On the Hilbert Series of Polarised Orbifolds

by

Michael Nicolas Selig

Thesis

Submitted to the University of Warwick

for the degree of

Doctor of Philosophy

Department of Mathematics

June 2015

THE UNIVERSITY OF WARWICK

Contents

Acknowledgments **iii**

Declarations **v**

Abstract **vi**

Chapter 1 Introduction and preliminaries **1**

1.1 Introduction, motivation and history 1

1.2 Basic concepts 5

1.3 Weighted projective space 7

1.4 Cyclic quotient singularities 9

1.5 The Hilbert syzygies theorem 14

Chapter 2 The isolated singularity case **18**

2.1 Preliminaries 18

2.2 Main result and proof 23

2.3 Examples and applications 32

Chapter 3 Generalising the formula to arbitrary rational functions **44**

3.1 Definitions, examples and observations in the curve orbifold locus case 44

3.2 The "isolated" case 48

3.3 The curve locus case 51

3.4 Generalisations of the "curve" locus case 60

Chapter 4 The curve orbifold locus case revisited **70**

4.1 Main theorem and examples 70

4.2 First parsing 80

4.3 Contributions from orbipoints 81

4.4 Contributions from orbicurves 82

4.5 End of proof 91

Chapter 5 Further problems and generalisations **94**

5.1 Understanding the curve orbifold locus more completely 94

5.2 Towards general results with arbitrary orbifold locus 97

5.3 General formulae without the symmetric assumption 102

Appendix A MAGMA code and functions **104**

Acknowledgments

First and foremost massive thanks are due to my supervisor Miles Reid for guidance, support and help throughout the course of my PhD, especially when things (perhaps inevitably) became difficult. His deep subject knowledge, ability to generate ideas and sheer enthusiasm for the topic have all been invaluable, as has his ability to navigate through the administrative loops when required.

I am also grateful to Gavin Brown for a number of useful discussions, and Shengtian Zhou for sharing her insight in these areas. Throughout my time at Warwick I have been fortunate enough to be backed by a team of colleagues and visiting experts, and I would like to thank all of them for the assistance they provided.

My work at Warwick was supported by an EPSRC grant.

I was fortunate enough to be able to visit places as far away as Moscow, Seoul and Shanghai in the course of my studies and would like to thank Yuri Prokhorov, Yongnam Lee and Meng Chen for inviting me and making sure I was suitably looked after. Thanks also must go to my friends Kuzma, Yongjoo and Yoonsuk for taking time out of their busy lives to help me feel welcome and show me around.

My journey as a budding Mathematician started a long time before I arrived at Warwick and I am indebted to everybody who has assisted me along the way, including my old school teachers and friends, and lecturers, friends and colleagues from Bath and Cambridge.

Thank you also to all the friends I have made at Warwick: my Maths colleagues including Sarah, Umar, Sohail, Eduardo, Seung-Jo, Taro, Andrew, Tom, Enrico, Sara et al; those I have made through fencing – Ian, Alasdair, Domnhall, Calum, Audoin, Lorenzo, Jon, Alex, Francine, Dan, Rob and Sierra amongst many others; and those from all other walks of life, too numerous to mention. Cricket has also been a huge part of my life and I am hugely proud of what our team achieved over the last few years, so you guys deserve my gratitude simply for being so much fun.

I cannot put into words how lucky I am to have the most brilliantly supportive family: my parents have been unwavering in their support, my grandparents have provided me with a second home in London and showered me with love, affection and food, whilst my two brothers Tom and Dan are absolutely everything one could wish for. I am delighted that Sisi has joined our family and am now hoping for nieces/nephews to corrupt shortly.

Last but very much not least my eternal thanks to my much better half Caroline, who has put up with more than anybody should ever have to, and without whose love and support I would simply have been unable to get through this.

Declarations

Chapters 1 and 2 are exposition material following the available literature (especially [Buckley, Reid, and Zhou, 2013]) with worked examples for practice. Chapter 4 builds on work done by Shengtian Zhou as written up in [Zhou, 2011]. Chapters 3 and 5 are completely my own and to the best of my knowledge original.

Abstract

We are interested in calculating the Hilbert series of a polarised orbifold (X, D) (that is D is an ample divisor on an orbifold X). Indeed, its numerical data is encoded in its Hilbert series, so that calculating this sometimes gives us information about the ring, notably possible generators and relations, using the Hilbert syzygies theorem. Vaguely, we have $P_X(t) = \frac{\text{Num}}{\text{Denom}}$ where Num is given by the relations and syzygies of R and Denom is given by the generators. Thus in particular we hope that we can use the numerical data of the ring to deduce possible explicit constructions.

A reasonable goal is therefore to calculate the Hilbert series of a polarised (X, D); we write it in closed form, where each term corresponds to an orbifold stratum, is Gorenstein symmetric and with integral numerator of "short support".

The study of the Hilbert series where the singular locus has dimension at most 1 leads to questions about more general rational functions of the form

$$\frac{N}{\prod(1 - t^{a_i})}$$

with N integral and symmetric. We prove various parsings in terms of the poles at the μ_{a_i}; each individual term is Gorenstein symmetric, with integral numerator of "short support" and geometrically corresponds to some orbifold locus.

Chapters 1 and 2 are expository material: Chapter 1 is basic introductory material whilst in Chapter 2 we explain the Hilbert series parsing in the isolated singularity case, as solved in Buckley et al. [2013] and Zhou [2011] and go over worked examples for practice. Chapter 3 uses the structure of the parsing in the isolated case and the expected structure in the non-isolated case to discuss generalisations to arbitrary rational functions with symmetry and poles only at certain roots of unity. We prove some special cases. Chapter 4 discusses the Hilbert series parsing in the curve orbifold locus case in a more geometrical setting. Chapter 5 discusses further generalisations and issues. In particular we discuss how the strategies used in Chapter 3 could work in a more general section, and the non symmetric case.

Chapter 1

Introduction and preliminaries

In this Chapter we first discuss some motivation and history, then go on to introduce all the necessary basic concepts and notation. We assume all varieties to be projective and normal over $\mathbb{C}$.

1.1 Introduction, motivation and history

We are interested in calculating the Hilbert series of a polarised orbifold (X, D) with given invariants. The Hilbert series should depend only on the numerical data of (X, D) such as the first few plurigenera, its basket of singularities, etc. and be parsed in such a way that each term clearly corresponds to an orbifold stratum and has numerator integral and symmetric. In other words we wish to pursue the "Ice cream functions" viewpoint adopted in Zhou [2011] and Buckley et al. [2013]. Ice cream functions have the same periodicity as the Dedekind sums introduced in Reid [1985] but have integral and symmetric numerators and are easy to calculate.

Our motivation comes from explicit problems in birational geometry, namely the construction and classification of 3-folds and 4-folds with given invariants and singularities. Indeed given a polarised orbifold (X, D), the graded ring

$$R(X, D) = \bigoplus_{m \geq 0} H^0(X, \mathcal{O}_X(mD))$$

gives an embedding of the variety $X \cong \mathrm{Proj}(R(X, D))$ into some weighted projective space. Finding generators and relations for the ring $R(X, D)$ therefore gives a possible construction for the variety X, and studying the Hilbert series of $R(X, D)$ is one way of finding generators and relations (via the Hilbert syzygies theorem as stated in 1.5.1).

Embedding X into some weighted projective space works similarly to embedding X into the usual $\mathbb{P}^n$ except that it will most often be singular and hence have some pathologies. We restrict ourselves in particular to quasismooth and well-formed orbifolds (see Section 1.3 for the relevant definitions). In practice moreover all our calculations restrict themselves to canonical (and nearly always terminal) singularities. Let us briefly here recally the definitions:

Definition 1.1.1. Let X be a normal variety and suppose the canonical class K_X is $\mathbb{Q}$-Cartier (see Section 1.2 for the relevant definition). Then X is said to have canonical (respectively terminal) singularities if for one (equivalently all) resolution of singularities $f : Y \to X$ with exceptional locus $\cup E_i$ we have

$$\mathrm{K}_Y = f^*(\mathrm{K}_X) + \sum a_i E_i$$

with all $a_i \geq 0$ (respectively > 0).

This restriction is natural from the point of view of birational geometry and in particular the minimal model program (MMP) where these types of singularities occur naturally in the course of contracting K_X-negative curves. More specifically, terminal singularities occur in the terminal objects in MMP (namely either minimal models or Mori fibre spaces), whilst canonical singularities occur in canonical models (of varieties of general type). See original papers by Mori [1982], Kawamata et al. [1987] or Matsuki [2013] for an excellent readable introduction to MMP; see Reid [1980] for the reference paper on canonical 3-folds.

In fact this restriction to the canonical or terminal case drives some of the motivation for our work. Indeed we note the following:

1. A surface is terminal if and only if it is smooth;
2. A surface is canonical if and only if it has Du Val singularities, namely a point isomorphic to one of the following hypersurface singularities:
    - A_n: $w^2 + x^2 + y^{n+1} = 0$ $(n \geq 1)$;
    - D_n: $w^2 + y(x^2 + y^{n-2} = 0$ $(n \geq 4)$;
    - E_6: $w^2 + x^3 + y^4 = 0$;
    - E_7: $w^2 + x(x^2 + y^3) = 0$;

- E_8: $w^2 + x^3 + y^5 = 0$.

3. Terminal singularities in dimension 3 are isolated (see for example Reid [1983] or Mori et al. [1985] for criteria for 3-fold singularities to be terminal).

Moreover by Iano-Fletcher [2000] lemmata 9.2 and 9.3 the cyclic quotient canonical surface singularities are just the A_n singularities, of type $\frac{1}{n+1}(1,-1)$, and the cyclic quotient terminal 3-fold singularities are of type $\frac{1}{r}(1,-1,a)$ for some a coprime to r. However 3-folds with canonical singularities and 4-folds with terminal singularities will usually have singular behaviour in dimension 1 (so not isolated) but no higher (see Reid [1980], Mori et al. [1988] or Morrison and Stevens [1984]). Thus any attempt to construct or classify such varieties needs to take into account such behaviour.

On the other hand there is no reason a priori for our methods to restrict to the terminal case, so that these results could lead us (or others) to consider similar non-terminal problems in the future.

Our work builds on previous attempts to use Riemann–Roch methods to construct varieties. The plurigenera forumlae in Reid [1985] could be used to construct orbifolds with isolated singularities, whilst those developed in Buckley [2003] and Buckley and Szendroi [2004] were used for 3-folds with curve orbifold locus. These took the point of view of using Dedekind sums in their calculations, which can be summarised in slogan form by the formula

$$P_X(t) = A(t) + \sum_{P \in \mathcal{B}} M_P$$

where the term $A(t)$ is a Riemann–Roch contribution obtained from the usual (see Borel and Serre [1958])

$$\mathrm{RR}(X, D) = (\mathrm{ch}(\mathcal{O}_X(D)) \cdot \mathrm{Td}(T_X))[n]$$

calculated on a resolution of singularities of X. In particular $A(t)$ deals exclusively with the growth of the plurigenera. The remaining terms M_P are local contributions from the basket of singularities $\mathcal{B}$ (P is not to be thought of as necessarily a point here) and are calculated via Dedekind sums. They have strict r periodicity corresponding to the type $\frac{1}{r}$ of the singularity. All the terms here are rational.

On the other hand more recent work in Zhou [2011] and Buckley et al. [2013] has built on this point of view to obtain a different style of formula which we summarise as

$$P(t) = P_I(t) + \sum_{P \in \mathcal{B}} P_{\text{orb}}(P)$$

where P_I should be thought of as a global contribution, which depends only on the first handful of plurigenera (in particular P_I does not control the growth of the plurigenera), and the P_{orb} contributions (sometimes called Ice cream functions) have the same r periodicity as the Dedekind sum contributions M_P earlier. This formula is obtained from the previous one by attributing a fractional part of the growth contribution to the local contributions. In doing so we can make all the terms integral and symmetric, whilst keeping the periodicity the same.

This approach was used in Zhou [2011] and Buckley et al. [2013], however their efforts focused on the isolated singularity case, with some progress towards the curve orbifold locus case. We build on their work, and find further results in the curve case, in Section 3.3 and Chapter 4. This is especially important since as mentioned before 3-folds with canonical singularities and 4-folds with terminal singularities will usually have orbifold behaviour in dimension 1 but no higher. Whilst we do not yet provide a complete formula in the curve case, we are really not far off, and the only information which is lacking is a collection of integers and therefore most assuredly countable.

In fact we go further. In Sections 3.2 and 3.4 in particular we strip away the geometry and study the Hilbert series P_X purely as a rational function with fixed poles at certain roots of unity, and strong symmetry (see definition 2.1.2). Looking at these poles, we are able to deduce a parsing in the way we wish by simply subtracting the part of P which has a maximal pole at these roots of unity and proceeding inductively. Our formulae are all constructive, and preserve integrality and symmetry as in the geometric case. Whilst in this work we still restrict ourselves to the "curve" case (see Sections 3.3 and 3.4) the higher dimensional cases should really follow from exactly the same methods (we discuss this in more detail in Section 5.2).

We make no apologies for the fact that our work contains numerous examples. We regularly refer to the lists drawn up in the second part of Iano-Fletcher [2000]; in particular we use some of the famous 95 families of Fano 3-fold hypersurfaces which were first discovered by Reid in 1979. In the beautiful paper of Corti, Pukhlikov and Reid Corti et al. [2000] it was shown that a general member of all these families is birationally rigid and therefore in particular irrational. A similar result was proved in Cheltsov and Park [to appear] in the quasismooth case. Other studies around the birational geometry of Fano 3-folds include Ahmadinezhad and Zucconi [2014], Ahmadinezhad and Okada [2015]. Moreover our results provide us with many examples of quasismooth Fano 4-fold hypersurfaces. It would be interesting to study their birational geometry similarly, for example questions over rigidity or rationality; it might be that some of the techniques used in the paper Corti et al. [2000] could be used for 4-fold Fano hypersurfaces. We don't do this here, but do dare to suggest that this would be an interesting area to take this research further.

More generally we believe that the work done here should link with other work on constructing and classifying terminal orbifolds (Fano, Calabi-Yau, etc.). We hope that these results may be used to further existing lists such as in Iano-Fletcher [2000], Altmok et al. [2002] or Brown and Kasprzyk [to appear], and possibly add to the graded ring database (http://www.grdb.co.uk/). There are also natural links as mentioned to explicit birational geometry of Fano varieties, rationality of Fano varieties. Finally, the study of Calabi-Yau varieties has natural links to mirror symmetry; see for example the papers Okada [2009], Okada [2013] or Coates et al. [2012]; for a more gentle introduction see Thomas [2005] or Cox and Katz [1999].

1.2 Basic concepts

Definition 1.2.1. Let X be a (normal, projective) variety. A Weil divisor is a formal linear combination of prime divisors (that is, irreducible codimension 1 subvarieties of X) with integer coefficients, i.e.

$$D = \sum_{i=1}^{N} n_i Z_i \quad \text{where } n_i \in \mathbb{Z} \text{ and } Z_i \subset X \text{ are prime divisors.}$$

A $\mathbb{Q}$-divisor is such a sum where we allow the coefficients $n_i \in \mathbb{Q}$.

A Cartier divisor is a Weil divisor D such that the sheaf of sections $\mathcal{O}_X(D)$ is invertible, so D is everywhere locally given as the divisor of some rational function f; thus we can view Cartier divisors as a collection (U_i, f_i) where the U_i form an open cover of X and the f_i are rational functions on X.

A $\mathbb{Q}$-Cartier ($\mathbb{Q}$-)divisor is a Weil ($\mathbb{Q}$-)divisor D such that mD is Cartier divisor for some integer $m > 0$.

We say X is $\mathbb{Q}$-factorial if every Weil divisor is $\mathbb{Q}$-Cartier.

The notion of $\mathbb{Q}$-factorial means we can define things like intersection numbers and pullbacks on all Weil divisors. Indeed, these are usually only well-defined for Cartier divisors, but if X is $\mathbb{Q}$-factorial, we can extend this linearly. For example, for a map $\phi\colon Y \to X$ where X is $\mathbb{Q}$-Cartier we define

$$\phi^*(D) = \frac{1}{m}\phi^*(mD)$$

where mD is a Cartier divisor, say given by (U_i, f_i) so $\phi^*(mD)$ is just the Cartier divisor given by the system $(\phi^{-1}(U_i), \phi^*(f_i))$.

Definition 1.2.2. Let D be a divisor on a variety X. We define as usual the Riemann-Roch space

$$H^0(X, \mathcal{O}_X(nD)) = \{f \in \mathbb{C}(X) : \mathrm{div}(f) + nD \geq 0\} \cup \{0\};$$

choosing a basis for $H^0(X, \mathcal{O}_X(nD))$ gives a rational map

$$\phi_{nD}\colon X \dashrightarrow \mathbb{P}(H^0(X, \mathcal{O}_X(D))).$$

We define the graded ring associated to D by

$$R(X, D) = \bigoplus_{n \geq 0} H^0(X, \mathcal{O}_X(nD)).$$

A divisor D is said to be *ample* if for some integer $n > 0$ the associated map ϕ_{nD} defines an embedding into projective space and $\mathcal{O}_X(nD) \cong \mathcal{O}_X(1)$ under this embedding.

A polarised variety (X, D) is a variety X together with a given choice of ample Weil divisor D. In this case $X \cong \mathrm{Proj}(R(X, D))$ and $\mathcal{O}_X(D) \cong \mathcal{O}(1)$ under this isomorphism.

Notation 1.2.3. We write as usual $h^i(X, \mathcal{O}_X(D))$, $h^i(\mathcal{O}_X(D))$ or even $h^i(D)$ (by abusive notation) for $\dim_{\mathbb{C}} H^i(X, \mathcal{O}_X(D))$.

Definition 1.2.4. We define the Hilbert series of a graded ring $R = \bigoplus_{n\geq 0} R_n$ over $\mathbb{C}$ as

$$P_R(t) = \sum_{n \geq 0} \dim_{\mathbb{C}}(R_n) t^n.$$

The Hilbert series of a divisor D on X is denoted by $P_{(X,D)}$ and is just the Hilbert series of $R(X, D)$. In particular for a polarised variety (X, D) we sometimes denote this by P_X or even P where there is no potential for confusion, so that

$$P(t) = \sum_{n \geq 0} h^0(X, \mathcal{O}_X(nD))t^n.$$

Notation 1.2.5. By abusive notation we refer to $h^0(X, \mathcal{O}_X(n))$ as the *n-th plurigenus* of X and sometimes write

$$P_n = h^0(X, \mathcal{O}_X(n))$$

so that for a polarised variety (X, D) we have

$$P_n = h^0(nD).$$

1.3 Weighted projective space

We now introduce the fundamental concept of *weighted projective space*; this is an analogue of the usual projective space, except we allow the coordinates to have different weights. In general if D is an ample divisor on X writing $R(X, D)$ in terms of generators and relations, the generators will have degrees bigger than 1. Consequently $X \cong \text{Proj}(R(X, D))$ will give an embedding into some weighted projective space, rather than the usual projective space.

The main reference for this part is [Iano-Fletcher, 2000].

Definition 1.3.1. Let $a_0, \ldots, a_n \in \mathbb{N}$ and define a $\mathbb{C}^\times$ action on $\mathbb{C}^{n+1}$ by

$$\lambda \cdot (x_0, \ldots, x_n) = (\lambda^{a_0} x_0, \ldots, \lambda^{a_n} x_n).$$

We then define the quotient

$$\mathbb{P}(a_0, \ldots, a_n) = \frac{\mathbb{C}^{n+1} \setminus \{0\}}{\mathbb{C}^\times}.$$

We call this quotient the *weighted projective space* with weights $a_0, \ldots, a_n$. It is a projective variety of dimension n.

To consider this as a geometric quotient, we need to define what the functions on it are. These are simply the $\mathbb{C}^\times$ invariant rational functions in $\mathbb{C}(x_0, \ldots, x_n)$, that is, a regular function on an open U can be written as a quotient $\frac{f}{g}$ where f, g are weighted homogeneous polynomials in $x_0, \dots, x_n$ (x_i has weight a_i) of same degree, and g is nowhere vanishing on U.

We note the following proposition (see Iano-Fletcher [2000] lemmata 5.5 and 5.7).

Proposition 1.3.2. *1.* $\mathbb{P}(a_0, \dots, a_n) \cong \mathbb{P}(qa_0, \dots, qa_n)$ *for any* $q \in \mathbb{N}$*;*

2. if $q = \mathrm{hcf}(a_0, \dots \widehat{a_i}, \dots, a_n)$ *then*

$$\mathbb{P}(a_0, \dots, a_n) \cong \mathbb{P}(a_0/q, \dots, a_i, \dots, a_n/q).$$

This motivates the following definition.

Definition 1.3.3. A weighted projective space $\mathbb{P}(a_0, \dots, a_n)$ is *well formed* if for any i

$$\mathrm{hcf}(a_0, \dots, \widehat{a_i}, \dots, a_n) = 1.$$

As a consequence of the previous proposition every weighted projective space is isomorphic to a well formed one. From now on, we assume all ambient weighted projective spaces to be well formed. We write $w\mathbb{P}$ or $w\mathbb{P}^n$ for a non specified weighted projective space, that is, where we do not write down the weights explicitly. We will also use the abbreviation WPS for weighted projective space, to lighten notation.

Definition 1.3.4. Let $X \subset w\mathbb{P}$ be a Zariski closed subvariety and $\pi \colon \mathbb{C}^{n+1} \setminus \{0\} \to w\mathbb{P}$ the quotient. The *affine cone* over X is denoted C_X and is the completion of $\pi^{-1}(X)$ in $\mathbb{C}^{n+1}$. In the case where $X = \mathrm{Proj}(R(X, D))$ we have simply $C_X = \mathrm{Spec}(R(X, D))$. We say X is *quasismooth* if C_X is nonsingular away from the origin.

Remark 1.3.5. We have

$$\mathrm{Proj}(R(X, D)) = \frac{\mathrm{Spec}(R(X, D)) \setminus \{0\}}{\mathbb{C}^\times}$$

where the $\mathbb{C}^\times$ action is induced by the action on the ambient $\mathbb{A}^{n+1} \setminus \{0\}$ which the cone lies in (this action takes $\mathbb{A}^{n+1}$ to $w\mathbb{P}^n$ in which X lies with this choice of polarisation). Thus in particular this implies that any singularity on X must come from the ambient WPS $w\mathbb{P}^n$. Moreover, assuming the ambient WPS is well-formed and X has dimension at least 3, X is nonsingular in codimensions 0 and 1 (in the terminology of [Iano-Fletcher, 2000] this says X is *well formed*).

1.4 Cyclic quotient singularities

We now introduce an important class of singularities.

Definition 1.4.1. Let X be a projective variety of dimension n and r, $a_1, \dots, a_n$ be natural numbers. A point $P \in X$ is said to be a *cyclic quotient singularity of type* $\frac{1}{r}(a_1, \dots, a_n)$ if there is a local analytic isomorphism

$$P \in X \cong 0 \in \frac{\mathbb{A}^n}{\mu_r}$$

where $\mu_r \subset \mathbb{C}^\times$ denotes the group of rth roots of unity and the action of μ_r on $\mathbb{A}^n$ is given by

$$\epsilon \cdot (x_1, \dots, x_n) = (\epsilon^{a_1} x_1, \dots, \epsilon^{a_n} x_n).$$

Again, this is a geometric quotient and the functions on the right are induced from μ_r invariant functions on $\mathbb{A}^n$, so we have

$$\frac{\mathbb{A}^n}{\mu_r} = \operatorname{Spec} \mathbb{C}[x_1, \dots, x_n]^{\mu_r}.$$

Such a singularity is called *isolated* if it lies on no singular locus of higher dimension.

The following is immediate.

Lemma 1.4.2. *With notation as above, a singular point P of type $\frac{1}{r}(a_1, \dots, a_n)$ is isolated if and only if all the a_i are pairwise coprime to r, that is for each i we have* $\mathrm{hcf}(r, a_i) = 1$.

Remark 1.4.3. For a cyclic quotient singularity of type $\frac{1}{r}(a_1, \dots, a_n)$ (not necessarily isolated) the orbinates x_i are local sections of $\mathcal{O}_X(a_i)(= \mathcal{O}_X(a_i D)$ in the setting of a polarised orbifold $(X, D))$ and the local index one cyclic cover given by the local isomorphism $\mathcal{O}_X(r) \cong \mathcal{O}_X$ is nonsingular.

The quotient map $\pi \colon \mathbb{A}^n \to \mathbb{A}^n/\mu_r$ induces a direct sum decomposition of $\pi_* \mathcal{O}_{\mathbb{A}^n}$ into eigensheaves

$$\mathcal{L}_i := \{ f : \epsilon(f) = \epsilon^i \cdot f \text{ for all } \epsilon \in \mu_r \} \quad \text{for } i \in \mathbb{Z}/r = \operatorname{Hom}(\mu_r, \mathbb{C}^\times).$$

Example 1.4.4. Consider $\mathbb{P}(1,1,2)$, with coordinates x, y, z respectively. The affine piece $z \neq 0$ has coordinates $\frac{x^2}{z}, \frac{xy}{z}, \frac{y^2}{z}$ so is isomorphic not to $\mathbb{A}^2$ but to the cone $(uw = v^2) \subset \mathbb{A}^3_{\langle u,v,w \rangle}$. You can check then that the coordinate point $P_z = (0:0:1)$ is a cyclic quotient singularity of type $\frac{1}{2}(1,1)$.

More generally, we write $P_{x_i} = P_i = (0 : \cdots : 0 : 1 : 0 : \cdots : 0)$ for the ith coordinate point of $\mathbb{P}(a_0, \ldots, a_n)$ (x_i is the homogeneous coordinate of weight a_i). Then P_i is of type $\frac{1}{a_i}(a_0, \ldots, \widehat{a_i}, \ldots, a_n)$, since the affine piece given by $x_i \neq 0$ is isomorphic to $\mathbb{A}^n/\mu_{a_i}$ where the group action is precisely given by $\epsilon \cdot z_j = \epsilon^{a_j} z_j$.

Let us now view some examples where $X \subset w\mathbb{P}$ is a variety embedded in some weighted projective space. The following calculations follow from the work in Iano-Fletcher [2000], in particular lemmata 9.4 and 9.5, and Section 10 which provides examples of similar calculations.

Example 1.4.5. Consider $X = X_6 = (f_6 = 0) \subset \mathbb{P}(1, 2, 3, 5)$ with coordinates x, y, z, t of weights $1, 2, 3, 5$ respectively. First let me explain the notation: take a hypersurface of weighted degree 6 inside the weighted projective space. There are of course many of these, for example $x^6 + y^3 = 0$ or $xt + z^2 = 0$. The collection of such hypersurfaces forms a family (in this case just a vector space), and we take a sufficiently general member of this family, i.e. choose coefficients for all the monomials of weighted degree 6 so that the equation doesn't induce any unnecessary singularities, and the resulting variety is quasismooth. Intuitively you can just think of this as something like having all the monomials appear with non-zero coefficients in the equation (the condition is not rigorously either necessary or sufficient).

Returning to $X_6 \subset \mathbb{P}(1, 2, 3, 5)$, the weights of x, y, z divide the degree of the equation, so the coordinate points $P_x, P_y, P_z \notin X$. To spell this out consider $P_z = (0 : 0 : 1 : 0)$; z^2 has weighted degree 6, but all other monomials of degree 6 vanish at P_z so that

$$f(P_z) = \text{coefficient of } z^2 \text{ in f} \neq 0 \text{ for sufficiently general } f.$$

However $P_t \in X$ since no pure power of t has degree 6, so all monomials of degree 6 vanish at P_t (since they all involve at least one other coordinate apart from t). As a point in the ambient WPS $P_t = \frac{1}{5}(1, 2, 3)$ but of course X is a surface so we only need two local coordinates. We notice that xt has degree 6, so

$$\frac{\partial f}{\partial x}(P_t) = \text{coefficient of } xt \text{ in f} \neq 0 \text{ for sufficiently general } f$$

thus by the implicit function theorem we can invert x. Intuitively, along the affine piece $t = 1$ we can write x as a function of the other coordinates so we can eliminate x. Thus $P_t \in X$ is of type $\frac{1}{5}(2, 3)$.

The general strategy for a coordinate point $P_i \in X \subset w\mathbb{P}^n$ is to find monomials $x_j x_i^s$ and for exactly the same reasons you can then eliminate x_j as a coordinate (again, setting $x_i = 1$ will give x_j as a function of the other coordinates, or apply the implicit function theorem since $\frac{\partial}{\partial x_j}(x_j x_i^s)(P_i) = $ coefficient of $x_j x_i^s \neq 0$).

Example 1.4.6. Consider now $X_{12,15} = (f_{12} = g_{15} = 0) \subset \mathbb{P}(1,3,4,5,7,8)$ where the coordinates are x, y, z, t, u, v. Assume as before X to be general; it has a $\frac{1}{7}(1,3,4)$ point at P_u and a $\frac{1}{8}(1,3,5)$ point at P_v.

Notice that the ambient WPS has a $\frac{1}{4}(1,-1,1)$ line of singularities along $L = \mathbb{P}(4,8)_{\langle z,v\rangle}$. However $L \not\subseteq X$ since $f|_L = z^3 + zv$ doesn't vanish identically along L (notice that $g|_L \equiv 0$). How many points of intersection does X have with L, i.e. how many zeroes does $z^3 + zv$ have along L? Notice that $L \cong \mathbb{P}^1_{\langle z^2, v\rangle}$ so the equation of f along L is $z(z^2 + v)$ which is linear in the coordinates of L viewed as a projective line (indeed we may assume $z \neq 0$ so $z = 1$ since $z = 0$ gives P_v). Thus X has one $\frac{1}{4}(1,-1,1)$ singularity at some general point of L.

We now introduce the concept of maximal Pfaffians of skew matrices, as an example of a 5×5 resolution of a codimension 3 ring.

Recall that a 5×5 skew matrix is given by exactly ten elements, since such a matrix must have 0s on the diagonal and the elements below the diagonal are determined by those above it. We write such a matrix M as

$$M = \begin{pmatrix} m_{12} & m_{13} & m_{14} & m_{15} \\ & m_{23} & m_{24} & m_{25} \\ & & m_{34} & m_{35} \\ & & & m_{45} \end{pmatrix}.$$

Then we define (up to sign) the Pfaffians of M as

$$\begin{aligned} \mathrm{Pf}_1 = \mathrm{Pf}_{2345} &= m_{23}m_{45} - m_{24}m_{35} + m_{25}m_{34} \\ \mathrm{Pf}_2 = \mathrm{Pf}_{1345} &= m_{13}m_{45} - m_{14}m_{35} + m_{15}m_{34} \\ \mathrm{Pf}_3 = \mathrm{Pf}_{1245} &= m_{12}m_{45} - m_{14}m_{25} + m_{15}m_{24} \end{aligned}$$

and so on. To calculate Pf_i we delete the ith row and column and from the resultant

$$\begin{pmatrix} a & b & c \\ & d & e \\ & & f \end{pmatrix}$$

calculate (up to sign) $af - be + cd$.

