{R}^n)$ is the homogenous space $\mathrm{SO}(n)/(\mathrm{SO}(k) \times \mathrm{SO}(n-k))$. In terms of spin groups this is the same as $\mathrm{Spin}(n)/(\mathrm{Spin}(k) \times_{C_2} \mathrm{Spin}(n-k)$ where C_2 is generated by $(-1,-1)$. It is the universal 2-fold cover of the unoriented Grassmannian $\mathrm{Gr}_k(\mathbb{R}^n) = \mathbb{O}(n)/(\mathbb{O}(k) \times \mathbb{O}(n-k)$ and many of the results concerning the unoriented Grassmannians carry over to the oriented Grassmannians with slight modifications.

Associated to $\mathrm{Gr}_k^+(\mathbb{R}^n)$ are 2 canonical oriented bundles which in this section will be referred to as $E_{k,n}$ and $F_{k,n}$ which are to the bundles associated to the usual vector representations V_k and V_{n-k} of $\mathrm{SO}(k)$ and $\mathrm{SO}(n-k)$. As the representation $V_k \oplus V_{n-k}$ is the pullback of the vector representation V_n of $\mathrm{SO}(n)$ the bundle $E_{k,n} \oplus F_{k,n}$ is trival.

The manifold $\mathrm{Gr}_k^+(\mathbb{R}^n)$ is $k(n-k)$ dimensional and tangent bundle of $\mathrm{Gr}_k^+(\mathbb{R}^n)$ is isomorphic to the oriented bundle $Hom(E_{k,n}, F_{k,n}) \cong E_{k,n} \otimes F_{k,n}$ by the same argument as in the unoriented case.

Let $g_n \in \mathrm{SO}(n)$ be the element defined by

$$g_n \cdot (x_1, x_2, \cdots, x_n) = (-1)^n (x_n, \cdots, x_2, x_1)$$

Conjugation by g_n interchanges the subgroups $\mathrm{SO}(k) \times \mathrm{SO}(n-k)$ and $\mathrm{SO}(n-k) \times \mathrm{SO}(k)$ and it defines isomorphism of the spaces $\mathrm{Gr}_k^+(\mathbb{R}^n)$ and $\mathrm{Gr}_{n-k}^+(\mathbb{R}^n)$ under which $E_{k,n}$ (resp $F_{k,n}$) pulls back to $F_{n-k,n}$ (resp $E_{n-k,n}$).

Finally in the case where $k = 1$ we have $\mathrm{Gr}_1^+(\mathbb{R}^n) = \mathrm{SO}(n)/\mathrm{SO}(n-1) = S^{n-1}$.

For $m < n$, the inclusion of $\mathrm{SO}(m) \times \mathrm{SO}(n-m)$ into $\mathrm{SO}(n)$ gives a natural action of $\mathrm{SO}(m) \times \mathrm{SO}(n-m)$ on the space $\mathrm{Gr}_k^+(\mathbb{R}^n)$. In particular if $m = 1$ this gives an action of $\mathrm{SO}(n-1)$ on $\mathrm{Gr}_k^+(\mathbb{R}^n)$ and in the case where $k = 2$ we obtain an action of $\mathrm{SO}(2) \times \mathrm{SO}(n-2) = S^1 \times \mathrm{SO}(n-2)$ on $\mathrm{Gr}_k^+(\mathbb{R}^n)$.

Let J be the matrix in $\mathrm{SO}(2n)$ such that $J_{2i,2j} = 1$, $J_{2i-1,2j-1} = -1$ and $J_{2i-1,2j} = J_{2i,2j-1} = 0$ for all $i, j \in \{1, \cdots n\}$ The unitary group can be embedded as a subspace of $\mathrm{SO}(2n)$ by the subset:

$$\{A \in \mathrm{SO}(2n) \mid AJ = JA\}$$

This induces the inclusion of the $U(n)$ in $\mathrm{SO}(2n)$ as the subspace

$$\{(\alpha_{i,j}) \in \mathrm{SO}(2n) \mid \alpha_{2i,2j} = \alpha_{2i+1,2j+1} \text{ and } \alpha_{2i,2j+1} = -\alpha_{2i+1,2j}\}.$$

This gives an action of $U(n)$ of the space $\mathrm{Gr}_k^+(\mathbb{R}^{2n})$. In the next sections we will

look at the decompositions associated to these actions.

3.3 The decomposition of $\mathrm{Gr}_k^+(\mathbb{R}^n)$ given by the action of $\mathrm{SO}(n-1)$

Consider the action of $\mathrm{SO}(n-1)$ on $\mathrm{Gr}_k^+(\mathbb{R}^n)$ assuming $n > k \geq 2$. There are two special cases.

The first special case to consider is the orbit of a point in $\mathrm{Gr}_k^+(\mathbb{R}^{n-1})$. In other words this is the orbit of an oriented k plane in $\mathbb{R}^{n-1} \times 0 \subset \mathbb{R}^n$. Since $\mathrm{SO}(n-1)$ acts transitively on $\mathrm{Gr}_k^+(\mathbb{R}^{n-1})$ this orbit is the same no matter which point we choose so we may as well choose the orbit of the k plane $\mathbb{R}^k \times 0 \subset \mathbb{R}^n$. As a point in $\mathrm{SO}(n)/(\mathrm{SO}(k) \times \mathrm{SO}(n-k))$ this is given by the point $[1]$ where, in the standard notation for $x \in \mathrm{SO}(n)$, $[x]$ is the coset

$$[x] = x \cdot (\mathrm{SO}(k) \times \mathrm{SO}(n-k)) \in \mathrm{SO}(n)/(\mathrm{SO}(k) \times \mathrm{SO}(n-k))$$

for the coset of $x \in \mathrm{SO}(n)$. Under the action of $\mathrm{SO}(n-1)$ the stabiliser of $[1]$ is

$$\mathrm{SO}(n-1) \cap \mathrm{SO}(k) \times \mathrm{SO}(n-k) = \mathrm{SO}(k-1) \times \mathrm{SO}(n-k)$$

So this orbit is isomorphic to $\mathrm{SO}(n-1)/\mathrm{SO}(k-1) \times \mathrm{SO}(n-k) \cong \mathrm{Gr}_{k-1}^+(\mathbb{R}^{n-1})$.

The second special case is orbit of a point in $\mathrm{Gr}_{n-k-1}^+(\mathbb{R}^{n-1})$. This corresponds to a k plane in $\mathbb{R}^n$ whose orthogonal complement is an $n-k$ plane in $\mathbb{R}^{n-1} \times 0$. Once more the orbit is independent of the choice of the point so we choose the k plane corresponding to the coset $[g_n]$. This time the stabiliser of $[g_n]$ point under the $\mathrm{SO}(n-1)$ action is

$$\mathrm{SO}(n-k-1) \times \mathrm{SO}(k) \cap \mathrm{SO}(n-1).$$

Finally we are left with the generic case where neither the k-plane nor its complement are contained in $\mathbb{R}^{n-1} \times 0$. In this case the stabiliser of the action of $\mathrm{SO}(n-1)$ on such a point is

$$\mathrm{SO}(k-1) \times \mathrm{SO}(n-k-1).$$

This orbit is

$$\mathrm{SO}(n-1)/(\mathrm{SO}(k-1) \times \mathrm{SO}(n-k-1))$$

which is diffeomorphic to both

$$S(F_{k-1,n-1}), \quad S(F_{n-k-1,n-1})$$

The fact that these sphere bundles are diffeomorphic comes from the diffeomorphism of $\mathrm{Gr}_{k-1}(\mathbb{R}^{n-1})$ to $\mathrm{Gr}_{n-k-1}(\mathbb{R}^{n-1})$ induced by conjugation by the element $g_{n-1} \in \mathrm{SO}(n-1)$ and the isomorphism of $F_{k-1,n-1}$ with $F_{n-k-1,n-1}$ covering this diffeomorphism.

As this orbit is codimension 1 we can use Mostert's theorem to conclude that:

Theorem 3.1. *For $n > k \geq 2$ the space $Gr_k^+(\mathbb{R}^n)$ is homeomorphic to the double mapping cylinder of the following diagram:*

$$Gr_{k-1}^+(\mathbb{R}^{n-1}) \longleftarrow S(F_{n-1,k-1}) \underset{g_{n-1}}{\cong} S(F_{n-k-1,n-1}) \longrightarrow Gr_{n-k-1}^+(\mathbb{R}^{n-1}).$$

The special case of this theorem where $k = 2$ gives the following corollary. To state the result let $\mathrm{V}_2(\mathbb{R}^n)$ be the Stiefel manifold of orthonormal 2 frames in $\mathbb{R}^n$. This Stiefel manifold is the homogeneous space

$$\mathrm{V}_2(\mathbb{R}^n) = \mathrm{SO}(n)/\mathrm{SO}(n-2).$$

The circle $\mathrm{SO}(2) = S^1$ acts on freely on the Stiefel manifold with quotient $\mathrm{Gr}_2^+(\mathbb{R}^n)$. Let $p : \mathrm{V}_2(\mathbb{R}^n) \to \mathrm{Gr}_2^+(\mathbb{R}^n)$ be the projection in this principal S^1 bundle. Another description of this Stiefel manifold is as the sphere bundle in the tangent bundle of S^{n-1}. Let $q : \mathrm{V}_2(\mathbb{R}^n) \to S^{n-1}$ be the projection in this sphere bundle.

Corollary 3.2. *The space $Gr_2^+(\mathbb{R}^{n+1})$ is homeomorphic to the double mapping cylinder of the following diagram:*

$$\mathrm{Gr}_2^+(\mathbb{R}^n) \xleftarrow{\ p\ } \tfrac{\mathrm{SO}(n)}{\mathrm{SO}(n-2)} \xrightarrow{\ q\ } S^{n-1}.$$

3.4 The decomposition of $Gr_2^+(\mathbb{R}^{2n})$ given by the action of $U(n)$

Choose a complex structure on $\mathbb{R}^{2n}$, that is an element $J \in \mathrm{SO}(2n)$ such that $J^2 = -1$. Then we identify $U(n)$ with the subgroup of $\mathrm{SO}(n)$ consisting of those matrices which commute with J.

We first make some notes in the unoriented case consider an unoriented 2 plane P such that $JP = P$. Such a P is a complex line in $\mathbb{R}^{2n}$ with complex

structure determined by J. Furthermore $U(n)$ acts transitively on the set of such P and so the orbit of P does not depend on the choice of P. Furthermore this orbit is just a copy of $\mathbb{CP}^{n-1}$ the space of complex lines in $\mathbb{R}^{2n}$ with complex structure determined by J.

Now consider the preimage of $\mathbb{CP}^{n-1}$ in $Gr_2^+(\mathbb{R}^{2n})$ under the map which forgets orientation. Then this preimage must be a 2-fold cover of $\mathbb{CP}^{n-1}$ and it follows, as $\mathbb{CP}^{n-1}$ is simply connected, that the preimage is isomorphic to 2 disjoint copies of $\mathbb{CP}^{n-1}$. Further as this map is equivarient with respect to the action of $U(n)$ we must have 2 disjoint orbits each isomorphic to $\mathbb{CP}^{n-1}$. We will refer to these as $\mathbb{CP}^{n-1}$ and $(\mathbb{CP}^{n-1})^*$ where the orbit $\mathbb{CP}^{n-1}$ contains $1 \in \mathrm{SO}(2n)$.

Restricting to $\mathbb{CP}^{n-1}$ the tangent bundle of $Gr_2^+(\mathbb{R}^{2n})$ splits as $L \otimes_{\mathbb{C}} (U_{n-1} + \overline{U_{n-1}})$. Where L is the canonical complex line bundle on $\mathbb{CP}^{n-1}$ and U_{n-1} is it's complement (so $L + U_{n-1} = n\mathbb{C}$). Then as the tangent bundle to $\mathbb{CP}^{n-1}$ is isomorpic to $L \otimes_{\mathbb{C}} \overline{U_{n-1}}$ the normal bundle to the embedding must be isomorphic to $L \otimes_{\mathbb{C}} U_{n-1}$. We will ref to this bundle as $\nu_{n-1} := L \otimes_{\mathbb{C}} U_{n-1}$.

Finally we come to the generic orbit where neither the oriented 2 plane nor its orthogonal complement is complex. In this case the orbit is isomorphic to

$$U(n)/(U(1) \times U(n-2))$$

This generic orbit has codimension 1 so once more we can apply Mostert's theorem to arrive at the following conclusion. To state it carefully we need to be clear about the difference between $\mathbb{CP}^n$ and $(\mathbb{CP}^n)^*$. These manifolds are diffeomorphic – complex conjugation defines a diffeomorphism

$$c : \mathbb{CP}^{n-1} \to (\mathbb{CP}^{n-1})^*.$$

It follows that $c^*(\nu((\mathbb{CP}^{n-1})^*)$ is isomorphic to $\nu(\mathbb{CP}^{n-1})$ as real vector bundles. Both bundles have complex structures but they are not isomorphic as complex bundles – one is the conjugate of the other. So there is a diffeomorphism $S(T\mathbb{CP}^n) \to S((T\mathbb{CP}^n)^*)$ which we continue to denote by c and refer to as the diffeomorphism defined by complex conjugation.

Theorem 3.3. *For $n > 2$, $Gr_2^+(\mathbb{R}^{2n})$ is isomorphic to the double mapping cylinder of the following diagram.*

$$\mathbb{CP}^{n-1} \longleftarrow S(\nu(\mathbb{CP}^{n-1})) \underset{c}{\cong} S(\nu(\mathbb{CP}^{n-1})^*) \longrightarrow \mathbb{CP}^{n-1}$$

Chapter 4

Cohomology of Oriented Grassmannians

We now use the decompositions in the previous section to compute the cohomology rings of some oriented Grassmannians. There are two reasons for doing this firstly we need these results as essential input for later calculations so it is worth giving a detailed account of them. The second reason is that they provide a very good example of how it is possible to use the decompositions of the previous sections to do cohomological calculations. Some of the results in this section can be found in part in [Zhou and Shi, 2008, Theorem 5.5] and [Lai, 1974, Theorem 2] in the case of [Lai, 1974, Theorem 2] the results in this section were derived independently. These results can also be derived from the integral K-theorey calculations given in Bott [1958] again our calculations were derived independently of this.

4.1 Cohomology of $\mathrm{Gr}_2^+(\mathbb{R}^{2n})$

For the rest of this section we will write Gr_{2n} for the Grassmannian $\mathrm{Gr}_2^+(\mathbb{R}^{2n})$. Then theorem 3.3 shows that Gr_{2n} is homeomorphic to the double mapping cylinder of the diagram

$$\mathbb{CP}^{n-1} \xleftarrow{p} S(E) \xrightarrow{q} \mathbb{CP}^{n-1}.$$

Here E is a complex bundle over $\mathbb{CP}^{n-1}$ whose underlying real bundle is the normal bundle of the embedding $i : \mathbb{CP}^{n-1} \to \mathrm{Gr}_{2n}$ and p is the projection of the sphere bundle of E. The map q is the composite

$$S(E) \to S(\bar{E}) \to \mathbb{CP}^{n-1}$$

where the first map is the diffeomorphism induced by complex conjugation and the second is the projection of the sphere bundle of $\bar{E}$.

It follows that there are two embeddings $i, j : \mathbb{CP}^{n-1} \to \mathrm{Gr}_{2n}$. The normal bundle to i is the bundle E and the normal bundle to j is the complex conjugate bundle $\bar{E}$. So we get cofibrations

$$\mathbb{CP}^{n-1} \xrightarrow{i} \mathrm{Gr}_{2n} \xrightarrow{\phi} \mathrm{Th}(\bar{E})$$

$$\mathbb{CP}^{n-1} \xrightarrow[j]{} \mathrm{Gr}_{2n} \xrightarrow[\theta]{} \mathrm{Th}(E)$$

These two cofibrations fit into the following commutative diagram

$$\begin{tikzcd}
S(E) \arrow[r, "p"'] \arrow[d, "q"'] & \mathbb{CP}^{n-1} \arrow[r] \arrow[d, "i"'] & \mathrm{Th}(E) \arrow[d] \\
\mathbb{CP}^{n-1} \arrow[r, "j"'] \arrow[d] & \mathrm{Gr}_{2n} \arrow[r, "\phi"'] \arrow[d, "\theta"'] & \mathrm{Th}(E) \arrow[d] \\
\mathrm{Th}(\bar{E}) \arrow[r] & \mathrm{Th}(\bar{E}) \arrow[r] & *
\end{tikzcd}$$

The maps p, q in this diagram are projections of sphere bundles and so their cofibres are homotopy equivalent to the corresponding Thom spaces.

Now let P be the tautological oriented two plane bundle over Gr_{2n}. Define

$$x = e(P) \in H^2(\mathrm{Gr}_{2n})$$

to be the Euler class of P. Now define

$$y = \theta^*(\mu_E) \in H^{2n-2}(\mathrm{Gr}_{2n})$$

where $\theta : \mathbb{CP}^{n-1} \to \mathrm{Th}(E)$ is the map occurring in the above diagram and μ_E is the Thom class of E.

Theorem 4.1. *If n is odd then integral cohomology ring of Gr_{2n} is given by:*

$$H^*(\mathrm{Gr}_{2n}) = \mathbb{Z}[x, y]/\langle x^n = 2xy, \quad y^2 = 0\rangle.$$

Otherwise if n is even then integral cohomology ring of Gr_{2n} is given by:

$$H^*(\mathrm{Gr}_{2n}) = \mathbb{Z}[x, y]/\langle x^n = 2xy, \quad y^2 = x^{n-1}y\rangle.$$

We will use the cofibration

$$\mathbb{CP}^{n-1} \xrightarrow{i} \mathrm{Gr}_{2n} \xrightarrow{\phi} \mathrm{Th}(\bar{E}).$$

First note that both $H^*(\mathbb{CP}^{n-1})$ and $H^*(\mathrm{Th}(\bar{E}))$ are concentrated in even dimensions. So it follows that the connecting homomorphism in the exact sequence of this cofibration is zero and we can read off the cohomology groups of Gr_{2n} and give canonical generators for them. To state the answer write $\mu_{\bar{E}} \in H^{2n-2}(\mathrm{Th}(\bar{E}))$ for the Thom class. Let $u \in H^2(\mathbb{CP}^{n-1})$ be the generator $i^*(x)$ and let $\bar{\mu}u^k$ be the elements of $H^{2(n-1+k)}(\mathrm{Th}(\bar{T}\mathbb{CP}^{n-1}))$ defined by the Thom isomorphism.

Lemma 4.2. *The integral cohomology groups of* Gr *are given by*

$$H^{2j}(\mathrm{Gr}_{2n}) = \begin{cases} \mathbb{Z} & \text{if } 0 \leq 2j \leq 2n-4 \text{ or } 2n \leq 2j \leq 4n-4 \\ \mathbb{Z} \oplus \mathbb{Z} & 2j = 2n-2 \\ 0 & \text{otherwise} \end{cases}$$

The following elements give a basis for $H^*(\mathrm{Gr}_{2n})$

$$\{x^j, \quad \theta^*(\bar{\mu}u^j) : 0 \leq j \leq n-1\}.$$

The proof of this proposition is a straightforward argument with the exact sequence of the above cofibration and the structure of the ring $H^*(\mathbb{CP}^{n-1})$.

Now we compute products. First we deal with the easiest products the first lemma which follows from the standard multiplicative properties of the Thom isomorphism.

Lemma 4.3.

$$\theta^*(u^j\bar{\mu}) = x^j y$$

Lemma 4.4. *If n is even then $y^2 = x^{n-1}y$ otherwise $y^2 = 0$.*

Proof. As y is the image of the Thom class $th(\nu)$ we have that:

$$y^2 = th(\nu)^2 = e(\nu)th(\nu) = e(\nu)y$$

So it will suffice to show that $e(\nu) = 0$ if n is odd and $e(\nu) = x^{n-1}$ if n is even. By elementary algebra this is the same as showing that $2e(\nu) - x^{n-1} = (-1)^{n-1}x^{n-1}$.

From the definition of ν we have that:

$$L \otimes_{\mathbb{C}} L + \nu = nL$$

Where L is the canonical complex line bundle on $\mathbb{CP}^{n-1}$ Thus we have that:

$$\begin{aligned} c(\nu) &= c(L)^n c(L \otimes_C CL)^{-1} \\ &= (1+x)^n (1+2x)^{-1} \\ &= \sum_{i=0}^{n-1} \sum_{j=0}^{n-i-1} \binom{n}{i} (-2)^j x^{i+j} \end{aligned}$$

And so $e(\nu) = c_{n-1}(\nu) = (\sum_{i=0}^{n-1} \binom{n}{i}(-2)^{n-1-i})x^{n-1}$ let the coefficient here be α we have that:

$$\begin{aligned} 2\alpha - 1 &= 2(\sum_{i=0}^{n-1} \binom{n}{i}(-2)^{n-1-i}) - 1 \\ &= -\sum_{i=0}^{n-1} \binom{n}{i}(-2)^{n-i} - \binom{n}{n}(-2)^0 \\ &= -\sum_{i=0}^{n} \binom{n}{i}(-2)^{n-i} \\ &= -(1-2)^n = (-1)^{n-1}. \end{aligned}$$

And so $2e(\nu) - x^{n-1} = (2\alpha - 1)x^{n-1} = (-1)^{n-1}x^{n-1}$ and the result follows.

□

Now we come to the final lemma required for the proof of Theorem 4.1.

Lemma 4.5.

$$nx^{2n-2} = 2nx^{n-1}y$$

To do this we need the following formula for the Euler class of Gr_{2n} coming from the general theory of characteristic classes.

Lemma 4.6.

$$e(T\mathrm{Gr}_{2n}) = nx^{n-2}$$

Assuming Lemma 4.6 for the moment we prove the formula in Lemma 4.5 and complete the proof of Theorem 4.1.

Proof. By Lemma 4.2 the Euler characteristic of Gr_{2n} is $2n$ and we know that $x^{n-1}y$ is a generator of $H^{4n-4}(\mathrm{Gr}_{2n})$ it follows that

$$e(T\mathrm{Gr}_{2n}) = \pm 2nx^{n-1}y \in H^{4n-4}(\mathrm{Gr}_{2n})$$

Using the above lemma it follows that

$$nx^{2n-2} = \pm 2nx^{n-1}y.$$

Now we can replace y by $-y$ if necessary to ensure the sign is $+$ in the previous equation, and then use the fact that $H^{4n-4}(\mathrm{Gr}_{2n})$ is torsion free to conclude that

$$x^{2n-2} = 2x^{n-1}y.$$

Combining this formula with Lemmas 4.2 - 4.4 completes the proof of Theorem 4.1. □

We now complete this subsection by giving the proof of Lemma 4.6.

Proof. Let P be the canonical oriented 2 plane bundle over Gr_{2n} and let Q be the complementary oriented $2n-2$ plane bundle. Then

$$P \oplus Q = \mathbb{R}^{2n}$$

and

$$T(\mathrm{Gr}_{2n}) = \mathrm{Hom}(P, Q) = P^* \otimes_{\mathbb{R}} Q \cong P \otimes_{\mathbb{R}} Q.$$

Now P has a complex structure and so we choose a complex line bundle E such that $E_{\mathbb{R}} = P$. Then $P \otimes_{\mathbb{R}} Q$ has a complex structure, indeed

$$(E \otimes_{\mathbb{C}} (Q \otimes_{\mathbb{R}} \mathbb{C}))_{\mathbb{R}} = P \otimes_{\mathbb{R}} Q.$$

To ease the notation write

$$F = (Q \otimes_{\mathbb{R}} \mathbb{C})$$

and then we see that

$$c_{2n-2}(E \otimes_{\mathbb{C}} F) = e(P \otimes_{\mathbb{R}} Q).$$

In addition by complexifying the equation $P \oplus Q = \mathbb{R}^{2n}$ and using the fact that $P \otimes_{\mathbb{R}} \mathbb{C} = E \oplus \bar{E}$ we see that

$$E \oplus \bar{E} \oplus F = \mathbb{C}^{2n}$$

So we look for the universal formula for $c_{2n-2}(E \otimes_{\mathbb{C}} F)$ where E is a complex line bundle and F is a complex bundle satisfying the above equation.

The first step is to tensor the relation of bundles with E to get

$$E \otimes_{\mathbb{C}} E \oplus 1_{\mathbb{C}} \oplus E \otimes_{\mathbb{C}} F = E^{2n}.$$

Now write

$$u = c_1(E)$$

and then we get the equation

$$(1+2u)c(E \otimes_{\mathbb{C}} F) = (1+u)^{2n}, \quad c(E \otimes_{\mathbb{C}} F) = (1+u)^{2n}(1+2u)^{-1}$$

It follows that

$$\begin{aligned} c_{2n-2}(E \otimes_{\mathbb{C}} F) &= \sum_{j=0}^{2n-2} \binom{2n}{j} u^j (-2u)^{2n-2-j} \\ &= \left(\sum_{j=0}^{2n-2} \binom{2n}{j} (-2)^{2n-2-j} \right) u^{2n-2}. \end{aligned}$$

By the binomial theorem we have

$$1 = (1+(-2))^{2n} = \sum_{j=0}^{2n} \binom{2n}{j} (-2)^{2n-j}.$$

Dividing by through by 4 this gives

$$\sum_{j=0}^{2n} \binom{2n}{j} (-2)^{2n-2-j} = \frac{1}{4}.$$

From this it easily follows that

$$\begin{aligned} \sum_{j=0}^{2n-2} \binom{2n}{j} (-2)^{2n-2-j} &= \sum_{j=0}^{2n} \binom{2n}{j} (-2)^{2n-2-j} - \binom{2n}{2n-1} (-2)^{-1} - \binom{2n}{2n} (-2)^{-2} \\ &= \frac{1}{4} - (2n)\frac{1}{-2} - \frac{1}{(-2)^2} \\ &= n \end{aligned}$$

It follows that

$$c_{2n-2}(E \otimes_{\mathbb{C}} F) = nu^{n-1}.$$

However

$$u = c_1(E) = e(P) = x, \quad c_{2n-2}(E \otimes_{\mathbb{C}} F) = e(T\mathrm{Gr}_{2n})$$

and this completes the proof of Lemma 4.6 $\square$

4.2 Cohomology of $\mathrm{Gr}_{2n+1} = \mathrm{Gr}_2^+(\mathbb{R}^{2n+1})$

We will use the decomposition of Gr_{2n+1} given by the action of by $\mathrm{SO}(2n)$ to compute the cohomology ring of Gr_{2n+1}. This time the decomposition 3.1 expresses Gr_{2n+1} as the double mapping cylinder

$$\mathrm{Gr}_{2n} \xleftarrow{p} \frac{\mathrm{SO}(2n)}{\mathrm{SO}(2n-2)} \xrightarrow{q} S^{2n-1}.$$

The middle space in this diagram is the Stiefel manifold, $\mathrm{V}_{2n} = \mathrm{V}_2(\mathbb{R}^{2n})$, of orthonormal two frames in $\mathbb{R}^{2n}$. The circle $\mathrm{SO}(2)$ acts on freely on the Stiefel manifold with quotient Gr_{2n} . The map p is the projection in this $\mathrm{SO}(2)$ bundle. Another description of this Stiefel manifold is as the sphere bundle in the tangent bundle of S^{2n-1} and the map q is the projection in this bundle.

