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LOWER BOUNDS ON BLOWING-UP SOLUTIONS OF THE THREE-DIMENSIONAL NAVIER–STOKES EQUATIONS IN $\dot{H}^{3/2}$, $\dot{H}^{5/2}$, AND $\dot{B}^{5/2}_{2,1}$*

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Abstract. If u is a smooth solution of the Navier–Stokes equations on $\mathbb{R}^3$ with first blowup time T, we prove lower bounds for u in the Sobolev spaces $\dot{H}^{3/2}$, $\dot{H}^{5/2}$, and the Besov space $\dot{B}^{5/2}_{2,1}$, with optimal rates of blowup: we prove the strong lower bounds $\|u(t)\|_{\dot{H}^{3/2}} \geq c(T-t)^{-1/2}$ and $\|u(t)\|_{\dot{B}^{5/2}_{2,1}} \geq c(T-t)^{-1}$; in $\dot{H}^{5/2}$ we obtain $\limsup_{t\to T^-}(T-t)\|u(t)\|_{\dot{H}^{5/2}} \geq c$, a weaker result. The proofs involve new inequalities for the nonlinear term in Sobolev and Besov spaces, both of which are obtained using a dyadic decomposition of u.

Key words. Navier–Stokes equations, blowup, commutator estimates

AMS subject classification. 35Q30

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1. Introduction.

The aim of this paper is to prove lower bounds on smooth solutions of the three-dimensional Navier–Stokes equations,

$$\partial_t u - \Delta u + (u\cdot\nabla)u + \nabla p = 0, \qquad \nabla\cdot u = 0, \tag{1.1}$$

posed on the entire space $\mathbb{R}^3$, under the assumption that there is a finite "first blowup time" T. Results of this type date back to Leray (1934), who showed that there exists an absolute constant c_1 such that

$$\|u(t)\|_{H^1} \geq \frac{c_1}{\sqrt{T-t}}.$$

In fact this result is a consequence of upper bounds on the local existence time for solutions with initial data in $\dot{H}^1$, a pattern of argument repeated for subsequent lower bounds in other spaces. Leray also stated (without proof) the lower bound

$$\|u(t)\|_{L^p} \geq \frac{c}{(T-t)^{(p-3)/2p}},$$

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a proof of which can be found in Giga (1986) and Robinson and Sadowski (2014). Lower bounds in L^p spaces are also discussed in some detail by Lorenz and Zingano (2015).

More recently there have been a number of papers that treat the problem of blowup in Sobolev spaces $\dot{H}^s$ for $s > 1/2$. Benameur (2010) (with a similar periodic analysis in 2013) showed that for $s > 5/2$

$$\|u(t)\|_{\dot{H}^s} \geq c_s \|u(T-t)\|_{L^2}^{(3-2s)/3}(T-t)^{-s/3},$$

which was improved by Robinson, Sadowski, and Silva (2012) to

$$\|u(t)\|_{\dot{H}^s} \geq \begin{cases} c(T-t)^{-(2s-1)/4}, & s \in (1/2, 5/2),\ s \neq 3/2, \\ c\|u_0\|_{L^2}^{(5-2s)/5}(T-t)^{-2s/5}, & s > 5/2. \end{cases} \tag{1.2}$$

Solutions of (1.1) have the following important scaling property: if $u(x,t)$ is a solution with initial data $u_1(x)$, then $\lambda u(\lambda x, \lambda^2 t)$ is a solution with initial data $u_\lambda(x) := \lambda u_1(\lambda x)$. We say that a space X scales with exponent α if $\|u_\lambda\|_X = \lambda^\alpha \|u_1\|_X$ (the space $\dot{H}^s$ scales with exponent $s - \frac{1}{2}$) and two spaces "have the same scaling" if they scale with the same exponent. Using these scaling considerations Robinson, Sadowski, and Silva (2012) argue that one would expect the bound

$$\|u(t)\|_{\dot{H}^s} \geq c(T-t)^{-(2s-1)/4}$$

for all $s > 1/2$; we refer to this here as the "optimal rate."

We note that in the bounds in (1.2) the cases $s = 3/2$ and $s = 5/2$ are excluded and that the bounds for $s > 5/2$ are not at the optimal rate. Although Benameur (2010) and Robinson, Sadowski, and Silva (2012) both obtained the lower bound

$$\|\hat{u}(t)\|_{L^1} \geq c(T-t)^{-1/2},$$

i.e., a bound with the optimal rate in a space with the same scaling as $\dot{H}^{3/2}$, no lower bound with the correct rate in any space scaling like $\dot{H}^{5/2}$ has previously been shown.

Recently, Cortissoz, Montero, and Pinilla (2014) proved lower bounds in $\dot{H}^{3/2}$ and $\dot{H}^{5/2}$ at the optimal rates but with logarithmic corrections,

$$\|u(t)\|_{\dot{H}^{3/2}} \geq \frac{c}{\sqrt{(T-t)|\log(T-t)|}}, \qquad \|u(t)\|_{\dot{H}^{5/2}} \geq \frac{c}{(T-t)|\log(T-t)|},$$

where in both cases c depends on $\|u_0\|_{L^2}$.

In this paper we fill some of these gaps. We will show that if u is a smooth solution with maximal existence time T, then

$$\|u(t)\|_{\dot{H}^{3/2}} \geq \frac{c}{(T-t)^{1/2}}, \tag{1.3}$$

which we refer to as a "strong blowup estimate," and

$$\limsup_{t \uparrow T^*} (T-t)\|u(t)\|_{\dot{H}^{5/2}} \geq c,$$

which we refer to as a "weak blowup estimate." We also prove a strong blowup estimate in the Besov space $\dot{B}^{5/2}_{2,1}$, which has the same scaling as $\dot{H}^{5/2}$,

$$\|u(t)\|_{\dot{B}^{5/2}_{2,1}} \geq \frac{c}{T-t}.$$

These bounds follow from two inequalities for the nonlinear term $B(u,u) := (u\cdot\nabla)u$. Both are proved using a dyadic decomposition of u. The first is the Sobolev space inequality

$$|(\Lambda^s B(u,u),\Lambda^s u)| \le c\|u\|_{\dot H^s}\|u\|_{\dot H^{s+1}}\|u\|_{\dot H^{3/2}}, \qquad s\ge 1,$$

valid whenever the right-hand side is finite (in fact we prove a more general commutator-type estimate in Proposition 5.1). The second is the Besov bound

$$|(\dot\triangle_k B(u,u),\dot\triangle_k u)| \le c d_k 2^{-5k/2}\|u\|^2_{\dot B^{5/2}_{2,1}}\|\dot\triangle_k u\|_{L^2},$$

where c does not depend on k and $\sum_k d_k = 1$. We present the proofs of these inequalities in sections 5 and 6, with the resulting blowup estimates given first in sections 3 and 4.

