INTERACTIONS BETWEEN KNOWLEDGE OF VARIABLES AND KNOWLEDGE ABOUT TEACHING VARIABLES

by

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ABSTRACT

The purpose of this study is to find out Turkish prospective teachers' subject matter knowledge of variables and pedagogical content knowledge of variables and also the nature of the interactions between these two types of knowledge. One hundred and eighty four students participated in the study. Questionnaires were distributed to 2nd, 3rd, 4th year mathematics education faculty students of three different universities. The questionnaire included 16 fixed and open-ended questions about (a) the principal uses of variables, (b) the awareness about different roles of variables, (c) the flexibility, versatility and connectedness among the different roles and uses, and (d) ways of presenting the subject matter, (e) curriculum knowledge. As a follow-up study, ten students of different year groups who completed this questionnaire were interviewed. The outcome of this study is that prospective teachers have different perceptions of the notion of the variable which are reflected in their pedagogical content knowledge in a complex way. Results indicate that the majority of prospective teachers are successful in manipulating variables; however they have problems in moving flexibly between different meanings and representations. Concrete objects and numbers are identified as two main forms of analogies that they would use to explain ideas relating to manipulation of symbols. The results indicate that there is a complex interaction between subject matter knowledge of variables and pedagogical content knowledge which may involve the prospective teachers' own learning experiences, general pedagogical knowledge and the robustness of one type of knowledge.

CHAPTER 1

Beginnings

INTRODUCTION

This thesis investigates Turkish prospective teachers' subject matter knowledge of variables, (from now on SMKv) and pedagogical content knowledge of variables, blend of content knowledge with general pedagogical knowledge which is required to teach variables to others, (PCKv). PCK differs from both the content knowledge and general pedagogical knowledge. Content knowledge is the knowledge which is held by a content expert, what the mathematician understands about the discipline of mathematics. General pedagogical knowledge is the knowledge of experienced teachers, such as knowledge of how to manage students and organise a classroom during instruction. These terms will be more fully explained in the next chapter.

In this study, the nature of the interactions between these two types of knowledge is also examined. One hundred and eighty four students participated in the study. Questionnaires were distributed to 2nd, 3rd, 4th year students from three different mathematics education faculties in Turkey. As a follow-up study, ten students of different year groups who completed this questionnaire were interviewed. In this first chapter, I will give an account of the need for such a study by discussing its importance. The case for the importance of this topic will be based on the importance of algebra, and the importance of variables. The lack of research on prospective teachers' knowledge about variables in Turkey will be also considered while discussing the need for this study. At the end of the chapter I will present the general and specific research questions and an outline of the thesis.

1.1 Importance of this study

Algebra is one of the most fundamental subjects in the school curriculum since it is used in other subjects, in technology and as a tool to solve real world problems. In some countries, the term "gatekeeper" has been applied to algebra since the successful completion of an algebra course is a prerequisite not only to further study in mathematics and other subjects but also to many jobs and later opportunities.

Although the word algebra may be interpreted broadly to encompass the diversity of possible definitions of the word, within the context of school mathematics, algebra deals with expressions and equations, e.g. simplification of expressions, solutions of equations, substitution within expressions, and the use of these to construct and solve word problems. Expressions and equations are composed of *numbers, operations, letters,* and (for equations) an *equality sign.* Within these components, it is commonly agreed that the use of literal symbols is among the most important building blocks for understanding algebra. According to Eisenberg (1991) understanding the use of variables is the basis of all abstractions in mathematics. It is so important that not understanding it may block students' success in algebra (Leitzel, 1989).

In discussing the richness and multiplicity of meanings of the notion of variable, Schoenfeld and Arcavi (1988) point out that it is almost impossible to capture the meaning of the term variable by a single word, furthermore it is very difficult to give a full definition that captures the essence of the term variable. They assert that this difficulty may be partly lessened by considering how variables are used and work. Philipp (1992) puts variables into seven categories depending on their usages,

1. Labels f, y in $3f = 1y$ (3 feet in 1 yard);
2. Constants Π, e, c;
3. Unknowns x in $5x\text{-}9\text{=}91$;
4. Generalised numbers a, b in $a + b = b + a$;
5. Varying quantities x, y in $y = 9x\text{-}2$;
6. Parameters m, b in $y = mx + b$;
7. Abstract symbols e, x in $e{*}x = x$.

In this study the word variable is used in a similar way, that is, by the word variable I will mean broadly 'letter standing for numbers'.

However, research conducted in many countries indicates that students experience difficulties on their journey to learning the concept of variable. Although it is so fundamental and so difficult to learn for some, we do not know enough about teachers' or prospective teachers' knowledge base for teaching this concept; in particular subject matter knowledge and pedagogical content knowledge.

1.2 Need for this study

The reasons for such a lack of research on teachers' subject matter and pedagogical content knowledge of variables are various. One of the reasons for scarcity of research in this field is that "pedagogical content knowledge" is relatively new notion (Even & Tirosh, 1995; Shulman, 1986). Another reason is that conceptions of the role of the teacher and its importance in relation to pupils' learning have undergone dramatic changes over the years. In the past, teacher had been regarded as the deliverers of facts and procedures. Within this conception, teachers' knowledge was regarded as very important in the process of learning. However, the subject matter knowledge of teachers was defined as the *number* of courses the teachers had taken in the natural sciences; and pedagogical content knowledge as *instructional outcome*, namely the marks students had taken from *standardised tests.* (Ball, 1991; Begle, 1979). Hence, this phase of research did not study the interactions between subject matter and pedagogical content knowledge.

During 1960s and 1970s the role of the teacher is regarded as implementing an expert made curriculum. That is, the teacher is seen as manager and facilitator who watch students while they are learning directly from ready made curriculum (Even & Tirosh, 1995).

One of the incentives that are related to such a conception of teaching is expressed by Herscovics (1989). Discussing the reasons why only a minority of students achieve a reasonable grasp of algebra content in secondary school, Herscovics (1989) focuses on cognitive obstacles. He goes on to claim that due to these obstacles most of the secondary school students are unable to understand the topics in elementary algebra. He moreover belittles those efforts to devise some learning environments in which students can overcome these obstacles. He writes that:

> Some people are convinced that it is due essentially to inadequate teaching. They strongly believe that the problem is mainly one of instruction and that "if only we could train teachers to teach well, most students would understand." This viewpoint is both **optimistic** and **simplistic**… (p. 60, emphasis added).

Hence, research during 1960s and 1970s did not study the teacher's knowledge rather they studied the teacher behaviours such as classroom management, questioning etc.(Even & Tirosh, 1995). On the other hand Sutherland (1991) and Tall (1989) discuss their concerns about the focus on cognitive obstacles in teaching and learning of algebra, and criticise the viewpoints similar to the ones mentioned above. Thus, they invite researchers to develop a variety of teaching strategies and examine the effects of each on the learning of algebra. The conceptions of teaching and its importance in relation to learning that are similar to Sutherland's view are now widely shared among educators. Support for this last claim comes from the following quote from Even and Tirosh (1995) and later in the quote from Fennema and Franke (1992).

It is now widely accepted that the teacher's role in promoting learning involves

> setting mathematical goals and creating classroom environments to pursue them; helping students understand subject matter by representing it in appropriate ways; asking questions, suggesting activities and conducting discussions. Subject matter knowledge is much more critical for this new role of the teacher (Even & Tirosh, 1995, p. 2).

As a result, currently "No one questions the idea that what a teacher knows is one of the most important influences on what is done in classrooms and ultimately on what students learn." (Fennema & Franke, 1992, p. 147). Therefore, teachers' knowledge is a critical concept in teaching, but it is a concept of varied definition.

Although common sense suggests that teacher knowledge is "a large, integrated, functioning system with each part difficult to isolate" (Fennema & Franke, 1992, p. 148) studying teacher knowledge by looking at its components is the most common approach taken in this field. However, the common assumption of these studies is that individual components of teacher knowledge are interrelated. That is, while they are studying the effects of individual components on students' learning, some of the researchers (e.g. Ball, 1991; Wilson, Shulman, Richert, 1987; Begle, 1979), hold the assumption that these different components are interrelated.

Even and Tirosh (1995) are the only researchers who have examined the relationships between SMK and PCK; they did so in the context of functions and undefined mathematical operations. As a result of this study, they conclude that their prospective teachers do not know the reasons behind the rules; therefore they opt to explain these as rules to be followed. Furthermore, they assert that the issue of the nature of the interactions between SMK and PCK needs further investigation. Hence, there is little evidence to support the existence of and illustrate the nature of the interactions between these different components.

One of the main focuses of this study therefore will be the nature of relationships between two components of teacher's knowledge; namely the nature of the relationship between subject matter knowledge and pedagogical content knowledge in the context of variables.

Another focus of this study is the state and nature of prospective teachers' SMKv and PCKv. In addition to the reasons so far discussed, the motives for this study comprising aims related to SMKv and PCKv can be given as followings. Sfard and Linchevski (1994) complain about the scarcity of research on older students' understanding of variables. In particular, they complain about the scarcity of studies that examine the passage from the 'algebra of a fixed value' to the 'functional algebra.' They assert that

> Among the issues that should be addressed are such questions as student's ability to think about algebraic formulae in terms of functions and his or her readiness to apply this outlook whenever appropriate (p. 108).

In addition, I considered that there had been no research done dealing with prospective teachers' SMKv and PCKv in Turkey, therefore I thought it was worth carrying out research in Turkey. The underlying reason for such a consideration is related to the social, cultural and pedagogical aspects of education system in Turkey which may be different from other countries.

Furthermore, research into prospective teachers' SMKv and PCKv is beneficial since it gives us an idea of future teachers' knowledge base for teaching variables. The results of this study can help teacher education programmes to rethink about their approaches to train prospective teachers.

That is, this study can be helpful for teacher instructors in Turkey. For example, teacher instructors should be able to get an idea about the state of prospective teachers' knowledge about learning and teaching this concept. They can consider those aspects that are lacking (if any) then they may try to adjust their training approaches according to these.

For such an investigation the following general and specific research questions which are deemed to be important are formed:

1.3 The purpose of this study

The main purpose of this research is to investigate Turkish prospective teachers' subject matter knowledge of variables as a source of their pedagogical content knowledge of variables. General and specific research questions according to this purpose are as follows:

1.3.1 General Research Questions

1. What is the state and nature of Turkish prospective mathematics secondary teachers' subject matter knowledge of variables?
2. What is the state and nature of Turkish prospective secondary mathematics teachers' pedagogical content knowledge of variables?
3. Is there any relationships between subject matter knowledge of variables and pedagogical content knowledge of variables, if so what is the nature of such relationships?

After reviewing the relevant literature, different aspects of SMKv and PCKv are identified and following specific research questions are formed.

1.3.2 Specific Research Questions that are related to Subject matter knowledge of variables

What is the state and nature of Turkish prospective secondary teachers' subject matter knowledge in;

1. The Principal uses of variables?
2. The Awareness of different roles of variables?
3. Flexibility, versatility and connectedness among different roles and uses?

1.3.3 Specific Research Questions that are related to Pedagogical content knowledge of variables

What is the state and nature of Turkish prospective secondary mathematics teachers' pedagogical content knowledge of;

4. Knowledge about ways of presenting the subject matter?
5. Curriculum knowledge?

1.3.3 Specific Research Questions that are related to the relationships between subject matter and pedagogical content knowledge

6. What is the nature of the relationships (if any) between SMKv and PCKv?

1.4 The structure of the thesis

This chapter then has offered a first quick look at what this study is about. Chapter Two will discuss integrated knowledge on several bodies of work; namely the studies on learning and teaching of variables, studies on teacher and prospective teacher subject matter knowledge and pedagogical content knowledge. The research methodology, which essentially derives from the use of a questionnaire with fixed and open questions supported by the use of selected interviews, that will be used to investigate research questions in the study will be discussed in Chapter Three. I will also provide accounts of procedure, size of sample, sample selection method, and choice of questions in Chapter Three.

Chapter Four will report and discuss the analysis of subject matter knowledge of variables (SMKv) and pedagogical content knowledge of variables (PCKv). The results suggest that many prospective teachers are successful in manipulating variables; however they have problems in moving flexibly between different meanings and representations. The pedagogical content knowledge began to develop prior to exposition to pedagogy courses and teaching experiences. Objects and numbers are identified as two main forms of representation that they would use to explain ideas relating to manipulation of symbols.

Chapter Five will report and discuss the findings in the context of the nature of the relationships between SMKv and PCKv. The results indicate that there is a complex interaction between subject matter knowledge of variables and pedagogical content knowledge which may involve the prospective teachers' own learning experiences, general pedagogical knowledge and the robustness of one type of knowledge.

Chapter Six will summarise the findings in the previous chapters and discuss the wider implications of this research as well as presenting recommendations for future research.

CHAPTER 2

A Review of the Literature

INTRODUCTION

In this chapter I will discuss the research reports and theoretical papers which I used to form my theoretical frameworks for subject matter knowledge and pedagogical content knowledge in the context of variables. I will begin by discussing what teacher's knowledge entails, in section 2.1. This discussion will be given in the context of the relationship between teacher's knowledge and instructional outcome in general. While giving this I will discuss the developments in the research field on teacher's knowledge. How this kind of inquiry started, how it has developed, and at what stage it is now will be the base points of this discussion.

After this, in section 2.2, I will continue by discussing research reports that examine the links between subject matter knowledge and instructional outcome in the context of mathematics. While discussing this, I will also consider a few research reports that particularly study the relationships between subject matter knowledge and pedagogical content knowledge. This review together with the review on knowledge of variables will shed light on defining the subject matter knowledge of variables (SMKv) (section 2.3), and pedagogical content knowledge of variables (PCKv) (section 2.4). Finally in section 2.5, I will present a summary and discuss the main points of the chapter.

2.1 Studies on the relationship between teachers' knowledge and instructional outcome

There have been very few studies that particularly study the relationship between subject matter knowledge and pedagogical content knowledge. In contrast, there have been many studies (e.g. Ball, 1991; Even, 1990; Shulman, 1987; Leinhardt & Smith, 1985) that try to find some answers to the question "How much knowledge is necessary to become an effective teacher?" Since there are different ways of defining "effective teaching" and also there are different ways of regarding knowledge, there have been different answers to the above question. These differences inevitably stem from different visions of learning, teaching, the role of the teacher in teaching mathematics, and of the student in learning mathematics. These different visions have been developed through research on learning, teaching and teachers' knowledge.

As Ball (1991) points out, the first attempts to identify the characteristics of effective teacher were based on pupils' assessments of their best teachers. These studies reported that good teachers were enthusiastic, helpful, and strict and also knew the subject matter well (Hart, 1934).

The weakness of such claims led researchers to define "effective teaching" as teaching that results in measurable student learning (Ball, 1991). The National Longitudinal Study of Mathematical Abilities, which followed 112,000 students from over 1500 school in 40 states of America during the 1960s, was conducted to classify teacher characteristics associated with student achievement (Fennema & Franke, 1992; Ball, 1991). In this study,

> [t]wenty teacher characteristics were studied, including years of teaching experience, credits in mathematics, having a major or minor in mathematics, personal enjoyment of mathematics, and philosophical orientation to learning. Overall, neither teacher background characteristics nor teacher attitudes were strongly related to student learning; significant positive relationships were found in fewer than 30 percent of the possible cases. No single teacher characteristic proved to be "consistently and significantly correlated with student achievement" (Begle and Geeslin, 1972). Begle (1979) concluded from these results that many widely held beliefs about good teaching "are false, or at the very best rest on shaky foundations" (p. 54). (Ball, p.2)

As a result of this research Begle (1979) claims that "the effects of a teacher's subject matter knowledge and attitudes on student learning seem to be far less powerful than many of us assumed," (p. 53, cited in Ball, 1991, p. 2-3).

With similar assumptions to those underlying the research of Begle, Eisenberg (1977) examined the links between teachers' subject matter knowledge and instructional outcome in the context of algebra. Eisenberg (1977) confirms the results of Begle (1972) after testing the knowledge of 28 teachers in algebra, looking for connections to students' performance on standardised tests. Both researchers report that what teachers know about mathematics is not a significant influence on what their students learn (Bromme, 1994).

Subsequent research (e.g. Rosenshine, 1979) questioned the assumptions underlying this research. For example, they questioned the reasonableness of using the number of courses in University level mathematics as a proxy for teachers' mathematical knowledge or using the students' performance on tests as a measure for teachers' effectiveness. Therefore, this phase of research shifted the focus to identifying specific behaviours associated with effective teaching.

Such a shift was also caused by the change of perception of the role of the teacher in the process of learning. During 1960s and 1970s the implementation of an expert made curriculum brought the assumption that "children could learn directly from ready-made curriculum materials while the teacher, instead of teaching, would adopt a role of manager and facilitator." (Even & Tirosh, 1995, p.2). Hence, during this phase most researchers who tried to identify characteristics of effective teaching went into classrooms and observed teachers' behaviours such as questioning and classroom management. As Sherin, Sherin and Madanes (2000) assert, the key driver for this research was the idea that adoption of certain behaviours enables teachers to affect students' learning. Therefore, once they identify and describe such behaviours, they prescribe these behaviours to teachers to use in their classrooms. Rosenshine (1979) summarises the result of this research phase in the following prescription:

> Large groups, decision making by the teacher, limited choice of materials and activities by students, orderliness, factual questions, limited exploration of ideas, drill, and high percentages of correct answers. (p. 47, as cited in Ball, 1991)

However, within this phase of research it became clearer that identifying such behaviours is difficult without reference to the given content of instructional interaction (Bromme, 1995, Even & Tirosh, 1995). They saw that teachers work with a broad range of students who have different learning styles and needs. They, therefore, come to appreciate the complexity of classrooms and the job of teaching (Ball, 1991).

Researching teaching from a cognitive perspective is the approach of later research for addressing these difficulties. The research in this phase not only examines teachers' actions but also the underlying reasons for such actions (Sherin *et al*, 2000). The goal of this phase is to explain why teachers do what they do while teaching. Therefore, they try to give answers to questions beginning with *why* and *how* (Brophy, 1991). Furthermore, while explaining such actions they use cognitive terms: teachers do what they do because they have (or don't have) certain *knowledge*. Thus, teaching was no longer regarded as a set of isolated behaviours but instead as a complex cognitive process (Sherin *et al*, 2000). Ball (1991) summarises the central issues of this phase of research on teacher knowledge from Clark and Yinger (1979) as:

> new approach to the study of teaching assumes that what teachers do is affected by what they think. This approach, which emphasises the processing of cognitive information, is concerned with the teachers' judgement, decision making, and planning. The study of the thinking processes of teachers--how they gather, organise, and interpret, and evaluate information--is expected to lead to understandings of the uniquely human processes that guide and determine their behavior. (p. 231)

This approach, which was new at that time, is later criticised by Bromme (1995) since within this phase, the given primary reference with respect to cognition was subject matter knowledge. By this criticism, he communicates the idea that in their necessary simplification of the complexities of classroom teaching, investigators focused on primarily one aspect of classroom life: the content of instruction, the subject matter.

Shulman and his colleagues at Stanford University pioneered a way of including other references to teacher cognition. As part of a research program which focuses on how beginning teachers learn to teach, the researchers divided the "knowledge base" for effective teaching into the following seven different categories: *subject matter knowledge, pedagogical content knowledge, general pedagogical knowledge, curriculum knowledge, knowledge of learners, knowledge of school contexts*, and *knowledge of educational aims*. In addition, they discuss the major "sources" of teacher knowledge- the routes that need to be taken to reach the knowledge base for teaching. According to them this base is developed by the teacher's own subject matter knowledge, knowledge of curriculum and other educational materials, materials based on the research literature, and actual teaching experience (Shulman, 1987).

The particular importance of Shulman and his colleagues' research lies in their notion of 'pedagogical content knowledge'. By stating that this category of knowledge exists they are actually making an important claim (Sherin *et al*, 2000). With this claim, they assert that teachers need to know not only some general pedagogy and the subject they teach but also subject specific pedagogy, which is "that special amalgam of content and pedagogy" (Shulman 1987, p. 8). This amalgamation is revealed by Shulman (1986) as "in a word, the ways of representing and formulating the subject that makes it comprehensible to others" (p.9), and he goes on to write that it includes:

> an understanding of what makes the learning of specific topics easy or difficult; the conceptions and preconceptions that students of different ages and backgrounds bring with them to the learning of those most frequently taught topics and lessons (p. 9).

Shulman also argues that pedagogical content knowledge is developed through "pedagogical reasoning"; by which novice teachers restructure their subject matter knowledge and become expert teachers (Wilson *et al*, 1987). That is, teachers employ this reasoning while thinking about the ways of communicating their subject matter knowledge to the students. In order to communicate their knowledge of subject matter, teachers have to consider many factors. They have to take into account students' preconceptions and misconceptions. They have to generate appropriate analogies, explanations and examples to explain the subject matter. They have to think about ways of engaging a group of students in an activity that facilitates learning (Gudmundsdottir, 1991).

The concept of 'pedagogical content knowledge' of Shulman and his colleagues has given rise to considerable interest among subsequent researchers (e.g. Even & Tirosh, 1995; Ball, 1988; Leinhardt & Smith, 1985) on teacher knowledge. It has influenced the research community significantly by helping them to refocus on the importance of the teacher's subject matter knowledge and its relationship to teaching. However, it should be pointed out that this important research has attracted its own share of criticism from the research community.

For example, Bromme (1995) examines the concept of pedagogical content knowledge "from a special perspective, namely as an empirically grounded, or yet to be grounded hypothetical construct." (p. 205). In his criticism he tries to imply that a major part of Shulman's work is done by his category labels without analysing them closely. That is, Bromme asserts that Shulman reaches such category labels without researching each aspect intensively. Shulman's theory is not alone in attracting criticism from Bromme. He also criticises another influential research program by the same token, which is conducted by Gaea Leinhardt and others. However, as Sherin *et al*, (2000) point out although Leinhardt and her colleagues' work is reminiscent of Shulman's in its early stages, later they closely analyse categories of teachers' knowledge by "pushing downward into each of these categories, in order to characterise some of the particular types of knowledge found there." (p. 361).

In fact, the complexity of relationships between teachers' subject matter knowledge and their teaching is more explicitly elaborated by a number of studies conducted by Leinhardt and her colleagues. For example, Leinhardt and Smith (1985) investigate the relationship between experts' subject matter knowledge and their classroom behaviour in the context of fractions. Their study is conducted within the expert-novice research tradition and based on cognitive psychology which includes both proposition (declarative) and procedural representation of knowledge as well as the use of goals and plans.

> "Declarative knowledge consists primarily of the known facts in a particular domain, while procedural knowledge represents the algorithms and heuristics that operate on those facts." (p. 248).

As can be seen in the above quote they go down into subject matter knowledge and give an account of the types of knowledge within this knowledge. They use the same approach for analysing other types of teacher knowledge as well. They do this by exploring the nature, level, and use of subject matter knowledge of expert and novice arithmetic teachers using interviews and sorting procedures and subsequently observing their lessons.

They perceive teaching as a cognitive skill. According to this perspective, teaching involves two core areas of knowledge: *lesson structure* and *subject matter.* Lesson structure is given as the skills needed to plan and run a lesson smoothly, to pass easily from one segment to another, and to explain material clearly. Subject matter knowledge is defined as "concepts, algorithmic operations, the connections among different algorithmic procedures, the subset of the number system being drawn upon, the understanding of classes of students errors, and curriculum presentation." (p. 247).

They state that there is substantial variability in teachers' knowledge of fundamental fractions concepts. They show differences between novices and experts as well as differences in levels of subject matter knowledge among experts. They report that the teachers in their sample present the concept of fractions differently. They write that there are differences both at conceptual level and at procedural level in the information presented. They report that some teachers have relatively rich conceptual knowledge of fractions, whilst some of them rely on precise knowledge of algorithms. Furthermore, teachers emphasise different things in their presentations and introduce fraction concept differently. They also use different representation systems: number line, regional, and numerical. As a result it is asserted that teachers rely heavily on their subject matter knowledge while presenting this knowledge to students. This study which analyses what it means to know mathematics contributes to our understanding of the essence of subject matter knowledge.

In a paper, in which Leinhardt and her colleagues report findings from most of their previous studies, Leinhardt, Putnam, Stein and Baxter (1991) consider good instruction as the one which expert teachers employ. According to them expert teachers are flexible, precise, and parsimonious planners. In examining how teachers use their subject matter knowledge in their instructional processes, they focus on *agendas, curriculum scripts, explanations,* and *representations*. According to them the agenda is the teacher's dynamic plan for a lesson.

> It is a mental plan that contains the goals and actions for the lesson. The function of the agenda is that of a map or chart for the flow and landscape markers of the lesson; it lays out the lesson segments and the strategy for explaining the mathematical topic to be taught. (p. 89)

A *curriculum script* provides the overall goal structure for the content presentation for a particular lesson; *explanation* is an activity in which teachers communicate subject-matter content to students by drawing on various *representations* of the target information (Leinhardt *et al*, 1991). They report that there are differences between experts' and novice teachers' agendas, scripts, explanations and representations since expert teachers have more highly organised systems of knowledge, and they therefore assert that subject matter knowledge plays a critical role in these instructional processes. For example, expert teachers are reported to have richer and more detailed agendas than novice teachers. Expert teachers produce twice as many lines of responses as the novices in describing lessons they are about to teach, they make more explicit references to testing and checking students' understanding as the lesson progresses. To sum up, this study helps us to conceptualise the intricate relationship between subject-matter knowledge and actual classroom instruction.

The studies that investigate SMK and PCK in the context of mathematics also have their foundations in the works of Shulman and Leinhardt. In the next section, I will discuss those studies which examine mathematics teachers' and prospective teachers' SMK and PCK.

2.2 PCK and SMK in the context of mathematics

In a study which is also embedded in the expert-novice paradigm, Even, Tirosh and Robinson (1993) examine differences in connectedness in instruction between two novice teachers and an expert teacher. They do this by means of lesson plans, lesson observations, and post-lesson interviews, in the context of equivalent algebraic expressions. They believe that creating classrooms where making connections is emphasised is essential to help the students construct understandings of the subject matter. They report findings related to differences between expert and novice teachers with respect to the connections in the representation of the subject matter, and to connections in relation to what is learnt. According to them these connections have two components:

1. Content connections *across* various concepts, representations, topics and procedures.
2. Lesson connections *within* different lesson segments and series of lessons (p. 51).

They report that although all three teachers' lessons are teacher-centred with no emphasis on students' investigations and communications, there are major differences in connectedness with respect to planning, teaching and post-lesson reflections between the expert and novice teachers. For example, only the expert teacher plans to connect the new lesson with the previous lesson through contexts which are familiar and interesting to students, whereas novice teachers do not seem to connect different lesson segments.

They claim that one of the possible reasons for the substantial differences in the occurrence of connectedness between expert and novice teachers might be subject-matter knowledge. Their view of subject matter knowledge of mathematics resembles Ball's (1988) characterisation of mathematical knowledge. According to Ball (1988) mathematical knowledge includes both knowledge *of*, and *about*, mathematics. Knowledge of mathematics refers to understandings of facts, ideas, theorems, mathematical definitions, concepts, procedures and connections among them. Knowledge about mathematics is a more general knowledge which includes

> "the nature of knowledge in the discipline— where it comes from, how it changes, and how truth is established; the relative centrality of different ideas, as well as what is conventional or socially agreed upon in mathematics versus what is necessary or logical." (Ball, 1988, p. 4).

She further asserts that these two types of knowledge are interconnected. Therefore, Even *et al* (1993) claim that "content connections" include connections not only within 'knowledge *of* mathematical knowledge' but also connections between the two types of mathematical knowledge. As a result they propose that the lack or weaknesses of such connections cause the differences in the occurrence of connectedness between expert and novice teachers.

Another factor which contributes to the differences in connectedness of instruction between expert and novice teachers is given as pedagogical content knowledge. Their perception of pedagogical content knowledge is acquired from Shulman's work which I discussed previously. However, their research approach is similar to Leinhardt's expert approach. Although this study uses the expert-novice research paradigm as a means of characterising teachers' dimensions of expertise, it examines the differences between expert and novice teachers in a specific context, namely equivalent algebraic expressions. In this respect, it belongs to a new trend of analysis of subject matter knowledge for teaching which concentrates on specific content areas. Paying attention to connectedness in teaching of a particular content area proves useful for analysing both subject matter knowledge and teaching of the subject matter.

In fact, in a project carried out by Askew, Brown, Rhodes, Johnson and Wiliam (1997) we see that distinguishing between approaches taken by teachers of numeracy by considering their orientations toward teaching mathematics as *connectionist, transmission,* and *discovery* helps to analyse the links between different components of teacher professional knowledge. Furthermore, it makes it easy to see which approach is more effective in terms of pupil gains. One of the results of this project is stated as that *connectionist* teachers are more successful than *transmission* and *discovery* teachers in helping their students understand numeracy.

> *[I]t was clear that those teachers with a strongly connectionist orientation were more likely to have classes that made greater gains over the two terms than those classes of teachers with strongly discovery or transmission orientations.* (p. 24, original italic).

These categories were built up from data gained from questionnaires completed by 90 teachers, observations of 54 mathematics lessons of 18 case study teachers and 30 lessons of 15 validation teachers, three interviews with each of the 18 case study teachers and two with each of the 15 validation teachers.

Although connectionist teachers' students gained more numeracy knowledge, they report that the aspects of teachers' subject matter knowledge that make a difference in terms of pupil gains are much harder to identify. Their framework for analysing numeracy subject knowledge consists of two aspects: *knowledge of content*-knowledge of facts, skills and concepts of the numeracy curriculum, and *knowledge of relationships*- knowledge of how different concepts of mathematical content relate to each other. Each of these aspects is measured along the dimensions which are given as fluency, scope, links, explanation, depth and understanding.

Only the variable 'depth' (the proportion of links which were explained in conceptual terms rather than by rule-based connections) is reported to be moderately related to pupil gains. None of the other subject matter knowledge variables are related to pupil gains. Similarly, they find no relationship between formal mathematical qualifications and pupil gains. Furthermore, they don't find any differences between teachers in their sample in terms of giving correct answers to straightforward problems based on the content of the numeracy curriculum. However, they report that this does not imply that mathematical subject knowledge is not important, in contrast the connectedness of their mathematical knowledge in terms of the depth and multi-faceted nature of their meanings is important. Once again the belief that not the level of formal qualification but the nature of the knowledge matters for teaching mathematics for understanding is confirmed by this research. However, this research is one of those few studies which conclude that there is link between teachers' subject matter knowledge and their students' gains.

As Darling-Hammond (1999) cites from Byrne's (1983) review of thirty studies which study the links between teachers' subject matter knowledge and student achievement, the results are mixed. 14 of these studies show no positive relationship between subject matter knowledge which is measured by standardised or researcher-constructed tests and student achievement, while 17 show a relationship. However, it should be noted that many of the "no relationship" studies, as Darling-Hammond notes from Byrne, have "so little variability in the teacher knowledge measure that insignificant findings were almost inevitable." (p. 6).

The categories of subject matter knowledge suggested by Askew *et al* (1997) are similar to Ball's (1991) proposed features for subject matter knowledge of mathematics. Ball (1988, 1991) argues that *connectedness*, *legitimacy* and *truth value* are required characteristics of substantive mathematical knowledge for effective teaching. Truth-value is characterised as mathematical correctness which entails knowing the conditions and limits of an idea. Legitimacy includes the justification and explanation. It entails being able to explain mathematics and being able to access those explanations when needed. Therefore, it requires explicit ways of knowing ideas and also connectedness among ideas.

Ball (1991) reports the analysis of a single question about place value taken from a series of interviews conducted with teacher education students, half of whom were mathematics majors intending to teach secondary school and half of whom were prospective elementary teachers with no academic majors.

As a result of this research, Ball (1991) reports that "seriously examining and analysing teachers' knowledge of mathematics is a complicated endeavor", and asserts that teachers' subject matter knowledge of mathematics "interacts with their ideas about the teaching and learning of mathematics and their ideas about pupils, teachers, and the context of classrooms" (p. 19). Therefore, she points out that the relationship between knowledge of mathematics and teaching is not direct. This point is similar to Askew *et al* findings about the relationship between subject matter knowledge and pupils' gains being nonlinear.

Chazan, Larriva and Sandow (1999) also agree that describing qualities of teachers' substantive knowledge of the mathematics they teach potentially is more useful than cataloguing the amount of coursework taken. In order to examine the issue of the qualities of teachers' knowledge of mathematical content, they explore pre-service teachers' knowledge related to teaching the solving of equations. Their sample is drawn from student teachers. They use tasks which have been explored in the existing literature on the solving of equations and questions about teaching students to solve linear equations in their interviews to elicit trainee teachers' understanding of solving equations, in particular whether they have a conceptual or procedural understanding. They present the data obtained only from one of the interviewees. They claim that although this student has a conceptual understanding related to solving equations, she lacks the resources to teach her students conceptually. Therefore, they assert that it is questionable to use descriptions such as conceptual or procedural understanding for an examination of teachers' substantive knowledge of mathematics.

They write that conceptual understanding is not something that one either has or does not have. Therefore, they raise the question whether distinctions such as conceptual, procedural are necessary for describing teachers' substantive knowledge of mathematics. However, it should be pointed out that they deduce that the student in their study has a conceptual understanding of solving equations since she uses a graphical representation of equations as lines on coordinate systems and then finds the intersection point of two lines to solve a system of two equations. When she is asked how she would explain solving equations to her students, she always tends to use lines even if it is not useful to use this representation. This implies that the researchers perceive conceptual understanding of solving equations as using lines. This perception on one side, and the student insufficiency in explaining solving equations to her student on the other side forces the researchers to conclude that characterising subject matter knowledge as conceptual and procedural is not sufficient for studying subject matter knowledge to teach for conceptual understanding. It is indeed not sufficient to characterise subject matter knowledge as just conceptual vs. procedural, since subject matter knowledge has many aspects.

As I have already suggested in previous reviews of papers, subject matter knowledge need to be connected across concepts by their meanings, representations, their places and importance within mathematics. In this case the student in their study seems to lack these connections; in particular connections between graphical representation and symbolic representation of equations. If she had those connections she could have used the appropriate representation when it was needed. As Even (1998) points out "Connectedness between different representations develops insights into understandings of the essence as well as the many facets of a concept." (p. 105). Therefore, analysing a teacher's subject matter knowledge of a topic requires careful attention to all interrelated aspects of that topic within mathematical knowledge.

In fact, as Tirosh (2000) illustrates, if the characterisation of subject matter knowledge is made appropriately not just as conceptual or procedural then it is indeed of use to describe and analyse teachers' knowledge of mathematics. In a project which was designed to firstly describe and then enhance prospective elementary school teachers' knowledge of rational numbers, Tirosh bases her framework for knowledge of rational numbers on three dimensions namely algorithmic, intuitive, and formal. These dimensions were obtained from relevant literature on students' understanding of rational numbers. That is, she uses an analysis of the body of knowledge on children's ways of thinking about this topic, and then differentiates different aspects of understanding rational numbers.

According to her, pedagogical content knowledge has many components. Her framework of pedagogical content knowledge is founded upon Shulman's theory. However in her analysis, she further divides pedagogical content knowledge into different components. In this paper she examines only two components: knowledge of different ways of presenting the concepts and operations of rational numbers and knowledge of common conceptions and misconceptions held by students and the possible sources of this knowledge. Her research proves useful to analyse and enhance her prospective teachers' knowledge of division of fractions.

Tirosh's perception of pedagogical content knowledge is similar to Even's understanding of pedagogical content knowledge. Even (1990, 1993) and Even and Tirosh's (1995) studies are among those few research studies which particularly investigate the links between subject matter knowledge and pedagogical content knowledge. Furthermore, Even's account of subject matter knowledge which she describes more comprehensively in Even (1990) is multifaceted. Even (1990, 1993) investigates teachers' subject-matter knowledge and its interrelations with pedagogical content knowledge in the context of teaching the concept of function. She collects data by means of an open-ended questionnaire concerning knowledge about function from 152 prospective secondary teachers. In the second phase, she interviewed an additional 10 prospective teachers after administering the questionnaire. She based her analysis of subject matter knowledge on these aspects: 1. *Essential features*, 2. *Different representations*, 3. *Alternative ways of approaching*, 4. *The strength of the concept*, 5. *Basic repertoire*, 6. *Knowledge and Understanding of a concept*, 7. *Knowledge about mathematics.*

In these aspects, she basically tries to include all different components of understanding and knowledge of a concept which may be required to teach that topic. For example, she claims that teachers' concept image (Tall and Vinner, 1981) of function must be "correct".

Concept image is explained by Tall & Vinner (1981) as

> "the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes. It is built up over the years through experiences of all kinds, changing as the individual meets new stimuli and matures."

By "correct" concept image Even means that teachers must be able to identify a concept in familiar and unfamiliar instances by using "an analytical judgement as opposed to a mere use of a prototypical judgement" (p.523). They must hold both conceptual and procedural understandings of the function concept in all different representations. Furthermore, according to her framework teachers must know the place where the function concept stands in mathematics and its importance.

Even's analysis shows that most of her subjects do not explain the importance and origin of the univalence requirement, and very few could appreciate the arbitrary nature of functions. According to her, this limited conception of function influenced the subjects' pedagogical thinking. Therefore, when describing functions for students, many chose to provide students with a rule to be followed without concern for understanding. In addition, many use their limited concept image and tended not to employ modern terms.

Even's papers referred to above, can also be classified amongst those research studies that focus mainly on studying teachers' understanding of specific mathematical topics which are included in the school curricula. The findings of these research studies are that many teachers do not have a solid understanding of the subject matter they teach. These kinds of studies are labelled as "disaster studies" later in a paper reported by Even and Tirosh (1995). In this paper, they point out that from this research we can't learn much about how such problems affect teachers' reactions to students' questions and ideas related to specific mathematical topics. They close this gap by investigating the interconnections between "subject-matter knowledge, knowledge about students, and knowledge about ways of presenting the subject-matter" in the context of functions and undefined mathematical operations (p.1).

Their characterisation of subject matter knowledge consists of two levels, namely "*knowing that*" and "*knowing why*", which are two kinds of understanding of the subject-matter that Shulman(1986) proposes as those which teachers need to have. Although this distinction does not correspond exactly to Skemp's (1976) distinction between 'instrumental mathematics' and 'relational mathematics', there would appear to be some similarities.

