Mathematics and Statistics
Centre for Doctoral Training

Sample path large deviations for the Laplacian model with pinning interaction in $(1+1)$-dimension

by

Alexander Karl Kister

Thesis

Submitted for the degree of

Doctor of Philosophy

Mathematics Institute

The University of Warwick

September 2015

MENS AGITAT MOLEM

UNIVERSITAS WARWICENSIS

Contents

Acknowledgments

I would like to thank my supervisor Stefan Adams for introducing me to the topic of this thesis and for his guidance throughout my studies. I also like to thank my second supervisor Hendrik Weber for his help and support in writing this thesis. Furthermore I like to thank Stefan Grosskinsky for joining some discussions at the beginning of my PhD.

Next, I would like to thank all my departmental friends and all my flatmates for their company during my time at the University of Warwick. Especially I like to thank my good friend Amal Alphonse for his help in proofreading some parts of this thesis. I also like to thank my parents, my sister and my grandmother for their encouragement.

Finally, I would like to thank the Engineering and Physical Sciences Research Council (EPSRC) for the funding they provide and everyone involved with the MASDOC doctoral training centre.

Declarations

The work presented here is my own, except where specifically stated otherwise, and was performed in the Department of Mathematics at the University of Warwick under the supervision of Dr. Stefan Adams and Dr. Hendrik Weber. I confirm that this thesis has not been submitted for a degree at another university.

Abstract

We consider the (1+1) dimensional Laplacian model with pinning interaction. This is a probabilistic model for a polymer or an interface that is attracted to the zero line. Without the pinning interaction, the Laplacian model is a Gaussian field $(\phi_i)_{i\in\Lambda_N}$, where $\Lambda_N = \{1, 2, \ldots, N-1\}$. The covariance matrix of this field is given by the inverse of $\phi \mapsto \frac{1}{2}\sum_{i=0}^{N}(\Delta\phi_i)^2$, where Δ is the discrete Laplacian. Furthermore the values at $\{-1, 0, N, N+1\}$ are fixed boundary values. The pinning interaction is introduced by giving the field a reward each time it touches the zero line.

Depending on the reward the model with pinning and the one without pinning show different behaviour. Caravenna and Deuschel [10] study the localisation behaviour of the polymer. The model is delocalised if the number of times a typical field touches the zero line is of order $o(N)$. The authors of [10] show that for zero boundary conditions there is a critical reward such that for smaller rewards the model is delocalised whilst for larger rewards the model is localised.

In this thesis we study the behaviour of the empirical profile of the field. We show that for non zero boundary conditions there is a critical reward such that for smaller rewards the empirical profile for the model with pinning and the one for the model without pinning behave in the same way whilst for larger rewards the empirical profile of the model with pinning interaction is attracted to the zero line.

Chapter 1

Introduction

Probability theory has been influenced by statistical physics for at least fifty years. The models that we study have their origin in statistical physics, too. In the mathematical literature they appear under the names *random interface* (see [16, 21, 17]) or *random polymer models* (see [22, 9]). Mathematically those models are random fields, i.e. families of countably many random variables.

The models appear under different names because they are used to explain different natural phenomena. A polymer is a long chain of repetitive units, called monomers. Random polymer models are designed to study the special arrangement of the monomers. Interface models describe the surface between two coexisting phases, for example, the one between water and ice at $0\,^{\circ}\mathrm{C}$ or the one between areas of positive and negative magnetisation in a ferromagnet. The atoms forming the surface are called *interface*. One approach to study these interfaces is to model the complete system, for example a volume of water or a ferromagnet, see [15]; another way - and this is the one where our models emerged from - is to model only the atoms that form the interface but such that this reduced model is still consistent with a model of the full system. This second approach leads to so-called *effective models*. These models are called effective because they only describe the location of the interface above a reference level and not the full system. The random interface models are such effective models.

The goal of the polymer and interface models is to understand how these systems interact with their environment. For example a polymer can be attracted to a membrane; if the membrane is penetrable we call this interaction *pinning* and if the membrane is not penetrable we call the interaction *wetting*. Depending on the strength of the interaction and the distance of the polymer from the membrane, the polymer might or might not be affected by the reward.

We study models with so-called Laplacian interaction and compare our results to models with gradient interaction.

The Laplacian model with pinning interaction

Laplacian models with pinning interaction are random fields ϕ on $\mathbb{Z}$. A random field on $\mathbb{Z}$ is a family of real valued random variables indexed by a subset of $\mathbb{Z}$. To define the distribution of the field ϕ in the subset $\Lambda_N := \{1, 2, \dots N-1\}$ we use the functions

$$\mathcal{H}_N(\phi) := \frac{1}{2} \sum_{i=0}^{N} (\Delta \phi_i)^2, \tag{1.1}$$

where $N \in \mathbb{N}$ and Δ is the discrete Laplacian given by

$$\Delta \phi_i := \phi_{i-1} - 2\phi_i + \phi_{i+1}.$$

In physics, for $N \geq 2$, the value $\mathcal{H}_N(\phi)$ is called the total energy of ϕ in Λ_N and $\mathcal{H}_N$ is the Hamiltonian in this set. The Hamiltonian models how the heights $(\phi_i)_{i \in \Lambda_N}$ interact with each other and with the boundary $(\phi_i)_{i \in \{-1,0,N,N+1\}}$.

To understand the interaction let $N = 2$ and fix the boundary condition $(\phi_i)_{i \in \{-1,0,2,3\}}$. So the only height which is not fixed is ϕ_1. The height ϕ_1 is energetically optimal if it minimises the total energy under the given boundary condition. The Hamiltonian $\mathcal{H}_2$ is the sum of the squares of the three Laplacians $\Delta\phi_0$, $\Delta\phi_1$ and $\Delta\phi_2$. Considering each of these terms separately, we see that it would be optimal if ϕ_1 is such that each of the heights ϕ_0, ϕ_1 and ϕ_2 coincides with the average of its neighbouring heights, because then each Laplacian would be zero. But for general boundary conditions $(\phi_i)_{i \in \{-1,0,2,3\}}$ a ϕ_1 that is such that ϕ_0 coincides with the average of ϕ_{-1} and ϕ_1 does not coincide with the average of ϕ_0 and ϕ_2. In general there is a trade off between choosing ϕ_1 such that ϕ_0 or ϕ_1 or ϕ_2 coincides with the average of its neighbouring heights. So ϕ_1 interacts with its nearest and next nearest neighbours.

To define the field ϕ we need a boundary condition $\psi \in \mathbb{R}^{\overline{\Lambda_N}}$, where the set $\overline{\Lambda_N} := \{-1, 0, \dots, N+1\}$. The field is given by the following probability distribution:

$$\gamma_N^{\psi,J}(\mathrm{d}\phi) := \frac{1}{\mathcal{Z}_N^{\psi,J}} e^{-\mathcal{H}_N(\phi)} \prod_{i=1}^{N-1} (\mathrm{d}\,\phi_i + e^J \delta_0(\mathrm{d}\,\phi_i)) \prod_{i \in \{-1,0,N,N+1\}} \delta_{\psi_i}(\mathrm{d}\,\phi_i),$$

where $\mathrm{d}\,\phi_i$ is the Lebesgue measure on $\mathbb{R}$ and δ_0 is the Dirac measure at zero and $J \in \mathbb{R}$ is the pinning strength; the normalisation constant $\mathcal{Z}_N^{\psi,J}$ is known in statistical physics as the partition function

$$\mathcal{Z}_N^{\psi,J} := \int_{\mathbb{R}^{\Lambda_N}} e^{-\mathcal{H}_N(\phi)} \prod_{i=1}^{N-1} (\mathrm{d}\,\phi_i + e^J \delta_0(\mathrm{d}\,\phi_i)) \prod_{i\in\{-1,0,N,N+1\}} \delta_{\psi_i}(\mathrm{d}\,\phi_i).$$

The terms $e^J\delta_0(\mathrm{d}\,\phi_i)$ attract ϕ to the zero line. Note that in physics the measure $\gamma_N^{\psi,J}$ is called the Gibbs distribution in Λ_N with boundary condition ψ, interaction potential $(\Delta\phi_i)^2$, and single spin measure $(\mathrm{d}\,\phi_i + e^J\delta_0(\mathrm{d}\,\phi_i)$ (see [20, Definition 2.9]).

To simplify the notation we use the conventions

$$\gamma_N^{\psi} := \gamma_N^{\psi,-\infty}, \quad \mathcal{Z}_N^{\psi} := \mathcal{Z}_N^{\psi,-\infty};$$

and in contexts where $J > -\infty$ is fixed we use the notation

$$\hat{\gamma}_N^{\psi} := \gamma_N^{\psi,J}, \quad \hat{\mathcal{Z}}_N^{\psi} := \mathcal{Z}_N^{\psi,J}.$$

Random walk representation

The random fields from above are related to a special class of random walks, the **integrated random walks (IRWs)**. First we consider the case $J = -\infty$. To define the **IRW** let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed (i.i.d.) standard normally distributed random variables on a probability space $(\Omega, \mathcal{E}, P^{\psi})$; we call the processes $(Y_n)_{n\in\mathbb{N}_0}$ and $(\zeta_n)_{n\in\mathbb{N}_0}$ given by

$$Y_0 = \psi_0 - \psi_{-1},\, Y_n = Y_0 + \sum_{i=1}^{n} X_i\,, \text{ for } n \geq 1; \;\; \zeta_0 = \psi_0,\, \zeta_n = \zeta_0 + \sum_{i=0}^{n} Y_i\,, \text{ for } n \geq 1, \tag{1.2}$$

random walk and **IRW**, respectively. The laws of $(\zeta_i)_{i\in\Lambda_N}$ under the measure $P_N^{\psi}(\cdot) := P^{\psi}(\cdot|\zeta_N = \psi_N, \zeta_{N+1} = \psi_{N+1})$ and of $(\phi_N)_{i\in\Lambda_N}$ under the measure γ_N^{ψ} coincide. To see this we study their densities. The most important observation for this study is that if two random variables have the joint density $f(x,y)$, then the density of the first random variable given that the second is zero coincides up to a multiplicative constant with $f(x,y)\delta_0(\mathrm{d}y)$. So for zero boundary conditions it is enough to show that the density of the **IRW** under $P^{\mathbf{0}}$ is up to a multiplicative constant equal to

$$e^{-\mathcal{H}_N(\phi)}\delta_0(\mathrm{d}\zeta_{-1})\delta_0(\zeta_0). \tag{1.3}$$

Since $\Delta\zeta_i = X_{i+1}$, we have for $Y_0 = 0$ and $\zeta_0 = 0$ that

$$
\begin{aligned}
&P^{\mathbf{0}}((\Delta\zeta_0, \Delta\zeta_1, \dots, \Delta\zeta_{N-1}, \zeta_{-1}, \zeta_0) \in (\mathrm{d}x_1, \mathrm{d}x_2, \dots, \mathrm{d}x_N, \mathrm{d}\zeta_{-1}, \mathrm{d}\zeta_0)) \\
&= P^{\mathbf{0}}((X_1, X_2, \dots, X_N, \zeta_{-1}, \zeta_0) \in (\mathrm{d}x_1, \mathrm{d}x_2, \dots, \mathrm{d}x_N, \mathrm{d}\zeta_{-1}, \mathrm{d}\zeta_0)) \\
&= \tfrac{1}{C} e^{-\frac{1}{2}\sum_{i=1}^N x_i^2} \prod_{i=1}^{N-1} (\mathrm{d}\, x_i) \prod_{i\in\{-1,0\}} \delta_0(\mathrm{d}\,\phi_i), \qquad (1.4)
\end{aligned}
$$

where C is a normalisation constant. Substituting x_i by $\Delta\zeta_{i-1}$ in the last equation we see that the density of the **IRW** under $P^{\mathbf{0}}$ is up to a multiplicative constant equal to (1.3). For the argument for non zero boundary conditions see [10, Lemma 2.1, Proposition 2.2].

For $J > -\infty$ relating the random field and the **IRW** requires a different argument. Note that the reference measure has the expansion

$$
\prod_{i\in\Lambda_N} (\mathrm{d}\,\phi(i) + e^J \delta_0(\mathrm{d}\,\phi(i))) = \sum_{S\subset\Lambda_N} e^{J|S^c|} \prod_{i\in S^c} \delta_0(\mathrm{d}\,\phi_i) \prod_{i\in S} (\mathrm{d}\,\phi_i), \qquad (1.5)
$$

where we use the convention $S^c := \Lambda_N \setminus S$ and where $|S|$ is the cardinality of S. Under the assumption that $\psi_i = 0$ for $i \in \Lambda_N$, the expansion (1.5) implies that for a set A that is measurable with respect to the σ-algebra generated by $\{\phi_i \mid i \in \Lambda_N\}$ we have:

$$
\hat{\gamma}_N^\psi(A) = \tfrac{1}{\hat{\mathcal{Z}}^\psi} \sum_{S\subset\Lambda_N} e^{J|S^c|} \mathcal{Z}_S^\psi \gamma_S^\psi(A), \qquad (1.6)
$$

where

$$
\gamma_S^\psi(\mathrm{d}\phi) := \frac{1}{\mathcal{Z}_S^\psi} e^{-\mathcal{H}_N(\phi)} \prod_{i\in S} (\mathrm{d}\,\phi_i) \prod_{i\in S^c} \delta_{\psi_i}(\mathrm{d}\,\phi_i),
$$

and

$$
\mathcal{Z}_S^\psi := \int_{\mathbb{R}^S} e^{-\mathcal{H}_N(\phi)} \prod_{i\in S} (\mathrm{d}\,\phi_i) \prod_{i\in S^c} \delta_{\psi_i}(\mathrm{d}\,\phi_i).
$$

A sample from γ_S^ψ coincides in $\mathcal{P} := S^c$ with ψ. We say the measure γ_S^ψ is pinned to ψ at the sites $\mathcal{P}$. The measures $\gamma_{\mathcal{P}^c}^\psi$ are related to the **IRW** as follows: analogously to (1.3) we see that the laws of $(\zeta_i)_{i\in\Lambda_N}$ under $P^\psi(\cdot|\zeta_i = \psi_i\,,\ \text{for}\ i \in \mathcal{P} \cup \{-1, 0, N, N+1\})$ and of $(\phi_i)_{i\in\Lambda_N}$ under $\gamma_{\mathcal{P}^c}^\psi$ coincide.

The reason for naming γ_S^ψ not after the sites where the reference measure has δ measures but after the sites S where the reference measure has Lebesgue measures is down to the fact that this is consistent with the definition of γ_N^ψ: just let $S = \Lambda_N$ and note that $\gamma_S^\psi = \gamma_N^\psi$.

By (1.6), sampling from $\hat{\gamma}_N^\psi$ can be realised using the following two stage procedure:

- **Stage 1:** Sample a subset of $\mathcal{P} \subset \Lambda_N$ according to the law

$$e^{J|\mathcal{P}|} \frac{\mathcal{Z}^{\psi}_{\mathcal{P}^c}}{\hat{\mathcal{Z}}^{\psi}_N}.$$

- **Stage 2:** Sample an element of $\mathbb{R}^{\Lambda_N}$ according to the law $\gamma^{\psi}_{\mathcal{P}^c}$.

The **IRW** does not satisfy the Markov condition but it satisfies a Markov condition with lag 2, which means that we need to know the current and the most recent past state in order to know the distribution of the future of the chain. For the measures γ^{ψ}_S and partition functions $\mathcal{Z}^{\psi}_S$ this has the consequence that

$$\begin{aligned} \gamma^{\psi}_S &= \gamma^{\psi}_{S_1} \gamma^{\psi}_{S_2} \\ \mathcal{Z}^{\psi}_S &= \mathcal{Z}^{\psi}_{S_1} \mathcal{Z}^{\psi}_{S_2} \end{aligned} \tag{1.7}$$

if $S = S_1 \cup S_2$ and $\min S_2 - \max S_1 \geq 3$ (note that this implies that the gap between S_1 and S_2 is at least two, but also note that also the sets S_1 and S_2 are not necessary connected). We call this the splitting property of lag 2.

The gradient model with pinning interaction

A related model is the gradient model (see [19]). This model differs from the Laplace model only by the Hamiltonian; the gradient model is defined with the Hamiltonian

$$\mathcal{H}^{\nabla}_N(\phi) := \frac{1}{2} \sum_{i=0}^{N-1} (\nabla \phi_i)^2 \text{, for } \Lambda \subset \mathbb{Z}, |\Lambda| < \infty \tag{1.8}$$

where

$$\nabla \phi_i := \phi_{i+1} - \phi_i.$$

With this Hamiltonian, each height ϕ_i interacts only with its nearest neighbours. We denote the probability distribution of the field with gradient interaction by $\gamma^{\nabla,\psi}_N$. For $J = -\infty$, the law of the gradient model coincides with the law of the random walk under the condition that $Y_0 = \psi_0$ and $Y_N = \psi_N$. The gradient model satisfies the splitting property (1.7) with lag 1, that means that (1.7) is satisfied already for $S = S_1 \cup S_2$ such that $\min S_2 - \max S_1 \geq 2$.

The random walk representation implies that for certain types of polymers, the so-called *semi-flexible polymers* [8], the gradient model is a less suitable choice than the Laplacian model. One characteristic of a semi flexible polymer is that the gradients of the polymer are correlated. For the gradient model, where the gradients are X_i, this is clearly not the case while for the Laplacian model, where the gradients are Y_i, this is the case. A model that is related to the Laplacian model but also captures other aspects of semi-flexible chains is studied in [24].

Localisation and delocalisation

For the Laplacian and the gradient model, we measure whether the reward has any effect by the expected fraction of pinned sites:

$$\mathrm{E}^J[\tfrac{|\mathcal{P}|}{N}],$$

where we write E^J for the expectation with respect to the measure $\gamma_N^{\mathbf{0},J}$, $\gamma_N^{\nabla,\mathbf{0},J}$, respectively. A quantity related to that expectation is the pinning free energy; which is defined as

$$\tau(J) := \lim_{N\to\infty} \tau_N(J), \quad \tau_N(J) := \frac{1}{N} \log \frac{\mathcal{Z}_N^{\mathbf{0},J}}{\mathcal{Z}_N^{\mathbf{0}}}. \tag{1.9}$$

To see this relation note that by (1.5) we have

$$\tfrac{\mathrm{d}}{\mathrm{d}J}\tau_N(J) = \tfrac{1}{\mathcal{Z}_N^{\mathbf{0},J}} \sum_{\mathcal{P}\subset\Lambda_N} \tfrac{|\mathcal{P}|}{N} e^{J|\mathcal{P}|} \mathcal{Z}_{\mathcal{P}^c}^{\mathbf{0}} = \mathrm{E}^J[|\mathcal{P}|/N].$$

So if $\tau(J)$ is identical to zero in an interval $(-\infty, J_c]$, then, for large N, the fraction of pinned sites is almost surely zero for all $J \le J_c$. If $\tau(J) > 0$, we call the model with reward J *localised* and otherwise we call it *delocalised*.

For the gradient model we have $\tau(J) > 0$ for all $J > -\infty$ (see [19, Remark 6.1]), while for the Laplacian model there is a $J_c > -\infty$ such that $\tau(J) = 0$ for $J \le J_c$ and $\tau(J) > 0$ if $J > J_c$ (see [10, Theorem 1.2]).

To prove that for the gradient model we have $J_c = -\infty$ Funaki and Sakagawa [19] show that for N large enough there is a constant C such that the following inequality is true:

$$\frac{\mathcal{Z}_N^{\mathbf{0},J}}{\mathcal{Z}_N^{\mathbf{0}}} = \sum_{\mathcal{P}\subset\Lambda_N} e^{J|\mathcal{P}|} \frac{\mathcal{Z}_{\mathcal{P}^c}^{\mathbf{0}}}{\mathcal{Z}_N^{\mathbf{0}}} \ge \sum_{\mathcal{P}\subset\Lambda_N} e^{J|\mathcal{P}|} e^{-C|\mathcal{P}|} = (1 + e^{J-C})^{N-1}. \tag{1.10}$$

The argument for this inequality uses that since the gradient model satisfies the splitting property with lag 1, the partition function $\mathcal{Z}_{\mathcal{P}^c}^{\mathbf{0}}$ is a product of partition functions of the form $\mathcal{Z}_S^{\mathbf{0}}$ where $S = \{s_* + 1, \ldots, s^* - 1\} =: (s_*, s^*)$ and that $\mathcal{Z}_S^{\mathbf{0}} = \mathcal{Z}_{|S|}^{\mathbf{0}}$. Since for the gradient model the $\mathcal{Z}_N^{\mathbf{0}}$ is equal to the square root of a polynomial in N times $e^{-\tilde{C}N}$, where $\tilde{C}$ is a constant, the inequality is true (for details see [19]).

For the Laplacian model the splitting property is not satisfied with lag 1. To determine the critical reward J_c for the Laplacian model Caravenna and Deuschel [10] consider only certain properties of the field $\hat{\gamma}_N^{\mathbf{0}}$: they consider the zero set $\mathcal{P}$ and heights before these zeros. To do so they use the density

$$\hat{\gamma}_N^{\mathbf{0}}(|\mathcal{P}| = k, \mathcal{P}_i = t_i, \mathrm{d}b_i \in \mathrm{d}y_i, i \in \{1, 2, \dots k\}),$$

where $\mathcal{P} = \{\mathcal{P}_1, \mathcal{P}_2, \dots \mathcal{P}_k\}$, $\mathcal{P}_i < \mathcal{P}_{i+1}$, and b_i are the values before the chain hits zero:

$$b_i := \phi_{\mathcal{P}_i - 1}.$$

They represent $(\mathcal{P}, b)$ with the help of a Markov renewal process (see [1, Chapter VII 4]).

Using this representation Caravenna and Deuschel [10] prove in addition to $J_c > -\infty$ also some properties of the function $J \mapsto \tau(J)$: For $J_c < J < \infty$ the function $J \mapsto \tau(J)$ is real analytic with $0 < \tau(J) < \infty$ and for $J \to \infty$, $\tau(J) = J + \log(1 + o(1))$, see [10, Theorem 1.2]. They also show that the first derivative of the function $J \mapsto \tau(J)$ at J_c is zero and that the second derivative does not exist, see [10, Theorem 1.4]. In statistical physics such a transition with a discontinuity in the second derivative is called *second order transition.* Furthermore the authors of [10] consider the number of sites at which a typical path picks reward: For $J \leq J_c$ this number is of order $o(N)$ while for $J > J_c$ this number increases at least linearly in N. Additionally they show that for $J > J_c$ the maximal gap between sites at which a typical path picks reward is of order $o(N)$.

In the following lemma we quote a result that Caravenna and Deuschel [10] obtain during their study of the free energy.

Lemma 1.1. *For each J, there is a renewal process $\chi = \{\chi_k\}_{k \in \mathbb{N}}$ such that*

$$\frac{\mathcal{Z}_N^{\mathbf{0},J}}{\mathcal{Z}_N^{\mathbf{0}}} = 2\pi\sqrt{p(N)}e^{\tau(J)N - 2J} P(N + 1 \in \chi), \tag{1.11}$$

where

$$p(N) = \tfrac{1}{6}N + \tfrac{5}{12}N^2 + \tfrac{1}{3}N^3 + \tfrac{1}{12}N^4.$$

For $J \geq J_c$, the process χ is non terminating.

Proof. See Proposition 5.1 and equation (5.3) in [10]. The factor $\sqrt{p(N)}(2\pi)$ appears because to obtain the Hamiltonian used by the authors of [10] we have to add $(N+1)\log(\sqrt{2\pi})$ to our Hamiltonian $\mathcal{H}_N$ and because by Proposition C.1

$$\mathcal{Z}_N^{\mathbf{0}} = \frac{\sqrt{2\pi}^{N-1}}{\sqrt{p(N)}}.$$

□

For a summary on the properties of χ see [11, Section 3.1.].

Large deviations of the empirical profile

The following results concern empirical profiles. They are given with the help of a function $h_N \colon \mathbb{R}^{\overline{\Lambda_N}} \to C(0,1)$, where $h_N(\phi)$ is the linear interpolation of a scaled version of $(\phi_{\xi N})_{\xi \in \overline{\Lambda_N}/N}$.

In this thesis we study for which reward level J the empirical profile of the Laplacian model with pinning behaves different than the one of the Laplacian model without pinning ($J = -\infty$). Furthermore we investigate the influence of the boundary condition on this critical reward J. Intuitively it is clear that for non zero boundary conditions the critical reward is larger than J_c, because if the interface does not start in zero it has to go down or up before it can touch zero.