Within the 10 days prior to the submission of this paper to arXiv, two other papers were submitted providing proofs of the lower bound in (1.3) for $\dot H^{3/2}$—one by Cheskidov and Zaya (using an alternative dyadic argument) and one by Montero (using a very neat interpolation argument).

2. Preliminaries. In this section we prove a simple ODE lemma that provides lower bounds on solutions that blow up, and we recall the dyadic decomposition that we will use to prove our Sobolev and Besov space inequalities.

2.1. Lower bounds and differential inequalities. Lower bounds on solutions that blow up at some time $T>0$ can be derived from differential inequalities for the norms of the solution (i.e., from upper bounds on the local existence time). The following simple ODE lemma makes this precise.

LEMMA 2.1. *If $\dot X \le cX^{1+\gamma}$ and $X(t)\to\infty$ as $t\to T$, then*

$$X(t) \ge \left(\frac{1}{\gamma c(T-t)}\right)^{1/\gamma}. \tag{2.1}$$

Proof. Write the differential inequality as

$$\frac{\mathrm{d}X}{X^{1+\gamma}} \le c\,\mathrm{d}t$$

and integrate from t to s to yield

$$\frac{1}{X(t)^\gamma} - \frac{1}{X(s)^\gamma} \le \gamma c(s-t).$$

Letting $s\to T$ yields (2.1). ∎

2.2. Homogeneous Sobolev spaces. We denote by $\dot H^s(\mathbb{R}^n)$ the space

$$\left\{u:\ \hat u\in L^1_{\mathrm{loc}}(\mathbb{R}^n):\ \int_{\mathbb{R}^n}|\xi|^{2s}|\hat u(\xi)|^2\,\mathrm{d}\xi<\infty\right\},$$

where

$$\mathscr{F}[u](\xi) = \hat u(\xi) = \int_{\mathbb{R}^n} \mathrm{e}^{-2\pi\mathrm{i}x\cdot\xi}u(x)\,\mathrm{d}x \tag{2.2}$$

is the Fourier transform of u. We denote by Λ^s the operator with Fourier multiplier $|\xi|^s$; then the norm in $\dot H^s$ is given by

$$\|u\|_{\dot H^s} = \|\Lambda^s u\|_{L^2} = \||\xi|^s\hat u(\xi)\|_{L^2} = \int_{\mathbb{R}^n}|\xi|^{2s}|\hat u(\xi)|^2\,\mathrm{d}k\xi.$$

2.3. Homogeneous Besov spaces. Here we recall some of the standard theory of homogeneous Besov spaces which we will use throughout the paper; we refer the reader to Bahouri, Chemin, and Danchin (2011) for proofs and many more details that we must omit.

For the purposes of this section, given a function ϕ and $j \in \mathbb{Z}$ we denote by ϕ_j the dilation

$$\phi_j(\xi) = \phi(2^{-j}\xi).$$

Let $\mathcal{C}$ be the annulus $\{\xi \in \mathbb{R}^n : 3/4 \le |\xi| \le 8/3\}$. There exist radial functions $\chi \in C_c^\infty(B(0,4/3))$ and $\varphi \in C_c^\infty(\mathcal{C})$ both taking values in $[0,1]$ such that

$$\forall \xi \in \mathbb{R}^n, \qquad \chi(\xi) + \sum_{j\ge 0} \varphi_j(\xi) = 1, \tag{2.3a}$$

$$\forall \xi \in \mathbb{R}^n \setminus \{0\}, \qquad \sum_{j\in\mathbb{Z}} \varphi_j(\xi) = 1, \tag{2.3b}$$

$$\text{if } |j-j'| \ge 2, \text{ then } \qquad \operatorname{supp}\varphi_j \cap \operatorname{supp}\varphi_{j'} = \varnothing, \tag{2.3c}$$

$$\text{if } j \ge 1, \text{ then } \qquad \operatorname{supp}\chi \cap \operatorname{supp}\varphi_j = \varnothing. \tag{2.3d}$$

We let $h = \mathscr{F}^{-1}\varphi$ and $\widetilde{h} = \mathscr{F}^{-1}\chi$, where $\mathscr{F}^{-1}$ is the inverse of the Fourier transform operator defined in (2.2).

Given a measurable function σ defined on $\mathbb{R}^n$ with at most polynomial growth at infinity, we define the Fourier multiplier operator M_σ by $M_\sigma u := \mathcal{F}^{-1}(\sigma\hat{u})$. For $j \in \mathbb{Z}$, the *homogeneous dyadic blocks* $\dot{\triangle}_j$ and the homogeneous cut-off operator $\dot{S}_j$ are defined by setting

$$\dot{\triangle}_j u = M_{\varphi_j} u = 2^{jn} \int_{\mathbb{R}^n} h(2^j y) u(x-y)\,\mathrm{d}y \qquad \text{and}$$
$$\dot{S}_j u = M_{\chi_j} u = 2^{jn} \int_{\mathbb{R}^n} \widetilde{h}(2^j y) u(x-y)\,\mathrm{d}y.$$

Formally, we can write the following *Littlewood–Paley decomposition*:

$$\mathrm{Id} = \sum_{j\in\mathbb{Z}} \dot{\triangle}_j.$$

We denote by $\mathscr{S}_h'(\mathbb{R}^n)$ the space of tempered distributions such that

$$\lim_{\lambda\to\infty} \|M_{\theta(\lambda\,\cdot\,)} u\|_{L^\infty} = 0 \qquad \text{for any } \theta \in C_c^\infty(\mathbb{R}^n).$$

