Submanifold Bridge Processes

by

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Acknowledgments

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Declaration

This thesis is submitted to the University of Warwick in support of my application for the degree of Doctor of Philosophy. It has been composed by myself in accordance with the regulations of the University of Warwick and has not been submitted for a degree at any other university.

Abstract

We introduce and study *submanifold bridge processes*. Our method involves proving a general formula for the integral over a submanifold of the minimal heat kernel on a complete Riemannian manifold. Our formula expresses this object in terms of a stochastic process whose trajectories terminate on the submanifold at a fixed positive time. We study this process and use the formula to derive lower bounds, an asymptotic relation and derivative estimates. Using these results we introduce and characterize *Brownian bridges to submanifolds.* Before doing so we prove necessary estimates on the Laplacian of the distance function and define a notion of local time on a hypersurface. These preliminary developments also lead to a study of the distance between Brownian motion and a submanifold, in which we prove exponential bounds and concentration inequalities. This work is motivated by the desire to extend the analysis of path and loop space to measures on paths which terminate on a submanifold.

Introduction

Brownian motion is an important stochastic process which can be naturally associated to any Riemannian manifold. The Brownian bridge is given by conditioning Brownian motion to hit a fixed point at a fixed positive time. We extend this concept by replacing the fixed point with a submanifold. More generally, we investigate *submanifold bridge processes*, by which we mean Brownian motions with drift which arrive in a fixed submanifold at a fixed positive time.

We hope this work will lead to a future study of the space of continuous paths which end on a submanifold, shedding light on the relationship between the geometry of the path space, the intrinsic geometry of the ambient manifold and the extrinsic geometry of the submanifold.

For the present study, the ambient manifold M will be a complete and connected Riemannian manifold of dimension m. While dealing with stochastic incompleteness is an important part of this thesis, let us assume for this introduction that M is compact. To complete the set-up, we suppose also that N is a closed embedded submanifold of M of dimension $n \in \{0, \ldots, m-1\}$ and we fix a positive time T.

Our first example of a submanifold bridge process, given in terms of the distance function $r_N(\cdot) := d(\cdot, N)$, is then the diffusion on M starting at x with time dependent generator

$$\frac{1}{2}\triangle - \frac{r_N}{T-t}\frac{\partial}{\partial r_N}$$

where $\frac{\partial}{\partial r_N}$ denotes differentiation in the radial direction. We refer to it as the *Fermi bridge between x and N in time T* and it coincides with the usual Brownian bridge if N is a point in $\mathbb{R}^m$. We study the *radial* part of the Fermi bridge in Chapter 4 and use it to access information about the heat kernel and its integral over the submanifold.

This leads to our next example of a submanifold bridge process. If we denote by $X(x)$ a Brownian motion on M starting at x and by p^M the heat kernel of M, then we define the integrated heat kernel by

$$p_t^M(x,N) := \int_N p_t^M(x,y)\, d\,\mathrm{vol}_N(y)$$

and prove that if $t \in [0,T)$ then for a bounded $\mathcal{F}_t^{X(x)}$-measurable random variable F we have

$$\mathbb{E}\left[F|X_T(x) \in N\right] = \frac{\mathbb{E}\left[p_{T-t}^M(X_t(x),N)F\right]}{p_T^M(x,N)}. \tag{1}$$

This gives rise to a diffusion on the time interval $[0,T)$ starting at x and arriving in N at time T, with time-dependent infinitesimal generator

$$\frac{1}{2}\triangle + \nabla \log p_{T-t}^M(\cdot, N).$$

We call this a *Brownian bridge to a submanifold* and study it in Chapter 6. To show that it is a semimartingale on $[0,T]$ we prove the gradient estimate

$$\|\nabla \log p_t^M(x,N)\|^2 \leq C\left(\frac{1}{t} + \frac{n}{t}\log\frac{1}{t} + \frac{d^2(x,N)}{t^2}\right) \tag{2}$$

and derive a Hessian estimate as corollary. These estimates are the main results in Chapter 6 and are given by Theorem 6.3.2 and Corollary 6.3.4.

A Heat Kernel Formula

We prove (2) using Bismut's formula and a lower bound on the integrated heat kernel. To deduce this lower bound we must first prove another of our main results, Theorem 5.2.1, which in terms of a Fermi bridge $\hat{X}(x)$ states that

$$p_T^M(x,N) = (2\pi t)^{-\frac{(m-n)}{2}} \exp\left[-\frac{d^2(x,N)}{2t}\right] \lim_{t\uparrow T} \mathbb{E}\left[\exp\left[\int_0^t \frac{r_N(\hat{X}_s(x))}{T-s}\Big(d\mathbf{A}_s + d\mathbf{L}_s\Big)\right]\right] \quad (3)$$

where $d\mathbf{A}$ denotes an absolutely continuous random measure given in terms of Jacobi fields while $d\mathbf{L}$ denotes a singular continuous random measure given in terms of the local time of $\hat{X}(x)$ on the cut locus of N. Formula (3) extends the formulae of Elworthy and Truman [1982] and Ndumu [1989]. The lower bound we derive from (3) is of the form

$$p_t^M(x,N) \geq Ct^{-\frac{(m-n)}{2}} \exp\left[-\frac{d^2(x,N)}{2t}\right] \quad (4)$$

which is stated more generally in Theorem 5.3.2. Combining this lower bound with a suitable upper bound we also prove the asymptotic relation

$$\lim_{t\downarrow 0} t \log p_t^M(x,N) = -\frac{d^2(x,N)}{2} \quad (5)$$

which is stated in Theorem 5.3.8. An exact expansion for $p_t^M(x,N)$ away from the cut locus was previously calculated by Ndumu [2011] using a submanifold bridge process called the *semiclassical bridge* which we also discuss in Chapter 4.

Local Time on a Hypersurface

The role of local time in formula (3) is to take in account the effect of the cut locus and Chapter 2 includes an investigation of the notion of local time on a hypersurface. This is based upon the work of Barden and Le [1995], who generalized Cranston, Kendall and March [1993]. In particular, we prove a Tanaka formula

$$dr_N(X_t(x)) = d\beta_t + \frac{1}{2}\triangle r_N(X_t(x))dt - d\mathbb{L}_t^{\mathrm{Cut}(N)}(X(x)) + dL_t^N(X(x)). \quad (6)$$

The local time $L^N(X(x))$ of $X(x)$ on N, given by the (symmetric) local time of $r_N(X(x))$ at zero, vanishes if $n \leq m-2$ while if $n = m-1$ we prove the occupation times approximation

$$L_t^N(X(x)) = \lim_{\epsilon \downarrow 0} \frac{1}{2\epsilon} \int_0^t \mathbf{1}_{B_\epsilon(N)}(X_s(x))\, ds. \tag{7}$$

We also deduce Theorem 2.4.1, which states that

$$\mathbb{E}\left[L_t^N(X(x))\right] = \int_0^t p_s^M(x, N)\, ds \tag{8}$$

which leads to the relation

$$\lim_{t \uparrow \infty} \frac{1}{t} \mathbb{E}\left[L_t^N(X(x))\right] = \frac{\mathrm{vol}_N(N)}{\mathrm{vol}_M(M)}. \tag{9}$$

A Laplacian Inequality

The Jacobi fields in formula (3) take into account the geometry of M in between N and the cut locus. This component of the theory is considered in Chapter 1 using the comparison theorem of Heintze and Karcher [1978]. Although for this introduction we have assumed that M is compact, the majority of our formulae and estimates do not require this assumption. We will usually only require the existence of constants $\nu \geq 1$ and $\lambda \in \mathbb{R}$ such that the Lyapunov condition

$$\frac{1}{2} \triangle r_N^2 \leq \nu + \lambda r_N^2 \tag{10}$$

holds off the cut locus. Giving geometric meaning to this assumption is a key objective in Chapter 1, the main result being Theorem 1.4.5. In particular, suppose that there exists a function $\kappa : [0, \infty) \to [0, \infty)$ such that one of the following conditions is satisfied off N and its cut locus:

(C1) $n \in \{0, \ldots, m-1\}$, the sectional curvature of planes containing the radial direction is bounded below by $-\kappa^2(r_N)$ and the absolute value of the principal curvature of N is bounded by a constant $\Lambda \geq 0$;

(C2) $n = 0$ and the Ricci curvature in the radial direction is bounded below by $-(m-1)\kappa^2(r_N)$;

(C3) $n = m - 1$, the Ricci curvature in the radial direction is bounded below by $-(m-1)\kappa^2(r_N)$ and the absolute value of the mean curvature of N is bounded by a constant $\Lambda \geq 0$.

Then we will prove that the inequality

$$\frac{1}{2}\triangle r_N^2 \leq (m-n) + (n\Lambda + (m-1)\kappa(r_N))\, r_N \tag{11}$$

holds off the cut locus pointwise and on the whole of M in the sense of distributions. It follows that inequality (10) holds if N is compact and the curvature in the radial direction is bounded below by $-C(1+r_N)^2$.

Radial Moment Estimates

In Chapter 3 we demonstrate a robust method of moment estimation based on inequality (10) and Laguerre polynomials. The main result Theorem 3.2.10 states that

$$\mathbb{E}\left[\mathbf{1}_{\{t<\zeta(x)\}} e^{\frac{\theta}{2} r_N^2(X_t(x))}\right] \leq (1-\theta t\lambda(t))^{-\frac{\nu}{2}} \exp\left[\frac{\theta r_N^2(x) e^{\lambda t}}{2(1-\theta t\lambda(t))}\right] \tag{12}$$

for all $t, \theta \geq 0$ with $\theta t\lambda(t) < 1$, where $\zeta(x)$ denotes the explosion time of $X(x)$ and

$$\lambda(t) := \begin{cases} (e^{\lambda t}-1)/(\lambda t) & \text{if } \lambda \neq 0 \\ 1 & \text{if } \lambda = 0. \end{cases}$$

This improves and generalizes a theorem of Stroock [2000] and can be used to deduce a comparison theorem, concentration inequalities and exit time estimates. Other results which are contained in this thesis and of independent interest include Theorem 1.4.8 on estimating the volume of tubes, Theorem 4.3.1 on the equivalence of bridge processes and Theorem 5.4.5 on the existence of solutions to the martingale problem for singular drift.

The organization of each chapter is described, alongside various literary remarks, in the introductions to each chapter. The six chapters are followed by four short appendices containing supplementary material on Hausdorff measure, the Fermi bridge and upper and lower bounds for the integrated heat kernel.

Chapter 1

Geometry of Submanifolds

Introduction

In this chapter we present a concise review of all geometric ideas relevant to later chapters. We prove several new results including the Jacobian inequalities (1.24), (1.25) and (1.26) which we use to deduce the main result in this chapter, Theorem 1.4.5, which gives an inequality for the Laplacian of the distance to a submanifold. We also use the Jacobian inequalities to obtain Theorem 1.4.8, which provides a estimate on the volume of tubes.

Section 1.1 is short, containing a few basic definitions on curvature and submanifolds which serve to clarify our notation. There are many excellent introductions to Riemannian geometry, including those by Chavel [1993], Sakai [1996], Lee [1997] and Petersen [1998].

Section 1.2 includes preliminary material on the exponential map and the cut locus of a submanifold. While the cut locus of a point has been studied by many authors, such as by Kobayashi [1961], Crittenden [1962], Warner [1967], Ozols [1974] and Hebda [1987], having been introduced as a concept originally by Poincaré [1905], it was not until after the development of the theory of viscosity solutions to Hamilton-Jacobi equations that the cut locus of a submanifold received full consideration, such as in the work of Mantegazza and Mennucci [2003].

Section 1.3 focusses on Jacobian comparison and the Laplacian of the distance function. A standard approach, as in Greene and Wu [1979] for the one point case, is to use Jacobi fields to prove a Hessian comparison theorem, from which a Laplacian comparison is derived as a corollary. There are alternative approaches which do not rely on Jacobi fields, including methods based on mean curvature or the Bochner identity, but we are primarily interested in the Laplacian of the distance to a submanifold and this seems best understood in terms of Jacobi fields.

Warner [1966] showed how Rauch's comparison theorem can be extended to a particular class of Jacobi fields associated to submanifolds, following Berger [1962] who considered the geodesic case. The comparison theorem which forms the basis of our geometric inequalities came later and is that of Heintze and Karcher [1978], which was soon after generalized slightly by Kasue [1982].

The Laplacian comparison implied by the Heinzte-Karcher theorem is too unwieldy for our purposes, so in Section 1.4 we deduce the secondary estimates (1.24), (1.25) and (1.26). These are used to prove Theorem 1.4.5 whose applications will be considered in later chapters. The secondary estimates will also be used to deduce volume estimates for tubular neighbourhoods. This topic was considered by Eschenburg [1987], who deduced a relative volume comparison for tubes around totally geodesic hypersurfaces of finite volume, and Gray [1982], who proved a comparison theorem for the volume of tubes generalizing Weyl's formula. The estimates we derive, given in Theorem 1.4.8, are more explicit than those found in these articles or in the book by Gray [2004]. The codimension one case is somewhat special, so a good example to have in mind throughout is the one where the submanifold is given by the boundary of a smooth domain.

1.1 Basic Riemannian Geometry

1.1.1 The Ambient Manifold

Suppose that M is a smooth manifold of finite dimension m and suppose that it is connected, metrizable and without boundary. Let $\pi : TM \to M$ denote the tangent bundle of M, equipped with the canonical smooth structure. For the remainder of this chapter suppose also that M is equipped with a Riemannian metric. Then, by Sasaki [1958, 1962], there is a canonical choice of Riemannian metric on TM, called the *Sasaki* metric, which takes the form of a Whitney sum metric and with respect to which TM is a $2m$-dimensional Riemannian manifold. Since these metrics will remain fixed throughout we will not refer to them explicitly. We will denote by vol_M the Riemannian volume measure associated to M and by $d(\cdot, \cdot)$ the Riemannian distance function and we will assume that M is complete with respect to this metric.

1.1.2 Sectional and Ricci Curvature

Adapted to the Riemannian structure on M there is a unique torsion-free affine connection, called the *Levi-Civita connection*, which we will denote by ∇ and in terms of which we define curvature. In particular, the *Riemann curvature tensor* is defined by

$$R(X, Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

for smooth vector fields X, Y and Z where $[X, Y]$ denotes the Lie bracket of the vector fields X and Y. If $m \geq 2$ suppose that $x \in M$ with σ_x a two-dimensional subspace of $T_x M$ spanned by orthogonal unit vectors v_1 and v_2. Then $K(\sigma_x)$, called the *sectional curvature of M at x associated to σ_x*, is defined by

$$K(\sigma_x) := \langle R_x(v_1, v_2)v_2, v_1 \rangle.$$

One can check that this definition is independent of the choice of v_1 and v_2 and one should note that if $m = 2$ then $K(T_x M)$ is just the Gaussian curvature of M at x. The *Ricci curvature* is the field of quadratic forms, denoted by Ric and given, for each $x \in M$ and $\xi \in T_x M$, by

$$\mathrm{Ric}_x(\xi, \xi) = \sum_{i=1}^{m-1} \langle R_x(\xi, e_i)e_i, \xi \rangle$$

where $\{e_i\}_{i=1}^m$ is any orthonormal basis of $T_x M$ with $\langle e_m, \xi \rangle = \|\xi\|$. The Ricci curvature, when acting on unit tangent vectors, is therefore given by the sum of $m-1$ sectional curvatures. When we later refer to bounds on the Ricci curvature, we will be referring to bounds on the restriction of Ric to unit tangent vectors.

1.1.3 The Submanifold

Now suppose that N is an closed embedded submanifold of M of dimension $n \in \{0, \ldots, m-1\}$, equipped with the Riemannian structure induced by the embedding. If $n = m-1$ then N is a called a closed embedded *hypersurface*. If the image of the embedding is a compact subset of M then we will refer to N as a *compactly* embedded submanifold. In either case we will identify N with its image under the embedding and we will assume that N has no boundary. We will denote by vol_N the induced Riemannian volume measure on N. If $n = 0$ then vol_N is simply a counting measure.

1.1.4 The Normal Bundle

For each $p \in N$ we view $T_p N$ as a subspace of $T_p M$ and denote its orthogonal complement by $T_p N^\perp$. If $TN^\perp := \bigsqcup_{p \in N} T_p N^\perp$ then $\pi_N := \pi|_{TN^\perp} : TN^\perp \to N$ has the structure of a vector bundle over N and is called the *normal bundle of* N. In this way $TN^\perp$ can be thought of as an m-dimensional Riemannian submanifold of TM, as in Borisenko and Yampol'skii [1987], and the restriction to N of the tangent bundle of M takes the form of a Whitney sum which can be written $TM|_N \cong TN \oplus TN^\perp$. For $\xi \in TM|_N$ we will denote by $\xi^\top$ and $\xi^\perp$ the projections of ξ onto TN and $TN^\perp$ respectively. When restricted to a common domain, the TN-component of the Levi-Civita connection on M agrees with the Levi-Civita connection on N.

1.1.5 The Shape Operator

If $p \in N$ with $\xi \in T_pN^\perp$ then ξ can be extended locally to a smooth normal vector field on N and the *shape operator* $\mathcal{S}_\xi : T_pN \to T_pN$ is defined for $v \in T_pN$ by $\mathcal{S}_\xi v = (\nabla_v \xi)^\top$. The shape operator can also be understood in terms of the *second fundamental form of* N. The latter object is a symmetric $(0,2)$-tensor field on N, denoted by $\mathrm{I\!I}$ and taking values in $TN^\perp$, which relates to the shape operator via the property $\langle \mathcal{S}_\xi v_1, v_2\rangle = -\langle \mathrm{I\!I}(v_1, v_2), \xi\rangle$ for all $v_1, v_2 \in T_pN$. Note that the operator $\mathcal{S}_\xi$ is also known as the *Weingarten map* and that it is trivial if $n = 0$.

1.1.6 Principal and Mean Curvature

If $\xi \in TN^\perp$ then the eigenvalues $\{\lambda_i(\xi)\}_{i=1}^n$ of $\mathcal{S}_\xi$ are called the *principal curvatures of* N *with respect to* ξ. Their arithmetic mean is called the *mean curvature of* N *with respect to* ξ and is denoted by $H(\xi)$. If the mean curvature is everywhere zero, then N is said to be *minimal.* The submanifold N is totally geodesic if and only if $\mathrm{I\!I}$ vanishes in which case $\mathcal{S}_\xi$ vanishes and the principal curvatures are all equal to zero. Consider, for example, the situation in which N is a closed geodesic.

1.1.7 A Connection on the Normal Bundle

There is a linear connection $\nabla^\perp$ on the normal bundle $\pi_N : TN^\perp \to N$, whose covariant derivative is defined on N for a smooth tangent vector field X and a smooth normal vector field ξ by $\nabla^\perp_X \xi := (\nabla_X \xi)^\perp$. For each $\xi \in T_pN^\perp$ the connection induces a direct sum decomposition $T_\xi TN^\perp = H_\xi \oplus V_\xi$ where H_ξ is isomorphic to T_pN and where V_ξ is isomorphic to $T_pN^\perp$. This allows us to identify $T_\xi TN^\perp$ with $T_pN \oplus T_pN^\perp$. Given $(A, B) \in T_pN \oplus T_pN^\perp$ and $\xi \in TN^\perp$ we will denote the corresponding element of $T_\xi TN^\perp$ by $(A, B)_\xi$.

1.2 The Exponential Map and Cut Locus

1.2.1 Gauss's Lemma

For $\xi \in TM$ denote by γ_ξ the unique maximal geodesic in M satisfying $\dot{\gamma}_\xi(0) = \xi$ and $\gamma_\xi(0) = \pi(\xi)$. Then, by the Hopf-Rinow theorem, γ_ξ is defined on the whole of the real line. This property is called *geodesic completeness.* The *exponential map* $\exp : TM \to M$ is defined by $\exp(\xi) := \gamma_\xi(1)$ and we call the smooth map $\exp_N := \exp|_{TN^\perp}$ the *normal exponential map.* A basic fact of Riemannian geometry concerning the exponential map is *Gauss's lemma.* It states, roughly speaking, that the normal exponential map is a radial isometry. More precisely and as in [Sakai, 1996, p.60], the lemma states that if $\xi \in TN^\perp$ then $D_{t\xi} \exp_N(0, t\xi)_{t\xi} = t\dot{\gamma}_\xi(t)$ and if $(A, B) \in T_pN \oplus T_pN^\perp$ then $\langle D_{t\xi} \exp_N(A, tB)_{t\xi}, \dot{\gamma}_\xi(t)\rangle = t\langle B, \xi\rangle$.

1.2.2 The Focal Locus

We will denote by $\mathcal{F}(N)$ the set of all critical points of $\exp_N$. This set is sometimes called the *tangential focus locus of* N. These are the points in $TN^\perp$ at which the differential $D_\cdot \exp_N : T.TN^\perp \to TM$ fails to be of maximal rank. It follows from basic existence and uniqueness theory for ordinary differential equations that $\exp_N$ is a local diffeomorphism around any point belonging to the zero section of $\pi_N : TN^\perp \to N$. In particular, if N is compact then there exists a tubular neighbourhood of the zero section on which the normal exponential map is a diffeomorphism onto its image. The *focal locus of* N, which we will denote by $F(N)$ and which is referred to as the *conjugate locus* if N is a point, is defined to be the image of $\mathcal{F}(N)$ under $\exp_N$. A consequence of Sard's theorem is that $F(N)$ has vol_M-measure zero. Since points outside $F(N)$ are regular values of $\exp_N$ it follows from the regular value theorem and our completeness assumption that the preimage of such a point is a countable collection of isolated points in $TN^\perp$. If $F(N)$ is empty then $\exp_N$ is a local diffeomorphism and therefore a covering map.

1.2.3 The Cut Locus

Now consider the unit normal bundle

$$UTN^\perp := \{\xi \in TN^\perp : \|\xi\| = 1\}$$

with fibre $UT_pN^\perp$. Then the function $f_N : UTN^\perp \to (0, \infty]$ defined by $f_N(\xi) := \inf\{t > 0 : t\xi \in \mathcal{F}(N)\}$ is the *the first focal time along the geodesic* γ_ξ. Itoh and Tanaka [2001] proved that for each $\xi \in UTN^\perp$ with $f_N(\xi) < \infty$ the function f_N is locally Lipschitz around ξ. The function $c_N : UTN^\perp \to (0, \infty]$ defined by $c_N(\xi) := \sup\{t > 0 : d(\gamma_\xi(t), N) = t\}$ is called *the distance to the cut locus of N along* γ_ξ. A consequence of Jacobi's criterion is that $c_N(\xi) \leq f_N(\xi)$ for all $\xi \in UTN^\perp$. We define the *tangential cut locus of N* by

$$\mathcal{C}(N) := \{c_N(\xi)\xi : c_N(\xi) < \infty,\ \xi \in UTN^\perp\}$$

and the *cut locus of N*, denoted by $\mathrm{Cut}(N)$, is defined to be the image of $\mathcal{C}(N)$ under $\exp_N$ (it is interesting to note that the tangential focal and cut loci need not have a point in common, as proved by Weinstein [1968]). If $\xi \in UTN^\perp$ with $t > 0$ then it follows from the triangle inequality that the geodesic segment $\gamma_\xi|_{[0,t]}$ is the *unique* length minimizing path between the points $\pi(\xi)$ and $\gamma_\xi(t)$ if $t < c_N(\xi)$ and that it fails to minimize if $t > c_N(\xi)$. Furthermore, if $c_N(\xi) < \infty$ then $\gamma_\xi|_{[0,c_N(\xi)]}$ is also length minimizing. Itoh and Tanaka [2001] proved also that for each $\xi \in UTN^\perp$ with $c_N(\xi) < \infty$ the function c_N is locally Lipschitz around ξ. Therefore if M is compact then c_N is globally Lipschitz which implies $\mathrm{Cut}(N)$ has finite $(m-1)$-dimensional Hausdorff measure (see Appendix A for a definition of Hausdorff measure). Other basic properties of $\mathrm{Cut}(N)$ are that it is closed with vol_M-measure zero and that if $q \in \mathrm{Cut}(N)$ then, in a non-exclusive sense, either $q \in F(N)$ or there are at least two distinct length minimizing geodesic segments connecting q with N.

Mantegazza and Mennucci [2003] used an approach based on Hamilton-Jacobi equations to prove that the cut locus $\mathrm{Cut}(N)$ can be expressed as the disjoint union of two sets $\mathring{C}(N)$ and $\check{C}(N)$, where the connected components of $\mathring{C}(N)$, of which there are at most countably many, are smooth two-sided $(m-1)$-dimensional submanifolds, and where $\check{C}(N)$ is a closed set of Hausdorff dimension at most $m-2$. Moreover, points in $\mathring{C}(N)$, which are referred to as *cleave points*, can be connected to N by precisely two length-minimizing geodesics segments both of which are non-focal (i.e. $f_N > c_N$ for the two associated normal tangent vectors). A more detailed description of the cut locus than this, upto sets of Hausdorff codimension three, is given by [Ardoy and Guijarro, 2011, Theorem 2.2.] but the one given here will suffice for our purposes. Now consider the set

$$\mathcal{M}(N) := \{t\xi : 0 \leq t < c_N(\xi),\ \xi \in UTN^{\perp}\}$$

with fibre at $p \in N$ denoted by $\mathcal{M}_p(N)$. Then $\mathcal{M}(N)$ is the largest domain in $TN^{\perp}$ whose fibres are star-like and such that $\exp_N|_{\mathcal{M}(N)}$ is a diffeomorphism onto its image. If we define the open domain $M(N)$ to be the image of $\mathcal{M}(N)$ under $\exp_N$ then $M(N) = M \setminus \mathrm{Cut}(N)$. We will define the *injectivity radius of* N by

$$\mathrm{inj}(N) := \inf\{c_N(\xi) : \xi \in UTN^{\perp}\}$$

so that $\mathrm{inj}(N) = \mathrm{dist}(N, \mathrm{Cut}(N))$ which could be equal to zero unless N is compact. For a simple example where $\mathrm{inj}(N) = 0$ consider a line in the product of $\mathbb{R}$ and $\mathbb{S}^1$ warped by the function $f : \mathbb{R} \to [0, \infty)$ given by $f(x) = e^x$.

1.2.4 Examples of Focal and Cut Loci

If $p \in M$ and the exponential map $\exp_p : T_pM \to M$ is a diffeomorphism then $\mathcal{M}(p) = T_pM$ and the cut locus of p is empty. In this case one says that p is a *pole for* M. The Cartan-Hadamard theorem states that if a complete Riemannian manifold has non-positive sectional curvature then the exponential map at any point is a covering map and that if such a manifold is simply connected then every one of its points is a pole. The function theory of non-positively curved manifolds with a pole was studied by Greene and Wu [1979].

For a more specific example, suppose that p is a point in the unit circle $\mathbb{S}^1$, equipped with the standard metric. Then $\mathrm{Cut}(p)$ is the point antipodal to p, which can be connected to p by precisely two length minimizing geodesic segments, and so in this case $\mathrm{Cut}(p) = \mathring{C}(p)$. Alternatively, suppose that $m \geq 2$ with p a point in the m-dimensional unit sphere $\mathbb{S}^m$, equipped with the standard round metric. Then $\mathrm{Cut}(p)$ is again the point antipodal to p, but which this time can be connected to p by *infinitely* many length minimizing geodesic segments. In this case $\mathrm{Cut}(p) = \check{C}(p)$ and the antipodal point is focal. For a point p in the cylinder $\mathbb{S}^1 \times \mathbb{R}$, equipped with the standard product metric, the cut locus of p is the line $\{q\} \times \mathbb{R}$, if q denotes the point in $\mathbb{S}^1$ antipodal to p, providing another simple example where the cut locus consists entirely of cleave points.

A less trivial example was considered by Gravesen, Markvorsen, Sinclair and Tanaka [2005] who provided a description and visualization of the cut locus of a point for a class of tori of revolution, which includes standard tori in three dimensional Euclidean space.

1.2.5 The Distance Function

We will define the distance function $r_N : M \to \mathbb{R}$ by

$$r_N(q) := d(q, N) = \inf\{d(p, q) : p \in N\}.$$

By the triangle inequality, r_N is Lipschitz continuous while r_N^2 is locally Lipschitz continuous. Mantegazza and Mennucci [2003] showed that r_N is a viscosity solution to the eikonal problem

$$\begin{cases} \|\nabla u\| = 1 & \text{in } M \setminus N \\ u = 0 & \text{on } N \end{cases} \tag{1.1}$$

and that it is the unique solution among continuous functions on M which are bounded from below. Similarly, they showed that r_N^2 is a viscosity solution to the problem

$$\begin{cases} \frac{1}{2}\|\nabla u\|^2 - 2u = 0 & \text{in } M \\ u = 0 & \text{on } N \end{cases} \tag{1.2}$$

and it is the unique solution among continuous functions on M whose zero set is contained in N.

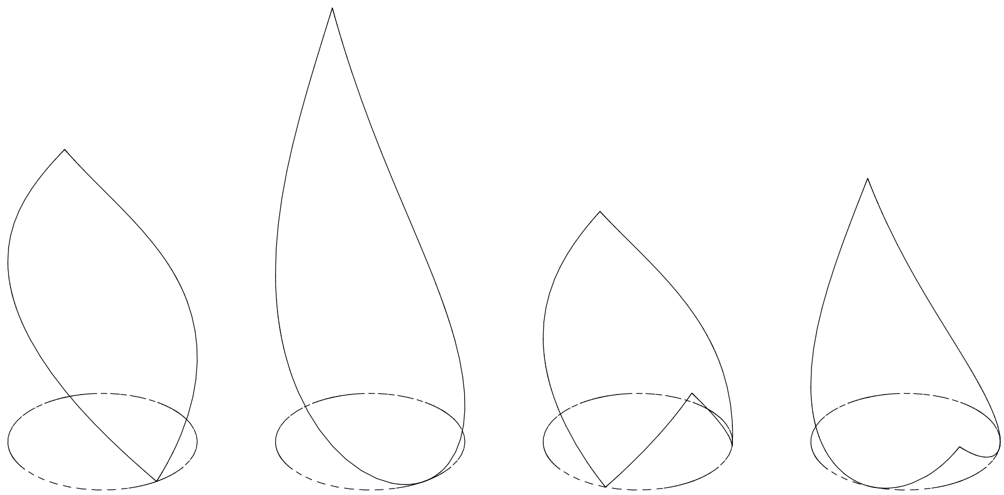

Figure 1: Suppose $M = \mathbb{S}^1$. Then on the left are graphs of r_N and $\frac{1}{2}r_N^2$ for the case in which N is a single point while, on the right, are graphs of r_N and $\frac{1}{2}r_N^2$ for the case in which N is given by the union of two points. The points of non-differentiability are evident in each case.

By first using the theory of viscosity solutions to show that r_N is locally semiconcave on $M \setminus N$, Mantegazza and Mennucci proved that $\mathrm{Cut}(N)$ is equal to the closure of the set of all points at which r_N^2 fails to be differentiable, that r_N is smooth on $M(N) \setminus N$ and that r_N^2 is smooth on $M(N)$. Therefore $\|\nabla r_N\| = 1$ on $M(N) \setminus N$ in the classical sense (which is also a consequence of Gauss's lemma).

1.2.6 The Radial Derivative

We will denote by $\frac{\partial}{\partial r_N}$ differentiation in the radial direction. In other words, $\frac{\partial}{\partial r_N}$ denotes the vector field on M which is equal to the gradient of r_N on $M(N) \setminus N$ and which vanishes elsewhere. Note that for $q \in M(N) \setminus N$ there is a unique $\xi_q \in UTN^{\perp}$

such that $\gamma_{\xi_q}(r_N(q)) = q$, in which case it follows that

$$\frac{\partial}{\partial r_N} f(q) = \frac{d}{dt} f(\gamma_{\xi_q}(t))|_{t=r_N(q)}$$

for any function f which is differentiable on $M(N) \setminus N$. Note also that according to these definitions we have $\frac{\partial}{\partial r_N} r_N^2 = 2r_N$ on $M(N)$.

1.2.7 The Laplacian of the Distance Function

The *Laplace-Beltrami* operator $\triangle$ is given by the trace of the Hessian, the latter object being defined in terms of the Levi-Civita connection in the usual way. While Wu [1979] studied the convexity and subharmonicity of distance functions on manifolds with nonnegative curvature, we wish to allow unbounded negative curvature. We will use the fact that the Laplacian of the distance function can be expressed in terms of the exponential map. Indeed, consider the function $\theta_N : TN^{\perp} \to \mathbb{R}$ defined by

$$\theta_N(\xi) := |\det D_\xi \exp_N| \tag{1.3}$$

for each $\xi \in TN^{\perp}$. Sometimes referred to as *Ruse's invariant*, θ_N is the Jacobian determinant of the normal exponential map (for the case in which N is a point, a remarkable probabilistic formula for this object, given in terms of an integral over loops, can be found in [Bismut, 1984, p.147]). If we let $\Theta_N : M(N) \to \mathbb{R}$ be defined by

$$\Theta_N := \theta_N \circ \left(\exp_N |_{\mathcal{M}(N)}\right)^{-1} \tag{1.4}$$

then, as proved in Gray [2004], there is the formula

$$\triangle r_N = \frac{m-n-1}{r_N} + \frac{\partial}{\partial r_N} \log \Theta_N \tag{1.5}$$

on $M(N) \setminus N$. The Laplacian of r_N can elsewhere be interpreted in the sense of distributions and extended to give a Radon measure. Indeed, in Savo [2001], who considers eigenvalue comparison and heat content on tubular neighbourhoods, it is proved that there exists a non-negative distribution $\triangle_{cut} r_N$ supported on $\mathrm{Cut}(N)$

such that

$$\triangle r_N = \begin{cases} \triangle_{reg} r_N - \triangle_{cut} r_N & \text{if } n \leq m-2 \\ \triangle_{reg} r_N - \triangle_{cut} r_N + 2\delta_N & \text{if } n = m-1 \end{cases} \tag{1.6}$$

where $\triangle_{reg} r_N$ is the distribution corresponding to the right-hand side of (1.5) supported on $M(N) \setminus N$ and where δ_N is the distribution corresponding to the Radon measure vol_N. Note that the function given by the right-hand side of (1.5) is absolutely continuous on $M(N) \setminus N$ and locally integrable (see Appendix A of Savo [2001] for proof). It follows from formula (1.5) that

$$\frac{1}{2} \triangle r_N^2 = m - n + r_N \frac{\partial}{\partial r_N} \log \Theta_N \tag{1.7}$$

on $M(N) = M \setminus \mathrm{Cut}(N)$.

1.2.8 Change of Variables Formulae

If one has an atlas for N then, by using either Cartesian or polar coordinates on $\mathcal{M}_p(N)$ for each $p \in N$, varying smoothly in p, one can obtain an atlas for $M(N)$. Coordinates belonging to such an atlas are called *Fermi coordinates*. If N is a point then these are called *geodesic normal coordinates*. In these terms we have the following change of variables formulae.

Theorem 1.2.1. *For any non-negative measurable $f : M \to \mathbb{R}$ we have*

$$\begin{aligned} &\int_M f(q)\, d\,\mathrm{vol}_M(q) \\ &= \int_{\mathcal{M}(N)} f(\exp_N(\xi))\theta_N(\xi)\, d\,\mathrm{vol}_{TN^\perp}(\xi) \\ &= \int_N \int_{\mathcal{M}_p(N)} f(\exp_N(\xi))\theta_N(\xi)\, d\xi\, d\,\mathrm{vol}_N(p) \\ &= \int_N \int_{UT_pN^\perp} \int_0^{c_N(\xi)} f(\exp_N(t\xi))\theta_N(t\xi) t^{m-n-1}\, dt\, d\sigma^{m-n-1}(\xi)\, d\,\mathrm{vol}_N(p) \end{aligned}$$

where σ^{m-n-1} denotes the natural spherical measure on $UT_pN^\perp$ for any $p \in N$ and where $\mathrm{vol}_{TN^\perp}$ denotes the natural measure on $TN^\perp$.

Proof. Since $\mathcal{M}(N)$ does not contain any critical points of $\exp_N$ and since $\mathrm{Cut}(N)$ has vol_M-measure zero, the first equality follows from the usual change of variables formula and the definition of θ_N. The second equality, whose right-hand side takes the form of an integral with respect to Cartesian Fermi coordinates, then follows from the smooth coarea formula (see Nicolaescu [2007]). The third equality, whose right-hand side takes the form of an integral with respect to polar Fermi coordinates, then follows by applying a standard change of variables on each $\mathcal{M}_p(N)$. $\square$

For $r > 0$ we will denote by $B_r(N)$ the tubular neighbourhood of radius r around N, which is to say $B_r(N) = \{q \in M : r_N(q) < r\}$.

Corollary 1.2.2. *Suppose that* $\mathrm{inj}(N) > 0$ *with* $r \in (0, \mathrm{inj}(N))$. *Then for any non-negative measurable function* $f : M \to \mathbb{R}$ *we have the change of variables formulae*

$$\begin{aligned}&\int_{B_r(N)} f(q)\, d\,\mathrm{vol}_M(q)\\ =\ &\int_N \int_{UT_pN^\perp} \int_0^r f(\exp_N(t\xi))\theta_N(t\xi)t^{m-n-1}\, dt\, d\sigma^{m-n-1}(\xi)\, d\,\mathrm{vol}_N(p);\end{aligned} \tag{1.8}$$

$$\begin{aligned}&\int_{\partial B_r(N)} f(q)\, d\,\mathrm{vol}_{\partial B_r(N)}(q)\\ =\ &\int_N \int_{UT_pN^\perp} f(\exp_N(r\xi))\theta_N(r\xi)r^{m-n-1}\, d\sigma^{m-n-1}(\xi)\, d\,\mathrm{vol}_N(p).\end{aligned} \tag{1.9}$$

Proof. Equation (1.8) follows immediately from Theorem 1.2.1 while equation (1.9) follows by applying equation (1.8) to the submanifold $\partial B_r(N)$ with a limiting argument. $\square$

Setting $f = 1$ in the previous corollary provides formulae for the volume and surface area of the tubular neighbourhood. We will return to these quantities, studied by Gray [2004], at the end of the chapter.

1.3 Jacobian Comparison

1.3.1 Jacobi Fields

For $p \in N$ and a unit normal vector $\xi \in T_pN^{\perp}$, a smooth vector field Y along the geodesic γ_ξ is called a *Jacobi field along* γ_ξ if it satisfies the *Jacobi equation* $D_t^2 Y + R(Y, \dot{\gamma_\xi})\,\dot{\gamma_\xi} = 0$ where D_t denotes covariant differentiation along γ_ξ with respect to the Levi-Civita connection. Given initial conditions $Y(0) \in T_pM$ and $D_tY(0) \in T_pM$ there exists a *unique* solution to the Jacobi equation which satisfies the initial conditions. If in addition we suppose that $A := Y(0) \in T_pN$ with $B := D_tY(0) - \mathcal{S}_\xi Y(0) \in T_pN^{\perp}$ then Y is called an N*-Jacobi field along* γ_ξ. In this case $Y(t) = D_{t\xi} \exp_N(A, tB)_{t\xi}$. The collection of all N-Jacobi fields along γ_ξ is a vector space of dimension m and will be denoted by $\mathcal{J}_N(\xi)$. For a nontrivial $Y \in \mathcal{J}_N(\xi)$ it follows that $Y(t) = 0$ if and only if $\exp_N(t\xi)$ is focal for N. The $(m-1)$-dimensional subspace of $\mathcal{J}_N(\xi)$ defined by the condition $Y \perp \dot{\gamma_\xi}$ will be denoted by $\mathcal{J}_N^{\perp}(\xi)$.