> We expect that the subject-matter content understanding of the teacher be at least equal to that of his or her lay colleague, the mere subject-matter major. The teacher

> need not only understand *that* something is so; the teacher must further understand *why* it is so (cited in Even and Tirosh ,1995, p. 3).

"Knowing that", according to Even and Tirosh (1995) is the most basic level of subject matter knowledge which includes "declarative knowledge of rules, algorithms, procedures and concepts related to specific mathematical topics in the school curriculum." (p. 7). "Knowing why" on the other hand refers to the knowledge which "pertains to the underlying meaning and understanding of why things are the way they are…" (p.9). However, it should be noted that these two terms are used not only for distinguishing between two kinds of understanding of the subject matter but also for distinguishing between two kinds of each of two components of pedagogical content knowledge.

For example, the "knowing that" aspect of the "knowledge about students" component of pedagogical content knowledge includes knowing "common students' conception of thinking related to specific mathematical topics." (p. 13). "Knowing why" includes being able to understand the reasoning behind students' conceptions and anticipate sources for common mistakes" (p. 13, ibid). As a result of this study they report that many participants of their study do not attempt to understand possible sources of students' mistakes ("knowing why"). Even when they are asked directly why a particular student has a particular misunderstanding, they find it hard to explain the reasons.

However, the teachers' ability to demonstrate this understanding might be significantly affected by the circumstances in which the situation takes place in real teaching (Wilson, Shulman and Richert, 1987). On the other hand, Even and Markovits (1993) point out, teachers in interview settings or in answering the questionnaires may consider some factors such as giving a "rich response directed towards the student's misconception in which s/he can emphasise meanings as opposed to rituals that can be actively constructed by the student." (p. 42). Being able to give such a response may sometimes depend on the teacher's own subject matter understanding.

The following table shows a summary of discussion so far.

Table 2.1 Development of research on teacher's knowledge

Era	Conception of Teacher's Knowledge
Begle 60s and 70s	Number of courses taken during university
Rosenshine (70s)	Teacher's behaviours (Classroom management, questioning)
Clark and Yinger (late 70s)	Teacher's actions with underlying reasons (subject matter knowledge is the primary reference to analyse teacher cognition)
Shulman mid 80s	Pedagogical content knowledge enters into the equation and other types of teacher's knowledge as well
Leinhardt mid 80s	Pushing downwards into each aspects of teacher's knowledge
Ball, Even 90s	Subject and Concept/topic specific teacher's knowledge.

As can be seen in the above table there has been significant development on research on teachers' knowledge. This progression has been towards making a better characterisation of teachers' knowledge. The common aim of this research is to portray the qualities of the effective teacher. In particular they want to know the relationships between teachers' knowledge and instructional outcome.

During 1960s and 1970s, researchers who employed a process-product research paradigm could not prove the assumption that a teacher's knowledge is an important factor of being an effective teacher since their perception of teachers' knowledge, particularly subject matter knowledge is based on the number of courses taken at university (e.g. Begle, 1979; Begle & Geeslin 1972). Beside this they bypassed the intervening processes between subject matter knowledge and students' performances. In order to reach the students' gains one needs to consider factors that may have an affect on students' performances, such as the school, the learner and the teacher on action of providing learning opportunities to learners.

New mathematics movement, during 1960s and 1970s, gave a new role for teachers. As Even and Tirosh (1995) point out, teacher-proofed curricula, introduced during these days, assumed that students "could learn directly from ready-made curriculum materials, while the teacher, instead of teaching, would adopt a role of manager and facilitator (p. 2). As a result, the approaches of the most studies of teachers that were conducted in those days are depicted as "it is what the teacher *does* rather than what a teacher *is* that matters" (Medley, 1979, p. 13, cited in Ball, 1991). Therefore, they observe teachers' behaviours whilst teaching, such as classroom management and questioning. They try to describe effective behaviours that result in students learning (Sherin et al, 2000; Even & Tirosh, 1995; Ball, 1991; Brophy & Good, 1986). The given response to this approach in later research studies (mid 1980s) is that what matters is both *what* the teacher does and *why* s/he does it. They try to find answers to these questions by examining teachers' cognition, mainly subject matter knowledge of teachers (Bromme, 1995). Starting with Shulman, teachers' cognition (knowledge base for teaching) is divided into different category labels. The notion of pedagogical content knowledge comes onto the scene. Leinhardt and her colleagues closely analyse these categories by pushing downwards into each one. The work of Shulman and Leinhardt set the stage for future research.

The work of Shulman and Leinhardt also marked a change in conceptions of the teacher's role in promoting learning. After Shulman and Leinhardt, basing their frameworks onto Shulman's and Leinhardt's theories, subsequent researchers could delve into topic specific issues of the teacher's knowledge, in particular subject matter knowledge (Askew et al, 1997; Even & Tirosh, 1995; Even, 1993; Ball, 1991; Even, 1990). One of the emerging results of these studies on subject matter knowledge for teaching is that subject matter knowledge is connected across topics, concepts and rules (Askew et al, 1997; Even & Tirosh, 1995; Even, 1993; Ball, 1991; Even, 1990). Having this type of knowledge is an advantage point for teachers, because its content can easily be reconstructed to make it communicable to students. That is, turning this kind of subject matter knowledge into pedagogical content knowledge is easier.

In most of those studies so far cited, the prevalent form for the collection and presentation of data are *case studies*. The common purpose of these studies is to *describe* teachers' knowledge, in particular subject matter knowledge. The researchers were interested in what *exists* in teachers' knowledge. For these researchers, it is difficult to use an analytical approach in the *absence* of a thorough understanding of the teacher knowledge. Therefore, case studies, due to their variety of possible interpretations, prove appropriate to characterise the complexity of teachers' knowledge.

In fact, the studies so far cited have been able to describe features of teachers' knowledge and show that subject matter knowledge is a critical variable in teaching. In order to explore teachers' knowledge, the new phase of research can base its framework on the findings of these studies and can develop a *more* thorough understanding. Therefore, in this study I will be both generating hypotheses and, I hope, pursuing a very economical path toward building a body of knowledge on prospective teachers' subject matter as a source of their pedagogical content knowledge of variables.

An extensive account of how I formed my research methodology will be presented in the next chapter. Nevertheless, in the remainder of this chapter, I will discuss how I base my framework on the previous body of knowledge on teachers' cognition and on students' learning of the concept of variable. Therefore, the next section takes up the question of what I mean by "knowledge of variables" in this study.

2.3 Subject matter knowledge of variables

In the previous section, I pointed out that subject matter knowledge is a critical variable in teaching, but a concept of varied definition; this variance is a reflection of the variance in various proposed definitions of knowledge and understanding in mathematics education. The only common ground of these definitions of subject matter is its being knowledge of rules, algorithms, procedures and concepts, namely knowledge and understanding of mathematical topics. However, when we consider knowledge and understanding in mathematics education we also encounter varied definitions, that is, knowledge and understanding are also terms with multi-meanings and interpretations. As Sierpinska (1994) points out there are at least four kinds of models or theories of understanding in mathematics education.

1. concerned with hierarchy of levels of understanding. e.g. Van Hiele (1958).
2. focusing on developing a 'mental model', 'conceptual model', 'cognitive structure' and the like. e.g. Greeno's mental models; Lesh, Landau, Hamilton, (1983); Arzarello (1989); Dubinsky and Lewin's (1986) genetic decomposition.
3. seeing the process of understanding as a dialectic game between two ways of grasping the object of understanding. e.g Skemp's (1978) relational versus instrumental understanding, Sfard's (1991) operational versus structural understanding.
4. historico-empirical approach (Piaget & Garcia, 1989).

These different theories show that there is not a unified theory of knowledge and understanding of a topic. Each one has its own strong points in explaining relevant concepts. For example, Van Hiele's theory of understanding geometrical topics has been very widely adopted in mathematics education to explain the development of geometrical concepts in students' minds. However, theories that have foundations in 'seeing the process of understanding as a dialectic game' seem most appropriate for explaining the knowledge and understanding of variables. That is, I will propose that knowledge and understanding of variables appears in different forms, and it is a combination and integration of the different forms of knowledge into one concept that is empowering. Integrating procedural and conceptual understanding by the aid of symbolism, gives one power to move flexibly between them.

This interplay of different forms of understanding of symbols is explained by the notion of "procept". It is formulated by Gray and Tall (1994) to elucidate the phenomenon, in which the symbols function dually as both process and product, a notion that embraces the ambiguous use of *notation* to stand either for a *pro*cess or for the con*cept* produced by that process. For instance, the expression $2x+1$ may be conceived both as procedurally (multiply x by 2 add 1) and conceptually (perceive $2x+1$ as it is as a whole object), but the integration and combination of these two types of understanding with the aid of symbols may give one an opportunity to move flexibly between them while manipulating variables.

Sfard and Linchevski (1994) open their paper with a salient illustration to exemplify different perceptions and understandings of symbols in algebra. They begin with asking what we see when we look at an algebraic expression such as $3(x + 5) + 1$. They point out that the answer depends on what we are able to perceive and are prepared to notice. In certain situations we may see it as a computational process; a sequence of instructions to be taken. In other situations we may see it as a number, when we regard it as a whole. Yet in other situations we may perceive it as a function which maps every number x into another. Finally, in some situation we may regard it in a much simpler way by looking at its face value, as a mere strings of symbols which represents nothing.

The issues which are brought out by Sfard and Linchevski (1994), and Gray and Tall (1994) explain a very important phenomenon in the development of mathematical knowledge. Within this development, once expressions can be regarded as a whole, as objects, they can be manipulated at a higher plane of thought. In the context of variables these manipulations do not involve just operations with symbols, but also analysis of their roles, meanings, (and) referents. That is 'procepts' enable us to consider variables as objects whose *roles*, *meanings* and other relevant features can be analysed.

As many others point out; there are different roles and conceptions of variables: variables as *generalised numbers*, variables as *unknowns* or *constants*, variables as *parameters* or *arguments in functional relationships*, variables as *abstract* symbols (Bills, 2001; Ursini and Trigueros, 1997; Philipp, 1992; Usiskin, 1988; Kuchemann, 1978). This shows, as Usiskin (1988) points out, that "trying to fit the idea of variable into a single conception oversimplifies the idea and in turn distorts the purposes of algebra." (p.10). However, the principal uses of variables are mostly considered as using variables as *unknowns*, as *generalised numbers* and as *arguments* in functional relationships (Kieran, 1997).

Understanding variables, therefore, involves being able to ascribe these different roles to symbols to represent problem situations and to manipulate them. It also involves, as Ursini and Trigerous (1997) point out, being able to "distinguish between the different roles and shift from one to another in a flexible way, integrating them as components of the same mathematical object." (p. 254).

Bills (2001) also agrees that being able to shift from one role of variables to another in a flexible way can be related to one's meaningful performance in solving problems.

These views show that knowledge of variables is made up of different aspects. Making these distinctions between these aspects proves useful for describing students' understanding of variables. Therefore, subject matter knowledge of variables will be discussed under three different headings in this section: knowledge of variables in;

1. *The principal uses of letters,*
2. *Awareness of different roles of variables,*
3. *Flexibility, versatility and connectedness among different roles and uses.*

2.3.1 Principal uses of variables

In this section I will discuss what knowledge of variables in 'Principal Uses of Variables' involves. Namely, the understanding of the use of variables as *unknowns*, as *generalised numbers*, and as *arguments in functional relationships* will be discussed.

The conception of variables as *unknowns* is inextricably related to regarding algebra as a study of *procedures* for *solving* certain kinds of problems (Usiskin, 1988). He asserts that the key instructions in this conception of algebra are *simplify* and *solve*. However, before simplifying one needs to recognise and to see that there is something unknown in the problem, and that this recognised unknown can be found by first constructing and then by simplifying the equation within the restrictions of the problem. This process is called 'symbolising a problem by posing an equation' (Ursini and Trigueros, 1997) and they add that understanding variables as unknown also involves being able to substitute to the variable the value or values that make the equation a true statement.

The literature on students' understanding of variables reports that beginning students have difficulties in using symbols as unknowns to solve equations (Filloy & Rojano, 1989; Herscovics & Linchevski, 1994). It is reported that students tend to use *reverse* operations instead of using *forward* operations that requires using symbols as unknowns. For instance, when it is asked to do this problem "When 3 is added to 5 times a certain number, the sum is 40. Find the number!" some students may solve this problem by undoing, using reverse operations: they first take away 3 from 40 and then divide the resulting number by 5 to find the number. This way of solving this problem is called the *arithmetical* method. However, to solve this problem using *algebra* it needs to be represented as $5x+3 = 40$ and forming this equation requires forward operations. The ability to form equations by a forward approach is called the 'algebraic method'. The lack of ability to use forward operations is termed a cognitive gap (or didactic cut) between arithmetic and algebra (Filloy & Rojano, Herscovics & Linchevski, 1994).

However, solving this equation to find the unknown x does not require an ability to operate spontaneously with or on the unknown, since x appears on only one side of the equation. For those equations where x appears on both sides, then it is required that x is manipulated to find its value. It is suggested that this difference in solution methods is another place where the cognitive gap between arithmetic and algebra occurs (Hercovics &, Linchevski, 1994).

Sfard and Linchevski (1994) view using a letter as an unknown as a stage in algebraic development which is termed as '*algebra of a fixed value*' as opposed to '*functional algebra*', where letters represent changing rather than constant magnitudes. Using symbols in a context in which any number may be substituted for the symbol marks a shift in algebra from a science of constant quantities into a science of changing magnitudes (Sfard and Linchevski, 1994).

Sfard and Linchevski (1994) suggest that this second stage of algebraic development involves using letters as variables but they don't mention explicitly using letters as generalised numbers. This may be due to the fact that both conceptions of variables require conceiving symbols as any number not a fixed number, thus symbols may assume a range or series of values. However, 'letter as variable' implies more than just a range of values. Understanding letters as variables implies conceiving how one set of numbers changes in relation to another. Therefore, it should be noted that using letters as generalised numbers is one of the aspects of understanding variables which should be dealt with in a separate category..

The conceptions of variables as *generalised* numbers involve using letters to generalise already recognised numerical or geometrical patterns which requires being aware that a symbol can stand for a general indeterminate object (Ursini and Trigueros, 1997). According to Usiskin (1988) this conception of algebra is algebra as *generalised* arithmetic, and the key instructions within this conception are *translate* and *generalise.*

> Generalizing is the process of exploring a given situation for patterns and relationships, organizing data systematically, recognizing the relations and expressing them verbally and symbolically, and seeking explanation and appropriate kinds of justification or proof according to level (Bell, 1995, p.50).

For instance, divisibility properties, such as 'the difference between the third power of a whole number and the number itself is always divisible by 6', provide an example of using letters as general numbers to prove general properties.

The research on children's understanding of variables show that difficulties arise in the students' journey to conceiving letters as generalised numbers. (Lee & Wheeler, 1989; Kuchemann, 1981; Booth, 1984). For example, Kuchemann (1981) reports that 25% of third-year students gave a correct answer to the question whether l+m+n=l+p+n is always, sometimes, never true. This shows that many children have difficulty in allowing two different letters to have the same value. After a follow up study, Booth (1984) reports that there is a strong resistance to the conception of letter as generalised number, even within the context of a teaching program specifically designed to address this aspect of algebra.

Goulding and Suggate (2001) report that 61% of 201 PGCE (Postgraduate Certificate in Education) students have difficulties using letters as generalised numbers to prove that the addition of two odd numbers results in an even number. 19% of these students use only specific numbers or diagrams, 16% of them use two equal odd numbers, and 32% of them cannot complete an algebraic proof.

The conceptions of variables *as arguments in functional relationships* involves recognising how one set of numbers changes in relation to another irrespective of the representation used (table, graph, formula, etc.), finding out the values of one variable given the value of the other one (independent or dependent), and being able to symbolise a relation based on the analysis of the data of a problem. (Ursini and Trigueros, 1997). According to Usiskin (1988) the fundamental distinction between this and previous conceptions of letters is that, here, variables *vary,* and two key instructions under this conception of algebra are *relate* and *graph.* However, not only graph but also other representations table, formula are also involved in understanding letters as variables.

Kuchemann (1981) reports that only 3% of 11 year old students tested gave a correct answer to the question "Which is larger, 2n or n+2? Explain", a question which tests the existence of the understanding of letters as *arguments in functional relationships.*

Ursini and Trigueros (1997) reports that, even after several algebra courses, starting university students still have difficulties in understanding the principal uses of variables. Their understanding of each use of the variable concept remains at an action level where they produce mechanistic answers to routine questions.

In order to recap what constitutes this aspect of subject matter knowledge of variables, I would like to mention that this aspect of subject matter includes an ability to interpret, symbolise and manipulate symbols as numbers, whether it is a fixed, or varying. However, each of the usages has its own related way of interpreting, manipulating and symbolising.

2.3.2 Awareness of different roles of variables

In order to integrate different yet relevant uses of variables in one concept, first of all one needs to be competent in each use of variables. However, being competent in each use alone may not guarantee that one can integrate them into one concept, therefore one needs to be consciously aware of that each use of variables is a different layer of the same concept. This awareness plays an important role in forming the concept image of variables.

As I mentioned earlier, Even (1990) asserts that teachers must have a "correct" concept image of functions while discussing subject matter knowledge of functions. I will use a different terminology to describe a similar phenomenon in the context of variables. I would assert that teacher's must not have a 'confused' concept image of variables. By this I mean that teachers must be able to state for sure what a letter can stand for in a given expression by using an analytical judgement as opposed to the mere use of a prototypical judgement.

For example, this involves recognising the different roles of variables in equations and expressions. In equations variables assume the role of the unknowns, whereas in the expressions they assume the role of generalised numbers. However, equations are constructed from expressions. They are related but the roles of variables are different.

In some cases there may be more than one variable with different roles in one context. Grasping what are the suitable roles of the variables in these contexts requires awareness of different roles of variables. And in some problems, one variable may even assume more than one role, and it may be necessary to ascribe these roles suitably throughout the solution processes.

Recognising the difference between usages of letters in manipulation of symbols and in other contexts can be another example of not having a confused concept image. For instance, 3m can be regarded as both 3 metres and 3 multiplied by *m*. In the first situation the role of the letter is not one of the principal uses of variables, it is used as an abbreviation of the word metre. However in the second case, the letter assumes the role of a number. Therefore, this aspect of understanding variables requires an awareness of these different usages of letters.

As Wagner (1981) points out, that "certain letters have acquired fixed connotations relative to particular contexts" (p. 126). That is, particular meanings of written symbols may be sometimes strongly attached to certain contexts. In these contexts, teachers need to be able to move themselves away from this attachment by setting prototypical judgement aside and to identify the role of the letter by analytical judgement.

Bills (1997) identifies how a stereotyped view of letters may affect one's problem solving by means of various problems. For instance, she gives the following task to teachers: "Write the equation of a straight line that passes through the point (m, c)." This question proves difficult for many teachers, since *m* and *c* have been already habituated to be used as the slope and the intercept in the ingrained form $y = m x + c$ for the equation of a straight line.

Cape (2000) conducted interviews with nine student teachers who completed a module called Teaching Early Algebra, in which she focused on their concept images regarding expressions, including the meaning of expressions and letters, equivalence, and the similarities between arithmetic calculations and algebraic expressions. She asked following question in the interviews "When I write down 3x+5, what does it mean to you?"

She reports that some of the student teachers have '*confused*' concept images of expressions and letters. For example some students could not say for sure whether x can stand for apples, people or only for numbers in the expression $3x+5$.

To sum up, this aspect of subject matter knowledge constitutes being aware that the concepts of variable have different uses and associated processes and these are the different layers and facets of one concept.

2.3.3 Flexibility, versatility and connectedness among different uses

Fluency in each use of variable and awareness of these uses alone is not sufficient for having complete subject matter knowledge of variables. Familiarity with different uses and the ability to translate and form linkages among them may create insights that allow a better, deeper, more powerful and more complete understanding of the concept of variable. For example, Hiebert (1988) writes that "meaning is carried in the relationships that students observe between symbols." (p. 342).

In a similar vein, Even (1998) points out

> The ability to identify and represent the same thing in different representations, and flexibility in moving from one representation to another, allow one to see rich relationships, develop a better conceptual understanding, broaden and deepen one's understanding, and strengthen one's ability to solve problems (p. 105).

In the context of variables, these connections can be formed through referents, meanings and roles of variables. For example, the number of referents of unknowns usually is one, whereas the number of referents of variables and generalised numbers is many. However, as Ursini and Trigueros (1997) point out, we recognise this only after having actually or mentally performed the necessary manipulations. Thus, considering the referents of variables allows us to, in a way, connect different uses of variables in one concept.

Similarly, considering different roles of variables in different contexts or within the same context also gives us an opportunity to connect different uses into one concept image. For example, a given symbol may assume more than a single role within a given problem.

The role of the variable may be interpreted at some point as unknown; later it may be required to interpret it as a parameter as in the case of finding the values of n that makes the equation $nx= 2$ is insoluble for x. The solution of this problem requires flexibility moving between parameter and unknown roles of n.

Bills (2001) illustrates how students' appreciation of the shifts in the role of variables can be related to their meaningful performance in routine tasks. The shifts she is pointing out involve: 1) 'variable' to 'unknown-to-be-found; 2) 'placeholder-in-a-form' to 'unknown-to-be-found'; 3) 'unknown-to-be-taken-as-given' to 'unknown-to-be-found'; 4) 'unknown-to-be-taken-as-given' to 'variable'.

In some contexts, variables can be used as one of a number of representations of some concept. For example, functions can be represented by using variables, graphics or using ordered pairs. Forming linkages between these representations is a sign of not only connected understanding of the function concept but also connected understanding of variables.

2.4 Pedagogical content knowledge of variables

Earlier, I pointed out that pedagogical content knowledge is the label used to designate one aspect of a teacher's knowledge. By referring to Shulman (1986, 1987), I assert that it is a blend of subject matter knowledge with pedagogical knowledge. Teachers need to do this blending in order to present subject-matter in such a way that it is comprehensible to students. As Kennedy (1990) points out, the key to pedagogical content knowledge is blend. Consequently, blending requires firstly knowing the subject matter and then knowing how to present that knowledge to students. In some sense pedagogical content knowledge is seen to include a restructured form of the subject matter knowledge. In fact, Leinhardt and her colleagues claim that subject matter knowledge means knowing a particular school topic in depth and it includes "knowledge about ways of representing and presenting content in order to foster student learning..." (Leinhardt et al, 1991, p. 88). Therefore, this restructured, in-depth knowledge is needed to communicate this knowledge to students and help them learn it. It is clear to Wilson, Shulman and Richert, (1987) that

> "teachers need more than a personal understanding of the subject matter they are expected to teach. They must also possess a specialized understanding of the subject matter, one that permits them to foster understanding in most of their students." (p.105).

One major issue related to this restructured specialised subject matter knowledge is its sources. It is commonly agreed that this kind of knowledge is influenced by many factors (Even & Tirosh, 1995; Wilson *et al*, 1987). For example, a teacher's own experiences, both as a learner and as a teacher, influence pedagogical content knowledge. Relevant developmental and cognitive research, including learning theories and interactions with students are given as other factors. Another source of pedagogical content knowledge is assumed to be a teacher's own subject matter knowledge of the material they teach. The existence of such an assumption can be clearly observed in Shulman's and Leinhardt's definitions of teachers' knowledge, in particular pedagogical content knowledge or subject matter knowledge. This assumption is investigated by Even and her colleagues in the context of functions and undefined mathematical operations. They report that prospective teachers' limited understanding of these concepts influenced their pedagogical content-specific decisions. Other than this relationship, they could not report anything about the nature of the interrelationships between SMK and PCK.

Another major issue related to pedagogical content knowledge is its components. It is also agreed that this kind of knowledge is made up of several components. For instance, Even and Tirosh (1995) claim that one of these components is a teacher's planned presentations of the subject matter. This component involves teachers' choices of presentations of the subject-matter; making appropriate decisions for helping and guiding students, and giving appropriate responses to students' questions, remarks and ideas. (Even & Tirosh, 1995; Even, 1993; Even and Markovits, 1993, Leinhardt *et al*, 1991)

Another component of pedagogical content knowledge is curriculum knowledge.

> This is the knowledge possessed by the teacher of how the almost infinite range of possible topics and skills that might be taught to students have been organised and arranged into systematic programs of instruction called curricula (Shulman & Sykes, 1986, p. 10).

This type of knowledge is necessary for teachers in making decisions on the teaching order of topics. Leinhardt and her colleagues include such a type of knowledge in the knowledge of *lesson structure*, and claim that it is "necessary for coordinating lesson segments, and for fitting lessons together within a day (across subject areas) and within a unit (across days)." (Leinhardt *et al*, 1991)

An issue which is not explicitly touched upon in the literature is knowledge of how to make an introduction to a topic. It should be pointed out that such kind of knowledge may be very important in fostering students' knowledge and understanding of a topic. As the U. S. National Council of Teachers of Mathematics (1989) envisages "Once introduced, a topic is used throughout the mathematics program." (p. 32). Therefore, this type of knowledge may also be included in curriculum knowledge.

The above discussions on PCK, especially on the components of PCK, arises from different categorisations of PCK in the literature. In the context of science teaching Tamir (1987, p.100) for example categorises PCK as follows.

1. Student
   - 1.1 Knowledge: Specific common conceptions and misconceptions in a given topic
   - 1.2 Skills: How to diagnose a student conceptual difficulty in a given topic
2. Curriculum
   - 2.1. Knowledge: The pre-requisite concepts needed for understanding photosynthesis
   - 2.2. Skills: How to design an inquiry oriented laboratory lesson
3. Instruction (Teaching and management)
   - Knowledge: A laboratory lesson consists of three phases: pre-lab discussion, performance, and post-laboratory discussion.

Skills: How to teach students to use a microscope.

4. Evaluation

Knowledge: The nature and composition of the Practical Tests assessment inventory

Skills: How to evaluate manipulation laboratory skills

Another categorisation comes from Grossman (1990) who divides PCK into four subcategories: conceptions of purposes for teaching subject matter, knowledge of students' understanding, curriculum knowledge, and knowledge of instructional strategies.

One immediate remark about these frameworks is that in both of them there is an attempt to cover all possible ways of helping students to learn the subject-matter. This is because both of them relate to Shulman's original definition of PCK. However, in their attempt, they try to distinguish PCK from SMK, from general pedagogical knowledge, in short, from other types of teachers' knowledge.

Another remark is that some of the categories of PCK mentioned in both frameworks can sometimes also be considered as a source of PCK. For example, the category which is labeled as 'knowledge of students' understanding' by Grossman (1990) (category 1.1 of Tamir's framework) can be considered as both a source and as a component of PCK. In fact, Even and Tirosh (1995) use knowledge about students' understanding as a source of one of the components of PCK. They call this component 'Knowledge about ways of presenting the subject matter'.

I chose these three frameworks because they represent diverse perspectives and therefore help me to more precisely illuminate the components of PCK for teaching variables. However, while considering components of PCKv, I also keep in mind my inquiry into the nature of the relationships between SMKv and PCKv. That is, I included in my framework for PCKv those components of PCKv which might have relationships with SMKv. Therefore, the following framework consists of the main components of PCKv that may have relationships with SMKv. It presents an attempt to organise and blend the categories suggested by the three frameworks presented above into a specific framework which can be used as a basis for the purpose of this study.

1. Knowledge about ways of presenting the subject matter,
    1.1 Reactions to pupils' comments and questions in the classroom
    1.2 Analysing a students' mistake
    1.3 Helping students to correct their misconception
2. Curriculum Knowledge
    2 .1 Knowledge about introducing a topic
    2. 2 Knowledge about ordering topics.

2.4.1 Knowledge about ways of presenting the subject matter

2.4.1.1 Reactions to pupils' comments and questions in the classroom

The findings of research on students' learning of variables indicate that students build their knowledge of variables in ways that often differ from what is assumed by the teachers. That is, students make sense of the variable concept in their own ways. For example, it is documented in the literature that students tend to conjoin or 'finish' algebraic expressions, that is they tend to write the expression $3x + 2$ as $5x$ or 5. (Tall & Thomas, 1991; Booth, 1988; Collis, 1975) Therefore, they ask such questions as "Why doesn't 4+2b equal 6b?" while learning to simplify algebraic expressions.

Giving appropriate reactions to such questions may depend on teacher's knowledge about students (Even & Tirosh, 1995). In fact, Even, Tirosh and Robinson (1998) report that since two experienced teachers in their study are aware of students' tendency to conjoin expressions, they try to apply appropriate teaching methods in helping students to learn to simplify algebraic expressions. On the other hand two novice teachers don't know about such a tendency, and so they do not take appropriate cautions and furthermore they do not respond to students' questions such the one given above.

Not responding to such questions from students is also related to teachers' limited subject matter knowledge. Even and Tirosh (1995) report that if teachers and prospective teachers do not know the reasons behind the rules, they cannot give appropriate answers or reactions to students' questions, ideas, and wrong comments. However, it should be pointed out that the literature does not investigate the interrelationships between knowing the reasons behind the rules and giving reactions to student questions.

2.4.1.2 Analysing a student's mistake

Analysing students' work is another area which requires pedagogical content knowledge. Tamir (1987) points out that although certain principles and methods of student assessment are common for all subject matter areas, some of these are unique to particular disciplines. Furthermore, there are recent attempts to regard these methods and principles as unique to particular topics, and concepts. For example, Even and Tirosh (1995) claim that a decision about whether a certain student's work is correct is based on subject matter knowledge of the topic being taught. Evaluating students' work only as either right or wrong is not sufficient to help the student construct his/her knowledge. Therefore they study teachers' and prospective teachers' knowledge about students as sources of their analysis of students' mistakes in the context of functions and undefined mathematical operations. By reporting an analysis of interviews, they seem to conclude that the single most important thing in understanding the reasoning behind students' work is the knowledge about students' conceptions and misconceptions of a concept.

Such a claim has its foundations in a frequently quoted remark of Ausubel (1968): "If I had to reduce all of educational psychology to just one principle, I would say this: "The most important single factor influencing learning is what the learner already knows. Ascertain this and teach him accordingly." (p. 4).

2.4.1.3 Helping a student to correct his/her mistake

Teaching students accordingly is especially important in helping students when they believe and claim that their way of understanding is correct when it is not correct. Finding ways of helping students understand that their claim is wrong sometimes may be a real challenge. Therefore, knowledge about ways of helping students is another aspect which may be unique to particular topics. This last assumption in fact implicitly says there may be a different type of pedagogical content knowledge of helping students for variables. Consequently, this aspect of pedagogical content knowledge in the context of variables involves knowledge about helping strategies for making students see their ways of conceptions of variables.

2.4.2 Curriculum Knowledge

It should be pointed out that the following two components of pedagogical content knowledge are not of direct concern within the study they are simply presented to give a full picture of the notion of PCK. In fact, I will present only the analysis of the question which elicits participants' knowledge about ordering topics in the next chapter.

2.4.2.1 Knowledge about introducing a topic

Another challenge for teachers is finding ways to introduce a topic. The teacher needs to take into account many factors to make an introduction to topics. This type of pedagogical content knowledge is especially important for introducing variables since with the introduction of variables students begin to enter a new domain of mathematics, that is they begin to take their first steps on their journey from the world of arithmetic to the world of algebra. Research on children's understanding of algebra indicates that these first steps are very important for future development of algebraic knowledge (Kieran, 1992).

Some of the possible ways of introducing variables can be given as follows. For example, they may be introduced in a context where letters assume the role of the unknowns. These contexts generally involve forming and solving equations. Or they may be introduced in the contexts where the letters assume the role of general numbers. In this case, introducing variables may involve generating activities in which students are invited to find the patterns, rules, relationships between numbers and then express these generalisations using letters.

Deciding on how to introduce variables may have many sources. For example, the relevant literature on beginning algebra students' learning may help teachers with ideas. Or the textbooks and teacher's handbooks may suggest a way of introducing the topic. But to make sense out of these tools the teacher must be able to interpret the purposes and meanings in those tools. While making sense, they may rely on their subject matter knowledge.

2.4.2.2 Knowledge about ordering topics

After the introduction of variables a teacher needs to order topics within the concept of variable. For example, he/she may need to decide whether she/he teaches substituting numbers in the expressions first or simplifying expressions first. While making this decision, she/he may take into account many factors. For example, the relationship of these topics to other topics in this concept, or the relationship between these two topics may be a point to be considered in this ordering.

Even if they do not consider such factors still it is expected of teachers that they make sense of how topics are organised and arranged into the curriculum. (Shulman & Sykes, 1986) Therefore, making sense of such organisation clearly requires understanding connections across topics.

As I quoted from Tirosh *et al* (1993) earlier, creating classrooms where making connections is emphasised is essential to help the students construct understandings of the subject matter. Therefore this aspect of PCKv is the point where teachers may need to think about connections in relation to what is learnt; content connections *across* different aspects of the concept, and lesson connections *within* different lesson segments and series of lessons which are devoted to teaching this concept.

2.5 Summary and conclusions

I began this chapter by discussing research papers and theories on teachers' knowledge. After discussing developments in the field I discussed papers which study the relationship between subject matter knowledge and instructional outcome, particularly the relationships between subject matter knowledge and pedagogical content knowledge. These reviews together with the review on students' understanding of variables helped me to construct a framework for subject matter knowledge of variables and pedagogical content knowledge of variables. Furthermore, they helped me to develop my research questions by considering missing issues in this research field. Although previous research in this field characterise teachers' knowledge, and show that SMK has an important role in teaching, the majority of these studies hold the assumption that SMK and PCK are related. Therefore they did not study these relationships. Even (1991) and Even and Tirosh (1995) are the only studies that make connections between SMK and PCK. However, the result of their studies can be summarised as if a teacher has limited knowledge of mathematics then their pedagogical knowledge is also limited. Put another way their results are an echo of other studies which state that if a teacher does not know a concept well s/he can not teach that concept well. Therefore, we still need to find explicitly the nature of the (inter)relationships between SMK and PCK.

Another missing area in this field is the fact that most of the studies in this field are conducted with primary school teachers or teacher candidates in the context of arithmetic. Again Even and her colleagues are among few researchers who conduct their studies with secondary and high school teachers or teacher candidates. However, they study teachers' knowledge in the context of the function concept and undefined mathematical operations. Therefore, we still need to know about teachers' or prospective teachers' SMK and PCK of *variables* and interrelationships between these two types of knowledge in the context of variables. In particular, we need to know about Turkish prospective teachers' knowledge about variables, since there is not a single study that examined this topic.

To study these issues the following research questions are formed:

1. What is the state and nature of Turkish prospective secondary mathematics teachers' subject matter knowledge of variables?

2. What is the state and nature of Turkish prospective secondary mathematics teachers' pedagogical content knowledge of variables?
3. Are there any relationships between subject matter knowledge of variables and pedagogical content knowledge of variables, if so what is the nature of such relationships?

In the next chapter, I will discuss how I chose an appropriate research strategy and methodology to investigate these research questions.

CHAPTER 3

Methodology

INTRODUCTION

In this chapter I will explain how my research questions were investigated and why particular research methods and techniques were employed. I will provide accounts of procedure, size of sample, sample selection method, choice of questions

3.1 Discussion of the Research Strategy

Before explaining the processes which I went through while choosing a particular research methodology, I would like to explain how I selected the appropriate *research strategy* to address my research questions. By research strategy, I mean the general approach I have taken in my enquiry. Although there are many different ways of classifying research strategies, the widely used classification is given as: 'Experiments', 'Surveys' and 'Case Studies.' (Robson, 1993)

As Robson (1993) points out, a clear idea of the central purpose of one's study and of the type of research questions that one intends to seek answers to, can be of great help in selecting one's research strategy. Actually this has become a rule in educational research, which states that "the research question(s) drives the methodology." I would like to add to this advice that the studies had been done previously in that research domain also can be useful in choosing a particular strategy.

Bearing these in mind, I decided that the most appropriate research strategy would be *cross-sectional survey study*, since one of the *purposes* of this study was to examine the relationships between *Subject Matter Knowledge of variables (SMKv)* and *Pedagogical content knowledge of variables (PCKv)*. In other words, since I only wanted to *portray* a profile of interactions between SMKv and PCKv, I thought that a cross-sectional survey strategy would be well suited to my study.

Alternative main strategies would be case studies or experimental designs. I rejected these approaches since I did not need to arrange for events to happen in order to look for some effects, which is the typical feature of an experimental design; or to observe the characteristics of an individual unit –a student, a class, a school, which is the typical feature of a case study. As I discussed in chapter two, case studies which are the prevalent research strategy in this field have helped us out to see the characteristics of teacher's knowledge. In a way these studies have produced operable definitions of aspects of teachers' knowledge. Now we are at a point to construct our frameworks for these aspects and investigate the relationships between them. Consequently a cross-sectional survey research strategy seemed an appropriate strategy for my concern in this study.

I am just concerned with gathering data at a particular point in time with the intention of describing SMKv, PCKv and the relationships between SMKv and PCKv. As Fraenkel and Wallen (1996) state, the cross-sectional survey is a type in which information is collected at just one point in time, although the time may vary from a day to a few weeks or more. In a longitudinal survey, on the other hand, the researcher collects information over a much longer period of time in order to study changes over time (Fraenkel and Wallen, 1996). I decided to conduct a cross-sectional survey, since it was my intention to elicit information from a single period of time, rather than to study changes over time. Thus, I thought that a cross-sectional survey would be a productive research strategy that could support a relational analysis.

3.2 Methodological approaches to research in examining the relationship between SMK and PCK:

In research on examining the links between SMK and PCK, there has been a 'moving away' from quantitative methodology to qualitative methodology. One of the reasons for this movement lies in the definitions of SMK and PCK. Earlier research in this domain defined the subject matter knowledge of teachers as the *number* of courses the teachers had taken in the natural sciences; and pedagogical content knowledge as *instructional outcome*, namely the marks students had taken from *standardised tests*.