Then the homogeneous decomposition makes sense in $\mathscr{S}_h'(\mathbb{R}^n)$: whenever $u \in \mathscr{S}_h'(\mathbb{R}^n)$, $u = \lim_{j\to\infty} \dot{S}_j u$ in $\mathscr{S}_h'(\mathbb{R}^n)$. Moreover, using the homogeneous decomposition, it is straightforward to show that

$$\dot{S}_j u = \sum_{j' \le j-1} \dot{\triangle}_{j'} u.$$

Given a real number s and two numbers $p, r \in [1,\infty]$, the *homogeneous Besov space* $\dot{B}^s_{p,r}(\mathbb{R}^n)$ consists of those distributions u in $\mathscr{S}_h'(\mathbb{R}^n)$ such that

$$\|u\|_{\dot{B}^s_{p,r}} := \left(\sum_{j\in\mathbb{Z}} 2^{rjs} \|\dot{\triangle}_j u\|^r_{L^p} \right)^{1/r} < \infty$$

if $r < \infty$ and

$$\|u\|_{\dot B^s_{p,\infty}} := \sup_{j\in\mathbb{Z}} 2^{js}\|\dot\triangle_j u\|_{L^p} < \infty$$

if $r = \infty$. For each of these spaces all choices of the function φ used to define the blocks $\dot\triangle_j$ lead to equivalent norms and hence to the same space.

Note that if $u \in \mathscr{S}'_h(\mathbb{R}^n)$ belongs to $\dot B^s_{p,r}(\mathbb{R}^n)$, then there exists a nonnegative sequence $(d_j)_{j\in\mathbb{Z}}$ such that

$$\|\dot\triangle_j u\|_{L^p} \le d_j 2^{-js}\|u\|_{\dot B^s_{p,r}} \;\forall\, j\in\mathbb{Z}, \qquad \text{where} \qquad \|(d_j)\|_{\ell^r} = 1. \tag{2.4}$$

3. Blowup estimates in $\dot H^{3/2}$ (strong) and $\dot H^{5/2}$ (weak). The proofs of the blowup results follows easily from upper bounds on the nonlinear term. We postpone a detailed presentation of the estimates and proofs of these bounds until section 5. In this section we assume those estimates and present a straightforward proof of the strong blowup estimate in $\dot H^{3/2}$ and, with an additional contradiction argument, of the weak blowup estimate in $\dot H^{5/2}$.

THEOREM 3.1. *Suppose that u is a classical solution of the Navier–Stokes equations with maximal existence time T. Then*

$$\|u(T-t)\|^2_{\dot H^{3/2}} \ge c_{3/2}^{-2} t^{-1}. \tag{3.1}$$

Proof. We take the inner product of the equation with u in $\dot H^{3/2}$, i.e., we apply $\Lambda^{3/2}$ and take the inner product with $\Lambda^{3/2}u$,

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2_{\dot H^{3/2}} + \|u\|^2_{\dot H^{5/2}} &= (\Lambda^{3/2}B(u,u), \Lambda^{3/2}u)\\ &\le c_{3/2}\|u\|^2_{\dot H^{3/2}}\|u\|_{\dot H^{5/2}},\end{aligned}$$

using the inequality

$$|(\Lambda^s[(u\cdot\nabla)u], \Lambda^s u)| \le c\|u\|_{\dot H^s}\|u\|_{\dot H^{s+1}}\|u\|_{\dot H^{3/2}}, \qquad s\ge 1,$$

from (5.5) with $s = 3/2$, which is proved in section 5. We use Young's inequality on the right-hand side to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2_{\dot H^{3/2}} + \|u\|^2_{\dot H^{5/2}} \le c_{3/2}^2\|u\|^4_{\dot H^{3/2}}.$$

Dropping the second term on the left-hand side, the required lower bound follows immediately from Lemma 2.1. ∎

We now use a contradiction argument to obtain a weak lower bound in $\dot H^{5/2}$ at the correct rate.

THEOREM 3.2. *Suppose that u is a classical solution of the Navier–Stokes equations with maximal existence time T. Then*

$$\limsup_{t\uparrow T}\,(T-t)\|u(t)\|_{\dot H^{5/2}} \ge c. \tag{3.2}$$

Proof. We proceed by contradiction and suppose that for $\tau \le t \le T$,

$$\|u(t)\|_{\dot H^{5/2}} \le \varepsilon(T-t)^{-1}, \tag{3.3}$$

where ε is chosen so that $2c_{3/2}\varepsilon < 1$. Then on this interval

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{\dot{H}^{3/2}}^2 \leq c_{3/2}\|u\|_{\dot{H}^{3/2}}^2\|u\|_{\dot{H}^{5/2}} - \|u\|_{\dot{H}^{5/2}}^2.$$

Since $ax - x^2$ is increasing in x while $x \leq a/2$, using our assumption (3.3) along with the result (3.1) from Theorem 3.1, we obtain

$$\|u(t)\|_{\dot{H}^{5/2}} \leq \frac{\varepsilon}{T-t} \leq \frac{\frac{1}{2}c_{3/2}^{-1}}{T-t} \leq \frac{1}{2}\left[c_{3/2}\|u(t)\|_{\dot{H}^{3/2}}^2\right].$$

It follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{\dot{H}^{3/2}}^2 \leq 2c_{3/2}\|u\|_{\dot{H}^{3/2}}^2\frac{\varepsilon}{T-t} - \frac{2\varepsilon^2}{(T-t)^2}.$$

Using the integrating factor $(T-t)^{2c_{3/2}\varepsilon}$ (note that the exponent is < 1) this becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u\|_{\dot{H}^{3/2}}^2(T-t)^{2c_{3/2}\varepsilon}\right) \leq -\varepsilon^2(T-t)^{-(2-2c_{3/2}\varepsilon)}.$$

Now drop the right-hand side and integrate from τ to t to conclude that

$$\begin{aligned}\|u(t)\|_{\dot{H}^{3/2}}^2 &\leq \|u(\tau)\|_{\dot{H}^{3/2}}^2(T-\tau)^{2c_{3/2}\varepsilon}(T-t)^{-2c_{3/2}\varepsilon}\\ &= C_\tau(T-t)^{-2c_{3/2}\varepsilon},\end{aligned}$$

which contradicts (3.1) provided that $2c_{3/2}\varepsilon < 1$, which we assumed above. It follows that there exist $t_k \to T$ such that

$$\|u(t_k)\|_{\dot{H}^{5/2}} \geq (4c_{3/2})^{-1}t_k^{-1}$$

and (3.2) follows. ∎

Note that this bound does not use directly any differential inequality governing the evolution of $\|u\|_{\dot{H}^{5/2}}$.