These Pfaffians satisfy

$$(\mathrm{Pf}_1, \mathrm{Pf}_2, \mathrm{Pf}_3, \mathrm{Pf}_4, \mathrm{Pf}_5)M \equiv 0 \equiv M(\mathrm{Pf}_1, \mathrm{Pf}_2, \mathrm{Pf}_3, \mathrm{Pf}_4, \mathrm{Pf}_5)^T,$$

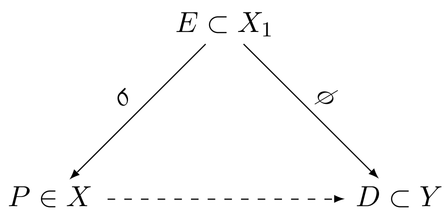
i.e. 5 linear combinations of the defining equations are identically 0.

Codimension 3 Gorenstein rings whose corresponding graded rings have a 5×5 resolution are given by 5 equations corresponding to Pfaffians of a 5×5 skew matrix (see Buchsbaum and Eisenbud [1977] theorem 2.1; in fact this theorem states that *all* codimension 3 Gorenstein rings arise as the maximal Pfaffians of $(2m+1) \times (2m+1)$ skew matrices; see the following section for more details; see also Altınok for examples of constructing codimension 3 varieties arising from the Pfaffians of 5×5 skew matrices).

Example 1.4.7. Define $X = (\mathrm{Pf}(M) = 0) \subset \mathbb{P}(1, 3, 4, 5, 7, 8, 11)$ (with coordinates x, y, z, t, u, v, s) where

$$M = \begin{pmatrix} s & a_9 & b_7 & c_4 \\ & d_{12} & e_{10} & f_7 \\ & & v & -t \\ & & & y \end{pmatrix}.$$

X is a codimension 3 Fano 3-fold, realised as the unprojection of $Y_{12,15} \subset \mathbb{P}(1, 3, 4, 5, 7, 8)$ along the divisor $D = \mathbb{P}(1, 4, 7)_{\langle y,t,v \rangle}$ with unprojection variable s, as given by the following commutative diagram:



Here, $\sigma \colon X_1 \longrightarrow X$ is an extremal extraction centred at P and $\phi \colon X_1 \longrightarrow Y$ is the anticanonical morphism; Y can be viewed as the midpoint of a Sarkisov link. See Brown et al. [2012] section 3.2 or [Papadakis and Reid, 2004] for more details on this point of view. The 5 equations of X are

$$F_{12} = a_9 y + b_7 t + c_4 v$$
$$G_{15} = d_{12} y + e_{10} t + f_7 v$$

and the 3 unprojection equations

$$sy = bf - ce$$
$$st = cd - af$$
$$sv = ae - bd.$$

Here a, b, c, d, e, f are general polynomials of the relevant degree. In fact we can replace $s, v, -t, y$ by more general polynomials of weights $11, 8, 5, 3$ respectively to get a more general variety (the reason we have written it in this form is to make the unprojection format clear).

As before, we look for the singularities of X. Things are different here in that the Pfaffian format of the equations gives a stronger requirement for the relation between the weight of the corresponding variable and the degrees of the equations: for $P_i \notin X$ it is no longer sufficient that the corresponding weight a_i divides one of the degrees of the equations (equivalently for $P_i \in X$ it is not necessary that the weight a_i divides none of the degrees of the equations; the condition of course remains sufficient). Indeed, each equation is made up of 3 terms, corresponding each to 2 entries in the matrix; for a coordinate x_i to appear as a pure power in the equation, it needs to appear as a pure power in (at least) one of these 3 terms, so in *both* of the corresponding 2 entries in the matrix.

Looking at our example, we see that $P_x \notin X$ since for sufficiently general a, b, c, d, e and f, x will appear as a pure power in the unprojection equations. Similarly $P_y \notin X$ since $ay \in F_{12}$ so all we require is for $y^3 \in a$ which is the case for sufficiently general a (or, if taking the bottom right entry to be also more general, we also need y to appear in it). Similarly $P_z \notin X$ by taking $z \in c, z^3 \in d$ so that z^4 appears in the equation of degree 16 (notice that as written z^3 can't appear in F_{12} but it can if we replace v by a polynomial involving z^2). Also $P_t \notin X$ using $et \in G_{15}$ (notice that this is the only possibility here, so we need t^2 to appear in e and t to appear in the entry where we have written $-t$ above), and $P_u \notin X$ using bf in the equation of degree 14 (again, the only possibility, so we need $u \in b \cap f$).

However $P_v \in X$ even though X has an equation of degree 16; indeed for $P_v \notin X$ we would need v^2 to appear in the equation $st - cd + af$ but the only way for this to happen is for $v \in (s \cap t) \cup (c \cap d) \cup (a \cap f)$ which can't happen for reasons of weights. Thus, *no matter how we choose our entries in the matrix,* $P_v \in X$. Now P_v is of type $\frac{1}{8}(1, 3, 5)$ since v eliminates z by $zv \in F_{12}$ (using $z \in c$), u by $uv \in G_{15}$ and s by sv in the equation of degree 19, exactly as before.

Exactly as in previous calculations we also have $P_s \in X$ (this time 11 doesn't divide any of the degrees of the equations) and s hits y, t, v by the unprojections equations, so P_s is of type $\frac{1}{11}(1,4,7)$. Notice for example that we can't eliminate z since $sz \notin G_{15}$ even by modifying the entries of the matrices, because of their weights; similarly $sx \notin F_{12}$.

Definition 1.4.8. An *orbifold* is a variety X which is everywhere locally the quotient of $\mathbb{A}^n$ by a finite group action, acting freely in codimension 1. In our cases, the group action will always be cyclic.

Example 1.4.9. All weighted projective spaces, as well as quasismooth varieties in weighted projective spaces, are orbifolds; indeed, they only have cyclic quotient singularities.

1.5 The Hilbert syzygies theorem

We now state the version of the Hilbert syzygies theorem relevant to our case. First recall the following definition (see for example Eisenbud [1995] for a general introduction or Bruns and Herzog [1998] for a more detailed exposition on Cohen-Macaulay and Gorenstein rings).

Theorem 1.5.1. *Let $S = \mathbb{C}[x_0, \ldots, x_N]$ with x_i (possibly) weighted variables be considered as a graded ring and $I \subset S$ a homogeneous ideal. Put $R = S/I$, which is both a graded ring and a graded S-module. Then R has a resolution by free S-modules, that is, there is an exact sequence*

$$0 \leftarrow R \leftarrow P_0 \leftarrow P_1 \leftarrow \cdots \leftarrow P_c \leftarrow 0,$$

with each $P_i = \bigoplus_{j=1}^{n_i} S(-d_{i,j})$.

Moreover if R is Gorenstein and the resolution is minimal, then

$$\begin{aligned} c &= \operatorname{codim}_S(I), \\ P_0 &= S, \\ P_c &= P_0^\vee \otimes S(-\alpha) = S(-\alpha), \end{aligned}$$

where α is the adjunction number and equals $k_R + \sum a_i$ with $\omega_R \cong R(k_R)$ and $\operatorname{wt} x_i = a_i$ for each i.

Finally, duality applies, so that

$$P_{c-i} = \mathrm{Hom}_S(P_i, P_c) = P_i^\vee \otimes P_c$$

thus if R is Gorenstein

$$P_{c-i} = \bigoplus_{j=1}^{n_i} S(d_{i,j} - \alpha) \quad i.e.$$

$$n_{c-i} = n_i \quad and \quad d_{c-i,j} = -d_{i,j} + \alpha.$$

Remark 1.5.2. There are many equivalent definitions for what it means for an abstract ring R to be Gorenstein, for example in terms of minimal injective resolutions, or the canonical module. We do not give the technical definitions here since they add little in the context of our work; the interested reader is invited to consult Eisenbud [1995] Chapters 18 and 21, or for a more concise treatment Matsumura and Reid [1989] Sections 17 and 18. Rather we point out that the case we are ultimately interested in, a polarised variety (X, D) where $R = R(X, D)$ is Gorenstein, is thus covered. In this case $X \cong \mathrm{Proj}(R) \subset \mathbb{P}(a_0, \ldots, a_n)$ and R being Gorenstein is equivalent to the conditions:

- $H^i(X, \mathcal{O}_X(mD)) = 0$ for all $0 < i < \dim X$ and all m;
- $K_X = \mathrm{k}_X D$ for some k_X

(see Goto and Watanabe [1978] 5.1.11 and 5.1.9 respectively). Then by the above theorem R has a free resolution of length c, the codimension of X, and $P_c = S(-(\mathrm{k}_X + \sum a_i))$.

Using the above resolution and additivity of the Hilbert series, we get the following:

Corollary 1.5.3. *With the notation as above, denoting by P_R the Hilbert series of R we have*

$$P_R(t) = \frac{\sum_{i,j} (-1)^i t^{d_{i,j}}}{\prod_l (1 - t^{a_l})}.$$

Notice that the above form of the Hilbert series doesn't uniquely determine the degrees of possible relations and syzygies for R (there could be cancellation between positive and negative terms in the sum).

Example 1.5.4. Assume in all that follows that $D = \mathcal{O}_X(1)$ and $R = R(X, D)$ is Gorenstein. If $X = (f_d = 0) \subset \mathbb{P}(a_0, \ldots, a_n)$ (in this case $\mathrm{k}_X = d - \sum a_i$) then R

has a resolution

$$0 \leftarrow R \leftarrow S \overset{f}{\leftarrow} S(-d) \leftarrow 0$$

and the Hilbert series of X is

$$P(t) = \frac{1 - t^d}{\prod(1 - t^{a_i})}.$$

If $X = (f_{d_1} = g_{d_2} = 0) \subset \mathbb{P}(a_0, \ldots, a_n)$ is a complete intersection of codimension 2 (in this case $\mathrm{k}_X = d_1 + d_2 - \sum a_i$) then R has a resolution

$$0 \leftarrow R \leftarrow S \xleftarrow{(f,g)} S(-d_1) \oplus S(-d_2) \xleftarrow{\binom{-g}{f}} S(-d_1 - d_2) \leftarrow 0$$

and the Hilbert series of X is

$$P(t) = \frac{1 - t^{d_1} - t^{d_2} + t^{d_1 + d_2}}{\prod(1 - t^{a_i})}.$$

More generally if X is a complete intersection of degrees $d_1, \ldots, d_c$ (in this case $\mathrm{k}_X = \sum d_j - \sum a_i$) then its Hilbert series is

$$P(t) = \frac{\prod(1 - t^{d_i})}{\prod(1 - t^{a_j})};$$

indeed, if we suppose $X = (f_1 = \cdots = f_c = 0)$ (assume f_i has degree d_i for each i) then the f_i form a regular sequence in R and so the resolution of R is the Koszul complex $K_\circ(f_1, \ldots, f_d)$ given by

$$K_l = \bigoplus S \cdot e_{i_1 \ldots i_l},$$

the free S-module of rank $\binom{c}{l}$ with basis $\{e_{i_1 \ldots i_l} | 1 \leq i_1 \leq \cdots \leq i_l \leq c\}$ if $0 \leq l \leq c$ and $K_l = 0$ otherwise. The differential is then given by

$$d(e_{i_1 \ldots i_l}) = \sum_{j=1}^{l} (-1)^{j+1} f_{i_j} e_{i_1 \ldots \hat{i_j} \ldots i_l}$$

(see for example Eisenbud [1995] Chapter 17 or Matsumura and Reid [1989] Section 16). In particular in the case $c = 2$ we recover the resolution given above, whilst for general c the resolution has length c and is

$$0 \leftarrow R \leftarrow S \leftarrow S^c \leftarrow \cdots \leftarrow S^c \leftarrow S \leftarrow 0.$$

Now suppose $X \cong \mathrm{Proj}(R)$ has codimension 3; then we know following Buchsbaum and Eisenbud [1977] that if R is Gorenstein then $X = (\mathrm{Pf}(M) = 0) \subset \mathbb{P}(a_0, \ldots, a_n)$ is given by the maximal $2m \times 2m$ Pfaffians of a $(2m+1) \times (2m+1)$ skew matrix (the complete intersection case corresponds to $m = 1$). Moreover if the Pfaffians have degree d_i for $1 \le i \le 2m+1$) then $\mathrm{k}_X = \sum_i (\alpha - d_i) - \sum_i d_i - \sum_j a_j$ and $\alpha = \mathrm{k}_X + \sum a_i$ and R has a resolution

$$0 \leftarrow R \leftarrow S \xleftarrow{(\mathrm{Pf}\, M)} \bigoplus_{i=1}^{2m+1} S(-d_i) \xleftarrow{M} \bigoplus_{i=1}^{2m+1} S(-\alpha + d_i) \xleftarrow{(\mathrm{Pf}\, M)^T} S(-\alpha) \leftarrow 0$$

and the Hilbert series of X is

$$P(t) = \frac{1 - \sum_{i=1}^{2m+1} t^{d_i} + \sum_{i=1}^{2m+1} t^{\alpha - d_i} - t^{\alpha}}{\prod (1 - t^{a_l})}.$$

Chapter 2

The isolated singularity case

In this Chapter we discuss a Hilbert series parsing for a polarised orbifold with only isolated orbifold points following [Buckley, Reid, and Zhou, 2013]. We introduce the necessary preliminary results and terminology, give the statement of the main result, explain the ideas behind the proof, and discuss examples and applications.

2.1 Preliminaries

The setting is as follows: let (X, D) be a projective polarised variety. Suppose $n = \dim X$, and denote by K_X the canonical class of X. Assume X is quasismooth and well-formed (as in definition 1.3.4); in particular recall this implies that any singularity on X must come from the ambient WPS $w\mathbb{P}^N$ and that X is nonsingular in codimensions 0 and 1. Thus all the singularities of X are cyclic quotient singularities and X is an orbifold.

Further assume (X, D) is *projectively Gorenstein*, that is, the ring $R(X, D)$ is Gorenstein. What this means is it is Cohen-Macaulay, so satisfies

$$H^j(X, \mathcal{O}_X(mD)) = 0 \qquad \text{for all } 0 < j < n \text{ and all } m$$

(see Goto and Watanabe [1978], 5.1.11), and $K_X = \mathrm{k}_X D$ for some $\mathrm{k}_X \in \mathbb{Z}$ (see Goto and Watanabe [1978], 5.1.9), called the *canonical weight* so in particular K_X is Cartier. We write $c = \mathrm{k}_X + n + 1$ for the *coindex.*

Moreover $h^n(mD) = h^0((k_X - m)D)$ for all integers m using Serre duality. Thus calculating h^0 is essentially the same as calculating the Euler characteristic χ which is given by a known (but not necessarily computable especially for $n \geq 4$) Riemann-Roch formula.

One consequence of R being Gorenstein is that using the Hilbert syzygies theorem and Serre duality, we can prove following Buckley et al. [2013] that it satisfies *Gorenstein symmetry,* namely:

Lemma 2.1.1. *Let $R = R(X, D)$ be as above (so in particular R is a graded Gorenstein ring of dimension $n+1$ and canonical weight k, that is $\omega_R \cong R(\mathrm{k})$).*

Then the Hilbert series of R (equivalently of (X, D)) satisfies

$$t^{\mathrm{k}} P\left(\tfrac{1}{t}\right) = (-1)^{n+1} P(t). \tag{2.1}$$

Although we state this for a polarised variety, the fact holds for all rings satisfying the emphasised hypotheses.

Definition 2.1.2. If a rational function P satisfies property (2.1) we say P is *Gorenstein symmetric of degree* k .

Proof. Suppose $R = S/I$ where $S = \mathbb{C}[x_0, \ldots, x_N]$, with $\operatorname{wt} x_i = a_i$ for all i. Then we can apply theorem 1.5.1 to get a free resolution of R

$$0 \leftarrow R \leftarrow P_0 \leftarrow P_1 \leftarrow \cdots \leftarrow P_c \leftarrow 0,$$

with

$$\begin{aligned} c &= N - n \\ P_0 &= S \\ P_i &= \bigoplus_{j=1}^{n_i} S(-d_{i,j}) \\ P_c &= S(-\alpha) \quad \text{where } \alpha = \mathrm{k} + \sum a_i \\ n_{c-i} &= n_i \\ d_{c-i,j} &= -d_{i,j} + \alpha. \end{aligned}$$

Then we calculate

$$P\left(\tfrac{1}{t}\right) = \frac{\sum_{i=0}^{c} (-1)^i \sum_{j=1}^{n_i} t^{-d_{i,j}}}{\prod_{l=0}^{N} (1 - t^{-a_l})}.$$

The denominator is then equal to

$$t^{-\sum a_l} (-1)^{N+1} \prod (1 - t^{a_l}),$$

while (using the duality properties given above) the numerator becomes

$$\begin{aligned}
&\sum_{i=0}^{c}(-1)^{c-i}\sum_{j=1}^{n_{c-i}=n_i} t^{-d_{c-i,j}} \\
&= t^{-\alpha}\sum_i(-1)^{c-i}\sum_j t^{d_{i,j}} \\
&= (-1)^c t^{-\alpha}\sum_{i,j}(-1)^i t^{d_{i,j}},
\end{aligned}$$

so that

$$P\left(\tfrac{1}{t}\right) = (-1)^{N+1-c} t^{-(\alpha-\sum_l a_l)} \frac{\sum_{i,j}(-1)^i t^{d_{i,j}}}{\prod_l (1-t^{a_l})} = (-1)^{N+1-c} t^{-(\alpha-\sum_l a_l)} P(t),$$

which yields the result, using $\mathrm{k} = \alpha - \sum_l a_l$ and $c = N - n$. $\square$

Remark 2.1.3. We are interested in proving Gorenstein symmetry of rational functions where the denominator is a product of $(N+1)$ factors of the form $(1-t^{a_i})$ $(a_i \in \mathbb{N})$. Under those circumstances it is easy to see directly from the definition that a rational function $\frac{N(t)}{\prod_{i=0}^{N}(1-t^{a_i})}$ is Gorenstein symmetric of degree k if and only if $N(t) = \sum_{i=k_1}^{k_2} b_i t^i$ is palindromic or antipalindromic, supported precisely on $[k_1, k_2]$ such that $k_1 + k_2 = \mathrm{k} + \sum a_i$.

Moreover, the sum and difference of two Gorenstein symmetric functions of same degree is again Gorenstein symmetric of same degree and with same parity of dimension (the $(-1)^n$ factors should be equal, so the dimensions agree mod 2).

The final ingredient we need before we can state the main result for the isolated singularities case is the existence of the inverse mod function for polynomials. The set-up is the following:

Lemma 2.1.4. *Let $F \in \mathbb{Q}[t]$ be monic of degree d with non-zero constant coefficient. Suppose $A \in \mathbb{Q}[t]$ is coprime to F.*

Then for any integer γ there is a unique Laurent polynomial $B \in \mathbb{Q}[t, t^{-1}]$ such that

1. *$AB \equiv 1$ (mod F);*
2. *B is supported on the interval $[t^\gamma, t^{\gamma+1}, \ldots, t^{\gamma+d-1}]$.*

We define $\mathrm{InvMod}(A, F, \gamma) := B$.

Proof. The quotient ring $\frac{\mathbb{Q}[t]}{(F)} =: V$ is a $\mathbb{Q}$-vector space of dimension d with basis $1, t, \dots, t^{d-1}$. As the constant term of F is non-zero, F and t are coprime so that t has an inverse mod F. That is, multiplication by t is an endomorphism of V hence so is multiplication by t^γ for any $\gamma \in \mathbb{Z}$; in particular such a map takes a basis of V to another basis, so that $t^\gamma, \dots, t^{\gamma+d-1}$ is a basis for V over $\mathbb{Q}$ for any $\gamma \in \mathbb{Z}$.

Finally, any A coprime to F is invertible mod F and its inverse B is uniquely determined in any basis of V. □

We give a construction in the case which interests us (namely cyclic quotient singularities). Let $r, a_1, \dots, a_n$ be natural numbers and put

$$A = \prod_{i=1}^{n} (1 - t^{a_i}),$$
$$h = \text{hcf}(1 - t^r, \prod_{i=1}^{n} (1 - t^{a_i})),$$
$$F = \frac{1 - t^r}{h}.$$

Notice that if all the a_i are pairwise coprime to r then $h = 1 - t$. Since we have factored out the hcf, A and F are coprime. We put $d = \deg(F) \leq r - 1$.

First assume $\gamma \geq 0$. Then $t^\gamma A$ and F are coprime polynomials, so applying Euclid's algorithm, we get a unique $\overline{B}$ of degree strictly less than d such that

$$t^\gamma A \overline{B} + FG = 1$$

and we can just set $B = t^\gamma \overline{B}$.

If now $\gamma < 0$, first notice that $t^r \equiv 1 \bmod F$, so that also $t^{mr} \equiv 1 \bmod F$ for all $m \in \mathbb{N}$.

Now choose m large enough so that $mr + \gamma \geq 0$.

Again, by Euclid's algorithm find $\overline{B}$ of degree less than d such that

$$t^{mr+\gamma} A \overline{B} + FG = 1$$

and set $B = t^\gamma \overline{B} = \frac{t^{mr+\gamma} \overline{B}}{t^{mr}}$.

For more general F you need to deal with powers of the matrix $M_t : V \to V$ corresponding to multiplication by t; in the above case we just have $M_t^r = 1$.

Example 2.1.5. Let

$$F = \frac{1-t^5}{1-t} = 1 + t + t^2 + t^3 + t^4$$

and

$$A = \frac{1-t^2}{1-t} = 1 + t.$$

Put $\gamma = -4 < 0$. In this case we have $r = 5$ so we can choose $m = 1$ in the previous notation. We are looking for $\overline{B}, G$ such that $tA\overline{B} + FG = 1$.

We proceed by long division: we have

$$F = A(t^2 + 1) + 1$$

hence

$$\overline{B} = -t^2 - 1$$

or equivalently

$$B = t^{-2} - t^{-4}.$$

Notice that B is Gorenstein symmetric of degree -6.

Example 2.1.6. Let

$$F = \frac{1-t^7}{1-t} = 1 + t + \cdots + t^6$$

and

$$A = \frac{1-t^5}{1-t} = 1 + t + \cdots + t^4.$$

Put $\gamma = -4 < 0$. In this case we have $r = 7$ so again we can choose $m = 1$. We are looking for $\overline{B}, G$ such that $t^3 A\overline{B} + FG = 1$. Long division yields the following calculations:

$$\begin{aligned} t^3 A &= tF + (-t - t^2) \\ t - t^2 &= tF - t^3 A \\ F &= (t + t^2)(t^4 + t^2 + 1) + 1 \end{aligned}$$

so that

$$\begin{aligned} 1 &= F + (t + t^2)(-t^4 - t^2 - 1) \\ &= F(1 + t) + t^3 A(t^4 + t^2 + 1) \end{aligned}$$

and thus

$$B = t^{-4} + t^{-2} + 1.$$

Again, notice that B is Gorenstein symmetric of degree -4.

2.2 Main result and proof

We now have the tools to state the main result in the isolated singularities case, following [Buckley, Reid, and Zhou, 2013].

Theorem 2.2.1. *Let (X, D) be a polarised variety of dimension $n \geq 2$. Suppose (X, D) is quasismooth, well-formed, projectively Gorenstein of canonical weight k_X and has orbifold locus consisting of only isolated points*

$$\mathcal{B} = \left\{Q \text{ of type } \tfrac{1}{r}(a_1, \ldots, a_n)\right\}$$

(recall that "isolated" means all the a_i are pairwise coprime to r).

Then we can write the Hilbert series $P(t) = P_{(X,D)}(t)$ as

$$P(t) = P_I(t) + \sum_{Q \in \mathcal{B}} P_{\mathrm{orb}}(Q, \mathrm{k}_X)(t) \tag{2.2}$$

where

1. *$P_I(t) = \frac{A(t)}{(1-t)^{n+1}}$ where $A(t)$ the unique integral palindromic polynomial of degree $c = \mathrm{k}_X + n + 1$ if $\mathrm{k}_X \geq 0$ or n if $\mathrm{k}_X < 0$, such that the series P_I and P coincide up to and including degree $\left\lfloor \frac{c}{2} \right\rfloor$;*

2. *the contribution from each orbifold point $Q \in \mathcal{B}$ of type $\frac{1}{r}(a_1, \ldots, a_n)$ is*

$$P_{\mathrm{orb}}(Q, \mathrm{k}_X) = \frac{B(t)}{(1-t)^n(1-t^r)}$$

   where

$$B(t) = \mathrm{InvMod}\left(\prod_{i=1}^{n} \frac{1-t^{a_i}}{1-t}, \frac{1-t^r}{1-t}, \left\lfloor \frac{c}{2} \right\rfloor + 1\right)$$

   has integral coefficients and is Gorenstein symmetric of degree $\mathrm{k}_X + n + r$ and is supported precisely on the interval

$$\left[\left\lceil \frac{c-1}{2} \right\rceil + 1, \left\lfloor \frac{c-1}{2} \right\rfloor + r - 1\right]$$

   (thus $P_{\mathrm{orb}}(Q, \mathrm{k}_X)$ is Gorenstein symmetric of degree k_X).

Remark 2.2.2. The initial term P_I deals with the plurigenera $P_1, \ldots, P_{\lfloor \frac{c}{2} \rfloor}$ but does not control the growth of the plurigenera; the orbifold points also contribute to the growth.

Definition 2.2.3. We define the ith *Dedekind sum* σ_i by

$$\sigma_i\left(\tfrac{1}{r}(a_1,\dots,a_n)\right) = \frac{1}{r} \sum_{\epsilon\in\mu_r;\epsilon^{a_j}\neq 1\forall j=1,\dots,n} \frac{\epsilon^i}{(1-\epsilon^{a_1})\dots(1-\epsilon^{a_n})}$$

so that ϵ runs over the roots of unity for which the denominator is non zero. We define the *Dedekind sum polynomial* as

$$\Delta\left(\frac{1}{r}(a_1,\dots,a_n)\right) = \sum_{i=1}^{r} \sigma_{r-i} t^i$$

which is supported in $[t,\dots,t^r]$.

Notice that $\sigma_{r+i} = \sigma_i$ for all i so that we need only concern ourselves with σ_i for $i = 0, 1, \dots, r-1$.

Recall the following result from [Reid, 1985]:

Proposition 2.2.4. *For $i = 0,\dots,r-1$, the Dedekind sums $\sigma_i\left(\frac{1}{r}(a_1,\dots,a_n)\right)$ are the unique solution to the $r\times r$ system of equations*

$$\sum_{i=0}^{r-1} \sigma_i \epsilon^i = \begin{cases} 0 & \text{if } \epsilon\in\mu_{a_j} \text{ for some } j, \\ \frac{1}{\prod_{j=1}^{n}(1-\epsilon^{-a_j})} & \text{otherwise.} \end{cases} \tag{2.3}$$

where the unknowns are the σ_i and the equations are indexed by $\epsilon\in\mu_r$.

Consider now again a cyclic quotient singularity $\frac{1}{r}(a_1,\dots,a_n)$ (not necessarily isolated) and put as before

$$A = \prod_{i=1}^{n}(1-t^{a_i}),$$
$$h = \operatorname{hcf}(1-t^r, A),$$
$$F = \frac{1-t^r}{h}.$$

Notice that in (2.3) the requirement for $\epsilon\in\mu_r\setminus(\cup\mu_{a_j})$ is exactly equivalent to ϵ being a root of F. In particular if the singularity is isolated this happens if and only if $\epsilon\neq 1$.

The following result is key.

Theorem 2.2.5. *With the notation as above we have*

$$\Delta = ht \cdot \mathrm{InvMod}(htA, F, 0). \tag{2.4}$$

Proof. F is coprime to t, h as well as A so that the InvMod defined in the right hand side makes sense. Now Δ is divisible by h (see Buckley et al. [2013] lemma 2.7) and t so the theorem is equivalent to proving

$$\frac{\Delta}{ht} = \mathrm{InvMod}(htA, F, 0).$$

Taking out a factor of t from Δ shifts it to the correct support, so that it is now enough to prove

$$\frac{\Delta}{ht} \cdot htA = \Delta \cdot A \equiv 1 \mod F.$$

Now let ϵ be a root of F (recall this means $\epsilon \in \mu_r \setminus (\cup \mu_{a_i})$). Then by (2.3) we see that

$$\begin{aligned} \Delta(\epsilon) &= \sum_{i=1}^{r} \sigma_{r-i}\epsilon^i \\ &= \sum_{i=0}^{r-1} \sigma_i \epsilon^{-i} \\ &= \sum_{i=0}^{r-1} \sigma_i (\epsilon^{-1})^i \\ &= \frac{1}{\prod_{j=1}^{n}(1-\epsilon^{a_j})}, \end{aligned}$$

where the second equality follows from substituting i for $r-i$ and $\epsilon^r = 1$ and the final equality follows from (2.3) applied to ϵ^{-1}. It then follows immediately that $A(\epsilon)\Delta(\epsilon) = 1$ and since this holds for all roots of F, F divides $A\Delta - 1$ which proves the claim. □

The following shows that the ice-cream functions do indeed have the correct periodicity.

Proposition 2.2.6. *With notation as above, suppose* $\mathrm{hcf}(a_i, r) = 1$ *for all* i *(that is, we are in the isolated singularity case) so that* $\mathrm{hcf}(1-t^{a_i}, 1-t^r) = 1-t$ *and*

$r-1=\deg F$. *Let γ be any integer. Then*

$$
\begin{aligned}
(1-t)^n \cdot \Delta &\equiv \operatorname{InvMod}\left(\prod\left(\frac{1-t^{a_i}}{1-t}\right), F, \gamma+1\right) \\
&= \operatorname{InvMod}\left(\frac{A}{(1-t)^n}, F, \gamma+1\right) \\
&= \sum_{j=\gamma+1}^{\gamma+r-1} \theta_j t^j
\end{aligned}
\tag{2.5}
$$

where the θ_j are integers calculated from Dedekind sums (see later for an explicit formula).

Proof. The first equality follows immediately from the fact that $A\Delta \equiv 1 \bmod \mathrm{F}$.

