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# Monochromatic Clique Decompositions of Graphs

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**Abstract:** Let  $G$  be a graph whose edges are colored with  $k$  colors, and  $\mathcal{H} = (H_1, \dots, H_k)$  be a  $k$ -tuple of graphs. A *monochromatic  $\mathcal{H}$ -decomposition* of  $G$  is a partition of the edge set of  $G$  such that each part is either a single edge or forms a monochromatic copy of  $H_i$  in color  $i$ , for some  $1 \leq i \leq k$ . Let  $\phi_k(n, \mathcal{H})$  be the smallest number  $\phi$ , such that, for every order- $n$  graph and every  $k$ -edge-coloring, there is a monochromatic  $\mathcal{H}$ -decomposition with at most  $\phi$  elements. Extending the previous results of Liu and Sousa [Monochromatic  $K_r$ -decompositions of graphs, *J Graph Theory* 76 (2014), 89–100], we solve this problem when each graph in  $\mathcal{H}$  is a clique and  $n \geq n_0(\mathcal{H})$  is sufficiently large. © 2015 The Authors Journal of Graph Theory

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**Keywords:** monochromatic graph decomposition; Turán number; Ramsey number

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## 1. INTRODUCTION

All graphs in this article are finite, undirected, and simple. For standard graph-theoretic terminology the reader is referred to [3].

Given two graphs  $G$  and  $H$ , an  $H$ -decomposition of  $G$  is a partition of the edge set of  $G$  such that each part is either a single edge or forms a subgraph isomorphic to  $H$ . Let  $\phi(G, H)$  be the smallest possible number of parts in an  $H$ -decomposition of  $G$ . It is easy to see that, if  $H$  is nonempty, we have  $\phi(G, H) = e(G) - v_H(G)(e(H) - 1)$ , where  $v_H(G)$  is the maximum number of pairwise edge-disjoint copies of  $H$  that can be packed into  $G$ . Dor and Tarsi [4] showed that if  $H$  has a component with at least three edges then it is NP-complete to determine if a graph  $G$  admits a partition into copies of  $H$ . Thus, it is NP-hard to compute the function  $\phi(G, H)$  for such  $H$ . Nonetheless, many exact results were proved about the extremal function

$$\phi(n, H) = \max\{\phi(G, H) \mid v(G) = n\},$$

which is the smallest number such that any graph  $G$  of order  $n$  admits an  $H$ -decomposition with at most  $\phi(n, H)$  elements.

This function was first studied, in 1966, by Erdős et al. [6], who proved that  $\phi(n, K_3) = t_2(n)$ , where  $K_s$  denotes the complete graph (clique) of order  $s$ , and  $t_{r-1}(n)$  denotes the number of edges in the Turán graph  $T_{r-1}(n)$ , which is the unique  $(r-1)$ -partite graph on  $n$  vertices that has the maximum number of edges. A decade later, Bollobás [2] proved that  $\phi(n, K_r) = t_{r-1}(n)$ , for all  $n \geq r \geq 3$ .

Recently, Pikhurko and Sousa [13] studied  $\phi(n, H)$  for arbitrary graphs  $H$ . Their result is the following.

**Theorem 1.1** [13]. *Let  $H$  be any fixed graph of chromatic number  $r \geq 3$ . Then,*

$$\phi(n, H) = t_{r-1}(n) + o(n^2).$$

Let  $\text{ex}(n, H)$  denote the maximum number of edges in a graph on  $n$  vertices not containing  $H$  as a subgraph. The result of Turán [20] states that  $T_{r-1}(n)$  is the unique extremal graph for  $\text{ex}(n, K_r)$ . The function  $\text{ex}(n, H)$  is usually called the *Turán function* for  $H$ . Pikhurko and Sousa [13] also made the following conjecture.

**Conjecture 1.2** [13]. *For any graph  $H$  of chromatic number  $r \geq 3$ , there exists  $n_0 = n_0(H)$  such that  $\phi(n, H) = \text{ex}(n, H)$  for all  $n \geq n_0$ .*

A graph  $H$  is *edge-critical* if there exists an edge  $e \in E(H)$  such that  $\chi(H) > \chi(H - e)$ , where  $\chi(H)$  denotes the *chromatic number* of  $H$ . For  $r \geq 4$ , a *clique-extension of order  $r$*  is a connected graph that consists of a  $K_{r-1}$  plus another vertex, say  $v$ , adjacent to at most  $r-2$  vertices of  $K_{r-1}$ . Conjecture 1.2 has been verified by Sousa for some edge-critical graphs, namely, clique-extensions of order  $r \geq 4$  ( $n \geq r$ ) [18] and the cycles of length 5 ( $n \geq 6$ ) and 7 ( $n \geq 10$ ) [17, 19]. Later, Özkahya and Person [12] verified the conjecture for all edge-critical graphs with chromatic number  $r \geq 3$ . Their result is the following.