This is why they mostly use quantitative methods to find out the relationship between SMK and PCK. Most of these studies look at correlation coefficients which are obtained by means of *statistical tests*. As Bromme (1994) points out “A large number of variables "interfere" with the effect the teacher's amount of subject-matter knowledge has on student performance.” (p. 76) In other words, there are a lot of mediating variables in between teachers’ knowledge and their students’ performances. This research therefore could not show that teachers' amount of knowledge contributes to the variance of their students’ performance. To give some examples concerning these mediating variables, student variables such as learning styles, motivation and attitudes may affect how they learn and consequently their performances on exams. Another example from the teachers’ side, one such variable might be teachers’ questioning. The type of questions teachers ask, and the way they ask them in the classroom may affect students’ learning, and thus their performance on exams.

Therefore, subsequent research criticises earlier research’s quantitative approach by saying that we need to approach the question in qualitative terms, since the earlier research conclusion was counterintuitive (Ball, 1991). That is, it does not make much sense to say that what teachers know does not contribute to what their students learn. This counterintuitive result of earlier research led subsequent researchers to examine the underlying assumptions of earlier research about knowledge of mathematics and learning. And the qualitative methodology begins to study this educational phenomenon. First of all, the definitions of knowledge as subject matter knowledge and pedagogical content knowledge have been changed by employing qualitative terms. Qualitative analysis of SMK and PCK took this research domain one step forward from a simplistic list of competencies that served as a criterion to knowledge. This consequently resulted in qualitative probing of subject matter knowledge and pedagogical content knowledge.

Interview data is frequently used, as are *questionnaires* and other forms of data such as *classroom observations* and *analyses of lesson transcripts*.

Defining subject matter and pedagogical content knowledge in qualitative terms was productive and this body of research has been able to show that there are indeed links between SMK and PCK. However, in the new approach there seems to be an antipathy towards the statistical paradigm. One can not see any statistical tests that are used to show those links between SMK and PCK in this research. One of the main reasons for this avoidance might be the apparent difficulty in quantifying the qualitative data and thereby using statistical tests.

Another difficulty arises from the fact that statistical tests require relatively large sample sizes. The rich data gathered by using interviews, observations or any other means from a large sample is very time consuming and difficult to analyse. Moreover, in order to use statistical tests to see relationships between two types of knowledge, at least two variables are needed; one for SMK and one for PCK. However, finding such variables requires a very sophisticated theory: a theory that could separate SMK from PCK in a way that would allow the researcher to find clear-cut questions to probe aspects of SMK and PCK.

In this study every caution was taken to avoid and overcome these difficulties. Firstly, SMK and PCK were defined in an appropriate way. By an appropriate way I mean not regarding SMK as the number of courses taken and PCK as the students' performance on exams. In this study, I define subject matter knowledge and pedagogical content knowledge of variables in qualitative terms as I described in the previous chapter while reviewing the literature. The theoretical framework which was used to find suitable questions for SMKv and PCKv was based on several bodies of work; the research on students' understanding of variables, the research on teachers' professional knowledge and the studies which examine the links between SMK and PCK. Having overcome these difficulties, I used statistical methods after taking pains to quantify rich qualitative data which was obtained by questionnaires. After analysing the questionnaire data, I conducted interviews to validate the responses given to questionnaire questions. In other words, my methodological approach can be regarded as a combination of previous approaches used in this research domain, that is an amalgamation of both quantitative and qualitative methodologies.

3.2.1 Discussion of limitations of studying pedagogical content knowledge only in prospective teachers

In this section I will discuss studying PCK only in *prospective teachers*. This discussion is related to the question where best to study the interactions between PCK and SMK. There are contrasting answers to this question. These contrasting answers stem from the views on when and how PCK is learned. Noddings (1992) states that most of pedagogical content knowledge is learned on the job, that is whilst teaching pupils. Fennema et al (1996) take a more extreme position. They claim that one of the components of PCK, -knowledge of pupils' thinking processes-, can only be acquired in the context of teaching mathematics. These kinds of claims imply that for *prospective* teachers acquisition of PCK is still in the future, therefore it is difficult to study PCK in intending teachers. Hence, this view clearly poses some limitations on studying PCK in *prospective* teachers.

Tirosh (2000), however, argues that *experience* acquired during teaching is the primary but not the only possible source of teachers' pedagogical content knowledge. She points out that prior to their teaching experiences, prospective teachers' own *SMK* together with their *experiences as learners* and *relevant developmental* and *cognitive research* could be used to enhance their *PCK*, in particular to their knowledge about students' thinking. Therefore, she points out that the viability of promoting development of prospective teachers' PCK should be systematically researched. Hence, she studies PCK in prospective teachers.

In line with Tirosh (2000), Grossmann (1990) argues that subject-specific method courses are the logical places for a researcher to look if they are interested in studying the acquisition of PCK, because it is here that prospective teachers' subject matter and pedagogical knowledge will definitely *interact* during the process of learning to teach a subject. On the other hand, some of those prospective teachers who had not yet taken any pedagogical courses might have encountered some pedagogical questions in the questionnaire for the first time. For the first time; during the process of answering the questionnaire, they might have made a pedagogical decision. This might have posed some limitations on the findings of this study. For example, their responses should be interpreted in a restricted way. We should not expect from them fully fledged pedagogical responses. We should not expect from them expert-like responses. However, we could think that their responses to PCK questions had not yet been affected by teaching experience. Therefore, it was considered that if there could be any relationships between PCK and SMK then they would possibly interact during the process of answering the questionnaire.

This last consideration helped me to assert that studying PCK in intending teachers is the ideal strategy to find out the interactions between SMK and PCK, because their thinking about pedagogical issues has not yet been affected by teaching experience. Therefore, it is thought that this is the ideal time to find out how SMK has an impact on students' thinking about pedagogical issues which are addressed in the questionnaire. Nevertheless, it should be pointed out that this kind of consideration is not conflicting to Shulman's definition of PCK, in particular its distinctive character.

Shulman points out that teachers should possess a special kind of knowledge which is required to make a subject understandable to the students. He claims that the knowledge for a subject matter specialist and the knowledge for a teacher are different. For example, the knowledge for solving problems, conducting experiments, passing courses etc. in a discipline is different from the knowledge which is required to teach that discipline to others. Shulman calls this knowledge PCK and considers it as distinctive to teachers. He claims this, because teachers teach; they explain, give examples, demonstrate lessons, etc. to help students learn. Their job is not only to learn that subject but also to help students learn it.

The common point of all teaching actions is that they are efforts of communicating one's knowledge to others. That is, teachers try to communicate their SMK in a suitable form that helps students understand the subject. These actions are related to practical aspects of PCK, and practical knowledge is highly context dependent. However, PCK also has formal aspects. These formal aspects are in the form of propositional knowledge (see Fenstermacher, pp. 32, 36, 1994, to see a discussion about the formal and practical elements of PCK). Therefore, formal elements of PCK include lists of sorts, of common misconceptions of particular topics among pupils that need correction and suggestions for ways of making a particular topic more understandable to the students. Prospective teachers may know a set of strategies, stories, and forms of representation for a content area. However, if PCK is only viewed as a kind of knowledge that only expert teachers have, then this knowledge must be developed through an integrative process rooted in classroom practice. This in turn, implies that prospective teachers usually have little or no PCK at their disposal. But again, it should be acknowledged that such knowledge is related to practical aspects of PCK and this can be based on formal aspects of PCK. That is, aspects of PCK which are formal in form can serve a basis for practical reasoning.

Fenstermacher (1994) points out that there may well be a "science for the advancement of practical knowledge as there is a science for the advancement of formal knowledge" (p. 36). In this study I regarded PCK as formal knowledge of ways of communicating one's SMK to others while constructing the questionnaire, in particular PCK questions. I had this regard also while I was analysing the responses to the PCK questions. While analysing, whether or not it is a 'suitable' explanation, I considered those responses to the PCK questions as instances of PCK which demands pedagogical considerations. I took these responses as they are without judging whether or not they indicate that the student has well-developed PCK. I looked at the similarities and differences between the responses and put them into categories.

3.3 Sample

In this section, I will discuss and present the sample selection methods. I will also write about the characteristics of the sample; the region, their educational status and background.

3.3.1 Questionnaire Sample

The sample comprised three different year groups of students from three different Universities in Ankara, the capital city of Turkey. This city was selected intentionally since I wanted to reach a large number of students. In this respect Ankara was the only city in Turkey in which I could carry out my research. After choosing the city, I visited all the Universities in this city that have a Mathematics Education department to explain my aim to the chair of department and to get permission to conduct my research. All universities gave permission and helped me to find out where and at what time I could get in touch with students to distribute the questionnaire.

The students ranged from the second year to the fourth year of courses in Mathematics Education Departments. One hundred and eighty-four students (184) completed the questionnaire. The following table shows the distribution of students between Universities and year groups. The letter of the variable, in the first column of the following table, denotes the name of the university and the number next to the letter denotes the year group. For example, A in A3 shows the university A, and 3 in A3 shows year 3. So from the table, it can be seen that there are 82 students from university A, and 81 students from university B, and there are 21 students from university C. The number of students from university C is smaller than the number of students from the other two universities, since very few students were studying in the Mathematics Education Department of university C. The students in group B5 were on their extension year to complete some of the courses from previous year. They were still following the 4^{th} year courses.

Table 3.2 Distribution of Students to Year Groups and Universities

	Frequency	Percent
A2	30	16.3
A3	24	13.0
A4	28	15.2
B2	38	20.7
B3	24	13.0
B4	14	7.6
B5	5	2.7
C4	21	11.4
Total	184	100.0

In order to explain what qualifications my sample brought from schools prior to University, I would like to briefly describe the Turkish educational system. In Turkey, formal education includes pre-school education, compulsory education, secondary education, and higher education. Pre-school education is the broad term applied to non-compulsory programs for children prior to the starting age for compulsory basic education. Compulsory education is for 8 years and it is for every Turkish citizen from the age of six to the age of fourteen, regardless of sex, and is free-of-charge in state schools. Prior to 1997, compulsory education was five years primary school (İlkokul). After primary school students go to three years of middle school, or junior high school (Ortaokul) in order to be able to continue their education in secondary schools (high schools, or lycee).

Secondary education comprises all schools providing a minimum of three years of general, vocational, or technical education. In secondary education, there are 6 subject areas as shown in the Table below. Students select their subject areas according to their performance in the first year of high school (9th grade), taking into account their interests, abilities, and achievement.

Table 3.3 Subject areas at high schools. (Source: YÖK, 2000)

Subject Areas	Feeder Subjects	Hours/Week
Science	Biology	2
	Physics	2
	Chemistry	2
	Mathematics	5
Social Studies	Turkish Language & Literature	4
	History	2
	Geography	2
Turkish-Mathematics	Turkish Language & Literature	4
	Mathematics	5
Foreign Language	Turkish Language & Literature	4
	Foreign Language	4
Fine Arts	Turkish Language & Literature	4
	Art or Music	2
Sports	Physical Education	2
	Biology	2

In Turkey, all lycee (high school) graduate students who want to continue their education in university should enter and pass a centralised nation-wide examination which is administrated by the Student Selection and Placement Centre (ÖSYM) every year. This entrance examination is comprised of multiple choice test questions which can cover all the subjects that exist in the Turkish high school curriculum. More precisely, there are questions that measure verbal and numerical ability, Turkish, mathematics, physics, chemistry, biology, history, and geography achievements. It is a time restricted test and four wrong answers cancel out one correct answer. The score taken from this exam is the only decisive factor for admission to a university. These scores are calculated by taking into account the score of the entrance examination as well as the high school grade-point averages, with different weights. Passing this exam and entering the universities is a great achievement, considering that the number of lycee graduates who could get a place in the universities is 414, 315 out of 1, 479, 326 applicants in 1999 (YÖK, 2000).

All universities in my sample draw students from all over Turkey. However, they accept students who achieve a higher score on the University Entrance Exam than other Turkish universities do. In this respect all three universities in my sample are in the top ten.

When the students in my sample are graduated from their departments, like all the graduates of mathematics education departments, they will be qualified to teach in middle and high schools. Beside these they can be employed by Mathematics Education Departments of universities as well as by the Ministry of Education and private schools as academics, supervisors, inspectors, curriculum consultants, test and evaluation specialists in mathematics education.

Participants in this study who were in the second or third year of all three universities or in the fourth year of university B (B4) had not taken any pedagogical courses at the time the questionnaire was completed, whereas students in the fourth year of university A (A4), university C (C4) and a few students of university B who failed to graduate in time and were denoted as B5 had taken pedagogy courses. Table 3.4 shows the distribution of students according to whether or not they had taken pedagogy courses.

Table 3.4 Distribution of Students according to Pedagogy Courses taken or not

	Frequency	Percent
no pedagogy	130	70.7
pedagogy	54	29.3
Total	184	100.0

As can be seen from this table there are 130 students who had not taken any pedagogy courses whereas 54 students had taken pedagogy courses. There are fewer students who had taken pedagogy courses because of the new system for mathematics education faculties in Turkey which was started four years ago in 1998. According to this new system, students cannot take pedagogy courses until they come to the second term of their fourth year. For this reason one of the universities in my sample stopped student admissions into education departments in 1998. However, students admitted before this date were allowed to follow the programs they originally registered for. All students who took pedagogy courses in my sample were admitted before this date and they were following the previous system, which is a four year programme.

The pedagogy courses which students in my sample had taken are generally named as Educational Psychology, Philosophy, Sociology, Teaching Methods, Measurement and Evaluation in Teaching, Introduction to Educational Sciences, Curriculum Development (Türk, 1999). In these courses students are taught different definitions and conceptions of curriculum, the role of mathematics in Turkish middle and high school curricula, concepts of measurement and evaluation as applied to behavioural sciences, classical test theory. Beside these courses, students have to go to schools in order to gain field experience and teaching practice including planning and preparation for teaching. They are required to have guided teaching practice in mathematics in high schools.

However, according to YÖK - the planning, coordinating and policy making body for higher education in Turkey is called the Council of Higher Education (Yükseköğretim Kurulu, YÖK)- pedagogical courses given in the previous system are mainly theoretical courses in educational sciences. This was criticised by saying that they did not perform the function of giving prospective teachers the knowledge, ability and vision that could be practiced in teaching (YÖK, 1998). Therefore, education departments began to educate teachers according to a new system.

3.4 Interview Design and Sample

After analysing the questionnaire responses; firstly by putting responses into different categories then using statistical methods to see if there are relationships between students' answers to questions in the SMKv group and questions in the PCKv group, it was decided to conduct interviews to explore the validity of the responses given in the questionnaire. That is, the interview sample was chosen to establish the range of responses given by students from all universities who completed the questionnaire. For this reason, ten students chosen from university B were interviewed.

The selection of students was done by mainly considering the relationships found from the analysis of questionnaire responses. That is, I chose such students who gave particular responses to particular questions.

The preliminary analysis consisted of putting questionnaire responses into categories. Then, I thought about the reasons for getting particular responses from PCK questions. In particular I considered whether prospective teachers' SMK could be a reason for their responses to PCK questions. In this examination, I set up some hypotheses which involved relationships between SMK questions and PCK questions. For example, I hypothesised that mentioning that variables could stand for objects may be related to explaining why $2a+5b$ is not equal to $7ab$ by using objects. As the reader may appreciate, the possible reasons for using objects to explain that the variables cannot be conjoined can be linked to the assumption that variables can stand for objects. Alternatively it can be linked to the belief that younger students could learn with concrete objects better. However, in this case, if one does not know the distinction between using variables as objects and using them as the number of objects, then this kind of misconception may also be related to that person's pedagogical content knowledge.

In order to see if the hypotheses I constructed from preliminary analyses could hold I carried out cross-tabulation analysis between SMKv and PCKv questions. From these cross-tabulations I saw consistencies between the responses in part I and part II of chapter 4. For example, students who talked about variables as objects in part 1 of the questionnaire were more likely to use objects in their explanation to why $2a+5b$ is not equal to $7ab$ in part II. Another example could be given as students who approach problems in different ways are more likely to give more than one way of helping to correct their student's misconception.

After hypothesising and checking these relationships between the questions by cross-tabulations, I chose 10 students from one of the universities where I collected my questionnaire data and who gave particular responses to the questions. As a result; 1 student from year 4; 5 students from year 3; and 4 students from year 2 were selected. For example, four students were chosen to represent the four different combinations of responses to questions 6

> 6. What different things might an algebraic expression such as, say $2x+1$, mean? What can x stand for?

and question 12a.

> 12 a) How would you react to your students' questions as below in the classroom? Explain!
>
> "Teacher, why does 2a+5b not equal 7ab?"

The first one of these students wrote in the questionnaire that x could be an object in $2x+1$ and used objects ('fruit-salad' algebra) to explain why variables can't be conjoined; the second one did not write x could be an object and did not use fruit salad algebra. The third student did not write that x could be an object, but mentioned that she would use a 'fruit-salad' explanation. The last student wrote that x could be an object, but did not mention she would use a 'fruit-salad' explanation. The selection of such students was accomplished by the help of SPSS. That is, I randomly chose those students who gave the types of responses required for the investigation.

3.5 Design of the Questionnaire

In this part of this section, I will explain the processes which I went through while designing the questionnaire. This stage of the design principally involved deciding on appropriate questions that were to be asked, and on their precise wording. The sequence, the form of response and the format of the whole questionnaire also had to be decided in order to ensure that the information would be gathered in an appropriate and efficient fashion.

The purpose of this study was to investigate relationships between subject matter knowledge (SMKv) of variables and pedagogical content knowledge (PCKv) of variables. To achieve this aim, a theoretical framework was established for analysing knowledge and understanding of variables and knowledge about teaching variables. This framework was based on integrated knowledge from several bodies of work which was presented in chapter 2.

Using this framework as a basis, I constructed a questionnaire, which had 15 free-response items. 12 of them were about subject matter knowledge of variables and 3 of them were about pedagogical content knowledge of variables. I piloted this questionnaire in July 2001 in Turkey. After the pilot analysis, I refined my questionnaire, and in October 2001, I collected my data by using the refined questionnaire which has 10 open-ended items about knowledge of variables and 6 open-ended items about teaching of variables.

While designing the questionnaire, I used open-ended questions (free-response or recall questions) rather than multiple-choice questions (recognition questions). In view of the fact that I did not have enough elicited information to write appropriate response categories, I preferred to use free-response questions. Furthermore, since I was interested in subjects' solution methods rather than whether or not they solved questions correctly, I chose to use free-response questions. In other words I wanted to obtain subjects` ideas and thoughts through their solution processes of subject matter knowledge questions, and I wanted to obtain their explanations and their reasons behind these explanations about teaching different aspects of variables. To clarify I would like to give the following two forms of the same question, the first one is free-response format and the second one is multiple-choice format:

> Five years ago, the sum of the ages of a mother and her daughter was 53. Now, if mother is 27 years older than her daughter, how old was the daughter 3 years ago?

> Five years ago, the sum of the ages of a mother and her daughter was 53. Now, if mother is 27 years older than her daughter, how old was the daughter 3 years ago? Please choose the correct answer from choices below.
>
> A) 15 B) 25 C) 30 D)35 E) 40

As can be appreciated from the different formats of the question, that I could not have acquired much information about the solution processes of subjects if I had asked this question in the second format. However, from the first format of the question, I could analyse their solution processes; I could see whether they solved this problem arithmetically or algebraically thereby I could categorise their responses.

3.6 Questions in the Questionnaire

In this section I will present the structure of the questionnaire and also the questions and their use in eliciting SMKv and PCKv.

The structure of the questionnaire is formed by use of the theoretical framework I presented in chapter two. More precisely, the main groups and related subgroups were based on the theoretical framework. This framework was based on integrated knowledge on several bodies of work. There were two main groups of questions in the questionnaire, namely subject matter knowledge questions and pedagogical content knowledge questions. And in each group there were sub-groups of questions for finding out different aspects of teaching about and knowledge of variables. The main groups and related sub-groups of the questionnaire are displayed in the following table (Table 3.5).

Table 3.5 The Structure of the Questionnaire

Codes	Main-Groups	*Sub-Groups*
SMKv	Subject Matter Knowledge	***Knowledge of variables in:*** • Principal uses of variables • Awareness of Different Roles of Variables • Flexibility, versatility and connectedness among different roles and uses
PCKv	Pedagogical Content Knowledge	***Knowledge about teaching variables in:*** • Knowledge about ways of presenting the subject matter • Curriculum Knowledge

The two main groups involve the research questions which were brought out by the review of the literature in chapter two. These are as follows:

SMKv: What is the state and nature of knowledge of variables of Turkish prospective mathematics teachers?

PCKv: What is the state and nature of knowledge about teaching of variables of Turkish prospective mathematics teachers?

To investigate these questions, the different aspects of SMKv and PCKv are reflected in the structure of the questionnaire. These aspects constituted the subgroups of the main groups. These subgroups and the relevant questions and their aims will be discussed in detail in the remainder of this section.

3.6.1 Subject Matter Knowledge Questions

3.6.1.1 Knowledge of variables in principal uses of variables

This group consists of questions that will exemplify prospective teachers' knowledge of variables in principal uses of variables. That is, the questions in this group are about understanding variable as unknown, as general number and as variable.

Question 1a

1 a) Five years ago, the sum of the ages of a mother and her daughter was 53. Now, if mother is 27 years older than her daughter, how old was the daughter 3 years ago?

Accordingly, understanding variable as unknown includes those questions that can represent the process of forming and solving an equation to solve the problem. These questions require the ability to recognise and to identify in the problem the presence of something unknown and after using appropriate symbolisation tools it remains to manipulate these symbols to get the value of the unknown. Question 1a was set up for this purpose. However, it should be acknowledged that the question 1a can be solved without using letters, by just using arithmetic. In this case, we could hypothesise only that the solver has the ability to recognise and identify in the problem the presence of something unknown but we can not be sure that the solver has the ability to use letters to denote the unknowns, and manipulate them to find the value of the letters. Hence, if this question had been constructed in a way that the only way to solve it was by employing letters for the unknown(s), then it would have better served the purpose for eliciting using variables as unknowns. It could not have been constructed like this because the researcher could not recognise that the question could be solved by just using arithmetic, since he was so accustomed to solve such problems using letters. Therefore, the analysis of this question will not be presented.

Question 1b

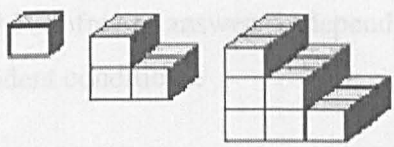
> 1) b) Prove that the sum of *n* consecutive even integers is divisible by *n*.

Question 1b was put in the questionnaire in order to illustrate subjects' understandings of variable as general number. In this question, the variable 'n' can be taken as a general number or as a *specific* but unknown number. Both could work to solve it. However, in order to give a general solution, it requires proving the assertion in the question for all cases. In the question, there are *any* 'n' even integers, and the sum of these 'n' integers is divisible by *any* 'n'. Solvers have to start the first number of the consecutive even integers at *any* number not at a *specific* number in order to give a general verification. Thus, they need to use a letter for the starting number of these consecutive even integers to show the generality of their solution. For example, can the sequence "2, 4, 6,….,2k" (where 'k' can be obtained by considering the number of integers in the sequence) be a candidate to represent all different variations of 'n' consecutive integers? If this sequence is divisible by 'n', can we claim that the sum of 'n' consecutive even integers is divisible by 'n'? How can we be sure that the sequence "12, 14, ….., 2m" (again 'm' can be obtained by considering the number of integers in the sequence) is also divisible by 'n'? As a result, if we prove that the sum of integers in the sequence "2a, 2a+2, 2a+4,…….,2m" (note that the starting number is represented by a letter) is divisible by 'n', then we can claim that the assertion in the question holds for any sequence of 'n' even integers.

Hence, prospective teachers' solution methods for this question can give clues about their thinking in dealing with problems which requires distinguishing between particularities and generalities. However, it should be acknowledged that the analysis of this question will not be presented in chapter 4, because it is believed that the purpose for asking this question is not clearly stated in the question.

Question 11

11) Examine the cube pattern below. (a) How many blocks would be needed to build the n^{th} item?



1. Item 2. Item 3. Item

Question 11 may also be regarded as of the same nature. But in question 11, it is firstly required to establish a generalisation, then to perform manipulations and finally to make and interpretation of the result of manipulations. More precisely, in this question it is required to go from figural pattern to number pattern to expressing the rule in general terms. Even if one can see the rule of the pattern from the figures, one still needs to express this rule by letters in order to express generality. Furthermore, since it is believed that these kinds of questions are absent from the Turkish curriculum, subjects are unfamiliar with this type of question. Therefore, this question also serves the purpose of assessing whether subjects had developed an intuitive feel for when to call on variables in the process of solving a non-routine generalisation problem. However, the analysis of this question will not be presented because this question may need more time to solve it in the questionnaire than assumed by the researcher.

3.6.1.2 Knowledge of variables in awareness of different roles of variables

This section of the questionnaire involves those questions that exemplify whether subjects are aware of the fact that variables have different meanings and roles depending on the effective context. Questions 2, 5 and 6 can be put into this group. However, question 2 and 5 can also be used for obtaining information about whether subjects can discriminate between different roles of variables; variables as unknown, as general number or as variable.

Question 2

2. Which of the following are equations for quadratic (second-degree) functions of one variable? Feel free to answer "it depends," and if you do, elaborate on the dependent conditions.
a) $y = ax^2 + bx + c$
b) $y = (1/2)\pi rd$
c) $E = mc^2$
d) $y = e^2$

Question 2 was adapted from Philipp (1992).This question highlights the importance that context plays in determining the role of a literal symbol. Whether one considers these equations to be quadratic or not depends on the context. For example, $y = e^2$ can either be quadratic or a constant function, depending on whether e varies or represents the base of the natural logarithm. Therefore, they don't have right or wrong answers, consequently in analysing responses to this question the emphasis was not on whether they arrived at "right" answers but rather on the reasons they selected the answers they did. However, it should be acknowledged that the respondents may have felt insecurity in responding to this question because they may think that there is a trap in the question. There is not enough crystal clear information in the question. Hence, such problems in the construction of the question may affect subjects' responses. As a result, the analysis of this question will not be presented in chapter 4.

Question 5

5. For each of the following expressions, indicate the role of the variable (the underlined letter). Please give your reasons!
If a variable is used as the *name for a number*, write A.
If a variable is used to represent a *specific unknown*, write B.
If a variable is used to represent a *general unknown* or a *pattern generaliser*, write C.
If a variable is used to represent a *varying value*, write D.
a) $y = \underline{x} + 10$, $\underline{x}$ is ...... since
b) $3\underline{p} = 9.42$, $\underline{p}$ is ...... since
c) $7 \bullet \underline{x} = 0$, $\underline{x}$ is ...... since
d) $7 + \underline{x} = 10$, $\underline{x}$ is ...... since
e) $0 \div \underline{n} = 0$, $\underline{n}$ is ...... since

The emphasis in question 5 is on whether students could give valid reasons why the variable assumes the role they claimed. It is similar to question 2, but in this question it is needed to decide on the role of the letter from the given equations whereas in question 5, the meaning of the equation is decided from elaboration of the roles of the letter. However, it should be acknowledged that the words 'specific unknown', 'general unknown' or pattern generaliser' may not be meaningful to the students, since in Turkey the difference between the different roles of the variables is not emphasised. Furthermore, in parts b), c), e) where p=3.14, x=0 and x=3 respectively, where the value of the letter is found, the students may think that the letter is used both as the 'name for a number' and as a 'specific unknown'. In part 'e)', 'n' can be considered as it is used as a 'general unknown' or as a 'pattern generaliser', however, the case n=0 distracts the generality in the equation $0 \div n = 0$. For these reasons, it is thought to concentrate on the given reasons for their choices while analysing the responses, but since there were not rich explanations for justification of their choices, I will not present the analysis of this question in chapter 4.

Question 6

6. What different things might an algebraic expression such as, say $2x+1$, mean? What can x stand for?

Question 6 was used to see what kind of things x is associated with in the minds of students. The context is again important in this problem. 'x' can stand for anything, however since it is used in the expression $2x+1$ it should stand for a number, or any other mathematical objects that can be multiplied by '2' and '1' can be added. I will discuss more about this question in chapters 4 and 5 while presenting the analysis and results.

3.6.1.3 Knowledge of variables in flexibility, versatility and connectedness among different roles and uses

This group consists of those questions that can illustrate subjects' flexibility and versatility while solving those questions where the variables assume more than one role in the same problem, as well as different roles than their accustomed roles: questions 3, 4, 7, 8, 9, 10. To respond to these questions it may be required to have rich relationships and connectedness between graphical representation and symbolic representation.

Question 8

8. If you substitute 1 for x in $ax^2 + bx + c$ (a, b and c are real numbers) you get a positive number. Substituting 6 gives a negative number. How many real solutions does the equation $ax^2 + bx + c = 0$ have? Explain.

This question is used in the literature for assessing the knowledge of functions by Even (1990). It was decided to change question 8 after analysing the pilot questionnaire, since I could not get information about whether subjects solved this question using or thinking about graphical representation due to fact that I could not see any graphics or mentioning of graphics in subjects' answers in pilot study. Question 8 in the pilot study read as "*Solve* $-x^2+2x-3>0$"; this was changed as above using an adaptation of a question from Even (1990).

In order to solve question 8, it was needed to work in the field of real numbers, and it requires using or visualising its graphic representation. Otherwise by symbolic manipulation it proves difficult to explain the solution of the problem. Graphical representation may give important insights into the solution of the problem. And this may call for rich relationships and connectedness between the symbolic representation and graphic representation. This, in turn, shows one's flexibility to give symbolic treatment to problems for better tools. I will discuss more about this question while presenting its analysis in chapter 4.

Question 9

> 9. For what values of a does the pair of equations
>
> $x^2-y^2 = 0$
>
> $(x-a)^2+y^2 = 1$
>
> have either 0, 2, 3, or 4 solutions?

Arcavi (1994) uses a question similar to above as an example to describe an aspect of 'symbol sense' which involves giving up symbolic treatment to problems for better tools. He points out that the above problem can be elegantly and easily solved by using Cartesian coordinates. The symbolic approach is less elegant and it needs more logical connections to the givens in the problem, and there is more chance of making mistakes while following technical manipulation. Therefore, this question assesses one's flexibility to move among different aspects of variables. It is similar to question 8 in the sense that both of them can be solved by working on Cartesian coordinates, either by imagining the graphs of the expression or by actually drawing them. The discussion on this question will be continued in chapter 4 while presenting its analysis.

Question 4

> 4. What *different* things could $y = 3$ mean if one derives it from $y = mx + 3$.

Question 4 was modified from a question which is used by Arcavi (1994), and it was also for flexibly oscillating between manipulations and objects involved in the question, such as line, point. The meanings of y=3 depend on how one finds it. If one finds this by substituting x=0, then y=3 is the y-intercept, if one finds y=3 by substituting m=0 then y=3 is the horizontal line with slope zero. I will discuss more about the purpose of this question in chapter 4.

Question 3

> 3. Explain two different solutions of the following problem to your students?
> " For which values of m does the equation $m(x-5) = m + 2x$ have no solutions?"

This question is adapted from Bills (1997). While solving this question the roles of m and x change through-out the solution process. The way the question is asked implies that x is the unknown, m is the parameter. In order to solve this question the solver must move flexibly between different roles of m and x. However, as Bills (1997) points out there is not only one way of solving this problem, and therefore each person can follow different routes while moving between different roles of m and x. Because of this fact, I wanted participants to solve this question at least in two different ways. I will not present the analysis of this question due to word limitation.

Question 10

> 10. Find the coordinates of the point where the line $m + 2n - 4 = 0$ meets the line $n = 2m - 2x + y$ in the m-n coordinate system.

Question 10 was used to see whether participants associate certain letters with certain roles in their minds. Although this kind of association may have both useful and counter-productive outcomes depending on the context, we can expect from prospective teachers of mathematics to work comfortably and solve problems in unfamiliar situations where the letters have different roles from their accustomed associative roles. I will not present the analysis of this question due to word limitation.

Question 7

7. Find the difference between the larger root and the smaller root of $p^2 - xp + (x^2 - 1)/4 = 0$.

Question 7 is similar to question 10, in the sense that in both questions the solver need to make shifts in the roles of variables. However, in this question the solver is free to choose which letter is the variable and which one is the parameter. Question 7 is different from question 10 because question 7 is written in a form that p can be taken as the variable but conventional use of x as the unknown may suggest that x is the variable. Hence, a student who is familiar only with using x as a variable and p as a parameter might have only one interpretation of this problem. It is difficult, if not impossible, for him/her to take p as a variable and x as a parameter. As a result, the 'flexibility' of moving among the different interpretations of the roles of the letters depends on the students' experiences and what they have been taught. Therefore, I will not present the analysis and results of this question.

3.6.2 Pedagogical Content Knowledge Questions

In this section I will present and discuss the questions which were put into the questionnaire to elicit prospective teachers' PCKv. As I pointed out earlier, the different aspects of PCK were justified by the literature review in chapter two. Therefore, asking such questions which will be presented in the remainder of this section serves the purpose of eliciting prospective teachers' PCKv.

3.6.2.1 Knowledge about ways of presenting the subject matter

The questions in this group are for eliciting prospective teachers' knowledge about ways of presenting the subject matter. This aspect of pedagogical content knowledge is investigated under three sub-aspects. These are namely "reactions to pupils' comments and questions in the classroom", "Analysing a student's mistake", and "Helping a student to correct his/her mistake".

3.6.2.1.1 Reactions to pupils' comments and questions in the classroom

Question 12

12. How would you react to your students' questions as below in the classroom? Explain! a) "Teacher, why does 2a+5b not equal 7ab?" b) "While solving equations, why does x change its sign when it is brought to the other side?"

The questions in this group are for eliciting prospective teachers' pedagogical knowledge on reacting to pupils' comments and questions in the classroom. They are for gathering data about reaction methods and approaches which are available to prospective teachers. They were constructed to reflect scenarios which generally happen in classrooms. The questions pupils ask in these scenarios are selected from the literature on learning and teaching of algebra. As documented in the literature, these questions arise from misconceptions and difficulties on pupils' journey to learn the concept of variable. It should be acknowledged that in question 12a, $2a+5b$ can be equal to $7ab$ if $a=b=0$ or 1. However, while constructing the question it is assumed that the pupil in the question does not ask why $2a+5b$ is not equal $7ab$ because s/he thinks that there are cases when the equality holds such as $a=b=0$ or 1. On the contrary, it is assumed that the pupil asks such a question because s/he has a misconception that the different variables can be conjoined. One of the underlying reasons for this misconception might be giving meaning to letters by referring to objects. That is, the pupil in the question might be thinking that 5 apples plus 2 bananas is 7 fruits. I will take this issue into account in my analysis of the students' responses to this question. I will discuss more about the issues related to this question in chapters 4 and 5.

In question 12b, it is believed that 'change sign, change side' is a method of teaching equation solving in Turkish schools. Hence, it is assumed that such questions can be raised by the pupils. It is clear that such method encourages rote learning although it gives short term success. I will not present the analysis of this question due to word limitation and also the results of responses to this question were not investigated deeply by interviews.

The following question 13 is also developed from literature and is also a common mistake pupils often make.

3.6.2.1.2 Analysing a students' mistake

Question 13

13. Ayşe was asked to simplify an algebraic expression, such as $2x + 4x + 6$. She wrote:

$$2x+4x+6=0$$

$$6x+6=0$$

$$6x=-6$$

$$x=-1$$

What do you think the student had in mind? Is she right? Explain.

In this question, the pupil made a mistake in recognising the correct role of the variable. She formed and solved an equation, rather than simplifying the given expression. I asked prospective teachers to analyse this mistake. However, I did not tell them the given work of the pupil is wrong. I wanted them to evaluate this work. They had to decide whether it was wrong or not. Then if they decided there was a mistake in this solution, they had to comment about the possible sources for this mistake. Hence it is a question about prospective teachers' knowledge about analysing a student's mistake.

In the question above I did not ask them to help the pupil to correct her mistake. However, in the following question, they are asked to help the pupil correct his mistake.

3.6.2.1.3 Helping students to correct their misconception

Question 15

15. According to you which one of these students is correct? How do you help the one who according to you is wrong understand his claim is wrong: " Ahmet claims that the letter S stands for students in the equation 6S=P which he wrote to represent a fact that "there are six times as many students as professors at this university". Mehmet claims that the letter S stands for the **number** of professors."

Analysing a pupil's mistake assists teachers to help that pupil in correcting that mistake. The scenario in the above question is also about a common mistake in solving the classical professor-student problem. In student-professor problem Clement, Lochhead, and Monk (1981) report that about 37% of college students makes a 'reversal error' where they write 6s=p rather than writing 6p=s. In this question prospective teachers need to firstly find which student is correct, and then help the other student to understand his solution is not correct. Therefore, they firstly solve the problem for themselves and then decide on a helping strategy. While deciding on a helping strategy, the respondents would have an opportunity on elaborating the possible sources of the mistake. However, the respondents could also think that their job is only to correct the pupil's mistake by teaching him how to get the equation right, without addressing the possible sources of the error. I will discuss more about the issues related to this question in part two of the next chapter.

By the help of these three questions, I wanted to elicit prospective teachers' pedagogical content knowledge about ways of presenting the subject matter. The following questions helped me to gather data on another component of pedagogical content knowledge, curriculum knowledge.

3.6.2.2 Curriculum knowledge

3.6.2.2.1 Introducing variables

Question 14

In the following question, prospective teachers are asked to explain how they would introduce variables. They need to write about their ideas on how to make an introduction to teaching the concept of variable.

> 14. How do you introduce your pupils to the idea of using letters to represent numbers?

The analysis of this question will not be presented due to word limitation and the results of this question were not investigated deeply by interviews. After introduction of a concept, teachers may need to order topics within a unit of teaching a topic. The following question asks prospective teachers to do this.

3.6.2.2.2 Ordering topics

Question 16

> 16. Which one of the following topics would you teach first? Why? Explain.
>
> - Substituting numbers in expressions
>
> - Simplifying expressions

This question can give us clues about prospective teachers' concerns in ordering topics. While ordering these topics, they would have an opportunity to consider several aspects. For example, they may think that one of the topics can be a base to teach another one.

They may think that one of them is more fundamental or easier than the other. Therefore, I am more interested on their reasons than their ordering in analysing the question. However, it should be acknowledged that it would have been better if the construction of the question had not forced the respondents to teach one of them before the other. They might think to teach them simultaneously. If I had constructed the question in this way, the respondents would not have thought that one should be taught before the other. I will discuss more about these issues concerning the question in the next chapter.