4. Strong blowup estimate in $\dot{B}^{5/2}_{2,1}$.

Although we have been unable to prove a strong lower bound in $\dot{H}^{5/2}$ at the correct rate (i.e., $\|u(t)\|_{\dot{H}^{5/2}} \geq c/(T-t)$) we can obtain such a bound in the Besov space $\dot{B}^{5/2}_{2,1}$, which has the same scaling (exponent 2). Again the proof relies on estimates of the nonlinear term, which we delay until section 6.

THEOREM 4.1. *Suppose that u is a classical solution of the Navier–Stokes equations with maximal existence time T. Then*

$$\|u(t)\|_{\dot{B}^{5/2}_{2,1}} \geq \frac{c}{T-t}. \tag{4.1}$$

Proof. We consider the equation for $\dot{\triangle}_k u$, which can be rewritten (by adding and subtracting the term involving the summation in i) as

$$\begin{aligned}\frac{\partial}{\partial t}\dot{\triangle}_k u - \Delta\dot{\triangle}_k u + \left[\dot{\triangle}_k((u\cdot\nabla)u) - \sum_i \dot{S}_{k-1}u_i\partial_i\dot{\triangle}_k u\right] + \sum_i \dot{S}_{k-1}u_i\partial_i\dot{\triangle}_k u&\\ + \nabla\dot{\triangle}_k p = 0,&\end{aligned}$$

since $\dot\triangle_k$ and Δ commute. Taking the inner product in L^2 with $\dot\triangle_k u$ yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\dot\triangle_k u\|_{L^2}^2 + \|\nabla\dot\triangle_k u\|_{L^2}^2 \le \left\|\dot\triangle_k((u\cdot\nabla)u) - \sum_i \dot S_{k-1}u_i\partial_i\dot\triangle_k u\right\|_{L^2}\|\dot\triangle_k u\|_{L^2}.$$

We drop the second term on the left-hand side and divide by $\|\dot\triangle_k u\|_{L^2}$ to yield

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|\dot\triangle_k u\|_{L^2} &\le \left\|\dot\triangle_k((u\cdot\nabla)u) - \sum_i \dot S_{k-1}u_i\partial_i\dot\triangle_k u\right\|_{L^2}\\ &\le c d_k(t)2^{-5k/2}\|u\|^2_{\dot B^{5/2}_{2,1}},\end{aligned}$$

using Proposition 6.6, and where $\sum d_k(t) = 1$ for each t.

We now multiply by $2^{5k/2}$ and sum to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{\dot B^{5/2}_{2,1}} \le c\|u\|^2_{\dot B^{5/2}_{2,1}},$$

from which (4.1) follows at once via Lemma 2.1. ∎

5. Bounds for the nonlinear term in Sobolev spaces.

In this section we will prove the bound on the nonlinear term that we used in the proof of Theorem 3.1, namely

$$|(\Lambda^{3/2}B(u,u),\Lambda^{3/2}u)| \le c_{3/2}\|u\|^2_{\dot H^{3/2}}\|u\|_{\dot H^{5/2}}.$$

In fact we prove a somewhat more general result in Corollary 5.4, which in turn is a consequence of the following commutator estimate (cf. Kato and Ponce (1988), Fefferman et al. (2014)).

PROPOSITION 5.1. *Take $s \ge 1$ and $s_1, s_2 > 0$ such that*

$$1 \le s_1 < \tfrac{n}{2}+1 \qquad \text{and} \qquad s_1+s_2 = s+\tfrac{n}{2}+1. \tag{5.1}$$

Then there exists a constant c such that for all $u, v \in \dot H^{s_1}(\mathbb{R}^n)\cap \dot H^{s_2}(\mathbb{R}^n)$,

$$\|\Lambda^s[(u\cdot\nabla)v] - (u\cdot\nabla)(\Lambda^s v)\|_{L^2} \le c(\|u\|_{\dot H^{s_1}}\|v\|_{\dot H^{s_2}} + \|u\|_{\dot H^{s_2}}\|v\|_{\dot H^{s_1}}).$$

To prove Proposition 5.1 we need two simple lemmas. A proof of the first can be found in Fefferman et al. (2014); the second is an immediate consequence of Bernstein's inequality (see McCormick, Robinson, and Rodrigo (2013), for example).

LEMMA 5.2. *If $s \ge 1$ and $|b| < |a|/2$, then*

$$||a|^s - |a-b|^s| \le c|a-b|^{s-1}|b|,$$

where $c = s3^{s-1}$.

LEMMA 5.3. *There exists a constant c such that for any $k\in\mathbb{Z}$ and any p, q with $1 \le p \le q \le \infty$, if $\dot\triangle_k u \in L^p(\mathbb{R}^n)$, then $\dot\triangle_k u \in L^q(\mathbb{R}^n)$ and*

$$\|\dot\triangle_k u\|_{L^q} \le c2^{kn(1/p-1/q)}\|\dot\triangle_k u\|_{L^p}.$$

We can now give the proof of Proposition 5.1.