**Theorem 1.3** [12]. *For any edge-critical graph  $H$  with chromatic number  $r \geq 3$ , there exists  $n_0 = n_0(H)$  such that  $\phi(n, H) = \text{ex}(n, H)$ , for all  $n \geq n_0$ . Moreover, the only graph attaining  $\text{ex}(n, H)$  is the Turán graph  $T_{r-1}(n)$ .*

Recently, as an extension of Özkahya and Person's work (and as further evidence supporting Conjecture 1.2), Allen et al. [1] improved the error term obtained by Pikhurko

and Sousa in Theorem 1.1. In fact, they proved that the error term  $o(n^2)$  can be replaced by  $O(n^{2-\alpha})$  for some  $\alpha > 0$ . Furthermore, they also showed that this error term has the correct order of magnitude. Their result is indeed an extension of Theorem 1.3 since the error term  $O(n^{2-\alpha})$  that they obtained vanishes for every edge-critical graph  $H$ .

Motivated by the recent work about  $H$ -decompositions of graphs, a natural problem to consider is the Ramsey (or colored) version of this problem. More precisely, let  $G$  be a graph on  $n$  vertices whose edges are colored with  $k$  colors, for some  $k \geq 2$  and let  $\mathcal{H} = (H_1, \dots, H_k)$  be a  $k$ -tuple of fixed graphs, where repetition is allowed. A *monochromatic  $\mathcal{H}$ -decomposition* of  $G$  is a partition of its edge set such that each part is either a single edge, or forms a monochromatic copy of  $H_i$  in color  $i$ , for some  $1 \leq i \leq k$ . Let  $\phi_k(G, \mathcal{H})$  be the smallest number, such that, for any  $k$ -edge-coloring of  $G$ , there exists a monochromatic  $\mathcal{H}$ -decomposition of  $G$  with at most  $\phi_k(G, \mathcal{H})$  elements. Our goal is to study the function

$$\phi_k(n, \mathcal{H}) = \max\{\phi_k(G, \mathcal{H}) \mid v(G) = n\},$$

which is the smallest number  $\phi$  such that, any  $k$ -edge-colored graph of order  $n$  admits a monochromatic  $\mathcal{H}$ -decomposition with at most  $\phi$  elements. In the case when  $H_i \cong H$  for every  $1 \leq i \leq k$ , we simply write  $\phi_k(G, H) = \phi_k(G, \mathcal{H})$  and  $\phi_k(n, H) = \phi_k(n, \mathcal{H})$ .

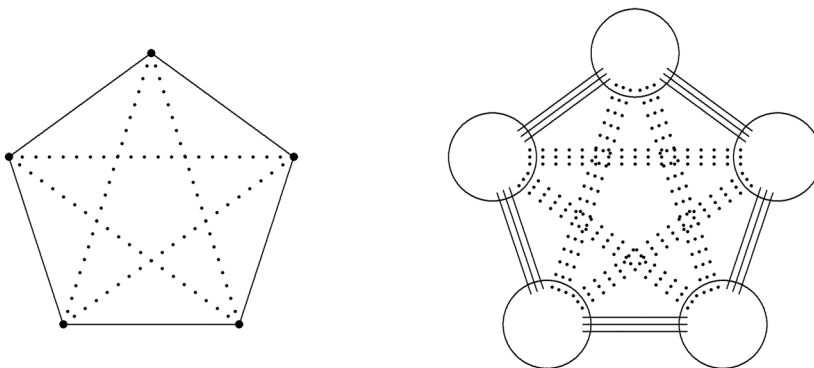
The function  $\phi_k(n, K_r)$ , for  $k \geq 2$  and  $r \geq 3$ , has been studied by Liu and Sousa [11], who obtained results involving the Ramsey numbers and the Turán numbers. Recall that for  $k \geq 2$  and integers  $r_1, \dots, r_k \geq 3$ , the *Ramsey number for  $K_{r_1}, \dots, K_{r_k}$* , denoted by  $R(r_1, \dots, r_k)$ , is the smallest value of  $s$ , such that, for every  $k$ -edge-coloring of  $K_s$ , there exists a monochromatic  $K_{r_i}$  in color  $i$ , for some  $1 \leq i \leq k$ . For the case when  $r_1 = \dots = r_k = r$ , for some  $r \geq 3$ , we simply write  $R_k(r) = R(r_1, \dots, r_k)$ . Since  $R(r_1, \dots, r_k)$  does not change under any permutation of  $r_1, \dots, r_k$ , without loss of generality, we assume throughout that  $3 \leq r_1 \leq \dots \leq r_k$ . The Ramsey numbers are notoriously difficult to calculate, even though, it is known that their values are finite [15]. To this date, the values of  $R(3, r_2)$  have been determined exactly only for  $3 \leq r_2 \leq 9$ , and these are shown in the following table [14].