3.7 Administration

Questionnaires were distributed and completed in the first semester of the 2001-2002 academic year over a period of three weeks. To prevent administration differences and biases, certain directions were given in the classroom by me and I observed each session of questionnaire administration. Beside this, detailed directions and explanations were written on the questionnaires. Prospective teachers' instructors/mentors were present at the time of the administration. There was no time restriction on the students to complete the questionnaires. However, on average approximately two hours were spent on answering the questionnaires.

3.8 Data analysis methods

In this section I will discuss the techniques and methods used in analysing the data. I will also give accounts of the validity and reliability of these methods.

3.8.1 Categorisation of Responses

Since all the questions in the first part of the questionnaire were designed in a way that could allow me to see solution processes, and questions in the second part require respondents to explain the reasons, what they did and whys; there were very different kinds of responses to each question. The categorisation of these responses for each question was strictly based on the entire range of given responses considering the similarities among them. In order to form categories, all responses from all year groups for each question were collated. Firstly, general categories were formed by putting similar answers together. Deciding on the similarity of responses is based on words, phrases or solution methods that seem similar. Although this first categorisation was multi-dimensional and descriptive, it formed a preliminary framework for analysis. These categories were gradually modified or replaced during the subsequent stages of analysis. Therefore, this allowed me to combine categories in a flexible way by identifying subcategories under these general categories. For those responses which could not be put into already formed categories, a new category called "other" or "miscellaneous" was formed.

Regrouping or linking of categories is accomplished by use of the literature review. These procedures, I think, will be more clearly understood if I give the following example. The first categorisation of the responses to question 16 which asks subjects to explain which of the topics, substituting numbers in expressions or simplifying expressions, first was as follows.

1. saying just simplification first without giving any reasons.
2. simplification first because it is easier to substitute values in simplified expressions.
3. simplification first because if we taught substitution first then they would try to do substitution even if when it is not needed.
4. simplification first because this is of use to teach features of letters.
5. simplification first because it is harder or more abstract or more important.
6. simplification first because it is needed for solving equations.
7. simplification first because it is used in substitutions.
8. saying just substitution first without giving any reasons.
9. substitution first because to understand simplification, equation solving or expressions they are required to understand substitution.

10. substitution first because this may be of use to teach them meaning of letters.

11. substitution first because they can check their answers before and after simplification.

12. substitution first because it is easier.

13. miscellaneous.

As can be seen, the above categorisation is multi-dimensional and descriptive. However, these categories can be regrouped at least in two ways. One of these is based on consideration of which topics would be taught first; another one is based on the aspects in ordering them, namely the given reasons for ordering. When the reasons are considered, the literature review helps me to regroup them. As I discussed in chapter two, making connections between different lesson segments is important. Furthermore, these connections can be made by considering what is already learnt, or what is going to be taught, or the mathematical connections in the content itself. Therefore, considering these aspects helped me to regroup the original categorisation under three subcategories, namely 'conceptual ordering', 'procedural ordering', and 'other'. For example, responses which suggest that the subject considers getting answers right such as "substitution first because they can check their answers before and after simplification" or "simplification first because it is easier to substitute values in simplified expressions" are put into the 'procedural ordering' category. Responses which suggest the subject considers one of topics can be a base to teach the other topic such as "substitution first because this may be of use to teach them meaning of letters" or "simplification first because it is needed for solving equations" are grouped into the 'conceptual category'. A similar approach to categorisation was used for all the questions in the questionnaire. Detailed accounts of these can be seen in the next chapter.

After categorisations I examined the relationship between subject matter knowledge and pedagogical content knowledge. This is accomplished by cross-tabulating SMKv responses with PCKv responses. Theoretical explanations are given for each of these relationships.

3.9 Validity

The validity of this study was improved by triangulating the data types; interview data was incorporated to the findings of questionnaire data. Semi-structured interviews were audio-taped and then transcribed. The students who were interviewed were selected as a subset of the students who were completed the questionnaires. Random selection procedures were used after putting students into different groups according to questionnaire data. This improved the validity of the study by preventing systematic bias. In this study, it is thought that the subjects did not bias their responses while answering the questionnaire. That is, I think they did not seek to give answers that might please the researcher. I thought like this, because I explained them my true aim in conducting this research. I also explained to them that I would not disclose their responses to anybody else. I told them I would not mark their responses and their responses would have no effect on their course marks. I asked them to write down what they really thought, and to give as much detail in their answers as possible.

The validity of this study was also improved by drawing from published literature. Both contradicting and substantiating studies will be considered while presenting the findings. If the findings do not correspond with previous studies, I will cite the contradictory literature and explain how and why the findings diverge while presenting the analysis and results in chapters 4 and 5.

Another measure which could inform us about the validity of this research would be inter-item reliability measure. That is, by using more than one question that gauges the same aspect of SMKv and PCKv. By doing this I could have seen the consistency among the responses to the questions in the same group. This could be regarded as an issue of construct validity. The researcher could not use this measure due to practical difficulties. For example, if there had been more than one question for each aspect then the total number of questions would have been very large, and this could require more time from subjects to answer the questionnaires. On the other hand, for some questions (question 8 and question 9) this measure is used. Furthermore, the use of the literature both in designing the questionnaire and in analysing the results helped me to guard against this problem. The conceptual framework for each tested construct was clear and well founded. This framework was founded on several bodies of work in the area, such as the work on the concept of variable, the work on SMK and PCK. This review of literature provided a basis for testable hypothesis concerning the constructs. These hypotheses are supported by empirical data. Furthermore, the constructs are related to purposes of the questionnaire.

3.10 Reliability

The reliability is used to measure the extent to which an instrument, scale or item will give similar results when administered in different times, locations, or populations, when the two administrations do not differ in relevant variables. It can be measured by obtaining a form of correlation coefficient which is used to establish the trustworthiness of an inquiry.

As Bryman and Cramer (1997) points out, the *kappa coefficient* is the most widely used method to measure the extent of agreement for categorical data between two judges. A kappa of 0.7 or more is usually considered to be an acceptable level of agreement. Therefore, this number measures the extent of agreement for categorical data between two judges. It measures the agreement by taking into account the proportion of agreements that may occur simply by chance (Bryman & Cramer, 1997).

Since my data obtained from questionnaires is in the form of nominal categories, I used the *kappa coefficient* to measure the reliability of the findings. I selected seven questions from the questionnaire, which are found to be the most difficult to analyse. I selected approximately 25 cases for each of these questions randomly by using SPSS© 10 for Windows.

I gave each of these questions together with the categories I formed and their descriptions to my colleagues and wanted them to match responses to given categories in one of the SUMINER (Σine^r, Seminars in Understanding Mathematics Involving New Education Researchers) sessions. SUMINERs are the regular discussion and presentation sessions of the postgraduate mathematics education students who are studying at Warwick University. I compared the degree of agreement between my categories and the categories these postgraduate students put by finding the kappa coefficient.

Kappa coefficients for the questions 6, 12a, 12b, 13, 14, 15, and 16 were found as 0.666, 0.699, 0.664, 1.000, 0.674 and 0.753 respectively. These kappa values indicate that they are reliable enough since they are approximately 0.7 or more. Therefore, it should be pointed out that the researcher in analysing the data did not bias the categorisation in line with his expectations.

3.10.1 More on reliability discussion

However, the reliability of this study could have been strengthened by incorporating some other measures. For example, it would have been desirable to control subject's errors by redistributing the questionnaires at different times. This could have told us the findings did not depend on subjects' tiredness, tension or illness. By retesting we could have observed the fluctuations in subjects' performances in answering the questionnaires.

3.11 A Discussion on Shortcomings of the Methodology

This study could have been carried out by using any of the available research techniques including interviews, questionnaires, and observation. In the present study, I could not use observation techniques to collect the required data, since it was difficult, if not impossible, to find prospective teachers who are teaching the concept of variable at school. In Turkey, under the old education system used in Education Faculties, prospective teachers are required to visit schools to make observations and one to two lesson hours teaching under the observation of class teacher. However, this is not seriously applied in most cases; the class teacher just signs the papers as if the prospective teacher has done observation and teaching. Therefore, it is impossible to conduct observations. Even if this rule is applied, it is impossible to match the dates when the concept of variable is taught with the dates prospective teachers go into schools. Thus, employing an observation method is considered as impracticable to collect information concerning the aims of this study. As I mentioned, my aim was to obtain the richest possible data, but also I wanted to get this data from a lot of respondents easily and quickly. This left me two choices, either written tests or interviews.

While interviewing is a flexible and adaptable way of finding things out, it also clearly poses problems in terms of the much longer time required to conduct individual interviews. Since the type of my target population (prospective teachers) was difficult to contact in large numbers, I decided against the interview technique as a main data gathering instrument. Since the participants include prospective teachers more than18 years old, I thought that a questionnaire would be suitable to obtain their thoughts since they are old enough to express their ideas in writing.

Although questionnaires lack the flexibility of the dialogue that interviewing entails, coding and analysing the data obtained from questionnaires is easier than coding and analysing the inevitably more complex data obtained from interviews.

I would like here to add some further points of clarification concerning the possible disadvantages of the questionnaire technique, and what steps I took avoid these dangers.

One of the disadvantages of questionnaires is the low response rate. Brown and Dowling (1998) state that low response rates can cause not only a drastic reduction in the number of samples, but can also be a source of unintentional bias since there is a connection between the reasons for non-response and the topic of the research. However, some measures can be taken to increase response rates.

One of the measures can be through careful design of the questionnaire. As Cohen, Manion and Morrison (2000) discuss, clear instructions, and the inclusion of interesting, clear and unambiguous questions in a questionnaire can encourage the co-operation of respondents.

Another measure that can be taken is to explain the actual aim of the research clearly to the respondents and ensure confidentiality, which can be explained in the cover page. The promise of confidentiality and a statement of the aim can help to decrease fears and thereby increase personal commitment (Brown and Dowling, 1998). As I pointed out previously I tried to lessen these disadvantages of the questionnaire by taking these steps. In fact, for almost all of the questions in the questionnaire the return rate is above 70%.

As well as questionnaires, I decided to use interviews to see the validity of the responses given in the questionnaire. Talking with the prospective teachers increased the richness of the data by clarifying misunderstandings in the questionnaire.

3.11.1 More on shortcomings of the methodology

Since PCK is partly an internal construct, there were some challenges to find a suitable methodology to study this construct. In this research I focussed on what the prospective teacher knew or said they would do in a hypothetical situation. I did not consider the translation of this knowledge into classroom practice. Such a focus informed us only about intending teachers' understanding of subject matter and pedagogy and how that understanding is organised. If the reader believes that actions are a more accurate representation of knowledge than questionnaires and interviews, then s/he may see the methodology used in this research as problematic.

Kagan (1990) identifies a number of complications for assessing teacher cognition; some of these difficulties apply to the study of PCK (Baxter & Lederman, 1999). First, PCK can't be observed directly. During observation, we could not notice why a teacher chooses to teach the way he/she teaches. We could not know why the teacher chooses to give some examples while avoiding others. Therefore, by only observations we could get a limited view of PCK. This will lead us to ask teachers to articulate their knowledge.

Asking teachers to state reasons for their teaching actions may risk changing their process of decision making (Kagan, 1990). As Baxter and Lederman (1999) point out when teachers are asked to articulate their reasoning behind their instructional actions they may tend to construct reasons that will sound "right" or logical to the researcher. This threat also exists for prospective teachers, furthermore for anybody. That is, when somebody is asked to articulate their reasoning behind their actions then they may try to find reasons or "good" reasons that could be relevant to their actions. Therefore, there are a lot of difficulties to find a suitable methodology to study PCK. However, it should be acknowledged that the methodology employed in this study was one of the reasonable ways of investigating PCKv of prospective teachers. If I collect the data by using classroom observations, still I need to ask prospective teachers to articulate their reasons behind their actions. In the questionnaires and interviews I ask teachers to articulate the reasons behind the actions they would take in the given hypothetical situations presented to them.

3.12 Summary

The general research strategy used in this thesis is a *cross-sectional survey study*. The methodological approach taken is regarded as the combination of previous approaches used in this research domain, which is an amalgamation of both quantitative and qualitative methodology. The main data was collected by means of a questionnaire which contains free-response, open-ended questions. 184 prospective mathematics teachers from three different Turkish Universities completed the questionnaire. 10 of these prospective teachers were later chosen to take part in semi-structured interviews.

Analysis of raw questionnaire data is accomplished by means of categorisation. The relationships between SMKv and PCKv questions are investigated by cross-tabulating the categories. Later, interview data is used to discuss the relationships.

The validity of this study is established by a theoretical framework which is developed through extensive investigation of the relevant literature. The measure of the reliability is obtained by comparing my categorisation with my colleagues' categorisation in order to find the Kappa reliability coefficient.

CHAPTER 4

Analysis and Results

INTRODUCTION

This chapter is broken down to two parts; in the first part I will present the analysis of the questions that are related to SMKv. In the second part, the analysis of the questions that are related to PCKv will be presented.

PART ONE

4.1 Analysis of Subject matter knowledge

In this first part of the chapter I will probe answers to the general research question, which reads as '*What is the state and nature of Turkish secondary mathematics teachers Subject Matter Knowledge of variables?*' This part is broken down into three subsections. In the first subsection I will present an analysis of responses to questions 8 and 9. These questions will deal with prospective teachers' flexibility to approach problems in different ways. The reasons for this flexibility will be discussed with the help of educational theories such as symbol sense (Arcavi, 1994), cognitive units (Barnard & Tall, 1997), visualisation (Krutetskii, 1976) and conceptual understanding (Hiebert and Carpenter,1992). Therefore, the title of this subsection will be "Aspects of flexibility in approaches to problems".

In the second subsection, I will present an analysis of responses to question 6 which probes prospective teachers' 'concept images' (Tall & Vinner, 1981) of variables. This subsection which will be called "Concept image of variables" will investigate prospective teachers' mental associations with variables. In particular, it will be examined whether these students associate variables with objects or the number of objects. Finally, I will present an analysis of responses to question 4 which deals with understanding that the roles of variables depend on the contexts in which they are used and the ability to sort out the multiplicity of their meanings. This subsection will be called "Ability to deal with different roles and meanings of variables".

4.1.1 Aspects of flexibility in approaches to problems

In this subsection, I will present an analysis of responses to questions 8 and 9. I will start with discussing question 8.

8. If you substitute 1 for x in $ax^2 + bx + c$ (a, b and c are real numbers) you get a positive number. Substituting 6 gives a negative number. How many real solutions does the equation $ax^2 + bx + c = 0$ have? Explain.

This question is taken from Even's (1988) study in which she investigates prospective teachers' knowledge of functions. She states that subject matter knowledge of functions involves connections between graphical representation and algebraic representations. However, this question may also provide evidence about subject matter knowledge of variables, because it assesses one's ability to approach problems in different ways. In particular, it assesses one's ability to decide when to call on symbolic manipulation and when to abandon it. This ability is involved in "symbol sense" (Arcavi, 1994).

Probably the most efficient way to solve this problem is to visualise or draw two graphs. While imagining or drawing these pictures, one might also recall that the given algebraic expression is a parabolic one, and hence its graph is symmetric. Having this information and visualising the possibilities or drawing the graphs one can argue that it has two real solutions since it has a zero in between its negative and positive values. Without making use of graphs either by drawing or visualising, it is very difficult, if not impossible, to solve this problem.

In the analysis of the responses to the question 8, I considered whether or not prospective teachers attempted to use graphs:

1. Graph; in which there is evidence of using graphs

2. <u>No Graph</u>; in which there is no evidence of opting for graphs

The following table presents the frequencies of these categories:

Table 4.1.6 Frequency distribution of Response Types for Question 8

		Frequency	Percent
Valid	Graph	18	9.8
	No Graph	148	80.4
	Total	166	90.2
Missing		18	9.8
Total		184	100.0

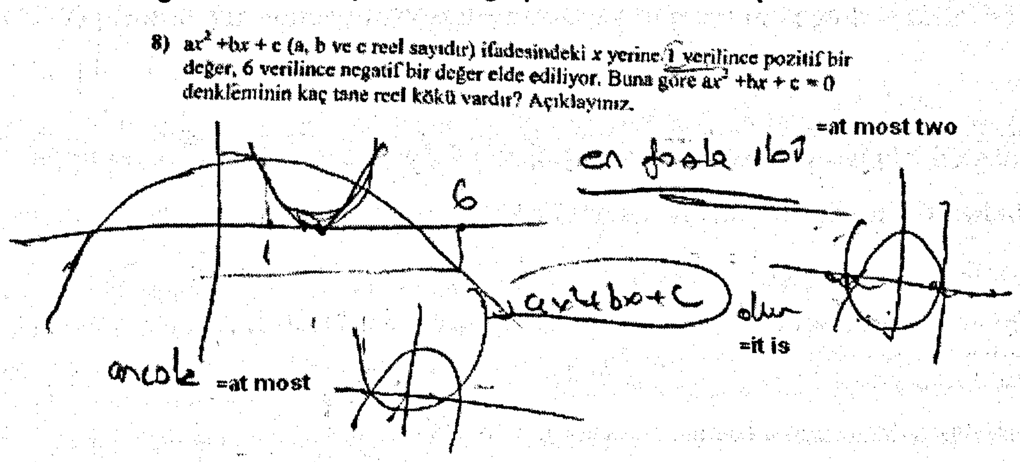
As can be seen from the table above very few prospective teachers (10%) make use of graphs. In some instances the graphs were incorrect and were erased but visible marks were left on the paper. However, even they could not reach the correct solution of the problem I included such answers in the 'Graph' category. The reason for such inclusion arises from my focus on obtaining evidence of an attempt to use a graph. Of those respondents who used a graph, fourteen reached a correct solution whereas four did not. All of these respondents also attempted to use non-graphical approaches, and then employed graphical approaches. However, it should be pointed out that this analysis was only able to identify those students who drew or attempted to draw the graph. In previous discussions of an appropriate solution method, I discussed about *visualising* or *drawing a graph*. Hence, we do not know whether other students attempted to visualise. As Presmeg (1986) points out:

> A visual method of solution is one which involves visual imagery, with or without a diagram, as an essential part of the method of solution, even if reasoning or algebraic methods are also employed.
>
> A nonvisual method of solution is one which involves no visual imagery as an essential part of the method of solution. (p. 298).

Hence, it is difficult to know from questionnaire responses whether or not a visual method may be used by those students even though they don't use a diagram. This is one of the shortcomings of questionnaires. It does not help the researcher to investigate more deeply what a student thinks while solving a problem. In order to lessen the disadvantages of the questionnaire, I asked another question (question 9) which is analysed in a similar way. Then, I cross-checked the answers of these questions in order to see the conformity between the responses.

The following figure (Figure 4.1.1) shows a prospective teacher's solution in which a graph was used. The respondent also attempted to solve the problem by substituting, but in this scan output, the non-graphical approach is not apparent, since it was erased before the questionnaire was handed in. This respondent concludes that the equation has at most two roots, after trying several cases according to the given information in the question.

Figure 4.1.1 A respondent's graphical solution to question 8



As can be seen in the above example solution, this student draws the graphs of $y = ax^2 + bx + c$. He started his solution by substituting the given numbers in the question. After a few manipulations of the inequalities he noticed that his way was unproductive and he made use of pictures. His solution method shows that he could return to a productive way of solving the problem. He tries to see how this parabola can be positive when $x=1$ and negative when $x=6$. This can be seen in top left corner of his solution. Beside this Cartesian graph he sketched two more. In these graphs there are two parabolas; one of which is upwards and the other one is downwards. It seems that he tries to see several other cases that satisfy the conditions given. From these several graphs, he may try to see all possible cases with conditions given in the question.

This solution method may require an aspect of 'symbol sense' which involves abandoning symbols when it is likely to drown in technical manipulations. The solver did not plunge into symbol pushing, since he realised that the algebraic manipulation could be quite laborious and prone to error. According to Arcavi (1994), "the decision to discard the almost unavoidable initial temptation to proceed mostly symbolically, in favour of the search for another approach, requires a healthy blend of "control" with symbol sense." (1994, p. 26). On the other hand, this blending may require having a rich "cognitive unit" which involves different aspects of a parabolic relationship between the variables x and y, and moving from one aspect to another in a flexible way. Barnard & Tall (1997) presents the notion of "cognitive unit" as "a piece of cognitive structure that can be held in the focus of attention all at one time" (p. 41).

The concept $y= ax^2 +bx + c$ contains a lot of different aspects. For example, its graph is symmetric, "c" is the intercept; the sign of "a" shows whether its graph is upwards or downwards; and any parabolic relationship can be represented by substituting numbers for the parameters a, b, c. All of these aspects may not be required for the solution of this problem. For example, one does not need to understand the roles of c and a in order to solve this problem. If someone has a very strong visual image of a quadratic graph, which he/she can link easily to the form $y= ax^2 +bx + c$ given in the question, then he/she may not require much more. She/he does not need to be conscious of the fact that the curve is symmetrical or the roles of c and a, for example. He/she just needs to picture the fact that it always 'goes back on itself' or has the same sign for large positive and large negative values of x. He/she does not need to be able to make this fact explicit to her/himself, or to verbalise it. However, she/he does need to be able to *connect* this strong visual image to the algebraic expressions for the function and the conditions given.

Some students can regard these different aspects as connected and bring the required aspect to their focus of attention when it is needed. They can compress all these different aspects into a single "cognitive unit" by using the power of symbols. Condensing separate ideas into a single idea with different aspects is called the "conceptual compression" (Thurston, 1990; Gray & Tall, 1994). This conceptual compression enables an individual to move flexibly from one aspect to another since the separate elements have close connections between them. Therefore, when it is compressed, the equation $y = ax^2 + bx + c$ and the "$\cup$"shaped parabolas describe exactly the same thing. That is, they are different aspects of the same idea.

If these different aspects are conceived as disconnected, then it is difficult, for the individual to bring the required one to the focus of attention when it is required. Since for such individuals different elements are not compressed into a flexible cognitive unit, these different aspects remain as split entities in a loosely connected structure. Students who have such loosely connected structures may lose their way while following procedural routes to the solution of the problem. The reason for this is that since they devote too much conscious thought to the procedures, they can't make cognitive links to solve the problem. For example, in order to solve question 8, they substitute 1, and 6, then try to find the values of a, b, c by manipulating the resulting inequalities. Then they lose their way in these manipulations and cannot find the solution. Another reason why these students plunge into symbol pushing is the fact that this problem is an unfamiliar problem. It is well documented in the literature that there is a tendency to evaluate or manipulate in the face of an unfamiliar problem.

The majority of prospective teachers' (80%) who were put into the 'No Graph' category seem to 'drown' in technical manipulations. None of these prospective teachers could reach the correct solution of the problem, but still they did not opt to employ graphical representation. Some of the respondents (7) in this group claimed that the number of real solutions is two since a polynomial of degree 2 has two real solutions. These students appear to have applied a misremembered rule.

The solutions of some of the prospective teachers (14%) who are in the 'No Graph' category imply that they were following memorised procedures or rules to cope with this problem. The reason for such an assertion is that these students attempted to use delta (Δ) formula from which they tried to guess the number of roots. The following figure (Figure 4.1.2) shows a student's solution who followed the technical manipulation route to solve question 8.

Figure 4.1.2 A respondent's symbolic solution to question 8

8) ax^2+bx+c (a, b ve c reel sayıdır) ifadesindeki x yerine 1 verilince pozitif bir değer, 6 verilince negatif bir değer elde ediliyor. Buna göre $ax^2+bx+c=0$ denkleminin kaç tane reel kökü vardır? Açıklayınız.

$a+b+c>0$

$36a+6b+c<0$

$30a-5c<0$

$5(6a-c)<0$

$\boxed{6a<c}$

$35a+5b<0$

$5(7a+b)<0$

$\boxed{7a<-b}$ $\quad -7a>b$

$\Delta = b^2-4ac \cong 49a^2-4.a.6a$

$\cong 49a^2-24a^2$

$\cong 25a^2 > 0$

olacağı için bu denklemin reel iki kökü vardır

as a result this equation has two roots

In the above solution method, the solver substitutes 1 and 6 for x (top left corner). He gets two inequalities with three unknowns. From these inequalities, he gets 7a<-b and 6a<c. Then, he writes the Δ formula. He substitutes 7a for b and 6a for c into the formula b^2-4ac. He substitutes these values because he already found that 7a<b and 6a<c. These inequalities are some sort of a relation between a and c and also between a and b. However, since he is not sure whether Δ is larger or smaller than this new expression that he gets as a result of substitutions, he puts the approximate symbol in between them. Finally, he gets $25a^2$, which is according to him approximately equal to Δ, and writes this is larger than zero. Therefore, he concludes that the equation has two real roots.

The general procedural strategy in Turkish schools to find out the roots or number of roots of 2nd degree equations is using the delta formula. In this formula the coefficients are used to find the 'Δ' which is b^2-4ac where a, b, c are the coefficients of ax^2+bx+c. Then what follows is to investigate the sign of 'Δ'. If it is negative then the equation has no real roots, if it is positive then it has two different real roots and if it is zero then it has two coinciding roots. The solutions of these prospective teachers may reveal that their goal was to find the signs of a, b, c in order to investigate the sign of 'Δ'. Their ultimate goal is to guess the number of roots from the sign of 'Δ'. In order to achieve these goals, they use a number of procedures and find it difficult to keep track of what they are doing. Hence, it seems that the most of the students in "No graph" category do not see where they are going because they are too busy doing the manipulations. It is likely that this prevents them from making necessary cognitive links to graphs.

The judgement of these prospective teachers might have been also clouded by another type of knowledge. This knowledge might be the fact that visual proofs are sometimes contradicting the idea of ultimate proving techniques in mathematics. Visual proofs are sometimes degraded in mathematics since they do not satisfy the ultimate formalism required by the formal school of thought. Therefore, these prospective teachers might have given up the validity of working with pictures on the grounds that it's a visual proof. Hence, they get into difficulties with the symbolism and break down.

Skemp (1979) conjectures that people may formulate goals to solve a specific problem, and sub-goals to solve different parts of the problem. He proposes that during the solution process, a *comparator* activity is used at various times to check whether the solution process is getting suitably close to the goal or to one of the intervening sub-goals. In a similar vein Schoenfeld (1985) states that:

> a category of behavior [which] deals with the way individuals use the information potentially at their disposal. If focuses on major decisions about what to do in a problem, decisions that in and of themselves may "make or break" an attempt to solve a problem. Behaviors of interest include making *plans, selecting goals and subgoals, monitoring and assessing solutions as they evolve, and revising or abandoning plans when the assessments indicate that such actions should be taken.* (p. 27, emphasis added).

Arcavi (1994) considers such behaviours in the context of symbol sense. He asserts that symbol sense includes abandoning symbols when we are likely "to drown" in the technical manipulations. He adds that "the managerial decision to change a course of action involves more than that:"

> it is also motivated and driven by a sense of aesthetics, elegance, efficiency, as well as an appreciation (or even belief) that mathematical work involves much more than the stoicism of embarking on "hairy" symbolic manipulations. (p. 26).

Hence, the search for another approach which may emerge from regarding the problem in a different way, or by changing the representation requires "a healthy blend of "control" with symbol sense" (Arcavi, 1994, p. 26).

Crowley and Tall (1999) theorise that when following a routine sequence of actions "the focus on successive remembered steps may be so great as to temporarily fill the focus of attention and suspend the activity of any comparator" (p. 4). Therefore, the reason why so many prospective teachers failed to solve question 8 may be their too much occupation with procedures which prevent them from using 'comparator activity' and hence making the relation between the given equation $ax^2 + bx + c = 0$ to a graphical representation of the function $f(x) = ax^2 + bx + c$. Ninety percent of the respondents did not (apparently) make the relation. They did not abandon the symbolic representations in favour of graphical representation. Even (1988) reports similar findings; only 10% of the prospective secondary mathematics teachers in her sample opted for the graphical method of solving this problem. As a result, Even (1988) argues that the majority of her prospective teachers compartmentalised knowledge related to functions without constructing connections among them. However, following Crowley and Tall's (1999) conjectures, I argue that prospective teachers who do not have compressed cognitive units with rich relationships resort to formulate sub-goals using lengthier procedures. This prevents them from looking for other ways of solving the problem. On the other hand prospective teachers who have compressed cognitive units encompassing the properties of algebra and the graph of a parabolic equation flexibly resort to make use of graphs.

The following question is a similar problem which serves a similar purpose to question 8. It assesses students' ability to regard a problem from different ways, in particular abandoning symbolic manipulations for better tools.

Question 9

9. For what values of a does the pair of equations $$x^2 - y^2 = 0$$ $$(x-a)^2 + y^2 = 1$$ have either 0, 2, 3, or 4 solutions?

Arcavi (1994) gives a very similar question as an example of approaching problems from different ways or by changing the representations. In particular, by this question he wants to explain that a premonitory feeling of symbol sense may prevent students from hard work in persistence with symbol manipulations. Probably the most efficient way to solve this question is to work on a Cartesian graph. Observing the number of intersections between two diagonals of the Cartesian plane ($x^2 - y^2 = 0$, namely $y = \pm x$) and a family of circles of radius 1 whose centres lie on the x-axis will take us to the solution of this problem.

This problem can also be treated algebraically, that is by technical symbol manipulations. However, this method involves heavy use of logical connectives, lots of technical work and a high probability of making mistakes. Instead of pushing symbols, working on a Cartesian graph saves one from "the stoicism of embarking on 'hairy' symbolic manipulations." (Arcavi, 1994, p. 26).

In the analysis of the responses to the question 8, I considered whether or not prospective teachers attempted to use graphs:

1. <u>Graph</u>; in which there is evidence of using graphs
2. <u>No Graph</u>; in which there is no evidence of opting for graphs

The following table presents the frequencies of these categories for question 9:

Table 4.1.7 Frequency distribution of Response Types for Question 9

Q9			
		Frequency	Percent
Valid	Graph	16	8.7
	No Graph	107	58.2
	Total	123	66.8
Missing		61	33.2
Total		184	100.0

As can be seen from the table above very few prospective teachers (about 9 %) make use of graphs. Of those respondents who used a graph, five reached a correct solution whereas eleven did not. All of these respondents also attempted use non-graphical approaches, and then employed graphical approaches. Even if they could not reach the correct solution of the problem I included such answers in the 'Graph' category. The reason for such inclusion is similar to the reasons given for analysis of the previous question, which was my focus on obtaining evidence of an attempt to use a graph. As I discussed before, using a graph to solve this question and the previous question may tell us that the solver may have an aspect of symbol sense which enables one to abandon symbols when he/she is likely to 'drown' in technical manipulations. This sense can be a result of having compressed cognitive units which allow the solver to move flexibly among different elements of that unit or in bigger network. However, it should be acknowledged again that in the analysis of this question we cannot know whether other students attempted to visualise. Therefore, in order to check the reliability of this analysis, I cross-tabulated this question with the previous one. This can be seen in the following table.

Table 4.1.8 Cross-tabulation of Question 9 with question 8

Q9 * Q8 Crosstabulation

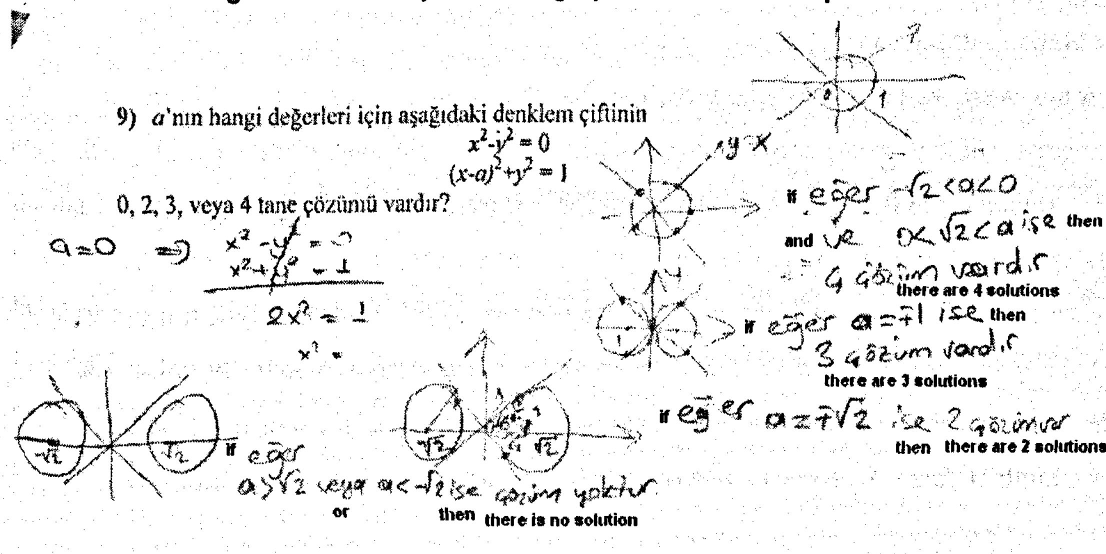
Count

		Q8		
		Graph	No graph	Total
Q9	Graph	14	2	16
	No Graph	3	97	100
Total		17	99	116

As can be seen in the above table very few students change their solution methods. For example, of those students (16) who used graph for question 9, 14 used graph for question 8 also. Only two of these students did not use graph for question 8. Similarly, of those students who used manipulation rather than graphs, 97 remained to do so for question 8 also, however only three of them used graph for solving question 8. These observations may tell us that there is conformity in students' solution methods for both questions. Hence, this enriches the reliability of the analysis employed for the responses of question 8 and 9.

The following figure (Figure 4.1.3) shows a respondent's solution that used a Cartesian graph.

Figure 4.1.3 A respondent's graphical solution to question 9



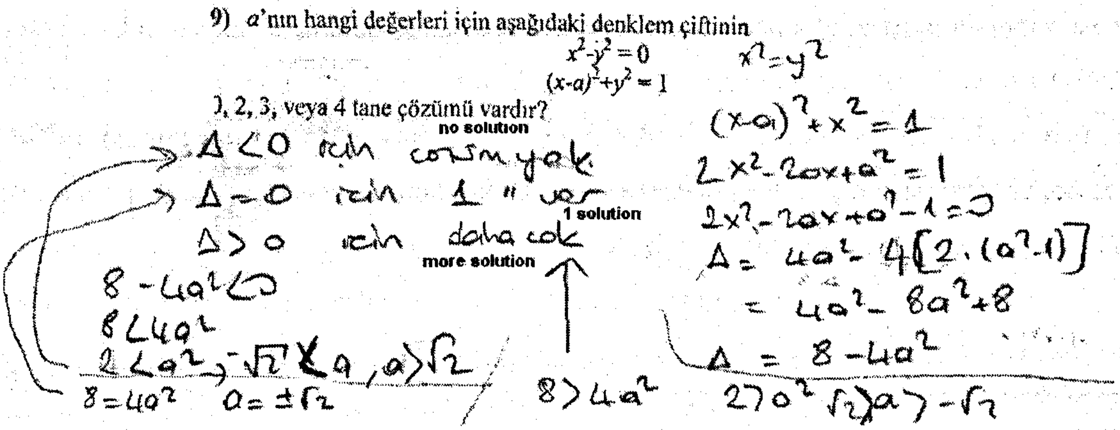
As can be seen in the above solution, the solver tries a case for the parameter 'a' (please see the top left corner). He takes a=0 and then finds a system of equations. From these equations he finds $2x^2=1$ and then he leaves the question there by leaving it as $x^2=$. This suggests that he saw his method is unproductive. Then, he also uses Cartesian graphs. In these graphs he shows the cases when the system has 0, 2, 3 and 4 solutions. From these graphs he finds that when $a=\pm\sqrt{2}$ the system has two solutions. He finds $\sqrt{2}$ by the help of a right-angled triangle. Since $y=x$ makes 45 degrees with $y=0$, he gets an isosceles triangle when $y=x$ becomes tangent to the circle. Using this triangle, he finds that the centre is located at $(\sqrt{2},0)$ from 1^2+1^2=(hypotenuse^2). Hence, he finds that the system has two solutions when $a=\pm\sqrt{2}$. It has three solutions when $a=\pm1$, four solutions when $-\sqrt{2}<a<0$ and $0<a<\sqrt{2}$, and no solution when $\sqrt{2}>a$ and $-\sqrt{2}<a$. His solution seems rather straightforward and elegant. It also seems very efficient. He solves the question by approaching it in a different way from his initial attempt. He changes the representation by giving up technical manipulation. Such remarks about his solution method may suggest that he developed a feeling for when to abandon symbols. However, it should be noted that not all students in this category could solve this problem even though they attempted to use a graph.

As I wrote earlier, 11 students in the 'Graph' category could not solve the problem. Even though these students attempted to use Cartesian graphs they could not continue to make use of these graphs. One of the reasons for such behaviour might be that they could not draw the graphs of the two diagonals (x^2-y^2 = 0, namely $y=\pm x$). They drew these as something like parabolic curves. As a result, they could not observe the intersection points and hence, they could not solve the question. But the most of these students could not reach the correct solution because they did not move the circle along the x-axis. That is, they did not find that the solution set consists of intervals. They wrote that the system has four solutions when a=0; two solutions when $a=\pm1$. However, since they tried to regard the problem in a different way by changing the representation, these students also might have developed reactions of the sort: "this involves too much hard, technical and uninteresting work, there *must* be another approach." (Arcavi, 1994). Such reactions may be the result of having compressed 'cognitive units' which have rich relationships. In the context of this problem the concepts x^2-y^2 = 0 and $(x\text{-a})^2+y^2 = 1$ have a lot of different aspects. For example, the graph of the first one is two diagonals on the Cartesian plane, the graph of the second one is a circle with radius 1 and centred at (a, 0). To follow a graphical approach, the solver needs to be able to *connect* the visual images to the algebraic expressions and the conditions given.

As I noted earlier, some students can regard these different aspects as connected and bring the required aspect to their focus of attention when it is needed. They can compress all these different aspects into a single "cognitive unit" by using the power of symbols. This conceptual compression enables an individual to move flexibly from one aspect to another since the separate elements have close connections between them. Therefore, when it is compressed, the equations x^2-y^2 = 0 and $(x\text{-a})^2+y^2 = 1$ and their graphs describe the same thing. That is, they are different aspects of the same idea. Hence, having connected cognitive structures might have helped the students to approach this problem in a different way.