As in the case of Gr_{2n} this leads to the following diagram of cofibrations.

$$\begin{array}{ccccc}
V_{2n} & \xrightarrow[p]{} & \mathrm{Gr}_{2n} & \longrightarrow & \mathrm{Th}(P) \\
{\scriptstyle q}\downarrow & & {\scriptstyle i}\downarrow & & \downarrow \\
S^{2n-1} & \xrightarrow{j} & \mathrm{Gr}_{2n+1} & \xrightarrow[\phi]{} & \mathrm{Th}(P) \\
\downarrow & & {\scriptstyle \theta}\downarrow & & \downarrow \\
\mathrm{Th}(T(S^{2n-1})) & \longrightarrow & \mathrm{Th}(T(S^{2n-1})) & \longrightarrow & * \\
\end{array}$$

The maps p, q in this diagram are the projections described in the previous paragraph. The maps i and j are the embeddings of the "ends" of the mapping cylinder; θ and ϕ are the Pontryagin-Thom maps defined by these embeddings; and P is the tautological oriented 2 plane bundle over Gr_{2n}, which is naturally isomorphic to the normal bundle of the embedding j.

As in the previous case we set

$$x = e(P) \in H^2(\mathrm{Gr}_{2n+1}).$$

Let μ_P be the Thom class in $H^2(\mathrm{Th}(P))$. Define

$$z = \phi^*(y \cdot \mu_P) \in H^{2n}(\mathrm{Gr}_{2n+1})$$

where $y \in H^{2n-2}(\mathrm{Gr}_{2n})$ is the class defined in the previous section and $y \cdot \mu_P$ is the class defined using the Thom isomorphism.

Theorem 4.7. *The integral cohomology ring of* Gr_{2n+1} *is given by*

$$H^*(\mathrm{Gr}) = \mathbb{Z}[x, z]/\langle x^n = 2z, \quad z^2 = 0\rangle.$$

Following the pattern of proof in the previous section we first establish the following lemma.

Lemma 4.8. *The integral cohomology groups of* Gr_{2n+1} *are given by*

$$H^{2j}(\mathrm{Gr}_{2n+1}) = \begin{cases} \mathbb{Z} & \text{if } 0 \leq 2j \leq 4n-2 \\ 0 & \text{otherwise.} \end{cases}$$

The following elements give a basis for $H^*(\mathrm{Gr}_{2n})$

$$\{x^j, \quad \theta^*(x^j z) : 0 \leq j \leq n-1\}.$$

Proof. The integral cohomology of the Stiefel manifold V_{2n+1} is given by

$$H^0(V_{2n+1}) = \mathbb{Z} \quad H^{2n}(V_{2n+1}) = \mathbb{Z}/2 \quad H^{4n-1}(V_{2n+1}) = \mathbb{Z}$$

and all other cohomology groups are zero. Since $H^r(V_{2n+1}) = 0$ for $r \leq 2n-1$ the Gysin sequence of the circle bundle $V_{2n+1} \to G_{2n+1}$ shows that

$$H^{r-2}(G_{2n+1}) = H^r(G_{2n+1}) \quad \text{if } r \leq 2n-1$$

Using the fact that $H^0(G_{2n+1}) = \mathbb{Z}$ and $H^1(G_{2n+1}) = 0$ we see that if $r \leq 2n-1$ then the cohomology group $H^r(G_{2n+1})$ is $\mathbb{Z}$ if r is even and 0 if r is odd. Now G_{2n+1} is a $4n-2$ dimensional closed oriented manifold and so, using Poincare if follows that the integral cohomology groups are as stated in the lemma. The Gysin sequence also shows that the generators of the groups $H^{2j}(\mathrm{Gr}_{2n+1})$ for $2j \leq 2n-2$ are x^j where x is the Euler class of P. Poincare duality then tells us that the generators of the groups $H^{2j}(\mathrm{Gr}_{2n+1})$ for $2j \geq 2n$ are $x^{2j-2n}z$. □

To complete the proof of Theorem 4.7 we simply need to prove the following lemma.

Lemma 4.9. *In* $H^{2n}(\mathrm{Gr}_{2n+1})$

$$x^n = 2z.$$

Proof. Let $\mu_P \in H^2(\mathrm{Th}(P))$ be the Thom class. Then in $H^{2n}(\mathrm{Th}(P))$ we have from the previous section $x^{n-1}\mu_P = 2y\mu_P$ where $x \in H^2(\mathrm{Gr}_{2n})$ and $y \in H^{2n-2}(\mathrm{Gr}_{2n})$ are the generators of $H^*(\mathrm{Gr}_{2n})$. Now $\phi : \mathrm{Gr}_{2n+1} \to \mathrm{Th}(P)$ is the Pontryagin Thom map associated to the embedding $i : \mathrm{Gr}_{2n} \to \mathrm{Gr}_{2n+1}$ with normal bundle P. Therefore it follows that

$$\phi^*(i^*(a)\mu_P) = a\phi^*(\mu_P), \quad a \in H^*(\mathrm{Gr}_{2n+1}).$$

From this formula it follows that $\phi^*(x^{n-1}\mu_P) = x^n$. We cannot use this formula to compute $\phi^*(y\mu_P)$ because y is not on the image of i^*. However, by definition $z = \phi^*(y\mu_P)$ and we get the relation $x^n = 2z$ in $H^{2n}(\mathrm{Gr}_{2n+1})$ by applying ϕ^* to the relation $x^{n-1}\mu_P = 2y\mu_P$ in $H^{2n}(\mathrm{Th}(P))$. $\square$

Chapter 5

Decomposition of E_6 spaces

In this section we give details for the decomposition of the spaces $\mathbb{P}^2(\mathbb{C} \otimes \mathbb{O})$ and $\mathbb{X}^2(\mathbb{C} \otimes \mathbb{O})$ as double mapping cylinders. These spaces were described in Chapter 2 as:

$$\mathbb{P}^2(\mathbb{C} \otimes \mathbb{O}) = \frac{E_6}{S^1 \times_{C_4} \mathrm{Spin}(10)}$$

$$\mathbb{X}^2(\mathbb{C} \otimes \mathbb{O}) = \frac{E_6}{S^3 \times_{C_2} \mathrm{SU}(6)}$$

In the case of the decomposition of $\mathbb{P}^2(\mathbb{C}\otimes\mathbb{O})$ this is principally an elaboration of the results of Berndt and Atiyah [2003] in particular Chapter 6 where we give further details as to the maps and spaces used and fill out details of the proofs.

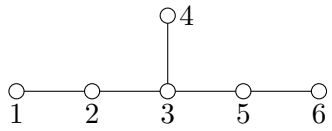
5.1 Lie Groups

We first give explicit forms for some of the Lie groups we will use and maps on them.

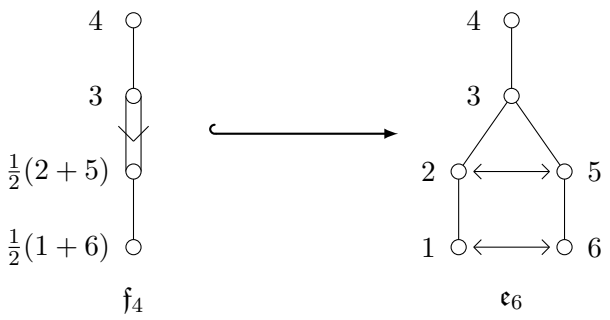
Following [Adams, 1996, pg. 57] we define the complex Lie algebra $\mathfrak{e}_6$ to be the rank 6 complex Lie algebra with simple root vectors given by:

1. $\frac{1}{2}(x_1 - x_2 - x_3 - x_4 + x_5 - \sqrt{3}x_6)$
2. $x_4 - x_5$
3. $x_3 - x_4$
4. $x_2 - x_3$
5. $x_4 + x_5$
6. $\frac{1}{2}(x_1 - x_2 - x_3 - x_4 - x_5 + \sqrt{3}x_6)$

With Dynkin diagram:



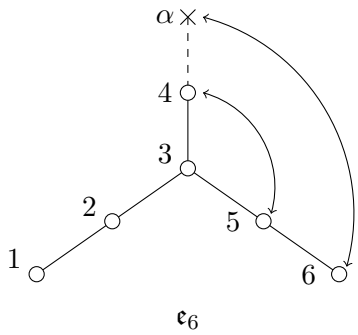
On $\mathfrak{e}_6$ we define 2 algebra involutions. The first we denote by ϕ is defined to to be the outer automorphism which acts on the simple root vectors by fixing roots 3 and 4 and exchanging root 1 with 6 and 2 with 5 respectively. This involution generates the group of outer automorphisms of $\mathfrak{e}_6$. The fixed point set of this involution is isomorphic to $\mathfrak{f}_4$ and thus gives an inclusion $\mathfrak{f}_4 \hookrightarrow \mathfrak{e}_6$ known as the folding inclusion.



A second involution ψ is given by taking the inner isomorphism with an element in the Weyl group of $\mathfrak{e}_6$ whose action on the Cartan subalgebra is given by:

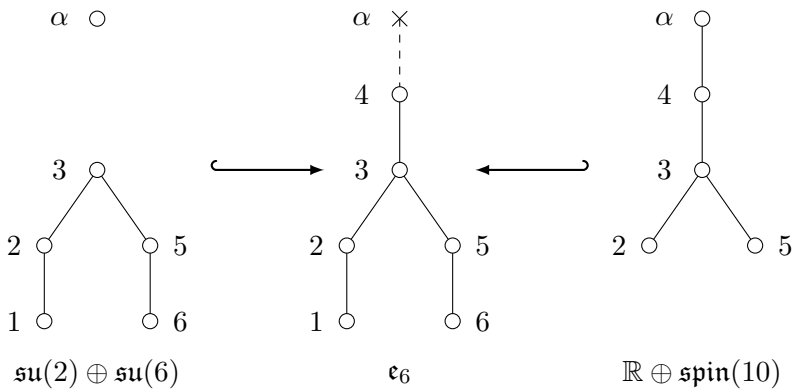
$$\begin{pmatrix} \frac{1}{2} & -1 & -1 & -1 & -1 & \frac{\sqrt{3}}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{3}{2} \end{pmatrix}$$

To show how this acts on $\mathfrak{e}_6$ we consider the extended Dynkin diagram of $\mathfrak{e}_6$ which is the Dynkin diagram with the highest weight root, $x_1 + x_2$, appended. Then ψ permutes the roots in this diagram as follows:



Where we label the highest weight root by α. This determines the action of ψ. Further as ϕ fixed the highest weight root, the pair ψ, ϕ generate the full symmetry group S_3 of the extended Dynkin diagram of $\mathfrak{e}_6$. Further on restriction to the subalgebra $\mathfrak{spin}8$ generated by roots 2,3,4,5 these generate the outer automorphism group S_3 of this space.

We can take the sub-root system generated by the simple roots (2,5) and the highest weight root α to get a subalgebra of $\mathfrak{e}_6$ which is isomorphic to $\mathbb{C} \oplus \mathfrak{so}(10)$. Finally we can take the sub-root system generated by the simple roots 1,2,3,5,6 and the highest weight root α to get a subalgebra isomorphic to $\mathfrak{su}(2) \oplus \mathfrak{su}(6)$.



Let E_6 be the compact Lie group defined in Adams [1996] then its complexified Lie algebra is isomorphic to $\mathfrak{e}_6$. The involutions ϕ and ψ induce involutions ϕ^* and ψ^* on E_6 whose derivatives are ϕ and ψ. The fixed point set of the involution ϕ^* is a Lie subgroup with Lie algebra isomorphic to $\mathfrak{f}_4$ this induces a mapping $F_4 \to E_6$ covering the inclusion $\mathfrak{f}_4 \hookrightarrow \mathfrak{e}_6$ in [Adams, 1996, Chapter 9] Adams shows this map to be an inclusion.

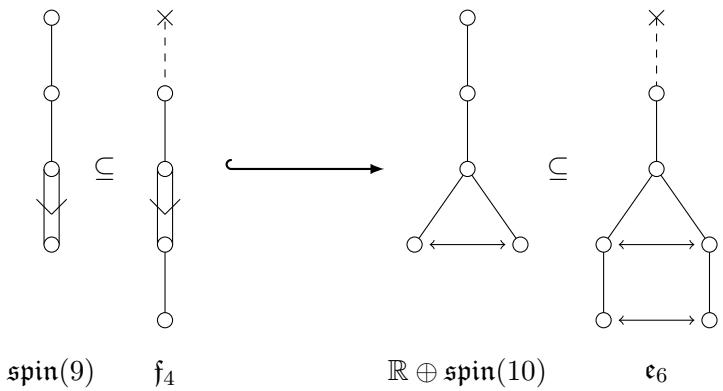
Taking the Lie subgroups of E_6 associated to the Lie subalgebras $\mathbb{R} \oplus \mathfrak{spin}(10)$ and $\mathfrak{su}(2) \oplus \mathfrak{su}(6)$ in $\mathfrak{e}_6$ induces maps $S^1 \times Spin(10) \to E_6$ and $S^3 \times \mathrm{SU}(6) \to E_6$. In [Adams, 1996, pg. 51] it is shown that the map $S^1 \times Spin(10) \to E_6$ has kernel

isomorphic to C_4 generated by the element (i, ω_{10}) and thus we have an inclusion $S^1 \times_{C_4} Spin(10) \subset E_6$. In Ishitoya [1977] it is shown that the map $S^3 \times \mathrm{SU}(6) \to E_6$ has kernel $\mathbb{Z}_2$ giving an inclusion $S^3 \times_{C_2} \mathrm{SU}(6) \to E_6$. This will be the inclusions of these groups for the remainder of this chapter unless specified otherwise, the space $\mathbb{P}^2(\mathbb{C} \otimes \mathbb{O})$ will be defined as $\frac{E_6}{S^1 \times_{C_4} Spin(10)}$ and the space $\mathbb{X}^2(\mathbb{C} \otimes \mathbb{O})$ will be defined as $\frac{E_6}{S^3 \times_{C_2} \mathrm{SU}(6)}$ with these groups and inclusions.

5.2 Decomposition of $\mathbb{P}^2(\mathbb{C} \otimes \mathbb{O})$ by F_4

We will decompose $\mathbb{P}^2(\mathbb{C} \otimes \mathbb{O})$ using the action F_4 as follows. First we will use the default embedding to determine an orbit of F_4. Then we use this to compute the generic orbit and show that the action of F_4 is cohomogeneity one. As $\mathbb{P}^2(\mathbb{C} \otimes \mathbb{O})$ is simply connected by the corollary to Mostert's theorem, it suffices to then find the other exceptional orbit and its normal bundle.

Consider the orbit of F_4 through the point $1 \cdot (S^1 \times_{C_4} Spin(10)) \in \frac{E_6}{S^1 \times_{C_4} Spin(10)}$. This orbit is stabilised by the group $F_4 \cap S^1 \times_{C_4} Spin(10) \subset E_6$ which is determined by the Lie algebra $\mathfrak{f}_4 \cap \mathbb{R} \oplus \mathfrak{so}(10) \subset \mathfrak{e}_6$. This algebra is the subalgebra of $\mathbb{R} \oplus \mathfrak{so}(10)$ fixed by the restriction of the involution ψ. The restriction of ψ to $\mathfrak{so}(10)$ is the standard outer involution on $\mathfrak{so}(2n)$ which fixes $\mathfrak{so}(2n-1)$. So $\mathfrak{f}_4 \cap \mathbb{R} \oplus \mathfrak{so}(10) \cong \mathfrak{so}(9) \subset \mathfrak{f}_4$ as follows:



This $\mathfrak{so}(9) \subset \mathfrak{f}_4$ induces a map $\mathrm{Spin}(9) \to F_4$ which is isomorphic to the Lie algebra map induced from inclusion of $\mathrm{Spin}(9)$ in F_4 used in Chapter 2. We therefore conclude that the orbit at $1 \cdot (S^1 \times_{C_4} Spin(10)) \in \frac{E_6}{S^1 \times_{C_4} Spin(10)}$ is isomorphic to the octonionic projective plane $\frac{F_4}{\mathrm{Spin}(9)}$.

The complex representation $\mathfrak{e}_6$ splits over $S^1 \times_{C_4} Spin(10)$ as $\mathbb{R} + \mathfrak{so}(10) + \xi^{-1} \otimes \Delta_{10}^+ + \xi \otimes \Delta_{10}^-$ thus the tangent bundle of $\mathbb{P}^2(\mathbb{C} \otimes \mathbb{O})$ is generated by the real representation which complexifies to $\xi^{-1} \otimes \Delta_{10}^+ + \xi \otimes \Delta_{10}^-$. This restricts to the

orbit $\frac{F_4}{\mathrm{Spin}(9)} \cong \mathbb{O}P^2$ as the bundle generated by $\Delta_9 + \Delta_9$. As the tangent bundle of this orbit is generated by Δ_9 the normal bundle must also be generated by the representation Δ_9.

Spin(9) acts transitively on S^{15} via the representation Δ_9 stabilized by $\mathrm{Spin}(7) \hookrightarrow \mathrm{Spin}(9)$. We denote this embedding by j^+, it is not isomorphic to the embedding of Spin(7) in Spin(9) which overlays the natural embedding $\mathrm{SO}(7) \hookrightarrow \mathrm{SO}(9)$ we denote this embedding by j^v.

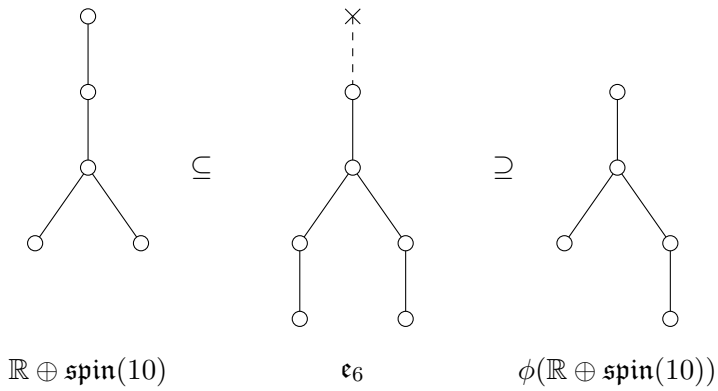
As Spin(9) acts transitively on a sphere in the normal bundle, F_4 acts transitively on the sphere bundle in the normal bundle. This gives rise to a codimension one generic orbit isomorphic to $\frac{F_4}{j^+(\mathrm{Spin}(7))}$

In particular this shows the action has cohomogeneity one and we can use the corollary to Mostert's theorem.

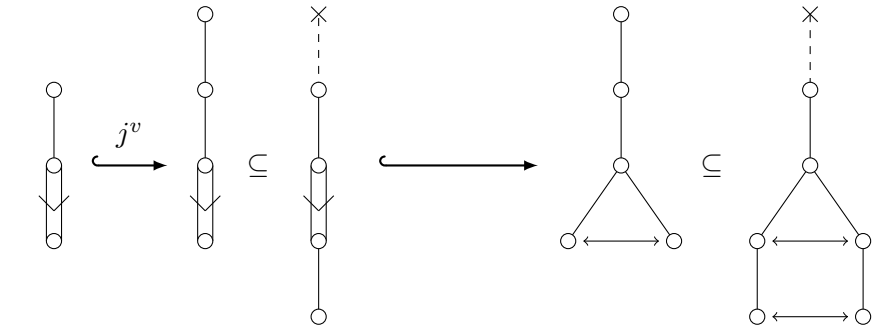
It remains to find the other exceptional orbit. Consider the involution ϕ, it is an inner automorphism and so there exists some $g_\phi \in E_6$ which generates it. In fact as ϕ preserves the maximal torus g_ϕ represents some element in the Weyl group of E_6. We consider the orbit through $g_\phi \cdot (S^1 \times_{C_4} Spin(10))$, the stabilizer of the action at this point has a Lie algebra isomorphic to

$$\mathfrak{f}_4 \cap ad_{g_\phi}(\mathbb{R} \oplus \mathfrak{so}(10)) \cong \mathfrak{f}_4 \cap \phi(\mathbb{R} \oplus \mathfrak{so}(10))$$

As $\mathbb{R} \oplus \mathfrak{so}(10)$ is given by a sub root system $\phi(\mathbb{R} \oplus \mathfrak{so}(10))$ is the sub root system generated by the action of ϕ on the generating roots of $\mathbb{R} \oplus \mathfrak{so}(10)$. Diagramatically this can be shown as follows



The intersection $\mathfrak{f}_4 \cap \phi(\mathbb{R} \oplus \mathfrak{so}(10))$ is then the fixed points of $\phi(\mathbb{R} \oplus \mathfrak{so}(10))$ under the involution ψ This is isomorphic to $\mathbb{R} \oplus \mathfrak{so}(7)$ where $\mathfrak{so}(7) \hookrightarrow \mathfrak{f}_4$ factors as $\mathfrak{so}(7) \underset{j^v}{\hookrightarrow} \mathfrak{so}(7) \hookrightarrow \mathfrak{f}_4$



$\mathfrak{spin}(7)$ $\mathfrak{spin}(9)$ $\mathfrak{f}_4$ $\mathbb{R} \oplus \mathfrak{spin}(10)$ $\mathfrak{e}_6$

The map $S^1 \times \mathrm{Spin}(7) \to \mathrm{Spin}(9)$ corresponding to j^v overlays the inclusion $\mathrm{SO}(2) \times \mathrm{SO}(7) \to \mathrm{SO}(9)$ and thus has kernel isomorphic to $\mathbb{Z}_2$ generated by $(-1, -1)$ where $-1 \in Spin(7)$ generates the kernel of the map $\mathrm{Spin}(7) \to \mathrm{SO}(7)$.

Thus passing to a quotient we obtain an embedding $S^1 \times_{C_2} Spin(7) \hookrightarrow F_4$ as the stabilizer is not isomorphic to the orbit at $1 \cdot (S^1 \times_{C_4} Spin(10))$ and is not isomorphic to the generic orbit, by Mostert's theorem, this must be the other exceptional orbit.

The normal bundle to the inclusion is isomorphic to the 2 plane real bundle generated from the representation ϵ of $S^1 \times_{C_2} j^v(\mathrm{Spin}(7))$ where ϵ pulls back to non-trivial 2-plane bundle on $S^1 \subset S^1 \times_{C_2} j^v(\mathrm{Spin}(7))$. The stabilizer of this action is $j^v(\mathrm{Spin}(7))$.

Taken together with previous results this implies the following

Theorem 5.1. *The space $\mathbb{P}^2(\mathbb{C} \otimes \mathbb{O})$ can be decomposed as the homotopy colimit of the following diagram*

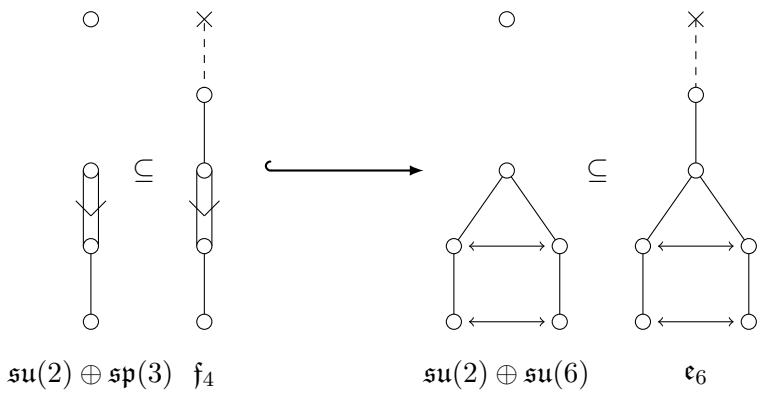
$$\frac{F_4}{\mathrm{Spin}(9)} \longleftarrow \frac{F_4}{j^+(\mathrm{Spin}(7))} \overset{\sigma}{\cong} \frac{F_4}{j^v(\mathrm{Spin}(7))} \longrightarrow \frac{F_4}{S^1 \times_{C_2} j^v(\mathrm{Spin}(7))}$$

5.3 Decomposition of $\mathbb{X}^2(\mathbb{C} \otimes \mathbb{O})$ by F_4

We will decompose $\mathbb{X}^2(\mathbb{C} \otimes \mathbb{O})$ using the action F_4 using the same procedure as in the case of $\mathbb{P}^2(\mathbb{C} \otimes \mathbb{O})$

Consider the orbit of F_4 through the point $1 \cdot S^3 \times_{C_2} \mathrm{SU}(6) \in \frac{E_6}{S^3 \times_{C_2} \mathrm{SU}(6)}$. This orbit is stabilised by the group $F_4 \cap S^3 \times_{C_2} \mathrm{SU}(6) \subset E_6$ which is determined by the Lie algebra $\mathfrak{f}_4 \cap \mathfrak{su}(2) \oplus \mathfrak{su}(6) \subset \mathfrak{e}_6$. This algebra is the subalgebra of $\mathfrak{su}(2) \oplus \mathfrak{su}(6)$ which is fixed by the restriction of the involution ψ. The restriction of ψ to $\mathfrak{su}(2) \oplus \mathfrak{su}(6)$ is the identity on $\mathfrak{su}2$ and is the standard outer involution on $\mathfrak{su}(6)$ which fixes $\mathfrak{sp}(3)$.