To study the effect of the pinning on the empirical profile we prove a **large deviations principle (LDP)** for this profile. For the Laplacian model the empirical profile is the linear interpolation of $(\frac{1}{N^2}\phi_{\xi N})_{\xi \in \overline{\Lambda_N}/N}$:

$$h_N(\phi)(\xi) := \tfrac{1}{N^2}\phi_{\lfloor N\xi \rfloor} + (\xi - \tfrac{\lfloor N\xi \rfloor}{N})\tfrac{1}{N^2}(\phi_{\lfloor N\xi \rfloor + 1} - \phi_{\lfloor N\xi \rfloor}) \quad \text{, for } \xi \in [0,1], \tag{1.12}$$

where for $x \in \mathbb{R}$ the value $\lfloor x \rfloor$ is the largest integer smaller than or equal to x. Let $\mathbf{r} := (a, \alpha, b, \beta) \in \mathbb{R}^4$, $\mathbf{a} = (a, \alpha)$, and

$$\psi^{\mathbf{r},N}(i) := \begin{cases} aN^2 - \alpha N & \text{, for } i = -1, \\ aN^2 & \text{, for } i = 0, \\ bN^2 & \text{, for } i = N, \\ bN^2 + \beta N & \text{, for } i = N+1, \\ 0 & \text{, otherwise.} \end{cases} \tag{1.13}$$

The scaling $\frac{1}{N^2}$ is motivated by Mogulskii's theorem (see Theorem A.3) and the **IRW** representation of the model with zero boundary condition. First note that by Mogulskii theorem the increments $(Y_i)_{i \in \mathbb{N}}$ of the **IRW** representation scaled by $\frac{1}{N}$ satisfy an **LDP**. Integrating $\xi \mapsto \frac{1}{N} Y_{\lfloor N\xi \rfloor}$ we obtain a function that coincides with

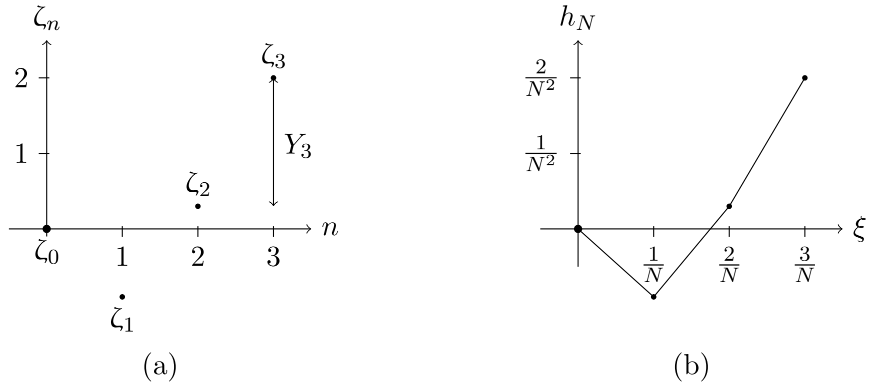

Figure 1.1: These are sketches of (a) the **IRW** (see (1.2)) and (b) the linear interpolation h_N (see (1.12)). Note that in (a) the jump heights correspond to $(Y_n)_{N\in\mathbb{N}}$, the third jump Y_3 is highlighted.

the linear interpolation of the scaled **IRW** $\frac{1}{N^2}\zeta$. So the scaling $\frac{1}{N^2}$ we obtain an **LDP** for the **IRW** if we use the scaling $\frac{1}{N^2}$. Note that the increments $(Y_i)_{i\in\mathbb{N}}$ have a variance of order N and that hence the **IRWs** has a variance of order N^3. Hence the **IRW** scaled by $\frac{1}{N^{3/2}}$ has a variance of order 1. In fact for $J=-\infty$ the **IRW** scaled by $\frac{1}{N^{3/2}}$ converges to the integrated Brownian motion, in [11] Caravenna and Deuschel use Donskers Invariance Principle to prove this. The integrated Brownian motion is a stochastic object and since we want to prove a principle of large deviations for the empirical profile on $(C,\|\cdot\|_\infty)$ we use the scaling $\frac{1}{N^2}$ instead of the scaling $\frac{1}{N^{3/2}}$.

We study the sequences

$$\gamma_N^{\mathbf{a}} := P^\psi \circ h_N^{-1},$$
$$\gamma_N^{\mathbf{r}} := \gamma_N^{\mathbf{r},-\infty} := \gamma_N^{\psi,-\infty} \circ h_N^{-1},$$
$$\hat{\gamma}_N^{\mathbf{r}} := \gamma_N^{\mathbf{r},J} := \gamma_N^{\psi,J} \circ h_N^{-1},$$

where $\psi=\psi^{\mathbf{r},N}$. For an illustration of the **IRW** and its linear interpolation see Figure 1.1. We say that the interface $(\gamma_N^{\mathbf{r}})_{N\in\mathbb{N}}$ has left boundary value a and right boundary value b because $N^{-2}\psi^{\mathbf{r},N}(0)\equiv a$ and $N^{-2}\psi^{\mathbf{r},N}(N)\equiv b$; we say that it has left gradient α and right gradient β because

$$\tfrac{N^{-2}\psi^{\mathbf{r},N}(0)-N^{-2}\psi^{\mathbf{r},N}(-1)}{N^{-1}} \equiv \alpha \quad \text{and} \quad \tfrac{N^{-2}\psi^{\mathbf{r},N}(N+1)-N^{-2}\psi^{\mathbf{r},N}(N)}{N^{-1}} \equiv \beta.$$

The first theorem in this thesis is concerned with the **LDP** for the case without pinning.

Theorem 1.2. *The sequences of measures* $(\gamma_N)_{N\in\mathbb{N}} = (\gamma_N^{\mathbf{a}})_{N\in\mathbb{N}}, (\gamma_N^{\mathbf{r}})_{N\in\mathbb{N}}$ *satisfy in* $(C(0,1), ||\cdot||_\infty)$ *the large deviation principles with speed* N *and good rate functions* $\Sigma^{\mathbf{a}}$ *and* $\Sigma^{\mathbf{r}}$ *of the form*

$$\Sigma(f) := \begin{cases} Q(f) - Q(H) & \text{, for } f \in H, \\ \infty & \text{, otherwise.} \end{cases} \tag{1.14}$$

Where $Q : H^2(0,1) \to \mathbb{R}$ *is the functional on the Sobolev space* H^2 *(the set of functions for which the second weak derivative is in* L^2*) defined by*

$$Q(f) := \frac{1}{2}\int_0^1 (\ddot{f}(\xi))^2 \,\mathrm{d}\,\xi,$$

where for every subset A *of* H^2 *the expression* $Q(A)$ *denotes the infimum of* Q *in* A*, and where* $H = H_{\mathbf{a}}$ *or* $H_{\mathbf{r}}$ *with*

$$\begin{aligned} H_{\mathbf{a}} &:= \{f \in H^2 \mid f(0) = a, \dot{f}(0) = \alpha\}, \\ H_{\mathbf{r}} &:= \{f \in H^2 \mid f(0) = a, \dot{f}(0) = \alpha, f(1) = b, \dot{f}(1) = \beta\}. \end{aligned} \tag{1.15}$$

For the case without terminal condition we can go beyond the case where the random variable X_1 from the random walk representation (1.2) has the standard normal distribution.

The central theorem of this thesis is the **LDP** for the model with pinning interaction, $(\gamma_N^{\mathbf{r};J})_{N\in\mathbb{N}}$. We will use the **LDP** for the model without pinning to prove the one for model with pinning. We will apply that the **LDP** for the model without pinning tells us that for a measurable subset A in $C(0,1)$ the probability of A under $\gamma_N^{\mathbf{r}}$ is for large N approximately $e^{-N\Sigma^{\mathbf{r}}(A)}$: Fix $\epsilon > 0$, by the **LDP** upper and lower bounds there is an N' such that for $N > N'$ we have

$$e^{-N(\Sigma^{\mathbf{r}}(A^o)+\epsilon)} \le \gamma_N^{\mathbf{r}}(A) \le e^{-N(\Sigma^{\mathbf{r}}(\bar{A})-\epsilon)}, \tag{1.16}$$

where $\bar{A}$ is the closure and A^o is the interior of the set A. Note that N' depends on the set A and the boundary condition $\mathbf{r}$. In our application of this approximation we have to take care of this dependence.

Theorem 1.3. *The sequence of measures* $(\gamma_N^{\mathbf{r};J})_{N\in\mathbb{N}}$ *satisfies in* $(C(0,1), ||\cdot||_\infty)$ *the* *large deviation principle with speed N and rate function*

$$\Sigma^{\mathbf{r},J}(f) := \begin{cases} \mathcal{E}^J(f) - \mathcal{E}^J(H_{\mathbf{r}}) & , \textit{for } f \in H_{\mathbf{r}}, \\ \infty & , \textit{otherwise}, \end{cases} \tag{1.17}$$

where $\mathcal{E}^J \colon H_{\mathbf{r}} \to \mathbb{R}$ is given by

$$\mathcal{E}^J(f) = \frac{1}{2}\int_0^1 (\ddot{f}(\xi))^2 \,\mathrm{d}\,\xi - \tau(J)|\mathcal{N}_f|, \tag{1.18}$$

where $|\mathcal{N}_f|$ is the Lebesgue measure of the zero set $\mathcal{N}_f := \{\xi \in [0,1] \mid f(\xi) = 0\}$ and $\mathcal{E}^J(H_{\mathbf{r}}) = \inf_{f\in H_{\mathbf{r}}} \mathcal{E}^J(f)$.

For the gradient model the **LDPs** for the corresponding models are known (see [5], [19]). To obtain them the scaled fields $(\frac{1}{N}\phi_{\xi N})_{\xi\in\overline{\Lambda_N}/N}$ are used and hence

$$h_N(\phi)(\xi) := \tfrac{1}{N}\phi_{\lfloor N\xi\rfloor} + (\xi - \tfrac{\lfloor N\xi\rfloor}{N})\tfrac{1}{N}(\phi_{\lfloor N\xi\rfloor+1} - \phi_{\lfloor N\xi\rfloor}) \quad , \text{for } \xi \in [0,1].$$

The boundary condition for the gradient model is given by

$$\psi^{\nabla,\mathbf{r},N}(i) := \begin{cases} aN & , \text{for } i = 0, \\ bN & , \text{for } i = N, \\ 0 & , \text{otherwise}, \end{cases}$$

where $\mathbf{r} = (a,b)$.

Funaki and Sakagawa [19] prove that the empirical profile of the gradient model $\gamma_N^{\nabla,J} \circ (h_N)^{-1}$ with boundary condition $\psi = \psi^{\nabla,\mathbf{r},N}$ satisfies an **LDP** in $(C(0,1), \|\cdot\|_\infty)$ with speed N and rate function

$$\Sigma^{\nabla,J}(f) := \begin{cases} \mathcal{E}^{\nabla,J}(f) - \mathcal{E}^{\nabla,J}(H) & , \text{for } f \in H, \\ \infty & , \text{otherwise}, \end{cases}$$

where H contains all functions from the Sobolev space H^1 that are equal to a at 0 and equal to b at 1, and where

$$\mathcal{E}^{\nabla,J}(f) = \frac{1}{2}\int_0^1 (\dot{f}(\xi))^2 \,\mathrm{d}\,\xi - \tau(J)|\mathcal{N}_f|.$$

The proof in [19] is based on an expansion of $\gamma_N^{\nabla,J}$ that is analogously to the expansion (1.6) and on the fact that the gradient model satisfies the splitting property with lag 1. These tools allow the authors of [19] to proof the **LDP** for $J > -\infty$ with the help of the **LDP** for $J = -\infty$. Funaki and Sakagawa [19] use the random walk representation to prove the **LDP** for $J = -\infty$ as follows: they use the Mogulskii theorem (see Theorem A.3) to derive an **LDP** for the random walk Y and since the process Y is Gaussian the well known Gaussian bridge (see Appendix B) allows the authors of [19] to implement the condition that $Y(N) = bN$ via the contraction principle.

To see the convenience of the splitting property with lag 1 for the proof of the **LDP** with $J > -\infty$, consider the term in the expansion that corresponds to the set $S = \Lambda_N \setminus \{p\}$: By the splitting property with lag 1 we have

$$e^J \mathcal{Z}_S^{\nabla,\psi} \gamma_S^{\nabla,\psi}(A) = e^J \mathcal{Z}_{\{1,2,\ldots,p-1\}}^{\nabla,\psi} \gamma_{\{1,2,\ldots,p-1\}}^{\nabla,\psi}(A) \mathcal{Z}_{\{p+1,p+2,\ldots,N\}}^{\nabla,\psi} \gamma_{\{p+1,p+2,\ldots,N\}}^{\nabla,\psi}(A).$$

The main reason why the authors of [19] can use the **LDP** for $J = -\infty$ to prove the **LDP** for $J > -\infty$ is that the Gibbs measure $\gamma_{\{1,2,\ldots,p-1\}}^{\nabla,\psi}$ where $\psi = \psi^{\nabla,\mathbf{r},N}$ coincides with the Gibbs measure $\gamma_{\Lambda_p}^{\nabla,\psi}$ where $\psi = \psi^{\nabla,\tilde{\mathbf{r}},p}$ and where $\tilde{\mathbf{r}} = (a\frac{N}{p}, 0)$. We can treat $\gamma_{\{p+1,p+2,\ldots,N\}}^{\nabla,\psi}$ analogously.

For the Laplacian model the measures only have a splitting property of lag 2 and hence using the **LDP** for $J = -\infty$ to prove the **LDP** for $J > -\infty$ is more complex.

In terms of the **LDP**, the main difference between the gradient and the Laplacian model is the smoothness of the minimisers of the rate functions: in the Laplacian case they have to be continuously differentiable (and the second weak derivative has to exists in L^2) whilst in the gradient case they do not need to be continuously differentiable.

Note that Funaki and Otobe [18] study the **LDP** for the gradient model under more general assumptions. For the models studied in [18] the random walk representation is not necessarily a Gaussian process. So the authors can not use the Gaussian bridge in their proof of the **LDP** for $J = -\infty$. Instead they use a change of measure approach. But to apply the change of measure the authors use that the gradient field satisfies a splitting property with lag 1. For a proof of the **LDP** for $J > -\infty$ the authors of [18] refer the reader to the proof of Funaki and Sakagawa [19].

The minimiser of the rate function is not unique

For our study whether the pinning reward J has an effect on the Laplacian model with non zero boundary conditions $\mathbf{r}$ we determine the set of minimisers $\mathcal{M}_{\mathbf{r}}^*$ of the rate function $\Sigma^{\mathbf{r},J}$ of the **LDP**. To see that the set $\mathcal{M}^*_{\mathbf{r}}$ contributes to our understanding of the behaviour of the interface model $(\hat{\gamma}^{\mathbf{r}}_N)_{N\in\mathbb{N}}$ for large N let $A \subset C(0,1)$ be an open set that contains $\mathcal{M}^*_{\mathbf{r}}$. By the **LDP** we have $\lim_{N\to\infty}\hat{\gamma}_N(A) = 1$. So if the set $\mathcal{M}^*_{\mathbf{r}}$ coincides with the minimiser of the rate function $\Sigma^{\mathbf{r},-\infty}$ of the model without pinning, then the pinning reward J has no effect on behaviour of the empirical profile. We will see that for non zero boundary condition $\mathbf{r} \neq 0$ there is always a critical reward strictly larger then J_c (recall that J_c is the reward until which the pinning free energy, with zero boundary conditions, is zero) such that for smaller rewards the pinning has no effect.

Furthermore, it turns out that there are boundary conditions such that the minimiser is not unique. In the extreme case the set of minimisers contains five functions, one of them has a zero set of Lebesgue measure zero and the other ones have a zero set of strictly positive Lebesgue measure. We present them in Section 4.

For the gradient case there are also boundary conditions such that the minimiser is not unique. Here up to two minimisers exist, one touching and one not touching zero.

Fixing boundary conditions such that two minimisers exist and considering for each minimiser an arbitrarily small ball around it we study the probability to observe an interface in one of these balls given that N is large. If for one ball the probability is arbitrarily close to one for large N, we say the empirical profile concentrates at the minimiser corresponding to this ball. For the gradient model Bolthausen, Funaki and Otobe [5] study the concentration behaviour of the empirical profile. They show that the probability to observe an interface in the neighbourhood of the minimiser that picks the reward is arbitrarily close to one while the probability to observe one near to the minimiser that does not pick the reward is close to zero.

For the Laplacian model we outline an approach for discussing the concentration in Chapter 5. We claim that concentration behaviour of the Laplacian model is completely opposite to the behaviour of the Gradient model: the probability to observe an interface in the neighbourhood of the minimiser that picks the reward is arbitrarily small while the probability to observe one near to the minimiser that does not pick the reward is close to one.

Fluctuations

For zero boundary conditions, the **LDP** tells us that the empirical profile for the Laplacian model is close to the zero line with a probability close to one. In other words it tells us that ϕ_i is typically smaller than N^2. For zero boundary conditions, Caravenna and Deuschel [11] show that the typical height of ϕ_i is $N^{3/2}$ for $J < J_c$, $O((\log N)^2)$ for $J > J_c$ and $O(\frac{N^{3/2}}{\log N})$ for $J = J_c$ (see [11, Theorem 1.2, Theorem 1.4]). In particular the authors of [11] show that the law of the linear interpolation of $\frac{1}{N^s}(\phi_{\xi N})_{\xi \in \overline{\Lambda_N}/N}$ where the scaling s is $\frac{3}{2}$ converges for $J < J_c$ in distribution to the law of the integrated Brownian motion bridge and for $J \geq J_c$ to the law concentrated on the constant function $f(\xi) = 0$ (see [11, Theorem 1.2]).

Related models

Models that are related to the (1+1)-dimensional models, are the (d+1)-dimensional models. The elements of the state space of these models are given by

$$\{\phi_i\}_{i \in \mathbb{Z}^d}, \text{ where } \phi_i \in \mathbb{R}.$$

Using the d dimensional discrete gradient and Laplacian, we can define $(d+1)$-dimensional models.

In statistical physics a natural next step after defining a family of Gibbs distributions is to study the existence of a so called *Gibbs measure* (see [20, Definition 2.9]). A Gibbs measure is a measure γ of a random field on $\mathbb{Z}^d$ that satisfies for all finite subsets $\Lambda \subset \mathbb{Z}^d$ that

$$\gamma(\cdot \mid \mathcal{F}_{\Lambda^c})[\psi] = \gamma_\Lambda^\psi \ \gamma\text{-a.s.},$$

where $\mathcal{F}_{\Lambda^c}$ is the σ-algebra generated by $\{\phi_j \mid j \notin \Lambda\}$. For the gradient model such a Gibbs measure exists if and only if $d \geq 3$ (see [20, Example 13.9]) whilst for the Laplacian model such a Gibbs measure exists if and only if $d \geq 5$ (see [27] or [25]). So for the gradient model the dimension $d = 2$ and for the Laplacian model the dimension $d = 4$ are the critical dimensions after which a Gibbs measure exists.

For $(d+1)$-dimensional models, localisation and delocalisation have been studied: for the gradient model we have $J_c = -\infty$ for all $d \geq 1$, see [29, Section 5], and for the Laplacian model it is shown in [28, Theorem 1] that $J_c = -\infty$ for $d \geq 4$.

An interesting phenomenon that has been studied for $(d+1)$-dimensional models is *entropic repulsion.* In statistical physics entropic repulsion referrers to the behaviour of an interface near to a wall (see [7]). An interface shows this behaviour if the interface is flat in the absence of a wall but divergent in the presence of a wall. Intuitively we explain this phenomenon by the fact that the interface that diverges from the wall has more space for fluctuation compared to the one that stays close to the wall. Mathematically, the wall is modelled by conditioning the Gibbs measure to stay positive above an area $\Lambda_N \subset \mathbb{Z}$, where the cardinality of Λ_N is of order N^d.

For the gradient model entropic repulsion has been studied in [4] for dimensions $d \geq 3$. Bolthausen and Deuschel [4] show that there is a constant C such that the interface $(\phi_i)_{i \in \Lambda_N}$ has a height of approximately $\sqrt{C \log N}$. In [4] the constant C is given explicitly. For the critical dimension $d = 3$ Bolthausen, Deuschel and Giacomin [3] prove that there is a constant C such that the maximum of the Gibbs distribution is pushed to the height $C \log N$.

For the Laplace model, the first rigorous results concerning entropic repulsion are in [27]. Sakagawa [27] proves lower and upper bounds for the probability to have positive heights. Kurt [25] proves an upper bound that asymptotically matches the lower bound of [27]. The author of [25] deduces that as for the gradient model there is a constant C such that for $d \geq 5$ the heights of the Laplacian model are approximately repelled to a level of $\sqrt{C \log N}$. Furthermore Kurt [26] considers the critical dimension $d = 4$: The local sample mean of the field is pushed to $C \log N$, where C is some constant.

A different generalisation are the $(1+s)$-dimensional models. The elements of the state space are given by

$$\{\phi_i\}_{i \in \mathbb{Z}}, \text{ where } \phi_i \in \mathbb{R}^s.$$

For the gradient model, localisation and delocalisation is studied in [5, Theorem 1.1].

A further direction of generalisation is to allow other interactions. For example Borecki [6] studies localisation and delocalisation for models with the Hamiltonian

$$\mathcal{H}(\phi) = \sum_i (\kappa_1 (\nabla \phi_i)^2 + \kappa_2 (\Delta \phi_i)^2),$$

where κ_1 and κ_2 are positive constants. The main observation for this $(\nabla + \Delta)$ model is that for $\kappa_1 > 0$ the model is localised for all $J > -\infty$, independently of the parameter κ_2. So if $\kappa_1 > 0$ the localisation behaviour of the $(\nabla + \Delta)$ model is the same as the one of the pure gradient model. The Laplacian interaction influences the $(\nabla + \Delta)$ model only if $\kappa_1 = 0$.

A further model that is related to the model with pinning interaction is the model with wetting interaction. The model with wetting interaction differs from the model with pinning interaction by the way the polymer interacts with the environment. Mathematically, the model with wetting interaction and the model with pinning interaction differ by the definition of the reference measure. We obtain the model for wetting by replacing all Lebesgue measures by Lebesgue measures on $[0, \infty)$. This is the same as conditioning the model with pinning interaction to stay in the positive half plane. Hence for models with wetting we should observe a competition between the effects of entropic repulsion and pinning. For the Laplacian model this was studied in [10] for $d = 1$. Caravenna and Deuschel [10] prove that the critical reward for the model with wetting interaction is strictly larger than the critical reward for the model with pinning interaction. Furthermore they show that the transition from delocalised to localised behaviour is of first order or in other words that the first derivative of the free energy of the model for wetting has a discontinuity, see [10, Theorem 1.3].

Overview

In Chapter 2 we prove the **LDPs** for the models without pinning, see Theorem 1.2. For the proof of the **LDP** for the model without terminal boundary conditions, $(\gamma_N^{\mathbf{a}})_{N\in\mathbb{N}}$, we use that, by the random walk representation (1.2), the the gradient of this field is a random walk with i.i.d. increments. We apply Mogulskii's theorem to get an **LDP** for this random walk. Then we use the contraction principle to extend the **LDP** for the random walk to an **LDP** for the integrated random walk.

The second model that we consider in Chapter 2 is the Laplacian model without pinning and with boundary condition zero on both sides. To obtain an **LDP** for this model we use the Gaussian bridge. We will see that the Gaussian bridge corresponds to a contraction map. In a last step we extend the **LDP** for zero boundary conditions to the **LDP** for non zero boundary condition. For this extension we show that for each non zero boundary condition there is a sequence of image measures of the measure with zero boundary conditions that is exponentially equivalent to the sequence with non zero boundary condition.

Funaki and Sakagawa [19] use this procedure to derive an **LDP** for the model with terminal boundary condition for the gradient model. But since the random walk representation of the gradient model without terminal boundary conditions is a random walk with i.i.d. increments, the **LDP** of the gradient model without terminal boundary conditions follows directly via Mogulskii's theorem. Furthermore the gradient model with terminal boundary conditions is only conditioned on one boundary point namely N while the Laplacian model is conditioned at two points namely at N and $N+1$. So the Gaussian bridge for the Laplacian model depends on $\phi(N)$ and $\phi(N+1)$, whilst the one for the gradient model does not depend on $\phi(N+1)$.

In Chapter 2 we also give extensions of Theorem 1.2 to **LDPs** for other intervals than $I = (0,1)$. We need these extensions in our proof of Theorem 1.3. In particular we need that the upper bounds that these **LDPs** imply for the probabilities of certain subsets of $C(0,1)$ (like the upper bound from (1.16)) hold uniformly for a certain family of intervals and boundary conditions.

In Chapter 3 we prove the **LDP** for the Laplacian model with pinning interaction, see Theorem 1.3. Therefore we use the two stage interpretation (1.6). For the gradient model, Funaki and Sakagawa [19] also use a two stage interpretation of the pinned measure. Since the gradient model satisfies the splitting property with lag 1, the authors of [19] can use the **LDP** for the model without pinning to prove the **LDP** for the model with pinning. For the Laplace model this property is not satisfied. Especially for the upper bound this forces us to use more complex methods than the authors of [19] did. In order to use the **LDP** for the model without pinning we apply a generalisation of the law of total expectation.