$r_2$	3	4	5	6	7	8	9
$R(3, r_2)$	6	9	14	18	23	28	36

The remaining Ramsey numbers that are known exactly are  $R(4, 4) = 18$ ,  $R(4, 5) = 25$ , and  $R(3, 3, 3) = 17$ . The gap between the lower bound and the upper bound for other Ramsey numbers is generally quite large.

For the case  $R(3, 3) = 6$ , it is easy to see that the only 2-edge-coloring of  $K_5$  not containing a monochromatic  $K_3$  is the one where each color induces a cycle of length 5. From this 2-edge-coloring, observe that we may take a “blow-up” to obtain a 2-edge-coloring of the Turán graph  $T_5(n)$ , and easily deduce that  $\phi_2(n, K_3) \geq t_5(n)$ . See Figure 1.

This example was the motivation for Liu and Sousa [11] to study  $K_r$ -monochromatic decompositions of graphs, for  $r \geq 3$  and  $k \geq 2$ . They have recently proved the following result.

FIGURE 1. The 2-edge-coloring of  $K_5$ , and its blow-up

**Theorem 1.4** [11].

- (a)  $\phi_k(n, K_3) = t_{R_k(3)-1}(n) + o(n^2)$ ;
- (b)  $\phi_k(n, K_3) = t_{R_k(3)-1}(n)$  for  $k = 2, 3$  and  $n$  sufficiently large;
- (c)  $\phi_k(n, K_r) = t_{R_k(r)-1}(n)$ , for  $k \geq 2$ ,  $r \geq 4$  and  $n$  sufficiently large.

Moreover, the only graph attaining  $\phi_k(n, K_r)$  in cases (b) and (c) is the Turán graph  $T_{R_k(r)-1}(n)$ .

They also made the following conjecture.

**Conjecture 1.5** [11]. Let  $k \geq 4$ . Then  $\phi_k(n, K_3) = t_{R_k(3)-1}(n)$  for  $n \geq R_k(3)$ .

Here, we will study an extension of the monochromatic  $K_r$ -decomposition problem when the clique  $K_r$  is replaced by a fixed  $k$ -tuple of cliques  $\mathcal{C} = (K_{r_1}, \dots, K_{r_k})$ . Our main result, stated in Theorem 1.6, is clearly an extension of Theorem 1.4. Also, it verifies Conjecture 1.5 for sufficiently large  $n$ .

**Theorem 1.6.** Let  $k \geq 2$ ,  $3 \leq r_1 \leq \dots \leq r_k$ , and  $R = R(r_1, \dots, r_k)$ . Let  $\mathcal{C} = (K_{r_1}, \dots, K_{r_k})$ . Then, there is an  $n_0 = n_0(r_1, \dots, r_k)$  such that, for all  $n \geq n_0$ , we have

$$\phi_k(n, \mathcal{C}) = t_{R-1}(n).$$

Moreover, the only order- $n$  graph attaining  $\phi_k(n, \mathcal{C})$  is the Turán graph  $T_{R-1}(n)$  (with a  $k$ -edge-coloring that does not contain a color- $i$  copy of  $K_{r_i}$  for any  $1 \leq i \leq k$ ).

The upper bound of Theorem 1.6 is proved in Section 2. The lower bound follows easily by the definition of the Ramsey number. Indeed, take a  $k$ -edge-coloring  $f'$  of the complete graph  $K_{R-1}$  without a monochromatic  $K_{r_i}$  in color  $i$ , for all  $1 \leq i \leq k$ . Note that  $f'$  exists by definition of the Ramsey number  $R = R(r_1, \dots, r_k)$ . Let  $u_1, \dots, u_{R-1}$  be the vertices of the  $K_{R-1}$ . Now, consider the Turán graph  $T_{R-1}(n)$  with a  $k$ -edge-coloring  $f$  that is a “blow-up” of  $f'$ . That is, if  $T_{R-1}(n)$  has partition classes  $V_1, \dots, V_{R-1}$ , then for  $v \in V_j$  and  $w \in V_\ell$  with  $j \neq \ell$ , we define  $f(vw) = f'(u_j u_\ell)$ . Then,  $T_{R-1}(n)$  with this  $k$ -edge-coloring has no monochromatic  $K_{r_i}$  in color  $i$ , for every  $1 \leq i \leq k$ . Therefore,  $\phi_k(n, \mathcal{C}) \geq \phi_k(T_{R-1}(n), \mathcal{C}) = t_{R-1}(n)$  and the lower bound in Theorem 1.6 follows.

In particular, when all the cliques in  $\mathcal{C}$  are equal, Theorem 1.6 completes the results obtained previously by Liu and Sousa in Theorem 1.4. In fact, we get the following direct corollary from Theorem 1.6.