1.3.2 Heintze-Karcher Comparison

Fix $p \in N$, a unit normal vector $\xi \in T_pN^{\perp}$ and a time $t_1 \in (0, f_N(\xi))$, where $f_N(\xi)$ is the first focal time in the direction ξ. Define

$$\underline{\kappa}_\xi(t_1) := \min \left\{ K(\sigma_{\gamma_\xi(t)}) \colon \begin{array}{l} \sigma_{\gamma_\xi(t)} \text{ is any two-dimensional subspace of} \\ T_{\gamma_\xi(t)}M \text{ containing } \dot{\gamma_\xi}(t) \text{ for any } t \in [0, t_1] \end{array} \right\} \tag{1.10}$$

and let $\kappa_\xi(t_1)$ be *any* constant which satisfies $\kappa_\xi(t_1) \leq \underline{\kappa}_\xi(t_1)$. Suppose that $\tilde{M}$ is a complete simply connected m-dimensional Riemannian manifold with constant sectional curvature equal to $\kappa_\xi(t_1)$ and let $\tilde{N}$ be a n-dimensional closed embedded submanifold of $\tilde{M}$ for which there exists $\tilde{p} \in \tilde{N}$ and $\tilde{\xi} \in T_{\tilde{p}}\tilde{N}^{\perp}$ such that $\mathcal{S}_{\tilde{\xi}}$ and $\mathcal{S}_\xi$ have the same set of eigenvalues $\{\lambda_i(\xi)\}_{i=1}^n$. Such an embedding can always be constructed, as noted by [Sakai, 1996, p.159]. Now choose a basis $\{Y_i\}_{i=1}^{m-1}$ of $\mathcal{J}_N^{\perp}(\xi)$ and for each $i = 1, \ldots, m-1$ define $A_i := Y_i(0)$ and $B_i := D_tY_i(0) - \mathcal{S}_\xi Y_i(0)$. Set

$U_i(t) = A_i + tB_i$ and define a function f by

$$f(t) := \frac{\|Y_1(t) \wedge \ldots \wedge Y_{m-1}(t)\|}{\|U_1(t) \wedge \ldots \wedge U_{m-1}(t)\|}$$

for $t \geq 0$. Choose also a basis $\{\tilde{Y}_i\}_{i=1}^{m-1}$ of $\mathcal{J}_{\tilde{N}}^{\perp}(\tilde{\xi})$ and define a function $\tilde{f}$ similarly. Then it follows, as proved by Heintze and Karcher [1978], that $t_1 < f_{\tilde{N}}(\tilde{\xi})$ and that for $0 < t \leq t_1$ we have

$$\frac{d}{dt} \log f(t) \leq \frac{d}{dt} \log \tilde{f}(t). \tag{1.11}$$

This implies a comparison theorem for the logarithmic derivative of the Jacobian determinant of the normal exponential map, since this object can be expressed in terms of N-Jacobi fields. Indeed, choose the N-Jacobi fields $\{Y_i\}_{i=1}^{m-1}$ so that the collection $\{A_i(0)\}_{i=1}^{n}$ forms an orthonormal basis of T_pN consisting of eigenvectors of $\mathcal{S}_\xi$ with $B_i = 0$ for $i = 1, \ldots, n$ and so that the collection $\{B_i\}_{i=n+1}^{m-1}$ forms an orthonormal basis of $T_pN^\perp \cap \xi^\perp$ with $A_i = 0$ for $i = n+1, \ldots, m-1$. Then it follows that

$$\begin{aligned}
f(t) &= \|Y_1(t) \wedge \cdots \wedge Y_{m-1}(t)\| \cdot \|U_1(t) \wedge \cdots \wedge U_{m-1}(t)\|^{-1} \\
&= \|D_{t\xi} \exp_N(A_1, 0)_{t\xi} \wedge \cdots \wedge D_{t\xi} \exp_N(A_n, 0)_{t\xi} \\
&\qquad \wedge D_{t\xi} \exp_N(0, tB_{n+1})_{t\xi} \wedge \cdots \wedge D_{t\xi} \exp_N(0, tB_{m-1})_{t\xi}\| \\
&\qquad\quad \cdot \|(A_1, 0) \wedge \cdots \wedge (A_n, 0) \wedge (0, tB_{n+1}) \wedge \cdots \wedge (0, tB_{m-1})\|^{-1} \\
&= |\det D_{t\xi} \exp_N| \\
&= \theta_N(t\xi)
\end{aligned} \tag{1.12}$$

for all $t > 0$. For comparison, let $\tilde{M}$, $\tilde{N}$, $\tilde{p}$ and $\tilde{\xi}$ be as above and for $\kappa, \lambda \in \mathbb{R}$ define functions S_κ, C_κ, G_κ and F_κ^λ by

$$S_\kappa(t) := \begin{cases} \frac{1}{\sqrt{\kappa}} \sin \sqrt{\kappa} t & \text{if } \kappa > 0 \\ t & \text{if } \kappa = 0 \\ \frac{1}{\sqrt{-\kappa}} \sinh \sqrt{-\kappa} t & \text{if } \kappa < 0 \end{cases}$$

$$C_\kappa(t) := \frac{d}{dt} S_\kappa(t)$$
$$G_\kappa(t) := \frac{d}{dt} \log(S_\kappa(t)/t)$$
$$F_\kappa^\lambda(t) := \frac{d}{dt} \log(C_\kappa(t) + \lambda S_\kappa(t)).$$

Suppose that $\{\tilde{E}_i\}_{i=1}^{m-1}$ is a collection of parallel vector fields along $\gamma_{\tilde{\xi}}$ such that $\{\tilde{E}_i(0)\}_{i=1}^n$ forms an orthonormal basis of $T_{\tilde{p}}\tilde{N}$ consisting of eigenvectors of $\mathcal{S}_{\tilde{\xi}}$ and such that $\{\tilde{E}_i(0)\}_{i=n+1}^{m-1}$ forms an orthonormal basis of $T_{\tilde{p}}\tilde{N}^\perp \cap \tilde{\xi}^\perp$. If we define

$$\tilde{Y}_i = \begin{cases} (C_{\kappa_\xi(t_1)} + \lambda_i(\xi) S_{\kappa_\xi(t_1)})\tilde{E}_i & \text{if } i \in \{1, \dots, n\} \\ S_{\kappa_\xi(t_1)}\tilde{E}_i & \text{if } i \in \{n+1, \dots, m-1\} \end{cases}$$

then it follows that $\{\tilde{Y}_i\}_{i=1}^{m-1}$ forms a basis for $\mathcal{J}_{\tilde{N}}^\perp(\tilde{\xi})$ and

$$\tilde{f}(t) = \left(\frac{S_{\kappa_\xi(t_1)}(t)}{t}\right)^{m-n-1} \prod_{i=1}^n \left(C_{\kappa_\xi(t_1)}(t) + \lambda_i(\xi) S_{\kappa_\xi(t_1)}(t)\right) \tag{1.13}$$

for all $t > 0$. It follows, by (1.11), (1.12) and (1.13), that

$$\frac{d}{dt} \log \theta_N(t\xi) \leq (m-n-1) G_{\kappa_\xi(t_1)}(t) + \sum_{i=1}^n F_{\kappa_\xi(t_1)}^{\lambda_i(\xi)}(t) \tag{1.14}$$

for all $0 \leq t \leq t_1$. Note that the right-hand side of inequality (1.14) is finite for all $0 \leq t \leq t_1$ since $t_1 < f_{\tilde{N}}(\tilde{\xi})$.

Recall that $n \in \{0, \dots, m-1\}$ and that inequality (1.14) was deduced using a lower bound on sectional curvature. If $n \in \{0, m-1\}$ then the above method can instead be formulated in terms of a lower bound on Ricci curvature and an upper bound on mean curvature. For this let $\underline{\rho}_\xi(t_1)$ satisfy

$$(m-1)\underline{\rho}_\xi(t_1) = \min\{\mathrm{Ric}(\dot{\gamma}_\xi(t), \dot{\gamma}_\xi(t)) : 0 \leq t \leq t_1\} \tag{1.15}$$

and let $\rho_\xi(t_1)$ be *any* constant which satisfies $\rho_\xi(t_1) \leq \underline{\rho}_\xi(t_1)$. Then, as explained by

Sakai [1996], for $n = 0$ it follows that

$$\frac{d}{dt} \log \theta_N(t\xi) \leq (m-1) G_{\rho_\xi(t_1)}(t) \tag{1.16}$$

for all $0 \leq t \leq t_1$ while if $n = m-1$ and $\lambda(\xi)$ is *any* constant which satisfies $H(\xi) \leq \lambda(\xi)$ then

$$\frac{d}{dt} \log \theta_N(t\xi) \leq (m-1) F^{\lambda(\xi)}_{\rho_\xi(t_1)}(t) \tag{1.17}$$

for all $0 \leq t \leq t_1$. Note that the right-hand sides of inequalities (1.16) and (1.17) are *finite* for all $0 \leq t \leq t_1$.

1.3.3 Hyperbolic, Euclidean and Spherical Spaces

By equation (1.13) it follows that if N is a submanifold of $\mathbb{R}^m$ then

$$\theta_N(t\xi) = \prod_{i=1}^{n} (1 + \lambda_i(\xi) t) \tag{1.18}$$

for any $\xi \in UTN^\perp$ and $t \geq 0$. In particular, if $p \in \mathbb{R}^m$ then $\theta_p(t\xi) = 1$. If $p \in \mathbb{S}^m_\kappa$, the m-dimensional sphere with constant sectional curvature $\kappa > 0$, then

$$\theta_p(t\xi) = \left(\frac{\sin(\sqrt{\kappa} t)}{\sqrt{\kappa} t} \right)^{m-1}$$

from which it follows, by expanding the cotangent function, that

$$\begin{aligned} t \frac{d}{dt} \log \theta_p(t\xi) &= 2(m-1) \sum_{k=1}^{\infty} \frac{\kappa t^2}{\kappa t^2 - \pi^2 k^2} \\ &\leq -\frac{2(m-1)\kappa t^2}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{k^2} \\ &= -\frac{(m-1)\kappa t^2}{3}. \end{aligned}$$

If $p \in \mathbb{H}^m_\kappa$, the m-dimensional hyperbolic space with constant sectional curvature $\kappa < 0$, then

$$\theta_p(t\xi) = \left(\frac{\sinh(\sqrt{-\kappa} t)}{\sqrt{-\kappa} t} \right)^{m-1} \tag{1.19}$$

from which it follows, again by Taylor expansion, that

$$t\frac{d}{dt}\log\theta_p(t\xi) \le -\frac{(m-1)\kappa t^2}{3}.$$

So this inequality holds on each of the model spaces, for which there is no dependence on ξ due to the radial symmetry. Note that we will usually write $\mathbb{H}^m = \mathbb{H}^m_{-1}$ and $\mathbb{S}^m = \mathbb{S}^m_1$.

1.3.4 Laplacian Comparison

For the unit normal vector ξ now suppose that $t_1 < c_N(\xi)$, where $c_N(\xi)$ is the distance to the cut locus of N in the direction ξ. For $q \in M(N) \setminus N$ we have

$$\frac{\partial}{\partial r_N}\log\Theta_N(q) = \frac{d}{dt}\log\theta_N(t\xi_q)|_{t=r_N(q)}$$

where ξ_q is the unique element in $UTN^\perp$ such that $\exp_N(r_N(p)\xi_q) = q$ so it follows from inequality (1.14) and formula (1.5) that we have the Laplacian comparison

$$\triangle r_N \le (m-n-1)\frac{C_{\underline{\kappa}_\xi(t_1)}(r_N)}{S_{\underline{\kappa}_\xi(t_1)}(r_N)} + \sum_{i=1}^n F^{\lambda_i(\xi)}_{\underline{\kappa}_\xi(t_1)}(r_N) \tag{1.20}$$

along $\gamma_\xi|_{(0,t_1]}$, where $\{\lambda_i(\xi)\}_{i=1}^n$ are the principal curvatures in the direction ξ. Note that we use the 'empty sum is zero' convention to cover the case $n = 0$ here and in the future. Inequalities (1.16) and (1.17) provide alternative comparisons for the case $n \in \{0, m-1\}$. In particular, if $n = 0$ then there is the well-known comparison

$$\triangle r_N \le (m-1)\frac{C_{\underline{\rho}_\xi(t_1)}(r_N)}{S_{\underline{\rho}_\xi(t_1)}(r_N)} \tag{1.21}$$

along $\gamma_\xi|_{(0,t_1]}$, while if $n = m-1$ then there is Kasue's comparison

$$\triangle r_N \le (m-1)F^{H(\xi)}_{\underline{\rho}_\xi(t_1)}(r_N) \tag{1.22}$$

along $\gamma_\xi|_{(0,t_1]}$, where $H(\xi)$ is the mean curvature of N with respect to ξ. Note that if $m = 1$ then there is no sectional curvature but by formulae (1.5) and (1.7) and Gauss's lemma it is nonetheless clear that in this case $\triangle r_N = 0$ on $M(N) \setminus N$ and $\frac{1}{2}\triangle r_N^2 = 1$ on $M(N)$.

1.4 Geometric Inequalities

1.4.1 Jacobian Inequalities

The objective now is to prove a *simple* inequality for the Laplacian of the distance function. As in Subsection 1.3.2, fix $p \in N$, a unit normal vector $\xi \in T_pN^\perp$, a time $t_1 \in (0, f_N(\xi))$ and let $\underline{\kappa}_\xi(t_1)$ be defined by (1.10).

Lemma 1.4.1. *If $\underline{\kappa}_\xi(t_1) \geq 0$ then for $0 < t \leq t_1$ we have*

$$\frac{d}{dt}\log\theta_N(t\xi) \leq \sum_{i=1}^{n} \lambda_i(\xi).$$

Proof. Setting $\kappa_\xi(t_1) = 0$ we have $S_{\kappa_\xi(t_1)}(t) = t$ and $C_{\kappa_\xi(t_1)}(t) = 1$ and from inequality (1.14) it follows that

$$\frac{d}{dt}\log\theta_N(t\xi) \leq \sum_{i=1}^{n} \frac{\lambda_i(\xi)}{1 + \lambda_i(\xi)t}$$

for all $0 \leq t \leq t_1$. The result follows by considering the cases $\lambda_i(\xi) \geq 0$ and $\lambda_i(\xi) < 0$ separately. $\square$

Lemma 1.4.2. *If $\underline{\kappa}_\xi(t_1) < 0$ then for $0 < t \leq t_1$ we have*

$$\frac{d}{dt}\log\theta_N(t\xi) \leq (m-n-1)\sqrt{-\underline{\kappa}_\xi(t_1)} + \sum_{i=1}^{n}\Big(\sqrt{-\underline{\kappa}_\xi(t_1)}\mathbf{1}_{\{|\lambda_i(\xi)| < \sqrt{-\underline{\kappa}_\xi(t_1)}\}}$$
$$+\lambda_i(\xi)\mathbf{1}_{\{|\lambda_i(\xi)| \geq \sqrt{-\underline{\kappa}_\xi(t_1)}\}}\Big).$$

Proof. Fix $\kappa < 0$ and $\lambda \in \mathbb{R}$. Note that

$$\lim_{t \downarrow 0} (\coth(t) - 1/t) = 0, \quad \lim_{t \uparrow \infty} (\coth(t) - 1/t) = 1$$

and that by Taylor's theorem the derivative of this function is strictly positive for positive t. Therefore $\coth(t) - 1/t \leq 1$ for $t \in (0, \infty)$ and $G_\kappa(t) \leq \sqrt{-\kappa}$. Note also that we have

$$-\frac{d}{dt} F_\kappa^\lambda(t) = \frac{\kappa + \lambda^2}{(C_\kappa(t) + \lambda S_\kappa(t))^2}$$

so F_κ^λ is increasing on $(0, t_1]$ if and only if $|\lambda| < \sqrt{-\kappa}$. If $|\lambda| \geq \sqrt{-\kappa}$ then F_λ^κ is non-increasing and $F_\lambda^\kappa(t) \leq \lim_{t \downarrow 0} F_\lambda^\kappa(t) = \lambda$. Conversely if $|\lambda| < \sqrt{-\kappa}$ then

$$C_\kappa(t) + \lambda S_\kappa(t) \geq \cosh\left(\sqrt{-\kappa} t\right) - \sinh\left(\sqrt{-\kappa} t\right) = e^{-\sqrt{-\kappa} t}$$

so F_λ^κ is defined on $(0, \infty)$ and

$$F_\lambda^\kappa(t) \leq \lim_{t \uparrow \infty} F_\lambda^\kappa(t) \leq \sqrt{-\kappa} \lim_{t \uparrow \infty} \left(\frac{\sinh(t) + \cosh(t)}{\cosh(t) - \sinh(t)} \right) = \sqrt{-\kappa}.$$

The lemma then follows from inequality (1.14) by setting $\kappa_\xi(t_1) = \underline{\kappa}_\xi(t_1)$. ☐

Proposition 1.4.3. *For* $0 < t \leq t_1$ *we have*

$$\frac{d}{dt} \log \theta_N(t\xi) \leq (m-1)\sqrt{|\underline{\kappa}_\xi(t_1) \wedge 0|} + \sum_{i=1}^{n} |\lambda_i(\xi)|. \tag{1.23}$$

Proof. By Lemmas 1.4.1 and 1.4.2 it follows that

$$\begin{aligned}
&\frac{d}{dt} \log \theta_N(t\xi) \\
&\leq \sum_{i=1}^{n} \lambda_i(\xi) \mathbf{1}_{\{\underline{\kappa}_\xi(t_1) \geq 0\}} + (m-n-1)\sqrt{-\underline{\kappa}_\xi(t_1)} \mathbf{1}_{\{\underline{\kappa}_\xi(t_1) < 0\}} \\
&\quad + \sum_{i=1}^{n} \left(\sqrt{-\underline{\kappa}_\xi(t_1)} \mathbf{1}_{\{|\lambda_i(\xi)| < \sqrt{-\underline{\kappa}_\xi(t_1)}\}} + \lambda_i(\xi) \mathbf{1}_{\{|\lambda_i(\xi)| \geq \sqrt{-\underline{\kappa}_\xi(t_1)}\}} \right) \mathbf{1}_{\{\underline{\kappa}_\xi(t_1) < 0\}} \\
&\leq \sum_{i=1}^{n} |\lambda_i(\xi)| \mathbf{1}_{\{\underline{\kappa}_\xi(t_1) \geq 0\}} + (m-n-1)\sqrt{|\underline{\kappa}_\xi(t_1) \wedge 0|}
\end{aligned}$$

$$+\, n\sqrt{|\underline{\kappa}_\xi(t_1)\wedge 0|} + \sum_{i=1}^{n} |\lambda_i(\xi)| \mathbf{1}_{\{\underline{\kappa}_\xi(t_1)<0\}}$$
$$= (m-1)\sqrt{|\underline{\kappa}_\xi(t_1)\wedge 0|} + \sum_{i=1}^{n} |\lambda_i(\xi)|$$

as required. □

Note that the factor $(m-1)$ appearing on the right-hand side of (1.23) is reasonable since an orthonormal basis of a tangent space $T_{\gamma_\xi}M$ gives rise to precisely $(m-1)$ orthogonal planes containing the radial direction $\dot{\gamma_\xi}$. In the next corollary we assume the existence of radially uniform lower bounds on curvature.

Corollary 1.4.4. *Suppose that there is a function $\kappa : [0,\infty) \to \mathbb{R}$ such that for each $\xi \in UTN^\perp$ and $t_1 \in (0, c_N(\xi))$ we have $\kappa(t_1) \leq \underline{\kappa}_\xi(t_1)$. Furthermore, suppose that the principal curvatures of N are bounded in modulus by a constant $\Lambda \geq 0$. Then*

$$\frac{\partial}{\partial r_N} \log \Theta_N \leq n\Lambda + (m-1)\sqrt{|\kappa(r_N)\wedge 0|} \tag{1.24}$$

on $M(N)$.

Proof. For each $\xi \in UTN^\perp$ and $t_1 \in (0, c_N(\xi))$ we see by Proposition 1.4.3 that

$$\frac{\partial}{\partial r_N} \log \Theta_N(\gamma_\xi(t_1)) = \frac{d}{dt} \log \theta_N(t\xi)\bigg|_{t=t_1} \leq n\Lambda + (m-1)\sqrt{|\kappa(t_1)\wedge 0|}.$$

Since for each $p \in M(N)\backslash N$ there exists a unique $\xi_p \in UTN^\perp$ such that $\gamma_{\xi_p}(r_N(p)) = p$, the result follows for such p by setting $t_1 = r_N(p)$. For $p \in N$ the radial derivative is set equal to zero in which case the result is trivial. □

Following from remarks made at the end of Subsection 1.3.2, for the case $n \in \{0, m-1\}$ there are alternative estimates available in terms of Ricci and mean curvature. In particular, recall that $\underline{\rho}_\xi(t_1)$ is defined by (1.15) and suppose that there is a function $\rho : [0,\infty) \to \mathbb{R}$ such that for each $\xi \in UTN^\perp$ and $t_1 \in (0, c_N(\xi))$ we have

$\rho(t_1) \leq \underline{\rho}_\xi(t_1)$. If $n = 0$ then

$$\frac{\partial}{\partial r_N} \log \Theta_N \leq (m-1)\sqrt{|\rho(r_N) \wedge 0|} \tag{1.25}$$

on $M(N)$ while if $n = m-1$ and for each $\xi \in UTN^\perp$ we have $|H(\xi)| \leq \Lambda$ then

$$\frac{\partial}{\partial r_N} \log \Theta_N \leq (m-1)\left(\sqrt{|\rho(r_N) \wedge 0|} + \Lambda\right) \tag{1.26}$$

on $M(N)$.

1.4.2 Laplacian Inequalities

The following theorem follows immediately from Corollary 1.4.4 and the remarks which followed it. It is our main result in this chapter, so we restate the hypotheses for clarity.

Theorem 1.4.5. *Suppose that M is complete and connected Riemannian manifold of dimension m and that N is a closed embedded submanifold of M of dimension $n \in \{0, \ldots, m-1\}$. Denote by* $\mathrm{Cut}(N)$ *the cut locus of N, by r_N the distance to N, by $\frac{\partial}{\partial r_N}$ the radial vector field and suppose that there exist constants $C_1, C_2 \geq 0$ such that one of the following conditions is satisfied off N and* $\mathrm{Cut}(N)$*:*

(C1) *$n \in \{0, \ldots, m-1\}$, the sectional curvature satisfies the lower bound*

$$K\left(\frac{\partial}{\partial r_N} \wedge \ \cdot \ \right) \geq -(C_1 + C_2 r_N)^2$$

and the absolute value of the principal curvature of N is bounded by a non-negative constant Λ;

(C2) *$n = 0$ and the Ricci curvature satisfies the lower bound*

$$\mathrm{Ric}\left(\frac{\partial}{\partial r_N}, \frac{\partial}{\partial r_N}\right) \geq -(m-1)(C_1 + C_2 r_N)^2; \tag{1.27}$$

(C3) *$n = m-1$, the Ricci curvature satisfies the lower bound (1.27) and the absolute value of the mean curvature of N is bounded by a non-negative constant Λ.*

Then, with Θ_N defined by (1.4), we have the inequality

$$\frac{\partial}{\partial r_N} \log \Theta_N \leq n\Lambda + (m-1)(C_1 + C_2 r_N) \tag{1.28}$$

from which it follows that we have the estimate

$$\frac{1}{2}\triangle r_N^2 \leq (m-n) + (n\Lambda + (m-1)C_1)r_N + (m-1)C_2 r_N^2 \tag{1.29}$$

on $M \setminus \mathrm{Cut}(N)$.

It is important to point out that the conditions of Theorem 1.4.5 refer only to curvature involving the radial direction $\frac{\partial}{\partial r_N}$. Of course, these conditions all hold if M is compact. Note also that if $n = 0$ then the mean curvature does not play a role and if $m = 1$ then the sectional curvatures do not play a role, but that in either case the above estimates still make sense.

Corollary 1.4.6. *The assumptions of Theorem 1.4.5 imply*

$$\frac{1}{2}\triangle r_N^2 \leq \nu + \lambda r_N^2 \tag{1.30}$$

on $M \setminus \mathrm{Cut}(N)$ where

$$\begin{cases} \nu = m - n + \frac{1}{2}\left(n\Lambda + (m-1)C_1\right) \\ \lambda = \frac{1}{2}\left(n\Lambda + (m-1)C_1\right) + (m-1)C_2. \end{cases}$$

For the particular case in which N is a point p, it was proved by Yau [1976] that if the Ricci curvature is bounded below by a constant R then the Laplacian of the distance r_p is bounded above by $(m-1)/r_p$ plus a constant depending on R. In Yau [1978] that bound was shown to imply the stochastic completeness of M. Yau used analytic techniques, whereas in Chapter 3 we use a more probabilistic approach. A relaxation of Yau's condition which allows the curvature to grow like a negative quadratic in the distance function is essentially optimal from the point of view of curvature and non-explosion. This is why we did not feel it necessary to present Theorem 1.4.5

in terms of a more general growth function, as we did in the introduction. We will return to this matter in Section 3.1. If in Yau's example we set $(m-1)\varrho = R$ then by inequality (1.16) and Taylor approximation, as explained in Subsection 1.3.3, it follows that $G_\varrho(t) \leq -\varrho t/3$, for all $t \geq 0$ if $\varrho \leq 0$ or for $t \in [0, \frac{\pi}{\sqrt{\varrho}})$ if $\varrho > 0$, which implies the simple estimate

$$\frac{\partial}{\partial r_p} \log \Theta_p \leq -\frac{Rr_p}{3} \tag{1.31}$$

on $M \setminus \mathrm{Cut}(p)$, having the advantage of taking into account the effect of positive curvature. This in turn yields the Laplacian estimate

$$\frac{1}{2}\triangle r_p^2 \leq m - \frac{Rr_p^2}{3} \tag{1.32}$$

on $M \setminus \mathrm{Cut}(p)$, which is different to Yau's bound. We will use the estimates (1.31) and (1.32) in a couple of examples.

Remark 1.4.7. *By equation (1.6) it follows that the Laplacian inequalities (1.29), (1.30) and (1.32) hold on all of M in the sense of distributions.*

1.4.3 Volume Inequalities

If $\xi \in UTN^\perp$ with $t_1 \in (0, f_N(\xi))$ with $\underline{\kappa}_\xi(t_1)$ defined by (1.10) then by applying Gronwall's inequality to the differential inequality (1.11) it follows, by equations (1.12) and (1.13), that

$$\theta_N(t\xi)t^{m-n-1} \leq S_{\underline{\kappa}_\xi^{m-n-1}(t_1)}(t) \prod_{i=1}^{n} \Big(C_{\underline{\kappa}_\xi(t_1)}(t) + \lambda_i(\xi) S_{\underline{\kappa}_\xi(t_1)}(t) \Big)$$

for all $0 \leq t \leq t_1$. By the inequality for arithmetic and geometric averages this implies

$$\theta_N(t\xi)t^{m-n-1} \leq S^{m-n-1}_{\underline{\kappa}_\xi(t_1)}(t) \Big(C_{\underline{\kappa}_\xi(t_1)}(t) + H(\xi) S_{\underline{\kappa}_\xi(t_1)}(t) \Big)^n$$

for all $0 \leq t \leq t_1$. Gray [2004] used inequalities of this type to deduce comparison theorems for the volume of tubular neighbourhoods. We can instead use the secondary estimates obtained in Subsections 1.4.1 and 1.4.2. For example, since $\theta_N|_N = 1$, the assumptions of Theorem 1.4.5 imply by Proposition 1.4.3 and Gronwall's inequality that

$$\theta_N(t\xi) \leq e^{(n\Lambda+(m-1)(C_1+C_2t))t} \tag{1.33}$$

for all $\xi \in UTN^\perp$ and $t \in [0, f_N(\xi))$. In order to estimate the volume of a tubular neighbourhood $B_r(N)$, however, we only need bounds on the curvature within the tube.

Theorem 1.4.8. *Suppose that N is compact with $r > 0$. If $n \in \{0, m-1\}$ then denote by $(m-1)\varrho(r)$ the minimum of the Ricci curvature on $B_r(N)$ and by Λ the maximum of the absolute value of the mean curvature of N. Otherwise denote by $\varrho(r)$ the minimum sectional curvature on $B_r(N)$ and by Λ the maximum of the absolute values of the principal curvatures of N. Then*

$$\mathrm{vol}_M(B_r(N)) \leq \mathrm{vol}_{\mathbb{R}^{m-n}}(B_{\bar{c}_N\wedge r}(0))\,\mathrm{vol}_N(N)e^{(n\Lambda+(m-1)\sqrt{|\varrho(r)\wedge 0|})(\bar{c}_N\wedge r)} \tag{1.34}$$

where $\bar{c}_N := \sup\{c_N(\xi) : \xi \in UTN^\perp\}$.

Proof. By Theorem 1.2.1, Proposition 1.4.3 and Gronwall's inequality we see that

$$\begin{aligned}
& \int_M \mathbf{1}_{B_r(N)}(q)\,d\,\mathrm{vol}_M(q) \\
= \;& \int_N \int_{UT_pN^\perp} \int_0^{c_N(\xi)} \mathbf{1}_{B_r(N)}(\exp_N(t\xi))\theta_N(t\xi)t^{m-n-1}\,dt\,d\sigma^{m-n-1}(\xi)\,d\,\mathrm{vol}_N(p) \\
\leq \;& \int_N \int_{UT_pN^\perp} \int_0^{\bar{c}_N\wedge r} e^{(n\Lambda+(m-1)\sqrt{|\varrho(r)\wedge 0|})t}t^{m-n-1}\,dt\,d\sigma^{m-n-1}(\xi)\,d\,\mathrm{vol}_N(p) \\
\leq \;& \mathrm{vol}_{\mathbb{R}^{m-n}}(B_{\bar{c}_N\wedge r}(0))\,\mathrm{vol}_N(N)e^{(n\Lambda+(m-1)\sqrt{|\varrho(r)\wedge 0|})(\bar{c}_N\wedge r)}
\end{aligned}$$

and so the proposition is proved. □

If M is non-negatively curved then from the original Heintze-Karcher inequalities it follows, by comparison with the flat case, that one has the superior estimate

$$\theta_N(t\xi) \leq (1+\Lambda t)^k \tag{1.35}$$

for all $\xi \in UTN^{\perp}$ and $t \in [0, c_N(\xi))$, which yields the superior bound

$$\operatorname{vol}_M(B_r(N)) \leq \operatorname{vol}_{\mathbb{R}^{m-n}}(B_{\bar{c}_N \wedge r}(0)) \operatorname{vol}_N(N)(1 + \Lambda(\bar{c}_N \wedge r))^k. \tag{1.36}$$

For the case $n = 0$ there are alternative estimates, given by the following proposition, which we will use on several occasions in Chapters 5 and 6.

Proposition 1.4.9. *Fix $p \in M$ and suppose that the Ricci curvature is bounded below by R. Then*

$$\theta_p(t\xi) \leq e^{-\frac{Rt^2}{3}} \tag{1.37}$$

for all $p \in M$, $\xi \in UT_pM$ and $t \in [0, f_p(\xi))$.

Proof. The proposition follows from inequality (1.31) and Gronwall's inequality. □

If $R > 0$ then the proposition implies, by a change of variables, that $\operatorname{vol}_M(M) \leq (3\pi/R)^{\frac{m}{2}}$. It also implies, by a different change of variables, that if $p \in M$ with $0 < r < \operatorname{inj}(p)$ and if $R(r)$ denotes the minimum Ricci curvature in the ball $B_r(p)$ then there is a Bishop-Gromov-type inequality

$$\operatorname{vol}_M(B_r(p)) \leq \operatorname{vol}_{\mathbb{R}^m}(B_r(0))e^{-\frac{R(r)r^2}{3}}. \tag{1.38}$$

One can also use the Jacobian estimates (1.33), (1.35) and (1.37), together with the change of variables formula (1.9) in Corollary (1.2.2), to obtain estimates for the *area of the boundary* of tubes around N of sufficiently small radius, simpler than those in Gray [2004] which are based directly on the Heintze-Karcher comparison.

Chapter 2

Semimartingales, Local Time and Brownian Motion

Introduction

Our main result in this chapter is the Tanaka formula given in Subsection 2.3.2, which leads to a concept of local time on a hypersurface. Applying this to Brownian motion will yield formula (2.14) for the radial part, which is used throughout Chapter 3 and which could have applications to the study of reflected processes. For the Brownian case we also prove an occupation times approximation, given by formula (2.16), a formula for the expected local time, given by Theorem 2.4.1, and a large time relation, given by Corollary 2.4.2.

Section 2.1 is a short review of basic definitions and notation for the local time of real semimartingales. See Revuz and Yor [1999].

Section 2.2 summarizes the basic theory of semimartingales on manifolds, including stochastic development and Itô's formula. See Elworthy [1982] and Émery [1989].

In Section 2.3 we define local time on a hypersurface, using the description of the cut locus given in Chapter 1 and the formula in Barden and Le [1995], which generalizes the formula of Cranston, Kendall and March [1993] to semimartingales. The one-dimensional Tanaka formula gives meaning to local time for real semimartingales, so it seems natural to consider semimartingales in the manifold setting. The approach based on Markov theory used by Cranston, Kendall and March is not as well suited to the bridge processes we introduce in Chapter 4, since it requires the existence of an excessive reference measure.

Section 2.4 considers the special case of Brownian motion, for which it suffices to consider the Markovian approach. This uses the fact, originally proved by Revuz [1970], that with respect to a suitable reference measure there is a one-to-one correspondence between continuous additive functionals and smooth Radon measures which do not charge semi-polar sets. Cranston, Kendall and March used this fact in conjunction with smooth approximation (see Azagra, Ferrera, López-Mesas and Rangel [2007] for more about the smooth approximation of Lipschitz functions) to derive their formula for the distance between Brownian motion and a point, which applies more generally to certain functions which are locally the difference of two convex functions (see Bačák and Borwein [2011] for more on such functions). Smooth approximation also features in Barden and Le's approach but not explicitly so, since it is used to derive the one-dimensional Tanaka formula on which their proof is based.

We conclude the chapter with a couple of examples. The first is an example of Corollary 2.4.2, based on the unit circle, while the second considers how the expected local time of an $\mathbb{R}^m$-valued Brownian motion on the boundary of a ball scales for large times as the radius of the ball remains fixed.

2.1 Local Time for Real Semimartingales

2.1.1 Real Semimartingales

Suppose that $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ is a filtered probability space satisfying the usual conditions. A process $X : [0,\infty) \times \Omega \to \mathbb{R}$ is a called an $(\mathcal{F}_t, \mathbb{P})$-*semimartingale* if X has the decomposition $X = M + V$ where M is an $(\mathcal{F}_t, \mathbb{P})$-local martingale and where V is an $(\mathcal{F}_t)$-adapted process of locally bounded variation. Our filtered probability space will remain fixed throughout this chapter, so we can safely drop reference to it from our terminology. If X is a *continuous* semimartingale, then the decomposition is unique with respect to the filtration and the processes into which X decomposes must also be continuous. We will only be concerned with continuous processes. If X is such a process then we denote by $[X]$ the quadratic variation of X and Itô's formula states that for $f \in C^2(\mathbb{R})$ we have

$$f(X_t) = f(X_0) + \int_0^t f'(X_s)dX_s + \frac{1}{2}\int_0^t f''(X_s)d\,[X]_s \tag{2.1}$$

for all $t \geq 0$, almost surely. This formula can be extended in a number of ways, summarized by Ghomrasni and Peskir [2003], including convex functions. Indeed, suppose that $f : \mathbb{R} \to \mathbb{R}$ is a convex function. Then f is differentiable at all but a countable set of points and the left and right derivatives of f, which we will denote by f'_- and f'_+ respectively, exist everywhere. For a continuous semimartingale X it can be proved, using Itô's formula and approximation by C^2 functions, that there exist continuous non-decreasing processes $A^{f,-}$ and $A^{f,+}$ such that the formulae

$$\begin{aligned} f(X_t) &= f(X_0) + \int_0^t f'_-(X_s)dX_s + A_t^{f,-} \\ f(X_t) &= f(X_0) + \int_0^t f'_+(X_s)dX_s + A_t^{f,+} \end{aligned}$$

hold for all $t \geq 0$, almost surely.

2.1.2 Local Time

Suppose that $f(x) = |x|$. If we define the two functions

$$\mathrm{sgn}^-(x) := \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x \leq 0 \end{cases}, \quad \mathrm{sgn}^+(x) := \begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0 \end{cases}$$

then $f'_-(x) = \mathrm{sgn}^-(x)$ and $f'_+(x) = \mathrm{sgn}^+(x)$ and it follows that for any $a \in \mathbb{R}$ there exist continuous, non-decreasing and non-negative processes $L^{a,-}(X)$ and $L^{a,+}(X)$

such that the formulae

$$|X_t - a| = |X_0 - a| + \int_0^t \operatorname{sgn}^-(X_s - a)dX_s + L_t^{a,-}(X) \tag{2.2}$$

$$|X_t - a| = |X_0 - a| + \int_0^t \operatorname{sgn}^+(X_s - a)dX_s + L_t^{a,+}(X) \tag{2.3}$$

hold for all $t \geq 0$, almost surely. Either of these equations can be referred to as *Tanaka's formula*. We will refer to the process $L^{a,-}(X)$ as the *left local time of X at a* and to the process $L^{a,+}(X)$ as the *right local time of X at a*. Roughly speaking, these processes record the amount of time spent by X at a but they do not in general agree with one another. In fact, by subtraction, equations (2.2) and (2.3) imply

$$\frac{1}{2}\left(L_t^{a,-}(X) - L_t^{a,+}(X)\right) = \int_0^t \mathbf{1}_{\{X_s = a\}} dX_s \tag{2.4}$$

for all $t \geq 0$, almost surely. Given a convex function f we can also consider the symmetric derivative given by $\frac{1}{2}\left(f'_- + f'_+\right)$. For example, if $f(x) = |x|$ then the symmetric derivative of f is given by the function

$$\operatorname{sgn}(x) := \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases} \cdot \tag{2.5}$$

With this in mind, suppose that X is a continuous semimartingale and consider the process

$$L^a(X) := \frac{1}{2}\left(L^{a,-}(X) + L^{a,+}(X)\right)$$

which will we refer to as the *symmetric local time of X at a*, and see by equations (2.2) and (2.3) that

$$|X_t - a| = |X_0 - a| + \int_0^t \operatorname{sgn}(X_s - a)dX_s + L_t^a(X) \tag{2.6}$$

for all $t \geq 0$, almost surely. In particular, if X is a continuous semimartingale then the Tanaka formula implies that $|X|$ is also a continuous semimartingale.

2.1.3 Basic Properties of Local Time

There exists a modification of the process $\{L_t^{a,-}(X) : a \in \mathbb{R}, t \in [0,\infty)\}$ such that the map $(a,t) \mapsto L_t^{a,-}(X)$ is continuous in t and càdlàg in a, almost surely, as shown in Revuz and Yor [1999]. This modification is the only version to which we will refer and it satisfies $L^{a-,-}(X) = L^{a,+}(X)$.

Lemma 2.1.1. *For any continuous semimartingale X we have $L^0(|X|) = L^0(X)$.*

Proof. First note that $L^{0,+}(|X|) = L^{0-,-}(|X|) = 0$. Furthermore, according to [Revuz and Yor, 1999, p.232], if $a \geq 0$ then $L^{a,-}(|X|) = L^{a,-}(X) + L^{(-a)-,-}(X)$ and therefore $L^{0,-}(|X|) = L^{0,-}(X) + L^{0,+}(X)$. So the lemma follows from the definition of the symmetric local time. $\square$

Associated to the processes $L^{a,-}(X)$ and $L^{a,+}(X)$ there are the random measures $dL^{a,-}(X)$ and $dL^{a,+}(X)$ whose support is contained in the set $\{t \geq 0 : X_t = a\}$.

The *occupation times formula* states, in terms of the left local time, that for any non-negative measurable function $f : \mathbb{R} \to \mathbb{R}$ we have

$$\int_0^t f(X_s) d\,[X]_s = \int_{\mathbb{R}} f(a) L_t^{a,-}(X) da$$

for all $t \geq 0$, almost surely. This formula and the right continuity of the left local time imply

$$L_t^{a,-}(X) = \lim_{\epsilon \downarrow 0} \frac{1}{\epsilon} \int_0^t \mathbf{1}_{[a,a+\epsilon)}(X_s) d\,[X]_s$$

from which it follows that

$$\begin{aligned} L_t^0(|X|) &= \frac{1}{2} L_t^{0,-}(|X|) \\ &= \lim_{\epsilon \downarrow 0} \frac{1}{2\epsilon} \int_0^t \mathbf{1}_{[0,\epsilon)}(|X_s|) d\,[|X|]_s \\ &= \lim_{\epsilon \downarrow 0} \frac{1}{2\epsilon} \int_0^t \mathbf{1}_{(-\epsilon,\epsilon)}(X_s) d\,[|X|]_s \end{aligned} \tag{2.7}$$

for all $t \geq 0$, almost surely. These equations imply that processes of locally bounded variation do not generate local time. The occupation times formula can be used to show that if X is a local martingale then the processes $L_t^{a,-}(X)$ and $L_t^{a,+}(X)$ agree with one another. If these two processes agree do with one another then $L^a(X)$ is continuous in a and can unambiguously be referred to as *local time of X at a*.

2.2 Semimartingales on Manifolds

2.2.1 Semimartingales on $\mathbb{R}^m$

We say that an $\mathbb{R}^m$-valued process $X = (X^1, \ldots, X^m)$ is a *semimartingale* if each of its components X^i is a semimartingale in the sense of the previous section. As mentioned above, we are concerned only with continuous semimartingales. If $X = \big(X^1, \ldots, X^m\big)$ is such a process then Itô's formula states that for $f \in C^2(\mathbb{R}^m)$ we have

$$f(X_t) = f(X_0) + \int_0^t D_i f(X_s) dX_s^i + \frac{1}{2} \int_0^t D_{jk} f(X_s) d[X^j, X^k]_s \tag{2.8}$$

for all $t \geq 0$, almost surely, where D_i and D_{jk} stand for the first and second partial derivatives of f and where we employ the usual summation convention over repeated up-down indices.

2.2.2 Semimartingales on M

Suppose that M is a smooth metrizable manifold of dimension m. We say that an M-valued process X is a *semimartingale* if for each $f \in C^2(M)$ the real-valued process $f(X)$ is a semimartingale in the sense of the previous section. Note that if $M = \mathbb{R}^m$ then this definition agrees with the one in the previous subsection. The collection of continuous semimartingales exhibits certain stability properties. For example, if $F : M \to \tilde{M}$ is a smooth map between manifolds and if X is a continuous semimartingale on M then $F(X)$ is a continuous semimartingale on $\tilde{M}$. Furthermore, if X is a continuous semimartingale with respect to $\mathbb{P}$ and if $\mathbb{Q}$ is a probability measure which is absolutely continuous with respect to $\mathbb{P}$ then X is a continuous semimartingale with respect to $\mathbb{Q}$. It follows that if $\mathbb{P}$ and $\mathbb{Q}$ are equivalent then the collection of processes which are continuous semimartingales with respect to $\mathbb{P}$ coincides with the collection of those which are with respect to $\mathbb{Q}$.

2.2.3 The Orthonormal Frame Bundle

Now suppose that M is a Riemannian manifold equipped with its Levi-Civita connection. In this setting we wish to write down a version of Itô's formula for M-valued continuous semimartingales. In order to do this we must first introduce some auxilliary objects. An *orthonormal frame* at $p \in M$ is an $\mathbb{R}$-linear isometry $u : \mathbb{R}^m \to T_pM$ and the collection of all orthonormal frames at a point p is denoted by $\mathcal{O}_p(M)$. The *orthonormal frame bundle* is then defined to be the disjoint union $\mathcal{O}(M) := \bigsqcup_{p \in M} \mathcal{O}_p(M)$. Since each fibre $\mathcal{O}_p(M)$ is diffeomorphic to the orthogonal group $O(m, \mathbb{R})$, it follows that $\mathcal{O}(M)$ can be made into a differentiable manifold of dimension $\frac{m}{2}(m+1)$ and the canonical projection $\Pi : \mathcal{O}(M) \to M$ is a smooth map between manifolds.

2.2.4 Horizontal Lifts and Antidevelopment

A smooth curve U taking values in $\mathcal{O}(M)$ is called *horizontal* if for each $e \in \mathbb{R}^m$ the vector field Ue is parallel along the curve ΠU. If $u \in \mathcal{O}(M)$ then a vector in $T_u\mathcal{O}(M)$ is called *horizontal* if it is the tangent vector to a horizontal curve starting at u. We denote by $H_u\mathcal{O}(M)$ the space of all horizontal tangent vectors at u. It follows that $T_u\mathcal{O}(M) = \ker(D_u\Pi) \oplus H_u\mathcal{O}(M)$ and for $u \in \mathcal{O}(M)$ there is a canonical lift map $H(u) : \mathbb{R}^m \to H_u\mathcal{O}(M)$ given by

$$H(u)e = (D_u\Pi|_{H_u\mathcal{O}(M)})^{-1}(ue).$$

An $\mathcal{O}(M)$-valued continuous semimartingale U is called *horizontal* if there exists an $\mathcal{F}_0$-measurable $\mathcal{O}(M)$-valued random variable U_0 and an $\mathbb{R}^m$-valued continuous semimartingale Z such that U solves the Stratonovich equation

$$dU_t = H(U_t) \circ dZ_t \tag{2.9}$$

with initial condition U_0. For the sense in which this equation should be interpreted, see Elworthy [1982]. The process Z, which if it exists can be shown to be unique, is called the *antidevelopment* of U. If X is an M-valued continuous semimartingale then an $\mathcal{O}(M)$-valued horizontal continuous semimartingale U is called a *horizontal lift* of X if $\Pi U = X$. If one specifies an $\mathcal{F}_0$-measurable $\mathcal{O}(M)$-valued random variable U_0 such that $\Pi U_0 = X_0$ then there exists a *unique* horizontal lift U of X with initial condition U_0 and the dependence of the process U (and of its antidevelopment Z) on the choice of U_0 commutes with the action of the orthogonal group. Alternatively, if we begin with an $\mathcal{F}_0$-measurable $\mathcal{O}(M)$-valued random variable U_0 and an $\mathbb{R}^m$-valued continuous semimartingale Z then we say that the projection onto M of the maximal solution to equation (2.9) with inital condition U_0 is the *development* of the semimartingale Z onto M with respect to U_0.