In Chapter 4 we study the minimisers of the rate function. For certain boundary conditions and rewards the minimiser is not unique and the set of minimisers contains up to five different minimisers.

In Chapter 5 we give an outlook and a conclusion. We present a possible approach for dealing with the concentration problem and describe some of the problems related to the wetting model.

We provide four appendices. In Appendix A we collect the results from large deviation theory that we apply in this thesis. Then, in Appendix B, we give some well known facts about finite dimensional Gaussian measures. Furthermore, in Appendix C, we analyse the partition function for the model without pinning. Finally, in Appendix D, we study the related minimisers of the rate function for the model without pinning and of the Hamiltonian.

Chapter 2

Integrated random walk

In this chapter we prove Theorem 1.2. In Section 2.1 we consider the empirical profiles of the models without terminal condition, $(\gamma_N^{\mathbf{a}})_{N\in\mathbb{N}}$. In Section 2.2 we consider the integrated random walk conditioned to have zero boundary conditions on both sides. In Section 2.3 we extend the **LDP** from Section 2.2 to models with none zero boundary condition. For this extension we use that the bridge of a Gaussian random walk is well known (see Appendix B). In our proof of Theorem 1.3 we use certain extensions of the Theorem 1.2, we present them in Section 2.4.

The approach to prove the **LDP** for the model with boundary conditions on both sides by first proving the one for the model with boundary conditions on only one side and then using the Gaussian bridge has been used already for the gradient model in [19]. In [12] the same procedure has been suggested for the Laplacian case.

2.1 Integrated random walk sample path large deviations

In this section we prove Theorem 1.2 for the models without terminal condition. Therefore we use the random walk representation (1.2). We study the empirical profile $h_N(\zeta)$ of the **IRW** ζ. Our proof works under a more general assumption than X_1 being Gaussian: It is enough if the log moment generating function $\Lambda(\lambda) := \log \mathbf{E}[e^{\lambda X_1}]$ is finite for all $\lambda \in \mathbb{R}$ and $\mathbf{E}[X_1] = 0$. We prove an **LDP** for the empirical profile of the **IRW** or in other words for $\vartheta_N^{\mathbf{a}} := P^{\psi} \circ h_N^{-1}$ where $\psi = \psi^{\mathbf{r},N}$ (for the definition of $\psi^{\mathbf{r},N}$ see (1.13)) and P^{ψ} is such that X_1 has a finite moment generating function.

Proposition 2.1. *The sequence of measures $(\vartheta_N^{\mathbf{a}})_{N\in\mathbb{N}}$ satisfies in $(C(0,1), \|\cdot\|_\infty)$ a* *large deviation principle with speed N and good rate function*

$$\Pi^{\mathbf{a}}(f) := \begin{cases} \int_0^1 \Lambda^*(\ddot{f}(\xi))\,\mathrm{d}\xi - \inf_{g\in\tilde{H}_{\mathbf{a}}} \int_0^1 \Lambda^*(\ddot{g}(\xi))\,\mathrm{d}\xi & , \textit{for } f \in \tilde{H}_{\mathbf{a}} \\ \infty & , \textit{otherwise}, \end{cases} \tag{2.1}$$

where $\tilde{H}_{\mathbf{a}}$ are the functions f such that the first derivative is absolutely continuous and such that $f(0) = a$ and $\dot{f}(0) = \alpha$, and where Λ^* is the Fenchel-Legendre transform of Λ:

$$\Lambda^*(\xi) := \sup_{\lambda\in\mathbb{R}}[\lambda\xi - \Lambda(\lambda)].$$

Proof. First we prove the proposition for $\mathbf{a} = \mathbf{0}$ and then we extend the proof to the cases $\mathbf{a} \in \mathbb{R}^2$. Finally we show the goodness of the rate function $\Pi^{\mathbf{a}}$.

Case $\mathbf{a} = \mathbf{0}$: For this case we use a small extension of Mogulskii's Theorem (see Theorem A.3) and the **contraction principle (CP)** (see Theorem A.4). Therefore note that $h_N(\zeta)$ is the image of $\xi \mapsto N^{-1}Y_{\lfloor N\xi\rfloor+1}$ under the integral operator:

$$h_N(\zeta)(\xi) = \tfrac{1}{N^2}\zeta_{\lfloor N\xi\rfloor} + \tfrac{1}{N^2}\int_{\frac{\lfloor N\xi\rfloor}{N}}^{\xi} (\zeta_{\lfloor Ns\rfloor+1} - \zeta_{\lfloor Ns\rfloor})\,\mathrm{d}s = \tfrac{1}{N}\int_0^{\xi} Y_{\lfloor Ns\rfloor+1}\,\mathrm{d}s. \tag{2.2}$$

The integral operator is a continuous map from $(L^\infty(0,1), \|\cdot\|_\infty)$ to $(C(0,1), \|\cdot\|_\infty)$ because it is linear and bounded. Additionally, by Proposition 2.2 below the sequence of laws of $(\xi \mapsto N^{-1}Y_{\lfloor N\xi\rfloor+1})_{N\in\mathbb{N}}$ satisfies an **LDP** in $(L^\infty(0,1), \|\cdot\|_\infty)$ with rate function I^M, where

$$I^M(f) := \begin{cases} \int_0^1 \Lambda^*(\dot{f}(\xi))\,\mathrm{d}\xi & , \text{for } f \in \mathcal{AC}, f(0) = 0, \\ \infty & , \text{otherwise}, \end{cases} \tag{2.3}$$

and where $\mathcal{AC}$ are the absolutely continuous functions. So, by the **CP**, the sequence $(\vartheta_N)_{N\in\mathbb{N}}$ satisfies an **LDP** with rate function

$$f \mapsto \inf_{g\in\mathcal{S}_f} I^M(g)\,, \text{ where } \mathcal{S}_f = \{g \in L^\infty(0,1) \mid \textstyle\int_0^\xi g(s)\,\mathrm{d}s = f(\xi)\,, \text{ for } \xi \in [0,1]\}. \tag{2.4}$$

By considering two cases we see that the functions (2.4) and $\Pi^{\mathbf{0}}$ are equal. If $f(0) \neq 0$ or if f is not differentiable, then we have $\mathcal{S}_f = \emptyset$; because the image of the integral operator is the set of differentiable functions with $f(0) = 0$. In the complementary case, $\mathcal{S}_f$ consists of the first derivative of f. So the values of the functions (2.4) and Π are equal for all $f \in C(0,1)$.

Case $\mathbf{a} \in \mathbb{R}^2$: For non zero boundary conditions we use that the **IRW** with zero boundary conditions and the one for the boundary condition $\psi^{\mathbf{r},N}$ differ by the linear trend $i \mapsto aN^2 + iN\alpha$. Hence for $N \to \infty$, the empirical profiles differ by $a + \xi\alpha$. This two observations allow us to prove the proposition by applying the exponential equivalence and the **CP**. Therefore we use the operators $\mathbb{K}_f \colon C(0,1) \to C(0,1)$ and $K_\varphi \colon \mathbb{R}^{\overline{\Lambda_N}} \to \mathbb{R}^{\overline{\Lambda_N}}$ given by

$$\mathbb{K}_f(h)(\xi) = h(\xi) - f(\xi)\,, \text{ for all}\, \xi \in [0,1]. \tag{2.5}$$

and

$$K_\varphi(\phi)_i = \phi_i - \varphi_i. \tag{2.6}$$

We show that $(\vartheta_N^{\mathbf{a}})_{N\in\mathbb{N}}$ and $(\vartheta_N^{\mathbf{0}} \circ \mathbb{K}_{h_{\mathbf{a}}})_{N\in\mathbb{N}}$, where $h_{\mathbf{a}}(\xi) = a + \xi\alpha$, are exponentially equivalent. Note

$$\vartheta_N^{\mathbf{a}} = P^\psi \circ h_N^{-1} = P^{\mathbf{0}} \circ K_{\phi_{\mathbf{a},N}} \circ h_N^{-1},$$

where $\phi_{\mathbf{a},N}(i) = aN^2 + iN\alpha$. The sequence $(P^{\mathbf{0}} \circ K_{\phi_{\mathbf{a},N}} \circ h_N^{-1})_{N\in\mathbb{N}}$ is exponentially equivalent with the sequence $(P^{\mathbf{0}} \circ h_N^{-1} \circ \mathbb{K}_{h_{\mathbf{a}}})_{N\in\mathbb{N}}$, because

$$\lim_{N\to\infty} \|K_{\phi_{\mathbf{a},N}} \circ (h_N)^{-1}(f) - (h_N)^{-1} \circ \mathbb{K}_{h_{\mathbf{a}}}(f)\|_\infty = 0 \quad \text{for all} \quad f \in C(0,1),$$

where the norm $\|\cdot\|_\infty$ is the uniform norm on $\mathbb{R}^{\overline{\Lambda_N}}$ (see Example A.6). Since we have $P^{\mathbf{0}} \circ h_N^{-1} \circ \mathbb{K}_{h_{\mathbf{a}}} = \vartheta_N^{\mathbf{0}} \circ \mathbb{K}_{h_{\mathbf{a}}}$ and since $(\vartheta_N^{\mathbf{0}} \circ \mathbb{K}_{h_{\mathbf{a}}})_{N\in\mathbb{N}}$ has, by the **CP**, the rate $\Pi^{\mathbf{a}}$ the exponential equivalence shows that $(\vartheta_N^{\mathbf{a}})_{N\in\mathbb{N}}$ also has the rate $\Pi^{\mathbf{a}}$.

Goodness of the rate function: The rate function $\Pi^{\mathbf{a}}$ is good because the rate function I^M is good and because goodness is preserved under the **CP**. (For the Gaussian case we present an alternative proof of the goodness in Section 2.4.1). □

Proof of Theorem 1.2 for models without terminal condition. Since for the normal distribution we have $\Lambda^*(x) = \frac{1}{2}x^2$, Proposition 2.1 implies that Theorem 1.2 is true for $(\gamma_N^{\mathbf{a}})_{N\in\mathbb{N}}$. □

The following proposition is a small adaptation of Mogulskii's theorem (see Theorem A.3).

Proposition 2.2. *Under $P^{\mathbf{0}}$, the sequence of laws of $(\xi \mapsto N^{-1}Y_{\lfloor N\xi\rfloor+1})_{N\in\mathbb{N}}$ satisfies an **LDP** on $(L^\infty(0,1), \|\cdot\|_\infty)$ with rate function I^M; where I^M is given in* (2.3) *above.*

Proof. Recall that by Mogulskii's theorem (see Theorem A.3), the sequence of laws of $(\xi \mapsto N^{-1}Y_{\lfloor N\xi\rfloor})_{N\in\mathbb{N}}$ satisfies in $(L^\infty(0,1), \|\cdot\|_\infty)$ the large deviation principle with the rate function I^M. We prove that I^M is also the rate function for the sequence of laws of $(\xi \mapsto N^{-1}Y_{\lfloor N\xi\rfloor+1})_{N\in\mathbb{N}}$, by proving that $(\xi \mapsto N^{-1}Y_{\lfloor N\xi\rfloor+1})_{N\in\mathbb{N}}$ and $(\xi \mapsto N^{-1}Y_{\lfloor N\xi\rfloor})_{N\in\mathbb{N}}$ are exponential equivalent (see Definition A.5 and Theorem A.7).

For all $\xi \leq 1$, the difference $|N^{-1}Y_{\lfloor N\xi\rfloor+1} - N^{-1}Y_{\lfloor N\xi\rfloor}| = |N^{-1}X_{\lfloor N\xi\rfloor+1}|$ is bounded from above by the maximum $N^{-1}\max_{i\leq N+1}|X_i|$, and hence for any $\eta > 0$, we have

$$P^{\mathbf{0}}(\|N^{-1}Y_{\lfloor N\xi\rfloor+1} - N^{-1}Y_{\lfloor N\xi\rfloor}\|_\infty > \eta) \leq P^{\mathbf{0}}(N^{-1}\max_{i\leq N+1}|X_i| > \eta).$$

Exponential equivalence follows by Proposition 2.3 below. □

We frequently use the following result to prove exponential equivalence.

Proposition 2.3. *For all $\eta > 0$,*

$$\limsup_{N\to\infty} \tfrac{1}{N} \log P^\psi(\max_{i\leq N+1}|X_i| > \eta N) = -\infty.$$

Proof. Since the distribution of the random variable X_1 is not effected by the boundary condition, we use the notation $P = P^\psi$ in this proof. For every $\lambda > 0$, by exponential Chebyshev's inequality,

$$P(\max_{i\leq N+1}|X_i| > \eta N) \leq (N+1)P(|X_1| > \eta N) \leq (N+1)\mathbf{E}[e^{\lambda|X_1|}]e^{-\lambda N\eta}.$$

Hence,

$$\begin{aligned}\limsup_{N\to\infty} \tfrac{1}{N} \log P(\max_{i\leq N}|X_i| > \eta N) &\leq \limsup_{N\to 0} \tfrac{1}{N} \log((N+1)\mathbf{E}[e^{\lambda|X_1|}]e^{-\lambda N\eta}) \\ &\leq -\lambda\eta,\end{aligned}$$

where we used $\mathbf{E}[e^{\lambda|X_1|}] < \infty$ for all λ. Let $\lambda \to \infty$ to finish the proof. □

2.2 Integrated random walk bridge sample path large deviations, with zero boundary condition

In this and the following section we prove Theorem 1.2 for $\gamma_N^{\mathbf{r}}$. In this section we assume $\mathbf{r} = \mathbf{0}$ and write $P := P^{\mathbf{0}}$.

Essential to our proof is that for the Gaussian measure P we can calculate explicitly a map $\mathbf{B}_N$ such that $P_N = P \circ \mathbf{B}_N^{-1}$ (see Section B). Note that for general distributions other methods to obtain $\mathbf{B}_N$ are necessary and hence we also need other methods to prove the **LDP**.

We use the maps $(\mathbf{B}_N)_{N\in\mathbb{N}}$ to show that there is a continuous map $\mathscr{B}$ such that

$$(P \circ \tilde{h}_N^{-1} \circ \mathscr{B}^{-1})_{N\in\mathbb{N}} \text{ and } (\gamma_N^{\mathbf{0}})_{N\in\mathbb{N}} = (P \circ \mathbf{B}_N^{-1} \circ h_N^{-1})_{N\in\mathbb{N}}$$

are exponentially equivalent, where $\tilde{h}_N : \mathbb{R}^{\mathbb{N}} \to C(0,1) \times \mathbb{R}$ is given by

$$\tilde{h}_N(\zeta) := (h_N(\zeta), \frac{1}{N}(\zeta(N+1) - \zeta(N)). \tag{2.7}$$

We present such a map $\mathscr{B}$ in the next proposition.

Proposition 2.4. *Let $\tilde{L} := C(0,1) \times \mathbb{R}$ with the norm $\|(f,v)\| = \|f\|_\infty + |v|$. The sequences $(\gamma_N)_{N\in\mathbb{N}}$ and $(\tilde{\vartheta}_N)_{N\in\mathbb{N}}$ are exponentially equivalent, where*

$$\tilde{\vartheta}_N := P \circ \tilde{h}_N^{-1} \circ \mathscr{B}^{-1},$$

and

1. *$\tilde{h}_N$ is given in (2.7),*
2. *$\mathscr{B}\colon \tilde{L} \to C(0,1)$ is the continuous map given by*

$$\mathscr{B}(f,v)(\xi) = f(\xi) - \mathcal{A}(\xi, f(1), v),$$

where $\mathcal{A}\colon \mathbb{R}^3 \to \mathbb{R}$ is given by

$$\mathcal{A}(\xi,u,v) = (3u - v)\xi^2 + (-2u + v)\xi^3.$$

We prove Proposition 2.4 below.

Proof of Theorem 1.2 for $\gamma_N^{\mathbf{r}}$ with $\mathbf{r} = \mathbf{0}$. By Proposition 2.4 it is enough to prove that $(\tilde{\vartheta}_N)_{N\in\mathbb{N}}$ has the rate $\Sigma^{\mathbf{0}}$. Since $\mathscr{B}$ is continuous we first obtain an **LDP** for $(P \circ \tilde{h}_N^{-1})_{N\in\mathbb{N}}$ in $\tilde{L}$ and apply the **CP** in a second step.
Step 1: We show that $(P \circ \tilde{h}_N^{-1})_{N\in\mathbb{N}}$ satisfies an **LDP** in $\tilde{L}$ with rate function

$$\tilde{\Sigma}(f,v) = \begin{cases} \frac{1}{2}\int_0^1 (\ddot{f}(\xi))^2 \,\mathrm{d}\,\xi & \text{, for } f \in H_{(0,0)}, \dot{f}(1) = v, \\ \infty & \text{, otherwise.} \end{cases} \tag{2.8}$$

We use the **CP** to prove (2.8) because $P \circ \tilde{h}_N^{-1}$ coincides with the law of $\xi \mapsto \Phi(N^{-1}Y_{\lfloor N\xi \rfloor + 1})$ where

$$\Phi(f) := ((\int_0^\xi f(s)\,\mathrm{d}s)_{\xi\in[0,1]}, f(1));$$

to verify this recall (2.2). Note that the linear map $\Phi\colon L^\infty \to \tilde{L}$ is continuous because it is bounded due to

$$\|\Phi(f)\| \le \|f\|_\infty + |f(1)| \le 2\|f\|_\infty.$$

So, by the **CP** and since $I^M(g) = \infty$ if $g \notin \mathcal{AC}$, the rate function of $(P \circ \tilde{h}_N^{-1})_{N\in\mathbb{N}}$ is given by $(f, v) \mapsto \inf_{g\in\mathcal{S}_{(f,v)}} I^M(g)$, where

$$\mathcal{S}_{(f,v)} := \{g \in \mathcal{AC} \mid \int_0^t g(\xi)\,\mathrm{d}\xi = f(t) \text{ for all } t \in [0,1], g(1) = v\}.$$

Since we have $\mathcal{S}_{(f,v)} \neq \emptyset$ only if there is a $g \in \mathcal{AC}$ such that $\dot{f} = g$ and hence such that in particular $\dot{f}(1) = g(1) = v$, we have

$$\mathcal{S}_{(f,v)} = \begin{cases} \{\dot{f}\} & \text{, for } \dot{f} \in \mathcal{AC}, f(0) = 0, \dot{f}(1) = v, \\ \emptyset & \text{, otherwise.} \end{cases} \tag{2.9}$$

Since for the normal distribution we have $\Lambda^*(x) = \frac{1}{2}x^2$ and $I^M(\dot{f}) = \infty$ if $\dot{f}(0) \neq 0$ the maps (2.8) and $(f, v) \mapsto \inf_{g\in\mathcal{S}_{(f,v)}} I^M(g)$ coincide.

Step 2: Now we use (2.8) to show that the rate function of $(\tilde{\vartheta}_N)_{N\in\mathbb{N}}$ coincides with $\Sigma^{\mathbf{0}}$. Since $\mathscr{B}$ is continuous the **CP** yields that the rate function of $(\tilde{\vartheta}_N)_{N\in\mathbb{N}}$ is given by $f \mapsto \inf_{(g,v)\in\mathcal{S}_f} \tilde{\Sigma}(g, v)$, where

$$\mathcal{S}_f = \{(g, v) \in \tilde{L} \mid \mathscr{B}(g, v) = f\}.$$

Since $\tilde{\Sigma}(g, v) = \infty$ if $g \notin H_{(0,0)}$ or if $\dot{g}(1) \neq v$, we have

$$\inf_{(g,v)\in\mathcal{S}_f} \tilde{\Sigma}(g, v) = \inf_{g\in\tilde{\mathcal{S}}_f} \tilde{\Sigma}(g, \dot{g}(1)),$$

where

$$\tilde{\mathcal{S}}_f := \{g \in H_{(0,0)} \mid \mathscr{B}(g, \dot{g}(1)) = f\}.$$

We show

$$\tilde{\mathcal{S}}_f = \begin{cases} \{f + \mathcal{A}(\cdot, u, v) \mid (u, v) \in \mathbb{R}^2\} & \text{, for } f \in H_{(0,0,0,0)}, \\ \emptyset & \text{, otherwise.} \end{cases} \tag{2.10}$$

Therefore note, that the map $f \mapsto \mathscr{B}(f, \dot{f}(1))$ is the orthogonal projection of

$(H_{(0,0)}, < \cdot, \cdot >)$ to $H_{(0,0,0,0)}$, where $< \cdot, \cdot >$ is the semi-inner-product given by

$$< f, g >:= \int_0^1 \ddot{f}(\xi)\ddot{g}(\xi)\, \mathrm{d}\xi.$$

To check this note that the range of the map is actually $H_{(0,0,0,0)}$ and that, by Proposition D.1, the difference $f - \mathscr{B}(f, \dot{f}(1)) = \mathcal{A}(\cdot, f(1), \dot{f}(1))$ minimises $f \mapsto < f, f >$ in $H_{(0,0,f(1),\dot{f}(1))}$.

Now we use this property of $\mathscr{B}$ to show that (2.10) is true. We first consider the case that f is not an element of $H_{(0,0,0,0)}$. Since the image of $H_{(0,0)}$ under $\mathscr{B}$ is $H_{(0,0,0,0)}$, the equation $\mathscr{B}(g, \dot{g}(1)) = f$ has no solution g in $H_{(0,0)}$ if f is not an element of $H_{(0,0,0,0)}$, and hence for those f we have $\tilde{\mathcal{S}}_f = \emptyset$. This implies that the left and right hand side of (2.10) coincide in the case that f is not an element of $H_{(0,0,0,0)}$. Now we consider the case $f \in H_{(0,0,0,0)}$. Since the kernel of the orthogonal projection $g \mapsto \mathscr{B}(g, \dot{g}(1))$ is $\{\mathcal{A}(\cdot, u, v) \mid (u, v) \in \mathbb{R}^2\}$, the left and right hand side of (2.10) also coincide if f is an element of $H_{(0,0,0,0)}$.