The students who did not use graphs and who only followed the technical manipulation route could not solve this question correctly. Some of them (about 4%) found those 'a' values when the system has no solution, and some of them (6%) found the 'a' values when the system has zero or two solutions. These students used the Δ formula to find these values. The following figure shows a students' response that used the Δ formula.

Figure 4.1.4 A respondent's symbolic solution to question 9



In the above response, the student writes the rules about the number of roots (top left corner). However, for Δ>0 she just writes there are more solutions, but she does not mention the exact number - whether it is 2, 3 or 4. On the right hand corner, she substitutes x^2 for y^2 in the second equation. By this substitution, she reduces the number of equations to one and number of unknowns also to one. She manipulates this equation in order to put it into the form ax^2+bx+c. When she puts it into the general form, she plugs the coefficients of this form into the formula $\Delta=b^2-4ac$. As a result she finds $\Delta=8-4a^2$. Then she investigates the sign of Δ. The interesting thing about this solution is that she writes that when $2>a^2$ the system has more than 1 solution. However, in the problem it is asked about the cases when the system has 3 or 4 solutions. The reason why she got stuck at this stage might be due to the fact that the delta formula only says when a second degree equation has 0, 1 or 2 solutions. Therefore, using this formula alone without logical connections to the conditions given in the problem does not help very much to find cases when the system has 3 or 4 solutions.

It seems that she has an overall strategy to solve the problem. Furthermore, it seems that she formulated goals to achieve all or part of the solution. However, her solution seems to involve too much technical manipulation. She applies a memorised rule to the question. It seems that she had built a mental action-schema by practising finding the roots of 2nd degree equations and systems of equations. Asiala, Brown, DeVries, Dubinsky, Mathews and Thomas (1996) describe an *action* as a transformation of objects perceived by the individual as essentially external and as requiring, either explicitly or from memory, step-by-step instructions on how to perform the operation. According to Asiala et al's (1996) APOS (action-process-object-schema) theory, when an action is interiorised then it becomes a process. Then the individual may encapsulate this process into an object. Finally, an individual may form a coherent collection of actions, processes, objects and other schemas that is her or his schema for the concept in question.

Finding the roots of 2nd degree equations may involve following actions: put the given equation into the general form $ax^2 + bx + c$; then, plug the coefficients a, b, and c into the formula $\Delta=b^2-4ac$; finally, investigate the sign of this last equation to comment about the number of roots or find the roots. She follows these actions in her solution. However, when something more is required she does not pursue other routes to the solution of the problem. As a result, she could solve half of the problem.

Her solution suggests that she has not developed an aspect of 'symbol sense' which includes when to invoke symbols and also when to abandon them. It seems that she does not reach to the goal of the problem as the solution evolves. The reason for this may be that she has not revised or abandoned her plans when assessments about the progression of her solution indicated that some different actions should be taken. It may even be the case that she did not revise her plans about the solution of the problem. One of the reasons why she did not take such actions may be that she did not want to spend too much time on the problem. Alternatively, regarding the problem in a different way may not be available to her. That is, she may have only one way of proceeding when faced with problems related to the roots of 2nd degree equations. As a result, she may resort to 'formulate sub-goals using lengthier procedures' (Crowley and Tall, 1999). If she had possessed compressed cognitive units encompassing the properties of algebra and the graph of equations, she might have flexibly resorted to make use of graphs. But, she concentrated on procedures that occupied most of her focus of attention. This may cause her to "lose touch with the ultimate goal and be faced with sequences of activity that are longer, more detailed, and more likely to break down." (Crowley and Tall, 1999).

However, it should also be noted Krutetskii's (1976) theories about the types of mathematical abilities of students. Krutetskii divides school children into three types with respect to mathematical abilities. He identifies the following groups:

1. students who use visualisation dominantly to solve abstract problems,
2. students who use logical, symbolic thinking dominantly to solve problems,
3. students who use both in equilibrium.

In light of Krutetskii's theories, the students who followed purely symbolic routes to solve questions 8 and 9 may prefer to employ non-visual methods to solve problems. That is these students may be non-visualisers who can be counted in the second group of Krutetskii's categorisation. The students who used graphs may be in third group who could use both methods in equilibrium. These students start to solve problems 8 and 9 with a symbolic approach. Then they were able to switch easily to another solution method.

All theories so far cited to explain students' solution methods to questions 8 and 9 do not contradict to each other. On the contrary it seems that they complement each other. For example, Krutetskii may see those students who switched their solution method to graphs as students who could use *both* symbolic and visual methods in *equilibrium*. On the other hand, Arcavi (1994) may regard these students as having an aspect of 'symbol sense' which involves *abandoning* symbolic approach *flexibly* for a better one when it is likely to drawn in technical manipulations . Even (1988) may probably claim that these students have connected knowledge, since they demonstrated *links* between graphical and symbolic representations. Crowley and Tall (2001) may claim that these students organised their mathematical knowledge into a network of *connected* cognitive units (Barnard & Tall, 1997) with *flexible links*. They are able to move among the cognitive units almost subconsciously as needed to solve the "problem—the network structure contains (usually multiple) procedures toward a solution, checking mechanisms, and links to a larger mathematical structure." (p. 1). It should be noted that Arcavi and Krutetskii describe behaviours that they observe in solution methods. On the other hand, Crowley and Tall describes more than this. They look into the reasons of such behaviours.

The number of students who prefer to use non-visual methods is very large. This may be due to the fact that the school curriculum and examinations reinforce this preference. In schools, students are taught ways of recalling formulae and procedures rapidly. The procedures and formulae are practised often in order to habituate these. This may lead students away from visual methods.

4.1.2 Concept image of variables

The following question will be used discuss prospective teachers' associations with variables. In particular, it will be used to explore whether they associate variables with objects or numbers.

Question 6

6. What different things might an algebraic expression such as, say $2x+1$, mean? What can x stand for?

In order to point out the different perceptions and understandings of symbols in algebra Sfard and Linchevski (1994) ask what we see when we look at an algebraic expression such as $3(x + 5) + 1$. They write, our answers depend on what we are *able* to *perceive* and are *prepared* to notice. Gray and Tall (1994) assert that the symbols play a very important role in what different things we can see when we look at such algebraic expressions. They use the term '*procept*' to point the pivotal role of symbols. An expression like $2x+1$ can be regarded as both as a process to carry out and as an object, the concept produced by that process. According to Gray and Tall (1994), a procedural student regards an expression such as $2x+1$ as a computational process; multiply 2 by x then add one. However, a proceptual student can perceive it flexibly either as a process or a concept depending on the task at hand. Furthermore, proceptual students could think about it and manipulate it mentally. They can embed different meanings of the expression in $2x+1$, such as it could be an odd number, or a function etc.

While analysing responses to question 6, I saw answers in which students mention either the process meaning or the concept meaning of $2x + 1$, not both. Although the way the question was formed encouraged students to give as many different meanings as possible, none of them wrote both of these meanings of $2x + 1$. If I could see both of them in one student's response, then I could have claimed that I saw a sign of proceptual thinking in that student's response. However, it is not that valid to claim a student is a procedural thinker from his/her response to this question even if she/he gives only the process meaning of $2x+1$. Even if s/he only writes that $2x+1$ means multiply 2 by x and add one, we can not be sure that this student is not able to perceive it as a concept. S/he may be able to perceive it as a concept, but at the time of answering the question her/his focus of attention may have been on the process meaning. Therefore, categorising responses according to the meanings of $2x+1$ was not seen to be very productive. Hence, I concentrated on the meanings of x in $2x+1$; on what it stands for.

The followings are the four categories for this question:

1. Number: Contains responses in which x is associated with numbers; e.g. x can be any number, or real number, or natural number etc.
2. Variable: Contains responses in which x is associated with one of the roles of the variables; e.g. x can be a variable, an unknown or a general number or a unit etc.
3. Object: Contains responses in which x is associated with objects; e.g. x can be anything here, x can be an apple.
4. Other: This category contains those responses which nothing is said or can be inferred about x. *e.g.* "It may be a linear line equation."

I will explain these categories following the frequency table below.

Table 4.1.9 Frequency distribution of Response Types for Question 6

		Frequency	Percent	Valid Percent
Valid	Number	84	45.7	53.2
	Object	20	10.9	12.7
	Variable	42	22.8	26.6
	Other	12	6.5	7.6
	Total	158	85.9	100.0
Missing		26	14.1	
Total		184	100.0	

Each student was allocated to only one category even though they sometimes gave several answers to the question. In analysing the responses, my focus is whether the student associates variables with objects or numbers. Since a student may have different associations with variables, he/she can mention more than one of these. However, if he/she mentions that x can be an object and also x can be a number together, then this student is put into the object category. The reason for this is that, here x can not be an object in $2x+1$, and such associations may cause problems for that student.

As can be seen from the frequency table above, the largest group of prospective teachers (46%) fall into the 'Number' category. These prospective teachers write that x can be a number or any number in $2x+1$. For example, they write;

> "In $2x+1$ for each value that is substituted for x, we get different values. Whatever number we want we can substitute it for x."

> "$2x+1$ may mean odd numbers, x may be used for natural numbers."

Those prospective teachers (about 23%) who are put into the 'Variable' category write that x can be a variable or an unknown. In some of the responses in this category they state that x could stand for a quantity rather than an object *e.g.* slope, age, or price of something. *e. g.*

> "if we change it to y=2x+1, x=buying price of a thing, y= selling price of it."

In this question the context dictates that x must stand for numbers. If the algebraic expression $2x+1$ was not mentioned in the question, then x could stand for anything. It could stand for an object, like apple or banana. It could be a multiplication sign or the 24th letter in the English alphabet. It could even be a cancellation sign or the sign for denoting unknown things. A person may have one or more of these interpretations at a given time. All of these meanings can be embedded in the symbol x. However, since here x is given in an algebraic expression, its non-mathematical meanings should be left out. A person who has an image of x as an object in $2x+1$ may have problems in manipulating symbols.

At first glance, those students who write that x can be a number or a variable seem not likely to have problems in manipulating symbols. However, we can not claim that they definitely do not confuse mathematical meanings of x in algebraic expressions with its non-mathematical meanings. The reason for this may be given as follows; although they write that x can be a number or a variable, we don't know if they understood that x cannot stand for an object in $2x+1$, or simply did not mention this interpretation. Therefore, I should point out that in this question if I explicitly asked whether or not x could stand for an object in $2x+1$, it would be better. I did not ask the question in this way because I did not want to suggest anything to the respondents. In order to get as many different meanings of x as possible, I did not use closed format questions. As a result, although those respondents who are put into the 'Number' and 'Variable' categories are more likely to regard variables as standing for numbers, there is still some doubt about their concept images of variables. However, we can be more definite in claiming that the students who are put into the 'Object' category may have problems in their image of variables, because, this small group of respondents (11%) write that x could stand for an object or for a thing. For example, they write

> "$2x+1$, with values we substitute for x, $2x+1$ always changes, x can be anything here. x can be an apple, 2apples+1."

As can be seen in the above response, these students both mention number meanings and object meanings together. The confusion about denoting the number of objects and objects themselves is more apparent in these responses. These students are more likely to make a 'reversal error' by confusing the number of objects with the objects themselves. Clement, Lochhead and Monk (1981) exposed the arguments about a misconception which is labelled as the 'reversal error' in conceptualisation of symbols in algebra. The "students and professors" problem asks the respondent to translate "There are six times as many students as professors" into symbols using S for the number of students and P for the number of professors. The analysis of the responses to this problem revealed that 37% of college students sampled were in error and two thirds of these wrote $6S = P$, rather than $S = 6P$. This reversal error is caused as a result of teaching the letters as standing for the objects (students, professors) rather than the *number* of objects (see, for example, Crowley, Thomas & Tall, 1994).

I put responses in which nothing was said or can be inferred about x into the 'Other' category (7%), for example responses like;

> "It may be a linear line equation."

As can be seen in the above response these students did not provide much information, so it is difficult to comment on their concept images of variables. Concept image is defined by Tall & Vinner (1981) as

> "the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes. It is built up over the years through experiences of all kinds, changing as the individual meets new stimuli and matures."

Cape (2000) reports that some of the prospective teachers in her sample could not say for sure whether the variables can stand for numbers or objects in algebraic expressions. For example, these teachers say x can stand for apples, or people. When the researcher asks again; then they say no it can't, it can stand only for numbers. Therefore, Cape (2000) asserts that these teachers have a confused concept image of x. According to Cape, the reason for this confusion is because of the teaching approach for early algebra used in

South Africa which overemphasises manipulative skills. In this approach, students are told what is right or wrong to do. According to Cape, the aim in this approach is to make students form their 'concept image' from 'concept definition'. He goes onto claim that in this teaching approach, the processes and properties associated with algebraic expressions in the mind of a student are intended to be something like; brackets then remove, collect like terms to simplify expressions, watch for order of operations, change sign change side etc.

Crowley, Thomas and Tall (1994) point out that in traditional teaching, "fruit salad" algebra is regarded as a way out to teach students manipulation of symbols. The expression "$2a+3b$" is explained to stand for "two apples and three bananas". This teaching approach helps some children to sort out "$2a+3b+a$" as "two apples and three bananas and an apple", which is "three apples and three bananas", or "$3a+3b$". So students seem to be able to simplify expressions. However, this may cause students to have an image of a letter as representing an object. As a result, these children are "ready to fall foul of the student-professor problem." (Crowley, Thomas & Tall, p.3, 1994).

Pimm (1987) also criticises 'fruit-salad' approach since "it leads to confusion between *a* being apples and *a* being the number of apples" (p. 132). Pimm goes onto assert that "the algebraic expression is not an analogue of 5 apples, nor is 5 apples a possible interpretation of 5...the letters themselves are standing for numbers" (p.132). Booth (1988) adds that this approach is severely flawed since "not only does it encourage an erroneous view of the meaning of letters, but it can also be used by students to justify their [wrong] simplification" (p. 26, cited in Tirosh et al. 1998, p. 60). For example, some students may simply write that "$2a+3b+a$" is "six apples and bananas". Since they have no mathematical symbol for "and", they may write the letters one after another, *conjoining* them as "6 *a b*". (Crowley, Thomas & Tall, p.3, 1994).

To sum up, a student who is not aware of the distinction between x denoting the number of objects and x standing for the objects themselves may have confusion about what the algebraic variables could stand for. This confusion sometimes may cause them to make mistakes while solving problems such as the famous student-professor problem. Therefore, those students who are put into the 'Object' category of this question may have difficulties in problems where they need to distinguish between letters representing the objects and the number of objects.

4.1.3 Aspects of flexibility in sorting out different roles and meanings of variables

The analysis here involves question 4.

Question 4

4. What *different* things could $y = 3$ mean if one derives it from $y = mx + 3$.

This question is a modified version of a problem which Arcavi (1994) gives as an example for discussing the importance of the context in deciding the role of variables. He suggests that "a desirable component of symbol sense consists of the *in situ* and operative recognition of the different (and yet similar) roles which symbols can play in high school algebra" (p. 30). As Arcavi (1994) points out, in $y = \mathrm{m}\,x + \mathrm{b}$ even though x, y and m, b represent numbers, "the kinds of mathematical objects one obtains by substituting in them are very different." (p. 30).

In question 4, therefore, the meanings of y=3 depend on how it is obtained from y=mx+3. If it is obtained by substituting x=0 then y=3 is the corresponding y-coordinates, *i.e.* the point where the lines y=mx+3 cut the y-axis. If, on the other hand, y =3 is obtained by substituting m=0, what we have found is the line y=3.

Therefore, one of the suitable ways of solving this question is to look at the possibilities that can produce y=3 from y=mx+3. Hence, if 3= mx+3 then mx=0, this means either m=0 or x=0 or both. Then the meanings of y=3 can be deduced from these last deductions. Therefore, there are two important factors in solving this question. One of them is the number of meanings of y=3, the other one is how these meanings are obtained. As I explained above, these meanings can be obtained by means of manipulations. These manipulations involve "sorting out the multiplicity of meanings symbols may have depending on the context and the ability to handle different mathematical objects and processes involved." (Arcavi, 1994, p. 30; cites from Sfard, 1992; Moschkovich, Schoenfeld and Arcavi, 1993). Therefore, in analysing the responses to this question I focused on the number of meanings and how these meanings are obtained.

The analysis of this question will be presented by using following categories:

1. Manipulation and two meanings: contains responses in which both the line and the point meanings are deduced by means of manipulations
2. Manipulation and one meaning: Either $y=3$ is a point or a line is deduced by means of manipulations.
3. Manipulation and no meaning: Contains responses in which manipulations are performed but neither line nor point meaning is mentioned.
4. No Manipulation: This category contains those responses in which either of the meanings is mentioned without manipulating the given information.
5. Other: Contains responses in which respondents confuse and deduce illegitimate meanings together with legitimate meanings.

I will explain these categories and discuss each of them after presenting their frequencies in the following table (Table 4.1.10).

Table 4.1.10 Frequency distribution of Response Types for Question 4

		Frequency	Percent	Valid Percent
	Manipulation and two meanings	16	8.7	8.8
	Manipulation and one meaning	26	14.1	14.4
	Manipulation and no meaning	85	46.2	47.0
	No manipulation	23	12.5	12.7
	Other	31	16.8	17.1
	Total	181	98.4	100.0
Missing		3	1.6	
Total		184	100.0	

About 9% of prospective teachers who are in the 'Manipulation and two meanings' category deduce from manipulations of given symbols that y=3 can be either a line or a point depending how one derives it. This can be seen in the following response:

> "If mx=0 then m=0 or x=0. If x=0 then the line cuts the y-axis at 3. If m=0 then since for all values of x we have y=3, this is a line parallel to x=axis."

As can be seen in the above response, the solver already got mx=0 in her mind. That is, she starts her solution by writing that "if mx=0 then...". This may imply that she had already equated y=3 to the mx+3 and gotten mx=0 mentally. Then she elaborates the cases when x=0 and m=0. When x=0, she writes that the line cuts the y-axis at 3. This implies the point meaning of y=3. She writes that when m=0, for all values of x, we get y=3 which is a line parallel to x-axis. To sum up, by means of manipulations she could deduce the different meanings of y= 3. Her response suggests that she was not confused by the usual meaning of y=3 which is a horizontal line. She used the available tools to extricate meanings of y=3. She did not just focus on y=3, but went beyond this by means of manipulations and then connected her manipulations to the meanings of y=3.

This suggests that she was able to make sense of the different roles symbols can play. She might have developed an aspect of symbol sense, which might have helped her to realise that she needs to check symbol meanings while solving the problem. During the solution process she could cope with the ambiguity of symbols. For example, the variables x, m, y

assume different roles throughout the solution. There are changes in the roles of each of these variables throughout solution process. In the equation, y=mx+3, x and y are changing values, m is constant or parameter. Then after starting to solve the problem, the role of m and x in mx become unknowns. I am mentioning these roles to point out the multiplicity of meanings involved in the problem. These multiple meanings are sorted out by the solver in a competent way. One of the underlying reasons for this competency might be having compressed cognitive units. Once again it should be pointed out that there are connections between having compressed 'cognitive units' and aspects of 'symbol sense'.

As Crowley and Tall (1999) point out compressed cognitive units allow one to flexibly move among different aspects of a concept. As I mentioned earlier, in this question, the solver needs to move flexibly among different aspects of the equation "y=mx+3". In order to have this flexibility, the solver must have intimate connections between the separate elements in their cognitive structures.

For example, they must move from usual meaning of y=3, which is a straight line, to an equation which is equivalent to y=mx+3. After equating these two expressions and after sorting out possibilities of mx=0, they need to come back to give meanings to y=3. While giving meanings to y=3, they need to deal with the particularities and generalities involved in the problem. For example, y=mx+3 stands as a particular form of a general form y=mx+n. It is also a general form because we don't know what m is. When m=0 is assumed then y=3 becomes one of the particular lines that are represented by y=mx+3. When x=0 is assumed then y=3 becomes a particular point that the lines y=mx+3 pass through. Being able to process these multiple aspects in the mind may become possible by bringing the required aspect into the focus of attention while suppressing other aspects.

In the responses that fall into the 'Manipulation and one meaning' category (14%), there are manipulations but from these manipulations only one of the meanings is deduced. For example as in the following response:

"If y=3 then 3 = mx+3 then m=0, if m=0 then it is a line which has slope zero and parallel to x-axis."

As can be seen in the above response, the students mentally found that mx=0 and deduce that m=0 in this case. Since he finds only that m=0, he correctly mentions the line meaning of y=3. His response suggests that he did not see the possibility that x=0 when mx=0. Like the student above, all the students in the 'Manipulation and one meaning' category, did not see that m=0 or x=0 when mx=0. They deduced either of m=0 or x=0, not both. As a result they could not mention both line meaning and point meaning of y=3. One of the reasons for this might be that they might have thought only one meaning is sufficient for the question; therefore they left the solution there. Alternatively, mx=0 may not trigger in their mind that either x=0 or m=0. Hence, they could not evaluate all possible meanings of y=3.

The students in the 'Manipulation and no meaning' category constitute the largest group of respondents (46%). These prospective teachers find that m=0 or x=0 when mx=0. However, they do not reach the different meanings of y=3 from these deductions. This can be seen in the following response:

"mx=0, m=0 or x=0, if m=0 then all numbers that are substituted for x make the equation true, if x=0 then all numbers that are substituted for m make the equation true"

As can be seen in the above response the student finds m=0 or x=0. Then, he writes the meanings of these cases in the context of the equation mx=0. That is, he mentions that if m=0 then whatever x is still mx=0. Similarly, if x=0 then whatever m is still mx=0. It should be noted that if either m or x tends to infinity then in this case mx may not tend to zero. Therefore, we may not conclude that whatever x or m is, mx=0 if either one is zero. The responses that are similar to the above and which are put into the "Manipulation and one meaning category" may suggest that these respondents might have forgotten that the question asked the meanings of y=3. They might have forgotten this due to many possible reasons. One of the reasons might be related to what these students understood from the question. In the question, I asked what different things y=3 could mean. For these students, y=3 may mean mx=0 and hence m=0 or x=0. For these students, these possibilities may be the meanings what the question asked. Alternatively, another reason why these students do not give line and point meanings of y=3 may be related to theories about how brain works.

As Crowley and Tall (1999) point out unrelated external sensations such as a mental process monitoring whether a longer-term goal is being achieved can interrupt and override the "focus of attention". For these students, this mental process did not warn them whether their solution process is getting suitably close to the goal of the solution process. That is, they could not use the '*comparator*' activity (Skemp, 1979) to check their solutions. Crowley and Tall (1999) conjecture that when following a routine sequence of actions, the activity of any comparator can be suspended if the focus on successive remembered steps is so great which temporarily fills the focus of attention. Hence, they point out that "the inflexibility of procedural thinking can become so dominant as to cause the individual to lose sight of the goal and so fail to solve the problem." (Crowley & Tall, 1999, p.4). Therefore, the procedures they used to attempt to solve this question might have taken up so much conscious thought and hence they could not make necessary cognitive links to complete the question.

A small group of the respondents (13%) bring up a meaning without manipulating the given information. These respondents seem to make use of accustomed meaning of y=3 which is a constant function or the horizontal line y=3:

> "y=3 is a constant function, this means that y=3 is a line with slope zero."
>
> "It can be a constant line equation"

The responses above may suggest that the students use the usual meaning of y=3 which is a horizontal line. They do not use available tools to extricate meanings of y=3. They just focus on y=3, but do not go beyond this by means of manipulations and then connect their manipulations to the meanings of y=3. This may suggest that they did not make sense of the different roles symbols can play. They might have lacked an aspect of "symbol sense" (Arcavi, 1994). As I pointed earlier, Arcavi (1994) suggests that "a desirable component of symbol sense consists of the *in situ* and operative recognition of the different (and yet similar) roles which symbols can play in high school algebra" (p. 30).

The students who are in the "Other" category (17%) gave very confused explanations which were a mix up of some illicit conclusions together with justifiable conclusions:

> "$mx+3=3$ then $mx=0$ so $m=0$ which means that it is a line which has slope zero and passes through the origin. And this is x-axis."

The above response may suggest that the students in this group could not cope with the multiplicity of meanings of the symbols in the question. Therefore, they gave very confused explanations. For example the student who gave the above response wrote indirectly that the line $y=3$ is x-axis. The reason why he had this confusion might be that he thought both that the slope is zero and the line passes through the origin. However, in this question one can claim that a line passes through the origin if he/she says that $y=3$ is the ordinate of the point (0, 3). In this case, only y- axis can go through the point (0, 3) and (0, 0). Moreover, in such a case the slope of the y-axis is not 0.

Overall, the largest group of respondents seem to focus on the procedures but from these procedures they do not bring up (all) of the meanings that are involved in the problem.

Those respondents who are in the 'Manipulation and two meanings' category seem to be able to sort out the multiplicity of meanings involved in the problem. They might have developed an aspect of symbol sense which involves the recognition of different (yet similar) roles of symbols. One of the underlying reasons for such a skill may be related to cognitive structures of the students. These students might have compressed cognitive units with flexible links and checking mechanisms. On the other hand, respondents who are in the other categories do not find different meanings of $y=3$. This suggests that these students do not organise different aspects of the concept $y=mx+n$ into a network of connected cognitive units (Barnard & Tall, 1997) with flexible links. This is because previous group of respondents do not focus on the object y=3, they unpack this object by means of manipulations and deduce two different kinds of objects. On the other hand, second group of respondents either concentrate on the object $y=3$ and mention its accustomed meaning or remain at manipulations and do not come back to the object $y=3$ though they already unpacked it.

PART TWO

4.3 Analysis of Pedagogical Content knowledge questions

This second part of the chapter will involve probing answers to my research question which reads as '*What is the state and nature of Turkish prospective secondary mathematics teachers' PCKv?*'

4.3.1 Knowledge about ways of presenting the subject matter

As I discussed in the literature review chapter this aspect of PCKv will involve the following three dimensions.

1. Reactions to pupils' comments and questions in the classroom (questions 12a)
2. Analysing a student's mistake (question 13)
3. Helping the student correct his/her mistake (question 15)

The analysis in this subsection involves question 12a.

Question 12a

12 a) How would you react to your students' questions as below in the classroom? Explain! "Teacher, why does 2a+5b not equal 7ab?"

This question is prepared considering one of the difficulties pupils face while learning manipulations of symbols. In the literature, it is often mentioned that pupils tend to conjoin or 'finish' algebraic variables when simplifying expressions (Booth, 1984; Tall & Thomas, 1991; Macgregor & Stacey, 1993). For example, some pupils tend to write the expression $3x+5$ as $8x$ or 8. In this section, I will investigate prospective teachers' familiarity with this tendency and its possible sources.

One of the possible ways that a teacher might choose to explain question 12a to a pupil might be by substituting different numbers for '*a*' and '*b*' in '$2a+5b$' and '$7ab$' then showing that resulting numbers are different.

Another explanation may be using objects, which is a common approach in mathematics teaching to refer to concrete objects when facing difficulties. It is generally called 'fruit-salad' algebra which assumes that an expression like $3a+4b$ could be regarded as 3 apples and 4 bananas. I pointed in an earlier part of this chapter that this approach cause more misconceptions in the long run. That is, this approach may cause pupils to confuse the number of objects with the numbers themselves.

A different explanation may be just telling the pupil that this is a rule in mathematics. That is, different variables can not be conjoined. However, such explanations may result in some pupils regarding Mathematics as a set of rules that should be accepted without questioning them. Hence, this may result in hating Mathematics. On the other hand, some pupils may get pleasure out of memorising the rules that give them correct answers in (procedural exams). Hence, this may cause them to believe that procedural capability is more important than other aspects.

Most of the prospective teachers replied to this question with explanations similar to the explanations given in the preceding paragraphs. In these explanations, a prevalent way of explaining was to refer to concrete objects. Some of the prospective teachers used number substitutions. Some of them mention it is a ruled to be followed. After studying the explanations offered by prospective teachers I separated them into the following categories:

1. Concrete Objects; contains responses in which there is a reference to concrete objects.

2. Substitution; contains responses in which there is substitution of values for '*a*' and '*b*', and finding they are not equal.

3. Rule-based; contains responses in which the rule about simplification of variables is repeated either by referring to variables or the rules about multiplication and summation.

4. Other; contains responses that can neither form a category nor be included in other categories.

The following table shows the frequencies of these categories.

Table 4.2.11 Frequency distribution of Response Types for Question 12a

		Frequency	Percent	Valid Percent
	Concrete Objects	68	37.0	42.8
	Substitution	33	17.9	20.8
	Rule-based	43	23.4	27.0
	Other	15	8.2	9.4
	Total	159	86.4	100.0
Missing		25	13.6	
Total		184	100.0	

As can be seen from the Table above, 37% of responses from prospective teachers fall into the 'Fruit-salad' category. This category is the largest group. The students in this category refer to objects like apples or bananas to explain why 2a+5b is not equal to 7ab. For example, they write;

> "First of all the student hadn't understood that the variables were different. I say to him/her that apples and pears can't be added together. I say that apples can be added to apples and pears can be added to pears."

> "The reason why the child thinks $2a+5b$ equals $7ab$ is that he/she thinks they can be added. I would say to him/her that a and b are different quantities that is $2a$ is 2 apples and $5b$ is 5 pears and that he/she can't write pear with apple as a summation."

As can be seen in the above responses, these prospective teachers try to give meaning to $2a + 5b$ by talking about objects in their explanations. Because two apples and five bananas do not make seven apples or seven bananas, these prospective teachers expect the student to see that $2a+5b$ is not $7ab$. However, do they know that in their explanations there is confusion between the numbers and objects? If they don't know this distinction, then this shows that they have misconception about variables. Can we say that this misconception is the cause of explaining this question by referring to objects? In other words, is their SMKv the cause of PCKv? If so what is the nature of this cause-effect relationship? I will discuss these issues in the next chapter.

One more issue should be acknowledged about the responses in this category. As I said before, this category consists of the largest group of students. One of the reasons for this might be due to the construction of the question. It may be wondered what would happen if only one variable was involved in the question. That is, if question was constructed something like this, why $3a+5$ is not equal to $8a$", would this group have been the largest? In the second case, using apple and bananas seems more difficult. Because, in this case there is only one variable, namely a. If a represents apples then what will 5 represent in $3a+5$? That is, the fruit-salad explanation would not be very suitable in this case?

The prospective teachers (about 18%) in the 'Substitution' category try to explain this question by a counter example in which they make use of numbers. Their examples suggest that they try to draw a connection between arithmetic and algebra. For example, they write:

> "Let's substitute numbers for a and b; let's try for a=2, b=3, $2a + 5b = 19 \neq 42 = 7ab$. So they are not equal"

As can be seen in the above response, this prospective teacher substitutes values for 'a' and 'b' in '$2a+5b$' and '$7ab$'. She gets 19 from '$2a+5b$' and 42 from '$7ab$'. Since these resulting values are different so are '$2a+5b$' and '$7ab$'. This kind of approach is also common in mathematics teaching. That is, explaining that algebraic expressions can not be conjoined by using numbers is another common approach that is frequently mentioned in textbooks (Tirosh, Even and Robinson, 1998).

The pedagogical reason given in the literature for this approach is that it helps students to see that the expressions are not equivalent. However, the downside of this approach is that it emphasises the process facet and does not encourage perceiving expressions as entities and as final answers (Tirosh, Even and Robinson, 1998). That is, the students may tend to see $2x+5$ as multiply x by two and add five. Hence, they can not conceive algebraic notations as flexible entities that can be regarded as either process or object depending on the task at hand. In Gray and Tall's (1994) terms, they do not achieve 'proceptual' understanding of variables.

While explaining this question by the use of numbers, do these prospective teachers themselves see expressions as processes to carry out? If this is the case then they themselves have problems in their SMKv. Hence, this may affect their PCKv. However, if they have 'proceptual' understanding then they can easily switch to process meaning of expressions when the task at hand requires, because 'procepts' gives one the flexibility of moving between process and objects meanings of expressions. As a result, whether they have 'proceptual' understanding of variables or only procedural understanding, this may cause them to give an explanation that are included in 'Substitution' category. On the other hand, it is difficult to make such claims just by their answers to this question. Therefore, in the next chapter I will incorporate interview data to discuss such issues while exploring the relationships between SMKv and PCKv.

The responses that fall into the 'Rule-based' category (about 23%) seem to repeat the rule about manipulation of symbols in different words. The following is an example that falls into this category.

> "Because *a* & *b* are two different variables we cannot add them together."
>
> "*a* and *b* are different variables. Therefore, while adding different variables, terms with the same variables are added together. Different variables can't be added together."

As can be seen in the above responses, these prospective teachers state a rule which tells two different variables cannot be added. By saying this they imply that two different variables cannot be conjoined. That is, they try to teach by telling the student how to do things. They try to transfer knowledge where the student is expected to stay relatively passive.

The responses of those prospective teachers (8%) who are put into 'Other' category could not be considered in the previous categories. Their responses suggest that these prospective teachers would take into account the factors such as the place and time where the situation takes place, the student's knowledge etc. before reacting to the student's question. For example they write:

> "My explanation depends on which year group they belong to" or
>
> "I have never taught. But, when they ask, I try to explain it patiently. I am here since I trust my patience and I like maths."

It may be wondered that why these prospective teachers gave different explanations to this question. The reasons underlying the differences in the responses will be discussed in the next chapter. Some of the possible reasons may be due to general pedagogical knowledge, learning experiences, subject matter knowledge. I will discuss these in chapter 5 in which I will explore my third research question which deals with the relationships between SMKv and PCKv.

In this question which is presented above prospective teachers were not explicitly asked to explain the possible sources of the student's difficulty in manipulating variables. In the following question, they are explicitly asked the student's way of thinking about manipulating symbols. Moreover, they are invited to pose assumptions about the sources of the student's mistake.

4.3.1.2 Analysing a students' mistake

This section involves the analysis of question 13.

13. Ayşe was asked to simplify an algebraic expression, such as $2x + 4x + 6$. She wrote: $$2x+4x+6=0$$ $$6x+6=0$$ $$6x=-6$$ $$x=-1$$ What do you think the student had in mind? Is she right? Explain.

This question is prepared by considering one of the mistakes students make while simplifying expressions. Philipp (1992) reports that such kind of mistakes is frequently encountered in the classrooms. As can be seen in the above question, the pupil equates the given expression to zero and finds x, although she was asked to simplify it. One of the possible sources for this mistake is due to the different roles symbols play in expressions. The equation $2x+4x+6=0$ treats the literal symbol as unknown waiting to be solved whereas the expression $2x + 4x + 6$ treats the literal symbol as a generalised number. For some pupils, using variables as unknowns can be psychologically easier than using them as generalised numbers. This may be related to not regarding the expression as something that is manipulable in itself, because it does not have an operational meaning. Alternatively, using variables as unknowns is emphasised more than using them as generalised numbers, and hence pupils may tend to regard simplifying expressions as finding the unknowns. That is, the focus on *solving equations* may lead the student to use procedures to solve equations.

Analysing a student's work starts with deciding about whether or not the response in that work is correct. Then, elaborating on the sources of this mistake (if any) and how to alleviate it can be planned.

The raw data of this question was led to the following categories. In this categorisation my focus was on whether the mistake was spotted and whether there was any explanation about what the student in the question had in mind:

1. Unaware of the mistake: This category contains responses where the respondent could not find/see the student's mistake.
2. Aware of the mistake with explanation: This category contains responses in which possible sources of the mistake of the student is explained.
3. Aware of the mistake with no explanation: This category contains responses in which possible sources of the mistake of the student is not explained though the mistake is mentioned. In the responses just what the student did or what she should have done is mentioned
4. Other: contains responses that can neither form a category nor be included in other categories.

Following table shows the frequencies of these categories.

Table 4.2.12 Frequency distribution of Response Types for Question 13

		Frequency	Percent	Valid Percent
	Unaware of the mistake	58	31.5	34.5
	Aware of the mistake with explanation	51	27.7	30.4
	Aware of the mistake with no explanation	48	26.1	28.6
	Other	11	6.0	6.5
	Total	168	91.3	100.0
Missing		16	8.7	
Total		184	100.0	

As can be seen in the above table about 32% of prospective teachers think that the pupil in the question did not make a mistake. For example, they write:

> "First of all collecting the coefficients of the same unknowns. Then making both sides of equation ready to be *simplified*. And then *simplifying*. Without doing all these operations by dividing both sides by 2, the *simplification* can be done. $x+2x+3=0$, $3x+3=0$ and then divide both sides by 3."

The respondent explains how to simplify an expression in the above response. According to him, the pupil correctly simplified the given expression. However, he explains how to solve an equation. Furthermore, he suggests another way of simplifying. He suggests that both sides of the equation can be divided by two, then add x-termed expressions together. Finally divide both sides by three. Since while solving equations the expressions on both sides are generally simplified by collecting like terms, this respondent might have confusion between simplifying and solving equations. In the interviews I found out one of the possible reasons for this confusion. One of the interviewees pointed out that generally when they are asked to find the roots of an equation, they are given only the expression part. They equate it to zero and then find the root(s). This common practice might have affected their perception of this question. That is, in this question they might have thought that the pupil was asked to find the roots. Therefore, she equated the given expression to zero and found x. However, this does not explain why they are explaining how to find the unknown even though they use the word *simplification*. It seems that these students don't know or don't notice the difference between *simplifying* and *solving* equations. It seems that they have a limited conception about the terminology. They do not know the difference between what simplification mean and what solving equations mean. Thus, this may affect their ability to analyse the pupil's mistake.

The respondents that fall into the second category (about 28%) analyse and give some explanations about the sources of the pupil's mistake. For example, they write:

> "She may think that every expression that contains x is an equation. Of course wrong. The correct way is $6x+6=6(x+1)$."

> "Ayşe might have got very used to equations and she does not understand the topic well or she is careless."

As can be seen in the above responses, these prospective teachers are aware of the mistake the pupil made in the question. Furthermore, they gave explanations about the pupil's thinking while performing the given task. In the first response, the respondent claims that the pupil in the question may regard every expression that contains x as an equation. She suggests that because of this perception the pupil equates the given expression to zero. In the second response, the respondent ties the pupil's mistake to her familiarity with solving equations. She moreover claims that the pupil does not understand the topic well or she was careless while solving the given problem.