2.2.5 Itô's Formula

We are now in a position to write down an intrinsic version of Itô's formula. In particular, if $f \in C^2(M)$ with X a continuous semimartingale on M with a horizontal lift U and anti-development Z and if $(e_1, \ldots, e_m)$ is an orthonormal basis for $\mathbb{R}^m$ then

$$f(X_t) = f(X_0) + \int_0^t U_s e_i f(X_s) dZ_s^i + \frac{1}{2} \int_0^t U_s e_j U_s e_k f(X_s) d[Z^j, Z^k]_s$$

for all $t \geq 0$, almost surely, which can can be written more succinctly as

$$f(X_t) = f(X_0) + \int_0^t \langle \nabla f(X_s), U_s dZ_s \rangle + \frac{1}{2} \int_0^t \operatorname{tr} \operatorname{Hess}_{X_s} f(U_s, U_s) d\,[Z]_s\,. \tag{2.10}$$

Note that if $M = \mathbb{R}^m$ then, after the usual identifications, we can choose $U_s = \mathrm{id}_{\mathbb{R}^m}$ and $Z = X$ and this formula reduces to formula (2.8). Just as the basic Itô formula (2.1) can be extended from C^2 functions to those which can be written as the difference of two convex functions, so formula (2.10) can be extended to a wider class of possibly non-differentiable functions.

2.3 Local Time for Semimartingales on Manifolds

2.3.1 Barden and Le's Formula

In this section we begin by stating a generalization of Itô's formula due to Barden and Le [1995], some consequences of which were considered in Barden and Le [1997]. Recalling that a connected $(m-1)$-dimensional submanifold N is called *two-sided* if its normal bundle is trivial, suppose that $f : M \to \mathbb{R}$ is a continuous function which fails to be C^2 on an at most countable disjoint union $\mathscr{L}$ of open subsets O_i of two-sided submanifolds of M. Suppose that for each i there is an open subset U_i of M such that $O_i = \mathscr{L} \cap U_i$ and such that $U_i \setminus O_i$ has two components. Choose a unit normal vector field $\mathbf{n}$ on $\mathscr{L}$ and for each i let U_i^+ be the component of $U_i \setminus O_i$ into which $\mathbf{n}$ points and let U_i^- be the other component. If we define $\mathscr{H}_i^{\pm} := O_i \cup U_i^{\pm}$ then suppose further that for each i there are C^2 functions $g_i^{\pm}$ on U_i such that $f|\mathscr{H}_i^{\pm} = g_i^{\pm}|\mathscr{H}_i^{\pm}$ and denote by P^i orthogonal projection onto O_i, uniquely defined on an open set containing O_i. If X is a continuous semimartingale on M with horizontal lift U and antidevelopment Z then Barden and Le proved that there exist two continuous, non-decreasing and non-negative predictable processes $L^{\pm\mathbf{n},\mathscr{L}}(X)$, whose associated random measures $dL^{\pm\mathbf{n},\mathscr{L}}(X)$ are supported by $\mathscr{L}$, almost surely, such that

$$
\begin{aligned}
f(X_t) = f(X_0) &+ \int_0^t \mathbf{1}_{\{X_s \notin \mathscr{L}\}} \langle \nabla f(X_s), U_s dZ_s \rangle \\
&+ \int_0^t \mathbf{1}_{\{X_s \in O_i\}} \langle \nabla (f \circ P^i)(X_s), U_s dZ_s \rangle \\
&+ \frac{1}{2} \int_0^t \mathbf{1}_{\{X_s \notin \mathscr{L}\}} \operatorname{tr} \operatorname{Hess}_{X_s} f(U_s, U_s) d\,[Z]_s \\
&+ \frac{1}{2} \int_0^t \mathbf{1}_{\{X_s \in O_i\}} \operatorname{tr} \operatorname{Hess}_{X_s} (f \circ P^i)(U_s, U_s) d\,[Z]_s \\
&+ \frac{1}{2} \int_0^t \Big(D^+_{X_s} f(\mathbf{n}) dL_s^{-\mathbf{n},\mathscr{L}}(X) - D^-_{X_s} f(\mathbf{n}) dL_s^{+\mathbf{n},\mathscr{L}}(X) \Big)
\end{aligned}
$$

for all $t \geq 0$, almost surely, where the summation convention applies to the index i and where the Gâteaux derivatives $D^{\pm} f$ are defined by

$$D_p^+ f(v) := \lim_{\epsilon \downarrow 0} \frac{1}{\epsilon} \left(f(\exp_p(\epsilon v)) - f(p) \right), \; D_p^- f(v) := -D_p^+ f(-v)$$

for $p \in M$ and $v \in T_p M$. The proof of this formula given in Barden and Le [1995] argues that one can assume, by localizing to a coordinate patch, that $M = \mathbb{R}^m$ equipped with the metric induced by the coordinates and that $\mathscr{L} = \{(x_1, \ldots, x_m) \in \mathbb{R}^m : x_1 = 0\}$ with $\mathscr{H}^+ = \{(x_1, \ldots, x_m) \in \mathbb{R}^m : x_1 \geq 0\}$. The proof then refers to the one-dimensional setting in order to obtain a version of the desired formula in local coordinates. Finally it is checked that the result is independent of the choice of coordinates. As noted by Barden and Le, the proof shows that if we define a process $Y^{\mathscr{L}}$ for times during which X is in a small neighbourhood of $\mathscr{L}$ to be $d(X, \mathscr{L})$ on the side of $\mathscr{L}$ into which $\mathbf{n}$ points and $-d(X, \mathscr{L})$ on the other, then $L^{\pm\mathbf{n}, \mathscr{L}}(X) = L^{0,\pm}(Y^{\mathscr{L}})$. Given this interpretation of the processes $L^{\pm\mathbf{n}, \mathscr{L}}(X)$ and the observation that the quantity $(D^+ - D^-) f(\mathbf{n})$ represents the gradient discontinuity of f on $\mathscr{L}$ in the normal direction, the intuition behind the above formula is hopefully clear.

2.3.2 Tanaka Formula

Definition 2.3.1. *Suppose that N is a closed embedded hypersurface and that X is a continuous semimartingale. If $r_N(X)$ is a continuous semimartingale then the process*

$$L^N(X) := L^0(r_N(X)) \tag{2.11}$$

will be called the local time of X on N*.*

Suppose in addition that N is two-sided. Then it follows, by the regularity properties of r_N and $\mathrm{Cut}(N)$ given in Chapter 1, that if $\check{C}(N)$ is polar for X (i.e. if the first hitting time of $\check{C}(N)$ by X is almost surely infinite) then we can apply Barden and Le's formula to the function r_N with $\mathscr{L} = \mathring{C}(N) \cup N$ to deduce that $r_N(X)$ is a continuous semimartingale. In this case Definition 2.3.1 is explained by the following remarks. Firstly, since the random measures $dL^{\pm\mathbf{n}, \mathring{C}(N) \cup N}(X)$ are supported by $\mathring{C}(N) \cup N$ almost surely and since N and $\mathring{C}(N)$ are disjoint it follows that we can write

$$L^{\pm\mathbf{n},\mathring{C}(N)\cup N}(X) = L^{\pm\mathbf{n},\mathring{C}(N)}(X) + L^{\pm\mathbf{n},N}(X)$$

where the processes $L^{\pm\mathbf{n},\mathring{C}(N)}(X)$ and $L^{\pm\mathbf{n},N}(X)$ have associated random measures whose supported is contained in the sets $\mathring{C}(N)$ and N, respectively. Now, if $\mathbf{n}$ is a unit normal vector field on $\mathring{C}(N) \cup N$ then

$$D^{+} r_N(\mathbf{n})|_N = -D^{-} r_N(\mathbf{n})|_N = 1$$

and so it follows that

$$\int_0^t \left(D^{+}_{X_s} r_N(\mathbf{n}) dL^{-\mathbf{n},N}_s(X) - D^{-}_{X_s} r_N(\mathbf{n}) dL^{+\mathbf{n},N}_s(X) \right) = L^{-\mathbf{n},N}_t(X) + L^{+\mathbf{n},N}_t(X)$$

for all $t \geq 0$, almost surely. If we define the process Y^N for N just as the process $Y^{\mathscr{L}}$ was defined for $\mathscr{L}$ at the end of the previous subsection then

$$\frac{1}{2} \left(L^{-\mathbf{n},N}(X) + L^{+\mathbf{n},N}(X) \right) = L^0(Y^N).$$

For times during which X is close to N we have $|Y^N| = r_N(X)$, so it follows from Lemma 2.1.1 that $L^0(Y^N) = L^0(r_N(X))$ and Definition 2.3.1 therefore seems reasonable. Denoting by $\mathring{C}_i(N)$ the connected components of $\mathring{C}(N)$ and by P^N orthogonal projection onto N, uniquely defined on a neighbourhood of N, it follows that we have a Tanaka formula

$$\begin{aligned}
r_N(X_t) = r_N(X_0) &+ \int_0^t \mathbf{1}_{\{X_s \notin N \cup \mathring{C}(N)\}} \langle \nabla r_N(X_s), U_s dZ_s \rangle \\
&+ \int_0^t \mathbf{1}_{\{X_s \in N\}} \langle \nabla (r_N \circ P^N)(X_s), U_s dZ_s \rangle \\
&+ \int_0^t \mathbf{1}_{\{X_s \in \mathring{C}_i(N)\}} \langle \nabla (r_N \circ P^i)(X_s), U_s dZ_s \rangle \\
&+ \frac{1}{2} \int_0^t \mathbf{1}_{\{X_s \notin N \cup \mathring{C}(N)\}} \operatorname{tr} \operatorname{Hess}_{X_s} r_N(U_s, U_s) d\,[Z]_s \\
&+ \frac{1}{2} \int_0^t \mathbf{1}_{\{X_s \in N\}} \operatorname{tr} \operatorname{Hess}_{X_s} (r_N \circ P^N)(U_s, U_s) d\,[Z]_s
\end{aligned}$$

$$+\frac{1}{2}\int_0^t \mathbf{1}_{\{X_s\in\mathring{C}_i(N)\}}\operatorname{tr}\mathrm{Hess}_{X_s}(r_N\circ P^i)(U_s,U_s)d\left[Z\right]_s$$
$$+\frac{1}{2}\int_0^t \Big(D^+_{X_s}r_N(\mathbf{n})dL_s^{-\mathbf{n},\mathring{C}(N)}(X)-D^-_{X_s}r_N(\mathbf{n})dL_s^{+\mathbf{n},\mathring{C}(N)}(X)\Big)$$
$$+L_t^N(X)$$

for all $t\geq 0$, almost surely. There does not seem to be a statement of this formula elsewhere in the literature, except for in the one point case. Note that the indicator functions used above correspond, in some sense, to the use of symmetric derivatives which is why the final correction term corresponds to a symmetric local time. Since $r_N(X)$ is evidently a continuous semimartingale one can apply equation (2.7) to see that

$$L_t^N(X)=\lim_{\epsilon\downarrow 0}\frac{1}{2\epsilon}\int_0^t \mathbf{1}_{B_\epsilon(N)}(X_s)d\left[r_N(X)\right]_s \tag{2.12}$$

for all $t\geq 0$, almost surely, where $B_\epsilon(N)$ denotes the tubular neighbourhood of N of radius ϵ. While the two basic assumptions made in this subsection were that N should be two-sided and that $\check{C}(N)$ should be polar for X, the latter assumption is not necessary in order for $L^N(X)$ to be well-defined. The two-sidedness assumption is also not necessary under certain circumstances, as in the next section.

Note that given a predictable stopping time ζ one can also define M-valued semimartingales on the stochastic time interval $[0,\zeta)$, using time-change. In particular, if X is a continuous semimartingale defined up to a predictable stopping time ζ and if τ is a stopping time with $0\leq\tau\leq\zeta$ and $\tau<\zeta$ if $\zeta>0$ then the formulae of this subsection also hold for the process X at the random times $t\wedge\tau$.

2.4 Local Time for Brownian Motion

2.4.1 Brownian Motion

A continuous stochastic process taking values on M with the strong Markov property and defined upto a predictable stopping time whose infinitesimal generator is of the form $\frac{1}{2}\triangle+b$, for some locally bounded measurable vector field b, is called a *Brownian* *motion with drift* b or simply a *Brownian motion* if $b = 0$. Any elliptic diffusion operator induces a Riemannian structure with respect to which it can be written in the form $\frac{1}{2}\triangle + b$ for some smooth vector field b. For general conditions on b under which this operator generates a strongly continuous Markovian semigroup in $L^2(M)$, see Shigekawa [2010].

If X is a Brownian motion with locally bounded and measurable drift b defined upto a predictable stopping time ζ with initial distribution X_0 then X is a continuous semimartingale and if τ is a stopping time with $0 \leq \tau \leq \zeta$ and $\tau < \zeta$ if $\zeta > 0$ then Itô's formula implies that for $f \in C^2(M)$ we have

$$f(X_{t\wedge\tau}) = f(X_0) + \int_0^{t\wedge\tau} \langle \nabla f(X_s), U_s dB_s \rangle + \int_0^{t\wedge\tau} \left(\frac{1}{2}\triangle + b\right) f(X_s)ds$$

for $t \geq 0$, almost surely, where $\{U_s : 0 \leq s < \xi\}$ is a horizontal lift of $\{X_s : 0 \leq s < \xi\}$ whose antidevelopment has martingale part given by an $\mathbb{R}^m$-valued Brownian motion B and finite variation part given by $\int_0^\cdot U_s^{-1} b(X_s)ds$. In fact, a continuous semimartingale on M is a Brownian motion if and only if it is the development of a Brownian motion on $\mathbb{R}^m$. In particular, if $x \in M$ with $U_0 \in \mathcal{O}_x(M)$ then the development of an $\mathbb{R}^m$-valued Brownian motion with respect to U_0 is an M-valued Brownian motion $X(x)$ starting at x defined upto an explosion time $\zeta(x)$.

The explosion time is the predictable stopping time at which $X(x)$ leaves all compact subsets of M. If $\zeta(x)$ is almost surely infinite and if M is connected then it follows that all Brownian motions on M are non-explosive. This property is called *stochastic completeness*. If M is stochastically complete with $m \geq 2$ then by removing a single point from M one obtains a manifold which is stochastically complete but not geodesically complete. As mentioned below in Section 3.1, there are plenty of manifolds which are geodesically complete but not stochastically complete.

2.4.2 Brownian Local Time

Suppose that M is a complete and connected Riemannian manifold of dimension m, that N is a closed embedded submanifold of M of dimension $n \in \{0, \ldots, m-1\}$

and that X is a Brownian motion on M with locally bounded and measurable drift b defined upto an explosion time ζ. It follows from [Taylor and Watson, 1985, p.331] that if $A \in \mathcal{B}(M)$ with $\dim_H(A) < m-1$ (where $\dim_H$ denotes Hausdorff dimension, as in Appendix A) then $\mathbb{P}\{X_t \in A \text{ for some } 0 < t < \zeta\} = 0$. Therefore $\check{C}(N)$ is polar for X since, as mentioned in Chapter 1, $\check{C}(N)$ is a set of Hausdorff dimension at most $m-2$. Now assume only that $\mathrm{vol}_M(A) = 0$. Then one can show, using Fubini's theorem and the existence of transistion densities for X, as in [Karatzas and Shreve, 1991, p.105], that for almost all $\omega \in \Omega$ the Lebesgue measure of $\{0 \leq t < \zeta(\omega) : X_t(\omega) \in A\}$ is zero. An example of such a set is given by $N \cup \mathring{C}(N)$. Since stochastic integrals with respect to a continuous semimartingale Z are only defined upto $d\,[Z]$-equivalence classes, sets of $d\,[Z]$-measure zero can be can be discarded from such integrals. In particular, since B is a Brownian motion it follows that the four terms on the right-hand side of the Tanaka formula in Subsection 2.3.2 involving orthogonal projection vanish, almost surely, and that

$$\int_0^{\cdot} \mathbf{1}_{\{X_s \notin N \cup \mathring{C}(N)\}} \langle \nabla r_N(X_s), U_s dB_s \rangle = \int_0^{\cdot} \left\langle \frac{\partial}{\partial r_N}, U_s dB_s \right\rangle = \beta. \tag{2.13}$$

where β is a standard one-dimensional Brownian motion, by Lévy's characterization and the fact that U consists of isometries. Consequently $d\,[r_N(X)]_s = ds$ and so by equation (2.4) it follows that the two processes $L^{\pm\mathbf{n}, \mathring{C}(N) \cup N}(X)$ agree with one another, almost surely. In this setting it is therefore not necessary to assume that N is two-sided in order for the Tanaka formula of Subsection 2.3.2 to be valid. It follows that if τ is a stopping time with $0 \leq \tau < \zeta$ then

$$r_N(X_{t\wedge\tau}) = r_N(X_0) + \beta_{t\wedge\tau} + \int_0^{t\wedge\tau} \left(\frac{1}{2}\triangle + b\right) r_N(X_s) ds - \mathbb{L}^{\mathrm{Cut}(N)}_{t\wedge\tau}(X) + L^N_{t\wedge\tau}(X) \tag{2.14}$$

for $t \geq 0$, almost surely, where the non-negative process $\mathbb{L}^{\mathrm{Cut}(N)}(X)$ is defined by

$$d\mathbb{L}^{\mathrm{Cut}(N)}(X) := -\frac{1}{2}\left(D^+_X - D^-_X\right) r_N(\mathbf{n})\, dL^{\mathring{C}(N)}(X). \tag{2.15}$$

Note that the integral on the right-hand side of (2.14) is well-defined since, as mentioned above, the set of times at which X takes values in $N \cup \mathrm{Cut}(N)$ has Lebesgue measure zero. Of course, if $n \leq m-2$ then N is polar for X and the local time $L^N(X)$ vanishes while if $n = m-1$ then

$$L^N_{t\wedge\tau}(X) = \lim_{\epsilon\downarrow 0} \frac{1}{2\epsilon} \int_0^{t\wedge\tau} \mathbf{1}_{B_\epsilon(N)}(X_s)\, ds \tag{2.16}$$

almost surely, by equations (2.12) and (2.13).

Before moving on to consider the Markovian approach, we should mention that the concept of local time on a submanifold has previously been considered in the context of reflected diffusions in a smooth domain, which relates to the Neumann problem and in which the local time appears as part of the solution to a corresponding Skorokhod problem. See [Wang, 2014, Chapter 3] and the references contained therein for more about this connection. For the case in which N is a point, formula (2.14) was proved in Cranston, Kendall and March [1993] using the approach mentioned in the next subsection.

2.4.3 Revuz Measure

We have thus far approached local time for the point of view of continuous semimartingales and the Tanaka formula. An alternative approach comes from the theory of Markov processes. This is not well suited to the processes considered in later chapters but it does work well for Brownian motion. Suppose that M is compact, that N is a closed embedded hypersurface and that $X(x)$ is a Brownian motion on M starting at x. Then the convexity based approach of Cranston, Kendall and March, which they applied to the one point case, adapts to our situation and implies that with respect to the invariant measure vol_M the non-negative process $\mathbb{L}^{\mathrm{Cut}(N)}(X(x))$, defined by (2.15), corresponds in the sense of Revuz [1970] to the measure

$$-\frac{1}{2}(D^+ - D^-)r_N(\mathbf{n})\, \mathcal{H}^{m-1}_M$$

which is Radon when restricted to $\mathrm{Cut}(N)$. Here $\mathcal{H}^{m-1}_M$ denotes the $(m-1)$-dimensional Hausdorff measure of M, normalized so as to agree with the induced measure on hypersurfaces (as in Appendix A). Similarly, the local time $L^N(X(x))$, defined by (2.11), corresponds to the induced measure vol_N, which ties in with Savo's decomposition of the distributional Laplacian given in Subsection 1.2.7. By a result of Fitzsimmons, Pitman and Yor [1993] this implies the following theorem, in which p^M denotes the transition density function of Brownian motion.

Theorem 2.4.1. *Suppose that M is compact, that N is a closed embedded hypersurface and that $X(x)$ is a Brownian motion on M starting at x. Then*

$$\mathbb{E}\left[L_t^N(X(x))\right] = \int_0^t \int_N p_s^M(x,y)d\,\mathrm{vol}_N(y)\,ds \tag{2.17}$$

for all $t \geq 0$.

In Chapter 5 we will calculate, bound and provide an asymptotic relation for the rate of change $\frac{d}{dt}\mathbb{E}\left[L_t^N(X(x))\right]$. Note that, by the change of variables formula (1.8), the expected value of the occupation times appearing inside the limit on the right-hand side of (2.16) converges to the right-hand side of (2.17) as $\epsilon \downarrow 0$.

Corollary 2.4.2. *Suppose that M is compact, that N is a closed embedded hypersurface and that X is a Brownian motion on M. Then*

$$\lim_{t\uparrow\infty} \frac{1}{t}\mathbb{E}\left[L_t^N(X)\right] = \frac{\mathrm{vol}_N(N)}{\mathrm{vol}_M(M)}. \tag{2.18}$$

Proof. Li [1986] proved that $\lim_{t\uparrow\infty} p_t^M = \mathrm{vol}_M(M)^{-1}$ so the corollary follows from Theorem 2.4.1. $\square$

Example 2.4.3. *Suppose $M = \mathbb{S}^1$ (the unit circle equipped with the standard metric) and let $X(x)$ be a Brownian motion starting at $x \in \mathbb{S}^1$. By formula (2.14), or equivalently by the formula of Cranston, Kendall and March [1993], it follows that*

$$r_x^2(X_t(x)) = r_x^2(x) + 2\int_0^t r_x(X_s(x))d\beta_s + t - 2\int_0^t r_x(X_s(x))dL_s^{\mathrm{Cut}(x)}(X(x))$$

for $t \geq 0$, where β is a standard one-dimensional Brownian motion. But $r_x(x) = 0$ and $\mathrm{Cut}(x)$ is antipodal to x, which is a distance π away from x, so as $dL^{\mathrm{Cut}(x)}(X(x))$

is supported on $\{s \geq 0 : X_s = \mathrm{Cut}(x)\}$ *we deduce that*

$$r_x^2(X_t(x)) = 2\int_0^t r_x(X_s(x))d\beta_s + t - 2\pi L_t^{\mathrm{Cut}(x)}(X(x)) \tag{2.19}$$

for $t \geq 0$. *Now* $p_t^{\mathbb{S}^1}(x,\cdot) \to (2\pi)^{-1}$ *as* $t \uparrow \infty$ *so*

$$\lim_{t\uparrow\infty} \mathbb{E}\left[r_x^2(X_t(x))\right] = \int_{\mathbb{S}^1} \frac{r_x^2(y)}{2\pi} d\,\mathrm{vol}_{\mathbb{S}^1}(y) = \int_{-\pi}^{\pi} \frac{v^2}{2\pi} dv = \frac{\pi^2}{3}. \tag{2.20}$$

Thus by equations (2.19) and (2.20) it follows that

$$\frac{\pi^2}{3} = \lim_{t\uparrow\infty}\left(t - 2\pi\mathbb{E}[L_t^{\mathrm{Cut}(x)}(X(x))]\right)$$

which implies that for large times t *we have the approximation*

$$\mathbb{E}[L_t^{\mathrm{Cut}(x)}(X(x))] \simeq \frac{t}{2\pi} - \frac{\pi}{6}. \tag{2.21}$$

This agrees with numerical approximation, as shown by Figure 2.

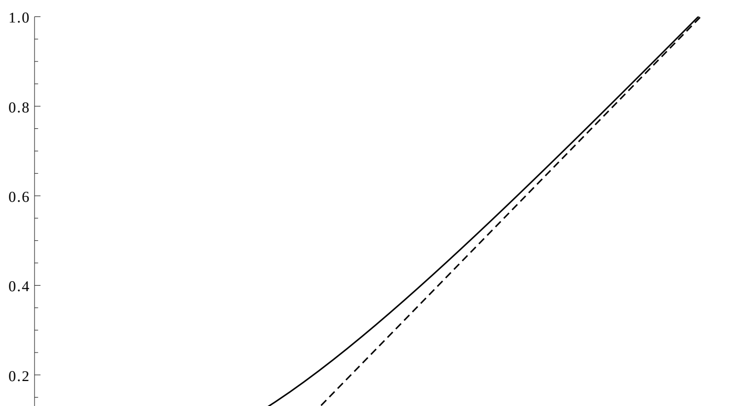

Figure 2: The solid curve represents the graph of the left-hand side of (2.21), calculated numerically using formula (2.17) by expressing $p^{\mathbb{S}^1}$ as a theta function. The dashed line represents the graph of the right-hand side of (2.21). The horizontal axis represents the time t.

Example 2.4.4. *For $r > 0$ denote by $\mathbb{S}^{m-1}(r)$ the boundary of the open ball in $\mathbb{R}^m$ of radius r centred at the origin. If X is a Brownian motion on $\mathbb{R}^m$ starting at the origin then, using polar coordinates, we deduce*

$$\frac{1}{r}\mathbb{E}\left[L_t^{\mathbb{S}^{m-1}(r)}(X)\right] = \frac{\Gamma\left(\frac{m}{2}-1, \frac{r^2}{2t}\right)}{\Gamma\left(\frac{m}{2}\right)}. \tag{2.22}$$

In particular, for the case $m = 2$ we obtain

$$\frac{1}{r}\mathbb{E}\left[L_t^{\mathbb{S}^{1}(r)}(X)\right] = \Gamma\left(0, \frac{r^2}{2t}\right). \tag{2.23}$$

By differentiating the exponential of the right-hand side of equation (2.23) one deduces the curious relation

$$\lim_{t\uparrow\infty}\left(\log\left(\frac{2t}{r^2}+1\right) - \frac{1}{r}\mathbb{E}\left[L_t^{\mathbb{S}^{1}(r)}(X)\right]\right) = \gamma,$$

where γ denotes the Euler-Mascheroni constant. Note that if $m = 1$ then the exponential of the left-hand side of (2.22) diverges in t exponentially, while for $m \geq 3$ it does so logarithmically. Only for $m = 2$ is the divergence linear.

Chapter 3

From Geometric Inequalities to Probabilistic Estimates

Introduction

The main results in this chapter are Theorem 3.2.6, which is a sharp radial moment estimate, and Theorem 3.2.10, which is an exponential estimate derived from the moment estimate. Theorem 3.2.10 improves and generalizes a theorem of Stroock and the improvements lead to a comparison theorem. In particular, the constants appearing in our estimates are given explicitly. We use Theorem 3.2.10 to derive a concentration inequality for tubes, given by Theorem 3.2.14, and various Feynman-Kac estimates, given in Subsection 3.2.9.

Section 3.1 presents a brief overview of stochastic completeness. See Grigor'yan [1999] and Li [2000] for excellent surveys of this topic.

Section 3.2 is a study of the distance between Brownian motion and a submanifold. This is something which has not been previously emphasised in the literature. We are particularly interested in exponential integrability, which in other contexts has been studied by Aida, Masuda and Shigekawa [1994] and Aida and Stroock [1994] using a log-Sobolev inequality. In Subsection 3.2.1 we use the heat kernel log-Sobolev inequality of Bakry and Ledoux [2006] to obtain exponential estimates under a lower bound on Ricci curvature. For the remainder of Section 3.2 we consider a more general situation and use induction on moments to obtain our estimates. Hu [1999] used a similar approach to study the uniform exponential integrability of $\mathbb{R}^m$-valued diffusion processes under C^2 functions satisfying a Lyapunov condition. Several of the results in this section also apply to such functions, but we focus on the distance function. Section 3.2 concludes with an estimate on the first exit time of a Brownian motion (with drift) from a tubular neighbourhood, the *mean* exit time having previously been studied by Gray, Karp and Pinsky [1986], who calculated an asymptotic expansion. The results of Section 3.2 are of independent interest, although the methods will also be used later in the thesis.

Section 3.3 includes the version of Girsanov's theorem to which we will later refer and several remarks which are of relevance to Chapter 4.

3.1 Stochastic Completeness

3.1.1 An Overview

Yau [1978] proved that a lower bound on Ricci curvature implies stochastic completeness. This was extend by Ichihara [1982] to allow the Ricci curvature to grow in the negative direction in a certain way, like for example a negative quadratic in the distance function. Conversely, if M has a pole p and there exist constants $C_3 \geq 0$, $C_4 > 0$ and $\epsilon > 0$ such that

$$\sup\{K(\sigma_x) : \sigma_x \text{ is a two-dimensional subspace of } T_xM\} \leq -C_3 - C_4 r_p(x)^{2+\epsilon}$$

for each $x \in M$ then M is not stochastically complete, as proved by Varopoulos [1983]. It follows that for the applications considered in the next section, the curvature assumptions appearing in Theorem 1.4.5 are essentially the best that one could hope for while discarding the effect of the cut locus.

More sophisticated conditions are given in terms of isomperimetric constants or volume growth. Ichihara [1982], for example, found a necessary and sufficient condition for the stochastic completeness of non-compact manifolds with radial symmetry, in terms of the growth of the ratio of the volume of a ball with the area of its boundary. Grigor'yan [1987] proved that if for some $p \in M$, with $V_r(p)$ denoting the volume of the ball $B_r(p)$, and a fixed $r_0 > 0$ one has

$$\int_{r_0}^{\infty} \frac{r dr}{\log V(p,r)} = \infty$$

then M is stochastically complete. Using this condition and Theorem 1.4.8 we therefore have another proof of the quadratic curvature condition. Note, however, that Grigor'yan's condition takes into account the effect of the cut locus in a way that the quadratic curvature condition does not. A weak uniform cover criterion was given in Li [1994a]. It would be interesting to know whether this criterion covers Grigor'yan's result. In Li [1994b] it was shown that completeness at one point implies completeness everywhere. In Subsection 3.3.3 we prove a condition for stochastic completeness given in terms of *infinitesimal* volume.

3.2 Radial Moment Estimates

3.2.1 A Log-Sobolev Approach

It is fairly standard practice to deduce exponential integrability using a log-Sobolev inequality. In this subsection we will show how this can be done for a special case of the more general situation considered later. In the following theorem $\{P_t : t \geq 0\}$ denotes the heat semigroup.

Theorem 3.2.1. *Suppose that M is a complete and connected Riemannian manifold of dimension m and that N is a closed embedded submanifold of M of dimension $n \in \{0, \ldots, m-1\}$. Assume that there exist constants $C_1, \Lambda \geq 0$ such that*

$$\mathrm{Ric} \geq -(m-1)C_1^2$$

*and such that at least one of the conditions **(C1)**, **(C2)** or **(C3)** of Theorem 1.4.5* *is satisfied with* $C_2 = 0$. *Then*

$$P_t(e^{\theta r_N})(x) \leq \exp\left[\theta\left(r_N^2(x) + (m-n)t\right)^{\frac{1}{2}} + (n\Lambda + (m-1)C_1)\theta t/2 + \theta^2 C(t)/2\right] \tag{3.1}$$

for all $\theta, t \geq 0$ *and*

$$P_t(e^{\frac{\theta}{2}r_N^2})(x) \leq \exp\left[\frac{\theta\left(\left(r_N^2(x) + (m-n)t\right)^{\frac{1}{2}} + (n\Lambda + (m-1)C_1)\theta t/2\right)^2}{2(1 - C(t)\theta)}\right] \tag{3.2}$$

for all $0 \leq \theta < C^{-1}(t)$, *where*

$$C(t) := \frac{e^{(m-1)C_1^2 t} - 1}{(m-1)C_1^2}.$$

Proof. Let $X(x)$ a Brownian motion starting at $x \in M$, let $\{D_i\}_{i=1}^{\infty}$ be an exhaustion of M by regular domains (which certainly exists, since we are assuming that M is connected) and denote by τ_{D_i} the first exit time of $X(x)$ from D_i. By Itô's formula (2.1), formula (2.14), Theorem 1.4.5 and Jensen's inequality, we see that

$$\mathbb{E}\left[r_N^2(X_{t\wedge\tau_{D_i}}(x)\right] \leq r_N^2(x) + (m-n)t + (n\Lambda + (m-1)C_1)\int_0^t \mathbb{E}\left[r_N^2(X_{s\wedge\tau_{D_i}}(x)\right]^{\frac{1}{2}} ds$$

for all $t \geq 0$. Bihari's inequality, which is a nonlinear integral form of Gronwall's inequality, then implies (the right-hand side of the following inequality is the exact solution to the corresponding Bernoulli differential equation)

$$\mathbb{E}\left[r_N^2(X_{t\wedge\tau_{D_i}}(x)\right] \leq \left(\left(r_N^2(x) + (m-n)t\right)^{\frac{1}{2}} + (n\Lambda + (m-1)C_1)t/2\right)^2$$

for all $t \geq 0$, from which it follows that

$$P_t(r_N^2)(x) \leq \left(\left(r_N^2(x) + (m-n)t\right)^{\frac{1}{2}} + (n\Lambda + (m-1)C_1)t/2\right)^2 \tag{3.3}$$

for all $t \geq 0$, by Fatou's lemma. Now, Bakry and Ledoux discovered (see Bakry and Ledoux [2006] or Driver and Hu [1996]) that the Ricci bound implies the heat kernel log-Sobolev inequality

$$\operatorname{Ent}_t(f^2)(x) \le 2C(t)P_t\left(\|\nabla f\|^2\right)(x) \tag{3.4}$$

for all $f \in C^\infty(M)$ and $t > 0$. By a slight generalization of the classical argument of Herbst (see Ledoux [1999]) it follows that for Lipschitz F with $\|F\|_{Lip} \le 1$ and $\theta \in \mathbb{R}$ we have

$$P_t(e^{\theta F})(x) \le \exp\left[\theta P_t F(x) + \theta^2 C(t)/2\right] \tag{3.5}$$

for all $t \ge 0$. Furthermore, it was proved by Aida, Masuda and Shigekawa [1994] that the log-Sobolev inequality (3.4) implies

$$P_t(e^{\frac{\theta}{2}F^2})(x) \le \exp\left[\frac{\theta P_t F^2(x)}{2(1 - C(t)\theta)}\right] \tag{3.6}$$

for all $0 \le \theta < C^{-1}(t)$. Since r_N is Lipschitz with $\|r_N\|_{Lip} = 1$, inequality (3.2) follows from (3.3) by the estimate (3.6) while inequality (3.1) is proved similarly, by applying Jensen's inequality to (3.3) and using the estimate (3.5). $\square$

An estimate given by [Stroock, 2000, Theorem 8.62], which concerns only the case $N = \{x\}$, suggests that the double exponentials in the estimates (3.1) and (3.2) are not actually necessary (but note that they are the inevitable result of using Herbst's argument and Bakry and Ledoux's log-Sobolev constant, as opposed to being a consequence of our moment estimates). To obtain exponential integrability for the heat kernel under relaxed curvature assumption we will use a different approach, which is developed in the next subsection. While the estimates (3.1) and (3.2) are, roughly speaking, the best we have under the conditions of Theorem 3.2.1, our later estimates will have the advantage of taking into account positive curvature. Thus our later estimates are preferable from the point of view of comparison.

3.2.2 Lyapunov Assumptions

For the remainder of this section we suppose that $X(x)$ is a Brownian motion on M with locally bounded and measurable drift b starting from $x \in M$, defined upto an explosion time $\zeta(x)$, and that N is a closed embedded submanifold of M of dimension $n \in \{0, \ldots, m-1\}$. We will assume that there exist constants $\nu \geq 1$ and $\lambda \in \mathbb{R}$ such that the inequality

$$\left(\frac{1}{2}\triangle + b\right) r_N^2 \leq \nu + \lambda r_N^2 \tag{3.7}$$

holds on $M(N)$ (i.e. off the cut locus). All statements made in this chapter regarding the validity of this inequality refer to it over the domain $M(N)$, unless otherwise stated. If b grows linearly in r_N then geometric conditions under which such an inequality arises were given by Theorem 1.4.5 (see Corollary 1.4.6), the content of which the reader might like to briefly review. In particular, there are various situations in which one can choose $\lambda = 0$. Alternatively, if N is a point with $b = 0$ and the Ricci curvature is bounded below by a constant R then inequality (3.7) holds with $\nu = m$ and $\lambda = -R/3$, as stated by inequality (1.32). If N is an affine subspace of $\mathbb{R}^m$ with $b = 0$ then inequality (3.7) holds as an *equality* with $\nu = m - n$ and $\lambda = 0$.

3.2.3 First and Second Radial Moments

We are now in a position to deduce two basic moment estimates.

Theorem 3.2.2. *Suppose there exists constants $\nu \geq 1$ and $\lambda \in \mathbb{R}$ such that inequality (3.7) holds. Then*

$$\mathbb{E}\left[\mathbf{1}_{\{t<\zeta(x)\}} r_N^2(X_t(x))\right] \leq \left(r_N^2(x) + \nu R(t)\right) e^{\lambda t} \tag{3.8}$$

for all $t \geq 0$, where $R(t) := (1 - e^{-\lambda t})/\lambda$.

Proof. Let $\{D_i\}_{i=1}^{\infty}$ be an exhaustion of M by regular domains and denote by τ_{D_i} the first exit time of $X(x)$ from D_i. These stopping times announce the explosion time $\zeta(x)$. By Itô's formula (2.1) and formula (2.14), it follows that

$$\begin{aligned} r_N^2(X_{t\wedge\tau_{D_i}}(x)) = r_N^2(x) + 2\int_0^{t\wedge\tau_{D_i}} r_N(X_s(x)) \left(d\beta_s - d\mathbb{L}_s^{\mathrm{Cut}(N)}(X(x))\right) \\ + \int_0^{t\wedge\tau_{D_i}} \left(\frac{1}{2}\triangle + b\right) r_N^2(X_s(x)) ds \end{aligned} \tag{3.9}$$

for all $t \geq 0$, almost surely. Since the domains D_i are of compact closure the Itô integral in (3.9) is a local martingale and it follows that

$$\begin{aligned}\mathbb{E}\left[r_N^2(X_{t\wedge\tau_{D_i}}(x))\right] = r_N^2(x) - 2\mathbb{E}\left[\int_0^{t\wedge\tau_{D_i}} r_N(X_s(x))d\mathbb{L}_s^{\mathrm{Cut}(N)}(X(x))\right] \\ + \int_0^t \mathbb{E}\left[\mathbf{1}_{\{s<\tau_{D_i}\}}\left(\frac{1}{2}\triangle + b\right) r_N^2(X_s(x))\right] ds\end{aligned} \tag{3.10}$$

for all $t \geq 0$, where exchanging the order of integrals in the last term is justified by the use of the stopping time and the assumptions of the theorem. Before applying Gronwall's inequality we should be careful, since we are allowing the coefficient λ to be negative. For this, note that

$$\mathbb{E}\left[r_N^2(X_{t\wedge\tau_{D_i}}(x))\right] = \mathbb{E}\left[\mathbf{1}_{\{t<\tau_{D_i}\}}r_N^2(X_t(x))\right] + \mathbb{E}\left[\mathbf{1}_{\{t\geq\tau_{D_i}\}}r_N^2(X_{\tau_{D_i}}(x))\right] \tag{3.11}$$

and that the two functions

$$t \mapsto \mathbb{E}\left[\int_0^{t\wedge\tau_{D_i}} r_N(X_s(x))d\mathbb{L}_s^{\mathrm{Cut}(N)}(X(x))\right], \quad t \mapsto \mathbb{E}\left[\mathbf{1}_{\{t\geq\tau_{D_i}\}}r_N^2(X_{\tau_{D_i}}(x))\right] \tag{3.12}$$

are non-decreasing. If we define a function $f_{x,i,2} : [0,\infty) \to [0,\infty)$ by

$$f_{x,i,2}(t) := \mathbb{E}\left[\mathbf{1}_{\{t<\tau_{D_i}\}}r_N^2(X_t(x))\right]$$

then $f_{x,i,2}$ is differentiable, since the boundaries of the D_i are smooth, and it follows from (3.10) and (3.11) that we have the differential inequality

$$\begin{cases} f'_{x,i,2}(t) \leq \nu + \lambda f_{x,i,2}(t) \\ f_{x,i,2}(0) = r_N^2(x) \end{cases} \tag{3.13}$$

for all $t \geq 0$, where we used the assumption $\nu \geq 0$. Applying Gronwall's inequality to (3.13) yields

$$\mathbb{E}\left[\mathbf{1}_{\{t<\tau_{D_i}\}}r_N^2(X_t(x))\right] \leq r_N^2(x)e^{\lambda t} + \nu\left(\frac{e^{\lambda t}-1}{\lambda}\right) \tag{3.14}$$

for all $t \geq 0$, from which the result follows by the monotone convergence theorem. $\square$

We will refer the object on the left-hand side of inequality (3.8) as *the second radial moment of $X(x)$ with respect to N*. Note that

$$\lim_{\lambda \to 0} (1 - e^{-\lambda t})/\lambda = t$$

and that this provides the sense in which Theorem 3.2.2 and similar statements should be interpreted if we set $\lambda = 0$. Note also that if $\lambda \geq 0$ then by comparing Taylor coefficients we see that $R(t) \leq t$, yielding a slightly simpler estimate, while if $\lambda < 0$ then there is the bound

$$\mathbb{E}\left[\mathbf{1}_{\{t<\zeta(x)\}} r_N^2(X_t(x))\right] \leq -\frac{\nu}{\lambda}$$

for all $t \geq 0$.

The *short time asymptotics* of the second radial moment in the one point case have been studied by Liao and Zheng [1995]. In particular, they proved that if $X(x)$ is a Brownian motion on M and if τ_ϵ denotes the first exit time of $X(x)$ from the ball of radius ϵ centred at x then

$$\mathbb{E}\left[r_x^2(X_{\tau_\epsilon \wedge t}(x))\right] = mt - \frac{1}{6}\,\mathrm{scal}(x) t^2 + o(t^2) \quad \text{as } t \downarrow 0, \tag{3.15}$$

where $o(t^2)$ might depend upon ϵ and where $\mathrm{scal}(x)$ denotes the scalar curvature at x and they proved that the stopping time can be dispensed with under certain conditions, such as when M is compact.

Example 3.2.3. *Suppose that $X(x)$ is a Brownian motion on $\mathbb{H}^3_\kappa$ starting at x. By the heat kernel formula (5.2) given below and the Jacobian formula (1.19) given above it follows that*

$$\mathbb{E}[r_x^2(X_t(x))] = 3t - \kappa t^2$$

for all $t \geq 0$. This ties in with Liao and Zheng's relation (3.15) since on $\mathbb{H}^3_\kappa$ the scalar curvature is constant and equal to 6κ. We will return to this example several *times in this chapter.*

Kim, Park and Jeon [2004] improved the original method to calculate the asymptotics upto order four. The asymptotics for the submanifold case have yet to be investigated; this is a direction for future research. For the hypersurface case $n = m - 1$ one can find an inequality for the first radial moment using an approach similar to the proof of Theorem 3.2.2, since in this case the factor involving the reciprocal of r_N in formula (1.5) disappears. For generality it will suffice to use Jensen's inequality to deduce the following corollary.

Corollary 3.2.4. *Suppose there exists constants $\nu \geq 1$ and $\lambda \in \mathbb{R}$ such that inequality (3.7) holds. Then*

$$\mathbb{E}\left[\mathbf{1}_{\{t<\zeta(x)\}} r_N(X_t(x))\right] \leq \left(\left(r_N^2(x) + \nu R(t)\right) e^{\lambda t}\right)^{\frac{1}{2}} \tag{3.16}$$

for all $t \geq 0$.