For $f \in H_{(0,0,0,0)}$ we actually have

$$\inf_{g \in \tilde{\mathcal{S}}_f} \tilde{\Sigma}(g, \dot{g}(1)) = \tfrac{1}{2} \int_0^1 (\ddot{f}(\xi))^2 \, \mathrm{d}\, \xi. \tag{2.11}$$

To check (2.11) fix $g \in \tilde{\mathcal{S}}_f$ and note that by (2.10) there is a vector $(u, v) \in \mathbb{R}^2$ such that $g(\xi) = f(\xi) + \mathcal{A}(\xi, u, v)$ and hence

$$\tilde{\Sigma}(g, \dot{g}(1)) = \tfrac{1}{2} \int_0^1 (\ddot{f}(\xi))^2 \, \mathrm{d}\, \xi + \int_0^1 \ddot{f}(\xi)\ddot{\mathcal{A}}(\xi, u, v) \, \mathrm{d}\, \xi + \tfrac{1}{2} \int_0^1 (\ddot{\mathcal{A}}(\xi))^2 \, \mathrm{d}\, \xi.$$

By Proposition D.1 the minimiser $\mathcal{A}(\cdot, u, v)$ is a polynomial of degree 3 and hence

$$\int_0^1 \ddot{f}(\xi)\ddot{\mathcal{A}}(\xi, u, v) \, \mathrm{d}\, \xi = 0 \text{ , for } f \in H_{(0,0,0,0)}.$$

Furthermore,

$$\int_0^1 (\ddot{\mathcal{A}}(\xi, u, v))^2 \, \mathrm{d}\, \xi \geq 0$$

for all $(u, v) \in \mathbb{R}^2$ with equality if and only if $(u, v) = (0, 0)$. Combining these three facts we see that (2.11) is true. □

Proof of Proposition 2.4. **Step 1:**

We show that there is a map $\mathbf{B}_N$ such that

$$P_N = P \circ \mathbf{B}_N^{-1}. \tag{2.12}$$

Therefore we use the well known formula for Gaussian bridges (see [13] and Appendix B). For this purpose the alternative definition of P_N given by

$$P_N(\cdot) = P(\cdot|\zeta_N = 0, \zeta_{N+1} - \zeta_N = 0) \tag{2.13}$$

is more convenient because $\zeta_{N+1} - \zeta_N = Y_{N+1}$. By (B.1), we have

$$\mathbf{B}_N(\zeta)(i) = \zeta_i - \begin{bmatrix} C_{i,N} & C_{i,N+1} \end{bmatrix} \begin{bmatrix} C_{N,N} & C_{N,N+1} \\ C_{N,N+1} & C_{N+1,N+1} \end{bmatrix}^{-1} \begin{bmatrix} \zeta_N \\ \zeta_{N+1} - \zeta_N \end{bmatrix}, \text{ for } i \le N,$$

where $C_{i,N} = \mathbf{E}[\zeta_i \zeta_N], C_{i,N+1} = \mathbf{E}[\zeta_i(\zeta_{N+1} - \zeta_N)]$ for all $i \in \{1, 2, \ldots, N\}$ and $C_{N+1,N+1} = \mathbf{E}[(\zeta_{N+1} - \zeta_N)^2] = (N+1)$ (where the expectations are with respect to P). Since

$$C_{i,N} = \mathbf{E}[\zeta_i \zeta_N] = \frac{1}{6}(-i^3 + 3Ni^2 + i(3N+1)) \text{ , for } i \le N$$

and $C_{i,N+1} = \mathbf{E}[Y_{N+1} \sum_{x=1}^{i} Y_x] = \frac{1}{2} i(i+1)$ for $i \le N$, we have

$$\mathbf{B}_N(\zeta)(x) = \zeta(x) - \mathbf{A}_N(x, \zeta(N), \zeta(N+1) - \zeta(N)), \text{ for all } x \in \{1, 2, \ldots, N, N+1\}, \tag{2.14}$$

where $\mathbf{A}_N \colon \{1, 2, \ldots, N, N+1\} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is given by

$$\begin{aligned} \mathbf{A}_N(x, U, V) := \frac{1}{N(N+1)(N+2)} \big\{ & x^3[-2U + NV] + x^2[3UN + VN - VN^2] \\ & + x[(2 + 3N)U - N^2 V] \big\} . \end{aligned} \tag{2.15}$$

Step 2 By (2.12), the sequences $(\gamma_N)_{N \in \mathbb{N}}$ and $(\tilde{\vartheta}_N)_{N \in \mathbb{N}}$ are exponential equivalent, if under P for any $\eta > 0$ the probability that

$$\mathscr{B}(\tilde{h}_N(\zeta))(\xi) - h_N(\mathbf{B}_N(\zeta))(\xi) \tag{2.16}$$

has a $\|\cdot\|_\infty$-norm larger than η decays with an logarithmic rate of $-\infty$. Let $u_N = N^{-2}\zeta(N)$ and $v_N = N^{-1}(\zeta(N+1) - \zeta(N))$. By definition, (2.16) is equal to

$$\begin{aligned} & h_N \left[\mathbf{A}_N(i, \zeta(N), \zeta(N+1) - \zeta(N)) \right] - \mathcal{A}\left(\xi, \tfrac{\zeta(N)}{N^2}, \tfrac{\zeta(N+1) - \zeta(N)}{N} \right) \\ & = \left[\tfrac{1}{N^2} \mathbf{A}_N(\xi N, \zeta(N), \zeta(N+1) - \zeta(N) v) - \mathcal{A}\left(\xi, \tfrac{\zeta(N)}{N^2}, \tfrac{\zeta(N+1) - \zeta(N)}{N} \right) \right] \\ & + \left[h_N \left[\mathbf{A}_N(i, \zeta(N), \zeta(N+1) - \zeta(N)) \right] - \tfrac{1}{N^2} \mathbf{A}_N(\xi N, \zeta(N), \zeta(N+1) - \zeta(N)) \right] . \end{aligned} \tag{2.17}$$

Hence, by applying Proposition 2.5 below to the first term on the right hand side of (2.17) and by applying the definition of h_N to the second term on the right hand side of (2.17), we see that (2.16) is

$$O(\tfrac{1}{N})\max\left(|\tfrac{\zeta(N)}{N^2}|, |\tfrac{\zeta(N+1)-\zeta(N)}{N}|\right) \quad , \text{for } N \to \infty.$$

Since $\zeta(N) \leq N^2 \max_{i\leq N}|X_i|$ and $\zeta(N+1)-\zeta(N) \leq N \max_{i\leq N+1}|X_i|$, (2.16) is $O(\frac{1}{N})\max_{i\leq N+1}|X_i|$ and hence there is a C such that

$$P(\|(\mathscr{B}(\tilde{h}_N(\zeta)))(\xi) - h_N(\mathbf{B}_N(\zeta))(\xi)\|_\infty > \eta) \leq P(CN^{-1}\max_{i\leq N+1}|X_i| > \eta).$$

Now the exponential equivalence follows by Proposition 2.3. □

Proposition 2.5. *For* $(u,v)\in\mathbb{R}^2$ *and* $\xi\in[0,1]$

$$|\tfrac{1}{N^2}\mathbf{A}_N(\xi N, N^2u, Nv) - \mathcal{A}(\xi,u,v)| = O(\tfrac{1}{N})\max(|u|,|v|)\,. \tag{2.18}$$

Proof. We consider the coefficients of the polynomial $\xi \mapsto N^{-2}\mathbf{A}_N(\xi N, N^2u, Nv)$: By definition of $\mathbf{A}_N$,

- the coefficient of the leading term is

$$\begin{aligned}\frac{N^3[-2uN^2+vN^2]}{N^3(N+1)(N+2)} &= [-2u+v]\frac{N^2}{(N+1)(N+2)}\\ &= (-2u+v)+O(\tfrac{1}{N})\max(|u|,|v|) \quad , \text{for } N\to\infty,\end{aligned} \tag{2.19}$$

- the second order term is

$$\begin{aligned}\frac{N^2[3uN^3+vN^2-vN^3]}{N^3(N+1)(N+2)} &= [3u-v+\tfrac{v}{N}]\frac{N^2}{(N+1)(N+2)}\\ &= (3u-v)+O(\tfrac{1}{N})\max(|u|,|v|) \quad , \text{for } N\to\infty,\end{aligned} \tag{2.20}$$

- the first order term is

$$\frac{N[2uN^2+3uN^3-vN^3]}{N^3(N+1)(N+2)} = O(\tfrac{1}{N})\max(|u|,|v|) \quad , \text{for } N\to\infty, \tag{2.21}$$

- and the constant term is zero.

Combining the definition of $\mathcal{A}$ with (2.19), (2.20) and (2.21), we see that (2.18) is true. □

2.3 Integrated random walk bridge sample path large deviations, non zero boundary condition

In this section we give the remainder of the proof of Theorem 1.2. We extend the part of Theorem 1.2 that we already proved to the case where the boundary conditions are not zero. In the following we provide the central tool for this extension.

Lemma 2.6. *The sequences* $(\gamma_N^{\mathbf{r}})_{N\in\mathbb{N}}$ *and* $(\gamma_N^{\mathbf{0}}\circ\mathbb{K}_{h_{\mathbf{r}}})_{N\in\mathbb{N}}$ *are exponentially equivalent in* $C(0,1)$*, where* $h_{\mathbf{r}}$ *is the unique minimiser of* Q *in the set* $H_{\mathbf{r}}$*, and where the operator* $\mathbb{K}_f\colon C(0,1)\to C(0,1)$ *is given in* (2.5)*.*

Proof. Central to our proof is that $\gamma_N^{\mathbf{r}}$ and $\gamma_N^{\mathbf{0}}\circ\mathbb{K}_{h_{\mathbf{r}}}$ are image measures of $\gamma_N^{\mathbf{0}}$: In fact, as we will show at the end of this proof, we have

$$
\begin{aligned}
(\gamma_N^{\mathbf{r}})_{N\in\mathbb{N}} &= (\gamma_N^{\mathbf{0}}\circ K_{\phi_{\mathbf{r}}}\circ(h_N)^{-1})_{N\in\mathbb{N}}\\
(\gamma_N^{\mathbf{0}}\circ\mathbb{K}_{h_{\mathbf{r}}})_{N\in\mathbb{N}} &= (\gamma_N^{\mathbf{0}}\circ(h_N)^{-1}\circ\mathbb{K}_{h_{\mathbf{r}}})_{N\in\mathbb{N}},
\end{aligned}
\tag{2.22}
$$

where $\phi_{\mathbf{r}}$ is the minimiser of $\mathcal{H}_N$ in the set that satisfies the boundary condition $\psi=\psi^{\mathbf{r},N}$ (see Proposition D.2, note that we drop the index N here and write $\phi_{\mathbf{r},N}=\phi_{\mathbf{r}}$) and where $K_\varphi\colon\mathbb{R}^{\overline{\Lambda_N}}\to\mathbb{R}^{\overline{\Lambda_N}}$ is given in (2.6).

Exponential equivalence follows from (2.22) because, by Proposition D.2, the sequence $(h_N(\phi_{\mathbf{r}}))_{N\in\mathbb{N}}$ converges to $h_{\mathbf{r}}$ and hence

$$
\lim_{N\to\infty}\|K_{\phi_{\mathbf{r}}}\circ(h_N)^{-1}(f)-(h_N)^{-1}\circ\mathbb{K}_{h_{\mathbf{r}}}(f)\|_\infty=0\quad\text{for all}\quad f\in C,
\tag{2.23}
$$

where the norm $\|\cdot\|_\infty$ is the uniform norm on $\mathbb{R}^{\overline{\Lambda_N}}$.

Statement (2.23) shows that (2.22) is sufficient to prove the exponential equivalence, now we show that (2.22) is actually satisfied. Therefore note that by the definition of $\gamma_N^{\mathbf{0}}$ the image measure $\gamma_N^{\mathbf{0}}\circ\mathbb{K}_{h_{\mathbf{r}}}$ has the claimed form. It remains to show that

$$
\gamma_N^{\psi}=\gamma_N^{\mathbf{0}}\circ K_{\phi_{\mathbf{r}}}\,,\ \text{for}\ \psi=\psi^{\mathbf{r},N}.
\tag{2.24}
$$

To prove (2.24) we apply the orthogonality property of $\phi_{\mathbf{r}}$ (see Proposition D.2): In the first place it implies that

$$\begin{aligned}
\mathcal{Z}_N^{\psi} &= \int_{\mathbb{R}^{\Lambda_N}} e^{-\mathcal{H}_N(\phi)} \prod_{i\in\Lambda_N} (\mathrm{d}\,\phi_i) \prod_{i\in\{-1,0,N,N+1\}} \delta_{\psi_i}(\mathrm{d}\,\phi_i) \\
&= \int_{\mathbb{R}^{\Lambda_N}} e^{-\mathcal{H}_N(\phi-\phi_{\mathbf{r}})-\mathcal{H}_N(\phi_{\mathbf{r}})} \prod_{i\in\Lambda_N} \mathrm{d}\,\phi_i \prod_{i\in\{-1,0,N,N+1\}} \delta_{\psi_i}(\mathrm{d}\,\phi_i) \\
&= e^{-\mathcal{H}_N(\phi_{\mathbf{r}})} \int_{\mathbb{R}^{\Lambda_N}} e^{-\mathcal{H}_N(\tilde{\phi})} \prod_{i\in\Lambda_N} (\mathrm{d}\,\tilde{\phi}_i) \prod_{i\in\{-1,0,N,N+1\}} \delta_0(\mathrm{d}\,\tilde{\phi}_i) \\
&= e^{-\mathcal{H}_N(\phi_{\mathbf{r}})} \mathcal{Z}_N^{\mathbf{0}},
\end{aligned} \tag{2.25}$$

where in the third line we substituted $\phi-\phi_{\mathbf{r}} = K_{\phi_{\mathbf{r}}}(\phi)$ by $\tilde{\phi}$; and in the same fashion the orthogonality property implies that

$$\begin{aligned}
&\gamma_N^{\psi}(A) \\
&= \frac{1}{\mathcal{Z}_N^{\psi}} \int_A e^{-\mathcal{H}_N(\phi)} \prod_{i\in\Lambda_N} (\mathrm{d}\,\phi_i) \prod_{i\in\{-1,0,N,N+1\}} \delta_{\psi_i}(\mathrm{d}\,\phi_i) \\
&= \frac{e^{\mathcal{H}_N(\phi_{\mathbf{r}})}}{\mathcal{Z}_N^{\mathbf{0}}} \int_A e^{-\mathcal{H}_N(\phi-\phi_{\mathbf{r}})-\mathcal{H}_N(\phi_{\mathbf{r}})} \prod_{i\in\Lambda_N} \mathrm{d}\,\phi_i \prod_{i\in\{-1,0,N,N+1\}} \delta_{\psi_i}(\mathrm{d}\,\phi_i) \\
&= \frac{1}{\mathcal{Z}_N^{\mathbf{0}}} \int_{K_{\phi_{\mathbf{r}}}(A)} e^{-\mathcal{H}_N(\phi)} \prod_{i\in\Lambda_N} (\mathrm{d}\,\phi_i) \prod_{i\in\{-1,0,N,N+1\}} \delta_0(\mathrm{d}\,\phi_i) \\
&= \gamma_N^{\mathbf{0}} \circ K_{\phi_{\mathbf{r}}}(A),
\end{aligned}$$

for any measurable set A. □

Proof of Theorem 1.2. By Lemma 2.6 and the **CP** the rate function of the model with $J = -\infty$ and boundary condition $\mathbf{r}$ is given by

$$f \mapsto \Sigma^{\mathbf{0}}(\mathbb{K}_{h_{\mathbf{r}}}^{-1}(f)) = \Sigma^{\mathbf{0}}(f - h_{\mathbf{r}}).$$

This rate function is only finite if $f-h_{\mathbf{r}} \in H_{\mathbf{0}}$ or in other words if $f \in H_{\mathbf{r}}$. For $f \in H_{\mathbf{r}}$ we have by definition of $\Sigma^{\mathbf{0}}$ that $\Sigma^{\mathbf{0}}(f-h_{\mathbf{r}}) = Q(f-h_{\mathbf{r}})$. By the orthogonality of $h_{\mathbf{r}}$ (see Proposition D.1) we see that this rate function and the one from Theorem 1.2 coincide.

□

2.4 Preparation for the proof of Theorem 1.3

In this section we present results that we use in our proof of Theorem 1.3. In Section 2.4.1, we prove the goodness of the rate functions $\Sigma^{\mathbf{a}}$ and $\Sigma^{\mathbf{r}}$. In our proof of Theorem 1.3, we use the expansion (1.6). Therefore we need upper and lower bounds to $\gamma_S^\psi(\mathcal{Q})$ where S is of the form $S = \{s_* + 1, s_* + 2, \ldots, s^* - 1\} =: (s_*, s^*)$ for $s_*, s^* \in \mathbb{N}$. In Section 2.4.2, we extend the **LDP** from Theorem 1.2 such that we can apply it to the measures γ_S^ψ. Finally we show an uniformity result for the upper bounds derived from this extension of the **LDP**, see Section 2.4.3.

2.4.1 Goodness of the rate function

We prove that $\Sigma^{\mathbf{a}}$ and $\Sigma^{\mathbf{r}}$ are good rate functions. Note that this follows already from the fact that the contraction principle preserves the goodness of the rate function. The main reason for presenting the proof as follows is that we use the same techniques to prove the goodness of $\Sigma^{\mathbf{r},J}$ for $J > -\infty$.

To show that $\Sigma^{\mathbf{a}}$ and $\Sigma^{\mathbf{r}}$ are good rate function we have to show (by definition of goodness) that they are lower semicontinuous function with compact level sets. By the definitions of $\Sigma^{\mathbf{a}}$ and $\Sigma^{\mathbf{r}}$ (see Theorem 1.2) it is sufficient to show that Q has compact level sets $(L_\kappa)_{\kappa\in\mathbb{R}}$ in $(C(0,1), \|\cdot\|_\infty)$, where

$$L_\kappa := \{f \in H_{\mathbf{a}} \mid Q(f) \le \kappa\}.$$

We start by showing that the sets $(L_\kappa)_{\kappa\in\mathbb{R}}$ are closed.

Lemma 2.7. *The level sets $(L_\kappa)_{\kappa\in\mathbb{R}}$ are closed in $(C(0,1), \|\cdot\|_\infty)$.*

Proof. Fix a level set L_κ and let $(h_n)_{n\in\mathbb{N}}$ be a uniformly converging sequence in L_κ with limit $h \in C(0,1)$. We prove that $h \in L_\kappa$. The sequence $(\|\ddot{h}_n\|_{L^2}^2)_{n\in\mathbb{N}}$ is bounded by 2κ because $(h_n)_{n\in\mathbb{N}} \subset L_\kappa$ and $Q(h_n) = \frac{1}{2}\|\ddot{h}_n\|_{L^2}^2$. Since L^2 is a reflexive space there is a subsequence such that $(\ddot{h}_{n_k})_{k\in\mathbb{N}}$ converges weakly in L^2 to a function g. By weak convergence we have

$$\|g\|_{L^2} \le \liminf \|\ddot{h}_{n_k}\|_{L^2}. \tag{2.26}$$

We show $g = \ddot{h}$: Since the boundedness of $(\|\ddot{h}_n\|_{L^2})_{n\in\mathbb{N}}$ implies by

$$[\dot{h}_n(\xi) - \dot{h}_n(0)]^2 = [\int_0^\xi \ddot{h}_n(s)\mathrm{d}\,s]^2 \le \|\ddot{h}_n\|_{L^2}^2$$

and by $h \in H_{\mathbf{r}}$ the boundedness of $(\|\dot{h}_{n_k}\|_{L^2})_{k\in\mathbb{N}}$, there is a weakly convergent sub sequence $(\dot{h}_{n_{k'}})_{k'\in\mathbb{N}}$ converging to a function $\tilde{g}$. By definition of the weak derivative we have

$$\int \dot{h}_{n_{k'}}(\xi)f(\xi)\,\mathrm{d}\xi = -\int h_{n_{k'}}(\xi)\dot{f}(\xi)\,\mathrm{d}\xi \quad \text{, for all} \quad f \in C_0^1. \tag{2.27}$$

Taking the limit $k' \to \infty$ on both sides of (2.27), yields

$$\int \tilde{g}(\xi)f(\xi)\,\mathrm{d}\xi = -\int h(\xi)\dot{f}(\xi)\,\mathrm{d}\xi \quad \text{, for all} \quad f \in C_0^1,$$

where we used that in L^2 the sequence $(h_{n_{k'}})_{k'\in\mathbb{N}}$ converges to h (since it does so in L_∞) and $(\dot{h}_{n_{k'}})_{k'\in\mathbb{N}}$ converges to $\tilde{g}$. So $\dot{h} = \tilde{g}$. Repeating this for $\ddot{h}_{n_k}$ we get $g = \ddot{h}$. Hence by (2.26)

$$\|\ddot{h}\|_{L^2} \leq \liminf \|\ddot{h}_{n_k}\|_{L^2}. \tag{2.28}$$

Applying the definition of Q to (2.28) yields

$$Q(h) \leq \liminf Q(h_{n_k}).$$

So $h \in L_\kappa$; and as h was arbitrary this implies that L_κ is closed. □

Now we show that L_κ is compact.

Lemma 2.8. *The level sets $(L_\kappa)_{\kappa\in\mathbb{R}}$ are compact.*

Proof. By Lemma 2.7 the level set L_κ is closed; so it suffice to show that L_κ is precompact. By Arzelà-Ascoli, L_κ is precompact if it is bounded and equicontinuous.

Boundedness follows after two application of the fundamental theorem of calculus: Note, for all $f \in L_\kappa$, the norm $\|\ddot{f}\|_{L^2}$ is bounded. Since $\dot{f}(x) = \dot{f}(0) + \int_0^x \ddot{f}(\xi)\,\mathrm{d}\xi$, the boundedness of $\|\dot{f}\|_\infty$ is a consequence of Jensen's inequality applied to $\int_0^x \ddot{f}\,\mathrm{d}\xi$: $|\int_0^x \ddot{f}\,\mathrm{d}\xi| \leq \|\ddot{f}\|_{L^2}$. Since $f(x) = f(0) + \int_0^x \dot{f}$, the norm $\|f\|_\infty$ is bounded as well.

To prove equicontinuity we have to show that for each $\epsilon > 0$ there is a $\delta > 0$ such that for all $f \in L_\kappa$

$$\text{if} \quad |y - x| < \delta, \quad \text{then} \quad |f(y) - f(x)| < \epsilon. \tag{2.29}$$

From the proof of boundedness, we know $\|\dot{f}\|_\infty$ is bounded by a constant C; so we have

$$f(y) - f(x) = |\int_x^y \dot{f}(s)\,\mathrm{d}s| \leq |y - x|\|\dot{f}\|_\infty \leq C|y - x| \quad \text{for all} \quad f \in L_\kappa.$$

Hence, for all $\epsilon > 0$ and for $\delta = \frac{\epsilon}{C}$, the inequality (2.29) is satisfied for all $f \in L_\kappa$. $\square$

2.4.2 Extension of Theorem 1.2

In our proof of Theorem 1.3, we use the expansion (1.6). Therefore we need upper and lower bounds to $\gamma_S^\psi(\mathcal{Q})$ where S is of the form $S = \{s_*+1, s_*+2, \ldots, s^*-1\} =: (s_*, s^*)$ for $s_*, s^* \in \mathbb{N}$. Motivated by the observation that a sample from $\gamma_S^\psi \circ h_N^{-1}$ is fixed outside of $I = ({s_*}/{N}, {s^*}/{N})$, we prove an **LDP** for

$$\gamma_{N,I}^{\mathbf{r}} := \gamma_{I_N}^{\psi} \circ \tilde{h}_N^{-1}\,, \text{ for } \psi = \psi^{\mathbf{r},N,I}, \tag{2.30}$$

where I is an interval in $[0,1]$ with rational end points $I_* < I^*$, where $I_N := IN \cap \mathbb{Z}$, where $\tilde{h}_N(\phi)$ is the restriction of $h_N(\phi)$ to I and where

$$\psi_i^{\mathbf{r},N,I} := \begin{cases} aN^2 - \alpha N & , \text{for } i = \lfloor NI_* \rfloor - 1, \\ aN^2 & , \text{for } i = \lfloor NI_* \rfloor\,, \\ bN^2 & , \text{for } i = \lceil NI^* \rceil\,, \\ bN^2 + \beta N & , \text{for } i = \lceil NI^* \rceil + 1, \\ 0 & , \text{otherwise}, \end{cases}$$

and where for $x \in \mathbb{R}$ the value $\lceil x \rceil$ is the smallest integer larger than or equal to x.

Note that the image of the interpolation $\tilde{h}_N$ is the space of continuous functions on I, we denote this space by $C(I)$. We frequently use the following notation: For $\mathcal{Q} \subset C(0,1)$, the measure $\gamma_{N,I}^{\mathbf{r}}(\mathcal{Q})$ is the measure of the set of restrictions of the functions in $\mathcal{Q}$ to functions in $C(I)$.

Proposition 2.9. *The sequence of measures $(\gamma_{N,I}^{\mathbf{r}})_{N\in\mathbb{N}}$ satisfies a large deviation principle in the space $(C(I), \|\cdot\|_\infty)$ with speed N and good rate function $\Sigma_I^{\mathbf{r}}$ that is given by*

$$\Sigma_I^{\mathbf{r}}(f) = \begin{cases} Q_I(f) - Q_I(H_{\mathbf{r}}(I)) & , \text{for } f \in H_{\mathbf{r}}(I), \\ \infty & , \text{otherwise}, \end{cases} \tag{2.31}$$

where

$$Q_I(h) = \tfrac{1}{2} \int_I (\ddot{h}(\xi))^2 \,\mathrm{d}\xi. \tag{2.32}$$

and

$$H_{\mathbf{r}}(I) := \{h \in H^2(I) \mid h(I_*) = a, h(I^*) = b, \dot{h}(I_*) = \alpha, \dot{h}(I^*) = \beta\}.$$

Proof. We provide tools to extend the **LDP** of $\gamma_N^{\tilde{\mathbf{r}}}$ for a suitable $\tilde{\mathbf{r}}$ to an **LDP** of $\gamma_{N,I}^{\mathbf{r}}$. Therefore let $N \in \mathbb{N}$ be such that NI_*, NI^* are in $\mathbb{N}$, too.

- we have
$$\gamma_{N,I}^{\mathbf{r}} = \gamma_{N,(0,|I|)}^{\mathbf{r}} \circ \mathbb{T}_x \text{ , for } x = -I_* \tag{2.33}$$
where for $x \in \mathbb{R}$ the **translation map** $\mathbb{T}_x \colon C(I) \to C(I + x)$ is given by $\mathbb{T}_x(h)(\xi) = h(\xi - x)$ for all $\xi \in I + x$;

- and
$$\gamma_{N,I}^{\mathbf{r}} = \gamma_{N|I|,(0,1)}^{\tilde{\mathbf{r}}} \circ S_{|I|} \tag{2.34}$$
where $\tilde{\mathbf{r}} = (\frac{a}{|I|^2}, \frac{\alpha}{|I|}, \frac{b}{|I|^2}, \frac{\beta}{|I|})$ and the **scaling map** $S_{|I|} \colon C(I) \to C(0,1)$ is given by $S_{|I|}(h)(\xi) = \frac{1}{|I|^2} h(|I|\xi)$ for all $\xi \in (0,1)$.