**Corollary 1.7.** *Let  $k \geq 2$ ,  $r \geq 3$  and  $n$  be sufficiently large. Then,*

$$\phi_k(n, K_r) = t_{R_k(r)-1}(n).$$

*Moreover, the only order- $n$  graph attaining  $\phi_k(n, K_r)$  is the Turán graph  $T_{R_k(r)-1}(n)$  (with a  $k$ -edge-coloring that does not contain a monochromatic copy of  $K_r$ ).*

## 2. PROOF OF THEOREM 1.6

In this section, we will prove the upper bound in Theorem 1.6. Before presenting the proof we need to introduce the tools. Throughout this section, let  $k \geq 2$ ,  $3 \leq r_1 \leq \dots \leq r_k$  be an increasing sequence of integers,  $R = R(r_1, \dots, r_k)$  be the Ramsey number for  $K_{r_1}, \dots, K_{r_k}$ , and  $\mathcal{C} = (K_{r_1}, \dots, K_{r_k})$  be a fixed  $k$ -tuple of cliques.

We first recall the following stability theorem of Erdős and Simonovits [5, 16].

**Theorem 2.1** (Stability Theorem [5,16]). *Let  $r \geq 3$ , and  $G$  be a graph on  $n$  vertices with  $e(G) \geq t_{r-1}(n) + o(n^2)$  and not containing  $K_r$  as a subgraph. Then, there exists an  $(r-1)$ -partite graph  $G'$  on  $n$  vertices with partition classes  $V_1, \dots, V_{r-1}$ , where  $|V_i| = \frac{n}{r-1} + o(n)$  for  $1 \leq i \leq r-1$ , that can be obtained from  $G$  by adding and subtracting  $o(n^2)$  edges.*

Next, we recall the following result of Győri [7, 8] about the existence of edge-disjoint copies of  $K_r$  in graphs on  $n$  vertices with more than  $t_{r-1}(n)$  edges.

**Theorem 2.2** [7,8]. *For every  $r \geq 3$  there is  $C$  such that every graph  $G$  with  $n \geq C$  vertices and  $e(G) = t_{r-1}(n) + m$  edges, where  $m \leq \binom{n}{2}/C$ , contains at least  $m - Cm^2/n^2$  edge-disjoint copies of  $K_r$ .*

Now, we will consider coverings and packings of cliques in graphs. Let  $r \geq 3$  and  $G$  be a graph. Let  $\mathcal{K}$  be the set of all  $K_r$ -subgraphs of  $G$ . A  $K_r$ -cover is a set of edges of  $G$  meeting all elements in  $\mathcal{K}$ , that is, the removal of a  $K_r$ -cover results in a  $K_r$ -free graph. A  $K_r$ -packing in  $G$  is a set of pairwise edge-disjoint copies of  $K_r$ . The  $K_r$ -covering number of  $G$ , denoted by  $\tau_r(G)$ , is the minimum size of a  $K_r$ -cover of  $G$ , and the  $K_r$ -packing number of  $G$ , denoted by  $\nu_r(G)$ , is the maximum size of a  $K_r$ -packing of  $G$ . Next, a fractional  $K_r$ -cover of  $G$  is a function  $f : E(G) \rightarrow \mathbb{R}_+$ , such that  $\sum_{e \in E(H)} f(e) \geq 1$  for every  $H \in \mathcal{K}$ , that is, for every copy of  $K_r$  in  $G$  the sum of the values of  $f$  on its edges is at least 1. A fractional  $K_r$ -packing of  $G$  is a function  $p : \mathcal{K} \rightarrow \mathbb{R}_+$  such that  $\sum_{H \in \mathcal{K}: e \in E(H)} p(H) \leq 1$  for every  $e \in E(G)$ , that is, the total weight of  $K_r$ 's that cover any edge is at most 1. Here,  $\mathbb{R}_+$  denotes the set of nonnegative real numbers. The fractional  $K_r$ -covering number of  $G$ , denoted by  $\tau_r^*(G)$ , is the minimum of  $\sum_{e \in E(G)} f(e)$  over all fractional  $K_r$ -covers  $f$ , and the fractional  $K_r$ -packing number of  $G$ , denoted by  $\nu_r^*(G)$ , is the maximum of  $\sum_{H \in \mathcal{K}} p(H)$  over all fractional  $K_r$ -packings  $p$ .

One can easily observe that

$$\nu_r(G) \leq \tau_r(G) \leq \binom{r}{2} \nu_r(G).$$

For  $r = 3$ , we have  $\tau_3(G) \leq 3\nu_3(G)$ . A long-standing conjecture of Tuza [21] from 1981 states that this inequality can be improved as follows.