We will refer the object on the left-hand side of inequality (3.16) as *the first radial moment of $X(x)$ with respect to N.*

3.2.4 Non-explosion

The quadratic curvature condition mentioned in Section 3.1 is implied by the following theorem, which is a simple consequence of our moment estimates.

Theorem 3.2.5. *Suppose that N is compact and that there exist constants $\nu \geq 1$ and $\lambda \in \mathbb{R}$ such that inequality (3.7) holds. Then $X(x)$ is non-explosive.*

Proof. For $i \in \mathbb{N}$ denote by $\tau_{B_i(N)}$ the first exit time of $X(x)$ from the tube $B_i(N)$. By following the proof of Theorem 3.2.2 with the stopping times τ_{D_i} replaced by $\tau_{B_i(N)}$ we deduce

$$\mathbb{P}\{\tau_{B_i(N)} \leq t\} \leq \frac{\left(r_N^2(x) + \nu R(t)\right) e^{\lambda t}}{i^2}$$

for all $t \geq 0$. This crude exit time estimate implies that $X(x)$ is non-explosive, since the compactness of N implies that the stopping times $\tau_{B_i(N)}$ announce the explosion time $\zeta(x)$. □

For non-compact N, however, the validity of inequality (3.7) is generally not a sufficient condition for the non-explosion of $X(x)$ since this does not rule out situations where $X(x)$ explodes in a direction tangential to N. Simple examples of such situations are given by the products of stochastically complete manifolds with ones which are not.

3.2.5 Higher Radial Moments

Recall that if X is a real-valued Gaussian random variable with mean μ and variance σ^2 then for $p \in \mathbb{N}$ the p-th absolute moment of X is given by the formula

$$\mathbb{E}\left[|X|^p\right] = 2^{\frac{p}{2}} \sigma^p \frac{\Gamma(\frac{1+p}{2})}{\sqrt{\pi}} {}_1F_1\left(-\frac{p}{2}, \frac{1}{2}, -\frac{\mu^2}{2\sigma^2}\right) \tag{3.17}$$

where Γ is the gamma function and where ${}_1F_1$ is the confluent hypergeometric function of the first kind. We note that for even moments equation (3.17) can be written

$$\mathbb{E}\left[|X|^{2p}\right] = \left(2\sigma^2\right)^p p! L_p^{-\frac{1}{2}}\left(-\frac{\mu^2}{2\sigma^2}\right) \tag{3.18}$$

where $L_p^\alpha(z)$ are the *Laguerre polynomials*, defined by the formula

$$L_p^\alpha(z) = e^z \frac{z^{-\alpha}}{p!} \frac{\partial^p}{\partial z^p}\left(e^{-z} z^{p+\alpha}\right)$$

for $p = 0, 1, 2, \ldots$ and $\alpha > -1$. For example, if $X(x)$ is a standard Brownian motion on $\mathbb{R}$ starting from $x \in \mathbb{R}$ then

$$\mathbb{E}\left[|X_t(x)|^{2p}\right] = (2t)^p p! L_p^{-\frac{1}{2}}\left(-\frac{|x|^2}{2t}\right) \tag{3.19}$$

for all $t \geq 0$. An important fact about Laguerre polynomials used in the proof of the next theorem is that

$$L_p^\alpha(z) = \sum_{k=0}^{p} \frac{\Gamma(p+\alpha+1)}{\Gamma(k+\alpha+1)} \frac{(-z)^k}{k!(p-k)!} \tag{3.20}$$

which can be proved using Leibniz's formula, as in Lebedev [1972]. Although Theorem 3.2.2 is a special case of the following theorem, which we will later use to obtain exponential estimates, we stated it separately both for clarity and because it constitutes the base case in an induction argument.

Theorem 3.2.6. *Suppose that there exist constants $\nu \geq 1$ and $\lambda \in \mathbb{R}$ such that inequality (3.7) holds. Then for each $p \in \mathbb{N}$ it follows that*

$$\mathbb{E}\left[\mathbf{1}_{\{t<\zeta(x)\}} r_N^{2p}(X_t(x))\right] \leq \left(2R(t)e^{\lambda t}\right)^p p! L_p^{\frac{\nu}{2}-1}\left(-\frac{r_N^2(x)}{2R(t)}\right) \tag{3.21}$$

for all $t \geq 0$, where $R(t) := (1-e^{-\lambda t})/\lambda$.

Proof. For $p \in \mathbb{N}$ we have, off the cut locus, that

$$\triangle r_N^{2p} = r_N^{2p-2}\left(p\triangle r_N^2 + 4p(p-1)\right) \tag{3.22}$$

and therefore

$$\left(\frac{1}{2}\triangle + b\right) r_N^{2p} \leq p\left(\nu + 2\left(p-1\right)\right) r_N^{2p-2} + p\lambda r_N^{2p}. \tag{3.23}$$

By Itô's formula (2.1) and formula (2.14) we see that

$$\begin{aligned} r_N^{2p}(X_{t\wedge\tau_{D_i}}(x)) = r_N^{2p}(x) + 2p\int_0^{t\wedge\tau_{D_i}} r_N^{2p-1}(X_s(x))\left(d\beta_s - d\mathbb{L}_s^{\mathrm{Cut}(N)}(X(x))\right) \\ + \int_0^{t\wedge\tau_{D_i}}\left(\frac{1}{2}\triangle + b\right) r_N^{2p}(X_s(x))ds \end{aligned} \tag{3.24}$$

for all $t \geq 0$, almost surely, where the stopping times τ_{D_i} are defined as in the proof of Theorem 3.2.2. If we define the function $f_{x,i,2p} : [0,\infty) \to [0,\infty)$ by

$$f_{x,i,2p}(t) := \mathbb{E}\left[\mathbf{1}_{\{t<\tau_{D_i}\}} r_N^{2p}(X_t(x))\right]$$

then, arguing as we did in the proof of Theorem 3.2.2, we deduce from (3.23) the differential inequality

$$\begin{cases} f'_{x,i,2p}(t) \leq p\left(\nu+2\left(p-1\right)\right) f_{x,i,2(p-1)}(t) + p\lambda f_{x,i,2p}(t) \\ f_{x,i,2p}(0) = r_N^{2p}(x) \end{cases} \tag{3.25}$$

for all $t \geq 0$. Applying Gronwall's inequality to (3.25) yields

$$f_{x,i,2p}(t) \leq \left(r_N^{2p}(x) + p\left(\nu+2\left(p-1\right)\right) \int_0^t f_{x,i,2(p-1)}(s) e^{-p\lambda s} ds \right) e^{p\lambda t} \tag{3.26}$$

for all $t \geq 0$ and $p \in \mathbb{N}$. The next step in the proof is to use induction on p to show that

$$f_{x,i,2p}(t) \leq \sum_{k=0}^{p} \binom{p}{k} \left(2R(t)\right)^{p-k} r_N^{2k}(x) \frac{\Gamma(\frac{\nu}{2}+p)}{\Gamma(\frac{\nu}{2}+k)} e^{p\lambda t} \tag{3.27}$$

for all $t \geq 0$ and $p \in \mathbb{N}$. Inequality (3.14) covers the base case $p = 1$. If we hypothesise that the inequality holds for some $p-1$ then by inequality (3.26) we have

$$f_{x,i,2p}(t) \leq \left(r_N^{2p}(x) + p\left(\nu+2\left(p-1\right)\right) \sum_{k=0}^{p-1} \binom{p-1}{k} r_N^{2k}(x) \frac{\Gamma(\frac{\nu}{2}+p-1)}{\Gamma(\frac{\nu}{2}+k)} \tilde{R}(t) \right) e^{p\lambda t} \tag{3.28}$$

for all $t \geq 0$, where

$$\tilde{R}(t) = \int_0^t \left(2R(s)\right)^{p-1-k} e^{-\lambda s} ds.$$

Using the fact that

$$2(p-k)\tilde{R}(t) = \left(2R(t)\right)^{p-k}$$

together with the relation

$$\frac{p}{(p-k)} \binom{p-1}{k} = \binom{p}{k}$$

and the definition of the Gamma function, inequality (3.27) follows from (3.28) and so the inductive argument is complete. Since $\nu \geq 1$ we then can apply relation (3.20) to see that

$$\sum_{k=0}^{p}\binom{p}{k}(2R(t))^{p-k}r_N^{2k}(x)\frac{\Gamma(\frac{\nu}{2}+p)}{\Gamma(\frac{\nu}{2}+k)}=(2R(t))^p p!L_p^{\frac{\nu}{2}-1}\left(-\frac{r_N^2(x)}{2R(t)}\right)$$

and so by inequality (3.27) it follows that

$$f_{x,i,2p}(t)\leq\left(2R(t)e^{\lambda t}\right)^p p!L_p^{\frac{\nu}{2}-1}\left(-\frac{r_N^2(x)}{2R(t)}\right) \tag{3.29}$$

for $t\geq 0$ and $i,p\in\mathbb{N}$. The result follows from this by the monotone convergence theorem. □

We will refer the object on the left-hand side of inequality (3.21) as *the* $(2p)$*-th radial moment of* $X(x)$ *with respect to* N. Note that if $M=\mathbb{R}$ with N the origin and $b=0$ with $\nu=1$ and $\lambda=0$ then the right-hand side of inequality (3.21) is *equal* to the right-hand side of equation (3.19). It is important to note that our estimates have this property; we will later sum the even moments to obtain a sharp exponential estimate.

Example 3.2.7. *Suppose that* $X(x)$ *is a Brownian motion on* $\mathbb{H}^3_\kappa$ *starting at* x. *Using the heat kernel formula (5.2) and the Jacobian formula (1.19) we calculate that for each* $p\in\mathbb{N}$ *we have*

$$\mathbb{E}[r_x^{2p}(X_t(x))]=(2t)^p\frac{\Gamma\left(\frac{3}{2}+p\right)}{\Gamma\left(\frac{3}{2}\right)}{}_1F_1\left(\frac{3}{2}+p,\frac{3}{2},-\frac{\kappa t}{2}\right)$$

for all $t\geq 0$.

One can also deduce from Theorem 3.2.6 an estimate for *the* $(2p-1)$*-th radial moment of* $X(x)$ *with respect to* N, again by Jensen's inequality.

3.2.6 Exponential Estimates

Before using the estimates of the previous subsection to derive exponential estimates, we need an inequality for Laguerre polynomials. Articles such as those by Love [1997] and Pogány and Srivastava [2007] include numerous inequalities for these polynomials but none which are suited to our purposes. We therefore include the following lemma.

Lemma 3.2.8. *For $\alpha, z \geq 0$ and $m = 1, 2, \ldots$ we have*

$$p! L_p^{\alpha}(-z) \leq (12\,(1+z))^p \,\frac{\Gamma\,(\alpha+1+p)}{\Gamma\,(\alpha+1)}.$$

Proof. By formula (3.20) it follows that

$$p! L_p^{\alpha}(-z) = \sum_{k=0}^{p} \binom{p}{k} z^k \frac{\Gamma\,(\alpha+1+p)}{\Gamma\,(\alpha+1+k)}.$$

Since $\alpha, z \geq 0$ it follows that $\Gamma\,(\alpha+1+k) \geq \Gamma\,(\alpha+1)$ and $z^k \leq (1+z)^p$ for all $k \in \{0, \ldots, p\}$. For $k \in \{1, \ldots, p\}$ there is the bound

$$\binom{p}{k} \leq \left(\frac{pe}{k}\right)^k$$

and since the largest binomial coefficient is 'the middle one' it follows that

$$\binom{p}{k} \leq \begin{cases} \left(\frac{2pe}{p+1}\right)^{\frac{p+1}{2}} & \text{if } p \text{ is odd} \\ (2e)^{\frac{p}{2}} & \text{if } p \text{ is even} \end{cases}$$

which yields the simple bound

$$\binom{p}{k} \leq 6^p$$

for $k \in \{0, \ldots, p\}$. Substituting these bounds into the equation above yields

$$p! L_p^{\alpha}(-z) \leq (p+1)(6(1+z))^p \frac{\Gamma\,(\alpha+1+p)}{\Gamma\,(\alpha+1)}$$

from which the lemma follows since $p+1 \leq 2^p$. $\square$

Theorem 3.2.9. *Suppose that there exist constants $\nu \geq 2$ and $\lambda \in \mathbb{R}$ such that inequality (3.7) holds. Then*

$$\mathbb{E}\left[\mathbf{1}_{\{t<\zeta(x)\}} e^{\theta r_N(X_t(x))}\right] \leq 1 + \left(1 + \mathbf{R}(t,\theta,x)^{-\frac{1}{2}}\right)\left({}_1F_1\left(\frac{\nu}{2}, \frac{1}{2}, \mathbf{R}(t,\theta,x)\right) - 1\right) \tag{3.30}$$

for all $t, \theta \geq 0$, where

$$\mathbf{R}(t,\theta,x) = 12\theta^2 \left(r_N^2(x) + 2R(t)\right) e^{\lambda t} \tag{3.31}$$

with $R(t) := (1 - e^{-\lambda t})/\lambda$ and where ${}_1F_1$ is the confluent hypergeometric function of the first kind.

Proof. With the stopping times τ_{D_i} defined as in the proof of Theorem 3.2.2, for $p \in \mathbb{N}$ with p even we see by inequality (3.29) that

$$\mathbb{E}\left[\mathbf{1}_{\{t<\tau_{D_i}\}} r_N^p(X_t(x))\right] \leq \left(2R(t)e^{\lambda t}\right)^{\frac{p}{2}} \Gamma\left(\frac{p}{2}+1\right) L_{\frac{p}{2}}^{\frac{\nu}{2}-1}\left(-\frac{r_N^2(x)}{2R(t)}\right)$$

and so, by Jensen's inequality, if p is odd that

$$\mathbb{E}\left[\mathbf{1}_{\{t<\tau_{D_i}\}} r_N^p(X_t(x))\right] \leq \left(2R(t)e^{\lambda t}\right)^{\frac{p}{2}} \left(\Gamma\left(\frac{p+1}{2}+1\right) L_{\frac{p+1}{2}}^{\frac{\nu}{2}-1}\left(-\frac{r_N^2(x)}{2R(t)}\right)\right)^{\frac{p}{p+1}}.$$

It follows from these estimates and Lemma 3.2.8, since $\nu \geq 2$, that

$$\begin{aligned}
&\mathbb{E}\left[\mathbf{1}_{\{t<\tau_{D_i}\}} e^{\theta r_N(X_t(x))}\right] \\
&\leq 1 + \sum_{p=1}^{\infty} \frac{\left(2\theta^2 R(t) e^{\lambda t}\right)^p}{(2p)!} p! L_p^{\frac{\nu}{2}-1}\left(-\frac{r_N^2(x)}{2R(t)}\right) \\
&\quad + \sum_{p=1}^{\infty} \frac{\left(2\theta^2 R(t) e^{\lambda t}\right)^{\frac{2p-1}{2}}}{(2p-1)!} \left(p! L_p^{\frac{\nu}{2}-1}\left(-\frac{r_N^2(x)}{2R(t)}\right)\right)^{\frac{2p-1}{2p}} \\
&\leq 1 + \sum_{p=1}^{\infty} \frac{\left(2\theta^2 R(t) e^{\lambda t}\right)^p}{(2p)!} \left(12\left(1 + \frac{r_N^2(x)}{2R(t)}\right)\right)^p \frac{\Gamma\left(\frac{\nu}{2}+p\right)}{\Gamma\left(\frac{\nu}{2}\right)} \\
&\quad + \sum_{p=1}^{\infty} \frac{\left(2\theta^2 R(t) e^{\lambda t}\right)^{\frac{2p-1}{2}}}{(2p-1)!} \left(\left(12\left(1 + \frac{r_N^2(x)}{2R(t)}\right)\right)^p \frac{\Gamma\left(\frac{\nu}{2}+p\right)}{\Gamma\left(\frac{\nu}{2}\right)}\right)^{\frac{2p-1}{2p}} \\
&= \sum_{p=0}^{\infty} \frac{\left(24\theta^2\left(R(t) + \frac{r_N^2(x)}{2}\right)e^{\lambda t}\right)^p}{(2p)!} \frac{\Gamma\left(\frac{\nu}{2}+p\right)}{\Gamma\left(\frac{\nu}{2}\right)} \\
&\quad + \sum_{p=1}^{\infty} \frac{\left(24\theta^2\left(R(t) + \frac{r_N^2(x)}{2}\right)e^{\lambda t}\right)^{\frac{2p-1}{2}}}{(2p-1)!} \left(\frac{\Gamma\left(\frac{\nu}{2}+p\right)}{\Gamma\left(\frac{\nu}{2}\right)}\right)^{\frac{2p-1}{2p}}.
\end{aligned}$$

Using $(2p)! = 2p(2p-1)!$, $2p \leq 4^p$ and $\Gamma\left(\frac{\nu}{2}+p\right) \geq \Gamma\left(\frac{\nu}{2}\right)$ the theorem follows from this by monotone convergence, since there is the relation

$$\sum_{p=0}^{\infty} \frac{z^p}{(2p)!} \frac{\Gamma\left(\frac{\nu}{2}+p\right)}{\Gamma\left(\frac{\nu}{2}\right)} = {}_1F_1\left(\frac{\nu}{2}, \frac{1}{2}, \frac{z}{4}\right) \tag{3.32}$$

which can be seen directly from the definition of ${}_1F_1$ as a generalized hypergeometric series. $\square$

The right-hand side of (3.30) is a continuous function of t, θ and x and since the function ${}_1F_1$ satisfies ${}_1F_1\left(\nu/2, 1/2, 0\right) = 1$ and

$$\lim_{r \downarrow 0} r^{-\frac{1}{2}} \left({}_1F_1\left(\frac{\nu}{2}, \frac{1}{2}, r\right) - 1\right) = 0$$

it follows that if $x \in N$ then inequality (3.30) provides a sharp estimate for small times. Furthermore, for the values of ν considered in the theorem the right-hand side of (3.30) grows exponentially with $\mathbf{R}(t, \theta, x)$ (in particular ${}_1F_1(1/2, 1/2, z) = e^z$). The theorem shows that under the given assumptions there is no positive time at which the left-hand side of (3.30) is infinite.

A further property of Laguerre polynomials that will be of use to us is the fact that

$$\sum_{p=0}^{\infty} \gamma^p L_p^{\alpha}(z) = (1-\gamma)^{-(\alpha+1)} e^{-\frac{z\gamma}{1-\gamma}} \tag{3.33}$$

for $|\gamma| < 1$, as proved in Lebedev [1972]. It follows from this and equation (3.18) that for a real-valued Gaussian random variable X with mean μ and variance σ^2 we have for $\theta \geq 0$ that

$$\mathbb{E}\left[e^{\frac{\theta}{2}|X|^2}\right] = \left(1-\theta\sigma^2\right)^{-\frac{1}{2}} \exp\left[\frac{\theta|\mu|^2}{2(1-\theta\sigma^2)}\right]$$

if $\theta\sigma^2 < 1$. A generalization of this formula for a Gaussian measures on Hilbert spaces is well-known, which proves of a special case of Fernique's theorem. If $X(x)$ is a standard Brownian motion on $\mathbb{R}$ starting from $x \in \mathbb{R}$ then for $t \geq 0$ it follows that

$$\mathbb{E}\left[e^{\frac{\theta}{2}|X_t(x)|^2}\right] = (1-\theta t)^{-\frac{1}{2}} \exp\left[\frac{\theta|x|^2}{2(1-\theta t)}\right] \tag{3.34}$$

so long as $\theta t < 1$.

Theorem 3.2.10. *Suppose there exists constants $\nu \geq 1$ and $\lambda \in \mathbb{R}$ such that inequality (3.7) holds. Then*

$$\mathbb{E}\left[\mathbf{1}_{\{t<\zeta(x)\}} e^{\frac{\theta}{2} r_N^2(X_t(x))}\right] \leq \left(1-\theta R(t)e^{\lambda t}\right)^{-\frac{\nu}{2}} \exp\left[\frac{\theta r_N^2(x) e^{\lambda t}}{2(1-\theta R(t)e^{\lambda t})}\right] \tag{3.35}$$

for all $t, \theta \geq 0$ such that $\theta R(t)e^{\lambda t} < 1$, where $R(t) := (1-e^{-\lambda t})/\lambda$.

Proof. Using inequality (3.29) and equation (3.33) we see that

$$\begin{aligned}
\mathbb{E}\left[\mathbf{1}_{\{t<\tau_{D_i}\}} e^{\frac{\theta}{2} r_N^2(X_t(x))}\right] &= \sum_{p=0}^{\infty} \frac{\theta^p}{2^p p!} f_{x,i,2p}(t) \\
&\leq \sum_{p=0}^{\infty} \left(\theta R(t)e^{\lambda t}\right)^p L_p^{\frac{\nu}{2}-1}\left(-\frac{r_N^2(x)}{2R(t)}\right) \\
&= \left(1-\theta R(t)e^{\lambda t}\right)^{-\frac{\nu}{2}} \exp\left[\frac{\theta r_N^2(x)e^{\lambda t}}{2(1-\theta R(t)e^{\lambda t})}\right]
\end{aligned}$$

where we safely switched the order of integration using the stopping time. The result follows by the monotone convergence theorem. □

Theorem 3.2.10 improves upon the estimate given by the second part of [Stroock, 2000, Theorem 5.40] since Stroock's estimate concerns only the one point case, does not take into account positive curvature or the possibility of drift and does not reduce to the correct expression in Euclidean space. If there is a version of Itô's formula to which inequality (3.7) can be applied in the sense of distributions, then Theorem 3.2.10 could itself be improved (in the proofs of Theorems 3.2.2 and 3.2.6 discard the local time and apply inequality (3.7) almost simultaneously).

Corollary 3.2.11. *Suppose that $X(x)$ is a Brownian motion on M starting at x and that one of the following conditions is satisfied:*

(I) $n \in \{0, \ldots, m-1\}$, *the sectional curvature of planes containing the radial direction is non-negative and N is totally geodesic;*

(II) $n \in \{0, m-1\}$*, the Ricci curvature in the radial direction is non-negative and* N *is minimal.*

If $B(y)$ *denotes a Brownian motion on* $\mathbb{R}^{m-n}$ *starting at* $y \in \mathbb{R}^{m-n}$ *with* $r_N^2(x) \leq \|y\|_{\mathbb{R}^{m-n}}$ *then*

$$\mathbb{E}\left[e^{\frac{\theta}{2} r_N^2(X_t(x))}\right] \leq \mathbb{E}\left[e^{\frac{\theta}{2}\|B_t(y)\|^2_{\mathbb{R}^{m-n}}}\right] \tag{3.36}$$

for all $t, \theta \geq 0$ *such that* $\theta t < 1$.

Proof. This follows directly from Theorems 1.4.5 and 3.2.10. □

The point here is that (3.34) provides an explicit formula for the right-hand side of (3.36). To find a comparison theorem which takes into account negative curvature seems harder. We can, however, perform an explicit calculation for the following special case, which compares favourably with our best estimate.

Example 3.2.12. *Suppose that* $X(x)$ *is a Brownian motion on* $\mathbb{H}^3_\kappa$ *starting at* x*, with* $\kappa < 0$*. Then by (5.2) and (1.19) we have*

$$\mathbb{E}[e^{\frac{\theta}{2} r_x^2(X_t(x))}] = (1-\theta t)^{-\frac{3}{2}} \exp\left[-\frac{\theta \kappa t^2}{2(1-\theta t)}\right] \tag{3.37}$$

for all $t, \theta \geq 0$ *such that* $\theta t < 1$*. Note that the explosion time of the right-hand side of (3.37) is independent of* κ*. One does not expect this to be true for the general situation of unbounded curvature considered in Theorem 3.2.10.*

3.2.7 Qualitative Comparison

For an illustration of the behaviour of the estimates given by Theorems 3.2.9 and 3.2.10, fix $x \in M$, suppose that $X(x)$ is a Brownian motion starting at x and suppose that the Ricci curvature of M is bounded below by a constant R. Then inequality (1.32) implies that the assumptions of Theorems 3.2.9 and 3.2.10 hold when $N = \{x\}$ with $\nu = m$ and $\lambda = -R/3$. For these parameters we plot the right-hand sides of the estimates (3.30) and (3.35) as functions of time for the three cases $R \in \{-1, 0, 1\}$ with $\theta = \frac{1}{6}$ and $m = 3$. Note that if $R > 0$ then the left-hand sides of the estimates (3.30) and (3.35) are bounded, by Myer's theorem.

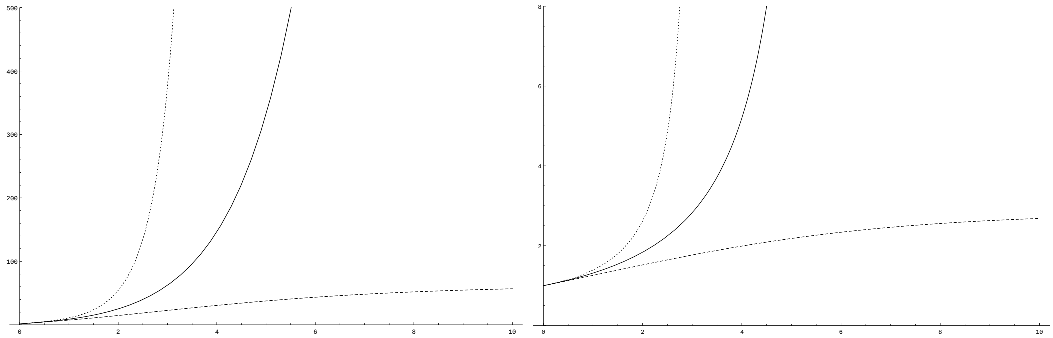

Figure 3: Suppose $N = \{x\}$ with $\nu = m$ and $\lambda = -R/3$. The solid curve on the left is the graph of the right-hand side of (3.30) for $R = 0$. Above it is a dotted curve, which is the graph for $R = -1$, and below it is a dashed curve, which is the graph for $R = 1$. The solid curve on the right is the graph of the right-hand side of (3.35) for $R = 0$. Above it is a dotted curve, which is the graph for $R = -1$, and below it is a dashed curve, which is the graph for $R = 1$. We have set $\theta = \frac{1}{6}$ and $m = 3$ in all cases and the horizontal axes represent the time t. Although not obvious from the two plots, the dotted and solid curves plotted on the left do not explode in finite time while the dotted and solid curves plotted on the right explode at times $t = 3\log 3 \simeq 3.3$ and $t = 6$ respectively.

3.2.8 Concentration Inequalities

If $X(x)$ is a Brownian motion on $\mathbb{R}^m$ starting at x then

$$\lim_{r\to\infty} \frac{1}{r^2} \log \mathbb{P}\{X_t(x) \notin B_r(x)\} = -\frac{1}{2t} \tag{3.38}$$

for all $t > 0$. Note that the right-hand side of the asymptotic relation (3.38) does not depend on the dimension m. Returning to the setting of Example 3.2.3, we find another situation where there is a relation of the type (3.38).

Example 3.2.13. *Suppose that $X(x)$ is a Brownian motion on $\mathbb{H}^3_\kappa$ starting at x. Then, by formula (5.2), we have*

$$\lim_{r\to\infty} \frac{1}{r^2} \log \mathbb{P}\{X_t(x) \notin B_r(x)\} = -\frac{1}{2t}$$

for all $t > 0$.

A heat kernel comparison argument would suggest that a relation of this type should hold in general for a Brownian motion $X(x)$ on M but as an inequality, so long as the Ricci curvature is bounded below by a constant. In fact, it follows from [Stroock, 2000, Theorem 8.62] that if the Ricci curvature is bounded below then there is the asymptotic estimate

$$\lim_{r\uparrow\infty}\frac{1}{r^2}\log\mathbb{P}\left\{\sup_{s\in[0,t]} r_x(X_t(x))\geq r\right\}\leq -\frac{1}{2t}. \tag{3.39}$$

For the general setting considered in this section, we have the following theorem.

Theorem 3.2.14. *Suppose there exists constants $\nu \geq 1$ and $\lambda \in \mathbb{R}$ such that inequality (3.7) holds and suppose that $X(x)$ is non-explosive. Then*

$$\lim_{r\to\infty}\frac{1}{r^2}\log\mathbb{P}\{X_t(x)\notin B_r(N)\}\leq -\frac{1}{2R(t)e^{\lambda t}}$$

for all $t>0$, where $R(t) := (1-e^{-\lambda t})/\lambda$.

Proof. For $\theta \geq 0$ and $r>0$, Markov's inequality and Theorem 3.2.10 imply

$$\begin{aligned}\mathbb{P}\{X_t(x)\notin B_r(N)\} &= \mathbb{P}\{r_N(X_t(x))\geq r\}\\ &= \mathbb{P}\left\{e^{\frac{\theta r_N^2(X_t(x))}{2}}\geq e^{\frac{\theta r^2}{2}}\right\}\\ &\leq e^{-\frac{\theta r^2}{2}}\mathbb{E}\left[e^{\frac{\theta r_N^2(X_t(x))}{2}}\right]\\ &\leq \left(1-\theta R(t)e^{\lambda t}\right)^{-\frac{\nu}{2}}\exp\left[\frac{\theta r_N^2(x)e^{\lambda t}}{2(1-\theta R(t)e^{\lambda t})}-\frac{\theta r^2}{2}\right]\end{aligned}$$

so long as $\theta R(t)e^{\lambda t}<1$. If $t>0$ then choosing $\theta=\delta(R(t)e^{\lambda t})^{-1}$ shows that for any $\delta\in[0,1)$ and $r>0$ we have the estimate

$$\mathbb{P}\{X_t(x)\notin B_r(N)\}\leq(1-\delta)^{-\frac{\nu}{2}}\exp\left[\frac{r_N^2(x)\delta}{2R(t)(1-\delta)}-\frac{\delta r^2}{2R(t)e^{\lambda t}}\right] \tag{3.40}$$

from which the theorem follows since δ can be chosen arbitrarily close to 1 after taking the limit. □

While Theorem 3.2.14 is trivial if M is compact, the concentration inequality (3.40) is valid in that setting and can be improved in certain circumstances. Indeed, for $r > 0$ suppose that $\nu \geq 1$ and $\lambda \geq 0$ are constants such that inequality (3.7) holds *on the tubular neighbourhood* $B_r(N)$ (such constants always exist if N is compact, by Corollary 1.4.4). Assuming $X(x)$ to be non-explosive (which would be the case if N is compact, by Theorem 3.2.5) then the methods of this chapter can also be used to estimate certain quantities involving the process $X(x)$ stopped on the boundary of the tubular neighbourhood. We will not include such calculations here, to avoid extensive repetition, but doing so yields the exit time estimate

$$\mathbb{P}\left\{ \sup_{s\in[0,t]} r_N(X_s(x)) \geq r \right\} \leq (1-\delta)^{-\frac{\nu}{2}} \exp\left[\frac{r_N^2(x)\delta}{2R(t)(1-\delta)} - \frac{\delta r^2}{2R(t)e^{\lambda t}}\right]$$

for all $t > 0$ and $\delta \in (0,1)$, which improves inequality (3.40) for the $\lambda \geq 0$ case.

3.2.9 Feynman-Kac Estimates

The following two proposition and their corollaries constitute simple applications of Theorems 3.2.9 and 3.2.10 and can be used to bound the operator norms of certain Feynman-Kac semigroups, acting on bounded functions.

Proposition 3.2.15. *Suppose there exists constants $\nu \geq 2$ and $\lambda \in \mathbb{R}$ such that inequality (3.7) holds. Then*

$$\mathbb{E}\left[\mathbf{1}_{\{t<\zeta(x)\}} e^{\theta\int_0^t r_N(X_s(x))ds}\right] \leq 1 + \left(1+\mathbf{R}(t,\theta t,x)^{-\frac{1}{2}}\right)\left({}_1F_1\left(\frac{\nu}{2},\frac{1}{2},\mathbf{R}(t,\theta t,x)\right)-1\right)$$

for all $t, \theta \geq 0$, where $\mathbf{R}$ is defined by (3.31).

Proof. Using stopping times τ_{D_i} to safely exchange the order of integrals, we see by Jensen's inequality that

$$\mathbb{E}\left[\mathbf{1}_{\{t<\tau_{D_i}\}} e^{\theta\int_0^t r_N(X_s(x))ds}\right] \leq \frac{1}{t}\int_0^t \mathbb{E}\left[\mathbf{1}_{\{s<\tau_{D_i}\}} e^{t\theta r_N(X_s(x))}\right] ds$$

and the result follows by the monotone convergence theorem and Theorem 3.2.9, since the right-hand side of inequality (3.30) is non-decreasing in t (which is evident from the way in which it was derived). $\square$

Corollary 3.2.16. *Suppose there exists constants $\nu \geq 2$ and $\lambda \in \mathbb{R}$ such that inequality (3.7) holds and that V is a measurable function on M such that $V \leq C(1 + r_N)$ for some constant $C \geq 0$. Then*

$$\mathbb{E}\left[\mathbf{1}_{\{t<\zeta(x)\}} e^{\int_0^t V(X_s(x))ds}\right] \leq e^{Ct}\left(1 + \left(1 + \mathbf{R}(t, Ct, x)^{-\frac{1}{2}}\right)\left({}_1F_1\left(\frac{\nu}{2}, \frac{1}{2}, \mathbf{R}(t, Ct, x)\right) - 1\right)\right)$$

for all $t \geq 0$, where $\mathbf{R}$ is defined by (3.31).

Using Theorem 3.2.10 the following proposition and its corollary can be proved in much the same way.

Proposition 3.2.17. *Suppose there exists constants $\nu \geq 1$ and $\lambda \in \mathbb{R}$ such that inequality (3.7) holds. Then*

$$\mathbb{E}\left[\mathbf{1}_{\{t<\zeta(x)\}} e^{\frac{\theta}{2}\int_0^t r_N^2(X_s(x))ds}\right] \leq \left(1 - \theta t R(t) e^{\lambda t}\right)^{-\frac{\nu}{2}} \exp\left[\frac{\theta r_N^2(x) t e^{\lambda t}}{2(1 - \theta t R(t) e^{\lambda t})}\right] \tag{3.41}$$

for all $t, \theta \geq 0$ such that $\theta t R(t) e^{\lambda t} < 1$.

An estimate found in [Wang, 2014, Subsection 2.6.1] on the left-hand side of (3.41) for the one point case when $\lambda \geq 0$ is implied by Proposition 3.2.17.

Corollary 3.2.18. *Suppose there exists constants $\nu \geq 1$ and $\lambda \in \mathbb{R}$ such that inequality (3.7) holds and that V is a measurable function on M such that $V \leq C(1 + \frac{1}{2}r_N^2)$ for some constant $C \geq 0$. Then*

$$\mathbb{E}\left[\mathbf{1}_{\{t<\zeta(x)\}} e^{\int_0^t V(X_s(x))ds}\right] \leq \left(1 - CtR(t)e^{\lambda t}\right)^{-\frac{\nu}{2}} \exp\left[Ct + \frac{C r_N^2(x) t e^{\lambda t}}{2(1 - CtR(t)e^{\lambda t})}\right]$$

for all $t \geq 0$ such that $CtR(t)e^{\lambda t} < 1$.

3.3 Additional Drift

3.3.1 Girsanov's Theorem

Suppose that $X(x)$ is a *non-explosive* Brownian motion on M starting from x with locally bounded and measurable drift b and suppose that c is a measurable vector field on M with

$$\mathbb{P}\left\{\int_0^t \|c(X_s(x))\|^2 ds < \infty\right\} = 1 \tag{3.42}$$

for all $t \geq 0$. If U is a horizontal lift of $X(x)$ whose antidevelopment has martingale part B then it follows that the stochastic integral $\int_0^{\cdot}\langle c(X_s(x)), U_s dB_s\rangle$ is a well-defined continuous local martingale. For $t \geq 0$ we can therefore set

$$Z_t(c(X(x))) := \exp\left[\int_0^t \langle c(X_s(x)), U_s dB_s\rangle - \frac{1}{2}\int_0^t \|c(X_s(x))\|^2 ds\right]$$

so that

$$Z_t(c(X(x))) = 1 + \int_0^t Z_s(c(X(x)))\langle c(X_s(x)), U_s dB_s\rangle$$

which shows that $Z(c(X(x)))$ is a continuous local martingale with $Z_0(c(X(x))) = 1$. In terms of these objects the version of Girsanov's theorem that will be of use to us is the following one, given by Elworthy [1982] and Karatzas and Shreve [1991].

Theorem 3.3.1. *Suppose that M is a complete Riemannian manifold and that $X(x)$ is a non-explosive Brownian motion on M with drift b starting at x and defined on the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$. Suppose that c is a measurable vector field on M satisfying (3.42) and assume that $Z(c(X(x)))$ is a martingale. If for $t \geq 0$ we define a new measure $\mathbb{Q}_t$ on $\mathcal{F}_t$ by*

$$\frac{d\mathbb{Q}_t}{d\mathbb{P}} := Z_t(c(X(x)))$$

then $\mathbb{Q}_t$ is a probability measure and with respect to $\mathbb{Q}_t$ the process $\{X_s(x) : 0 \leq s \leq t\}$ is identical to a Brownian motion on M with drift $b + c$ starting at x.

3.3.2 A Martingale Criterion

If $\|c\|$ is bounded then $Z(c(X(x))$ is obviously a martingale, by Novikov's criterion. For more generality we deduce the following proposition, which is applied in the next chapter.

Proposition 3.3.2. *Suppose that there exist constants $\nu \geq 1$ and $\lambda \in \mathbb{R}$ such that* *inequality (3.7) holds (in case N is non-compact we assume also that $X(x)$ is non-explosive). If c is a measurable vector field such that $\|c\| \leq C(1 + r_N)$ for some constant $C \geq 0$ then $Z(c(X(x)))$ is martingale.*

Proof. Condition (3.42) is checked using inequality (3.14) and the monotone convergence theorem. The rest follows from Proposition 3.2.17, together with Novikov's criterion and the fact that Brownian motion is a time-homogeneous Markov process. □

Note that the condition on c which appears in Proposition 3.3.2 is less general than the condition on b used in the previous section. Indeed, while the question of non-explosion depends only on the radial part of the vector field, the total magnitude of the vector field must be controlled if we wish to deduce the martingale property using the above approach. Note also that these results apply to suitable time-dependent vector fields, since one recovers time-homogeneity in the space-time setting.

3.3.3 A Non-explosive Diffusion

Suppose now that $\exp_N : TN^\perp \to M$ is a diffeomorphism. Then $\mathrm{Cut}(N)$ is empty and $\nabla \log \Theta_N$ is smooth. A diffusion on M whose infinitesimal generator is given by

$$\frac{1}{2}\triangle + \nabla \log \Theta_N^{-\frac{1}{2}}$$

will be called a *Brownian-Riemannian motion*, by analogy with the terminology of Elworthy [1982]. Using Itô's formula and equation (1.5) we see, modulo initial conditions, that the radial part of a Brownian-Riemannian motion satisfies the same stochastic differential equation as that which is satisfied by the radial part of a Brownian motion in $\mathbb{R}^{m-n}$. It follows that if N is compact then a Brownian-Riemannian motion Y never explodes and thus, by Girsanov's theorem, we conclude that if $Z(\nabla \log \Theta_N^{-\frac{1}{2}}(Y))$ is a martingale then M is stochastically complete. This is, for example, true if $\|\nabla \log \Theta_N\| \leq C(1 + r_N)$ for some constant $C \geq 0$. The non-explosive diffusion Y serves as a precursor to the semiclassical bridge considered in the next chapter.

Chapter 4

Semiclassical and Fermi Bridges

Introduction

In this chapter we present two preliminary examples of submanifold bridge processes.

In order to do so, we suppose that M is a complete and connected Riemannian manifold of dimension m and that N is a closed embedded submanifold of M of dimension $n \in \{0, \ldots, m-1\}$. The two processes introduced in this chapter will be defined in terms of the function $q_\cdot(\cdot, N) : (0, \infty) \times M \to \mathbb{R}$ given by

$$q_t(x, N) := (2\pi t)^{-\frac{(m-n)}{2}} \exp\left[-\frac{r_N^2(x)}{2t}\right] \tag{4.1}$$

for $t > 0$ and $x \in M$.

Section 4.1 describes the first of these processes, called the *semiclassical bridge*, which is defined also in terms of the Jacobian determinant of the exponential map. It was studied by Elworthy [1982] and Watling [1986], who considered the one point case, and by Ndumu [1989], who considered the general case. They used it to derive heat kernel formulae, which we will state in Chapter 5.

Section 4.2 introduces a different process, which we call the *Fermi bridge* and in terms of which we will formulate the main theorem of Chapter 5. The key estimates in this section are Lemmas 4.2.3 and 4.2.4.

Section 4.3 compares the semiclassical bridge with a conditioned diffusion and includes an observation regarding the Riemann zeta function.

Throughout this chapter we fix the vector field b and assume that it is locally bounded and measurable. Both the terminal time T and target submanifold N will also remain fixed.

4.1 Semiclassical Bridges

4.1.1 Definition

Throughout this section suppose that $\exp_N : TN^\perp \to M$ is a diffeomorphism and fix $T > 0$. Then the cut locus of N is empty and for each $t \in [0, T)$ the function $q_{T-t}(\cdot, N)$ is smooth on M. For $y \notin N$ denote by $\gamma_{y,N}$ the unique length-minimizing geodesic segment between y and N parametrized to take unit time with $\gamma_{y,N}(0) = y$ and $\gamma_{y,N}(1) \in N$ and define the smooth function $S_N : M \to \mathbb{R}$ by

$$S_N(y) := \exp\left[\int_0^1 \langle \dot{\gamma}_{y,N}(s), b(\gamma_{y,N}(s)\rangle \, ds\right]$$

if $y \notin N$ and by $S_N(y) = 1$ if $y \in N$. Then the function $C_N : M \to \mathbb{R}$ defined by $C_N := S_N \Theta_N^{-\frac{1}{2}}$ is smooth with $C_N|_N = 1$ and for each $t \in [0, T)$ the vector field

$$\nabla \log C_N q_{T-t}(\cdot, N)$$

is smooth (and if N is a point with b a gradient vector field then $\nabla \log S_N = -b$). A diffusion on M starting at $x \in M$ defined upto a predictable stopping time which is less than or equal to T whose time-dependent infinitesimal generator is of the form

$$\frac{1}{2}\triangle + b + \nabla \log C_N q_{T-t}(\cdot, N)$$

is called a *semiclassical bridge between x and N in time T*. Such processes were once referred to as *Brownian-Riemannian bridges*, which explains the terminology of Subsection 3.3.3. If we do not assume that $\exp_N$ is a diffeomorphism then such processes can still be defined upto the exit time from a domain whose closure is contained in $M \setminus \mathrm{Cut}(N)$.

4.1.2 Radial Part and Bridge Property

The proofs we give for the following lemma and proposition are similar to those found in Watling [1986] and Ndumu [1989].