Proof of Proposition 5.1. Write $u = \sum_{i\in\mathbb{Z}} \dot{\triangle}_i u$ and $v = \sum_{j\in\mathbb{Z}} \dot{\triangle}_j v$; then

$$
\begin{aligned}
f &= \Lambda^s[(u\cdot\nabla)v] - (u\cdot\nabla)(\Lambda^s v) \\
&= \sum_{j\in\mathbb{Z}} \left\{ \Lambda^s \left[\left(\sum_{i\in\mathbb{Z}} \dot{\triangle}_i u \right) \nabla \dot{\triangle}_j v \right] - \left(\sum_{i\in\mathbb{Z}} \dot{\triangle}_i u \right) \nabla \Lambda^s \dot{\triangle}_j v \right\} \\
&= \sum_{j\in\mathbb{Z}} \left\{ \Lambda^s \left[\left(\sum_{i=-\infty}^{j-10} \dot{\triangle}_i u \right) \nabla \dot{\triangle}_j v \right] - \left(\sum_{i=-\infty}^{j-10} \dot{\triangle}_i u \right) \nabla \Lambda^s \dot{\triangle}_j v \right\} \\
&\quad + \sum_{j\in\mathbb{Z}} \left\{ \Lambda^s \left[\left(\sum_{i=j-9}^{j+9} \dot{\triangle}_i u \right) \nabla \dot{\triangle}_j v \right] - \left(\sum_{i=j-9}^{j+9} \dot{\triangle}_i u \right) \nabla \Lambda^s \dot{\triangle}_j v \right\} \\
&\quad + \sum_{i\in\mathbb{Z}} \left\{ \Lambda^s \left[\dot{\triangle}_i u \left(\sum_{j=-\infty}^{i-10} \nabla \dot{\triangle}_j v \right) \right] - \dot{\triangle}_i u \left(\sum_{j=-\infty}^{i-10} \nabla \Lambda^s \dot{\triangle}_j v \right) \right\} \\
&=: \sum_{j\in\mathbb{Z}} f_{1,j} + \sum_{j\in\mathbb{Z}} f_{2,j} + \sum_{i\in\mathbb{Z}} f_{3,i}.
\end{aligned}
$$

Taking the Fourier transform of $f_{1,j}$, we have

$$
\hat{f}_{1,j}(\xi) = \int_{\mathbb{R}^n} (|\xi|^s - |\eta|^s) \sum_{i=-\infty}^{j-10} \widehat{\dot{\triangle}_i u}(\xi-\eta)\, \mathrm{i}\eta \widehat{\dot{\triangle}_j v}(\eta)\, \mathrm{d}\eta.
$$

Since $i \le j-10$, $|\xi-\eta| < |\eta|/2$, so by Lemma 5.2 we have

$$
|\hat{f}_{1,j}(\xi)| \le \int_{\mathbb{R}^n} |\xi-\eta| \left| \sum_{i=-\infty}^{j-10} \widehat{\dot{\triangle}_i u}(\xi-\eta) \right| |\eta|^s \left| \widehat{\dot{\triangle}_j v}(\eta) \right| \mathrm{d}\eta.
$$

Let q_1, q_2 satisfy $\frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{2}$ and $2 < q_1 < \frac{n}{s_1-1}$, and let p_1, p_2 satisfy $\frac{1}{p_i} = \frac{1}{q_i} + \frac{1}{2}$. Noting that $1 + \frac{1}{2} = \frac{1}{p_1} + \frac{1}{p_2}$, by Young's inequality for convolutions we have

$$
\|\hat{f}_{1,j}\|_{L^2} \le \left\| |\zeta| \left| \sum_{i=-\infty}^{j-10} \widehat{\dot{\triangle}_i u}(\zeta) \right| \right\|_{L^{p_1}} \left\| |\eta|^s \left| \widehat{\dot{\triangle}_j v}(\eta) \right| \right\|_{L^{p_2}}.
$$

As $1 - s_1 + n/q_1 > 0$, by Hölder's inequality we have

$$
\begin{aligned}
\left\| |\zeta| \left| \sum_{i=-\infty}^{j-10} \widehat{\dot{\triangle}_i u}(\zeta) \right| \right\|_{L^{p_1}} &\le \left\| |\zeta|^{1-s_1} \mathbf{1}_{\{|\zeta| \le 2^{j-10}\}} \right\|_{L^{q_1}} \left\| |\zeta|^{s_1} \left| \sum_{i=-\infty}^{j-10} \widehat{\dot{\triangle}_i u}(\zeta) \right| \right\|_{L^2} \\
&\le c 2^{j(1-s_1+n/q_1)} \|u\|_{\dot{H}^{s_1}}.
\end{aligned}
$$

For the other term, by Hölder's inequality,

$$
\begin{aligned}
\left\| |\eta|^s \widehat{\dot{\triangle}_j v}(\eta) \right\|_{L^{p_2}} &\le \left\| |\eta|^s \mathbf{1}_{\{2^{j-1} \le |\zeta| \le 2^{j+1}\}} \right\|_{L^{q_2}} \left\| \widehat{\dot{\triangle}_j v}(\eta) \right\|_{L^2} \\
&\le c 2^{j(s+n/q_2)} \left\| \dot{\triangle}_j v \right\|_{L^2},
\end{aligned}
$$

hence

$$\begin{aligned}\|f_{1,j}\|_{L^2} &\le c\,\|u\|_{\dot H^{s_1}}\, 2^{j(s-s_1+n/q_1+n/q_2+1)} \left\|\dot\triangle_j v\right\|_{L^2}\\ &\le c\,\|u\|_{\dot H^{s_1}}\, 2^{js_2} \left\|\dot\triangle_j v\right\|_{L^2}\end{aligned}$$

and thus

$$\sum_{j\in\mathbb{Z}} \|f_{1,j}\|_{L^2}^2 \le c\,\|u\|_{\dot H^{s_1}}^2\, \|v\|_{\dot H^{s_2}}^2 . \tag{5.2}$$

