THE UNIVERSITY OF WARWICK

University of Warwick institutional repository: http://go.warwick.ac.uk/wrap

A Thesis Submitted for the Degree of PhD at the University of Warwick

http://go.warwick.ac.uk/wrap/77523

This thesis is made available online and is protected by original copyright.

Please scroll down to view the document itself.

Please refer to the repository record for this item for information to help you to cite it. Our policy information is available from the repository home page.

Dimension theory and multifractal analysis via thermodynamic formalism

by

Dintle Nelson Kagiso

Thesis

Submitted to the University of Warwick

for the degree of

Doctor of Philosophy

Mathematics Institute

June 2015

THE UNIVERSITY OF WARWICK

Contents

Acknowledgments

I would like to thank my supervisor Professor Mark Pollicott for his dedication and patience throughout my studies. To my wife Khumo who has been a solid support I can not thank you more. To my daughters Aone and Tsaone for their cheerfulness which kept me going, a big thank you.

To my colleagues Tom Kempton, Andy Ferguson, Dave Franco, Polina Vytnova, Dayal Strub, Georg Ostrovski, Aljalila Al Abri, George Kenison, Rhiannon Dougall, Italo Cipriano and Philip Felton your shoulders have been a much needed relief to lean on during hard times, thank you also for captivating discussions we have had.

Declarations

Chapter 5 is a collaboration with Mark Pollicott and has been submitted for publication.

Abstract

The thesis deals with dimension theory and ergodic theory. We are interested in applying thermodynamic formalism to give explicit values. Mainly we study dimension of sets with different ergodic averages. An extension to the case of level sets for Gibbs measures of hyperbolic dynamical system are investigated. This leads to very accurate numerical averages.

Chapter 1

Introduction

1.1 Ergodic Theory

In the late nineteenth century Boltzmann and Gibbs working on statistical mechanics raised a problem which turned out to be rather more mathematical than physical. In today's context the problem can be restated as follows: given a measure preserving mapping $T : X \to X$ of a space (X, μ) and integrable function $f : X \to \mathbb{R}$, can one find conditions under which the limit

$$\lim_{n \to +\infty} \frac{f(x) + f(T(x)) + \cdots + f\left(T^{n-1}(x)\right)}{n}$$

exists and is constant everywhere ?
In 1931 Birkhoff in his paper [8] proved that the limit in question exists almost everywhere and so became a theorem, an important one for that matter. Almost at the same time and independently von Neumann proved the ergodic theorem for L^p spaces for $1 \leq p < \infty$.

1.2 Thermodynamic Formalism

At the heart of thermodynamic formalism lies the two main pillars in the form of entropy and topological pressure. The notion of entropy was introduced into ergodic theory in 1958 by Kolgomorov. It is one of the most important invariants in dynamical systems and hence is a tool for classifying any two dynamical systems. The definition of entropy which has generally been adopted by ergodic theorists is a slight modification from Kolgomorov's and it is due to Sinai since 1959.

The notion of topological pressure was introduced by Ruelle in the paper [41] for expansive transformations and Walters in [49] for the general case.

1.3 Dimension Theory

The notion of dimension has been known for a while for topological space taking integral values. In this form the dimension was introduced by Urysohn in [48] and Menger in [32]. However for the non-integral valued dimension the initial formulation is due to Caratheodory in [13]. A firm footing of the definition of dimension was introduced by Felix Hausdorff in 1919 in his paper [23]. This notion of dimension later became known as Hausdorff dimension and it is arguably the most popular among experts in the area.

Besicovitch and collaborators took the Hausdorff dimension to another level and were heavily involved in function theory side of studies. Further down the line of contributions to the dimension theory was made by Falconer and his book [15] has been adopted as the standard text in many fractal geometry modules across universities.

1.4 Multifractal Analysis

Some of the pioneers in multifractal analysis were the authors Halsey, Jensen, Kadanoff, Procaccia and Shraiman in their paper [22], who were working on multiscaling behaviour of physical measures on strange attractors and related problems. Almost immediately a different set of authors Collet, Lebowitz and Porzio in their work [14] gave a rigorous treatment to the multifractal analysis. Their work involved studying analysis of a certain measures invariant for some interval Markov maps.

1.5 A preview of the chapters

The main theoretical areas are given a broad introduction in Chapter 1 together with a mention of the main early and current contributors to these areas. The thesis is structured in a way such that there is a gradual build up towards main result and therefore each chapter is dependent on the preceding ones.

Chapter 2 is a collection of basic objects that we require either as definitions or as theorems within areas of dimension theory, thermodynamic formalism and multifractal analysis to carry out computations for the latter chapters.

In Chapter 3 we begin with the transfer operators which are linear operators on the Banach space $F_\theta(\mathbb{C})$. The underlying dynamics associated to the transfer operator is the shift map acting on the shift space. By finding an appropriate projection, the shift of finite type can be applied to model many examples of dynamical systems. The transfer operator is pivotal to the Theorem (3.1.4).

Chapter 4 is survey of the method to compute Hausdorff dimension of dynamically defined sets of conformal iterative function scheme by finding periodic points of period up to some N. This method was introduced by Jenkinson and Pollicott in [25].

The main result is presented in Chapter 5. An algorithm to calculate the Hausdorff dimension for the Birkhoff average level sets is presented. By applying the technique of determinants of the nuclear operators, the method is shown to exhibit a high degree of accuracy. A numerical estimation is presented as a consequence. We show that the result holds for the levels sets resulting from the local dimension.

Chapter 6 covers possible projects and generalisation to the main results. The projects may include among other examples to investigate if a similar result holds true for the perturbations of the Arnold CAT map on the 2-torus.

Chapter 2

Preliminaries

2.1 Dimension Theory

The central idea of dimension of set lies in investigating how much space for each point in a set is occupied by other points of the set in its neighbourhood. Hausdorff dimension is probably the most popular thus far of the many definitions of dimension of a set. Besicovitch developed the Hausdorff dimension further by studying its properties using tools from function theory (see [7] for example).

A civil war broke out in Russian and this impacted negatively on the education system. After being prevented from leaving the country Besicovitch finally escaped in 1924. He later in 1925 settled at Cambridge University where he established most of his work.

2.1.1 Covers

We begin by introducing the idea of Hausdorff dimension and describing its main properties. We start by recalling the definition of an open cover.

Definition 2.1.1. An *open cover* of a subset $U \subseteq X$, for a topological space X is a collection of open subsets $\{Y_i\}_{i=1}^{\infty}$ of X such that $Y \subseteq \bigcup_{i=1}^{\infty} Y_i$.

A useful concept for open covers is that of the refinement.

Definition 2.1.2. A *refinement* of an open cover C of $U \subseteq X$, for a topological space X is a new cover D of U such that every set in D is contained in some set in C.

For a metric space (X, d), the *diameter* of the set $A \subset X$ is

$$|A| := \sup \{d(x, y) : \quad x, y \in A\}.$$

Definition 2.1.3. The $\delta - cover$ of a set $V \subset X$ is a collection of subsets $\{V_i\}_{i=1}^{\infty} \subset X$ such that $V \subset \bigcup_{i=1}^{\infty} V_i$ and $|V_i| < \delta$ for all i.

We now give the definition of a *normal space*.

Definition 2.1.4. A topological space X is called *normal* if for any two closed disjoint subsets A, B of X then there exists open neighbourhoods $O_A \supset A$ and $O_B \supset B$ such that $O_A \cap O_B = \emptyset$.

One of the questions to be asked early in geometry is how to give a numerical value to depict the 'amount' of the space occupied by any subset of a topological space.

Some sets have a natural idea of integer dimension, for example for widely used sets like intervals and smooth manifolds this is well known. In general for more complicated sets one needs to introduce a more formal definition. However there are different definitions depending on the properties we require.

Lebesgue Covering Dimension

Lebesgue covering dimension or *topological dimension* of a topological space is defined to be the minimum value of n, where $n \in \mathbb{N}$, such that any open cover has a refinement in which no point is included in more than $n + 1$ elements. If no such n exists then the space is said to have infinite dimension. This quantity will be denoted by $\dim_{top}$. Among the pioneers for topological dimension are Urysohn in his paper [48] and Menger in [32].

The Lebesgue covering dimension is attributed to Henri Lebesgue. The Euclidean space $\mathbb{R}^n$ has topological dimension n (cf. Pears in [35]). One also would expect that any two homeomorphic spaces to have the same topological dimension. This is because of the bijection between open sets of the two homeomorphic spaces. It will be appealing to establish how the subsets are related to the space they live in as illustrated by the following theorem.

Theorem 2.1.1. *Every closed subspace M of a normal space X we have $\dim_{top} M \leq \dim_{top} X$.*

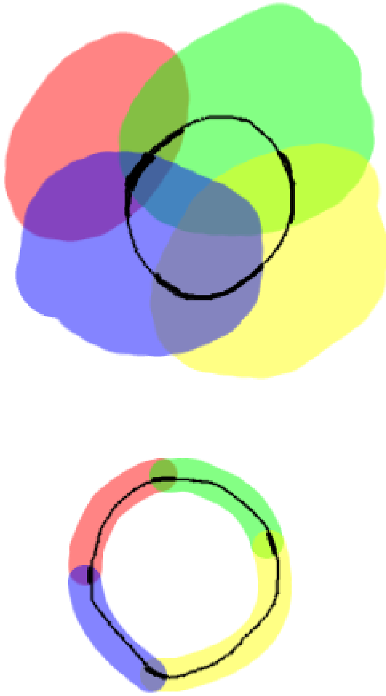

Figure 2.1: Refinement of circle cover at the top by bottom picture covers.

Proof. The theorem is obvious if $\dim_{top} X = \infty$, so that we can assume that $\dim_{top} X = n < \infty$.
Consider a finite open cover $\{U_i\}_{i=1}^{k}$ of the space M. For $i = 1, 2, ..., k$ let W_i be an open subset of X such that $U_i = M \cap W_i$. The family $X \setminus M \cup W_{i=1}^{k}$ is an open cover of the space X and since $\dim_{top} \leq n$ it has a finite open refinement γ which no point is included in more than $n+1$ elements of γ. One easily sees that the family $\gamma \setminus M$ is a finite open cover of space M, refines U and has no point of M is included in more than $n+1$ elements of $\gamma \setminus M$, so that $\dim_{top} M < n = \dim_{top} X$. $\square$

Example 2.1.1. Let S^1 be the unit circle then $\dim_{top} S^1 = 1$. This is evident if we consider any cover for S^1. It is always possible to refine the cover until where any point, say $x \in S^1$ is contained in at most 2 arcs. Figure (2.1) shows the covers of the black circle in the top diagram refined into sets in the neighbourhood of the circle curve of the bottom diagram.

2.1.2 Measure

A $\sigma - algebra$ on a set X is a collection of subsets of X that contains the empty set $\emptyset$ and the whole space X, and is closed under complements, countable unions, and countable intersections.

Let X be any set (which will be called our space), and let $\mathcal{A}$ be a collection of subsets of X. Suppose

1. $\emptyset, X \in \mathcal{A}$,
2. $A^c \in \mathcal{A}$ whenever $A \in \mathcal{A}$,

And in addition if $\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$ whenever $A_n \in \mathcal{A}$ for every $n \in \mathbb{N}$ then $\mathcal{A}$ is called $\sigma - algebra$.

We will refer to the elements of $\mathcal{A}$ as *measurable* sets.

Definition 2.1.5. A *measurable space* $(X, \mathcal{A})$ is a non-empty set X equipped with a σ-algebra $\mathcal{A}$ on X.

It turns out that measurable spaces are a rich source of domain for 'measures' which will be discussed shortly.

Definition 2.1.6. Let μ be a function on the subsets of a metric space (X, d) into the interval of positive real numbers and possibly taking the value ∞. If in addition μ satisfies

1. if $A = \emptyset$ then $\mu(A) = 0$,
2. if $A \subseteq B$ then $\mu(A) \leq \mu(B)$,
3. if $A_1, A_2, ...$ are disjoint countable sets then

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i).$$

The function μ with these properties will be called a *measure*.

A measure is yet another tool for distinguishing among the sets of the same metric space according to their numerical 'size'. For a measure we require that if we group together a countable number of sets in a reasonable manner then the size of the whole space occupied by the group is equal to the sum of the size of the pieces making it. We require that the empty set occupies a 'space' of size zero. We would also expect that a set be at most as big as any of its subsets.

Example 2.1.2. Let X be any set, and let $\mathcal{A} = 2^X$ be the power set of X, that is, the collection of all subsets of X. $\mathcal{A}$ is a $\sigma - algebra$ (in fact, it is the largest $\sigma - algebra$ on X), we may define a measure ν

$$\nu(A) = \begin{cases} \text{card}(A) & A \text{ is finite,} \\ \infty & \text{otherwise.} \end{cases}$$

So that if A is a finite set, ν counts the number of points in A; otherwise gives ∞. This is known as the *counting measure* on X.

2.1.3 Lebesgue Measure

Each of the subsets of $\mathbb{R}$ inherits a metric, and thus a topology, from the real line, as indeed does every subset of $\mathbb{R}^n$. The topological notions provide one, very coarse, way of classifying subsets of $\mathbb{R}$. On the other hand the *Lebesgue measure*, which generalises the notion of "length" to sets which are not intervals on $\mathbb{R}$ gives finer tool.

Definition 2.1.7. Let $A \subset \mathbb{R}^n$, for $n \in \mathbb{N}$, the $n-dimensional$ *Lebesgue measure* of set A is defined

$$\mathcal{L}^n(A) = \inf\left\{ \mathrm{vol}^{\mathrm{n}}(A_i) : \ A \subset \bigcup_{i=1}^{\infty} A_i, \text{ where } \{A_i\} \text{ is a box cover of } A \right\}.$$

The *box cover* of set A in $\mathbb{R}^n$ is the 'cubes ' covering of A with sides parallel to each corresponding axis of $\mathbb{R}$.

Examples of Lebesgue measure zero sets in $\mathbb{R}$:

We define one of the simplest families of 'fractal' sets, which will serve to demonstrate the definitions that follow.
Let $0 < \lambda < 1$. The *middle* $- \lambda$ *Cantor set* $C_\lambda \subseteq [0, 1]$ is defined by the limit of a recursive procedure. For $n = 0, 1, 2, ...$ we construct a set C_λ^n which is a union of 2^n closed intervals, indexed by sequences $i = i_1, ...i_n \in \{0, 1\}^n$ and each of length

$((1-\lambda)/2)^n$. To begin let $C_\lambda^0 = [0,1]$ and $I = [0,1]$ (indexed by the unique empty sequence). Assuming that C_λ^n has been defined and is the disjoint union of the 2^n closed intervals $I_{i_1 \ldots i_n}$, $i = i_1, \ldots i_n \in \{0,1\}^n$, divide each of the intervals into the two subintervals, $I_{i_1 \ldots i_n 0}, I_{i_1 \ldots i_n 1} \subseteq I_{i_1 \ldots i_n}$ which remain after removing from I_i the open subinterval with the same center as $I_{i_1 \ldots i_n}$ and λ times shorter. Finally let

$$C_\lambda^{n+1} = \bigcup_{i \in \{0,1\}^{n+1}} I_i$$

Clearly $C_\lambda^0 \supseteq C_\lambda^1 \supseteq \ldots$, and since the sets are compact,

$$C_\lambda = \bigcap_{n=0}^{\infty} C_\lambda^n$$

is compact and nonempty.

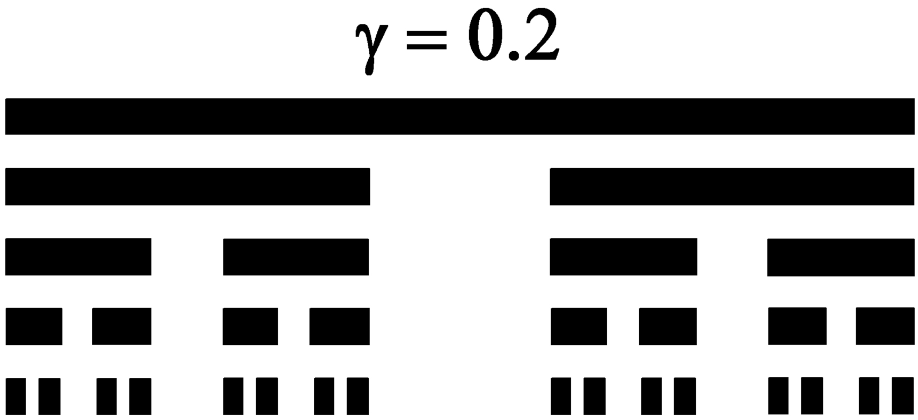

Figure 2.2: Middle λ Cantor set for $\lambda = 0.2$.

All of the sets C_λ for $0 < \lambda < 1$ are mutually homeomorphic, since all are topologically Cantor sets (i.e. compact and totally disconnected without isolated points). They all are of first Baire category. And they all have Lebesgue measure 0, since one may verify that $\mathcal{L}\left(C_\lambda^n\right) = (1-\lambda)^n \to 0$ as $n \to \infty$. Hence none of these theories can distinguish between them.

Nevertheless qualitatively it is clear that C_λ becomes "larger " as $\lambda \to 0$, since decreasing λ results in removing shorter intervals at each step. In order to quantify this one uses dimension.

2.1.4 Box Dimension

One of the widely accepted notion of determining how 'big' a set is the box-dimension also known as the Minkowski dimension or Minkowski - Bouligand dimension. In a more usable form the box dimension was reformulated by Pontrjagin and Schnirelman [39] in 1932. The box-dimension may be thought of as the most efficient way of covering a set by small sets of equal size.

Definition 2.1.8. Let E be any non-empty bounded subset of of $\mathbb{R}^n$ and let $N_\delta(E)$ be the smallest number of sets of diameter at most δ which can cover E. The *lower* and *upper box – counting dimensions* of E respectively are defined as

$$\begin{aligned}
\underline{\dim}_B E = &\quad \underline{\lim}_{\delta\to 0}\frac{\log N_\delta(E)}{-\log\delta}, \\
\overline{\dim}_B E = &\quad \overline{\lim}_{\delta\to 0}\frac{\log N_\delta(E)}{-\log\delta}.
\end{aligned}$$

If these are equal we refer to the common value as the *box – counting dimension* or *box dimension* of E

$$\dim_B E := \lim_{\delta\to 0}\frac{\log N_\delta(E)}{-\log\delta}.$$

Proposition 2.1.2. *Let $\overline{E}$ denote the closure of E in $\mathbb{R}^n$ then*

$$\dim_B \overline{E} = \dim_B E.$$

Proof. Let $B_1, \cdots, B_k$ be a finite collection of closed balls, each with a radius of δ. Suppose that the closed set $\bigcup_{i=1}^k B_i$ contains E, then it also contains $\overline{E}$. If $N_\delta(E)$ is the smallest number of closed balls of radius δ that cover E then we have

$$\underline{\lim}_{\delta\to 0}\frac{\log N_\delta(E)}{-\log\delta} = \underline{\lim}_{\delta\to 0}\frac{\log N_\delta\left(\overline{E}\right)}{-\log\delta}.$$

and the results follow. □

Let dim be a general definition of dimension. We will classify a definition of dimension as satisfying the *countable stability* if for any countable collection of subsets $\{E_i\}_{i=1}^{\infty}$ then

$$\dim\left(\bigcup_{i=1}^{\infty} E_i\right) = \sup_{1\leq i<\infty} \dim E_i.$$

The result of the proposition may seem to be an attractive at first, nevertheless it shows how restrictive the application of box dimension is, since the countable stability fails as illustrated by the following example.

Example 2.1.3. The set of rational numbers $\mathbb{Q}$ in the interval $[0,1]$ has $\dim_B(\mathbb{Q}\cap[0,1]) = 1$. By proposition (2.1.2) since $\overline{\mathbb{Q}\cap[0,1]} = [0,1]$, we see that $\dim_B(\mathbb{Q}\cap[0,1]) = \dim_B[0,1]$.
On the other hand $\mathbb{Q}\cap[0,1]$ is a set of countable union of singletons, each having $\dim_B(\{x\}) = 0$.

Box dimension calculations for C_λ

For C_λ as before, $\dim_B C_\lambda = \log 2/\log(2/(1-\lambda))$. Let us demonstrate this. To get an upper bound, we note that for $\delta_n = ((1-\lambda)/2)^n$ the sets C_λ^n are covers of C_λ by 2^n intervals of length δ_n, hence $N_{\delta_n}(C_\lambda) \leq 2^n$. If $\delta_{n+1}\leq\delta<\delta_n$ then clearly

$$N_\delta(C_\lambda) \leq N_{\delta_{n+1}}(C_\lambda) \leq 2^{n+1}.$$

On the other hand every set of diameter $\leq \delta$ can intersect at most two maximal intervals in C_λ^{n+1}, hence

$$N_\delta(C_\lambda) \geq \frac{1}{2}\cdot 2^n,$$

so for $\delta_{n+1}\leq\delta<\delta_n$

$$\frac{(n-1)\log 2}{(n+1)\log(2/(1-\lambda))} \leq \frac{\log N_\delta(C_\lambda)}{\log 1/\delta} \leq \frac{(n+1)\log 2}{n\log(2/(1-\lambda))},$$

and so, taking $\delta\to 0$,

$$\dim_B C_\lambda = \log 2/\log(2/(1-\lambda)).$$

In general for any Cantor set like set $A\subseteq\mathbb{R}^n$ in the sense that their box dimension is less than n are not seen by Lebesgue measure, that is, if $\dim_B A < n$ then we have $\mathcal{L}(A) = 0$. To see this, let us choose

$$\epsilon = \frac{1}{2}(n - \dim_B A).$$

From the definition of $\dim_B A$ we can find sufficiently small δ and

$$\delta^{-(\dim_B A + \epsilon)}$$

sets of diameter less than δ that cover A. In addition we can cover a set of diameter less than δ by a set of volume strictly less $c \cdot \delta^n$ for some $c > 0$. In total the cover of A in question has a volume

$$c\delta^n \cdot \delta^{-(\dim_B A + \epsilon)} = c\delta^\epsilon.$$

Since this holds for arbitrary small δ, we conclude that $\mathcal{L}(A) = 0$.

2.1.5 Hausdorff Dimension

This notion of dimension is named after Felix Hausdorff, a Polish mathematician born in 1868 in Breslau, Germany which later became Wroclaw, part of Poland. In introducing the definition, Hausdorff in his 1919 paper [23], generalised the results of Caratheodory [13]. Life because unbearable for Hausdorff being a Jew when he was forced to resign from his position at Bonn in 1935. Having being marked and his transfer to Cologne imminent, Hausdorff committed suicide with his wife, Charlotte on 26th January 1942.

Definition 2.1.9. The Hausdorff *pre-measure* is a function on subsets A of a metric space (X, d) given by,

$$\mathcal{H}^t_\delta(A) := \inf \left\{ \sum_{i=1}^{\infty} |A_i|^t : \ \{A_i\}_{i=1}^{\infty} \text{ is a } \ \delta - \textit{cover} \ \text{ of } A \right\}.$$

The infimum in the definition exists and this can be easily verified. Let $\delta_1 < \delta$ then $\mathcal{H}^t_{\delta_1}(A) \geq \mathcal{H}^t_\delta(A)$ since the infimum over the larger class of covers is lower. The function $\mathcal{H}^t_\delta$ is decreasing in δ.
We will also define $\sum |U_i|^s$ to be the $s-$value for the $\delta-$cover $\{U_i\}_{i=1}^{\infty}$.