Lemma 4.1.1. *For all $y \in M$ we have*

$$\left\langle \frac{\partial}{\partial r_N}, b(y) + \nabla \log S_N(y) \right\rangle = 0.$$

Proof. It suffices to assume that $y \notin N$. For such y note that

$$\frac{1}{2}\nabla r_N^2(y) = -\dot{\gamma}_{y,N}(0)$$

so we have

$$\left\langle \nabla \log S_N(y), \frac{1}{2}\nabla r_N^2(y) \right\rangle = -\frac{d}{ds}\log S_N(\gamma_{y,N}(s))\bigg|_{s=0}.$$

On the other hand, from the definition of S_N we have

$$\log S_N(\gamma_{y,N}(s)) = \int_s^1 \langle \dot{\gamma}_{y,N}(u), b(\gamma_{y,N}(u))\rangle du,$$

from which it follows that

$$-\frac{d}{ds}\log S_N(\gamma_{y,N}(s))\bigg|_{s=0} = \langle \dot{\gamma}_{y,N}(0), b(y)\rangle = -\frac{1}{2}\langle \nabla r_N^2(y), b(y)\rangle.$$

We have therefore deduced that

$$\left\langle b(y) + \nabla \log S_N(y), \frac{1}{2}\nabla r_N^2(y) \right\rangle = 0$$

from which the lemma follows since $\frac{1}{2}\nabla r_N^2 = r_N \frac{\partial}{\partial r_N}$. $\square$

Proposition 4.1.2. *Suppose that $m-n \geq 2$ and that $\hat{Y}(x)$ is a semiclassical bridge between x and N in time T. Then the radial part of $\hat{Y}(x)$ is identical to that of a Brownian bridge on $\mathbb{R}^{m-n}$ between a point of distance $r_N(x)$ from the origin and the origin in time T.*

Proof. Since $m - n \geq 2$ it follows that N is polar for $\hat{Y}(x)$ and so by Itô's formula we have

$$\begin{aligned} r_N(\hat{Y}_t(x)) = r_N(x) + \beta_t + \int_0^t \Big\langle \frac{\partial}{\partial r_N}, b(\hat{Y}_s(x)) + \nabla \log S_N(\hat{Y}_s(x)) \Big\rangle \\ + \frac{\partial}{\partial r_N} \log \Theta_N^{-\frac{1}{2}}(\hat{Y}_s(x)) + \frac{1}{2} \triangle r_N(\hat{Y}_s(x)) \\ + \frac{\partial}{\partial r_N} \log q_{T-s}(\hat{Y}_s(x), N)\, ds \end{aligned}$$

for all $t \in [0, T)$, almost surely, where β is a standard one-dimensional Brownian motion. Using Lemma 4.1.1, formula (1.5) and the definition of $q_{\cdot}(\cdot, N)$ we see that

$$r_N(\hat{Y}_t(x)) = r_N(x) + \beta_t + \int_0^t \frac{m-n-1}{2 r_N(\hat{Y}_s(x))} - \frac{r_N(\hat{Y}_s(x))}{T-s}\, ds$$

for all $t \in [0, T)$, almost surely, which by comparing to the Euclidean case yields the claim. □

Besides characterizing the radial part of the semiclassical bridge, Proposition 4.1.2 tells us that a semiclassical bridge does not explode prior to the terminal time T. In particular, the proposition implies the bridge property

$$\lim_{t \uparrow T} r_N(\hat{Y}_t(x)) = 0$$

almost surely. Therefore, if N is a point $p \in M$ then a semiclassical bridge $\hat{Y}(x)$ can be extended to give a continuous process defined on the closed time interval $[0, T]$ by setting $\hat{Y}_T(x) := p$.

4.2 Fermi Bridges

4.2.1 Definition

In the previous section we assumed that the normal exponential map was a global diffeomorphism. In this section we do not make this assumption. In general, for $T > 0$ fixed and with the function $q_\cdot(\cdot, N)$ defined by equation (4.1), the time-dependent vector field

$$\mathbf{1}_{M(N)} \nabla \log q_{T-t}(\cdot, N) = -\frac{r_N}{T-t}\frac{\partial}{\partial r_N}$$

with $t \in [0, T)$ is smooth away from the cut locus but generally not continuous on the cut locus. One imagines the deterministic flow associated to this vector field as being one for which $\mathrm{Cut}(N)$ is a source and for which N is a sink. The strength of the flow increases dramatically as the terminal time T is approached, while the vector field vanishes on N. Given the locally bounded and measurable vector field b fixed at the beginning of this chapter, a diffusion on M starting at $x \in M$ and defined upto a predictable stopping time which is less than or equal to T whose time-dependent infinitesimal generator is of the form

$$\frac{1}{2}\triangle + b - \frac{r_N}{T-t}\frac{\partial}{\partial r_N}$$

will be called a *Fermi bridge between x and N in time T*. We call such a process a *Fermi bridge* since the time-dependent part of the drift acts in the normal direction, which corresponds to the radial part of polar Fermi coordinates, and since under suitable conditions we will prove that such a process arrives at N at time T almost surely. To help understand the effect of the drift discontinuity, two real-valued diffusions whose drifts exhibit similar jump discontinuities are considered in Appendix B. It should be noted that our definition is related to one which appeared in Stroock [2001], in which the *Ornstein-Uhlenbeck* process on M was defined to be any diffusion starting at $x \in M$ whose infinitesimal generator is of the form

$$\frac{1}{2}\triangle - \alpha r_x \frac{\partial}{\partial r_x}$$

for some $\alpha > 0$. Stroock proved that the corresponding Markov semigroup is hypercontractive, provided the Ricci curvature is bounded below. We are, however, more interested in the behaviour of the radial part of the Fermi bridge, since suitable bounds on that object will imply lower bounds on integrals of the heat kernel.

Of course, if $\exp_N$ is a diffeomorphism with $\nabla \log \Theta_N = 0$ then the definitions of the semiclassical and Fermi bridges coincide. In particular, if $M = \mathbb{R}^m$ with N a point and $b = 0$ then the definitions of these two processes reduce to that of a standard Brownian bridge.

4.2.2 Radial Part

Suppose that $\hat{X}(x)$ is a Fermi bridge between x and N in time T defined upto the minimum of T and its explosion time. Suppose that D is a regular domain in M and denote by $\hat{\tau}_D$ the first exit time of $\hat{X}(x)$ from D. Then $\hat{X}(x)$ is, in particular, a continuous semimartingale upto time $T \wedge \hat{\tau}_D$ and, since $\check{C}(N)$ is polar for $\hat{X}(x)$ and since the martingale part of any antidevelopment of $\hat{X}(x)$ is a standard Brownian motion, it follows from the Tanaka formula of Subsection 2.3.2 that

$$\begin{aligned} r_N(\hat{X}_{t\wedge\hat{\tau}_D}(x)) = r_N(x) + \beta_{t\wedge\hat{\tau}_D} + \int_0^{t\wedge\hat{\tau}_D} \left(\frac{1}{2}\triangle + b\right) r_N(\hat{X}_s(x))\, ds \\ - \int_0^{t\wedge\hat{\tau}_D} \frac{r_N(\hat{X}_s(x))}{T-s}\, ds - \mathbb{L}^{\mathrm{Cut}(N)}_{t\wedge\hat{\tau}_D}(\hat{X}(x)) + L^N_{t\wedge\hat{\tau}_D}(\hat{X}(x)) \end{aligned} \tag{4.2}$$

for all $t \in [0, T)$, almost surely, where β is a standard one-dimensional Brownian motion and where the non-negative non-decreasing continuous process $\mathbb{L}^{\mathrm{Cut}(N)}(\hat{X}(x))$ is defined by

$$d\mathbb{L}^{\mathrm{Cut}(N)}(\hat{X}(x)) := -\frac{1}{2}\left(D^+_{\hat{X}(x)} - D^-_{\hat{X}(x)}\right) r_N(\mathbf{n})\, dL^{\mathring{C}(N)}(\hat{X}(x)) \tag{4.3}$$

using the notation of Subsection 2.3.2. With this formula we can estimate the radial moments of the Fermi bridge, as we did for Brownian motion in the previous chapter. For now we restrict our attention to the domain D.

Theorem 4.2.1. *Let $\nu \geq 1$ and $\lambda \geq 0$ be any constants such that inequality (3.7) holds on $D \setminus \mathrm{Cut}(N)$. Then we have*

$$\mathbb{E}\left[\mathbf{1}_{\{t<\hat{\tau}_D\}} r_N^2(\hat{X}_t(x))\right] \leq \left(r_N^2(x)\left(\frac{T-t}{T}\right) + \nu t\right)\left(\frac{T-t}{T}\right) e^{\lambda t} \tag{4.4}$$

for all $t \in [0,T)$.

Proof. Define the function $\hat{f}_{x,2} : [0,T) \to \mathbb{R}$ by

$$\hat{f}_{x,2}(t) := \mathbb{E}[\mathbf{1}_{\{t<\hat{\tau}_D\}} r_N^2(\hat{X}_t(x))]$$

for $t \in [0,T)$. By Itô's formula and formula (4.2) and using an argument similar to that given for the proof of Theorem 3.2.2 we deduce the differential inequality

$$\begin{cases} \hat{f}'_{x,2}(t) \leq \nu + \left(\lambda - \frac{2}{T-t}\right) \hat{f}_{x,2}(t) \\ \hat{f}_{x,2}(0) = r_N^2(x) \end{cases}$$

for all $t \in [0,T)$. Applying Gronwall's inequality to it yields

$$\begin{aligned} \hat{f}_{x,2}(t) &\leq \left(r_N^2(x) + \nu \int_0^t \left(\frac{T}{T-s}\right)^2 e^{-s\lambda} ds\right)\left(\frac{T-t}{t}\right)^2 e^{\lambda t} \\ &\leq \left(r_N^2(x) + \nu t\left(\frac{t}{T-t}\right)\right)\left(\frac{T-t}{t}\right)^2 e^{\lambda t} \end{aligned}$$

where we used the assumption $\lambda \geq 0$ for the second inequality. □

Notice that we now assume $\lambda \geq 0$ while in the previous chapter we allowed for negative values of λ. This extra assumption is to avoid the future occurrence of certain exponential integrals that cannot be evaluated explicitly. Note that for $M = \mathbb{R}^m$ with N a linear subspace and $b = 0$, with $\nu = m - n$ and $\lambda = 0$, one can set $D = M$ and inequality (4.4) holds as an *equality.* Also, Jensen's inequality implies the following estimate on the first radial moment.

Corollary 4.2.2. *Let $\nu \geq 1$ and $\lambda \geq 0$ be any constants such that inequality (3.7) holds on $D \setminus \mathrm{Cut}(N)$. Then we have*

$$\mathbb{E}[\mathbf{1}_{\{t<\hat{\tau}_D\}} r_N(\hat{X}_t(x))] \leq \left(\left(r_N^2(x)\left(\frac{T-t}{T}\right)+\nu t\right)\left(\frac{T-t}{T}\right)\right)^{\frac{1}{2}} e^{\frac{\lambda t}{2}}$$

for all $t \in [0,T)$.

Additional estimates can be found in Section C.1 of Appendix C. These include an exponential estimate, a concentration inequality and an integrability theorem, whose relevance is explained in Section C.2. In the next subsection we consider the case where there exist constants $\nu \geq 1$ and $\lambda \geq 0$ such that inequality (3.7) holds on the whole of $M \setminus \mathrm{Cut}(N)$.

4.2.3 Bridge Property

Suppose for this subsection that b satisfies $\|b\| \leq C(1+r_N)$, for some $C \geq 0$, and that there exists constants $\nu \geq 1$ and $\lambda \geq 0$ such that inequality (3.7) holds on $M \setminus \mathrm{Cut}(N)$. Suppose also that $X(x)$ is a *non-explosive* Brownian motion on M with drift b starting at x, defined on the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$. Recall that if N is compact then inequality (3.7) *implies* the non-explosion of $X(x)$. For $t \in [0,T)$ then define

$$M_t := \exp\left[-\int_0^t \frac{r_N(X_s(x))}{T-s}\left\langle \frac{\partial}{\partial r_N}, U_s dB_s\right\rangle - \frac{1}{2}\int_0^t \frac{r_N^2(X_s(x))}{(T-s)^2}ds\right]$$

where U is a horizontal lift of $X(x)$ whose antidevelopment has martingale part given by a Brownian motion B. It follows from Proposition 3.3.2 that M is a martingale up to time t for each $t \in [0,T)$. We can therefore define a probability measure $\mathbb{Q}_{T-}$ on $\mathcal{F}_{T-}$ by

$$\frac{d\mathbb{Q}_{T-}|_{\mathcal{F}_t}}{d\mathbb{P}} = M_t$$

for each $t \in [0,T)$. It follows by Girsanov's theorem that the process $X(x)$ when restricted to the time interval $[0,T)$ and considered on the filtered probability space

$$\left(\Omega, \mathcal{F}_{T-}, \{\mathcal{F}_t\}_{t\in[0,T)}, \mathbb{Q}_{T-}\right)$$

is a Fermi bridge between x and N in time T. Since $\mathbb{Q}_{T-}$ and $\mathbb{P}$ are equivalent on $\mathcal{F}_{T-}$ it follows that this process is a continuous semimartingale upto time T, by the stability properties of such processes discussed in Chapter 2, and that it does not explode prior to this terminal time. The equivalence also implies the existence of transition densities and the polarity of $\check{C}(N)$. If this new process is denoted by $\hat{X}(x)$ then, by considering an exhaustion of M by regular domains, Theorem 4.2.1 and the monotone convergence theorem imply the bridge property

$$\lim_{t\uparrow T} r_N(\hat{X}_t(x)) = 0$$

almost surely. Therefore, if N is a point $p \in M$ then one then can extend $\hat{X}(x)$ to a continuous process on the time interval $[0,T]$ by setting $\hat{X}_T(x) := p$.

4.2.4 Two Integrability Lemmas

Returning to the setting of Subsection 4.2.2, suppose once more that D is a regular domain in M and that $\hat{X}(x)$ is a Fermi bridge between x and N in time T defined upto the minimum of T and its explosion time. The following two lemmas are the basic integrability estimates that we will use in Chapters 5 and 6 to deduce heat kernel lower bounds and gradient estimates.

Lemma 4.2.3. *Let $\nu \geq 1$ and $\lambda \geq 0$ be any constants such that inequality (3.7) holds on $D \setminus \mathrm{Cut}(N)$. Then we have*

$$\sup_{t\in[0,T)} \mathbb{E}\left[\mathbf{1}_{\{t<\hat{\tau}_D\}} \int_0^t \frac{r_N^2(\hat{X}_s(x))}{T-s}\, ds\right] \leq \left(r_N^2(x) + \nu T\right) e^{\lambda T}.$$

Proof. By Theorem 4.2.1 we see that

$$\begin{aligned}\mathbb{E}\left[\mathbf{1}_{\{t<\hat{\tau}_D\}}\int_0^t \frac{r_N^2(\hat{X}_s(x))}{T-s}ds\right] &\leq \int_0^t \frac{\mathbb{E}[\mathbf{1}_{\{s<\hat{\tau}_D\}}r_N^2(\hat{X}_s(x))]}{T-s}ds\\ &\leq t\left(\frac{r_N^2(x)+\nu t}{T}\right)e^{\lambda t}\end{aligned}$$

for all $t \in [0,T)$ and so the lemma is proved. $\square$

Lemma 4.2.4. *Let $\nu \geq 1$ and $\lambda \geq 0$ be any constants such that inequality (3.7) holds on $D \setminus \mathrm{Cut}(N)$. Then we have*

$$\sup_{t\in[0,T)}\mathbb{E}\left[\mathbf{1}_{\{t<\hat{\tau}_D\}}\int_0^t \frac{r_N(\hat{X}_s(x))}{T-s}\,ds\right] \leq 2\left(r_N^2(x)+\nu T\right)^{\frac{1}{2}}e^{\frac{\lambda T}{2}}.$$

Proof. By Corollary 4.2.2 we see that

$$\begin{aligned}\mathbb{E}\left[\mathbf{1}_{\{t<\hat{\tau}_D\}}\int_0^t \frac{r_N(\hat{X}_s(x))}{T-s}ds\right] &\leq \int_0^t \frac{\mathbb{E}[\mathbf{1}_{\{s<\hat{\tau}_D\}}r_N(\hat{X}_s(x))]}{T-s}ds\\ &\leq \left(\frac{r_N^2(x)+\nu t}{T}\right)^{\frac{1}{2}}e^{\frac{\lambda t}{2}}\int_0^t (T-s)^{-\frac{1}{2}}\,ds\\ &= 2\left(r_N^2(x)+\nu t\right)^{\frac{1}{2}}e^{\frac{\lambda t}{2}}\left(1-\left(\frac{T-t}{T}\right)^{\frac{1}{2}}\right)\end{aligned}$$

for all $t \in [0,T)$ and so the lemma is proved. $\square$

4.3 Comparison of Bridge Processes

4.3.1 An Equivalence Theorem

Suppose that N is a point p and that p is a pole for M. In this setting, the difference between the semiclassical and Fermi bridges is evident since their generators are given explicitly. So let us compare the semiclassical bridge with a conditioned diffusion. For this, fix $T > 0$, $x \in M$, suppose that the vector field b is smooth and suppose that $\hat{Y}(x)$ is a semiclassical bridge between x and p in time T, defined as in Subsection 4.1.1 on the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\leq t\leq T}, \mathbb{P})$. Recall that the process

$\hat{Y}(x)$ has time-dependent infinitesimal generator

$$\frac{1}{2}\triangle + b + \nabla W_t$$

for $t \in [0, T)$ where $W_t(\cdot) := \log C_p q_{T-t}(\cdot, p)$ and for $t \in [0, T]$ define the random variable $h_t(x, p)$ by

$$h_t(x,p) = \exp\left[\int_0^t (C_p^{-1}\left(\frac{1}{2}\triangle + b\right) C_p)(\hat{Y}_s(x))ds\right]$$

whenever the expression on the right-hand side is finite.

Theorem 4.3.1. *Suppose that p is a pole for M, that b is smooth and that*

$$C_p^{-1}\left(\frac{1}{2}\triangle + b\right) C_p$$

is bounded above on M. Then there exists a probability measure $\mathbb{P}'$, defined on $\mathcal{F}_T$ by

$$\frac{d\mathbb{P}'|_{\mathcal{F}_T}}{d\mathbb{P}} = \frac{h_T(x,p)}{\mathbb{E}^{\mathbb{P}}\left[h_T(x,p)\right]},$$

under which the process $\hat{Y}(x)$ is identical to a Brownian motion with drift b started at x and conditioned to arrive at p at time T.

Proof. For $t \in [0, T)$ define a measure $\mathbb{P}'_t$ on $\mathcal{F}_t$ by

$$\frac{d\mathbb{P}'_t}{d\mathbb{P}} = \frac{p^{M,b}_{T-t}(\hat{Y}_t(x), p)}{p^{M,b}_T(x,p)} \exp\left[-\int_0^t \langle \nabla W_s(\hat{Y}_s(x)), \hat{U}_s dB_s\rangle - \frac{1}{2}\int_0^t |\nabla W_s(\hat{Y}_s(x))|^2 ds\right] \tag{4.5}$$

where $\hat{U}$ is a horizontal lift of $\hat{Y}(x)$ whose antidevelopment has martingale part B and where $p^{M,b}$ denotes the transition densities of a Brownian motion with drift b. By Girsanov's theorem it follows that $\mathbb{P}'_t$ is a probability measure under which the process $\hat{Y}(x)$ is, upto time t, identical to a Brownian motion with drift b started at x and conditioned to arrive at p at time T. The stochastic integral on the right-hand side of equation (4.5) can be eliminated using Itô's formula, which yields

$$\frac{d\mathbb{P}'_t}{d\mathbb{P}} = \frac{p^{M,b}_{T-t}(\hat{Y}_t(x),p)}{p^{M,b}_T(x,p)} \exp\left[W_0(x) - W_t(\hat{Y}_t(x)) + \int_0^t \langle \nabla W_s(\hat{Y}_s(x)), b(\hat{Y}_s(x))\rangle ds \right.$$
$$\left. + \int_0^t \frac{\partial}{\partial s} W_s(\hat{Y}_s(x)) + \frac{1}{2}\triangle W_s(\hat{Y}_s(x)) + \frac{1}{2}|\nabla W_s(\hat{Y}_s(x))|^2 ds\right]. \tag{4.6}$$

For the term inside the first integral we calculate

$$\langle \nabla W_s, b\rangle = C_p^{-1} b\, C_p - \frac{\langle \nabla r_p^2, b\rangle}{2(T-s)}.$$

For the first term inside the second integral we calculate

$$\frac{\partial}{\partial s} W_s = \frac{m}{2(T-s)} - \frac{r_p^2}{2(T-s)^2}.$$

For the second term inside the second integral we calculate

$$\frac{1}{2}\triangle W_s = -\frac{m}{2(T-s)} + \frac{\langle \nabla r_p^2, \nabla \log \Theta_p^{-\frac{1}{2}}\rangle}{2(T-s)}$$
$$+ \frac{1}{2} C_p^{-1} \triangle C_p - \frac{1}{2}\|\nabla \log C_p\|^2.$$

For the third term inside the second integral we calculate

$$\frac{1}{2}\|\nabla W_s\|^2 = \frac{1}{2}\|\nabla \log C_p\|^2 - \frac{\langle \nabla r_p^2, \nabla \log S_p\rangle}{2(T-s)}$$
$$+ \frac{r_p^2}{2(T-s)^2} - \frac{\langle \nabla \log \Theta_p^{-\frac{1}{2}}, \nabla r_p^2\rangle}{2(T-s)}.$$

Using these calculations and Lemma 4.1.1 we deduce from equation (4.6) that

$$\frac{d\mathbb{P}'_t}{d\mathbb{P}} = \frac{p^{M,b}_{T-t}(\hat{Y}_t(x),p)}{p^{M,b}_T(x,p)} \exp\left[W_0(x) - W_t(\hat{Y}_t(x)) + \int_0^t (C_p^{-1}\left(\frac{1}{2}\triangle + b\right) C_p)(\hat{Y}_s(x)) ds\right].$$

Substituting in Elworthy, Truman and Watling's formula for $p^{M,b}$, given below by

Theorem 5.1.4, yields

$$\frac{d\mathbb{P}_t'}{d\mathbb{P}} = \frac{\mathbb{E}^{\mathbb{P}}\left[h_{T-t}(\hat{Y}_t(x), p)\right] h_t(x,p)}{\mathbb{E}^{\mathbb{P}}\left[h_T(x,p)\right]}$$

for $0 \leq t < T$. Since we assume that $C_p^{-1}\left(\frac{1}{2}\triangle + b\right) C_p$ is bounded above it follows immediately that

$$\lim_{t \uparrow T} \mathbb{E}^{\mathbb{P}}[h_{T-t}(\hat{Y}_t(x), p)] = 1$$

from which the result follows. □

Brownian motions with drift b started at x and conditioned to arrive at p at time T, on manifolds with non-negative Ricci curvature, were the object of study in Qian [1994]. We will consider processes similar to these in Chapter 6.

4.3.2 A Consequence of the Equivalence

It follows from our equivalence theorem and the heat kernel formula (5.2) given below that semiclassical bridges are identical to Brownian bridges in $\mathbb{H}^3_\kappa$. This can also be seen directly from formula (5.2), while Theorem 4.3.1 describes the difference between these two processes in $\mathbb{H}^m_\kappa$ for dimensions other than (one and) three. In the study of the maximum of Bessel bridges it turns out that dimensions one and three have special significance; see [Pitman and Yor, 1999, p.18]. Let us focus on dimension three and suppose that $X(0)$ is a Brownian bridge in $\mathbb{R}^3$ starting at the origin and returning to it at time 1. If we define

$$M_3 := \sup_{0 \leq t \leq 1} r_0(X_t(0))$$

then for $\theta > 0$ it was observed by Chung [1982] that there is the relation

$$\mathbb{E}\left[e^{-\frac{\theta^2}{2}M_3^2}\right] = \left(\frac{\frac{\pi\theta}{2}}{\sinh\left(\frac{\pi\theta}{2}\right)}\right)^2$$

and that a similar formula exists in dimension one. Note that the right-hand side of this formula is similar to that of formula (1.19) in dimension three. Furthermore, for any complex number z there is also the remarkable formula

$$\mathbb{E}[M_3^z] = 2^{-\frac{z}{2}} z(z-1)\Gamma\left(\frac{z}{2}\right)\zeta(z) \tag{4.7}$$

where ζ denotes the Riemann zeta function, proved in Biane and Yor [1987]. A similar formula exists in dimension one. See the articles Biane, Pitman and Yor [2001] and Williams [1990] for more about the relationship between Brownian bridges and the Riemann zeta function. We observe that a consequence of Biane and Yor's formula and our Theorem 4.3.1 is that if $X(p)$ is a Brownian bridge in $\mathbb{H}^3_\kappa$ starting at p and returning to it at time 1 and if

$$M_{3,\kappa} := \sup_{0 \le t \le 1} r_p(X_t(p))$$

then for any complex number z it follows that

$$\mathbb{E}[M_{3,\kappa}^z] = 2^{-\frac{z}{2}} z(z-1)\Gamma\left(\frac{z}{2}\right)\zeta(z). \tag{4.8}$$

It is slightly intriguing that this should be true for *any* $\kappa < 0$.

Chapter 5

Heat Kernel Formulae and Estimates

Introduction

In this chapter we prove formulae and estimates for the integral of the heat kernel over a submanifold. The main results, given in Sections 5.2 and 5.3, will be used in the next chapter to study Brownian bridges to submanifolds. They are also connected to local time, by formula (2.17).

A separate reason why we study this object relates to the splitting of heat kernels. The heat kernel on $\mathbb{R}^m$ is essentially the product of heat kernels on $\mathbb{R}^n$ and $\mathbb{R}^{m-n}$, so we might wonder: is a sense in which this property can be expressed for the heat kernel on a Riemannian manifold? We will show that integrating the heat kernel over a totally geodesic or minimal submanifold yields a formula in which only the effect of the cut locus and the curvature *in the radial direction* appear explicitly. We also prove Theorem 5.3.8, which shows how *asymptotic* splitting occurs in a more explicit sense.

The main result in this chapter is Theorem 5.2.1, in which we express the integral of the heat kernel over a submanifold in terms of an integral over the paths of a Fermi bridge. A special case is thus a formula for the heat kernel itself. We use Theorem

5.2.1 and results from the previous chapter to prove Theorem 5.3.2 which gives lower bounds (needed for the final chapter). Our approach to upper bounds assumes N is compact and obtains them from the lower bounds using a result of Grigor'yan, Hu and Lau [2008]. This simple approach is reasonable since the distance to the submanifold is defined as an infimum.

Before proving these results, we discuss the elementary formula of Elworthy and Truman [1982]. An extension of this formula to a Hamiltonian setting, motivated by analogy with the Schrödinger equation for a magnetic field, was developed by Watling [1986, 1988, 1992]. Applied to the heat kernel, Watling's assumptions imply the existence of a pole, as required by the elementary formula. A different extension has been considered by Ndumu [1989, 1991, 1996, 2011], who places emphasis on the integral of certain Dirichlet heat kernels over a submanifold. We discuss these formulae in Section 5.1.

Section 5.4 is something of an aside. We apply Gaussian upper bounds on the heat kernel and a Jacobian estimate from Chapter 1 to obtain estimates on the $L^q_{q'}$-norm of the heat kernel, extending a result of Krylov and Röckner [2005] to the manifold setting. We use our estimates to prove the existence of solutions to a martingale problem for singular drift. This is the content of Theoerm 5.4.5.

5.1 The Heat Kernel

5.1.1 Dirichlet and Minimal Heat Kernels

Suppose that M is a Riemannian manifold of dimension m and that $X(x)$ is a Brownian motion on M starting at x defined up to an explosion time $\zeta(x)$. Recall that an open connected subset D of M is called a *regular domain* if it has smooth boundary and compact closure. If p^D denotes the Dirichlet heat kernel on D then p^D is the unique positive fundamental solution to the heat equation on D with Dirichlet boundary conditions. A probabilistic interpretation of p^D is that if f is a non-negative measurable function then

$$\mathbb{E}\left[\mathbf{1}_{\{t<\tau_D\}}f(X_t(x))\right] = \int_M f(y)p_t^D(x,y)d\,\mathrm{vol}_M(y)$$

for all $t > 0$, where τ_D denotes the first exit time of $X(x)$ from D. We will define p^D on $(0,\infty)\times M\times M$ by setting this function equal to zero if at least one of the space variables is not contained in D. Now suppose that M is connected and recall that a collection $\{D_i\}_{i=1}^\infty$ of subsets of M is called an *exhaustion of M by regular domains* if each D_i is a regular domain with $\overline{D}_i \subset D_{i+1}$ and $\cup_{i=1}^\infty D_i = M$. If p^{D_i} denotes the Dirichlet heat kernel on D_i then the *minimal heat kernel on M* is denoted by p^M and defined by the increasing limit $p^M := \lim_{i\uparrow\infty} p^{D_i}$ which is independent of the choice of exhaustion. The minimal heat kernel is the minimal positive fundamental solution of the heat equation on M and coincides with the transition densities of Brownian motion. In particular, if f is a non-negative measurable function then

$$\mathbb{E}\left[\mathbf{1}_{\{t<\zeta(x)\}}f(X_t(x))\right] = \int_M f(y)p_t^M(x,y)d\,\mathrm{vol}_M(y)$$

for all $t > 0$. It follows that M is stochastically complete if and only if

$$\int_M p_t^M(x,y)d\,\mathrm{vol}_M(y) = 1$$

for all $t > 0$, in which case p^M can unambiguously be referred to as the *heat kernel of M*. For more about p^M see the article Yau [1978], the book Chavel [1984] or the survey Saloff-Coste [2010].

Example 5.1.1. *On the Euclidean space $\mathbb{R}^m$ the heat kernel is given by the Gauss-Weierstrass kernel*

$$p_t^{\mathbb{R}^m}(x,y) = (2\pi t)^{-\frac{m}{2}}\exp\left(-\frac{r_y^2(x)}{2t}\right) \tag{5.1}$$

for $x, y \in \mathbb{R}^m$ and $t > 0$.

Example 5.1.2. *If $\mathbb{H}_\kappa^3$ denotes the hyperbolic space of dimension 3 with constant* *sectional curvature κ then*

$$p_t^{\mathbb{H}^3_\kappa}(x,y) = (2\pi t)^{-\frac{3}{2}} \exp\left[-\frac{r_y^2(x)}{2t}\right] \underbrace{\frac{\sqrt{-\kappa} r_y(x) e^{\frac{\kappa t}{2}}}{\sinh\left(\sqrt{-\kappa} r_y(x)\right)}}_{\leq 1} \tag{5.2}$$

for $x, y \in \mathbb{H}^3_\kappa$ and $t > 0$.

Example 5.1.3. *If $\mathbb{S}^1$ denotes the unit circle then*

$$p_t^{\mathbb{S}^1}(x,y) = (2\pi t)^{-\frac{1}{2}} \exp\left[-\frac{r_y^2(x)}{2t}\right] \underbrace{\sum_{k\in\mathbb{Z}} \exp\left[-\frac{2\pi k(r_y(x) + \pi t)}{t}\right]}_{\geq 1} \tag{5.3}$$

for $x, y \in \mathbb{S}^1$ and $t > 0$.

Grigor'yan and Noguchi [1998] and Nagase [2010] provide iterative formulae for the heat kernels on the standard hyperbolic spaces and spheres, respectively, of arbitrary dimension.

5.1.2 Elworthy, Truman and Watling's Formula

Suppose that M is connected, that b is a smooth vector field on M and that V is a smooth function on M which is bounded above. Then the Riemannian Schrödinger equation

$$\frac{\partial}{\partial t} u = \left(\frac{1}{2}\triangle + b + V\right) u \tag{5.4}$$

has a unique minimal fundamental solution, defined for positive times, which we will will denote by $p^{M,b,V}$. Suppose that $X(x)$ is a Brownian motion on M with drift b starting at x and defined up to an explosion time $\zeta(x)$. If f is a non-negative measurable function on M then a probabilistic interpretation of $p^{M,b,V}$ is that

$$\mathbb{E}\left[\mathbf{1}_{\{t<\zeta(x)\}} f(X_t(x)) \exp\left[\int_0^t V(X_s(x)) ds\right]\right] = \int_M f(y) p_t^{M,b,V}(x,y) d\,\mathrm{vol}_M(y) \tag{5.5}$$

for all $t > 0$. This is the Feynman-Kac formula. In certain circumstances it is possible to obtain a probabilistic formula for $p^{M,b,V}$. For the case in which M has a pole there is the following one, due to Elworthy and Truman [1982] and [Watling, 1986, Section 7].

Theorem 5.1.4. *Suppose that p is a pole for M and that $C_p^{-1}\left(\frac{1}{2}\triangle + b\right)C_p$ is bounded above. Then for $T > 0$ and $x \in M$ we have*

$$p_T^{M,b,V}(x,p) = q_T(x,p)C_p(x)\mathbb{E}\left[\exp\left[\int_0^T (C_p^{-1}\left(\frac{1}{2}\triangle + b + V\right)C_p)(\hat{Y}_s(x))ds\right]\right] \tag{5.6}$$

where $\hat{Y}(x)$ is a semiclassical bridge between x and p in time T.

Elworthy and Truman proved the formula for the case $b = 0$, using a Feynman-Kac-Girsanov transformation and Itô's formula, while Watling allowed for the drift b. Remarks on this formula can be found in [Elworthy, 1988, Chapter V].

Example 5.1.5. *On $\mathbb{H}_\kappa^m$ we have*

$$\frac{1}{2}\Theta_y^{\frac{1}{2}}(x)\triangle\Theta_y^{-\frac{1}{2}}(x) = \frac{(m-1)^2\kappa}{8} + \frac{(m-1)(m-3)}{8r_y^2(x)}\left(1 - \left(\frac{\sqrt{-\kappa}r_y(x)}{\sinh\left(\sqrt{-\kappa}r_y(x)\right)}\right)^2\right)$$

for $x, y \in \mathbb{H}_\kappa^m$ so for $m = 3$ (and in the absence of a drift and potential) formula (5.6) reduces to formula (5.2).

We would like to point out that a geometric interpretation of the boundedness assumption in Theorem 5.1.4 is not entirely obvious. Nonetheless, Aida [2004] used formula (5.6) to obtain gradient and Hessian estimates for the heat kernel on manifolds with a pole by differentiating it directly. Unfortunately this required rather heavy assumptions of asymptotic flatness.

One can also use Theorem 5.1.4 to obtain a formula which is valid under more general assumptions. Following [Elworthy, 1988, p.389], suppose only that there exists $p \in M$ for which $\exp_p$ is a *local* diffeomorphism. Then $\exp_p$ makes T_pM into a Riemannian manifold by pulling back the metric on M. When endowed with this metric the tangent space T_pM will be denoted by $\tilde{M}$. The Laplacian on $\tilde{M}$ will be denoted by $\tilde{\triangle}$ and the Riemannian distance function by $\tilde{d}$. It follows that $\exp_p$ is a local isometry when considered as a map from $\tilde{M}$ to M and the origin is a pole for $\tilde{M}$. Supposing that $b = 0$ (so that $p^{M,V}$ is symmetric in its space variables) and that V is a function on M which is smooth and bounded above, denote by $P^{M,V}$ the minimal semigroup for the operator $\frac{1}{2}\triangle + V$ and by $p^{M,V}$ the associated kernel. Defining a function $\tilde{V}$ on $\tilde{M}$ by $\tilde{V} = V \circ \exp_p$, denote by $P^{\tilde{M},\tilde{V}}$ the minimal semigroup for the operator $\frac{1}{2}\tilde{\triangle} + \tilde{V}$ and by $p^{\tilde{M},\tilde{V}}$ the associated kernel. Then, since our assumptions imply that $\exp_p$ is a covering map, it follows that for each $x \in M$ there are at most countably many elements in the preimage of x under $\exp_p$. Denoting by $\tilde{\Theta}_0$ the Jacobian determinant of the inverse exponential map of $\tilde{M}$ based at the origin, we therefore have the following corollary of Theorem 5.1.4.

Corollary 5.1.6. *Suppose that there exists $p \in M$ such that $\exp_p$ is a local diffeomorphism with $\tilde{\Theta}_0^{\frac{1}{2}} \left(\frac{1}{2}\tilde{\triangle} + \tilde{V}\right) \tilde{\Theta}_0^{-\frac{1}{2}}$ bounded above on $\tilde{M}$. Then for all $T > 0$ and $x \in M$ we have*

$$p_T^{M,V}(x,p) = \sum_{\xi \in \exp_p^{-1}(x)} \tilde{q}_T(\xi,0)\tilde{\Theta}_0^{-\frac{1}{2}}(\xi)\mathbb{E}\left[\exp\left[\int_0^T (\tilde{\Theta}_0^{\frac{1}{2}}\left(\frac{1}{2}\tilde{\triangle} + \tilde{V}\right)\tilde{\Theta}_0^{-\frac{1}{2}})(\hat{Y}_s(\xi))ds\right]\right]$$

where for each $\xi \in \exp_p^{-1}(x)$ the process $\hat{Y}(\xi)$ is a semiclassical bridge on $\tilde{M}$ between ξ and the origin in time T and where the function $\tilde{q}_T(\cdot,0)$ is defined by

$$\tilde{q}_T(\xi,0) := (2\pi T)^{-\frac{m}{2}} \exp\left(-\frac{\tilde{d}^2(\xi,0)}{2T}\right)$$

for $\xi \in \tilde{M}$.

Proof. The following argument is due to Elworthy [1988]. Since $\exp_p$ is a covering map it follows that if $U(x)$ is a sufficiently small open neighbourhood of x then its preimage under $\exp_p$ is a countable collection of pairwise disjoint open sets $\{U_\xi(x)\}_{\xi \in \exp_p^{-1}(x)}$ and each $U_\xi(x)$ has the same volume as $U(x)$, by the local isometry. It follows that

$$P_T^{M,V}\mathbf{1}_{U(x)}(p) = \sum_{\xi \in \exp_p^{-1}(x)} P_T^{\tilde{M},\tilde{V}}\mathbf{1}_{U_\xi(x)}(0),$$

by the Feynman-Kac formula, and if we choose $U(x)$ to be a ball around x with suitably small radius then we can let the radius of the ball tend to zero to see that

$$p_T^{M,V}(x,p) = \sum_{\xi \in \exp_p^{-1}(x)} p_T^{\tilde{M},\tilde{V}}(\xi, 0)$$

by the symmetry of $p_T^{M,V}$. The corollary follows by Theorem 5.1.4. □

If the exponential map is a local diffeomorphism then there is a one-to-one correspondence between points in preimage of x under $\exp_p$ and geodesic segments in M which connect p with x in unit time. Expressed in these terms, Corollary 5.1.6 becomes the *sum over geodesics* formula found in Arede [1985] and Elworthy [1988]. In the latter the function V is merely assumed to be *continuous* and bounded above.

5.1.3 The Integrated Heat Kernel

Now suppose that N is a closed embedded submanifold of M of dimension $n \in \{0, \ldots, m-1\}$. For a regular domain D consider the integrated Dirichlet heat kernel $p_\cdot^D(\cdot, N) : [0,\infty) \times M \to \mathbb{R}$ defined by

$$p_T^D(x,N) := \int_N p_T^D(x,y)\, d\,\mathrm{vol}_N(y) \tag{5.7}$$

for $T > 0$ and $x \in M$ and the integrated minimal heat kernel $p_\cdot^M(\cdot, N) : [0,\infty) \times M \to \mathbb{R}$ defined by

$$p_T^M(x,N) := \int_N p_T^M(x,y)\, d\,\mathrm{vol}_N(y) \tag{5.8}$$

for $T > 0$ and $x \in M$. For $y \in M$ one can think of $p_\cdot^M(\cdot, y)$ as a solution to the heat equation on M with a measure-valued initial condition given by the Dirac measure based at y. Similarly, the integrated heat kernel, considered as a function of time and space, can be thought of as a solution to the heat equation on M for the measure-valued initial condition vol_N. For example, if N is a closed embedded surface in $\mathbb{R}^3$ uniformly heated at time zero then the integrated heat kernel describes how the heat diffuses for positive times. If N is a closed embedded loop in $\mathbb{R}^3$ then it could be modelling a hot metal wire. A *probabilistic* interpretation of the integrated heat kernel will be given by Theorem 6.1.1 in the next chapter.

Remark 5.1.7. *Using a non-negative measurable function f with $\mathrm{vol}_N(f) < \infty$ one could more generally replace vol_N with the measure $f \cdot \mathrm{vol}_N$. As remarked upon below, the main result of this chapter, Theorem 5.2.1, extends to measures of this form. It follows that if N is compact then one could instead consider the probability measure $\mathrm{vol}_N(N)^{-1} \cdot \mathrm{vol}_N$. All results obtained in Chapter 6 for the measure vol_N can be applied to the normalized measure too, upto a constant.*

Example 5.1.8. *For $r > 0$ denote by $\mathbb{S}^1(r)$ the circle of radius r in $\mathbb{R}^2$ and suppose that ϕ is a continuous function on $\mathbb{R}^2$ with compact support. Then, using the polar coordinates (r, θ), we see that*

$$\frac{1}{2\pi r} \int_{\mathbb{S}^1(r)} \phi(y) d\,\mathrm{vol}_{\mathbb{S}^1(r)}(y) = \frac{1}{2\pi} \int_0^{2\pi} \phi((r,\theta)) d\theta \to_{r\downarrow 0} \phi(0) \tag{5.9}$$

so that

$$\lim_{r\downarrow 0} \frac{1}{2\pi r} \int_{\mathbb{S}^1(r)} \phi(y) d\,\mathrm{vol}_{\mathbb{S}^1(r)}(y) = \int_{\{0\}} \phi(y) d\,\mathrm{vol}_{\{0\}}(y). \tag{5.10}$$

In this and other examples where N is of finite volume, normalizing therefore leads to additional continuity properties, but in this chapter we wish to avoid extra assumptions and study the integrated heat kernel for the reasons given in the introduction. In particular, we wish to compare it to the function $q_{\cdot}(\cdot, N)$ defined by (4.1), motivated by the fact that if $\mathbb{R}^n$ is viewed as an affine subspace of $\mathbb{R}^m$ then

$$p_T^{\mathbb{R}^m}(x, \mathbb{R}^n) = q_T(x, \mathbb{R}^n). \tag{5.11}$$

If $n = 0$ then vol_N is a counting measure so the results we prove for the integrated heat kernel apply also to the heat kernel itself.