To see the integrality of the coefficients, note that each of the individual factors $\frac{1-t^{a_j}}{1-t}$ are coprime to F so that

$$\operatorname{InvMod}\left(\prod\left(\frac{1-t^{a_j}}{1-t}\right), F\right) \equiv \prod \operatorname{InvMod}\left(\frac{1-t^{a_j}}{1-t}, F\right).$$

Since changing the support doesn't modify the integrality of the coefficients, it is therefore enough to show that each $\operatorname{InvMod}(\frac{1-t^{a_j}}{1-t}, F)$ has integral coefficients. Write a for a_j to lighten notation. Since $(a,r)=1$, let b be such that $ab \equiv 1 \bmod r$. We claim that

$$\sum_{i=0}^{b-1} t^{ai} = \frac{1-t^{ab}}{1-t^a} \equiv \operatorname{InvMod}\left(\frac{1-t^a}{1-t}, F\right),$$

whence the result. Indeed we have

$$
\begin{aligned}
\frac{1-t^a}{1-t} \cdot \frac{1-t^{ab}}{1-t^a} &= \frac{1-t^{ab}}{1-t} \\
&= \frac{1-t^{1+lr}}{1-t},
\end{aligned}
$$

so that evaluating at any root of F (equivalently any $\epsilon \in \mu_r \setminus \{1\}$) gives 1, which implies the claim as before. □

Corollary 2.2.7. *For an isolated singularity P of type $\frac{1}{r}(a_1, \ldots, a_n)$,*

$$P_{\mathrm{orb}}\left(\frac{1}{r}(a_1, \ldots, a_n), \mathrm{k}_X\right) - \frac{\sum_{i=1}^{r-1}(\sigma_{r-i}-\sigma_0)t^i}{1-t^r} = \frac{I_P(t)}{(1-t)^{n+1}} \tag{2.6}$$

where the numerator I_P is a polynomial with rational coefficients.

Proof. Put

$$B(t) = \mathrm{InvMod}\left(\prod_{j=1}^{n}\left(\frac{1-t^{a_j}}{1-t}\right), F, \left\lfloor\frac{c}{2}\right\rfloor\right)$$

and then compute

$$\begin{aligned}
&P_{\mathrm{orb}}\left(\frac{1}{r}(a_1,\dots,a_n),\mathrm{k}_X\right) - \frac{\sum_{i=1}^{r-1}(\sigma_{r-i}-\sigma_0)t^i}{1-t^r}\\
&= \frac{B(t)-(1-t)^n(\sum_{i=0}\sigma_{r-i}t^i - \sum_{i=0}^{r-1}\sigma_0 t^i)}{(1-t)^n(1-t^r)}\\
&= \frac{B(t)-(1-t)^n\Delta - \sigma_0\sum_{i=0}^{r-1}t^i}{(1-t)^n(1-t^r)}\\
&= \frac{FG-\sigma_0 F}{(1-t)^n(1-t^r)}\\
&= \frac{G-\sigma_0}{(1-t)^{n+1}}
\end{aligned}$$

as required. □

Computing explicitly mod F we see that with the notation as in Proposition 2.2.6

$$\theta_j = \sum_{l=0}^{n}(-1)^l\binom{n}{l}(\sigma_{l-j}-\sigma_{l-\gamma})$$

so that in particular for $\gamma = \lfloor\frac{c}{2}\rfloor$ the numerator $B(t) = \mathrm{InvMod}\left(\prod_{j=1}^{n}\left(\frac{1-t^{a_j}}{1-t}\right), F, \lfloor\frac{c}{2}\rfloor\right)$ is palindromic, with top term vanishing if the coindex c is even, so that by the remark about Gorenstein symmetry,

$$P_{\mathrm{orb}}\left(\frac{1}{r}(a_1,\dots,a_n),\mathrm{k}_X\right) = \frac{B(t)}{(1-t)^n(1-t^r)}$$

is Gorenstein symmetric in dimension n of degree

$$d = \begin{cases} \lfloor\frac{c}{2}\rfloor + 1 + \lfloor\frac{c}{2}\rfloor + r - 2 - r - n & \text{if } c \text{ is even}\\ \lfloor\frac{c}{2}\rfloor + 1 + \lfloor\frac{c}{2}\rfloor + r - 1 - r - n & \text{if } c \text{ is odd}\end{cases}$$

so that in either case $d = c - 1 - n = \mathrm{k}_X$. That is, we have proved:

Corollary 2.2.8. *For an isolated singularity of type $\frac{1}{r}(a_1,\dots,a_n)$, $P_{\mathrm{orb}}\left(\frac{1}{r}(a_1,\dots,a_n),\mathrm{k}_X\right)$ is Gorenstein symmetric of degreee k_X, with numerator supported precisely in the interval*

$$\left[\left\lceil\frac{c-1}{2}\right\rceil + 1, \left\lfloor\frac{c-1}{2}\right\rfloor + r - 1\right].$$

We are now in a position to prove the main theorem. The key is that the P_{orb} function has the same r periodicity as $\frac{\Delta}{1-t^r}$. But we know that the orbifold strata give a contribution of the form

$$\frac{\sum_1^{r-1}(\sigma_{r-i}-\sigma_0)t^i}{1-t^r} \tag{2.7}$$

to the Riemann–Roch formula, by [Reid, 1985], so these should combine to give a contribution which looks like P_{orb} plus some stuff; forcing this stuff into the RR contribution gives the initial term in the format required. The proof now follows by the following steps.

Step 1: reduce to a usual Riemann–Roch formula. Give an interpretation of the usual Riemann–Roch formula (see Borel and Serre [1958])

$$\mathrm{RR}(X,D) = (\mathrm{ch}(\mathcal{O}_X(D)) \cdot \mathrm{Td}(T_X))[n]$$

for a singular X. We do this by considering a resolution of singularities $f : Y \to X$ and pulling back the relevant expressions to Y. As in the non-singular case $\mathrm{RR}(mD)$ is a polynomial in m of degree n.

We then prove that

$$\chi(X, \mathcal{O}_X(D)) - \mathrm{RR}(X,D) = \sum_{Q \in \mathrm{Sing}\, X} c_Q(D)$$

is a sum of fractional terms $c_Q(D) \in \mathbb{Q}$ which depend only on the local analytic type of X and D at each singular point $Q \in \mathrm{Sing}\, X$. Thus in particular we may assume for fixed Q an orbifold point of type $\frac{1}{r}(a_1,\ldots,a_n)$ and any $m \geq 1$ that X is a global quotient $\mathbb{A}^n/\mathbb{C}^*$ where $mD \cong \mathcal{O}_X(1)$ so that by [Reid, 1985] we have $c_Q(mD) = (\sigma_{r-m} - \sigma_0)\left(\frac{1}{r}(a_1,\ldots,a_n)\right)$.

Remark 2.2.9. This trick of assuming the local quotient is a global one is really a “stacky process”; the proof that we have a suitable Riemann-Roch contribution so that $\chi(D) - \mathrm{RR}(D)$ depends only on the local geometry of (X,D) near each orbifold point is what allows us in this case to circumvent the language of stacks. See [Buckley et al., 2013] for a more detailed explanation about this.

Step 2: split the Hilbert series into a RR component and a Dedekind sum component. Using that (X,D) is Cohen-Macaulay and Serre duality, we get that for any $m \geq 0$

$$\begin{aligned}\chi(mD) &= \sum_{i=0}^{n}(-1)^i h^i(mD)\\ &= h^0(mD) + (-1)^n h^n(mD)\\ &= h^0(mD) + (-1)^n h^0((\mathrm{k}_X - m)D),\end{aligned}$$

so that

$$\begin{aligned}P(t) &= \sum_{m\geq 0}(\chi(mD) + (-1)^{n+1}h^0((\mathrm{k}_X - m)D))t^m\\ &= \sum_{m=0}^{\mathrm{k}_X}(\mathrm{RR}(mD) + (-1)^{n+1}h^0((\mathrm{k}_X - m)D))t^m\\ &+ \sum_{m>\mathrm{k}_X}\mathrm{RR}(mD)t^m + \sum_{m\geq 0, Q} c_Q(mD)t^m\end{aligned}$$

if $\mathrm{k}_X \geq 0$, or just

$$\sum_{m\geq 0}\mathrm{RR}(mD)t^m + \sum_{m\geq 0, Q} c_Q(mD)t^m$$

otherwise.

Step 3: reduce the RR contribution to that of a rational function with denominator $(1-t)^{n+1}$ by "differencing". We use the following lemma:

Lemma 2.2.10. *For $l \geq 0$ let $Q(t) = \sum_{i\geq l} h(i)t^i$ be a series where h is a polynomial of degree n. Then $(1-t)^{n+1}Q$ is a polynomial of degree $l+n$.*

Proof. This is a straightforward induction on n. First if h is a constant then $Q(t) = h\sum_{i\geq l} t^i = h \cdot \frac{t^l}{1-t}$ so that $(1-t)Q = h \cdot t^l$ as claimed.

Now for the inductive step: we have that

$$\begin{aligned}(1-t)Q &= h(l)t^l + (h(l+1) - h(l))t^{l+1} + \dots\\ &= h(l)t^l + \sum_{i\geq l+1}(h(i) - h(i-1))t^i,\end{aligned}$$

where $h(i) - h(i-1)$ is a polynomial of degree at most $n-1$ in i so that by induction

$$L(t) := (1-t)^n \sum_{i\geq l+1}(h(i) - h(i-1))t^i$$

is a polynomial of degree at most $l+1+n-1 = l+n$ hence

$$(1-t)^{n+1}Q = (1-t)^n h(l)t^l + L(t)$$

is a polynomial of degree $l+n$ as claimed. □

Applying this lemma to the polynomial $\mathrm{RR}(mD)$ we have that if $\mathrm{k}_X \geq 0$ then

$$R(t) := (1-t)^{n+1} \sum_{m>\mathrm{k}_X} \mathrm{RR}(mD)t^m$$

is a polynomial of degree $\mathrm{k}_X + n + 1 = c$ hence so is

$$(1-t)^{n+1}(P(t) - \sum_{m,Q} c_Q(mD)t^m) = (1-t)^{n+1}S(t) + R(t)$$

where we have written

$$S(t) := \sum_{m=0}^{\mathrm{k}_X} (\mathrm{RR}(mD) + (-1)^{n+1}h^0((\mathrm{k}_X - m)D))t^m$$

for the terms of degree up to k_X of P not depending on the singularities of X.

If $\mathrm{k}_X < 0$ then we have

$$(1-t)^{n+1}(P(t) - \sum_{m,Q} c_Q(mD)t^m) = (1-t)^{n+1} \sum_{m\geq 0} \mathrm{RR}(mD)t^m$$

which is a polynomial of degree n by the lemma.

Thus we have proved the following proposition:

Proposition 2.2.11.

$$P(t) = \frac{N(t)}{(1-t)^{n+1}} + \sum_{m\geq 0, Q\in \mathrm{Sing}\, X} c_Q(mD)t^m$$

where N is a polynomial with rational coefficients, of degree $c = \mathrm{k}_X + n + 1$ if $\mathrm{k}_X \geq 0$, or n otherwise.

Step 4: force the Dedekind sum contributions into P_{orb} contributions. Fix a singularity Q of type $\frac{1}{r}(a_1, \ldots, a_n)$ and use the r periodicity of the Dedekind sums. We have

$$\begin{aligned}\sum_{m\geq 0} c_Q(mD)t^m &= \sum_{m\geq 0}(\sigma_{r-m}-\sigma_0)t^m \\ &= \frac{\sum_{m=1}^{r-1}(\sigma_{r-m}-\sigma_0)t^m}{1-t^r} \quad \text{since the } m=0 \text{ term gives } \sigma_r-\sigma_0=0 \\ &= P_{\mathrm{orb}}\left(\frac{1}{r}(a_1,\dots,a_n),\mathrm{k}_X\right)-\frac{I_Q(t)}{(1-t)^{n+1}}\end{aligned}$$

using Corollary 2.2.7. Thus we get the following proposition:

Proposition 2.2.12.

$$P(t)=\frac{A(t)}{1-t^{n+1}}+\sum_{Q\in\mathrm{Sing}\,X}P_{\mathrm{orb}}(Q,\mathrm{k_X}),$$

where $A(t)=N(t)-\sum_Q I_Q(t)$ *with notation as before.*

Step 5: show that A is as claimed in the theorem. Notice that A is palindromic, since $P(t)$ and all the P_{orb} contributions are Gorenstein symmetric in dimension n of degree k_X hence so is $\frac{A(t)}{1-t^{n+1}}$ which shows by the remark about Gorenstein symmetry that $A=\sum_{k_1}^{k_2} b_j t^j$ is palindromic with $k_1+k_2=n+1+\mathrm{k}_X$. It therefore remains to show that $k_1=0$ (we have $k_1\geq 0$ since A is a genuine polynomial, not just a Laurent polynomial) and A has integer coefficients.

If $c<0$ then $A(t)=0$ (as it is a polynomial). If $c\geq 0$ then notice that each P_{orb} contribution starts in degree $\left\lfloor\frac{c}{2}\right\rfloor+1$. We write

$$A_0(t):=\sum_{i=0}^{\lfloor\frac{c}{2}\rfloor}P_i t^i,$$

so that

$$\frac{A(t)}{(1-t)^{n+1}}=A_0(t)+\sum_{i>\lfloor\frac{c}{2}\rfloor}Q_i t^i$$

for some (integral) coefficients Q_i where A_0 has integral coefficients.

Now since A is palindromic it depends only on its first $\left\lfloor\frac{c}{2}\right\rfloor+1$ coefficients. By the above we have

$$A(t)=(1-t)^{n+1}A_0(t)+(1-t)^{n+1}\sum_{i>\lfloor\frac{c}{2}\rfloor}Q_i t^i$$

where the second term on the right hand side only involves terms of degrees greater than $\lfloor \frac{c}{2} \rfloor$ so A is determined by the first term which has integral coefficients. Moreover the degree 0 term is $P_0 = 1 \neq 0$. This completes the proof.

Remark 2.2.13. In fact from the above we deduce that $A(t) = \sum_{j=1}^{c} I_j(t)t^j$ where

$$I_j = \begin{cases} \sum_{l=0}^{j}(-1)^{j-l}P_l\binom{n+1}{j-l} & \text{if } j \leq \lfloor \frac{c}{2} \rfloor \\ \sum_{l=0}^{j}(-1)^{c-j-l}P_{j-l}\binom{n+1}{c-j-l} & \text{if } j > \lfloor \frac{c}{2} \rfloor. \end{cases}$$

Thus writing a program to calculate P_I given $P_1, \ldots, P_{\lfloor \frac{c}{2} \rfloor}$ is relatively straightforward; we reuse the program written for MAGMA from [Zhou, 2011]. For completion, we reproduce this program as well as that for calculating the P_{orb} contributions (including the non-isolated case) in the appendices.

2.3 Examples and applications

We now look at examples and applications.

Example 2.3.1. Consider the weighted projective line $\mathbb{P}(5,7)$; notice that this is not well-formed. The canonical weight is $\mathrm{k} = -12$ so that the coindex is $c = -12+1+1 = -10 < 0$, and so $\gamma = \lfloor \frac{c}{2} \rfloor + 1 = -4$. Thus in this case $P_I = 0$.

It has singularities $\frac{1}{5}(2)$ and $\frac{1}{7}(5)$. We thus have

$$P(t) = \frac{1}{(1-t^5)(1-t^7)}$$

or in terms of the theorem

$$P(t) = \frac{-t^{-4} - t^{-2}}{(1-t)(1-t^5)} + \frac{t^{-4} + t^{-2} + 1}{(1-t)(1-t^7)}.$$

Indeed, previously we had established that

$$-t^{-4} - t^{-2} = \text{InvMod}\left(\frac{1-t^2}{1-t}, \frac{1-t^5}{1-t}, -4\right)$$
$$t^{-4} + t^{-2} + 1 = \text{InvMod}\left(\frac{1-t^5}{1-t}, \frac{1-t^7}{1-t}, -4\right).$$

Example 2.3.2. Consider now the surface $S_{17} \subset \mathbb{P}(2,3,5,7)$. Recall that we assume S_{17} to be quasismooth, and so adjunction gives that S is a K3 surface, and so $\mathrm{k}_X = 0$ and the coindex $c = 3$. Thus to determine the initial term P_I we need only the first plurigenus, and because the ambient weighted projective space has no coordinate of weight 1, $P_1 = 0$. Thus using the formula for P_I we get

$$P_I(t) = \frac{1 - 3t - 3t^2 + t^3}{(1-t)^3}.$$

Exactly the same singularity analysis as before then shows that S has basket of singularities

$$\mathcal{B} = \left\{ \tfrac{1}{2}(1,1), \tfrac{1}{3}(1,2), \tfrac{1}{5}(2,3), \tfrac{1}{7}(2,5) \right\}$$

corresponding to the 4 coordinate points. As before we can calculate

$$\begin{aligned}
P_{\text{orb}}\left(\tfrac{1}{2}(1,1),0\right) &= \frac{t^2}{(1-t^2)(1-t)^2} \\
P_{\text{orb}}\left(\tfrac{1}{3}(1,2),0\right) &= \frac{t^2+t^3}{(1-t^3)(1-t)^2} \\
P_{\text{orb}}\left(\tfrac{1}{5}(2,3),0\right) &= \frac{2t^2+t^3+t^4+2t^5}{(1-t^5)(1-t)^2} \\
P_{\text{orb}}\left(\tfrac{1}{7}(2,5),0\right) &= \frac{3t^2+t^3+2t^4+2t^5+t^6+3t^7}{(1-t^7)(1-t)^2}.
\end{aligned}$$

Then

$$P_S(t) = P_I(t) + P_{\text{orb}}\left(\tfrac{1}{2}(1,1),0\right) + P_{\text{orb}}\left(\tfrac{1}{3}(1,2),0\right) + P_{\text{orb}}\left(\tfrac{1}{5}(2,3),0\right) + P_{\text{orb}}\left(\tfrac{1}{7}(2,5),0\right)$$

which we can check is equal to

$$\frac{1-t^{17}}{(1-t^2)(1-t^3)(1-t^5)(1-t^7)}$$

as expected.

Example 2.3.3. Consider another K3 surface $S_{11} \subset \mathbb{P}(1,2,3,5)$. It has singularities $\frac{1}{2}(1,1)$, $\frac{1}{3}(1,2)$, $\frac{1}{5}(2,3)$, and $P_0 = P_1 = 1$. Then

$$P_I(t) = \frac{1 - 2t - 2t^2 + t^3}{(1-t)^3}$$

and exactly the same type of InvMod and P_{orb} calculations as before yield

$$\begin{aligned}P_S(t) &= \frac{1-2t-2t^2+t^3}{(1-t)^3} \\ &+ \frac{t^2}{(1-t)^2(1-t^2)} + \frac{t^2+t^3}{(1-t)^2(1-t^3)} + \frac{2t^2+t^3+t^4+2t^5}{(1-t)^2(1-t^5)} \\ &= \frac{1-t^{11}}{(1-t)(1-t^2)(1-t^3)(1-t)^5}\end{aligned}$$

as expected.

All we have done in these examples so far is illustrate how the theorem works; we are starting with a given variety and an embedding, and checking that the parsing as written above is correct. However our objective is to reverse this: we want to start from a (hypothetical) variety with given invariants and see whether we can construct it explicitly.

Example 2.3.4. Starting from the surface $S_{17} \subset \mathbb{P}(2,3,5,7)$, assume we just want to modify the singularities a bit. For example, say we want to try to construct a quasismooth polarised K3 surface (so $\mathrm{k}_X = 0$) still with $P_1 = 0$ (so in particular the initial term P_I is unchanged) but this time assume it has as singularities

$$\mathcal{B} = \left\{2 \times \tfrac{1}{2}(1,1), \tfrac{1}{3}(1,2), \tfrac{1}{4}(1,3), \tfrac{1}{7}(2,5)\right\}.$$

The good thing is we already know all but one of the P_{orb} contributions from these points, as well as P_I, so the calculations we have to do are very short. We get

$$\begin{aligned}P(t) &= P_I(t) + 2P_{\text{orb}}\left(\tfrac{1}{2}(1,1),0\right) + P_{\text{orb}}\left(\tfrac{1}{3}(1,2),0\right) \\ &+ P_{\text{orb}}\left(\tfrac{1}{4}(1,3),0\right) + P_{\text{orb}}\left(\tfrac{1}{7}(2,5),0\right) \\ &= \frac{1-t^{10}-t^{11}+t^{21}}{(1-t^2)(1-t^3)(1-t^4)(1-t^5)(1-t^7)}\end{aligned}$$

which is exactly the same Hilbert series as $X_{10,11} \subset \mathbb{P}(2,3,4,5,7)$, which is also a K3 surface with $P_1 = 0$ so we have a chance. It remains to check that X has the singularities we are after.

Consider $X_{10,11} \subset \mathbb{P}(2,3,4,5,7)$ with equations (f_{10}, g_{11}) and coordinates x, y, z, t, u respectively. Suppose X is sufficiently general (see Chapter 1) so that $P_x, P_t \notin X$. On the other hand $P_y, P_z, P_v \in X$ are of type $\frac{1}{3}(1,2), \frac{1}{4}(1,3), \frac{1}{7}(2,5)$ respectively.

Now consider the line $L_{<x,z>} = \mathbb{P}(2,4) \cong \mathbb{P}^1_{<x^2,z>}$. Then $g|_L \equiv 0$ but

$$\begin{aligned} f|_L &= x^5 + x^3 z + z^2 \\ &= x((x^2)^2 + x^2 z + z^2) \end{aligned}$$

which is a quadratic expression in the coordinates of the $\mathbb{P}^1$ (where $x \neq 0$; if $x = 0$ we recover P_z), so that $L \cap X$ consists of 2 points of type $\frac{1}{2}(1,1)$.

Thus the singularities do coincide, so that the variety $X_{10,11} \subset \mathbb{P}(2,3,4,5,7)$ *is* a K3 surface with the required properties. Thus this numerical model exists.

Example 2.3.5. Consider a Fano 3-fold X with $k_X = -1$ and so $c = 3$. Suppose X has $P_1 = 2$ (equivalently genus 0) and basket of singularities

$$\mathcal{B} = \left\{ \tfrac{1}{3}(1,1,2), \tfrac{1}{4}(1,1,3), \tfrac{1}{7}(1,1,6) \right\}.$$

Then

$$\begin{aligned} P_I &= \frac{1 - 2t - 2t^2 + t^3}{(1-t)^4} \\ P_{\mathrm{orb}}\left(\tfrac{1}{3}(1,1,2), -1\right) &= \frac{t^2 + t^3}{(1-t^3)(1-t)^3} \\ P_{\mathrm{orb}}\left(\tfrac{1}{4}(1,1,3), -1\right) &= \frac{t^2 + t^3 + t^4}{(1-t^4)(1-t)^3} \\ P_{\mathrm{orb}}\left(\tfrac{1}{7}(1,1,6), -1\right) &= \frac{t^2 + t^3 + t^4 + t^5 + t^6 + t^7}{(1-t^7)(1-t)^3} \end{aligned}$$

so that

$$\begin{aligned} P &= P_I + P_{\mathrm{orb}}\left(\tfrac{1}{3}(1,1,2), -1\right) + P_{\mathrm{orb}}\left(\tfrac{1}{4}(1,1,3), -1\right) + P_{\mathrm{orb}}\left(\tfrac{1}{7}(1,1,6), -1\right) \\ &= \frac{1 - t^9 - 2t^{10} - t^{11} - t^{12} + t^{14} + t^{15} + 2t^{16} + t^{17} - t^{26}}{(1-t)^2(1-t^3)(1-t^4)(1-t^5)(1-t^6)(1-t^7)} \end{aligned}$$

which is the Hilbert series of a 5×5 resolution, corresponding to the codimension 3 orbifold $X_{9,10,10,11,12} \subset \mathbb{P}(1,1,3,4,5,6,7)$. Such an X could be given by the maximal 4×4 Pfaffians of a 5×5 skew matrix with weights

$$\begin{pmatrix} 7 & 7 & 6 & 5 \\ & 6 & 5 & 4 \\ & & 5 & 4 \\ & & & 3 \end{pmatrix}.$$

Can we now construct a Pfaffian with these weights having the correct singularities?

Let x, y, z, t, u, v, s be the coordinates of weights $1, 1, 3, 4, 5, 6, 7$ respectively. The first thing to notice is that if we restrict our attention to terminal singularities (see the introduction for a discussion on this) we need each of these coordinates to appear in the matrix (else X would be a cone, which is not terminal), and as a pure power in at least one of the entries (else X is a coordinate point of embedding dimension 6, and thus not terminal).

Now we want $P_x, P_y, P_u, P_v \notin X$ so that a pure power of x, y, u, v must appear in the equations of X; however $P_s, P_t, P_z \in X$ and of the singularity types required; in particular then s must hit each of t, u, z in the equations (so as to eliminate t, u, z by the implicit function theorem). Because s has degree 7, we may as well take $m_{12} = s$ and then this occurs in the Pfaffians $\mathrm{Pf}_3, \mathrm{Pf}_4, \mathrm{Pf}_5$ hitting m_{45}, m_{35}, m_{34} respectively, so these entries must contain z, t, u respectively; since m_{45} is the only entry of weight 3 we may as well assume $m_{45} = z$ (for the others note there are several entries of the required weight).

For x, y this can happen in any of the equations so we leave these for the end, but for u the only possibility is that u^2 occurs in one of the equations of degree 10 (these are Pf_2 and Pf_3). In other words $u \in m_{15} \cap (m_{24} \cup m_{34})$ but remember s has to hit u so we need $u \in m_{34}$.

Now consider $P_v \notin X$. Again v^2 must appear in the equation of degree 12, namely Pf_5. Looking at the weights the only possibility is $v \in m_{14} \cap m_{23}$. Thus far, we have reduced the question to a matrix

$$\begin{pmatrix} s & M_7 & v + \dots & u + \dots \\ & v + \dots & L_5 & C_4 \\ & & u + \dots & t + \dots \\ & & & z \end{pmatrix}.$$

Now consider $P_z \in X$. This means I can't have z^3 appearing in Pf_1 or z^4 appearing in Pf_5. The former excludes $z^2 \in m_{23}$ so we may as well take $m_{23} = v$ and now the latter is excluded already. However we must be able to eliminate t, v from P_z (we have already eliminated s from $sz \in \mathrm{Pf}_3$); since $zv \in \mathrm{Pf}_1$ takes care of v it remains to consider t so we may as well have $z^2 \in m_{14}$ and $t \in C_4 \cap D_4$. Adding in powers of x, y where we previously had "$+\dots$" we are left with

$$\begin{pmatrix} s & M_7 & v+z^2 & u \\ & v & L_5 & t \\ & & u+x^5 & t+y^4 \\ & & & z \end{pmatrix}.$$

It now remains to choose M_7 and L_5 suitably to as to get powers of x, y appearing in the equations. It is easy to see for entirely similar reasons that this forces $x^5, y^5 \in L_5$ and $x^7 \in M_7$. Thus the matrix can be chosen as

$$\begin{pmatrix} s & s+x^7 & v+z^2 & u \\ & v & x^5+y^5 & t \\ & & ux^5 & t+y^4 \\ & & & z \end{pmatrix}.$$

One checks the Pfaffians to give rise to the correct singularities (however this is straightforward, as we have constructed our matrix in such a way). By modifying the entries of the matrix in a suitable fashion, we do still get a family of varieties with these properties.

These examples though are really too restrictive. What often happens in practice is we fix an initial term, and then allow the singularities to vary slightly, and "hunt" recognisable Hilbert series amongst those you obtain.

More precisely, we fix a choice of k_X and dimension n (hence we also fix the coindex c), and the first $\lfloor \frac{c}{2} \rfloor$ plurigenera. This fixes the initial term P_I. We now fix some choices of $\frac{1}{r_i}$ singularities, with $1 \le i \le d$ say. Now for each i, we allow X to have up to $j_i \times \frac{1}{r_i}$ singularities. Now we compute a possible Hilbert series

$$P = P_I + \sum_{i=1}^{d} k_i \times P_{\mathrm{orb}}\left(\tfrac{1}{r_i}\right)$$

for every possible $0 \le k_i \le j_i$ (recall we allow X to have *up to* j_i singularities of type $\frac{1}{r_i}$).

For a fixed choice of k_X, n, plurigenera, and singularity types, it is easy enough to write up programs in the computer algebra program MAGMA which will do this (using programs to calculate each P_{orb} and the P_I beforehand). The question is then whether we can recognise any of the resulting Hilbert series. Recall from the introduction we can recognise the form of a Hilbert series in codimension 1, 2 or 3 easily enough (see later remark about the codimension 4 case). If we are very lucky, MAGMA will give as an output a Hilbert series already in recognisable form (we will only need to factorise the denominator).

Most of the time though even when the orbifold does exist in low codimension, MAGMA will have done some cancellation which makes it hard. The strategy is then to multiply the resulting Hilbert series by a collection of terms of the form $(1-t^l)$ until you get a recognisable Hilbert numerator (the $(1-t^l)$ terms correspond to a coordinate of weight l in the ambient projective space). To do this, we run through the following algorithm.

1. The first plurigenera P_1 gives the number of generators of degree 1 of the ambient WPS. Therefore multiply P by $(1-t)^{P_1}$.

2. If there is a single singularity of type $\frac{1}{r}$, then this corresponds either to a coordinate point in the ambient $w\mathbb{P}$ where the corresponding coordinate has weight r or a coordinate stratum $L \subset w\mathbb{P}$ of dimension at least 1, giving at least 2 coordinates with weight divisible by r. In either case, the denominator of P has at least one $(1-t^r)$ factor, so multiply P by $(1-t^r)$.

3. If there are more than one singularity of same type $\frac{1}{r}$ appears more than once, then this corresponds to a coordinate strata $L \subset w\mathbb{P}$ of dimension at least 1 (at least a line) giving 2 coordinates with weight divisible by r. In this case, the denominator of P has at least 2 $(1-t^r)$ factors, so multiply P by $(1-t^r)^2$.

Again, it is easy enough to get MAGMA to do this or even to do it by hand. What comes out from this algorithm is a polynomial. We refer to this output as a "Hilbert almost numerator". If we are lucky, it is exactly the numerator of a Hilbert series, and the $(1-t^l)$ terms we have multiplied the Hilbert series by gives exactly the weights of the coordinates in the ambient $w\mathbb{P}^n$. For example, if looking with a 3-fold with $P_1 = 2$ and precisely 1 type of each of 3 singularity types (say $\frac{1}{r_1}$, $\frac{1}{r_2}$ and $\frac{1}{r_3}$), then the output may return the numerator $1-t^m$ which says that the possible Hilbert series is

$$\frac{1-t^m}{(1-t)^2(1-t^{r_1})(1-t^{r_2})(1-t^{r_3})}$$

which is that of a hypersurface of degree m in $\mathbb{P}(1,1,r_1,r_2,r_3)$ (of course we still have to check that this hypersurface has the required singularities, which in this case is just the requirement that all the r_i are pairwise coprime and divide m).