For the second term, since $\left(\sum_{i=j-9}^{j+9}\dot\triangle_i u\right)\nabla\dot\triangle_j v$ is localized in Fourier space in an annulus centered at radius 2^j, we obtain

$$\begin{aligned}\|f_{2,j}\|_{L^2} &\le \left\|\Lambda^s\left[\left(\sum_{i=j-9}^{j+9}\dot\triangle_i u\right)\nabla\dot\triangle_j v\right]\right\|_{L^2} + \left\|\left(\sum_{i=j-9}^{j+9}\dot\triangle_i u\right)\nabla\Lambda^s\dot\triangle_j v\right\|_{L^2}\\ &\le c2^{js}\sum_{i=j-9}^{j+9}\|\dot\triangle_i u\|_{L^4}\|\nabla\dot\triangle_j v\|_{L^4} + \sum_{i=j-9}^{j+9}\|\dot\triangle_i u\|_{L^4}\|\nabla\Lambda^s\dot\triangle_j v\|_{L^4}\\ &\le c2^{j(s+n/4)}\|\nabla\dot\triangle_j v\|_{L^2}\sum_{i=j-9}^{j+9}2^{in/4}\|\dot\triangle_i u\|_{L^2}\\ &\le c2^{j(s+n/2-s_1)}\|\nabla\dot\triangle_j v\|_{L^2}\sum_{i=j-9}^{j+9}2^{j(s_1-n/4)}2^{in/4}\|\dot\triangle_i u\|_{L^2}\end{aligned}$$

using Bernstein's inequality (Lemma 5.3). Since $|i-j|\le 9$, $2^{j(s_1-n/4)}\le c2^{i(s_1-n/4)}$, so

$$\|f_{2,j}\|_{L^2} \le c2^{j(s_2-1)}\|\nabla\dot\triangle_j v\|_{L^2}\sum_{i=j-9}^{j+9}2^{is_1}\|\dot\triangle_i u\|_{L^2},$$

and thus

$$\sum_{j\in\mathbb{Z}} \|f_{2,j}\|_{L^2}^2 \le c\,\|u\|_{\dot H^{s_1}}^2\, \|v\|_{\dot H^{s_2}}^2 . \tag{5.3}$$

For the third term, we use the Sobolev embedding

$$\|\nabla u\|_{L^p} \le c\|u\|_{\dot H^{s_1}}$$

provided $p=\frac{2n}{n-2s_1+2}$. Using Hölder's inequality, we obtain

$$\begin{aligned}\|f_{3,i}\|_{L^2} &\le \left\|\Lambda^s\left[\dot\triangle_i u\left(\sum_{j=-\infty}^{i-10}\nabla\dot\triangle_j v\right)\right]\right\|_{L^2} + \left\|\dot\triangle_i u\left(\sum_{j=-\infty}^{i-10}\nabla\Lambda^s\dot\triangle_j v\right)\right\|_{L^2}\\ &\le 2^{is}\|\dot\triangle_i u\|_{L^{n/(s_1-1)}}\left\|\sum_{j=-\infty}^{i-10}\nabla\dot\triangle_j v\right\|_{L^{2n/(n-2s_1+2)}}\\ &\quad + \|\dot\triangle_i u\|_{L^{n/(s_1-1)}}\left\|\sum_{j=-\infty}^{i-10}\nabla\Lambda^s\dot\triangle_j v\right\|_{L^{2n/(n-2s_1+2)}}\end{aligned}$$

$$\le c2^{i(s+n/2+1-s_1)}\|\dot{\triangle}_i u\|_{L^2}\|v\|_{\dot H^{s_1}}$$
$$\le c2^{is_2}\|\dot{\triangle}_i u\|_{L^2}\|v\|_{\dot H^{s_1}}$$

using Bernstein's inequality (Lemma 5.3) and the fact that $2^{js} \le 2^{is}$. Hence

$$\sum_{i\in\mathbb{Z}} \|f_{3,i}\|_{L^2}^2 \le c\,\|u\|_{\dot H^{s_2}}^2\,\|v\|_{\dot H^{s_1}}^2\,. \tag{5.4}$$

Combining (5.2), (5.3), and (5.4) yields the desired result. ∎

In particular, taking $s = s_1 = n/2$ and $s_2 = n/2 + 1$ in Proposition 5.1 yields

$$\|\Lambda^{n/2}[(u\cdot\nabla)v] - (u\cdot\nabla)(\Lambda^{n/2}v)\|_{L^2} \le c(\|\nabla u\|_{\dot H^{n/2}}\|v\|_{\dot H^{n/2}} + \|u\|_{\dot H^{n/2}}\|\nabla v\|_{\dot H^{n/2}}).$$

The counterexample in the appendix to Fefferman et al. (2014) shows that one cannot remove the second term on the right-hand side, at least in the case $n = 2$.

We will use this estimate in the form of the following corollary, which provides a partial generalisation of Lemma 1.1 from Chemin (1992).

COROLLARY 5.4. *Take $s \ge 1$ and $s_1, s_2 > 0$ such that*

$$1 \le s_1 < \tfrac{n}{2} + 1 \qquad \text{and} \qquad s_1 + s_2 = s + \tfrac{n}{2} + 1.$$

Then there exists a constant c such that for all $u, v \in \dot H^{s_1}(\mathbb{R}^n)\cap\dot H^{s_2}(\mathbb{R}^n)$ with $\nabla\cdot u = 0$,

$$|(\Lambda^s[(u\cdot\nabla)v], \Lambda^s v)| \le c(\|u\|_{\dot H^{s_1}}\|v\|_{\dot H^{s_2}} + \|u\|_{\dot H^{s_2}}\|v\|_{\dot H^{s_1}})\|v\|_{\dot H^s}.$$

Proof. Observe that since

$$((u\cdot\nabla)\Lambda^s v, \Lambda^s v) = 0$$

it follows that

$$(\Lambda^s[(u\cdot\nabla)v], \Lambda^s v) = (\Lambda^s[(u\cdot\nabla)v] - (u\cdot\nabla)\Lambda^s v, \Lambda^s v)$$

and the inequality is an immediate consequence of Proposition 5.1. ∎

Note that in particular for any $s \ge 1$, if $\nabla\cdot u = 0$, then

$$|(\Lambda^s[(u\cdot\nabla)u], \Lambda^s u)| \le c\|u\|_{\dot H^s}\|u\|_{\dot H^{s+1}}\|u\|_{\dot H^{n/2}} \tag{5.5}$$

whenever the right-hand side is finite.

6. Bounds for the nonlinear term in Besov spaces.

Much like the Sobolev embeddings, Besov spaces enjoy certain embeddings with the correct exponents. We quote the two embeddings we will use most frequently.