Example 5.1.9. *For $r > 0$ denote by $\mathbb{S}^1(r)$ the circle of radius r in $\mathbb{R}^2$. Then for $t > 0$ and $x \in \mathbb{R}^2$ calculation yields*

$$p_t^{\mathbb{R}^2}(x, \mathbb{S}^1(r)) = r t^{-1} \exp\left[-\frac{(r^2 + \|x\|^2)}{2t}\right] \mathrm{BesselI}\left(0, \frac{r\|x\|}{t}\right) \tag{5.12}$$

where BesselI *denotes the modified Bessel function of the first kind. In Figure 4 below we use this expression to produce a density plot of the integrated heat kernel at a small positive time for the case $r = 1$.*

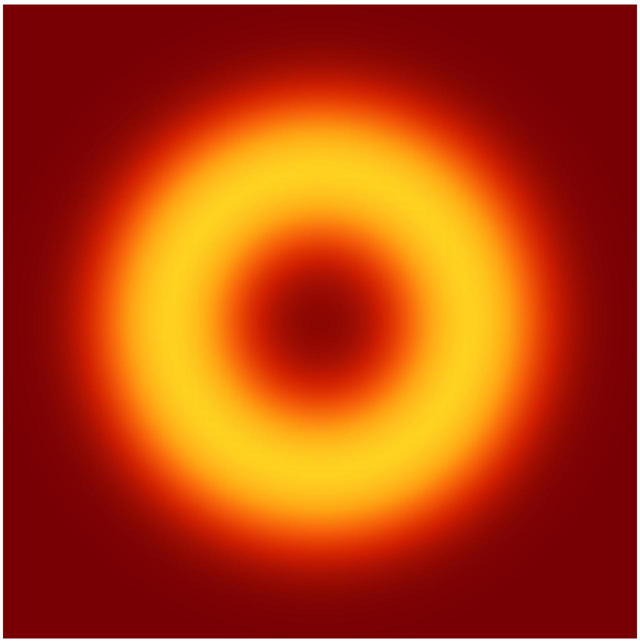

Figure 4: A density plot of the right-hand side of (5.12) for $r = 1$ at a fixed small time $t > 0$. The origin is located at the center of the image.

5.1.4 Ndumu's Formula

Ndumu [1989] proved a formula which generalizes Theorem 5.1.4. Note that Ndumu's formula, and those we obtain below in Theorems 5.2.1 and 5.2.2, can all be extended to the case in which vol_N is replaced by $f \cdot \mathrm{vol}_N$ for a suitable function f.

Theorem 5.1.10. *Suppose that N is compact and that b and V are smooth. Let D be a regular domain compactly contained in the connected open set $M \setminus \mathrm{Cut}(N)$ and let $x \in D$. Let $\hat{Y}(x)$ be a semiclassical bridge between x and N in time T and denote by $\hat{\tau}_D$ the first exit time of this process from D. If $p^{D,b,V}$ denotes the fundamental solution to equation (5.4) on D with Dirichlet boundary conditions then*

$$p_T^{D,b,V}(x, N) = q_T(x, N)C_N(x)\mathbb{E}\left[\mathbf{1}_{\{T<\hat{\tau}_D\}} \exp\left[\int_0^T \left(C_N^{-1}\left(\frac{1}{2}\triangle + b + V\right)C_N\right)(\hat{Y}_s(x))ds\right]\right].$$

Unfortunately, it is not generally possible to construct an exhaustion of M using regular domains contained in $M \setminus \mathrm{Cut}(N)$. Furthermore, the regularity of Θ_N on the cut locus and around $\check{C}(N)$ is not known. It is therefore difficult to use Ndumu's formula to access information about the behaviour of p^M on the cut locus. While the 'quantum potential' appearing in Ndumu's formula is rather difficult to work with, as remarked upon earlier, Ndumu used this formula to deduce an exact expansion which he extended in Ndumu [2011] using a theorem of Azencott. We will mention Ndumu's expansion later in the chapter but we will not state it.

Example 5.1.11. *With the* 2*-dimensional hyperbolic space* $\mathbb{H}^2$ *viewed as a totally geodesic embedded submanifold of* $\mathbb{H}^3$*, Ndumu used Theorem 5.1.10 and formula (5.2) for the case* $\kappa = -1$ *to show that*

$$p_t^{\mathbb{H}^3}(x, \mathbb{H}^2) = (2\pi t)^{-\frac{1}{2}} \exp\left[-\frac{r^2_{\mathbb{H}^2}(x)}{2t}\right] \frac{e^{-\frac{t}{2}}}{\cosh(r_{\mathbb{H}^2}(x)))}$$

for $t > 0$ *and* $x \in \mathbb{H}^3$*. If* $X(x)$ *is a Brownian motion in* $\mathbb{H}^3$ *starting at* x *then it follows, by formula (2.17), that*

$$\begin{aligned} \lim_{t\uparrow\infty} \mathbb{E}\left[L_t^{\mathbb{H}^2}(X(x))\right] &= \mathrm{sech}(r_{\mathbb{H}^2}(x))) \int_0^\infty (2\pi t)^{-\frac{1}{2}} \exp\left[-\frac{r^2_{\mathbb{H}^2}(x)}{2t} - \frac{t}{2}\right] dt \\ &= \mathrm{sech}(r_{\mathbb{H}^2}(x)) \exp\left[-r_{\mathbb{H}^2}(x)\right]. \end{aligned}$$

In contrast, if $\mathbb{R}^2$ *is embedded as a linear subspace of* $\mathbb{R}^3$ *with* $X(x)$ *is a Brownian motion in* $\mathbb{R}^3$ *starting at* x *then*

$$\lim_{t\uparrow\infty} \mathbb{E}\left[L_t^{\mathbb{R}^2}(X(x))\right] = \infty.$$

The formula we prove in the next section is one in which the effect of the cut locus is not neglected and in which the 'quantum potential' appearing in the formulae of Elworthy, Truman, Watling and Ndumu is replaced by something more amenable to analysis.

5.2 General Formulae

5.2.1 A Formula the Integrated Heat Kernel

In this subsection we take $b = 0$ and $V = 0$ (we consider a drift and potential in the next subsection). The following is the main result of this chapter.

Theorem 5.2.1. *Suppose that M is a complete and connected Riemannian manifold of dimension m, that N is a closed embedded submanifold of M of dimension $n \in \{0, \ldots, m-1\}$ and that D is a regular domain in M. Suppose that $x \in M$ with $T > 0$ and that $\hat{X}(x)$ is a Fermi bridge between x and N in time T, defined upto the minimum of T and its explosion time, and denote by $\hat{\tau}_D$ the first exit time of this process from D. Then, with $q_\cdot(\cdot, N)$ and $p_\cdot^D(\cdot, N)$ defined by (4.1) and (5.7) respectively, we have*

$$p_T^D(x, N) = q_T(x, N) \lim_{t \uparrow T} \mathbb{E}\left[\mathbf{1}_{\{t < \hat{\tau}_D\}} \exp\left[\int_0^t \frac{r_N(\hat{X}_s(x))}{T - s}\left(d\mathbf{A}_s + d\mathbf{L}_s\right)\right]\right] \tag{5.13}$$

where

$$d\mathbf{A}_s := \frac{\partial}{\partial r_N} \log \Theta_N^{-\frac{1}{2}}(\hat{X}_s(x))ds, \quad d\mathbf{L}_s := d\mathbb{L}_s^{\mathrm{Cut}(N)}(\hat{X}(x)). \tag{5.14}$$

Proof. We begin by using Theorem 1.2.1 to see that

$$\begin{aligned}
&\lim_{t\uparrow T} \int_M p_t^D(x, y) q_{T-t}(y, N) d\operatorname{vol}_M(y) \\
=\ &\lim_{t\uparrow T} \int_N \int_{T_pN^\perp} (p_t^D(x, \exp_N)\mathbf{1}_{\mathcal{M}_p(N)}\theta_N)(\xi)(2\pi(T-t))^{-\frac{(m-n)}{2}} \exp\left[-\frac{\|\xi\|^2}{2(T-t)}\right] d\xi d\operatorname{vol}_N(p) \\
=\ &\lim_{t\uparrow T} \int_N \int_{T_pN^\perp} (p_t^D(x, \exp_N)\mathbf{1}_{\mathcal{M}_p(N)}\theta_N)(\sqrt{T-t}\xi)(2\pi)^{-\frac{(m-n)}{2}} \exp\left[-\frac{\|\xi\|^2}{2}\right] d\xi d\operatorname{vol}_N(p) \\
=\ &\int_N \int_{T_pN^\perp} (p_t^D(x, \exp_N)\mathbf{1}_{\mathcal{M}_p(N)}\theta_N)(0_p)(2\pi)^{-\frac{(m-n)}{2}} \exp\left[-\frac{\|\xi\|^2}{2}\right] d\xi d\operatorname{vol}_N(p) \\
=\ &\int_N p_T^D(x, p) d\operatorname{vol}_N(p)
\end{aligned}$$

where 0_p denotes the origin of the vector space $T_pN^\perp$ and where the third equality is justified by the compactness of the closure of D, the dominated convergence theorem and the fact that that for each $p \in N$ the indicator function $\mathbf{1}_{\mathcal{M}_p(N)}$ is continuous on $T_pN^\perp$ in a neighbourhood of the origin. Then, denoting by $\{P_t^D : t \geq 0\}$ the

Dirichlet heat semigroup for the domain D, it follows from Girsanov's theorem that

$$\begin{aligned}
&\int_N p_T^D(x,y)\,d\,\mathrm{vol}_N(y)\\
&= \lim_{t\uparrow T}\int_M p_t^D(x,y)q_{T-t}(y,N)\,d\,\mathrm{vol}_M(y)\\
&= \lim_{t\uparrow T} P_t^D q_{T-t}(\cdot,N)(x)\\
&= \lim_{t\uparrow T}\mathbb{E}\left[\mathbf{1}_{\{t<\hat\tau_D\}}q_{T-t}(\hat X_t(x),N)\hat M_t\right]
\end{aligned}$$

where

$$\hat M_{t\wedge\hat\tau_D} = \exp\left[\int_0^{t\wedge\hat\tau_D}\frac{r_N(\hat X_s(x))}{T-s}\left\langle\frac{\partial}{\partial r_N},\hat U_s dB_s\right\rangle - \frac{1}{2}\int_0^{t\wedge\hat\tau_D}\frac{r_N^2(\hat X_s(x))}{(T-s)^2}ds\right] \quad (5.15)$$

with $\hat U$ a horizontal lift of $\hat X(x)$ whose antidevelopment has martingale part given by an $\mathbb{R}^m$-valued Brownian motion B. Itô's formula and formula (4.2) imply

$$\begin{aligned}
&\log q_{T-(t\wedge\hat\tau_D)}(\hat X_{t\wedge\hat\tau_D}(x),N)\\
=\ &\log q_T(x,N) - \int_0^{t\wedge\hat\tau_D}\frac{r_N(\hat X_s(x))}{T-s}\left\langle\frac{\partial}{\partial r_N},\hat U_s dB_s\right\rangle\\
&+\int_0^{t\wedge\hat\tau_D}\frac{\partial}{\partial s}\log q_{T-s}(\hat X_s(x),N)ds + \int_0^{t\wedge\hat\tau_D}\frac{r_N^2(\hat X_s(x))}{(T-s)^2}ds\\
&+\frac{1}{2}\int_0^{t\wedge\hat\tau_D}\triangle\log q_{T-s}(\hat X_s(x),N)ds\\
&+\int_0^{t\wedge\hat\tau_D}\frac{r_N(\hat X_s(x))}{T-s}d\mathbb{L}_s^{\mathrm{Cut}(N)}(\hat X(x)).
\end{aligned}$$

and so we can eliminate the stochastic integral in (5.15) by rearrangement and substitution. Finally, using the fact that

$$\frac{\partial}{\partial s}\log q_{T-s}(\cdot,N) = \frac{m-n}{2(T-s)} - \frac{r_N^2(\cdot)}{2(T-s)^2}$$

and also that

$$\triangle\log q_{T-s}(\cdot,N) = -\frac{\triangle r_N^2(\cdot)}{2(T-s)}$$

on $M(N)$ together with equation (1.7), we can further simplify the resulting formula so as to obtain the desired expression. $\square$

Theorem 5.2.2. *Suppose that $\{D_i\}_{i=1}^{\infty}$ is an exhaustion of M by regular domains. Then we have*

$$p_T^M(x,N) = q_T(x,N) \lim_{i\uparrow\infty} \lim_{t\uparrow T} \mathbb{E}\left[\mathbf{1}_{\{t<\hat{\tau}_{D_i}\}} \exp\left[\int_0^t \frac{r_N(\hat{X}_s(x))}{T-s}\left(d\mathbf{A}_s + d\mathbf{L}_s\right)\right]\right] \tag{5.16}$$

where $\mathbf{A}$ *and* $\mathbf{L}$ *are defined by (5.14).*

Proof. Recalling that p^M is given as the limit of the increasing sequence of Dirichlet heat kernels p^{D_i}, it follows from the monotone convergence theorem that

$$p_T^M(x,N) = \lim_{i\uparrow\infty} p_T^{D_i}(x,N)$$

and so the result follows by Theorem 5.2.1. $\square$

Note that the integrator in the exponent in formula (5.16) is given as the sum of an absolutely continuous part

$$d\mathbf{A} = \frac{\partial}{\partial r_N} \log \Theta_N^{-\frac{1}{2}}(\hat{X}(x))\, ds,$$

whose support is contained in the set of times at which $\hat{X}(x)$ is in $M \setminus \mathrm{Cut}(N)$ (i.e. off the cut locus) and a singular part

$$d\mathbf{L} = d\mathbb{L}^{\mathrm{Cut}(N)}(\hat{X}(x)) = -\frac{1}{2}\left(D^+_{\hat{X}(x)} - D^-_{\hat{X}(x)}\right) r_N(\mathbf{n})\, dL^{\mathring{C}(N)}(\hat{X}(x))$$

whose support is contained in the set of times at which $\hat{X}(x)$ is in $\mathrm{Cut}(N)$ (i.e. on the cut locus). These two random measures describe how the kernel $q_T(x,N)$ differs from the true integrated heat kernel $p_T^M(x,N)$. The comparison theorem of Heintze and Karcher [1978] (see Section 1.3) and the monotone convergence theorem imply the following corollary.

Corollary 5.2.3. *Suppose that M is stochastically complete, that the cut locus of N has Hausdorff dimension at most $m-2$ and that one of the following conditions is* *satisfied:*

($\mathbf{I_0}$) $n \in \{0, \ldots, m-1\}$*, the sectional curvature of planes containing the radial direction vanishes and* N *is totally geodesic;*

($\mathbf{II_0}$) $n \in \{0, m-1\}$*, the Ricci curvature in the radial direction vanishes and* N *is minimal.*

Then we have

$$p_T^M(x, N) = q_T(x, N)$$

for all $x \in M$ *and* $T > 0$*.*

Consequently one recovers the identity (5.11) in the Euclidean setting.

Example 5.2.4. *Theorem 5.2.2 implies that for the* m*-dimensional sphere* $\mathbb{S}^m_\kappa$ *with constant sectional curvature* κ *we have*

$$p_T^{\mathbb{S}^m_\kappa}(x, y) = q_T(x, y)\mathbb{E}\left[\prod_{k=1}^{\infty} \exp\left[\int_0^T \frac{(m-1)\kappa r_y^2(\hat{X}_s(x))}{(T-s)(\pi^2 k^2 - \kappa r_y^2(\hat{X}_s(x)))}\, ds\right]\right]$$

by the monotone convergence theorem and the expansion for the cotangent function given in Subsection 1.3.3. Note that the set of times at which the denominator of the integrand vanishes has Lebesgue measure zero.

While passing the two limits in formula (5.16) through the integral does not seem to be a hugely important thing to do, this matter has been considered in Section C.2, where a few additional remarks about the formula itself can also be found.

5.2.2 A Formula for the Feynman-Kac Kernel

One can include a smooth drift b and a smooth potential V using a similar approach. In particular, for the special case in which N is a point, the Feynman-Kac formula

(5.5) yields the formula

$$p_T^{M,b,V}(x,y) = q_T(x,y) \lim_{i\uparrow\infty} \lim_{t\uparrow T} \mathbb{E}\left[\mathbf{1}_{\{t<\hat{\tau}_{D_i}\}} \exp\left[\int_0^t V(\hat{X}_s(x))ds \right.\right.$$

$$+ \int_0^t \frac{r_y(\hat{X}_s(x))}{(T-s)} \frac{\partial}{\partial r_y}\left(\log \Theta_y^{-\frac{1}{2}}(\hat{X}_s(x)) - b(\hat{X}_s(x))\right) ds$$

$$\left.\left. + \int_0^t \frac{r_y(\hat{X}_s(x))}{(T-s)} d\mathbb{L}_s^{C(y)}(\hat{X}(x))\right]\right]$$

where for convenience we have defined $\frac{\partial}{\partial r_y} b := \left\langle \frac{\partial}{\partial r_y}, b\right\rangle$ and where we should recall that $\hat{X}(x)$ is a diffusion on M starting at x with time-dependent infinitesimal generator

$$\frac{1}{2}\triangle + b - \frac{r_y}{T-s}\frac{\partial}{\partial r_y}.$$

In particular, if $M = \mathbb{R}^m$ and b is bounded then by Corollary C.1.4 we have the representation formula

$$p_T^{\mathbb{R}^m,b}(x,y) = p_T^{\mathbb{R}^m}(x,y)\mathbb{E}\left[\exp\left[\int_0^T \frac{r_y(\hat{X}_s(x))}{T-s}\left\langle \frac{\partial}{\partial r_y}, b(\hat{X}_s(x))\right\rangle ds\right]\right].$$

Estimation and comparison of the transition densities of a Brownian motion with drift b, whether it be smooth and bounded or only measurable and under a growth condition, has already been considered in the series of articles Qian and Wei [1991], Qian [1994, 1995] and Qian and Zheng [2004]. For the remainder of this thesis we will therefore focus on how our approach can be applied to the submanifold generalization and assume that $b = 0$ for simplicity. Under suitable assumptions on b it should nevertheless be possible to use our formulae, and the upper bounds of Qian [1994] where appropriate, to deduce estimates and asymptotic relations for $p^{M,b}$ and its integrals which are similar to those for p^M given later in the chapter.

5.3 Estimates and Asymptotic Relations

5.3.1 Lower Bounds

The history of Gaussian heat kernel estimates begins with the famous work of Nash [1958], on the local Hölder continuity of solutions of second order uniformly parabolic equations in $\mathbb{R}^m$ with non-smooth coefficients. Nash derived regularity properties of the general solutions to these equations using properties of the fundamental solutions, the key result being a moment estimate similar to the one we proved in Corollary 3.2.4. The upper and lower bounds which Nash proved in the appendix to that article were later improved with a Harnack inequality by Aronson [1967, 1968] before Fabes and Stroock [1986] demonstrated that the method of Nash can be improved without using a Harnack inequality.

The heat kernel lower bounds of Cheeger and Yau [1981], for balls in Riemannian manifolds, were proved using a bound on the Ricci curvature in the radial direction and a Laplacian comparison theorem. These are similar to the objects we will use, but our method is quite different. Our method is closer in spirit to that of Wang [1997], who also used stochastic techniques with unbounded curvature (but only for the one point case). Our lower bounds for the *integrated* heat kernel will be deduced from the following proposition, for which we recall that the functions $q_{\cdot}(\cdot, N)$ and $p_{\cdot}^M(\cdot, N)$ are defined by (4.1) and (5.8) respectively.

Proposition 5.3.1. *Suppose that M is stochastically complete and that there exist constants $\alpha, \beta \geq 0$ such that*

$$\frac{\partial}{\partial r_N} \log \Theta_N \leq \alpha + \beta r_N. \tag{5.17}$$

Then for any $x \in M$ and $T > 0$ we have the lower bound

$$p_T^M(x, N) \geq q_T(x, N) \exp\left[-\alpha\left(r_N^2(x) + \nu T\right)^{\frac{1}{2}} e^{\frac{\lambda T}{2}} - \frac{\beta}{2}\left(r_N^2(x) + \nu T\right) e^{\lambda T}\right]$$

where $\nu = m - n + \frac{\alpha}{2}$ and $\lambda = \frac{\alpha}{2} + \beta$.

Proof. Using the non-explosive Fermi bridge $\hat{X}(x)$ constructed in Subsection 4.2.3 we see, by Theorem 5.2.2 and the fact that $\mathbb{L}^{\mathrm{Cut}(N)}(\hat{X}(x))$ is non-decreasing, that

$$p_T^M(x,N) \geq q_T(x,N) \lim_{i\uparrow\infty} \lim_{t\uparrow T} \mathbb{E}\left[\mathbf{1}_{\{t<\hat{\tau}_{D_i}\}} \exp\left[-\int_0^t \frac{f(\hat{X}_s(x))}{T-s} ds\right]\right]$$

where the non-negative function f is defined by $f(x) = \frac{1}{2}\left(\alpha r_N(x) + \beta r_N^2(x)\right)$ for $x \in M$. For $t \in [0,T)$ we see that

$$\begin{aligned}
&\mathbf{1}_{\{t<\hat{\tau}_{D_i}\}} \exp\left[-\int_0^t \frac{f(\hat{X}_s(x))}{T-s} ds\right] \\
&= \mathbf{1}_{\{t<\hat{\tau}_{D_i}\}} \sum_{p=0}^{\infty} \frac{\left(-\int_0^t \frac{f(\hat{X}_s(x))}{T-s} ds\right)^p}{p!} \\
&= \mathbf{1}_{\{t<\hat{\tau}_{D_i}\}} - 1 + \sum_{p=0}^{\infty} \frac{\left(-\mathbf{1}_{\{t<\hat{\tau}_{D_i}\}} \int_0^t \frac{f(\hat{X}_s(x))}{T-s} ds\right)^p}{p!} \\
&= \mathbf{1}_{\{t<\hat{\tau}_{D_i}\}} - 1 + \exp\left[-\mathbf{1}_{\{t<\hat{\tau}_{D_i}\}} \int_0^t \frac{f(\hat{X}_s(x))}{T-s} ds\right]
\end{aligned}$$

from which it follows, by Jensen's inequality, that

$$\begin{aligned}
&\mathbb{E}\left[\mathbf{1}_{\{t<\hat{\tau}_{D_i}\}} \exp\left[-\int_0^t \frac{f(\hat{X}_s(x))}{T-s} ds\right]\right] \\
&\geq \mathbb{Q}_{T-}\{t < \hat{\tau}_{D_i}\} - 1 + \exp\left[-\mathbb{E}\left[\mathbf{1}_{\{t<\hat{\tau}_{D_i}\}} \int_0^t \frac{f(\hat{X}_s(x))}{T-s} ds\right]\right].
\end{aligned} \tag{5.18}$$

For the exponential term on the right-hand side of inequality (5.18), our assumptions imply that (3.7) holds with ν and λ as given so by Lemmas 4.2.3 and 4.2.4 we have

$$\begin{aligned}
&\mathbb{E}\left[\mathbf{1}_{\{t<\hat{\tau}_{D_i}\}} \int_0^t \frac{f(\hat{X}_s)}{T-s} ds\right] \\
&\leq \frac{\alpha}{2}\mathbb{E}\left[\int_0^t \frac{r_N(\hat{X}_s(x))}{T-s} ds\right] + \frac{\beta}{2}\mathbb{E}\left[\int_0^t \frac{r_N^2(\hat{X}_s(x))}{T-s} ds\right] \\
&\leq \alpha\left(r_N^2(x) + \nu T\right)^{\frac{1}{2}} e^{\frac{\lambda T}{2}} + \frac{\beta}{2}\left(r_N^2(x) + \nu T\right) e^{\lambda T}.
\end{aligned} \tag{5.19}$$

For the first term on the right-hand side of inequality (5.18), we note that

$$
\begin{aligned}
\lim_{i\uparrow\infty}\lim_{t\uparrow T}\mathbb{Q}_{T-}\{t<\hat{\tau}_{D_i}\} &= \lim_{i\uparrow\infty}\mathbb{Q}_{T-}\{\hat{X}_s(x)\in D_i, \forall s\in[0,T)\}\\
&= \mathbb{Q}_{T-}\{\hat{X}_s(x)\in M, \forall s\in[0,T)\} \qquad (5.20)\\
&= 1
\end{aligned}
$$

by the dominated convergence theorem and non-explosion property. Combining (5.18) with (5.19) and (5.20) yields the desired estimate. □

Theorem 5.3.2. *Suppose that M is stochastically complete and that there exist constants $C_1, C_2, \Lambda \geq 0$ with respect to which at least one of the conditions **(C1)**, **(C2)** or **(C3)** of Theorem 1.4.5 is satisfied. Then for any $x \in M$ and $T > 0$ the assumptions of Proposition 5.3.1 hold with $\alpha = n\Lambda + (m-1)C_1$ and $\beta = (m-1)C_2$. In particular, for each $T > 0$ there exists a constant $C \geq 0$, depending only on T, C_1, C_2, Λ, m and n, such that*

$$
p_t^M(x,N) \geq t^{-\frac{(m-n)}{2}} \exp\left[-\frac{r_N^2(x)}{2t} - C(1+r_N^2(x))\right]
$$

for all $x \in M$ and $t \in (0,T]$.

Proof. The theorem follows from Theorem 1.4.5 and Proposition 5.3.1, since they imply the lower bound

$$
\begin{aligned}
p_t^M(x,N) \geq q_t(x,N)\exp\Big[&-(n\Lambda+(m-1)C_1)\left(r_N^2(x)+\nu t\right)^{\frac{1}{2}} e^{\frac{\lambda t}{2}}\\
&-\frac{(m-1)C_2}{2}\left(r_N^2(x)+\nu t\right)e^{\lambda t}\Big]
\end{aligned}
$$

where $\nu = m - n + \frac{n\Lambda+(m-1)C_1}{2}$ and $\lambda = \frac{n\Lambda+(m-1)C_1}{2} + (m-1)C_2$. □

Appendix D includes an alternative lower bound for the case $C_2 = 0$, which is less explicit but has better large time behaviour. Since we we are primarily interested in applications to the study of bridge processes, our focus will be on short time estimates.

Corollary 5.3.3. *Suppose that M is stochastically complete and that one of the following conditions is satisfied:*

(I) $n \in \{0, \ldots, m-1\}$, *the sectional curvature of planes containing the radial direction is non-negative and N is totally geodesic;*

(II) $n \in \{0, m-1\}$, *the Ricci curvature in the radial direction is non-negative and N is minimal.*

Then we have the comparison

$$p_T^M(x, N) \geq q_T(x, N)$$

for all $x \in M$ and $T > 0$.

Proof. The curvature assumptions imply $\frac{\partial}{\partial r_N} \log \Theta_N \leq 0$, by Theorem 1.4.5, so the assertion follows from Theorem 5.3.2. □

The constants C_1, C_2 and Λ typically depend upon N. For lower bounds on the heat kernel which are uniform in both space variables we have the following corollary of Theorem 5.3.2, which we include for completeness.

Corollary 5.3.4. *Suppose* $\operatorname{Ric} \geq -(m-1)C_1^2$, *for some constant $C_1 \geq 0$. Then there exists a constant $C \geq 0$, depending only on T, C_1 and m, such that*

$$p_t^M(x, y) \geq t^{-\frac{m}{2}} \exp\left[-\frac{r_y^2(x)}{2t} - C(1 + r_y(x))\right]$$

for all $x, y \in M$ and $t \in (0, T]$.

Proof. The corollary follows from Theorems 1.4.5 and 5.3.2, since they imply the lower bound

$$p_t^M(x, y) \geq q_t(x, y) \exp\left[-(m-1)C\left(r_y^2(x) + \left(m + \frac{(m-1)C}{2}\right)t\right)^{\frac{1}{2}} e^{\frac{(m-1)Ct}{4}}\right].$$

□

The following corollary shows that the large-time behaviour of our estimates can be improved in the compact case.

Corollary 5.3.5. *Suppose that M is compact, let $-C_1^2$ be a lower bound on the sectional curvatures of M with $C_1 \geq 0$ and let Λ be an upper bound on the absolute value of the principal curvatures of N. Then*

$$p_T^M(x,N) \geq q_T(x,N) \exp\left[-\alpha\left(r_N^2(x)+\nu T\right)^{\frac{1}{2}}\right]$$

for all $x \in M$ and $T > 0$, where $\alpha = n\Lambda + (m-1)C_1$ and $\nu = m - n + \operatorname{diam}(M)\alpha$.

Proof. By Theorem 1.4.5 the assumptions of Proposition 5.3.1 are satisfied with $\alpha = n\Lambda + (m-1)C_1$ and $\beta = 0$, while inequality (3.7) holds with $\nu = m-n+\operatorname{diam}(M)\alpha$ and $\lambda = 0$. $\square$

Example 5.3.6. *Suppose that $M = \mathbb{R}^m$ with*

$$N = \{(x_1,\ldots,x_m) \in \mathbb{R}^m : x_1^2 + \cdots + x_{n+1}^2 = 1, x_{n+2} = \cdots = x_m = 0\}$$

equipped with the induced metric. By Theorem 5.3.2 there is the lower bound

$$p_T^M(0,N) \geq (2\pi T)^{-\frac{m-n}{2}} \exp\left[-\frac{1}{2T} - n\left(1+\left(m-\frac{n}{2}\right)T\right)^{\frac{1}{2}} e^{\frac{nT}{4}}\right] \tag{5.21}$$

for all $T > 0$. On the other hand, we know that

$$p_T^M(0,N) = (2\pi T)^{-\frac{m}{2}} \exp\left[-\frac{1}{2T}\right] \operatorname{vol}_N(N)$$

for all $T > 0$. So the difference between the powers of T appearing in the two prefactors is balanced by the exponential factor on the right-hand side of (5.21), which contains information about curvature. If we define

$$f_T(x,N) := \frac{p_T^M(x,N)}{q_T(x,N)}$$

then in this example, where the origin is focal for N, we see that

$$f_T(0, N) = \mathrm{vol}_N(N)(2\pi T)^{-\frac{n}{2}}$$

for all $T > 0$. In the general setting, it would be interesting to determine the order in T of the ratio $f_T(x, N)$ when $x \in \mathrm{Cut}(N)$ is a focal point.

5.3.2 Local Time Comparison

Given formula (2.17), the lower bounds of the previous subsection imply lower bounds on the expected value of the local time of Brownian motion on a hypersurface. In particular, we have the following comparison.

Theorem 5.3.7. *Suppose that N is a minimal hypersurface, that the Ricci curvature in the radial direction is non-negative and that $X(x)$ is a non-explosive Brownian motion on M starting at x. Then*

$$\mathbb{E}[L_t^N(X(x))] \geq \mathbb{E}[L_t^0(B(r_N(x)))]$$

for all $t \geq 0$, where $B(r_N(x))$ denotes a Brownian motion on $\mathbb{R}$ starting at $r_N(x)$ (or at $-r_N(x)$).

Proof. The comparison follows from Corollary 5.3.3 and formula (2.17). $\square$

Note that the factor $s^{-\frac{(m-n)}{2}}$ appearing in the definition of $q_s(x, N)$ is integrable only in the hypersurface case and that in the notation of the above theorem we have

$$\mathbb{E}\left[L_t^0(B(0))\right] = \sqrt{\frac{2t}{\pi}}$$

for all $t \geq 0$.

5.3.3 Asymptotic Relations

Since the heat kernel is a positive fundamental solution to the heat equation, fixing one of the spatial variables for small times will result in densities whose mass is localized around that fixed point. Riemannian manifolds are locally Euclidean, so we might expect that the resulting densities should, for these small times, look like the Gauss-Weierstrass kernel (5.1). The precise sense in which this is true is given by Varadhan's asymptotic relation, proved originally in the articles Varadhan [1967a,b]. In particular, for the minimal heat kernel p^M of a complete Riemannian manifold M it was proved by Varadhan that

$$\lim_{t \downarrow 0} t \log p_t^M(x,y) = -\frac{d^2(x,y)}{2} \tag{5.22}$$

uniformly on compact subsets of $M \times M$. Hsu [1990] found the best conditions under which Varadhan's relation holds on *noncomplete* Riemannian manifolds.

Similarly, the embedding in M of a submanifold N is locally diffeomorphic to an affine embedding of $\mathbb{R}^n$ in $\mathbb{R}^m$ and so one might expect that for small times the integrated heat kernel $p^M_\cdot(\cdot, N)$ should look something like the kernel $q_\cdot(\cdot, N)$. Our lower bounds on the integrated heat kernel, combined with the pointwise relation (5.22), allow us to deduce an asymptotic relation for the integrated heat kernel which makes this intuition precise.

Theorem 5.3.8. *Suppose that M is a complete and connected Riemannian manifold of dimension m and that N is a compactly embedded submanifold of M of dimension $n \in \{0, \ldots, m-1\}$. Then*

$$\lim_{t \downarrow 0} t \log p_t^M(x,N) = -\frac{d^2(x,N)}{2} \tag{5.23}$$

uniformly on compact subsets of M.

Proof. It is a simple matter to show that the left-hand side of (5.23) is less than or equal to the right-hand side, using Varadhan's relation and the fact that $r_N(x) \leq r_y(x)$ for all $y \in N$. To prove the other inequality first assume that M is compact. Then the result follows immediately from Corollary 5.3.5. So let us assume that M is non-compact, let K be any compact subset of M and for $x \in K$ and $y \in N$ denote by $\Gamma_{x,y}$ the set of all length-minizing geodesic segments between x and y, viewed as a subset of M. Then $\Gamma_{x,y}$ contains (the image of) at least one such geodesic and by the triangle inequality the set

$$\Gamma_{K,N} := \bigcup_{x \in K, y \in N} \Gamma_{x,y}$$

is a bounded subset of M. Now let D be any regular domain in M containing $\Gamma_{K,N}$. Modify M outside of D so as to obtain a compact Riemannian manifold M_D (by doubling, for example) and suppose that D is sufficiently large so that

$$\lim_{t \downarrow 0} \frac{p_t^D(x,y)}{p_t^{M_D}(x,y)} = 1$$

uniformly for $x \in K$ and $y \in N$. This is the principle of *not feeling the boundary* (see Hsu [1995]). Such D can always be found since we are assuming that M is non-compact (see Norris [1997]). Then for all $\epsilon > 0$ there exists $t_{\epsilon,K} > 0$ such that for $t \in (0, t_{\epsilon,K})$ we have

$$(1-\epsilon) p_t^{M_D}(x,N) \leq p_t^D(x,N) \leq p_t^M(x,N)$$

for all $x \in K$. It follows from this and the result in the compact case that

$$\lim_{t \downarrow 0} t \log p_t^M(x,N) \geq -\frac{d_{M_D}^2(x,N)}{2}$$

where d_{M_D} denotes the distance function on M_D. But since $\Gamma_{K,N}$ is contained in D it follows that $x \in K$ and $y \in N$ implies $d_{M_D}(x,y) \leq d(x,y)$. Therefore $d_{M_D}(x,N) \leq d(x,N)$ and the result follows. $\square$

The expansion of Ndumu [2011], while only valid away from the cut locus, could be used as an alternative to our lower bounds by connecting points in K to N with smooth curves, covering them with small balls and invoking the Markov property. Alternatively, note that Hino and Ramírez [2003] proved, in the context of Dirichlet spaces, that

$$\lim_{t \downarrow 0} t \log \mathbb{P}\{X_0 \in A; X_t \in B\} = -\frac{d^2(A,B)}{2}$$

for all measurable sets A and B of positive measure, where X denotes the Markov process associated with the underlying local regular Dirichlet form and where d is the associated intrinsic distance. Using the upper bounds of Sturm [1995] and pointwise lower bounds, one can deduce from this the pointwise relation of Varadhan, as shown by [Ramírez, 2001, Theorem 4.1]. A modification of this approach, replacing balls with tubular neighbourhoods and pointwise lower bounds with our integrated heat kernel lower bounds, can also be used to deduce a relation similar to (5.23).

Example 5.3.9. *Suppose that D is a regular domain in M; then ∂D is a compactly embedded hypersurface. According to [Hsu, 2002, Theorem 5.2.6] one has*

$$\lim_{t\downarrow 0} t \log \mathbb{P}\{\tau_D(x) < t\} = -\frac{d^2(x, \partial D)}{2}$$

for all $x \in D$, where $\tau_D(x)$ denotes the first exit time from D of a Brownian motion $X(x)$ starting at x. On the other hand, according to [Norris, 1997, Theorem 1.2] one has

$$\lim_{t\downarrow 0} t \log p_t^M(x, D, x) = -\frac{d^2(x, \partial D)}{2}$$

for all $x \notin D$, where $p_t^M(x, D, y) := p_t^M(x, y) - p_t^{M\setminus\overline{D}}(x, y)$ is the measure of heat passing through D. Theorem 5.3.8 implies, in either case, that

$$\lim_{t\downarrow 0} t \log \frac{d}{dt}\mathbb{E}\left[L_t^{\partial D}(X(x))\right] = -\frac{d^2(x, \partial D)}{2}$$

by Theorem 2.4.1.

Example 5.3.10. *If TM is equipped with the Sasaki metric then Theorem 5.3.8 implies*

$$\lim_{t\downarrow 0} t \log p_t^{TM}(\xi, M) = -\frac{\|\xi\|^2}{2}$$

uniformly for ξ in compact subsets of TM.

5.3.4 Obtaining Upper Bounds from Lower Bounds

The compact case implies that in general one should only expect a Gaussian upper bound on the heat kernel to hold over a finite time interval. Such bounds were proved by Cheng, Li and Yau [1981] assuming bounded curvature and later extended to a wider class of kernels by Cheeger, Gromov and Taylor [1982]. Soon after, Li and Yau [1986] used their famous gradient estimates to derive upper and lower bounds via a Harnack inequality.

In the series of articles Davies [1987a,b, 1988] and Davies and Pang [1989] it was shown that a uniform bound of the type $p_t(x,y) \leq c(t)$ on the kernel of a second order hypoelliptic operator implies a Gaussian estimate and that no further hypotheses are needed. In particular, for the case of the Laplace-Beltrami operator on a complete Riemannian manifold M, if for $T > 0$ there exists a constant c_1 such that the on-diagonal upper bound

$$p_t^M(x,x) \leq c_1 t^{-\frac{m}{2}} \tag{5.24}$$

holds for all $t \in (0,T]$ then for arbitrary $\delta > 0$ there exists a constant c_δ, which might depend on T, such that

$$p_t^M(x,y) \leq c_\delta t^{-\frac{m}{2}} \exp\left[-\frac{d^2(x,y)}{2(1+\delta)t}\right] \tag{5.25}$$

for all $t \in (0,T]$. As noted in Davies [1987a], the on-diagonal estimate is known to hold if the Ricci curvature is bounded below with the injectivity radius positive.

While the upper bound (5.25) will suffice, we obtain a more complete argument by referring to the work of Grigor'yan, Hu and Lau [2008], who showed how upper bounds can be obtained from lower bounds. Lower bounds are frequently obtained from upper bounds, as in the method of Aronson, while theirs was the first result to go in the other direction. To use their result we need the following definition.

Definition 5.3.11. *We say that* vol_M *is* lower regular *if there exist constants* $T_0, C > 0$ *such that*

$$V_r(x) \geq Cr^m$$

for all $x \in M$ *and* $0 < r < \sqrt{T_0}$.

If the injectivity radius of M is positive and if the Ricci curvature is bounded above by a constant (for example if M is compact) then vol_M is lower regular. For geometric assumptions which imply a positive injectivity radius, see Cheeger, Gromov and Taylor [1982]. Let us assume that vol_M is lower regular and also that the Ricci curvature of M is bounded below by a constant. Then the lower bound of Corollary 5.3.4 holds and implies a *near-diagonal lower estimate* of the form

$$p_t^M(x,y) \geq C' t^{-\frac{m}{2}}$$

for all $0 < t < T_0$ and $x, y \in M$ satisfying $d(x,y) < t^{\frac{1}{2}}$. It follows from [Grigor'yan, Hu and Lau, 2008, Corollary 3.5] that we have, automatically, the existence of constants $c, \sigma^2 > 0$ (which might depend upon T_0) such that

$$p_t^M(x,y) \leq c t^{-\frac{m}{2}} \exp\left[-\frac{d^2(x,y)}{\sigma^2 t}\right] \tag{5.26}$$

for all $t \in (0, T_0)$ and $x, y \in M$. We therefore have the following theorem, by observing that $y \in N$ implies $r_y(x) \geq r_N(x)$.

Theorem 5.3.12. *Suppose that the Ricci curvature of* M *is bounded below by a constant, that* vol_M *is lower regular and that* N *is compact. Then there exist constants* $c, \sigma^2 > 0$ *(which might depend upon* T_0*) such that*

$$p_t^M(x,N) \leq c t^{-\frac{m}{2}} \exp\left[-\frac{r_N^2(x)}{\sigma^2 t}\right]$$

for all $x \in M$ *and* $t \in (0, T_0)$.

5.4 The Martingale Problem for Singular Drift

5.4.1 $L^q_{q'}$ Estimates for the Heat Kernel

For this section, in which we apply the Gaussian upper bounds and Proposition 1.4.9 to a problem of independent interest, suppose that M is a complete and connected

Riemannian manifold of dimension m with

$$R := \inf\{\mathrm{Ric}(\xi,\xi) : \xi \in UTM\} > -\infty$$

and suppose also that the volume measure vol_M is lower regular. These conditions guarantee the existence, upto time T_0, of the Gaussian upper bound (5.26) and $c, \sigma^2 > 0$ will denote the constants appearing in that bound (these constants, as previously mentioned, might depend upon T_0). If for $q \in [1,\infty)$ we denote by $\|\cdot\|_{L^q(M)}$ the usual norm on $L^q(M)$ then we have the following lemma.

Lemma 5.4.1. *Assume that* $0 < t < T_0$ *and that*

$$q > \sigma^2 \left(-\frac{R}{3} \vee 0 \right) t. \tag{5.27}$$

Then for all $x \in M$ *and* $0 < s \le t$ *we have*

$$\|p_s^M(x,\cdot)\|_{L^q(M)} \le C(c,\sigma^2,R,q,m) s^{\frac{m}{2}\left(\frac{1}{q}-1\right)} \tag{5.28}$$

where $C(c,\sigma^2,R,q,m) > 0$ *is a constant.*

Proof. Using the fact that if $a > 0$ then

$$\int_0^\infty \exp\left[-ar^2\right] r^{m-1} ds = \frac{\Gamma(m/2)}{2a^{\frac{m}{2}}}, \tag{5.29}$$

the heat kernel upper bound (5.26) and Proposition 1.4.9 we see that

$$\begin{aligned}
&\|p_s^M(x,\cdot)\|^q_{L^q(M)} \\
\le\ & \int_M \left(cs^{-\frac{m}{2}} \exp\left[-\frac{r_y^2(x)}{\sigma^2 s} \right] \right)^q d\,\mathrm{vol}_M(y) \\
=\ & c^q s^{-\frac{mq}{2}} \int_{UT_xM} \int_0^{c_x(\xi)} \exp\left[-\frac{qr^2}{\sigma^2 s} \right] \theta_x(r\xi) r^{m-1}\, dr\, d\sigma^{m-1}(\xi) \\
\le\ & c^q s^{-\frac{mq}{2}} \frac{m\pi^{\frac{m}{2}}}{\Gamma(\frac{m}{2}+1)} \int_0^\infty \exp\left[\left(\frac{-q}{\sigma^2 s} - \frac{R}{3} \right) r^2 \right] r^{m-1} dr \\
\le\ & c^q s^{-\frac{mq}{2}} \frac{m\pi^{\frac{m}{2}}}{\Gamma(\frac{m}{2}+1)} \int_0^\infty \exp\left[\left(\frac{-q + \sigma^2\left(-\frac{R}{3} \vee 0\right) t}{\sigma^2 s} \right) r^2 \right] r^{m-1} dr
\end{aligned}$$

$$\leq c^q s^{-\frac{mq}{2}} \frac{m\pi^{\frac{m}{2}}}{\Gamma(\frac{m}{2}+1)} \frac{\Gamma(m/2)}{2} \left(\frac{q - \sigma^2 \left(-\frac{R}{3} \vee 0\right) t}{\sigma^2 s} \right)^{-\frac{m}{2}}$$

$$= c^q \left(\frac{\sigma^2 \pi}{q - \sigma^2 \left(-\frac{R}{3} \vee 0\right) t} \right)^{\frac{m}{2}} s^{\frac{m}{2}(1-q)}$$

where $UT_xM := \{\xi \in T_xM : \|\xi\| = 1\}$. □

Note that condition (5.27) always holds for sufficiently small t and that for $M = \mathbb{R}^m$ the inequality (5.28) holds as an equality for the constant $C(q,m) = (2\pi)^{\frac{m}{2}(\frac{1}{q}-1)} q^{-\frac{m}{2q}}$. For $T > 0$, $q, q' \in [1, \infty)$ and a non-negative measurable function $V : [0,T] \times M \to \mathbb{R}$ define

$$\|V\|_{L^q_{q'}([0,t]\times M)} := \left(\int_0^t \|V(s,\cdot)\|^{q'}_{L^q(M)} ds \right)^{\frac{1}{q'}}$$

for $t \in (0,T]$.