Most of the time though we are not so lucky, and the output is not immediately recognisable as a Hilbert numerator. Sometimes the output is just not extractable, X may not exist or may lie in too high codimension for us to be able to compute it. Sometimes the "Hilbert almost numerator" needs just a bit of tweaking to make it into a Hilbert numerator. Indeed, we cannot expect the coordinates of the ambient $w\mathbb{P}^n$ to be those given; instead we can expect:

1. we may simply have other generators of the ambient $w\mathbb{P}^n$ corresponding to non singular coordinate points, so not appearing in our Hilbert series parsing. That is, we might need to multiply the output by a $(1-t^s)$ factor where s is not one of the r_i (in practice this is the easiest case to spot and fix);

2. a type of singularity appearing once may actually correspond to a line of singularities in the ambient $w\mathbb{P}^n$ (intersecting X in exactly one point). So we try for a single $\frac{1}{r}$ singularity to multiply the output by another factor of $(1-t^r)$;

3. a line of singularities may correspond to coordinates strictly divisible by r, not equal to r. For example, a line of $\frac{1}{2}$ singularities may correspond to coordinates of weights $2, 4$ or even $6, 10$. So we may have to try multiplying the output by factors like $(1+t^s)$ or slightly more complicated combinations;

4. it could be that the ambient $w\mathbb{P}^n$ has two lines of singularities (say $\frac{1}{r}$ and $\frac{1}{s}$) which intersect. For example it could have coordinates of weights $2, 3, 6$ so that the denominator of the Hilbert series does have $2 \times (1-t^2)$ and $2 \times (1-t^3)$ as factors, but not independently. In this case we will need to divide our output by the extra $(1-t)$ factor, and then multiplying by some other factor. In our example, we divide by $(1-t)$ and multiply by $(1-t+t^2)$. Note that in practice it is often simpler to divide by an entire $(1-t^r)(1-t^s)$ and then multiply by the remaining $(1-t^a)$ (in our example this would mean dividing by $(1-t^2)(1-t^r)$ and then multiplying by $(1-t^6)$).

Example 2.3.6. To illustrate this method, suppose for example we want to construct a Fano 3-fold with $P_1 = 2$ and with singularities of type $\frac{1}{2}(1,1,1)$, $\frac{1}{3}(1,1,2)$ and $\frac{1}{5}(1,2,3)$. Suppose we allow our Fano to have 0, 1 or 2 of the singularity types given and no other singularities. We can now search for such Fanos using MAGMA.

Indeed, we have

$$\begin{aligned} P_I &= \frac{1-2t-2t^2+t^3}{(1-t)^4} \\ P_{\mathrm{orb}}\left(\tfrac{1}{2}(1,1,1),-1\right) &= \frac{t^2}{(1-t^2)(1-t)^3} \\ P_{\mathrm{orb}}\left(\tfrac{1}{3}(1,1,2),-1\right) &= \frac{t^2+t^3}{(1-t^3)(1-t)^3} \\ P_{\mathrm{orb}}\left(\tfrac{1}{5}(1,1,4),-1\right) &= \frac{t^2+t^3+t^4+t^5}{(1-t^5)(1-t)^3}. \end{aligned}$$

We then use MAGMA to produce a list of possible "Hilbert almost numerators". To emphasise the point, what we are doing is producing a list of possible candidates for Fano 3-folds (27 candidates in total). Not all these candidates occur, and some will be unextractable (too high codimension, or equivalently, a too complicated numerator of the Hilbert series). However if the Fano does occur, then its Hilbert series is

$$P = P_I + iP_{\mathrm{orb}}\left(\tfrac{1}{2}(1,1,1),-1\right) + jP_{\mathrm{orb}}\left(\tfrac{1}{3}(1,1,2),-1\right) + kP_{\mathrm{orb}}\left(\tfrac{1}{5}(1,1,4),-1\right)$$

where i,j,k run over $[0,1,2]$.

Better than that, for each of these candidates, if this Hilbert series is right, then we know the denominator must have as factors $(1-t)^2$ (because $P_1 = 2$ so that the ambient weighted projective space has 2 coordinates of weight 1), and $(1-t^2)^i(1-t^3)^j(1-t^5)^k$ (here the fact we are restricting to $i,j,k \leq 2$ means this is correct as written; in general we would be looking at m_i, m_j and m_k where $m_i = 2$ if $i \geq 2$ and i otherwise; similarly for m_j, m_k).

Write the following code in MAGMA, using the functions $P_{\mathrm{orb}}(r,[a_1,\dots,a_n],-\mathrm{k}_X)$ and initial$([P_0,\dots,P_{\lfloor\frac{c}{2}\rfloor}])$ as given in the appendices.

```
>Px := Porb(2,[1,1,1],-1);
>Py := Porb(3,[1,1,2],-1);
>Pz := Porb(5,[1,1,4],-1);
>
>for i,j,k in [0..2] do
>P := initial([1,2],-1,3)+i*Px+j*Py+k*Pz;
>P*(1-t)^2*(1-t^2)^i*(1-t^3)^j*(1-t^5)^k; [i,j,k];
>end for
```

We ask it to return not only the "Hilbert almost numerator" but also the $[i, j, k]$ tuple which has given it.

We now analyse the outputs. Ignoring the $[0, 0, 0]$ case (X would be smooth), the first manageable case seems to be $[0, 1, 2]$ corresponding to $1 \times \frac{1}{3}(1, 1, 2)$ and $2 \times \frac{1}{5}(1, 1, 4)$. In this case the "Hilbert almost numerator" is

$$Q = 1 + t^4 - t^{10} - t^{14}.$$

This looks similar to a codimension 2 complete intersection, except the degrees don't match up, and we would need one more coordinate. However, it is easy enough to see that

$$(1 - t^4)Q = 1 - t^8 - t^{10} + t^{18} =: N$$

which is the Hilbert numerator of a complete intersection of degrees $8, 10$. Thus we are in case 1 above and the Hilbert series is

$$P = \frac{1 - t^8 - t^{10} + t^{18}}{(1 - t)^2(1 - t^3)(1 - t^4)(1 - t^5)^2}$$

which gives as possible candidate $X_{8,10} \subset \mathbb{P}(1, 1, 3, 4, 5, 5)$. One checks entirely as before that X has the required singularities (the $2 \times \frac{1}{5}(1, 1, 4)$ points correspond to the equation of degree 10 which is quadratic in the coordinates of the line of $\frac{1}{5}$ points).

Now look at the output corresponding to $2 \times \frac{1}{3}(1, 1, 2)$ and $1 \times \frac{1}{5}(1, 1, 4)$. We have

$$Q = 1 - t^3 + t^9 - t^{12}.$$

In this case again it looks like a codimension 2 output, and we see that the cancellation required is

$$(1 - t^9)Q = 1 - t^3 - t^{18} + t^{21}$$

suggesting $X_{3,18} \subset \mathbb{P}(1, 1, 3, 3, 5, 9)$. Here we have an equation of degree 3 and a coordinate of weight 3. Thus X would be a linear cone which is excluded, so we eliminate one of our coordinates of degree, to obtain $X_{18} \subset \mathbb{P}(1, 1, 3, 5, 9)$. On the level of the Hilbert series we are using

$$\frac{1-t^3-t^{18}+t^{21}}{1-t^3} = 1-t^{18}$$

so that

$$P = \frac{1-t^3-t^{18}+t^{21}}{(1-t)^2(1-t^3)^2(1-t^5)(1-t^9)} = \frac{1-t^{18}}{(1-t)^2(1-t^3)(1-t^5)(1-t^9)}.$$

In other words we are in the case 3 above (the line of $\frac{1}{3}$ points corresponds to coordinates of weights 3 and 9) and we have actually done $(1+t^3+t^6)Q$ to get the correct Hilbert numerator. We check again that $X_{18} \subset \mathbb{P}(1,1,3,4,9)$ has the required singularities.

Proceeding in a similar way we get the following:

- the output $[2,1,1]$ gives $X_{7,8} \subset \mathbb{P}(1,1,2,3,4,5)$ with basket

  $$\mathcal{B} = \left\{2 \times \tfrac{1}{2}(1,1,1), 1 \times \tfrac{1}{3}(1,1,2), 1 \times \tfrac{1}{5}(1,1,4)\right\}.$$

  We mutiply the "Hilbert almost numerator" Q by $(1+t^2)$ (the line of $\frac{1}{2}$ points corresponds to coordinates of weights 2 and 4);

- the output $[2,2,0]$ eventually gives $X_{12} \subset \mathbb{P}(1,1,2,3,6)$ with basket

  $$\mathcal{B} = \left\{2 \times \tfrac{1}{2}(1,1,1), 2 \times \tfrac{1}{3}(1,1,2)\right\}.$$

  Because we necessarily have 2 lines of singularities, we guess we may be in case 4 above, so mutiply by $\frac{1+t^3}{1-t^2}$ (we know that if our guess is correct then there will be a coordinate of weight divisible by 6, which will hence have a $(1-t^6)$ as a factor; we make this factor by multiplying one of our $(1-t^3)$ factors by $(1+t^3)$; we divide by $(1-t^2)$ because we know the second coordinate of weight divisible by 2 has been taken care of). In this case, this immediately gives us the result we were after (we would normally expect to maybe apply case 3 above as well);

- the output $[1,0,2]$ eventually gives $X_{20} \subset \mathbb{P}(1,1,4,5,10)$ with basket

  $$\mathcal{B} = \left\{1 \times \tfrac{1}{2}(1,1,1), 2 \times \tfrac{1}{5}(1,1,4)\right\}.$$

  This is in fact a hybrid of cases 2, 3 and 4 above and a bit more work, but in the end we multiply Q by $(1+t^5)(1+t^2)$.

There is one further example of interest. The output $[1,2,1]$ gives

$$Q = 1 + t^4 - t^6 - 2t^7 - t^8 + t^{10} + t^{14}$$

which looks like it could be in Pfaffian form although missing one coordinate. Multiplying by $(1-t^4)$ gives as possible Hilbert series

$$P = \frac{1 - t^6 - 2t^7 - 2t^8 + 2t^{10} + 2t^{11} + t^{12} - t^{18}}{(1-t)^2(1-t^2)(1-t^3)^2(1-t^4)(1-t^5)}$$

which is the Hilbert series of the codimension 3 $X_{6,7,7,8,8} \subset \mathbb{P}(1,1,2,3,3,4,5)$. We look therefore for a suitable matrix of weights

$$\begin{pmatrix} 5 & 5 & 4 & 4 \\ & 4 & 3 & 3 \\ & & 3 & 3 \\ & & & 2 \end{pmatrix}.$$

Let x, y, z, t, u, v, s be the coordinates of weights $1,1,2,3,3,4,5$ respectively. An entirely similar calculation to that in the previous Pfaffian example gives a possible

$$M = \begin{pmatrix} s & s + x^5 & v & v + y^4 \\ & v + z^2 & t & u + x^3 \\ & & t + y^3 & u \\ & & & z \end{pmatrix}.$$

The only real difficulty is caused by the 1 point along the line $L_{<z,v>} \cong \mathbb{P}^1_{<z^2,v>}$ of $\frac{1}{2}$ points. The resulting equations of degree 6, 8 and 8 restricted to the line give $z(v+z^2)$ and $v(v+z^2)$ (the latter for both equations of degree 8). Thus neither P_z nor P_v are in X so that the equation of degree 6 restricts to a linear equation in v, z^2 (since $z \neq 0$ on $L \cap X$) and this vanishes also on the equations of degree 8. The check that the other singularities coincide is exactly as in the previous example (in particular s hits z, t, u but not v for example); notice that for this particular construction the $2 \times \frac{1}{3}$ points occur at P_t, P_u.

Chapter 3

Generalising the formula to arbitrary rational functions

In this Chapter we wish to generalise the formula obtained in the previous section. A main motivation is the construction of 4-folds since whilst terminal singularities in 4 dimensions are not well known, in general quasismooth terminal orbifolds will have curve orbifold behaviour, so that any attempt at making lists in the spirit of [Iano-Fletcher, 2000] or [Altmok, Brown, and Reid, 2002] (for example as in [Brown and Kasprzyk, to appear]) will have to take such behaviour into account.

3.1 Definitions, examples and observations in the curve orbifold locus case

Definition 3.1.1. Let $X \subset w\mathbb{P}$ be an orbifold of dimension n. We say a curve $C \subset X$ is of (generic) type $\frac{1}{s}(b_1, \ldots, b_{n-1})$ if for a general hyperplane $H \subset w\mathbb{P}$, the point $C \cap H \subset X \cap H$ is of this type.

A point $P \in C$ on such a curve is said to be a *dissident point* if its singularity type is different from that of the curve (so that P will be of type $\frac{1}{r}$ where $s \mid r$).

Example 3.1.2. The (not well-formed) weighted projective surface $\mathbb{P}(1, 2, 6)$ contains the weighted projective line $\mathbb{P}(2, 6)$ as a locus of $\frac{1}{2}(1)$ singularities (i.e. ordinary nodes), with the dissident point P_z of type $\frac{1}{6}(1, 2)$.

The variety $X_{18} \subset \mathbb{P}(1, 2, 2, 4, 9)$ contains the curve $C = X \cap \mathbb{P}(2, 2, 4)$ of type $\frac{1}{2}(1, 1)$ with a dissident point of type $\frac{1}{4}(1, 1, 2)$.

More generally a point of type $\frac{1}{r}(a_1, \ldots, a_n)$ where $s_i = \mathrm{hcf}(r, a_i) > 1$ for some i is a dissident point on precisely the corresponding curve(s) of type

$\frac{1}{s_i}(\overline{a_1}, \ldots, \widehat{\overline{a_i}}, \ldots, \overline{a_n})$ where $\overline{a_j}$ denotes the residue of a_j mod s.

As in the isolated case we expect each orbipoint to contribute a P_{orb} type function to the Hilbert series; in practice (see following sections for more details) we do this by attributing to the strictly periodic Dedekind sum not only some of the growth contribution, but also for the dissident points some part of the contribution from the curve(s) on which it lies. This is a continuation of the dichotomy of parsing the Hilbert series into a collection of fractional periodic terms (Dedekind sum contributions), or integral symmetric terms that in fact have the same periodicity (ice cream functions).

In practice we find that the orbicurves contribute two terms to the Hilbert series:

1. a term corresponding to the degree of the curve A_C; this looks like a P_{orb} contribution, except that the denominator is of the form $(1-t^s)^2(1-t)^{n-1}$. Indeed this generalises to arbitrary dimensional strata, since if X has a $\frac{1}{\sigma}$ orbifold strata of dimension d then its graded ring $R(X, D)$ has at least $d+1$ generators divisible by σ so that its Hilbert series has a pole of order $d+1$ at the σth roots of unity. In the curve case, multiplying by $(1-t^{ms})$ corresponds to taking a transverse hyperplane section in $|msD|$ for some m and transforms this term into that of an isolated orbifold point (on a variety of dimension $n-1$);

2. a term corresponding to the normal bundle of the curve B_C, with denominator $(1-t^s)(1-t)^n$; this term is killed by taking a hyperplane section (since we can no longer see the normal bundle) and you can't recover it afterwards.

Both terms depend not only on properties of the underlying curve, but also the dissident points on it, as we shall see explicitly and explain later on.

Remark 3.1.3. Both observations above are key to our philosophy of generalising these types of formulae.

1. The fact that we expect a $(1-t^s)^2$ contribution in the denominator because the Hilbert series has a double pole at sth roots of unity is what suggests that a more general result does exist. Indeed, fitting together combinations of the various orbifold terms to recover the Hilbert series is essentially a "crossword puzzle" involving the partial fraction decomposition of the series and the various terms involved. On the face of it, the fact that we can solve this puzzle doesn't seem to depend on the geometry of X (although the results can be interpreted as such). Based on this approach, the key ingredients of the Hilbert series are

- the degree of its Gorenstein symmetry
- the number of factors in the denominator
- its poles and their orders.

On the face of it it should therefore be possible to generalise our results to rational functions with these properties.

2. The fact that we wish to split the curve contribution into two contributions corresponds to a wish to keep the numerator supported on as short an interval as possible (in this case, a residue mod $\frac{1-t^s}{1-t}$). We could otherwise have grouped both terms together, but then the interval of support would have length $2s-1$ rather than $s-1$ as is currently the case.

Let us now look at some explicit examples illustrating the parsing in various curve orbifold cases.

Example 3.1.4. Consider $X_{13} \subset \mathbb{P}(1,1,3,3,5)$. This is a Calabi-Yau 3-fold, so $\mathrm{k}_X = 0$ and $c = 4$. The first plurigenera are $P_0 = 1, P_1 = 2, P_2 = 3$ so that

$$P_I = \frac{1 - 2t + t^2 - 2t^3 + t^4}{(1-t)^4}.$$

It has an orbipoint of type $\frac{1}{5}(1,1,3)$ at P_u and an orbicurve $C = \mathbb{P}(3,3)_{<z,t>}$ of transverse type $\frac{1}{3}(1,2)$ and of degree $\frac{1}{3}$. The contribution from the orbipoint is

$$P_{\mathrm{orb}}(P_u, 0) = \frac{t^3 + t^5}{(1-t^5)(1-t)^3};$$

the A_C contribution is

$$A_C = \frac{P_{\mathrm{orb}}(C,3)}{1-t^3} = \frac{-t^4}{(1-t^3)^2(1-t)^2}.$$

The B_C contribution is

$$B_C = \frac{3t^3}{(1-t^3)(1-t)^3}$$

and we recover

$$P = \frac{1-t^{13}}{(1-t)^2(1-t^3)^2(1-t^5)} = P_I + P_{\mathrm{orb}}(P_u) + A_C + B_C.$$

Example 3.1.5. Consider now $X_{13} \subset \mathbb{P}(1,1,2,3,3,4)$. This is a Fano 4-fold with

$k_X = -1$ and $c = 4$. The first plurigenera are $P_0 = 1, P_1 = 2, P_2 = 4$ so that

$$P_I = \frac{1 - 3t + 4t^2 - 3t^3 + t^4}{(1-t)^5}.$$

It has an orbipoint of type $\frac{1}{4}(1, 2, 3, 3)$ at P_v and two orbicurves $C_1 = \mathbb{P}(2, 4)_{<z,v>}$ of type $\frac{1}{2}(1, 1, 1)$ and degree $\frac{1}{4}$ and $C_2 = \mathbb{P}(3, 3)_{<t,u>}$ of type $\frac{1}{3}(1, 1, 2)$ and degree $\frac{1}{3}$. On this occasion the orbipoint is a dissident point on C_1, whilst C_2 has no dissident points. The contribution from P_v is

$$P_{\text{orb}}(P_v, -1) = \frac{-t^4}{(1-t^4)(1-t^2)(1-t)^2}$$

(notice the $(1 - t^2)$ factor in the denominator, corresponding to the orbinate z of weight $2 \mid 4$; we will discuss this phenomenon in more detail later). For the orbicurve C_1, we have that $B_{C_1} = 0$ whilst

$$A_{C_1} = \frac{-t^3}{(1-t^2)^2(1-t)^3}.$$

For the orbicurve C_2, the A_{C_2} contribution is just

$$A_{C_2} = \frac{P_{\text{orb}}(C_2, 2)}{1 - t^3} = \frac{-t^4}{(1-t^3)^2(1-t)^3}.$$

The B_{C_2} contribution is

$$B_C = \frac{2t^3}{(1-t^3)(1-t)^3}$$

and we recover

$$P = \frac{1 - t^{13}}{(1-t)^2(1-t^2)(1-t^3)^2(1-t^4)} = P_I + P_{\text{orb}}(P_v) + A_{C_1} + A_{C_2} + B_{C_2}.$$

Example 3.1.6. Let us now look at a dissident point lying at the intersection of two orbicurves. Consider $X_{13} \subset \mathbb{P}(1, 1, 2, 3, 6)$, a Calabi-Yau 3 fold. $P_0 = 1, P_1 = 2, P_2 = 4$ give

$$P_I = \frac{1 - 2t + 2t^2 - 2t^3 + t^4}{(1-t)^4}.$$

X has an orbipoint of type $\frac{1}{6}(1, 2, 3)$ at P_u lying at the intersection of the two orbicurves $C_1 = \mathbb{P}(2, 6)_{<z,u>}$ of type $\frac{1}{2}(1, 1)$ and degree $\frac{1}{6}$ and $C_2 = \mathbb{P}(3, 6)_{<t,u>}$ of type $\frac{1}{3}(1, 2)$ and degree $\frac{1}{6}$. We find

$$A_{C_1} = B_{C_1} = A_{C_2} = 0$$

for example by noticing in the partial fraction decomposition that the contribution

$$P_{\mathrm{orb}}(P_u) = \frac{t^6}{(1-t)^6(1-t)(1-t^2)(1-t^3)}$$

has taken care of all terms whose denominator have factor $(1-t^2)^2$ or $(1-t^3)^2$ in the partial fraction decomposition of P. Thus we see that

$$B_{C_2} = \frac{t^3}{(1-t^3)(1-t)^3}$$

and again check that

$$P = \frac{1-t^{13}}{(1-t)^2(1-t^2)(1-t^3)(1-t^6)} = P_I + P_{\mathrm{orb}}(P_u) + A_{C_1} + A_{C_2} + B_{C_2}.$$

Remark 3.1.7. The calculations to find the parsing described in the above examples are somewhat involved. We can do this either using the geometric formulae of the following section, or by looking at the corresponding partial fraction decompositions, and fitting them together. The key in this case is not the calculation, but merely the observation that such a partial fraction decomposition does exist with each individual term being Gorenstein symmetric of same degree, and with numerator integral and of "short support". As explained above, this leads to the conjecture that the key properties of the Hilbert series P may not be its geometry, but rather its algebra, and specifically its symmetry and poles, and the relations between the poles. This is what leads to the hunt for a more general result on rational functions which need not be Hilbert series, and occupies the remainder of this chapter.

3.2 The "isolated" case

We have an analogous result to the isolated orbifold locus case as follows.

Theorem 3.2.1. *Let* $a_0, \ldots, a_m \in \mathbb{N}$ *be pairwise coprime and*

$$P(t) = \frac{N(t)}{\prod_i (1-t^{a_i})}$$

be Gorenstein symmetric of degree k. *Suppose that* N *has a zero of order exactly* d *at* 1 *(*d *could be 0); let* $n = m - d$ *and put as before* $c = \mathrm{k} + n + 1$. *Then there is a* *partial fraction decomposition*

$$P(t) = \frac{I(t)}{(1-t)^{n+1}} + \sum_{i=0}^{m} \frac{N_i(t)}{((1-t^{a_i})(1-t)^n}$$

where each term is Gorenstein symmetric of degree k *and each numerator* N_i *is integral and supported on*

$$\left[\left\lfloor\frac{c}{2}\right\rfloor+1, \left\lfloor\frac{c-1}{2}\right\rfloor + a_i - 1\right],$$

an interval of length at most $a_i - 1$.

Remark 3.2.2. In the main theorem of the previous section, the rational function P had denominator the product of $N+1$ terms $(1-t^{a_i})$ whilst each term in the partial fraction decomposition had denominator the product of $n+1$ such terms, where n was the dimension of the variety and in general $n \leq N$. Thus you can think of n as being the analogue of the dimension; saying that the polynomial N has a zero of order d at 1 is the same as saying that $(1-t)^d \mid N$ and d is the highest such power; geometrically you should think of d as the codimension. Shrinking m by d (equivalently removing all $(1-t)$ factors from the numerators) is what allows us to keep the short support assumption for the numerators I, N_i.

Notation 3.2.3. We refer to the denominator $(1-t)^{n+1}$ as the *initial denominator* (in correspondance with the initial term in the Hilbert series parsing).

Let us give a couple of illustrative examples.

Example 3.2.4.

1. Consider

$$P(t) = \frac{1+t^2+t^3+t^5}{(1-t)^2(1-t^3)};$$

P is Gorenstein symmetric of degree 0 in the sense of Definition 2.1.2. There is only one non-initial denominator $[1,1,3]$ and we have that P splits as

$$P = \frac{1-t-t^2+t^3}{(1-t)^3} + \frac{2t^2+2t^3}{(1-t)^2(1-t^3)}.$$

2.

$$P = \frac{1+t^3+t^6}{(1-t)(1-t^2)(1-t^3)};$$

P is Gorenstein symmetric of degree 0. There are two non-initial denominator

$[1,1,2],[1,1,3]$ and we have that P splits as

$$P = \frac{1-2t-2t^2+t^3}{(1-t)^3} + \frac{t^2}{(1-t)^2(1-t^2)} + \frac{3t^2+3t^3}{(1-t)^2(1-t^3)}.$$

The proof revolves around the following lemma.

Lemma 3.2.5. *Suppose $P(t)$ is as in the theorem above. Fix i. Then there is a polynomial B_i, constructed as a residue mod $\frac{1-t^{a_i}}{1-t}$ such that the rational function*

$$P - \frac{B_i}{(1-t)^n(1-t^{a_i})}$$

has no poles on $\mu_{a_i} \setminus \{1\}$.

Proof. To lighten notation, set

$$A_i = \prod_{j \neq i} \left(\frac{1-t^{a_j}}{1-t} \right)$$
$$F_i = \frac{1-t^{a_i}}{1-t}.$$

Now put

$$N_i = \frac{N}{(1-t)^d} \cdot \mathrm{InvMod}(A_i, F_i)$$

and let

$$B_i \equiv N_i \mod F_i.$$

Then by construction we have that

$$(1-t)^d A_i B_i \equiv N \mod F_i$$

and

$$\frac{B_i}{(1-t^{a_i})(1-t)^n} = \frac{(1-t)^d A_i B_i}{\prod_{j=0}^m (1-t^{a_j})}$$

so that

$$P - \frac{B_i}{(1-t^{a_i})(1-t)^n} = \frac{N-(1-t)^d A_i B_i}{\prod_{j=0}^m (1-t^{a_j})};$$

the numerator is divisible by F_i (equivalently vanishes at all $\epsilon \in \mu_{a_i} \setminus \{1\}$) whereas the denominator has a pole of order (at most) 1 at these, so that

$$P - \frac{B_i}{(1-t^{a_i})(1-t)^n}$$

has no poles on $\mu_{a_i} \setminus \{1\}$, as claimed. □

Proposition 3.2.6. *With notation as above, we can choose* $N_i \equiv B_i$ *with support on*

$$\left[\left\lfloor\frac{c}{2}\right\rfloor + 1, \left\lfloor\frac{c-1}{2}\right\rfloor + a_i - 1\right].$$

Moreover N_i *is integral, symmetric so that*

$$\frac{N_i}{(1-t^{a_i})(1-t)^n}$$

is Gorenstein symmetric of degree k.

Proof. Use notation as in the proof of the previous lemma. By the results of chapter 2 B_i is integral, and symmetric. Now the lemma 4.5.6 of Zhou [2011] tells us that we can shift B_i until it is centred at $\frac{c+a_i}{2}$. If we now trim B_i from either end then we maintain its symmetry and end up with a N_i supported on the desired interval. □

We now finish off the proof of the theorem.

Proof. For each i we construct N_i as above. Then

$$A = P - \sum_{i=0}^{m} \frac{N_i}{(1-t^{a_i})(1-t)^n}$$

has no poles at any of the $\mu_{a_i} \setminus \{0\}$. The only pole of A is therefore at 1 and of order at most $n+1$ so that

$$A = P - \sum_{i=0}^{m} \frac{N_i}{(1-t^{a_i})(1-t)^n} = \frac{I}{(1-t)^{n+1}}.$$

Now A is Gorenstein symmetric of degree k since all the other terms are. Moreover I has integer coefficients by exactly the same reasoning as in Step 5 of Chapter 2. This completes the proof. □

3.3 The curve locus case

We now consider the case analogous to the curve singularity. Here we have come across some difficulties. Recall that geometrically a point singularity will be of the form $\frac{1}{r}(b_1, \ldots, b_n)$. Put $s_i = \mathrm{hcf}(r, b_i)$; then the s_i are pairwise coprime and possibly some $s_i > 1$. The P_{orb} contribution from such a point typically has denominator

$$(1-t^r)\prod(1-t^{s_i})$$

and deals with the maximal (single) poles at $\mu_r \setminus \bigcup \mu_{s_i}$. However this term also contributes to some of the poles (of order at most 2) at the s_ith roots of unity.

Trying to generalise this to the case of an arbitrary rational function proves to be difficult. The issue here is that we have a function with denominator

$$\prod_{i=0}^{m}(1 - t^{a_i})$$

which we want to parse into terms with denominator some product of $n+1$ $(1-t^{b_j})$ terms where usually $n < m$. For a "main period" r (suppose say $r = a_0$ for now) we wish to find an analogue of a $\frac{1}{r}(b_1, \ldots, b_n)$ (the b_i should be thought of as a_i reduced mod r) to tell us how to deal with the maximal poles at μ_r; this should give us a term with denominator

$$(1 - t^r)\prod_{i=1}^{n}(1 - t^{s_i})$$

where $s_i = \text{hcf}(r, b_i) = \text{hcf}(r, a_i)$ for the corresponding a_i. However in general letting $s_i = \text{hcf}(r, a_i)$ we could have many more than n of the s_i being nontrivial, and we simply do not know which of these will occur in the denominator. This means that we cannot as of yet prove a parsing for a general P without reference to a concrete orbifold by first dealing with the "point" contributions.

On the other hand, consider the "curve" contributions: these are the terms we expect in the partial fraction decomposition to have denominator of the form

$$(1 - t^s)^2(1 - t)^{n-1} \quad \text{or} \quad (1 - t^s)(1 - t)^n.$$

Where the curve is a pure curve (no dissident points) then this works because these terms deal entirely with the poles at sth roots of unity. In general however the curve has dissident points which modify the curve contribution and in particular take care of the fractional part of the curve contribution.

It turns out there are two potential ways around this:

1. ignore issues about the dimension. Allow all the s_i to appear in the denominators so that the denominators appear to be "too large";
2. instead of insisting that $(1 - t^s)^2$ appears in the denominator explicitly, parse the curve contribution into a collection of $(1-t^s)(1-t^{a_i})$ for all the a_i of which s is a factor. In this case the point contributions appear as denominators

$(1-t^{a_i})(1-t)^n$, that is the point contributions stand on their own, at the expense of not seeing a curve contribution, but rather a collection of contributions belonging to the same curve.

We discuss both these strategies in more detail in the next section.