Definition 2.1.10. Formally we write

$$\mathcal{H}^t(A) = \lim_{\delta \to 0} \mathcal{H}^t_\delta(A)$$

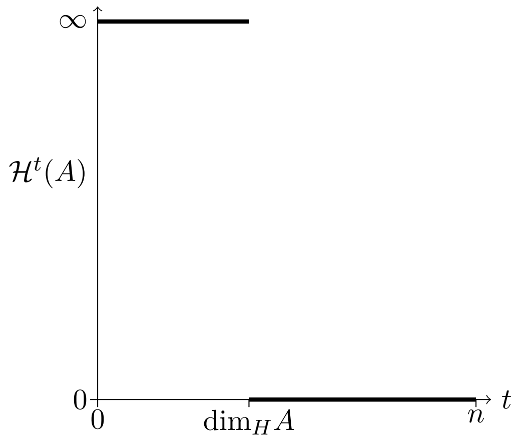

Figure 2.3: Graph of $\mathcal{H}^t(A)$ against t for a set A. The Hausdorff dimension is the value of t at which the 'jump' from ∞ to 0 occurs.

and we call this quantity the *t-dimensional Hausdorff measure* of the set A.

Definition 2.1.11. The Hausdorff dimension of a set A is defined by

$$\begin{aligned}\dim_H A &= \inf\left\{t \ : \ \mathcal{H}^t(A) = 0\right\} \\ &= \sup\left\{t \ : \ \mathcal{H}^t(A) = \infty\right\}.\end{aligned}$$

The following Lemma asserts that the Hausdorff dimension has desirable properties of a dimension function.

Lemma 2.1.3. *Let $A, B \in \mathbb{R}^n$ and $\{A_i\}$ be a sequence of sets in $\mathbb{R}^n$,*

1. *If $A = \emptyset$ then* $\dim_H A = 0$,
2. *If $A \subseteq B$ then* $\dim_H A \le \dim_H B$,
3. $\dim_H\left(\bigcup_{i=1}^{\infty} A_i\right) = \sup_{1\le i<\infty} \dim_H A_i$.
4. *If $\{x_i\}_{i=1}^{\infty}$ is a countable set then* $\dim_H A = 0$.

Proof. (1). For a set A then if $A = \emptyset$ it is clear from the fact that $\mathcal{H}^t(A) \le |A|^t$.

(2). Suppose that $A \subset B$ and let $s > \dim_H B$. We have that $\mathcal{H}^s(B) = 0$. This implies that $\mathcal{H}^s(A) = 0$. Hence $\dim_H A \le s$. Since s is arbitrary $\mathcal{H}^s(A) \le \mathcal{H}^s(B)$.

(3). We recall that $\dim_H A = \inf\left\{t \mid \mathcal{H}^t(A) = 0\right\}$. It suffices to only show the less than side since the other side of equality is a consequence of point (2) of the

Lemma. Let $s > \dim_H A_i$ for all i, we have

$$\mathcal{H}^t\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mathcal{H}^t\left(A_i\right) = 0.$$

Hence $\dim_H\left(\bigcup_{i=1}^{\infty} A_i\right) \leq s$.

(4). For any t and any singleton subset $\{x_i\}$ of A we note that $\mathcal{H}^t\left(\{x_i\}\right) \leq |\{x_i\}|^t = 0$. On the optimum δ- cover is $\{x_i, 0, 0, 0, 0, 0, 0, 0, 0, \cdots\}$ giving us $\mathcal{H}^0\left(\{x_i\}\right) = 1$. Hence $\dim_H\left(\{x_i\}\right) = 0$. By part 3 of the Lemma we deduce that $\dim_H(A) = 0$. $\square$

Proposition 2.1.4. *Let $A \subset \mathbb{R}^n$ and suppose that $f : A \to \mathbb{R}^m$ satisfies a Hölder condition*

$$|f(x) - f(y)| \leq c|x - y|^{\alpha} \qquad (x, y \in F).$$

then $\dim_H f(A) \leq (1/\alpha)\dim_H A$.

Proof. If $s > \dim_H A$. Suppose that $\{U_i\}$ is a δ-cover of A, then, since

$$|f(A \cap U_i)| \leq c|A \cap U_i|^{\alpha} \leq c|U_i|^{\alpha},$$

it follows that $\{f(A \cap U_i)\}$ is a ϵ-cover of $f(A)$, where $\epsilon = c\delta^{\alpha}$. Thus

$$\sum_i |f(A \cap U_i)|^{s/\alpha} \leq c^{s/\alpha}\sum_i |U_i|^s,$$

so that

$$\mathcal{H}_{\epsilon}^{s/\alpha}(f(A)) \leq c^{s/\alpha}\mathcal{H}_{\delta}^{s}(A).$$

Taking $\delta \to 0$ we have

$$\mathcal{H}^{s/\alpha}(f(A)) \leq c^{s/\alpha}\mathcal{H}^{s}(A).$$

Since $\mathcal{H}^s(A) = 0$ then $\mathcal{H}^{s/\alpha}(f(A)) = 0$, which implies that $\dim_H f(A) \leq (s/\alpha)\dim_H A$ for all $s > \dim_H A$. $\square$

Corollary 2.1.5. *(a) If $f : A \to \mathbb{R}^m$ is a Lipschitz transformation then* $\dim_H f(A) \leq \dim_H A$.
(b) If $f : A \to \mathbb{R}^m$ is a bi-Lipschitz, that is,

$$c_1|x - y| \leq |f(x) - f(y)| \leq c_2|x - y| \qquad (x, y \in A)$$

where $0 < c_1 \leq c_2 < \infty$ *, then* $\dim_H f(A) = \dim_H A$.

Proof. Part (a) follows from Proposition 2.1.4 taking $\alpha = 1$. Applying this to $f^{-1} : f(A) \to A$ gives the other inequality required for (b). $\square$

Example 2.1.4. $\dim_H (\mathbb{Q} \cap [0,1]) = 0$ and $\dim_B (\mathbb{Q} \cap [0,1]) = 1$

Some techniques for determining the Hausdorff dimension

For an upper estimate of the Hausdorff dimension of a given set it is enough to find one sufficiently efficient cover. For a lower estimate of the Hausdorff dimension one has to prove something for all conceivable covers. The main method to do so is to introduce an appropriate measure on the set. This method was suggested by Frostman in [17] and it is widely used. We define the mass distribution on the set.

Definition 2.1.12. Let E be a subset of a general metric space (X,d). A measure μ on the measurable space $(X,\mathcal{A})$ is called *mass distribution* over E if there is a compact subset $A \subset E$ such that

$$\mu(A^c) = 0 \qquad \text{and} \qquad 0 < \mu(A) < \infty.$$

Theorem 2.1.6. *(Mass distribution principle)*
Let μ be a mass distribution over E such that for some $\alpha \geq 0$ and some positive constants c and δ we have

$$\mu(U) \leq c\,|U|^{\alpha} \text{ for all } U \text{ with } |U| < \delta.$$

Then

$$\mathcal{H}^{\alpha}(E) \geq \frac{\mu(E)}{c} \text{ and } \dim_H E \geq \alpha.$$

Proof. We may suppose without loss of generality $\mathcal{H}^{\alpha}(E) < \infty$. Let U_i be a δ-cover of E satisfying

$$\sum_{i=1}^{\infty} |U_i|^{\alpha} \leq \mathcal{H}^{\alpha}(E) + \epsilon.$$

Then

$$0 < \mu(E) \leq \mu\left(\bigcup_{i=1}^{\infty} U_i\right) \leq \sum_{i=1}^{\infty} \mu(U_i) \leq \sum_{i=1}^{\infty} c\,|U_i|^{\alpha} \leq c\mathcal{H}^{\alpha}(E) + c\epsilon$$

and hence $\infty > \mathcal{H}^{\alpha}(E) \geq \frac{\mu(E)}{c} > 0$ which implies the first assertion. In particular we have $\dim_H E \geq \alpha$. $\square$

Box dimension as an upper bound for Hausdorff dimension

Choose any $\delta > 0$. Let $s > \underline{\dim}_B E$. Choose $\eta \in (\underline{\dim}_B E, s)$. Then for each sufficiently small number $\epsilon > 0$ with $\epsilon < \delta$ there is a covering of E by k balls of diameter ϵ such that

$$\log k = \log (N_\epsilon (E)) \leq \eta \log \frac{1}{\epsilon} = \log \left(\frac{1}{\epsilon}\right)^\eta .$$

Hence the $s-$value of this

$$\delta - \text{cover is } \leq k\epsilon^s \leq \frac{\epsilon^s}{\epsilon^\eta} = \epsilon^{s-\eta},$$

which converges to zero as $\epsilon \to 0$. This implies $\mathcal{H}^s_\delta (E) = 0$ and passing to the limit as δ tends to 0 we therefore have $\dim_H E \leq s$. This completes the proof.

Hausdorff dimension calculations for C_λ

We can now complete the calculation of the dimension of C_λ. Write

$$\beta = \frac{\log 2}{\log (2/ (1-\lambda))}.$$

We have already seen that

$$\dim_B C_\lambda \leq \beta$$

so, since

$$\dim_H C_\lambda \leq \dim_B C_\lambda,$$

we have an upper bound of β on $\dim_H C_\lambda$.
Let $\mu = \mu_\lambda$ on C_λ denote the measure which gives equal mass to each of the 2^n intervals in the set C^n_λ introduced in the construction of C_λ. Let

$$\delta_n = ((1-\lambda/ 2))^n$$

be the length of these intervals. Then for every $x \in C_\lambda$, one sees that $B_{\delta_n} (x)$ contains one of these intervals and at most a part of one other interval, so

$$\mu (B_{\delta_n} (x)) \leq 2 \cdot 2^{-n} = C \cdot \delta_n^\beta.$$

Using the fact that

$$B_{\delta_{n+1}} (x) \subseteq B_r (x) \subseteq B_{\delta_n} (x)$$

whenever

$$\delta_{n+1} \leq r < \delta_n \text{ for } x \in C_\lambda$$

we have

$$\mu\left(B_r\left(x\right)\right) \leq \mu\left(B_{\delta_n}\left(x\right)\right) \leq C \cdot \delta_n^{\beta} \leq C \cdot \left(\frac{2}{1-\lambda}\right)^{\beta} \cdot \delta_{n+1}^{\beta} \leq C' r^{\beta}.$$

Hence by the mass distribution principle, $\dim_H C_\lambda \geq \beta$. Since this is the same as the upper bound, we conclude $\dim_H C_\lambda = \beta$.

2.2 Shift Spaces

In various examples of dynamical systems the orbits are usually described through its *itinerary.* As an alternative we introduce some symbolic spaces that allow to describe the dynamics of more maps using itineraries.
Let us decompose the space X in finitely many pieces, that is,

$$X = P_1 \cup P_2 \cup \cdots P_N.$$

If P_i are pairwise disjoint, we say that $\{P_1, \ldots, P_N\}$ is a *finite partition* of X. Let $x \in X$. Since P_i covers X, for each $i \in \mathbb{N}$ there exists $1 \leq a_i \leq N$ such that $f^i\left(x\right) \in P_{a_i}$. The sequence $a_0, a_1, \ldots, a_n, \ldots,$ is the itinerary of x with respect to $\{P_1, \ldots, P_N\}$.
We can *code* the (forward) orbit $\mathcal{O}_f^+\left(x\right)$ and write the sequence $\underline{a} = (a_i)_{i=0}^{\infty}$. The sequence belongs to

$$\Sigma_N^+ = \{1, \ldots, N\}^{\mathbb{N}} = \{\underline{a} = (a_i)_{i=0}^{\infty}, \qquad 1 \leq a_i \leq N\},$$

that is, the space of (one-sided) sequences in the digits $1, \ldots, N$.
If f is invertible, for each $i \in \mathbb{Z}$ there exits $1 \leq a_i \leq N$ such that $f^i\left(x\right) \in P_{a_i}$ and we can code the full orbit $\mathcal{O}_f\left(x\right)$ with the full (past and the future) itinerary $\left\{\underline{a} = (a_i)_{i=-\infty}^{\infty}\right\}$, which belongs to the space

$$\Sigma_N = \{1, \ldots, N\}^{\mathbb{Z}} = \left\{\underline{a} = (a_i)_{i=-\infty}^{\infty}, \qquad 1 \leq a_i \leq N\right\},$$

that is the space of bi-sided sequences in the digits $1, \ldots, N$.
In both cases, $f\left(x\right)$ is coded by the *shifted sequence:* since $f^i\left(f\left(x\right)\right) = f^{i+1}\left(x\right) \in P_{a_{i+1}}$ by definition of itinerary of x, the itinerary of $f\left(x\right)$, and hence the coding of $\mathcal{O}_f^+\left(f\left(x\right)\right)$ is given by

$$\sigma^+\left((a_i)_{i=0}^{+\infty}\right) = (a_{i+1})_{i=0}^{+\infty},$$

or, when f is invertible, by

$$\sigma\left((a_i)_{i=-\infty}^{+\infty}\right) = (a_{i+1})_{i=-\infty}^{+\infty}.$$

The maps,

$$\sigma^+ : \Sigma_N^+ \to \Sigma_N^+, \qquad\qquad \sigma : \Sigma_N \to \Sigma_N,$$

are known as *full* (one-sided) *shifted* on N symbols and *full bi-sided shift* on N symbols.
If $\psi : X \to \Sigma_N^+$ (or $\psi : X \to \Sigma_N$ in the invertible case) is the *coding map* which assign to each point its itinerary, the previous relation shows that for $x \in X$

$$\psi(f(x)) = \sigma^+(\psi(x)) \qquad (\text{or } \ \psi(f(x)) \ \text{ if } f \text{ is invertible}).$$

in order to give a conjugacy, though, the coding ψ should be both injective and surjective. Thus, it is natural to ask: (1) Is the coding unique? and (2) Do all sequences in Σ_N^+ (or in Σ_N) occur as possible itineraries?
If consider examples of the baker map and the doubling map the answer to the first questions is no when coding by $\sigma : \Sigma_2 \to \Sigma_2$ and $\sigma^+ : \Sigma_2^+ \to \Sigma_2^+$ respectively. However for both systems any sequence can occur as a possible itinerary. The coding of the Gauss map by a countable many digits $\{1, \ldots, n, \ldots,\}^{\mathbb{N}}$ the coding is also not always unique.

Example 2.2.1. Consider the map $\psi : \Sigma_2^+ \to [0,1]$. For each $(a_i)_{i=1}^{\infty} \in \Sigma_2^+$ we define

$$\psi\left((a_i)_{i=1}^{\infty}\right) = \sum_{i=1}^{\infty} \frac{a_i}{2^i} \in [0,1].$$

The rational numbers of the form $\frac{k}{2^n}$ has two different codings or expansions one ending with a sequence of 0's and the with 1's.

To be able to describe the subset of the shift that captures itineraries of this form is one of the reasons to study *subshifts of finite type* of the following form.

Definition 2.2.1. An $N \times N$ matrix is called a *transition matrix* if all entries $A_{i,j}$, $1 \le i,j \ \le N$, are either 0 or 1.

One can use a matrix A to encode the information of which pairs of consecutive digits can appear in the itinerary: the digit i can be followed by digit j if and only if the entry $A_{i,j}$ is equal to 1. More formally, we can consider the following subspaces $\Sigma_A^+ \subset \Sigma_N^+$ and $\Sigma_A \subset \Sigma_N$ of sequences.

Definition 2.2.2. The shift spaces associated to a transition matrix A are:

$$\Sigma_A^+ = \left\{(a_i)_{i=0}^{+\infty} \in \Sigma_N^+, \quad A_{a_i a_{i+1}} = 1 \text{ for all } i \in \mathbb{N}\right\}$$
$$\Sigma_A = \left\{(a_i)_{i=-\infty}^{+\infty} \in \Sigma_N, \quad A_{a_i a_{i+1}} = 1 \text{ for all } i \in \mathbb{Z}\right\}.$$

The spaces Σ_A and Σ_A^+ are *invariant* under the shift and we can consider the restriction of σ^+ and σ to these subspaces respectively.

Definition 2.2.3. The restriction of the shift maps to

$$\sigma^+ : \Sigma_A^+ \to \Sigma_A^+, \qquad \sigma : \Sigma_A \to \Sigma_A.$$

are called *subshift of finite type* (also called a *topological Markov chain*).

Often the cylinder sets are crucial for doing manipulations of the shifts. The cylinder sets are also the open sets for the shifts when endowed with an appropriate metric.

Definition 2.2.4. The *cylinder sets* are the sets with points which have a fixed initial coordinates , we define

$$C_k(x) = \{y \in \Sigma_A : x_i = y_i, \quad -l \leq j < k\},$$

as for one sided shift we define

$$C_k(x) = \left\{y \in \Sigma_A^+ : x_i = y_i, \quad 0 \leq j < k\right\}.$$

Example 2.2.2. Let us consider a one sided shift on k symbols. Then we see that there are k^n periodic points of period n.
This is evident since the *periodic points* of period n in Σ_A^+ are exactly the points of the form $x_i = x_{i+n}$ for all $i \in \mathbb{N}$. This implies that these are the points which satisfy $(\sigma^+)^n x = x$. In other words how many members of $C_n(x)$ are there?

2.3 Ergodicity

In this section we state a theorem which bears Birkhoff's name. The theorem transformed Maxwell-Boltzmann kinetic theory of gases into a rigorous principle through the use of Measure Theory in 1931.

Definition 2.3.1. Let $(X, \mathcal{B}, \mu)$ be a probability space and let $T : X \to X$ be a measure-preserving transformation. T is said to be *ergodic* if for any $B \in \mathcal{B}$ then

$$T^{-1}B = B \Longrightarrow \mu(B) = 0 \text{ or } 1.$$

Remark 2.3.1. It is important to note that ergodicity implies indecomposability condition. This means that if T is ergodic then we can not have $T^{-1}A = A$ with $0 < \mu(A) < 1$.

Theorem 2.3.2. *(Birkhoff's Ergodic Theorem) Let T be an ergodic transformation of the probability space $(X, \mathcal{B}, \mu)$ and let $f \in L^1(X, \mathcal{B}, \mu)$. Then*

$$\frac{1}{n}\sum_{j=0}^{n-1} f\left(T^j x\right) \to \int f d\mu$$

for $\mu -$ almost every $x \in X$.

The proof to this theorem can found in Birkhoff's paper [8] and the books of Walters [50] and Mañé [28].

2.4 Thermodynamic Formalism

2.4.1 Topological Pressure

The notion of topological pressure, which is the most basic notion of thermodynamic formalism, was introduced by Ruelle for expansive transformations and Walters in the general case.
It is worthwhile to notice that the possibility of coding repellers and hyperbolic sets via symbolic dynamics often allows one to give simpler proofs. Thus, it is of interest to have explicit formula for the topological pressure with respect to the shift map.

There are a number of different ways of defining the topological pressure. The most general one (as when considering continuous maps defined over compact spaces) is using (n, ϵ) - generating sets. Here we will consider a different definition, which coincides with the classical one for dynamical systems that are sufficiently hyperbolic. Even though the following definition holds in greater generality, we will restrict ourselves to symbolic systems and to piecewise expanding interval maps with full branches.

Let $\sigma^+ : \Sigma_A^+ \to \Sigma_A^+$ be a one-sided sub-shift and $\sigma : \Sigma_A \to \Sigma_A$ be a two-sided sub-shift of finite type. These will be dynamical systems under consideration. Let $\phi : \Sigma_A \to \mathbb{R}$ or $\phi : \Sigma_A^+ \to \mathbb{R}$ be a Holder continuous function, that we will call a *potential*.

Definition 2.4.1. The topological pressure of the map σ at the potential ϕ is defined by

$$P(\phi) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{\sigma^n x = x} \exp\left(\sum_{i=0}^{n-1} \phi\left(\sigma^i x\right)\right).$$

Remark 2.4.1. The pressure for an expanding map with full branches is defined in the same way, the only difference being that we sum over periodic points of period $n \in \mathbb{N}$, that is the set $\{x \in X : T^n x = x\}$.

Let us note that using sub-additivity arguments it is possible to prove that the above limit exists.

In order to understand the definition, let us start by considering the null-potential, that $\phi \equiv 0$. In this case we obtain

$$P(\phi) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{\sigma^n x = x} 1.$$

We introduce a useful asymptotic notation,

Definition 2.4.2. Suppose f and g are complex valued functions defined on the same space, we write $f \asymp g$ if there exists a constant $C > 0$ such that

$$\frac{1}{C} |f| \leq |g| \leq C |f|.$$

Therefore, $P(0)$ quantifies the exponential growth of periodic orbits,

$$\sum_{\sigma^n x = x} 1 \asymp \exp(nP(0)).$$

The number $P(0)$ is usually called topological entropy and it is denoted by $h_{top}(\sigma)$. The topological pressure can be thought of as a weighted topological entropy. Indeed, each point on the periodic orbit $\left\{x, \sigma x, \sigma^2 x, ..., \sigma^{n-1} x\right\}$, is given weight $\phi\left(\sigma^i x\right)$. The topological pressure quantifies exponential growth of the weighted periodic orbits, that is,

$$\sum_{\sigma^n x = x} \exp\left(\sum_{i=0}^{n-1} \phi\left(\sigma^i x\right)\right) \asymp \exp(nP(\phi)).$$

There are several properties of the pressure that can easily be deduced from the definition, for instance

1. If $\phi \leq \psi$ then $P(\phi) \leq P(\psi)$
2. The pressure function $P(\cdot)$ is convex (with respect to the potential)
3. If $c \in \mathbb{R}$ then $P(\phi + c) = P(\phi) + c$
4. $P(\phi) = P(\phi + \psi \circ \sigma - \psi)$.

These properties will be discussed in Theorem (2.4.4).

Example 2.4.1. Let I_1, I_2 be subintervals of the unit interval $[0,1]$ and let $T : I_1 \cup I_2 \to [0,1]$ be an affine cookie cutter, then T restricted to each interval I_i is piecewise linear, $T|_{I_i} = a_i x + c_i$. The topological pressure of this system is given by

$$\begin{aligned}
P\left(-t \log |T'|\right) &= \lim_{n\to\infty} \frac{1}{n} \log \sum_{T^n x = x} \exp\left(\sum_{i=0}^{n-1} -t \log \left|\left(T'\left(T^i x\right)\right)\right|\right) \\
&= \lim_{n\to\infty} \frac{1}{n} \log \sum_{T^n x = x} \prod_{i=0}^{n-1} \left|T'\left(T^i x\right)\right| \\
&= \lim_{n\to\infty} \frac{1}{n} \log \sum_{\substack{i_j \in \{1,2\} \\ j \in \{1,\ldots,n\}}} \left(a_{i_1} \cdots a_{i_n}\right)^{-t} \\
&= \lim_{n\to\infty} \frac{1}{n} \log \left(a_1^{-t} + a_2^{-t}\right)^n \\
&= \log\left(a_1^{-t} + a_2^{-t}\right).
\end{aligned}$$

2.4.2 Entropy

The notion of entropy was introduced into ergodic theory in 1958 by Kolgomorov. It is one of the most important invariants in dynamical systems. The definition of entropy used now is a slightly different from Kolgomorov's and it is due to Sinai (1959).

In this section we briefly sketch the definition of entropy, for a thorough account see for example Walters [50] and Pesin [38]. Entropy can be thought of as a measure of the disorder of the system. In other words, the entropy of an invariant measure quantifies the amount of disorder of the system realised by the measure μ. The definition of entropy is done in several steps. Let $T : X \to X$ be a continuous map of the compact metric space X and let μ be a T-invariant probability measure.