Applying these tools, the **CP** and the **LDP** of $\gamma_N^{\tilde{\mathbf{r}}}$, we obtain the **LDP** of $\gamma_{N,I}^{\mathbf{r}}$. □

2.4.3 Uniformity

First we show the uniformity for the interval $I = (0,1)$ and the boundary condition $\mathbf{r} = \mathbf{0}$. Then we extend this to general intervals and boundary conditions.

Proposition 2.10. *For all compact $K \subset C(0,1)$, all $r > 0$ and all $\epsilon > 0$, there is a $N' \in \mathbb{N}$ such that for all $N > N'$, all $g \in K$ and all $\delta > 0$ such that $B(g,\delta) \subset K$ we have*
$$\gamma_{N,I}^{\mathbf{r}}(B(g,\delta)) \leq e^{-N(\Sigma_I^{\mathbf{r}}(\bar{B}(g,\delta+r))-\epsilon)}, \tag{2.35}$$
where $I = (0,1)$, $\mathbf{r} = \mathbf{0}$ and $\bar{B}(g,\delta+r)$ is the closure of the ball $B(g,\delta+r)$.

Proof. Fix K, r, ϵ. In this proof we drop the index $\mathbf{r}$ and I, because for this proposition we assume $\mathbf{r} = \mathbf{0}$ and $I = (0,1)$. In particular we write $\Sigma = \Sigma_{(0,1)}^{\mathbf{0}}$. We use a finite cover $\mathcal{K} = \{B(g_i, r_i)\}_{i=1}^k$ of K that satisfies
$$\Sigma(\bar{B}(g_i, r_i)) \geq \Sigma(g_i) - \frac{\epsilon}{3}. \tag{2.36}$$

Such a cover exists by lower semicontinuity of Σ. To see this note that for each $f \in K$ there is a $r_f < \frac{r}{2}$ such that
$$\Sigma(\bar{B}(f, r_f)) \geq \Sigma(f) - \frac{\epsilon}{3};$$

and since K is compact we can pick out of the cover $K \subset \cup_{f \in K} B(f, r_f)$ a finite cover $\mathcal{K}$ that satisfies (2.36) .

Let $\tilde{N} \in \mathbb{N}$ be such that for all $N > \tilde{N}$ we have that

$$\gamma_N(B(g_i, r_i)) \leq e^{-N(\Sigma(\bar{B}(g_i, r_i)) - \frac{\epsilon}{3})},$$

for all $i = 1, 2, \ldots, k$. Such a value $\tilde{N}$ exists because $(\gamma_N)_{N \in \mathbb{N}}$ satisfies an **LDP** with rate function Σ.

Let $g \in K$ and $\delta > 0$ be such that the ball $B(g, \delta)$ is a subset of K and let $\mathcal{U} = \{i \mid B(g_i, r_i) \cap B(g, \delta) \neq \emptyset\}$ be the indices of the balls in $\mathcal{K}$ that cover $B(g, \delta)$, we have

$$\begin{aligned}
\gamma_N(B(g, \delta)) \leq \gamma_N(\cup_{i \in \mathcal{U}} B(g_i, r_i)) &\leq \sum_{i \in \mathcal{U}} \gamma_N(B(g_i, r_i)) \\
&\leq k \max_{i \in \mathcal{U}} \gamma_N(B(g_i, r_i)) \\
&\leq k \max_{i \in \mathcal{U}} e^{-N(\Sigma(\bar{B}(g_i, r_i)) - \frac{\epsilon}{3})} \\
&= k e^{-N(\min_{i \in \mathcal{U}} \Sigma(\bar{B}(g_i, r_i)) - \frac{\epsilon}{3})}.
\end{aligned}$$

By (2.36) we have $\min_{i \in \mathcal{U}} \Sigma(\bar{B}(g_i, r_i)) \geq \min_{i \in \mathcal{U}} \Sigma(g_i) - \frac{\epsilon}{3} \geq \inf_{f \in B(g, \delta + r)} \Sigma(f) - \frac{\epsilon}{3}$. So we get the upper bound

$$\gamma_N(B(g, \delta)) \leq e^{-N(\Sigma(B(g, \delta + r)) - \frac{2\epsilon}{3} - \frac{\log(k)}{N})}.$$

To obtain (2.35) set $N' = \max(\tilde{N}, \frac{3 \log(k)}{\epsilon})$. □

Now we expand Proposition 2.10 by allowing other boundary conditions than $\mathbf{r} = \mathbf{0}$.

Corollary 2.11. *Fix $K \subset C(0, 1)$, $\rho > 0$ and $r > 0$. For every $\epsilon > 0$ there is a $N' \in \mathbb{N}$ such that if $N > N'$ and $B(g, \delta) \subset K$ we have*

$$\gamma^{\mathbf{r}}_{N,I}(B(g, \delta)) \leq e^{-N(\Sigma^{\mathbf{r}}_I(\bar{B}(g, \delta + r)) - \epsilon)}, \tag{2.37}$$

holds for $I = (0, 1)$ and every $\mathbf{r}$ such that $\|\mathbf{r}\|_{\mathbb{R}^4} < \rho$.

Proof. Recall that in the proof of Lemma 2.6, the convergence of $(h_N(\phi_{\mathbf{r}}))_{N \in \mathbb{N}}$ to $h_{\mathbf{r}}$ was central. By Proposition D.2 this convergence is uniform in $\|\mathbf{r}\|_{\mathbb{R}^4} < \rho$. Plugging this into (4.2.20) of [14] we conclude the uniformity of the large deviation principle. □

Now we expand Corollary 2.11 by allowing other intervals than $I = (0, 1)$.

Corollary 2.12. *Fix $\rho > 0$, $l > 0$ and $r > 0$. For every $\epsilon > 0$ there is a N' such that if $N > N'$ and $B(g, \delta) \subset K$ we have*

$$\gamma^{\mathbf{r}}_{N,I}(B(g,\delta)) \le e^{-N(\Sigma^{\mathbf{r}}_I(\bar{B}(g,\delta+r))-\epsilon)}, \tag{2.38}$$

holds for all intervals I such that $l < |I| \le 1$ and every boundary condition $\mathbf{r}$ such that $\|\mathbf{r}\|_{\mathbb{R}^4} < \rho$.

Proof. Fix $\epsilon > 0$. We use Corollary 2.11, because by (2.33) and (2.34),

$$\gamma^{\mathbf{r}}_{N,I} = \gamma^{\tilde{\mathbf{r}}}_{N|I|,(0,1)} \circ S_{|I|} \circ \mathbb{T}_{-I_*}. \tag{2.39}$$

Recall $\tilde{\mathbf{r}} = (\frac{a}{|I|^2}, \frac{\alpha}{|I|}, \frac{b}{|I|^2}, \frac{\beta}{|I|})$. Before we can apply Corollary 2.11 we have to check that there is a compact set $\tilde{K}$ such that

$$S_{|I|} \circ \mathbb{T}_{-I_*}(B(g,\delta)) \subset \tilde{K} \quad \text{for all} \quad B(g,\delta) \subset K, \quad \text{and for all} \quad I \quad \text{s.t.} \quad l < |I| \le 1, \tag{2.40}$$

and that there is a $\tilde{\rho} > 0$ such that

$$|\tilde{\mathbf{r}}| < \tilde{\rho} \quad \text{for all} \quad I \quad \text{s.t.} \quad l < |I| \le 1. \tag{2.41}$$

The conditions are satisfied because $l < |I|$ and because of the definition of $S_{|I|}$.

Now by Corollary 2.11 for each $\tilde{r} > 0$, there is constant $\tilde{N}$ such that for $N|I| > \tilde{N}$

$$\gamma^{\tilde{\mathbf{r}}}_{N|I|,(0,1)}(\tilde{B}(g,\delta)) \le e^{-N|I|(\Sigma^{\tilde{\mathbf{r}}}_{(0,1)}(\tilde{B}(g,\delta+\tilde{r}))-\epsilon)}, \text{ for all } \quad \tilde{B}(g,\delta) \subset \tilde{K}.$$

For the moment let $\tilde{r} > 0$ be arbitrary but fixed. Since $|I| \le 1$, we have $|I|\epsilon \le \epsilon$ and hence

$$\gamma^{\tilde{\mathbf{r}}}_{N|I|,(0,1)}(\tilde{B}(g,\delta)) \le e^{-N(|I|\Sigma^{\tilde{\mathbf{r}}}_{(0,1)}(\tilde{B}(g,\delta+\tilde{r}))-\epsilon)}, \text{ for all } \quad \tilde{B}(g,\delta) \subset \tilde{K}. \tag{2.42}$$

We have $|I|\Sigma^{\tilde{\mathbf{r}}}_{(0,1)}(\mathbb{T}_{-I_*}(S_{|I|}(g))) = \Sigma^{\mathbf{r}}_I(g)$ for all $g \in H_{\mathbf{r}}(I)$ and hence combining (2.42) and (2.39) yields

$$\gamma^{\mathbf{r}}_{N,|I|}(B(g,\delta)) \le e^{-N(\Sigma^{\mathbf{r}}_I(\bar{B}(g,\delta+\bar{r}))-\epsilon)},$$

where $\bar{r}$ depends on $\tilde{r}$ and I. So there is suitable $\tilde{r}$ such that $\bar{r} < r$ and for this $\tilde{r}$ setting $N' = \frac{\tilde{N}}{l}$ implies (2.38).

□

Chapter 3

Large deviation principle for the model with pinning interaction

In this chapter we prove the **LDP** for the model with pinning interaction (see Theorem 1.3). Therefore we show that the sequence of measures $(\gamma_N^{\mathbf{r},J})_{N\in\mathbb{N}}$ satisfies an **LDP** with rate function $\Sigma^{\mathbf{r},J}$. In Section 3.1 we prove that $\Sigma^{\mathbf{r},J}$ is actually a good rate function and in Section 3.2 we show that $(\gamma_N^{\mathbf{r},J})_{N\in\mathbb{N}}$ satisfies an **LDP** with this rate function.

Our basic tools for the proof of the **LDP** are the two stage interpretation (1.6) and the splitting (1.7). For the gradient model the authors of [19] use that the gradient model has a two stage interpretation that is analogously to (1.6) and that the gradient model satisfies a splitting property with lag 1. The splitting property allows them to use the **LDP** for $J = -\infty$ to prove the **LDP** for $J > -\infty$. The Laplacian model only satisfies a splitting property of lag 2. This means that only for sets $S \subset \Lambda_N$ such that for each element of the complement $p \in S^c$ at least one neighbour of p is also in the complement S^c, the measure γ_S^ψ coincides with the product of $(\gamma_{S_i}^\psi)_{i\in\{1,2,\ldots K\}}$, where $(S_i)_{i\in\{1,2,\ldots K\}}$ are such that their union coincides with S. It turns out that we can not ignore the sets S that do not allow to write γ_S^ψ as such a product. Especially for the upper bound this forces us to use more complex methods than the authors of [19] did.

3.1 Goodness of the rate function

We prove the goodness of the rate function $\Sigma^{\mathbf{r},J}$. By definition of goodness and by the definition of $\Sigma^{\mathbf{r},J}$ (see Theorem 1.3), it is sufficient to prove that $\mathcal{E}^J$ (defined in (1.18)) is lower semicontinuous with compact level sets.

As long as $J \leq J^c$ proving goodness is easy: since $\tau(J) = 0$ we have $\mathcal{E}^J = \mathcal{E}^{-\infty} = Q$ and we know, from Lemma 2.8, that Q is lower semicontinuous with compact level sets. But if $J > J^c$, we have to deal with the term $\tau(J)|\mathcal{N}_f|$. In this case, already the proof of lower semicontinuity is not trivial. We have to show that the map $f \mapsto |\mathcal{N}_f|$ is upper semicontinuous with respect to the $\|\cdot\|_\infty$ norm. This is done in Lemma 3.1. But note that $f \mapsto |\mathcal{N}_f|$ is clearly not continuous: While the sequence $(f_n)_{n\in\mathbb{N}}$ with $f_n \equiv \frac{1}{n}$ converges uniformly to $f \equiv 0$, we have $0 = \lim_{n\to\infty}|\mathcal{N}_{f_n}| < |\mathcal{N}_f| = 1$.

Lemma 3.1. *The map from $(C(0,1), \|\cdot\|_\infty)$ to $\mathbb{R}$ given by $f \mapsto |\mathcal{N}_f|$ is upper semicontinuous.*

Proof. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence that converges uniformly to f. By switching to a subsequence, we may assume

$$\|f_n - f\|_\infty \leq \frac{1}{n} \quad \text{for all } n \in \mathbb{N}. \tag{3.1}$$

Fix $n \in \mathbb{N}$. We conclude from (3.1) that if ξ is a zero of f_n, then $|f(\xi)| \leq \frac{1}{n}$. So

$$\mathcal{N}_{f_n} \subset \{\xi \in [0,1] \mid |f(\xi)| \leq \tfrac{1}{n}\} \quad \text{for all } n \in \mathbb{N}. \tag{3.2}$$

Since $\lim_{n\to\infty}|\{\xi \in [0,1] \mid |f(\xi)| \leq \frac{1}{n}\}| = |\mathcal{N}_f|$, the upper semicontinuity of the map $f \mapsto |\mathcal{N}_f|$ follows from (3.2) by the monotonicity of the Lebesgue measure. □

In order to prove that $\Sigma^{\mathbf{r},J}$ is good, we still have to prove that the level sets

$$L_\kappa := \{f \in H_{\mathbf{r}} \mid \mathcal{E}^J(f) \leq \kappa\}$$

are compact in $(C(0,1), \|\cdot\|_\infty)$.

Lemma 3.2. *The function $\mathcal{E}^J$ has compact level sets $(L_\kappa)_{\kappa\in\mathbb{R}}$ in $(C(0,1), \|\cdot\|_\infty)$.*

Proof. Fix $\kappa \in \mathbb{R}$. We have the inclusion

$$L_\kappa = \{f \in H_{\mathbf{r}} \mid \mathcal{E}^J(f) \leq \kappa\} \subset \{f \in H_{\mathbf{r}} \mid \mathcal{E}^{-\infty}(f) \leq \kappa + \tau(J)\},$$

the later set being compact by Lemma 2.8. By Lemma 3.1 the set L_κ is also closed and therefore compact. □

3.2 The large deviation principle for the model with pinning interaction

For the proof that $(\gamma_N^{\mathbf{r},J})_{N\in\mathbb{N}}$ satisfies an **LDP** with rate function $\Sigma^{\mathbf{r},J}$, we use the two stage interpretation (see (1.6)). In the following we omit the index $\mathbf{r}$ in our notation of the measures and we write $\hat{\gamma}_N = \gamma_N^{\mathbf{r},J}$. To see the advantage of the two stage interpretation consider a set $\mathcal{Q} \subset C(0,1)$ containing only functions that have no zero in $[0,1]$. If the result of the first experiment is not $\mathcal{P} = \emptyset$, the probability to observe $\mathcal{Q}$ in the second stage is zero; so

$$\hat{\gamma}_N(\mathcal{Q}) = \frac{\mathcal{Z}_N^{\mathbf{r}}}{\hat{\mathcal{Z}}_N^{\mathbf{r}}}\gamma_N(\mathcal{Q}), \tag{3.3}$$

where we use the notation $\gamma_N = \gamma_N^{\mathbf{r}}$ for the measure $\gamma_N^{\mathbf{r},-\infty}$. If we already knew the rate $\lim_{N\to\infty}(\frac{1}{N}\log\hat{\mathcal{Z}}_N^{\mathbf{r}})$, equation (3.3) and the **LDP** for $(\gamma_N(\mathcal{Q}))_{N\in\mathbb{N}}$ would suffice to verify that Theorem 1.3 holds for $\mathcal{Q}$.

We will encounter this problem of not knowing the rate $\lim_{N\to\infty}(\frac{1}{N}\log\hat{\mathcal{Z}}_N^{\mathbf{r}})$ also for general sets. To postpone calculating this rate we show that

$$\liminf_{N\to\infty}\frac{1}{N}\log\left(\frac{\hat{\mathcal{Z}}_N^{\mathbf{r}}}{\mathcal{Z}_N^{\mathbf{0}}}\hat{\gamma}_N(B(g,\delta))\right) \geq -\mathcal{E}^{\mathbf{r},J}(g), \tag{3.4}$$

holds for all $\delta > 0$ and all $g \in C(0,1)$ and that

$$\limsup_{N\to\infty}\frac{1}{N}\log\left(\frac{\hat{\mathcal{Z}}_N}{\mathcal{Z}_N^{\mathbf{0}}}\hat{\gamma}_N(\mathcal{C})\right) \leq -\inf_{f\in\mathcal{C}}\mathcal{E}^{\mathbf{r},J}(f), \tag{3.5}$$

holds for all closed sets $\mathcal{C}$ (closed with respect to the L^∞ norm), where

$$\mathcal{E}^{\mathbf{r},J}(f) := \begin{cases} \mathcal{E}^J(f) & \text{, for } f \in H_{\mathbf{r}}, \\ \infty & \text{, otherwise,} \end{cases}$$

and where $\mathcal{E}^J$ is defined in (1.18).

These two statements imply Theorem 1.3, because we have

$$\hat{\gamma}_N(\mathcal{Q}) = \frac{\mathcal{Z}_N^{\mathbf{0}}}{\hat{\mathcal{Z}}_N^{\mathbf{r}}}\left(\frac{\hat{\mathcal{Z}}_N^{\mathbf{r}}}{\mathcal{Z}_N^{\mathbf{0}}}\hat{\gamma}_N(\mathcal{Q})\right),$$

and because we can determine the rate $\lim_{N\to\infty} N^{-1}\log(\frac{\mathcal{Z}_N^{\mathbf{0}}}{\hat{\mathcal{Z}}_N^{\mathbf{r}}})$ as follows: We consider the sequence $(\hat{\mathcal{Z}}_N^{\mathbf{r}}/\mathcal{Z}_N^{\mathbf{0}})_{N\in\mathbb{N}}$. We use the upper and lower bounds (3.5) and (3.4) to show that the sequence has the rate $-\mathcal{E}^{\mathbf{r},J}(H_{\mathbf{r}})$: By applying the upper bound (3.5) to the closed set $C(0,1)$ we see that the rate of the sequence has $-\mathcal{E}^{\mathbf{r},J}(C(0,1)) = -\mathcal{E}^{\mathbf{r},J}(H_{\mathbf{r}})$ as upper bound and by applying the lower bound (3.4) to the ball $B(\tilde{h}_{\mathbf{r}}, \delta) \subset C(0,1)$, where $\tilde{h}_{\mathbf{r}}$ is a minimiser of $\mathcal{E}$ in $H_{\mathbf{r}}$, we see that the rate of the sequence has $-\mathcal{E}^{\mathbf{r},J}(H_{\mathbf{r}})$ also as lower bound.

Clearly, by (3.3), the lower and upper bound from above are satisfied for sets $\mathcal{Q} \subset H_{\mathbf{r}}$ that contain only functions that have no zero in $[0,1]$. For other sets we need more sophisticated arguments. We fix $\mathbf{r}$ and J. In the following two sections we denote $\mathcal{E}^{\mathbf{r},J}$ by $\mathcal{E}$. We split the proof into two parts: the proof of the lower and the upper bound. We prove the lower bound in Section 3.2.1 and the upper bound in Section 3.2.2.

3.2.1 Lower bound

We have to prove

$$\liminf_{N\to\infty} \tfrac{1}{N} \log\left(\frac{\hat{\mathcal{Z}}_N^{\mathbf{r}}}{\mathcal{Z}_N^{\mathbf{0}}} \hat{\gamma}_N(B(g,\delta)) \right) \geq -\mathcal{E}(g), \tag{3.6}$$

for all $\delta > 0$ and all $g \in C(0,1)$. As (3.6) holds trivially if $\mathcal{E}(g) = \infty$, let $g \in H_{\mathbf{r}}$. The next lemma implies that it is sufficient to prove (3.6) only for a subset of $H_{\mathbf{r}}$, we call this subset the well-behaved functions.

Definition 3.3. We call a function $f \in H_{\mathbf{r}}$ **well-behaved** if $\mathcal{N}_f$ is an union of finitely many intervals and or isolated points. We say that ξ is an isolated point of $\mathcal{N}_f$ if there is a $\delta > 0$ such that $(\xi - \delta, \xi + \delta) \cap \mathcal{N}_f = \{\xi\}$.

Lemma 3.4. *For any function $g \in H_{\mathbf{r}}$ and $\delta > 0$, there is a well-behaved function $f \in B(g,\delta)$ such that*

$$\mathcal{E}^{\mathbf{r},J}(g) \geq \mathcal{E}^{\mathbf{r},J}(f). \tag{3.7}$$

We give the proof at the end of this section. Lemma 3.4 is useful because it implies that once we have established that (3.6) holds for all well-behaved functions, then (3.6) holds automatically also for all $g \in H_{\mathbf{r}}$. To see this, let f be a well-behaved function such that $f \in B(g,\delta)$ and $\mathcal{E}(f) \leq \mathcal{E}(g)$, furthermore let $\delta' > 0$ be such that

$B(f,\delta') \subset B(g,\delta)$. If f satisfies (3.6), then

$$\begin{aligned}\liminf_{N\to\infty} \tfrac{1}{N} \log\left(\frac{\hat{\mathcal{Z}}^{\mathbf{r}}_N}{\mathcal{Z}^{\mathbf{0}}_N}\hat{\gamma}_N(B(g,\delta))\right) &\geq \liminf_{N\to\infty} \tfrac{1}{N}\log\left(\frac{\hat{\mathcal{Z}}^{\mathbf{r}}_N}{\mathcal{Z}^{\mathbf{0}}_N}\hat{\gamma}_N(B(f,\delta'))\right)\\ &\geq -\mathcal{E}(f)\\ &\geq -\mathcal{E}(g).\end{aligned}$$

Let g be a well-behaved function. We use the two stage interpretation (1.6) to prove (3.6). If $\mathcal{N}_g$ consists only of isolated points, then, since $|\mathcal{N}_g| = 0$ and since hence $\mathcal{E}(g) = Q(g)$, the proof of (3.6) is straightforward: We drop all terms on the right hand side of (1.6) except of the one for $\mathcal{P} = \emptyset$ and get

$$\hat{\gamma}_N(\mathcal{Q}) \geq \tfrac{1}{\hat{\mathcal{Z}}^{\mathbf{r}}_N}\mathcal{Z}^{\psi}_N\gamma^{\psi}_N(\mathcal{Q}) \text{ for all } \mathcal{Q}\in H_{\mathbf{r}},$$

where $\psi = \psi^{\mathbf{r},N}$, for the definition of $\psi^{\mathbf{r},N}$ see (1.13). We multiply both sides of the previous inequality by $\hat{z}^{\mathbf{r}}_N/z^{\mathbf{0}}_N$, and use that, by Theorem 1.2 the sequence $(\gamma^{\psi}_N)_{N\in\mathbb{N}} = (\gamma^{\mathbf{r}}_N)_{N\in\mathbb{N}}$ has the rate $\Sigma^{\mathbf{r}}$ and that by Proposition D.2, we have $\mathcal{Z}^{\mathbf{r}}_N = \mathcal{Z}^{\mathbf{0}}_N e^{-\mathcal{H}_N(\phi_{\mathbf{r}})}$. To see that for $\mathcal{Q} = B(g,\delta)$ we have (3.6) we combine that, by Lemma D.3, we have $\mathcal{H}_N(\phi_{\mathbf{r}}) = NQ(h_{\mathbf{r}})(1+o(N))$ for $N\to\infty$ and that by definition of $\Sigma^{\mathbf{r}}$ we have $\Sigma^{\mathbf{r}}(g) = Q(g) - Q(h_{\mathbf{r}})$.

As soon as $\mathcal{N}_g$ contains intervals we have to argue more carefully. We present the argument for the case where $\mathcal{N}_g$ contains only one interval $\mathcal{I}$; the other cases just need a more complex notation. After we discus the case where $\mathcal{N}_g$ contains only one interval, we briefly discuss how to approach the proof if $\mathcal{N}_g$ contains more than one interval.

Let $g \in H_{\mathbf{r}}$ be a well-behaved function such that $\mathcal{N}_g$ contains one interval $\mathcal{I}$ and such that $\mathcal{N}_g \setminus \mathcal{I}$ are only finitely many isolated points. To obtain a lower bound we drop terms from the right hand side of (1.6):

$$\hat{\gamma}_N(\mathcal{Q}) \geq \tfrac{1}{\hat{\mathcal{Z}}^{\mathbf{r}}_N}\sum_{\mathcal{P}\subset\mathcal{I}_N} e^{J|\mathcal{P}|}\mathcal{Z}^{\psi}_{\mathcal{P}^c}\gamma^{\psi}_{\mathcal{P}^c}(\mathcal{Q}),$$

where $\mathcal{I}_N = N\mathcal{I}\cap\mathbb{Z}$, and $\psi = \psi^{\mathbf{r},N}$ as previously. Note that the inequality is strict if the zero set of g contains isolated points.