For now therefore we discuss the case where the rational function is a Hilbert series of a polarised orbifold. Here we have the following result:

Theorem 3.3.1. *Let (X, D) be a quasismooth, well-formed, projectively Gorenstein polarised orbifold of dimension $n \geq 2$; we again denote its canonical weight by k_X and its coindex by c. Suppose X has orbifold strata*

$$\mathcal{B} = \{\text{points } Q \text{ of type } \tfrac{1}{r}(a_1, \ldots, a_n)\} \text{ (not necessarily isolated)}$$
$$\mathcal{B}' = \{\text{curves } C \text{ of transverse type } \tfrac{1}{s}(b_1, \ldots, b_{n-1})\ \}.$$

Then its Hilbert series has the form

$$P_X(t) = P_I(t) + \sum_{Q \in \mathcal{B}} P_{\mathrm{orb}}(Q, \mathrm{k}_X) + \sum_{C \in \mathcal{B}'} (A_C + B_C) \tag{3.1}$$

where the terms on the right are as follows:

1. *P_I deals with the first $\lfloor \frac{c}{2} \rfloor$ plurigenera;*
2. *for each $Q \in \mathcal{B}$ we set $s_i = \mathrm{hcf}(a_i, r)$, $A = \prod_i \frac{1-t^{a_i}}{1-t^{s_i}}$ and $F = \frac{1-t^r}{h}$ where $h = \mathrm{hcf}\left(\prod(1-t^{a_i}), 1-t^r\right) = (1-t)\prod\left(\frac{1-t^{s_i}}{1-t}\right)$; then*

$$P_{\mathrm{orb}}(Q, \mathrm{k}_X) = \frac{\mathrm{InvMod}(A, F, \gamma)}{(1-t^r)\prod_i(1-t^{s_i})},$$

   where $\gamma = \lfloor \frac{c}{2} \rfloor + \deg h$ so that the numerator is a residue mod $\frac{1-t^r}{h}$ supported in

$$\left[\left\lfloor \frac{c}{2} \right\rfloor + \deg h, \left\lfloor \frac{c-1}{2} \right\rfloor + r - 1\right];$$

3. $$A_C = \frac{D_C}{(1-t)^{n-1}(1-t^s)^2}$$

   where D_C is a residue mod $\frac{1-t^s}{1-t}$ supported on

$$\left[\left\lfloor \frac{c}{2} \right\rfloor + \left\lfloor \frac{s-1}{2} \right\rfloor + 1, \left\lfloor \frac{c-1}{2} \right\rfloor + \left\lfloor \frac{s}{2} \right\rfloor + s - 1\right]$$

4.

$$B_C = \frac{N_C}{(1-t)^n(1-t^s)}$$

where N_C *is also a residue mod* $\frac{1-t^s}{1-t}$ *supported on*

$$\left[\left\lfloor\frac{c}{2}\right\rfloor + 1, \left\lfloor\frac{c-1}{2}\right\rfloor + s - 1\right].$$

The numerators N_C, D_C *and* $\mathrm{InvMod}(A, F, \gamma)$ *are all integral and palindromic, so that each individual term is Gorenstein symmetric of degree* k_X.

In this Chapter we prove the result using the rational function properties similar to the previous section. Our strategy is therefore first to resolve singularities so as to deduce a first parsing, then deal with the point contributions similarly to the previous Chapter; what remains is a rational function whose denominator is a product of $(1-t^s)^2$ terms where all the s are pairwise coprime, and we deal with this in the same way as we have done the general "isolated" case in this Chapter.

In the following Chapter we will focus more on the geometric properties, and offer a partial interpretation for the curve contributions.

Resolution of singularities and first parsing Assume (X, D) is as in the theorem. First, using similar methods to those in [Buckley, 2003] and [Buckley, Reid, and Zhou, 2013] (see [Buckley and Szendroi, 2004] section 2.2 in particular) we resolve the singularities on X and deduce a Riemann–Roch type formula.

Theorem 3.3.2. *Let* (X, D) *be as above. Then for any integer* m

$$\chi(\mathcal{O}_X(mD)) = \mathrm{RR}(X, mD) + \sum_{P\in\mathcal{B}} c_P(mD) + \sum_{C\in\mathcal{B}'} s_C(mD)$$

where the contributions c_P *and* s_C *depend only on the type of singularities, i.e. on a local neighborhood of* P *(respectively of a generic point on* C*).*

The terms c_P and c_S were calculated in [Zhou, 2011], section 5.1. Using the above theorem and the same "differencing" technique as in Chapter 2, we deduce a first parsing of the Hilbert series as follows.

Theorem 3.3.3. *We have*

$$P_X(t) = I(t) + \sum_{P\in\mathcal{B}} P_{\mathrm{per}}(P)(t) + \sum_{C\in\mathcal{B}'} P_{\mathrm{per}}(C)(t)$$

where

1. $$I(t) = \frac{A_I(t)}{(1-t)^{n+1}}$$

   where A_I is rational, of degree c if $\mathrm{k}_X \geq 0$ or n otherwise;

2. *for P of type $\frac{1}{r}(a_1, \ldots, a_n)$ we have*

   $$P_{\mathrm{per}}(P)(t) = \frac{\sum_{i=1}^{r-1}(\sigma_{r-i} - \sigma_0)(P)t^i}{1-t^r}$$

   with the generalised Dedekind sums σ_j defined as in Chapter 2;

3. *for a curve C of type $\frac{1}{s}(b_1, \ldots b_{n-1})$ we have that*

   $$P_{\mathrm{per}}(C)(t) = \frac{Q_C}{(1-t^s)^2}$$

   where Q_C is rational.

Remark 3.3.4. In the following Chapter we give a more explicit formula for $P_{\mathrm{per}}(C)$. However at this stage all that interests us is that it has poles (of order ≤ 2) only at μ_s.

Contributions from orbipoints Consider a point of type $\frac{1}{r}(a_1, \ldots, a_n)$ not necessarily isolated. Denote $s_i = \mathrm{hcf}(a_i, r)$ for each i, let σ_i and Δ be as in the previous Chapter, and put

$$A = \prod_{i=1}^{n}(1 - t^{a_i})$$
$$h = \mathrm{hcf}(A, 1 - t^r) = \prod_i \left(\frac{1 - t^{s_i}}{1 - t}\right) \cdot (1 - t)$$
$$F = \frac{1 - t^r}{h}.$$

We start with the following analogue to proposition 2.2.6:

Proposition 3.3.5. *With the notation as above let $d = \deg F$ and let γ be any* *integer. Then*

$$
\begin{aligned}
\prod(1-t^{s_i})\cdot\Delta &\equiv \operatorname{InvMod}\left(\prod\left(\frac{1-t^{a_i}}{1-t^{s_i}}\right), F, \gamma+1\right)\\
&= \operatorname{InvMod}\left(\frac{A}{\prod(1-t^{s_i})}, F, \gamma+1\right)\\
&= \sum_{j=\gamma+1}^{\gamma+d}\theta_j t^j
\end{aligned}
\tag{3.2}
$$

where the θ_j are integers calculated from Dedekind sums.

Proof. The first equality follows from the fact that $\Delta\cdot A\equiv 1 \bmod F$ as proved in theorem 2.2.5. Therefore all that remains to be shown is the integrality of the coefficients.

Again as in the proof of proposition 2.2.6 we notice that each $\frac{1-t^{a_i}}{1-t^{s_i}}$ is pairwise coprime to F so we have

$$\operatorname{InvMod}\left(\prod\left(\frac{1-t^{a_i}}{1-t^{s_i}}\right), F\right) = \prod\operatorname{InvMod}\left(\frac{1-t^{a_i}}{1-t^{s_i}}, F\right)$$

and shifting the support does not change integrality, so that it is sufficient to prove that $\operatorname{InvMod}\left(\frac{1-t^a}{1-t^s}, \frac{1-t^r}{(1-t^s)g}\right)$ has integral coefficients, where r, a are integers, $s = \operatorname{hcf}(a, r)$ and g is a polynomial coprime to $1-t^s$. We let b, l be such that $ab = s + lr$ by the Euclidean algorithm, and claim that

$$\sum_{i=0}^{b-1} t^{ai} = \frac{1-t^{ab}}{1-t^a} \equiv \operatorname{InvMod}\left(\frac{1-t^a}{1-t^s}, \frac{1-t^r}{(1-t^s)g}\right).$$

This is because

$$\frac{1-t^{ab}}{1-t^a}\cdot\frac{1-t^a}{1-t^s} = \frac{1-t^{s+lr}}{1-t^s}$$

which is equal to 1 at all $\epsilon\in\mu_r\setminus\mu_s$ so in particular at all roots of $\frac{1-t^r}{(1-t^s)g}$. □

Using this result we have the following:

Proposition 3.3.6. *With notation as above*

$$P_{\text{orb}}\left(\tfrac{1}{r}(a_1,\dots,a_n)\right) = \frac{\Delta}{1-t^r} + \frac{A(t)}{(1-t)^{n+1}} + \sum_{j=1}^{n}\frac{Q_j(t)}{(1-t^{s_j})^2(1-t)^{n-1}} + \frac{R_j(t)}{(1-t^{s_j})(1-t)^n}.$$

Moreover, A, Q_j, R_j all have rational coefficients.

Proof. For arbitrary A, R_j, Q_j, multiply the right hand side by

$$(1-t^r)\prod(1-t^{s_i}) = (1-t^r)h(1-t)^{n-1};$$

we get

$$\begin{aligned}\overline{B} &= \Delta \cdot \prod(1-t^{s_i}) + A(1-t^r)\frac{h}{(1-t)^2} \\ &+ \sum_j \frac{Q_j}{(1-t^{s_j})^2(1-t)^{n-1}} h(1-t)^{n-1}(1-t^r) \\ &+ \sum_j \frac{R_j}{(1-t^{s_j})(1-t)^n} h(1-t)^{n-1}(1-t^r)\end{aligned}$$

that is,

$$\overline{B} = \Delta \cdot \prod(1-t^{s_i}) + F \cdot \left(A \cdot \frac{h^2}{(1-t)^2} + \sum_j Q_j \cdot \frac{h^2}{(1-t^{s_j})^2} + \sum_j R_j \cdot \frac{h^2}{(1-t^{s_j})(1-t)} \right)$$

where we let the sum run over those j such that $s_j \neq 1$ (where $s_j = 1$ the Q_j and R_j terms get absorbed by the A term). This then becomes

$$\overline{B} = \Delta \cdot \prod(1-t^{s_i}) + F \cdot \left(A \cdot \frac{h^2}{(1-t)^2} + \sum_j \prod_{l \neq j} \left(\frac{1-t^{s_l}}{1-t}\right)^2 S_j \right)$$

where

$$S_j = Q_j + \left(\frac{1-t^{s_j}}{1-t}\right) R_j,$$

where we have used that

$$\frac{h}{1-t^{s_j}} = \prod_{l \neq j} \frac{1-t^{s_l}}{1-t}.$$

Now taking residues modulo F we obtain

$$\overline{B} \equiv \Delta \cdot \prod(1-t^{s_i}) \equiv \operatorname{InvMod}\left(\frac{A}{\prod(1-t^{s_i})}, F, \gamma\right)$$

by proposition 3.3.1. Since all the $\left(\frac{h}{1-t}\right)^2, \left(\frac{h}{1-t^{s_j}}\right)^2$ (where $s_j > 1$) are all coprime, we can find A, S_j to move $\overline{B}$ to the correct support. The result now follows from writing

$$S_j = Q_j + \left(\frac{1-t^{s_j}}{1-t}\right) R_j$$

for suitable Q_j, R_j. □

Proposition 3.3.7. P_{orb} *is Gorenstein symmetric of degree* k_X.

Proof. We need to show that the numerator

$$B(t) := \mathrm{InvMod}\left(\prod\left(\frac{1-t^{a_i}}{1-t^{s_i}}\right), \frac{1-t^r}{h}, \left\lfloor\frac{c}{2}\right\rfloor + \deg h\right)$$

is palindromic.

Recall from the proof of proposition 3.3.1 that $B(t)$ is the residue of $\overline{B}(t) := \prod\left(\frac{1-t^{a_ib_i}}{1-t^{a_i}}\right)$ mod F where b_i is the smallest integer such that $a_ib_i \equiv s_i \bmod r$, then moved to the correct support.

First notice that $\overline{B}$ is palindromic, supported on $[0, \sum a_i(b_i - 1)]$. The idea is to first reduce this polynomial mod F by trimming from both sides so as to keep it palindromic, then move to the correct interval of support. This is done entirely similarly as in the previous section; it requires a case-by-case analysis because the intervals we get along the way depend on the parity of $\sum a_i(b_i - 1), r, \deg h$. In fact we first trim mod $\frac{1-t^r}{1-t}$ then shift the support along by a multiple of r (which keeps the palindromic format, because we are working mod $\frac{1-t^r}{1-t}$) and then finally trimming further mod F to get the required interval; in particular we again use lemma 4.5.6 from Zhou [2011] repeatedly. □

Curve contributions and end of proof Putting together the results we have so far, we get the following:

Proposition 3.3.8. *Let* (X, D) *be as above. Then*

$$P_X(t) - \sum_{Q\in\mathcal{B}} P_{\mathrm{orb}}(Q, \mathrm{k}_X) = \frac{N}{(1-t)^{n+1}\prod_s(1-t^s)^2}$$

where the s *run over all the types of curve in* $\mathcal{B}'$. *Moreover* N *is integral and the right hand side is Gorenstein symmetric of degree* k_X.

Proof. By the results proved previously

$$P_X(t) - \sum_{Q\in\mathcal{B}} P_{\mathrm{orb}}(Q, \mathrm{k}_X)$$

is a rational function with poles of order at most 2 at $\bigcup_{\mathcal{B}'} \mu_s \setminus \{1\}$ and a pole of order at most $n+1$ at 1; moreover it is Gorenstein symmetric as the sum of symmetric functions. Its numerator is integral since the rational function has integer coefficients when viewed as a power series (it is the sum of power series who are known to have integer coefficients. □

To complete the proof of the theorem, it remains to show the following.

Proposition 3.3.9. *Let $s_1, \dots, s_m$ be pairwise coprime and*

$$P = \frac{N}{(1-t)^{n+1}\prod_i (1-t^{s_i})^2}$$

be Gorenstein symmetric of degree k *in dimension n, where N has a zero of order $2m$ at 1. Put $c = \mathrm{k} + n + 1$. Then there is a partial fraction decomposition*

$$P = \frac{I}{(1-t)^{n+1}} + \sum_i \frac{M_i}{(1-t)^{n-1}(1-t^{s_i})^2} + \frac{N_i}{(1-t)^n(1-t^{s_i})}$$

where each term is integral, symmetric, M_i, N_i are both residues mod $\frac{1-t^{s_i}}{1-t}$ and thus supported on intervals of length at most $s_i - 1$. More precisely:

- *M_i is supported on*

$$\left[\left\lfloor\frac{c}{2}\right\rfloor + \left\lfloor\frac{s_i - 1}{2}\right\rfloor + 1, \left\lfloor\frac{c-1}{2}\right\rfloor + \left\lfloor\frac{s_i}{2}\right\rfloor + s_i\right];$$

- *$N_{i,j}$ is supported on*

$$\left[\left\lfloor\frac{c}{2}\right\rfloor + 1, \left\lfloor\frac{c-1}{2}\right\rfloor + s_i - 1\right]$$

Proof. We refer to the more general Lemma 3.4.6 (which argues purely algebraically, along similar lines to the results in Section 3.3.2). In particular applying it successively to each i we find M_i and N_i supported on the required intervals such that

$$P - \sum_i \frac{M_i}{(1-t)^{n-1}(1-t^{s_i})^2} + \frac{N_i}{(1-t)^n(1-t^{s_i})}$$

has no poles at any of the $\mu_{s_i} \setminus \{1\}$. The difference has therefore only poles at 1 of order at most $n+1$ and can thus be written as

$$\frac{I}{(1-t)^{n+1}}.$$

It is Gorenstein symmetric as the difference of Gorenstein symmetric functions. □

3.4 Generalisations of the "curve" locus case

Following the results of the previous section, we give a conjecture.

Conjecture 3.4.1. *Let* $a_0, \ldots, a_m \in \mathbb{N}$*; for each* i, j *set* $\operatorname{hcf}(a_i, a_j) = s_{i,j}$*. Suppose all the* $s_{i,j}$ *are pairwise coprime. Let*

$$P(t) = \frac{N(t)}{\prod_i (1 - t^{a_i})}$$

be Gorenstein symmetric of degree k*. Assume that* N *has a zero of order* d *at* 1*, let* $n = m - d$ *and* $c = \mathrm{k} + n + 1$*. Then there is a partial fraction decomposition*

$$P = \frac{I}{(1-t)^{n+1}} + \sum_i \sum_{J_i} \frac{N_{i,J}}{(1-t^{a_i}) \prod_{j \in J_i} (1 - t^{s_{i,j}})}$$
$$+ \sum_{i,j : s_{i,j} > 1} \frac{M_{i,j}}{(1-t)^{n-1}(1-t^{s_{i,j}})^2} + \frac{N_{i,j}}{(1-t)^n (1 - t^{s_{i,j}})}$$

where the first sum runs over every n*-tuple* $J_i = \{a_{l_1}, \ldots, a_{l_n}\}$ *where for all* j *we have* $a_{l_j} \neq a_i$*. Moreover, for each* i, J_i *let*

$$h_{i,J} = \operatorname{hcf}\left(1 - t^{a_i}, \prod_{j \in J_i} (1 - t^{a_j})\right) = (1-t) \cdot \prod_{j \in J_i} \left(\frac{1 - t^{s_{i,j}}}{1-t}\right).$$

Each term is Gorenstein symmetric of degree k*, each* $N_{i,J}, M_{i,j}, N_{i,j}$ *is integral, and moreover:*

- $N_{i,J}$ *is supported on*

$$\left[\left\lfloor \frac{c}{2} \right\rfloor + \deg h_{i,J}, \left\lfloor \frac{c-1}{2} \right\rfloor + a_i - 1\right],$$

  a residue mod $\frac{1-t^{a_i}}{h_{i,J}}$*;*

- $M_{i,j}$ *is supported on*

$$\left[\left\lfloor \frac{c}{2} \right\rfloor + \left\lfloor \frac{s_{i,j}-1}{2} \right\rfloor + 1, \left\lfloor \frac{c-1}{2} \right\rfloor + \left\lfloor \frac{s_{i,j}}{2} \right\rfloor + s_{i,j}\right];$$

- $N_{i,j}$ *is supported on*

$$\left[\left\lfloor \frac{c}{2} \right\rfloor + 1, \left\lfloor \frac{c-1}{2} \right\rfloor + s_{i,j} - 1\right]$$

(both $M_{i,j}, N_{i,j}$ are residues mod $\frac{1-t^{s_{i,j}}}{1-t}$ and thus intervals of length at most $s_{i,j} - 1$).

Remark 3.4.2. The introduction of an extra summation over n-tuples J_i reflects that we do not know which $s_{i,j}$ appear in the denominator of a main period a_i; by writing this extra sum we are allowing all possible combinations of the $s_{i,j}$ to occur. In practice we expect many (all but one?) of the $N_{i,J}$ to be zero.

Remark 3.4.3. We note the same ideas and reasoning for shrinking n by d as in the previous section still hold.

Let us first look at some examples illustrating this case.

Example 3.4.4. Consider

$$P = \frac{1-t^{23}}{(1-t^2)(1-t^3)(1-t^5)(1-t^6)(1-t^7)}.$$

In this case the non trivial $s_{i,j}$ are $s_{0,3} = 2$ and $s_{1,3} = 3$. According to the conjecture we get

$$\begin{aligned} P = &\frac{1-4t+7t^2-4t^3+t^4}{(1-t)^4} \\ &+ \frac{-t^3+t^4-t^5}{(1-t^5)(1-t)^3} + \frac{t^6}{(1-t^6)(1-t)(1-t^2)(1-t^3)} + \frac{-t^4+t^5-t^6}{(1-t^7)(1-t)^3} \\ &+ \frac{0 \cdot t^3}{(1-t^2)^2(1-t)^2} + \frac{0}{(1-t^2)(1-t)^2} \\ &+ \frac{0 \cdot t^4}{(1-t^3)^2(1-t)^2} + \frac{-2t^3}{(1-t^3)(1-t)^2} \end{aligned}$$

where we include the terms which are equal to 0 for clarity of illustration purposes.

To work out what the contributions for each denominator, we use the computer algebra program MAGMA. The initial term agrees with P as a power series up to and including degree $\left\lfloor \frac{c}{2} \right\rfloor$. To check this, we can use the following code:

```
Q:=Rationals();
R<t>:=PolynomialRing(Q);
S<s>:=PowerSeriesRing(Q,50);
```

and then the command

```
S!(P-PI);
```

where we have defined P as the Hilbert series we are trying to parse, and PI is our guess at the initial term. We try various PI (say by running through a for loop) and win when MAGMA returns a power series (in s) which starts in degree $> \lfloor \frac{c}{2} \rfloor$.

For the non initial terms, the theorem gives us the support of the numerator in each case, and we use MAGMA to work out each coefficient individually, either by looking at the partial fraction decompositions, or running through a for loop. For the example above, to work out the contribution with denominator $(1-t^5)(1-t)^3$ we write the following code

```
P:= (1-t^23)/Denom([2,3,5,6,7]);
PI := (1-4*t+7*t^2-4*t^3+t^4)/(1-t)^4;
P5 := (t^3+t^5)/Denom([1,1,1,5]);
Q5 := t^4/Denom([1,1,1,5]);

for i,j in [-5..5] do
(P - PI - i*P5 - j*Q5)*Denom([2,3,6,7]);
[i,j];
end for;
```

To spell out what is going on, we know that the numerator is supported on the interval $[t^3, t^5]$ and the coefficient of t^3 and t^5 are equal by symmetry. We win when MAGMA returns a polynomial, since then we know our combination of i, j has taken care of the poles at $\mu_5 \setminus \{1\}$. In this case this happens for $[i, j] = [-1, 1]$ so we know that the contribution with denominator $[1, 1, 1, 5]$ is

$$\frac{-t^3 + t^4 - t^5}{(1-t^5)(1-t)^3}.$$

We then proceed in a similar way for the subsequent denominators, proceeding successively.

In fact, when some $s_{i,j} = 2$ we always have the second term $N_{i,j} = 0$. Indeed in this case $M_{i,j}$ will be a residue mod $(1+t)$ and thus consist of only one term, hence shrinking it from either side won't modify it. That is:

Proposition 3.4.5. *Suppose $s_{i,j} = 2$ for some i, j. Then in the notation of the theorem above $N_{i,j} = 0$.*

Proof of this conjecture remains beyond us, for the reasons expanded on at the start of the previous Section. However we can make some progress if we no longer worry about the number of factors in the denominator, thus allowing denominators to be products of $N+1$ giving terms of the form

$$\frac{M_i}{(1-t^{a_i})\prod_{j\neq i}(1-t^{s_{i,j}})}.$$

Now in reality these terms have at worst simple poles at the $\mu_{a_i} \setminus \cup\mu_{s_{i,j}}$, double poles at the $\mu_{s_{i,j}} \setminus \{1\}$ and poles of order $n+1$ at 1. So there should be a way of parsing each of these expressions as

$$\sum_{J_i} \frac{N_{i,J}}{(1-t^{a_i})\prod_{j\in J_i}(1-t^{s_{i,j}})}$$

as written in the conjecture. At the moment this further parsing is beyond us, although we have examples in which it works.

If however we no longer worry about the number of factors in the denominators, we do get some results.

First, we note the following lemma:

Lemma 3.4.6. *Let P be as in the above conjecture. Fix i,j such that $s = s_{i,j} > 1$. Then we can find $M_{i,j}, N_{i,j}$ as in the above result such that*

$$P - \left(\frac{M_{i,j}}{(1-t)^{n-1}(1-t^{s_{i,j}})^2} + \frac{N_{i,j}}{(1-t)^n(1-t^{s_{i,j}})}\right)$$

has no poles at $\mu_s \setminus \{1\}$.

Proof. We proceed similarly to the isolated case. Let

$$A = \prod_{l\neq i,j} \frac{1-t^{a_l}}{1-t}$$
$$L = \frac{1-t^{a_i}}{1-t^s}\cdot\frac{1-t^{a_j}}{1-t^s}$$
$$F = \left(\frac{1-t^s}{1-t}\right)^2;$$

then by construction A and L are coprime to F. Now let

$$B_{i,j} \equiv \frac{N}{(1-t)^d}\cdot \mathrm{InvMod}(AL,F) \mod F$$

so that

$$\frac{B_{i,j}}{(1-t^s)^2(1-t)^{n-1}} = \frac{(1-t)^d AFB_{i,j}}{\prod_{l=0}^m (1-t^{a_l})}$$

and

$$(1-t)^d AFB_{i,j} \equiv N \mod F.$$

Thus

$$P - \frac{B_{i,j}}{(1-t^s)^2(1-t)^{n-1}} = \frac{N-(1-t)^d AFB_{i,j}}{\prod_{l=0}^m (1-t^{a_l})}$$

which has no poles at $\mu_s \setminus \{1\}$ because F divides the numerator.

Moreover as before we can choose $B_{i,j}$ such that it is symmetric and centred at $\frac{\mathrm{k}+n+1+2s}{2}$. Now reduce $B_{i,j}$ further mod $\frac{1-t^s}{1-t}$ from both ends so write

$$B_{i,j} = M_{i,j} + N_{i,j}\frac{1-t^s}{1-t};$$

then $M_{i,j}, N_{i,j}$ are supported precisely on the intervals claimed in the conjecture, which completes the proof. □

Using this lemma, if we allow ourselves to have potentially larger denominator, we get the following theorem.

Theorem 3.4.7. *Let P be as in the statement of the main conjecture. Then there is a partial fraction decomposition*

$$P = \frac{I}{(1-t)^{m+1}} + \sum_i \frac{N_i}{(1-t^{a_i})\prod_{j\neq i}(1-t^{s_{i,j}})} + \sum_{i,j:s_{i,j}>1} \frac{M_{i,j}}{(1-t)^{n-1}(1-t^{s_{i,j}})^2} + \frac{N_{i,j}}{(1-t)^n(1-t^{s_{i,j}})};$$

moreover, for each i let

$$h_i = \mathrm{hcf}\left(1-t^{a_i}, \prod_{j\neq i}(1-t^{a_j})\right) = (1-t)\cdot\prod_{j\neq i}\left(\frac{1-t^{s_{i,j}}}{1-t}\right).$$

Each term is Gorenstein symmetric of degree k*, each $N_i, M_{i,j}, N_{i,j}$ is integral, and moreover:*

- *N_i is supported on*

$$\left[\left\lfloor\frac{c}{2}\right\rfloor + \deg h_i, \left\lfloor\frac{c-1}{2}\right\rfloor + a_i - 1\right],$$

a residue mod $\frac{1-t^{a_i}}{h_i}$*;*

- $M_{i,j}$ *is supported on*

$$\left[\left\lfloor\frac{c}{2}\right\rfloor + \left\lfloor\frac{s_{i,j}-1}{2}\right\rfloor + 1, \left\lfloor\frac{c-1}{2}\right\rfloor + \left\lfloor\frac{s_{i,j}}{2}\right\rfloor + s_{i,j}\right];$$

- $N_{i,j}$ *is supported on*

$$\left[\left\lfloor\frac{c}{2}\right\rfloor + 1, \left\lfloor\frac{c-1}{2}\right\rfloor + s_{i,j} - 1\right]$$

(both $M_{i,j}, N_{i,j}$ *are residues mod* $\frac{1-t^{s_{i,j}}}{1-t}$ *and thus intervals of length at most* $s_{i,j} - 1$*).*

Proof. We first apply the previous lemma successively for each i, j. After subtracting off all of the

$$\frac{M_{i,j}}{(1-t)^{n-1}(1-t^{s_{i,j}})^2} + \frac{N_{i,j}}{(1-t)^n(1-t^{s_{i,j}})}$$

we are left with an expression

$$P_1 = \frac{N_1}{\prod(1-t^{a_j})}$$

with only simply poles at each $\mu_{a_i} \setminus \bigcup_{j\neq i} \mu_{s_{i,j}}$. If we now fix an a_i and put

$$\begin{aligned} A &= \prod_{j\neq i} \frac{1-t^{a_j}}{1-t^{s_{i,j}}} \\ h &= \operatorname{hcf}\left(1-t^{a_i}, \prod_{j\neq i}(1-t^{a_j})\right) = (1-t)\cdot\prod_{j\neq i}\left(\frac{1-t^{s_{i,j}}}{1-t}\right) \\ F &= \frac{1-t^r}{1-t}, \end{aligned}$$

then A, F are coprime by definition, so we can let

$$N_i \equiv N_1 \cdot \operatorname{InvMod}(A, F) \mod F.$$

Now by construction

$$P_1 - \frac{N_i}{(1-t^{a_i})\prod_{j\neq i}(1-t^{s_{ij}})} = \frac{N_1 - N_i A}{\prod_{j=0}^m (1-t^{a_j})}$$

where the numerator is divisible by F. Thus what remains has no poles at $\mu_{a_i} \setminus \{1\}$

since we had already removed all the poles at all $\mu_{s_{i,j}}$.

As before we can choose the InvMod so that N_i is centred at the correct interval (trimming from both sides to preserve symmetry). Doing this successively for each i proves the theorem. $\square$

Let us now turn our attention to a general result in this case in slightly different form, namely that instead of seeing the contribution of a $\frac{1}{s}$ curve as a term whose denominator has explicitly $(1-t^s)^2$ as a factor, we see it as a sum of contributions $(1-t^s)(1-t^{a_i})$ for all of the original a_i of which s is a factor. We start with the following lemma.

Lemma 3.4.8. *Let all notation be as in the conjecture above. Let i, j be such that $r = s_{i,j} > 1$. Then there is a symmetric, integral polynomial $B_{i,j}$ calculated as a residue mod $\frac{1-t^r}{1-t}$ such that*

$$P - \frac{B_{i,j}}{(1-t)^{n-1}(1-t^{a_i})(1-t^{a_j})}$$

has at worst simple poles at $\mu_r \setminus \{1\}$.

Proof. As in the isolated case, set

$$A = \prod_{l \neq i,j} \left(\frac{1-t^{a_l}}{1-t} \right)$$
$$F = \frac{1-t^r}{1-t}.$$

Now put

$$N_{i,j} = \frac{N}{(1-t)^d} \cdot \mathrm{InvMod}(A, F)$$

and let

$$B_{i,j} \equiv N_{i,j} \mod F.$$

Then by construction we have that

$$(1-t)^d A B_{i,j} \equiv N \mod F$$

and

$$\frac{B_{i,j}}{(1-t^{a_i})(1-t^{a_j})(1-t)^{n-1}} = \frac{(1-t)^d A B_{i,j}}{\prod_{l=0}^{m}(1-t^{a_l})}$$

so that

$$P - \frac{B_{i,j}}{(1-t^{a_i})(1-t^{a_j})(1-t)^{n-1}} = \frac{N - (1-t)^d AB_{i,j}}{\prod_{l=0}^{m}(1-t^{a_l})};$$

the numerator is divisible by F (equivalently vanishes at all $\epsilon \in \mu_r \setminus \{1\}$) whereas the denominator has a zero of order (at most) 2 at these, so that

$$P - \frac{B_{i,j}}{(1-t^{a_i})(1-t^{a_j})(1-t)^{n-1}}$$

has a pole of order at most 1 on $\mu_r \setminus \{1\}$, as claimed. □

In exactly the same way as before, by choosing a suitable interval for InvMod we can assume $B_{i,j}$ is supported on an interval centred at $\frac{\mathrm{k}+n+a_i+a_j}{2}$ so that

$$\frac{B_{i,j}}{(1-t^{a_i})(1-t^{a_j})(1-t)^{n-1}}$$

is Gorenstein symmetric of degree k.

Using this lemma, we are able to prove the following main theorem.