Entropy of a Partition

Suppose that P_1 and P_2 are partitions of X in the Borel σ-algebra $\mathcal{B}$. If $P_1 = \{A_1, \cdots, A_n\}$ and $P_2 = \{B_1, \cdots, B_m\}$ are finite partitions of X then their join is the partition

$$P_1 \vee P_2 := \{A_j \cap B_j : 1 \leq i \leq n \text{ and } 1 \leq j \leq m\}.$$

Definition 2.4.3. Let $P_1 = \{A_1, \cdots, A_n\}$ be a partition of X. The entropy of P_1 with respect to μ is defined by

$$H(P_1) = -\sum_{i=1}^{n} \mu(A_i) \log \mu(A_i).$$

Entropy of measure preserving transformation

Let $P_1 = \{A_1, \cdots, A_n\}$ be a partition of X. This partition can be refined using dynamics. Indeed, we can consider the partition

$$\bigvee_{i=0}^{n-1} T^{-i} P_1 = \left\{ \cap_{j=0}^{n-1} A_{i_j} : i_j \in \{1, ..., n\} \right\}.$$

The entropy of (T, μ) with respect to P_1 is defined by

$$h(T, P_1) = \lim_{n \to \infty} \frac{1}{n} H\left(\bigvee_{i=0}^{n-1} T^{-i} P_1 \right).$$

Finally,

Definition 2.4.4. Let $T : X \to X$ be a continuous map of the compact metric space X and let $\mu \in \mathcal{M}_T(X)$ then the entropy of T with respect to μ is defined by

$$h_\mu(T) = \sup \{h(T, P) : P \text{ finite partition of } X\},$$

where $\mathcal{M}_T(X)$ is the set of all T-invariant probability measures.

2.4.3 Gibbs Measures

Gibbs measure were were first introduced into ergodic theory and dynamical systems by Sinai in the paper [47].

Definition 2.4.5. Let T be a continuous transformation from compact metric space X to itself. The transformation T is said to be *topologically mixing* if for any U, V nonempty open subsets of X, there is an N so that $T^{-m}U \cap V \neq 0 \ \forall \ m \geq N$.

Lemma 2.4.2. *$\sigma : \Sigma_A \to \Sigma_A$ is topologically mixing if and only if $A^M > 0$ (that is, $A^M_{i,j} > 0 \ \forall \ i, j$) for some M.*

Definition 2.4.6. Let T be a continuous transformation from a compact metric space X to itself. Suppose μ is a probability measure on X and let $\phi \in C(X, \mathbb{R})$. The measure μ is called *Gibbs measure with respect to* ϕ if there exits constants $A, B > 0$ and $P \in \mathbb{R}$ such that

$$A \leq \frac{\mu[x_1 \cdots x_n]}{\exp\left(-Pn + \sum_{k=0}^{n-1} \phi(T^k x)\right)} \leq B \qquad \forall x \in X, \quad \forall n \geq 0 \tag{2.1}$$

where $[x_1 \cdots x_n]$ are the cylinder sets.

In Section 3.2 the Gibbs measure will be shown to be unique where we will appeal to the properties of the transfer operator on space of functions on the shift space Σ_A and the shift map σ to achieve this. The introduction of the transfer operator is deferred until Section 3.1 where it will be discussed.

2.4.4 Variational Principle

Let $f : \Sigma_A^+ \to \mathbb{C}$ be complex valued function defined on Σ_A^+. Define

$$\operatorname{var}_n(f) = \sup\left\{|f(x) - f(y)| : x, y \in \Sigma_A^+ \text{ and } x_i = y_i \text{ for } i = 0, \ldots, n-1\right\}$$

to be the nth variation of f. One observes that $\operatorname{var}_n(f)$ measures how much f can vary on cylinders of length n.

Definition 2.4.7. For $f : \Sigma_A^+ \to \mathbb{C}$ define

$$|f|_\theta = \sup\left\{\frac{\operatorname{var}_n(f)}{\theta^n} : n = 0, 1, 2, \ldots\right\}$$

to be the least Hölder of f.

We define

$$F_\theta(\mathbb{C}) = \left\{f : \Sigma_A^+ \to \mathbb{C} : |f|_\theta < \infty\right\}$$

similarly define

$$F_\theta(\mathbb{R}) = \left\{f : \Sigma_A^+ \to \mathbb{R} : |f|_\theta < \infty\right\}.$$

Definition 2.4.8. Suppose μ is a probability measure on the space Σ_A^+ satisfying $\mu(B) = \mu\left(\sigma^{-1}B\right)$ for any $B \in \Sigma_A^+$ then μ is said to be $\sigma-$ *invariant.* We denote $\mathcal{M}_\sigma\left(\Sigma_A^+\right)$ for the set of all $\sigma-$invariant probability measures on Σ_A^+.

Theorem 2.4.3. *[11] Let $\phi \in F_\theta(\mathbb{R})$, Σ_A^+ is topologically mixing and μ_ϕ the Gibbs* *measure of ϕ. Then ϕ is the unique $\mu \in \mathcal{M}_\sigma\left(\Sigma_A^+\right)$ for which*

$$P(\phi) = h_\mu(\sigma) + \int \phi d\mu. \tag{2.2}$$

We will define the measure $\mu \in \mathcal{M}_\sigma\left(\Sigma_A^+\right)$ satisfying equation(2.2) to be the *equilibrium measure* for the potential ϕ.
We will see in the next chapter that the pressure $P(f)$ for $f \in F_\theta(\mathbb{R})$ can be expressed as $P(f) = \log \lambda$ where λ is the maximal eigenvalue for the transfer operator L_f. We can regard pressure as a functional $F_\theta(\mathbb{R}) \to \mathbb{R}$. By using the variational principle, one can prove the following properties of this functional.

Theorem 2.4.4. *1. Pressure is monotone: if $f, g \in F_\theta(\mathbb{R})$ and $f \leq g$ then $P(f) \leq P(f)$.*

2. Pressure is convex: if $f, g \in F_\theta(\mathbb{R})$ and $\alpha \in [0,1]$ then

$$P(\alpha f + (1-\alpha) g) \leq \alpha P(f) + (1-\alpha) P(f).$$

3. If f is cohomologous to $g + c$, where $f, g \in F_\theta(\mathbb{R})$ and $c \in \mathbb{R}$ then

$$P(f) = P(g) + c.$$

Proof. Throughout, let $f, g \in F_\theta(\mathbb{R})$.
If $f \leq g$ then

$$\begin{aligned} P(f) &= \sup\left\{h_\mu(\sigma) + \int f d\mu\right\} \\ &\leq \sup\left\{h_\mu(\sigma) + \int g d\mu\right\} \\ &= P(g). \end{aligned}$$

where the suprema are taken over all σ-invariant probability measures; hence 1. holds.
Statement 2. follows by noting that, for $\alpha \in [0,1]$,

$$\begin{aligned} P(\alpha f + (1-\alpha) g) &= \sup\left\{h_\mu(\sigma) + \int \alpha f + (1-\alpha) g d\mu\right\} \\ &\leq \sup\left\{\alpha\left(h_\mu(\sigma) + \int f d\mu\right)\right\} \\ &+ \sup\left\{(1-\alpha)\left(h_\mu(\sigma) + \int g d\mu\right)\right\} \\ &= \alpha P(f) + (1-\alpha) P(g). \end{aligned}$$

where the suprema is taken over all σ-invariant probability measures.
Statements 3 follow immediately from the variational principle. $\square$

2.5 Multifractal Analysis

The theory of multifractal analysis can be traced back to the authors, Hasley et al. in [22]. The early work in multifractal analysis of dynamical systems was done by Collet et al. in [14] for invariant measures for interval Markov maps.
In this section we will present a motivation for the study of a general theory of multifractal analysis by restricting ourselves to the multifractal analysis for Gibbs measures which are invariant for conformal repellers.

Pointwise Dimension

Definition 2.5.1. Let $x \in X$ and suppose we have a Borel probability measure ν on X, we define the *pointwise dimension* at point $x \in X$ by

$$d_\nu(x) = \lim_{r \to 0} \frac{\log \nu(B(x,r))}{\log r}.$$

The limit exists at a point x, if the following two limits exist and are equal,

$$\overline{d}_\nu(x) = \lim_{r \to 0} \sup \frac{\log \nu(B(x,r))}{\log r},$$

and

$$\underline{d}_\nu(x) = \lim_{r \to 0} \inf \frac{\log \nu(B(x,r))}{\log r}.$$

The two limits are called *upper pointwise dimension* and *lower pointwise dimension* respectively.

However most of the measures will be nicely behaved in that we have existence for almost every point of the space under consideration.

Let ν be a finite Borel measure in a metric space X, of particular interest are the level sets

$$K_\alpha = \{x \in X : d_\nu(x) = \alpha\}$$

for each $\alpha \in [-\infty, +\infty]$. Denote by Y the set of points in X which do not attain any of the α's. There is a natural *multifractal decomposition* of X by,

$$X = Y \cup \bigcup_{\alpha \in [-\infty, +\infty]} K_\alpha.$$

Definition 2.5.2. Let M be a smooth manifold and $T : M \to M$ be a $C^{1+\epsilon}$ transformation. A compact subset J of M is *repeller* if

1. $T(J) = J$,
2. there exists $C > 0$ and $\tau > 1$ such that

   $$\|D_x T^n u\| \geq C\tau^n \|u\|$$

   for all $x \in J$ and $u \in T_x M$,
3. there exists an open subset $V \subset M$ such that $V \supset J$ and $J \cap T^n V = \emptyset$ for all $n \geq 1$.

The transformation T is called *conformal* if $D_x T$ is a multiple of an isometry for any $x \in J$.

Definition 2.5.3. The *dimension spectrum* of the measure ν is the function

$$f_\nu(\alpha) = \dim_H K_\alpha.$$

Let J be a conformal repeller of a $C^{1+\epsilon}$ transformation T, for some $\epsilon > 0$. Suppose also that $\psi : J \to \mathbb{R}$ is a Holder continuous function satisfying $P(\psi) = 0$ for the topological pressure function P defined on J.

Bowen's dimension formula

We present a criterion of calculating Hausdorff dimension for conformal repellers which was originally discovered by Bowen on his work for quasi-circles in [12].

Theorem 2.5.1. *If J is a repeller for a $C^{1+\epsilon}$ transformation T, for some $\epsilon \in (0, 1]$, such that T is conformal on J,*

$$\dim_H J = s,$$

where s is the unique real number such that $P(s\phi)$, for the function $\phi : J \to \mathbb{R}$ defined by

$$\phi(x) = -\log \|D_x T\|.$$

For the proof we refer the reader to Barreira's book [5].

Multifractal theorem

Continuing with the formulation in the multifractal analysis from Definition (2.5.3) we note that there is a function $\kappa : \mathbb{R} \to \mathbb{R}$ for the pressure function such that

$$P\left(-\kappa\left(q\right)\log\left\|DT\right\| + q\psi\right) = 0$$

for every $q \in \mathbb{R}$.
It is easy to show that the function $\phi_q = -\kappa\left(q\right)\log\left\|DT\right\| + q\psi$ is Holder continuous. One establishes that the function κ is well defined from the real analyticity of the pressure function P (see [43]) and then invoking the implicit function theorem.

We define the bounds for α, $\underline{\alpha}$ and $\overline{\alpha}$.

Definition 2.5.4. Let

$$\overline{\alpha} = \sup_{\nu_q} \frac{\int_J d\nu_q}{\int_J \log\left\|DT\right\| d\nu_q}$$

and

$$\underline{\alpha} = \inf_{\nu_q} \frac{\int_J d\nu_q}{\int_J \log\left\|DT\right\| d\nu_q}.$$

Let μ be the equilibrium measure of $-\left(\dim_H J\right)\log\left\|DT\right\|$ and ν_q be the equilibrium measure of the function ϕ_q. We now state the main multifractal analysis theorem for equilibrium measures which is due to Pesin and Weiss in [37].

Theorem 2.5.2. *Let J be a repeller of a $C^{1+\epsilon}$ transformation T, for some $\epsilon > 0$, such that T is a conformal and topologically mixing on J. If ν is the equilibrium measure of a Holder continuous function $\psi : J \to \mathbb{R}$ with $P\left(\psi\right) = 0$, then:*

1. *the set $K_{\alpha(q)}$ is T - invariant and dense for every $q \in \mathbb{R}$;*

2. *if $\nu = \mu$, then $\underline{\alpha} = \overline{\alpha} = \dim_H J$ and f_ν is a delta function;*

3. *$f_\nu : \left(\underline{\alpha}, \overline{\alpha}\right) \to \mathbb{R}$ is analytic and strictly convex;*

4. *if f_ν is the Legendre transform of κ, that is, for each $q \in \mathbb{R}$ we have*

$$f_\nu\left(\alpha\left(q\right)\right) = \kappa\left(q\right) + q\alpha\left(q\right);$$

5. *for each* $q \in \mathbb{R}$ *we have* $\nu_q\left(K_{\alpha(q)}\right) = 1$ *and*

$$\lim_{r \to 0} \frac{\log \nu_q\left(B\left(x, r\right)\right)}{\log r} = \kappa\left(q\right) + q\alpha\left(q\right),$$

for ν_q*-almost every* $x \in K_{\alpha(q)}$.

A detailed account of the proof of this theorem can be found in the paper [37] for Gibbs measures and the book of Barreira in [4]. A similar result was earlier done also by Pesin and Weiss for conformal expanding maps and Moran-like geometric constructions in [36].

2.6 Linear Operators on Banach spaces

2.6.1 Spectral radius

Let $\mathcal{B}\left(X, Y\right)$ be the space of bounded linear operators from a normed linear space X to another normed linear space Y.

We state a well known proposition.

Proposition 2.6.1. *Given* $\mathcal{B}\left(X, Y\right)$ *described above. If in addition* Y *is a complete normed linear space then* $\mathcal{B}\left(X, Y\right)$ *is a Banach space.*

In particular we will write $\mathcal{B}\left(X\right) = \mathcal{B}\left(X, X\right)$ if X is a complete normed linear space.

Definition 2.6.1. Let $T \in \mathcal{B}\left(X\right)$. The set

$\sigma\left(T\right) = \{\lambda \in \mathbb{C} \mid T - \lambda I$ is not invertible$\}$ is called *the spectrum* of T.

The *spectral radius* for $T \in \mathcal{B}\left(X\right)$ is defined by

$$R\left(T\right) = \sup_{\lambda \in \sigma(T)} \left|\lambda\right|.$$

2.6.2 Riesz Representation Theorem

Theorem 2.6.2. *Let* X *be a compact metric space and* $J : C\left(X, \mathbb{C}\right) \to \mathbb{C}$ *a continuous linear map such that* J *is a positive operator and* $J\left(1\right) = 1$. *Then there exists* $\mu \in \mathcal{M}\left(X\right)$, *where* $\mathcal{M}\left(X\right)$ *is the set of Borel probability measures on* X, *such that* $J\left(f\right) = \int_X f d\mu$ *for all* $f \in C\left(X, \mathbb{C}\right)$.

The theorem provides an convenient procedure to construct Borel probability measures on the space X. The measure constructed in this fashion is unique.
The proof can be found in the book of Parthasarathy [33].

2.6.3 Schauder Tychonov Theorem

Julius Schauder was born of a Jewish family in 1899 in Lemberg, The Austrian Empire now Lviv Ukraine. During World War I Schauder served in the Austro-Hungarian and Polish armies. Schauder completed his thesis *The theory of surface measure* at Jan Kazimierz University under Steinhaus in 1923.
Schauder generalised Brouwer's fixed point theorems on finite dimensional spaces to Banach spaces in 1930.

At the beginning of World War II in 1939 Schauder enjoyed the protection of the new Soviet administration but things turned for the worst when the Germans entered Poland in 1941. Schauder was executed by the Gestapo in 1943.

Schauder generalised Brouwer's fixed point theorems on finite dimensional spaces to Banach spaces in 1930 which we state below.

Theorem 2.6.3. *Let K be a convex compact subset of a locally convex topological vector space, with ϕ a continuous function from K to K. Then ϕ has a fixed point.*

We refer the reader to the book of Rudin [40] for the proof of this theorem.

2.6.4 Arzela-Ascouli Theorem

Giudo Ascoli was born in Livorno, Italy in 1887 to a Jewish family. Ascoli worked as a secondary school teacher for several years in different schools in Italy before the break out of World War I. His poor health would at first seem to be a blessing in disguise as military conscription intensified, however Ascoli was later send to fight in the frontline as an officer in the 44th Field Artillery Regiment in March 1917. After the war in 1920 Ascoli moved to Turin to take up a teaching position at the Technical Institute. With an improved health and better research environment Ascoli began his research in Analysis at Turin. The quality of his work earned him a position at the University of Cagliari in 1930 as chair of algebraic analysis. The second of interruptions to his research came when the the Fascist government passed a law barring people with Jewish descent from government institutions and education establishments in 1938. Ascoli would later work at the University of Milan from

1945 to 1948 where he again produced several research papers.

The concept of equicontinuity was studied by both Giulio Ascoli and Cesare Arzela and the equicontinuity theorem bears both their names.

Definition 2.6.2. A subset $\mathcal{F}$ of $\mathcal{B}(X, Y)$ is said to be *equibounded*, or *uniformly bounded*, by some constant $M > 0$ if we have $\|f(x)\|_Y \leq M$ $\forall x \in X$ and $\forall f \in \mathcal{F}$.

Definition 2.6.3. A subset $\mathcal{F}$ of $\mathcal{B}(X, Y)$ of functions is said to be *equicontinuous* if for all $\epsilon > 0$ there is $\delta > 0$ such that

$$\|f(x) - f(y)\|_Y < \epsilon \qquad \forall x, y \in X \quad with \quad d_X(x, y) < \delta, \quad and \ \ \forall f \in \mathcal{F}.$$

We now present the theorem.

Theorem 2.6.4. *Let X be a compact metric space. A subset $\mathcal{F}$ of $C(X, \mathbb{R})$ is relatively compact if and if it is uniformly bounded and equicontinuous.*

For the proof of this theorem we refer the reader to the books [19], [40].

2.6.5 Perturbation Theorem

Kato survived the World War II living in Japan where he was forced to live in the countryside from 1941 after completing B.S. in Physics. After the war Kato graduated from the University of Tokyo in 1951 with a doctorate thesis titled *On the convergence of the perturbation method*. Kato would be promoted to professorship of Physics at the University of Tokyo in 1958. He moved to the University of California, Berkeley , USA where he continued to produce important results including his work on introducing "techniques for studying the partial differential equations of incompressible fluid mechanics, the Navier-Stokes equations."

Theorem 2.6.5. *(Kato, Rellich) Let V be a complex Banach space and L(V) be a Banach space of linear operators on V. If $S_0 \in L(V)$ has a simple eigenvalue α_0 which is isolated point of the spectrum of S_0 with the associated eigenvector v_0, then for every $\epsilon > 0$ there exists $\delta > 0$ such that $\|S - S_0\| < \delta$ then the operator S has a simple eigenvalue $\alpha(S)$ and associated eigenvector $v(S)$ such that*

1. *the functions $S \mapsto \alpha(S)$ and $S \mapsto v(S)$ are holomorphic on $\{\|S - S_0\| \leq \delta\}$, and*

2. *if $\{\|S - S_0\| \leq \delta\}$ the spectrum$(S) \cap D(\alpha_0, \epsilon) = \{\alpha(S)\}$.*

where $D(\alpha_0, \epsilon)$ is a disk in the complex plane of radius ϵ with centre α_0.

The proof of this theorem can be found in the book of Kato [26].

Chapter 3

Transfer Operator for shifts

3.1 Transfer operator

We will be interested in spaces of functions defined on Σ_A^+. Among the continuous functions defined on Σ_A^+ we restrict ourselves to functions satisfying the Hölder continuity condition.
The $|\cdot|_\theta$ defined in the Definition(2.4.7) has the property that $|f|_\theta < \infty$ if and only if there exists $C > 0$ such that

$$|f(x) - f(y)| \leq Cd_\theta(x, y) \quad \text{for all} \quad x, y \in \Sigma_A^+. \tag{3.1}$$

and $|f|_\theta$ is the least such $C > 0$ for which inequality (3.1) holds. Note that the condition (3.1) says that f is Lipschitz continuous with respect to the metric d_θ. It is customary in thermodynamic formalism to say instead that f is Hölder of exponent θ. This is because $|f(x) - f(y)| \leq Cd_\theta(x, y)^\alpha$, then f is Lipschitz continuous with respect to d_{θ^α}. One observes that if f is necessarily continuous. We shall also be interested in the space $F_\theta(\mathbb{R})$ of real-valued Holder functions.

It is worth to note that $|\cdot|_\theta$ is a semi-norm, but not a norm. This is because $|f|_\theta = 0$ if f is a constant function. We define a norm on $F_\theta(\mathbb{C})$ by setting

$$\|f\|_\theta = |f|_\infty + |f|_\theta$$

where $|f|_\infty = \sup|f(x)|$ is the uniform norm of f. We have the following important result.

Proposition 3.1.1. *The space $F_\theta(\mathbb{C})$ is a complex Banach space with respect to the norm $\|\cdot\|_\theta$.*

A particularly important and tractable class of functions are those which only depend on finitely many co-ordinates. Let $f : \Sigma_A^+ \to \mathbb{C}$. We say that f is *locally constant* if f depends on only finitely many co-ordinates of Σ_A^+. That is, there exists $n \geq 0$ such that $f(x) = f(x_0, x_1, \ldots, x_{n-1})$. Equivalently, f is constant on cylinders of length n.
Clearly, if f is locally constant, then with n as above, $var_m(f) = var_n(f)$ for all $m \geq n$. Hence $|f|_\theta < \infty$ for any $\theta \in (0,1)$. Hence $f \in F_\theta(\mathbb{C})$ for all $\theta \in (0,1)$.

Transfer operators for the shift spaces

Let $f \in F_\theta(\mathbb{R})$. We define the transfer operator or Ruelle operator to be the map

$$\mathcal{L}_f : F_\theta(\mathbb{C}) \to F_\theta(\mathbb{C})$$

given by

$$\mathcal{L}_f w(x) = \sum_{\sigma(y)=x} e^{f(y)} w(y).$$

It is clear that $\mathcal{L}_f$ is a linear operator on the Banach space $F_\theta(\mathbb{C})$. It is also straightforward to check that $\mathcal{L}_f$ is bounded.

Proposition 3.1.2. *Let $f \in F_\theta(\mathbb{C})$. Then the transfer operator $\mathcal{L}_f : F_\theta(\mathbb{C}) \to F_\theta(\mathbb{C})$ is a bounded linear operator.*

We will be interested in understanding the spectral properties of $\mathcal{L}_f$, in particular we want to determine the eigenvalues of $\mathcal{L}_f$.

Ruelle's Perron-Frobenius Theorem

In this section we study the case where the weight function f is real and we consider the associated transfer operator acting on the real Banach space $F_\theta(\mathbb{R})$ of real-valued functions.
The first step to this is the following result. We assume, for convenience, that $\mathcal{L}_f 1 = 1$.

Proposition 3.1.3. *(Lasota-Yorke inequality) Let $f \in F_\theta(\mathbb{C})$ and suppose that $\mathcal{L}_f 1 = 1$. Then for all $w \in F_\theta(\mathbb{C})$ and $n \geq 0$ we have*

$$\left|\mathcal{L}_f^n w\right|_\theta \leq C\,|w|_\infty + \theta^n\,|w|_\theta$$

where $C > 0$ depends only on f and θ.