**Conjecture 2.3** [21]. *For every graph  $G$ , we have  $\tau_3(G) \leq 2v_3(G)$ .*

Conjecture 2.3 remains open although many partial results have been proved. By using the earlier results of Krivelevich [10], and Haxell and Rödl [9], Yuster [22] proved the following theorem which will be crucial to the proof of Theorem 1.6. In the case  $r = 3$ , it is an asymptotic solution of Tuza's conjecture.

**Theorem 2.4** [22]. *Let  $r \geq 3$  and  $G$  be a graph on  $n$  vertices. Then*

$$\tau_r(G) \leq \left\lfloor \frac{r^2}{4} \right\rfloor v_r(G) + o(n^2). \quad (1)$$

We now prove the following lemma that states that a graph  $G$  with  $n$  vertices and at least  $t_{R-1}(n) + \Omega(n^2)$  edges falls quite short of being optimal.

**Lemma 2.5.** *For every  $k \geq 2$  and  $c_0 > 0$  there are  $c_1 > 0$  and  $n_0$  such that for every graph  $G$  of order  $n \geq n_0$  with at least  $t_{R-1}(n) + c_0 n^2$  edges, we have  $\phi_k(G, \mathcal{C}) \leq t_{R-1}(n) - c_1 n^2$ .*

**Proof.** Suppose that the lemma is false, that is, there is  $c_0 > 0$  such that for some increasing sequence of  $n$  there is a graph  $G$  on  $n$  vertices with  $e(G) \geq t_{R-1}(n) + c_0 n^2$  and  $\phi_k(G, \mathcal{C}) \geq t_{R-1}(n) + o(n^2)$ . Fix a  $k$ -edge-coloring of  $G$  and, for  $1 \leq i \leq k$ , let  $G_i$  be the subgraph of  $G$  on  $n$  vertices that contains all edges with color  $i$ .

Let  $m = e(G) - t_{R-1}(n)$ , and let  $s \in \{0, \dots, k\}$  be the maximum such that

$$r_1 = \dots = r_s = 3.$$

Let us very briefly recall the argument from [11] that shows  $\phi_k(G, \mathcal{C}) \leq t_{R-1}(n) + o(n^2)$ , adopted to our purposes. If we remove a  $K_{r_i}$ -cover from  $G_i$  for every  $1 \leq i \leq k$ , then we destroy all copies of  $K_R$  in  $G$ . By Turán's theorem, at most  $t_{R-1}(n)$  edges remain. Thus,

$$\sum_{i=1}^k \tau_{r_i}(G_i) \geq m. \quad (2)$$

By Theorem 2.4, if we decompose  $G$  into a maximum  $K_{r_i}$ -packing in each  $G_i$  and the remaining edges, we obtain that

$$\begin{aligned} \phi_k(G, \mathcal{C}) &\leq e(G) - \sum_{i=1}^k \left( \binom{r_i}{2} - 1 \right) v_{r_i}(G_i) \\ &\leq t_{R-1}(n) + m - \sum_{i=1}^k \frac{\binom{r_i}{2} - 1}{\lfloor r_i^2/4 \rfloor} \tau_{r_i}(G_i) + o(n^2) \\ &\leq t_{R-1}(n) + m - \sum_{i=1}^k \tau_{r_i}(G_i) - \frac{1}{4} \sum_{i=s+1}^k \tau_{r_i}(G_i) + o(n^2) \leq t_{R-1}(n) + o(n^2). \end{aligned} \quad (3)$$

The third inequality holds since  $(\binom{r}{2} - 1)/\lfloor r^2/4 \rfloor \geq 5/4$  for  $r \geq 4$  and is equal to 1 for  $r = 3$ .



Let us derive a contradiction from this by looking at the properties of our hypothetical counterexample  $G$ . First, all inequalities that we saw have to be equalities within an additive term  $o(n^2)$ . In particular, the slack in (2) is  $o(n^2)$ , that is,

$$\sum_{i=1}^k \tau_{r_i}(G_i) = m + o(n^2). \quad (4)$$

Also,  $\sum_{i=s+1}^k \tau_{r_i}(G_i) = o(n^2)$ . In particular, we have that  $s \geq 1$ . To simplify the later calculations, let us redefine  $G$  by removing a maximum  $K_{r_i}$ -packing from  $G_i$  for each  $i \geq s+1$ . The new graph is still a counterexample to the lemma if we decrease  $c_0$  slightly, since the number of edges removed is at most  $\sum_{i=s+1}^k \binom{r_i}{2} \tau_{r_i}(G_i) = o(n^2)$ .