Theorem 3.4.9. *Let $a_0, \ldots, a_m \in \mathbb{N}$; for each i, j set $\mathrm{hcf}(a_i, a_j) = s_{i,j}$. Suppose all the $s_{i,j}$ are pairwise coprime. Let*

$$P(t) = \frac{N(t)}{\prod_i (1-t^{a_i})}$$

be Gorenstein symmetric of degree k*. Then there is a partial fraction decomposition*

$$P = \frac{I}{(1-t)^{n+1}} + \sum_{i=0}^{m}\left(\frac{B_i}{(1-t^{a_i})(1-t)^n} + \sum_{j\neq i | s_{i,j}>1} \frac{N_{i,j}}{(1-t^{a_i})(1-t^{s_{i,j}})(1-t)^{n-1}}\right)$$

where all the individual terms are Gorenstein symmetric of degree k*, the B_i are calculated as residues mod $\frac{1-t^{a_i}}{1-t}$ and the $N_{i,j}$ as residues mod $\frac{1-t^{s_{i,j}}}{1-t}$ (that is, the "short support" requirement holds).*

Proof. Fix a_i. Similarly to before let

$$A_i = \prod_{j\neq i}\left(\frac{1-t^{a_j}}{1-t^{s_{i,j}}}\right)$$

$$F_i = \frac{1-t^{a_i}}{1-t}.$$

Now put

$$M_i \equiv N \cdot \mathrm{InvMod}(A_i, F_i) \mod F_i.$$

Then exactly as before

$$Q - \frac{M_i}{(1-t^{a_i})\prod_{j\neq i}(1-t^{s_{i,j}})} = \frac{N - M_i A_i}{\prod_{l=0}^{m}(1-t^{a_l})};$$

the numerator is divisible by F_i and crucially $(1-t)^d$ so that we can rewrite the expression as

$$\frac{N_i}{(1-t)\prod_{j\neq i}(1-t^{a_j})},$$

where we can assume M_i is symmetric and supported on the required interval, so that N_i is now also symmetric. Doing this inductively for each i we obtain a first parsing as follows: consider first

$$\sum_i \frac{M_i}{(1-t^{a_i})\prod_{j\neq i}(1-t^{s_{i,j}})}.$$

Subtracting this off P the only possible pole is at 1 at 1 and since $(1-t)^d$ divides each of the M_i this pole has order at most $n+1$, that is

$$P = \frac{I_1}{(1-t)^{n+1}} + \sum_i \frac{M_i}{(1-t^{a_i})\prod_{j\neq i}(1-t^{s_{i,j}})}$$

where all the terms are symmetric.

Fix i again and to lighten notation denote $r = a_i$, $M = M_i$ and for each j $s_j = s_{i,j}$. We are interested in further parsing the term

$$Q := \frac{M}{(1-t^r)\prod_{j\neq i}(1-t^{s_j})},$$

where each s_j divides $r = q\prod_j s_j$ and the s_j are pairwise coprime. Recall that $(1-t)^d$ still divides M. Thus we can apply the previous lemma to this term: we find

$$M_j \equiv M \cdot \mathrm{InvMod}\left(\prod_{l\neq j}\left(\frac{1-t^{s_l}}{1-t}\right), \frac{1-t^{s_j}}{1-t}\right)$$

such that

$$Q - \frac{M_j}{(1-t^r)(1-t^{s_j})(1-t)^{n-1}}$$

has no double pole at μ_{s_j}; as usual M_j is symmetric and calculated as a residue mod $\frac{1-t^{s_j}}{1-t}$. Doing this successively for each j what remains has only simple poles at $\mu_r \setminus \{1\}$ and a pole of order at most $n+1$ (because all the other terms do) at 1. Thus we have

$$Q = \frac{B}{(1-t^r)(1-t)^n} + \sum_j \frac{M_j}{(1-t^r)(1-t^{s_j})(1-t)^{n-1}}$$

where B is symmetric since all the terms are. Reducing B mod $\frac{1-t^r}{1-t}$ we deduce

$$Q = \frac{I_2}{(1-t)^{n+1}} + \frac{B}{(1-t^r)(1-t)^n} + \sum_j \frac{M_j}{(1-t^r)(1-t^{s_j})(1-t)^{n-1}}$$

where each term is integral, symmetric, of short support. Doing this for each a_i in term and then combining the terms with initial denominator proves the result. $\square$

Remark 3.4.10. Note that the terms

$$\frac{M_i}{(1-t^{a_i})\prod_{j\neq i}(1-t^{s_{i,j}})}$$

in the first parsing may well have denominators with more than $n+1$ factors. At the moment there seems to be no way around this and no way of predicting which $s_{i,j}$ could occur.

Remark 3.4.11. The second parsing represents a slight departure from our conjectured attempts to parse the series "geometrically"; in general there is no obvious contribution from a $\frac{1}{s}$ curve, since there need not be any term with denominator featuring $(1-t^s)$ or $(1-t^s)^2$; rather there will be a number of terms with denominators $(1-t^s)(1-t^{a_i})(1-t)^{m-1}$ which we view as all contributing to the curve rather than the individual dissidents points. The only contribution we attribute to the dissident point is the one with denominator $(1-t^{a_i})(1-t)^n$.

On the other hand the strategy used during the proof of this theorem is obviously repeatable so that similar results must happen without restricting ourselves to the case of at most pairs of the a_i having common prime divisor. We discuss possible generalisations further in Chapter 5.

Chapter 4

The curve orbifold locus case revisited

In this Chapter, we revisit the case of the curve orbifold locus of the previous Chapter, focusing more on the geometric interpretations especially of the curve locus contribution. We build on the work of [Zhou, 2011]. We restate the theorem from the previous Chapter but offer further descriptions of the A_C and B_C terms. More precisely we want to understand what fractional parts of the curve contributions we need to assign to its dissident points to recover the P_{orb} contributions for these points from their Dedekind sums.

4.1 Main theorem and examples

Let us start by stating the theorem.

Theorem 4.1.1. *Let (X, D) be a quasismooth, projectively Gorenstein polarised orbifold of dimension $n \geq 3$; we again denote its canonical weight by k_X and its coindex by c. Suppose X has orbifold strata*

$$\mathcal{B} = \{\text{points } Q \text{ of type } \tfrac{1}{r}(a_1, \ldots, a_n)\} \text{ (not necessarily isolated)}$$
$$\mathcal{B}' = \{\text{curves } C \text{ of transverse type } \tfrac{1}{s}(b_1, \ldots, b_{n-1})\ \}.$$

Then its Hilbert series has the form

$$P_X(t) = P_I(t) + \sum_{Q \in \mathcal{B}} P_{\mathrm{orb}}(Q, \mathrm{k}_X) + \sum_{C \in \mathcal{B}'} (A_C + B_C) \tag{4.1}$$

where the terms on the right are as follows:

1. *P_I deals with the first $\lfloor \frac{c}{2} \rfloor$ plurigenera;*

2. *for each $Q \in \mathcal{B}$ we set $s_i = \operatorname{hcf}(a_i, r)$, $A = \prod_i \frac{1-t^{a_i}}{1-t^{s_i}}$ and $F = \frac{1-t^r}{h}$ where $h = \operatorname{hcf}\left(\prod(1-t^{a_i}), 1-t^r\right) = (1-t)\prod\left(\frac{1-t^{s_i}}{1-t}\right)$; then*

$$P_{\mathrm{orb}}(Q, \mathrm{k}_X) = \frac{\operatorname{InvMod}(A, F, \gamma)}{(1-t^r)\prod_i(1-t^{s_i})},$$

where $\gamma = \lfloor \frac{c}{2} \rfloor + \deg h$ so that the numerator is supported in

$$\left[\left\lfloor \frac{c}{2} \right\rfloor + \deg h, \left\lfloor \frac{c-1}{2} \right\rfloor + r - 1\right];$$

3. $$A_C = \frac{D_C}{(1-t)^{n-1}(1-t^s)^2}$$

where D_C is integral, palindromic, supported on

$$\left[\left\lfloor \frac{c}{2} \right\rfloor + \left\lfloor \frac{s-1}{2} \right\rfloor + 1, \left\lfloor \frac{c-1}{2} \right\rfloor + \left\lfloor \frac{s}{2} \right\rfloor + s - 1\right]$$

i.e. on precisely the same interval as the numerator of

$$\frac{P_{\mathrm{orb}}(\frac{1}{s}(b_1, \ldots, b_{n-1}, \mathrm{k}_X + s)}{1-t^s};$$

4. $$B_C = \frac{N_C}{(1-t)^n(1-t^s)}$$

where N_C depends on the isotypical components of the normal bundle as modified by the dissident points, and is supported on the interval

$$\left[\left\lfloor \frac{c}{2} \right\rfloor + 1, \left\lfloor \frac{c-1}{2} \right\rfloor + s - 1\right].$$

The numerators N_C, D_C and $\operatorname{InvMod}(A, F, \gamma)$ are all integral and palindromic, so that each individual term is Gorenstein symmetric of degree k_X.

Notice that the existence of the partial fraction decomposition follows immediately from the theorems of the previous chapter. What changes here is that we make a start on giving geometric interpretations (and an alternative proof) for the quantities A_C, B_C.

Remark 4.1.2. 1. The P_{orb} contributions from dissident points contain terms with denominators $(1-t^{s_i})^2(1-t)^{n-1}$ and $(1-t^{s_i})(1-t)^n$ that seem native to the curves of type $\frac{1}{s_i}$. This is because a dissident $\frac{1}{r}$ point on an orbicurve forces its degree and isotypical components to become fractional (with denominator divisible by r) which introduces fractional terms into our Hilbert series parsing. Attributing these fractional terms to the dissident points is essentially the same process as attributing a fractional part of the global growth contribution to the orbifold points in the isolated case (in slogan form $P_{\mathrm{orb}} = P_{\mathrm{per}} + P_{\mathrm{gro}}$). It allows us to keep our coefficients integral, but introduces a choice into our Hilbert series parsing. For the curve orbifold locus, the choice we make may seem logical, but in the higher dimensional cases this could be a stumbling block.

2. Continuing with this reasoning, we see that the numerator D_C is of the form

$$\mathrm{InvMod}\left(\prod_i \left(\frac{1-t^{b_i}}{1-t}\right), \frac{1-t^s}{1-t}, \left\lfloor\frac{c}{2}\right\rfloor + \left\lfloor\frac{s-1}{2}\right\rfloor + 1\right),$$

but we may modify some of its coefficients. This modification depends on the degree of the curve, and any dissident points on it.

3. In fact, both terms corresponding to curve contributions depend not only on properties of the curve, but also its dissident points, so curves of the same type and same degree will usually give different contributions towards the Hilbert series.

We note the following two special cases of curve contributions.

Proposition 4.1.3. *1. If C is a $\frac{1}{2}(1,\dots,1)$ curve then $B_C = 0$ and the modification done in A_C is to multiply each term by the same integer (with the notation as in the following sections, g_C is an integer – this is because the bite off terms are calculated mod $1+t$).*

2. If C is a $\frac{1}{s}$ curve without dissident points, then

$$A_C = s\cdot\deg C\cdot\frac{P_{\mathrm{orb}}(C,\mathrm{k}_X+s)}{1-t^s},$$

that is the modification is to multiply each term by $s\deg C$.

Let us now revisit the examples seen in the previous section with the above theorem in mind.

Example 4.1.4. Consider $X_{13} \subset \mathbb{P}(1,1,3,3,5)$ a Calabi-Yau 3-fold, with $\mathrm{k}_X = 0$, $c = 4$ and first plurigenera $P_0 = 1, P_1 = 2, P_2 = 3$ so that

$$P_I = \frac{1 - 2t + t^2 - 2t^3 + t^4}{(1-t)^4}.$$

Recall the singularities are an orbipoint $\frac{1}{5}(1,1,3)$ and an orbicurve $\frac{1}{3}(1,2)$ of degree $\frac{1}{3}$. The contribution from the orbipoint is

$$P_{\mathrm{orb}}(P_u, 0) = \frac{t^3 + t^5}{(1-t^5)(1-t)^3}$$

whilst since the orbicurve has no dissident points, its A_C contribution is modified by $3 \cdot \frac{1}{3} = 1$ so is just

$$A_C = \frac{P_{\mathrm{orb}}(C, 3)}{1 - t^3} = \frac{-t^4}{(1-t^3)^2(1-t)^2}.$$

The B_C contribution has denominator $(1-t^3)(1-t)^3$ so that the short support assumption together with Gorenstein symmetry shows that its numerator is supported only at t^3. An easy check then shows that

$$B_C = \frac{3t^3}{(1-t^3)(1-t)^3}$$

and we recover

$$P = \frac{1 - t^{13}}{(1-t)^2(1-t^3)^2(1-t^5)} = P_I + P_{\mathrm{orb}}(P_u) + A_C + B_C.$$

Example 4.1.5. Consider now $X_{13} \subset \mathbb{P}(1,1,2,3,3,4)$ a Fano 4-fold with $\mathrm{k}_X = -1$, $c = 4$ and first plurigenera are $P_0 = 1, P_1 = 2, P_2 = 4$ so that

$$P_I = \frac{1 - 3t + 4t^2 - 3t^3 + t^4}{(1-t)^5}.$$

It has an orbipoint $\frac{1}{4}(1,2,3,3)$ at P_v two orbicurves C_1 of type $\frac{1}{2}(1,1,1)$ and degree $\frac{1}{4}$ and C_2 of type $\frac{1}{3}(1,1,2)$ and degree $\frac{1}{3}$. On this occasion the orbipoint is a dissident point on C_1, whilst C_2 has no dissident points. The contribution from P_v is

$$P_{\mathrm{orb}}(P_v, -1) = \frac{-t^4}{(1-t^4)(1-t^2)(1-t)^2}.$$

For the orbicurve C_1, we know that $B_{C_1} = 0$ whilst

$$A_{C_1} = \delta_{C_1} \cdot \frac{P_{\mathrm{orb}}(C_1, 1)}{1-t^2} = \delta_{C_1} \cdot \frac{-t^3}{(1-t^2)^2(1-t)^3};$$

we calculate easily enough that $\delta_{C_1} = 1$ (either by trial and error, or by a "bite off" calculation as explained later – this can be done as all the terms have small degree).

For the orbicurve C_2, there are no dissident points, so the A_{C_2} contribution remains unmodified again and is just

$$A_{C_2} = \frac{P_{\mathrm{orb}}(C_2, 2)}{1-t^3} = \frac{-t^4}{(1-t^3)^2(1-t)^3}.$$

The B_{C_2} contribution has denominator $(1-t^3)(1-t)^4$ so that again the short support assumption together with Gorenstein symmetry shows that its numerator is supported only at t^3. An easy check then shows that

$$B_{C_2} = \frac{2t^3}{(1-t^3)(1-t)^3}$$

and we recover

$$P = \frac{1-t^{13}}{(1-t)^2(1-t^2)(1-t^3)^2(1-t^4)} = P_I + P_{\mathrm{orb}}(P_v) + A_{C_1} + A_{C_2} + B_{C_2}.$$

Example 4.1.6. Consider now $X_{13} \subset \mathbb{P}(1,1,2,3,6)$, a Calabi-Yau 3 fold with $P_0 = 1, P_1 = 2, P_2 = 4$ so

$$P_I = \frac{1-2t+2t^2-2t^3+t^4}{(1-t)^4}.$$

X has an orbipoint $\frac{1}{6}(1,2,3)$ lying at the intersection of the two orbicurves C_1 of type $\frac{1}{2}(1,1)$ and degree $\frac{1}{6}$ and C_2 of type $\frac{1}{3}(1,2)$ and degree $\frac{1}{6}$. By the previous calculations, we have

$$\begin{aligned} A_{C_1} &= \delta_{C_1} \cdot \frac{-t^3}{(1-t^2)^2(1-t)^2} \\ B_{C_1} &= 0 \\ A_{C_2} &= \delta_{C_2} \cdot \frac{-t^4}{(1-t^3)^2(1-t)^2} \end{aligned}$$

for suitable modification terms δ_{C_i}. Again, B_{C_2} has denominator $(1-t^3)(1-t)^3$ so its numerator is supported in t^3. In this case we calculate

$\delta_{C_1} = \delta_{C_2} = 0$ either by "bite off" type calculations or by noticing in the partial fraction decomposition that the contribution

$$P_{\mathrm{orb}}(P_u) = \frac{t^6}{(1-t)^6(1-t)(1-t^2)(1-t^3)}$$

has taken care of all terms whose denominator have factor $(1-t^2)^2$ or $(1-t^3)^2$ in the partial fraction decomposition of P. Thus we see that

$$B_{C_2} = \frac{1 \cdot t^3}{(1-t^3)(1-t)^3}$$

and again check that

$$P = \frac{1-t^{13}}{(1-t)^2(1-t^2)(1-t^3)(1-t^6)} = P_I + P_{\mathrm{orb}}(P_u) + A_{C_1} + A_{C_2} + B_{C_2}.$$

Remark 4.1.7. We will see later following [Zhou, 2011] that for a dissident point Q of type $\frac{1}{r}(a_1, \dots, a_n)$ lying on an orbicurve of type $\frac{1}{s}$ where $s = s_i = \mathrm{hcf}(r, a_i)$ for some i the point "bites off"

$$\frac{s}{r} \,\mathrm{InvMod}\left(\prod_j \frac{1-t^{a_j}}{1-t^{s_j}}, \frac{1-t^r}{\mathrm{hcf}\left(\prod(1-t^{a_j}), 1-t^r\right)}, \gamma \right) \cdot \prod_{j \neq i} \frac{1-t^{a_j}}{1-t^{s_j}} \mod \frac{1-t^s}{1-t}$$

from the curve P_{orb} contribution, where the inverse mod is supported on an interval of length $r-2$ or $r-1$. Sometimes however it is easier to use our method of forcing all the numerators to have short support, where finding the various terms only involves solving a "crossword puzzle" by looking at partial fractions decompositions of the terms involved.

Example 4.1.8. Let $X_{85} \subset \mathbb{P}(1,1,14,17,21,32)$ be a hypersurface where the WPS has coordinates x, y, z, t, u, v. X is a Fano 4-fold with plurigenera $P_0 = 1$, $P_1 = 2, P_2 = 3$ so that the initial term is

$$P_I = \frac{1-3t+3t^2-3t^3+t^4}{(1-t)^5}.$$

The orbicurves are

$$C_1 = \mathbb{P}(14,32)_{\langle z,v\rangle} = \tfrac{1}{2}(1,1,1)$$
$$C_2 = \mathbb{P}(14,21)_{\langle z,u\rangle} = \tfrac{1}{7}(1,3,4)$$

of degrees $\frac{1}{224}, \frac{1}{42}$ respectively.

X has orbipoints

$$
\begin{aligned}
P_u &= \frac{1}{21}(1,11,14,17) \in C_2, \\
P_v &= \frac{1}{32}(1,1,14,17) \in C_1, \\
P_z &= \frac{1}{14}(1,3,4,7) \in C_1 \cap C_2.
\end{aligned}
$$

The point contributions give

$$
\begin{aligned}
P_{\text{orb}}(P_z) &= \frac{-t^{11}+2t^{12}-t^{13}}{(1-t^{14})(1-t)^2(1-t^2)(1-t^7)} \\
P_{\text{orb}}(P_u) &= \frac{t^9-t^{12}+t^{13}+t^{17}-t^{18}+t^{21}}{(1-t^{21})(1-t)^3(1-t^7)} \\
P_{\text{orb}}(P_v) &= \frac{t^5+t^9+t^{13}+t^{17}-t^{18}+t^{19}+t^{23}+t^{27}+t^{31}}{(1-t^{32})(1-t)^3(1-t^2)}.
\end{aligned}
$$

The curve contributions from C_1 are $A_{C_1} = \delta_{C_1} \cdot \frac{-t^3}{(1-t^2)^2(1-t)^3}$ and $B_{C_1} = 0$. The curve contributions from C_2 are

$$
\begin{aligned}
A_{C_2} &= \frac{\delta_{C_2,1}(t^6+t^{10})+\delta_{C_2,2}(t^7+t^9)+\delta_{C_2,3}t^8}{(1-t^7)^2(1-t)^3} \\
B_{C_2} &= \frac{N_{C_2,1}(t^3+t^7)+N_{C_2,2}(t^4+t^6)+N_{C_2,3}t^5}{(1-t^7)(1-t)^4}
\end{aligned}
$$

so that we have a total of 7 coefficients we need to calculate. We can do this by looking at the partial fraction decompositions of P and the various terms involved, or by doing the bite off calculations and then folding the extreme terms into the correct support.

In this case the points P_z, P_v bite off $\frac{4}{7}, -\frac{9}{16}$ from C_1 respectively, so that $\delta_{C_1} = 0$.

For C_2 we get that P_z bites off $t^2+\frac{3}{2}+t^{-2}$ and P_u bites off $-t^2-2t-\frac{7}{3}-2t^{-1}-t^{-2}$ so that the $P_{\text{orb}}(C_2)$ contribution is multiplied by $g_{C_2} = 2t^{-1}+1+2t$ to give

$$
\frac{2t^5+3t^6+t^7+3t^8+t^9+3t^{10}+2t^{11}}{(1-t)^3(1-t^7)^2}
$$

and then the normal bundle contribution is

$$
\frac{2t^3+t^4-2t^5+t^6+2t^7}{(1-t)^4(1-t^7)}.
$$

Folding the degree contribution into the shortest support attributes an additional $2t^5$ to the normal bundle contribution so that in the end we have

$$A_{C_2} = \frac{t^6 - t^7 + t^8 - t^9 + t^{10}}{(1-t^7)^2(1-t)^3}$$
$$B_{C_2} = \frac{2t^3 + t^4 + t^6 + 2t^7}{(1-t^7)(1-t)^4},$$

i.e. $\delta_{C_1} = 0, \delta_{C_2,1} = 1, \delta_{C_2,2} = -1, \delta_{C_2,,3} = 1, N_{C_2,1} = 2, N_{C_2,,2} = 1, N_{C_2,,3} = 0.$

Example 4.1.9. Consider now the Fano 4-fold $X_{152} \subset \mathbb{P}(2,3,5,17,50,76)$ with coordinates x, y, z, t, u, v. It has plurigenera $P_0 = 1, P_1 = 0, P_2 = 1$ so that the initial term is

$$P_I = \frac{1 - 5t + 11t^2 - 5t^3 + t^4}{(1-t)^5}.$$

It has two isolated orbifold points $P_y = \frac{1}{3}(1,2,2,2)$ and $P_t = \frac{1}{17}(2,3,5,8)$ contributing

$$\frac{-t^3}{(1-t^3)(1-t)^4}$$

and

$$\frac{-t^3 - t^4 - t^5 - 2t^6 + t^8 - 4t^9 + 4t^{10} + \text{sym} \cdots - t^{17}}{(1-t^{17})(1-t)^4}$$

towards the Hilbert series, respectively.

X also has a dissident point $P_u = \frac{1}{50}(3,5,17,26)$ lying at the intersection of the two orbicurves $C_1 = \mathbb{P}(2,50,76)_{<x,u,v>} \cap X = \frac{1}{2}(1,1,1)$ of degree $\frac{1}{25}$ and $C_2 = \mathbb{P}(5,50)_{<z,u>} = \frac{1}{5}(1,2,3)$ of degree $\frac{1}{50}$. In this case, $B_{C_1} = 0$ (as C_1 is a $\frac{1}{2}$ curve). A_{C_1}, A_{C_2} are modifications of

$$\frac{-t^3}{(1-t^2)^2(1-t)^3}, \frac{t^5 - t^6 + t^7}{(1-t^5)^2(1-t)^3}$$

respectively. The denominator of B_{C_2} is $(1-t^5)(1-t)^4$ and its numerator is supported in $[3,4,5]$ by the short support and Gorenstein symmetry properties. We thus have 1,2,2 coefficients to find for $A_{C_1}, A_{C_2}, B_{C_2}$ respectively. As before we can find these coefficients by trial and error, looking at the partial fraction decompositions of P and

$$\frac{t^3}{(1-t^2)^2(1-t)^3}, \frac{t^5 + t^7}{(1-t^5)^2(1-t)^3}, \frac{t^6}{(1-t^5)^2(1-t)^3},$$
$$\frac{t^3 + t^5}{(1-t^5)(1-t)^4}, \frac{t^4}{(1-t^5)(1-t)^4}$$

and obtaining the former as linear combinations of the latter, or by doing the bite off calculations. After some work, we obtain

$$\begin{aligned} A_{C_1} &= \frac{-3t^3}{(1-t^2)^2(1-t)^3} \\ A_{C_2} &= \frac{6t^5 - 3t^6 + 6t^7}{(1-t^5)^2(1-t)^3} \\ B_{C_2} &= \frac{-4t^3 - t^4 - 4t^5}{(1-t^5)(1-t)^4}. \end{aligned}$$

In terms of bite-offs, P_u bites off $-\frac{73}{25}$ from C_1 so that $\delta_{C_1} = 3$. For C_2, P_u bites off $3t^{-1} + \frac{1}{10} + 3t$ so that $g_{C_2} = -3t^{-1} - 3t$. The original numerator of the degree contribution for C_2 is $-3t^4 + 3t^5 - 6t^6 + 3t^7 - 3t^8$ so that the folding requires attributing a further $-3t^4$ to the normal bundle contribution which gives the result as above.

The modifications to the $P_{\mathrm{orb}}(C)$ contributions as well as the B_C contribution both depend on the topological properties of (X, C) (degree, normal bundle) which aren't a priori directly extractable from the numerical data (plurigenera, singularity basket). Thus, unlike the isolated case, we cannot calculate a unique Hilbert series candidate based only on the numerical model of a variety.

However, each modification and each B_C contribution is given by only a few typically small integers, and for each choice of such integers, the resulting series has the right symmetry and poles (with given order). In other words, whilst we can't calculate the contributions A_C and B_C of a curve by its type, for each type of curve we can draw up a list of potential A_C, B_C (this list is hypothetically infinite, so that bounds on the modifications would be a useful result; at the moment we introduce an arbitrary cut-off point), each giving a potential Hilbert series. Studying this series in the same way as in the previous section we can then see whether any of these are constructible.

Example 4.1.10. We look for Calabi-Yau 3-folds with two orbicurves C_1, C_2 of type $\frac{1}{2}$, $\frac{1}{3}$, intersecting at a dissident point $P = \frac{1}{6}(1, 2, 3)$. We calculate

$$\begin{aligned} P_{\mathrm{orb}}(P, 0) &= \frac{t^6}{(1-t)(1-t^2)(1-t^3)(1-t^6)} \\ A_{C_1} &= \delta_{C_1} \cdot \frac{-t^3}{(1-t^2)^2(1-t)^2} \\ A_{C_2} &= \delta_{C_2} \cdot \frac{-t^4}{(1-t^3)^2(1-t)^2}. \end{aligned}$$

Moreover we have $B_{C_1} = 0$ and by the short support assumption

$$B_{C_2} = N_{C_2} \cdot \frac{t^3}{(1-t^3)(1-t)^3}.$$

The initial term depends on P_1, P_2 and each choice of $P_1, P_2, \delta_{C_1}, \delta_{C_2}, N_{C_2}$ (all integers) gives a potential Hilbert series.

Fix for now $P_1 = 1$ and $P_2 = 2$ so that

$$P_I = \frac{1 - 3t + 4t^2 - 3t^3 + t^4}{(1-t)^4}.$$

The graded ring R has exactly one generator in degrees 1 and 2, and because of the singularity assumption it must have also at least one generator of degree divisible by 3, and another of degree divisible by 6. We therefore set

$$\begin{aligned} P &= P_I + P_{\text{orb}}(P, 0) + \delta_{C_1} \cdot \frac{-t^3}{(1-t^2)^2(1-t)^2} \\ &+ \delta_{C_2} \cdot \frac{-t^3}{(1-t^2)^2(1-t)^2} + N_{C_2} \cdot \frac{t^3}{(1-t^3)(1-t)^3} \\ Q &= (1-t)(1-t^2)(1-t^3)(1-t^6)P, \end{aligned}$$

where we let $\delta_{C_1}, \delta_{c_2}, N_{C_2}$ run over a sequence of integers, to recover a series of "Hilbert almost numerators". We then get the following data:

- For $\delta_{C_1} = \delta_{C_2} = N_{C_2} = 0$, $Q = 1 - t^6 + t^{12}$ so that $(1+t^6)(1-t^{18})Q = 1 - t^{36}$. This gives the Hilbert series of $X_{36} \subset \mathbb{P}(1, 2, 3, 12, 18)$ which has the given singularities;
- For $\delta_{C_1} = 0, \delta_{C_2} = 2, N_{C_2} = 1$, $Q = 1+t^3+t^6+t^9+t^{12}$ so that $(1-t^3)Q = 1-t^{15}$, giving the Hilbert series of $X_{15} \subset \mathbb{P}(1, 2, 3, 3, 6)$;
- For $\delta_{C_1} = 0, \delta_{C_2} = 4, N_{C_2} = 2$, $Q = 1 + 2t^3 + 3t^6 + 2t^9 + t^{12}$ so that $(1-t^3)^2 Q = 1 - 2t^9 + t^{18}$, giving the Hilbert series of $X_{9,9} \subset \mathbb{P}(1, 2, 3, 3, 3, 6)$;
- For $\delta_{C_1} = 1, \delta_{C_2} = 0, N_{C_2} = 1$, $Q = 1 + t^4 + t^8 + t^{12}$ so that $(1-t^4)Q = 1 - t^{16}$, giving the Hilbert series of $X_{16} \subset \mathbb{P}(1, 2, 3, 4, 6)$.

In each of these examples we can check that the "bite off" calculations recover the given δ_{C_i}, N_{C_2}. The output $\delta_{C_1} = 2, \delta_{C_2} = 0, N_{C_2} = 2$ gives $Q = 1+2t^4+t^6+2t^8+t^{12}$ so that $(1-t^4)^2(1-t^6)Q = 1 - t^8 - 2t^{10} - 2t^{12} + 2t^{14} + 2t^{16} + t^{18} - t^{26}$, which gives a potential codimension 3 Pfaffian construction $X_{8,10,10,12,12} \subset \mathbb{P}(1,2,3,4,4,6,6)$. We therefore look for a matrix of degrees

$$\begin{pmatrix} 8 & 8 & 6 & 6 \\ & 6 & 4 & 4 \\ & & 4 & 4 \\ & & & 2 \end{pmatrix}.$$

Let x, y, z, t, u, v, s be the coordinates of weights $1,2,3,4,4,6,6$ respectively. The difficulties are caused by:

1. the line of singularities $L = \mathbb{P}(4,4)_{\langle t,u \rangle}$ cannot intersect X;
2. the line $\Lambda = \mathbb{P}(6,6)_{\langle v,s \rangle}$ must intersect X at one point, whilst the equations of degree 12 will be quadratic along that line.