Proof. Throughout, if $x = (x_0, x_1, \ldots)$ then $ix = (i, x_0, x_1, \ldots)$ (and we assume that $A_{i,x_0} = 1$). Note that if $x, y \in \Sigma_A^+$ then $d_\theta(ix, iy) \le \theta d_\theta(x, y)$.
The proof is by induction on n. When $n = 1$, we estimate

$$\begin{aligned} |\mathcal{L}_f w(x) - \mathcal{L}_f w(y)| &\le \sum |e^{f(ix)} w(ix) - e^{f(iy)} w(iy)| \\ &\le \sum e^{f(iy)} |e^{f(ix)-f(iy)} - 1| |w(ix)| \\ &\quad + \sum e^{f(iy)} |w(ix) - w(iy)| \end{aligned}$$

(the sums are all over i for which $A_{i,\ x_0} = 1$). Noting that

$$\sup_{x \ne y} \frac{\left|e^{f(ix)-f(iy)} - 1\right|}{d(x, y)} \le \sum_{r=1}^{\infty} \frac{\theta^r |f|_\theta^r d(x, y)^{r-1}}{r!} \le C_0$$

for some constant $C_0 > 0$ and recalling that $\sum e^{f(ix)} = 1$ (as $\mathcal{L}_f 1 = 1$), we obtain

$$|\mathcal{L}_f w|_\theta \le C_0 |w|_\infty + \theta |w|_\theta .$$

Using induction, we assume that $\left|\mathcal{L}_f^n w\right|_\theta \le C |w|_\infty + \theta^n |w|_\theta$. Then

$$\begin{aligned} \left|\mathcal{L}_f^{n+1} w\right|_\theta &= \left|\mathcal{L}_f \left(\mathcal{L}_f^n w\right)\right|_\theta \\ &\le C_0 \left|\mathcal{L}_f^n w\right|_\infty + \theta \left|\mathcal{L}_f^n w\right|_\theta \end{aligned}$$

By induction we have

$$\begin{aligned} \left|\mathcal{L}_f^{n+1} w\right|_\theta &\le C_0 |w|_\infty + \theta \left(C |w|_\infty + \theta^n |w|_\theta\right) \\ &= (C_0 + C\theta) |w|_\infty + \theta^{n+1} |w|_\theta \\ &= C_1 |w|_\infty + \theta^{n+1} |w|_\theta . \end{aligned}$$

□

We study the spectrum of L_f in the case where $f \in F_\theta(\mathbb{R})$ is real-valued.

Theorem 3.1.4. *(Ruelle's Perron-Frobenius Theorem) Let A be an aperiodic matrix with entries in $\{0, 1\}$, with associated shift of finite type Σ_A^+. Let $f \in F_\theta(\mathbb{R})$.*

1. *There is a simple maximal positive eigenvalue λ of $\mathcal{L}_f : F_\theta(\mathbb{C}) \to F_\theta(\mathbb{C})$ with a corresponding strictly positive eigenfunction h.*

2. *The remainder of the spectrum is contained inside a disc in $\mathbb{C}$ of radius strictly smaller than λ.*

3. *There is a unique probability measure μ such that*

$$\int \mathcal{L}_f v d\nu = \lambda \int v d\nu \quad for \quad all \quad v \in C\left(\Sigma_A^+, \mathbb{R}\right).$$

Moreover, if h is as in 1. and $\int h d\nu = 1$ then the measure μ defined by $d\mu = h d\nu$ is a σ-invariant probability measure.

4. *If h is as in 1. and $\int h d\nu = 1$ then*

$$\frac{1}{\lambda^n} \mathcal{L}_f^n v \to \int v d\nu$$

uniformly for all $v \in C\left(\Sigma_A^+, \mathbb{R}\right)$.

We give the proof which follows the proof in book of Parry and Pollicott [34].

Proof. We first focus on the Schauder-Tychonov fixed point theorem. The Schauder-Tychonov theorem is a surprisingly general fixed point theorem in the context of convex sets that generalises the Brouwer fixed point theorem. It says the following: Let Λ be a convex compact subset of a normed vector space X and suppose that $L : \Lambda \to \Lambda$ is a continuous transformation. Then L has a fixed point in Λ.

The second step is to show the existence of λ and h. Let

$$\Lambda = \left\{ g \in C\left(\Sigma_A^+, \mathbb{R}\right) : 0 \leq g(x) \leq 1, \; g(x) \leq g(y) \exp\left(\frac{|f|_\theta \theta^n}{1-\theta}\right) \right.$$
$$\left. \text{whenever } x_j = y_j, \; j = 0, 1, \ldots, n-1 \right\}.$$

We claim that Λ is convex and uniformly closed. Suppose that $x, y \in \Sigma_A^+$ are such that $x_j = y_j$, $j = 0, 1, \ldots, n-1$. Then from the definition of Λ it follows that

$$\begin{aligned} |g(x) - g(y)| &\leq |g(y)| \left(\exp\left(\frac{|f|_\theta \theta^n}{1-\theta}\right) - 1 \right), \\ &\leq |g|_\infty \frac{|f|_\theta \theta^n}{1-\theta} \exp\left(\frac{|f|_\theta \theta^n}{1-\theta}\right). \end{aligned}$$

Hence Λ is uniformly equicontinuous.
As Λ is uniformly closed and uniformly equicontinuous, it follows from the Arzela-Ascoli theorem that Λ is uniformly compact. It is also worth to note that $\Lambda \subset F_\theta(\mathbb{R})$.

Define a family of linear operators $\Lambda \to \Lambda$ as follows. For $n \geq 1$ define

$$\mathcal{L}_n g(x) = \frac{\mathcal{L}_f\left(g(x) + \frac{1}{n}\right)}{\left|\mathcal{L}_f\left(g(x) + \frac{1}{n}\right)\right|_\infty}.$$

Clearly $|\mathcal{L}_n g|_\infty = 1$. Suppose $x, y \in \Sigma_A^+$ are such that $x_j = y_j$ for $0 \leq j \leq k$. Then

$$\mathcal{L}_f\left(g + \frac{1}{n}\right)(x) \leq \mathcal{L}_f\left(g + \frac{1}{n}\right)(y)\exp\left(\frac{\theta^k}{1-\theta}|f|_\theta\right).$$

In particular,

$$\mathcal{L}_n g(x) \leq \mathcal{L}_n g(y)\exp\left(\frac{\theta^k}{1-\theta}|f|_\theta\right),$$

so that $\mathcal{L}_n$ is well-defined operator $\Lambda \to \Lambda$, for each n.
Since Λ is a complex uniformly compact subset of $C\left(\Sigma_A^+, \mathbb{R}\right)$, we can apply the Schauder-Tychonov theorem for each $\mathcal{L}_n : \Lambda \to \Lambda$. Hence, for each $n \geq 1$, there exists $h_n \in \Lambda$ with $\mathcal{L}_f\left(h_n + \frac{1}{n}\right) = \lambda_n h_n$, where $\lambda_n = \left|\mathcal{L}_f\left(h_n + \frac{1}{n}\right)\right|_\infty$.
As Λ is uniformly compact, h_n has a uniformly convergent subsequence with limit $h \in \Lambda$. As $\Lambda \subset F_\theta(\mathbb{R})$, we have $h \in F_\theta(\mathbb{R})$.
By continuity, $\mathcal{L}_f h = \lambda h$ where $\lambda = |\mathcal{L}_f h|_\infty$.

In the third stage we show that the eigenvalue λ is positive .

To see that λ is positive, we note that

$$\begin{aligned}\lambda_n h_n(x) &= \sum_{\sigma(y)=x} e^{f(y)}\left(h_n(y) + \frac{1}{n}\right)\\ &\geq \left(\inf h_n + \frac{1}{n}\right)e^{-|f|_\infty}.\end{aligned}$$

Hence $\lambda_n \geq e^{-|f|_\infty}$ for each $n \geq 1$. Hence $\lambda \geq e^{-|f|_\infty}$.

In the fourth stage we show that the eigenfunction h can be taken to be strictly positive.
As $h_n \in \Lambda$ it follows from the definition of Λ that $h_n(x) \geq 0$. Hence $h_n(x) \geq 0$ for all $x \in \Sigma_A^+$. Suppose for a contradiction that there exists x_0 for which $h(x_0) = 0$. Iterating the eigenvalue equation $\mathcal{L}_f h = \lambda h$ gives that

$$\sum_{\sigma^n y = x_0} e^{f^n y} h(y) = \lambda^n h(x_0) = 0.$$

In particular, $h(y) = 0$ whenever $\sigma^n y = x_0$. As the set of such y is dense in Σ_A^+, by the aperiodicity of A, it follows that h is identically zero, a contradiction.

In the fifth step we show that λ is simple.
We know that $\mathcal{L}_f h = \lambda h$. Suppose that g is another continuous eigenfunction for $\mathcal{L}_f$ corresponding to the eigenvalue λ. Let $t = \inf g(x)/h(x)$. By compactness this infimum is achieved at some point: $t = g(x_0)/h(x_0)$, say. Then $g(x_0) - th(x_0) = 0$. Repeating the argument from Step 4 shows that $g(x) - th(x) = 0$ whenever $y \in \Sigma_A^+$ is such that $\sigma^n y = x$. Again, by aperiodicity the set of such y is dense, hence $g(x) - th(x) = 0$ for all x, that is, g is a scalar multiple of h. Hence the eigenspace corresponding to λ is one-dimensional.

In the sixth step we show that $\mathcal{L}_f$ can be reduced to a normalised form.
Let h, λ be as above, so that $\mathcal{L}_f h = \lambda h$ and $h > 0$. Define

$$g = f - \log h\sigma + \log h - \log \lambda.$$

Then

$$\begin{aligned}\mathcal{L}_g w(x) &= \sum_{\sigma(y)=x} e^{g(y)} w(y) \\ &= \frac{1}{\lambda} \sum_{\sigma(y)=x} e^{f(y)} \frac{h(y)}{h(\sigma(y))} w(y) \\ &= \frac{1}{\lambda} \frac{1}{h(x)} \sum_{\sigma(y)=x} e^{f(y)} h(y) w(y).\end{aligned}$$

Hence if we let M_h denote the linear operator that multiplies a function by h, that is

$$(M_h w)(x) = h(x) w(x),$$

then

$$\mathcal{L}_g = \lambda^{-1} M_h^{-1} \mathcal{L}_f M_h. \tag{3.2}$$

As $\mathcal{L}_f h = \lambda h$, it follows from relation(3.2) that $\mathcal{L}_g 1 = 1$, that is, g is normalised. Since the spectrum of $\mathcal{L}_f$ is the spectrum of $\mathcal{L}_g$ scaled by a factor of $1/\lambda$, it is sufficient to prove the remainder of the theorem under the hypothesis that $\mathcal{L}_f 1 = 1$.

In the seventh step we show existence of ν.
The operator $\mathcal{L}_f^*$ acts on $C(\Sigma_A^+, \mathbb{R})^*$ and preserves the convex weak-$*$ compact

subset of functionals that correspond to σ-invariant probability measures. By the Schauder-Tychonov theorem, $\mathcal{L}_f^*$ has a fixed point ν.

In the eighth step we show the uniqueness of ν.
We note that

$$\mathrm{var}_k \mathcal{L}_f^n w \leq \left|\mathcal{L}_f^n w\right|_\theta \theta^k \leq C\theta^k \left|w\right|_\infty + \theta^{n+k} \left|w\right|_\theta$$

by Proposition (3.1.3). Hence, for fixed $w \in F_\theta(\mathbb{R})$, the set $\left\{\mathcal{L}_f^n w\right\}_{n=1}^{\infty}$ is a uniformly equicontinuous subset of $C\left(\Sigma_A^+, \mathbb{R}\right)$ and so has a convergent subsequence, $\mathcal{L}_f^{n_k} w \to w^*$ uniformly. We claim that w^* is constant. To see that w^* is constant note that, as $\mathcal{L}_f$ is a convex combination of preimages, we have that $\sup w \geq \sup \mathcal{L}_f w \geq \cdots$. Hence $\sup \mathcal{L}_f^{n_k} w^* = \sup w^*$. Choose $x_{n_k} \in \Sigma_A^+$ such that $\mathcal{L}_f^{n_k} w^*\left(x_{n_k}\right) = \sup w^*$ (so that, in particular $w^*\left(x_0\right) = \sup w^*$). Then

$$\mathcal{L}_f^{n_k} w^*\left(x_{n_k}\right) = \sum_{\sigma^{n_k} y = x_{n_k}} e^{f^{n_k}(y)} w^*\left(y\right) = w^*\left(x_0\right).$$

This is a convex combination of the points $w^*\left(y\right)$. Hence $w^*\left(y\right) = w^*\left(x_0\right)$ whenever $\sigma^{n_k}\left(y\right) = x_{n_k}$. As the set of such y is dense, it follows that w^* is constant.
To see that ν is unique, we note that

$$w^* = \int w^* d\nu = \lim_{k\to\infty} \int \mathcal{L}_f^{n_k} w d\nu = \int w d\nu.$$

We can repeat this argument through any subsequence to see that $\mathcal{L}_f^n w \to \int w d\nu$ for all $w \in F_\theta(\mathbb{R})$. By approximation, this is also true for all $w \in C\left(\Sigma_A^+, \mathbb{R}\right)$. Hence by the Riesz Representation Theorem, ν is uniquely determined by the condition that $\mathcal{L}_f^* \nu = \nu$.

In the ninth step the remainder of the spectrum is estimated.
We have seen in Step 8 that if $w \in F_\theta(\mathbb{R})$ then $\mathcal{L}_f^n w \to \int w d\nu$. Thus the constant functions are eigenfunctions with eigenvalue 1. To show that the remainder of the spectrum of $\mathcal{L}_f : F_\theta(\mathbb{C}) \to F_\theta(\mathbb{C})$ lies in a disc of radius strictly less than 1 it is sufficient to prove that $\mathcal{L}_f$, acting on the space $\mathbb{C}^\perp = \left\{w \in F_\theta(\mathbb{R}) : \int w d\nu = 0\right\}$, has a spectral radius strictly less than 1. By Proposition (3.1.3), we have

$$\left|\mathcal{L}_f^{n+k} w\right|_\theta \leq C\left|\mathcal{L}_f^k w\right|_\infty + \theta^n \left|\mathcal{L}_f^k w\right|_\theta \leq C\left|\mathcal{L}_f^k w\right|_\infty + C\theta^n \left|w\right|_\infty + \theta^{n+k}\left|w\right|_\theta.$$

Moreover, by Step 8, as $w \in \mathbb{C}^\perp$ we have that $\mathcal{L}_f^n w \to 0$ on the uniformly compact set $\{w \in \mathbb{C}^{\perp} : |w|_{\theta} < 1\}$. Fix a choice of $\epsilon > 0$. Then $\left|\mathcal{L}_f^k w\right|_{\theta} < \epsilon$ provided k is sufficiently large. The spectral radius formula tells us that the spectral radius of $\mathcal{L}_f$ on $\mathbb{C}^{\perp}$ is bounded above by

$$\inf\left\{\left|\mathcal{L}_f^{n+k} w\right|_{\theta}^{1/(n+k)} : w \in \mathbb{C}^{\perp}, |w|_{\theta} \leq 1\right\} \leq \epsilon^{1/(n+k)}.$$

The claim follows. $\square$

3.2 Transfer operator and Gibbs measures

Theorem 3.2.1. *Suppose Σ_A^+ is topologically mixing and $f \in F_{\theta}(\mathbb{R})$. The σ-invariant Gibbs measure μ on Σ_A^+ is unique and one can find constants $c_1 > 0$, $c_2 > 0$ and P such that*

$$c_1 \leq \frac{\mu\{y : y_i = x_i \ \forall i = 0, \ldots, m\}}{\exp\left(-Pm + \sum_{k=0}^{m-1} f(\sigma^k x)\right)} \leq c_2 \tag{3.3}$$

for every $x \in \Sigma_A^+$ and $m \geq 0$.

We state the sketch of the proof of this theorem by Bowen [11].

Proof. Let $f \in F_{\theta}(\mathbb{R})$ and we adopt the μ, ν and h given by Ruelle's theorem (3.1.4). These assumptions also mean that μ is a σ-invariant probability measure which is given by $\mu(w) = \int w(x) h(x) d\nu(x)$, for any $w \in F_{\theta}(\mathbb{R})$ and $x \in \Sigma_A^+$ once we know that μ is mixing which is the other ingredient.

But first we establish the two characteristics of the measure μ on the space Σ_A^+ under the shift map σ, which are that this measure is both invariant and mixing.

Let $w \in C\left(\Sigma_A^+, \mathbb{R}\right)$ and write $S_m f(x) = \sum_{k=0}^{m-1} f\left(\sigma^k x\right)$ and recalling that after m iterations

$$\left(\mathcal{L}_f^m w\right)(x) = \sum_{\sigma^m y = x} e^{S_m f(y)} w(y) \tag{3.4}$$

allowing us to rewrite $\left(\left(\mathcal{L}_f^m w\right) \cdot g\right)(x) = \mathcal{L}_f^m\left(w \cdot (g \circ \sigma^m)\right)$ for any $g \in C\left(\Sigma_A^+, \mathbb{R}\right)$.

We then observe that

$$
\begin{aligned}
\mu(w) &= \nu(hw) \\
&= \nu\left(\lambda^{-1}\mathcal{L}_f h \cdot w\right) \\
&= \lambda^{-1}\nu\left(\mathcal{L}_f\left(h \cdot (w \circ \sigma)\right)\right) \\
&= \lambda^{-1}\left(\mathcal{L}_f^{*}\nu\right)\left(h\left(w \circ \sigma\right)\right) \\
&= \nu\left(h \cdot (w \circ \sigma)\right) \\
&= \mu\left(w \circ \sigma\right)
\end{aligned}
$$

and hence the invariance follows.

To establish the mixing requirement we look at the cylinder sets

$$
\begin{aligned}
E &= \left\{y \in \Sigma_A^{+} : y_i = a_i, 0 \le i \le r\right\}, \\
F &= \left\{y \in \Sigma_A^{+} : y_i = b_i, 0 \le i \le s\right\}.
\end{aligned}
$$

To estimate $|\mu(E \cap \sigma^{-n}F) - \mu(E)\mu(F)|$, the terms within can be interpreted as

$$
\begin{aligned}
\mu\left(E \cap \sigma^{-n}F\right) &= \mu\left(\chi_E \cdot \chi_{\sigma^{-n}F}\right) \\
&= \lambda^{-n}\mathcal{L}_\phi^{*n}\nu\left(h\chi_E \cdot \left(\chi_F \circ \sigma^n\right)\right) \\
&= \nu\left(\lambda^{-n}\mathcal{L}_\phi^{n}\left(h\chi_E\right) \cdot \chi_F\right)
\end{aligned}
$$

and

$$
\mu(E)\,\mu(F) = \nu\left(h\chi_E\right)\nu\left(h\chi_F\right).
$$

giving us

$$
\begin{aligned}
\left|\mu\left(E \cap \sigma^{-n}F\right) - \mu(E)\,\mu(F)\right| &= \left|\nu\left(\lambda^{-n}\mathcal{L}_\phi^{n}\left(h\chi_E\right) \cdot \chi_F\right) - \nu\left(h\chi_E\right)\nu\left(h\chi_F\right)\right| \\
&\le \left\|\lambda^{-n}\mathcal{L}_\phi^{n}\left(h\chi_E\right) - \nu\left(h\chi_E\right)h\right\|_\infty \nu(F).
\end{aligned}
$$

Since $\chi_E \in \Sigma_A^{+}$, also satisfies $\mathrm{var}_k\chi_E = 0$ for $k > 0$, we can further estimate

$$
\left|\lambda^{-n}\mathcal{L}_\phi^{n}\left(h\chi_E\right) - \nu\left(h\chi_E\right)h\right|_\infty \le A\mu(E)\,\beta^{n-k}
$$

where $n \ge k,\ A > 0$, and $\beta \in (0,1)$. Hence

$$
\left|\mu\left(E \cap \sigma^{-n}F\right) - \mu(E)\,\mu(F)\right| \le A\left(\inf h\right)^{-1}\mu(E)\,\mu(F)\,\beta^{n-k}. \tag{3.5}
$$

Passing to the limit in the inequality (3.5) as $n \to \infty$ gives us the desired result.

We turn our attention to showing that we can always find the constants $c_1 > 0, c_2 > 0$ and P satisfying the double inequality (2.1 and 3.3) as well as the measure μ of the Ruelle theorem.

Let $x \in \Sigma_A$ and consider the cylinder set $E = \{y \in \Sigma : y_i = x_i \text{ for } i = 0, \cdots, m-1\}$. Set $a = \sum_{k=0}^{\infty} \mathrm{var}_k \phi < \infty$. We have

$$\begin{aligned} \mu(E) &= \nu(h\chi_E) \\ &= \lambda^{-m} \nu(\mathcal{L}^m(h\chi_E)) \\ &\leq \lambda^{-m} e^{S_m \phi(x)} e^a \|h\|_\infty. \end{aligned} \tag{3.6}$$

Since there can be no more than one $y_1 \in E$ such that $\sigma^m y_1 = x_1$ for any $x_1 \in \Sigma_A^+$ in the formula (3.4) with f replaced by $h\chi_E$ hence we can find a bound,

$$\mathcal{L}_\phi^m(h\chi_E)(x_1) \leq e^{S_m \phi(x_1)} e^a \|h\|_\infty.$$

Let $c_2 = e^a \|h\|_\infty$ in the inequality (3.6) to give us one side.

The other side of the inequality can be obtained by first noting that there are possibly more than one y_1 such that $\sigma^{m+M} y_1 = x_1$ for any $x_1 \in \Sigma_A^+$ giving us a lower bound,

$$\begin{aligned} \mathcal{L}_\phi^{m+M}(h\chi_E)(x_1) &\geq e^{S_{m+M}\phi(y_1)} h(y_1) \\ &\geq e^{-M\|\phi\|-a} (\inf h)\, e^{S_m \phi(x)} \end{aligned}$$

Hence

$$\mu(E) = \lambda^{-m-M} \nu\left(\mathcal{L}_\phi^{m+M}(h\chi_E)\right) \geq \lambda^{-M} e^{-M\|\phi\|-a} \lambda^{-m} e^{S_m \phi(x)}.$$

If we take $c_1 = \lambda^{-M} e^{-M\|\phi\|-a}$ and letting $P = \log \lambda$ we get the desired double inequality (3.3).

We now show that the measure μ is unique.
Let T_m be a finite set of representatives of each cylinder set. We subdivide Σ_A into disjoint union of subsets $\bigcup_{x \in T_m} E_m(x)$ where $E_m(x)$ are cylinder sets containing x for $i = 0, \ldots, m-1$. In addition suppose that μ', c_1', c_2' and P_1 are another measure and constants satisfying (3.3).

We observe that

$$c_1' e^{-P'm} \sum_{x \in T_m} e^{S_m \phi(x)} \leq \sum_{x \in T_m} \mu' \left(E_m \left(x \right) \right) \leq c_2' e^{-P'm} \sum_{x \in T_m} e^{S_m \phi(x)} \tag{3.7}$$

Since $\sum_{x \in T_m} \mu' \left(E_m \left(x \right) \right) = 1$, passing to the limit in (3.7) we obtain

$$P' = \lim_{m \to \infty} \frac{1}{m} \log \left(\sum_{x \in T_m} e^{S_m \phi(x)} \right).$$

The above treatment can be applied to the measure μ and the corresponding constants c_1, c_2 and P to obtain explicit expression for P. Hence we see that $P' = P$. Up to a constant depending on c_1 and c_2 the measure μ' is absolutely continuous with respect to μ. By Radon-Nikodym theorem $\mu' = f\mu$ where f is μ-integrable. On the hand $\mu' = \left(f \circ \sigma \right) \mu$. By uniqueness of the Radon-Nikodym derivative we have $f \circ \sigma = f$ a.e.
Since μ is ergodic then f can be identified with a constant function, c say.
We see that $\mu' \left(\Sigma_A \right) = \int c \; d\mu = c = 1$ giving us equality of μ' and μ. $\square$

Chapter 4

Computing Hausdorff dimension of Julia sets and Schottky group limit sets

This section deals with the calculation of Hausdorff dimension of sets resulting from conformal dynamical systems. The method of calculations depend on Bowen's thermodynamic pressure criterion [12]. However the main tool is the Fredholm determinant of nuclear transfer operators. For a complete literature on determinants of nuclear operators the reader is referred to Atiyah and Bott [1], Baladi [2], Grothendieck [20], [21], Mayer [29] and Ruelle [44]. The technique provides a motivation to the calculations in the multifractal analysis of the level sets defined in some manner in chapter 5 which stand on these two modes of approach.