Lemma 5.4.2. *In addition to the assumptions of Lemma 5.4.1 suppose also that*

$$\frac{1}{q} + \frac{2}{mq'} > 1. \tag{5.30}$$

Then it follows that

$$\sup_{x\in M} \|p^M_\cdot(x,\cdot)\|_{L^q_{q'}([0,t]\times M)} \leq C(c,\sigma^2,R,q,q',m) t^{\gamma(m,q,q')}$$

where $C(c,\sigma^2,R,q,q',m) \geq 0$ and $\gamma(m,q,q') > 0$ are constants.

Proof. Using Lemma 5.4.1 we see that

$$\begin{aligned} \|p^M_\cdot(x,\cdot)\|^{q'}_{L^q_{q'}([0,t]\times M)} &\leq C(c,\sigma^2,R,q,m)^{q'} \int_0^t s^{\frac{mq'}{2}\left(\frac{1}{q}-1\right)} ds \\ &= C(c,\sigma^2,R,q,m)^{q'} \frac{t^{\frac{mq'}{2}\left(\frac{1}{q}-1\right)+1}}{\frac{mq'}{2}\left(\frac{1}{q}-1\right)+1}. \end{aligned}$$

The right-hand side of this inequality is finite by assumption (5.30) and independent of x and so the lemma follows. □

A consequence of Lemma (5.4.2) is that, under the conditions of the lemma, we have

$$\lim_{t \downarrow 0} \| p_{\cdot}^{M}(x, \cdot) \|_{L^{q}_{q'}([0,t] \times M)} = 0.$$

This observation will be put to use in the following subsection.

5.4.2 Feynman-Kac Potentials in $L^p_{p'}([0,T] \times M)$

Suppose that $V : [0,T] \times M \to \mathbb{R}$ is a non-negative and measurable function and for each $x \in M$ let $X(x)$ be a Brownian motion on M starting at x. Then Khasminskii's lemma (see Fitzsimmons and Pitman [1999]) implies for each $t \in [0,T]$ that if there is a constant $0 \le \alpha < 1$ such that

$$\sup_{(t_0,x) \in [0,t] \times M} \mathbb{E} \left[\int_0^{t-t_0} V(t_0 + s, X_s(x)) ds \right] = \alpha$$

then it follows that

$$\sup_{(t_0,x) \in [0,t] \times M} \mathbb{E} \left[\exp \left[\int_0^{t-t_0} V(t_0 + s, X_s(x)) ds \right] \right] \le \frac{1}{1-\alpha}.$$

We use this to deduce the next proposition.

Proposition 5.4.3. *Assume that there exist $p, p', q, q' \in [1, \infty]$ such that*

i) $\frac{1}{p} + \frac{1}{q} = \frac{1}{p'} + \frac{1}{q'} = 1$*;*

ii) $\frac{1}{q} + \frac{2}{mq'} > 1$*;*

iii) $\|V\|_{L^p_{p'}([0,T] \times M)} < \infty$*.*

Then there exists $0 < \tilde{t} < T$ such that

$$\sup_{x \in M} \mathbb{E} \left[\exp \left[\int_0^{\tilde{t}} V(s, X_s(x))\, ds \right] \right] < \infty. \tag{5.31}$$

Proof. Suppose that $0 < t < T$ and that $p, p', q, q' \in (1, \infty)$. Then for $(t_0, x) \in$

$[0,t]\times M$ we see, by Tonelli's theorem and Hölder's inequality, that

$$
\begin{aligned}
&\mathbb{E}\left[\int_0^{t-t_0} V(s+t_0, X_s(x)\,)ds\right] \\
&= \int_0^{t-t_0}\int_M V(s+t_0,y)p_s^M(x,y)\,d\,\mathrm{vol}_M(y)\,ds \\
&\leq \left(\int_0^{t-t_0}\|V(t_0+s,\cdot)\|_{L^p(M)}^{p'}\,ds\right)^{\frac{1}{p'}}\left(\int_0^{t-t_0}\|p_s^M(x,\cdot)\|_{L^q(M)}^{q'}\,ds\right)^{\frac{1}{q'}} \\
&\leq \|V\|_{L^p_{p'}([0,T]\times M)}\|p_\cdot^M(x,\cdot)\|_{L^q_{q'}([0,t]\times M)}.
\end{aligned}
$$

We next wish to apply Lemma 5.4.2. For this we must satisfy condition (5.27). If $R\geq 0$ then this condition is clearly satisfied for all $0<t<T_0\wedge T$ but if $R<0$ then we should additionally assume that

$$
t<\frac{3}{\sigma^2|R|}
$$

which implies condition (5.27) since $q\geq 1$. Applying Lemma 5.4.2 to such t implies

$$
\begin{aligned}
&\sup_{(t_0,x)\in[0,t]\times M}\mathbb{E}\left[\int_0^{t-t_0}V(t_0+s,X_s(x))ds\right] \\
&\qquad\qquad\leq \|V\|_{L^p_{p'}([0,T]\times M)}C(c,\sigma^2,R,q,q',m)t^{\gamma(m,q,q')}. \qquad (5.32)
\end{aligned}
$$

There exists $\alpha\in[0,1)$ and $\tilde{t}\in(0,T_0\wedge T\wedge 3(\sigma^2|R|)^{-1})$ such that for all $t\in\left(0,\tilde{t}\right]$ the right-hand side of inequality (5.32) is less than or equal to α, so it follows by Khasminskii's lemma that

$$
\sup_{(t_0,x)\in[0,\tilde{t}]\times M}\mathbb{E}\left[\exp\left[\int_0^{\tilde{t}-t_0}V(t_0+s,X_s(x))ds\right]\right]<\infty.
$$

In particular, this implies inequality (5.31) which is what we wanted to prove. The remaining cases for the exponents p,p',q and q' can be dealt with similarly. $\square$

Proposition 5.4.3 can be improved, as shown by the following corollary.

Corollary 5.4.4. *Under the same conditions as Proposition 5.4.3 it follows that*

$$\sup_{x\in M} \mathbb{E}\left[\exp\left[\int_0^T V(s, X_s(x))ds\right]\right] < \infty.$$

Proof. For each $k \in \mathbb{N}$ and $i \in \{1, \ldots, k\}$ consider the function $V_k^{(i)} : \left[0, \frac{T}{k}\right] \times M \to \mathbb{R}$ defined by $V_k^{(i)}(s, y) := V(s + T(1 - \frac{i}{k}), y)$. Then, by the Markov property of Brownian motion, we see that

$$\begin{aligned}
&\mathbb{E}\left[\exp\left[\int_0^T V(s, X_s(x))ds\right]\right] \\
&\leq \mathbb{E}\left[\exp\left[\int_0^{T(1-\frac{1}{k})} V(s, X_s(x))\, ds\right]\right] \sup_{x\in M} \mathbb{E}\left[\exp\left[\int_0^{\frac{T}{k}} V_k^{(1)}(s, X_s(x))\, ds\right]\right]
\end{aligned}$$

for all $x \in M$. Proceeding inductively we find that

$$\sup_{x\in M} \mathbb{E}\left[\exp\left[\int_0^T V(s, X_s(x))\, ds\right]\right] \leq \prod_{i=1}^{k} \sup_{x\in M} \mathbb{E}\left[\exp\left[\int_0^{\frac{T}{k}} V_k^{(i)}(s, X_s(x))\, ds\right]\right]. \tag{5.33}$$

Now, since

$$\|V_k^i\|_{L^p_{p'}([0,\frac{T}{k}]\times M)} \leq \|V\|_{L^p_{p'}([0,T]\times M)} < \infty$$

for each $k \in \mathbb{N}$ and $i \in \{1, \ldots, k\}$, we can apply Proposition 5.4.3 to deduce that there exists $\tilde{k} \in \mathbb{N}$ such that

$$\sup_{x\in M} \mathbb{E}\left[\exp\left[\int_0^{\frac{T}{\tilde{k}}} V_{\tilde{k}}^{(i)}(s, X_s(x))ds\right]\right] < \infty$$

for each $i \in \{1, \ldots, \tilde{k}\}$. Using this value of k, each term in the product in the right-hand side of inequality (5.33) is finite and since there are only a finite number of terms in the product it follows that the product itself is finite, which yields the desired result. □

5.4.3 Solving the Martingale Problem

Theorem 5.4.5. *Suppose that M is a complete and connected Riemannian manifold of dimension m. Suppose that the Ricci curvature of M is bounded below by* *a constant and that the volume measure is lower regular. Let $X(x)$ be a Brownian motion on M starting at $x \in M$, U a horizontal lift of X and B the associated antidevelopment, all defined on a suitable filtered probability space. Fix $T > 0$ and suppose that $b : [0, T] \times M \to TM$ is a measurable (time-dependent) vector field on M. Suppose that there exist $p, q, p', q' \in [1, \infty]$ such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{p'} + \frac{1}{q'} = 1$ with $\|b\|^2 \in L^p_{p'}([0, T] \times M)$ and $\frac{1}{q} + \frac{2}{mq'} > 1$. Then*

$$\left\{ \exp\left[\int_0^t \langle b(s, X_s), U_s dB_s \rangle - \frac{1}{2}\int_0^t \|b(s, X_s)\|^2 ds\right], t \in [0, T] \right\}$$

is a martingale.

Proof. By Novikov's condition (see Stummer [1993] for the appropriate version), it suffices to check

$$\mathbb{E}^x \left[\exp\left[\frac{1}{2}\int_0^T \|b(s, X_s)\|^2 ds\right]\right] < \infty.$$

The theorem thus follows from Corollary 5.4.4 by setting $V = \frac{1}{2}\|b\|^2$. □

Corollary 5.4.6. *Under the conditions of Theorem 5.4.5, there exists a solution to the martingale problem for the time-dependent generator $\frac{1}{2}\triangle + b$ up to time T.*

Proof. The corollary follows by Girsanov's theorem and Theorem 5.4.5. □

Chapter 6

Conditioning and Derivative Estimates

Introduction

In this chapter we introduce and establish the basic properties of *Brownian bridges to submanifolds*. There has so far been almost no research on such processes, even for the case in which the ambient space is Euclidean, although they have appeared in the context of Wiener measure approximation.

In particular, Smolyanov, Weizsäcker and Wittich [2000] considered a compact submanifold embedded in an Euclidean space. They proved that the law of a Brownian motion started on the submanifold and conditioned to return to it at the end of each interval of a partition of the unit interval converges in law as the mesh of the partition goes to zero to the law of a Brownian motion on the submanifold. On each interval of the partition this conditioned process is an example of the type of process we have in mind.

In Section 6.1 we construct the bridge measure on path space, using upper bounds on the heat kernel and a Jacobian estimate. We then prove a concentration inequality and identify the generator of the conditioned process in terms of the logarithmic derivative of the integrated heat kernel, motivating the next two sections.

Our main result in Section 6.2 is a formula for the derivative of the integrated heat kernel, valid in a polar setting and given by Theorem 6.2.1. Different formulae for the derivatives of the heat kernel, upto any order, were proved by Norris [1993], using an extension of the method of Bismut [1984]. Norris used them to deduce that for x and y not in one another's cut locus there is the asymptotic relation

$$\lim_{t \downarrow 0} t^l \frac{\nabla^l p_t^M(x,y)}{p_t(x,y)} = \dot{\gamma}(0)^{\otimes l}$$

where γ is the unique geodesic from x to y in time 1, which Bismut had previously deduced for $l = 1$. Malliavin and Stroock [1996] showed that away from the cut locus there is actually cancellation between powers of t so that

$$\lim_{t \downarrow 0} t \operatorname{Hess} \log p_t^M(x,y) = -\frac{1}{2} \operatorname{Hess} d^2(x,y)$$

uniformly on compact subsets of $M \setminus \mathrm{Cut}(y)$, with versions of this relation also being valid for higher derivatives. The small-time asymptotics of the gradient and Hessian of the logarithm of the heat kernel on the cut locus were completely studied by Neel and Stroock [2004] and Neel [2007] who showed that the cut locus is precisely the set of points where the Hessian blows up faster than t^{-1}.

Section 6.3 includes our estimates on the gradient and Hessian of the logarithm of the integrated heat kernel. These are given by Theorem 6.3.2 and Corollary 6.3.4, respectively. The small time behaviour of our estimates is explained in the one point case by the asymptotics discussed in the previous paragraph. We prove them using the method of Stroock [1996], the inductive element of which had previously been discovered by Cheng, Li and Yau [1981]. Our results generalize the main theorem of Engoulatov [2006] and the gradient and Hessian estimates of Hsu [1999], who considered only the one point case. We apply Theorem 6.3.2 to prove that Brownian bridges to submanifolds are, under appropriate conditions, semimartingales.

6.1 Brownian Bridges to Submanifolds

6.1.1 The Canonical Probability Space

Suppose that M is stochastically complete, fix $T > 0$ and $x \in M$ and consider the associated canonical probability space $(W(M), \mathcal{B}(W(M)), \mathbb{P}^x)$ equipped with canonical filtration $\{\mathcal{B}_t(W(M))\}_{0 \leq t \leq T}$. Here $W(M)$ denotes the space of continuous paths defined on $[0,T]$ taking values in M, $\mathscr{B}_t(W(M))$ denotes the σ-algebra generated by the coordinate maps upto time t and $\mathbb{P}^x$ denotes Wiener measure, with respect to which the coordinate process $\{X_t : t \in [0,T]\}$ is a Brownian motion on M starting at x.

6.1.2 Conditioning on the Distance Function

Now suppose that N is a compactly embedded submanifold of M of dimension $n \in \{0, \ldots, m-1\}$.

Theorem 6.1.1. *Choose $t \in [0,T)$ and suppose that F is a bounded $\mathscr{B}_t(W(M))$-measurable random variable on $W(M)$. Then*

$$\mathbb{E}^x \left[F(X) | X_T \in N \right] = \frac{\mathbb{E}^x \left[p^M_{T-t}(X_t, N) F(X) \right]}{p^M_T(x, N)}. \tag{6.1}$$

Proof. For small $\epsilon > 0$ it follows from the definition of conditional expectation, the Markov property, Fubini's theorem and Corollary 1.2.2 that

$$\begin{aligned}
& \mathbb{E}^x \left[F(X) | r_N(X_T) < \epsilon \right] \\
= & \frac{\mathbb{E}^x \left[\mathbf{1}_{\{r_N(X_T) < \epsilon\}} F(X) \right]}{\mathbb{P}^x \{ r_N(X_T) < \epsilon \}} \\
= & \frac{\mathbb{E}^x \left[\mathbb{E}^{X_t} \left[\mathbf{1}_{\{r_N(X_{T-t}) < \epsilon\}} \right] F(X) \right]}{\mathbb{P}^x \{ r_N(X_T) < \epsilon \}} \\
= & \frac{\int_{B_\epsilon(N)} \mathbb{E}^x \left[p^M_{T-t}(X_t, y) F(X) \right] \, d\,\mathrm{vol}_M(y)}{\int_{B_\epsilon(N)} p^M_T(x, y) \, d\,\mathrm{vol}_M(y)} \\
= & \frac{\int_N \int_{B^p_\epsilon(0)} \mathbb{E}^x \left[p^M_{T-t}(X_t, \exp_N(\xi)) F(X) \right] \theta_N(\xi) \, d\xi \, d\,\mathrm{vol}_N(p)}{\int_N \int_{B^p_\epsilon(0)} p^M_T(x, \exp_N(\xi)) \theta_N(\xi) \, d\xi \, d\,\mathrm{vol}_N(p)}
\end{aligned}$$

where $B_\epsilon^p(0)$ denotes the open ball in $T_pN^\perp$ of radius ϵ centred at the origin. Since the volume of these balls is constant and independent of p, it follows by the continuity of the above integrands and the fact that $\theta_N|_N = 1$ that

$$\lim_{\epsilon \downarrow 0} \mathbb{E}^x \left[F(X)|r_N(X_T) < \epsilon\right] = \frac{\int_N \mathbb{E}^x \left[p_{T-t}^M(X_t, p)F(X)\right] d\,\mathrm{vol}_N(p)}{\int_N p_T^M(x,p) d\,\mathrm{vol}_N(p)}$$

from which the result follows, by the definition of the left-hand side of (6.1) as a Radon-Nikodym derivative. □

For each $t \in [0, T)$ it follows, by Theorem 6.1.1 and Corollary 1.2.2, that conditioning Brownian motion to be in the *interior* of a tubular neighbourhood of N of radius r at time T while separately conditioning Brownian motion to belong to the *boundary* of that tubular neighbourhood at time T results in two measures on $\mathcal{B}_t(W(M))$ which converge weakly to the same limit as $\epsilon \downarrow 0$.

6.1.3 Existence of the Bridge Measure

Suppose, temporarily, that N is a point $y \in M$. If we define a measure $\mathbb{P}^{x,y;T}$ on $\mathcal{B}(W(M))$ by $\mathbb{P}^{x,y;T}\{A\} = \mathbb{P}\{A|X_T = y\}$, for $A \in \mathcal{B}(W(M))$, then Theorem 6.1.1 implies that $\mathbb{P}^{x,y;T}$ is absolutely continuous with respect to $\mathbb{P}^x$ on $\mathcal{B}_t(W(M))$ for any $t \in [0, T)$ and that the Radon-Nikodym derivative is given by

$$\frac{d\mathbb{P}^{x,y;T}|_{\mathcal{B}_t(W(M))}}{d\mathbb{P}^x} = \frac{p_{T-t}^M(X_t, y)}{p_T^M(x,y)}.$$

In particular, if $\mathbb{P}^{x,y;T}$ exists as a probability measure on the space of continuous paths starting at x and terminating at y at time T then under $\mathbb{P}^{x,y;T}$ and for $0 < t_1 < \cdots < t_k < T$ the joint density function of $X_{t_1}, \ldots, X_{t_k}$, denoted by $p_{t_1,\ldots,t_k}^M(x, x_1, \ldots, x_k, y)$, is given by

$$p_{t_1,\ldots,t_k}^M(x, x_1, \ldots, x_k, y) = \frac{p_{t_1}^M(x, x_1)p_{t_2-t_1}^M(x_1, x_2) \cdots p_{T-t_k}^M(x_k, y)}{p_T^M(x,y)}. \tag{6.2}$$

To prove the existence of $\mathbb{P}^{x,y;T}$ we will assume that there exist constants $c, \sigma^2 > 0$ such that

$$p_t^M(w,z) \leq ct^{-\frac{m}{2}} \exp\left[-\frac{d^2(w,z)}{\sigma^2 t}\right] \tag{6.3}$$

for all $w, z \in M$ and $t \in (0,T]$ and a constant $\beta \geq 0$ such that

$$\theta_w(\xi) \leq \exp\left[\beta(1 + \|\xi\|^2)\right] \tag{6.4}$$

for all $w \in M$ and $\xi \in T_w M$. By Proposition 1.4.9 and Theorem 5.3.12, such bounds exist if the Ricci curvature is bounded below by a constant with vol_M lower regular. Alternatively, by Proposition 1.4.9 and comments made in Subsection 5.3.4, such bounds also exist if the Ricci curvature is bounded with M having positive injectivity radius, in which case the constants c and σ^2 can be chosen so that σ^2 is arbitrarily close to 2. For the case in which M is compact, a simple version of the following lemma was proved by Driver [1994].

Lemma 6.1.2. *Assume (6.3) and (6.4), suppose $0 \leq s < t \leq T$ and without loss of generality assume $\beta > 0$. Then for all $\gamma \in (0,1)$ there exists a constant $C(m, c, \sigma^2, \beta, \gamma, T) > 0$ such that for all $p > 0$ we have*

$$\mathbb{E}^{x,y;T}\left[d^p(X_s, X_t)\right] \leq \frac{C(m,c,\sigma^2,\beta,\gamma,T)}{p_T^M(x,y)} \frac{\Gamma\left(\frac{m+p}{2}\right)}{\Gamma\left(\frac{m}{2}\right)} \left(\frac{\sigma^2(t-s)}{1-\gamma}\right)^{\frac{p}{2}} \tag{6.5}$$

so long as $t - s < \gamma(\sigma^2\beta)^{-1}$.

Proof. First assume $0 < s < t < 2T/3$. Then, by (5.29), for $w \in M$ and $p > 0$ we have

$$\begin{aligned}
&\int_M p_{t-s}^M(w,z) d^p(w,z)\, d\,\mathrm{vol}_M(z) \\
&\leq c(t-s)^{-\frac{m}{2}} \int_M \exp\left[-\frac{d^2(w,z)}{\sigma^2(t-s)}\right] d^p(w,z)\, d\,\mathrm{vol}_M(z) \\
&\leq ce^{\beta}(t-s)^{-\frac{m}{2}} \int_{T_w M} \|\xi\|^p \exp\left[\left(\beta - \frac{1}{\sigma^2(t-s)}\right)\|\xi\|^2\right] dv \\
&= ce^{\beta} \frac{m\pi^{\frac{m}{2}}(t-s)^{-\frac{m}{2}}}{\Gamma\left(\frac{m}{2}+1\right)} \int_0^\infty r^{p+m-1} \exp\left[\left(\beta - \frac{1}{\sigma^2(t-s)}\right) r^2\right] dr
\end{aligned}$$

$$
\begin{aligned}
&= ce^{\beta} \frac{\pi^{\frac{m}{2}} \Gamma\left(\frac{m+p}{2}\right)(t-s)^{-\frac{m}{2}}}{\Gamma\left(\frac{m}{2}\right)} \left(\frac{1-\beta\sigma^2(t-s)}{\sigma^2(t-s)}\right)^{-\frac{(m+p)}{2}} \\
&\leq ce^{\beta} \frac{\Gamma\left(\frac{m+p}{2}\right)}{\Gamma\left(\frac{m}{2}\right)} \left(\frac{\sigma^2\pi}{1-\gamma}\right)^{\frac{m}{2}} \left(\frac{\sigma^2(t-s)}{1-\gamma}\right)^{\frac{p}{2}}.
\end{aligned}
$$

Thus there exists a constant $C_0(m, c, \sigma^2, \beta, \gamma) > 0$ such that

$$
\int_M p_{t-s}^M(w,z) d^p(w,z) d\,\mathrm{vol}_M(z) \leq C_0(m, c, \sigma^2, \beta, \gamma) \frac{\Gamma\left(\frac{m+p}{2}\right)}{\Gamma\left(\frac{m}{2}\right)} \left(\frac{\sigma^2(t-s)}{1-\gamma}\right)^{\frac{p}{2}}
$$

for all $p > 0$, $w \in M$ and s, t satisfying $t - s < \gamma(\sigma^2\beta)^{-1}$. For such s, t we see that

$$
\begin{aligned}
&\mathbb{E}^{x,y;T}\left[d^p(X_s, X_t)\right] \\
&= \int_M \int_M \frac{p_s^M(x,w) p_{t-s}^M(w,z) d^p(w,z) p_{T-t}^M(z,y)}{p_T^M(x,y)} d\,\mathrm{vol}_M(w) d\,\mathrm{vol}_M(z) \\
&\leq (T/3)^{-m/2} \frac{C_1(m, c, \sigma^2, \beta, \gamma)}{p_T^M(x,y)} \frac{\Gamma\left(\frac{m+p}{2}\right)}{\Gamma\left(\frac{m}{2}\right)} \left(\frac{\sigma^2(t-s)}{1-\gamma}\right)^{\frac{p}{2}}.
\end{aligned}
$$

The same result is obtained for $T/3 < s < t < T$ while the cases $s = 0$ or $t = T$ can be treated similarly. $\square$

Now suppose that N is a compactly embedded submanifold, as in Subsection 6.1.2. If F is a bounded $\mathscr{B}_t(W(M))$-measurable function on $W(M)$ for some $t \in [0, T)$ then it follows from Theorem 6.1.1 that

$$
\mathbb{E}^x\left[F(X) | X_T \in N\right] = \frac{\int_N p_T^M(x,y) \mathbb{E}^{x,y;T}\left[F(X)\right] d\,\mathrm{vol}_N(y)}{p_T^M(x,N)}. \tag{6.6}
$$

This implies, by Lemma 6.1.2, that for all $p \geq 2$ there exists $\epsilon, C_\epsilon > 0$ such that

$$
\mathbb{E}^{x;N,T}[d^p(X_s, X_t)] \leq C_\epsilon (t-s)^{\frac{p}{2}}
$$

for all $0 \leq s < t \leq T$ with $t - s < \epsilon$. It follows by Kolmogorov's continuity theorem, by covering the interval $[0, T]$ with finitely many closed intervals each of length less that ϵ, that there exists a probability measure $\mathbb{P}^{x;N,T}$ on the bridge space

$$L_{x;N,T}(M) := \{\omega \in W(M) : X_0(\omega) = x,\ X_T(\omega) \in N\}$$

which satisfies $\mathbb{P}^{x;N,T}\{A\} = \mathbb{P}^x\{A|X_T \in N\}$ for $A \in \mathscr{B}(W(M))$. The finite-dimensional distributions of this measure can be easily deduced from equations (6.2) and (6.6) and the asymptotic behaviour of the density of X_t under $\mathbb{P}^{x;N,T}$ as $t \uparrow T$ follows from Theorem 5.3.8. In particular, if $\mathcal{L}(X_T)$ denotes the law of the random variable X_T under the measure $\mathbb{P}^{x;N,T}$ then

$$\mathcal{L}(X_T) = \frac{p_T^M(x,\cdot)}{p_T^M(x,N)} \operatorname{vol}_N . \tag{6.7}$$

Example 6.1.3. *If $M = \mathbb{R}^m$ with N given by the unit $(m-1)$-sphere, embedded in $\mathbb{R}^m$ in the usual way, and $x = 0$ then the terminal law $\mathcal{L}(X_T)$ is given by the uniform measure on N.*

Example 6.1.4. *If $M = \mathbb{R}^m$ with N given by an n-dimensional subspace and $x = 0$ then the terminal law $\mathcal{L}(X_T)$ is given by the heat kernel measure on N. Although in this example N is non-compact, one can check that the above results also apply to this and similar examples.*

6.1.4 Concentration Inequality

Using Lemma 6.1.2 we deduce a concentration inequality for tubular neighbourhoods.

Theorem 6.1.5. *Assuming (6.3) and (6.4), for all $\gamma \in (0,1)$ there exists $\epsilon > 0$ such that*

$$\lim_{r\uparrow\infty} \frac{1}{r^2} \log \mathbb{P}^{x;N,T}\{X_t \notin B_r(N)\} \leq -\frac{1-\gamma}{\sigma^2(T-t)} \tag{6.8}$$

for all $t \in (T-\epsilon, T]$.

Proof. Without loss of generality, assume $\beta > 0$. Since the distance function r_N minimizes over points belonging to N, it follows from (6.6) and Lemma 6.1.2 that for all $\gamma \in (0,1)$ there exists $C(m, c, \sigma^2, \beta, \gamma, N, T) > 0$ such that

$$\mathbb{E}^{x;N,T}[r_N^p(X_t)] \leq \frac{C(m, c, \sigma^2, \beta, \gamma, N, T)}{p_T^M(x, N)} \frac{\Gamma\left(\frac{m+p}{2}\right)}{\Gamma\left(\frac{m}{2}\right)} \left(\frac{\sigma^2(T-t)}{1-\gamma}\right)^{\frac{p}{2}}$$

for all $0 \leq t \leq T$ with $T - t < \gamma(\sigma^2\beta)^{-1}$. For such t, choosing $\theta > 0$, applying this bound to the case where p is an even integer and summing yields

$$\mathbb{E}^{x;N,T}\left[e^{\frac{\theta}{2}r_N^2(X_t)}\right] \leq C(m, c, \sigma^2, \beta, \gamma, x, N, T)\left(1 - \frac{\theta\sigma^2(T-t)}{2(1-\gamma)}\right)^{-\frac{m}{2}}$$

so long as $t > T - 2(1-\gamma)(\theta\sigma^2)^{-1}$. Under these conditions on t, it follows from Markov's inequality that for all $r > 0$ there is the estimate

$$\mathbb{P}^{x;N,T}\{X_t \notin B_r(N)\} \leq C(m, \delta, \gamma, c_R, x, N, T)\left(1 - \frac{\theta\sigma^2(T-t)}{2(1-\gamma)}\right)^{-\frac{m}{2}} e^{-\frac{\theta r^2}{2}}.$$

Fixing $\delta \in [0,1)$ and choosing $\theta = 2\delta(1-\gamma)(\sigma^2(T-t))^{-1}$ yields

$$\lim_{r\uparrow\infty} \frac{1}{r^2} \log \mathbb{P}^{x;N,T}\{X_t \notin B_r(N)\} \leq -\frac{\delta(1-\gamma)}{\sigma^2(T-t)}$$

from which the result follows since δ can be chosen arbitrarily close to 1. $\square$

If in addition to the assumptions of Theorem 6.1.5 we suppose that the Ricci curvature is bounded below by a constant then the asymptotic estimate (3.39), which as we commented earlier follows from Theorem 8.62 of Stroock [2000], implies that the concentration inequality (6.8) actually holds with $\sigma^2 = 2$ and $\gamma = 0$.

6.1.5 Semimartingale Property

It follows from formula (6.6) and Girsanov's theorem that under the measure $\mathbb{P}^{x;N,T}$ the coordinate process on $W(M)$ is a diffusion process on the half-open time interval $[0,T)$ starting at x with time-dependent infinitesimal generator

$$\frac{1}{2}\triangle + \nabla \log p_{T-t}^M(\cdot, N) \tag{6.9}$$

for $t \in [0,T)$. To show that the coordinate process is a semimartingale under this measure on the closed time interval $[0,T]$ requires a suitable estimate on the logarithmic derivative of the integrated heat kernel. We will deduce such an estimate using curvature assumptions to access Bismut's formula and the lower bounds of the previous chapter. This motivates the final two sections, although the next section contains numerous remarks which are of independent interest.

6.2 Derivative Formulae

6.2.1 Bismut's Formula

A formula for the derivative of the heat semigroup was proved by Bismut [1984]. A simple proof was given in Li [1992], generalizing Bismut's formula to non-compact manifolds, further developed in Elworthy and Li [1994] and Elworthy and Li [1996]. Other authors then derived similar formulae using methods based on local martingales, including Thalmaier [1997], Arnaudon and Thalmaier [1999], Driver and Thalmaier [2001] and Arnaudon, Plank and Thalmaier [2003]. The formula can be stated as follows, as in Thalmaier [1997].

Suppose that M is a complete and connected Riemannian manifold of dimension m with Ricci curvature bounded below. Denote by $X(x)$ a Brownian motion on M starting at $x \in M$ and by U a horizontal lift of with antidevelopment B. If we denote by $\{\mathcal{Q}_s : s \geq 0\}$ the solution the ordinary differential equation

$$\begin{cases} \dot{\mathcal{Q}}_s = -\frac{1}{2}\operatorname{Ric}_{U_s}\mathcal{Q}_s \\ \mathcal{Q}_0 = U_0^{-1} \end{cases} \tag{6.10}$$

with $\operatorname{Ric}_{U_s} := U_s^{-1}\operatorname{Ric}^\sharp U_s$ then for any bounded measurable function $f : M \to \mathbb{R}$ there is the formula

$$d(P_t f)_x(v) = \mathbb{E}\left[f(X_t(x))\frac{1}{t}\int_0^t \langle \mathcal{Q}_s v, dB_s\rangle\right] \tag{6.11}$$

for all $t > 0$ and $v \in T_xM$.

6.2.2 Derivative Formulae in a Polar Setting

Li [2013] uses a Bismut-Elworthy-Li type formula to deduce formulae for the derivative of the heat kernel by extending the method of Elworthy and Truman [1982]. We will now generalize this approach to prove a formula for the derivative of the *integrated* heat kernel, also in a polar setting.

In particular, we will suppose that N is a compactly embedded submanifold of M of dimension $n \in \{0, \ldots, m-1\}$ and suppose that $\exp_N : TN^\perp \to M$ is a diffeomorphism. For the case in which N is a point this is to say that the point is a pole for M. If we define a smooth function $V_N : M \to \mathbb{R}$ by

$$V_N := \frac{1}{2}\Theta_N^{\frac{1}{2}} \triangle \Theta_N^{-\frac{1}{2}}$$

and suppose that V_N is bounded, then by Theorem 5.1.10 and a limiting argument we see that for $T > 0$ and $x \in M$ we have

$$p_T^M(x, N) = q_T(x, N)\Theta_N^{-\frac{1}{2}}(x)\mathbb{E}\left[\exp\left[\int_0^T V_N(\hat{Y}_s(x))ds\right]\right] \tag{6.12}$$

where $\hat{Y}(x)$ is a semiclassical bridge between x and N in time T (defined as in Subsection 4.1.1 with $b = 0$). Rather than differentiating (6.12) directly, like Aida [2004] did for the one point case, we will use a different approach. We will denote by $\hat{U}$ a horizontal lift of $\hat{Y}(x)$, by B the Brownian motion given by the martingale part of its antidevelopment and by $\{\hat{\mathcal{Q}}_s : s \in [0, T)\}$ the process which solves

$$\begin{cases} \dot{\hat{\mathcal{Q}}}_s = -\frac{1}{2}\operatorname{Ric}_{\hat{U}_s} \hat{\mathcal{Q}}_s \\ \hat{\mathcal{Q}}_0 = \hat{U}_0^{-1}. \end{cases} \tag{6.13}$$

In these terms we have the following theorem.

Theorem 6.2.1. *Suppose that V_N and $\nabla \log \Theta_N$ are bounded. Then for $T > 0$,*

$x \in M$ and $v \in T_xM$ we have

$$dp_T^M(\cdot, N)_x(v) = q_T(x,N)\Theta_N^{-\frac{1}{2}}(x)\mathbb{E}\left[\exp\left[\int_0^T V_N(\hat{Y}_s(x))\,ds\right]\frac{1}{T}\int_0^T \langle \hat{\mathcal{Q}}_s v, d\hat{B}_s\rangle\right] \tag{6.14}$$

where $\hat{B}$ satisfies

$$d\hat{B}_s = dB_s + \hat{U}_s^{-1}\nabla \log q_{T-s}(\hat{Y}_s(x), N)\,ds + \hat{U}_s^{-1}\nabla \log \Theta_N^{-\frac{1}{2}}(\hat{Y}_s(x))\,ds$$

for $s \in [0,T)$.

Proof. Suppose that D is a regular domain containing N and for $t \in (0,T)$ apply formula (6.11) to the function $f(\cdot) = \mathbf{1}_D(\cdot)q_{T-t}(\cdot, N)\Theta_N^{-\frac{1}{2}}(\cdot)$. By applying Girsanov's theorem and Itô's formula this yields

$$\begin{aligned}
&d(P_t(\mathbf{1}_D(\cdot)q_{T-t}(\cdot,N)\Theta_N^{-\frac{1}{2}}(\cdot)))_x(v)\\
=\ & q_T(x,N)\Theta_N^{-\frac{1}{2}}(x)\mathbb{E}\left[\mathbf{1}_D(\hat{Y}_t(x))\exp\left[\int_0^t V_N(\hat{Y}_s(x))ds\right]\frac{1}{t}\int_0^t \langle \hat{\mathcal{Q}}_s v, d\hat{B}_s\rangle\right]
\end{aligned} \tag{6.15}$$

for all $t \in (0,T)$. Since the closure of D and N are both compact with $\Theta_N|_N = 1$ it follows that

$$\begin{aligned}
&\lim_{t\uparrow T} d(P_t(\mathbf{1}_D(\cdot)q_{T-t}(\cdot,N)\Theta_N^{-\frac{1}{2}}(\cdot)))_x(v)\\
&= \lim_{t\uparrow T} d\left(\int_D p_t^M(\cdot,y)q_{T-t}(y,N)\Theta_N^{-\frac{1}{2}}(y)d\,\mathrm{vol}_M(y)\right)_x(v)\\
&= \lim_{t\uparrow T}\int_D dp_t^M(\cdot,y)_x(v)q_{T-t}(y,N)\Theta_N^{-\frac{1}{2}}(y)d\,\mathrm{vol}_M(y)\\
&= \int_N dp_T^M(\cdot,y)_x(v)d\,\mathrm{vol}_N(y)\\
&= dp_T^M(\cdot,N)_x(v)
\end{aligned} \tag{6.16}$$

where the third equality is justified by the argument used in the proof of Theorem 5.2.1. The result then follows from (6.15) and (6.16) since the boundedness and curvature assumptions allow for the remaining limit to be passed through the expectation on the right-hand side of (6.15). □

Elworthy and Li [1994] proved a formula for the Hessian of the heat semigroup, so a formula for the Hessian of the heat kernel can also be obtained, using a similar method under additional assumptions, the investigation of which is a topic for future research.

Corollary 6.2.2. *Under the conditions of Theorem 6.2.1 we have*

$$d\log p_T^M(\cdot,N)_x(v) = \frac{\mathbb{E}\left[\exp\left[\int_0^T V_N(\hat{Y}_s(x))ds\right]\frac{1}{T}\int_0^T\langle\hat{\mathcal{Q}}_s v, d\hat{B}_s\rangle\right]}{\mathbb{E}\left[\exp\left[\int_0^T V_N(\hat{Y}_s(x))ds\right]\right]}. \tag{6.17}$$

Proof. The corollary follows directly from Theorem 6.2.1 and Ndumu's formula (6.12). □

6.2.3 A Conjecture on Asymptotics

The asymptotic expansion of Ndumu [2011] implies, essentially by Varadhan's relation, that

$$\lim_{t\downarrow 0}\left|\log p_t^M(\cdot,N) - \log q_t(\cdot,N) - \log\Theta_N^{-\frac{1}{2}}\right| = 0$$

uniformly on compact subsets of $M\setminus\mathrm{Cut}(N)$. Furthermore, if D is a regular domain whose closure is contained in $M\setminus\mathrm{Cut}(N)$ then the assumptions of Subsection 6.2.2, which were that the Ricci curvature be bounded below with V_N and $\nabla\log\Theta_N$ bounded, hold in D. These observations together with Corollary 6.2.2 lead the author to make the following conjecture.

Conjecture 6.2.3. *Suppose that M is a complete and connected Riemannian manifold of dimension m and that N is a closed embedded submanifold of M of dimension $n\in\{0,\dots,m-1\}$. Then*

$$\lim_{t\downarrow 0}\left\|\nabla\log p_t^M(\cdot,N) - \nabla\log q_t(\cdot,N) - \nabla\log\Theta_N^{-\frac{1}{2}}\right\| = 0$$

uniformly on compact subsets of $M\setminus\mathrm{Cut}(N)$.

The author has proved the conjecture under additional assumptions. For now, we will simply provide a couple of examples where the conjecture can be proved directly. For instance, it is straightforward to check that the conjecture holds for the situation considered in Example 5.1.11. The following example requires a little more calculation.

Example 6.2.4. *If $\mathbb{S}^1$ denotes the unit circle in $\mathbb{R}^2$ then for $t > 0$ and $x \in \mathbb{R}^2$ it follows from formula (5.12) for the case $r = 1$ that*

$$\nabla \log p_t^{\mathbb{R}^2}(x, \mathbb{S}^1) = \frac{1}{t}\left(\frac{\mathrm{BesselI}(1, r_0(x)/t)}{\mathrm{BesselI}(0, r_0(x)/t)} - r_0(x)\right)\frac{\partial}{\partial r_0}(x)$$

where r_0 denotes the distance to the origin (i.e. the radial part of standard polar coordinates). We also have

$$\nabla \log q_t(x, \mathbb{S}^1) = \frac{(1 - r_0(x))}{t}\frac{\partial}{\partial r_0}(x)$$

and therefore

$$\nabla \log p_t^{\mathbb{R}^2}(x, \mathbb{S}^1) - \nabla \log q_t(x, \mathbb{S}^1) = \frac{1}{t}\left(\frac{\mathrm{BesselI}(1, r_0(x)/t)}{\mathrm{BesselI}(0, r_0(x)/t)} - 1\right)\frac{\partial}{\partial r_0}(x).$$

From this we deduce that if $x \neq 0$ (note that the origin is the cut locus of $\mathbb{S}^1$) then

$$\lim_{t \downarrow 0}\left(\nabla \log p_t^{\mathbb{R}^2}(x, \mathbb{S}^1) - \nabla \log q_t(x, \mathbb{S}^1)\right) = -\frac{1}{2r_0(x)}\frac{\partial}{\partial r_0}(x).$$

On the other hand, by formula (1.18) we have

$$\nabla \log \Theta_{\mathbb{S}^1}(x) = \frac{1}{r_0(x)}\frac{\partial}{\partial r_0}(x)$$

which agrees with the conjecture. Note that $1/r_0(x)$ is the curvature of the level set of $r_{\mathbb{S}^1}$ to which x belongs, which is the circle of radius $r_0(x)$.

6.3 Derivative Estimates

6.3.1 Gradient Estimate

In the polar setting one can use Corollary 6.2.2 to obtain estimates on the logarithmic derivative of $p_T^M(\cdot, N)$. For a more general result, we will use Bismut's formula and the method of Stroock [1996]. This approach requires the following lemma, proved with Jensen's inequality.