Fix one set $\mathcal{P}\subset\mathcal{I}_N$, we consider the Gaussian measure $\gamma^{\psi}_{\mathcal{P}^c}$. Certain sets $\mathcal{P}$ are particularly convenient (not just for this proof but also in the proof of the upper bound), we call them **good** sets: A set $\mathcal{P}$ is good if $\mathcal{P}$ is not empty and beside $p_* := \min\mathcal{P}$ and $p^* := \max\mathcal{P}$ it also contains p_*+1 and p^*-1 or if $\mathcal{P} = \emptyset$. The set of all good sets is denoted by $\mathcal{G}$. The convenience of non-empty good sets is that by (1.7) the measure $\gamma^{\psi}_{\mathcal{P}^c}$ corresponding to these sets split as follows:

$$\gamma^{\psi}_{\mathcal{P}^c} = \gamma^{\psi}_{(0,p_*)}\gamma^{\psi}_{(p_*+1,p^*-1)\setminus\mathcal{P}}\gamma^{\psi}_{(p^*,N)}, \tag{3.8}$$

where we again use the notation $(s_*, s^*) = \{s_*+1, s_*+2, \dots, s^*-1\}$ for $s_*, s^* \in \mathbb{Z}$.

Before we use (3.8), we drop more terms from the right hand side of (1.6):

$$\hat{\gamma}_N(\mathcal{Q}) \geq \frac{1}{\hat{\mathcal{Z}}^{\mathbf{r}}_N} \sum_{\substack{\mathcal{P}\subset\mathcal{I}_N, \mathcal{P}\in\mathcal{G}\setminus\{\emptyset\} \\ \text{s.t. } p_*=\mathcal{I}_{N,*}, p^*=\mathcal{I}^*_N}} e^{J|\mathcal{P}|}\mathcal{Z}^{\psi}_{\mathcal{P}^c}\gamma^{\psi}_{\mathcal{P}^c}(\mathcal{Q}), \tag{3.9}$$

where $\mathcal{I}_{N,*} := \min \mathcal{I}_N$ and $\mathcal{I}^*_N := \max \mathcal{I}_N$.

The next Lemma uses (3.8) to determine the rate of $\gamma^{\psi}_{\mathcal{P}^c}$.

Lemma 3.5. *Let $g \in H_{\mathbf{r}}$ be a function such that the zero set $\mathcal{N}_g$ contains only one interval $\mathcal{I}$. For all $\epsilon > 0$, there is a N' such that if $N \geq N'$ we have*

$$\gamma^{\psi}_{\mathcal{P}^c}(B(g,\delta)) \geq e^{-N(\Sigma(g)+\epsilon)} \tag{3.10}$$

for all non empty good sets $\mathcal{P} \subset \mathcal{I}_N$ such that $p_* = \mathcal{I}_{N,*}, p^* = \mathcal{I}^*_N$, where $\Sigma = \Sigma^{\mathbf{r},-\infty}$.

We give the proof at the end of the section. Applying Lemma 3.5 and $\mathcal{Z}^{\mathbf{r}}_N = \mathcal{Z}^{\mathbf{0}}_N e^{-\mathcal{H}_N(\phi_{\mathbf{r}})}$ together with the asymptotic $\mathcal{H}_N(\phi_{\mathbf{r}}) = NQ(h_{\mathbf{r}})(1+o(N))$ for $N \to \infty$ (see Lemma (D.3)) and that by definition of $\Sigma^{\mathbf{r}}$ we have $\Sigma^{\mathbf{r}}(g) = Q(g) - Q(h_{\mathbf{r}})$ to (3.9), we obtain

$$\begin{aligned} \hat{\mathcal{Z}}^{\mathbf{r}}_N\hat{\gamma}_N(B(g,\delta)) &\geq e^{-N(Q(g)+\epsilon)} \sum_{\substack{\mathcal{P}\subset\mathcal{I}_N, \mathcal{P}\in\mathcal{G}\setminus\{\emptyset\} \\ \text{s.t. } p_*=\mathcal{I}_{N,*}, p^*=\mathcal{I}^*_N}} e^{J|\mathcal{P}|}\mathcal{Z}^{\mathbf{0}}_{(0,p_*)}\mathcal{Z}^{\mathbf{0}}_{(p_*+1,p^*-1)\setminus\mathcal{P}}\mathcal{Z}^{\mathbf{0}}_{(p^*,N)} \\ &= e^{-N(Q(g)+\epsilon)}\mathcal{Z}^{\mathbf{0}}_{(0,p_*)}\mathcal{Z}^{\mathbf{0}}_{(p^*,N)} \sum_{\substack{\mathcal{P}\subset\mathcal{I}_N, \mathcal{P}\in\mathcal{G}\setminus\{\emptyset\} \\ \text{s.t. } p_*=\mathcal{I}_{N,*}, p^*=\mathcal{I}^*_N}} e^{J|\mathcal{P}|}\mathcal{Z}^{\mathbf{0}}_{(p_*+1,p^*-1)\setminus\mathcal{P}}. \end{aligned} \tag{3.11}$$

We apply

$$\hat{\mathcal{Z}}^{\mathbf{0}}_N = \sum_{\mathcal{P}\subset\Lambda_N} e^{J|\mathcal{P}|}\mathcal{Z}^{\mathbf{0}}_{\Lambda_N\setminus\mathcal{P}}, \tag{3.12}$$

which follows analogously to (1.6), to the sum on the right hand side of (3.11):

$$\hat{\mathcal{Z}}^{\mathbf{r}}_N\hat{\gamma}_N(B(g,\delta)) \geq e^{-N(Q(g)+\epsilon)}\mathcal{Z}^{\mathbf{0}}_{(0,\mathcal{I}_{N,*})}\mathcal{Z}^{\mathbf{0}}_{(\mathcal{I}^*_N,N)}\hat{\mathcal{Z}}^{\mathbf{0}}_{(\mathcal{I}_{N,*}+1,\mathcal{I}^*_N-1)}. \tag{3.13}$$

Since by Lemma 1.1 the limit of $\frac{1}{N}\log(\frac{\hat{\mathcal{Z}}^{\mathbf{0}}_{\mathcal{I}_N}}{\mathcal{Z}^{\mathbf{0}}_{\mathcal{I}_N}})$ for $N \to \infty$ is $\tau(J)|\mathcal{I}|$, taking the limit $\liminf_{N\to\infty} \frac{1}{N}\log(\frac{1}{\mathcal{Z}^{\mathbf{0}}_N}\cdot)$ on both sides yields:

$$\begin{aligned}
&\liminf_{N\to\infty} \tfrac{1}{N}\log\left(\frac{\hat{\mathcal{Z}}^{\mathbf{r}}_N}{\mathcal{Z}^{\mathbf{0}}_N}\hat{\gamma}_N(B(g,\delta))\right) \\
&\quad \geq -(Q(g)+\epsilon) + \liminf_{N\to\infty} \tfrac{1}{N}\log\left(\frac{\mathcal{Z}^{\mathbf{0}}_{(0,\mathcal{I}_{N,*})}\hat{\mathcal{Z}}^{\mathbf{0}}_{(\mathcal{I}_{N,*}+1,\mathcal{I}^*_N-1)}\mathcal{Z}^{\mathbf{0}}_{(\mathcal{I}^*_N,N)}}{\mathcal{Z}^{\mathbf{0}}_N}\right) \\
&\quad = -(Q(g)+\epsilon) + \liminf_{N\to\infty} \tfrac{1}{N}\log\left(\frac{\hat{\mathcal{Z}}^{\mathbf{0}}_{(\mathcal{I}_{N,*}+1,\mathcal{I}^*_N-1)}}{\mathcal{Z}^{\mathbf{0}}_{(\mathcal{I}_{N,*}+1,\mathcal{I}^*_N-1)}}\frac{\mathcal{Z}^{\mathbf{0}}_{(0,\mathcal{I}_{N,*})}\mathcal{Z}^{\mathbf{0}}_{(\mathcal{I}_{N,*}+1,\mathcal{I}^*_N-1)}\mathcal{Z}^{\mathbf{0}}_{(\mathcal{I}^*_N,N)}}{\mathcal{Z}^{\mathbf{0}}_N}\right) \\
&\quad = -Q(g) + \tau(J)|\mathcal{I}| - \epsilon,
\end{aligned}$$

where we used that, since the cardinality of $\Lambda_N \backslash [(0,\mathcal{I}_{N,*}) \cup (\mathcal{I}_{N,*}+1, \mathcal{I}^*_N - 1) \cup (\mathcal{I}^*_N, N)]$ is 4 for all N, we have by Proposition C.1 that

$$\liminf_{N\to\infty} \tfrac{1}{N}\log\left(\frac{\mathcal{Z}^{\mathbf{0}}_{(0,\mathcal{I}_{N,*})}\mathcal{Z}^{\mathbf{0}}_{(\mathcal{I}_{N,*}+1,\mathcal{I}^*_N-1)}\mathcal{Z}^{\mathbf{0}}_{(\mathcal{I}^*_N,N)}}{\mathcal{Z}^{\mathbf{0}}_N}\right) = 0. \tag{3.14}$$

As ϵ was arbitrary, we proved inequality (3.6) for well-behaved g such that $\mathcal{N}_g$ contains only one interval.

In the next remark we briefly discuss how we have to change our approach if the well-behaved function g has a zero set $\mathcal{N}_g$ that contains more than one interval.

Remark 3.6. Assume g is such that there are $M \in \mathbb{N}$ intervals $(\mathcal{I}_i)_{i\in\{1,2,\dots,M\}}$ in $\mathcal{N}_g$ and such that the zeros that are not in this intervals are isolated zeros. Let

$$\mathcal{I}_{i,N} := N\mathcal{I}_i \cap \mathbb{Z}. \tag{3.15}$$

In (3.9), we drop all $\mathcal{P}$ except of the ones where the maximum and minimum of $\mathcal{P} \cap \mathcal{I}_{i,N}$ coincide with the maximum and minimum of $\mathcal{I}_{i,N}$ and where one of the nearest neighbours of these extrema is also in $\mathcal{P} \cap \mathcal{I}_{i,N}$ (see Figure 3.1). For such $\mathcal{P}$ the expression on the right hand side of (3.8) is a product of $2 + M$ measures. The rest follows analogously.

Collection of remaining proofs

In this section we give the proofs for Lemma 3.4 and Lemma 3.5.

Proof of Lemma 3.4. Fix $g \in H_{\mathbf{r}}$, we construct a well-behaved function $f \in B(g,\delta)$ satisfying (3.7). (Note that there are functions in $H_{\mathbf{r}}$ that are not well-behaved; this

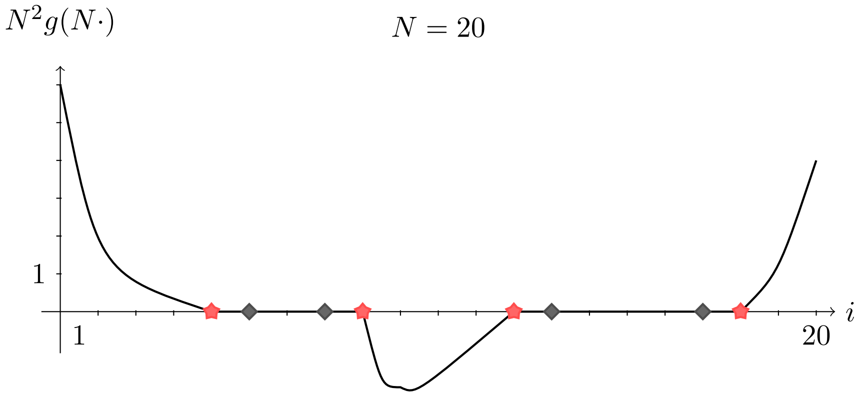

Figure 3.1: This figure illustrates Remark 3.6: we consider a function g that is zero in two intervals $\mathcal{I}_1$ and $\mathcal{I}_2$. The red stars are the maxima and minima of $\mathcal{I}_{1,N}$ and $\mathcal{I}_{2,N}$ (these sets are defined in (3.15)). The black diamonds are the nearest neighbours of the extrema. As described in Remark 3.6: if $\mathcal{P} \subset \{1, 2, \ldots, N-1\}$ does not contain all red stars and black diamonds we drop the term corresponding to $\mathcal{P}$ from the expansion.

is because for example $x \mapsto x^4 \sin(1/x)$ is an element of H^2.) To construct f we use a parameter $n \in \mathbb{N}$. For now fix some $n \in \mathbb{N}$; later we will see which n is suitable for our purpose. Let K be the collection of values $k \in \{1, 2, \ldots, n\}$ such that

$$\mathcal{N}_g \cap [(k-1)\tfrac{1}{n}, k\tfrac{1}{n}] \, \text{contains infinitely many points}$$

and let l_k be the smallest and r_k the largest accumulation point of $\mathcal{N}_g \cap [(k-1)\frac{1}{n}, k\frac{1}{n}]$ for $k \in K$.

We show that there is a $n \in \mathbb{N}$ such that the function

$$f(\xi) := \begin{cases} 0 & , \text{for } \xi \in [l_k, r_k] \text{ and } k \in K, \\ g(\xi) & , \text{otherwise}, \end{cases}$$

is a well-behaved function that is an element of $B(g, \delta)$ and satisfies (3.7).

We consider the three conditions that f has to satisfy separately. First of all note that for all $n \in \mathbb{N}$ the function f is well-behaved. Therefore fix $n \in \mathbb{N}$. By definition of f, the set of points $\xi \in [0, 1]$ such that $f(\xi) = g(\xi)$ contains finitely many zeros of f and the set where $f(\xi) \neq g(\xi)$ consists of $|K|$ intervals in which $f(\xi) = 0$. So the zero set of f is a union of finitely many intervals and isolated points and hence f is well-behaved (see Definition 3.3).

Now we show that for all $n \in \mathbb{N}$ inequality (3.7) is satisfied. Therefore note that, since (by Lemma 3.7 below) $g(l_k) = \dot{g}(l_k) = g(r_k) = \dot{g}(r_k) = 0$ and since hence

$\dot{f}(l_k) = \dot{g}(l_k) = 0$ and $\dot{f}(r_k) = \dot{g}(r_k) = 0$ for $k \in K$, we have

$$
\begin{aligned}
2Q(f) = \int_0^1 (\ddot{f}(\xi))^2 \,\mathrm{d}\xi &= \sum_{k=1}^{n} \int_{(k-1)/n}^{k/n} (\ddot{f}(\xi))^2 \,\mathrm{d}\xi \\
&= \sum_{k \in K} \int_{(k-1)/n}^{k/n} (\ddot{f}(\xi))^2 \,\mathrm{d}\xi + \sum_{k \in K^c} \int_{(k-1)/n}^{k/n} (\ddot{f}(\xi))^2 \,\mathrm{d}\xi \\
&= \sum_{k \in K} \int_{[(k-1)/n, k/n] \setminus [l_k, r_k]} (\ddot{f}(\xi))^2 \,\mathrm{d}\xi + \sum_{k \in K^c} \int_{(k-1)/n}^{k/n} (\ddot{f}(\xi))^2 \,\mathrm{d}\xi \\
&= \sum_{k \in K} \int_{[(k-1)/n, k/n] \setminus [l_k, r_k]} (\ddot{g}(\xi))^2 \,\mathrm{d}\xi + \sum_{k \in K^c} \int_{(k-1)/n}^{k/n} (\ddot{g}(\xi))^2 \,\mathrm{d}\xi \\
&\leq \int_0^1 (\ddot{g}(\xi))^2 \,\mathrm{d}\xi = 2Q(g).
\end{aligned}
$$

We also have

$$|\mathcal{N}_f| \geq |\mathcal{N}_g|.$$

By definition of $\mathcal{E}^J$ (see (1.18)), these two observations prove that (3.7) is true for all $n \in \mathbb{N}$.

We use the uniform continuity of g to obtain a value n such that $f \in B(g, \delta)$: But first of all note that by uniform continuity, there is a n such that

$$\text{for all } \xi_1, \xi_2 \text{ s.t. } |\xi_2 - \xi_1| \leq \tfrac{1}{n} \quad \text{we have} \quad |g(\xi_2) - g(\xi_1)| < \delta. \tag{3.16}$$

Now we show that for this n we have $f \in B(g, \delta)$. Therefore we prove for each $k \in \{1, 2, \ldots, n\}$ that

$$|f(\xi) - g(\xi)| < \delta \,, \text{ for all } \xi \in [(k-1)\tfrac{1}{n}, k\tfrac{1}{n}]. \tag{3.17}$$

The values of f and g differ only if ξ is an element of one of the intervals $[l_k, r_k]$ for $k \in K$. So for $k \in K^c$ condition (3.17) is satisfied. And for $k \in K$ we only need to consider $\xi \in [l_k, r_k]$. Since $|[l_k, r_k]| \leq \frac{1}{n}$ and since g has at least one zero in $[l_k, r_k]$, uniform continuity (see (3.16)) implies $|g(\xi)| < \delta$ for $\xi \in [l_k, r_k]$. Since $f(\xi) = 0$ for $\xi \in [l_k, r_k]$ condition (3.17) is also satisfied for $k \in K$.

□

Lemma 3.7. *For $f \in C^1(0,1)$, we have*

$$\dot{f}(\xi) = 0 \tag{3.18}$$

for all accumulation points of $\mathcal{N}_f$.

Proof. We consider the difference quotient of f at ξ: For any sequence $(\xi_n)_{n\in\mathbb{N}}$ converging to ξ we have

$$\dot{f}(\xi) = \lim_{n\to\infty} \frac{f(\xi) - f(\xi_n)}{\xi - \xi_n}. \tag{3.19}$$

Since ξ is an accumulation point of the closed set $\mathcal{N}_f$ we have $f(\xi) = 0$. Furthermore there is at least one sequence $(\xi_n)_{n\in\mathbb{N}}$ converging to ξ that stays in $\mathcal{N}_f$ so that $f(\xi_n) = 0$ for all $n \in \mathbb{N}$. So (3.19) implies (3.18). □

Proof of Lemma 3.5. As said, we use that $\gamma^{\psi}_{\mathcal{P}^c}$ could be written as a product of three measures, see (3.8). To begin with, note that the **LDP** from Proposition 2.9 is applicable to two of the measures in (3.8), namely to $\gamma^{\psi}_{(0,p_*)}$ and $\gamma^{\psi}_{(p^*,N)}$. Therefore we define the two intervals $S_1 = (0, {p_*}/{N})$ and $S_2 = ({p^*}/{N}, 1)$ and the boundary values $\mathbf{r}_1 = (\mathbf{a}, \mathbf{0})$ and $\mathbf{r}_2 = (\mathbf{0}, \mathbf{b})$. By the definition of $\gamma^{\mathbf{r}}_{N,I}$ (see (2.30)) and since $\psi = \psi^{\mathbf{r},N}$ we have $\gamma^{\psi}_{(0,p_*)} = \gamma^{\mathbf{r}_1}_{N,S_1}$ and $\gamma^{\psi}_{(p^*,N)} = \gamma^{\mathbf{r}_2}_{N,S_2}$. So by Proposition 2.9, for every $\epsilon > 0$ there is a N' such that if $N \geq N'$ we have that

$$\gamma^{\psi}_{(0,p_*)}(B(g,\delta))\gamma^{\psi}_{(p^*,N)}(B(g,\delta)) \geq e^{-N(\Sigma_1(g)+\Sigma_2(g))+\epsilon}, \tag{3.20}$$

where for $j \in \{1,2\}$ we let $\Sigma_j = \Sigma^{\mathbf{r}_j}_{S_j}$. This is a valid application of the large deviation principle, because $p_* = \mathcal{I}_{N,*}$ and $N - p^* = N - \mathcal{I}^*_N$ are of order N.

For the third term we have $\gamma^{\psi}_{(p_*+1,p^*-1)\setminus\mathcal{P}}(B(g,\delta)) = \gamma^{\mathbf{0}}_{(p_*+1,p^*-1)\setminus\mathcal{P}}(B(g,\delta))$. In the following we omit the index $\mathbf{0}$. We show that we make an error that decays exponentially if we replace the third term by the constant 1. Therefore we show that $B(g,\delta)$ is not a tail event of the measure $\gamma_{(p_*+1,p^*-1)\setminus\mathcal{P}}$. We prove

$$\gamma_{(p_*+1,p^*-1)\setminus\mathcal{P}}(B(g,\delta)) \geq 1-|\mathcal{I}_N|O(e^{-CN}) = 1-O(e^{-CN}), \text{ for all } \quad \mathcal{P} \subset \mathcal{I}_N. \tag{3.21}$$

To check (3.21) note that the Gaussian measure $\gamma_{(p_*+1,p^*-1)\setminus\mathcal{P}}$ has expectation zero and that the ball $B(g,\delta)$ restricted to $\mathcal{I}$ is centred in zero. Furthermore under $\gamma_{(p_*+1,p^*-1)\setminus\mathcal{P}}$ each random variable $(\phi_i)_{i\in(p_*+1,p^*-1)}$ is Gaussian with a variance of order N^3 (here we use that conditioning only decreases the variance (see B.2)). So by Lemma B.1 we have

$$\gamma_{(p_*+1,p^*-1)\setminus\mathcal{P}}(\phi(i) \notin [-\delta N^2, \delta N^2]) = O(e^{-CN}) \text{ , for all } \quad \mathcal{P} \subset \mathcal{I}_N. \tag{3.22}$$

Since

$$\gamma_{(p_*+1,p^*-1)\setminus\mathcal{P}}(B(g,\delta)^c) \leq \sum_{i\in\mathcal{I}_N} \gamma_{(p_*+1,p^*-1)\setminus\mathcal{P}}(\phi(i)\notin[-\delta N^2,\delta N^2]), \tag{3.23}$$

the asymptotic (3.22) yields (3.21).

To obtain (3.10) we establish a lower bound for the exponent of the right hand side of (3.20) by proving the following upper bound:

$$\Sigma_1(g)+\Sigma_2(g)\leq\Sigma_{(0,1)}(g). \tag{3.24}$$

Therefore we use that by definition of $\Sigma_I^{\mathbf{r}}$ (see (2.31)) we have

$$\Sigma_{S_j}^{\mathbf{r}_j}(g) = Q_{S_j}(g) - Q_{S_j}(h_j)\ , \text{ for } \quad j\in\{1,2\},$$

where h_j is the minimiser of Q_{S_j} in $H_{\mathbf{r}_j}(S_j)$. Since $g(\xi)=0$ in the complement of $S_1\cup S_2$, we have

$$Q_{(0,1)}(g) = Q_{S_1}(g)+Q_{S_2}(g).$$

Let h be the function that coincides with h_j in S_j and that is zero otherwise, then

$$Q_{(0,1)}(h) = Q_{S_1}(h)+Q_{S_2}(h).$$

So the left hand side of (3.24) coincides with $Q_{(0,1)}(g)-Q_{(0,1)}(h)$ and the right hand side of (3.24) coincides with $Q_{(0,1)}(g)-Q_{(0,1)}(h_{\mathbf{r}})$ where $h_{\mathbf{r}}$ is a minimiser of Q in $H_{\mathbf{r}}$. Since $Q_{(0,1)}(h_{\mathbf{r}})\leq Q_{(0,1)}(h)$, the inequality (3.24) is valid.