In 2002 Jenkinson and Pollicott in their paper [25] introduced a new algorithm which relies entirely on all the periodic points of order less than some given N for a conformal dynamical system. Our discussion will follow closely their result in this chapter. Different algorithms to calculate Hausdorff dimension of conformal limit sets have been formulated in the past, in particular the work of Bodart and Zinsmeister in [9], Widom, Bensimon, Kadanoff and Shenker in [51], Garnett in [18], Saupe in [45] and McMullen in [31]. These algorithms come short of the super-exponential convergence achieved in the periodic point method.

4.1 General theory for conformal iterated function scheme

Proposition 4.1.1. *Given an iterated function system scheme, the Hausdorff dimension* $\dim_H \Lambda$ *of the limit set* Λ *is the largest zero of the function* $s \mapsto \det(I - \mathcal{L}_s)$.

We define the function Δ_N. Let $D_z T^n$ be the derivative of T^n at the fixed point z. Define

$$a_n = \frac{1}{n} \sum_{T^n z = z} \frac{|D_z T^n|^{-s}}{\det\left(I - (D_z T^n)^{-1}\right)}$$

and

$$\Delta_N(s) = 1 + \sum_{n=1}^{N} \sum_{\substack{(n_1, \ldots, n_m) \\ n_1 + \cdots + n_m = n}} \frac{(-1)^n}{m!} a_{n_1} \cdots a_{n_m}.$$

Theorem 4.1.2. *Let* Λ *be the limit set for a real-analytic conformal iterated function scheme. Suppose* Λ *lies in a d-dimensional real analytic manifold. Then there exists* $C > 0$ *and* $0 < \delta < 1$ *such that if* $0 \leq s_N \leq d$ *is the largest real zero for* Δ_N *then*

$$|\dim_H \Lambda - s_N| \leq C\delta^{N^{1+1/d}}.$$

Theorem 4.1.3. *Let* $X \subset M$ *be a locally maximal compact invariant set for a conformal real-analytic hyperbolic Markov map* $T : X \to X$, *where* M *is a* C^ω *manifold of dimension* $d \in \mathbb{N}$. *For each* $N \geq 1$ *we can explicitly define a function* Δ_N, *using only the derivatives* $DT^n(z)$ *evaluated at period-n points z, for* $1 \leq n \leq N$, *and associate* $C > 0$ *and* $0 < \delta < 1$ *such that if* s_N *is the largest real zero of* Δ_N *then*

$$|\dim_H \Lambda - s_N| \leq C\delta^{N^{1+1/d}}.$$

4.2 Applications

4.2.1 Schottky groups

Theorem 4.2.1. *(Kleinian groups) Let* Γ *be a finitely generated non-elementary convex co-compact Schottky or quasifuchsian group, with associated limit set* Λ. *Let* $T : \Lambda \to \Lambda$ *be the associated dynamical system. For each* $N \geq 1$ *we can explicitly define a function* Δ_N, *using only the derivatives* $DT^n(z)$ *evaluated at period-n points z, for* $1 \leq n \leq N$, *and associate* $C > 0$ *and* $0 < \delta < 1$ *such that if* s_N *is the largest real zero of* Δ_N *then*

$$|dim_H(\Lambda) - s_N| \leq C\delta^{N^{3/2}}.$$

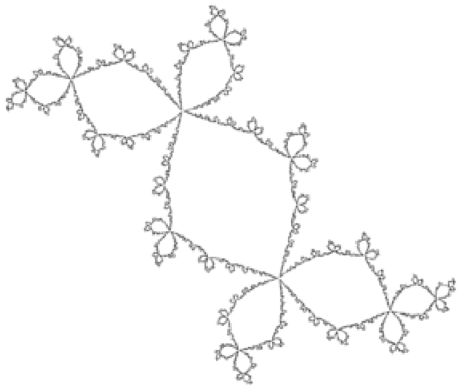

Figure 4.1: Julia set for $f_c(z) = z^2 + c$, for $c = -0.123 + 0.745i$.

4.2.2 Julia sets

Proposition 4.2.2. *Let $f : \mathcal{J} \to \mathcal{J}$ be a hyperbolic holomorphic Markov map, with Julia set $\mathcal{J}$. Let $\mathcal{L}_s$ be the associated transfer operator. Then*

$$\operatorname{tr}(\mathcal{L}_s^n) = \sum_{\substack{f^n z = z \\ z \in \mathcal{J}}} \left|(f^n)'\right|^{-s} \left(1 + \frac{1 - 2\mathrm{Re}\left((f^n)'(z)\right)}{\left|(f^n)'(z)\right|^2}\right)^{-1}$$

Corollary 4.2.3. *Let $f : \mathcal{J} \to \mathcal{J}$ be a hyperbolic holomorphic map, with Julia set $\mathcal{J}$. The corresponding functions Δ_N, whose leading zeros give a sequence of approximations to $\dim_H(\mathcal{J})$, are given by the formula*

$$\Delta_N(s) = 1 + \sum_{n=1}^{N} \sum_{\substack{(n_1,\dots,n_m) \\ n_1+\dots+n_m=n}} \frac{(-1)^m}{m!} \prod_{l=1}^{m} \frac{1}{n_l} \sum_{\substack{f^{n_l} z = z \\ z \in \mathcal{J}}} \left|(f^{n_l})'\right|^{-s} \left(1 + \frac{1 - 2\mathrm{Re}\left((f^{n_l})'(z)\right)}{\left|(f^{n_l})'(z)\right|^2}\right)^{-1}.$$

Corollary 4.2.4. *Let $f_c : \mathcal{J}_c \to \mathcal{J}_c$ be the quadratic map $f_c(z) = z^2 + c$ restricted to the Julia set $\mathcal{J}_c$, where the real parameter $c < -2$. For each $N \geq 1$ we can explicitly define a function Δ_N, using only the derivatives $(f_c^n)'(z)$ evaluated at period-n points z, for $1 \leq n \leq N$, and associate $C > 0$ and $0 < \delta < 1$ such that if s_N is the largest real zero of Δ_N then*

$$|\dim_H(\mathcal{J}_c) - s_N| \leq C\delta^{N^2}.$$

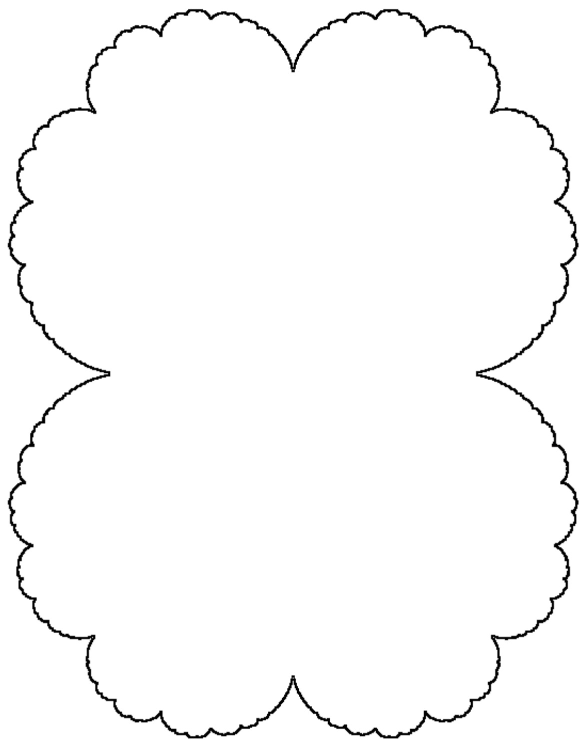

Figure 4.2: Julia set for $f_c(z) = z^2 + c$, for $c = 0.25$.

Chapter 5

Computing multifractal spectra

The famous Birkhoff Ergodic Theorem shows that given an ergodic measure the averages of an integrable function along typical orbits converges to the integral of the function. The multifractal spectrum describes the sets of points for which the averages converge to another limit. In this note we will consider the specific setting of conformal repellers and show how to estimate the Hausdorff Dimension of such sets via approximations to their alternative characterizations as zeros of appropriate determinant functions.

5.1 Introduction

Given a measurable transformation $T : X \to X$ and an ergodic probability measure μ the Birkhoff Ergodic Theorem tells us that for almost every point the Cesàro averages (or Birkhoff averages) along an orbit converge to the integral. We summarise this as follows (cf. Theorem 2.3.2).

Theorem 5.1.1 (Birkhoff, 1931)**.** *Let $f \in L^1(X, \mu)$ then for a.e. (μ) we have that*

$$\frac{1}{N}\sum_{n=0}^{N-1} f(T^n x) \to \int f d\mu \text{ as } N \to +\infty.$$

However, there is a still a set of zero measure about which Birkhoff's theorem gives no information. It is a natural question to ask what is the "size" of the set of points of zero measure which don't converge to $\int f d\mu$, but to some other given limit, as $N \to +\infty$. We will be interested in a particularly well known family of maps and invariant probability measures.

Definition 5.1.1. We say that a C^2 conformal map $T : X \to X$ ($X \subset \mathbb{R}^d$) is a conformal repeller if

1. T is expanding, i.e., there exists $c > 0$, $\lambda > 1$ such that $\|DT^n v\| \geq c\lambda^n \|v\|$, for all $v \neq 0$ and $n \geq 1$;

2. X is a repeller, i.e., there exists an open set $U \supset X$ with $X = \cap_{n=0}^{\infty} T^{-n} U$.

The conformality automatically holds for interval maps and hyperbolic rational maps restricted to their Julia sets, for example. Let $f : X \to \mathbb{R}$ be a C^ω function. Upper and lower bounds for the range of values of the accumulation points of the Cesàro averages $\left\{ \frac{1}{n} \sum_{k=0}^{n-1} f(T^k x) \,:\, n \geq 1 \right\}$ of this function come from the following quantities. We denote

$$\alpha_+ = \sup \left\{ \int f d\mu \,:\, \mu = T - \text{invariant probability} \right\} \text{ and}$$
$$\alpha_- = \inf \left\{ \int f d\mu \,:\, \mu = T - \text{invariant probability} \right\}.$$

In particular, for any $x \in [0,1]$ we have that

$$\alpha_- \leq \liminf_{N \to +\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(T^n x) \leq \limsup_{N \to +\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(T^n x) \leq \alpha_+.$$

It is natural to ask about the size of the set of points for which the limit exists for α in the range (α_-, α_+). This leads to the following definition.

Definition 5.1.2. Given $\alpha_- < \alpha < \alpha_+$ we let

$$\Lambda_\alpha^{(f)} = \left\{ x \in X \,:\, \lim_{n \to +\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(T^n x) = \alpha \right\}.$$

We can now state our main theorem on approximating the values of the multifractal spectra. In the context of C^ω conformal expanding repellers it provides a very efficient algorithm for the numerical computation of the multifractal spectrum. We write $\sum_{n=0}^{N-1} f\left(T^n x\right) = S_N f\left(x\right)$.

Theorem 5.1.2. *Let $T : X \to X$ be a C^ω conformal expanding repeller and let $f : X \to \mathbb{R}$ be a C^ω function. There exists $0 < \theta < 1$ such that given $\alpha \in (\alpha_-, \alpha_+)$ we can associate to the set of values $\mathcal{D}_N = \{S_n f\left(x\right) \,:\, T^n x = x, n \leq N\}$ an approximation $d_N = d_N(\mathcal{D}_N)$ such that*

$$\dim_H \left(\Lambda_\alpha^{(f)} \right) = d_N + O\left(\theta^{N^{\left(1+\frac{1}{d}\right)}} \right). \tag{5.1}$$

The significance of the super exponential error term in (5.1) is that it dominates the number of values to be computed in $\mathcal{D}_N$, which grows exponentially with order $O(e^{hN})$ (where $h > 0$ denotes the topological entropy of the map). In the particular case of expanding Markov interval maps we will have $d = 1$ and we have the following corollary.

Corollary 5.1.3. *Let $T : X \to X$ be a C^ω expanding Markov interval map and let $f : X \to \mathbb{R}$ be a C^ω function. There exists $0 < \theta < 1$ such that given $\alpha \in (\alpha_-, \alpha_+)$ we can associate to the set of values $\mathcal{D}_N = \{S_n f(x) : T^n x = x, n \le N\}$ an approximation $d_N = d_N(\mathcal{D}_N)$ such that*

$$\dim_H \left(\Lambda_\alpha^{(f)}\right) = d_N + O\left(\theta^{N^2}\right).$$

Remark 5.1.4. The C^ω hypothesis is crucial in proving these results. If we only assumed that T is C^∞ then we could only establish an exponential error term in the approximation in Theorem 5.1.2 (i.e., $O(\theta^N)$ for some $0 < \theta < 1$). We write $\mathcal{F}^{(f)}(\alpha) := \dim_H \left(\Lambda_\alpha^{(f)}\right)$.

We will describe the precise algorithm(s) later. In practice, there are two different approaches:

1. We can solve for α and $\mathcal{F}^{(f)}(\alpha)$ independently in terms of a third variable t;
2. We can solve for $\mathcal{F}^{(f)}(\alpha)$ in terms of α.

The value d_N comes from approximating an exact implicit expression for $\dim_H \left(\Lambda_\alpha^{(f)}\right)$. In the specific setting of expanding interval maps, we define a determinant function of two variables:

$$d_2(s,t) = \exp\left(-\sum_{n=1}^{\infty} \frac{1}{n} \sum_{T^n x = x} \frac{\exp\left(-t\sum_{i=0}^{n-1} f(T^i x)\right) |(T^n)'(x)|^{-s}}{1 - (T^n)'(x)^{-1}}\right).$$

This converges for s and t sufficiently large and extends to all values (as an analytic function). Assume without loss of generality that that $d_2(1,1) = 0$ (otherwise we can add a constant to f such that this hypothesis then holds, as we explain in detail later in §3, and then the multifractal spectrum is merely translated by this constant). The following gives an exact implicit characterization of the multifractal spectrum $\dim_H \left(\Lambda_\alpha^{(f)}\right)$ for $\alpha_- < \alpha < \alpha_+$.

Theorem 5.1.5. *Given α there is s_α and t_α such that:*

1. *$d_2(s_\alpha, t_\alpha) = 0$; and*

2. $\frac{\partial d_2(s_\alpha,t)}{\partial t}|_{t=t_\alpha} = \alpha\frac{\partial d_2(s,t_\alpha)}{\partial s}|_{s=s_\alpha}$,

and then we can write that $\dim_H\left(\Lambda_\alpha^{(f)}\right) = s_\alpha + \alpha t_\alpha$.

Explicit estimates on $d_2(s,t)$ allow us to deduce the approximation result in the previous theorem.

In section 2 we illustrate the numerical application with two concrete examples, the details of which will be given in a later section. In sections 2-8 we will restrict to the simpler case that T is the doubling map. In the subsequent sections we will generalise to where T is a C^ω Markov expanding map.

5.2 Two examples

For the moment let us consider two specific examples to help illustrate this result. In both examples we take X to be the unit interval and let $T : [0,1] \to [0,1]$ be the doubling map defined by $Tx = 2x \pmod 1$. This preserves the Lebesgue measure μ. The two examples will correspond to the specific choices of functions $f(x) = \cos(2\pi x)$ or $f(x) = \sin(2\pi x)$

Remark 5.2.1. In these particular cases, it is very easy to see the pointwise convergence to zero for the Cesáro averages without resorting to the use of the full weight of the Birkhoff theorem. More precisely, we can explicitly compute

$$\int_0^1 \left(\frac{1}{N}\sum_{n=0}^{N-1} f(T^n x)\right)^4 dx = \frac{1}{N^4}\sum_{n_1,n_2,n_3,n_4=0}^{N-1}\int_0^1 f(T^{n_1+n_2+n_3+n_4}x)dx = \frac{2N^2+N}{8N^4}.$$

In particular, we can deduce that $\int_0^1 \sum_{N=1}^\infty \left(\frac{1}{N}\sum_{n=0}^{N-1} f(T^n x)\right)^4 dx < +\infty$ and thus we conclude that for a.e. (μ) x we have that $\frac{1}{N}\sum_{n=0}^{N-1} f(T^n x) \to 0$ as $N \to +\infty$.

Example 5.2.1 ($f(x) = \cos 2\pi x$)**.** Let $f : [0,1] \to \mathbb{R}$ be defined by $f(x) = \cos(2\pi x)$. Clearly $\int f(x)d\mu(x) = 0$ and thus for almost all points (with respect to Lebesgue measure) we have the average converges to zero. In this case it is easy to check that $\alpha_+ = 1$ and $\alpha_- = -\frac{1}{2}$ For any other value $\alpha \neq 0$ in this range the Lebesgue measure of the set of points converging to $\alpha \neq 0$ with be zero, and in fact it will have Hausdorff Dimension strictly less than 1. We can the estimate the Hausdorff Dimension of the set of points for which the Birkhoff averages converge to $\frac{1}{2}$ is:

$$\dim_H\left\{x \in [0,1] : \lim_{N\to+\infty}\frac{1}{N}\sum_{n=0}^{N-1}\cos(2^{n+1}\pi x) = \frac{1}{2}\right\}$$
$$= 0.739882772328498106813775738 56\ldots$$

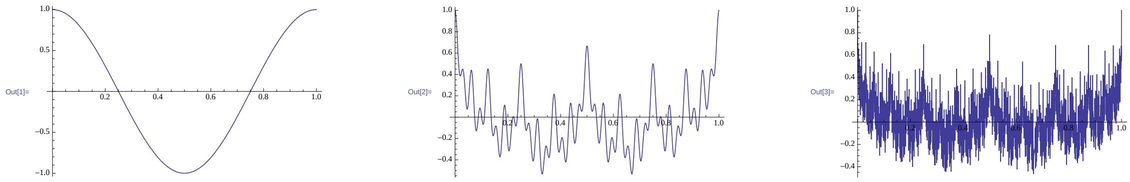

Figure 5.1: Plots of the Cesàro averages $\frac{1}{N}\sum_{n=0}^{N-1}\cos(2^{n+1}\pi x)$ for (i) $N = 1$; (ii) $N = 6$; and (iii) $N = 12$.

Example 5.2.2 ($f(x) = \sin(2\pi x)$)**.** Let $f : [0,1] \to \mathbb{R}$ be defined by $f(x) = \sin 2\pi x$. Clearly $\int f(x)d\mu(x) = 0$ and thus for almost all points (with respect to Lebesgue measure) we have the average converges to zero. In this case $\alpha_+ = \sqrt{15}/8 = 0.4841\cdots$ and $\alpha_- = -\sqrt{15}/8 = -0.4841\cdots$ (by a result of Bousch [10]). For any value $\alpha \neq$ in this range the Lebesgue measure of the set of points converging to α with be zero, and in fact it will have Hausdorff Dimension strictly less than 1. We can the estimate the Hausdorff Dimension of the set of points for which the Birkhoff averages converge to $\frac{1}{4}$ is:

$$\begin{aligned}\dim_H &\left\{x \in [0,1] : \lim_{N\to+\infty} \tfrac{1}{N}\textstyle\sum_{n=0}^{N-1}\sin(2^{n+1}\pi x) = \tfrac{1}{4}\right\} \\ &= 0.90143475104318749821613891644\ldots\end{aligned}$$

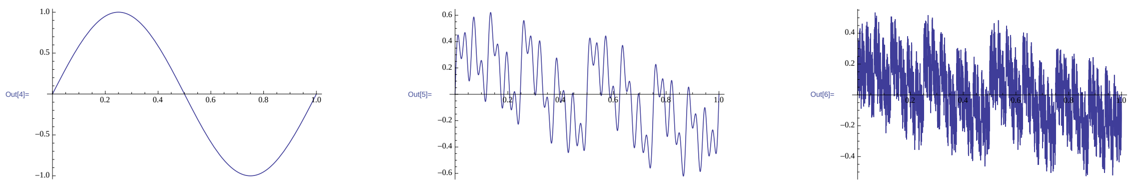

Figure 5.2: Plots of the Cesàro averages $\frac{1}{N}\sum_{n=0}^{N-1}\sin(2^{n+1}\pi x)$ for (i) $N = 1$; (ii) $N = 6$; and (iii) $N = 12$.

In both of these simple examples we have considered the specific case of the doubling map. If $T^n x = x$ is a periodic point of period n then we trivially see that $|(T^n)'(x)| = 2^n$ and then we have the simplification

$$d_2(s,t) = \exp\left(-\sum_{n=1}^{\infty}\frac{2^{-ns}}{n}\sum_{T^n x = x}\frac{\exp\left(-t\sum_{i=0}^{n-1} f(T^i x)\right)}{1-(T^n)'(x)^{-1}}\right) \tag{5.2}$$

in the complex function used in Theorem 5.1.5. We will return to this point in §6.

5.3 Hausdorff dimension

We want to begin by describing a standard approach using thermodynamic formalism. For the purposes of exposition, we will first consider the simplified case that T is the doubling map. We will then explain the modifications needed for the general case in a later section.

5.3.1 The pressure

We now introduce some notation and recall some standard results. Let $g : [0, 1] \to \mathbb{R}$ be a continuous function.

Definition 5.3.1. We define the *pressure function* of g by:

$$P(g) = \lim_{n\to+\infty} \frac{1}{n} \log \left(\sum_{j=0}^{2^n-1} \exp \left(\sum_{k=0}^{n-1} g \left(\left\{ \frac{2^k}{2^n-1} j \right\} \right) \right) \right)$$

Remark 5.3.1. The definition is in terms of the periodic points for the doubling map. It is well known that the 2^n periodic points of period n are of the form

$$\frac{j}{2^n-1}, \quad j = 0, 1, \cdots, 2^n - 1.$$

The summation of the function g around the points of the orbit

$$\left\{ \frac{2^k j}{2^n-1} \right\} \text{ for } k = 0, \cdots, n-1$$

where $\{\cdot\}$ is the fractional part contributes to the pressure function the weight

$$\sum_{k=0}^{n-1} g \left(\left\{ \frac{2^k}{2^n-1} j \right\} \right).$$

There is an alternative formulation of the pressure using the variational principle:

Lemma 5.3.2. *We can write*

$$P(g) = \sup_m \left\{ h(m) + \int g dm \right\}$$

where the supremum is over all T-invariant probability measures m. Moreover, providing g is Hölder continuous there is a unique T-invariant probability measure realising the supremum, and called the equilibrium state for g.

Remark 5.3.3. We observe that when μ is Lebesgue measure we have that $h(\mu) = \log 2$. Moreover, for either $f(x) = \sin(2\pi x)$ or $f(x) = \cos(2\pi x)$ we have that $\int f d\mu = 0$. Thus by the variational principle we have that $P(-f) \geq \log 2$.

Remark 5.3.4. If we replace g by $g + C$ then we see that $P(g + C) = P(g) + C$.

In principle, we would like to restrict to functions f for which $P(-f) = 0$. We would therefore like to "normalise" the function f by adding a constant so as to obtain a new function $\overline{f}$ which indeed has this property. This is achieved in the next lemma.

Lemma 5.3.5. *If define $\bar{f} := f + P(-f)$ then $P(-\bar{f}) = 0$.*

Proof. This follows easily from the definition of pressure, or from the variational principle. In particular, $P(-\bar{f}) = P(-(f + P(-f)) = P(-f) - P(-f) = 0$. □

We now trivially see that points for which the Cesaro averages of f converge to α are precisely those points for which the Cesaro averages of $\overline{f}$ converge to $\overline{\alpha} = \alpha + P(-f)$ for $\bar{f}$, i.e., $\mathcal{F}^{(f)}(\alpha) = \mathcal{F}^{(\bar{f})}(\bar{\alpha})$. We are particularly interested in the value of the pressure for the following two examples.