So $\mathfrak{f}_4 \cap \mathfrak{su}(2) \oplus \mathfrak{su}(6) \cong \mathfrak{su}(2) \oplus \mathfrak{sp}(3) \subset \mathfrak{f}_4$ as follows where this is the space $\mathbb{X}^2(\mathbb{O})$ discussed in Chapter 2.



We now proceed to calculate the normal bundle to the embedding. The roots of $\mathfrak{e}_6$ are as follows:

$$\pm(x_i \pm x_j) \text{ for } 1 \leq i < j \leq 5$$

$$\pm\frac{1}{2}(x_1 \pm x_2 \pm x_3 \pm x_4 \pm x_5 \pm \sqrt{3}x_6)$$

Where in the number of negative signs inside the parentheses in the second form is even. The roots of $\mathfrak{e}_6$ which restrict to the subroot system $\mathfrak{su}(2) \oplus \mathfrak{su}(6)$ are:

$$\pm(x_1 \pm x_2)$$

$$\pm(x_i \pm x_j) \text{ for } 3 \leq i < j \leq 5$$

$$\pm\frac{1}{2}(x_1 - x_2 \pm x_3 \pm x_4 \pm x_5 \pm \sqrt{3}x_6)$$

Where in the number of negative signs inside the parentheses in the third form is even. The simple roots of this set are thus just the generating roots:

1. $x_1 + x_2$
2. $\frac{1}{2}(x_1 - x_2 - x_3 - x_4 - x_5 + \sqrt{3}x_6)$
3. $x_4 - x_5$
4. $x_3 - x_4$
5. $x_4 + x_5$
6. $\frac{1}{2}(x_1 - x_2 - x_3 - x_4 - x_5 + \sqrt{3}x_6)$

The remaining roots of $\mathfrak{e}_6$ which do not restrict to $\mathfrak{su}(2) \oplus \mathfrak{su}(6)$ are:

$$\pm(x_1 \pm x_i) \text{ for } 3 \leq i \leq 5$$

$$\pm\frac{1}{2}(x_1 + x_2 \pm x_3 \pm x_4 \pm x_5 \pm \sqrt{3}x_6) \text{ where the number of negative signs is even.}$$

Of which the highest weight root is $x_1 + x_3$. The remaining roots form a representation of $\mathfrak{su}(2)\oplus\mathfrak{su}(6)$ with dominant weight given by x_1+x_3. Let $w_1, \cdots, w_6$ be the fundamental weights of $\mathfrak{su}(2) \oplus \mathfrak{su}(6)$ corresponding to the simple roots (1-6) above then we have the following

$$\langle w_1, x_1 + x_3\rangle = \langle x_1 + x_2, x_1 + x_3\rangle = 1$$

$$\langle w_2, x_1 + x_3\rangle = \langle \frac{1}{2}(x_1 - x_2 - x_3 - x_4 - x_5 + \sqrt{3}x_6), x_1 + x_3\rangle = 0$$

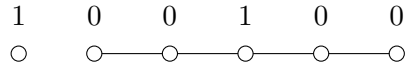
$$\langle w_3, x_1 + x_3\rangle = \langle x_4 - x_5, x_1 + x_3\rangle = 0$$

$$\langle w_4, x_1 + x_3\rangle = \langle x_3 - x_4, x_1 + x_3\rangle = 1$$

$$\langle w_5, x_1 + x_3\rangle = \langle x_4 + x_5, x_1 + x_3\rangle = 0$$

$$\langle w_6, x_1 + x_3\rangle = \langle \frac{1}{2}(x_1 - x_2 - x_3 - x_4 + x_5 - \sqrt{3}x_6), x_1 + x_3\rangle = 0$$

The representation formed by the remaining roots thus has an irreducible component isomorphic to the representation with weight vector $(1, 0, 0, 1, 0, 0)$ from [Bröcker and Dieck, 1985, 5.1, pg. 265] we conclude that this is the representation $U_2 \otimes \bigwedge^3 U_6$ where $\bigwedge^3 U_6$ is the standard representation of 3rd exterior power of the representation U_6. As both these representations are of complex dimension 40 they are thus isomorphic.



Thus pulling back the representation $\mathfrak{e}_6$ from E_6 to $S^3 \times \mathrm{SU}(6)$ the representation splits as $\mathfrak{su}(2)+\mathfrak{su}(6)+U_2\otimes\bigwedge^3 U_6$. And thus the tangent bundle of $\mathbb{X}^2(\mathbb{C}\otimes\mathbb{O})$ is generated by the representation $U_2 \otimes \bigwedge^3 U_6$.

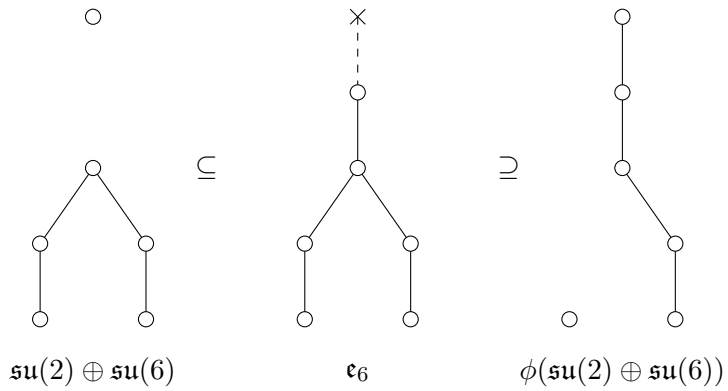
From [Bröcker and Dieck, 1985, 5.3, pg. 269] we know that under the map $Sp(n) \to \mathrm{SU}(2n)$ the representation U_{2n} pulls back to H_n thus the representation $U_2 \otimes \bigwedge^3 U_6$ pulls back to $S^3 \times Sp(3)$ as $H_1 \otimes_{\mathbb{H}} \bigwedge^3 H_3$. By restriction of roots we know that the tangent bundle to $F_4/S^3 \times \mathrm{Sp}(3)$ is generated by the irreducible representation $H_1 \otimes_{\mathbb{H}} \Delta$ where Δ is such that $\Delta + H_3 \cong \bigwedge^3 H_3$. Thus the normal bundle is generated by the representation $H_1 \otimes_{\mathbb{H}} H_3$ of $S^3 \times \mathrm{Sp}(3)$.

$S^3 \times \mathrm{Sp}(3)$ acts transitively of the sphere S^{11} via the representation $H_1 \otimes_{\mathbb{H}} H_3$ and is stablized by $S^3 \times Sp(2)$. Thus F_4 acts transitively on the sphere bundle in the normal bundle stabalized by $S^3 \times Sp(2) \subset S^3 \times Sp(3) \subset F_4$. Hence the generic orbit is $\frac{F_4}{S^3 \times Sp(2)}$ as this is codimension 1 and we can apply the corollary to Mostert's theorem and it remains to find the other exceptional orbit.

We consider the orbit through $g_\phi \cdot (S^3 \times_{C_2} \mathrm{SU}(6))$ where g_ϕ is as defined in the previous subsection. The stabilizer of the action at this point has a Lie algebra isomorphic to

$$\mathfrak{f}_4 \cap ad_{g_\phi}(\mathfrak{su}(2) \oplus \mathfrak{su}(6)) \cong \mathfrak{f}_4 \cap \phi(\mathfrak{su}(2) \oplus \mathfrak{su}(6))$$

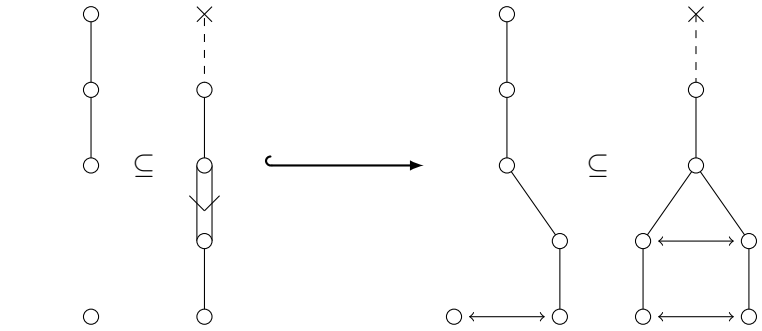
As $\mathfrak{su}(2) \oplus \mathfrak{su}(6)$ is given by a sub root system $\phi(\mathfrak{su}(2) \oplus \mathfrak{su}(6))$ is the sub root system generated by the action of ϕ on the generating roots of $\mathfrak{su}(2) \oplus \mathfrak{su}(6)$. Diagramatically this can be shown as follows:



The intersection $\mathfrak{f}_4 \cap \phi(\mathfrak{su}(2) \oplus \mathfrak{su}(6))$ is then the fixed points of $\phi(\mathfrak{su}(2) \oplus \mathfrak{su}(6))$ under the involution ψ, this is isomorphic to $\mathfrak{su}(2) \oplus \mathfrak{su}(4)$ where:

$$\mathfrak{su}(2) \oplus \mathfrak{su}(4) \hookrightarrow \mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(4) \hookrightarrow \mathfrak{su}(2) \oplus \mathfrak{su}(6)$$

Where the first map is the diagonal embedding of $\mathfrak{su}(2) \hookrightarrow \mathfrak{su}(2) \oplus \mathfrak{su}(2)$, $x \mapsto (x, x)$ and the second is the natural map from $\mathfrak{su}(n) \oplus \mathfrak{su}(m) \hookrightarrow \mathfrak{su}(n+m)$.



$\mathfrak{su}(2) \oplus \mathfrak{su}(4)$ $\mathfrak{f}_4$ $\mathfrak{su}(2) \oplus \mathfrak{su}(4)$ $\mathfrak{e}_6$

Let $\mathrm{SU}(2) \times \mathrm{SU}(4) \to F_4$ be the map of Lie groups associated to the inclusion $\mathfrak{su}(2) \oplus \mathfrak{su}(4) \subseteq \mathfrak{f}_4$ as the centre of $S^3 \times \mathrm{SU}(4)$ is trivial this map is an inclusion and ths the orbit at $g_\phi \cdot (S^3 \times_{C_2} \mathrm{SU}(6))$ is isomorphic to $\frac{F_4}{S^3 \times \mathrm{SU}(4)}$. As this is not isomorphic to either the orbit at $1 \cdot (S^3 \times_{C_2} \mathrm{SU}(6))$ or the generic orbit for dimensional reasons this must be the other exceptional orbit. Finally we compute the normal bundle to the embedding. As the orbit $\frac{F_4}{S^3 \times \mathrm{SU}(4)}$ is of codimension 6 the representation generating the normal bundle must be of dimension 6 and $S^3 \times \mathrm{SU}(4)$ by Theorem 7.15 this must be a representation such that either S^3 or $\mathrm{SU}(4)$ act transitively on the restriction. By Theorem 7.15 there is only 1 such representation, which is the representation V_6 of $\mathrm{Spin}(6) \cong \mathrm{SU}(4)$, and thus the normal bundle is generated by this representation.

Taken together with previous results this implies the following

Theorem 5.2. *The space $\mathbb{X}^2(\mathbb{C} \otimes \mathbb{O})$ can be decomposied as the homotopy colimit of the following diagram*

$$\frac{F_4}{S^3 \times \mathrm{Sp}(3)} \longleftarrow \frac{F_4}{S^3 \times \mathrm{Sp}(2)} \overset{\sigma}{\cong} \frac{F_4}{S^3 \times \mathrm{Spin}(5)} \longrightarrow \frac{F_4}{S^3 \times \mathrm{Spin}(6)}$$

Where we note that, like in the case of $\mathbb{P}^2(\mathbb{C} \otimes \mathbb{O})$, one exceptional orbit of the decomposition of $\mathbb{X}^2(\mathbb{C} \otimes \mathbb{O})$ is $\mathbb{X}^2(\mathbb{O})$.

Chapter 6

Cohomology of $E_6/S^1 \times_{C_4} \mathrm{Spin}(10)$

We will compute the cohomology of the space $E_6/S^1 \times_{C_2} Spin(10)$ using the decomposition in Chapter 5.

This will recover the following result due in [Toda and Watanabe, 1974, Corollary C]

Theorem 6.1.

$$H^*(E_6/S^1 \times_{C_4} Spin(10)) \cong \frac{\mathbb{Z}[s,w]}{\langle 3w^2s = s^9, w^3 = 9ws^8 - 15w^2s^4\rangle}$$

For 2 generators s and w with $|s|$=2 and $|w| = 8$

We first obtain the cohomology of the 2 exceptional orbits of $\mathbb{O}P^2$ and $F_4/S^1 \times_{C_2} \mathrm{Spin}(7)$ and then use the diagram in 5.1 to compute the cohomology of the total space $E_6/S_1 \times_{C_4} \mathrm{Spin}(10)$.

6.1 Cohomology of $\mathbb{O}P^2$

We first recall the following result due to Borel in [Borel and Hirzebruch, 1958, Theorem 19.4 ,pg. 535].

Theorem 6.2. *The cohomology of $\mathbb{O}P^2$ is isomorphic to $\mathbb{Z}[x_8]/ < x_8^3 >$ for a generator x_8 in degree 8. The Pontryagin class of the tangent bundle is given by $1 + 6x_8 + 39x_8^2$*

As by [Adams, 1996, pg. 51] the representation $\mathfrak{f}_4$ restricts to Spin(9) as $\mathfrak{spin}(9) + \delta_9$ the tangent bundle to $\mathbb{O}P^2$ is generated by the representation Δ_9. Let V_9 be the vector bundle on $\mathbb{O}P^2$ generated by the representation V_9 then by [Adams, 1996, Corollary 8.1, pg. 52] the 26-dimensional representation Δ_{26} of F_4 restricts to

Spin(9) as $1 + \delta_9 + \Delta_9$. This implies that the sum of a trivial bundle, V_9 and the tangent bundle is trivial.

Lemma 6.3. $p(V_9) = 1 - 6x_8 - 3x_8^2$

Proof. As the cohomology of $\mathbb{O}P^2$ is only non zero in the degrees $0, 8, 16$ we have that $p(V_9) = 1 + \lambda x_8 + \mu x_8^2$ for some $\lambda, \mu \in \mathbb{Z}$. As $1 + V_9 + T(\mathbb{O}P^2)$ is trivial we have that

$$p(1)p(V_9)p(T\mathbb{O}P^2) = p(27)$$

and thus

$$(1 + \lambda x_8 + \mu x_8^2)(1 + 6x_8 + 39x_8^2) = 1$$

and therefore $\lambda x_8 + 6x_8 = 0$ which implies that $\lambda = -6$ and $39x_8^2 - 36x_8^2 + \mu x_8^2 = 0$ this gives that $\mu = -3$.

□

6.2 Cohomology of $F_4/S^1 \times_{C_2} Spin(7)$

We compute the cohomology of the other exceptional orbit $F_4/S^1 \times_{C_2} Spin(7)$. By composing with the map ρ from Theorem 5.1 we have an inclusion of $F_4/S^1 \times_{C_2} Spin(7) \hookrightarrow F_4/\mathrm{Spin}(9)$ and this gives a fibration:

$$Gr_2^+(\mathbb{R}^9) \underset{i}{\hookrightarrow} F_4/S^1 \times_{C_2} Spin(7) \underset{\pi}{\to} \mathbb{O}P^2$$

We have the cohomology of $\mathbb{O}P^2$ in the previous subsection and the cohomology of $Gr_2^+(\mathbb{R}^9)$ is given by Theorem 4.7 as $\mathbb{Z}[e, f]/ < f^2, 2e^4 - f >$ where e is the euler class of the oriental 2-plane bundle $E_{2,9}$ and f is of degree 8.

Lemma 6.4. *Let $t := e(V_2)$, where V_2 is the oriented 2-plane bundle generated by the representation ξ of $S^1 \subset S^1 \times_{C_2} Spin(7)$ and y_8 is the image of $x_8 \in H^*(\mathbb{O}P^2)$. Then there exists $b \in H^8(F_4/S^1 \times_{C_2} Spin(7))$ such that t, b and y_8 generate $H^*(F_4/S^1 \times_{C_2} Spin(7))$ with the following relations:*

1. $2b - t^4 = \lambda y_8$ *for some* $\lambda \in \{0, 1\}$.
2. $b^2 = \alpha y_8 b + \beta y_8^2$ *for some* $\alpha, \beta \in \mathbb{Z}$.
3. $y_8^3 = 0$.

Proof. As both $H^*(\mathbb{O}P^2)$ and $H^*(Gr_2^+(\mathbb{R}^9))$ are concentrated in an even dimension the Lerry-Serre spectral sequence collapses at the first page see McCleary [2001].

Under the map $\mathrm{Gr}_2(\mathbb{R}^9) \cong \mathrm{Spin}(9)/S^1 \times_{C_2} \mathrm{Spin}(7) \hookrightarrow F_4/S^1 \times_{C_2} \mathrm{Spin}(7)$ the bundle V_2 pulls back to the bundle $E_{2,9}$ in particular this implies that $t = e(V_2)$ pulls back to $e = e(E_{2,9})$ in $H^*(Gr_2^+(\mathbb{R}^9))$.

Choose $b' \in H^8(F_4/S^1 \times_{C_2} Spin(7))$ such that b' pulls back to f in $H^8(Gr_2^+(\mathbb{R}^9))$ such a b exists as by Leray-Hirsch this map is a surjection, as the spectral sequence collapses at the first page.

By the Leray-Hirsch Theorem $H^*(F_4/S^1 \times_{C_2} Spin(7))$ is a free $H^*(\mathbb{O}P^2)$ - module generated by 1, t, t^2, t^3, b', $b't$,$b't^3$. The element $2b' - t^4$ pulls back to $2f - e^4 = 0$ in $H^*(Gr_2^+(\mathbb{R}^9))$ and thus $2b' - t^4 = ay_8$ for some $a \in \mathbb{Z}$.

Now there exists $\mu \in \mathbb{Z}, \lambda \in \{0, 1\}$ such that $a = 2\mu + \lambda$. Define $b = b' - \mu y_8$ then we have that $2b - t^4 = 2b' - t^4 - 2\mu y_8 = \lambda y_8$.

Also as the image of y_8 is 0 in $H^8(G_2^+(\mathbb{R}^9))$ we have that $H^*(F_4/S^1 \times_{C_2} Spin(7))$ is a free module over $H^*(\mathbb{O}P^2)$ generated by $1, t, t^2, t^3, b, bt, bt^2, bt^3$. In particular as y_8 generates $H^*(\mathbb{O}P^2)$, the classes t, b, y_8 generate $H^*(F_4/S^1 \times_{C_2} Spin(7))$.

We know $2b - t^4 = \lambda y_8$ for $\lambda \in \{0, 1\}$ and that $y_8^3 = 0$ as y_8 is the image of x_8, it only remains to compute b^2 as this maps to 0 in $Gr_2^+(\mathbb{R}^9)$ it must equal $\alpha y_8 b + \beta y_8^2$ for some $\alpha, \beta \in \mathbb{Z}$.

□

Thus it remains to compute the values of λ, α and β

Lemma 6.5.

$$H^8(F_4/\mathrm{Spin}(7)) = \mathbb{Z}$$

Proof. As $F_4/j^v(\mathrm{Spin}(7)) \cong_\rho F_4/j^+(\mathrm{Spin}(7))$ we have that $H^8(F_4/j^+(\mathrm{Spin}(7))) \cong \mathbb{Z}$ if and only if $H^8(F_4/j^v(\mathrm{Spin}(7))) = \mathbb{Z}$ but we have a 15-sphere bundle $S^{15} \to F^4/j^*(\mathrm{Spin}(7)) \to F^4/\mathrm{Spin}(9) \cong \mathbb{O}P^2$ and thus $H^8(F_4/j^+(\mathrm{Spin}(7))) \cong H^8(\mathbb{O}P^2) \cong \mathbb{Z}$.

□

Lemma 6.6. *There does not exist $z \in H^8(F_4/S^1 \times_{C_2} Spin(7))$ such that $2z = t^4$.*

Proof. Consider the circle bundle

$$S^1 \hookrightarrow F_4/Spin(7) \to F_4/S^1 \times_{C_2} Spin(7)$$

Then by the Wang exact sequence we have the following:

$$H^6\left(\frac{F_4}{S^1\times_{C_2}\mathrm{Spin}(7)}\right)\underset{\times e(V_2)}{\longrightarrow}H^8\left(\frac{F_4}{S^1\times_{C_2}\mathrm{Spin}(7)}\right)\to H^8\left(\frac{F_4}{\mathrm{Spin}(7)}\right)\to H^7\left(\frac{F_4}{S^1\times_{C_2}\mathrm{Spin}(7)}\right)$$

As $H^7(F_4/S^1\times_{C_2}Spin(7))=0$ we have that $H^8(F_4/\mathrm{Spin}(7))\cong H^8(F_4/S^1\times_{C_2}Spin(7))/ImH^6(F_4/S^1\times_{C_2}Spin(7))$. But $H^6\cong\mathbb{Z}$ and is generated by t^3 as the eular class of the bundle V_2 is t the image of $H^6(F_4/S^1\times_{C_2}Spin(7))$ in $H^8(F_4/S^1\times_{C_2}Spin(7))$ is generated by t^4 if there was a $z\in H^8(F_4/S^1\times_{C_2}Spin(7))$ with $2z=t^4$ as $H^8(F_4/S^1\times_{C_2}Spin(7))$ is torsion free $z\in ImH^6(F_4/S^1\times_{C_2}Spin(7))$ and thus $H^8(F_4/\mathrm{Spin}(7))$ would have a torsion component but $H^8(F_4/\mathrm{Spin}(7))$ is torsion free by Lemma 6.4

□

Corollary 6.7. *In Lemma 6.4* $\lambda=1$.

Proof. Suppose instead $\lambda=0$ then $2b=t^4$ which contradicts Lemma 6.6.

□

Lemma 6.8.

$$t^8-6t^4y_8=3y_8^2$$

Proof. Under the map $S^1\times_{C_2}\mathrm{Spin}(7)\to\mathrm{Spin}(9)$ the representation δ_9 pulls back to $\delta_2+\delta_7$. Let V_2,V_7 also represent the bundles on $F_4/S^1\times_{C_2}\mathrm{Spin}(7)$ generated by these representations and recall that V_9 also represents the 9-dimensional bundle on $\mathbb{O}P^2$ generated by the representation δ_9 of Spin(9).

As V_2 is 2-dimensional it only has a 4-dimensional Pontryagin class which is the square of its euler class thus $p(V_2)=1-t^2$. As V_7 is 7-dimensional it has 3-Pointragin classes in dimension 4,8 and 12.

Finally as $p(V_9)=1-6x_8-3x_8^2$ we have that

$$\begin{aligned}1-6y_8-3y_8^2&=\pi^*(1-6x_8-3x_8^2)\\&=\pi^*(p(V_9))\\&=p(\pi^*(V_9))\\&=p(V_2+V_9)\\&=p(V_2)p(V_7)\\&=(1-t^2)(1+p_1(V_7)+p_2(V_7)+p_3(V_7))\end{aligned}$$

This gives the following set of equations

$$p_1(V_7) - t^2 = 0$$

$$p_2(V_7) - p_1(V_7)t^2 = -6y_8$$

$$p_3(V_7) - p_2(V_7)t^2 = 0$$

$$-t^2 p_3(V_7) = -3y_8^2$$

We conclude that following relations

$$p_1(V_7) = t^2$$

$$p_2(V_7) = -6y_8 + t^4$$

$$p_3(V_7) = t^6 - 6y_8 t^2$$

and thus we get that $t^8 - 6y_8 t^4 = 3y_8^2$.

□

Lemma 6.9. $3b^2 = t^8$

Proof. Substituting $2b - t^4 = y_8$ into $t^8 - 6t^4 y_8 = 3y_8^2$ we obtain $t^8 + 12t^4 b - bt^8 = 12b^2 - 12t^4 b = 3t^8$ which simplifies to $4t^8 = 12b^2$ as $H^*(F_4/S^1 \times_{C_2} Spin(7))$ is torsion-free this implies that $3b^2 = t^8$.

□

Lemma 6.10. $b^2 = y_8^2 + 2t^4 y_8$

Proof. As $y_8 = 2b - t^4$ we have $y_8^2 = 4b^2 - 4bt^4 + t^8$ thus we have that $4b^2 = y_8^2 + 4bt^4 - t^8$ combining with the equation $3b^2 = t^8$ we obtain

$$\begin{aligned} b^2 &= y_8^2 + 4bt^4 - 2t^8 \\ &= y_8^2 + 2t^4(2b - t^4) \\ &= y_8^2 + 2t^4 y_8 \end{aligned}$$

□

This gives us the following theorem:

Theorem 6.11.

$$H^*(F_4/S^1 \times_{C_2} \mathrm{Spin}(7)) = \frac{\mathbb{Z}[t, b]}{< 3b^2 = t^8, b^3 + 15b^2 t^4 - 9bt^8 >}$$

Where $t = e(V_2)$ *and* $|b| = 8$.

Proof. We have from the Lemmas 6.4, 6.7 and 6.10 that

$$H^*(F_4/S^1 \times_{C_2} Spin(7)) \cong \frac{\mathbb{Z}[t, b, y_8]}{< y_8^3, 2b - t^4 = y_8, b^2 = y_8^2 - 2t^4 y_8 >}$$

As $y_8 = 2b - t^4$ we can eliminate it from the ring giving the result as

$$\frac{\mathbb{Z}[t, b]}{< 3b^2 = t^8, 8b^3 - 12b^2t^4 + 6bt^8 - t^{12} >}$$

We can substitute the relation $b^3 + 15b^2t^4 - 9bt^8$ for $8b^3 - 12b^2t^4 + 6bt^8 - t^{12}$ in this ring as we have that:

$$b^3 + 15b^2t^4 - 9bt^8 = (3b - t^4)[3b^2 - t^8] - [8b^3 - 12b^2t^4 + 6bt^8 - t^{12}]$$

. □

We now move on to the computation of the cohomology of $E_6/S^1 \times_{C_4} Spin(10)$.