PROPOSITION 6.1 (Proposition 2.20 in Bahouri, Chemin, and Danchin (2011)). *Let $1 \le p_1 \le p_2 \le \infty$ and $1 \le r_1 \le r_2 \le \infty$. For any real number s, we have the continuous embedding*

$$\dot B^s_{p_1,r_1}(\mathbb{R}^n) \hookrightarrow \dot B^{s-n(1/p_1-1/p_2)}_{p_2,r_2}(\mathbb{R}^n).$$

PROPOSITION 6.2 (Proposition 2.39 in Bahouri, Chemin, and Danchin (2011)). *For $1 \le p \le q \le \infty$, we have the continuous embedding*

$$\dot B^{n/p-n/q}_{p,1}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n).$$

6.1. Homogeneous paradifferential calculus. Let u and v be tempered distributions in $\mathscr{S}'_h(\mathbb{R}^n)$. We have

$$u = \sum_{j'\in\mathbb{Z}} \dot{\triangle}_{j'} u \quad \text{and} \quad v = \sum_{j\in\mathbb{Z}} \dot{\triangle}_j v,$$

so, at least formally,

$$uv = \sum_{j,j'\in\mathbb{Z}} \dot{\triangle}_{j'} u \dot{\triangle}_j v.$$

One of the key techniques of paradifferential calculus is to break the above sum into three parts, as follows: define

$$\dot{T}_u v := \sum_{j\in\mathbb{Z}} \dot{S}_{j-1} u \dot{\triangle}_j v$$

and

$$\dot{R}(u,v) := \sum_{|k-j|\le 1} \dot{\triangle}_k u \dot{\triangle}_j v.$$

At least formally, the following *Bony decomposition* holds true:

$$uv = \dot{T}_u v + \dot{T}_v u + \dot{R}(u,v).$$

We now state two standard estimates on $\dot{T}$ and $\dot{R}$ that we will use in proving our a priori estimates.

LEMMA 6.3 (Theorem 2.47 from Bahouri, Chemin, and Danchin (2011)). *There exists a constant C such that for any real number s and any $p, r \in [1,\infty]$ we have, for any $u \in L^\infty$ and $v \in \dot{B}^s_{p,r}$,*

$$\|\dot{T}_u v\|_{\dot{B}^s_{p,r}} \le C^{1+|s|} \|u\|_{L^\infty} \|v\|_{\dot{B}^s_{p,r}}.$$

LEMMA 6.4 (Theorem 2.52 from Bahouri, Chemin, and Danchin (2011)). *Let $s_1, s_2 \in \mathbb{R}$ such that $s_1 + s_2 > 0$. There exists a constant $C = C(s_1, s_2)$ such that for any $p_1, p_2, r_1, r_2 \in [1,\infty]$, $u \in \dot{B}^{s_1}_{p_1,r_1}$, and $v \in \dot{B}^{s_2}_{p_2,r_2}$,*

$$\|\dot{R}(u,v)\|_{\dot{B}^{s_1+s_2}_{p,r}} \le C \|u\|_{\dot{B}^{s_1}_{p_1,r_1}} \|v\|_{\dot{B}^{s_2}_{p_2,r_2}}$$

provided that

$$\frac{1}{p} := \frac{1}{p_1} + \frac{1}{p_2} \le 1 \quad \text{and} \quad \frac{1}{r} := \frac{1}{r_1} + \frac{1}{r_2} \le 1.$$

We also require the following result, a particular case of Lemma 2.100 from Bahouri, Chemin, and Danchin (2011).

LEMMA 6.5. *Let $-1 - n/2 < \sigma < 1 + n/2$ and $1 \le r \le \infty$. Let $\boldsymbol{v}$ be a divergence-free vector field on $\mathbb{R}^n$, and set $Q_j := [(v\cdot\nabla), \dot{\triangle}_j] f$. There exists a constant $C = C(\sigma, n)$ such that*

$$\left\| \left(2^{j\sigma} \|Q_j\|_{L^2}\right)_j \right\|_{\ell^r} \le C \|\nabla v\|_{\dot{B}^{n/2}_{2,\infty}\cap L^\infty} \|f\|_{\dot{B}^{\sigma}_{2,r}}.$$

6.2. Main estimate in Besov spaces. We are now ready for the main estimate in Besov spaces.

PROPOSITION 6.6. *There exists a constant $c > 0$ such that if $u \in \dot{B}^{n/2+1}_{2,1}$, then*

$$\left\| \dot{\triangle}_k((u \cdot \nabla)u) - \sum_i \dot{S}_{k-1} u_i \partial_i \dot{\triangle}_k u \right\|_{L^2} \le c\, d_k 2^{-k(n/2+1)} \|u\|^2_{\dot{B}^{n/2+1}_{2,1}} \tag{6.1}$$

with $\sum_k d_k = 1$.

Throughout the proof we use $\lesssim$ to denote that the inequality holds up to a multiplicative constant, which may vary from line to line.

Proof. Notice that the lth coordinate of $(u \cdot \nabla)u$ is given by $\sum_i u_i \partial_i u_l$, and so we have

$$(u \cdot \nabla u)_l = \sum_i \dot{T}_{u_i} \partial_i u_l + \sum_i \dot{T}_{\partial_i u_l} u_i + \sum_i \dot{R}(u_i, \partial_i u_l).$$

Recall that by definition

$$\dot{T}_{u_i} \partial_i u_l = \sum_j \dot{S}_{j-1} u_i \dot{\triangle}_j \partial_i u_l,$$

and so we can rewrite $\dot{\triangle}_k \dot{T}_u \nabla u_l$ as follows:

$$\sum_i \dot{\triangle}_k \dot{T}_{u_i} \partial_i u_l = \sum_i \dot{S}_{k-1} u_i \partial_i \dot{\triangle}_k u_l \tag{6.2a}$$

$$+ \sum_i \sum_j (\dot{S}_{j-1} u_i - \dot{S}_{k-1} u_i) \partial_i \dot{\triangle}_k \dot{\triangle}_j u_l \tag{6.2b}$$

$$+ \sum_i \sum_j [\dot{\triangle}_k, \dot{S}_{j-1} u_i \partial_i] \dot{\triangle}_j u_l. \tag{6.2c}$$

Rearranging this we obtain the following expression for the lth component of the term we want to estimate:

$$\Big(\dot{\triangle}_k((u \cdot \nabla)u) - \sum_i \dot{S}_{k-1} u_i \partial_i \dot{\triangle}_k u \Big)_l$$

$$= \sum_i \sum_j (\dot{S}_{j-1} u_i - \dot{S}_{k-1} u_i) \partial_i \dot{\triangle}_k \dot{\triangle}_j u_l \tag{6.3a}$$

$$+ \sum_i \sum_j [\dot{\triangle}_k, \dot{S}_{j-1} u_i \partial_i] \dot{\triangle}_j u_l \tag{6.3b}$$

$$+ \sum_i \dot{\triangle}_k \dot{T}_{\partial_i u_l} u_i \tag{6.3c}$$

$$+ \sum_i \dot{\triangle}_k \dot{R}(u_i, \partial_i u_l). \tag{6.3d}$$

We will show that L^2 norm of each of the four terms in the right-hand side is controlled by a constant multiple of $d_k 2^{-k(n/2+1)} \|u\|^2_{\dot{B}^{n/2+1}_{2,1}}$, hence obtaining the result.