Example 5.3.1 (Cosine function)**.** In the particular case $f(x) = \cos(2\pi x) \in [-1, 1]$ we can explicitly compute $P(-f) = 0.8575307\cdots$ and thus we can replace $f(x)$ by the normalised function $\overline{f}\,(x) = f\,(x) + 0.8575307\cdots$. The range of $\overline{f}$ is $[-0.1424693\cdots, 1.8575307\cdots]$.

Example 5.3.2 (Sine function)**.** In the particular case $f(x) = \sin(2\pi x) \in [-1, 1]$ we can explicitly compute $P(-f) = 0.8933924\cdots$ and thus we can replace $f(x)$ by the normalised function $\overline{f}\,(x) = f\,(x) + 0.8933924\cdots$. The range of $\overline{f}$ is $[-0.1066076\cdots, 1.8933924\cdots]$.

However, obtaining estimates on the Hausdorff Dimension for $\overline{f}$ is equivalent to obtaining estimates on f since we see from the definitions that

$$\dim_H(\Lambda_\alpha^{(f)}) = \dim_H(\Lambda_{\alpha+P(-f)}^{(\overline{f})}).$$

We complete this section by recalling an important property of the pressure function.

Lemma 5.3.6. *There is an analytic dependence of the pressure function $P(g)$ where g is an element of the Banach space of Hölder continuous functions (with a fixed Hölder exponent).*

5.3.2 Pressure and the Hausdorff dimension

The pressure is an important ingredient in the general theory of thermodynamic formalism. Moreover, it plays an important role in the computation of Hausdorff dimension of certain sets, as is known from the work of Bowen [12] and Ruelle [43] (cf. [52] and [16]).

We begin by recasting the pressure function in a more convenient form. Let us assume that f is not cohomologous to a constant.

Definition 5.3.2. We can consider the function $P : \mathbb{R} \to \mathbb{R}$ defined by

$$P(t) := P(-t\overline{f}) \, (= P(-tf) - tP(-f))$$

We can consider the unique equilibrium measure μ_t for the potential $-tf$.

Lemma 5.3.7. *The function $P(t)$ is analytic with $P(0) = \log 2$ and $P(1) = 0$. Moreover,*

1. $P'(t) = -\int \bar{f} d\mu_t$*;*
2. $-P'(t)$ *obtains all the values in* $(\alpha_{min}, \alpha_{max})$*; and*
3. $P(t)$ *is strictly convex and* $P''(t) > 0$

Proof. These properties follow easily from those of the pressure. For part (1) we recall that $\frac{\partial P(-tf)}{\partial t} = -\int f d\mu_t$ from [34], for example. For part (2) we refer to [3]. For part (3), the convexity is well known since f is not cohomologous to a constant [34]. $\square$

The connection between the function $P(t)$ and the Hausdorff Dimension $\mathcal{F}^{(\bar{f})}(\alpha)$ of the level set is given by the following:

Lemma 5.3.8. *Given α let us choose the unique $t = t_\alpha$ such that $P'(t_\alpha) = -\alpha$. We then have that*

$$\mathcal{F}^{(\bar{f})}(\alpha) = \frac{P(t_\alpha) + t_\alpha \alpha}{\log 2}.$$

We state a connecting Lemma relating the Hausdorff dimension of μ_t to the Hausdorff dimension of $\Lambda_\alpha^{(f)}$.

Lemma 5.3.9. $\dim_H(\mu_t) = \dim_H\left(\Lambda_\alpha^{(f)}\right)$.

The proof depends on the following general proposition of Pesin and Weiss [36].

Proposition 5.3.10. *Let (X, ρ) be a complete separable metric space of finite topological dimension with metric ρ, and let μ be a Borel probability measure. If $A_\beta = \{x \in X \mid d_\mu(x) = \beta\}$ and $\mu(A_\beta) > 0$, then $\dim_H A_\beta = \beta$.*

Proof. By definition $\dim_H(\mu_t)$ is the infimum of $\dim_H A$ taken over all subsets A with $\mu_t(A) = 1$. But μ_t satisfies $\mu\left(\Lambda_\alpha^{(f)}\right) = 1$. Hence $\mu\left(\Lambda_\alpha^{(f)}\right) > 0$. By Proposition (5.3.10) $\dim_H\left(\Lambda_\alpha^{(f)}\right) = \alpha$. On the other hand we have $d_{\mu_t}(x) = \alpha$ for all $x \in \Lambda_\alpha^{(f)}$ we conclude that $\dim_H(\mu_t) = \alpha$. $\square$

We revert back to the proof of Lemma (5.3.8).

Proof. This is well explained in the article of Pesin and Weiss [37] and the book of Pesin [38]. The starting point is that we have $P(-t\bar{f} - P(t)) = 0$ for any t. By Lemma (5.3.9) we only need to calculate $\dim_H(\mu_t)$. The dimension of the measure μ_t satisfies

$$\begin{aligned}\dim_H(\mu_t) &:= \frac{h(\mu_t)}{\log 2} \\ &= \frac{\int (t\bar{f} + P(t))d\mu_t}{\log 2} \\ &= \frac{P(t) - tP'(t)}{\log 2}\end{aligned} \tag{5.3}$$

using the variational principle and part (1) of Lemma 5.3.7. Moreover, for almost all points with respect to μ_t we have that

$$\lim_{n\to+\infty} \frac{1}{n}\sum_{k=0}^{n-1} \bar{f}(T^k x) = \int \bar{f} d\mu_t = -P'(t),$$

by using part (1) of Lemma 5.3.7 again. In particular, setting $t = t_\alpha$ and then substituting $\alpha = -P'(t_\alpha)$ into (5.3) gives the required result. $\square$

5.4 Determinants and spectra

To address the problem of computing the we need to compute the pressure and the derivative of the pressure. We can characterize the pressure using the zeta function and determinant, which in the context that T is the doubling map takes a simple form.

5.4.1 Determinant of a single variable

We begin with a complex function of one variable which is useful in estimating $P(-f)$.

Definition 5.4.1. We formally define a function (for the doubling map) by

$$d_0(z) = \exp\left(-\sum_{n=1}^{\infty} \frac{z^n}{n} \frac{2^n}{(2^n-1)} \sum_{k=0}^{2^n-1} \exp\left(-\sum_{m=0}^{n-1} f\left(\left\{\frac{2^m k}{2^n-1}\right\}\right)\right)\right)$$

where $z \in \mathbb{C}$.

In particular, we have the following properties for $d_0(z)$ which are useful in estimating $P(-f)$.

Lemma 5.4.1. *We have the following properties:*

1. *The function $d_0(z)$ converges to a non-zero analytic function for $|z| < e^{-P(-f)}$;*
2. *The value $e^{-P(-f)}$ is a simple zero for $d_0(z)$;*
3. *For any $\epsilon > 0$, there exists $C > 0$ such that we can expand*

$$d_0(z) = 1 + \sum_{n=1}^{\infty} a_n z^n$$

where $|a_n| \leq C\left(\frac{1}{2}+\epsilon\right)^{n^2}$, for $n \geq 1$ and a_n depends on the values

$$\bigcup_{m=1}^{n} \left\{ f\left(\left\{\frac{2^k}{2^m-1}\right\}\right) \ : 0 \leq k \leq m-1 \right\};\ \textit{and}$$

4. *The function $d_0(z)$ has an analytic extension to $\mathbb{C}$ as an entire function.*

Proof. These results can be deduced from Ruelle's original article [**?**]. We briefly explain the construction in the present setting.

Part 1 follows easily from the definitions of $P(-f)$ and $d_0(z)$.

For $r > 0$ the disk $D(r) = \left\{z \in \mathbb{C} : |z - \frac{1}{2}| < r\right\}$ and then the inverse branches $T_0(z) = \frac{z}{2}$ and $T_1(z) = \frac{z+1}{2}$ satisfy $T_0(D(r)) \cup T_1(D(r)) \subset D(\frac{r}{2} + \frac{1}{4})$. Given $\epsilon > 0$, we choose r sufficiently large that $\frac{r}{2} + \frac{1}{4} < \left(\frac{1}{2} + \frac{\epsilon}{2}\right) r$. Let $\mathcal{B}$ be the Banach space of bounded analytic functions on D with the supremum norm. The operator $\mathcal{L} : \mathcal{B} \to \mathcal{B}$ defined by $\mathcal{L}w(z) = e^{-f(T_0 z)}w(T_0 z) + e^{-f(T_1 z)}w(T_1 z)$ is a nuclear operator since we can expand

$$\mathcal{L}w(z) = \frac{1}{2\pi i}\int_\Gamma \frac{\mathcal{L}w(\xi)}{z-\xi}d\xi = \sum_{n=0}^{\infty} z^n \left(\frac{1}{2\pi i}\int_\Gamma \frac{\mathcal{L}w(\xi)}{\xi^{n+1}}d\xi\right)$$

where $\Gamma = \{z \in \mathbb{C} : |z-\frac{1}{2}| = r-\epsilon\}$. In particular, we can write $\mathcal{L}w = \sum_{n=0}^{\infty} \lambda_n w_n l_n(w)$ where $w_n \in \mathcal{B}$, $l_n \in \mathcal{B}^*$ with $\|w_n\|_{\mathcal{B}} = \|l_n\|_{\mathcal{B}^*} = 1$ and $|\lambda_n| \leq C\left(\frac{1}{2}+\epsilon\right)^n$, for $n \geq 0$. It then follows that for $z \in \mathbb{C}$ we can write

$$\det(I - z\mathcal{L}) = 1 + \sum_{n=1}^{\infty} z^n \sum_{k_1<\cdots<k_n} \lambda_{k_1}\cdots\lambda_{k_n} \det\left(l_{n_i}(w_{n_j})\right)_{i,j=1}^n$$

Moreover, $\det(I - z\mathcal{L}) = \exp\left(-\sum_{n=1}^{\infty}\frac{z^n}{n}\mathrm{tr}(\mathcal{L}^n)\right)$ can be identified with $d_0(z)$ by explicitly computing

$$\mathrm{tr}(\mathcal{L}^n) = \sum_{k=0}^{2^n-1}\mathrm{tr}(\mathcal{L}_k^{(n))}) = \frac{2^n}{2^n-1}\sum_{k=0}^{2^n-1}\exp\left(-\sum_{m=0}^{n-1} f\left(\left\{\frac{2^m k}{2^n-1}\right\}\right)\right)$$

by using the Taylor expansion at the periodic point $\frac{2^m k}{2^n-1}$ to compute the eigenvalues of $\mathcal{L}_k^{(n)}$ and thus, by summing, the trace $\mathrm{tr}(\mathcal{L}_k^{(n))})$. This completes the sketch of parts 3 and 4.

Finally, $\mathcal{L}$ has a maximal eigenvalue $e^{P(-f)}$ [43][34] from which part 2 follows.

□

As an immediate consequence we have the following.

Corollary 5.4.2. *If $z_0 > 0$ is the largest zero for $d_0(z)$ then $P(-f) = -\log z_0$*

Revisiting our previous examples, we can estimate the following.

Example 5.4.1 (Sine function)**.** If $f(x) = \sin(2\pi x)$ then we can estimate

$$z_0 = 0.409264981980930309113375642482\cdots \text{ and}$$
$$P = -\log z_0 = 0.893392455017504971692687831819\cdots$$

In particular, we generate a sequence $p_m = -\log z_m$, $m \geq 2$, converging to $P(-\sin(2\pi\cdot))$, where z_m is a zero of the polynomial

$$d_0^{(m)}(z) = 1 + \sum_{n=1}^{m} a_n z^n$$

given by truncating the expansion for $d_0(z)$. These approximations are illustrated in

Table 1 (a), where we can easily see the super exponential convergence coming from Lemma 5.4.1(3). The implied level of accuracy is already achieved when $m = 9$.

m	p_m	m	p_m
2	0.693147180559945286226763982995	2	0.815264689983832058217672056344
3	0.875751917382645683751718479471	3	0.867255866395854169148549317470
4	0.892841703325299884674848271970	4	0.857241229966777096294094917539
5	0.893392687557898357297858638049	5	0.857525895896090961656454965123
6	0.893392428470504484927516841708	6	0.857530725856244013804996484396
7	0.893392455043265032443855488964	7	0.857530739837428890304238393583
8	0.893392455017498754443749930942	8	0.857530739821697252089904850436
9	0.893392455017504971692687831819	9	0.857530739821700693781281188421
10	0.893392455017504971692687831819	10	0.857530739821700693781281188421
11	0.893392455017504971692687831819	11	0.857530739821700693781281188421
12	0.893392455017504971692687831819	12	0.857530739821700693781281188421

Table 5.1: (a) Approximations p_m to $P(-\sin(\cdot))$; and (b) Approximations p_m to $P(-\cos(\cdot))$

Example 5.4.2 (Cosine function)**.** If $f(x) = \cos(2\pi x)$ then we can estimate

$$\begin{aligned} z_0 &= 0.424208270720691171806748798190 \cdots \text{ and} \\ P &= -\log z_0 = 0.857530739821700693781281188421 \cdots \end{aligned}$$

We again generate a sequence $p_m = -\log z_m$, $m \geq 2$, converging to $P(-\cos(2\pi\cdot))$, where z_m is a zero of the polynomial

$$d_0^{(m)}(z) = 1 + \sum_{n=1}^{m} a_n z^n$$

given by truncating the expansion for $d_0(z)$. These approximations are illustrated in Table 5.1 (b), where we can again see the super exponential convergence coming from Lemma 5.4.1 (3). The inferred level of accuracy is already achieved when $m = 9$.

5.4.2 Determinant of two variables

Once we replace f by $\bar{f} := f + P(-f) = f - \log z_0$ we can consider a second function depending on two variables. This is the function used in giving an expression for the Hausdorff dimension.

Definition 5.4.2. We formally define a second complex function (for the doubling map) by

$$d_1(z,t) = \exp\left(-\sum_{n=1}^{\infty} \frac{z^n}{n} \frac{2^n}{2^n-1} \sum_{k=0}^{2^n-1} \exp\left(-t \sum_{m=0}^{n-1} \bar{f}\left(\left\{ \frac{2^m k}{2^n-1} \right\} \right) \right) \right)$$

where $z \in \mathbb{C}$ and $t \in \mathbb{R}$.

This converges provided t is sufficiently large and $|z|$ is sufficiently small. We observe that when $t = 1$ this reduces to the previous function, i.e., $d_0(z) = d_1(z, 1)$.

Remark 5.4.3. In the particular case that T is the doubling map then one can also write $z = 2^{-s}$ and then we will consider instead the complex function $d_2(z,t)$ which satisfies $d_2(s,t) = d_1(2^{-s}, t)$, as in (5.2).

In particular, we have the following properties for $d_1(z,t)$ which are analogous to those for $d_0(z)$.

Lemma 5.4.4. *We have the following properties:*

1. *The function z converges to a non-zero analytic function for $|z| < e^{-P\left(-t\bar{f}\right)}$;*
2. *The value $e^{-P\left(-t\bar{f}\right)}$ is a simple zero for $d_1(z,t)$;*
3. *For any $\epsilon > 0$, there exists $C > 0$ such that we can expand*

$$d_1(z,t) = 1 + \sum_{n=1}^{\infty} a_n(t) z^n$$

where $|a_n(t)| \leq C\left(\frac{1}{2} + \epsilon\right)^{n^2}$, for $n \geq 1$ and $a_n(t)$ depends on the values

$$\bigcup_{m=1}^{n} \left\{ \bar{f}\left(\left\{ \frac{2^k}{2^m-1} \right\} \right) : 0 \leq k \leq m-1 \right\}; \text{ and}$$

4. *The function $d_1(z,t)$ has an analytic extension to $\mathbb{C}$ as an entire function.*

Proof. These results too can be deduced from Ruelle's original article [**?**], as in the proof of Lemma 5.4.1. The only difference is that we need to replace the operator $\mathcal{L}$ by the operators $\mathcal{L}_t$ defined by $\mathcal{L}_t w(z) = e^{-tf(T_0 z)} w(T_0 z) + e^{-tf(T_1 z)} w(T_1 z)$. This is again a nuclear operator on $\mathcal{L}$ and we can identify $d_1(z,t) = \det(I - z\mathcal{L}_t)$. The estimates are very similar to those in Lemma 5.4.1. $\square$

5.5 The first algorithm

We can now recast for formulae for the dimension in Lemma 5.3.8 using Lemma 5.4.1. This leads to two different approaches to the dimension. In this section we consider the first algorithm, where given a parameter t we can associate $\alpha = \alpha(t)$ and $\mathcal{F}^{(\bar{f})}(\alpha(t))$.

5.5.1 The algorithm

Given t we can solve implicitly for $d_1(z(t), t) = 0$. In particular, we can then differentiate this identity in t and write

$$\frac{\partial d_1}{\partial z}(z(t), t)\frac{\partial z}{\partial t}(t) + \frac{\partial d_1}{\partial t}(z(t), t) = 0. \tag{5.4}$$

Since by part (2) of Lemma 5.4.4 we have that $z(t) = e^{-P(t)}$ we can differentiate in t and write

$$\frac{\partial z}{\partial t}(t) = -z(t)P'(t) > 0. \tag{5.5}$$

Comparing (5.4) and (5.5) we can now write

$$P'(t) = \frac{1}{z(t)}\frac{\frac{\partial d_1}{\partial t}(z(t), t)}{\frac{\partial d_1}{\partial z}(z(t), t)} < 0.$$

In particular, can associate to t the value $\alpha = \alpha(t)$ given by $\alpha := -P'(t)$. We can then use Lemma 5.4.4 to write

$$\mathcal{F}^{(\bar{f})}(\alpha) = \frac{P(t) + t\alpha}{\log 2} = \frac{-\log z(t) + t\alpha}{\log 2}$$

We can now consider different choices of t. For each value of t in a suitable range we can associate the corresponding value $\alpha = \alpha(t)$ in the multifractal analysis. We can similarly associate to t the value $\mathcal{F} = \mathcal{F}^{(\bar{f})}(\alpha(t))$. In particular, given t we can consider the truncations

$$d_1^{(m)}(z, t) = 1 + \sum_{n=1}^{m} a_n(t)z^n,$$

where we can approximate the zero $z(t)$ by the zero $z^{(m)}(t)$ for $d_1^{(m)}(z, t)$. We can then generate two approximating sequences:

1. A sequence

$$\alpha_m := -\frac{1}{z^{(m)}(t)} \frac{\frac{\partial d_1^{(m)}}{\partial t}(z^{(m)}(t), t)}{\frac{\partial d_1^{(m)}}{\partial z}(z^{(m)}(t), t)}, \quad m \geq 2$$

converging to α; and

2. A sequence

$$\mathcal{F}_m := \frac{-\log z^{(m)}(t) + t\alpha^{(m)}}{\log 2}, \quad m \geq 2$$

converging to $\mathcal{F}$.

5.5.2 Examples

We can consider our two functions $f(x) = \cos(2\pi x)$ and $f(x) = \sin(2\pi x)$ and examples for different t. In particular, we will consider the cases $t = 0.1$ and 1.0 for $m = 2, \cdots, 9$. We begin with the function $f(x) = \cos(2\pi x)$.

Example 5.5.1 ($f(x) = \cos(2\pi x)$ and $t = 0.1$)**.** When we choose $t = 0.1$ we can estimate $\alpha = -0.046437472850649252897881069 66\cdots$ then $\mathcal{F}(\alpha) = 0.99673365723505025215445130015 7\cdots$. This approximation can be seen in Table 5.2.

m	$\mathcal{F}_m$	α_m
2	0.99921878521968787723039895354 3	-0.00831264815020349434604440830
3	0.99671894485227907622970633383 4	-0.04657526329466277204005564272
4	0.99673366738698176980904008814 8	-0.04643738789236173669650042939
5	0.99673365723532980631205191457 4	-0.04643747284846799772140002460
6	0.99673365723505114033287100028 2	-0.04643747285064514507268995658
7	0.99673365723505025215445130015 7	-0.04643747285064936392018353217
8	0.99673365723505025215445130015 7	-0.04643747285064925289788106966
9	0.99673365723505025215445130015 7	-0.04643747285064925289788106966

Table 5.2: Approximations when $t = 0.1$

Example 5.5.2 ($f(x) = \cos(2\pi x)$ and $t = 1.0$)**.** When we choose $t = 1.0$ we can estimate $\alpha = -0.272615452696240434526941953 66\cdots$ then $\mathcal{F}(\alpha) = 0.84385438407604596022792975418 3\cdots$. This approximation can be seen in Table 5.3.

We next turn to the function $f(x) = \sin(2\pi x)$.

Example 5.5.3 ($f(x) = \sin(2\pi x)$ and $t = 0.1$)**.** When we choose $t = 0.1$ we can estimate $\alpha = -0.049690425429048112881957877 11\cdots$ and

m	$\mathcal{F}_m$	α_m
2	0.784291089580260947222711820359	-0.27163553250298655417083182328
3	0.820382435542054921917554111133	-0.29861009421897766191733580854
4	0.845225458720281763724813117733	-0.27137558631732749958587191941
5	0.843897526688114574255905608879	-0.27258070459071259694638911242
6	0.843854558785547714805375107971	-0.27261531763138413531066817086
7	0.843854383791804329995045463875	-0.27261545290898747673224988830
8	0.843854384076114016899339276279	-0.27261545269618747688866733370
9	0.843854384076035524131498277711	-0.27261545269624520848594784184
10	0.843854384076045405116417441604	-0.27261545269624087861615180373
11	0.843854384076045960227929754183	-0.27261545269624043452694195366
12	0.843854384076045960227929754183	-0.27261545269624043452694195366

Table 5.3: Approximations when $t = 1.0$

$\mathcal{F}(\alpha) = 0.996426724035268218671035356238\cdots$. This approximation can be seen in Table 5.4.

m	$\mathcal{F}_m$	α_m
2	0.999999999999999888977697537480	0.0
3	0.996444380964240594700243036641	-0.04952689509082119911909103394
4	0.996426736287734260422155330161	-0.04969032340952528326027959338
5	0.996426724035267330492615656112	-0.04969042542905410808629085295
6	0.996426724035269106849455056363	-0.04969042542904245074453228881
7	0.996426724035268440715640281269	-0.04969042542904711368123571447
8	0.996426724035268218671035356238	-0.04969042542904811288195787711
9	0.996426724035268218671035356238	-0.04969042542904811288195787711

Table 5.4: Approximations when $t = 0.1$

Example 5.5.4 ($f(x) = \sin(2\pi x)$ and $t = 1.0$)**.** When we choose $t = 1.0$ we can estimate $\alpha = -0.334519887980170\cdots$ then $\mathcal{F}(\alpha) = 0.806282680953646\cdots$. This approximation can be seen in Table 5.5.