6.3 Cohomology of $E_6/S^1 \times_{C_4} Spin(10)$

As in the case of oriented Grassmanians in Chapter 4 and the decomposition in the previous chapter we have the following diagram of spaces:

$$\begin{tikzcd}
\frac{F_4}{\mathrm{Spin}(7)} \arrow[r] \arrow[d] & \mathbb{O}P^2 \arrow[r] \arrow[d, "i_1"'] & \mathrm{Th}(T(\mathbb{O}P^2)) \arrow[d] \\
\frac{F_4}{S^1 \times_{C_2} \mathrm{Spin}(7)} \arrow[r, "i_2"'] \arrow[d] & \frac{E_6}{S^1 \times_{C_4} \mathrm{Spin}(10)} \arrow[r, "j_2"'] \arrow[d, "j_1"'] & \mathrm{Th}(T(\mathbb{O}P^2)) \arrow[d] \\
\mathrm{Th}(V_2) \arrow[r] & \mathrm{Th}(V_2) \arrow[r] & *
\end{tikzcd}$$

Which induces the following commuting diagram in cohomology

$$
\begin{array}{ccccc}
H^*\left(\frac{F_4}{\mathrm{Spin}(7)}\right) & \longleftarrow & H^*(\mathbb{O}P^2) & \longleftarrow & H^*(\mathrm{Th}(T(\mathbb{O}P^2))) \\
\uparrow & & i_1^* \uparrow & & \uparrow \\
H^*\left(\frac{F_4}{S^1\times_{C_2}\mathrm{Spin}(7)}\right) & \xleftarrow[i_2^*]{} & H^*\left(\frac{E_6}{S^1\times_{C_4}\mathrm{Spin}(10)}\right) & \xleftarrow[j_2^*]{} & H^*(\mathrm{Th}(T(\mathbb{O}P^2))) \\
\uparrow & & j_1^* \uparrow & & \uparrow \\
H^*(\mathrm{Th}(V_2)) & \longleftarrow & H^*(\mathrm{Th}(V_2)) & \longleftarrow & 0
\end{array}
$$

Lemma 6.12. *The map*

$$i_2^* : H^n(E_6/S^1 \times_{C_4} Spin(10)) \to H^n(F_4/S^1 \times_{C_2} \mathrm{Spin}(7))$$

is an isomorphism for $0 < n < 15$

Proof. The pair of maps

$$F_4/S^1 \times_{C_2} \mathrm{Spin}(7) \underset{i_2}{\longrightarrow} E6/S^1 \times_{C_4} \mathrm{Spin}(10) \underset{j_2}{\longrightarrow} Th(T(\mathbb{O}P^2))$$

forms a cofibration sequence. Thus the long exact cofibration sequence can be formed, but if $0 < n \leq 15$ we have that $H^n(Th(\mathbb{O}P^2)) = 0$ as it is the Thom space of a 16-dimensional vector bundle and hence in these dimensions i_2^* is an isomorphism.

□

Definition 6.13. *Let* $s = i_2^{*-1}(t)$ *and* $w = i_2^{*-1}(b)$ *as* $|t| = 2$ *and* $|b| = 8$ *we note that by lemma 6.12 both of these inverses exist.*

Lemma 6.14. *Up to dimension 15 we have that* $H^*(E_6/S^1 \times_{C_4} Spin(10))$ *is isomorphic to* $\mathbb{Z}[s, w]$.

Proof. As i_2^* is an isomorphism up to dimension 15 $H^*(E_6/S^1 \times_{C_4} Spin(10))$ is isomorphic to $H^*(F^4/S^1 \times_{C_2} \mathrm{Spin}(7))$ up to this dimension but $H^*(F^4/S^1 \times_{C_2} \mathrm{Spin}(7))$ has no relations of dimension less than 15 thus up to dimension 15 we have that $H^*(E_6/S^1 \times_{C_4} Spin(10)) = (i_2^*)^{-1}\mathbb{Z}[t, b] = \mathbb{Z}[s, w]$.

□

Definition 6.15. *Let* U_1 *and* U_2 *be the Thom classes of the bundles* V_2 *and* $T(\mathbb{O}P^2)$ *in* $H^2(Th(V_2))$ *and* $H^{16}(Th(\mathbb{O}P^2))$.

Lemma 6.16. *The relations $j_1^*(U_1) = s$ and $j_1^*(bU_1) = sw$ both hold.*

Proof. The composition $j_1 \circ i_2$ is equal to the inclusion in the Thom exact sequence thus $(i_2^* \circ j_1^*)(U_1) = e(V_2) = t$. But $(i_2^*)^{-1}(t) = s$ and thus $j_1^*(U_1)$ must also equal s.

Next $(i_2^* \circ j_1^*)(bU_1) = bt$ but i_2^* is an isomorphism in this degree and so $i_2^*(sw) = bt$ giving $sw = j_1^*(bU_1)$.

□

Lemma 6.17. *Given $k \geq 0$ the following equations hold:*

$$j_1^*(t^k U_1) = s^{k+1}$$

$$j_1^*(t^k bU_1) = s^{k+1} w$$

$$j_1^*(t^{k+1} b^2 U_1) = s^{k+2} w^2$$

$$j_1^*(t^{k+2} b^3 U_1) = s^{k+3} w^3$$

Proof. These follow by applying j_1^* to the following relations:

$$t^k U_1 = U_1^{k+1}$$

$$t^k bU_1 = U_1^k (bU_1)$$

$$t^{k+1} b^2 U_1 = U_1^k (bU_1)^2$$

$$t^{k+2} b^3 U_1 = U_1^k (bU_1)^3$$

□

Lemma 6.18. *The relation $i_1^*(w) = \pm x_8$ holds.*

Proof. From the cofibration sequence

$$F_4/\mathrm{Spin}(9) \xrightarrow[i_1]{} E_6/S^1 \times_{C_4} Spin(10) \xrightarrow[j_1]{} Th(V_2)$$

We obtain the following exact sequence

$$\begin{array}{ccccc} 0 = H^7(\mathbb{O}P^2) & \longrightarrow & H^8(\mathrm{Th}(V_2)) & \longrightarrow & H^8(\frac{E_6}{S^1 \times_{C_4} \mathrm{Spin}(10)}) \\ & & & & \downarrow \\ & & & & H^8(\mathbb{O}P^2) \qquad \longrightarrow \quad H^9(\mathrm{Th}(V_2)) = 0 \end{array}$$

From Lemma 6.14 we have that $H^8(E_6/S^1 \times_{C_4} Spin(10))$ is generated by s^4, w. From the Thom isomorphism $H^8(Th(V_2))$ is generated by a class $t^3 U_1$ such

that $(i_2^* \circ j_2^*)(t^3U_1) = t^4 \in H^8(F_4/S^1 \times_{C_2} \mathrm{Spin}(7))$ but by lemma 6.14 $(i_2^*)^{-1}(t^4) = s^4$ and thus by exactness $i_1^*(s^4) = 0$ and so $i_1^*(w) = \pm x^8$ as i_1^* surjects.

□

Lemma 6.19. *$H^{16}(E_6/S^1 \times_{C_4} \mathrm{Spin}(10)) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ and is generated by s^8, s^4w and w^2*

Proof. Again we can use the i_1 cofibration sequence to obtain:

$$0 \to H^{16}(Th(V_2)) \underset{j_1^*}{\to} H^{16}(E6/S^1 \times_{C_4} \mathrm{Spin}(10)) \underset{i_1^*}{\to} H^{16}(\mathbb{O}P^2) \to 0$$

Thus as $H^{16}(Th(V_2)) \cong \mathbb{Z} \oplus \mathbb{Z}$ and $H^{16}(F^4/\mathrm{Spin}(9)) \cong \mathbb{Z}$ we have that $H^{16}(E6/S^1 \times_{C_4} \mathrm{Spin}(10)) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$. As $H^{16}(Th(V_2))$ is generated by the 2 classes t^7U_1 and t^3bU_1 we have that $H^{16}(E_6/S^1 \times_{C_4} \mathrm{Spin}(10))$ is generated by $j_1^*(t^7U_1)$, $j_1^*(t^3bU_1)$ and a class z such that $i_1^*(z)$ generates $H^{16}(\mathbb{O}P^2)$.

By Lemma 6.17 we have that $j_1^*(t^7U_1) = s^8$ and $j_1^*(t^3bU_1) = s^4w$. Finally as by Lemma 6.18 $i_1^*(w) = \pm x_8$ it follows that $i_1^*(w^2) = x_8^2$ and thus we can take $z = w^2$.

□

Lemma 6.20. *$H^{32}(\frac{E_6}{S^1 \times_{c4} \mathrm{Spin}(10)}) \cong \mathbb{Z}$ generated by some orientation class $\tau = 4s^4w^3 - 4s^8w^2 + s^{12}w$.*

Proof. From the spectral sequence of the $Gr_2^+(\mathbb{R}^9)$ fibration over $\mathbb{O}P^2$ we have that $H^{30}(\frac{F_4}{S^1 \times_{C_2} \mathrm{Spin}(7)}) \cong \mathbb{Z}$ and is generated by $y_8^2bt^3$. Eliminating y_8 this gives the top class of $\frac{F_4}{S^1 \times_{c2} \mathrm{Spin}(7)}$ as $4t^3b^3 - 4t^7b^2 + c^{11}b$. Thus $H^{32}(Th(V_2)) \cong \mathbb{Z}$ and is generated by the class $4t^3B^3U_1 - 4t^7b^2U_1 + t^{11}bU_1$.

As $H^*(\mathbb{O}P^2) = 0$ in dimension greater than 16 we have that $j_1^* : H^{32}(Th(V_2)) \to H^{32}(E_6/S^1 \times_{C_4} Spin(10))$ is an isomorphism and by Lemma 6.17 the result follows.

□

Theorem 6.21. *$H^{32}(E_6/S^1 \times_{C_4} Spin(10))$ is generated by s and w.*

Proof. We have shown that this holds in dimensions less than or equal to the middle dimension. Further as $H^*(\mathbb{O}P^2) = 0$ in dimension greater than 16 we have that $j_1^* : H^k(Th(V_2)) \to H^k(E_6/S^1 \times_{C_4} Spin(10))$ is an isomorphism for $k > 16$. As $H^k(Th(V_2))$ is additivly generated by classes of the form $t^pb^qU_1$ the result follows by Lemma 6.17.

□

It now remains to find relations between these classes.

Lemma 6.22. $s^9 = 3sw^2$

Proof. As $t^8 = 3b^2$ in Lemma 6.9 we have that $t^8U_1 = 3b^2U_1$ and hence by Lemma 6.17 $s^9 = 3w^2s$. □

We next attempt to find a relation in dimension 24 to do this we first make some notes about the 24-dimensional cohomology groups of some related spaces.

Lemma 6.23. *$H^{24}(F_4/S^1 \times_{C_2} Spin(7))$ is spanned by the classes t^{12}, bt^8, b^2t^4 and b^3 with relations:*

- $3b^3 - bt^8$
- $3b^2t^4 - t^{12}$
- $b^3 + 15b^2t^4 - 9bt^8$

It is isomorphic to $\mathbb{Z}$ with a generator given by the element $19b^2t^4 - 11bt^8$.

$H^{24}(Th(V_2))$ is spanned by the classes $t^{11}U_1, bt^7U_1$ and $b^2t^3U_1$ with the single relation $3b^2t^3U_1 - t^{11}U_1$. It is isomorphic to $\mathbb{Z}\oplus\mathbb{Z}$ with a basis given by the elements bt^7U_1 and $b^2t^3U_1$.

Proof. This follows directly from the definition of the cohomology ring in terms of generators and relations given in Theorem 6.11 specialised to dimension 22 and 24 along with the Thom isomorphism.

□

Lemma 6.24. *Under the map $i_2^* \circ j_1^* : H^{24}(Th(V_2)) \to H^{24}(F_4/S^1 \times_{C_2} Spin(7))$ we have that:*

- $i_2^* \circ j_1^*(bt^7U_1) = -45[19b^2t^4 - 11bt^8]$
- $i_2^* \circ j_1^*(b^2t^3U_1) = -26[19b^2t^4 - 11bt^8]$

Thus $26bt^7U_1 - 45b^2t^3U_1$ generates the kernel of this map.

Proof.

$$\begin{aligned} i_2^* \circ j_1^*(bt^7U_1) &= bt^8 \\ &= -19[3b^3 - b^2t^4] \\ &+ 57[b^3 + 15b^2t^4 - 9bt^8] \\ &- 45[19b^2t^4 - 11bt^8] \\ &= -45[19b^2t^4 - 11bt^8] \end{aligned}$$

$$
\begin{aligned}
i_2^* \circ j_1^*(b^2t^3U_1) &= b^2t^4 \\
&= -11[3b^3 - b^2t^4] \\
&+ 33[b^3 + 15b^2t^4 - 9bt^8] \\
&- 26[19b^2t^4 - 11bt^8] \\
&= -26[19b^2t^4 - 11bt^8]
\end{aligned}
$$

As both $3b^3 - b^2t^4$ and $b^3 + 15b^2t^4 - 9bt^8$ are 0 in $H^{24}(F_4/S^1 \times_{C_2} Spin(7))$ by Lemma 6.23

□

Lemma 6.25. *$H^{24}(Th(T\mathbb{OP}2))$ is isomorphic to $\mathbb{Z}$ generated by x_8U_2 and $j_2^*(x_8U_2) = 3w^3 - ws^8$.*

Proof. The structure of $H^{24}(Th(T\mathbb{OP}2))$ follows directly from the cohomology of $\mathbb{OP}2$ given in Theorem 6.2 and the Thom isomorphism. We next show that $j_2^*(U_2) = 3w^2 - s^8$ the result will then follow from the Thom isomorphism.

First we note that by exactness $j_2^*(U_2)$ is in the kernel of i_2^*. In dimension 16 this kernel in isomorphic to $\mathbb{Z}$ and generated by $3w^2 - s^8$ which is the pull back of the 16-dimensional relation in the presentation of $H^*(F_4/S^1 \times_{C_2} Spin(7))$ given in Theorem 6.11. We thus conclude that $j_2^*(U_2) = n[3w^2 - s^8]$ for some $n \in \mathbb{Z}$. Finally from the Thom isomorphism we have that

$$3nx_8 = i_1^*(n[3w^2 - s^8]) = i_1^* \circ j_2^*(U_2) = e(T\mathbb{OP}2) = 3x_8$$

And hence $n = 1$.

□

Lemma 6.26. $w^3 = 9ws^8 - 15w^2s^4$.

Proof. First as $i_2^*(w^3 + 15w^2s^4 - 9ws^8) = b^3 + 15bt^4 - 9bt^8$ which is a relation in $H^{24}(F_4/S^1 \times_{C_2} Spin(7))$ by Lemma 6.23 we have that $w^3 + 15w^2s^4 - 9ws^8 \in Ker i_2^*$ hence by exactness $w^3 + 15w^2s^4 - 9ws^8 \in Im j_2^*$. From Lemma 6.25 this image is generated by $3w^3 - ws^8$ and so we have that for some $n \in \mathbb{Z}$ we have that $w^3 + 15w^2s^4 - 9ws^8 = n[3w^3 - ws^8]$.

It remains to show that $n = 0$ which will complete the proof. As j_1^* is an isomorphism in dimension 24, as $H^{24}(\mathbb{OP}2) = 0$, the kernel of i_2^* is the image of the

kernel of $i_2^* \circ j_1^*$. Thus the we have by Lemma 6.24 that:

$$j_1^*(26bt^7U_1 - 45b^2t^3U_1) = 26ws^8 - 45w^2s^4$$

generates the kernel of i_2^* and hence the image of j_2^*. As the image of j_2^* is also generated by $3w^3 - ws^8$ this implies that:

$$26ws^8 - 45w^2s^4 = \pm 3w^3 - ws^8$$

But we have the following:

$$\begin{aligned} 3n[3w^3 - ws^8] &= 3[w^3 + 15w^2s^4 - 9ws^8] \\ &= 3w^3 + 45w^2s^4 - 27ws^8 \\ &= [3w^3 - ws^8] + [26ws^8 - 45w^2s^4] \\ &= [3w^3 - ws^8] \pm [3w^3 - ws^8] \end{aligned}$$

As $3w^3 - ws^8$ generates the image of j_1^* which is non-zero and torsion-free we thus have that $3n \in \{0, 2\}$ and hence that $n = 0$.

□

We can now compete the proof of the main theorem of this section:

Proof. We have shown s, w generate the cohomology and both $3w^2s = s^9, w^3 = 9ws^8 - 15w^2s^4$ hold. It only remains to show that there are no further relations needed. We define $r_1 := 3w^2 - s^9$ and $r_2 := w^3 - 9ws^8 + 15w^2s^4$ for notational convienience.

We show that for any $i \in \mathbb{N}$ the group $H^i(E_6/S^1 \times_{C_4} Spin(10))$ and the i-dimensional part of $\mathbb{Z}[s, w]/ < r_1, r_2 >$ are the same which will show that no further relations are needed. We note that as $H^(E_6/S^1 \times_{C_4} Spin(10))$ and $\mathbb{Z}[s, w]/ < r_1, r_2 >$ have only even dimensional generators we can restrict to the case where i is even. We note that as $H^i(E_6/S^1 \times_{C_4} Spin(10)) \cong H^i(F_4/S^1 \times_{C_2} Spin(7))$ which is torision free in dimension less than the middle dimension and $H^{16}(E_6/S^1 \times_{C_4} Spin(10))$ is torision free by Lemma 6.19 Poincarre duality shows that the whole ring $H^*(E_6/S^1 \times_{C_4} Spin(10))$ is torision free.

We proceed by case analysis on i to show the equality holds in all cases:

$i < 16$ In dimension less than 16 we have that, by Lemma 6.14, $H^i(E_6/S^1 \times_{C_4} Spin(10))$ is isomorphic to the i-dimensional part of $\mathbb{Z}[s, w]$ which is isomorphic to the i-dimensional part of $\mathbb{Z}[s, w]/ < r_1, r_2 >$ as both r_1 and r_2 have

dimension greater than 16.

$i = 16$ By Lemma 6.19 $H^16(E_6/S^1 \times_{C_4} Spin(10))$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ which is isomorphic to the 16-dimensional part of $\mathbb{Z}[s,w]/ < r_1, r_2 >$ as both r_1 and r_2 have dimension greater than 16.

$i \in \{18, 20, 22\}$ By Poincarre duality we have that $H^i(E_6/S^1 \times_{C_4} Spin(10)) \cong H^{32-i}(E_6/S^1 \times_{C_4} Spin(10))$ which is isomorphic to the $32 - i$-dimensional part of $\mathbb{Z}[s,w]$ as $32 - i < 16$. In dimensions 10, 12 and 14 $\mathbb{Z}[s,w]$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$ and hence so is $H^i(E_6/S^1 \times_{C_4} Spin(10))$.

Let $i = 18 + k$ with $k \in \{0, 2, 4\}$ then the i-dimensional part of $\mathbb{Z}[s,w]/ < r_1, r_2 >$ is additivly spanned by the elements $s^{9+k}, s^{5+k}w$ and $s^{1+k}w^2$ with a relation given by $s^k r_1$. As $s^k r_1$ is can be extended to a basis with the elements $s^{5+k}w$ and $s^{1+k}w^2$ the quotient group is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$ and thus equality holds.

$i = 24$ By Poincarre duality we have that $H^{24}(E_6/S^1 \times_{C_4} Spin(10)) \cong H^8(E_6/S^1 \times_{C_4} Spin(10))$ which is isomorphic to the 8-dimensional part of $\mathbb{Z}[s,w]$ as $8 < 16$ this is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$ and therefore so is $H^{24}(E_6/S^1 \times_{C_4} Spin(10))$.

The 24-dimensional part of $\mathbb{Z}[s,w]/ < r_1, r_2 >$ is additivly spanned by the elements s^12, s^8w, s^4w^2 and w^3 with relations given by s^3r_1 and r_2. As s^3r_1 and r_2 are can be extended to a basis with s^8w and s^4w^2 we have that the quotient is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$ and thus equality holds.

$i \in \{26, 28, 30\}$ By Poincarre duality we have that $H^i(E_6/S^1 \times_{C_4} Spin(10)) \cong H^{32-i}(E_6/S^1 \times_{C_4} Spin(10))$ which is isomorphic to the $32 - i$-dimensional part of $\mathbb{Z}[s,w]$ as $32 - i < 16$. In dimensions 2, 4 and 6 $\mathbb{Z}[s,w]$ is isomorphic to $\mathbb{Z}$ and hence so is $H^i(E_6/S^1 \times_{C_4} Spin(10))$.

Let $i = 26 + k$ with $k \in \{0, 2, 4\}$ then the i-dimensional part of $\mathbb{Z}[s,w]/ < r_1, r_2 >$ is additivly spanned by the elements $s^{13+k}, s^{9+k}w, s^{5+k}w^2$ and $s^{1+k}w^3$ with relations given by $s^{k+4}r_1, s^k w r_1$ and $s^{1+k}r_2$. As the relations can be extended to a basis with the element $19w^2s^{5+k} - 11ws^{9+k}$ the quotient group is isomorphic to $\mathbb{Z}$ and thus equality holds.

$i = 32$ As $E_6/S^1 \times_{C_4} Spin(10)$ is a 32-dimensional manifold we have that $H^{32}(E_6/S^1 \times_{C_4} Spin(10)) = \mathbb{Z}$.

The 32-dimensional part of $\mathbb{Z}[s,w]/ < r_1, r_2 >$ is additivly spanned by the elements $s^{16}, s^{12}w, s^8w^2, s^4w^3$ and w^4 with relations given by s^7r_1, s^3wr_1, s^4r_2

and wr_2. As the relations are can be extended to a basis with the element $19s^8w^2 - 11s^{12}w$ the quotient group is isomorphic to $\mathbb{Z}$ and thus equality holds.

$i = 34$ As $E_6/S^1 \times_{C_4} Spin(10)$ is a 32-dimensional manifold we have that $H^{34}(E_6/S^1 \times_{C_4} Spin(10)) = 0$.

The 34-dimensional part of $\mathbb{Z}[s, w]/ < r_1, r_2 >$ is additivly spanned by the elements $s^{17}, s^{13}w, s^9w^2, s^5w^3$ and sw^4 with relations given by $s^8r_1, s^4wr_1, w^2r_1, s^5r_2$ and swr_2. As the relations are form a basis the quotient group is isomorphic to 0 and thus equality holds.

$i > 34$ As $E_6/S^1 \times_{C_4} Spin(10)$ is a 32-dimensional manifold we have that $H^i(E_6/S^1 \times_{C_4} Spin(10)) = 0$.

Take the sum of all the i-dimensional parts of $\mathbb{Z}[s, w]$ where $i > 32$ then this is additivly spanned by the elements s^nw^m for $n, m \in \mathbb{N}$ with $2n + 8m > 32$. Let $b = s^nw^m$ for some $n, m \in \mathbb{N}$ with $2n + 4m > 32$ we will show that $b \in < r_1, r_2 >$ which completes the proof.

We proceed by case analysis on m, first suppose that $m \leq 4$ then as $2n + 8m > 32$ we have that $2n \geq 34 - 8m$ and so $n \geq 16 - 4m$ in particular $b = s^{n-16+4m}(s^{16-4m}w^m)$ but $s^{16-4m}w^m$ is 34 dimensional and previously we have shown that this group has rank 0 and so $s^{16-4m}w^m \in < r_1, r_2 >$ hence $b \in < r_1, r_2 >$.

Next suppose that $m > 4$ then $b = (s^nw^{5-m})w^5$ but we have that:

$$w^5 = w^2r_2 + 9s^8w^3 - 15s^4w^4 = w^2r_2 + s^3(9s^5w^3 - 15sw^4)$$

but $9s^5w^3 - 15sw^4$ is 34 dimensional and hence $9s^5w^3 - 15sw^4 \in < r_1, r_2 >$ thus $w^5 \in < r_1, r_2 >$ and finally $b \in < r_1, r_2 >$.

These all match proving the theorem. □

Chapter 7

The decomposition of some spheres associated to representations

In this section we will study some decompositions of spheres associated to representations. Given a real m-dimensional representation of some Lie group G, say $\phi : G \to \mathrm{SO}(m)$ then the group $\phi(G) \subseteq \mathrm{SO}(m)$ acts on the sphere $S^{m-1} \cong \mathrm{SO}(m)/\mathrm{SO}(m-1)$. It is these decompositions that we will study in this section. In particular we will investigate the conditions for such a respresentation to be transitive or codimension one.

This has a relation to more general decompositions as if G acts on a manifold M given as an orbit G/H then G acts with codimension k if and only if H acts with codimension $k-1$ on the sphere by the representation of H which induces the normal bundle. In particular for codimension one actions such as in Chapter 5 we have a transitive action on the sphere in the normal representation and for codimension two actions such as in Chapter 9 we have a codimension one action on the sphere in the normal representation.