The responses in 'Aware of the mistake with explanation' category indicate that these prospective teachers could comment on the possible sources of specific students' work in this question. The data indicate that most prospective teachers in this category attributed these mistakes to familiarity to solving equations. This suggests that these teachers themselves can know the difference between simplification and solving equations. This helps them to spot the pupil's mistake in the question. Although, it is very difficult to know why a pupil made such a mistake, these prospective teachers could elaborate on the possible sources of the mistake. Hence, this may show a relationship between SMKv and PCKv. I will discuss this relationship in chapter five more fully by incorporating the interview data.

About 26% of the respondents, although spotting the pupil's mistake, did not give any explanation about the sources of this mistake. For example, they write explanations similar to the below:

> "Rather than simplifying $2x+4x+6$, she solved it and wrote it as an equation. She should not have equated it to 0. She should have done this $2x+4x+6=6x+6=6(x+1)$.

As can be seen in the above response, the prospective teacher writes that the pupil made a mistake. He describes this mistake. He moreover mentions what the pupil should not have done and what she should have done. This response may suggest that the prospective teachers in this category have the most basic level of pedagogical content knowledge about analysing students' mistakes. However, they don't mention about the sources of this mistake. They don't elaborate on what the pupil had in mind while she was carrying out the given task. The reason why they did not elaborate on the possible sources of this mistake may be attributed to just not spending too much time on the question. Alternatively, they could not comment on the mistake the pupil made.

There are eleven responses (6%) which could not be considered in the first three categories, for example:

> "Ayşe should have done this; $2x+4x+6=a$, $6x+6=a$, $3x+3=a/2$, $3(x+1)=a/2 \rightarrow x+1=a/6$."

To sum up, the analysis of this question revealed that the prospective teachers' answers varied with respect to spotting the mistake and explaining the possible sources of this mistake. We saw that some of the students (about 32%) could not even spot the pupil's mistake in the question, whereas more than half of the students (about 54%) spot this mistake. Of those students who are aware of the mistake the pupil made, about half of them gave some sort of explanation about what the pupil might have had in mind, whereas about half of them did not give any explanation about the pupil's thinking processes.

The responses in the first category may provide evidence to the relationships between subject matter knowledge and pedagogical content knowledge. The students in this group have confusion between simplifying and solving equations, therefore this prevents them from spotting the pupil's mistake. However, it should be acknowledged that this confusion seems to stem from the terminology. That is, these prospective teachers don't know what the term simplification means. It seems that what they understand from simplification is finding the roots. Therefore, they describe how to find the roots even though they use the term simplification. This may suggest that knowledge of the Mathematical terminology can have relation to PCKv. This suggests that there is interaction between SMKv and PCKv. I will discuss these interactions in the next chapter more.

Within above question the prospective teachers were not asked to help the student. In the following question they were explicitly asked to do this.

4.3.1.3 Helping students to correct their misconception

Question 15

> 15. According to you which one of these students is correct? How do you help the one who according to you is wrong understand his claim is wrong:
>
> "Ahmet claims that the letter S stands for students in the equation 6S=P which he wrote to represent a fact that "there are six times as many students as professors at this university". Mehmet claims that the letter S stands for the **number** of professors."

This question is inspired by the famous Student-Professor problem that has received a lot of interest from researchers in the past twenty years. I mentioned about this 'Student-Professor' problem a couple of times in previous parts. Clement, Lochhead and Monk (1981) were the first to expose the arguments about a misconception which is labelled as the reversal error (see Laborde 1990 for a summary). They report that more than 30% of first-year college students made a reversal error and wrote 6S=P instead of writing 6P=S to the question that reads as:

> Write an equation using the variables S and P to represent the following statement: "There are six times as many students as professors at this university." Use S for the number of students and P for the number of professors. (p.208)

They call this misconception as 'reversal error' because it is the reverse of the true solution. However, it is questionable whether or not there is actually 'reverse' anything in the students' mind, since the students regard the equations with a totally different meaning (see Crowley, Thomas & Tall; 1994).

The findings of Clement, Lochhead and Monk are consistent with findings in later studies that investigated this problem. These findings motivated many studies to remedy the reversal error in college students since they believed this error is a manifestation of a deep misconception about the notion of variable (e.g. Philipp, 1992; Rosnick & Clement, 1980). The methods used to help students avoid this error in these studies have been mixed and the results have been mixed as well. However, in general most of the studies have been reported to be not very successful in remedying this error.

Crowley, Thomas and Tall (1994) discuss about the sources of the 'reversal error' and report that it is caused by children's orientations to notations. That is, they report that process-oriented children are more likely to make this error. They describe such behaviours with the notion of 'procepts' (Gray & Tall, 1994) and relate it to children's "learning experiences in mathematics, their use of symbols, and the natural process of compression of knowledge with growing maturity" (p. 2).

In the analysis of the responses to this question, I focus on the helping methods. That is, I identified the different methods of helping students that exist in the respondents' comments. These helping methods give clues about the respondent's SMKv of variables. Then, this helps us me to see the interactions between SMKv and PCKv. I will begin by presenting the result of this analysis. This analysis brought out the following seven different helping strategies:

1. Reread the question slowly and carefully; this method involves recommending the student (Mehmet) to reread the question again. This kind of response suggests that the respondents try to allow the student to check his thinking again. Hence, this might suggest that the respondent considers factors that might have distracted the student's perception when he solved the problem. Consequently, this response may also suggest that the respondent try to allow the student to construct his knowledge actively. There are only seven respondents who chose to use only this method:

> "Mehmet is right. I would tell Ahmet to read the question again and control what S represents."

Others who chose this method also incorporate other methods (3, 4, 5, and 6).

2. Asking which is more; this method involves asking the student to observe which is the larger group in the problem, students or professors. This method is pedagogically very

close to the first method presented above. These respondents also try to allow the student to construct his knowledge by encouraging him to check for possible distracters when he solved the question. There are 14 respondents who chose to employ only this method without incorporating other methods:

> "Mehmet is right. I would ask Ahmet whether there are more students or professors in the given fact."

3. Substituting values for S and P; this method involves giving values for S and P and observing the equation. This method is also pedagogically close to the first two methods presented above:

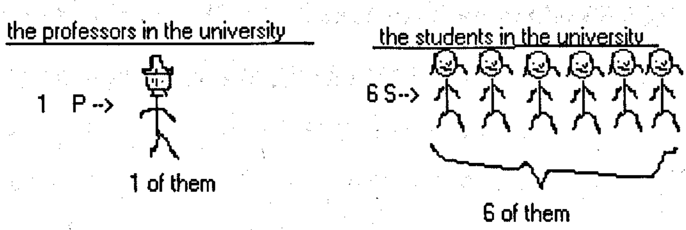
> "In the equation S: number of professors, P: number of students. Therefore Ahmet is wrong. I would help Ahmet by giving his equation as an example. I would show him his mistake. In the equation, 6S=P, if S=1 then the number of professors is 6. Hence he understands his mistake. This is somewhat a trial and error method. And this shows that the correct is exactly the opposite."

There are 21 respondents who chose only this method.

4. Trying to concretise and contextualise the situation by either calling 6 boys and 1 girl to the front in the classroom or using pictorial representation, etc. This is another method which is pedagogically close to the first two methods described above. However, in these responses the respondents try to coach the student while constructing his knowledge by giving examples that are equivalent to the distribution in the problem. Therefore, the respondents' aims are more didactic in this group. This can be seen in the following response:

> "Mehmet is right. I would help Ahmet by drawing pictures. If we put 6 of professors next to each other we can reach the number of students; therefore, 6xP=S"

Figure 4.1. 5 A respondent's pictorial representation for helping a student correct a student's mistake



"

<u>5. By suggesting to use S as the number of students and P as the number of professors.</u> These respondents try to help both students by using the first letters of the professors and students. There are 15 respondents who chose to employ only this method:

> "Ahmet writes mathematical expression wrong. But Mehmet writes the expression correct and corrects Ahmet's expression by his method. To solve this confusion first letters of words should be written."

<u>6. Telling what to do</u>. These respondents choose to tell the students what they should do. Therefore, these responses seem to be teacher-centred, where the student stays passive while the knowledge is *transformed*. There are 16 respondents who chose to employ only this method:

> "Mehmet is correct. I would tell Ahmet that the number of students is six times of the number of professors. That is when we multiply the number of professors by 6 it should give the number of students."

<u>7. Use general letters rather than using S and P which are first letters of students and professors respectively</u>. The responses that fall into this group suggest that the respondent thinks that the student's difficulty in this problem is about the choice of letters. Therefore, these responses may suggest that the respondent is concentrating on the student's difficulty. There are three respondents who provided this method only:

> "I would recommend him using general letters rather than special letters for representing numbers."

46 % of respondents who chose only one of above helping methods and 26% of respondents who chose any two of above helping methods. There are only three respondents (about 2%) who write that Mehmet is wrong, and 21 respondents (11%) did not mention any helping methods. The following table (Table 4.2.13) presents the summary of the discussion and analysis so far presented. In this table the numbers which are diagonally situated and which are underlined present the number of respondents who gave only one corresponding method. For example, the number '7' which is situated at second row and column shows there are 7 respondents that use the helping method 1 only. The number '14' which is at third row and column shows there are 14 respondents who use helping method 2 only. The other numbers that are not situated diagonally and which are italic show the number of respondents who gave two helping methods. For example, the number '4' which is at row two and column three shows that four respondents use helping methods 1 and 2 together. The number '6' which is at row four and column five shows that six respondents use helping methods 3 and 4 together.

Table 4.2.13 Summary frequencies of helping methods for question 15

Helping methods	1	2	3	4	5	6	7		Number of students who use two methods
1	7 [a]	*4* [b]	*3*	*2*	*3*	*1*			13
2		14	*3*	*2*	*3*	*7*			15
3			21	*6*	*1*	*1*			8
4				11	*2*	*1*			3
5					15	*5*			5
6						14	*3*		3
7							3		
Total number of students who use only one method								85	
Total number of students who use two methods									47

a. Underlined numbers show the number of students who use only one method that is corresponding to its row or column.

b. Italic numbers that are in shaded region show the number of students who use methods that are corresponding to its row and column.

As can be seen from this table those respondents who use two methods generally start helping by means of the first or the second method in the above list. Then, they go on to incorporate other methods.

It should be pointed out that some of the helping strategies prospective teachers mentioned are similar to the behaviour patterns used by students to construct equations for student-professor type problems. For instance, Clement (1982) reports that one of these approaches is multiplying the number of professors by 6 in order to equal to the number of students. In helping method 3, we see such behaviour. That is, these respondents first substitute numbers for trial equations and then explain the meaning of the equation by multiplying the number of professors by 6. This shows that the respondents in this category understand the meaning of the variables and how equations are used in mathematics. This kind of understanding clearly helps them to use this method. Hence, this shows a relationship between SMKv and PCKv.

However, the helping method number 2, which asks who is more in the problem, may lead to an incorrect response. This approach is similar to the behaviour pattern which is characterised as placing the coefficient next to the letter associated with the larger group. The reason for this placement stems from the fact that there are more students than professors. Using such fact, the students who use helping method number 2 try to explain the equation by saying that there are "6 students for every professor." In helping method 4 also respondents try to explain the equation by showing that there are "6 students for every professor" by pictures or by calling 6 boys and 1 girl to the front. Hence, these two methods are similar in justifying the semantics of the problem situation. However, these methods also carry some misconceptions with them. For example, these methods may cause a misconception about the referents of the variables; it may lead to view 6S as six students rather than viewing it as "six times the *number* of students". As a result, these strategies (number 2 and 4) may give us some clues about the respondents' SMKv.

The respondents who use the strategies that are grouped in category 2 and 4 may themselves have the misconceptions that are accompanied by their methods. That is, these respondents may also have confusion between variables referring to number of objects and variables referring to numbers. If this is the case, this clearly shows a relationship between SMKv and PCKv.

Another remark about these results would be that one of these helping strategies has been tried by researchers to remedy the reversal error. For example, Kaput (cited in Clement, 1982) compares the effects of using arbitrary letters, such as X for students and Y for professors against the effects of using first letters such as S for students and P for professors. Kaput (cited in Clement 1982) finds that the performance using arbitrary letters is slightly worse; whereas Mestre and Lochhead (1983) report that using X and Y instead of S and P result in more correct answers. However, the results of these papers are not supported by careful interviews with the students concerned. I cited these papers, because it is interesting to note that the helping strategy that the prospective teachers would use exist in the literature. It may be wondered whether these prospective teachers such helping method from the literature or they use their subject matter knowledge to invent it.

This strategy gives us some clues about the respondents' SMKv. It may be hypothesised that these respondents may think that the reversal error in this problem is related to the ambiguous information conveyed by the variables S and P. P can refer to the professors; also, it can refer to the *number* of professors. If these prospective teachers think as hypothesised than this shows that they know the difference between variables denoting the objects and the number of objects. Hence, this shows an interaction between SMKv and PCKv.

4.3.2 Curriculum Knowledge

In this section, I will present an analysis of question 16 that is involved in my general research question "What is the state and nature of Turkish prospective mathematics secondary teachers' pedagogical content knowledge in the 'Curriculum knowledge?'"

4.3.2.1 Ordering Topics

This section will deal with the analysis of the following question:

Question 16

> 16. Which one of the following topics would you teach first? Why? Explain.
>
> - Substituting numbers in expressions
>
> - Simplifying expressions

In analysis of the data obtained from this question, I did not focus on which of the topics respondents chose to teach first, rather I chose to concentrate on the respondents' reasons for choosing one of the topics. In particular, I focused on whether or not they make connections in their provided reasons. Even, Tirosh and Robinson (1993) suggest creating classrooms where making connections is emphasised is essential to help the students construct understandings of the subject matter. Ball (1990) argues that "teachers must appreciate and understand the connections among mathematical ideas". Askew, Brown, Rhodes, Johnson and Wiliam (1997) report that connectionist teachers are more successful than transmission and discovery teachers in helping their students understand numeracy.

Substituting numbers in given expressions assumes regarding expressions as processes to carry out, whereas simplifying expressions assumes regarding expressions as manipulable objects (Sfard, 1994; Tall & Thomas, 1991). Sfard (1991, 1994) theorises that three *stages* that characterise the development of mathematical understanding in any area of mathematics, not just algebra: *interiorization*, *condensation*, and *reification*. In the first stage, interiorization, students can only carry out processes on already familiar objects. In the second stage, condensation, they become automatic while carrying out these processes, since steps in these processes are already squeezed into more manageable units. And finally in the third stage, processes are reified into mathematical objects, and they are now ready to be operated upon. Therefore, if this question was posed to Sfard, probably she would choose to teach substitutions first. On the other hand, Tall and Thomas (1991) would teach these two topics simultaneously, since children could observe when expressions behave as processes to carry out and when they behave as objects. In Gray and Tall's (1994) term children could have an opportunity to become a 'proceptual' thinker, so that they could conceive algebraic symbols both as processes and as concepts. The discussion so far presented about this question shows that which topics to teach first do not have a definite answer. Therefore, concentrating on the reasons for ordering the given topics is regarded as more productive. Within the given reasons I observed that prospective teachers connect these topics from different perspectives. Some of them connect them conceptually; some of them connect them procedurally. Therefore, two most common connections identified within the data were as follows:

1. Conceptual; emphasising conceptual benefits of teaching one topic before another.
2. Procedural; emphasising procedural benefits of teaching one topic before another.

Table 4.2.14 Frequency distribution of Response Types for Question 14

		Frequency	Percent	Valid Percent
Valid	Conceptual	29	15.8	18.2
	Procedural	89	48.4	56.0
	Other	41	22.3	25.8
	Total	159	86.4	100.0
Missing		25	13.6	
Total		184	100.0	

As can be seen in the table above the largest group of prospective teachers (48%) emphasises procedural connections in the ordering, whereas few of them (about 16%) consider conceptual connections. Those responses (22%) which are put into the 'Other' category do not give reasons for their choices or they mention one of the topics easier than the other. It is difficult to infer what they mean by the word 'easier' from their responses.

I considered conceptual connections as the one in which teaching either of topics first can be beneficial to the understanding of the other topic. For example;

> "I would teach substitution first because without grasping the meanings of letters, they may get confused while doing simplification. They have to understand what letters are, what for they are used and how they are used, and so that they understand simplification. Otherwise they may not understand reason for simplification, because they may not know that the letters in just simplified expressions actually have the same values before simplification."

As can be seen in the above response, the prospective teachers try to give importance to meaningful learning in which the students construct the meaning of the concept of variable. That is, he points out that teaching one of the topics first help students to understand what the variables are for, how they are used and why they are used. His response suggests that substitutions help students to use variables as standing for numbers. Furthermore, he mentions that when children understand that variables represent numbers then they can understand that simplification does not change the value of the expressions. Hence, this response suggests that these prospective teachers give importance to meaningful learning in which students could develop concepts. However, it should be acknowledged that what they understand from the concept development is not parallel to concept development expressed in the literature. For example, they do not mean that teaching substitution first helps children to regard expressions as processes to carry out which can lead them through interiorization, condensation and reification stages of Sfard (1994). According to Sfard (1994), 'structural conception' (which she regards it as conceptual understanding) occurs following the stages presented above. However, as Tall, Thomas, Davis, Gray & Simpson (2000) state that Sfard's notion of structural confuses two meanings, which involves conceptual structure on the one hand (the overall structure of an object) and mathematical structure on the other (the axioms and definitions used in formal mathematics). Therefore, the conceptual in this analysis is more close to the term 'relational understanding' which means "knowing both what to do and why" (Skemp, p.20, 1976) which emphasises connections among different topics. Skemp (1976) mentions that "Ideas required for understanding a particular topic turn out to be basic for understanding many other topics too." (p. 24).

Similarly, the term 'Procedural' that labels the second group of responses does not mean 'operational conception' (Sfard, 1994). 'Procedural' in this context means that responses in the second group emphasise rituals in which teaching one topic makes manipulations easier for getting the results of problems. In other words, it has connotations with 'instrumental understanding' which is described as "rules without reasons" (Skemp, p.20, 1976). For example;

> "I teach simplification first. Because this saves a lot of effort. By getting the expression in its simplest form and then substituting values may save life and prevent risk of doing computation errors."
>
> "I would teach substitution first. Because when they see they can reach the solution more quickly in simplifications by using substitution, and when the simplification takes more time, they can not put up with listening long topic. They may say 'Sir why are we lengthening solution though we have a shorter way of solving it'. Simplification comes later."

As can be seen in the above responses, the prospective teachers' reasons for their choices are related to procedural benefits. That is, these prospective teachers teach one topic before another due to certain advantages of 'instrumental mathematics'. Skemp (1976) gives three advantages of instrumental mathematics: it is usually *easier* to understand, therefore, its rewards are *more immediate* and *more apparent* and by instrumental thinking the *right answer* can be gotten *more quickly and reliably*. The responses given above as an example suggest that the prospective teachers in this group favour teaching one topic before another due to advantages quoted above.

Why do these prospective teachers give importance to these advantages? Can this be related to their SMKv. That is, can we claim that these prospective teachers are more likely to view mathematics as procedures to carry out without understanding underlying reasons? Therefore, do they themselves learn mathematics instrumentally?

Similarly, can we claim that prospective teachers who take into account the conceptual benefits view mathematics as meaningful procedures, concepts and rules with underlying reasons? Therefore, do this group of teachers themselves enjoy learning relational mathematics? The answers to these questions definitely shed some light on the interactions between SMKv and PCKv. However, it is difficult to comment about the answers from the data obtained from the questionnaire alone. On the other hand, as Skemp (1976) point out that a teacher might choose to teach for instrumental understanding on one or more of the following grounds:

> 1.That relational understanding would take too long to achieve, and to be able to use a particular technique is all that these pupils are likely to need.
>
> 2.That relational understanding of a particular topic is too difficult, but the pupils still need it for examination reasons.
>
> 3.That a skill is needed for use in another subject (e.g. science) before it can be understood relationally with schemas presently available to the pupils.
>
> 4. That he is a junior teacher in a school where all the other mathematics teaching is instrumental. (p. 24)

Although we may not be very sure that some of the above reasons affect the prospective teachers whose responses included in the 'Procedural category' while answering this question, we can claim from their written answers that they are ordering the topics considering the advantages of instrumental teaching. This gives clues about these prospective teachers' SMKv. That is, these prospective teachers themselves are more likely to favour instrumental understanding, thus they order topics instrumentally. On the other hand, prospective teachers in the second category are more likely to favour relational understanding, thus they order topics conceptually.

4.3.2.3 A Summary of the analysis of the 'Curriculum Knowledge'

Within this section I analysed prospective teachers' knowledge about how the two topics involved in teaching variables may be organised and arranged into the curriculum. Their responses were analysed by focusing on the nature of the connections they made. It is found that 48% prospective teachers consider procedural benefits of teaching one topic before another. In contrast, very few of them (16%) consider conceptual benefits of teaching one of the topics first.

4.4 Conclusions of Part Two

In this part, I examined the general research question which reads as "What is the state and nature of Turkish prospective secondary mathematics teachers pedagogical content knowledge of variables?" In order to examine this knowledge several different components were identified and several different tasks were used to deal with examination of this knowledge.

In this conclusion I will take a general look at the integration of the responses provided to different questions. In many cases the prospective teachers' responses varied. In cases where their responses were analysed by focusing on the representations they use it is found that they would use either objects or numbers to give meaning to letters. They would organise topics in the curricula either by considering procedural benefits or conceptual benefits.

4.5 Conclusions

In part one of this chapter, I presented the analysis and results of 184 Turkish prospective teachers' knowledge of variables. In part two, the analysis and results of these prospective teachers' knowledge on teaching the concept of variable were presented. I regarded the first type of knowledge as subject matter knowledge of variables which is the knowledge for oneself. The second type is the kind of knowledge which is required to teach the content to others, in particular to students. This second type, pedagogical content knowledge, has many sources as suggested in the literature.

According to Shulman (1987) teachers transform SMK into PCK by means of pedagogical reasoning. This assumes SMK is one of the sources of PCK. In the next chapter, I will examine whether SMKv of prospective teachers is one of the sources of their PCKv. Thus, the next chapter will analyse and discuss the nature of relationships between SMKv and PCKv. In this analysis, I will look at possible reasons for getting particular types of responses to PCKv questions. In particular, question 12a and question 6 will be used to make this analysis.

CHAPTER 5
Interactions between SMKv and PCKv

INTRODUCTION

In this chapter, I will present and discuss my analysis of the relationships between Subject matter knowledge of variables (SMKv) and pedagogical content knowledge of variables (PCKv). In this discussion I will draw on the analysis of questionnaire data from chapter 4, and use interview data to support my hypotheses. I explained how I chose my interview sample from the questionnaire sample in chapter 3, section 3.4. Now, I will explain my method of choosing the interview transcripts of the students I will use in this chapter. In this chapter, I mainly discuss the underlying reasons for response types of question 12a which asks

> 12 a) How would you react to your students' questions as below in the classroom? Explain!
>
> "Teacher, why does 2a+5b not equal 7ab?"

Therefore, I will use the interview transcripts of four students who are chosen for this investigation. As I discussed in chapter 3, section 3.4, prospective teachers' own perception of variables can be one of the reasons for different types of responses to question 12a. In the questionnaire, question 6 which read as

> What different things might an algebraic expression such as, say $2x+1$, mean? What can x stand for?

is used to investigate prospective teachers' perceptions of variables; whether or not variables stand for objects.

However, it should be noted that in the interviews I asked questions related to questions 12a, and 6 to all ten students. Then, I chose four students from these ten students; two of these students changed their solution methods from technical manipulations to visual, graphical methods in solving questions 8 and 9, and two of them did not change their solution methods and demonstrated no sign of using visual methods. I also took into account the responses of these prospective teachers to questions 12a and 6 while choosing them. The first one of these students wrote in the questionnaire that x could be an object in $2x+1$ and used concrete objects ('fruit-salad' algebra) to explain why variables can't be conjoined; the second one did not write x could be an object and did not use concrete objects. The third student did not write that x could be an object, but mentioned that she would use concrete objects. The last student wrote that x could be an object, but did not mention she would use concrete objects. I will use the interview data of these four students to investigate more fully about their thinking formed from looking at the questionnaire responses. In the following transcript extracts the names of the interviewees are not their real names. These extracts are chosen from the complete transcripts of the interviews by considering the relevancy of questions asked in the interview to the discussion presented in this chapter. While discussing the reasons for response types of question 12a, I will investigate whether prospective teachers SMK can be one of these reasons for their explanations to why variables cannot be conjoined.

5.1 Amalgamation of different kinds of knowledge in developing PCKv

In this section, I will present the analysis of how general pedagogical knowledge, subject matter knowledge is amalgamated by prospective teachers to give a pedagogical explanation. In this discussion, I will use interviews of two students; Fatma and Yezdan. I will explore why Fatma would use 'fruit-salad' approach to explain $2a+5b$ is not equal to $7ab$ even though she does not mention that x could be an object in $2x+1$. Then I will explore why Yezdan would use number substitution to explain $2a+5b$ is not equal to $7ab$ even though she mentions that x could be an object in $2x+1$. These cases will present when there may not be a direct link between SMKv and PCKv.

5.1.1 The case of Fatma

Following is a transcript of a prospective teacher who explained that she would introduce variables to pupils by using concrete examples. This student called Fatma was in her third year of university education. She had not taken any pedagogical courses at the time I conducted my interview.

In order to give more information about this student, I would like to briefly present how she replied the questions in the questionnaire. As I discussed in chapter four, part one, four questions were used to discuss SMKv of prospective teachers. Fatma responded to question 8 by making use of graphs. As the reader may recall question 8 was difficult to solve by just using technical manipulations. Fatma starts to solve this question by using technical manipulations. Then, she gives up this approach and flexibly changes her approach to using graphs. As I discussed in chapter 4, this may be a sign of flexibility, which is included in the aspects of 'symbol sense' (Arcavi, 1994), and which is a result of having compressed 'cognitive units' (Barnard & Tall, 1997). She solves question 9 in a similar way. Question 9 is also more difficult to solve by using technical manipulations than using graphs. Therefore, both questions (question 8 and question 9) are similar. Fatma, starts to solve question 9 also by using technical manipulations, then she changes her solution method to using graphs. Hence, from these results we can claim that Fatma is a flexible student, who achieved an aspect of symbol sense. She has conceptual thinking in solving algebra problems. She does not just concentrate on procedures. The reason for this may be having flexible compressed cognitive units.

She also shows flexibility in solving question 4. As the reader may check from chapter 4, question 4 asks different meanings of y=3 that is obtained from y=mx+3. In order to obtain different meanings of y=3 from the givens in the question, one must have developed an aspect of 'symbol sense' which helps one to be aware of the different roles the variables can take in different contexts. Furthermore, this awareness includes ability to sort out different procedures and objects involved in a problem. Fatma solves this question in a competent way. This adds support to the view that Fatma does not concentrate on procedures, but she could flexibly oscillate between object and process meanings of symbols.

To investigate more about Fatma's flexibility, I ask her in the interview to write the equation of a straight line that passes through a pair of known points. These kinds of questions are usually solved by using the general equation of a straight line:

> N: for example, you know two points that a line passes through. Can you find the equation of that line?
>
> F: Yes generally I solve such questions like this (she writes the equation of a straight line $y=mx+n$) if I know the points which it passes.
>
> N: let's say it passes through the points (1,3) and (1, 2)
>
> F: You are asking the equation that these points meet? The line equation?
>
> N: Yes
>
> F: From those points I would find the slope, m value. I would do like this. (she goes on to find the equation, she writes the following on the paper)

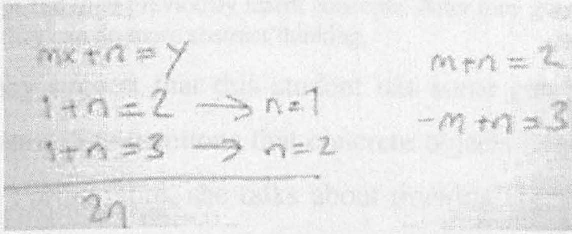
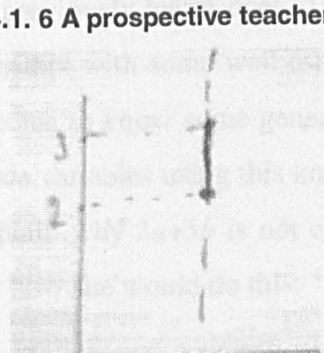


Figure 4.1. 6 A prospective teacher's graphical solution to find a line equation.



> N: Can you think loud while you are solving…
>
> F: Is the equation like this (she points to her solution on the paper, which is shown above that presents the line equation on the Cartesian graph, which she writes after she saw above usual method is of no use)… the line is $x=1$?

Fatma changes her usual solution technique when she realises that her method is not of use to find the equation of the line that passes through the given points. She flexibly chooses to think it on Cartesian graph and from this she finds the equation.

In the questionnaire, she writes that she would use concrete examples to introduce variables, and also to explain why $2a+5b$ is not equal to $7ab$. For example in the questionnaire she replies to the question how you would introduce using letters to stand for numbers by writing:

> *"I would approach introducing letter manipulations from concrete examples, such as apples, bananas".*

In the interview I ask her why she would use concrete examples. She replies:

> F: Since this topic is taught to children who are young, they can have difficulty understanding abstract concepts. If it is introduced by using concrete objects then the pupils could draw connections between this new concept and their previously learnt concepts. After they got used to these then they can do more abstract thinking.

The transcript above may suggest that this student has some general knowledge about how younger students learn. She mentions that concrete objects help younger students to learn abstract concepts. Furthermore, she talks about drawing connections between new concepts and previously learnt ones. The ideas which she expresses about how students learn are consistent with some well established theories of concept development. Hence, this student seems to know some general pedagogical knowledge. She mentions that she would introduce variables using this knowledge. In the questionnaire, she also writes that she would explain why $2a+5b$ is not equal to $7ab$ by concretising the situation, but she does not write how she would do this:

> *"I would explain addition and multiplication concepts. I would give concrete examples".*

Let's see how she would concretise the situation to explain why $2a+5b$ not equal to $7ab$ in the following interview transcript:

> N: (by pointing to her response in the questionnaire) here you wrote that I would give examples containing concrete things. What do you mean by this?

F: by that I mean for example if you add 2 pears with 5 apples, what happens? Two pears and five apples, seven fruits...(pause)

N: yes on the left and on the right seven fruits?

F: Yes on both sides there are seven fruits, himm, so the pupil may not see these can not be equal. That is, that pupil may not see that the sum of $2a$ and $5b$ can't be equal to this multiplication; $7ab$. The number values of 'a' and 'b' can be different, if 'a' is 2 and 'b' is five then 2x2+5x5= 29 but 7x2x5 is 70. This way of explaining is more suitable I think. Hence, using fruits as an example may not be right.

N: Why not?

F: the student must think 'a' and 'b' as different numbers. There, two numbers like 'a' and 'b'. They are both numbers but different numbers because they are represented by different letters.

N: can't 'a' and 'b' be equal?

F: no if $a=b$ then they are the same numbers. But I am calling one of them 'a' the other one 'b', therefore they are different numbers. Hence, I would tell different numbers must be represented by different letters.

When I ask Fatma what she means by using concrete things, she mentions fruit-salad algebra; that is she would use the example of pears and apples to explain. She starts off asking what happens if you add two apples and five pears; she gives the answer as seven fruits and stops. At this moment, I think she understands that there is something wrong with her example. As a matter of fact two apples and five pears are seven fruits. That is, the pupil who asked the question may think "two apples and five pears on the left and seven fruits on the right, so they are equal". Hence, that pupil could think that $2a+5b$ can make $7ab$. In order to ascertain whether she understands that there is something wrong with her example, I say "yes on the left and on the right we have seven fruits." Then, she explains why the two sides can be different by using numbers. That is, she mentions that 'a' and 'b' can represent different numbers and therefore the equation $2a+5b=7ab$ does not hold. Here, her subject matter knowledge about variables helps her to correct her example containing objects in which she uses her general pedagogical knowledge. She knows that the variables stand for numbers. She also knows that younger students could learn with the help of concrete objects easily. Firstly, she tries to use her general pedagogical knowledge; when she sees her example is not helpful, she proposes another example in which she uses her subject matter knowledge. In order to investigate more about her subject matter knowledge involving whether variables stand for numbers or objects I ask another question.

Fatma writes in the questionnaire that x could be a real number in $2x+1$. Fatma replies as follows when I ask her "if I write down 5a here, what does '*a*' stand for, what can it be, what can it be used instead of, what do you think?"

> F: First of all, there is a multiplication; five times *a*; '*a*' is a number here.
>
> N: Can it be anything different from numbers? For example, can '*a*' be an apple in five times *a*?
>
> F: pear can be represented by *a*, for example (the first letter of pear in Turkish is *a*)
>
> N: then does 'a' stand for pear?
>
> F: let's call pear as *a*, if there are five pears we say five times *a* but...
>
> N: is it a number?
>
> F: it is not a number but it corresponds to a number, we use a letter, '*a*' instead of a number. Even if we say it is a pear, there the pear is not the pear we know as an object, it is used as a letter. It does not mean it is a pear itself. We can't multiply pears.

In the above interview, transcript Fatma mentions that the letter '*a*' stands for a number in $5a$, since '*a*' is multiplied by 5. I asked her whether it could stand for objects. She mentions that even if we called '*a*' as pear in $5a$, the word pear would play the role of the letter '*a*'. That is, the word pear would stand for numbers, not the objects themselves. The reason for this, according to her, is that we can't multiply objects. Her comments suggest that she knows that variables stand for numbers, not for objects. This knowledge helps her to use substitutions to explain to a pupil that $2a+5b$ is not equal to $7ab$. However, before using this knowledge she tries to use her general pedagogical knowledge to explain it. She tries to use objects. However, she notices that her explanation containing objects would not explain why $2a+5b$ is not equal to $7ab$. According to her, if '*a*' is pears and '*b*' is apples then $2a+5b$ may be regarded as 7 fruits, and $7ab$ could also be regarded like that. In this comment she uses her 'pedagogical reasoning'. So both $2a+5b$ and $7ab$ may mean seven fruits. This view is supported by Tall and Thomas (1991) who suggest that the meanings of 'and' and 'plus' are similar in natural language and therefore that pupils regard '*ab*' as meaning the same as '*a+b*'. This way of seeing ab and $a+b$ may lead pupils to believe that on both sides there is same number of fruits. Fatma could see this and gives up her idea of using objects, and returns to using numbers. Here we see how general pedagogical knowledge, subject matter knowledge and pedagogical content knowledge come together in Fatma's thinking about this problem. Fatma's decision to use a number substitution to explain why $2a+5b$ is not the same as $7ab$ is evidence of her developing pedagogical content knowledge. The decision grows out of a blend of subject matter knowledge (knowing that variables stand for numbers) and general pedagogical knowledge (which suggests to her that concretising the situation will be a good strategy). This adds supports to Shulman's (1987) assertions about pedagogical content knowledge. Shulman suggests that pedagogical content knowledge is the special amalgamation of different kinds of knowledge by the help of pedagogical reasoning.

Fatma is chosen to represent those students who writes in the questionnaire that x could be a number in $2x+1$ and who writes that they would use concrete objects to explain why $2a+5b$ is not equal to $7ab$. That is, she represents those students who are in the 'Number' category of question 6 and 'Fruit-salad category' of question 12a. Please see chapter 4 in order to remember what these categories mean.

5.1.2 The case of Yezdan

Yezdan is another year-3 student. She is chosen to represent those students who wrote in the questionnaire that x could be an object and who wrote that they would use number substitution to explain why $2a+5b$ is not equal to $7ab$. She represents those students who are in the 'Object' category of question 6 and 'Substitution category' of question 12a. Please see chapter 4 in order to remember what these categories mean.

She writes in the questionnaire that

> *"x could be anything. It can be apple, pear."*

She explains question 12a by making use of numbers. She starts to explain it by mentioning the properties of addition and multiplication. She mentions that

> *"multiplication gives us the total number of elements in sets that are made of same kind of elements".*

Then she writes

> *"since a and b are different kinds of variables. They represent different numbers. Let's a=4 and b=5 then 2a+5b=33 and 7ab=140. Therefore, they are not equal"*

In order to give more idea about Yezdan, I would like to describe briefly how she responds to SMKv questions in the questionnaire. She starts to solve question 8 using technical manipulation. She substitutes given numbers in the question and obtained two inequalities with three unknowns. Then she leaves the question there. In question 9 she also uses technical manipulation. She adds the given two equations and reduces the number of equations to one with one unknown. Then she substitutes three values for '*a*'. She tries for the case a=0, a=1, and a=-1. For each of these cases she finds x values. Since for each of these values the equation she finds gives two different solutions for x, she writes the given system has two solutions for a=1, a=-1 and a=0. As a result, she does not completely solve the problem. From her solutions to these two questions, it can be claimed that she has one way of approaching problems. That is, she does not have flexible cognitive structures to cope with problems that need to be approached from the accustomed way. In order to ascertain this last claim I asked her how to find the equation of a straight line that passes through two given points. She replies:

> Y: In y=mx+n, I substitute values for x and y or...
>
> N: for example it passes through (1,3) and (1, 2), can you show me how you find the equation of this line
>
> Y: if we substitute values in this equation, (she starts to substitute values into the following equation) x-1, noo, hang on a minute, in the similar way x-1 divide by... if we give values to these x_1 , y_1 x_2 , y_2, x-1 divided by zero, no it is $\frac{x-x_1}{x_1-x_2}=\frac{y-y_1}{y_1-y_2}$ undefined, since we can't have zero in the denominator therefore I can't write such equation...(pause) ...
>
> N: what is the equation, then?
>
> Y: I can't find it now, I forgot the formula.

Yezdan, firstly uses her standard way of finding the equation of straight lines that passes through given two points. Then, she finds that this standard form does not give the equation she needs because the denominators give zero. When I ask her again about the equation, she returns to use standard formula back again by substituting numbers. She cannot find the equation and she begins to think that the formula she uses may be wrong. Therefore, she gives up by saying that she forgot the formula. This shows that she is not flexible enough to consider the given problem in a different way. Furthermore, she approaches problems procedurally. This shows her cognitive structures are diffuse.