5.6 The second algorithm

In the second algorithm we assume that we are given suitable α and then we want to solve for $\mathcal{F}^{(\bar{f})}(\alpha)$.

m	$\mathcal{F}_m$	α_m
2	1.00000000000000000000000000000	0.0
3	0.85120672064042302018549435160 8	-0.2857403788970594993301688191 4
4	0.80885638470467358107640620801 2	-0.3321851807893448915720568948 0
5	0.80628045768589018393868173006 9	-0.3345216615723408670746152893 1
6	0.80628301166709304137469871420 8	-0.3345196322000779654004531948 9
7	0.80628268049672680017891934767 2	-0.3345198883226436770144118781 9
8	0.80628268095378763380409736782 9	-0.33451988798006671 022022828765
9	0.80628268095364430401161826012 0	-0.3345198879801720703852652150 1
10	0.80628268095364685752457489798 0	-0.3345198879801707381176356648 2
11	0.80628268095364696854687736049 5	-0.3345198879801706270953332023 1
12	0.80628268095364685752457489798 0	-0.3345198879801707381176356648 2

Table 5.5: Approximations when $t = 1.0$

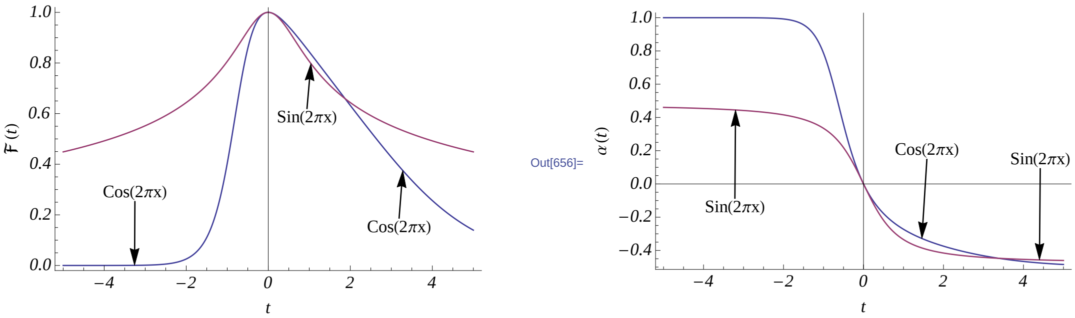

Figure 5.3: Superimposed plots for $\cos(2\pi x)$ and $\sin(2\pi x)$ for (i) $\mathcal{F}(t)$ as a function of t; and (ii) $\alpha(t)$ as a function of t.

5.6.1 The algorithm

Given α we can define

$$P(z,t) = d_1(z,t) \text{ and}$$
$$Q(z,t) = \frac{\partial d_1}{\partial z}(z,t)z\alpha + \frac{\partial d_1}{\partial t}(z,t)$$

The following simple result gives the Hausdorff Dimension of the level set.

Lemma 5.6.1. *Given the solution* $(z_\alpha, t_\alpha) \in \mathbb{R}^2$ *for*

$$P(z_\alpha, t_\alpha) = Q(z_\alpha, t_\alpha) = 0$$

we can write

$$\mathcal{F}^{(\bar{f})}(\alpha) = \frac{-\log z_\alpha + t_\alpha \alpha}{\log 2}.$$

Proof. By part (2) of Lemma 5.4.1, the identity $P(z_\alpha, t_\alpha) = 0 = d_1(z_\alpha, t_\alpha)$ ensures that $z_\alpha = e^{-P(t_\alpha)}$. In particular, by taking logarithms we can write

$$P(t_\alpha) = -\log z_\alpha. \tag{5.6}$$

Locally we have an implicit solution $z(t)$ for $d_1(z(t), t) = 0$ with $z(t_\alpha) = z_\alpha$. In particular, given t we have by part (2) of Lemma 5.4.4 that $z(t) = e^{-P(t)}$. Differentiating this identity with respect to t at t_α gives:

$$\frac{\partial z}{\partial t}(t_\alpha) = -z_\alpha P'(t_\alpha) > 0. \tag{5.7}$$

Differentiating the identity $d_1(z(t), t) = 0$ with respect to t at t_α we can write:

$$\frac{\partial d_1}{\partial z}(z_\alpha, t_\alpha)\frac{\partial z}{\partial t}(t_\alpha) + \frac{\partial d_1}{\partial t}(z_\alpha, t_\alpha) = 0. \tag{5.8}$$

Comparing (5.7) and (5.8) gives:

$$\frac{\partial d_1}{\partial z}(z_\alpha, t_\alpha)z_\alpha \alpha + \frac{\partial d_1}{\partial t}(z(t), t_\alpha) = Q(z_\alpha, t_\alpha) = 0, \tag{5.9}$$

providing $\alpha = -P'(t_\alpha)$. Finally, by Lemma 5.3.8 and (5.6) we have that

$$\mathcal{F}^{(\bar{f})}(\alpha) = \frac{P(t_\alpha) + t_\alpha \alpha}{\log 2} = \frac{-\log z_\alpha + t_\alpha \alpha}{\log 2}$$

as required. □

We can consider the truncations of the Taylor series in z for $P(z,t)$ and $Q(z,t)$.

Definition 5.6.1. Given α and $m \geq 1$ we can define

$$P^{(m)}(z,t) = \sum_{n=1}^{m} a_n(t) z^n \text{ and}$$
$$Q_\alpha^{(m)}(z,t) = \sum_{n=1}^{m} b_n^\alpha(t) z^n$$

where $b_n^\alpha(t) = a_n(t) n\alpha + \frac{\partial a_n}{\partial t}(t)$ (corresponding to the terms of the Taylor series expansion for (5.9)).

We can then approximate the solution (z_α, t_α) for $P(z,t) = Q(z,t) = 0$ by a solution $(z_\alpha^{(m)}, t_\alpha^{(m)})$ for $P^{(m)}(z,t) = Q^{(m)}(z,t) = 0$. We can then generate an approximating sequence

$$\mathcal{F}_m := \frac{-\log z^{(m)} + t^{(m)}\alpha}{\log 2}, \quad m \geq 2,$$

converging to $\mathcal{F}^{(\bar{f})}(\alpha)$.

5.6.2 Examples

We will again concentrate on the two basic functions $f(x) = \cos(2\pi x)$ and $f(x) = \sin(2\pi x)$ We will consider the cases $\alpha = 0.1$ and 0.25 for $m = 2, \cdots, 12$.

We begin with the function $f(x) = \cos(2\pi x)$.

Example 5.6.1 ($f(x) = \cos(2\pi x)$ and $\alpha = 0.1$)**.** If we let $\alpha = 0.1$ then we see that $\mathcal{F}(0.1) = 0.986826533447210\cdots$. The approximations can be seen in Table 5.6(a).

Example 5.6.2 ($f(x) = \cos(2\pi x)$ and $\alpha = 0.25$)**.** If we let $\alpha = 0.25$ then we see that $\mathcal{F}(0.25) = 0.92613854650709\cdots$. The approximations can be seen in Table 5.6 (b).

Example 5.6.3 ($f(x) = \cos(2\pi x)$ and $\alpha = 0.5$)**.** If we let $\alpha = 0.5$ then we see that $\mathcal{F}(0.5) = 0.73988277232849\cdots$. The approximations can be seen in Table 5.7.

m	$\mathcal{F}_m$
3	0.986897682342242281687820788 86
4	0.98682643801924750477021109379
5	0.98682653345543771433541984014
6	0.98682653344721068897338782873
7	0.98682653344721057084995889897
8	0.98682653344721001358904053031
9	0.98682653344721089386354338843
10	0.98682653344721040539922453469
11	0.98682653344721047612613835983
12	0.98682653344721013507321708529

m	$\mathcal{F}_m$
3	0.92794105886194732473057235603
4	0.92612628563107682553298155683
5	0.92613855638089299462695116826
6	0.92613854650484694052564962604
7	0.92613854650710013162835988062
8	0.92613854650709843282271627189
9	0.92613854650709765562223538437
10	0.92613854650709812219678862622
11	0.92613854650709837436195088373
12	0.92613854650709750369835003272

Table 5.6: Approximations when: (a) $\alpha = 0.1$ and $\alpha = 0.25$

m	$\mathcal{F}_m$
3	0.75563703473136142018614163259
4	0.73952399234525433961288453076
5	0.73988403788470085510734083708
6	0.73988277093889877336055887469
7	0.73988277232873744452984157569
8	0.73988277232849432938865800530
9	0.73988277232849558424408253207
10	0.73988277232849866572825436467
11	0.73988277232849862849718837786
12	0.73988277232849810681377573856

Table 5.7: Approximations when: $\alpha = 0.5$

We now consider the function $f(x) = \sin(2\pi x)$.

Example 5.6.4 ($f(x) = \sin(2\pi x)$ and $\alpha = 0.1$)**.** If we let $\alpha = 0.1$ then we see that $\mathcal{F}(0.1) = 0.985388104329581\cdots$. The approximations can be seen in Table 5.8 (a).

Example 5.6.5 ($f(x) = \sin(2\pi x)$ and $\alpha = 0.25$)**.** If we let $\alpha = 0.25$ then we see that $\mathcal{F}(0.25) = 0.90143475104318\cdots$. The approximations can be seen in Table 5.8 (b).

Example 5.6.6 ($f(x) = \sin(2\pi x)$ and $\alpha = 0.45$)**.** If we let $\alpha = 0.45$ then we see that $\mathcal{F}(0.25) = 0.512808947\cdots$. The approximations can be seen in Table 5.9.

m	$\mathcal{F}_m$
3	0.98528409921769489544983732690
4	0.98538792910437795495127142083
5	0.98538810432976287379448653780
6	0.98538810432958044919196928747
7	0.98538810432958056746737725478
8	0.98538810432958068073480286105
9	0.98538810432958165767639685827
10	0.98538810432958101283614678557
11	0.98538810432958118930599334707

m	$\mathcal{F}_m$
3	0.89451246656456310702542523567
4	0.90136138862288395497325369619
5	0.90143475603409668847921458858
6	0.90143475083506846869909133680
7	0.90143475104321903950257055274
8	0.90143475104318901948902623184
9	0.90143475104319124889630543175
10	0.90143475104318781565644510264
11	0.90143475104318749821613891644

Table 5.8: Approximations when: (a) $\alpha = 0.1$ and (b) $\alpha = 0.25$

m	$\mathcal{F}_m$
3	0.181222960135624937547726313
4	0.484094722819625782862013387 6
5	0.512873783498801130047410227 4
6	0.512340804174366925974662910 3
7	0.512835906578455147063261046 0
8	0.512808591994503571023122086 1
9	0.512808946416812858444089916 1
10	0.512808947085190666522611433 6
11	0.512808947100468180994584710 4

Table 5.9: Approximations when: $\alpha = 0.45$

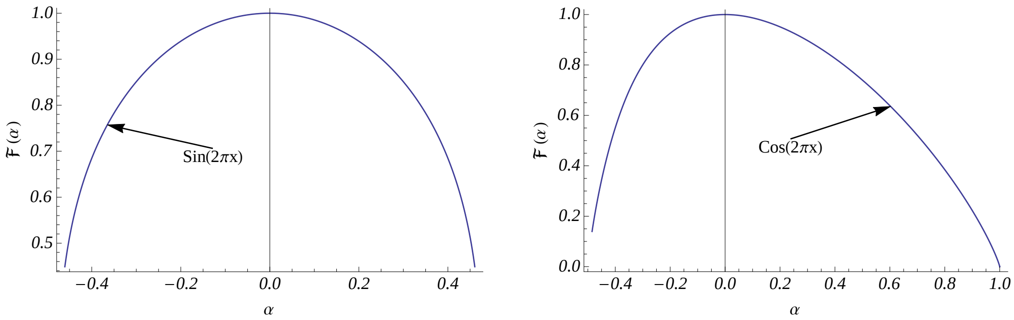

Figure 5.4: Plots of $\mathcal{F}(\alpha)$ as a function of α for: (a) $\sin(2\pi x)$; and (b) $\cos(2\pi x)$

5.7 Speed of approximation

We can consider the approximation using some simple estimates. We can approximate $d_1(z,t)$ by the complex function

$$d_1^{(m)}(z,t) = 1 + \sum_{n=1}^{m} a_n(t) z^n,$$

for $m \geq 2$, and observe that by the bounds in Lemma 5.4.1 (3), we have

$$d_1(z,t) - d_1^{(N)}(z,t) = O\left(\left(\frac{1}{2} + \epsilon\right)^{m^2}\right)$$

on any compact region. Moreover, by a simple application of Cauchy's theorem we can bound the derivatives

$$\frac{\partial d_1(z,t)}{\partial z} - \frac{\partial d_1^{(m)}(z,t)}{\partial z} = O\left(\left(\frac{1}{2}+\epsilon\right)^{m^2}\right) \text{ and}$$
$$\frac{\partial d_1(z,t)}{\partial t} - \frac{\partial d_1^{(m)}(z,t)}{\partial t} = O\left(\left(\frac{1}{2}+\epsilon\right)^{m^2}\right).$$

We now have the following result.

Proposition 5.7.1. *Given solutions* $(z_\alpha^{(m)}, t^{(m)})$ *for*

$$Q_\alpha^{(m)}(z,t) = P^{(m)}(z,t) = 0.$$

we can then write

$$\mathcal{F}(\alpha) = \frac{\log z_\alpha^{(m)}}{\log 2} + \frac{t_\alpha^{(m)}\alpha}{\log 2} + O\left(\left(\frac{1}{2}+\epsilon\right)^{m^2}\right)$$

Remark 5.7.2. Although Proposition 5.7.1 gives that the error term tends to zero at the same super exponential rate, the inferred constant in the Landau O term may vary. From the proof of Lemma 5.4.1 we can get bounds of the form $|a_n| \leq C_n(\frac{1}{2}+\frac{1}{4r})^{n^2/2}$ and $C_n \leq n^{n/2}\left(\sup_{|z-\frac{1}{2}|<r}|e^{-tf(z)}|\right)^n$, for any $r > \frac{1}{2}$. Thus, for $|t|$ larger the estimate on C, and consequently for a_n, may be worse. In particular, the approximation to $d_1(z,t)$ by truncating to a given number of terms may give a worse estimate.

In the explicit case that $f(z) = \cos(2\pi z)$ (or $f(z) = \sin(2\pi z)$) we can bound $|\bar{f}(z)| \leq \frac{1}{2}\left(e^{2\pi r}+e^{-2\pi r}\right)$. Thus we can bound the nth term in $d_1(e^{-P(-t\bar{f})}, t)$ by

$$n^{n/2}\left(e^{-P(-t\bar{f})}\exp\left(e^{2\pi r}+e^{-2\pi r}\right)\right)^n\left(\frac{1}{2}+\frac{1}{4r}\right)^{n^2/2} \tag{5.10}$$

This explains the observed variation in the accuracy in the numerical approximations in the examples and can be quantified with a little extra work (to optimise the choice of $r > 0$ so as to minimise the contributions from the bound (5.10) for $n \geq m$, for a given m).

5.8 Generalizations

In this final section, we will indicate how these results can be generalised and applied to other problems.

5.8.1 The case of general expanding interval maps

Thus far, we have concentrated on the simpler case that the underlying transformation is the doubling map. Let us now outline the modifications for the case of a general C^ω expanding Markov map T.

Definition 5.8.1. We can consider the function $Q : \mathbb{R} \to \mathbb{R}$ defined implicitly by

$$P(-t\overline{f} - Q(t)\log|T'|) = 0 \big(= P(-tf - Q(t)\log|T'|) - tP(-f)\big)$$

Remark 5.8.1. It is well known that we always have $P(-\log|T'|) = 0$.

We can consider the unique equilibrium measure m_t for the potential $-t\bar{f} - Q(t)\log|T'|$.

Lemma 5.8.2. *The function $Q(t)$ is analytic with $Q(0) = 1$ and $Q(1) = 0$. Moreover,*

1. $Q'(t) = -\frac{\int \overline{f} dm_t}{\int \log|T'| dm_t}$;
2. $-Q'(t)$ *attains all the values in* $(\alpha_{min}, \alpha_{max})$*; and*
3. $Q''(t) > 0$

Proof. These properties are easily checked. In particular, Part (1) follows from the implicit function theorem. □

The connection between the function $P_2(t)$ and the Hausdorff Dimension $\mathcal{F}(\alpha)$ of the level set is given by the following:

Theorem 5.8.3. *Given α let us choose the unique $t = t_\alpha$ such that $Q'(t_\alpha) = -\alpha$. We then have that*

$$\mathcal{F}^{(\bar{f})}(\alpha) = Q(t_\alpha) + t_\alpha \alpha.$$

Proof. This is well explained in the article of Pesin and Weiss [37] and the book of Pesin [38]. We can write

$$Q'(t_\alpha) = -\frac{\frac{\partial P(-t_\alpha \overline{f} - u\log|T'|)}{\partial u}\big|_{u=Q(t_\alpha)}}{\frac{\partial P(-t\overline{f} - Q(t_\alpha)\log|T'|)}{\partial t}\big|_{t=t_\alpha}} = -\frac{\int \overline{f} dm_\alpha}{\int \log|T'| dm_{t_\alpha}} = -\alpha \tag{5.11}$$

for $t = t_\alpha$. The essential idea is that for any t we have from the variational principle that

$$P(-t\overline{f} - Q(t)\log|T'|) = 0 = h(m_t) + \int(-t\overline{f} - Q(t)\log|T'|)dm_t \tag{5.12}$$

and that the dimension of the measure m_t satisfies

$$\begin{aligned}
\dim_H(m_t) &:= \frac{h(m_t)}{\int \log|T'|dm_t} \\
&= \frac{\int(t\overline{f} + Q(t)\log|T'|)dm_t}{\int \log|T'|dm_t} \\
&= Q(t) + t\frac{\int \overline{f}dm_t}{\int \log|T'|dm_t} \\
&= Q(t) + t\alpha
\end{aligned}$$

using (5.11) and (5.12) and Part (1) of Lemma 5.8.2. □

We can define a complex function by:

$$d_0(z) = \exp\left(-\sum_{n=1}^{\infty}\frac{z^n}{n}\sum_{T^nx=x}\frac{\exp\left(-\sum_{i=0}^{n-1} f(T^ix)\right)}{1-(T^n)'(x)^{-1}}\right)$$

which converges for $|z|$ sufficiently small. In fact, when T and f are C^ω it follows by work of Ruelle (after Grothendieck) that $d_0(z)$ is entire in $\mathbb{C}^2$. There is a zero at $z_0 = e^{-P(-f)}$ and we can replace f by $\overline{f} = f + P(-f) = f - \log z_0$.

As explained in the introduction, we can define a complex function by:

$$d_2(s,t) = \exp\left(-\sum_{n=1}^{\infty}\frac{1}{n}\sum_{T^nx=x}\frac{\exp\left(-t\sum_{i=0}^{n-1} f(T^ix)\right)|(T^n)'(x)|^{-s}}{1-(T^n)'(x)^{-1}}\right)$$

which converges for s, t sufficiently large. In fact, when T and f are C^ω it follows by work of Ruelle (after Grothendieck) that $d_2(s,t)$ is entire in $\mathbb{C}^2$.

Given α there is t such that there is zero at $(s(t),t)$, i.e., $d_2(s(t),t) = 0$. Using the Implicit Function Theorem we can write

$$\frac{\partial d_2(s_\alpha,t)}{\partial t}|_{t=t_\alpha} + \frac{\partial d_2(s,t_\alpha)}{\partial t}|_{s=s_\alpha}\frac{\partial s}{\partial t} = 0$$

and we want to solve (s_α, t_α) such that

$$\frac{\partial d_2(s_\alpha, t)}{\partial t}|_{t=t_\alpha} - \alpha \frac{\partial d_2(s, t_\alpha)}{\partial s}|_{s=s_\alpha} = 0$$

In particular, we can write that

$$\mathcal{F}^{(\bar{f})}(\alpha) = s_\alpha + \alpha t_\alpha.$$

Example 5.8.1. We can consider the simple example $T : [0, 1] \to [0, 1]$ given by

$$T(x) = 2x + \frac{1}{4\pi} \sin(2\pi x)$$

and the function $f(x) = \sin(2\pi x)$.

5.8.2 Pointwise dimension of measures

There is a further and natural generalisation of the multifractal spectrum can be extended to the case that we look at measures and pairs of functions, following the full version of the analysis of Pesin and Weiss in [37], [38].

The previous ideas are based on the the different possible limits of the Birkhoff averages $\frac{1}{N} \sum_{n=1}^{\infty} \bar{f}(T^n x)$ as $n \to +\infty$. The distinction now is that we fix a reference measure μ and consider instead the limits of

$$\frac{\log \mu(B(x, r))}{\log r}$$

as $r \to +\infty$. Replacing the Birkhoff ergodic theorem we want the following limiting result associated to measures

Definition 5.8.2. We say that μ has pointwise dimension α if for almost all (μ) $x \in X$ we have that

$$d_\mu(x) := \lim_{r \to 0} \frac{\log \mu(B(x, r))}{\log r} = \alpha$$

.

Let us assume that μ is a Gibbs measure for a Hölder continuous function ψ. We can assume without loss of generality that $P(-\psi) = 0$.

Proposition 5.8.4. *If μ is a Gibbs measure for a Hölder continuous function ψ then there exists $\alpha = \alpha_\mu$ such that for almost all (μ) $x \in X$ we have that $d_\mu(x) = \alpha$.*

Assume that we have a real value α and that we want to choose a value q

such that

$$\alpha = \frac{\int \psi dm_q}{\int \log |T'| dm_q}$$

where $t(q)$ satisfies $P(-t(q)\log|T'| - q\psi) = 0$ and m_q is the unique equilibrium state for $-t(q)\log|T'| - q\psi$. In particular, if

$$\Lambda_\alpha = \{x : d_\mu(z) = \alpha\}$$

then we see that for a.e.(m_q), $x \in \Lambda_\alpha$ we have that

$$d_\mu(x) = \lim_{n\to+\infty} \frac{\log \mu(I_n(x))}{\log |(T^n)'(x)|} = \frac{\int \psi dm_q}{\int \log |T'| dm_q} = \alpha$$

where $I_n(x)$ is the dyadic interval containing x, by using the Birkhoff ergodic theorem for μ_q.

In particular, we see that $m_q(\Lambda_\alpha) = 1$ and then we deduce that $\dim_H(\Lambda_\alpha) \geq \dim_H(m_q)$. In fact, a simple estimate shows that there is an equality: $\dim_H(\Lambda_\alpha) = \dim_H(m_q)$. Moreover, we know that:

Lemma 5.8.5. $\dim_H(m_q) = t(q) + q\alpha$.

Proof. We know that

$$P(-t(q)\log|T'| - q\psi) = 0 = h(m_q) + \int (-t(q)\log|T'| - q\psi) dm_q$$

which allows us to rewrite

$$\dim_H(m_q) = \frac{h(m_q)}{\int \log |T'| dm_q} = t(q) + q\alpha.$$

□

Example 5.8.2. A trivial example would be where we took the measure μ to be the $(p, 1-p)$-Bernoulli measure. In this case we let $T(x) = 2x(\text{mod } 1)$ and see that

$$\phi(x) = \begin{cases} \log p & \text{if } 0 \leq x \leq \frac{1}{2} \\ \log(1-p) & \text{if } \frac{1}{2} \leq x \leq 1 \end{cases}$$

5.8.3 Other examples of conformal repellers

Finally, we briefly mention other familiar examples of repellers to which our results apply.

Hyperbolic Julia sets

We let $T : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a rational map. We define the Julia set to be $\mathcal{J}$ to be the closure of the the periodic points. We say that $T : \mathcal{J} \to \mathcal{J}$ is a hyperbolic rational map if there exists $c > 1$ such that $|T'(z)| > c$, for all $z \in \mathcal{J}$ [52]. We can apply the algorithm(s) to $T : \mathcal{J} \to \mathcal{J}$ and any real analytic map $f : \mathcal{J} \to \mathbb{R}$.