After a fair amount of work, we find for example

$$M = \begin{pmatrix} u^2 & x^8 & v + z^2 & s \\ & v & t & y^2 + x^4 \\ & & u & t + x^4 \\ & & & y \end{pmatrix}.$$

Then P_s is a point of type $\frac{1}{6}(1,2,3)$ whilst the lines P_sP_z and P_sP_y are of type $\frac{1}{3}(1,2)$ and $\frac{1}{2}(1,1)$ respectively. Notice that:

- $\mathrm{Pf}_5 |_\Lambda = v^2$ so that $\Lambda \cap X = P_s$;
- $\mathrm{Pf}_5 |_L = u^3$ whilst $\mathrm{Pf}_1 |_L = t^2$ so that $L \cap X = \emptyset$ as required.

4.2 First parsing

We now turn our attention to the proof of the theorem, so from now on assume (X, D) satisfies the hypotheses of the theorem. For simplicity we restate the intermediate results proved in the previous section, before turning our attention to the curve locus in more detail.

The first parsing was as follows:

Theorem 4.2.1. *With (X, D) as in the theorem*

$$P_X(t) = I(t) + \sum_{P \in \mathcal{B}} P_{\mathrm{per}}(P)(t) + \sum_{C \in \mathcal{B}'} P_{\mathrm{per}}(C)(t)$$

where

1. $$I(t) = \frac{A_I(t)}{(1-t)^{n+1}}$$

   where A_I is rational, of degree c is $\mathrm{k}_X \geq 0$ or n otherwise;

2. *for P of type $\frac{1}{r}(a_1, \dots, a_n)$ we have*

   $$P_{\mathrm{per}}(P)(t) = \frac{\sum_{i=1}^{r-1} (\sigma_{r-i} - \sigma_0)(P) t^i}{1-t^r}$$

   with the σ_j defined as in section 2;

3. *for a curve C of type $\frac{1}{s}(b_1, \dots b_{n-1})$ we have*

   $$P_{\mathrm{per}}(C)(t) = \left(\frac{st^s \Delta}{(1-t^s)^2} + \frac{t\Delta'}{1-t^s} + \frac{\mathrm{k}_X}{2} \cdot \frac{\Delta}{1-t^s} \right) \deg C + \sum_{j=1}^{n-1} \frac{\sum_{i=0}^{s-1} \delta_{i,j} t^i}{1-t^s} \frac{\deg \gamma_j}{2}$$

   where $\Delta = \Delta\left(\frac{1}{s}(b_1, \dots, b_{n-1})\right)$ as in section 2, $\Delta' = \frac{\mathrm{d}\Delta}{\mathrm{dt}}$, the γ_j are the Chern roots of the normal bundle of C and

   $$\delta_{i,j} = \frac{1}{s} \sum_{\epsilon \in \mu_s \setminus 1} \frac{\epsilon^i (1 + \epsilon^{-b_j})}{(1-\epsilon^{-b_j}) \prod_l (1 - \epsilon^{-b_l})} = \sigma_{s-i} \frac{1+\epsilon^{-b_j}}{1-\epsilon^{-b_j}}.$$

Remark 4.2.2. As well as following from a resolution of singularities similar to those in Buckley [2003] we could also get this theorem from the stack methods of Zhou [2011].

4.3 Contributions from orbipoints

Recall the results for a point of type $\frac{1}{r}(a_1, \dots, a_n)$ not necessarily isolated, where $s_i = \mathrm{hcf}(a_i, r)$ for each i, and σ_i and Δ are as in the previous section, and we have put

$$A = \prod_{i=1}^{n} (1 - t^{a_i})$$
$$h = \mathrm{hcf}(A, 1-t^r) = \prod_i \left(\frac{1-t^{s_i}}{1-t} \right) \cdot (1-t)$$
$$F = \frac{1-t^r}{h}.$$

Proposition 4.3.1. *With the notation as above let $d = \deg F$ and γ be any integer. Then*

$$\begin{aligned}\prod(1-t^{s_i})\cdot\Delta &\equiv \operatorname{InvMod}\left(\prod\left(\frac{1-t^{a_i}}{1-t^{s_i}}\right), F, \gamma+1\right)\\ &= \operatorname{InvMod}\left(\frac{A}{\prod(1-t^{s_i})}, F, \gamma+1\right)\\ &= \sum_{j=\gamma+1}^{\gamma+d}\theta_j t^j\end{aligned}\tag{4.2}$$

where the θ_j are integers calculated from Dedekind sums.

Proposition 4.3.2. *With notation as above*

$$P_{\mathrm{orb}}\left(\tfrac{1}{r}(a_1,\dots,a_n)\right) = \frac{\Delta}{1-t^r} + \frac{A(t)}{(1-t)^{n+1}} + \sum_{j=1}^{n}\frac{Q_j(t)}{(1-t^{s_j})^2(1-t)^{n-1}} + \frac{R_j(t)}{(1-t^{s_j})(1-t)^n}.\tag{4.3}$$

Moreover, A, Q_j, R_j all have rational coefficients.

Proposition 4.3.3. *P_{orb} is Gorenstein symmetric of degree k_X and with integral numerator.*

4.4 Contributions from orbicurves

We now look at the curve contributions more geometrically. Let C be an orbicurve of type $\frac{1}{s}(b_1,\dots,b_{n-1})$. Recall that the contribution from orbicurves is

$$P_{\mathrm{per}}(C)(t) = \left(\frac{st^s\Delta}{(1-t^s)^2} + \frac{t\Delta'}{1-t^s} - \frac{\mathrm{k}_X}{2}\cdot\frac{\Delta}{1-t^s}\right)\deg C + \sum_{j=1}^{n-1}\frac{\sum_{i=0}^{s-1}\delta_{i,j}t^i}{1-t^s}\frac{\deg\gamma_j}{2}$$

where $\Delta = \Delta\left(\frac{1}{s}(b_1,\dots,b_n)\right)$ and

$$\delta_{i,j} = \sigma_{s-i}\frac{1+\epsilon^{-b_j}}{1-\epsilon^{-b_j}}.$$

By a similar reasoning as in the previous section we deal with the first term.

Proposition 4.4.1. *Keeping our notation the same as before, we have*

$$\frac{P_{\mathrm{orb}}(C, \mathrm{k}_X + s)}{1-t^s} = \frac{t^s\Delta}{(1-t^s)^2} + \frac{G_C}{1-t^s} + \frac{H_C}{(1-t)^{n+1}}$$

where the numerators G_C and H_C have rational coefficients.

Proof. This follows from the same result in the isolated case. Working mod $\frac{1-t^s}{1-t}$ we have

$$(1-t^n)t^s\Delta \equiv t^s \operatorname{InvMod}\left(\prod\left(\frac{1-t^{b_i}}{1-t}\right), \frac{1-t^s}{1-t}, \gamma\right).$$

Now the t^s factor just folds the InvMod into the correct interval of support for the numerator of $P_{\mathrm{orb}}(C, \mathrm{k}_X + s)$. Thus we get

$$\frac{P_{\mathrm{orb}}(C, \mathrm{k}_X + s)}{1-t^s} - \frac{t^s\Delta}{(1-t^s)^2} = \frac{L}{(1-t^s)(1-t)^n}$$

for some L with rational coefficients, whence the result by splitting L into its contribution with single poles at sth roots of unity, and possibly multiple poles at 1, using entirely similar methods to those used in the previous Chapter. □

Grouping together the first three terms in $P_{\mathrm{per}}(C)$ we have

Lemma 4.4.2.

$$P_{\mathrm{per}}(C) = \frac{N(t)}{(1-t)^{n-1}(1-t^s)^2}\deg C + \sum_{j=1}^{n-1}\frac{N_j(t)}{(1-t)^n(1-t^s)}\frac{\deg\gamma_j}{2} + \frac{I_C(t)}{(1-t)^{n+1}}$$

where

- *N is palindromic (i.e. symmetric) with rational coefficients, supported on*

$$\left[\left\lfloor\frac{c}{2}\right\rfloor + 1, \left\lfloor\frac{c-1}{2}\right\rfloor + 2s - 2\right];$$

- *each N_j is integral, palindromic, supported on*

$$\left[\left\lfloor\frac{c}{2}\right\rfloor + 1, \left\lfloor\frac{c-1}{2}\right\rfloor + s - 1\right];$$

- *I_C has rational coefficients.*

Proof. First let

$$C_j = \sum_{i=0}^{s-1}\delta_{i,j}t^i = \Delta\frac{1+\epsilon^{-b_j}}{1-\epsilon^{-b_j}}.$$

Then notice that

$$C_j(1-t^{b_j})\prod_{l=1}^{n-1}(1-t^{b_l}) \equiv 1 + t^{b_j} \mod \frac{1-t^s}{1-t}$$

by evaluating both sides at all $\epsilon \in \mu_s \setminus 1$ as before. Then as usual we can reduce $(1-t)^n C_j \mod \frac{1-t^s}{1-t}$ and then move it to the correct support to get an N_j in the correct support such that

$$N_j - (1-t)^n C_j = \frac{1-t^s}{1-t} A_j$$

or equivalently

$$\frac{N_j}{(1-t)^n(1-t^s)} = \frac{C_j}{1-t^s} + \frac{A_j}{(1-t)^{n+1}}.$$

Then moreover we have

$$N_j \left(\frac{1-t^{b_j}}{1-t}\right) \prod_{l=1}^{n-1} \left(\frac{1-t^{b_l}}{1-t}\right) \equiv 1 + t^{b_j} \mod \frac{1-t^s}{1-t};$$

thus

$$N_j \equiv (1+t^{b_j}) \left(\frac{1-t^{a_j b_j}}{1-t^{b_j}}\right) \prod_l \left(\frac{1-t^{a_l b_l}}{1-t^{b_l}}\right) \mod \frac{1-t^s}{1-t},$$

where the a_j were chosen such that $a_j b_j \equiv 1 \mod s$. The right hand side is palindromic and with integral coefficients, and this remains after taking residues and moving to the correct support similarly to before.

Now focus our attention on the first three terms in the sum of P_{per}: putting them over a common factor and regrouping terms together gives

$$\frac{\sum_{i=0}^{s-1}\left(i - \frac{\mathrm{k}_X}{2}\right)\sigma_{s-i}t^i + \sum_{i=0}^{s-i}\left(s-i+\frac{\mathrm{k}_X}{2}\right)\sigma_{s-i}t^{s+i}}{(1-t^s)^2} \deg C;$$

by writing

$$\overline{N}(t) = (1-t)^{n-1}\left(\sum_{i=0}^{s-1}\left(i - \frac{\mathrm{k}_X}{2}\right)\sigma_{s-i}t^i + \sum_{i=0}^{s-i}\left(s-i+\frac{\mathrm{k}_X}{2}\right)\sigma_{s-i}t^{s+i}\right)$$

and trimming $\overline{N} \mod \left(\frac{1-t^s}{1-t}\right)^2$ then moving it to the right support we get and $N(t) \equiv \overline{N}(t)$ supported on the right interval such that

$$\frac{N(t)}{(1-t)^{n-1}(1-t^s)^2} = \frac{\sum_{i=0}^{s-1}\left(i - \frac{\mathrm{k}_X}{2}\right)\sigma_{s-i}t^i + \sum_{i=0}^{s-i}\left(s-i+\frac{\mathrm{k}_X}{2}\right)\sigma_{s-i}t^{s+i}}{(1-t^s)^2} + \frac{A(t)}{(1-t)^{n+1}}.$$

Such an N has rational coefficients. It was proved to be rational by an explicit calculation in Zhou [2011] pp 72-75. □

Combining this with proposition 4.4.1 we get

Proposition 4.4.3.

$$P_{\text{per}}(C) = s \cdot \deg C \cdot \frac{P_{\text{orb}}(C, \mathrm{k_X} + \mathrm{s})}{1 - t^s} + \frac{M}{(1-t)^n(1-t^s)} + \frac{I_C}{(1-t)^{n+1}}.$$

Moreover M has rational coefficients and $\frac{M}{(1-t)^n(1-t^s)}$ is Gorenstein symmetric of degree k_X.

Proof. Consider

$$\frac{N(t)}{(1-t)^{n-1}(1-t^s)^2}\deg C + \sum_{j=1}^{n-1} \frac{N_j(t)}{(1-t)^n(1-t^s)} \frac{\deg \gamma_j}{2} - s \deg C \frac{P_{\text{orb}}(C, \mathrm{k_X} + \mathrm{s})}{1-t^s};$$

since each individual term is Gorenstein symmetric of degree k_X then so is the total, and using proposition 4.4.1 it has as poles at most a pole of order $n+1$ at 1 and a simple pole at the other sth roots of unity (since the $s \deg C \frac{P_{\text{orb}}(C, \mathrm{k_X}+\mathrm{s})}{1-t^s}$ term takes care of the double pole at sth roots of unity). In other words we may write it as

$$\frac{\overline{M}(t)}{(1-t)^n(1-t^s)};$$

writing $M = \overline{M} + \sum N_j$ we get that

$$\frac{M}{(1-t)^n(1-t^s)} = \frac{\overline{M}}{(1-t)^n(1-t^s)} + \sum_j \frac{N_j}{(1-t)^n(1-t^s)}$$

is again Gorenstein symmetric, whence the result. □

The following proposition formalises the notion of a dissident point "biting off" some part of the curve contribution.

Proposition 4.4.4. *Suppose we have a point P of type $\frac{1}{r}(a_1, \dots, a_n)$ and put as usual $s_i = \mathrm{hcf}(a_i, r)$. Suppose $s_i > 1$ for some i so that P is a dissident point on a $\frac{1}{s_i}$ curve C. Put*

$$Q_{P,C}(t) \equiv \frac{s_i}{r} \operatorname{InvMod}\left(\prod_{j=1}^{n} \frac{1-t^{a_j}}{1-t^{s_j}}, \frac{1-t^r}{\mathrm{hcf}\left(\prod(1-t^{a_j}), 1-t^r\right)}, \gamma\right) \cdot \prod_{j \neq i} \frac{1-t^{a_j}}{1-t^{s_j}} \quad \mathit{mod}\ \frac{1-t^s}{1-t},$$

such that $Q_{P,C}$ is a Laurent polynomial centred at 0. Then

$$(1-t)(1-t^r)\prod_{j\neq i}(1-t^{s_j})\left(P_{\mathrm{orb}}(P,\mathrm{k}_X)-Q_{P,C}(t)\frac{P_{\mathrm{orb}}(C,\mathrm{k}_X+s_i)}{1-t^{s_i}}\right)$$

is a polynomial.

Proof. Suppose P lies on the intersection of curves C_j for $j\in J$. Notice that $i\in J$ with $C=C_i$ by assumption. Put B,B_j for the numerators of $P_{\mathrm{orb}}(P,\mathrm{k}_X)$, $P_{\mathrm{orb}}(C_j,\mathrm{k}_X+s_j)$ for each j respectively, put as usual $h=\mathrm{hcf}\,(1-t^r,\prod(1-t^{a_j}))$, $F=\frac{1-t^r}{h}$ and rewrite 4.3 from proposition 4.3.2 as

$$B=\Delta\cdot\prod(1-t^{s_i})+F\cdot\left(A\cdot\frac{h^2}{(1-t)^2}+\sum_{j\in J}\prod_{l\neq j}\left(\frac{1-t^{s_l}}{1-t}\right)^2\left(Q_j+R_j\frac{1-t^{s_j}}{1-t}\right)\right).$$

We can put $Q_j=\overline{Q_j}B_j$ because B_j is invertible mod $\frac{1-t^{s_j}}{1-t}$ and modify $\overline{Q_j}$ mod $\frac{1-t^{s_j}}{1-t}$ as we want. Then an explicit calculation mod $\frac{1-t^{s_i}}{1-t}$ shows that $\overline{Q_i}\equiv Q_{P,C_i}$ as follows: reduce the above equation mod $\frac{1-t^{s_i}}{1-t}$ to get

$$B\equiv F\prod_{l\neq i}\left(\frac{1-t^{s_l}}{1-t}\right)^2\overline{Q_i}B_i;$$

since B_i is the inverse of $\prod_{j\neq i}\frac{1-t^{a_j}}{1-t}$ mod $\frac{1-t^{s_i}}{1-t}$ the above implies

$$\begin{aligned}\overline{Q_i}&\equiv\frac{h}{1-t^r}\prod_{l\neq i}\left(\frac{1-t}{1-t^{s_l}}\right)^2B\prod_{j\neq i}\frac{1-t^{a_j}}{1-t}\\&\equiv\frac{h}{1-t^r}\prod_{l\neq i}\left(\frac{1-t}{1-t^{s_l}}\right)B\prod_{j\neq i}\frac{1-t^{a_j}}{1-t^{s_j}};\end{aligned}$$

since $h=(1-t)\prod_{l=1}^n\frac{1-t^{s_l}}{1-t}$ we deduce that

$$\begin{aligned}\overline{Q_i}&\equiv\frac{1-t^{s_i}}{1-t}B\prod_{j\neq i}\frac{1-t^{a_j}}{1-t^{s_j}}\\&\equiv\frac{1}{1+t^{s_i}+\cdots+tr-s_i}B\prod_{j\neq i}\frac{1-t^{a_j}}{1-t^{s_j}}.\end{aligned}$$

By evaluating at s_ith roots of unity, we see that

$$\frac{1}{1+t^{s_i}+\cdots+tr-s_i}\equiv\frac{s_i}{r}$$

mod $\frac{1-t^{s_i}}{1-t}$ which implies that $\overline{Q_i} \equiv Q_{P,C_i}$ as claimed. Since $B, \prod_{j\neq i} \frac{1-t^{a_j}}{1-t^{s_j}}$ are both symmetric we can get $Q_{C_i,P}$ symmetric centred at 0.

Now consider

$$R = \left(P_{\mathrm{orb}}(P, \mathrm{k}_X) - Q_{P,C_i}(t) \frac{P_{\mathrm{orb}}(C_i, \mathrm{k}_X + s_i)}{1 - t^{s_i}} \right);$$

By proposition 4.3.2 and the above

$$\begin{aligned} R = &\frac{\Delta}{1-t^r} + \frac{A(t)}{(1-t)^{n+1}} \\ &+ \sum_{j \in J \setminus \{i\}} \left(\frac{Q_j}{(1-t^{s_j})^2 (1-t)^{n-1}} + \frac{R_j}{(1-t^{s_j})(1-t)^n} \right) \\ &+ \frac{R_i}{(1-t^{s_i})(1-t)^n}. \end{aligned}$$

Multiplying R by

$$(1-t)(1-t^r) \prod_{j \neq i} (1 - t^{s_j})$$

then kills all the poles of R looking at each term in the RHS of the above. This proves the claim. □

In other words, subtracting $Q_{P,C}(t) \frac{P_{\mathrm{orb}}(C, \mathrm{k}_X + s_i)}{1 - t^{s_i}}$ kills the $(1 - t^{s_i})$ factor in the denominator of $P_{\mathrm{orb}}(P, \mathrm{k}_X)$, and this is exactly the contribution from the curve C that P "bites off", that is, for a point P which lies on the intersection of curves $C_j, j \in J$ of type $\frac{1}{s_j}$ respectively we have

$$P_{\mathrm{orb}}(P, \mathrm{k}_X) = P_{\mathrm{per}}(P) + \frac{I_P}{(1-t)^{n+1}} + \sum_{j \in J} Q_{P,C_j} \cdot \frac{P_{\mathrm{orb}}(C, \mathrm{k}_X + s_j)}{1 - t^{s_j}} + \sum_{j \in J} \frac{R_{C_j}(P)}{(1-t)^{s_j}(1-t)^n} \tag{4.4}$$

for a suitable growth term I_P, where $R_{C_j}(P)$ is the R_j of proposition 4.3.2. Moreover

Proposition 4.4.5. *For a curve C of type $\frac{1}{s}(b_1, \dots, b_{n-1})$ with dissident points $P_j, j \in J$ we have that*

$$g_C := s \cdot \deg C - \sum_{j \in J} Q_{C,P_j}(t)$$

is a Laurent polynomial, supported in $\left[-\lfloor \frac{s}{2} \rfloor, \lfloor \frac{s}{2} \rfloor\right]$ with integer coefficients.

Proof. Each Q_{C,P_j} is supported in the required interval and symmetric. We will show that all the non constant coefficients of Q_{C,P_j} are integers and then deal with the constant term. Fix for now $P = P_i$ for some i and suppose P is of type $\frac{1}{r}(a_1, \dots, a_n)$; we know that there is an a_l such that $\mathrm{hcf}(a_l, r) = s$ and for simplicity and clarity reorder so that $l = i$ (to make it clear that this l is just dependent on i). Let B be as in the proof of proposition 4.4.4, and $s_j = \mathrm{hcf}(a_j, r)$ so that $s_j > 1$ for all $j \in J$ and $s_i = s$. Then mod $\frac{1-t^s}{1-t}$ we have

$$Q_{C,P} \equiv \frac{s}{r} B \prod_{j \neq i} \frac{1-t^{a_j}}{1-t^{s_j}}$$

where the product takes place over $1, \dots, \hat{i}, \dots, n$ (that is, not just over J); moreover recall from the proof of proposition 3.3.5 that mod $\frac{1-t^r}{h}$ (where

$$h = \mathrm{hcf}(\prod_{j=1}^{n}(1-t^{a_j}), 1-t^r) = (1-t)\prod_{j \in J}\frac{1-t^{s_j}}{1-t})$$

we have

$$B \equiv \prod_{j=1}^{n} \frac{1-t^{a_j b_j}}{1-t^{a_j}},$$

where b_j is chosen such that $a_j b_j \equiv s_j \mod r$. Therefore we can write

$$B = \prod_{j=1}^{n} \frac{1-t^{a_j b_j}}{1-t^{a_j}} + R(t)\frac{1-t^r}{h}.$$

Putting this into our formula for $Q_{C,P}$ we obtain

$$\begin{aligned} Q_{C,P} &\equiv \frac{s}{r}\prod_{j=1}^{n}\frac{1-t^{a_j b_j}}{1-t^{a_j}}\prod_{j\neq i}\frac{1-t^{a_j}}{1-t^{s_j}} + \frac{s}{r}\frac{1-t^r}{h}\prod_{j \neq i}\frac{1-t^{a_j}}{1-t^{s_j}} \\ &\equiv \frac{s}{r}\frac{1-t^{a_i b_i}}{1-t^{a_i}}\prod_{j\neq i}\frac{1-t^{a_j b_j}}{1-t^{s_j}} + \frac{s}{r}R(t)\frac{1-t^r}{1-t^s}\prod_{j\neq i}\frac{1-t}{1-t^{s_j}}\prod_{j\neq i}\frac{1-t^{a_j}}{1-t^{s_j}}. \end{aligned}$$

Now for each $j \neq i$

$$\frac{1-t^{a_j b_j}}{1-t^{s_j}} = \frac{1-t^{s_j + q_j r}}{1-t^{s_j}} \equiv 1 \mod \frac{1-t^s}{1-t}$$

by evaluating at sth roots of unity other than 1 (these are distinct from the s_jth roots of unity because $\mathrm{hcf}(s, s_j) = 1$, and these are also rth roots of unity because $s \mid r$). Moreover

$$\frac{1-t^{a_i b_i}}{1-t^{a_i}} = 1 + t^{a_i} + \dots + t^{a_i(b_i - 1)} \equiv b_i$$

again by evaluating at sth roots of unity and using that $s \mid a_i$. Thus the first part of the above sum is simply congruent to $\frac{b_i s}{r}$ mod $\frac{1-t^s}{1-t}$.

For the second part, since each s_j is coprime to s we let r_j be such that $s_j r_j \equiv 1$ mod s. Now working mod $\frac{1-t^s}{1-t}$ we see that

$$\frac{1-t^r}{1-t^s} = 1 + t^s + \cdots + t^{s(\frac{r}{s}-1)} \equiv \frac{r}{s}$$

and so the second part of the sum becomes

$$R(t) \prod_{j \neq i} \frac{1-t^{r_j s_j}}{1-t^{s_j}} \prod_{j \neq i} \frac{1-t^{a_j}}{1-t^{s_j}}$$

which is a polynomial with integer coefficients. Thus $Q_{C,P}$ is a Laurent polynomial, centred at 0, with all non constant coefficients integers, and with constant coefficient $\frac{bs}{r} + m$ for some integer m.

To prove the theorem, it therefore remains to show that

$$s \deg C - \sum_{j \in J} \frac{b_j s}{r_j}$$

is an integer, where we are assuming P_j is of type $\frac{1}{r_j}(a_{j,1}, \ldots, a_{j,n})$ and for each j we have a (unique) l_j such that $\mathrm{hcf}(a_{j,l_j}, r_j) = s$ and b_j chosen such that $a_{j,l_j} b_j \equiv s$ mod r_j. We reproduce the reasoning used in [Zhou, 2011] for completion.

Because the problem restricts itself to one on the curve, we may assume without loss of generality that $C \subset \mathbb{P}(a_0, \ldots, a_m)$ where $s \mid a_i$ for all i; in this case each dissident point can be viewed simply as an orbipoint of type $\frac{1}{a_j}(a_{j,l_j})$. We need to prove that

$$s \deg C - \sum_{j=0}^{m} \frac{b_j s}{a_j}$$

is an integer, where b_j was chosen such that $a_{j,l_j} b_j \equiv s$ mod r_j. Notice that if we consider the corresponding curve $C' \subset \mathbb{P}(\frac{a_0}{s}, \ldots \frac{a_m}{s})$, then $\deg C' = s \deg C$ and the $\frac{1}{a_j}(a_{j,l_j})$ dissident point on C reduces to a point of type $\frac{1}{a_j/s}(\frac{a_{j,l_j}}{s})$ on C'. Then by [Zhou, 2011] section 3.3.1 we have that

$$\chi(\mathcal{O}_{C'}(1)) = \chi(\mathcal{O}_{C'}) + \deg C' + \sum_j \frac{b_j}{a_j/s}$$

where the sum runs over all orbipoints of type $\frac{1}{a_j/s}(\frac{a_{j,l_j}}{s})$ and b_j is such that

$$b_j \frac{a_{j,l_j}}{s} \equiv 1 \mod \frac{a_j}{s}$$

which implies in particular

$$b_j a_{j,l_j} \equiv s \mod a_j.$$

Rearranging the above equation shows that

$$s \deg C - \sum_{j=0}^{m} \frac{b_j s}{a_j} = \chi(\mathcal{O}_{C'}(1)) - \chi(\mathcal{O}_{C'})$$

which is an integer, as required. □

By grouping various terms together we therefore have the following result.

Theorem 4.4.6. *For a suitable growth term G_C, we have*

$$P_{\text{per}}(C) + \frac{G_C}{(1-t)^{n+1}} - \sum_j Q_{P_j} \frac{P_{\text{orb}}(C)}{1-t^s} - \sum_j \frac{R_C(P_j)}{(1-t^s)(1-t)^n} = g_C \cdot \frac{P_{\text{orb}}(C)}{1-t^s} + B_{C,2}$$

where $B_{C_2} = \frac{N_{C,2}}{(1-t)^n(1-t^s)}$ is such that $N_{C,2}$ is integral, palindromic, of short support making $B_{C,2}$ Gorenstein symmetric of degree k_X.

We still have to show that $B_{C,2}$ is as claimed, but we leave this for the end for now.

Consequently, by attributing the end terms of $g_C \cdot \frac{P_{\text{orb}}(C)}{1-t^s}$ to $B_{C,2}$ we obtain the following.

Corollary 4.4.7.

$$P_{\text{per}}(C) + \frac{G_C}{(1-t)^{n+1}} - \sum_j Q_{P_j} \frac{P_{\text{orb}}(C)}{1-t^s} - \sum_j \frac{R_C(P_j)}{(1-t^s)(1-t)^n} = A_C + B_C$$

for A_C and B_C as in the theorem.

Again, the proof that B_C is as claimed is still to do (it is equivalent to the proof for $B_{C,2}$).

However assuming this, summing over all orbicurves and orbipoints and putting all the initial terms together, we obtain the following result

Corollary 4.4.8. *With notation as in the theorem*

$$P(t) = P_J(t) + \sum_P P_{\text{orb}}(P, \mathrm{k}_X) + \sum_C (A_C + B_C)$$

where $P_{\mathrm{orb}}, A_C, B_C$ are all as claimed, and $P_J = \frac{J}{(1-t)^{n+1}}$ where J is a polynomial with rational coefficients.

4.5 End of proof

As well as dealing with the B_C term, it remains to prove that the initial term is exactly of the required form, that is that J as given above is integral and palindromic of the correct degree. This is done entirely analogously to the isolated case and indeed gives the same formula for P_I.

Let us now deal with the B_C term. We have

Lemma 4.5.1. *For each orbicurve C, the rational function $B_{C,2}$ is Gorenstein symmetric of degree k_X.*

Proof. Suppose that C has dissident points $P_1, \ldots, P_l$; with notation as in the previous section we have

$$B_{C,2} = \frac{M}{(1-t^s)(1-t)^n} - \sum_{j=1}^{l} \frac{R_{C,P_j}}{(1-t^s)(1-t)^n}$$

so that as $\frac{M}{(1-t^s)(1-t)^n}$ is Gorenstein symmetric of degree k_X it is enough to show the same thing for each $\frac{R_{C,P_j}}{(1-t^s)(1-t)^n}$.

Now fix an orbipoint P of type $\frac{1}{r}(a_1, \ldots, a_n)$ and let s_i, h, etc be as before; suppose P lies on the intersection of orbicurves $C_j, j \in J$. Again with notation as in the previous section recall that

$$P_{\mathrm{orb}}(P, \mathrm{k}_X) = P_{\mathrm{per}}(P) + \frac{I_P}{(1-t)^{n+1}} + \sum_{j \in J} \left(\frac{Q_{C_j,P} B_j}{(1-t^{s_j})^2 (1-t)^{n-1}} + \frac{R_{C_j,P}}{(1-t^{s_j})(1-t)^n} \right).$$

We let

$$\frac{R(t)}{(1-t)^{n+1-|J|} \prod_{j \in J} (1-t^{s_j})} = \sum_{j \in J} \frac{R_{C_j,P}}{(1-t^{s_j})(1-t)^n};$$

fixing $i \in J$ and work mod $\frac{1-t^{s_i}}{1-t}$ we have

$$\begin{aligned} R &= \sum_{j \in J} R_{C_j,P} \prod_{l \in J, l \neq j} \frac{1-t^{s_l}}{1-t} \\ &\equiv R_{C_i,P} \prod_{l \in J, l \neq i} \frac{1-t^{s_l}}{1-t}. \end{aligned}$$

Thus still mod $\frac{1-t^{s_i}}{1-t}$ we have that

$$R_{C_i,P} \equiv R \prod_{l\in J, l\neq i} \frac{1-t^{s_l r_{l,i}}}{1-t^{s_l}}$$

with $r_{l,i}s_l \equiv 1 \mod s_i$. Thus it is sufficient to show that R is symmetric. But this follows from the fact that if we let

$$P_{\text{per}}(P) + \frac{I_P}{(1-t)^{n+1}} = \frac{N}{(1-t)^{n+1}F}$$

(where $F = \frac{1-t^r}{h}$) we see that

$$N \equiv \text{InvMod}\left(\prod_{i=1}^{n} \frac{1-t^{a_i}}{1-t}, F\right)$$

which shows that $\frac{N}{(1-t)^{n+1}F}$ is Gorenstein symmetric of degree k_X, hence so is

$$P_{\text{orb}}(P, \text{k}_X) - \frac{N}{(1-t)^{n+1}F} - \sum_{j\in J} \frac{Q_{C_j,P}B_j}{(1-t^{s_j})^2(1-t)^{n-1}} = \frac{R(t)}{(1-t)^{n+1-|J|}\prod_{j\in J}(1-t^{s_j})}.$$

This completes the proof. □

We still need to prove that $B_{C,2}$ has integral numerator. Recall that this is equivalent to showing B_C does.