Suppose that we remove, for each  $i \leq s$ , an arbitrary (not necessarily minimum)  $K_3$ -cover  $F_i$  from  $G_i$  such that

$$\sum_{i=1}^s |F_i| \leq m + o(n^2). \quad (5)$$

Let  $G' \subseteq G$  be the obtained  $K_R$ -free graph. (Recall that we assumed that  $G_i$  is  $K_{r_i}$ -free for all  $i \geq s+1$ .) Let  $G'_i \subseteq G_i$  be the color classes of  $G'$ . We know by (5) that  $e(G') \geq t_{R-1}(n) + o(n^2)$ . Since  $G'$  is  $K_R$ -free, we conclude by the Stability Theorem (Theorem 2.1) that there is a partition  $V(G) = V(G') = V_1 \dot{\cup} \dots \dot{\cup} V_{R-1}$  such that

$$\forall i \in \{1, \dots, R-1\}, \quad |V_i| = \frac{n}{R-1} + o(n) \quad \text{and} \quad |E(T) \setminus E(G')| = o(n^2), \quad (6)$$

where  $T$  is the complete  $(R-1)$ -partite graph with parts  $V_1, \dots, V_{R-1}$ .

Next, we essentially expand the proof of (1) for  $r = 3$  and transform it into an algorithm that produces  $K_3$ -coverings  $F_i$  of  $G_i$ , with  $1 \leq i \leq s$ , in such a way that (5) holds but (6) is impossible whatever  $V_1, \dots, V_{R-1}$  we take, giving the desired contradiction.

Let  $H$  be an arbitrary graph of order  $n$ . By the LP duality, we have that

$$\tau_r^*(H) = v_r^*(H). \quad (7)$$

By the result of Haxell and Rödl [9] we have that

$$v_r^*(H) = v_r(H) + o(n^2). \quad (8)$$

Krivelevich [10] showed that

$$\tau_3(H) \leq 2\tau_3^*(H). \quad (9)$$

Thus,  $\tau_3(H) \leq 2v_3(H) + o(n^2)$  giving (1) for  $r = 3$ .

The proof of Krivelevich [10] of (9) is based on the following result.

**Lemma 2.6.** *Let  $H$  be an arbitrary graph and  $f : E(H) \rightarrow \mathbb{R}_+$  be a minimum fractional  $K_3$ -cover. Then  $\tau_3(H) \leq \frac{3}{2} \tau_3^*(H)$  or there is  $xy \in E(H)$  with  $f(xy) = 0$  that belongs to at least one triangle of  $H$ .*

**Proof.** If there is an edge  $xy \in E(H)$  that does not belong to a triangle, then necessarily  $f(xy) = 0$  and  $xy$  does not belong to any optimal fractional or integer  $K_3$ -cover. We can remove  $xy$  from  $E(H)$  without changing the validity of the lemma. Thus, we can assume that every edge of  $H$  belongs to a triangle.

Suppose that  $f(xy) > 0$  for every edge  $xy$  of  $H$ , for otherwise we are done. Take a maximum fractional  $K_3$ -packing  $p$ . Recall that it is a function that assigns a weight



$p(xyz) \in \mathbb{R}_+$  to each triangle  $xyz$  of  $H$  such that for every edge  $xy$  the sum of weights over all  $K_3$ 's of  $H$  containing  $xy$  is at most 1, that is,

$$\sum_{z \in \Gamma(x) \cap \Gamma(y)} p(xyz) \leq 1, \quad (10)$$

where  $\Gamma(v)$  denotes the set of neighbors of the vertex  $v$  in  $H$ .

This is the dual LP to the minimum fractional  $K_3$ -cover problem. By the complementary slackness condition (since  $f$  and  $p$  are optimal solutions), we have equality in (10) for every  $xy \in E(H)$ . This and the LP duality imply that

$$\tau_3^*(H) = \nu_3^*(H) = \sum_{\text{triangle } xyz} p(xyz) = \frac{1}{3} \sum_{xy \in E(H)} \sum_{z \in \Gamma(x) \cap \Gamma(y)} p(xyz) = \frac{1}{3} e(H).$$

On the other hand  $\tau_3(H) \leq \frac{1}{2} e(H)$ : take a bipartite subgraph of  $H$  with at least half of the edges; then the remaining edges form a  $K_3$ -cover. Putting the last two inequalities together, we obtain the required result. ■

Let  $1 \leq i \leq s$ . We now describe an algorithm for finding a  $K_3$ -cover  $F_i$  in  $G_i$ . Initially, let  $H = G_i$  and  $F_i = \emptyset$ . Repeat the following.

Take a minimum fractional  $K_3$ -cover  $f$  of  $H$ . If the first alternative of Lemma 2.6 is true, pick a  $K_3$ -cover of  $H$  of size at most  $\frac{3}{2} \tau_3^*(H)$ , add it to  $F_i$  and stop. Otherwise, fix some edge  $xy \in E(H)$  returned by Lemma 2.6. Let  $F'$  consist of all pairs  $xz$  and  $yz$  over  $z \in \Gamma(x) \cap \Gamma(y)$ . Add  $F'$  to  $F_i$  and remove  $F'$  from  $E(H)$ . Repeat the whole step (with the new  $H$  and  $f$ ).