7.1 Irreducibility

We first recall some standard lemmas of irreducible real real representations and note some corollaries for representations to be either transitive or codimension 1. Let G be a semi simple Lie group then by definition there is a map $\pi : \prod_{i=0} G_i \to G$ whose kernel is a finite group and such that each G_i is simple. Then any representation of G can be decomposed into the sum of irreducible representations and we have the

following standard Lemmas see for example [Bröcker and Dieck, 1985, Chapter II,1]

Lemma 7.1. *Let $\Delta : G \to \mathrm{SO}(m)$ be a real irreducible representation then either:*

1. *Δ is a complex irreducible representation*
2. *There exists a complex irreducible representation Γ such that $\Delta = \Gamma_{\mathbb{R}}$.*

Lemma 7.2. *If $\Delta : G \to \mathrm{SU}(m)$ is a complex irreducible representation then there are complex irreducible respresentations $\Delta_i : G_i \to \mathrm{SU}(m_i)$ such that we have $\pi^*(\Delta) = \otimes_{i=0}^{k} \Delta_i$*

The following theorem allows us to related the reducibility of a representation to codimension of the associated action:

Theorem 7.3. *For a Lie group G with 2 representations $\Delta : G \to \mathrm{SO}(n)$ and $\Gamma : G \to \mathrm{SO}(m)$ and suppose, G acts with codimension k on S^{n-1} via Δ and with codimension l on S^{m-1} via Γ. Then G acts on S^{n+m-1} via the representation $\Delta \oplus \Gamma$ and the codemension of this action is greater than or equal to $1 + k + l$.*

Proof. Embed $S^{n-1} \times S^{m-1}$ into S^{n+m-1} as $(v, w) \to (\frac{1}{\sqrt{2}}v, \frac{1}{\sqrt{2}}w)$ then the representation $\Delta + \Gamma$ fixes this subspace. The action of G on $S^{n-1} \times S^{m-1}$ factors as $G \to G \times G \to \mathrm{SO}(n) \times \mathrm{SO}(m)$ where the first map is the diagonal.

Consider the action of $G \times G$ on $S^{n-1} \times S^{m-1}$ then the orbits are just the products of the orbits of the G actions on S^{n-1} and S^{m-1}. Therefore the generic orbit is just the product of the generic orbits of G on S^{n-1} and S^{m-1}. Hence the $G \times G$ action on $S^{n-1} \times S^{m-1}$ is codimension $k + l$. As the G action factors through this $G \times G$ action the codimension of the G action on $S^{n-1} \times S^{m-1}$ must be at least $k + l$.

As $S^{n-1} \times S^{m-1}$ is codimension 1 in S^{n+m-1} it follows that the action of G on S^{n+m-1} is at least $k + l + 1$.

□

This immediately gives the following corrolaries

Corollary 7.4. *For a Lie group G and a representation $\Delta : G \to \mathrm{SO}(n)$ if G acts transitively on S^{n-1} via Δ then Δ is irreducible.*

Corollary 7.5. *For a Lie group G and a representation $\Delta : G \to \mathrm{SO}(n)$ if G act on S^{n-1} with codimension 1 via Δ then either:*

1. *Δ is irreducible*
2. *$\Delta = \Delta_1 \oplus \Delta_2$ for 2 irreducible representations Δ_1, Δ_2 such that G acts transitively on the spheres $S(\Delta_1)$ and $S(\Delta_2)$.*

7.2 Tensor products

We now note a few results of actions on spheres coming from the tensor product of 2 representations.

Suppose we have 2 real representations $\Delta_1 : G \to \mathrm{SO}(n)$ and $\Delta_2 : G \to \mathrm{SO}(m)$ then the tensor representation $\Delta_1 \otimes \Delta_2 : G \to \mathrm{SO}(n \times m)$ factors through the representation $\delta_n \otimes \delta_m : \mathrm{SO}(n) \times \mathrm{SO}(m) \to \mathrm{SO}(n \times m)$. Likewise any complex tensor of complex representation factors through the representation $\mu_n \otimes_{\mathbb{C}} \mu_m : U(n) \times U(m) \to U(n \times m)$. We thus study the tensors $\delta_n \otimes \delta_m$ and $\mu_n \otimes_{\mathbb{C}} \mu_m$ first.

Lemma 7.6. *Let $1 \leq n \leq m$ then the action of $\mathrm{SO}(n) \times \mathrm{SO}(m)$ on $S^{n \times m-1}$ is transitive if and only if n is 1 and is codimension 1 if and only if n is 2.*

Proof. We proceed by induction on n. First suppose that $n = 1$ then $\mathrm{SO}(1) = 1$ and the representation $\delta_1 \otimes \delta_m$ is isomorphic to δ_m which is known to be transitive and hence codimension 0.

In general assume the result holds for $n-1$ we show it holds for n. Consider the orbit at $v \in S(\mathbb{R}^n \otimes \mathbb{R}^m)$

$$v = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & \underline{0} & \\ 0 & & & \end{pmatrix}$$

Then this is stabilized by $\mathrm{SO}(n-1) \times \mathrm{SO}(m-1)$ with normal bundle generated by $\delta_{n-1} \otimes \delta_{m-1}$. By induction $\mathrm{SO}(n-1) \times \mathrm{SO}(m-1)$ acts via the representation $\delta_{n-1} \otimes \delta_{m-1}$ with codimension $n-2$ hence $\mathrm{SO}(n) \times \mathrm{SO}(m)$ acts on the sphere bundle in the normal bundle of the orbit at v with codimension $n-2$.

As this subspace is codimension 1 and $\mathrm{SO}(n) \times \mathrm{SO}(m)$ acts with codimension $(n-1)$.

□

Lemma 7.7. *Let $1 \leq n \leq m$ then the action of $U(n) \times U(m)$ on the S^{4nm-1} via $\mu_n \otimes_{\mathbb{C}} \mu_m$ is transitive if and only if n is 1 and is codimension 1 if and only if n is 2.*

Proof. We proceed by induction on n. First suppose that $n = 1$ then $U(1) = S^1$ and the representation $\xi \otimes_{\mathbb{C}} \mu_m$ is transitive as μ_m is and hence the result holds.

In general assume the result holds for $n-1$ we show it holds for n. Consider the orbit at $w \in S(\mathbb{C}^n \otimes \mathbb{C}^m)$

$$w = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & \underline{0} & \\ 0 & & & \end{pmatrix}$$

Then this is stabilized by $U(n-1) \times U(m-1)$ with normal bundle generated by $\mu_{n-1} \otimes_{\mathbb{C}} \mu_{m-1}$. By induction $U(n-1) \times U(m-1)$ acts via the representation $\mu_{n-1} \otimes_{\mathbb{C}} \mu_{m-1}$ with codimension $n-2$ hence $U(n) \times U(m)$ acts on the sphere bundle in the normal bundle of the orbit at w with codimension $n-2$.

As this subspace is codimension 1 the group $U(n) \times U(m)$ acts with codimension $(n-1)$.

□

We next focus more closely on the two codimension 1 decompositors $\delta_2 \otimes \delta_m$ and $\mu_2 \otimes_{\mathbb{C}} \mu_m$

Lemma 7.8. *For $m > 1$ the sphere S^{2m-1} decomposes under the $\delta_2 \otimes \delta_m$ action of $SO(2) \times SO(m)$ as*

$$\frac{SO(2) \times SO(m)}{O(1) \times SO(m-1))} \longleftarrow \frac{SO(2) \times SO(m)}{O(1) \times SO(m-2)} \longrightarrow \frac{SO(2) \times SO(m)}{SO(2) \times SO(m-2)}$$

Where these groups are embedded as:

1. $SO(2) \times SO(m-2) \underset{(Diag,id)}{\rightarrow} SO(2) \times SO(2) \times SO(m-2) \rightarrow SO(2) \times SO(m)$
2. $O(1) \times SO(m-1) \underset{(Diag,id)}{\rightarrow} O(1) \times O(1) \times SO(m-1) \rightarrow SO(2) \times SO(m)$

Proof. We first consider the orbit containing the element $V \in S(\mathbb{R}^2 \otimes \mathbb{R}^m)$ where

$$V = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \end{pmatrix}$$

then the orbit at V is stabilized by $O(1) \times SO(m-1)$ as in the embedding 2. The normal bundle to this orbit is generated by normal bundle to $\delta_1 \otimes \delta_{m-1}$ $O(1) \times SO(m-1)$ acts transitively via the representation $\delta_1 \otimes \delta_{m-1}$ stabilized by $O(1) \times SO(m-2)$.

In particular this implies the generic orbit is codimension one and stabilized by $O(1) \times SO(m-2)$. As S^{2m-1} is simply connected for $(m > 1)$ we can apply Mostart's theorem and it suffices to find the other exceptional orbit.

Consider the orbit at the element $W \in S(\mathbb{R}^2 \otimes \mathbb{R}^m)$ where

$$W = \begin{pmatrix} 1/\sqrt{2} & 0 & \dots & 0 \\ 0 & 1/\sqrt{2} & \dots & 0 \end{pmatrix}$$

Then this orbit is stabilized by $SO(2) \times SO(m-2)$ embedded in $SO(2) \times SO(m)$ as in map (1). From counting the dimension of the stabilizer this is distinct from the orbit of W and the generic orbit this by Mostart this must be the other exceptional orbit completing the proof.

□

Lemma 7.9. *For $m > 1$ the sphere S^{4m-1} decomposes under the $\mu_2 \otimes_{\mathbb{C}} \mu_m$ action of $U(2) \times U(m)$ as:*

$$\frac{U(2) \times U(m)}{U(1) \times U(1) \times U(m-1)} \longleftarrow \frac{U(2) \times U(m)}{U(1) \times U(m-2)} \longrightarrow \frac{U(2) \times U(m)}{U(2) \times U(m-2)}$$

Where we have that

1. $U(2) \times U(m-2) \underset{(Diag,id)}{\rightarrow} U(2) \times U(2) \times U(m-2) \rightarrow U(2) \times U(m)$
2. $U(1) \times U(1) \times U(m-1) \underset{(Diag,id,id)}{\rightarrow} U(1) \times U(1) \times U(1) \times U(m-1) \rightarrow U(2) \times U(m)$

Proof. We first consider the orbit containing the element $V \in S(\mathbb{C}^2 \otimes \mathbb{C}^m)$ where

$$V = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \end{pmatrix}$$

then the orbit at V is stabilized by $U(1) \times U(1) \times U(m-1)$ as in the embedding 2. The normal bundle to this orbit is generated by $\xi \otimes 1 \otimes \mu_{m-1}$ and $U(1) \times U(1) \times U(m-1)$ acts transitively via the representation $\xi \otimes 1 \otimes \mu_{m-1}$ stabilized by $U(1) \times U(1) \times U(m-2)$. In particular this implies the generic orbit is codimension one and stabilized by $U(1) \times U(1) \times U(m-2)$. As S^{4m-1} is simply connected we can apply Mostart's theorem and it suffices to find the other exceptional orbit.

Consider the orbit at the element $W \in S(\mathbb{C}^2 \otimes \mathbb{C}^m)$ where

$$W = \begin{pmatrix} 1/\sqrt{2} & 0 & \dots & 0 \\ 0 & 1/\sqrt{2} & \dots & 0 \end{pmatrix}$$

then this orbit is stabilized by $U(2) \times U(m-2)$ embedded in $U(2) \times U(m)$ as in map (1). From the fundamental group of the stabilizer this is distinct from the orbit of W and the generic orbit thus by Mostart this must be the other exceptional orbit completing the proof.

□

We now move to the more general case. We first generalise the results on transitive actions.

Lemma 7.10. *Given a compact, connected Lie group G and 2 real representations Δ_1 and Δ_2 then $\Delta_1 \otimes \Delta_2$ acts transitively only if at least one of Δ_1 or Δ_2 is trivial.*

Proof. As $\Delta_1 \otimes \Delta_2$ factors as $(\delta_{\dim \Delta_1} \otimes \delta_{\dim \Delta_2}) \circ (\Delta_1, \Delta_2)$ we have that $\Delta_1 \otimes \Delta_2$ is only transitive if $\delta_{\dim \Delta_1} \otimes \delta_{\dim \Delta_2}$ is by 7.6 this implies that $\dim \Delta_1$ or $\dim \Delta_2$ is 1 but then Δ_1 or Δ_2 is trivial by connectedness.

The converse is obvious. □

Lemma 7.11. *Given a compact Lie group G and 2 complex representations Δ_1 and Δ_2 with $dim\Delta_1 \leq dim\Delta_2$ then G acts transitively via $\Delta_1 \otimes_{\mathbb{C}} \Delta_2$ if and only if either:*

1. *Δ_1 is trivial and G acts transitively via Δ_2*
2. *$G \cong S^1 \times_{C_k} H$ for some H such that $\Delta_1 \cong \xi$ and H acts transitively via $\Delta_2|_H$*

Proof. As $\Delta_1 \otimes_{\mathbb{C}} \Delta_2$ factors as $(\mu_{dim\Delta_1} \otimes \mu_{dim\Delta_2}) \circ (\Delta_1, \Delta_2)$ we have that $\mu_{dim\Delta_1} \otimes_{\mathbb{C}} \mu_{dim\Delta_2}$ must be transitive and hence by 7.7 that $dim\Delta_1 = 1$. Either Δ_1 is trivial or G has a non-trivial 1 dimensional complex representation.

Suppose Δ_1 is trivial then $\Delta_1 \otimes_{\mathbb{C}} \Delta_2$ is isomorphic to Δ_2 and hence $\Delta_1 \otimes \Delta_2$ is transitive if and only if Δ_2 is.

Suppose Δ_1 is a non-trivial representation then as S^1 is the only simple group with a non-trivial 1 dimensional complex representation and G is semi simple it must split as $S^1 \times_K H$ with $\Delta_1 \cong \xi^m$ and K a finite group.

As S^1 is cyclic we must have that $K \cong C_k$ for some $k \in \mathbb{N}$

Finally as ξ commutes with any complex representation we have that the action of Δ_2 is isomorphic to the action of $\xi^m \otimes \Delta_2$ for some $m \in \mathbb{Z}$ hence as $\Delta_2|_H \cong \xi^n \otimes \Delta_2$ we have that $\Delta_1 \otimes_{\mathbb{C}} \Delta_2$ is transitive if and only if $\Delta_2|_H$ is.

□

We next generalise when tensor products have codimension one actions.

Lemma 7.12. *Given a connected Lie groups G and 2 real representations Δ_1 and Δ_2 with $dim\Delta_1 \leq dim\Delta_2$ then G acts with codimension one on $S(\Delta_1 \otimes \Delta_2)$ if and only if one of the following hold:*

1. *Δ_1 is trivial and G acts with codimension 1 on the sphere $S(\Delta_2)$*

2. *There exists a Lie group H and $k \in N$ such that $G \cong \mathrm{SO}(2) \times_{C_k} H$, $\Delta_1 = \delta_2$ on $\mathrm{SO}(2)$ and H acts transitively on the sphere $S(\Delta_2|_H)$ stabilized by some $K \subseteq H$ such that $\Delta_2|_k = 1 + \Delta_K$ where K acts transitively on the sphere $S(\Delta_K)$*

Proof. As $\Delta_1 \otimes \Delta_2$ factors through $\delta_{\dim \Delta_1} \otimes \delta_{\dim \Delta_2}$ we must have that $\delta_{\dim \Delta_1} \otimes \delta_{\dim \Delta_2}$ must give an action of $\mathrm{SO}(\dim \Delta_1) \times \mathrm{SO}(\dim \Delta_2)$ whose codimension is at most 1. By 7.6 we must therefore have that $\dim \Delta_1 \in \{1, 2\}$.

First suppose $\dim \Delta_1 = 1$ thus Δ_1 must be trivial and $\Delta_1 \otimes \Delta_2 \cong \Delta_2$ and $\Delta_1 \otimes \Delta_2$ acts with codimension 1 if and only if Δ_2 does.

Next suppose that $\dim \Delta_1 = 2$ as Δ_1 is irreducible it must be non-trivial. As the only simple group with a 2 dimensional irreducible real representation is $\mathrm{SO}(2)$ and δ_2 we must have that the Cartan type of G contains an $\mathrm{SO}(2)$ and Δ_1 is generated by δ_2. Let $G = \mathrm{SO}(2) \times_{C_k} H$ be such a decomposition.

We next show that $\Delta_2|_H$ is transitive on $S(\Delta_2|_H)$.

First suppose that Δ_2 is not transitive on $S(\Delta_2)$ then G cannot act transitively on the orbit passing through the element V where

$$V = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \end{pmatrix}$$

As $\mathrm{SO}(2) \times \mathrm{SO}(\dim \Delta_2)$ acts transitively on this orbit and with codimension one on the whole space G must act transively on this orbit. This orbit is isomorphic to $S^1 \times S^{\dim \Delta_2 - 1}$ as the $\mathrm{SO}(2)$ acts freely on the S^1 part of this G acting transitively imples that the restriction of the action to H is transitive on $S^{dim \Delta_2 - 1}$ as required.

Let K be the stabilizer of the transitive H action on $S^{\dim \Delta_2 - 1}$. The stabilizer of the G action at this orbit is K as the $\mathrm{SO}(2)$ action is free. As this action is codimension one the normal bundle must be generated by some representation Δ_K of K such that K acts transitively on $S(\Delta_K)$ it remains to show that $\Delta_2|_K = 1 + \Delta_K$. This follows as by transitivity Δ_2 and Δ_K are the pull backs of $\delta_{\dim \Delta_2}$ and $\delta_{\dim \Delta_2 - 1}$ to H and K respectively.

□

Lemma 7.13. *Given a Lie group G and 2 complex representations Δ_1 and Δ_2 with $dim \Delta_1 \subseteq dim \Delta_2$ then G acts with codimension one on $S(\Delta_1 \otimes_{\mathbb{C}} \Delta_2)$ if and only if one of the following hold.*

1. *Δ_1 is trivial and G acts with codimension 1 on $S(\Delta_2)$*

2. *There exists a Lie group H and kEN such that $G \cong S^1 \times_{C_2} H$, $\Delta_1 = \mu_1$ and H acts transitively on $S(\Delta_2|_H)$.*

3. *There exists a Lie group H and $k \in N$ such that $G \cong \mathrm{SU}(2) \times_{C_k} H$, $\Delta_1 = \mu_2$ and H acts transitively on $S(\Delta_2|_H)$ stabilized by some$K \in H$ such that $\Delta_2|_K = 1 + \Delta_K$ for some representation Δ_K such that K acts transitively on $S(\Delta_K)$.*

Proof. Same as previous with minor changes for complex numbers

□

7.3 Semisimple Groups

We now extend the previous results to the case of representations of semisimple groups in terms of the irreducible representations of simple groups. The irreducible representations of compact simple Lie groups with transitive actions has been shown by Simons [1962] to be the same as the Berger classification f Holonomy groups [Berger, 1955, Theorem 3]. A more direct reference to the classification can be found in [Besse, 1987, 10.94, pg 301]

Theorem 7.14. *If G is a simple Lie group and Δ an irreducible representation such that G acts transitively then one of the following cases hold.*

1. *G is of Cartan type $A(n)$ and Δ is the complex representation μ_n*

2. *G is of Cartan type $B(n)$ for some n and Δ is the real representation δ_{2n+1}*

3. *G is of Cartan type $C(n)$ for some n and Δ is the quatmionic h_n*

4. *G is of Cartan type $D(n)$ for some n and Δ is the real representation δ_{2n}*

5. *G is of Cartan type $G(2)$ and Δ if the 8-dimensional real irreducible representation*

6. *G is of Cartan type $B(n)$ for $n \leq 4$ and Δ is the spin representation Δ_{2n+1}*

7. *G is of Cartan type $D(n)$ for $n \leq 4$ and Δ is the spin representation Δ_{2n}*

Theorem 7.15. *Let G be a semi simple group and Δ a representation then G acts transitively on $S(\Delta)$ if and only if one of the following holds*

1. *There exists a map $\pi : H \times K \to G$ with finite kernel such that H is simple and Δ pulls back under π to a real irreducible representation Δ_H of H such that H acts transitively on $S(\Delta_H)$.*

2. *There exists a map* $\pi : (S^1)^k \times K \to G$ *with finite kernel such that* H *is simple and* Δ *pulls back under* π *to a complex representation* $\xi^{l_1} \otimes \cdots \otimes \xi^{l_r} \otimes \Delta_H$ *for an irreducible complex representation* Δ_H *of* H *such that* H *acts transitively on* $S(\Delta_H)$

Proof. By Corollary 7.4 the representation must be irreducible using Lemma 7.1 on form of the irreducible representations as tensors and the Lemma 7.10 and Lemma 7.11 on the real and complex tensors that can induce transitive actions. □

We proceed to the following partial result about semi simple groups which act via representations with cohomogenaity one.

Theorem 7.16. *Let* G *be a semi-simple group and* Δ *a representation of* G *such that* G *acts with codimension 1 on* $S(\Delta)$ *then one of the following holds*

1. *There exist* Δ_1 *and* Δ_2 *real irreducible representations of* G *such that* G *acts transitively on* $S(\Delta_1)$ *and* $S(\Delta_2)$ *and that* $\Delta \cong \Delta_1 + \Delta_2$.

2. *There exists a map* $\pi : S^1 \times H \times K \to G$ *with finite kernel such that* H *is simple and* Δ *pulls back under* π *to* $\delta_2 \otimes_{\mathbb{R}} \Delta_K$ *for some real irreducible representation of* Δ_K *of* K *such that* K *acts transitively on* $S(\Delta_K)$

3. *There exists a map* $\pi : (S^1)^k \times \mathrm{SU}(2) \times H \times K \to G$ *with finite kernel such that* H *is simple and* Δ *pulls back under* π *to* $\xi^{l_1} \otimes \cdots \otimes \xi^{l_k} \otimes \Delta_H$ *where* Δ_k *is a complex irreducible representation of* K *and* K *acts transitively on* $S(\Delta_K)$

4. *There exists a map* $\pi : H \times K \to G$ *with finite kernel such that* H *is simple and* Δ *pulls back under* π *to* Δ_K *such that* Δ *is a irreducible representation of* K *and* K *acts with codimension 1 on* $S(\Delta_K)$

5. *There exists a map* $\pi : (S^1)^k \times K \to G$ *with finite kernel such that* H *is simple and* Δ *pulls back under* π *to* $\xi^{l_1} \otimes \cdots \xi^{l_r} \otimes \Delta_K$ *for a complex irreducible representation* Δ_K *of* K *where* K *acts with codimension 1 on* $S(\Delta_K)$.

Proof. By the corrollary 7.4 the representation must be irreducible or the sum of 2 irreducible representations For case 1 assume it is the sum of 2 irreducible representations. Otherwise use Lemma 7.1 on form of the irreducible representations as tensors and the Lemma 7.12 and Lemma 7.13 on the tensors that can be codimension 1. □

Chapter 8

Weyl Group computations

In this section we give a description of the Weyl group of E_8 in terms of the Weyl group of the maximal subgroup Spin(16). This will allow us to compute the double quotients $W(E_6 \times S^1) \setminus W(E_7)/W(S^3 \times \mathrm{Spin}(12))$ and $W(E_7 \times S^3) \setminus W(E_8)/W(\mathrm{Spin}(16))$ which we will need in Chapter 9 these eunmerate various orbit types of the double quotient $E_6 \times S^1 \setminus E_7/S^3 \times \mathrm{Spin}(12)$. We show that in both cases the double quotients have 2 distinct classes and we give representative elements of these.

We perform these computations by giving the structure of $W(E_8)$ with respect to the subgroup $W(\mathrm{Spin}(16))$ to give a description of the cosets $W(E_8)/W(\mathrm{Spin}(16))$. While theoretically these can be computed by computer as $|W(E_8)| = 696729600$ both the computational packages GAP and van Leeuwen et al. [1992] fail to compute them within reasonable levels of resources.

8.1 Weyl group of E_8

Definition 8.1. *Let $\Pi^i(n)$ be the subset of the group $\{1, -1\}^n$ whose elements contain (-1) exactly i times, and let $\Pi(n)$ be the sub group of $\{1, -1\}^n$ where $\Pi(n) = \coprod_{i=0}^{i=n/2} \Pi^{2i}(n)$*

Then $S(n)$ acts on $\Pi(n)$ by permutation of coordinates and with this action we have the following theorem from [Bröcker and Dieck, 1985, Theorem 3.6,pg. 171]

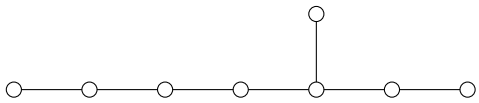
Theorem 8.2.

$$W(\mathrm{Spin}(2n)) \cong \Pi(n) \rtimes S(n)$$

Where the element $(\psi, \rho) \in \Pi(n) \rtimes S(n)$ acts on the maximal torus $T^n \subset \mathrm{Spin}(2n)$ by the composition of the natural actions of the ψ and ρ on the standard basis.

For this section we use the root system for the $\mathfrak{e}_8$ in [Adams, 1996, pg. 56] generated by the following simple roots:

1. $x_2 - x_3$
2. $x_3 - x_4$
3. $x_4 - x_5$
4. $x_5 - x_6$
5. $x_6 - x_7$
6. $x_7 - x_8$
7. $x_7 + x_8$
8. $\frac{1}{2}(x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8)$



E_8

If we take the subroot system generated by the roots $(1-7)$ and the root $(x_1 - x_2)$ we obtain a root system of type $D(8)$ giving an embedding of $\mathrm{Spin}(16) \hookrightarrow E_8$ and $W(\mathrm{Spin}(16)) \subset W(E_8)$ it is this subgroup of $W(\mathrm{Spin}(16))$ in $W(E_8)$ we will use in the rest of this section. It gives the standard representation of $W(\mathrm{Spin}(16))$ on the maximal torus in $\mathrm{Spin}(16)$

Further for the rest of this section we will refer to the Weyl element generated by the reflection in the root 8 by s. Then s will act on the coroot space by the matrix

$$S_m = \frac{1}{4}\begin{pmatrix} 3 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 & -1 & -1 & -1 & -1 \\ -1 & -1 & 3 & -1 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & 3 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 & 3 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 & -1 & 3 & -1 & -1 \\ -1 & -1 & -1 & -1 & -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & 3 \end{pmatrix}$$

Lemma 8.3. *Let $\rho \in S(8) \subset W(\mathrm{Spin}(16)) \subset W(E_8)$ then*

$$s \circ \rho = \rho \circ s$$

Proof. Let $\mathbb{1}_8$ be the 8×8 matrix consisting of only 1's and ψ the representation of ρ then

$$\begin{aligned}
\psi S_m &= \psi(Id - \frac{1}{4}\mathbb{1}_8) \\
&= \psi - \frac{1}{4}\mathbb{1}_8 \\
&= Id\psi - \frac{1}{4}\mathbb{1}_8\psi \\
&= S_m\psi
\end{aligned}$$

From linearity and the fact that $\mathbb{1}_8$ is invariant under symmetry action.