Combing the statements (3.20), (3.24) and (3.21) we see that (3.10) is true. □

3.2.2 Upper bound

We have to prove

$$\limsup_{N\to\infty}\frac{1}{N}\log\left(\frac{\hat{\mathcal{Z}}_N^{\mathbf{r}}}{\mathcal{Z}_N^{\mathbf{0}}}\hat{\gamma}_N(\mathcal{C})\right)\leq-\inf_{f\in\mathcal{C}}\mathcal{E}(f) \tag{3.25}$$

for all closed sets $\mathcal{C}$. But conveniently, as we will show in Lemma 3.13 below, the family $(\hat{\gamma}_N)_{N\in\mathbb{N}}$ is exponentially tight and the rate $\lim_{N\to\infty}{}^1/_N\log(\frac{\hat{\mathcal{Z}}_N^{\mathbf{r}}}{\mathcal{Z}_N^{\mathbf{0}}})$ is finite. So it suffices to prove (3.25) for compact sets K. Therefore we use the following local to global approach: Fix $\epsilon>0$. First, we show that for all functions g in a dense subset of $H_{\mathbf{r}}$, we call the elements of this subset very-well-behaved functions (see

Definition 3.8 below), there is a $\delta_g > 0$ such that

$$\limsup_{N\to\infty} \tfrac{1}{N} \log\left(\frac{\hat{\mathcal{Z}}_N^{\mathbf{r}}}{\mathcal{Z}_N^{\mathbf{0}}}\hat{\gamma}_N(B(g,\delta_g))\right) \leq -\mathcal{E}(g) + \epsilon. \tag{3.26}$$

Then we boost the local control (3.26) to a global control: Since the very-well-behaved functions are dense in $H_{\mathbf{r}}$ (see Lemma 3.14 below) the compact set K has a finite subcover $\bigcup_{i=1}^{k} B(g_i, \delta_i)$, where $g_i \in K$ and $\delta_i = \delta_{g_i}$; and hence

$$\begin{aligned}
\limsup_{N\to\infty} \tfrac{1}{N} \log\left(\frac{\hat{\mathcal{Z}}_N^{\mathbf{r}}}{\mathcal{Z}_N^{\mathbf{0}}}\hat{\gamma}_N(K)\right) &\leq \limsup_{N\to\infty} \tfrac{1}{N} \log\left(\sum_{i=1}^{k}\frac{\hat{\mathcal{Z}}_N^{\mathbf{r}}}{\mathcal{Z}_N^{\mathbf{0}}}\hat{\gamma}_N(B(g_i,\delta_i))\right) \\
&\leq \max_{i=1,2,\ldots,k} \limsup_{N\to\infty} \tfrac{1}{N} \log\left(\frac{\hat{\mathcal{Z}}_N^{\mathbf{r}}}{\mathcal{Z}_N^{\mathbf{0}}}\hat{\gamma}_N(B(g_i,\delta_i))\right) \\
&\leq \max_{i=1,2,\ldots,k} (-\mathcal{E}(g_i) + \epsilon) = -\min_{i=1,2,\ldots,k} (\mathcal{E}(g_i)) + \epsilon \\
&\leq -\inf_{g\in K} \mathcal{E}(g) + \epsilon.
\end{aligned}$$

To obtain (3.25), note that $\epsilon > 0$ was arbitrary.

Definition 3.8. A function $g \in C(0,1)$ is very-well-behaved if for all sufficiently small $\delta \geq 0$ the set

$$\mathcal{I}(\delta) := \{\xi \in [0,1] \mid |g(\xi)| \leq \delta\} \tag{3.27}$$

is the union of finitely many intervals.

In Lemma 3.14, at the end of this section, we prove that with respect to the $\|\cdot\|_\infty$ norm the very-well-behaved functions are a dense subset of $H_{\mathbf{r}}$.

Now we prove (3.26). To keep the notation simple, we present the argument only for very-well-behaved functions g such that for all $\delta > 0$ small enough $\mathcal{I}(\delta)$ is one interval. At the end of this section we briefly discuss what has to be changed for general very-well-behaved functions.

We use the two stage interpretation (1.6). For many choices of the set $\mathcal{P} \subset \Lambda_N$ the probability to observe $B(g,\delta)$ in the second experiment is zero: Since for any $\xi \in (\mathcal{I}(\delta))^c$ we have $h(\xi) \neq 0$ for all $h \in B(g,\delta)$, we have

$$\gamma_{\mathcal{P}^c}^{\psi}(B(g,\delta)) = 0 \text{ if } \mathcal{P} \cap (\mathcal{I}(\delta))_N^c \neq \emptyset. \tag{3.28}$$

Let $\mathcal{I}_N(\delta) := N\mathcal{I}(\delta) \cap \mathbb{Z}$. Applying (3.28) to (1.6) we get

$$\hat{\gamma}_N(B(g,\delta)) = \tfrac{1}{\hat{\mathcal{Z}}_N^{\mathbf{r}}} \sum_{\mathcal{P}\subset\mathcal{I}_N(\delta)} e^{J|\mathcal{P}|} \mathcal{Z}_{\mathcal{P}^c}^{\psi} \gamma_{\mathcal{P}^c}^{\psi}(B(g,\delta)), \tag{3.29}$$

where $\psi = \psi^{\mathbf{r},N}$, for the definition of $\psi^{\mathbf{r},N}$ see (1.13).

In the next lemma we consider $\mathcal{Z}^{\psi}_{\mathcal{P}^c}\gamma^{\psi}_{\mathcal{P}^c}(B(g,\delta))$.

Lemma 3.9. *Let g be a very-well-behaved function such that for all sufficiently small $\delta > 0$ the set $\mathcal{I}(\delta)$ is one interval. Fix $\epsilon > 0$. There is a radius $\delta' > 0$ and a value $N' \in \mathbb{N}$ such that for all $N > N'$, $\delta \le \delta'$ and non empty $\mathcal{P} \subset \mathcal{I}_N(\delta)$ we have*

$$\mathcal{Z}^{\psi}_{\mathcal{P}^c}\gamma^{\psi}_{\mathcal{P}^c}(B(g,\delta)) \le \mathcal{Z}^{\mathbf{0}}_{\mathbf{c}(\mathcal{P})^c} e^{-N(Q(g)-\epsilon)}, \tag{3.30}$$

where

$$\mathbf{c}(\mathcal{P}) := \begin{cases} \mathcal{P} \cup \{p_* + 1, p^* - 1\} & \text{, for } p^* - p_* \ge 2, \\ \mathcal{P} & \text{, for } p^* - p_* = 1, \\ \mathcal{P} \cup \{p_* + 1\} & \text{, for } p^* - p_* = 0, \\ \emptyset & \text{, for } \mathcal{P} = \emptyset, \end{cases} \tag{3.31}$$

recall that $p_* = \min \mathcal{P}$ and $p^* = \max \mathcal{P}$ if $\mathcal{P} \neq \emptyset$.

We give the proof of the lemma at the end of this section. We now determine the value of δ_g for which (3.26) holds. Let δ_g satisfy the following three conditions:

$$\begin{aligned} &\text{a) } \mathcal{I}(\delta_g) \text{ is an interval,} \\ &\text{b) } \delta_g < \delta' \text{ (where } \delta' \text{ is defined in Lemma 3.9),} \\ &\text{c) } |\mathcal{I}(\delta_g)| - |\mathcal{N}_g| < \epsilon. \end{aligned} \tag{3.32}$$

To show that for $\delta = \delta_g$ the upper bound (3.26) is satisfied, we apply Lemma 3.9 to (3.29) to get

$$\begin{aligned} \frac{\hat{\mathcal{Z}}^{\mathbf{r}}_N}{\mathcal{Z}^{\mathbf{0}}_N}\hat{\gamma}_N(B(g,\delta)) &\le e^{-N(Q(g)-\epsilon)} \frac{1}{\mathcal{Z}^{\mathbf{0}}_N} \sum_{\mathcal{P} \subset \mathcal{I}_N(\delta)} e^{J|\mathcal{P}|} \mathcal{Z}^{\mathbf{0}}_{\mathbf{c}(\mathcal{P})^c} \\ &\le e^{-N(Q(g)-\epsilon)} \frac{1}{\mathcal{Z}^{\mathbf{0}}_N} 4 \sum_{\mathcal{P} \subset \mathcal{I}_N(\delta), \mathcal{P} \in \mathcal{G}} e^{J|\mathcal{P}|} \mathcal{Z}^{\mathbf{0}}_{\mathcal{P}^c}, \end{aligned} \tag{3.33}$$

where we have used that $\mathbf{c}(\mathcal{P})$ is a good set for all $\mathcal{P} \subset \Lambda_N$ and that for each good set $\mathcal{P} \in \mathcal{G}$ there are up to 4 sets $\tilde{\mathcal{P}} \subset \Lambda_N$ such that $\mathbf{c}(\tilde{\mathcal{P}}) = \mathcal{P}$. As for the lower bound we make use of (3.12). Therefore we temporarily ignore the term for $\mathcal{P} = \emptyset$ and group the remaining terms in the sum on the right hand side of (3.33) appropriately:

$$\begin{aligned}\frac{1}{\mathcal{Z}_N^{\mathbf{0}}}\sum_{\mathcal{P}\subset\mathcal{I}_N,\mathcal{P}\in\mathcal{G}\setminus\{\emptyset\}} e^{J|\mathcal{P}|}\mathcal{Z}_{\mathcal{P}^c}^{\mathbf{0}} &= \frac{1}{\mathcal{Z}_N^{\mathbf{0}}}\sum_{0<i<k<N}\sum_{\substack{\mathcal{P}\subset\mathcal{I}_N(\delta),\\ \mathcal{P}\in\mathcal{G}\setminus\{\emptyset\}\\ \text{s.t. } p_*=i,p^*=k}} e^{J|\mathcal{P}|}\mathcal{Z}_{(0,i)}^{\mathbf{0}}\mathcal{Z}_{(i+1,k-1)\setminus\mathcal{P}}^{\mathbf{0}}\mathcal{Z}_{(k,N)}^{\mathbf{0}}\\ &= \frac{1}{\mathcal{Z}_N^{\mathbf{0}}}\sum_{\mathcal{I}_{N,*}\le i<k\le\mathcal{I}_N^*}\mathcal{Z}_{(0,i)}^{\mathbf{0}}\hat{\mathcal{Z}}_{(i+1,k-1)}^{\mathbf{0}}\mathcal{Z}_{(k,N)}^{\mathbf{0}}\\ &\le \frac{1}{\mathcal{Z}_N^{\mathbf{0}}}N^2\max_{\mathcal{I}_{N,*}\le i<k\le\mathcal{I}_N^*}\mathcal{Z}_{(0,i)}^{\mathbf{0}}\hat{\mathcal{Z}}_{(i+1,k-1)}^{\mathbf{0}}\mathcal{Z}_{(k,N)}^{\mathbf{0}},\end{aligned}\tag{3.34}$$

where $\mathcal{I}_{N,*} := \min\mathcal{I}_N(\delta)$ and $\mathcal{I}_N^* := \max\mathcal{I}_N(\delta)$. Recall that we deal with a product like $\mathcal{Z}_{(0,i)}^{\mathbf{0}}\hat{\mathcal{Z}}_{(i+1,k-1)}^{\mathbf{0}}\mathcal{Z}_{(k,N)}^{\mathbf{0}}$ already in (3.13). So by Lemma 1.1 and Proposition C.1 there is a N'' such that for all $N > N''$

$$\frac{\mathcal{Z}_{(0,i)}^{\mathbf{0}}\hat{\mathcal{Z}}_{(i+1,k-1)}^{\mathbf{0}}\mathcal{Z}_{(k,N)}^{\mathbf{0}}}{\mathcal{Z}_N^{\mathbf{0}}} \le e^{\tau(J)(k-i-2)+N\epsilon}, \text{ for all } \mathcal{I}_{N,*}\le i<k\le\mathcal{I}_N^*.\tag{3.35}$$

The term that we ignored was the term for $\mathcal{P}=\emptyset$. Since $\mathcal{Z}_{\{\emptyset\}^c}^{\mathbf{0}} = \mathcal{Z}_N^{\mathbf{0}}$ and $\tau(J)\ge 0$, we have by (3.34) and (3.35) that

$$\frac{1}{\mathcal{Z}_N^{\mathbf{0}}}\sum_{\mathcal{P}\subset\mathcal{I}_N(\delta),\mathcal{P}\in\mathcal{G}} e^{J|\mathcal{P}|}\mathcal{Z}_{\mathcal{P}^c}^{\mathbf{0}} \le N^2 e^{\tau N[|\mathcal{I}(\delta)|+\epsilon]},\tag{3.36}$$

for N large enough.

Combining statements (3.33) and (3.36) we see that (3.26) is true for all very-well-behaved g and for a δ_g that satisfies (3.32).

In the next remark we briefly discuss how we have to change our approach if the well-behaved function g is such that for all sufficiently small $\delta > 0$ the set $\mathcal{I}(\delta)$ is not an interval.

Remark 3.10. We consider a very-well-behaved g such that for all sufficiently small $\delta > 0$ the set $\mathcal{I}(\delta)$ is the union of $M\in\mathbb{N}$ intervals $(\mathcal{I}_i)_{i\in\{1,\dots,M\}}$. The main difference to the case where the set $\mathcal{I}(\delta)$ is only one interval is the definition of $\mathbf{c}(\mathcal{P})$ in Lemma 3.9. Let $\mathcal{I}_{i,N} = N\mathcal{I}_i\cap\mathbb{Z}$. We let $\mathbf{c}(\mathcal{P})$ be the set $\mathbf{c}(\mathcal{P})\supset\mathcal{P}$ such that $\mathbf{c}(\mathcal{P})\cap\mathcal{I}_{i,N}$ contains the left nearest neighbour of its maximum and the right nearest neighbour of its minimum. We give a sketch of this situation in Figure 3.2. In the upper bound (3.33) we have to replace the constant 4 by 4^M.

Collection of remaining proofs

In this section we give the proofs of Lemma 3.9 (including two auxiliary results),

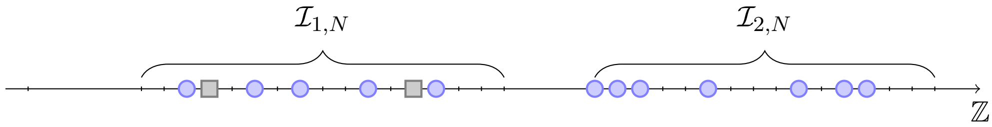

Figure 3.2: This figure illustrates Remark 3.10: This is a sketch of the situation when $\mathcal{I}(\delta)$ (see (3.27)) consists of two intervals $\mathcal{I}_1$ and $\mathcal{I}_2$. In Remark 3.10 we describe how the definition of $\mathbf{c}(\mathcal{P})$ (see (3.31)) has to be changed if $\mathcal{I}(\delta)$ consists of more than one interval: If the blue circles are the elements of $\mathcal{P}$ then the corrected set $\mathbf{c}(\mathcal{P})$ contains the blue circles and the grey squares. The grey squares are added because, by Remark 3.10, $\mathbf{c}(\mathcal{P}) \cap \mathcal{I}_{1,N}$ must contain the left nearest neighbour of its maximum and the right nearest neighbour of its minimum.

the exponential tightness of the family $(\hat{\gamma}_N)_{N\in\mathbb{N}}$ (see Lemma 3.13) and that the very-well-behaved functions are dense in $(C(0,1), \|\cdot\|_\infty)$ (see Lemma 3.14).

Proof of Lemma 3.9. In this proof we let $B_\delta := B(g,\delta)$. We have to show that there are $\delta' > 0$ and $N' \in \mathbb{N}$ such that (3.30) holds for all $\delta \le \delta'$ and $\mathcal{P} \subset \mathcal{I}_N(\delta)$.

The proof is simplified by the observation that if (3.30) holds for $\delta = \delta'$ and all $\mathcal{P} \subset \mathcal{I}_N(\delta')$, then (3.30) also holds for $\delta < \delta'$ and all $\mathcal{P} \subset \mathcal{I}_N(\delta)$. To see this let $\delta < \delta'$ then, since $B(g,\delta) \subset B(g,\delta')$, we have $\mathcal{Z}^{\psi}_{\mathcal{P}^c}\gamma^{\psi}_{\mathcal{P}^c}(B(g,\delta)) \le \mathcal{Z}^{\psi}_{\mathcal{P}^c}\gamma^{\psi}_{\mathcal{P}^c}(B(g,\delta'))$ and note that the upper bound in (3.30) does not depend on δ. Furthermore note that $\mathcal{I}_N(\delta) \subset \mathcal{I}_N(\delta')$.

Depending on which one of the following equations is satisfied

$$\begin{aligned} \mathbf{c}(\mathcal{P}) &= \mathcal{P}, \\ \mathbf{c}(\mathcal{P}) \setminus \mathcal{P} &= \{p_* + 1\}, \\ \mathbf{c}(\mathcal{P}) \setminus \mathcal{P} &= \{p^* - 1\}, \\ \mathbf{c}(\mathcal{P}) \setminus \mathcal{P} &= \{p_* + 1, p^* - 1\}, \end{aligned} \tag{3.37}$$

we distinguish four types of sets $\mathcal{P} \subset \{1, 2, \ldots, N-1\}$. For each type we show that there are $\delta' > 0$ and $N' \in \mathbb{N}$ such that (3.30) holds for $\delta = \delta'$ and all $\mathcal{P} \subset \mathcal{I}_N(\delta')$ of that type. We illustrate these four types in Figure 3.3

In the following it is not necessary to differentiate between cases for g. But phenomenologically there is a difference between functions g that have an interval as zero set and functions that cross zero once: Intuitively, if g has an interval as zero set, then $\mathcal{P}$ typically satisfies $\mathbf{c}(\mathcal{P}) = \mathcal{P}$ or $p_* \neq p^*$ and if g crosses zero, then $\mathcal{P}$ typically satisfies $p_* = p^*$.

Type $\mathbf{c}(\mathcal{P}) = \mathcal{P}$: Note that $\mathbf{c}(\mathcal{P}) = \mathcal{P}$ if and only if $\mathcal{P}$ is a good set. We

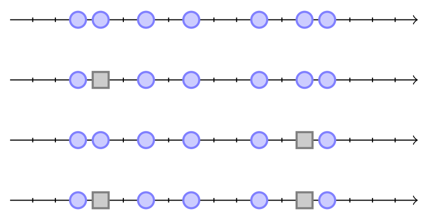

Figure 3.3: These figures illustrate the four possible ways how the map $\mathbf{c}$ corrects a set $\mathcal{P}$ (see (3.37)). Each of this four lines is one example for a set $\mathcal{P}$, where the elements of $\mathcal{P}$ are the blue circles. The grey squares are the points that are added to $\mathcal{P}$ in order to obtain $\mathbf{c}(\mathcal{P})$. The first set is good and hence no point is added, in the second set the neighbour of the minimum of $\mathcal{P}$ has to be added, in the third set the neighbour of the maximum of $\mathcal{P}$ has to be added and in the fourth set the neighbours of the maximum and the minimum of $\mathcal{P}$ have to be added.

seek $N' \in \mathbb{N}$ and $\delta' > 0$ such that (3.30) holds for all $\mathcal{P} \subset \mathcal{I}_N(\delta')$ where $\mathcal{P}$ is a non empty good set. Since $\mathcal{I}_N(\delta')$ depends on δ' we start by defining δ'. Therefore let $\delta'_1 > 0$ be such that for all $\delta < \delta'_1$ the set $\mathcal{I}(\delta)$ is one interval and we have

$$Q^{(\mathbf{a},\mathbf{0})}_{(0,l)}(B_\delta) + Q^{(\mathbf{0},\mathbf{b})}_{(r,1)}(B_\delta) \geq Q^{(\mathbf{a},\mathbf{b})}_{(0,1)}(g) - \epsilon \text{ for all } l, r \in \mathcal{I}(\delta), l < r, \tag{3.38}$$

where

$$Q^{\mathbf{r}}_I(g) := \begin{cases} Q_I(g) & \text{, for } g \in H_{\mathbf{r}}(I), \\ \infty & \text{, otherwise.} \end{cases}$$

Note that such a δ'_1 exists by Lemma 3.11 below. Fix δ' such that $\delta' < \frac{1}{2}\delta'_1$.

Fix a good set $\mathcal{P} \subset \mathcal{I}_N(\delta')$. By the splitting property (1.7), we have

$$\begin{aligned} \mathcal{Z}^\psi_{\mathcal{P}^c}\gamma^\psi_{\mathcal{P}^c}(B_{\delta'}) = {} & \mathcal{Z}^\psi_{(0,p_*)}\mathcal{Z}^\psi_{(p_*+1,p^*-1)\setminus\mathcal{P}}\mathcal{Z}^\psi_{(p^*,N)} \\ & \times \gamma^\psi_{(0,p_*)}(B_{\delta'})\gamma^\psi_{(p_*+1,p^*-1)\setminus\mathcal{P}}(B_{\delta'})\gamma^\psi_{(p^*,N)}(B_{\delta'}). \end{aligned}$$

Recall $\psi = \psi^{\mathbf{r},N}$ (see (1.13)). Using Proposition D.2 and the trivial upper bound 1 for the second probability, we see

$$\begin{aligned} \mathcal{Z}^\psi_{\mathcal{P}^c}\gamma^\psi_{\mathcal{P}^c}(B_{\delta'}) \leq {} & \mathcal{Z}^{\mathbf{0}}_{\mathcal{P}^c} e^{-(\mathcal{H}_{(0,p_*)}(\phi_L)+\mathcal{H}_{(p^*,N)}(\phi_R))} \\ & \times \gamma^\psi_{(0,p_*)}(B_{\delta'})\gamma^\psi_{(p^*,N)}(B_{\delta'}), \end{aligned} \tag{3.39}$$

where ϕ_L is the minimiser of $\mathcal{H}_{(0,p_*)}$ for the boundary condition $\phi(-1) = N^2a - N\alpha$, $\phi(0) = N^2a$, $\phi(p_*) = 0$ and $\phi(p_*+1) = 0$ and where ϕ_R is the minimiser of $\mathcal{H}_{(p^*,N)}$ for the boundary condition $\phi(p^*) = 0$, $\phi(p^*+1) = 0$, $\phi(N) = N^2b$ and

$\phi(N+1) = N^2 b + N\beta$.

To obtain upper bounds for $(\gamma^{\psi}_{(0,p_*)}(B_{\delta'}))_{p_* \in \mathcal{I}_N(\delta')}$ and $(\gamma^{\psi}_{(p^*,N)}(B_{\delta'}))_{p^* \in \mathcal{I}_N(\delta')}$ we use the uniform **LDP** from Corollary 2.12; the reason why we need Corollary 2.12 (and why the **LDP** is not sufficient) is that we need an upper bound for a family of measures that is parameterized by the location p_*. To apply Corollary 2.12 there has to be a constant $l > 0$ such that for all $p_*, p^* \in \mathcal{I}_N(\delta')$ the lengths of the intervals $(0, {}^{p_*}/_N)$ and $({}^{p^*}/_N, 1)$ are bounded below by l. Note that this is the case if the set $\mathcal{I}(\delta')$ does not contain 0 or 1. The special case that there is no such l (even for $\delta' > 0$ sufficiently small) is treated at the end of this case; for now we assume that for all $p_*, p^* \in \mathcal{I}_N(\delta')$ the lengths of the intervals $(0, {}^{p_*}/_N)$ and $({}^{p^*}/_N, 1)$ are bounded below by l. To apply Corollary 2.12 we also have to fix a value $r > 0$. We will see that $r = \delta'$ is a suitable choice.

By Corollary 2.12 and the definition of $\Sigma^{\mathbf{r}}_I$ (see (2.31)), there is a $N_1'(r)$ such that

$$\begin{aligned}
\gamma^{\psi}_{(0,p_*)}(B_{\delta'}) &\leq e^{-N(\Sigma^{(\mathbf{a},\mathbf{0})}_{(0,p_*/N)}(B_{\delta'+r}) - \epsilon/3)} = e^{-N(Q^{(\mathbf{a},\mathbf{0})}_{(0,p_*/N)}(B_{\delta'+r}) - Q^{(\mathbf{a},\mathbf{0})}_{(0,p_*/N)}(h_L) - \epsilon/3)}, \\
\gamma^{\psi}_{(p^*,N)}(B_{\delta'}) &\leq e^{-N(\Sigma^{(\mathbf{0},\mathbf{b})}_{(p^*/N,1)}(B_{\delta'+r}) - \epsilon/3)} = e^{-N(Q^{(\mathbf{0},\mathbf{b})}_{(p^*/N,1)}(B_{\delta'+r}) - Q^{(\mathbf{0},\mathbf{b})}_{(p^*/N,1)}(h_R) - \epsilon/3)},
\end{aligned} \tag{3.40}$$

for all $p_*, p^* \subset \mathcal{I}_N(\delta')$ and all $N > N_1'(r)$; where h_L minimises $Q_{(0,p_*/N)}$ in $H_{(\mathbf{a},\mathbf{0})}(0, {}^{p_*}/_N)$ and h_R minimises $Q_{(p^*/N,1)}$ in $H_{(\mathbf{0},\mathbf{b})}({}^{p^*}/_N, 1)$.

The Hamiltonians appearing on the right hand side of (3.39) and the terms depending on h_L and h_R on the right hand sides of (3.40) approximately cancel each other out. To see this note that by Lemma D.3, there is a N_2' such that

$$\tfrac{1}{N}(\mathcal{H}_{(0,p_*)}(\phi_L) + \mathcal{H}_{(p^*,N)}(\phi_R)) \geq Q^{(\mathbf{a},\mathbf{0})}_{(0,p_*/N)}(h_L) + Q^{(\mathbf{0},\mathbf{b})}_{(p^*/N,1)}(h_R) - \tfrac{\epsilon}{3}, \tag{3.41}$$

for all p_* and p^* and all $N > N_2'$.

Now we consider the terms on the right hand sides of (3.40) that depend on the ball $B_{\delta'+r}$. Since $\delta' + r < \delta_1'$ we have

$$Q^{(\mathbf{a},\mathbf{0})}_{(0,p_*/N)}(B_{\delta'+r}) + Q^{(\mathbf{0},\mathbf{b})}_{(p^*/N,1)}(B_{\delta'+r}) \geq Q(g) - \tfrac{\epsilon}{3}. \tag{3.42}$$

Let $N' = \max(N_1'(r), N_2')$; by (3.39), (3.40), (3.41) and (3.42) we see that (3.30) holds for non empty good sets $\mathcal{P} \subset \mathcal{I}_N(\delta')$ if $\mathcal{I}_N(\delta')$ does not contain 0 or 1.