Consider any moment during this algorithm, when we had  $f(xy) = 0$  for some edge  $xy$  of  $H$ . Since  $f$  is a fractional  $K_3$ -cover, we have that  $f(xz) + f(yz) \geq 1$  for every  $z \in \Gamma(x) \cap \Gamma(y)$ . Thus, if  $H'$  is obtained from  $H$  by removing  $2\ell$  such pairs, where  $\ell = |\Gamma(x) \cap \Gamma(y)|$ , then  $\tau_3^*(H') \leq \tau_3^*(H) - \ell$  because  $f$  when restricted to  $E(H')$  is still a fractional cover (although not necessarily an optimal one). Clearly,  $|F_i|$  increases by  $2\ell$  during this operation. Thus, indeed we obtain, at the end, a  $K_3$ -cover  $F_i$  of  $G_i$  of size at most  $2\tau_3^*(G_i)$ .

Also, by (7) and (8) we have that

$$\sum_{i=1}^s |F_i| \leq 2 \sum_{i=1}^s \nu_3(G_i) + o(n^2).$$

Now, since all slacks in (3) are  $o(n^2)$ , we conclude that

$$\sum_{i=1}^s \nu_3(G_i) \leq \frac{m}{2} + o(n^2)$$

and (5) holds. In fact, (5) is equality by (4).

Recall that  $G'_i$  is obtained from  $G_i$  by removing all edges of  $F_i$  and  $G'$  is the edge-disjoint union of the graphs  $G'_i$ . Suppose that there exist  $V_1, \dots, V_{R-1}$  satisfying (6). Let  $M = E(T) \setminus E(G')$  consist of *missing* edges. Thus,  $|M| = o(n^2)$ .

Let

$$X = \{x \in V(T) \mid \deg_M(x) \geq c_2 n\},$$

where we define  $c_2 = (4(R-1))^{-1}$ . Clearly,

$$|X| \leq 2|M|/c_2n = o(n).$$

Observe that, for every  $1 \leq i \leq s$ , if the first alternative of Lemma 2.6 holds at some point, then the remaining graph  $H$  satisfies  $\tau_3^*(H) = o(n^2)$ . Indeed, otherwise by  $\tau_3(G_i) \leq 2\tau_3^*(G_i) - \tau_3^*(H)/2 + o(n^2)$  we get a strictly smaller constant than 2 in (9) and thus a gap of  $\Omega(n^2)$  in (3), a contradiction. Therefore, all but  $o(n^2)$  edges in  $F_i$  come from some *parent edge*  $xy$  that had  $f$ -weight 0 at some point.

When our algorithm adds pairs  $xz$  and  $yz$  to  $F_i$  with the same parent  $xy$ , then it adds the same number of pairs incident to  $x$  as those incident to  $y$ . Let  $\mathcal{P}$  consist of pairs  $xy$  that are disjoint from  $X$  and were a parent edge during the run of the algorithm. Since the total number of pairs in  $F_i$  incident to  $X$  is at most  $n|X| = o(n^2)$ , there are  $|F_i| - o(n^2)$  pairs in  $F_i$  such that their parent is in  $\mathcal{P}$ .

Let us show that  $y_0$  and  $y_1$  belong to different parts  $V_j$  for every pair  $y_0y_1 \in \mathcal{P}$ . Suppose on the contrary that, say,  $y_0, y_1 \in V_1$ . For each  $2 \leq j \leq R-1$  pick an arbitrary  $y_j \in V_j \setminus (\Gamma_M(y_0) \cup \Gamma_M(y_1))$ . Since  $y_0, y_1 \notin X$ , the possible number of choices for  $y_j$  is at least

$$\frac{n}{R-1} - 2c_2n + o(n) \geq \frac{n}{R-1} - 3c_2n.$$

Let

$$Y = \{y_0, \dots, y_{R-1}\}.$$

By the above, we have at least  $(\frac{n}{R-1} - 3c_2n)^{R-2} = \Omega(n^{R-2})$  choices of  $Y$ . Note that by the definition, all edges between  $\{y_0, y_1\}$  and the rest of  $Y$  are present in  $E(G')$ . Thus, the number of sets  $Y$  containing at least one edge of  $M$  different from  $y_0y_1$  is at most