□

Lemma 8.4. *Let $x = (x_1, \cdots, x_8) \in \Pi^2(8) \subset W(\mathrm{Spin}(16)) \subset W(E_8)$. As $x \in \Pi^2(8)$ there exists unique $i < j$ with $x_i = x_j = -1$ let ρ_x be the 2-cycle in $S(8) \subset W(\mathrm{Spin}(16))$ which exchanges i and j then:*

$$sxs = xsx\rho_x$$

Proof. First assume $i = 1, j = 2$ so $x = (-1, -1, 1, 1, 1, 1, 1, 1)$ if $D(x)$ is the mtirx corresponding to x then direct computation shows that:

$$S_m D(x) S_m D(x) S_m D(x)$$

is the matrix corresponding to ρ which proves in this lemma in this case. In general let p be a permutation exchanging 1 with i and 2 with j then $pxp^{-1} = (-1, -1, 1, 1, 1, 1, 1, 1)^T$ and the result above shows that

$$spxp^{-1}s = pxp^{-1}spxp^{-1}spxp^{-1}p\rho_x p^{-1}$$

as both p and p^{-1} commute with s by Lemma 8.3 the result follows.

□

Lemma 8.5. *For $x \in \Pi^8(8)$ we have $sx = xs$.*

Proof. As $x \in \Pi^8(8)$ we have that $x = (-1, -1 \cdots, -1)$ so it acts on $\mathbb{R}^8$ as multiplication by (-1), as S_m acts linearly the result follows.

□

Finally we will need the following result found in [Adams, 1996, Theorem 10.1]

Lemma 8.6.

$$|W(E_8)| = 696729600$$

Definition 8.7. *There is an embedding of* $\Pi^i(n) \hookrightarrow \Pi^i(n+1)$ *by taking* $(x_1, \cdots, x_n) \mapsto (x_1, \cdots, x_n, 1)$ *denote this by* $\Pi^i_+(n) \subset \Pi^i(n)$.

This gives rise to a subgroup $\Pi_+(n) = \coprod_{i=0}^{n/2} \Pi^{2i}_+(n) \subset \Pi(n)$

Definition 8.8. *Given* $x \in \Pi^i_+(n)$ *where* $i > 0$ *and* $n > 1$. *Let* j *be* $max\{j \in \{2, \cdots, n\} \mid x_j = -1\}$.

Then we define $\mathrm{tw}x \in \Pi^2(n)$ *to be such that* $(\mathrm{tw}x)_n = -1, (\mathrm{tw}x)_j = -1$

This allows us to state the main theorem of this section:

Theorem 8.9. *For any* $\omega \in W(E_8)$ *exactly one of the following hold:*

1. $\omega \in W(\mathrm{Spin}(16))$,
2. $\omega = xsa$ *for some* $x \in \Pi_+(8)$ *and* $a \in W(\mathrm{Spin}(16))$,
3. $\omega = (\mathrm{tw}x)^i sxsa$ *for* $x \in \Pi^4_+(8)$ *and* $a \in W(\mathrm{Spin}(16))$ *with* $i \in \{0, 1\}$.

And this data uniquely defines ω.

Proof. We will first show that at most one of $(1) - (3)$ can hold. We will then show uniqueness and finally we will use a counting argument to show that all $w \in W(E_8)$ have such a form. To do this we will use the following proposition:

□

Proposition 8.10. *Let* $w \in W(\mathrm{Spin}(16))$ *with matrix* W_m *and* M *an* 8×8*-matrix with coefficients in a subset* $S \subset \mathbb{R}$ *which is closed under negation then:*

1. $W_m M$ *and* MW_m *are matrices with coefficients in* $S \subset \mathbb{R}$,
2. $\det(M) = \det(W_m M) = \det(MW_m)$.

Proof. Let $M = \{\alpha_{i,j}\}_{i,j=1}^8$ and $w = (x, \rho) \in W(\mathrm{Spin}(16))$ then for $1 \leq i, j \leq 8$ we have that:

$$(W_m M)_{i,j} = x_{\rho(j)} \alpha_{i,\rho(j)}$$

$$(MW_m)_{i,j} = x_i \alpha_{\rho(i),j}$$

But $x_{\rho(j)}\alpha_{i,\rho(j)} = \pm\alpha_{i,\rho(j)} \subset S$ and $x_i\alpha_{\rho(i),j} = \pm\alpha_{\rho(i),j} \subset S$ due to the fact that S is closed under negation.

Finally we have that $\det(W_m) = \det(\rho W_m \rho^{-1}) = \det(\mathrm{Diag}(x, \cdots, x_n))$ and so $\det(W_m) = \coprod_{i=1}^{n} x_i = 1$ as $x \in \Pi(8)$.

□

Continuing the proof of Theorem 8.9, let W_m be the action of w on the coroot space with standard basis. First suppose that case 1 holds and $w \in W(\mathrm{Spin}(16))$ then, by the proposition as $W_m = W_m \cdot Id_8$ and Id_8 has coefficient in $S_1 := \{-1, 0, 1\}$ and thus W_m has coefficients in S_1

Next suppose that case 2 holds and thus $w = xsa$ for some $x \in \Pi_+(8)$ and $a \in W(\mathrm{Spin}(16))$.

As S_m, the matrix associated to s, has coefficients in $S_2 := \{-\frac{3}{4}, -\frac{1}{4}, \frac{1}{4}, \frac{3}{4}\}$ and $a, x \in W(\mathrm{Spin}(16))$ by the proposition W_m has coefficients in S_2.

Now suppose that case 3 holds and $w = (\mathrm{tw}x)^i sxsa$ for some $x \in \Pi_+^4(8)$, $a \in W(\mathrm{Spin}(16))$ and $i \in \{0, 1\}$.

Further let $i < j < k < l$ be such that $x_i = x_j = x_k = x_l = 1$ which exist by definition as $x \in \Pi_+^4(8)$ and let ρ be the permutation given by cycles $(1, i)(2, j)(3, k)(4, l)$ then $\rho x \rho^{-1} = (1, 1, 1, 1, -1, -1, -1, -1) =: \alpha \in \Pi_+^4(8)$.

Now $s\alpha s$ acts on the coroot space via the matrix:

$$S_m \cdot Diag(1, 1, 1, 1, -1, -1, -1, -1) \cdot S_m$$

Which direct computation show to be equal to the matrix:

$$\frac{1}{2}\begin{pmatrix} 1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 1 & -1 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & -1 \end{pmatrix}$$

So $S\alpha S$ has a matrix form with all the coefficients in the set $S_3 := \{-\frac{1}{2}, 0, \frac{1}{2}\}$ As we

have

$$\begin{aligned} w &= (\mathrm{tw}x)^i sxsa \\ &= (\mathrm{tw}x^i)s\rho x\rho^{-1}sa \\ &= ((\mathrm{tw}x)^i\rho)s\alpha s(\rho^{-1}a) \end{aligned}$$

and both $\rho^{-1}a$ and $(\mathrm{tw}x)^i\rho$ are in $W(\mathrm{Spin}(16))$, we have from the proposition that W_m has coefficients in S_3.

As w is invertible we must have that $W_m \neq 0_8$ and thus it can have coefficients in at most one of the sets S_1, S_2 or S_3 and thus at most one of the cases $(1), (2)$ or (3) may hold.

We next show that in each case the data uniquely classifies w. In case (1) this is trivial. Then we show uniqueness in case 2, suppose case 2 holds let $x, y \in \Pi_+(8)$ and $a, b \in W(\mathrm{Spin}(16))$ such that

$$xsa = ysb$$

We want to show that $x = y$ and $a = b$. As $s^{-1} = s$, we have that $sy^{-1}xs = a^{-1}b$. Further as $y^{-1}x \in \Pi_+(8)$ there exists $i \in \{0, 1, 2, 3\}$ such that $y^{-1}x \in \Pi_+^{2i}(8)$ we proceed by case analysis on i.

First, suppose that $i = 0$ and $y^{-1}x \in \Pi_+^0(8)$ then as $\Pi_+^0(8) = \{Id\}$ we have that $x = y$ and thus $sa = sb$ therefore $a = b$.

Next, suppose that $i = 1$ and $y^{-1}x \in \Pi_+^2(8)$ then by lemma 8.4 there exists $\rho_x \in S(8)$ such that $a^{-1}b = sy^{-1}xs = (y^{-1}x)s(y^{-1}x\rho_x)$ so $a^{-1}b$ is of type 2 but as $ab^{-1} \in W(\mathrm{Spin}(16))$ we obtain a contradiction.

Next suppose that $i = 2$ and $y^{-1}x \in \Pi_+^4(8)$ then $ab^{-1} = sy^{-1}xs$ is of type 3 but as $ab^{-1} \in W(\mathrm{Spin}(16))$ we obtain a contradiction

Finally suppose that $y^{-1}x \in \Pi_+^6(8)$ and let (-1) be the unique element of $\Pi^8(8)$ then $(-1)y^{-1}x \in \Pi^2(8)$ and so

$$\begin{aligned} (-1)a^{-1}b &= (-1)sy^{-1}xs \\ &= s(-1)y^{-1}xs \\ &= (-1)y^{-1}xs(-1)y^{-1}x \end{aligned}$$

Thus $a^{-1}b = (-1)y^{-1}xs(-1)y^{-1}x$ is of type 2 but $ab^{-1} \in W(\mathrm{Spin}(16))$ and we again obtain a contradiction. This proves uniqueness in case 2.

Finally we show uniqueness in case 3, suppose that case 3 holds and we have

$i, j, \in \{0, 1\}$, $x, y, \in \Pi^4_+(8)$ and $a, b, \in W(\mathrm{Spin}(16))$ such that:

$$(\mathrm{tw}x)^i sxsa = (\mathrm{tw}y)^j sysb$$

We want to show that $i = j$, $x = y$ and $a = b$. We proceed in this by case analysis on i and j. First take $i = j = 0$ then $sxsa = sysb$ and so $xsa = ysb$ but both these elements are of type 2 thus by uniqueness in case 2 we have that $x = y$ and $a = b$.

Next suppose $i \neq j$ without loss of generality we can assume $i = 0$ and $j = 1$, so that $sxsa = (\mathrm{tw}y)sysb$ and thus we have that:

$$\begin{aligned} xsa &= s(\mathrm{tw}y)sysb \\ &= (\mathrm{tw}y)s(\mathrm{tw}y)\rho_{\mathrm{tw}y}ysb \\ &= (\mathrm{tw}y)sysb \end{aligned}$$

But $(\mathrm{tw}y)sysb$ is of type 3 as opposed to xsa which is of type 2 which gives a contradiction.

Finally in the case where $i = 1$ and $j = 1$ we have $(\mathrm{tw}x)sxsa = (\mathrm{tw}y)sysb$. If $\mathrm{tw}x = \mathrm{tw}y$ then $sxsa = sysb$ and by the case where $i = 0, j = 0$ we have $x = y$ and $a = b$. On the other hand if $\mathrm{tw}x \neq \mathrm{tw}y$ then $\nu = (\mathrm{tw}x)(\mathrm{tw}y) \in \Pi^2_+(8)$.

By assumption $(\mathrm{tw}x)sxsa = (\mathrm{tw}y)sysb$ thus as $(\mathrm{tw}x)^{-1} = \mathrm{tw}x$

$$\begin{aligned} xsa &= ((\mathrm{tw}x)s)^{-1}(\mathrm{tw}y)sysb \\ &= s(\mathrm{tw}x)(\mathrm{tw}y)sysb \\ &= s\nu sysb \\ &= \nu s\nu\rho_\nu ysb \\ &= \nu sys\rho_\nu b \\ &= (\mathrm{tw}x)(\mathrm{tw}y)sys(\rho_\nu b) \end{aligned}$$

And so $(\mathrm{tw}x)xsa = (\mathrm{tw}y)sys(\rho_\nu b)$ but $(\mathrm{tw}x)xsa$ of type 2 and $(\mathrm{tw}y)sys(\rho_\nu b)$ is of type 3 which is a contradiction.

This shows that if w is of type $(1) - (3)$ then it is uniquely determined by the data.

We finally show that the elements of type $(1) - (3)$ constitute all the elements

of $W(\mathrm{Spin}(16))$ by simple counting.

First there are $2^7 \times 8! = 5160960$ elements of $W(\mathrm{Spin}(16))$ and so 5160960 there are elements of type 1. Next, we have that

$$|\Pi_+(8)| = 1 + \binom{7}{2} + \binom{7}{4} + \binom{7}{6} = 64$$

and so there are $64 \times 5160960 = 330301440$ elements of type 2. Finally as $|\Pi_+^4(8)| = \binom{7}{4} = 35$ there are $2 \times 35 \times 5160960 = 361267200$ elements of type 3.

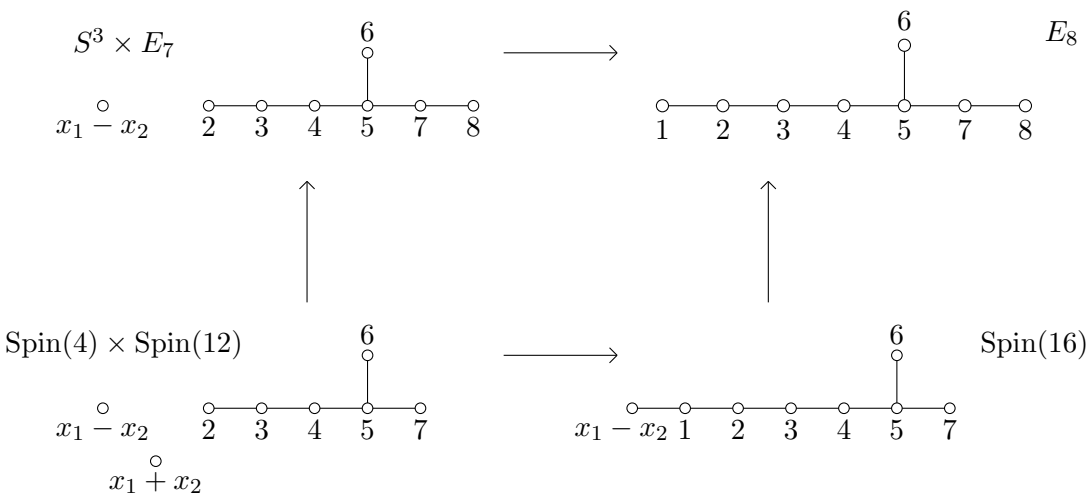
Thus in total there are $5160960 + 330301440 + 361267200 = 696729600$ unique elements of types 1-3 as the total number of elements is 696729600 from Theorem 8.6 we see that all of the elements must be of some type.

8.2 Weyl group of $E_7 \times S^3$

We now consider the subgroup $E_7 \times S^3 \subset E_8$

This embedding is induced from the sub-root system consisting of the roots $(2) - (8)$ and the root $(x_1 - x_2)$

The intersection of $E_7 \times S^3$ and $\mathrm{Spin}(16)$ within E_8 is given by the sub root system consisting of the roots $(2)-(7)$ along with the roots (x_1-x_2) and (x_1+x_2) this is of type $D(2)D(6)$ and corresponds to the natural subgroup $\mathrm{Spin}(4) \times_{C_2} \mathrm{Spin}(12) \hookrightarrow \mathrm{Spin}(16)$ as in the following diagram:



We wish to describe $W(S^3 \times E_7)$ in terms of $W(\mathrm{Spin}(4) \times \mathrm{Spin}(12))$ in the same way as our description of $W(E_8)$ in terms of $W(\mathrm{Spin}(16))$. To do this we first investigate the map $W(\mathrm{Spin}(4) \times \mathrm{Spin}(12)) \hookrightarrow W(\mathrm{Spin}(16))$

Definition 8.11. *The map* $\pi : \{-1,1\}^n \times \{-1,1\}^m \hookrightarrow \{-1,1\}^{n+m}$ *descends to give an embedding*

$$\pi : \Pi(n) \times \Pi(m) \hookrightarrow \Pi(n+m)$$

We will use $\Pi(n,m), \Pi_+(n,m)$ *and* $\Pi^i_+(n,m)$ *respectively to denote the intersection of the image of* π *with* $\Pi(n+m), \Pi_+(n+m)$ *and* $\Pi^i_+(n+m)$ *respectively.*

As in the case in theorem 8.2 by following the map of tori we have the following theorem.

Theorem 8.12.

$$\begin{aligned} W(\mathrm{Spin}(2n) \times \mathrm{Spin}(2m)) &\cong \Pi(n,m) \rtimes S(n) \times S(m) \\ &\subset \Pi(n+m) \rtimes S(n+m) \\ &\cong W(\mathrm{Spin}(2n+2m)). \end{aligned}$$

This allows us to state the following theorem:

Theorem 8.13. *For any* $w \in W(S^3 \times E_7)$ *exactly one of the following hold:*

1. $w \in W(\mathrm{Spin}(4) \times \mathrm{Spin}(12))$.
2. $w = xsa$ *for* $x \in \Pi_+(2,6)$ *and* $a \in W(\mathrm{Spin}(4) \times \mathrm{Spin}(12))$.
3. $w = (\mathrm{tw}x)^i sxsa$ *for* $x \in \Pi^4_+(2,6)$ *and* $a \in W(\mathrm{Spin}(4) \times \mathrm{Spin}(12))$ *with* $i \in \{0,1\}$.

And this data uniquely defines w.

Proof. As $W(\mathrm{Spin}(4) \times \mathrm{Spin}(12))$ and hence $\Pi_+(2,6)$ along with s are contained in $W(E_7 \times S^3)$ any element of type $(1)-(2)$ must also be an element of $W(S^3 \times E_7)$. Now as any element $x \in \Pi^4_+(2,6) \subset \{1,-1\}^8$ has exactly 4 coefficients with value -1 we must have that the maximum index of such a coefficient must be greater than 2.

This implies that $\mathrm{tw}x \in \Pi^2_+(2,6) \subset \Pi^2_+ + (8)$ and so any element of type (3) must also be an element of $W(S^3 \times E_7)$.

From the proof of Theorem 8.9 we see that any such w is uniquely defined by the data and it remains to show that all $w \in W(S^3 \times E_7)$ have such an expression.

We use a simple counting argument as we have shown all such elements of type $(1)-(3)$ must be elements of $W(S^3 \times E_7)$. Firstly as:

$$
\begin{aligned}
|W(\mathrm{Spin}(4) \times \mathrm{Spin}(12))| &= |W(\mathrm{Spin}(4))| \times |W(\mathrm{Spin}(12)| \\
&= 4 \times 23040 \\
&= 92160
\end{aligned}
$$

So those are 92160 elements of type 1. Next we have that:

$$
\begin{aligned}
|\Pi_4(2,6)| &= |\Pi(2)| \times |\Pi_+(6)| \\
&= 2 \times \left(\binom{5}{0} + \binom{5}{2} + \binom{5}{4} \right) = 2 \times 16 = 32
\end{aligned}
$$

and thus there are $32 * 92160 = 2949120$ elements of type 2. Finally as:

$$
\begin{aligned}
|\Pi_+^4(2,6)| &= |\Pi^0(2)| \times |\Pi_+^4(6)| + |\Pi^2(2)| \times |\Pi_+^2(6)| && (8.1) \\
&= 1 \times \binom{5}{4} + 1 \times \binom{5}{2} && (8.2) \\
&= 15 && (8.3)
\end{aligned}
$$

Thus there are $2 \times 15 \times 92160 = 2764800$ elements of type 3.

This implies there are $92160 + 2949120 + 2764800 = 5806080$ elements of types $(1) - (3)$ but as $|W(S^3 \times E_7)| = 5806080$ Adams [1996] and each of these elements is distinct, all the elements of $W(S^3 \times E_7)$ must be of types $(1) - (3)$.

□

8.3 Computation of $W(E_7 \times S^3) \setminus W(E_8)/W(\mathrm{Spin}(16))$

We now use these results to compute the required double quotients. First we note an additional lemma on the structure of $\Pi(n+m)$.

Lemma 8.14. *For $m > 1$ let $x \in \Pi_+^2(n+m)$ be such that $x_1 = -1$ and $x_{n+1} = -1$ then*

1. $\Pi(n+m) = \Pi(n,m) \coprod x\Pi(n,m)$
2. $\Pi_+(n,m) = \Pi_+(n,m) \coprod x\Pi_+(n,m)$

Proof. Let $a \in \Pi(n+m)$, if a has an even number of (-1)'s for indexes between 1 and n then it also has an even number of (-1)'s between $n+1$ and m hence

$a \in \Pi(n,m)$. Otherwise the number of (-1)'s between 1 and n and $n+1$ and m are odd and hence $xa \in \Pi(n,m)$ thus $a \in x\Pi(n,m)$.

For (2) we have that $\Pi_+(n+m) = \Pi(n+(m-1))$ and $\Pi_+(n,m) = \Pi(n,m-1)$ and the result follows from (1).

□

We can now prove the main theorem of this section.

Theorem 8.15. *Let x be the element of $\Pi^2_+(8)$ such that $x_1 = -1, x_3 = -1$ then the double quotient then $W(E_7 \times S^3) \setminus W(E_8)/W(\mathrm{Spin}(16))$ has two equivalence classes with representative elements $1 \in W(E_8)$ and $xs \in W(E_8)$.*

Proof. We show that for every element $w \in W(E_8)$ there exist $a \in W(S^3 \times E_7)$ and $b \in W(\mathrm{Spin}(16))$ such that either $w = ab$ or $w = axsb$ so $w \sim 1$ or $w \sim xs$ respectively.

Suppose $w \in W(\mathrm{Spin}(16))$ is of type 1 then $w = 1w$ thus $w \sim 1$. Next suppose $w = \lambda s\mu$ is of type 2 with $\mu \in W(\mathrm{Spin}(16))$ and $\lambda \in \Pi_+(8)$. By Lemma 8.14 either $\lambda \in \Pi_+(2,6)$ or $\lambda x \in \Pi_+(2,6)$ first suppose $\lambda \in \Pi_+(2,6)$ then as both λ and s are in $W(S^3 \times E_7)$ so is λs and hence $w = (\lambda s)\mu$, which implies $w \sim 1$.

Otherwise assume $\lambda x \in \Pi_+(2,6)$ then we have that $w = (\lambda x)(xs)\mu$ and as $\lambda x \in W(S^3 \times E_7)$ we have that $w \sim xs$.

Finally suppose that $w = (\mathrm{tw}\lambda)s\lambda s\mu$ for some $\mu in W(\mathrm{Spin}(16))$ and $\lambda \in \Pi^4_+(8)$ then by Lemma 8.14 either $\lambda \in \Pi^4_+(2,6)$ or $x\lambda \in \Pi_+(2,6)$. Suppose $\lambda \in \Pi^4_+(2,6)$ as $w\lambda, \lambda, s \in W(S_3 \times E_7)$ we have that $w = ((\mathrm{tw}\lambda)s\lambda s)\mu$ and so $w \sim 1$.

Otherwise $x\lambda \in \Pi_+(2,6)$ then as $\mathrm{tw}\lambda, \lambda x, \in W(S_3 \times E_7)$ we have that $w = (\mathrm{tw}\lambda s\lambda x)xs\mu$ and so $w \sim xs$.

It remains to show $1 \not\sim xs$. Suppose that this were the case, then there would exist $a \in W(S^3 \times E_7), b \in W(\mathrm{Spin}(16))$ such that $a = xsb$ which implies by Theorem 8.9 that a is a type 2 element of $W(E_8)$ and thus a type 2 element of $W(S^3 \times E_7)$ but this is a contradiction as $x \notin \Pi^2_+(2,6)$ and Theorem 8.13 has uniqueness of the representations.

□

8.4 Computation of $W(E_6 \times S^1) \setminus W(E_7)/W(S^3 \times \mathrm{Spin}(16))$

Given the maximal torus of E_7 and the sub root system generated by roots $(3)-(8)$ gives an embedding of $S^1 \times E_6 \hookrightarrow E_7$ found in [Adams, 1996, Chapter 8].

Thus the roots $(3)-(8)$ and $(x_1 - x_2)$ with the maximum torus give an embedding of $S^3 \times S^1 \times E_6 \hookrightarrow S^3 \times E_7 \subset E_8$. The intersection of $\mathrm{Spin}(4) \times_{C_2} \mathrm{Spin}(12)$

with $S^3 \times S^1 \times E_6$ is $S^3 \times S^1 \times S^1 \times \mathrm{Spin}(10)$ where $S^3 \times S^1 \times S^1 \times \mathrm{Spin}(10) \hookrightarrow \mathrm{Spin}(4) \times \mathrm{Spin}(12)$ is the map such that $S^1 \times \mathrm{Spin}(10) \hookrightarrow \mathrm{Spin}(12)$ is the natural map of $\mathrm{Spin}(2) \times \mathrm{Spin}(10) \hookrightarrow \mathrm{Spin}(12$ and the map $S^3 \times S^1 \hookrightarrow \mathrm{Spin}(4)$ is the inclusion of $S^3 \times S^1 \hookrightarrow S^3 \times S^3 \cong \mathrm{Spin}(4)$.

This gives a map $S^3 \times S^1 \times S^1 \times \mathrm{Spin}(10) \hookrightarrow S^3 \times E_7$, where $S^1 \times S^1 \times \mathrm{Spin}(10) \hookrightarrow S^1 \times E_6 \hookrightarrow E_7$ and $S^1 \times \mathrm{Spin}(10) \hookrightarrow E_6$ is the embedding in Chapter 5

For Chapter 9 we want to compute $W(S^1 \times E_6) \setminus W(E_7)/W(S^3 \times \mathrm{Spin}(12))$ which is isomorphic to the set $W(S^3 \times S^1 \times E_6) \setminus W(S^3 \times E_7)/W(\mathrm{Spin}(4) \times \mathrm{Spin}(12))$. We can describe $W(S^3 \times S^1 \times E_6)$ in terms of $W(S^3 \times S^1 \times S^1 \times \mathrm{Spin}(10))$ analogously to theorem 8.9.

We note that $\Pi_+^k(2,n) = \{x \in \Pi_+^k(2+n) | x_1 = x_2\}$ in this direction we get the following definition.