Lemma 6.3.1. *Suppose* $(\Omega, \mathcal{F}, \mathbb{P})$ *is a probability space and* ϕ *a non-negative measurable function on* Ω *with* $\mathbb{E}[\phi] = 1$*. If* ϕ *is a measurable function on* Ω *such that* $\phi\psi$ *is integrable, then* $\mathbb{E}[\phi\psi] \leq \mathbb{E}[\phi \log \phi] + \log \mathbb{E}[e^{\psi}]$*.*

Proof. See [Stroock, 2000, Lemma 6.45]. □

Theorem 6.3.2. *Suppose that* M *is a complete and connected Riemannian manifold of dimension* m *and that* N *is a compactly embedded submanifold of* M *of dimension* $n \in \{0, \ldots, m-1\}$*. Suppose that the Ricci curvature is bounded below and that* vol_M *is lower regular (see Definition 5.3.11). Furthermore, if* $n \in \{1, \ldots, m-2\}$ *then additionally assume that there exist constants* $C_1, C_2 \geq 0$ *such that the sectional curvatures of planes containing the radial direction are bounded below by* $-(C_1 + C_2 r_N)^2$*. Then for all* $T > 0$ *there exists* $C > 0$ *such that*

$$\|\nabla \log p_t^M(\cdot, N)_x\|^2 \leq C\left(\frac{1}{t} + \frac{n}{t}\log\frac{1}{t} + \frac{d^2(x,N)}{t^2}\right) \tag{6.18}$$

for all $x \in M$ *and* $t \in (0, T]$*.*

Proof. Using the notation of Subsection 6.2.1, for a bounded and positive measurable function f and $\gamma \in \mathbb{R}$ set

$$\phi := \frac{f(X_t(x))}{P_t f(x)}, \quad \psi := \gamma \int_0^t \langle Q_s v, dB_s \rangle.$$

By Lemma 6.3.1 and formula (6.11) it follows that

$$\gamma t \frac{d(P_t f)_x(v)}{P_t f(x)} \leq h_t(x; f) + \log \mathbb{E}\left[\exp\left[\gamma \int_0^t \langle Q_s v, dB_s \rangle\right]\right]$$

where

$$h_t(x; f) := \mathbb{E}\left[\frac{f(X_t(x))}{P_t f(x)} \log \frac{f(X_t(x))}{P_t f(x)}\right]. \tag{6.19}$$

Standard estimates for Brownian integrals imply

$$\log \mathbb{E}\left[\exp\left[\gamma \int_0^t \langle \mathcal{Q}_s v, dB_s\rangle\right]\right] \leq \frac{\gamma^2}{2}\int_0^t e^{-Rs} ds \|v\|^2$$

where R denotes the minimum of the Ricci curvature on M, so after minimizing over γ we deduce

$$\left|\frac{d(P_t f)_x(v)}{P_t f(x)}\right| \leq \frac{1}{t}\left(2h_t(x;f)\int_0^t e^{-Rs} ds\right)^{\frac{1}{2}} \|v\|.$$

Now choose $f = p_t^M(\cdot, N)$. Then $P_t f(z) = p_{2t}^M(z, N)$, by Tonelli's theorem, and for all $z \in M$ it follows that

$$h_t(x; p_t^M(\cdot, N)) \leq \sup_{z\in M} \log\left(\frac{p_t^M(z,N)}{p_{2t}^M(x,N)}\right). \tag{6.20}$$

The assumptions of the theorem imply, by Theorem 5.3.2, that there exists a constant $c_1 \geq 0$, depending only on T, R, C_1, C_2, m and n, such that

$$p_{2t}^M(x,N) \geq (2t)^{-\frac{(m-n)}{2}} \exp\left[-\frac{r_N^2(x)}{4t} - c_1(1 + r_N^2(x))\right] \tag{6.21}$$

for all $x \in M$ and $t \in (0, T]$. The assumptions also imply, by Theorem 5.3.12 and the Chapman-Kolmogorov equation, that there exist $c_2 > 0$ such that

$$p_t^M(z,N) \leq c_2 t^{-\frac{m}{2}} \tag{6.22}$$

for all $t \in (0, T]$ and $z \in M$. Substituting the estimates (6.21) and (6.22) in to (6.20) yields the theorem. □

It follows that the gradient estimate (6.18) holds automatically if M is compact. More generally, the lower regularity of vol_M can be discarded simply by assuming the on-diagonal bound (5.24) instead.

Corollary 6.3.3. *Under the assumptions of Theorem 6.3.2, the coordinate process on the bridge space $L_{x;N,T}$ is a semimartingale with respect to the measure $\mathbb{P}^{x;N,T}$.*

Proof. It suffices to control the singularity in the drift close to the terminal time.

Since the distance function r_N minimizes over points belonging to N, it follows from (6.6) and Lemma 6.1.2 that there exists $\epsilon, C_\epsilon > 0$ such that

$$\mathbb{E}^{x;N,T}[r_N^2(X_t)] \leq C_\epsilon (T-t)$$

for all $t \in (T-\epsilon, T]$. Therefore, by Theorem 6.3.2, there exists $C > 0$ such that

$$\begin{aligned}
&\mathbb{E}^{x;N,T}\left[\int_{T-\epsilon}^{T} \|\nabla \log p_{T-t}^M(X_t, N)\| dt\right] \\
&\leq \int_{T-\epsilon}^{T} \mathbb{E}^{x;N,T}\left[\|\nabla \log p_{T-t}^M(X_t, N)\|^2\right]^{\frac{1}{2}} dt \\
&\leq \sqrt{C}\int_{T-\epsilon}^{T} \left(\frac{1}{T-t} + \frac{n}{T-t}\log\frac{1}{T-t} + \frac{\mathbb{E}^{x;N,T}\left[r_N^2(X_t)\right]}{(T-t)^2}\right)^{\frac{1}{2}} dt \\
&\leq \sqrt{C}\int_{T-\epsilon}^{T} \left(\frac{1}{T-t} + \frac{n}{T-t}\log\frac{1}{T-t} + \frac{C_\epsilon}{T-t}\right)^{\frac{1}{2}} dt \\
&< \infty
\end{aligned}$$

and the result follows. $\square$

6.3.2 Hessian Estimate

For the case in which M is compact we have the following corollary of Theorem 6.3.2.

Corollary 6.3.4. *Suppose that M is a compact and connected Riemannian manifold of dimension m and that N is a closed embedded submanifold of M of dimension $n \in \{0, \ldots, m-1\}$. Then for all $T > 0$ there exists $C > 0$ such that*

$$\|\operatorname{Hess}\log p_t^M(\cdot, N)_x\| \leq C\left(\frac{1}{t} + \frac{n}{t}\log\frac{1}{t} + \frac{d^2(x,N)}{t^2}\right) \tag{6.23}$$

for all $x \in M$ and $t \in (0, T]$.

Proof. Stroock [1996] proved that for any continuous positive function f there exists $C > 0$ such that

$$t\frac{\|\operatorname{Hess}(P_t f)_x\|}{P_t f(x)} \leq C\left(1 + h_t(x; f)\right)$$

for all $x \in M$ and $t \in (0,T]$ where $h_t(x;f)$ is defined by (6.19). Choosing $f = p_t^M(\cdot, N)$, using the lower bound (6.21) and the on-diagonal upper bound (6.22) yields the corollary, by Theorem 6.3.2 and the fact that

$$\operatorname{Hess}\log P_t f = \frac{\operatorname{Hess} P_t f}{P_t f} - d\log P_t f \otimes d\log P_t f$$

for all $t > 0$. □

Appendix A

Hausdorff Measure and Dimension

A.1 Hausdorff Measure

Suppose that S is a subset of a separable metric space (E, d) and denote by $\mathrm{diam}(S)$ the *diameter of* S, defined by

$$\mathrm{diam}(S) := \sup\{d(x, y) : x, y \in S\}$$

with $\mathrm{diam}(\emptyset) := 0$. For any subsets $S_1, S_2 \subseteq E$ denote by $\mathrm{dist}(S_1, S_2)$ the *distance between* S_1 *and* S_2, defined by

$$\mathrm{dist}(S_1, S_2) := \inf\{d(x, y) : x \in S_1, y \in S_2\}.$$

For $k \geq 0$ fixed denote by ω_k the volume of the unit ball in k-dimensional Euclidean space and for $\delta > 0$ define a set function $\mathcal{H}_\delta^k$ by

$$\mathcal{H}_\delta^k(S) := \frac{\omega_k}{2^k} \inf\left\{\sum_{j=1}^{\infty} (\mathrm{diam}(S_j))^k : S \subseteq \bigcup_{j=1}^{\infty} S_j,\ \ \mathrm{diam}(S_j) < \delta\right\}$$

for $S \subseteq E$. Note that $\mathcal{H}_\delta^k(S)$ is monotone decreasing in δ so the limit $\lim_{\delta \downarrow 0} \mathcal{H}_\delta^k(S)$ either exists or is infinite. Thus we can define

$$\mathcal{H}_+^k(S) := \lim_{\delta \downarrow 0} \mathcal{H}_\delta^k(S)$$

and see that $\mathcal{H}_+^k$ has the following properties, for any collection $\{S_j\}_{j=1}^\infty$ with $S_j \subseteq E$:

1. $\mathcal{H}_+^k(\emptyset) = 0$;
2. $\mathcal{H}_+^k\left(\bigcup_{j=1}^\infty S_j\right) \leq \sum_{j=1}^\infty \mathcal{H}_+^k(S_j)$;
3. if $S_1 \subseteq S_2 \subseteq E$ then $\mathcal{H}_+^k(S_1) \leq \mathcal{H}_+^k(S_2)$;
4. if $\operatorname{dist}(S_1, S_2) > 0$ then $\mathcal{H}_+^k(S_1 \cup S_2) = \mathcal{H}_+^k(S_1) + \mathcal{H}_+^k(S_2)$.

Thus $\mathcal{H}_+^k$ is a *metric outer measure* and so, by general theory, it is a measure when restricted to the σ-algebra of Carathéodory-measurable sets, a σ-algebra which contains all of the Borel sets $\mathcal{B}(E)$. This measure, called the *k-dimensional Hausdorff measure*, is denoted by $\mathcal{H}_E^k$. Note that we have defined this measure in such a way that if M is a Riemannian manifold of dimension m then for any $B \in \mathcal{B}(M)$ we have $\mathcal{H}_M^m(B) = \operatorname{vol}_M(B)$. Similarly, if N is a smooth n-dimensional submanifold of M then for any $B \in \mathcal{B}(N)$ we have $\mathcal{H}_M^n(B) = \operatorname{vol}_N(B)$.

A.2 Hausdorff Dimension

For $S \in \mathcal{B}(E)$ the *Hausdorff dimension of S* is given by

$$\dim_H S := \inf\{k : H_E^k(S) = 0\} = \sup\{k : \mathcal{H}_E^k(S) = \infty\}.$$

There are many examples of fractals whose Hausdorff dimension strictly exceeds their topological dimension. If S is a subset of E with finite Hausdorff dimension k then $\mathcal{H}_E^k(S) > 0$ while if S is a subset of E with $\mathcal{H}_E^{k+1}(S) = 0$ then $\dim_H S \leq k$. These and other basic facts can be found in Hurewicz and Wallman [1941].

Appendix B

Discontinuous Drift

B.1 Away from the Origin

The Fermi bridge defined in Subsection 4.2.1 consists of a Brownian motion with drift that is discontinuous on the cut locus and which, roughly speaking, points away from the cut locus. In this appendix we consider a couple of similar one-dimensional processes. First, suppose that X is a standard Brownian motion on $\mathbb{R}$ starting at the origin and defined on the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$. In terms of the function sgn, which was defined by equation (2.5) in Chapter 2, let

$$Z_t := \exp\left[\int_0^t \operatorname{sgn}(X_s)dX_s - \frac{1}{2}\int_0^t |\operatorname{sgn}(X_s)|^2 ds\right]$$

for $t \geq 0$. Then Z is a martingale so if for each $T > 0$ we define a new measure $\hat{\mathbb{P}}_T$ by $d\hat{\mathbb{P}}_T = Z_T\, d\mathbb{P}$ then, by Girsanov's theorem, the triple (X, B), $(\Omega, \mathcal{F}, \hat{\mathbb{P}}_T)$, $\{\mathcal{F}_t\}_{t\geq 0}$ with

$$B_t := X_t - \int_0^t \operatorname{sgn}(X_s)\, ds$$

is a weak solution to the stochastic differential equation

$$\begin{cases} dX_t = dB_t + \operatorname{sgn}(X_t)dt \\ X_0 = 0 \end{cases} \tag{B.1}$$

for $0 \leq t \leq T$, which is unique in the sense of probability law, and the Tanaka formula implies

$$\hat{\mathbb{P}}_T\{X_t \in A\} = \mathbb{E}^{\mathbb{P}}\left[\mathbf{1}_A(X_t)\exp\left[-\frac{t}{2} + |X_t| - L_t^0(X)\right]\right]$$

for all $0 \leq t \leq T$. According to [Karatzas and Shreve, 1991, p.420] there is the joint density formula

$$\mathbb{P}\{X_t \in da, L_t^0(X) \in db\} = \frac{b + |a|}{\sqrt{2\pi t^3}}\exp\left[-\frac{(b+|a|)^2}{2t}\right] da\, db$$

for $a \in \mathbb{R}$ and $b > 0$. It follows that

$$\hat{\mathbb{P}}_T\{X_t \in A\} = \int_A \hat{p}_t(a)\, da$$

where

$$\hat{p}_t(a) := \frac{1}{\sqrt{2\pi t}}\exp\left[-\frac{(|a| - t)^2}{2t}\right] - \frac{e^{2|a|}}{2}\operatorname{erfc}\left[\frac{t + |a|}{\sqrt{2t}}\right]$$

where erfc denotes the complementary error function. The density for time $t = 1$ is illustrated below in Figure 5. Note that the densities $\hat{p}_t(a)$ are *smooth* in t, *continuous* in a and smooth in a away from the origin (since erfc is an analytic function) despite the discontinuity at the origin of the drift in equation (B.1). One therefore expects the densities of the Fermi bridge, for times strictly less than the terminal one, to be smooth in time, continuous in space and smooth away from the cut locus.

B.2 Towards the Origin

If we had instead considered weak solutions to the stochastic differential equation

$$\begin{cases} dX_t = dB_t - \operatorname{sgn}(X_t)dt \\ X_0 = 0 \end{cases} \tag{B.2}$$

then the same approach as above reveals a formula for the densities of such solutions which are illustrated below in Figure 6 for the time $t = 1$.

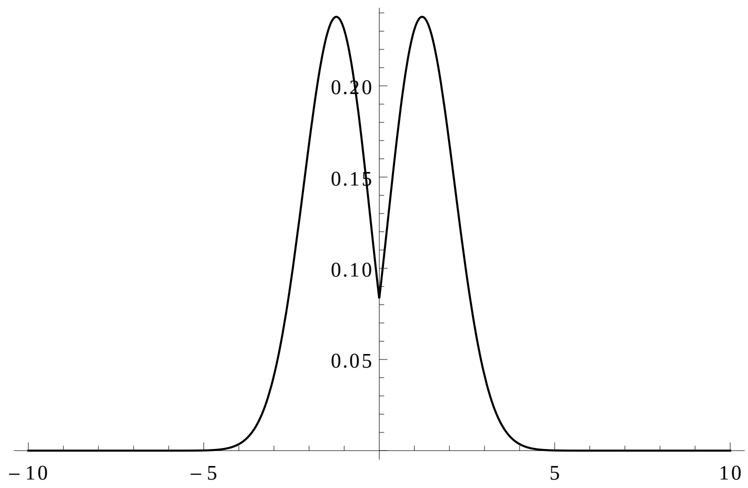

Figure 5: The graph of the density $\hat{p}_1$. The drift pushes mass away from the origin.

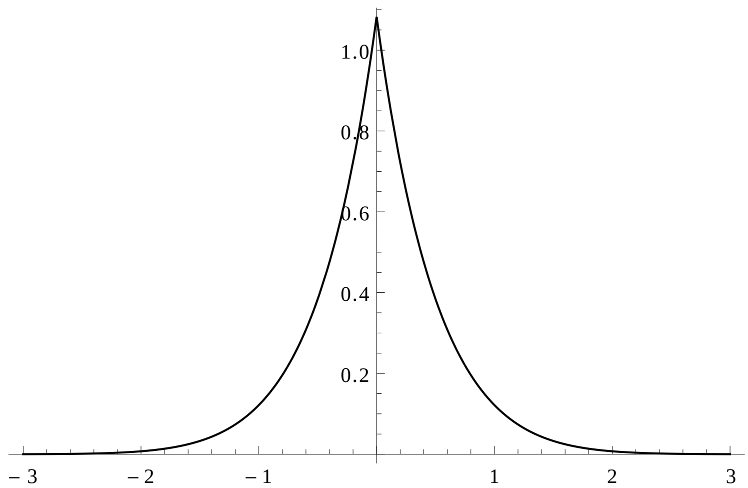

Figure 6: The graph of the density of a solution to equation (B.2) at time $t = 1$. The drift now pushes mass towards the origin.

Appendix C

Limits and Integrals

C.1 Supplementary Estimates

As in Subsection 4.2.2, suppose that D is a regular domain in M and that $\hat{X}(x)$ is a Fermi bridge between x and N in time T defined upto the minimum of T and its explosion time, whose first exit time from D is denoted by $\hat{\tau}_D$.

Theorem C.1.1. *Let $\nu \geq 1$ and $\lambda \geq 0$ be any constants such that inequality (3.7) holds on $D \setminus \mathrm{Cut}(N)$. Then for positive even integers p we have*

$$\mathbb{E}[\mathbf{1}_{\{t<\hat{\tau}_D\}} r_N^p(\hat{X}_t(x))] \leq \left(\frac{2t(T-t)e^{\lambda t}}{T}\right)^{\frac{p}{2}} \Gamma\left(\frac{p}{2}+1\right) L_{\frac{p}{2}}^{\frac{\nu}{2}-1}\left(-\frac{r_N^2(x)}{2}\left(\frac{T-t}{Tt}\right)\right)$$

for all $t \in [0, T)$.

Proof. Define the function $\hat{f}_{x,2p} : [0, T) \to \mathbb{R}$ by

$$\hat{f}_{x,2p}(t) := \mathbb{E}[\mathbf{1}_{\{t<\hat{\tau}_D\}} r_N^{2p}(\hat{X}_t(x))]$$

for $t \in [0, T)$. By Itô's formula, formula (4.2) and inequality (3.22) we deduce the differential inequality

$$\begin{cases} f'_{x,2p}(t) \leq p(\nu + 2(p-1)) f_{x,2(p-1)}(t) + p\left(\lambda - \frac{2}{T-t}\right) f_{x,2p}(t) \\ f_{x,2p}(0) = r_N^{2p}(x) \end{cases}$$

and thus by Gronwall's inequality we have

$$
\begin{aligned}
f_{x,2p}(t) &\leq \left(r_N^{2p}(x) + p(\nu + 2(p-1)) \int_0^t f_{x,2(p-1)}(s) e^{-\int_0^s p\left(\lambda - \frac{2}{T-u}\right)du} ds \right) e^{\int_0^t p\left(\lambda - \frac{2}{T-u}\right)du} \\
&= \left(r_N^{2p}(x) + p(\nu + 2(p-1)) \int_0^t f_{x,2(p-1)}(s) e^{-p\lambda s} \left(\frac{T}{T-s}\right)^{2p} ds \right) e^{p\lambda t} \left(\frac{T-t}{T}\right)^{2p}.
\end{aligned}
$$

By induction, with Theorem 4.2.1 serving as the base case, it follows that

$$
f_{x,2p}(t) \leq \left(\frac{2t(T-t)e^{\lambda t}}{T}\right)^p \sum_{j=0}^p \binom{p}{j} \left(\frac{r_N^2(x)}{2}\left(\frac{T-t}{Tt}\right)\right)^j \frac{\Gamma\left(\frac{\nu}{2}+p\right)}{\Gamma\left(\frac{\nu}{2}+j\right)}
$$

by the fact that $e^{(p-1)\lambda s - p\lambda s} = e^{-\lambda s} \leq 1$. The result follows from this by formula (3.20). $\square$

In the case $M = \mathbb{R}^m$ with N a subspace and $b = 0$, with $\nu = m-n$ and $\lambda = 0$, one can set $D = M$ and the inequality provided by Theorem C.1.1 holds as an *equality*. In particular, it follows by Proposition 4.1.2 that versions of this and certain subsequent results also hold for the semiclassical bridge of Section 4.1.

Corollary C.1.2. *Let $\nu \geq 1$ and $\lambda \geq 0$ be any constants such that inequality (3.7) holds on $D \setminus \mathrm{Cut}(N)$. Then for positive odd integers p we have*

$$
\mathbb{E}[\mathbf{1}_{\{t<\hat{\tau}_D\}} r_N^p(\hat{X}_t(x))] \leq \left(\frac{2t(T-t)e^{\lambda t}}{T}\right)^{\frac{p}{2}} \left(\Gamma\left(\frac{p+1}{2}+1\right) L_{\frac{p+1}{2}}^{\frac{\nu}{2}-1}\left(-\frac{r_N^2(x)}{2}\left(\frac{T-t}{Tt}\right)\right)\right)^{\frac{p}{p+1}}
$$

for all $t \in [0, T)$.

Note that by the summation formula (3.33), Theorem C.1.1 implies the exponential estimate

$$
\mathbb{E}\left[\mathbf{1}_{\{t<\hat{\tau}_D\}} e^{\frac{\theta}{2} r_N^2(\hat{X}_t(x))}\right] \leq \left(1 - \frac{\theta t(T-t)e^{\lambda t}}{T}\right)^{-\frac{\nu}{2}} \exp\left[\frac{\theta r_N^2(x)(T-t)^2 e^{\lambda t}}{2T(T - t(T-t)\theta e^{\lambda t})}\right]
$$

for all $\theta \geq 0$ and $t \in [0, T)$ such that $T - \theta t(T-t)e^{\lambda t} > 0$ (a condition which for $\theta \geq 0$ fixed is always satisfied for t sufficiently close to either 0 or T). Under the assumptions of Subsection 4.2.3, this implies by Markov's inequality that for all

$\delta \in [0,1)$ there is the concentration inequality

$$\mathbb{Q}_{T-}\{\hat{X}_t(x) \notin B_r(N)\} \leq (1-\delta)^{-\frac{\nu}{2}} \exp\left[\frac{\delta r_N^2(x)(T-t)}{2Tt(1-\delta)} - \frac{\delta r^2 T}{2t(T-t)e^{\lambda t}}\right]$$

and therefore the asymptotic estimate

$$\lim_{r\to\infty} \frac{1}{r^2} \log \mathbb{Q}_{T-}\{\hat{X}_t(x) \notin B_r(N)\} \leq -\frac{T}{2t(T-t)e^{\lambda t}}$$

for all $t \in (0,T)$. For the semiclassical bridge this relation holds with $\lambda = 0$ as an *equality.*

The next theorem is proved using Theorem C.1.1 and Corollary C.1.2 and implies

$$\sup_{t\in[0,T)} \mathbb{E}\left[\mathbf{1}_{\{t<\hat{\tau}_D\}} \exp\left[\theta \int_0^t \|\nabla \log q_{T-s}(\hat{X}_s(x), N)\| ds\right]\right] < \infty \tag{C.1}$$

for each $\theta \geq 0$. To put this in context, note that if we replaced $\hat{X}(x)$ by a Brownian motion with drift b conditioned to arrive at a point y at time T and replaced q by the transition density $p^{M,b}$ of a Brownian motion with drift b, evaluated at y rather than integrated over N, then the left-hand side of (C.1) would be the object estimated by Lyons and Zheng [1990] and by Qian [1994]. The latter estimate was used by Qian and Zheng [2004] to establish a formula for kernels of the form $p^{M,b+c}$ in terms of $p^{M,b}$ and an integral involving c over the paths of a conditioned diffusion. The following theorem can similarly be used to verify uniform integrability and obtain kernel estimates, as we will see in the next section, albeit in a rather special setting.

Theorem C.1.3. *Let $\nu \geq 2$ and $\lambda \geq 0$ be any constants such that inequality (3.7) holds on $D \setminus \mathrm{Cut}(N)$. Then we have*

$$\mathbb{E}\left[\mathbf{1}_{\{t<\hat{\tau}_D\}} \exp\left[\theta \int_0^t \frac{r_N(\hat{X}_s(x))}{T-s} ds\right]\right] \leq 1 + \left(1 + \hat{\mathbf{R}}(T,\theta,x)^{-\frac{1}{2}}\right)\left({}_1F_1\left(\frac{\nu}{2}, \frac{1}{2}, \hat{\mathbf{R}}(T,\theta,x)\right) - 1\right)$$

for all $\theta \geq 0$ and $t \in [0,T)$, where $\hat{\mathbf{R}}(T,\theta,x) = 48\theta^2 \left(2T + r_N^2(x)\right) e^{\lambda T}$.

Proof. First note by Tonelli's theorem and Hölder's inequality that

$$\begin{aligned}
&\mathbb{E}\left[\mathbf{1}_{\{t<\hat{\tau}_D\}} \exp\left[\theta \int_0^t \frac{r_N(\hat{X}_s(x))}{T-s} ds\right]\right] \\
&\leq 1 + \mathbb{E}\left[\sum_{p=1}^{\infty} \frac{\theta^p}{p!} \left(\int_0^t \frac{\mathbf{1}_{\{s<\hat{\tau}_D\}} r_N(\hat{X}_s(x))}{T-s} ds\right)^p\right] \\
&= 1 + \mathbb{E}\left[\sum_{p=1}^{\infty} \frac{\theta^p}{p!} \prod_{j=1}^{p} \int_0^t \frac{\mathbf{1}_{\{s_j<\hat{\tau}_D\}} r_N(\hat{X}_{s_j}(x))}{T-s_j} ds_j\right] \\
&= 1 + \sum_{p=1}^{\infty} \frac{\theta^p}{p!} \int_0^t \cdots \int_0^t \frac{\mathbb{E}[\prod_{j=1}^p \mathbf{1}_{\{s_j<\hat{\tau}_D\}} r_N(\hat{X}_{s_j}(x))]}{\prod_{j=1}^p (T-s_j)} ds_1 \cdots ds_p \\
&\leq 1 + \sum_{p=1}^{\infty} \frac{\theta^p}{p!} \int_0^t \cdots \int_0^t \frac{\prod_{j=1}^p \mathbb{E}[\mathbf{1}_{\{s_j<\hat{\tau}_D\}} r_N^p(\hat{X}_{s_j}(x))]^{\frac{1}{p}}}{\prod_{j=1}^p (T-s_j)} ds_1 \cdots ds_p \\
&= 1 + \sum_{p=1}^{\infty} \frac{\theta^p}{p!} \prod_{j=1}^{p} \int_0^t \frac{\mathbb{E}[\mathbf{1}_{\{s_j<\hat{\tau}_D\}} r_N^p(\hat{X}_{s_j}(x))]^{\frac{1}{p}}}{T-s_j} ds_j \\
&= 1 + \sum_{p=1}^{\infty} \frac{\theta^p}{p!} \left(\int_0^t \frac{\mathbb{E}[\mathbf{1}_{\{s<\hat{\tau}_D\}} r_N^p(\hat{X}_s(x))]^{\frac{1}{p}}}{T-s} ds\right)^p.
\end{aligned}$$

Now, by Theorem C.1.1 and Corollary C.1.2 we see that

$$\begin{aligned}
&\sum_{p=1}^{\infty} \frac{\theta^p}{p!} \left(\int_0^t \frac{\mathbb{E}[\mathbf{1}_{\{s<\hat{\tau}_D\}} r_N^p(\hat{X}_s(x))]^{\frac{1}{p}}}{T-s} ds\right)^p \\
&\leq \sum_{\substack{p=1,\\ p \text{ even}}}^{\infty} \frac{\theta^p}{p!} \left(\int_0^t \left(\left(\frac{2se^{\lambda s}}{T(T-s)}\right)^{\frac{p}{2}} \Gamma\left(\frac{p}{2}+1\right) L_{\frac{p}{2}}^{\frac{\nu}{2}-1}\left(-\frac{r_N^2(x)}{2}\left(\frac{T-s}{Ts}\right)\right)\right)^{\frac{1}{p}} ds\right)^p \\
&+ \sum_{\substack{p=1,\\ p \text{ odd}}}^{\infty} \frac{\theta^p}{p!} \left(\int_0^t \left(\left(\frac{2se^{\lambda s}}{T(T-s)}\right)^{\frac{p+1}{2}} \Gamma\left(\frac{p+1}{2}+1\right) L_{\frac{p+1}{2}}^{\frac{\nu}{2}-1}\left(-\frac{r_N^2(x)}{2}\left(\frac{T-s}{Ts}\right)\right)\right)^{\frac{1}{p+1}} ds\right)^p \\
&= \sum_{p=1}^{\infty} \frac{\theta^{2p}}{(2p)!} \left(\int_0^t \left(\left(\frac{2se^{\lambda s}}{T(T-s)}\right)^{p} p! L_{p}^{\frac{\nu}{2}-1}\left(-\frac{r_N^2(x)}{2}\left(\frac{T-s}{Ts}\right)\right)\right)^{\frac{1}{2p}} ds\right)^{2p} \\
&+ \sum_{p=1}^{\infty} \frac{\theta^{2p-1}}{(2p-1)!} \left(\int_0^t \left(\left(\frac{2se^{\lambda s}}{T(T-s)}\right)^{p} p! L_{p}^{\frac{\nu}{2}-1}\left(-\frac{r_N^2(x)}{2}\left(\frac{T-s}{Ts}\right)\right)\right)^{\frac{1}{2p}} ds\right)^{2p-1}.
\end{aligned}$$

By formula (3.20) and Lemma 3.2.8 we see that for $s \in [0, t)$ and $p = 1, 2, \ldots$ we have

$$
\begin{aligned}
&\left(\frac{2se^{\lambda s}}{T(T-s)}\right)^p p!L_p^{\frac{\nu}{2}-1}\left(-\frac{r_N^2(x)}{2}\left(\frac{T-s}{Ts}\right)\right)\\
&= \left(\frac{2e^{\lambda s}}{T-s}\right)^p \sum_{j=0}^p \binom{p}{j}\frac{\Gamma\left(\frac{\nu}{2}+p\right)}{\Gamma\left(\frac{\nu}{2}+j\right)}\left(\frac{r_N^2(x)}{2T}\right)^j\left(\frac{T-s}{T}\right)^j\left(\frac{s}{T}\right)^{p-j}\\
&\leq \left(\frac{2e^{\lambda T}}{T-s}\right)^p \sum_{j=0}^p \binom{p}{j}\frac{\Gamma\left(\frac{\nu}{2}+p\right)}{\Gamma\left(\frac{\nu}{2}+j\right)}\left(\frac{r_N^2(x)}{2T}\right)^j\\
&= \left(\frac{2e^{\lambda T}}{T-s}\right)^p p!L_p^{\frac{\nu}{2}-1}\left(-\frac{r_N^2(x)}{2T}\right)\\
&\leq \left(\frac{24e^{\lambda T}}{T-s}\left(1+\frac{r_N^2(x)}{2T}\right)\right)^p\frac{\Gamma\left(\frac{\nu}{2}+p\right)}{\Gamma\left(\frac{\nu}{2}\right)}
\end{aligned}
$$

and therefore

$$
\begin{aligned}
&\sum_{p=1}^{\infty}\frac{\theta^p}{p!}\left(\int_0^t\frac{\mathbb{E}[\mathbf{1}_{\{s<\hat{\tau}_D\}}r_N^p(\hat{X}_s(x))]^{\frac{1}{p}}}{T-s}ds\right)^p\\
&\leq \sum_{p=1}^{\infty}\frac{\left(24\theta^2e^{\lambda T}\left(1+\frac{r_N^2(x)}{2T}\right)\right)^p}{(2p)!}\frac{\Gamma\left(\frac{\nu}{2}+p\right)}{\Gamma\left(\frac{\nu}{2}\right)}\left(\int_0^t(T-s)^{-\frac{1}{2}}ds\right)^{2p}\\
&\quad+\sum_{p=1}^{\infty}\frac{\left(24\theta^2e^{\lambda T}\left(1+\frac{r_N^2(x)}{2T}\right)\right)^{\frac{2p-1}{2}}}{(2p-1)!}\left(\frac{\Gamma\left(\frac{\nu}{2}+p\right)}{\Gamma\left(\frac{\nu}{2}\right)}\right)^{\frac{2p-1}{2p}}\left(\int_0^t(T-s)^{-\frac{1}{2}}ds\right)^{2p-1}\\
&\leq \sum_{p=1}^{\infty}\frac{\left(48\theta^2e^{\lambda T}\left(2T+r_N^2(x)\right)\right)^p}{(2p)!}\frac{\Gamma\left(\frac{\nu}{2}+p\right)}{\Gamma\left(\frac{\nu}{2}\right)}\\
&\quad+\sum_{p=1}^{\infty}\frac{\left(48\theta^2e^{\lambda T}\left(2T+r_N^2(x)\right)\right)^{\frac{2p-1}{2}}}{(2p-1)!}\left(\frac{\Gamma\left(\frac{\nu}{2}+p\right)}{\Gamma\left(\frac{\nu}{2}\right)}\right)^{\frac{2p-1}{2p}}\\
&\leq \left(1+\left(48\theta^2e^{\lambda T}\left(2T+r_N^2(x)\right)\right)^{-\frac{1}{2}}\right)\sum_{p=1}^{\infty}\frac{\left(192\theta^2e^{\lambda T}\left(2T+r_N^2(x)\right)\right)^p}{2p!}\frac{\Gamma\left(\frac{\nu}{2}+p\right)}{\Gamma\left(\frac{\nu}{2}\right)}
\end{aligned}
$$

where for the final inequality we used the fact that $\Gamma\left(\frac{\nu}{2}+p\right)\geq\Gamma\left(\frac{\nu}{2}\right)$ since $\nu\geq 2$. The result now follows by the relation (3.32), as in the proof of Theorem 3.2.9. $\square$

Uniform square-integrability implies uniform integrability, so the previous theorem implies the following corollary.

Corollary C.1.4. *Let $\nu \geq 1$ and $\lambda \geq 0$ be any constants such that inequality (3.7) holds on $D \setminus \mathrm{Cut}(N)$. Then for each $\theta \geq 0$ the random variables*

$$\left\{ \mathbf{1}_{\{t<\hat{\tau}_D\}} \exp \left[\theta \int_0^t \frac{r_N(\hat{X}_s(x))}{T-s} ds \right] : t \in [0,T) \right\}$$

are uniformly integrable.

C.2 Uniform Integrability

Recall that Theorem 5.2.2 stated that if $\{D_i\}_{i=1}^{\infty}$ is an exhaustion of M by regular domains then

$$p_T^M(x,N) = q_T(x,N) \lim_{i\uparrow\infty} \lim_{t\uparrow T} \mathbb{E}\left[\mathbf{1}_{\{t<\hat{\tau}_{D_i}\}} \exp\left[\int_0^t \frac{r_N(\hat{X}_s(x))}{T-s} \left(d\mathbf{A}_s + d\mathbf{L}_s \right) \right] \right] \tag{C.2}$$

where $\mathbf{A}$ and $\mathbf{L}$ are defined by (5.14). The reader might, for aesthetic reasons or otherwise, wish to pass these two limits through the expectation. If one can pass the inner limit through the expectation then the outer limit can be dealt with by the monotone convergence theorem. There are various circumstances in which it is easy to justify passing the inner limit through the expectation. For example, if N is totally geodesic (or minimal if $n = m-1$) and if the sectional curvature of planes containing the radial direction is non-negative (or if $n \in \{0, m-1\}$ and the Ricci curvature in the radial direction is non-negative) then

$$\frac{\partial}{\partial r_N} \log \Theta_N^{-\frac{1}{2}} \geq 0$$

and one can apply the monotone convergence theorem. Conversely, if N is totally geodesic (or minimal if $n = m-1$) and if the sectional curvature of planes containing the radial direction is non-positive (or if $n \in \{0, m-1\}$ and the Ricci curvature in the radial direction is non-positive) then

$$\frac{\partial}{\partial r_N} \log \Theta_N^{-\frac{1}{2}} \leq 0$$

and one can apply the dominated convergence theorem, if in addition the cut locus of N is polar for $\hat{X}(x)$. In particular, if N is a point then we can justify passing both limits through the expectation for any of the simply connected space forms. In general, however, one must first verify the uniform integrability of the random variables

$$\left\{\mathbf{1}_{\{t<\hat{\tau}_{D_i}\}} \exp\left[\int_0^t \frac{r_N(\hat{X}_s(x))}{T-s}\Big(d\mathbf{A}_s + d\mathbf{L}_s\Big)\right] : t \in [0,T)\right\}. \tag{C.3}$$

Let us explain a general strategy for verifying this property, using Corollary C.1.4, and how it can be applied to a special case.

Firstly, since $\check{C}(N)$ is polar for Brownian motion with $M\setminus\check{C}(N)$ open and connected, it follows from the theorems of [Chavel, 1984, Chapter IX] that for all $x \notin \check{C}(N)$, $y \in N$ and $T > 0$ we have

$$p_T^{M\setminus\check{C}(N)}(x,y) = p_T^M(x,y).$$

Furthermore, since $\check{C}(N)$ has vol_M-measure zero it follows that p^M is the unique continuous extension of $p^{M\setminus\check{C}(N)}$ to $(0,\infty)\times M\times M$. So let us choose an exhaustion $\{D_i\}_{i=1}^\infty$ of $M\setminus\check{C}(N)$ by regular domains. Since the part of the cut locus contained in $M\setminus\check{C}(N)$ is exactly $\mathring{C}(N)$, which consists of points which can be connected to N by precisely two length-minimizing geodesic segments, both of which are non-focal, this sequence of domains has the property that for each $i\in\mathbb{N}$ there exists a constant $K_i \geq 0$ such that

$$\frac{\partial}{\partial r_N}\log\Theta_N^{-\frac{1}{2}} \leq K_i$$

on D_i. For the case in which $\mathrm{Cut}(N)$ is polar with $x \notin \check{C}(N)$ this implies, by Corollary C.1.4, the uniform integrability of the random variables (C.3). For the general case it thus suffices, by the Cauchy-Schwarz inequality, to verify the uniform square-integrability of the random variables

$$\left\{\mathbf{1}_{\{t<\hat{\tau}_{D_i}\}} \exp\left[\int_0^t \frac{r_N(\hat{X}_s(x))}{T-s} d\mathbf{L}_s\right] : t \in [0,T)\right\}. \tag{C.4}$$

If the uniform square-integrability of these random variables can be verified then both limits in formula (C.2) can be passed through the expectation, so long as $x \notin \check{C}(N)$. While this problem is open, it might be helpful to observe that our choice of D_i implies that $\mathrm{Cut}(N) \cap D_i$ is given by the union of only *finitely* many smooth $(m-1)$-dimensional submanifolds and that the density $(D^+ - D^-)\, r_N(\mathbf{n})$ is bounded on $\mathrm{Cut}(N) \cap D_i$.

A related problem is to deduce upper bounds for the integrated heat kernel directly from formula (C.2). If $\mathrm{Cut}(N)$ is polar and there exist constants $\alpha, \beta, \gamma \geq 0$ such that

$$-2\gamma \leq \frac{\partial}{\partial r_N} \log \Theta_N \leq \alpha + \beta r_N$$

on M then Theorems 5.2.2 and C.1.3 imply the upper bound

$$p_T^M(x, N) \leq q_T(x, N) \left(1 + \left(1 + \hat{\mathbf{R}}(T, \gamma, x)^{-\frac{1}{2}}\right) \left({}_1F_1\left(\frac{\nu}{2}, \frac{1}{2}, \hat{\mathbf{R}}(T, \gamma, x)\right) - 1\right)\right) \tag{C.5}$$

for all $x \in M$ and $T > 0$, where $\lambda = \alpha/2 + \beta$ and $\nu = m - n + 1 + \alpha/2$ with

$$\hat{\mathbf{R}}(T, \gamma, x) = 48\gamma^2(2T + r_N^2(x))e^{\lambda T}.$$

Note that the limit as $T \downarrow 0$ of the largest term in parentheses on the right-hand side of (C.5) is strictly greater than 1 unless $r_N(x) = 0$, which is to be expected by comparing Theorems 5.1.4 and 5.2.2.

This may not be the best approach to upper bounds, so we should look for an alternative. Avoiding Girsanov's theorem altogether by applying Jensen's inequality directly to $P_t(q_{T-t}(\cdot, N))(x)$ doesn't quite work, but there might be a way to use the concentration inequalities of Subsection 3.2.8 instead. For the time being, the approach to upper bounds described in Subsection 5.3.4 seems to be the most satisfactory.

Appendix D

Large Time Behaviour

D.1 A Spectral Gap Inequality

Suppose that M is stochastically complete and that there exist constants $C_1, \Lambda \geq 0$ such that at least one of the three conditions **(C1)**, **(C2)** or **(C3)** of Theorem 1.4.5 is satisfied with $C_2 = 0$. Fix $x \in M$ and $T > 0$, let $\nu = m-n$ and $\mu = n\Lambda+(m-1)C_1$ and suppose that $u : [0,T] \to [0,\infty)$ solves the ordinary differential equation

$$\begin{cases} u'(t) = \nu + \mu\sqrt{u(t)} - \frac{2}{T-t}u(t) \\ u(0) = r_N^2(x) \end{cases} \tag{D.1}$$

for $t \in [0,T)$. It follows from a nonlinear version of Gronwall's inequality and Jensen's inequality that the first radial moment of a Fermi bridge between x and N in time T is bounded above by $\sqrt{u}$. Using this observation and inequality (1.28), an approach similar to the one used for the proof of Proposition 5.3.1 yields the implicit lower bound

$$p_T^M(x,N) \geq q_T(x,N) \exp\left[-\frac{\mu}{2}\int_0^T \frac{\sqrt{u(s)}}{T-s}\,ds\right] \tag{D.2}$$

for all $x \in M$ and $T > 0$. Unfortunately, we are not aware of an explicit formula for the solution to equation (D.1), except in the pathological case $\nu = 0$ when the equation becomes a Bernoulli differential equation. We expect that inequality (D.2) improves the large time behaviour of the lower bounds proved in Subsection 5.3.1

when $C_2 = 0$. To see this, note first that (D.2) implies

$$\lim_{T\to\infty} \frac{1}{T} \log p_T^M(x, N) \geq \lim_{T\to\infty} -\frac{\mu}{2T} \int_0^T \frac{\sqrt{u(s)}}{T-s}\, ds$$

for any $x \in M$. It was proved by Li [1986] that if $\lambda_1(M)$ denotes the bottom of the spectrum of $-\triangle$ then

$$\lim_{T\to\infty} \frac{1}{T} \log p_T^M(x, y) = -\frac{\lambda_1(M)}{2}$$

for any $x, y \in M$ so if $\mathrm{Ric} \geq -(m-1)C_1^2$ we have a spectral gap inequality

$$\lambda_1(M) \leq \lim_{T\to\infty} \frac{(m-1)C_1^2}{T} \int_0^T \frac{\sqrt{u_T(s)}}{T-s}\, ds.$$

For the hyperbolic space $\mathbb{H}_\kappa^m$ with $\kappa < 0$ it is known, as in [Chavel, 1984, p.46], that

$$\lambda_1(\mathbb{H}_\kappa^m) = -\frac{(m-1)^2\kappa}{4}$$

while numerical approximation suggests that

$$\lim_{T\to\infty} -\frac{(m-1)\kappa}{T} \int_0^T \frac{\sqrt{u_T(s)}}{T-s}\, ds = \frac{(m-1)^2\kappa^2}{2}.$$

Thus we should expect the large time behaviour of the lower bound (D.2) to be generally quite favourable. The extra factor of -2κ appearing in the hyperbolic example can probably be attributed to the use of Jensen's inequality, which an alternative approach, such as a Laplace-type method, might be able to eliminate.

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