For (6.3a), ignoring the summation in i for now we have

$$\sum_j (\dot{S}_{j-1} u_i - \dot{S}_{k-1} u_i) \partial_i \dot{\triangle}_k \dot{\triangle}_j u_l = \dot{\triangle}_{k-1} u_i \dot{\triangle}_k \dot{\triangle}_{k+1} \partial_i u_l - \dot{\triangle}_{k-2} u_i \dot{\triangle}_k \dot{\triangle}_{k-1} \partial_i u_l,$$

and so (now summing in i as well)

$$\begin{aligned}\|\text{expression (6.3a)}\|_{L^2} &\lesssim 2^k\|\dot{\triangle}_{k-1}u\|_{L^\infty}\|\dot{\triangle}_k u_l\|_{L^2}\\ &\quad + 2^k\|\dot{\triangle}_{k-2}u\|_{L^\infty}\|\dot{\triangle}_k u_l\|_{L^2}\\ &\lesssim \|\dot{\triangle}_k u_l\|_{L^2}\|u\|_{\dot{B}^{n/2+1}_{2,1}}\\ &\lesssim d_k 2^{-k(n/2+1)}\|u\|_{\dot{B}^{n/2+1}_{2,1}}\|u_l\|_{\dot{B}^{n/2+1}_{2,1}}\end{aligned}$$

since

$$2^k\|\dot{\triangle}_k u\|_{L^\infty} \le \|u\|_{\dot{B}^1_{\infty,\infty}} \lesssim \|u\|_{\dot{B}^{n/2+1}_{2,1}}.$$

Above we have used the definition of $\dot{B}^1_{\infty,\infty}$ and the embedding

$$\dot{B}^{n/2+1}_{2,1}(\mathbb{R}^n) \hookrightarrow \dot{B}^1_{\infty,\infty}(\mathbb{R}^n). \tag{6.4}$$

from Proposition 6.1, and also (2.4), to find

$$\|\dot{\triangle}_k u\|_{L^2} \lesssim d_k 2^{-k(n/2+1)}\|u\|_{\dot{B}^{n/2+1}_{2,1}}.$$

To treat (6.3b), define $Q_k = \sum_j [\dot{\triangle}_k, \dot{S}_{j-1}u_i\partial_i]\dot{\triangle}_j u_l$ and apply Lemma 6.5 to give

$$\left\|2^{k(n/2+1)}\|Q_k\|_{L^2}\right\|_{\ell^1} \lesssim \|\nabla u\|_{\dot{B}^{n/2}_{2,\infty}\cap L^\infty}\|u\|_{\dot{B}^{n/2+1}_{2,1}} \lesssim \|u\|^2_{\dot{B}^{n/2+1}_{2,1}},$$

since $\dot{B}^{n/2}_{2,1}$ embeds continuously into L^∞ and $\dot{B}^{n/2}_{2,\infty}$ (see Propositions 6.1 and 6.2). It follows that

$$\|Q_k\|_{L^2} \lesssim d_k 2^{-k(n/2+1)}\|u\|^2_{\dot{B}^{n/2+1}_{2,1}}.$$

To estimate (6.3c) we use Lemma 6.3 and the embeddings from Proposition 6.2; we have

$$\begin{aligned}\|\dot{T}_{\partial_i u_l}u_i\|_{\dot{B}^{n/2+1}_{2,1}} &\lesssim \|\nabla u_l\|_{L^\infty}\|u_i\|_{\dot{B}^{n/2+1}_{2,1}}\\ &\lesssim \|u\|^2_{\dot{B}^{n/2+1}_{2,1}}.\end{aligned}$$

Using (2.4) we find

$$\|\dot{\triangle}_k\dot{T}_{\partial_i u_l}u_i\|^2_L \le d_k 2^{-k(n/2+1)}\|u\|^2_{\dot{B}^{n/2+1}_{2,1}}.$$

Finally we consider (6.3d); using Lemma 6.4 with $p = 2$, $(p_1, p_2) = (\infty, 2)$, $r = 1$, $(r_1, r_2) = (\infty, 1)$, $(s_1, s_2) = (1, n/2)$, we obtain

$$\begin{aligned}\|\dot{R}(u_i, \partial_i u_l)\|_{\dot{B}^{n/2+1}_{2,1}} &\lesssim \|u_i\|_{\dot{B}^1_{\infty,\infty}}\|\nabla u_l\|_{\dot{B}^{n/2}_{2,1}}\\ &\lesssim \|u\|^2_{\dot{B}^{n/2+1}_{2,1}},\end{aligned}$$

using the embedding (6.4) once more. Again, by (2.4) we find

$$\|\dot{\triangle}_k\dot{R}(u_i, \partial_i u_l)\|_{L^2} \lesssim d_k 2^{-k(n/2+1)}\|u\|^2_{\dot{B}^{n/2+1}_{2,1}}.$$

Combining these estimates yields (6.1). ∎

7. Conclusion. Lower bounds in $\dot{H}^{3/2}$ are now available from a number of sources. It remains an interesting open question whether it is possible to obtain a strong lower bound in $\dot{H}^{5/2}$, and any type of lower bound at the optimal rate in $\dot{H}^s$ with $s > 5/2$. If not, it would be worthwhile to develop an understanding of the qualitative change of the initial value problem for $u_0 \in \dot{H}^s$ from $s < 5/2$ to $s > 5/2$.

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