$$|M| \times n^{R-4} = o(n^{R-2}).$$

This is  $o(1)$  times the number of choices of  $Y$ . Thus, for almost every  $Y$ ,  $H = G'[Y]$  is a clique (except perhaps the pair  $y_0y_1$ ). In particular, there is at least one such choice of  $Y$ ; fix it. Let  $i \in \{1, \dots, k\}$  be arbitrary. Adding back the pair  $y_0y_1$  colored  $i$  to  $H$  (if it is not there already), we obtain a  $k$ -edge-coloring of the complete graph  $H$  of order  $R$ . By the definition of  $R = R(r_1, \dots, r_k)$ , there must be a monochromatic triangle on  $abc$  of color  $h \leq s$ . (Recall that we assumed at the beginning that  $G_j$  is  $K_{r_j}$ -free for each  $j > s$ .) But  $abc$  has to contain an edge from the  $K_3$ -cover  $F_h$ , say  $ab$ . This edge  $ab$  is not in  $G'$  (it was removed from  $G$ ). If  $a, b$  lie in different parts  $V_j$ , then  $ab \in M$ , a contradiction to the choice of  $Y$ . The only possibility is that  $ab = y_0y_1$ . Then  $h = i$ . Since both  $y_0c$  and  $y_1c$  are in  $G'_i$ , they were never added to the  $K_3$ -cover  $F_i$  by our algorithm. Therefore,  $y_0y_1$  was never a parent, which is the desired contradiction.

Thus, every  $xy \in \mathcal{P}$  connects two different parts  $V_j$ . For every such parent  $xy$ , the number of its children in  $M$  is at least half of all its children. Indeed, for every pair of children  $xz$  and  $yz$ , at least one connects two different parts; this child necessarily belongs to  $M$ . Thus,

$$|F_i \cap M| \geq \frac{1}{2} |F_i| + o(n^2).$$

(Recall that parent edges that intersect  $X$  produce at most  $2n|X| = o(n^2)$  children.) Therefore,

$$|M| \geq \frac{1}{2} \sum_{i=1}^s |F_i| + o(n^2) \geq \frac{m}{2} + o(n^2) = \Omega(n^2),$$

contradicting (6). This contradiction proves Lemma 2.5. ■

We are now able to prove Theorem 1.6.

**Proof of the upper bound in Theorem 1.6.** Let  $C$  be the constant returned by Theorem 2.2 for  $r = R$ . Let  $n_0 = n_0(r_1, \dots, r_k)$  be sufficiently large to satisfy all the inequalities we will encounter. Let  $G$  be a  $k$ -edge-colored graph on  $n \geq n_0$  vertices. We will show that  $\phi_k(G, \mathcal{C}) \leq t_{R-1}(n)$  with equality if and only if  $G = T_{R-1}(n)$ , and  $G$  does not contain a monochromatic copy of  $K_{r_i}$  in color  $i$  for every  $1 \leq i \leq k$ .

Let  $e(G) = t_{R-1}(n) + m$ , where  $m$  is an integer. If  $m < 0$ , we can decompose  $G$  into single edges and there is nothing to prove.

Suppose  $m = 0$ . If  $G$  contains a monochromatic copy of  $K_{r_i}$  in color  $i$  for some  $1 \leq i \leq k$ , then  $G$  admits a monochromatic  $\mathcal{C}$ -decomposition with at most  $t_{R-1}(n) - \binom{r_i}{2} + 1 < t_{R-1}(n)$  parts and we are done. Otherwise, the definition of  $R$  implies that  $G$  does not contain a copy of  $K_R$ . Therefore,  $G = T_{R-1}(n)$  by Turán's theorem and  $\phi_k(G, \mathcal{C}) = t_{R-1}(n)$  as required.

Now suppose  $m > 0$ . We can also assume that  $m < \binom{n}{2}/C$  for otherwise we are done:  $\phi_k(G, \mathcal{C}) < t_{R-1}(n)$  by Lemma 2.5. Thus, by Theorem 2.2, the graph  $G$  contains at least  $m - Cm^2/n^2 > \frac{m}{2}$  edge-disjoint copies of  $K_R$ . Since each  $K_R$  contains a monochromatic copy of  $K_{r_i}$  in the color- $i$  graph  $G_i$ , for some  $1 \leq i \leq k$ , we conclude that  $\sum_{i=1}^k v_{r_i}(G_i) > \frac{m}{2}$ , so that  $\sum_{i=1}^k \left(\binom{r_i}{2} - 1\right) v_{r_i}(G_i) \geq \sum_{i=1}^k 2v_{r_i}(G_i) > m$ . We have

$$\phi_k(G, \mathcal{C}) = e(G) - \sum_{i=1}^k \binom{r_i}{2} v_{r_i}(G_i) + \sum_{i=1}^k v_{r_i}(G_i) < t_{R-1}(n),$$

giving the required. ■

**Remark.** By analyzing the above argument, one can also derive the following stability property for every fixed family  $\mathcal{C}$  of cliques as  $n \rightarrow \infty$ : every graph  $G$  on  $n$  vertices with  $\phi_k(G, \mathcal{C}) = t_{R-1}(n) + o(n^2)$  is  $o(n^2)$ -close to the Turán graph  $T_{R-1}(n)$  in the edit distance.

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