Definition 8.16. $\Pi_+^k(a;n) = \{x \in \Pi_+^k(a+n) \mid \ x_1 = \cdots = x_a\}$ *and* $\Pi_+(a;n) = \{x \in \Pi_+^k(a+n) \mid \ x_1 = \cdots = x_a\}$

Then the embedding of $S^3 \times S^1 \times S^1 \times \mathrm{Spin}(10) \hookrightarrow \mathrm{Spin}(6) \times \mathrm{Spin}(10) \hookrightarrow \mathrm{Spin}(16)$ gives an embedding of $\mathbb{Z}_2 \times \Pi_+(5) \subset W(S^3 \times S^1 \times S^1 \times \mathrm{Spin}(10))$ into $\Pi_+(8) \subset W(\mathrm{Spin}(16))$.

This allows us to state our theorem for this section.

Theorem 8.17. *Let $w \in W(S^3 \times S^1 \times E_6)$ then exactly one of the following hold:*

1. $w \in W(S^3 \times S^1 \times S^1 \times \mathrm{Spin}(10))$.
2. $w = xsa$ *for* $x \in \Pi_+(3;5)$ *and* $a \in W(S^3 \times S^1 \times S^1 \times \mathrm{Spin}(10))$.
3. $w = (\mathrm{tw}x)^i sxsa$ *for* $a \in W(S^3 \times S^1 \times S^1 \times \mathrm{Spin}(10))$ *and* $i \in \{0,1\}$.

And this data uniquely determines w.

Proof. As shown $\Pi_+(3;5) \subset W(S^3 \times S^1 \times E_6)$ we also note that for any $x \in \Pi_+^4(3;5)twx \in \Pi_+^2(3;5)$ as there must be both 1 in x with indices in the range $4-8$. As $S \in W(S^3 \times S^1 \times E_6)$ this implies all elements of type $(1)-(3)$ are elements of $W(S^3 \times S^1 \times E_6)$. Further as all these elements are distinct as elements of $W(E_8)$, they are distinct elements of $W(S^3 \times S^1 \times E^6)$.

We show that all $w \in W(S^3 \times S^1 \times E_6)$ have such a form again via a counting argument.

There are $|W(S^3 \times S^1 \times S^1 \times \mathrm{Spin}(10))| = |W(S^3)| \times |W(\mathrm{Spin}(10))| = 2 \times 2^4 \times 5! = 3840$ elements of type 1.

As $|\Pi_+(3;5)| = 2\times|\Pi_+(5)| = 2\times(1+6+1) = 16$ there are $16\times 3840 = 61440$ elements of type 2.

Finally as $|\Pi^4_+(3;5)| = |\Pi^4_+(5)| + |\Pi^1_+(5)| = 1+4 = 5$ thus there are $2\times 5\times 3840 = 38400$ elements of types $(1)-(3)$ as from Adams [1996] we have that $|W(E_6)| = 51840$ we thus have $|W(S^3\times S^1\times E_6)| = 103680$ so all elements are of types $(1)-(3)$.

□

We give a lemma analogous to Lemma 8.14.

Lemma 8.18. *Let $x\in\Pi^2_+(2,6)$ be such that $x_3=-1, x_4=-1$ then $\Pi^2_+(2,6)$ is the disjoint union $\Pi_+(3,5)\coprod x\Pi(3;5)$*

Proof. Take $a=(a_1,\cdots,a_8)\in\Pi^2_+(2,6)$ then $a_1=a_2$ as $(a_1,a_2)\in\Pi(2)$

Suppose $a_1=a_3$ then $a\in\Pi_+(3;5)$ otherwise if $a_1\neq a_3$ then $(xa)_1=(xa)_3$ and $xa\in\Pi_+(3;5)$. □

We are now in a position to compute the double quotient $W(S^3\times S^1\times E_6)\setminus W(S^3\times E_7)/W(\mathrm{Spin}(4)\times\mathrm{Spin}(12)$

Theorem 8.19. *The set $W(S^3\times S^1\times E_6)\setminus W(S^3\times E_7)/W(\mathrm{Spin}(4)\times\mathrm{Spin}(12))$ has 2 elements with representative elements 1 and xs where $x\in\Pi^2_+(2,6)$ has $x_4=-1, x_5=-1$.*

Proof. We show that for any $w\in W(S^3\times E_6)$ either $w\sim 1$ or $w\sim xs$. Firstly suppose w is of type 1 then $w\in W(\mathrm{Spin}(4)\times\mathrm{Spin}(12))$ thus $w\sim 1$.

Next, suppose w is of type 2 so that $w=\lambda s\mu\lambda\in\Pi^4_+(2,6)$ and $\mu\in W(\mathrm{Spin}(4)\times\mathrm{Spin}(12)$ then by Lemma 8.18, either λ or $x\lambda$ are in $\Pi_+(3;5)\subset w(S^3\times S^1\times E_6)$. If $\lambda\in\Pi_+(3;5)$ then as $s\in W(S^3\times S^1\times E_6)w=(\lambda s)\mu$ thus $w\sim 1$ otherwise if we have that $\lambda x\in\Pi_+(3;5)$ then $w=\lambda s\mu=(\lambda x)xs\mu$ and so $w\sim xs$.

Finally, if w is of type 3 then there exists a λ such that $(\mathrm{tw}\lambda)s\lambda s\mu\lambda\in\Pi^4_+(2,6)$ with $\mu\in W(\mathrm{Spin}(4)\times\mathrm{Spin}(12)$ by then Lemma 8.18, either λ or $x\lambda$ are in $\Pi_+(3;5)$.

Suppose $\lambda\in\Pi_+(3;5)$ then as $\mathrm{tw}\lambda\in\Pi^2_+(3;5)$ we have that $w=((\mathrm{tw}\lambda)s\lambda s)\mu$ and $w\sim 1$. Otherwise if $x\lambda\in\Pi_+(3;5)$ then we have that $w=(\mathrm{tw}\lambda)s\lambda s\mu=((\mathrm{tw}\lambda)s(\lambda x))xs(\mu)$ and we conclude that $w\sim xs$.

It remains to show that $1\not\sim xs$, suppose the contrary, then from the counting argument there exists $a\in W(\mathrm{Spin}(4)\times\mathrm{Spin}(12))$ such that $xsa\in W(S^3\times S^1\times E_6)$. We have xsa is of type 2 in $W(S^3\times E_7)$ and so must also be of type 2 in $W(S^3\times S^1\times E_6)$ but $x\notin\Pi_+(3;5)$, a contradiction.

□

Chapter 9

The decomposition of $E_7/S^3 \times \mathrm{Spin}(12)$

In this section we give partial results on the decomposition of the space $\mathbb{P}^2(\mathbb{H} \otimes \mathbb{O})$ with respect to an action of $S^1 \times E_6$. We will show that one of the orbits of this action is isomorphic to $\mathbb{P}^2(\mathbb{C} \otimes \mathbb{O})$ and another is isomorphic to $\mathbb{X}^2(\mathbb{C} \otimes \mathbb{O})$ which we have discussed in Chapter 5. We describe particular orbits of this action and sketch at the end a method to compute the complete form of the decomposition if time would allow.

We note that unlike the decomposition in Chapter 5 the action is of codimension 2.

9.1 Lie Groups

We first give explicit forms for some of the Lie groups that we will use in this section. We first define $\mathfrak{e}_7$ and 2 Lie subalgebras isomorphic to $\mathbb{R} \oplus \mathfrak{e}_6$ and $\mathfrak{su}(2) \oplus \mathfrak{so}(12)$. These correspond to the natural subalgebras of the algebras $\mathfrak{su}(2) \oplus \mathfrak{e}_7$, $\mathfrak{su}2 \oplus \mathfrak{so}(12) \oplus \mathfrak{e}_6$ and $\mathfrak{so}(4) \oplus \mathfrak{so}(12)$ used in Chapter 8. For clarity we give explicit definitions here.

Definition 9.1. *Following Adams [1996][pg. 56] we define $\mathfrak{e}_7$ to be the Lie-algebra with simple root vectors given by:*

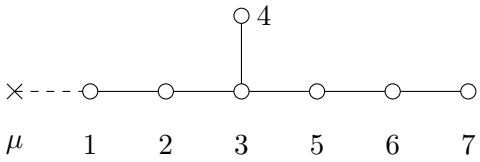
1. $x_2 - x_3$
2. $x_3 - x_4$
3. $x_4 - x_5$
4. $x_5 + x_6$

5. $x_5 - x_6$

6. $\frac{1}{2}(x_1 - x_2 - x_3 - x_4 - x_5 + x_6 - \sqrt{2}x_7)$

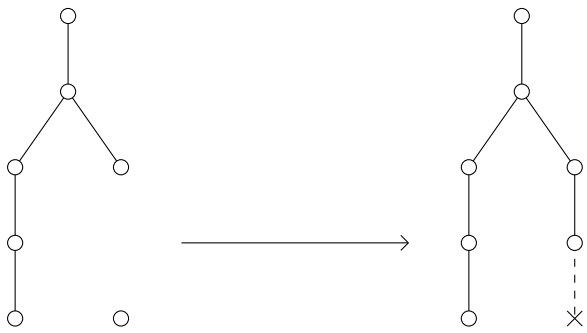
7. $\sqrt{2}x_7$

The highest root of $\mathfrak{e}_7$ is $x_1 - x_2$, with this root we get the extended Dynkin diagram:



E_7

Where we label the highest root as μ. We can form 2 Lie subalgebras of $\mathfrak{e}_7$ by taking sub root systems. First we can take the sub root system associated to the roots $(1, 2, 3, 4, 5, 7)$ and the highest root $x_1 - x_2$. This gives an embedding of $\mathfrak{su}(2) \oplus \mathfrak{so}(12)$ as follows:

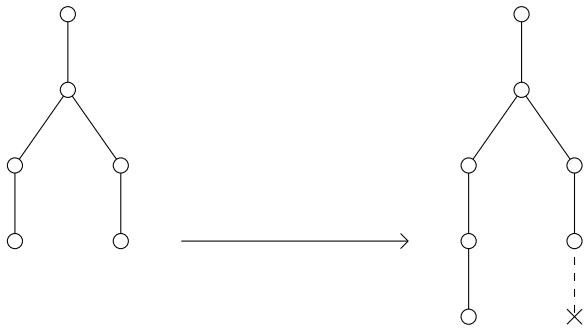


$\mathfrak{su}(2) \oplus \mathfrak{so}(12)$ $\qquad$ $\mathfrak{e}_7$

Where the roots of $\mathfrak{su}(2) \oplus \mathfrak{so}(12)$ are given by:

- $\pm(x_i \pm x_j)$ for $1 \leq i < j \leq 6$
- $\pm\sqrt{2}x_7$

Secondly we can take the subroot system associated to the roots $(1-6)$ along with the maximal torus. This gives an embedding of $\mathbb{R} \oplus \mathfrak{e}_6$ as follows:



$\mathbb{R} \oplus \mathfrak{e}_6$ $\qquad$ $\mathfrak{e}_7$

Where the roots of $\mathbb{R} \oplus \mathfrak{e}_6$ are given by

- $\pm(x_i \pm x_j)$ for $1 \leq i < j \leq 5$
- $\frac{1}{2}(\pm x_1 \pm x_2 \pm x_3 \pm x_4 \pm x_5 \pm (x_6 + \sqrt{2}x_7))$

Where there are an even number of $-$ signs in the second type of roots. The embedding of $\mathbb{R} \oplus \mathfrak{e}_6$ and $\mathfrak{su}(2) \oplus \mathfrak{so}(12)$ in $\mathfrak{e}_7$ induce maps of $S^1 \times E_6$ and $S^3 \times \mathrm{Spin}(12)$ into E_7. The map of $S^1 \times E_6$ is isomorphic to an embedding in Mimura and Toda [1991] and thus is an embedding. The maps of $S^3 \times \mathrm{Spin}(12)$ into E_7 is isomorphic to the embedding in [Adams, 1996, Chapter 8] and thus has a kernel isomorphic to C_2 generated by $(-1, -1) \in S^3 \times \mathrm{Spin}(12)$.

This embedding of $S^3 \times_{C_2} \mathrm{Spin}(12)$ into E_7 gives us the space $E_7/S^3 \times_{C_2} \mathrm{Spin}(12)$ which is $\mathbb{P}^2(\mathbb{H} \otimes \mathbb{O})$ discussed in Chapter 2. The embedding of $S^1 \times E_6$ in E_7 thus gives us an action of $S^1 \times E_6$ on $\mathbb{P}^2(\mathbb{H} \otimes \mathbb{O})$, we will study the orbits of the decomposition of $\mathbb{P}^2(\mathbb{H} \otimes \mathbb{O})$ under this action for the remainder of this section.

We will build up the decomposition of $E_7/S^3 \times \mathrm{Spin}(12)$ by classifiying the orbits with stabilizers of different ranks. In doing this we will use the notion of co-ranks defined below:

Definition 9.2. *Let H and K be Lie subgroups of a Lie Group G and let HgK be an orbit in the double quotient $H \setminus G/K$ then we define its rank to be the rank of the stabilizer $gKg^{-1} \cap H$.*

As $gKg^{-1} \cap H \subseteq G$ the rank of an orbit is at most the rank of G, define the corank of an orbit to be the difference between its rank and the rank of G. We say an orbit is full rank if its corank is 0.

9.2 Full rank orbits

We first consider the full rank orbits of the decomposition we first prove the following theorem to relate this to the computations in Chapter 8. We will relate the full rank orbits of a double quotient of Lie groups to the double quotient of the associated Weyl groups. We first prove a related lemma.

Lemma 9.3. *Let H and K be full rank Lie subgroups of a Lie Group G and suppose $gKg^{-1} \cap H \subseteq G$ is full rank then there exists a $g' \in N(G)$, the normalizer of the maximal torus in G, such that $HgK = Hg'K$.*

Proof. As $gKg^{-1} \cap H$ is full rank it has some maximal torus $T' \subseteq gKg^{-1} \cap H$ but as $gKg^{-1} \cap H \subseteq H$ we have that T' is a maximal torus of H as well and thus there exists $h \in H$ such that $T = hT'h^{-1}$.

Further we have the following:

$$\begin{aligned} g^{-1}T'g \subseteq & g^{-1}(gKg^{-1} \cap H)g \\ \subseteq & K \cap g^{-1}Hg \\ \subseteq & K \end{aligned}$$

and so $g^{-1}T'g$ is a maximal torus of K and thus there exists $k \in K$ with $g^{-1}T'g = kTk^{-1}$. Let $g' = hgk$ then $Hg'K = HgK$ and we have that:

$$\begin{aligned} g'T(g')^{-1} = & (hgk)T(k^{-1}g^{-1}h^{-1}) \\ = & hg(kTk^{-1})g^{-1}h^{-1} \\ = & hg(g^{-1}T'g)g^{-1}h^{-1} \\ = & hT'h^{-1} \\ = & T \end{aligned}$$

And thus $g' \in N(G)$. □

Theorem 9.4. *Let H and K be a full rank Lie subgroups of a Lie group G and let $i : W(H) \setminus W(G)/W(K) \to H \setminus G/K$ take $W(H)gW(K) \mapsto HgK$. Then this map is well defined and surjects onto the full rank orbits in $H \setminus G/K$.*

Proof. We first prove the map is well-defined. Take the map $j : N_T(G) \to H \setminus G/K$ induced from the inclusion $N_T(G) \subseteq G$.

Then j descends to a map $W(G) \to H \setminus G/K$ as $W(G) := N_T(G)/T$ and $T \subseteq K$, therefore $j(w) = j(wt)$ for all $t \in T$. Next we show that j descends to a map $W(H) \setminus W(G)/W(K) \to H \setminus G/K$ we have that:

$$W(H) = (N_T(G) \cap H)/T = W(G) \cap H$$

$$W(K) = (N_T(G) \cap K)/T = W(G) \cap K$$

Thus we have that $j(W(H)) \subseteq H$ and $j(W(K)) \subseteq K$. This implies that $j(hgk) = j(g)$ for all $h \in W(H), k \in W(K)$. This surffices to prove the map is well-defined.

To prove surjectivity it surffices to note that by Lemma 9.3 j is surjective and j factors through i. $\square$

From Theorem 8.19 we know that the set $W(S^1 \times E_6) \setminus W(E_7)/W(S^3 \times \mathrm{Spin}(12))$ has 2 elements and thus the action of $S^1 \times E_6$ on $E_7/S^3 \times \mathrm{Spin}(12)$ has at most 2 full rank orbits. We will next describe these orbits and show that they are distinct. From the second orbit we will deduce the codimension of the action and the type of the generic orbit.

9.2.1 The orbit at one

We first deal with the orbit containing $1 \in E_7$, it is stabilized by the group $S^1 \times E_6 \cap S^3 \times \mathrm{Spin}(12)$ which has Lie algebra $\mathbb{R} \oplus \mathfrak{e}_6 \cap \mathfrak{su}(2) \oplus \mathfrak{so}(10)$. As both $\mathbb{R} \oplus \mathfrak{e}_6$ and $\mathfrak{su}(2) \oplus \mathfrak{so}(10)$ are subroot systems of $\mathfrak{e}_7$, the intersection consists of the maximal torus and the common roots.

Comparing the roots of $\mathbb{R} \oplus \mathfrak{e}_6$ and $\mathfrak{su}(2) \oplus \mathfrak{so}(10)$ we see the common roots are the following:

$$\pm(x_i \pm x_j) \text{ for } 1 \leq i < j \leq 5$$

Thus the stabilizer has Lie algebra of type $\mathbb{R} \oplus \mathbb{R} \oplus \mathfrak{so}(10)$ as the inclusion of the $\mathbb{R} \oplus \mathfrak{so}(10)$ into $\mathfrak{e}_6$ is the same as the one in Chapter 5, we see the stabilizer is thus equal to $S^1 \times S^1 \times_{C_4} \mathrm{Spin}(10)$ and the orbit is thus isomorphic to $\mathbb{P}^2(\mathbb{C} \otimes \mathbb{O})$.

It remains to compute the normal bundle to this orbit. The tangent bundle to $E_7/S^3 \times \mathrm{Spin}(12)$ is generated by the representation bundle $\mu_2 \otimes_{\mathbb{C}} \Delta_{12}$ this has weights given by:

$$\frac{1}{2}(\pm x_1 \pm x_2 \pm x_3 \pm x_4 \pm x_5 \pm x_6 \pm \sqrt{2}x_7)$$

Where there are an even number of $-$ signs in the variables x_1 to x_6. On restriction to $\mathbb{R} \oplus \mathbb{R} \oplus \mathfrak{so}(10)$ these weights naturally spilt into 2 classes dependent on the sign of x_7. On restriction to the first 6 variables these representations are the

same, thus in particular on their restriciton to $S^1 \times \mathrm{Spin}(10) \subset E_6 \subset S^1 \times E_6 \subset E_7$ they are the same and are isomorphic to the representation $\xi \otimes_{\mathbb{C}} \Delta_{10}$ which generates the tangent bundle of $\mathbb{P}^2(\mathbb{C} \otimes \mathbb{O})$. This implies that the tangent and normal bundles at this orbit are isomorphic as real vector bundles.

9.2.2 The other full rank orbit

We now move on the investigate the other possible full rank orbit this orbit exists as all the elements of the Weyl group are contained in E_7. From [Adams, 1996, Chapter 10] we know that $W(E_8) \subset E_8$ thus $W(E_7) \subset E_7$ and hence this orbit contains an element. We will refer to this element as $x \in E_7$. This element acts on the maximal torus as the composition of a reflection in the root $\frac{1}{2}(x_1+x_2+x_3+x_4+x_5+x_6-\sqrt{2}x_7)$ and a change of signs in the 5th and 6th coordinate. This action is equivelent to a reflection in the root $\frac{1}{2}(x_1 + x_2 + x_3 + x_4 - x_5 - x_6 - \sqrt{2}x_7)$

The orbit at x is stabilized by $x(S^3 \times \mathrm{Spin}(12))x^{-1} \cap S^1 \times E_6$ which has a Lie algebra given by $ad_x(\mathfrak{su}(2) \oplus \mathfrak{so}(12)) \cap \mathbb{R} \oplus \mathfrak{e}_6$. As x is in the Weyl group of E_7 and $\mathfrak{su}(2) \oplus \mathfrak{so}(12)$ is a subroot system, $ad_x(\mathfrak{su}(2) \oplus \mathfrak{so}(12))$ is determined by the action of x on the roots. Direct computation gives that the roots of $ad_x(\mathfrak{su}(2) \oplus \mathfrak{so}(12))$ are:

- $\pm(x_i - x_j)$ for $1 \leq i < j \leq 4$ coming from $\pm(x_i - x_j)$,
- $\pm\frac{1}{2}(x_1 + \cdots - x_i + \cdots + x_5 - x_6 - \sqrt{2}x_7)$ for $1 \leq i \leq 4$ coming from $\pm(x_i - x_5)$,
- $\pm\frac{1}{2}(x_1 + \cdots - x_i + \cdots - x_5 + x_6 - \sqrt{2}x_7)$ for $1 \leq i \leq 4$ coming from $\pm(x_i - x_6)$,
- $\pm(x_5 - x_6)$ coming from $\pm(x_5 - x_6)$,
- $\pm\frac{1}{2}(x_1 + \cdots - x_i + \cdots - x_j + \cdots - x_5 - x_6 - \sqrt{2}x_7)$ for $1 \leq i < j \leq 4$ coming from $\pm(x_i - x_j)$,
- $\pm(x_i + x_5)$ for $1 \leq i \leq 4$ coming from $\pm(x_i + x_5)$,
- $\pm(x_i + x_6)$ for $1 \leq i \leq 4$ coming from $\pm(x_i - x_6)$,
- $\pm\frac{1}{2}(x_1 + x_2 + x_3 + x_4 + x_5 + x_6 - \sqrt{2}x_7)$ coming from $\pm(x_5 + x_6)$,
- $\pm(\frac{1}{2}(x_1 + x_2 + x_3 + x_4 - x_5 - x_6 + \sqrt{2}x_7))$ coming from $\pm\sqrt{2}x_7$.

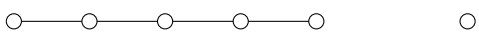
As both $ad_x(\mathfrak{su}(2) \oplus \mathfrak{so}(12))$ and $\mathbb{R} \oplus \mathfrak{e}_6$ are subroot systems, to find there intersection it surffices to take the subroot system consisting of the common roots. This gives that the Lie algebra of the stabilizer is the subroot system generated by the roots:

- $\pm(x_i - x_j)$ for $1 \leq i < j \leq 4$ coming from $\pm(x_i - x_j)$,
- $\pm\frac{1}{2}(x_1 + \cdots - x_i + \cdots + x_5 - x_6 - \sqrt{2}x_7)$ for $1 \leq i \leq 4$ coming from $\pm(x_i - x_5)$,
- $\pm\frac{1}{2}(x_1 + \cdots - x_i + \cdots - x_j + \cdots - x_5 - x_6 - \sqrt{2}x_7)$ for $1 \leq i < j \leq 4$ coming from $\pm(x_i - x_j)$,
- $\pm(x_i + x_5)$ for $1 \leq i \leq 4$ coming from $\pm(x_i + x_5)$.

The simple roots of this system can then be calculated to be the following:

1. $x_4 + x_5$,
2. $x_3 - x_4$,
3. $x_2 - x_3$,
4. $\frac{1}{2}(x_1 - x_2 + x_3 + x_4 - x_5 + x_6 + \sqrt{2}x_7)$,
5. $\frac{1}{2}(x_1 - x_2 - x_3 - x_4 + x_5 - x_6 - \sqrt{2}x_7)$,
6. $\frac{1}{2}(x_1 - x_2 - x_3 - x_4 + x_5 - x_6 - \sqrt{2}x_7)$.

Giving the Dynkin diagram:



And hence the Lie algebra of the stabilizer is isomorphic to $\mathbb{R} \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(6)$. This embeds $\mathfrak{su}(2) \oplus \mathfrak{su}(6)$ into $\mathfrak{e}_6$ as in $\mathbb{X}^2(\mathbb{C} \otimes \mathbb{O})$ and hence this orbit is stabilized by $S^1 \times S^3 \times \mathrm{SU}(6)$. Therefore this orbit is isomorphic to $\mathbb{X}^2(\mathbb{C} \otimes \mathbb{O})$. As this stabilizer is distinct from the stabilizer at the other full rank orbit, we see that these orbits are distinct.

We next compute the normal bundle at this orbit. We first compute its weights, as this orbit is full rank, these are the weights of the representation $x^*(\xi \otimes_{\mathbb{C}} \Delta_{10})$ which do not restrict to roots of $S^1 \times E_6$.

But as $S^3 \times \mathrm{Spin}(12)$ is a subroot system the weights of $x^*(\xi \otimes_{\mathbb{C}} \Delta_{10})$ are just the roots of E_7 which are not roots of $x(S^3 \times \mathrm{Spin}(12))x^{-1}$. So the weights of the normal bundle are given as $ad_x(\mathfrak{su}(2) \oplus \mathfrak{so}(12))^{\perp} \cap \mathbb{R} \oplus \mathfrak{e}_6^{\perp}$ where the perpendicular is taken in $\mathfrak{e}_7$.

The roots of E_7 not in $S^1 \times E_6$ are the elements

- $\pm x_i \pm x_6$ for $1 \leq i \leq 5$,
- $\frac{1}{2}(\pm x_1 \pm x_2 \pm x_3 \pm x_4 \pm x_5 \pm (x_6 - \sqrt{2}x_7))$.

Where elements of the second type have an even number of negative signs in the variables x_1 to x_6. The weights of the normal bundle are thus the elements of the list which are not roots of $ad_x(\mathfrak{su}(2) \oplus \mathfrak{so}(12))$, these are:

- $\pm(x_i - x_6)$ for $1 \leq i \leq 4$,
- $\pm(x_5 + x_6)$,
- $\pm\frac{1}{2}(x_1 + \cdots - x_i + \cdots + x_4 - x_5 + (x_6 - \sqrt{2}x_7))$ for $1 \leq i \leq 4$,
- $\pm\frac{1}{2}(x_1 + x_2 + x_3 + x_4 + x_5 + (x_6 - \sqrt{2}x_7))$.