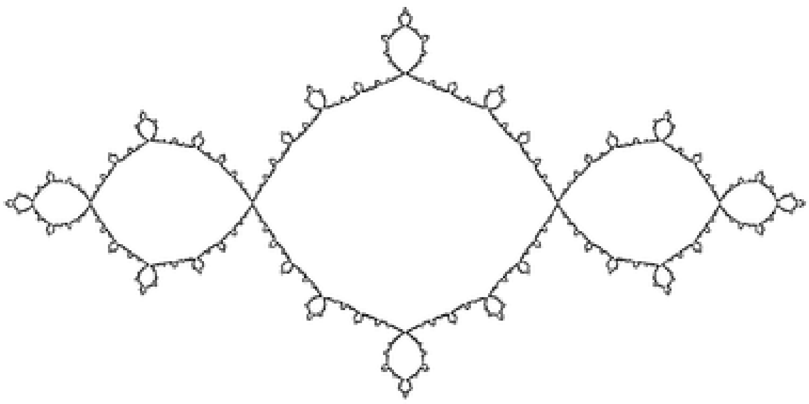

Figure 5.5: Julia set for $f_c(z) = z^2 + c$, for $c = -1$

Schottky Groups

Let $C_1, \cdots, C_k, C_{k+1}, \cdots, C_{2k}$ be circles in $\mathbb{C}$ with disjoint interiors $D_1, \cdots, D_{2k}$. We can let $\gamma_i : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a linear fractional transformation which maps C_i to C_{k+i}, for $i = 1, \cdots, k$. A Schottky group Γ is generated by $\gamma_1^{\pm 1}, \cdots, \gamma_k^{\pm 1}$ and we denote by Λ the associated Limit set (i.e., the set of accumulation points for the orbits γz_0, $g \in \Gamma$, for any fixed reference point z_0).

We can consider the transformation $T : \Lambda \to \Lambda$ defined by

$$T(z) = \begin{cases} \gamma_i(z) & \text{if } z \in D_i \\ \gamma_i^{-1}(z) & \text{if } z \in D_{k+i} \end{cases}$$

for $i = 1, \cdots, k$. There is also associated to this a natural conformal measure μ such that $\mu \circ T = |T'|^\delta \mu$, where δ is the Hausdorff Dimension of Λ. It is the possible to apply the algorithm(s) for $T : \Lambda \to \Lambda$ and any real analytic function $f : \Lambda \to \mathbb{R}$. It is also natural to apply the results on point wise dimension multifractal spectrum to the measure μ.

Remark 5.8.6. In the case of non-conformal expanding maps it is possible to recover many of these results by replacing the Hausdorff dimension of the sets Λ_α^f by their

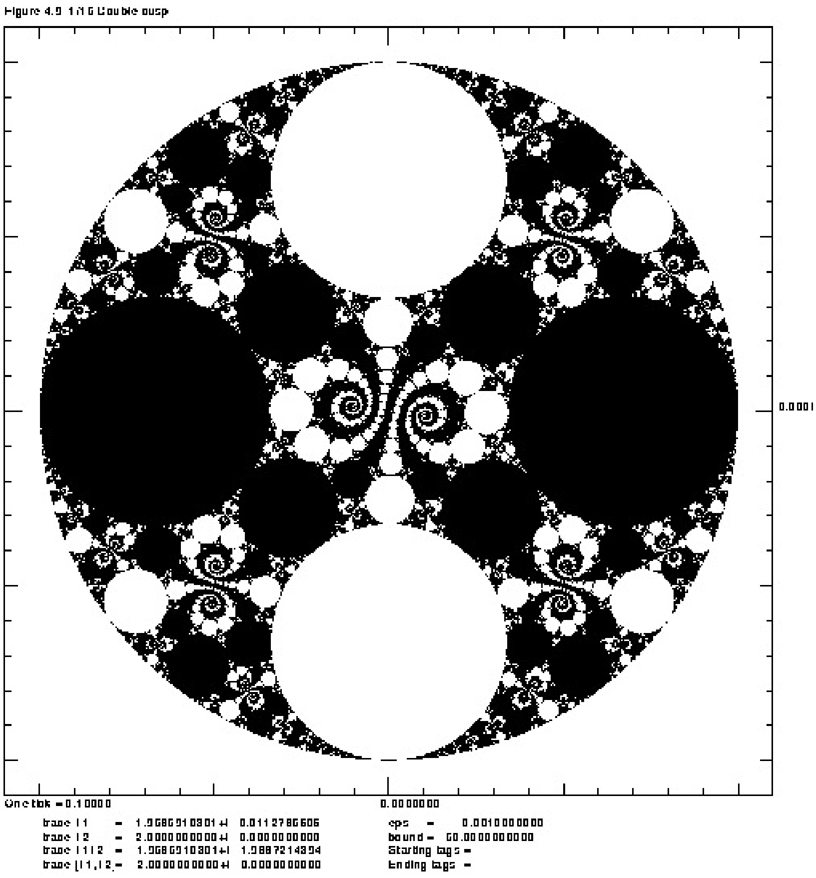

Figure 5.6: A double cusp Fuchsian group

entropy (which is defined in terms of covers by dynamical Bowen-balls, rather than the standard definition). We have also considered only discrete transformations. However, for real analytic (semi-)flows many of the results can be modified by using Markov sections.

5.9 Thermodynamic Formalism

Following the approach of Pesin and Weiss, we can characterise the dimension $\mathcal{F}(\alpha) = \dim_H(\Lambda_\alpha)$ as follows.

Definition 5.9.1. Given q we can consider the function $qf - \beta(q)\log|T'|$ where $\beta(q)$ is chosen to satisfy $P(qf - \beta(q)\log|T'|) = 0$.

We can consider the unique equilibrium measure μ for the potential $qf - \beta(q)\log|T'|$, i.e., $h(\mu) + \int(qf - \beta(q)\log|T'|)d\mu = 0$.

We can write

$$\frac{\partial d(z,q)}{\partial z} = \sum_{n=1}^{\infty} a_n(q) n z^{n-1} \text{ and } \frac{\partial d(z,q)}{\partial q} = \sum_{n=1}^{\infty} \frac{\partial a_n(q)}{\partial q} z^n$$

and then substituting these expansions into (1) we have

$$\sum_{n=1}^{\infty} \underbrace{\left(a_n(q_\alpha) n(\alpha \log 2) + \frac{\partial a_n(q)}{\partial q}|_{q=q_\alpha} \right)}_{=:b_n^\alpha(t)} e^{nP(-q_\alpha f)} = 0. \tag{2}$$

Lemma 5.9.1. *The function $\beta(q)$ is analytic and strictly convex. Moreover,*

1. $\beta'(q) = \frac{\int f d\mu}{\int \log |T'| d\mu}$;
2. $-\beta'(q)$ *obtains all the values in* $(\alpha_{min}, \alpha_{max})$; *and*
3. $\beta''(q) > 0$

Lemma 5.9.2. *We have that* $\mathcal{F}(\alpha) = \frac{h(\mu)}{\int |T'| d\mu} = \beta(q) + q\alpha$.

We can specialise to the case of the doubling map, where $\log |T'| = \log 2$.

Lemma 5.9.3. *In the particular case of the doubling map, we can write* $\beta'(q) = \frac{1}{\log 2} \frac{\partial P}{\partial q}(-qf)$.

This leads to the following.

Example 5.9.1. In the case of the doubling map: Given α

1. Find q such that $\frac{\partial P}{\partial q}(-qf) = \alpha \log 2$; and
2. Estimate $\mathcal{F}(\alpha) = \beta(q) + q\alpha = \frac{P(-qf)}{\log 2} + q\alpha$.

5.10 Computing the spectra

To address the original question of computing the we need to compute the pressure and the derivative of the pressure. We can characterize the pressure using the zeta function and determinant.

Definition 5.10.1. We formally define a complex function by

$$d(z,q) = \exp\left(- \sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{k=0}^{2^n-1} \exp\left(-q \sum_{m=1}^{n} f\left(\left\{ \frac{2^m k}{2^n - 1} \right\} \right) \right) \right)$$

$z \in \mathbb{C}$.

In particular, we have the following properties for $d(z,q)$.

Theorem 5.10.1. *We have the following properties:*

1. *The function z converges to a non-zero analytic function for $|z| < e^{-P(-qf)}$;*
2. *The value $e^{-P(-qf)}$ is a simple zero for $d(z,q)$; and*
3. *For any $\epsilon > 0$, there exists $C > 0$ such that we can expand*

$$d(z,q) = 1 + \sum_{n=1}^{\infty} a_n(q) z^n$$

where $|a_n(q)| \leq C \left(\frac{1}{2} + \epsilon\right)^{n^2}$, for $n \geq 1$ and $a_n(q)$ depends on the values

$$\left\{ f\left(\left\{\frac{2^k}{2^m - 1}\right\}\right) \ : 0 \leq k \leq m-1, m \leq n \right\};$$

and

4. *The function $d(z,q)$ has an analytic extension to $\mathbb{C}$ as an entire function.*

We can use the implicit function theorem to write

$$-\frac{\partial d(e^{-P(qf)}, q)}{\partial z} \underbrace{\frac{\partial P(-qf)}{\partial q}}_{=\alpha \log 2} e^{-P(-qf)} + \frac{\partial d(e^{-P(-qf)}, q)}{\partial q} = 0. \tag{1}$$

We can write

$$\frac{\partial d(z,q)}{\partial z} = 1 + \sum_{n=1}^{\infty} a_n(q) n z^{n-1} \text{ and } \frac{\partial d(z,q)}{\partial q} = 1 + \sum_{n=1}^{\infty} \frac{\partial a_n(q)}{\partial q} z^n$$

then we can substitute into (1) to write

$$\sum_{n=1}^{\infty} \underbrace{(a_n(t) n \alpha \log 2 - a_n'(t)) e^{nP(tf)}}_{=:b_n^{\alpha}(t)} = 0. \tag{2}$$

Definition 5.10.2. Given α we can define

$$Q_\alpha(z,t) = \sum_{n=1}^{\infty} b_n^{\alpha}(t) z^n \text{ and } P(z,t) = \sum_{n=1}^{\infty} a_n(t) z^n$$

We can deduce the following:

Lemma 5.10.2. *Given α, the solution (z_0, t_0) for $Q_\alpha(z,t) = P(z,t) = 0$, value of t satisfies $\frac{\partial P(-qf)}{\partial q}|_{q=t} = \alpha$. In particular,*

$$\mathcal{F}(\alpha) = \frac{P(-qf)}{\log 2} + t_0\alpha = \frac{\log z_0}{\log 2} + t_0\alpha.$$

In order to to this we can approximate $d(z)$ by the complex function

$$d_N(z,q) = 1 + \sum_{n=1}^{N} a_n(q) z^n$$

and observe that $d(z,q) - d_N(z,q) = O\left(\left(\frac{1}{2}+\epsilon\right)^{N^2}\right)$, on any compact region.

We can write

$$\frac{\partial d_N(z,q)}{\partial z} = 1 + \sum_{n=1}^{N} a_n(q) n z^{n-1} \text{ and } \frac{\partial d_N(z,q)}{\partial q} = 1 + \sum_{n=1}^{N} \frac{\partial a_n(q)}{\partial q} z^n$$

then

$$\frac{\partial d(z,q)}{\partial z} - \frac{\partial d_N(z,q)}{\partial z} = O\left(\left(\frac{1}{2}+\epsilon\right)^{N^2}\right) \text{ and } \frac{\partial d(z,q)}{\partial q} - \frac{\partial d_N(z,q)}{\partial q} = O\left(\left(\frac{1}{2}+\epsilon\right)^{N^2}\right).$$

In particular, we now proceed as follows.

Definition 5.10.3. Given α and N we can define

$$Q_\alpha^{(N)}(z,t) = \sum_{n=1}^{N} b_n^\alpha(t) z^n \text{ and } P^{(N)}(z,t) = \sum_{n=1}^{N} a_n(t) z^n$$

- we want to solve for (z_N, t_N) such that

$$Q_\alpha^{(N)}(z,t) = P^{(N)}(z,t) = 0.$$

- In particular, we can then write

$$\mathcal{F}(\alpha) = \frac{P(-qf)}{\log 2} + t_0\alpha = \frac{\log z_0}{\log 2} + t_0\alpha.$$

  In particular, $z(q_N) = z(q) + O\left(\left(\frac{1}{2}+\epsilon\right)^{N^2}\right)$ and $t_N = t + O\left(\left(\frac{1}{2}+\epsilon\right)^{N^2}\right)$.

- We estimate $\mathcal{F}(\alpha) = \frac{P(-q_N f)}{\log 2} + q_N\alpha + O\left(\left(\frac{1}{2}+\epsilon\right)^{N^2}\right)$.

There is an implicit characterization of the exact value of the dimension

$\dim_H(\Lambda_\alpha^{(f)})$ using a function defined in terms of periodic points. For simplicity of exposition, let us

$$d_1(z,t) = \exp\left(-\sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{T^n x = x} \frac{\exp\left(-t\sum_{i=0}^{n-1} f(T^i x)\right) |(T^n)'(x)|^{-q}}{1 - |(T^n)'(x)|^{-1}}\right)$$

which converges for t sufficiently large and $|z|$ sufficiently small. In fact, when T and f are C^ω it follows by work of Ruelle (after Grothendieck) that $d(z,t)$ is entire in $\mathbb{C}^2$.

Theorem 5.10.3. *Assume for simplicity that $d(1,1) = 0$ and T is the doubling map. Given $\alpha \in (\alpha_-, \alpha_+)$ there is a solution $(z_\alpha, q_\alpha) \in \mathbb{R}^2$ to the equations*

$$d_1(t,q) = 0 \text{ and}$$
$$\frac{\partial d_1(t,q_\alpha)}{\partial z}|_{t=z_\alpha} - \alpha \frac{\partial d_1(t_\alpha,q)}{\partial q}|_{q=q_\alpha} = 0$$

and then we can write

$$\dim_H(\Lambda_\alpha^{(f)}) = q_\alpha + t_\alpha \alpha$$

Chapter 6

Further generalisations and projects

Finally there are several possible further investigations to be carried out in the light of the main results of Chapter 6.

- Can the multifractal results be generalised to invertible maps - for example, the perturbations of Arnold CAT map on the torus ($\mathbb{T}^2$)?

  Consider the matrix
  $$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix},$$
  and the 2 - torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, where any two points (x_1, y_1) and $(x_2, y_2) \in \mathbb{R}^2$ belong to the same class if there exists an integer vector $(a, b) \in \mathbb{Z}^2$ such that
  $$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} n \\ m \end{pmatrix} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}.$$
  The matrix A induces an automorphism of the torus $\mathbb{T}^2$ and we denote by $T_A : \mathbb{T}^2 \to \mathbb{T}^2$ given by,
  $$T_A\,(x, y) = (2x + y, x + y) \mod 1.$$

- Can the multifractal results be generalised to conformal maps in higher dimensions - for example Julia sets in higher dimension?

  At the risk of losing conformality in higher dimension the following theorem by Jenkinson and Pollicott in [25] gives a motivation to apply the main result for these Julia sets.

Theorem 6.0.4. *Let $X \subset M$ be a locally maximal compact invariant set for a conformal real-analytic hyperbolic Markov map $T : X \to X$, where M is a C^{ω} manifold of dimension $d \in \mathbb{N}$. For each $N \geq 1$ we can explicitly define a function Δ_N, using only the derivatives $DT^n(z)$ evaluated at period-n points z, for $1 \leq n \leq N$, and associate $C > 0$ and $0 < \delta < 1$ such that if s_N is the largest real zero of Δ_N then*

$$|\dim_H \Lambda - s_N| \leq C\delta^{N^{1+1/d}}.$$

- Can the multifractal results be generalised to non-conformal dynamical systems or repellers but " regular examples" such as the Bedford-McMullen and related problems?

The one generalisation of the Cantor set on the interval is that of Bedford-McMullen construction. Consider two nonzero positive integers $n \geq m$. We define the limit set resulting from the following construction,

$$\overline{\mathrm{R}} = \left\{ \left(\sum_{k=1}^{\infty} \frac{x_k}{n^k}, \sum_{k=1}^{\infty} \frac{y_k}{m^k} \right) : (x_k, y_k) \in R \text{ where } 0 \leq x_k < n \text{ and } 0 \leq y_k < m \right\}.$$

This follows the presentations of Bedford in his thesis [6] , McMullen in the paper [30], Sierpinski in [46] and Mandelbrot's book [27] on Sierpinski carpets.

Bibliography

[1] M. Atiyah, R. Bott, A Lefschetz fixed point formula for elliptic complexes, Annals of Mathematics, (2) 86 (1967), 374 - 407.

[2] V. Baladi, *Positive Transfer Operators and Decay of Correlations.* World Scientific, Singapore, 2000.

[3] A. T. Baraviera, R. Leplaideur, A. O. Lopes, Ergodic optimization, zero temperature limits and the max-plus algebra, mini-course in XXIX Coloquio Brasileiro de Matemtica, 2013.

[4] L. Barreira, *Dimension and Recurrence in Hyperbolic Dynamics.* Birkhauser, Basel, 2008.

[5] L. Barreira, *Ergodic Theory, Hyperbolic Dynamics and Dimension Theory.* Springer, Berlin, 2012.

[6] T. Bedford, *Crinkly curves, Markov partitions and dimension.* PhD thesis, University of Warwick, 1984.

[7] A.S. Besicovitch, Sets of fractional dimensions IV: on the rational approximation to real numbers. Journal of London Mathematical Society, 9 (1934), 126 - 131.

[8] G. D. Birkhoff, Proof of the Ergodic Theorem. Proc. Nat. Acad. of Sciences, 17 (1931), 656 - 660.

[9] O. Bodart, M. Zinsmeister, Dimension de Hausdorff des ensembles de Julia, Fundamenta Mathematicae, 151 (1996), 121 - 137.

[10] T. Bousch, Le poisson n'a pas d'arêtes. Annales de l'Institut Henri Poincar (probabilités et statistiques), 36 (2000), 489-508.

[11] R. Bowen, *Equilibrium States and the Ergodic Theory of Anosov Diffeomorphism.* Springer Lecture Notes in Mathematics, 470, 1975.

[12] R. Bowen, Hausdorff dimension of quasi-circles. Publications (I.H.E.S. Paris) 50, 1979, 11-26.

[13] C. Caratheodory, Uber das Lineare Mass. Gottingen Nachr. (1914), 406 - 426.

[14] P. Collet, J .L. Lebowitz, A. Porzio, The Dimension spectrum of some dynamical systems. Journal of statistical physics, 47 (1987), 609 - 644.

[15] K. Falconer, *Fractal Geometry, A mathematical Foundations and Applications.* Wiley, Chichester, 1990.

[16] K. Falconer, *Techniques in Fractal Geometry.* Wiley, New York, 1997.

[17] O. Frostman, Potential d′Équilibre et Capacité des Ensembles avec Quelques Applications à la Théorie des Fonctions. Meddel. Lunds Univ. Math. Sem. 3 (1935), 1-118.

[18] L. Garnett, A computer algorithm for determining the Hausdorff dimension of certain fractals. Mathematics of Computations, (1988), 291 - 300.

[19] M. Giaquinta and G. Modica, *Mathematical analysis [electronic resource] : linear and metric structures and continuity.* Birkhaeauser, Boston, 2007.

[20] A. Grothendieck, Produits tensoriels topologiques et espaces nucleaires. (Mem. Amer. Math. Soc.), 16, 1955.

[21] A. Grothendieck, La theorie de Fredholm, Bull. Soc. Math. France, 84, 1956, 319-384.

[22] T. Hasley, M. Jensen, L. Kadanoff, I. Procaccia, B. Shraiman, Fractal measures and their singularities: The characterization of strange sets. Physical Review A, 33, N2 (1986), 1141 - 1151.

[23] F. Hausdorff, Dimension and ausseres Mass. Math. Annalen, 79 (1919), 157-179.

[24] C. Ionescu-Tulcea and G. Marinescu, Theorie ergodique pour les classes d'operations non complemetent continues. Annals of Mathematics, 52 (1950), 140-147.

[25] O. Jenkinson and M. Pollicott, Calculating Hausdorff dimension of Julia sets and Kleinian limit sets. American Journal of Mathematics, 124, 3 (2002), 495-545.

[26] T. Kato, *Perturbation Theory for Linear Operators*. Springer-Verlag, New York, 1976.

[27] B.B. Mandelbrot, *Fractals*, San Francisco, 1977.

[28] R. Mañé, *Ergodic Theory and Differentiable Dynamics*. Springer-Verlag, Berlin, 1980.

[29] D. Mayer, Continued fractions and related transformations, in Proceedings of Ergodic Theory, symbolic dynamics and hyperbolic spaces. Oxford University Press, (1991) 175-222.

[30] C. McMullen, The Hausdorff dimension of general Sierpinski Carpet., Nagoya Math. J. , 96 (1984), 1-9.

[31] C. McMullen, Hausdorff dimension and conformal dynamics. III. Computation of dimension. American Journal of Mathematics, 120 (1998), 691-721.

[32] K. Menger, Über die Dimensionulitatvon Punktmengen I. Monatsh. für Math. und Phys. 33(1923), 148 - 160.

[33] K.R. Parthasarathy, *Introduction to Probability and Measure*. Hindustan Book Agency, New Delhi, 2005.

[34] W. Parry and M. Pollicott, Zeta Functions and the Periodic Orbit Structure of Hyperbolic Dynamics. Société mathématique de France, 1990.

[35] A.R. Pears, *Dimension Theory of General Spaces*. Cambridge University Press, London, 1975.

[36] Y. Pesin and H. Weiss, A multi fractal analysis of equilibrium measures for conformal expanding maps and Markov Moran geometric constructions. Journal of Statistical Physics, 86:1-2 (1997), 233-275.

[37] Y. Pesin and H. Weiss, The multifractal analysis of Gibbs measures: motivation, mathematical foundation, and examples. Chaos: An Interdisciplinary Journal of Nonlinear Science, 7 (1997) 89-106.

[38] Y. Pesin, Dimension theory in dynamical systems: contemporary views and applications. University of Chicago Press, Chicago, 1997.

[39] L. Pontrjagin and L. Schnirelman, Sur une propriete metrique de la dimension. Annals of Mathematics, 33 (1932), 156 - 162.

[40] W. Rudin, *Functional analysis*. McGraw-Hill, New York, 1973.

[41] D. Ruelle, Statistical Mechanics on a compact set with Z^{ν} action satisfying expansiveness and specification. Trans. Amer. Math. Soc., 185 (1973), 237 - 251.

[42] D. Ruelle, Thermodynamic formalism. The mathematical structures of classical statistical mechanics. Encyclopedia of Mathematics and its Applications. 5. Addison-Waley, Reading, Massachusetts, 1978.

[43] D. Ruelle, Repellers for real analytic maps. Ergod. Th. and Dynam. Sys., 2 (1982) 99-107.

[44] D. Ruelle, An extension of the theory of Fredholm determinants. Inst. Hautes Estudes Sci. Publ. Math. 72, (1990) 175-193.

[45] D. Saupe, Efficient computation of Julia sets and their fractal dimension. Physica, 28D (1987), 358 - 370.

[46] W. Sierpinski, Sur une courbe cantalienne qui contient une image biunivoque et continue de toute courbe donne. Comptes Rendus, 162 (1916), 629 - 842.

[47] Y.G. Sinai, Gibbs measures in ergodic theory. Russ. Math. Surv., 27 (1972), 21 - 70.

[48] P. Urysohn, Les multiplicites Cantoriennes. C.R. Acad. Paris, 175 (1922), 440-442.

[49] P. Walters, A variational principle for the pressure of continuous transformations. American Journal of Mathematics, 17 (1976), 937 - 971.

[50] P. Walters, *An Introduction to ergodic theory.* Springer-Verlag, New York, 1982.

[51] M. Widom, D. Bensimon, L.P. Kadanoff and S.J. Shenker, Strange objects in the complex plane, Journal of Statistical Physics, 32 (1983), 443 - 454.

[52] M. Zinsmeister, *Formalisme thermodynamique et Systémes dynamiques holomorphes.* Panorama et Synthses no 4, 1996.