As I presented before, Yezdan does not use 'fruit-salad' explanation to explain why variables can not be conjoined rather she uses number substitutions, although she writes that x could be an object in $2x+1$. One of the reasons why she uses number substitutions can be related to Yezdan's procedural approaches to the problems. That is, as I described before Yezdan does not show flexibility in approaching problems. She tries to use perfectly legitimate procedures, but even she sees that her procedures is not of use, she does not find other ways. Therefore, it can be claimed that she procedural approaches to algebraic problems. This may be one of the reasons why she uses number substitution to explain question 12a even though she writes that x can be an object. If she has confusion about what variables stand for, then how this knowledge can affect her pedagogical content knowledge. Can this be a reason why she uses fruit-salad explanation to introduce variables?

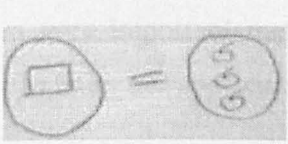
In the interview, she mentions that she teaches a year-2 primary school student. In her questionnaire responses I saw that she would also use concrete objects to introduce variables to pupils. When I ask her the reasons why she would try to use concrete objects, she mentions that she discovers that in this way the pupil whom she teaches understands better. Then, I ask her to explain more about how she would introduce variables. She replies:

> Y: If I had taught my students the concept of equality then I would teach them the unknown concept using sets. Teaching students that two sets are equivalent if their number of elements is equal is easy. They can learn this very easily. If they know that two sets are equivalent if their number of elements are equal then I draw two Venn diagrams as here (she points to the Venn diagrams in her questionnaire response, please see Figure 4.1.7) and tell $\square=3$; that is this square here corresponds to three apples there (she points again to the Venn diagrams in her questionnaire response)

N: you say that the square diagram corresponds to three apple pictures.

Y: yes, that box is three apples, I help the students to think that in order to have the sets equivalent there must be three apples in this box also. As a matter of fact, in primary schools, at the beginnings generally, symbolic notations are not used; rather diagrams like square, triangle, and circle are used to introduce variables. This way of teaching is used because it is thought that later students could understand if the letters are used instead of those squares. That is, introducing variables is started by concrete objects. If we don't use concrete objects, it is very difficult to teach something to a primary school student.

Figure 4.1. 7 A prospective teacher's pictorial method for introducing variables.



In the above interview transcript, Yezdan mentions that she would introduce variables by using sets. When we look at the reasons why she would use sets, we see interesting facts about the relationships between SMK and PCK. Beside her general pedagogical knowledge which includes a belief that younger students learn with concrete objects, one of her underlying reasons for using sets is that teaching that two sets are equivalent is easy. According to her, pupils could learn easily that two sets are equal if their number of elements is equal. Once they learn this fact, she could use it to introduce variables. She draws two Venn diagrams and put a square in to the first one and three apples into to the second one. She mentions that if these two sets are equal then the square in the first one must be equal to the three apples in the second one. Hence, she claims that one square corresponds to three apples; as a result the square symbol corresponds to three. However, it should be acknowledged that in her representation, it is very difficult to distinguish between the number of objects and the object themselves. It seems that these two are very mixed up in her explanation.

Another underlying reason why she would use such an explanation can be seen in the second paragraph of her interview. Here she mentions how variables are introduced in the textbooks. She says that before using letters some geometrical figures are used to help children to learn about using letters instead of numbers. Therefore, she uses the square symbol instead of a letter to introduce variables. According to her, her approach is therefore parallel to the approaches taken in textbooks. However, it should be acknowledged that the parallelism between these approaches seems to be only pragmatic. That is, in both approaches geometrical figures are used, but their contexts are very different. In Yezdan's approach, the use of boxes is in the context of sets, in the textbooks the use of geometrical figures in the context of equations such as $\square+3=5$. According to Yezdan, using these kinds of figures instead of letters helps children, because they are more concrete than letters. Hence, for her, concretising concepts, making them more tangible, is a very important method for teaching primary school children. She uses this knowledge to tailor her approach to introducing variables.

To sum up, two main reasons caused her to use such an approach to introducing variables. One of them is the ease of using sets, and the second one is that her approach, according to her, is parallel to the approaches used in textbooks. The underlying reason for using geometrical figures is that these figures make the concepts more tangible, more concrete for children. However, while she is tailoring this knowledge to fit the context of introducing variables, we see that she is using her subject matter knowledge. As I mentioned before, in her explanation there is confusion between variables standing for objects and variables standing for the number of objects. This confusion may be caused by her subject matter knowledge. In order to investigate this, I asked her "if I write down 3*a* here, what does *a* stand for, what can it be, what can it be used instead of, what do you think?" She replied:

> Y: It might be any object, 3 times of that object, as in the previous example, it might be 3 times of professors, or 3 times of the length of a line segment.
>
> N: If I said -3a, then what would it be?
>
> Y: (short pause) -3a?
>
> N: Would it stand for objects?
>
> Y: No, it would not stand for objects, for example if "a" is an apple, it can't be minus three apples.
>
> N: OK, can you explain why it can't?
>
> Y: Objects are concrete, if we say minus; we are doing some abstract thinking. There, the term "minus" is a bit abstract.
>
> N: Can you explain a bit more what do you mean by abstract.

> Y: For example, when we put 2 on the number line, this is possible, but -2, when we put -2, we don't know this one, that is, that negative numbers exist.
>
> N: So you are saying that if it is minus it can't be apple, but if it is plus it can be?
>
> Y: Yes because "plus" denotes a bulkiness of objects.
>
> N: OK, you are saying that it is possible if it is plus, but if it is minus, since it is abstract, it is not possible?
>
> Y: No, if we say it is object, for example, there is a table but we can't say there is a minus table, (short pause) ha... perhaps if the table is taken away from here, then it disappears from here, there, minus one, then it might be. But in daily life most people don't use minus for objects, this is established practice. In other words if there is a table, if it is gone then there is not, it is zero, that is, we don't say there is a minus table.
>
> N: So you say since it is not used in daily life, it is not accepted saying minus object.
>
> Y: For example, natural numbers, these came from nature, people, in early stages of foundation of mathematics, natural numbers are found since they are in the nature. But they could not find the number zero. 1, 2, 3... these were accepted by the people, but even existence of zero was accepted by people hesitatingly.

Yezdan claims that if there is a negative sign before a variable, the variable can not be an object, but if there is no negative sign then the variable can be an object. According to her, the reason for this is that the negative sign is used for abstract things. That is, for her minus is nothing, it denotes non-existence. However, objects are concrete and exist; therefore when a negative sign appears before variables, then it may mean for her moving away an object, making it disappear. These comments clearly show that she has a big confusion between variables denoting the number of objects and the variables denoting the objects themselves. This may show a relationship between subject matter knowledge and pedagogical content knowledge. This relationship can be explained as follows: if one has a misconception about a concept, then this misconception can be passed onto that person's pedagogical content knowledge. In the above transcript, Yezdan blends three pieces of information to give a pedagogical explanation; knowing that younger students learn by the help of concrete objects easily, more difficult topics/concepts can be built on relatively easier topics/concepts, and assuming that the variables could stand for objects.

When she blends all of these, she gives a pedagogical explanation that contains the misconception she has about variables.

5.2 Cases when SMKv alone can be a source of PCKv

In this section, I will discuss the cases when SMKv alone can be a source of PCKv. In this discussion, I will use the interview data of Hakan and Mehmet. The data I present will show the cases when there can be a direct relationship between SMKv and PCKv.

5.2.1 The case of Hakan

Following is the transcript of the interview with Hakan. Hakan is a year-4 student. He started to take pedagogical content knowledge courses at the time I carried out the interview. He replies to question 8 by making use of graphs. He starts to solve the question by technical manipulations, and then he returns to use graphs. Therefore, he has a flexibility of approaching problems from different ways. He writes in his solution to question 8 that:

> *"It has two real solutions, because the graph of this equation is a parabola and it has a shape either ∩ or ∪. Therefore, in order to cut the x-axis at one point, it has to cut it again at another point".*

From his solutions to the questions in the questionnaire, it can be claimed that he has conceptual understanding of variables. He is not driven by procedures and rules.

When it is required, he can find other ways of solving problems. Hakan writes in the questionnaire that "*x can be a real or complex number in 2x+1.*" He answers question 4 (please see chapter 4) in a competent way. He deduces two different meanings of $y=3$ by means of manipulations. This shows that he is aware of the importance of the context in deciding the role of symbols. Furthermore, he is able to flexibly move between objects and processes in order to sort out different meanings of symbols involved.

He explains question 12a (please see chapter 4) by making use of numbers:

> *"Let's substitute numbers. Let 'a' be 2 and 'b' be 3. Then 2a+5b=19 and 7ab=42. Therefore, these are not equal."*

In order to investigate the reasons why he does not use fruit-salad explanation to question 12a, I wondered whether he knows the distinction between letters denoting numbers and letters denoting objects. Therefore, I asked him in the interview; "What would '*a*' be, what would it stand for if I write down here 5a?"

He replied:

> H: if nothing is mentioned, I would say it would be a number in mathematics.
>
> N: what else?
>
> H: a physician can see it as five bricks; a student can see it as five frogs.
>
> N: does it matter if I say five multiplied by a (5·a)?
>
> H: yes (pause)
>
> N: can '*a*' be an object? Can it be an apple?
>
> H: yes, it depends on the question?
>
> N: can we say five multiplied by apple?
>
> H: no we can't multiply apples by five. Therefore, we can't say five multiplied by apple but we can say five apples. If $5a$ is five apples, then a can't be a number. If $5a$ is 5 multiplied by '*a*' then '*a*' can't be an object.

The above transcript shows that Hakan knows the difference between letters representing the numbers and the letters representing the objects. He is aware of the fact that the meaning of letters depend on the context they are used. This may be one of the reasons why he does not use objects to explain why $2a+5b$ is not equal to $7ab$. This claim can be supported more by the last paragraph of the transcript. He mentions that if a is apple in $5a$ then a can't be a number, but if $5a$ is five multiplied by a then a can't be an object. Therefore, this may be the reason why he does not describe $2a+5b$ as 2 apples and five bananas, because in $2a+5b$, it is clear that 2 is multiplied by a and 5 is multiplied by b. As a result, it can be claimed that not giving fruit-salad explanation to question 12a may be related to knowing that the variables stand for numbers. Hence, this shows how SMK can be related to PCK. Another relationship I observed will be discussed as below.

One of the questions I posed in the interview asked him to explain more fully about the mistake the pupil made in question 13. In question 13 a pupil called Ayse was asked to simplify the expression $2x+4x+6$. However, this pupil equated the expression to zero and solved this equation. I asked the prospective teachers whether Ayse was right, and what she might have been thinking while performing the given task. Hakan, in the questionnaire, wrote that Ayse was wrong, and did not give any explanation about her thinking. In the interview, I asked Hakan explain more fully about Ayse's thinking.

> H: Here she equated the expression to zero and solved the equation. There should not be an equation. There is only an expression. This expression can be simplified by considering the common coefficients. We have two coefficients in this expression. Furthermore, we can add x's together and we get $6x$ which has coefficient 6. Actually there is no simplifying the coefficients. The thing what the pupil made here is to solve the equation but there is nothing there that we can simplify, if there was something we could simplify it according to that. In order to do simplification, first of all we have to do a division. Simplification... we have to make the coefficients smaller however, here if we make the coefficients smaller the expression changes. It has to be equal to something. By making both sides smaller, you could do simplification.
>
> N: what do you mean by equality?
>
> H: it must be equal to something, there is an expression $6x+6$.
>
> N: if we leave it there as $6x+6$, is it not simplified?
>
> H: yes, that is, you are shortening the expression.
>
> N: you were saying that it must be equal to something?
>
> H: by being equal to something...I thought about the rational numbers, how I do simplification there. For example, if it is not in its simplest form, for example $\frac{6}{4}$ can be reduced to $\frac{3}{2}$. I thought like this.
>
> N: you are saying that this simplification is confused with the simplifications in rational numbers?
>
> H: yes, when simplification is said, cancelling out by drawing the numbers in rational numbers comes to mind first. Here the only thing which she could have done was summation. I could not draw connection between this expression and simplification.

In the above transcript Hakan correctly claims that the pupil made a mistake. Then he comments about the reasons why such a mistake was made. According to him, the given expression does not behave the way he expected. He mentions that to do a simplification, we need to work with the coefficients. By making the coefficients smaller, we can make both sides smaller. Or in rational numbers, we cancel common coefficient from both the denominator and nominator. He mentions that in the given expression we can not do cancelling, because if we do this then the expression changes; the equivalency is not satisfied. This kind of knowledge about simplification affects his elaboration of the pupil's thinking given in the question. Hence, this shows another relationship between SMK and PCK. That is, SMK can affect one's understanding about analysing pupils' mistakes, misconceptions.

5.2.2 The case of Mehmet

Mehmet is another year- 3 student. He was chosen to represent those prospective teachers who writes in the questionnaire that x could stand for an object and who explains question 12a by using fruit-salad algebra.

In order to give more idea about Mehmet, I would like to present how he responds to SMKv questions in the questionnaire. He starts question 8 by substituting given values into the equation $ax^2+bx+c=0$. He obtains two inequalities with three unknowns. He reduces the number of inequalities to one and number of unknowns to two by subtracting the second inequality from the first inequality. Hence, he obtains $7a+b<0$. Then he leaves the question there. Similarly, in his solution to question 9, he adds two equations in order to eliminate y^2 and gets $x^2+(x\text{-}a)^2=1$. Then, he leaves his solution there. His solutions to these two questions show that he does not cope with the problems that need to be approached by using different strategies from his usual solution methods. He uses legitimate procedures to solve the problem; however, he does not show the flexibility that he needs to cope with different problems in new contexts. In order to ascertain this last claim, in the interview, I ask him how he finds the equation of a straight line that passes through given two points:

> M: Generally, I use this formula that gives the equation (he writes the formula $\dfrac{x-x_1}{x_1-x_2}=\dfrac{y-y_1}{y_1-y_2}$ on the paper)
>
> N: let's find the equation of the line that passes trough (2, 3) and (2, 4)
>
> M: Ok let's substitute these values into the formula (he starts to plug in the values into the formula, while he is putting these; he is reading out what he is doing). x minus two over two minus two equals y minus three over four minus three. Then, x-2 over zero… (pause) but we can't have zero in the denominator… (pause) infinity equals y minus three. I can't find it… it does not work.

N: Why can't you find it?

M: I don't know, there is something wrong with the formula I use, I think. I don't know other ways of finding the equation.

This interview shows that Mehmet faces difficulties when he tries to use his accustomed strategy of dealing with finding the equations of straight lines. When he finds that his strategy does not work, he gives up and begins to suspect that his formula is wrong. This suggests that his cognitive structures is too diffuse, therefore he does not make flexible links to the concepts that gives the formula he uses.

Mehmet solves the question 4 (please see chapter 4) as below:

$y=mx+3$ $3=mx+3$, $mx=0$ if $m=0$ $\forall x \in R$, there is a solution. Or if $x=0$ $\forall m \in R$ there is a solution.

His above solution suggests that he does not sort out the different meanings of $y=3$ that is obtained from $y=mx+3$. Alternatively, he may think that $y=3$ may mean only the deductions he makes above. In either case, his solution to this question together with the solutions I presented above suggests that he has procedural approaches to the problems.

As I mentioned before Mehmet uses 'fruit-salad' explanation to describe why variables cannot be conjoined. That is, he writes in the questionnaire that:

You can't add apples with pears. Two apples and five pears don't make seven apples or pears, therefore, 2a+5b is not equal to 7ab.

As can be seen in the above response Mehmet uses apples and pears to explain why $2a+5b$ is not equal to $7ab$. I asked him in the interview why he would use such approach:

M: Because, that is the reason why $2a+5b$ cannot be equal to $7ab$. This is a very well known fact that I have known since primary school. Our teacher told us you can't add different kinds together. Same kind can be added with the same kind. Everybody knows this fact. Two apples and five pears don't make seven apples or pears.

As can be seen in the above transcript Mehmet strongly believes that the underlying reason why different variables can't be conjoined is because they are different kinds. He mentions that everybody knows this fact and that he has known this since primary school. In chapter 4, I discuss that this approach helps children to manipulate variables. That is, they can understand by this approach that $2a+a+5b=3a+5b$. However, such approach can cause difficulties for students in the long run. For example, they can not be aware of the distinction between variables denoting the number of objects and objects themselves. As a matter of fact the above transcript suggests that Mehmet himself has such confusion. In order to find out whether Mehmet is aware of this distinction, I asked him "what would 'a' stand for if I write down here 5*a*?" he replies:

> M: five pencils, five real numbers, rational numbers etc. it can be everything five atoms.
>
> N: if I say here 5*a* means five multiplied by *a*, then what do you think?
>
> M: if it is five multiplied by *a*, then again as I said before it can be everything, like atoms, pencils etc. it does not differ in multiplication case.

This interview shows that Mehmet does not know the difference between letters denoting the numbers and letters denoting the objects. That is, for him the roles of letters in 5m where m is metres and 5·m where m is any number are difficult to distinguish. This may show that he does not cope with the different meanings of symbols in a flexible manner. He himself believes that different variables cannot be conjoined because different kinds of objects cannot be added together. The knowledge which he has known since primary school causes him to confuse between denoting the objects and the numbers. Hence, this may be one of the reasons why he uses fruit-salad explanation to teach why variables can't be conjoined. However, it should be acknowledged that I am not claiming that the misconception Mehmet has about variables is a direct reason for him to explain why $2a+5b$ is not equal to $7ab$. That is, it is not claimed that this misconception is the cause of his pedagogical explanation. On the other hand, since he has such a misconception, it is difficult for him to see the possible misconceptions that can occur in the future learning of his students as a result of the pedagogical explanation he provides. Hence, it may be claimed that at some level his misconception about variables can have an implicit effect on his 'pedagogical reasoning' (Shulman, 1987). As I discussed in chapter 2, pedagogical content knowledge is developed by the help of pedagogical reasoning. As a result, the implicit affect of his misconception on pedagogical reasoning can have an effect on his pedagogical content knowledge. Therefore, this shows another relationship between subject matter knowledge of variables and pedagogical content knowledge of variables.

Mehmet writes in the questionnaire that he would teach simplifying expressions before teaching substitutions because after simplifying substituting values, getting results is easier. Therefore, he is put into the procedural category of question 16 (please see chapter 4). In the interview, I asked him to explain the reasons more fully about teaching simplification first:

> M: for example we have an expression like a^2-b^2 if we substitute values for a and b in this expression then it is difficult to find a^2 and b^2, for example if a is 11 and b is 25 and if we write a^2-b^2 as (a-b) times (a+b) now then if we take away 11 from 25, 14 left and add 11 to 25 it makes 36. Now it is easier to multiply 25 by 36. But if we consider the former case, I will take the square of a then square of b, then subtract second one from the first one, to me this is lengthier. Therefore, firstly, a simplification here is easier. However, if we have an expression like a^2+b^2, then in this case you cannot simplify this expression therefore you have to do substitution first.
>
> N: then, which one would you prefer to teach first?
>
> M: again I prefer to teach simplification first, because in most cases it helps us to perform substitutions more easily. It makes the job easier. It will give us the result quicker and easier.

In above transcript, Mehmet emphasises procedural benefits of teaching simplification first. It should be acknowledged that in his example, Mehmet does not simplify the expression a^2-b^2 rather he opens it. However, he regards it as a simplification. According to him, in most cases after simplifying an expression it is *easier* and *quicker* to substitute values and find the result of the substitution, this shows that Mehmet really concentrates on procedures in algebra. He does not mention any of conceptual, relational benefits of teaching one topic first. The emphasise in his response is on procedures. As I presented before Mehmet does not demonstrate flexibility in solving SMKv questions. As a result it is claimed that Mehmet has diffuse cognitive structures that prevent him approaching problems from different ways. One of the underlying reasons for having diffuse cognitive structures is concentrating on procedures without considering underlying reasons why these procedures work. Such an approach may be one of the reasons why Mehmet give procedural reasons about teaching simplification first. Hence, his way of learning affects his pedagogical reasoning and therefore this shows how SMKv can be related to PCKv.

5.3 Summary and discussions

In this chapter, I discussed the interactions between SMKv and PCKv by using interviews of four students. Two of these students are identified as having flexible 'cognitive units' (Barnard & Tall, 1997) who has conceptual understanding of variables, whereas two of them are identified as having diffuse cognitive structures who have procedural understanding of variables. In the discussion I focused on mainly whether having confusion between variables denoting the numbers and variables denoting the objects may have a relation to explaining why different variables can't be conjoined by using objects. The data I presented, both from the questionnaires and from the interviews, suggests that there is such a relationship. However, this relation occurs in an indirect manner, in a complex way. For example, Fatma knows that children learn with concrete objects easily. She moreover knows that new concepts can be built on previously learnt concepts. When she amalgamates all this knowledge she attempts to use 'fruit-salad' explanation in which she gives meanings to variables by referring to objects, namely apples and pears. Then, she realises that her approach does not explain the reason why different variables cannot be conjoined. She thinks of another approach where she uses her SMKv which is knowing that variables stand for numbers. Hence, she returns to use number substitutions. In Fatma's case, different kinds of knowledge come together in her thinking to give a pedagogical explanation in which she uses substitutions. That is, knowing that variables stand for numbers alone is not the reason to use number substitution. Similarly, at first using objects is not caused because she has confusion between variables denoting the numbers and objects themselves. She could be able to change her strategy when she understands that there is something wrong with her explanation. Such change can be facilitated by her being flexible. All this shows that the relationship between SMKv and PCKv is not *always* direct. Supports come for this claim from Yezdan's case also.

As I discussed previous part of this chapter, although Yezdan has confusion between variables denoting numbers and objects, she uses number substitution to explain why different variables cannot be conjoined. Similarly, she has a general pedagogical knowledge, which she has achieved by teaching a year-2 primary school children, that says younger children learn with concrete objects. At first glance, it may be thought that both of these types of knowledge can cause her to explain why variables cannot be conjoined by using 'fruit-salad' algebra. However, she does not use this type of explanation instead she uses number substitution. Hence, there must be another reason why she uses number substitution. In her case one of the reasons for such explanation may be related her procedural approaches to problems. That is, she does not have flexible 'cognitive units' which may help her to regard problems from different ways. She has learnt procedural rules that can be used to solve specific, accustomed problems. As I discuss in chapter 4 and elsewhere, substituting numbers is equivalent to approach expressions from procedural side. Hence, Yezdan's being a procedural student might have caused her to use number substitution to explain why different variables cannot be conjoined.

However, it should be acknowledged that Yezdan's misconception about variables that regarding them as labels cause her to carry this misconception in her explanation to how she introduces variables. She supports her choice of using objects in her explanation by referring to approaches in textbooks. She mentions that in textbooks before letters are introduced, pupils are shown geometrical figures such as squares, triangles etc. standing for numbers instead of letters. She also supports her approach by mentioning that harder concepts can be built on relatively easier concepts. When she amalgamates all this knowledge together, she gives an explanation in which appears the misconception she has.

These examples suggest that the relationship between SMKv and PCKv is not *always* direct. They also suggest that PCKv is not something that is automatic amalgamation of different kinds of knowledge. It depends on the person who has those types of knowledge. It is on her/his disposal to amalgamate them, use some of them and discard others or use only one of them as a basis to give a pedagogical explanation specific to relevant to context at hand. When s/he decides to use only one of them as a basis and if this knowledge is SMK then the relationship between SMKv and PCKv can be direct. I see such instances in the case of Mehmet and Hakan. Hakan is identified as flexible student who has conceptual approaches to problems who has organised his knowledge of variables into linked 'cognitive units'. He does not concentrate on procedures rather on concepts and links between them. He is also aware of the distinction between variables as labels and variables as denoting numbers. He clearly explains that if a is an apple in 5a then it cannot be a number similarly if it is a number then it cannot be an object. This awareness is the most probable reason why he does not use 'fruit-salad' explanation to teach that different variables cannot be conjoined. Hence, this may show a direct relationship between SMKv and PCKv.

Another direct relationship occurs when Hakan chooses his SMKv to analyse a pupil's mistake. The pupil finds the root of an expression even though she is asked to simplify it. Hakan, elaborates this mistake by using his SMKv. He mentions that the given expression is not a suitable expression to be simplified. According to him expressions that can be divided by the same coefficient are the most suitable candidate for simplification tasks. The reason for this is that, according to him, simplifications are mostly used in rational numbers. Hence, such kinds of beliefs and knowledge become a source for Hakan to analyse the pupil's mistake given in the question.

Direct relationship between SMKv and PCKv is also seen in Mehmet's case. Mehmet strongly believes that the underlying reason for variables cannot be conjoined is because apples and pears cannot be added together. He mentions that he has known this fact since primary school, and everybody knows it. Therefore, two different variables cannot be conjoined. This knowledge can be both considered as SMKv and also a pedagogical knowledge that he has learnt from his teachers. He himself does not see the difference between variables denoting the objects and variables denoting the numbers. This confusion not only affects his pedagogical reasoning but also his SMKv. These together affect his explanation to variables cannot be conjoined. Furthermore, another direct relationship is that Mehmet himself concentrates on procedures, for him the important thing is getting answers right, knowing how to do. Such behaviours affect him to decide which topic to teach first. It affects him because he does not use any different type of knowledge to reason about which topic to teach first. Hence, these examples suggest that SMKv and PCKv can be related directly if the person decides to do so. A person may decide to use his SMKv alone because that person may not have any other kinds of knowledge such as general pedagogical knowledge to restructure his SMKv. In this case a more direct relationship between SMKv and PCKv can occur.

CHAPTER 6
Summary and Conclusions

INTRODUCTION

In this chapter, I will present the summary and discussions of the main themes by referring back to the research questions raised in Chapter I and I will consider directions for further research.

6.2 Summary of the themes

This study was carried out to probe answers to the following general research questions:

1. What is the state and nature of Turkish prospective mathematics secondary teachers' subject matter knowledge of variables?
2. What is the state and nature of Turkish prospective secondary mathematics teachers' pedagogical content knowledge of variables?
3. Is there any relationships between subject matter knowledge of variables and pedagogical content knowledge of variables, if so what is the nature of such relationships?

In this study one hundred and eighty four Turkish prospective teachers' subject matter knowledge of variables and their pedagogical content knowledge of variables were investigated by the means of an open-ended questionnaire and follow-up semi structured interviews. Also in this study, the nature of the interaction between these two types of knowledge was examined.

The summary of the themes will be presented along three lines; subject matter knowledge of variables, pedagogical content knowledge of variables, and finally the nature of the relationships between these two types of knowledge.

6.2.1 Main themes of subject matter knowledge of variables

The concept of variable is one of the basic topics taught from late primary education to high school education in the Turkish Mathematics Curriculum. It is introduced in the 7^{th} grade in the context of mathematical expressions and continued in the 8^{th}, 9^{th} (1^{st} grade of high school) grades with more emphasis on solving equations.

The 12-13 year old children are introduced to variables by teaching them how to write mathematical expressions; expressing different numerical relationships using letters. They start learning to write these expressions in the context of propositional formulas. Propositional formulas are sentences that contain a proposition. For example; "Ataturk is the founder of Turkish Republic" is given as a starting example in the textbooks for propositions. Then, propositions that contain number relationships start to appear later. Students are taught to become competent in solving equations in one variable, and then two variables in systems of equations, and finally second degree equations. One of the main aims of the algebra curricula is to teach students to manipulate symbols to arrive at results. Students are taught to memorise and use rules involved in symbol manipulations. They are given automatic skills which enable them to find the solutions of problems quickly.

The findings of this study, in a way, reflect the situation in the Turkish curriculum discussed above. In chapter 4, the analysis of questions 8 and 9 revealed that very few prospective teachers (about 10 %) have a variety of approaches to problems, checking mechanisms and links between algebraic expressions and their graphs. These prospective teachers start to solve a problem by using technical manipulations. When they see their methods are unproductive, they flexibly resort to use a different approach; visual, graphical approach. This shows that they have a flexibility of approaching problems from different ways.

Such flexibility, in particular abandoning technical manipulations for better tools is included in the aspects of 'symbol sense' (Arcavi, 1994). This sense can stem from having compressed 'cognitive units' (Barnard & Tall, 1997), which allow one to move flexibly among different aspects of a concept. In this case, such 'cognitive units' help these students to move from algebraic representations to visual, graphical representations. In other words, for these students algebraic expressions and their graphs mean exactly the same thing. In order to ascertain this claim, firstly I cross-tabulated the results of two questions and I saw consistencies among the responses. Then, I interviewed with one of these students. The interview results show that the student has, in fact, "various approaches to problems, checking mechanisms and an overall grasp" of algebraic equations to "build up links as if it were a 'cognitive unit'" (Crowley & Tall, 2001, p. 1). When she sees that the general formula for finding the equation of a straight line that passes through two given points is of no use for finding the required equation, she flexibly resorts to work on Cartesian coordinates and finds the equation. However, two of those students who do not demonstrate flexibility on solving questions 8 and 9 in the questionnaire, do not find the required line equation. The reason for this is that the general formula for finding the line equation does not work for the given points in the interview. They have to work on Cartesian coordinates to find out the required line equation. However, they do not think of working on Cartesian coordinates, and begin to be suspicious about the formula they have always used. They give up and utter they cannot find. This shows that these students have learnt procedural techniques to find the equation of straight lines and have not organised their cognitive structures into a network of flexible 'cognitive units'. They did not demonstrate flexibility in choosing an alternative route to a solution, which is evidence of missing links between graphs, formulas, and other aspects of a problem. However, the analysis of similar questions I presented in chapter 4 was presented in the light of some different theories in the literature.

For example, Even (1988) claims that prospective teachers who solve question 8 (please see chapter 4) by use of graphs have connected knowledge. Those students who do not use graphs and follow technical manipulation route are claimed to have compartmentalised knowledge related to functions. In the light of Krutetskii's (1976) constructs those students who use first technical manipulations then resort to make use of graphs can be identified as using both visual and symbolic thinking in equilibrium. On the other hand the other group of students who do not use graphs may be identified as non-visualisers. However, one of the underlying reasons for being able to move flexibly among different aspects of a concept can be explained by the theories suggested by Gray and Tall (1994). Gray and Tall (1994) suggest that there is a fundamental role of the student's perception of notation in mathematics.

In arithmetic, proceptual students could regard notations both as a process and product of that process. They coin the term '*pro*c**ept**' to point out the pivotal role of symbols which functions dually both as a *pro*cess and the con**cept** produced by that process. In arithmetic, procepts give a proceptual thinker flexibility of embedding different kinds of meanings to numbers. For example, a proceptual student could view the symbol 5 as 3+2, or 4+1 or 6-1 *etc*. However, procedural student may view this symbol 5 as counting from one to five. Such qualitatively different way of interpreting notation exists also in algebra.

As Tall (1993) point out in algebra, notation functions dually both as a process and concept, but in algebra the process is potentially successful. Without knowing the value of a symbol, an expression like $2x+1$ can not be evaluated, even though it can be interpreted procedurally as multiply x by two add one. Since x is not known a procedural thinker may have difficulty in interpreting such expressions. In algebra, the relationships between two or more variables can be expressed by using algebraic expressions, or by graphs. A proceptual student could embed these different representations as different aspects of the same idea by giving it a name or symbol. The process of embedding different aspects of a concept into a single idea is accomplished by compressing them, and making them small enough to be held in the focus of attention. Barnard and Tall (1997) call such small enough entities as 'cognitive units'. According to Crowley and Tall (1999), such cognitive unit "has rich interiority through carrying "within" it many powerful links that enable it to be manipulated and invoked to solve problems." (p. 3).

The theories suggested by Tall and his colleagues are more plausible to explain the reasons underlying behaviours that are demonstrated by prospective teachers in dealing with problems 8 and 9 (please see chapter 4). They explain why some prospective teachers are flexible enough to approach these problems from different ways. This suggests that they have developed links between different aspects of a problem. They also explain underlying reasons for behaviours that are included in the 'symbol sense' of Arcavi (1994). Arcavi (1994) suggests being able to approach problems from alternative ways is a sign of having symbol sense.

Ursini and Trigueros (1997) report similar findings on 164 starting Mexican college students' understanding of variables. They report that the majority of their students' understanding of variables is still restricted at an *action* concept of variable. In this kind of understanding, students' actions are caused by *stimuli*, which allow them to answer routine questions by means of mechanistic approaches. Ursini and Trigueros suggest that such an understanding may prevent students from "attaining a level of abstraction which will enable them to consider variables as objects whose role can be analysed." (p. 254). Similarly, it should be suggested that rote learning may prevent students from having flexible 'cognitive units' (Barnard & Tall, 1997) involving variables.

To sum up, the analysis presented in chapter four which investigated prospective teachers' subject matter of variables suggest that very few of them have developed flexible links between different aspects of the concept of variable. A large majority of them have mechanisation of procedures rather than the conceptualisation of these procedures. They demonstrated that they know the procedures that are related to different aspects of variables, such as solving first order, second order equations, simplifying expressions. However, when these procedures are of no use, very few of them try different approaches to problems. This suggests that they store collection of procedural techniques with no flexibility and checking mechanisms. Therefore, the data presented in chapter 4 support the theories suggested by Tall and his colleagues.

6.2.2 Main themes of pedagogical content knowledge of variables

In chapter four, part two, I presented the analysis of pedagogical content knowledge of variables. In this section I will present the summary of main themes of pedagogical content knowledge. In this thesis, pedagogical content knowledge is defined in line with Shulman's (1987) definition. Shulman (1987) defines PCK as the type of knowledge that is required to teach topics. It is neither SMK nor general pedagogical knowledge; rather it is a blend of subject matter knowledge and general pedagogical knowledge. It also includes knowledge about students, their preconceptions and misconceptions about concepts. As I explained in the methodology chapter, in this study, I have regarded PCK as formal knowledge of ways of communicating one's SMK to others while I was analysing the responses to the PCK questions. While using the word "communicating", I don't imply SMK can be transmitted from teacher to students. By communicating SMK I mean the type of knowledge that is required for teachers to help their students construct the knowledge that are presented to them. This means that PCK can be exhibited not only in the ways that teachers choose to explain, perhaps the most direct form of 'communication', but also in the way they choose to design classroom activities and to sequence topics. In this study I have treated the desire to make SMK accessible to pupils as evidence of PCK. However, how can we know whether that person has such a desire from his/her responses? We can suggest that if that person uses analogies, metaphors, examples, and different kinds of representations in his/her communication then s/he tries to make the concept comprehensible. Furthermore, if s/he is taking into account the common preconceptions, misconceptions, how knowledge is developed, then s/he tries to make the concept comprehensible. Another question arises from this last assertion. How can we know whether that person is taking into account such issues? We can suggest that by asking for the underlying reasons behind his/her explanation, we can examine whether s/he takes such issues into account. However, it is difficult to know whether s/he constructs reasons that will sound "right" or logical to the researcher when s/he is asked to articulate reasoning behind his/her explanations.

The analysis presented in chapter four, part two, showed that there are various kinds of pedagogical responses given by these students. In their responses, I saw the signs of their efforts to make the various aspects of the concept of variable comprehensible to the students. For example, 37% of them use 'fruit-salad' explanations in which they give meaning to variables by referring to objects in order to explain why variables cannot be conjoined. About 18% of them use number substitution in which they substitute numbers for the variables. Tall (1993) reports that when he poses, in a professional course, the question of introducing variables to pupils, a prospective teacher replies he would use 'fruit-salad' algebra. Then other prospective teachers accept such a suggestion. The reason for such suggestion is given as that the prospective teacher himself is taught like that. Furthermore, 'fruit-salad' algebra provides a guide to manipulate variables. In 'fruit-salad' algebra, $2a+3a+b$ is explained as add two *apples* plus three *apples* plus a banana, that gives five apples and a banana. However, as Tall (1993) points out this method of teaching although giving success for a while, in the long run causes damage to the development of algebraic knowledge, because its meaning is inflexible and fails when it does not apply. Furthermore, Pimm (1987) criticises this approach because "it leads to confusion between *a* being apples and *a* being 'the number of apples" (p. 132). Hence, he suggests that it is necessary to distinguish between the objects themselves and the number of them. "The algebraic expression is not an analogue of 5 apples, nor is 5 apples a possible interpretation of 5a... the letters themselves are standing for numbers" (p. 132).

The 'substitution' strategy can also be criticised by saying that it emphasises the process facet of expressions and it does not encourage viewing them as objects. However, both of these methods of showing that variables cannot be conjoined are used by textbooks and experienced teachers. If we refer to Shulman's (1987) definition of PCK that says PCK is knowledge of making topics comprehensible to students, we can claim that these methods (substitution, fruit-salad) are efforts towards making variables comprehensible to students. However, it can be questioned whether these methods result from the amalgamation of SMKv and general pedagogical knowledge. This question can be raised from Shulman's (1987) definition of PCK in which he claims PCK is also the amalgamation of SMK and general pedagogical knowledge. I will summarise the results of the investigation whether the responses given by prospective teachers to PCKv questions are affected by their SMK and general pedagogical knowledge in the next section.

Various efforts of making variables comprehensible are also seen in the responses of other questions analysed in chapter four part two. For example, prospective teachers analysed a pupil's mistake which involves rather than simplifying a given expression, finding its roots and possible sources of that mistake. However, not all prospective teachers could spot this mistake. About 32% of them wrote that the pupil correctly simplified the expression. The possible reason for these students not to spot the mistake is that they have problems in terminology. They do not know the difference between the terms 'simplifying' and 'solving equations'. This also shows how SMKv can be related to PCKv; I will summarise main themes about these relationships in the next section. For the moment, I am continuing to discuss main themes about PCKv. About 54% the prospective teachers spot the student mistake however only about half of these students comment on the possible sources of this mistake. The questionnaire responses suggested that the most of the students who comment on the pupil's mistake attribute this mistake to familiarity of the pupil with solving equations due to overemphasis on solving equations. Philipp (1992) attributes this mistake to the difficulty in grasping different roles of variables. According to him, the letter x assumes the role of generalised number in simplifying expression $2x+4x+6$, whereas it assumes the role of the specific unknown in solving the equation $2x+4x+6=0$. However, Philipp (1992) does not discuss why using letters as generalised numbers is more difficult than using them as specific unknowns. Kieran (1997) asserts that we don't have enough evidence to suggest that using letters as specific unknowns is psychologically primitive to using them as generalised numbers. On the other hand, Sfard and Linchevski (1994) suggest that algebra of a fixed value where the letters are used as specific unknowns should precede the functional algebra where the letters play the role of variables and generalised numbers, because the second one is more difficult.