The highest weight of this representation is $\frac{1}{2}(x_1+x_2+x_3+x_4+x_5+x_6-\sqrt{2}x_7)$ we denote this by ν. Let $w_1, \cdots, w_6$ be the fundamental weights of $\mathbb{R} \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(6)$ corresponding to the simple roots (where w_1 corresponds to the remaining principal weight element of the maximal torus). Then we have the following:

$$\begin{aligned}
\langle w_1, \nu \rangle &= (x_4 + x_5, \nu) &&= 1 \\
\langle w_2, \nu \rangle &= (x_3 - x_4, \nu) &&= 0 \\
\langle w_3, \nu \rangle &= (x_2 - x_3, \nu) &&= 0 \\
\langle w_4, \nu \rangle &= \left(\frac{1}{2}(x_1 - x_2 + x_3 + x_4 - x_5 + x_6 + \sqrt{2}x_7), \nu\right) &&= 0 \\
\langle w_5, \nu \rangle &= \left(\frac{1}{2}(x_1 - x_2 - x_3 - x_4 + x_5 - x_6 - \sqrt{2}x_7), \nu\right) &&= 0 \\
\langle w_6, \nu \rangle &= \left(\frac{1}{2}(x_1 - x_2 - x_3 - x_4 + x_5 - x_6 - \sqrt{2}x_7), \nu\right) &&= 1
\end{aligned}$$

Where $\langle . \rangle$ is the weight inner product and $(.)$ is the inner product in the root space. Thus the representation associated to the complexified normal bundle has an irreducible component isomorphic to $\mu_2 \otimes_{\mathbb{C}} \mu_6$. But the normal bundle is the complexification of a bundle with a complex structure and the real dimension of the representation $\mu_2 \otimes_{\mathbb{C}} \mu_6$ is the same as the real dimension of the normal bundle. Thus we must have that the normal bundle is generated by the representation $\mu_2 \otimes_{\mathbb{C}} \mu_6$.

9.2.3 The generic orbit

This allows us to compute the generic orbit for this action as the generic orbit of the $S^1 \times E_6$ action on the space is the same as the generic orbit of the action of $S^1 \times E_6$ on the sphere bundle in the normal bundle at the other full rank orbit. But the stabilizer of the generic orbit on the sphere bundle in the normal bundle is the same as the stabilizer of the generic orbit of the $S^1 \times \mathrm{SU}(2) \times \mathrm{SU}(6)$ action on S^{23}

via the representation $\mu_2 \otimes_{\mathbb{C}} \mu_6$. But this is shown in Chapter 7 to be stabilized by the group $S^1 \times S^1 \times \mathrm{SU}(4)$.

Thus the generic orbit is stabilized by the group $S^1 \times S^1 \times \mathrm{SU}(4)$. In particular it is corank 2 and codimension 2. This also implies that the action is codimension 2 on the whole space. It also implies that the maximal corank for any orbit of this action is 2.

9.3 Corank one orbits

We now consider what can be said about corank 1 orbits in $H \setminus G/K$ where H and K are full rank in G.

Definition 9.5. *Let G be a semisimple compact Lie group and α a positive root then there is a natural inclusion of* $\mathrm{SU}(2)$ *into G taking the unique positive roots of* $\mathrm{SU}(2)$ *to α. This induces a map $S_\alpha : S^3 \cong \mathrm{SU}(2) \to G$.*

We first prove a lemma relating corank one orbits to full rank orbits.

Lemma 9.6. *Let G be a semisimple compact Lie group and let H, K be 2 full rank subgroups such that there is a common maximal torus $T \subseteq H, K, G$. Suppose HgK is an orbit of $H \setminus G/K$ of corank 1.*

Then there exists an element $g' \in G$, a positive root α of G and an s in the image of S_α such that:

- *$HgK = Hg'K$.*
- *The orbit $H(s.g')K$ is of full rank with maximal torus T.*

Proof. We will first show that there exists a $g' \in G$ with $HgK = Hg'K$ with the property that $g'T(g')^{-1} \cap T$ is corank 1 in T. As the orbit HgK is corank 1 there is a maximal torus T' in $gKg^{-1} \cap H$ with dimension one less than T. As T' is a non maximal torus in H which has T as a maximal torus there exists some (non-unique) $h \in H$ such that $hT'h^{-1} \subseteq T$. Also we have that

$$\begin{aligned} g^{-1}T'g \subseteq & g^{-1}(gKg^{-1} \cap H)g \\ \subseteq & K \cap g^{-1}Hg \\ \subseteq & K \end{aligned}$$

As $g^{-1}T'g$ is a non-maximal torus in K it is contained inside some (non-unique) maximal torus. As T is maximal in K this maximal torus containing $g^{-1}T'g$ must be kTk^{-1} for some $k \in K$. We thus have that $g^{-1}T'g \subseteq kTk^{-1}$.

Let $g' = hgk$ then we have that $Hg'K = HgK$ and further that:

$$\begin{aligned} g'T(g')^{-1} &= hgkTk^{-1}g^{-1}h^{-1} \\ &= h(gkTk^{-1}g^{-1})h^{-1} \\ &\supseteq hT'h^{-1} \end{aligned}$$

But $hT'h^{-1}$ is corank 1 in T thus $g'Tg'^{-1} \cap T$ is at most corank 1. It is not full rank as $g'Tg'^{-1} \cap T \subseteq g'Hg'^{-1} \cap K$ and the orbit at HgK is corank one, thus $g'Tg'^{-1} \cap T$ must be corank 1 in T.

Let $\mathfrak{t}$ be the Lie algebra of T in $\mathfrak{g}$ then as $g'Tg'^{-1} \cap T$ is corank 1 we have that $ad_{g'}(\mathfrak{t}) \cap \mathfrak{t}$ is codimension 1 in $\mathfrak{t}$. From the root space decomposition of $\mathfrak{g}$ we have that $ad_{g'}(\mathfrak{t})/(ad_{g'}(\mathfrak{t}) \cap \mathfrak{t})$ is one dimensional and generated by an element $v = \sum_{\alpha \in \Gamma} c_\alpha \alpha$ where Γ is the set of roots of G and $c_\alpha \in \mathbb{R}$.

We next prove that c_α must be zero if the coroot of α is not in the orthogonal complement of $(ad_{g'}(\mathfrak{t}) \cap \mathfrak{t})$ in $\mathfrak{t}$. Suppose then there exists $t \in \mathfrak{t}$ and a root β such that $\beta(t) \neq 0$ and $c_\beta \neq 0$. But $ad_{g'}(\mathfrak{t})$ is a Cartan subalgebra of $\mathfrak{g}$ thus we must have that $[t, x] = 0$ in particular $< [t, x], \beta >= 0$ but we have that:

$$\begin{aligned} < [t, x], \beta > &= \langle [t, \sum_{\alpha \in \Gamma} c_\alpha \alpha], \beta \rangle \\ &= \sum_{t, \alpha \in \Gamma} \langle [t, c_\alpha \alpha], \beta \rangle \\ &= \sum_{t, \alpha \in \Gamma} \langle c_\alpha < t, \alpha > \alpha, \beta \rangle \\ &= < c_\beta < t, \beta > \beta, \beta > \\ &= c_\beta < t, \beta > \neq 0 \end{aligned}$$

Thus we must have that c_α is non zero only for those roots α whose coroot is in the orthogonal complement of $(ad_{g'}(\mathfrak{t}) \cap \mathfrak{t})$. But this is one dimensional, thus there exists a unique positive root ξ whose coroot generates the orthogonal complement of $(ad_{g'}(\mathfrak{t}) \cap \mathfrak{t})$ and $v \in < \xi, \xi^{-1} >$ where ξ^{-1} is the negative root associated to ξ.

Take $S_\xi : S^3 \to G$ we want to show that there exists an element s, in the image of S_ξ such that $H(s \cdot g')K$ is full rank with maximal torus T. As $g'Tg'^{-1} \cap T$ is corank 1 in T it suffces to find an s such that $sg'Tg'^{-1}s^{-1} = T$ or equivalently that $ad_s(ad_{g'}(\mathfrak{t})) = \mathfrak{t})$. As the coroot of ξ is orthogonal to all the elements of $(ad_{g'}(\mathfrak{t}) \cap \mathfrak{t})$ we must have that for all s in the image of S_ξ the map ad_s restricts to $(ad_{g'}(\mathfrak{t}) \cap \mathfrak{t})$ as the identity.

Finally as S^3 acts transitively on on its Lie algebra via the representation ad

we must also have that the image of S_ξ acts transively via ad on the restriction of $\mathfrak{g}$ to the image of $\mathfrak{su}(2)$ under S_ξ. Thus there exists an element s such that ad_stakes v to the coroot of ξ. This s completes the proof.

□

This lemma simplifies finding corank 1 orbits to finding full rank orbits (see Theorem 9.4) and acting via some element in the image S_α for a positive root α.

As both H and K are full rank we have that the map $S^3 \to H \setminus G/K$ will factor through $S^1 \setminus S^3/S^1$, we will next give a brief description of this space.

Lemma 9.7. $S^1 \setminus S^3/S^1 \cong I$.

Proof. We will show that the S^1 action on $S^3/S^1 \cong S^2$ is codimension one and hence the lemma follows by Mostert's theorem.

But the orbit at the identity is stabilized by $S^1 \cap S^1 = S^1$ and is thus isomorphic to a point. The tangent bundle of $S^2 = \mathrm{Spin}(3)/\mathrm{Spin}(2)$ is generated by the representation δ_2, this restricts to give the normal bundle as the point has trivial tangent bundle. As S^1 acts transitively via the representation δ_2 the space is codimension 1 and the result follows.

□

Let $\psi : [0,1] \to S^3 \cong Sp(1)$ be the map such that:

$$\psi(t) = \cos(\pi t) + \sin(\pi t)j$$

Then ψ descends to an injective map $\psi : [0,1] \to [0,1] \cong S^1 \setminus S^3/S^1$. We show that this map is surjective. The orbit $\psi(0) = 1$ was shown to be one of the exceptional orbits of the S^1 action on S^2, if we show that $\psi(1)$ is the other then, by connectedness the map ψ must be surjective. But the stabilizer of the orbit at $\psi(1) = j$ is a point by the anti-commutivity of j and $\mathbb{C}$ inside $\mathbb{H}$. This shows that ψ descends to give an isomorphism of $S^1 \setminus S^3/S^1$ to $[0,1]$.

We restrict our attention to the Lie groups which have roots $\alpha \neq \beta$ such that $< \alpha, \beta > \in \{-1, 0, 1\}$ as these are the only cases relevant to this thesis. We note the following general lemma from Lie theory.

Lemma 9.8. *If G is a semisimple compact Lie group whose simple decomposition only contains simple algebras of Cartan type of A_n or D_n for any positive n or E_6, E_7 or E_8 then for any positive roots α, β of G with $\alpha \neq \beta$ we have that $< \alpha, \beta > \in \{-1, 0, 1\}$.*

Proof. Directly from the types of bonds in the Dynkin diagram of these groups. □

Definition 9.9. *We denote groups as in Lemma 9.8 as groups of type 1.*

To compute the orbit type of an orbit of the form $x.g$ where x in the image of some S_α it is necessary to compute (ad_x).

Lemma 9.10. *Let G be a compact semisimple Lie group of type 1 and let α, β be 2 distinct positive roots of G and x be in the image of $S\alpha$ then the following hold:*

1. *If $< \alpha, \beta >= 0$ then $ad_x\beta = \beta$.*
2. *If $< \alpha, \beta >= -1$ then $ad_x\beta \in < \beta, \beta \pm \alpha >$ where $ad_x\beta = \pm\beta$ if and only if $\psi(x) = 0$ and $ad_x\beta \in < \beta + \alpha >$ if and only if $\psi(S\alpha^{-1}x) = 1$.*
3. *If $< \alpha, \beta >= 1$ then $ad_x\beta \in < \beta, \alpha \pm \beta >$ with $ad_x\beta = \pm\beta$ if and only if $\psi(x) = 0$ and $ad_x\beta \in < \beta - \alpha >$ if and only if $\psi(S\alpha^{-1}x) = 1$.*

Proof. In the first case consider the embedding then consider the embedding $S_\alpha \times S_\beta$ then x is in the image of this and the pullback of the restriction of ad_x at β to $S_\alpha \times S_\beta$ is the identity proving the result.

In the second case consider the embedding ϕ^* of SU(3) into G induced by the map $\phi : \mathfrak{su}(3) \to \mathfrak{g}$ in which the roots $x_1 - x_2$ and $x_2 - x_3$ of $\mathfrak{su}(3)$ map to α and β respectivly. Then the map S_α factors through the map $S_{x_1-x_2} : S^3 \to \mathrm{SU}(3)$ and x is in the image of SU(3). The action of ad_x can be restricted to the image of $\mathfrak{su}(3)$ under ϕ which contains β. So we have that $ad_x(\beta) = \phi(ad_{\phi^*(-1)_x}(x_2 - x_3))$. The action of the image of $S_{x_1-x_2}$ by ad on $\mathfrak{su}(3)$ restricts to the action of the representation U_2 on the complex subspace spanned by the roots $x_2 - x_3$ and $x_1 - x_3$ as under ϕ the element $x_1 - x_2$ maps to $\alpha + \beta$ the result follows from the description of ψ.

The third case follows the same line as the second but the map ϕ takes α and β to $x_1 - x_2$ and $x_1 - x_3$ respectivly

□

In particular this shows the following:

Lemma 9.11. *Let G be a compact semisimple Lie group of type 1 and let H, K be full rank sub Lie groups generated by sub root systems. Further let $g \in G$, α is a positive root of G and $s, s' \in S^3$ with $\psi(s), \psi(s') \in (0, 1)$ then the orbits $H(S_\alpha(s) \cdot g)K$ and $H(S_\alpha(s') \cdot g)K$ are of the same type.*

Proof. We first suppose that α is also root of H then the image of S_α must also be contained in H and so $H(S_\alpha(s) \cdot g)K = HgK = H(S_\alpha(s') \cdot g)K$ in particular the orbits are of the same type.

Now suppose that α is not a root of H. By passing to Lie algebras showing $H(S_\alpha(s)\cdot g)K$ and $H(S_\alpha(s')\cdot g)K$ are of the same orbit type is equivilent to showing that $ad_{S_\alpha(s)\cdot g}\mathfrak{h}\cap\mathfrak{k}$ is equal to $ad_{S_\alpha(s')\cdot g}\mathfrak{h}\cap\mathfrak{k}$. Rearraging this is equivelent to showing that:

$$ad_{S_\alpha(s)}\mathfrak{h}\cap ad_{g^{-1}}\mathfrak{k} = ad_{S_\alpha(s')}\mathfrak{h}\cap ad_{g^{-1}}\mathfrak{k}$$

In particular it is sufficient to show that the action of $ad_{S_\alpha(s)}$ and $ad_{S_\alpha(s')}$ on $\mathfrak{h}$ is the same. As H is generated by a sub root system it surfices to show that the action of $ad_{S_\alpha(s)}$ and $ad_{S_\alpha(s')}$ on the roots of G which are also roots of H are the same. But as α is not a root of H and G is of type 1 for any root β of H we must have that $<\alpha,\beta>\in\{-1,0,1\}$ and the result follows from Lemma 9.10. □

This allows us to prove the main theorem we will use on corank 1 orbits. Here we restrict to the case where H, K are subroot systems.

Theorem 9.12. *Let G be a compact semisimple Lie group of type 1 and let H, K be full rank sub Lie groups generated by sub root systems. Then for any orbit HgK of corank 1 there exists an embedding $\psi : (-\epsilon,\epsilon)\hookrightarrow H\backslash G/K$ with $\psi(0) = HgK$ and such that the orbit type ψ is constant.*

Proof. As H, K are full rank sub Lie groups generated by sub root systems of G they have a common maximl torus T By Lemma 9.6 we have the existence of a root α of G and s in the image of S_α such that $HgK = Hg'K$ and $Hs\cdot g'K$ is a full rank orbit with maximal torus T in the stabilizer.

To simplifiy notation we define $r := s^{-1}$ and $f := sg'$ then we have that r is in the image of S_α, HfK is a full rank orbit and $HgK = Hr\cdot fK$.

Choose some $x\in S\alpha^{-1}(r)\in S^3$ we show that $\phi(x)\in(0,1)$. First suppose that $\phi(x)\in\{0,1\}$ then x normalizes $S^1\subset S^3$. We can view the maximal torus T as the direct sum of a rank 1 torus in the image of $S\alpha$ and a corank 1 torus. Then r commutes with the corank 1 torus as it is in the image of S_α and as we have shown normalizes the rank 1 torus hence r must normalize T. As T is in the stabilizer of the orbit HfK and r stabilizes T we must also have that T is in the stabilizer of the orbit $Hr\cdot fK$ but this is the same at the orbit HgK which is corank 1 a contradiction. Hence we conclude that $\phi(x)\in(0,1)$.

As $\phi : S^3\to[0,1]$ is the projection map of a double mapping cylinder by the description given in Lemma 9.7 we have $\phi^{-1}(0,1)\cong(0,1)\times S^1\times S^1$ where ϕ is just the projection to the first coordinate. Thus as $\phi(x)\in(0,1)$ there exists some $\epsilon>0$ and $\theta : (-\epsilon,\epsilon)\to S^3$ such that $\theta(0) = x$ and $\phi\circ\theta$ is an isomorphism from $(-\epsilon,\epsilon)$ to $(-\epsilon+\phi(x),\epsilon+\phi(x))$.

We now define $\psi(t) = S_\alpha(\theta(t))f$ then by $\psi(0) = S_\alpha(x)f = rf$ and so the orbit through $\psi(0)$ is the same as the orbit through g as required. Finally for any $t \in (-\epsilon, \epsilon)$ as $\phi(\theta(t)), \phi(\theta(0)) \in (0,1)$ by Lemma 9.11 the orbits $HgK = H\psi(0)K = HS_\alpha(\theta(0)) \cdot fK$ and $H\psi(t)K = HS_\alpha(\theta(t)) \cdot fK$ are of the same orbit type this completes the proof.

□

Specifying down to the case we are studying in this section we have that:

Lemma 9.13. *In the decomposition of $E_7/S^3 \times \mathrm{Spin}(12)$ by $S^1 \times E_6$ then if $g \cdot (S^3 \times \mathrm{Spin}(12))$ is an orbit with corank 1 there exists an embedding $\psi : (-\epsilon, \epsilon) \hookrightarrow E_7/S^3 \times \mathrm{Spin}(12)$ with $\psi(0) = g \cdot (S^3 \times \mathrm{Spin}(12))$ and such that the orbit type ψ is constant.*

In particular this shows that the normal bundle of any corank one orbit must be generated by a representation which splits as $1 + \Delta$ for some representation Δ.

As the action of $S^1 \times E_6$ on $E_7/S^3 \times \mathrm{Spin}(12)$ is codimension 2 we must have that the action on the sphere bundle in the normal bundle must be codimension one. This implies that the action of the stabilizer on a sphere via the representation must also be codimension one. If the orbit is corank one and has a representation of the form $1 + \Delta$ the results in Chapter 7 show that for this action to be codimension one the action on $S(\Delta)$ must be transitive.

In particular the stabilizer for the generic orbit will be the same as the stabilizer of the transitive action on $S(\Delta)$. As this stabilizer is known to be $S^1 \times S^1 \times \mathrm{SU}(4)$ the classification of transitive actions in Theorem 7.15 can be used to show that the only possible stabilizers for corank one orbits must be isomorphic to:

$$S^1 \times S^1 \times \mathrm{SU}(5)$$

up to quotient by some finite group.

9.4 Corank 2 orbits

We were unable to present the full theory of the corank 2 orbits in this thesis. But a sketch of our approach is as follows:

1. To enumerate the possible groups which are corank 2 inside $S^1 \times E_6$.

2. Using the list in (1) and Theorem 7.16 to find those groups with representations on which they act with codimension one and such that the generic orbit of

such an action is stabilized by the group $S^1 \times S^1 \times \mathrm{SU}(4)$. This is a neccesary requirement for any such orbit.

3. The use of Theorem 8.6 in [Bredon, 1972, pg. 211] on the action of a compact Lie group on a compact manifold with codimension 2 to prove that the orbit space $S^1 \times E_6 \setminus E_7/S^3 \times \mathrm{Spin}(12)$ admits a description as a polytope.

4. We identify 2 of the vertices of this polytope with the full rank orbits and the interior with the generic orbit. Lemma 9.13 shows that corank one orbits cannot be vertices and so any remaining vertices must be of corank 2.

5. We show using a cohomological argument on dimension that $H^2(E_7/S^3 \times \mathrm{Spin}(12))$ is isomorphic to the disjoint union of the second cohomology groups of the orbits which are both vertices on the polytope and corank 2. We use this to show there is exactly one corank 2 vertex orbit with stabilizer $S^1 \times S^3 \times \mathrm{Spin}(7)$.

6. We analyse the decomposition of these actions in the sphere bundle to the normal bundle at the vertex bundle to obtain the orbit type of the edges of the polytope

7. We conclude with a complete description of the decomposition as the homotopy colimit of a diagram obtained as a barycentric subdivision of the 2-simplex.

Full details are not given of this method as while I have completed many of the computations I have not been able to describe the argument in the time I had to finish this thesis.

Bibliography

J F Adams. *Lectures on exceptional Lie groups.* Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 1996. ISBN 0-226-00526-7; 0-226-00527-5.

J Baez. The Octonions. *Bulletin of the American Mathematical Society*, 2002.

C.H. Barton and A. Sudbery. Magic squares and matrix models of Lie algebras. *Advances in Mathematics*, 180(2):596–647, December 2003. ISSN 00018708.

Marcel Berger. Sur les groupes d'holonomie homognes de varits connexion affine et des varits riemanniennes. *Bulletin de la Socit Mathmatique de France*, 83:279–330, 1955.

J Berndt and M Atiyah. *Projective planes, Severi varieties and spheres*, pages 1 – 27. International Press, 2003.

Arthur L. Besse. *Einstein manifolds*, volume 10 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1987. ISBN 3-540-15279-2.

A. Borel and F. Hirzebruch. Characteristic classes and homogeneous spaces. i. *Amer. J. Math.*, 80:458–538, 1958. ISSN 0002-9327.

Raoul Bott. The space of loops on a lie group. *Michigan Math. J.*, 5:35–61, 1958. ISSN 0026-2285.

G.E. Bredon. *Introduction to compact transformation groups.* Pure and Applied Mathematics. Elsevier Science, 1972. ISBN 9780080873596.

T. Bröcker and T. Dieck. *Representations of Compact Lie Groups.* Graduate Texts in Mathematics. Springer, 1985. ISBN 9783540136781.

E. Cartan. *Sur la structure des groupes de transformations finis et continus.* Thèses présentées a la Faculté des Sciences de Paris pour obtenir le grade de docteur ès sciences mathématiques. Nony, 1894.

J.H. Conway and D.A. Smith. *On Quaternions and Octonions.* Taylor & Francis, 2003. ISBN 9781568811345.

L. E. Dickson. On quaternions and their generalization and the history of the eight square theorem. *Annals of Mathematics*, 20(3):pp. 155–171, 1919. ISSN 0003486X.

H Freudenthal. Lie groups in the foundations of geometry. *Advances in Mathematics*, 69(1):3, January 1964. ISSN 00318108.

GAP. *GAP – Groups, Algorithms, and Programming, Version 4.7.7.* The GAP Group, 2015. URL `http://www.gap-system.org`.

Yongdong Huang and Naichung Conan Leung. A uniform description of compact symmetric spaces as grassmannians using the magic square. *Mathematische Annalen*, 350(1):79–106, 2011. ISSN 0025-5831.

A. Hurwitz. Ueber die composition der quadratischen formen von belibig vielen variablen. *Nachrichten von der Gesellschaft der Wissenschaften zu Gttingen, Mathematisch-Physikalische Klasse*, 1898:309–316, 1898.

Kiminao Ishitoya. Integral cohomology ring of the symmetric-space *eii*. *J. Math. Kyoto Univ.*, 17(2):375–397, 1977. URL `http://projecteuclid.org/euclid.kjm/1250522773`.

HF Lai. On the topology of the even-dimensional complex quadrics. *Proceedings of the American Mathematical Society*, 46(3):419–425, 1974.

John McCleary. *A user's guide to spectral sequences*, volume 58 of *Cambridge Studies in Advanced Mathematics.* Cambridge University Press, Cambridge, second edition, 2001. ISBN 0-521-56759-9.

Mamoru Mimura and Hirosi Toda. *Topology of Lie groups. I, II*, volume 91 of *Translations of Mathematical Monographs.* American Mathematical Society, Providence, RI, 1991. ISBN 0-8218-4541-1.

Paul S. Mostert. On a compact lie group acting on a manifold. *Annals of Mathematics*, 65(3):pp. 447–455, 1957a. ISSN 0003486X.

Paul S. Mostert. Errata, on a compact lie group acting on a manifold. *Ann. of Math. (2)*, 66:589, 1957b. ISSN 0003-486X.

B. Rosenfeld. *Geometry of Lie Groups*. Mathematics and Its Applications. Springer, 1997. ISBN 9780792343905.

James Simons. On the transitivity of holonomy systems. *Annals of Mathematics*, 76(2):pp. 213–234, 1962.

J Tits. Algèbres alternatives, algèbres de Jordan et algèbres de Lie exceptionnelles. *Nederl. Akad. Wetensch. Proc. Ser. A*, 28:223–237, 1966.

Hirosi Toda and Takashi Watanabe. The integral cohomology ring of f_4/t and e_6/t. *J. Math. Kyoto Univ.*, 14:257–286, 1974. ISSN 0023-608X.

Marc van Leeuwen, Arjeh M. Cohen, and Bert Lisser. *LiE, A package for Lie group computations*. Computer Algebra Nederland, Amsterdam, 1992.

Jianwei Zhou and Jin Shi. Characteristic Classes on Grassmann Manifolds. *arXiv preprint arXiv:0809.0808*, pages 1–38, 2008. URL `http://arxiv.org/abs/0809.0808`.