Lemma 4.5.2. *The numerator of B_C has integer coefficients.*

Proof. Suppose $\mathcal{B}'$ consists of a collection of m curves of type $\frac{1}{s_1} \dots \frac{1}{s_m}$. Recall the s_i must be pairwise coprime. Consider the expression

$$\sum_{C\in\mathcal{B}'} B_C = \frac{B}{(1-t)^d \prod(1-t^{s_i})}$$

where $d = \min(0, n+1-m)$ (so that the denominator has at least $n+1$ factors). This is a power series constructed as the difference of other power series, each with integer coefficients, so that B must have integer coefficients. But we are now in the isolated case, so that if C is of type $\frac{1}{s_i}$

$$B_C \equiv \frac{B}{(1-t)^d} \cdot \text{InvMod}\left(\prod_{j\neq i} \frac{1-t^{a_j}}{1-t}, \frac{1-t^{a_i}}{1-t}\right)$$

using lemma 3.2.5. □

We can now also prove the special case of the $\frac{1}{2}$ curve.

Proposition 4.5.3. *Suppose C is a curve of type $\frac{1}{2}(1,\dots,1)$. Then $B_C = 0$.*

Proof. Using that $n - 1 + \mathrm{k}_X \equiv 0 \mod 2$ we see that the coindex c is necessarily even. But then

$$B_C = \frac{N_C}{(1-t^2)(1-t)^n}$$

where the numerator N_C is supported purely on $\frac{c}{2} + 1$; this cannot be Gorenstein symmetric of degree k_X so must be equal to 0. □

Chapter 5

Further problems and generalisations

In this Chapter we look at issues arrising from the rest of our work, and offer tentative solutions as to how these may be solved in time.

5.1 Understanding the curve orbifold locus more completely

We have a couple of issues with the theorem in the curve orbifold locus case as stated in the previous Chapters:

1. there seems to be no way around the dichotomy that we either attribute to a $\frac{1}{r}$ dissident point a term with denominator $(1-t^r)\prod(1-t^{s_i})$ where the point lies on the intersection of $\frac{1}{s_i}$ curves (in which case we can view the curve contributions as having denominators $(1-t^{s_i})^2(1-t)^{n-1}$ or $(1-t^{s_i})(1-t)^n$), or we have to attribute to the $\frac{1}{s}$ curve a collection of terms with denominators $(1-t^s)(1-t^{a_i})(1-t)^{n-1}$. There is no way of completely separating dissident points from the curves on which they lie;

2. whilst in Chapter 4 we have an explicit formula for the P_{orb} terms, and effectively a formula for the A_C contribution, we don't have an explicit formula for the normal bundle contribution B_C (equivalently the terms $N_{i,j}$ in Chapter 3).

We are however capable of understanding the "bite off" phenomenom in more detail. The key lemma 3.4.6 provides a way to parse the dissident points P_{orb} contributions so that it is apparent what part is contributed by which orbicurve, namely:

Proposition 5.1.1. *Let P be a point of type $\frac{1}{r}(a_1, \dots a_n)$ where for each i $\mathrm{hcf}(r, a_i) = s_i \geq 1$, and all the s_i are pairwise coprime. That is, we can write $r = q\prod_i s_i$, $a_i = r_i s_i$ where q is coprime to the s_i. Put k such that $\mathrm{k} + \sum a_i \equiv 0 \mod r$. Then there is a partial fraction decomposition*

$$P_{\mathrm{orb}}(P, \mathrm{k}) = \frac{N}{(1-t)^n(1-t^r)} + \sum_i \frac{N_i}{(1-t)^{n-1}(1-t^{s_i})(1-t^r)}$$

where each N, N_i are symmetric of short support, so that each term is Gorenstein symmetric of degree k with integral numerator. Moreover each N_i is such that

$$\frac{N_i}{(1-t)^{n-1}(1-t^{s_i})} = g_i \cdot P_{\mathrm{orb}}\left(\tfrac{1}{s_i}(a_1, \dots, \widehat{a_i}, \dots, a_n), \mathrm{k} + r\right)$$

with g_i some symmetric Laurent polynomial centered about 0.

Example 5.1.2. Consider a point of type $\frac{1}{140}(1, 4, 5, 7)$ which lies at the intersection of curves of type $\frac{1}{4}(1, 5, 7)$, $\frac{1}{5}(1, 4, 7)$, $\frac{1}{7}(1, 4, 5)$. We have that

$$\begin{aligned} P_{\mathrm{orb}}\left(\tfrac{1}{140}(1,4,5,7), -17\right) = {} & \frac{N}{(1-t)^4(1-t^{140})} + g_4 \cdot \frac{P_{\mathrm{orb}}\left(\frac{1}{4}(1,5,7), 123\right)}{1-t^{140}} \\ & + g_5 \cdot \frac{P_{\mathrm{orb}}\left(\frac{1}{5}(1,4,7), 123\right)}{1-t^{140}} + g_7 \cdot \frac{P_{\mathrm{orb}}\left(\frac{1}{7}(1,4,5), 123\right)}{1-t^{140}} \end{aligned}$$

where

$$\begin{aligned} g_4 &= -34 \\ g_5 &= 28t^{-1} + 1 + 28t \\ g_7 &= 20t^{-1} + 1 + 20t \end{aligned}$$

and $N(t)$ is some polynomial of degree 119.

An interesting observation is that if we consider a singularity of type $\frac{1}{140}(i, 4, 5, 7)$ for any i coprime to 140 then the g_i don't change (i plays no part in the construction in the lemma).

As regards calculating the $N_{i,j}$ contributions, we offer no conjecture for general formulae, but note only that there appears to be a link between them if you modify the numerator of the series, but leave the denominator in tact. Indeed, consider the following example.

Example 5.1.3. Consider the Fano 3-fold $X_{11} \subset \mathbb{P}(1, 1, 2, 5, 5)$. Its orbifold locus consists of a $\frac{1}{2}(1, 1, 1)$ point and a $\frac{1}{5}(1, 2)$ curve C. The Hilbert series contributions are given by

$$\begin{aligned}
P_I &= \frac{1+t}{(1-t)^4} \\
P_{\text{orb}}\left(\tfrac{1}{2}(1,1,1),-3\right) &= \frac{-t}{(1-t)^3(1-t^2)} \\
A_C &= \frac{-t^3-t^6}{(1-t)^2(1-t^5)^2} \\
B_C &= \frac{-2t-t^2-t^3-2t^4}{(1-t)^3(1-t^5)}
\end{aligned}$$

so that in this case we have

$$N_C = -2t - t^2 - t^3 - 2t^4 =: N_{C,1}.$$

Consider now the orbifolds $X_{16}, X_{21} \subset \mathbb{P}(1,1,2,5,5)$. They too have a $\frac{1}{5}(1,2)$ curve of singularities and we get respectively

$$\begin{aligned}
N_{C,2} &= 2t^5 \\
N_{C,3} &= -3t^6 - 2t^7 - 2t^8 - 3t^9.
\end{aligned}$$

We look at the relations between the $N_{C,i}$. All of these are residues and we proceed mod $\frac{1-t^5}{1-t}$. We see that

$$\begin{aligned}
N_{C,1} - (t^2+t^3) &= -2t - 2t^2 - 2t^3 - 2t^4 \\
&\equiv 2t^5 = N_{C,2}.
\end{aligned}$$

Moreover

$$\begin{aligned}
N_{C,2} - (t+t^4) \equiv N_{C,1} &- (t+t^2+t^3+t^4) \\
&= -3t - 2t^2 - 2t^3 - 3t^4 \\
&\equiv t^5(-3t - 2t^2 - 2t^3 - 3t^4) = N_{C,3}
\end{aligned}$$

so that combining the two

$$N_{C,1} + 1 \equiv N_{C,3}.$$

This pattern continues so that for $X_{26} \subset \mathbb{P}(1,1,2,5,5)$ we have

$$N_{C,4} = 3t^{10},$$

for $X_{31} \subset \mathbb{P}(1,1,2,5,5)$ we get

$$N_{C,5} = -4t^{11} - 3t^{12} - 3t^{13} - 4t^{14},$$

and so on.

Similar patterns emerge looking at other examples. Thus whilst we don't as of yet offer a general formula for the normal bundle contribution, we do think that if we can calculate it for a specific series we should be able to deduce it for all series with the same denominator.

5.2 Towards general results with arbitrary orbifold locus

The underlying philosophy of Ice cream functions is the following: traditionally we have viewed Riemann-Roch theorems as of the form

$$\chi(\mathcal{O}_X(D)) = \mathrm{RR}(X, D) + \sum_{P \in \mathcal{B}} c_P(D)$$

where $\mathcal{B}$ consists of the orbifold locus of X (points, curves, etc.) with

- $\mathrm{RR}(X, D) = (\mathrm{ch}(\mathcal{O}_X(D)) \cdot \mathrm{Td}(T_X))$

- $c_P(D)$ a local contribution depending only on the local type of singularity $P = \frac{1}{r}$ along D; this contribution has strict r periodicity.

Whilst the above approach yields formulae and results in the case where $\dim X = 1, 2, 3$ for higher dimensional cases it quickly becomes untractable because the entities $\mathrm{ch}(\mathcal{O}_X(D))$ and $\mathrm{Td}(T_X)$ are not well understood, so that although there are general statements (often using the language of stacks, such as in [Toen, 1999]) explicit computations often remain beyond us.

On the other hand, we should be able to generalise the results presented in the previous Chapters to the higher dimensional orbifold locus case. The Hilbert series should parse into an initial term (which deals with the first few plurigenera) and a sum of orbifold contributions. Which form these orbifold contributions now take is a continuation of the dichotomy introduced in Chapter 3.

1. On the one hand our initial approach was that we wanted to explicitly see for an orbifold locus $\frac{1}{r}$ of dimension d a term with denominator in which $(1-t^r)^d$ appeared (times some factors $(1-t^{r_i})$ where $r_i \mid d$ – the r_i correspond to higher dimensional loci on which the $\frac{1}{r}$ singularity lies). Thus in particular the highest dimensional orbifold locus has simply $(1-t^r)^d(1-t)^{n+1-d}$ as denominator. In this case the obstruction is that too many r_i may occur. Moreover the orbifold locus seems on the face of it to contain some information native to the higher dimensional loci on which it lies.

2. On the other hand our second approach was based on seeing the dimension of the orbifold locus clearly (namely by the number of nontrivial factors appearing in the denominator). In this case an orbifold locus $\frac{1}{r}$ of dimension d has as contribution terms with denominator $(1-t^r)(1-t^{s_1})\dots(1-t^{s_d})(1-t)^{n-d}$ where we have a string of divisors $r \mid s_1 \mid \cdots \mid s_d$ and $s_d = a_i$ for one of the original a_i; these s_j correspond to lower dimensional loci which lie on the $\frac{1}{r}$ singularity. In this case we don't need more factors in the denominator, because $d \leq n$.

In both cases each numerator should be integral symmetric of short support, so that each term Gorenstein is symmetric of degree k_X. Crucially we retain the r periodicity (even if it is now somewhat muddled) and the terms should all be easier to calculate using computer algebra.

We summarise the overall idea in the following conjecture.

Conjecture 5.2.1. *Let $b_0, \dots, b_m \in \mathbb{N}$ and $N(t)$ be integral, palindromic such that*

$$P(t) = \frac{N(t)}{\prod_{i=0}^{m}(1-t^{b_i})}$$

is Gorenstein symmetric of degree k. Then there is a unique partial fraction decomposition

$$\begin{aligned} P(t) &= \sum_A P_A(t) \\ &= \sum \frac{N_A}{\prod_{a\in A}(1-t^a)} \end{aligned} \tag{5.1}$$

where each $A = \{a_0, \dots, a_n\}$ consists of some main period $a_i = r$ and some divisors $a_j \mid r$, each a_l divides a corresponding b_m (so in particular finitely many A occur).

Each N_A is integral, palindromic, of short support (this should mean that the interval of support has length $< \deg F_A$ where $F_A = \frac{1-t^r}{\mathrm{hcf}\left(1-t^r, \prod_{a\neq r}(1-t^a)\right)}$) and is centred at some $\frac{k_A}{2}$ so that each P_A is Gorenstein symmetric of degree k.

Remark 5.2.2. Again, we refer to the denominator $(1-t)^{n+1}$ as the *initial denominator* . We have that $n = m - d$ where d is the order of the zero of the numerator N at 1.

We give some simple illustrative examples of the conjecture.

Example 5.2.3. As a simple example of a "surface singularity", let

$$P = \frac{1-t^{13}}{(1-t)^3(1-t^2)(1-t^4)(1-t^6)}$$

which is Gorenstein symmetric of degree 2 in "dimension" 4. In this case we have a "surface" singularity of type $\frac{1}{2}$ with "dissident points" $\frac{1}{4}$ and $\frac{1}{6}$. We expect to see as denominators therefore $[1,1,1,1,1], [1,1,2,2,4], [1,1,2,2,6], [1,1,2,2,2]$, $[1,1,1,2,2], [1,1,1,1,2]$. In correspondence with the "curve" singularity case we find that there can be no term with denominator $(1-t^2)^2(1-t)^3$ and we find that we can parse

$$\begin{aligned} P = {} & \frac{1-2t-2t^2+t^3}{(1-t)^5} \\ & + \frac{t^4}{(1-t^4)(1-t)^2(1-t^2)^2} + \frac{t^4+0\cdot t^5+t^6}{(1-t^6)(1-t)^2(1-t^2)^2} \\ & + \frac{t^3}{(1-t^2)^3(1-t)^2} + \frac{4t^2}{(1-t^2)(1-t)^4}. \end{aligned}$$

The existence of the partial fraction decompositions is not in doubt: it follows from separating the rational function into sums based on its poles. What is less obvious is the symmetry and integrality of the corresponding numerator for the ones which we are interested in. In practice we have proven this in a few cases by taking an appropriate residue of the numerator and noticing it is essentially an Inverse Mod type function possibly times some clearly symmetric functions.

Whilst we expect precise formulae for the P_A to be quite involved (if we can find explicit formulae at all), working out the interval of support is in practice not too hard, so that for any given example the problem reduces to essentially working out a number of integers which is a priori doable at least by computers. An overall but at the moment distant goal could therefore be to write an algorithm which will work out the parsing for any given series of this form.

Which factors appear non-trivially in the sum in 5.1 depends to some extent on the relations between the b_i and the terms in $N(t)$. It may be that this reduces to a combinatorial problem, and it would be interesting to see how our work relates to work done on the combinatorics of Hilbert series.

In the first scenario, the strategy is to first remove the maximal poles and then induct, whilst in the second we start first by parsing the series into point contributions and then induct from there. We discuss the second scenario in a bit more detail. In this case, the result is:

Theorem 5.2.4. *Let* $a_0, \ldots, a_m \in \mathbb{N}$ *and* $N(t)$ *be integral, palindromic such that*

$$P(t) = \frac{N(t)}{\prod_{i=0}^{m}(1 - t^{a_i})}$$

is Gorenstein symmetric of degree k.

For each i *we denote by* $I_i = \{i_1, \ldots, i_n\}$ *an ordered permutation of* $\{0, \ldots, \widehat{i}, \ldots, n\}$ *and we consider all such permutations. For each such* I *we let*

$$s_{i,I,k} = \mathrm{hcf}(a_i, a_{i_1}, \ldots, a_{i_k})$$

and denote $s_{i,I,0} = a_i$*; let* $k_{i,I}$ *be the largest* k *such that* $s_{i,I,k} > 1$*, with the convention that* $k_i = 0$ *if* $s_{i,I,1} = 1$*.*

Then there is a partial fraction decomposition

$$P = \frac{I}{(1-t)^{m+1}} + \sum_{i,I_i} \frac{N_{i,I}}{(1 - t^{a_i}) \prod_{k \in I_i}(1 - t^{s_{i,I,k}})}$$

where the numerators N_{i,I_i} *are all integral, symmetric (so that each individual term is Gorenstein symmetric of degree* k*) and a residue mod* $\frac{1-t^{s_{i,I,k_i}}}{1-t}$*.*

Remark 5.2.5. The set-up of the notation is a bit cumbersome. The point is that each individual term is of the form

$$\frac{N_B}{\prod_{b \in B}(1 - t^b)}$$

where each $B = \{b_0, \ldots, b_n\}$ contributes towards an orbifold locus of dimension d say. In this case we have $b_0 = s$ and a string of divisors $s \mid b_1 \mid \cdots \mid b_d$ where $b_d = a_i$ (one of the original a_i) and $b_{d+1} = \cdots = b_n = 1$. It is crucial to understand that we are viewing this as a contribution to the $\frac{1}{s}$ singularity (of dimension d) and not as a contribution to the point $\frac{1}{a_i}$ (the contribution to this point is the unique term with denominator $(1-t^{a_i})(1-t)^n$).

The proof is an inductive argument. We first parse P into

$$\frac{I}{(1-t)^{m+1}} + \sum_{i=0}^{m} \frac{N_i}{(1-t^{a_i})\prod_{j\neq i}(1-t^{s_{i,j}})}$$

where $s_{i,j} = \mathrm{hcf}(a_i, a_j)$. Then for each term

$$Q_i = \frac{N_i}{(1-t^{a_i})\prod_{j\neq i}(1-t^{s_{i,j}})}$$

we can proceed inductively. The key is that the $s_{i,j}$ share fewer common factors than the original a_i so we can use induction on these collections.

Let us illustrate this process in the case where no more than three of the a_i share a common factor (this corresponds geometrically to the case of surface orbifold locus, although as we have discussed we are now fairly far removed from the geometrical situation). The first step is to prove the equivalent of Lemma 3.4.6; in this case then our original parsing gives us the Hilbert series as a sum of an initial term and terms of the form

$$Q = \frac{N}{(1-t^r)\prod_{i=1}^{m}(1-t^{s_i})}$$

for some main period r (one of the original a_i) and divisors s_i of r. We then get a further parsing

$$Q = \sum_{i=1}^{m} \frac{M_i}{(1-t^r)(1-t^{s_i})(1-t^{s_{i,j}})}$$

and using the lemma and the curve case this eventually becomes

$$\begin{aligned} Q = &\sum_{i,j} \frac{N_{i,j}}{(1-t^r)(1-t^{a_i})(1-t^{s_{i,j}})(1-t)^{m-2}} \\ &+ \sum_{i} \frac{N_i}{(1-t^r)(1-t^{a_i})(1-t)^{m-1}} \\ &+ \frac{R}{(1-t^r)(1-t)^m}; \end{aligned}$$

after subtracting an initial term we may assumed that $N_{i,j}$ is a residue mod $\frac{1-t^{s_{i,j}}}{1-t}$, N_i mod $\frac{1-t^{s_i}}{1-t}$ and R mod $\frac{1-t^r}{1-t}$, with all terms symmetric and integral.

5.3 General formulae without the symmetric assumption

This final section discusses potential generalisations to rational functions where we omit the symmetric assumption. A main motivation for this is the construction of polarised varieties which are Cohen-Macaulay but not projectively Gorenstein (that is, the corresponding graded ring is not assumed to be Gorenstein). We believe that a similar partial fraction decomposition exists, but the issue is we have no natural interval of support for the numerators anymore (it is easy enough to see why we can find all the "orbifold" numerators of short support and integral, but which precise interval is not clear). Thus when we get to the initial term, we have no guarantee that the numerator is of sensible length (it is integral since it is the difference of power series with integer coefficients).

Example 5.3.1. Let

$$P = \frac{1 - t^6 - t^7 - t^8 + t^{13} + t^{15}}{(1-t^3)(1-t^5)(1-t^7)};$$

Experimenting about shows that we can write P as

$$\begin{aligned} P = &\frac{2-t}{(1-t)^2} - \frac{t}{(1-t)*(1-t^3)} \\ &+ \frac{2t^3 - t^2 + t}{(1-t)(1-t^5)} \\ &- \frac{1 + 2t + 2t^3 + t^4 + 3t^6}{(1-t)(1-t^7)}. \end{aligned}$$

Remark 5.3.2. Notice that whilst the terms are no longer symmetric, the numerators remain integral and of short support. However they no longer are supported in the same interval they would be if the terms were symmetric (we have gone from terms of "degree" 0 to "degree" -1 or -2). It is this difficulty in defining an interval of support which proves to be a major obstacle to obtaining a general formula.

On the other hand if we work purely naively, then we should be able to get at least an existence result. Consider for example the isolated case, i.e. let

$$P = \frac{N}{\prod_{i=0}^{m}(1-t^{a_i})}$$

where the a_i are pairwise coprime positive integers. Exactly as in the Gorenstein case assume N has a zero of order d at 1, let $n = m - d$ and put

$$A_i = \prod_{j \neq i} \left(\frac{1 - t^{a_j}}{1 - t} \right)$$
$$F_i = \frac{1 - t^{a_i}}{1 - t}.$$

Now put

$$N_i = \frac{N}{(1 - t)^d} \cdot \mathrm{InvMod}(A_i, F_i)$$

and let

$$B_i \equiv N_i \mod F_i.$$

Then exactly as in the Gorenstein case

$$P - \frac{B_i}{(1 - t^{a_i})(1 - t)^n}$$

has no poles at $\mu_{a_i} \setminus \{1\}$. Doing this successively we are left with

$$P - \sum_i \frac{B_i}{(1 - t^{a_i})(1 - t)^n} = \frac{I}{(1 - t)^{n+1}}.$$

As before the B_i can be chosen of short support, a residue mod $\frac{1-t^{a_i}}{1-t}$. The difference is that there is no obvious choice of support, so we may as well choose as support for example $[0, 1, \ldots, a_i - 1]$. The issue is now because of this we have no control over the support (and in particular the length of the support) of I. In practice we expect I to quickly get out of hand (remember from a constructing point of view, we need I to recover P; if I depends potentially on a very large number of integers this makes the problem significantly harder).

It could be that there are intervals of support that we can choose so that we can find intervals for the N_i giving an analogue of the statement that I deals with the first few plurigenera (that is P and $\frac{I}{(1-t)^{n+1}}$ should coincide as power series for the first few terms), but at the moment we do not know.

Appendix A

MAGMA code and functions

In this appendix we write down for completion the various MAGMA functions we have used in the course of our calculations. For more information see All the calculations we run include as preamble

```
Q:=Rationals();
R<t>:=PolynomialRing(Q);
K:=FieldOfFractions(R);
S<s>:=PowerSeriesRing(Q,50);
```

The following is a useful shortcut to return a denominator $\prod(1-t^{a_i})$ given as input a string L of the a_i.

```
function Denom(L)
 return &*[1-t^i : i in L];
end function;
```

To calculate the initial numerator given the first $\left\lfloor \frac{c}{2} \right\rfloor$ plurigenera (encoded in the string L), the canonical weight k and the dimension n:

```
function initial(L,k,n)
f := &+[L[i]*t^(i-1): i in [1..#L]];
pp := R!(f*(1-t)^(n+1));
c := k+n+1;
if IsEven(c) eq true then
return (&+[Coefficient(pp,i)*(t^i+t^(c-i)) : i in [0..c d
iv 2-1]]+
Coefficient(pp,c div 2)*t^(Floor(c/2)))/(1-t)^(n+1);
else
```

```
return &+[Coefficient(pp,i)*(t^i+t^(c-i)) : i in [0..Flo
or(c/2)]]
/(1-t)^(n+1);
end if;
end function;
```

To calculate P_{orb} of some (not necessarily isolated) orbifold locus $\frac{1}{r}(a_1, \ldots, a_n)$ (the a_is are encoded in the string L) with canonical weight k:

```
function Porb(r,LL,k)
 L := [Integers() | i : i in LL]; // this allows empty list
 if (k + &+L) mod r ne 0
    then error "Error: Canonical weight not compatible";
 end if;
n := #L;
S := [GCD(a,r) : a in L];
D := (1-t^r) * &*[1-t^s : s in S]; // Denom
A := &*[(1-t^(L[i])) div (1-t^(S[i])) : i in [1..n]];
F := (1-t^r) div GCD(1-t^r, &*[1-t^s: s in S]);
dF := Degree(F);
shift0 := Ceiling((k + 1 + &+[s : s in S] + r - dF)/2);
de := Maximum(0,Ceiling(-shift0/r));
shift := shift0+de*r;
 G, al, be := XGCD(t^shift*A, F);
 return t^shift*al/(D*t^(de*r));
end function;
```

Bibliography

Hamid Ahmadinezhad and Takuzo Okada. Birationally rigid Pfaffian Fano 3-folds. *arXiv preprint arXiv:1508.02974*, 2015.

Hamid Ahmadinezhad and Francesco Zucconi. Birational rigidity of Fano 3-folds and Mori dream spaces. *arXiv preprint arXiv:1407.3724*, 2014.

Selma Altınok. *Graded rings corresponding to polarised K3 surfaces and $\mathbb{Q}$-Fano 3-folds, Univ. of Warwick Ph. D.* PhD thesis, thesis, Sep. 1998, 93+ vii pp.

Selma Altmok, Gavin Brown, and Miles Reid. Fano 3-folds, K3 surfaces and graded rings. In *Topology and Geometry: Commemorating SISTAG: Singapore International Symposium in Topology and Geometry,(SISTAG) July 2-6, 2001, National University of Singapore, Singapore*, volume 314, page 25. American Mathematical Soc., 2002.

Armand Borel and Jean-Pierre Serre. Le théorème de Riemann–Roch. *Bulletin de la Société mathématique de France*, 86:97–136, 1958.

Gavin Brown and Alexander Kasprzyk. Four-dimensional projective orbifold hypersurfaces. *Exp. Math.*, to appear.

Gavin Brown, Michael Kerber, and Miles Reid. Fano 3-folds in codimension 4, Tom and Jerry. part I. *Compositio Mathematica*, 148(04):1171–1194, 2012.

Winfried Bruns and H. Jürgen Herzog. *Cohen–Macaulay rings.* Cambridge University Press, 1998.

David A Buchsbaum and David Eisenbud. Algebra structures for finite free resolutions, and some structure theorems for ideals of codimension 3. *American Journal of Mathematics*, pages 447–485, 1977.

Anita Buckley. *Orbifold Riemann–Roch for threefolds and applications to Calabi–Yaus.* PhD thesis, University of Warwick, 2003.

Anita Buckley and Balázs Szendroi. Orbifold Riemann–Roch for threefolds with an application to Calabi–Yau geometry. *arXiv preprint math/0309033*, 2004.

Anita Buckley, Miles Reid, and Shengtian Zhou. Ice cream and orbifold Riemann–Roch. *Izvestiya: Mathematics*, 77(3):461, 2013.

Ivan Cheltsov and Jihun Park. Birationally rigid Fano threefold hypersurfaces. *Memoirs of the American Mathematical Society*, to appear.

Tom Coates, Alessio Corti, Sergey Galkin, Vasily Golyshev, and Alexander Kasprzyk. Mirror symmetry and Fano manifolds. *arXiv preprint arXiv:1212.1722*, 2012.

Alessio Corti, Alexandr Pukhlikov, and Miles Reid. Fano 3-fold hypersurfaces. In Alessio Corti and Miles Reid, editors, *Explicit birational geometry of 3-folds*. CUP, June 2000.

David A Cox and Sheldon Katz. *Mirror symmetry and algebraic geometry*. Number 68. American Mathematical Soc., 1999.

David Eisenbud. *Commutative algebra: with a view toward algebraic geometry*, volume 150. Springer Science & Business Media, 1995.

Shiro Goto and Keiichi Watanabe. On graded rings. I. *J. Math. Soc. Japan*, 30 (2):179–213, 1978. ISSN 0025-5645. doi: 10.2969/jmsj/03020179. URL `http://dx.doi.org/10.2969/jmsj/03020179`.

Robin Hartshorne. *Algebraic geometry*, volume 52. Springer Science & Business Media, 1977.

http://www.grdb.co.uk/. Graded ring database. website.

Anthony R Iano-Fletcher. Working with weighted complete intersections. In Alessio Corti and Miles Reid, editors, *Explicit birational geometry of 3-folds*. CUP, June 2000.

Yujiro Kawamata, Katsumi Matsuda, and Kenji Matsuki. Introduction to the minimal model problem. *Adv. Stud. Pure Math*, 10:283–360, 1987.

Kenji Matsuki. *Introduction to the Mori program*. Springer Science & Business Media, 2013.

Hideyuki Matsumura and Miles Reid. *Commutative ring theory*, volume 8. Cambridge university press, 1989.

S Mori. Threefolds whose canonical bundles are not numerically effective. *Ann. of Math.*, 116:133–176, 1982.

Shigefumi Mori, David R Morrison, and Ian Morrison. On four-dimensional terminal quotient singularities. *Mathematics of computation*, 51(184):769–786, 1988.

Shigefumi Mori et al. On 3-dimensional terminal singularities. *Nagoya Mathematical Journal*, 98:43–66, 1985.

David R Morrison and Glenn Stevens. Terminal quotient singularities in dimensions three and four. *Proceedings of the American Mathematical Society*, 90(1):15–20, 1984.

So Okada. Homological mirror symmetry of Fermat polynomials. *arXiv preprint arXiv:0910.2014*, 2009.

So Okada. Quintic periods and stability conditions via homological mirror symmetry. *arXiv preprint arXiv:1304.0937*, 2013.

Stavros Papadakis and Miles Reid. Kustin–Miller unprojection without complexes. *Journal of Algebraic Geometry*, 13(3):563–577, 2004.

Miles Reid. Canonical 3-folds. In *Journes de gomtrie algbrique d'Angers*, pages 273–310. Alphen, 1980.

Miles Reid. Minimal models of canonical 3-folds. *Adv. Stud. Pure Math*, 1:131–180, 1983.

Miles Reid. Young persons guide to canonical singularities. *Algebraic geometry, Bowdoin*, pages 345–414, 1985.

Richard P Thomas. The geometry of mirror symmetry. *arXiv preprint math/0512412*, 2005.

Bertrand Toen. Théorèmes de Riemann–Roch pour les champs de Deligne–Mumford. *K-theory*, 18(1):33–76, 1999.

Shengtian Zhou. *Orbifold Riemann-Roch and Hilbert series*. PhD thesis, University of Warwick, 2011.