I referred to these research reports in order to point out the various comments that can be made about the underlying reasons behind a particular mistake a pupil makes. It is difficult to know why a pupil makes a particular mistake without talking with him, knowing his/her background, knowing research results in the literature. However, to start elaborating a mistake, teachers may figure out some hypothesis about the reasons of that mistake, and then they can talk with the pupils to verify or discard their hypothesis. In the questionnaire and interview settings, it is difficult to give prospective teachers the opportunity to talk with the pupil that made the mistake in the question. However, from the questionnaire responses we can be informed at least about their hypothesis on the underlying reasons of the mistake. As I mentioned earlier, the most of the prospective teachers who commented (26%) about the sources of the mistake hypothesised that the pupil got used to equation solving due to overemphasis of this topic in textbooks and classrooms. These hypotheses, although they are different from the comments I cited earlier, show that these prospective teachers could analyse a pupil's mistake by spotting firstly the mistake then commenting on the possible sources of that mistake. This adds supports to the claim that there are signs of the desire to make SMK accessible to pupils in prospective teachers' responses.

The responses given to help a student who makes a 'reversal error' also suggest that prospective teachers exhibit their PCK in various ways they choose to help the student. Some of the helping methods prospective teachers suggest are even used in the literature to alleviate the 'reversal error'. For example, using arbitrary letters rather than using first letters, or using pictorial representations can be given as methods that are tried by researchers (e.g. Mestre & Lochhead,1983) to alleviate the 'reversal error'. When we look at how the prospective teachers choose to teach a topic first, -simplification or substitutions- we see that 16% of these prospective teachers choose either of the topics first because their ordering provides conceptual benefits. For example, they write that they would teach substitution first so that pupils could see that variables are denoting numbers hence they can see what variables are for, what they are used for. However, about half of prospective teachers order these topics considering 'procedural', 'instrumental' benefits. For example, they consider 'getting answers right', or 'quickly and easily finding results'. The question of teaching simplification or substitution first does not have definite answer. For example, Sfard and Linchevski (1994) may prefer to teach substitutions first because it gives an operational conception of variables and the operational conception should precede the structural conception. On the other hand, Tall and Thomas (1991) would prefer to teach them simultaneously because in this way students could achieve versatile learning of variables. They use different activities that can allow students to focus on either processes or concepts; which in turn allow students to interlink them to give a connection between processes and concepts. Hence, the important thing in deciding on which topic to teach first lies in the underlying reasons for such a decision. In the given reasons provided by Sfard and Linchevski (1994) and Tall and Thomas (1991) we see that they want to make variables understandable to the pupils. They are considering the development of algebraic knowledge, pupils' difficulties with this concept. In short, they are using their pedagogical content knowledge which is defined by Shulman (1987) as knowing ways of making a topic comprehensible to students as well as knowing common students' misconceptions and difficulties.

In responses provided by prospective teachers in this research, we see that there are various kinds of pedagogical responses given by these students. In their responses, I saw the signs of their efforts of making the various aspects of the concept of variable comprehensible to the students. This result sheds some lights onto the discussions surrounding when and with whom to study PCK; expert teachers, novice teachers or prospective teachers. One can study PCK with any of these, the important thing is being aware of the context of the sample.

6.2.3 Main themes of the interactions between SMKv and PCKv

In this section, I will present main themes about the interactions between SMKv and PCKv. I will start with summarising issues presented in the literature about the interactions between SMK and PCK.

Leinhardt and her colleagues (e. g. Leinhardt, Putnam, Stein and Baxter, 1991; Leinhardt & Smith, 1985, etc.) report that there are differences between experts' and novice teachers' representations since expert teachers have more highly organised systems of knowledge than novice teachers. Within such reports they assume that subject matter knowledge and pedagogical content knowledge are interrelated.

Similarly, Even and her colleagues (Even & Tirosh, 1995; Even, 1991 etc.) suggest that since their prospective teachers did not know the reasons behind the rules they opted to present mathematics as a set of rules to be followed. These research reports suggest that the interaction between SMK and PCK is direct and clear. This research can be seen as an answer to earlier research which suggests that what teachers know about mathematics is not a significant influence on what their students learn (Begle, 1979; Eisenberg, 1977). That is, earlier research suggest that "the effects of a teacher's subject matter knowledge and attitudes on student learning seem to be far less powerful than many of us assumed," (p. 53, Begle, 1979 cited in Ball, 1991, p. 2-3). At some level, earlier research suggestions can be regarded as that there is no relationship between SMK and PCK. This is in contrast to following researches which suggest that there is a direct relationship between SMK and PCK.

However, as Prestage (1999) reports that even many experienced teachers use their own learning experiences to make pedagogical decisions. This report suggests that there may be some intervening knowledge that can affect the interactions between SMK and PCK. Skemp (1976) asserts that "nothing else but a relational understanding can ever be adequate for a teacher." However, as Skemp himself points out, even if one has relational understanding; he/she may not teach relationally, because the students may want to learn instrumentally. Within this remark, Skemp might have suggested that the relationship between subject matter knowledge and pedagogical content knowledge is not as clear cut as he suggests.

As a matter of fact, the analysis and results of this study suggest that the interaction between subject matter and pedagogical content knowledge are more complex than is suggested in the literature (Even & Tirosh, 1995; Even, 1991; Begle, 1979; Eisenberg, 1977). In chapter five, I presented the analysis that examines the interactions between SMKv and PCKv. This analysis suggested that the relationship between SMKv and PCKv is not *always* direct. They also suggest that PCKv is not something that is automatic amalgamation of different kinds of knowledge. It depends on the person who has those types of knowledge. It is on her/his disposal to amalgamate them, use some of them and discard others or use only one of them as a basis to give a pedagogical explanation specific to context at hand. When s/he decides to use only one of them as a basis and if this knowledge is SMK then the relationship between SMKv and PCKv can be more direct. In other words, when SMKv is the predominant influence on a pedagogic explanation then the relationship between PCK and SMK is more direct. I use the word "more" because there is a question of whether the relationship between any action and its causes is ever direct.

For example, although a prospective teacher mentions that x could be a number, she chooses to use 'fruit-salad' explanation to respond to a pupil's question who asks why $2a+5b$ does not equal to $7ab$. The first response of this student suggests that she knows that variables are standing for numbers. As a matter of fact the analysis of interview with her suggests that she knows the distinction between letters standing for numbers and letters standing for objects, as labels. However, she mentions that she would use 'fruit-salad' algebra in which the algebraic expressions are analogised to apples and bananas. The given reason by her for this analogy is that younger students learn with concrete objects easily. That is, she is under the influence of her general pedagogical knowledge while she is suggesting using 'fruit-salad' algebra. When she explains how she would use 'fruit-salad' algebra, she notices that 'fruit-salad' analogy does not explain why variables cannot be conjoined. However, her observation does not result from knowing that the letters stand for numbers, rather it results from noticing that on both sides there are seven fruits. That is, on the left there are two apples and five bananas which makes seven fruits on the right. Then, she returns to use number substitution. This shows that the interaction between SMKv and PCKv may not always be direct.

On the other hand, such relationship can be direct, when the person who possesses the knowledge does not know the difference between letters standing for numbers and letters as labels. Also when that person is taught by 'fruit-salad' methods, that person may strongly believe that the reason why 2a+5b does not equal to 7ab is because apples and bananas don't add up together. These last assertions result from the responses of a prospective teacher who writes that *x* can be an object and who would use 'fruit-salad' algebra. In the interview, this student strongly mentions that the reason why different variables cannot be conjoined is because different objects don't add up together. This student also does not know the difference between letters standing for numbers and letters standing for objects. As a result this misconception causes him to use 'fruit-salad' algebra.

To sum up, the data I presented, both from the questionnaires and from the interviews, suggests that there is a relationship between SMKv and PCKv. However, this relation occurs in an indirect manner, in a complex way. SMKv and PCKv can be related more directly if the person decides to do so. A person may decide to use his SMKv alone because that person may not have any other kinds of knowledge such as general pedagogical knowledge to restructure his SMKv. On the other hand, this kind of decision making may not be always consciously. That is, one type of knowledge may have predominance in one's thinking while giving a pedagogical explanation and that person can unconsciously use that type of knowledge. Especially, prospective teachers may not often 'choose' consciously which forms of knowledge to use.

6.3 Recommendations for future research

The possible interactions suggested in this study were examined at a particular point of time with prospective mathematics teachers. If knowledge is regarded as a dynamically growing entity, it might be wondered what nature of relationships between SMK and PCK would be observed in the knowledge base of experienced teachers. Furthermore, it might be also wondered about the nature of such interactions while teachers are teaching in the classrooms. Therefore, it would be desirable for this study to be replicated with teachers in action. Classroom observations could be used to replicate this study with teachers in actions. Interview data and questionnaire data can be used as a back up in order to triangulate the data sources.

It would be desirable to study more whether a person who has "instrumental knowledge" about mathematics would emphasise procedural aspects and whether a person who has "relational knowledge" would emphasise conceptual aspects while teaching. That is, the effects of "relational" and "instrumental knowledge" on one's pedagogical content knowledge would be researched. Askew, Brown, Rhodes, Johnson and Wiliam (1997) report that the proportion of links which were explained in conceptual terms rather than by rule-based connections is reported to be moderately related to pupil gains in numeracy. Even and Tirosh (1995) report that if one does not know underlying reasons of the rules in mathematics then s/he opts to teach instrumentally (teaching mathematics as a set of rules to be memorised). However, in order to see the nature of interactions between SMK and PCK, it would be more productive to study what happens if one has relational understanding of topics. In that case, we can see the interactions between SMK, PCK, general pedagogical knowledge and knowledge about students more clearly.

Similarly, it would be desirable to study the effects of possessing flexible 'cognitive units' on one's PCK. It would be researched whether those who possess flexible 'cognitive units' could have various helping methods to alleviate a pupil's particular mistake. Would flexible 'cognitive units' give one an advantage of giving rich explanations (explanations that contain various aspects of a concept, rule etc.)? There are some hints about the answer of the last question above in the research reports of Leinhardt, Putnam, Stein and Baxter (1991). These researchers suggest that since expert teachers have more highly organised systems of knowledge; they give richer and more detailed explanations than novice teachers.

The data for this study was collected from the students attending Universities in Ankara, the capital city of Turkey. It would be desirable for this study to be replicated in the other regions of Turkey in order to provide some comparisons.

The challenge for future studies is to develop a methodology and framework for investigating the interactions. For future studies, the theoretical framework used in this study seems promising for analysing subject matter knowledge and pedagogical content knowledge of variables. As I explained in methodology chapter, it is difficult to investigate PCKv because the conceptual difficulties surrounding its definition and its being partly an internal construct. There is still need for an operable definition of PCK. There is not a methodology that gives the researcher an accurate representation of knowledge. I believe that there are some hints in the definition used for PCK in this study can be expanded to make it operable. Furthermore, triangulation of data sources may provide more accurate representation of knowledge.

6.4 Limitations of the study

This research study had some limitations that need to be mentioned.

- Although there is a question whether the meaning is ever conveyed directly, in this study, the problems of interpretation that can exist in every research are compounded by the need to translate from one language to another. While carrying out this research a lot of language translation between English and Turkish and vice versa has been made, therefore in these processes there is the possibility that some of the intended meanings in the original language might have been lost.

- The main data collection method of this study was questionnaire technique which lacks the flexibility and adaptability that interviewing entails. Although, to see the validity of the responses given in the questionnaire, talking with the prospective teachers increased the richness of the data by clarifying misunderstandings in the questionnaire, there are still some limitations posed by using the questionnaires as a main data collection method. For example, the construct validity of the questionnaire might have been strengthened by using more than one question for each aspect of SMKv and PCKv. Secondly, as a backup to the questionnaires and interviews, the data sources can be triangulated by employing classroom observations as well as lesson plans.

- The results of this study are restricted to the context of pre-service teachers. Since the data is gathered by using hypothetical situations that can be encountered in the classrooms, we cannot say for sure that the behaviours that are mentioned by prospective teachers would be shown by them also in classrooms while teaching. We cannot generalise these results.
- This study was conducted in only one city of Turkey. Therefore, it is difficult to generalise the results of this study although they give a general insight about the state and nature of Turkish prospective teachers' SMKv and PCK.

References

Arcavi, A. (1994). Symbol sense: Informal sense-making in formal mathematics. *For the Learning of Mathematics, 14*(3), 24-35.

Asiala, M., Brown, A., DeVries, D., Dubinsky, E., Mathews, D. & Thomas, K. (1996). A framework for research and curriculum development in undergraduate mathematics education, *Research in Collegiate Mathematics Education II, CBMS Issues in Mathematics Education*, 6, 1-32.

Askew, M., Brown, M., Rhodes, V., Wiliam, D. & Johnson, D. (1997). *Effective Teachers of Numeracy*. London: King's College London.

Ausubel, D. P. (1968). *Educational Psychology: A Cognitive View.* New York: Holt, Rinehart and Winston.

Ball, D. (1991). Research on teaching mathematics: Making subject-matter knowledge part of the equation. In J. Brophy (Ed.), *Advances in research on teaching* (Vol. 2, pp. 1–48). Greenwich, CT: JAI Press.

Ball, D. (1988). The subject matter preparation of prospective mathematics teachers: Challenging the myths. (Research Report, 88-3), East Lansing, MI: Michigan State University, National Center for Research of Teacher education.

Barnard, T. & Tall, D. O. (1997). Cognitive Units, Connections, and Mathematical Proof. In E. Pehkonen, (Ed.), *Proceedings of the 21st Annual Conference for the Psychology of Mathematics Education,* Vol. 2. Lahti, Finland, 41–48.

Baxter, J. A & Lederman N. G. (1999). Assessment and Measurement of Pedagogical Content Knowledge. In Gess-Newsome, J. & N. B. Lederman (Eds.), *Examining pedagogical content knowledge*, (147-161). Dordrecht, Boston, London: Kluwer.

Bell, A. (1995). Purpose in school algebra. *The Journal of Mathematical Behaviour, 14*(1), 41-73.

Begle, E. G. (1979). *Critical variables in mathematics education: Findings from a survey of empirical literature*. Washington, DC: Mathematics Association of America and the National Council of Teachers of Mathematics.

Begle E. G. & Geeslin, W. (1972). Teacher effectiveness in mathematics instruction (National Longitudinal Study of Mathematical Abilities Reports: No. 28). Washington, DC: Mathematical Association of America and National Council of Teachers of Mathematics.

Bills, L. (2001). Shifts in the meanings of literal symbols, in M. van den Heuvel-Panhuizen, (Ed.) *Twenty fifth Conference of the International Group for the Psychology of Mathematics Education,* (Vol. 2, pp.161-168). Utrecht: The Netherlands.

Bills, E. J. (1997) *Shifting sands: Students' understanding of the roles of variables in A-level mathematics.* Unpublished PhD Thesis, Open University, UK.

Booth, L.R. (1984). Algebra: children's strategies and errors. Windsor: NFER-Nelson

Bromme, R. (1995). What exactly is pedagogical content knowledge? Critical remarks regarding a fruitful research program. In S. Hopmann & K. Riquarts (Eds.), *Didaktik and/or curriculum. IPN Schriftenreihe,* (Vol. 147, pp. 205-216). Kiel: IPN.

Bromme, R. (1994). Beyond subject matter: a psychological topology of teachers' professional knowledge, In R. W. Scholz, R. Strasser & B. Winkelmann (Eds.), *Mathematics didactics as a scientific discipline: The state of the art* (pp. 73-88). Dordrecht: Kluwer

Brophy, J. (Ed.). 1991. Advances in research on teaching (Vol. 2) Greenwich, CT: JAI Press.

Brown, A & Dowling, P. (1998). *Doing Research/Reading Research,* London: Falmer Press

Bryman, A., & Cramer, D. (1997). *Quantitative Data Analysis with SPSS for Windows: A Guide for Social Scientists,* London & New York: Routledge.

Byrne, C. J. (1983). *Teacher knowledge and teacher effectiveness: A literature review, theoretical analysis and discussion of research strategy.* Paper presented at the meeting of the Northwestern Educational Research Association, Ellenville, NY.

Cape, N., V. (2000). Student teachers' concept images of algebraic expressions, *Twenty fourth Conference of the International Group for the Psychology of Mathematics Education,* (Vol. 4, 257-265). Hiroshima: Japan .

Chazan, D., Larriva, C. & Sandow, D. (1999). What kind of mathematical knowledge supports teaching for "conceptual understanding"? *Preservice teachers and the solving of equations, Twenty third Conference of the International Group for the Psychology of Mathematics Education,* (Vol. 2, 193-201). Israel.

Clark, C., & Yinger, R. (1979). Teachers' thinking. In P. L. Peterson and H. Walberg (Eds.), *Research on teaching: Concepts, findings, and implications* (pp. 231-263). Berkeley: McCutchan.

Clement, J. (1982). Algebra word problem solutions: thought processes underlying a common misconception. *Journal for Research in Mathematics Education, 13*(1), 16-30.

Clement, J., Lochhead, J. & Monk, G.S. (1981). Translation Difficulties in Learning Mathematics, *American Mathematics Monthly,* 4, 286–290.

Cohen, L., Manion, L., & Morrison, K. (2000). *Research Methods in Education* (5 ed.). London: Routledge Falmer.

Collis, K. F. (1975). *The development of formal reasoning,* Report of a social science research council sponsored project (HR 2434/1) carried out at the University of Nottingham, University of Newcastle, NSW: Australia.

Crowley, L. R. F., & Tall, D. O. (2001). Attainment and Potential: Procedures, Cognitive Kit-Bags and Cognitive Units, *Proceedings of PME* 25.

Crowley, L. R. F., & Tall, D. O. (1999). The Roles of Cognitive Units, Connections and Procedures in achieving Goals in College Algebra. In Zaslavsky, O. (Ed.), *Proceedings of the 23rd Conference for the Psychology of Mathematics Education,* Vol. 2 (pp. 225-232). Haifa, Israel.

Crowley, L. R. F., Thomas, M. O. J., & Tall, D. O. (1994). Algebra, Symbols, and Translation of Meaning, *Proceedings of PME18*, Lisbon, II, 240–247.

Darling-Hammond, L. (1999). Teacher quality and student achievement: A review of state policy evidence. *Educational Policy Analysis Archives, 8*(1).

Eisenberg, T. A. (1977). Begle revisited: Teacher knowledge and student achievement in algebra. *Journal for Research in Mathematics Education*, 8, 216-222.

Eisenberg, T. (1991). Functions and associated learning difficulties, in D. Tall, (Ed.) *Advanced Mathematical Thinking*, Dordrecht / Boston / London: Kluwer Academic Publishers, 140-171.

Even, R. (1988). Pre-service teachers conceptions of the relationships between functions and equations, in Borbas (Ed.), *Proceedings of the Twelfth International Conference for the Psychology of Mathematics Education* (Vol. 1, pp. 304-311). Vesprem: Hungary.

Even, R. (1990). Subject-matter knowledge for teaching and the case of functions, *Educational Studies in Mathematics*, 21, 521-544.

Even, R. (1993). Subject-matter knowledge and pedagogical content knowledge prospective secondary teachers and the function concept, *Journal of Research in Mathematics Education, 24* (2), 94-116.

Even, R. (1998). Factors involved in linking representations of functions. *Journal of Mathematical Behavior, 17*(1), 105-121.

Even, R. & Tirosh, D. (1995). Subject-matter knowledge and knowledge about students as sources of teacher presentations of the subject-matter, *Educational Studies in Mathematics,* 29, 1-20.

Even, R., Markovits, Z. (1993) Teachers' pedagogical content knowledge of functions: Characterization and Applications, *Journal of Structural Learning, 12* (1), 35-51.

Even, R., Tirosh, D. & Robinson, N. (1993). Connectedness in teaching equivalent algebraic expressions: Novice versus expert teachers. *Mathematics Education Research Journal, 5*(1), 50-59.

Fennema, E. & Franke, M. L. (1992). Teachers' knowledge and its impact. In D. A. Grouws, *NCTM Handbook of Research on Mathematics Teaching and Learning,* (pp. 147-164). New York: Macmillan.

Fenstermacher, G. D. (1994). The knower and the known: The nature of knowledge in research on teaching. In L. Darling-Hammon (Ed.), Review of research in education (pp. 3–56). Washington, DC: American Educational Research Association.

Filloy, E. & Rojano, T. (1989). Solving equations: The transition from arithmetic to algebra, *For the Learning of Mathematics*, 9(2), 19-25.

Fraenkel, J., R.& Wallen, N., E. (1996). *How to Design and Evaluate Research in Education.* McGraw –Hill Inc.

Goulding, M. & Suggate, J. (2001) Opening a can of worms: Investigating primary teachers' subject knowledge in mathematics, *Mathematics Education Review,* 13, 41-53.

Gray, E. M. & Tall, D. O. (1994). Duality, ambiguity and flexibility: a proceptual view of simple arithmetic. *Journal of Research in Mathematics Education*, *26* (2), 115–141.

Grossman, P. L. (1990). *The making of a teacher: Teacher knowledge and teacher education.* New York: Teachers College Press.

Gudmundsdottir, S. (1991). Pedagogical Models of Subject Matter. In J. Brophy (Ed.) *Advances in Research on Teaching,* (2, pp. 265-304). Greenwich, CT: JAI Press.

Harper, E. (1987). Ghosts of Diophantus. *Educational Studies in Mathematics*, 18, 75-90.

Hart, F. (1934). *Teachers and teaching: By ten thousand high school seniors*. New York: Macmillan.

Hiebert, J. (1988). A theory of developing competence with written mathematical symbols, *Educational Studies in Mathematics,* 19, 333-355.

Hiebert, J. & Carpenter, T. P. (1992), Learning and Teaching with Understanding. In D. Grouws (Ed.),*Handbook of Research on Mathematics Teaching and Learning* (pp. 65–97), MacMillan, NY.

Herscovics, N. (1989). Cognitive obstacles in the learning of algebra. In Wagner, S. & Kieran, C. (Eds.), *Research issues in the learning and teaching of algebra,* (pp. 60-86). Hillsdale, NJ: Lawrence Erlbaum.

Herscovics, N., & Linchevski, L. (1994). A cognitive gap between arithmetic and algebra. *Educational Studies in Mathematics*, 27, 59-78.

Kagan, D. M. (1990). Ways of evaluating teacher cognition: Inferences concerning the Goldilocks Principle. *Review of Educational Research, 60*(3), 419-469.

Kaput, J.J., & Clement, J. (1979). Letter to the editor. *Journal of Children's mathematical Behaviour*, *2*(2), p. 208.

Kennedy, M. (1990). *Trends and Issues in: Teachers' Subject Matter Knowledge. Trends and Issues Paper No. 1.* ERIC Clearinghouse on Teacher Education, Washington, DC. (ED#322 100)

Kieran, C. (1997). Mathematical concepts at the secondary school level: The learning of algebra and functions. T. Nunes & P. Bryant (Eds.). *Learning and Teaching Mathematics: An International Perspective* (pp.133-158). East Sussex: Psychology Press.

Kieran, C. (1992). The learning and teaching of school algebra. In D.A. Grouws (Ed.), *The Handbook of research on mathematics teaching and learning* (pp. 390- 419). New York, NY: Macmillan.

Kieran, C. & Chalouh, L. (1993). The transition from arithmetic to algebra, in D. T. Owens (Ed.), *Research ideas for the classroom: Middle grades mathematics* (pp. 179-198), New York: Macmillan.

Krutetskii, V.A. (1976). *The Psychology of Mathematical Abilities in Schoolchildren,* translated from the Russian by Joan Teller, edited by Jeremy Kilpatrick and Izaak Wirszup. Chicago, IL: University of Chicago Press.

Küchemann, D. (1981). Algebra. In K. Hart (Ed.), *Children's understanding of mathematics: 11-16* (pp. 102-119). London: John Murray.

Küchemann D.E., (1978) Children's understanding of numerical variables, *Mathematics in School, 7* (4), 23-26.

Laborde, C. (1990). Language and Mathematics, in P.Nesher & J. Kilpatrick (Eds), *Mathematics and Cognition*, Cambridge: CUP, 53–69.

Lee, L. & Wheeler, D. (1989). The arithmetic connection. *Educational Studies in Mathematics.* 20, 41-54

Leinhardt, G. P, Ralph T., Stein, M. K. & Baxter, J. (1991). Where subject knowledge matters. In J. Brophy (Ed.), Advances in research on teaching (Vol. 2, pp. 87-113). Greenwich, CT: JAI Press.

Leinhardt, G., & Smith, D. (1985). Expertise in mathematics instruction: Subject matter knowledge. *Journal of Educational Psychology*, *77*, 247-271.

Leitzel, J. R. (1989). Critical Considerations for the Future of Algebra Instruction, In S. Wagner & C. Kieran (Eds.), *Research issues in the learning and teaching of algebra* (pp. 25-32). Hillsdale, NJ: Lawrence Erlbaum.

Macgregor, M. & Stacey, K. (1993). Cognitive Models Underlying students' Formulation of simple Linear Equations, *Journal for Research in Mathematics Education,* 24(3), 217–232.

Mason, J., & Pimm, D. (1984). Generic Examples: Seeing the General in the Particular. *Educational Studies in Mathematics 15*(3), 277289.

Medley, D. (1979). The effectiveness of teachers. In P. L. Peterson and H. Walberg (Eds.), *Research on teaching: Concepts, findings, and implications* (pp. 11-26). Berkeley: McCutchan.

Moschkovich, J., Schoenfeld, A. H. & Arcavi, A. (1993) Aspects of understanding: multiple perspectives and representations of linear relations and connections among them. In Romberg, T. A. E. Fennema & Carpenter, T. P. (Eds.) *Integrating research on the graphical representations of functions.* L. Erlbaum associates, Inc. Hillsdale, NJ. Pp. 69- 100.

National Council of Teachers of Mathematics (1989). *Curriculum and Evaluation Standards for School Mathematics*. Reston, VA: Author.

Philipp, R. A. (1992). The many uses of algebraic variables. *Mathematics Teacher, 85*(7), 557-561.

Philipp, R. A. (1992a). A study of algebraic variables: Beyond the student-professor problem, *Journal of Mathematical Behavior*, 11, 161-176.

Pimm, D. (1987). Pupils' written mathematical records, in D. Pimm (Ed.), *Speaking mathematically*, London & New York: Routledge & Kegan Paul.

Presmeg, N.C., (1986), Visualisation and mathematical giftedness, *Educational Studies in Mathematics*, 17, 297-311.

Prestage, S. (1999). *An exploration of teachers' mathematical subject knowledge.* Unpublished doctoral thesis. London: King's College, University of London.

Robson, C. (1993). *Real World Research: A Resource for Social Scientists and Practitioner Researchers*. Oxford: Blackwell.

Rosenshine, B. (1979). Content, time, and direct instruction. In P. L. Peterson and H. Walberg (Eds.), *Research on teaching: Concepts, findings, and implications*, (pp. 28-56). Berkeley: McCutchan.

Rosnick, P., & Clement, J. (1980). Learning without understanding: The effect of tutoring strategies on algebra misconceptions. *Journal of Mathematical Behaviour, 3*(1), 3-27.

Schoenfeld, A. (1985). *Mathematical Problem Solving*. New York: Academic Press.

Schoenfeld, A. H., & Arcavi, A. (1988). On the meaning of variable. *Mathematics Teacher, 81*(6), 420-427.

Sfard, A. (1992). Operational origins of mathematical objects and the quandary of reification: the case of function. Harel G. & E. Dubinsky (Eds.) *The Concept of Function: aspects of epistemology and pedagogy,* (MAA notes, Vol. 25), Mathematical association of America, Washington DC, pp. 261- 289.

Sfard, A., & Linchevski, L. (1994). The gains and the pitfalls of reification - the case of algebra. *Educational Studies in Mathematics*, 26, 191-228.

Sherin, M. G., Sherin, B.L. & Madanes, R. (2000). Exploring Diverse Accounts of Teacher Knowledge. *Journal of Mathematical Behavior, 18* (3), 357- 375

Shulman, L. S. (1987). Knowledge and teaching: Foundations of the new reform. *Harvard Educational Review*, 57, 1-22.

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher*, *15*(2), 4-14.

Shulman, L. S. & Sykes, G. (1986). *A national board for teaching? In search of a bold standard. A report for the task force on teaching as a profession.* New York: Carnegie Corporation.

Sierpinska, A. (1994). *Understanding in Mathematics*, London / Washington D.C.: The Falmer Press

Sierpinska, A. (1992). On understanding the notion of function, in G. Harel & E. Dubinsky (Eds.) *The Concept of Function: aspects of epistemology and pedagogy,* (MAA notes, Vol. 25), Mathematical association of America, Washington DC, pp. 25-28.

Skemp, R. R. (1979). *Intelligence, Learning, and Action.* New York: John Wiley & Sons.

Skemp, R. R. (1976). Relational understanding and instrumental understanding. *Mathematics Teaching,* 77, 20-26.

Sutherland, R. (1989). Providing a computer-based framework for algebraic thinking. *Educational Studies in Mathematics,* 20 (3), 317-344

Tall, D., O., (1993). The Transition from Arithmetic to Algebra: Number Patterns, or Proceptual Programming? *New Directions in Algebra Education*, Queensland University of Technology, Brisbane, 213–231.

Tall, D. O. (1991). *Advanced Mathematical Thinking*, Dordrecht / Boston / London: Kluwer Academic Publishers.

Tall, D. O. (1989). Different cognitive obstacles in a technological paradigm *or* A reaction to "Cognitive obstacles encountered in the learning of algebra". In S.Wagner & C. Kieran (Eds.), *Research issues in the learning and teaching of algebra* (pp. 87-92). Hillsdale, NJ: Lawrence Erlbaum.

Tall, D. O., Thomas, M. O. J., Davis, G., Gray, E. & Simpson, A. (2000).What is the object of the encapsulation of a process?, *Journal of Mathematical Behavior*, 18 (2), 1–19.

Tall D. O. & Thomas M. O. J. (1991). Encouraging Versatile Thinking in Algebra using the Computer, *Educational Studies in Mathematics*, *22*(2), 125–147.

Tall, D. & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity, *Educational Studies in Mathematics*, 12, 151-169

Tamir, P. (1988). Subject matter and related pedagogical content knowledge in teacher education, *Teaching & Teacher Education*, *4*(2), 99-110.

Thurston, W. (1990). Mathematical education. *Notices of the American Mathematical Society, 37*, 844–850.

Tirosh, D. (2000). Enhancing prospective teachers' knowledge of children's conceptions: The case of division of fractions. *Journal for Research in Mathematics Education, 31*(1), 5-25.

Tirosh, D., Even, R. & Robinson, N. (1998). Simplifying algebraic expressions: Teacher Awareness and teaching approaches, *Educational Studies in Mathematics*, 35, 51-64.

Türk, E. (1999). *Milli Eğitim Bakanlığı'nda Yapısal Değişmeler ve Türk Eğitim Sistemi.* Ankara: Nobel.

Ursini, S. & Trigueros, M. (2001). *A model for the uses of variable in elementary algebra*, in M. van den Heuvel-Panhuizen, (Ed.) *Twenty fifth Conference of the International Group for the Psychology of Mathematics Education,* (Vol. 4, pp. 327-333). Utrecht: The Netherlands.

Ursini, S., & Trigueros, M. (1997). Understanding of different uses of variable: A study with starting college students. *Proceedings of the Twenty-First International Conference for the Psychology of Mathematics Education* (Vol. 4, pp. 254-261). Lahti, Finland.

Usiskin, Z. (1988). Conceptions of school algebra and uses of variables. In A. Coxford & A. P. Schulte (Eds.), *Ideas of algebra, K-12 (1988 Yearbook: 8-19).* Reston, VA: National Council of Teachers of Mathematics.

Wagner, S. (1981). An Analytical Framework for Mathematical Variables. In *Proceedings of the Fifth Conference of the Psychology of Mathematics Education*, Grenoble, France: Laboratoire I.M.A.G., 65-70.

Wilson, S. M., Shulman, L. S., & Richert, A. E. (1987). 150 different ways of knowing: Representations of knowledge in teaching. In James Calderhead (Ed.), Exploring teachers' thinking (pp. 104-124). London: Cassell.

YÖK 2000, *Selection and Placement of Students in Higher Education Institutions in Turkey,* Retrieved 2003, from http://www.osym.gov.tr/eng/index.html.

YÖK 1998, Eğitim fakülteleri öğretmen yetiştirme proğramlarının yeniden düzenlenmesi, (Report), T.C. Yüksek Eğitim Kurulu Başkanlığı, Retrieved 2003, from http://www.meb.gov.tr.

APPENDIX

QUESTIONNAIRE

RELATIONSHIPS BETWEEN KNOWING AND TEACHING OF VARIABLES

Name-Surname:............................ In which year you are now:.........................
The year you started your university education:...............

EXPLANATIONS

- The questions in this questionnaire were prepared in order to investigate the relationships between knowing about and teaching of variables.
- In this questionnaire there are 16 questions, 10 of them are about knowledge and understanding of variables, and 6 of them are about teaching this concept.
- It is not so important whether you give the right or wrong answers, the important thing is how your solution processes are. Hence, please show as much as all your work.
- There is no time restriction for completing the questionnaire.
- This questionnaire was prepared only for gathering information for my doctoral theses study. Your answers will completely be kept secret.
- I wish you good luck, and thank you for your time.

QUESTIONS

1) a) Five years ago, the sum of the ages of a mother and her daughter was 53. Now, if mother is 27 years older than her daughter, how old was the daughter 3 years ago?

b) Prove that the sum of n consecutive even integers is divisible by n.

c)* Two mobile phone companies rent line. Both of them take some money for line rental per month and some money for calls per minute. The following tables give some information about their pricing for some particular number of call minutes and for some number of months.

i. For what number of call minutes per month would the price be the same?

ii. if you are a big user of mobile phones which company is preferable?

First Company

	10 Mins	20 Mins	40 Mins	60 Mins	70 Mins	100Mins	200mins
1 Month	35	50	80	110	125	170	320
2 Months	55	70	100	130	145	190	340
3 Months	75	90	120	150	165	210	360
4 Months	95	110	140	170	185	230	380
5 Months	115	130	160	190	205	250	400

Second Company

	10 Mins	20 Mins	40 Mins	60 Mins	70 Mins	100Mins	200mins
1 Month	38	58	98	138	158	218	418
2 Months	53	73	113	153	173	233	433
3 Months	68	88	128	168	188	248	448
4 Months	83	103	143	183	203	263	463
5 Months	98	118	158	198	218	278	478

i)

ii)

* This question is not analysed and presented in the thesis, because the numbers in the table do not allow the solver to express a first-degree functional relationship.

2) Which of the following are equations for quadratic (second-degree) functions of one variable? Feel free to answer "it depends," and if you do, elaborate on the dependent conditions.

a) $y = ax^2 + bx + c$

b) $y = (1/2)\pi rd$

c) $E = mc^2$

d) $y = e^2$

3) Explain two different solutions of the following problem to your students?
" For which values of m does the equation $m(x - 5) = m + 2x$ have no solutions?"

First Solution

Second Solution

4) What *different* things could $y = 3$ mean if one derives it from $y = mx + 3$.

5) For each of the following expressions, indicate the role of the variable (the underlined letter). Please give your reasons!

If a variable is used as the *name for a number*, write A.
If a variable is used to represent a *specific un-known*, write B.
If a variable is used to represent a *general un-known* or a *pattern generaliser*, write C.
If a variable is used to represent a *varying value*, write D.

a) $y = \underline{x} + 10$, $\underline{x}$ is ____ since
b) $3\underline{p} = 9.42$, $\underline{p}$ is ____ since
c) $7 \bullet \underline{x} = 0$, $\underline{x}$ is ____ since
d) $7 + \underline{x} = 10$, $\underline{x}$ is ____ since
e) $0 \div \underline{n} = 0$, $\underline{n}$ is ____ since

6) What different things might an algebraic expression such as, say $2x+1$, mean? What can x stand for?

7) Find the difference between the larger root and the smaller root of $p^2 - xp + (x^2 - 1)/4 = 0$.

8) If you substitute 1 for x in $ax^2 + bx + c$ (a, b and c are real numbers) you get a positive number. Substituting 6 gives a negative number. How many real solutions does the equation $ax^2 + bx + c = 0$ have? Explain.

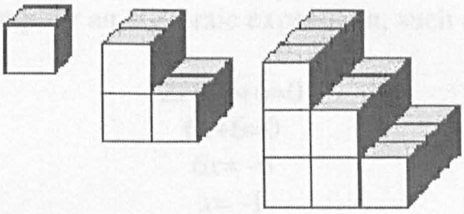
9) For what values of a does the pair of equations

$$x^2 - y^2 = 0$$
$$(x-a)^2 + y^2 = 1$$

have either 0, 2, 3, or 4 solutions?

10) Find the coordinates of the point where the line $m + 2n - 4 = 0$ meets the line $n = 2m - 2x + y$ in the m-n coordinate system.

11) Examine the cube pattern below. (a) How many blocks would be needed to build the nth item?



1. Item 2. Item 3. Item

12) How would you react to your students’ questions as below in the classroom? Explain!

a. “Teacher, why does 2a+5b not equal 7ab?”

b. “While solving equations, why does *x* change its sign when it is brought to the other side?”

13) Ayse was asked to simplify an algebraic expression, such as $2x + 4x + 6$. She wrote :

$$2x+4x+6=0$$
$$6x+6=0$$
$$6x=-6$$
$$x=-1$$

What do you think the student had in mind? Is she right? Explain.

14) How would you introduce your pupils the idea of using letters to represent numbers?

15) According to you which one of these students is correct? How would you help the one who according to you is wrong understand his claim is wrong:

" Ahmet claims that the letter S stands for students in the equation 6S=P which he wrote to represent a fact that "there are six times as many students as professors at this university". Mehmet claims that the letter S stands for the **number** of professors."

16) Which one of the following topics would you teach first? Why? Explain.

- Substituting numbers in expressions
- Simplifying expressions