Now we consider the special case where $\mathcal{I}(\delta')$ contains 0 or 1. Here we only need to discuss the cases where g has a zero in 0 or 1; because if not then we can choose a smaller value for δ' for which $\mathcal{I}(\delta')$ does not contain 0 and 1. So assume that g has a zero in 0 or 1. If 0 and 1 are both elements of the zero set of g, then, since $\mathcal{I}(\delta)$ is one interval for all $\delta \le \delta'$, the function g is equal to zero. In this case, since $Q(g) = 0$, we obtain (3.30) from (3.39). The cases where either 0 or 1 is an element of the zero set of g are symmetric. Here we only discuss the case where 0 is an element of the zero set of g.

The proof of this special case is also based on (3.39). Since g is not zero at 1, the lengths of the intervals $({p^*}/{N}, 1)$ have a common minimal lengths and hence we can apply Corollary 2.12 to $(\gamma^{\psi}_{(p^*,N)}(B_{\delta'}))_{p^*\in\mathcal{I}_N(\delta')}$. To deal with the family $(\gamma^{\psi}_{(0,p_*)}(B_{\delta'}))_{p_*\in\mathcal{I}_N(\delta')}$, we use that for a possibly smaller δ' we have $Q_{(0,l)}(g) < \frac{2}{3}\epsilon$ for all $l \le \sup \mathcal{I}(\delta')$. Such a δ' exists because $Q_{(0,l)}(g) \to 0$ for $l \to 0$.

So for $l \le \sup \mathcal{I}(\delta')$ we have

$$1 \le e^{-N(Q_{(0,l)}(g) - {2\epsilon}/{3})} \le e^{-N(Q_{(0,l)}(B_{\delta'+r}) - {2\epsilon}/{3})}.$$

Since Corollary 2.12 is applicable to all $(\gamma^{\psi}_{(p^*,N)}(B_{\delta'}))_{p^*\in\mathcal{I}_N(\delta')}$ there is a $N_1'(r)$ such that

$$\begin{aligned} \gamma^{\psi}_{(0,p_*)}(B_{\delta'}) e^{-\mathcal{H}_{(0,p_*)}(\phi_L)} &\le 1 \le e^{-N(Q^{(\mathbf{a},\mathbf{0})}_{(0,{p_*}/{N})}(B_{\delta'+r}) - {2\epsilon}/{3})}, \\ \gamma^{\psi}_{(p^*,N)}(B_{\delta'}) e^{-\mathcal{H}_{(p^*,N)}(\phi_R)} &\le e^{-N(Q^{(\mathbf{0},\mathbf{b})}_{({p^*}/{N},1)}(B_{\delta'+r}) - {2\epsilon}/{3})}, \end{aligned} \tag{3.43}$$

for all $p_*, p^* \in \mathcal{I}_N(\delta)$ and $N > N_1'(r)$.

Combining (3.43) and (3.42), we see that for $N' = N_1'(r)$ statement (3.30) holds for non empty good sets $\mathcal{P} \subset \mathcal{I}_N(\delta')$ also if $\mathcal{I}_N(\delta')$ contain 0 or 1.

Type $\mathbf{c}(\mathcal{P}) \setminus \mathcal{P} = \{p_* + 1\}$, Case 1: We restrict ourselves to $|\mathcal{P}| = 1$: We seek $N' \in \mathbb{N}$ and $\delta' > 0$ such that (3.30) holds for $\delta = \delta'$ and $N > N'$ and all $\mathcal{P} = \{p_*\}$ such that $p_* \in \mathcal{I}_N(\delta')$.

As before we first fix a suitable δ'. The δ' for this type is possibly smaller than the one for the good sets because the value $\phi(p_* + 1)$ is not fixed to zero. To fix δ' let δ_1' be such that for all $\delta < \delta_1'$ the set $\mathcal{I}(\delta)$ is one interval and we have

$$Q^{(\mathbf{a},\mathbf{c})}_{(0,l)}(B_\delta) + Q^{(\mathbf{d},\mathbf{b})}_{(r,1)}(B_\delta) \ge Q^{(\mathbf{a},\mathbf{b})}_{(0,1)}(g) - \epsilon \text{ for all } l, r \in \mathcal{I}(\delta), l < r \text{ and all } \mathbf{c}, \mathbf{d} \in \mathbb{R}^2. \tag{3.44}$$

Such a δ_1' exists by Lemma 3.11 below. Let δ' be such that for all $\delta \le \delta'$ the set $\mathcal{I}(\delta)$ is an interval and such that $\delta' < \frac{1}{2}\delta_1'$. Now we show that there is a $N' \in \mathbb{N}$ such that for $\delta = \delta'$ the inequality (3.30) is satisfied for all $N > N'$ and all $\mathcal{P} = \{p_*\}$ such that $p_* \in \mathcal{I}_N(\delta')$.

We express $\mathcal{Z}^{\psi}_{\{p_*\}^c}\gamma^{\psi}_{\{p_*\}^c}(B_{\delta'})$ by partition functions and measures that are conditioned at p_* and p_*+1, therefore we use a generalisation of the law of total expectation: Since $\gamma^{\psi}_{\{p_*,p_*+1\}^c}$ coincides with the measure $\gamma^{\psi}_{\{p_*\}^c}$ given that $\phi(p_*+1)=\psi(p_*+1)$ and since $\mathcal{Z}^{\psi}_{\{p_*,p_*+1\}^c}$ satisfies an analogous property we have

$$\mathcal{Z}^{\psi}_{\{p_*\}^c}\gamma^{\psi}_{\{p_*\}^c}(B_{\delta'}) = \int_D \mathcal{Z}^{v}_{\{p_*,p_*+1\}^c}\gamma^{v}_{\{p_*,p_*+1\}^c}(B_{\delta'})\,\mathrm{d}v, \tag{3.45}$$

where $\mathcal{Z}^{v}_{\{p_*,p_*+1\}^c} := \mathcal{Z}^{\psi_v}_{\{p_*,p_*+1\}^c}$ with

$$\psi_v(i) := \begin{cases} v & \text{, for } i = p_*+1, \\ \psi^{\mathbf{r},N}(i) & \text{, otherwise,} \end{cases} \tag{3.46}$$

and where $\gamma^{v}_{\{p_*,p_*+1\}^c} := \gamma^{\psi_v}_{\{p_*,p_*+1\}^c}$ and $D = N^2[g(^{(p_*+1)}/_N)-\delta', g(^{(p_*+1)}/_N)+\delta']$. Recall that $\psi^{\mathbf{r},N}(i)=0$ for $i \in \Lambda_N$. Since for any profile which attains the value 0 at p_* and the value Nv at p_*+1 the empirical profile has the gradient v at $^{p_*}/_N$, we express (3.45) as follows

$$\mathcal{Z}^{\psi}_{\{p_*\}^c}\gamma^{\psi}_{\{p_*\}^c}(B_{\delta'}) = N\int_{D/N} \mathcal{Z}^{Nv}_{\{p_*,p_*+1\}^c}\gamma^{Nv}_{\{p_*,p_*+1\}^c}(B_{\delta'})\,\mathrm{d}v. \tag{3.47}$$

Note that $g(^{(p_*+1)}/_N)$ is not necessarily equal to zero, but, by definition of $\mathcal{I}(\delta')$, we have $|g(^{p_*}/_N)| < \delta'$ and since g is continuous we have $|g(^{(p_*+1)}/_N)| < 2\delta'$ for N large enough. So

$$\begin{aligned} \mathcal{Z}^{\psi}_{\{p_*\}^c}\gamma^{\psi}_{\{p_*\}^c}(B_{\delta'}) &\le N\int_{-2\delta' N}^{2\delta' N} \mathcal{Z}^{Nv}_{\{p_*,p_*+1\}^c}\gamma^{Nv}_{\{p_*,p_*+1\}^c}(B_{\delta'})\,\mathrm{d}v \\ &= N\mathcal{Z}^{\mathbf{0}}_{\{p_*,p_*+1\}^c}\int_{-2\delta' N}^{2\delta' N} e^{-\mathcal{H}_N(\phi^*(v))}\gamma^{Nv}_{\{p_*,p_*+1\}^c}(B_{\delta'})\,\mathrm{d}v, \end{aligned} \tag{3.48}$$

where $\phi^*(v)$ minimises $\mathcal{H}_N$ in $H^N_{\mathbf{r}}$ under the additional condition that $\phi(p_*)=0$ and $\phi(p_*+1)=Nv$.

Now we consider the integral in (3.48). We show that there is a $w>0$ and a $\bar{N} \in \mathbb{N}$ such that for $N > \bar{N}$ the two terms on the right hand side of the following equation

$$\begin{aligned} \int_{-2\delta' N}^{2\delta' N} e^{-\mathcal{H}_N(\phi^*(v))}\gamma^{Nv}_{\{p_*,p_*+1\}^c}(B_{\delta'})\,\mathrm{d}v &= \int_{-w}^{w} e^{-\mathcal{H}_N(\phi^*(v))}\gamma^{Nv}_{\{p_*,p_*+1\}^c}(B_{\delta'})\,\mathrm{d}v \\ &\quad + \int_{w<|v|<2\delta' N} e^{-\mathcal{H}_N(\phi^*(v))}\gamma^{Nv}_{\{p_*,p_*+1\}^c}(B_{\delta'})\,\mathrm{d}v \end{aligned} \tag{3.49}$$

are both bounded from above by

$$e^{-N(Q(g)-\epsilon/3)}. \tag{3.50}$$

For the moment let w be arbitrary but fixed. First we consider the second term on the right hand side of (3.49). We use that

$$e^{-\mathcal{H}_N(\phi^*(v))}\gamma^{Nv}_{\{p_*,p_*+1\}^c}(B_{\delta'}) \leq e^{-\mathcal{H}_N(\phi^*(v))} \leq e^{-\mathcal{H}_{(0,p_*)}(\phi^*(v))}.$$

We claim that

$$\mathcal{H}_{(0,p_*)}(\phi^*(v)) \geq p_* \tfrac{N^2}{(p_*+1)^2}(\alpha - v)^2. \tag{3.51}$$

Indeed, the right hand side can be obtained as the value of $\mathcal{H}_{(0,p_*)}(\phi^{**})$, where ϕ^{**} minimises $\mathcal{H}_{(0,p_*)}$ under the constraint that $\phi(0)-\phi(-1) = N\alpha$ and $\nabla\phi(p_*) = Nv$ (but without constraints for $\phi(0)$ and $\phi(N)$). Since by (D.9) a minimiser of $\mathcal{H}_{(0,p_*)}$ is a discrete polynomial of degree 3 the discrete Laplacian of the minimiser ϕ^{**} is constant $i \mapsto \frac{N}{p_*+1}(\alpha - v)$ (because $\sum_{i=0}^{p_*} \Delta\phi_i = \nabla\phi(p_*) - [\phi(0)-\phi(-1)]$). Thus the Hamiltonian of ϕ^{**} coincides with the right hand side of (3.51).

Since $\lim_{|v|\to\infty}(\alpha - v)^2 = \infty$ and since there is a constant C such that $\frac{N^2 p_*}{(p_*+1)^2} \geq CN$ for all $p_* \in \Lambda_N$, there is a w such that $C(\alpha - v)^2 \geq Q(g)$ for $|v| > w$ and hence for this w we have

$$\mathcal{H}_{(0,p_*)}(\phi^*(v)) \geq NQ(g), \text{ for all } v \text{ such that } |v| > w.$$

For this w the second term on the right hand side of (3.49) has the following upper bound:

$$\int_{w<|v|<2\delta' N} e^{-\mathcal{H}_N(\phi^*(v))}\gamma^{Nv}_{\{p_*,p_*+1\}^c}(B_{\delta'})\,\mathrm{d}v \leq 2\delta' N e^{-NQ(g)}.$$

So clearly there is a $\bar{N}_1$ and a w such that the second term on the right hand side of (3.49) satisfies the bound (3.50) for $N > \bar{N}_1$.

Now we consider the first term on the right hand side of (3.49). The following argument is valid for any choice of $w > 0$, because we will only use that $[-w, w]$ is compact. By the splitting property with lag 2 (see (1.7)), we have

$$\begin{aligned} e^{-\mathcal{H}_N(\phi^*(v))}\gamma^{Nv}_{\{p_*,p_*+1\}^c}(B_{\delta'}) &= e^{-\mathcal{H}_{(0,p_*)}(\phi^*(v))}\gamma^{Nv}_{(0,p_*)}(B_{\delta'}) \\ &\quad \times e^{-\mathcal{H}_{(p_*+1,N)}(\phi^*(v))}\gamma^{Nv}_{(p_*+1,N)}(B_{\delta'}). \end{aligned} \tag{3.52}$$

The following argument is similar to our argument for good sets. We apply Corollary 2.12 and Lemma D.3. Since v is an element of a compact set, Corollary 2.12 is applicable if there is a $l > 0$ such that $|\frac{1}{N}p_*| > l$ and such that $|1 - \frac{1}{N}p_*| > l$. For the following discussion we assume that this is the case. During this discussion we will see that we can treat the special case where there is no such l as above.

To apply Corollary 2.12 we also have to fix a value $r > 0$. Let $r = \frac{1}{2}\delta_1'$. Now we apply Corollary 2.12 and Lemma D.3 in a way that is similar to the one that we use in the two steps (3.40) and (3.41). Here we combine these two steps into one. So by Corollary 2.12 and Lemma D.3 there is a $N_2(r)$ such that for $N > N_2(r)$

$$\begin{aligned} e^{-\mathcal{H}_{(0,p_*)}(\phi^*(v))}\gamma^{Nv}_{(0,p_*)}(B_{\delta'}) &\leq e^{-N(Q^{(\mathbf{a},0,v)}_{(0,p_*/N)}(B_{\delta'+r})-\epsilon)}, \\ e^{-\mathcal{H}_{(p_*+1,N)}(\phi^*(v))}\gamma^{Nv}_{(p_*+1,N)}(B_{\delta'}) &\leq e^{-N(Q^{(0,v,\mathbf{b})}_{((p_*+1)/N,1)}(B_{\delta'+r})-\epsilon)}, \end{aligned} \tag{3.53}$$

for all $v \in [-w, w]$ and p_* such that $\frac{1}{N}p_* \in \mathcal{I}(\delta')$.

We consider the terms on the right hand sides of (3.53) that depend on $B_{\delta'+r}$. Since $\delta' + r < \delta_1'$ we have by (3.44) that

$$Q^{(\mathbf{a},0,v)}_{(0,p_*/N)}(B_{\delta'+r}) + Q^{(0,v,\mathbf{b})}_{((p_*+1)/N,1)}(B_{\delta'+r}) \geq Q_{(0,1)}(g) - \epsilon. \tag{3.54}$$

Combining (3.54) and (3.53) we see that

$$e^{-\mathcal{H}_N(\phi^*(v))}\gamma^{Nv}_{\{p_*,p_*+1\}^c}(B_{\delta'}) \leq e^{-N(Q_{(0,1)}(g)-3\epsilon)}, \text{ for } N > N_2(r).$$

So the first term on the right hand side of (3.49) satisfies

$$\int_{-w}^{w} e^{-\mathcal{H}_N(\phi^*(v))}\gamma^{Nv}_{\{p_*,p_*+1\}^c}(B_{\delta'})\,\mathrm{d}v \leq 2we^{-N(Q_{(0,1)}(g)-3\epsilon)}, \text{ for } N > N_2(r).$$

Clearly there is a $\bar{N}_2$ such that the previous upper bound is bounded by (3.50).

To summarise the discussion, let $\bar{N} = \max(\bar{N}_1, \bar{N}_2)$ then for $N > \bar{N}$ the bound (3.50) is a bound to both terms on the right hand side of (3.49). Applying this to (3.48), we see that there is a $N' > \bar{N}$ such that (3.30) holds for $N > N'$ and $\mathcal{P} = \{p_*\}$.

Type $\mathbf{c}(\mathcal{P}) \setminus \mathcal{P} = \{p_* + 1\}$, Case 2: We seek $N' \in \mathbb{N}$ and $\delta' > 0$ such that (3.30) holds for all $\mathcal{P} \subset \mathcal{I}_N(\delta')$ such that $\mathbf{c}(\mathcal{P}) \setminus \mathcal{P} = \{p_* + 1\}$ and $\mathcal{P} \neq \{p_*\}$. We treat this cases in the same way as we treat Case 1. In the formulas from Case 1 we only need to replace $\{p_*\}$ by $\mathcal{P}$ and $\{p_*, p_* + 1\}$ by $\mathbf{c}(\mathcal{P})$. The major difference is that instead of the splitting (3.52) we have

$$
\begin{aligned}
e^{-\mathcal{H}_N(\phi^*(v))}\gamma^{Nv}_{\mathbf{c}(\mathcal{P})^c}(B_{\delta'}) = {} & e^{-\mathcal{H}_{(0,p_*)}(\phi^*(v))}\gamma^{\psi_v}_{(0,p_*)}(B_{\delta'}) \\
& \times e^{-\mathcal{H}_{(p_*,p^*)}(\phi^*(v))}\gamma^{\psi_v}_{(p_*,p^*)\setminus\mathcal{P}}(B_{\delta'}) \\
& \times e^{-\mathcal{H}_{(p^*,N)}(\phi^*(v))}\gamma^{\psi_v}_{(p^*,N)}(B_{\delta'}).
\end{aligned}
$$

But using the trivial upper bound 1 for the term in the second line, we can proceed as in Case 1.

Type: $\mathbf{c}(\mathcal{P}) \setminus \mathcal{P} = \{p^* - 1\}$: We only need to change the notation in our discussion of Type $\mathbf{c}(\mathcal{P}) \setminus \mathcal{P} = \{p_* + 1\}$ to obtain a proof for the current type.

Type: $\mathbf{c}(\mathcal{P}) \setminus \mathcal{P} = \{p_* + 1, p^* - 1\}$: For this type we can adapt the proof of Type $\mathbf{c}(\mathcal{P}) \setminus \mathcal{P} = \{p_* + 1\}$, Case 2. The main difference is that instead of equation (3.45) we have to use the following

$$
\mathcal{Z}_{\mathcal{P}^c}\gamma^{\psi}_{\mathcal{P}^c}(B_\delta) = \int_D \mathcal{Z}^{u,v}_{\mathbf{c}(\mathcal{P})^c}\gamma^{u,v}_{\mathbf{c}(\mathcal{P})^c}(B_\delta)\,\mathrm{d}u\,\mathrm{d}v, \tag{3.55}
$$

where $\mathcal{Z}^{u,v}_{\mathbf{c}(\mathcal{P})^c} := \mathcal{Z}^{\psi_{u,v}}_{\mathbf{c}(\mathcal{P})^c}$ with

$$
\psi_{u,v}(i) := \begin{cases} u & \text{, for } i = p_* + 1, \\ v & \text{, for } i = p^* - 1, \\ \psi(i) & \text{, otherwise,} \end{cases}
$$

and where $\gamma^{u,v}_{\mathbf{c}(\mathcal{P})^c} := \gamma^{\psi_{u,v}}_{\mathbf{c}(\mathcal{P})^c}$.

□

Lemma 3.11. *Let g be a very-well-behaved function such that for all sufficiently small $\delta > 0$ the set $\mathcal{I}(\delta)$ is an interval. For each $\epsilon > 0$ there is a $\delta' > 0$ such that for $\delta < \delta'$ we have*

$$
Q^{(\mathbf{a},\mathbf{c})}_{(0,l)}(B_\delta) + Q^{(\mathbf{d},\mathbf{b})}_{(r,1)}(B_\delta) \geq Q^{(\mathbf{a},\mathbf{b})}_{(0,1)}(g) - \epsilon \quad \textit{for all } l, r \in \mathcal{I}(\delta), l < r \textit{ and all } \mathbf{c}, \mathbf{d} \in \mathbb{R}^2. \tag{3.56}
$$

Proof. First of all note that removing boundary conditions leads to a possibly smaller infimum:

$$
Q^{(\mathbf{a},\mathbf{c})}_{(0,l)}(B_\delta) + Q^{(\mathbf{d},\mathbf{b})}_{(r,1)}(B_\delta) \geq Q^{(\mathbf{a},-)}_{(0,l)}(B_\delta) + Q^{(-,\mathbf{b})}_{(r,1)}(B_\delta),
$$

where

$$Q^{(\mathbf{a},-)}_{(0,l)}(f) := \begin{cases} Q_{(0,l)}(f) & \text{, for } f \in H_{\mathbf{a}}(0,l), \\ \infty & \text{, otherwise,} \end{cases} \tag{3.57}$$

and where $Q^{(-,\mathbf{b})}_{(r,1)}$ is the corresponding function without boundary conditions on the left.

Now we show that there is a $\bar{\delta}$ such that for $\delta < \bar{\delta}$ we have

$$Q_{(l,r)}(g) \leq \tfrac{1}{3}\epsilon \text{, for all } l, r \in \mathcal{I}(\delta) \text{ such that } l < r. \tag{3.58}$$

To show that $\bar{\delta}$ exists we combine that the Lebesgue measure of $\mathcal{I}(\delta) \setminus \mathcal{N}_g$ converges to zero for $\delta \to 0$ and that, since $g(\xi) = 0$ for $\xi \in \mathcal{N}_g$, we have $Q_{\mathcal{N}_g}(g) = 0$. By Lemma 3.12 below, there is a $0 < \delta' < \bar{\delta}$ such that

$$Q^{(\mathbf{a},-)}_{(0,l)}(B_\delta) \geq Q_{(0,l)}(g) - \tfrac{1}{3}\epsilon \quad \text{and} \quad Q^{(-,\mathbf{b})}_{(r,1)}(B_\delta) \geq Q_{(r,1)}(g) - \tfrac{1}{3}\epsilon, \tag{3.59}$$

for all $\delta < \delta'$ such that $l, r \in \mathcal{I}(\delta)$. Since $Q_{(0,1)}(g) = Q_{(0,l)}(g) + Q_{(l,r)}(g) + Q_{(r,1)}(g)$ and since $Q_{(0,1)}(g) = Q^{(\mathbf{a},\mathbf{b})}_{(0,1)}(g)$ statements (3.58) and (3.59) imply statement (3.56).

□

Lemma 3.12. *Let $g \in H_{\mathbf{r}}$. If $\mathcal{N}_g = \mathcal{I}$ for an interval $\mathcal{I} \subset [0,1]$, then there is a $\delta' > 0$ such that*

$$Q^{(\mathbf{a},-)}_{(0,l)}(B_\delta) \geq Q_{(0,l)}(g) - \epsilon \text{, for } \delta < \delta' \text{ and } l \in \mathcal{I}(\delta), \tag{3.60}$$

where $Q^{(\mathbf{a},-)}_{(0,l)}$ is defined in (3.57).

Proof. We use that $\mathcal{I}$ is a compact set. To begin with we show that for each $\xi \in [0,1]$ there is a $s > 0$ and a $\delta' > 0$ such that

$$Q^{(\mathbf{a},-)}_{(0,l)}(B_\delta) \geq Q_{(0,l)}(g) - \epsilon \text{, for } \delta < \delta' \text{ and } l \in (\xi - s, \xi + s). \tag{3.61}$$

Therefore let s be such that

$$Q_{(\xi-s,\xi+s)}(g) \leq \tfrac{\epsilon}{2} \tag{3.62}$$

and let $\tilde{\delta}$ be such that

$$Q^{(\mathbf{a},-)}_{(0,\xi-s)}(B_\delta) \geq Q_{(0,\xi-s)}(g) - \tfrac{\epsilon}{2} \text{, for all } \delta < \tilde{\delta}. \tag{3.63}$$

Note that s exists because $s \mapsto Q_{(\xi-s,\xi+s)}(g)$ is continuous and that $\tilde{\delta}$ exists because for fixed s the map $g \mapsto Q^{(\mathbf{a},-)}_{(0,\xi-s)}(g)$ is lower semicontinuous (see Lemma 2.7). Since

$l \mapsto Q^{(\mathbf{a},-)}_{(0,l)}(B_\delta)$ is an increasing function for all δ, we have

$$Q^{(\mathbf{a},-)}_{(0,l)}(B_\delta) \geq Q^{(\mathbf{a},-)}_{(0,\xi-s)}(B_\delta) \geq Q_{(0,\xi-s)}(g) - \tfrac{\epsilon}{2}\,, \text{ for } \delta < \tilde{\delta} \,\text{and}\, l \in (\xi - s, \xi + s).$$

Since $Q_{(0,l)}(g) = Q_{(0,\xi-s)}(g) + Q_{(\xi-s,l)}(g)$, the bound (3.62) implies that (3.61) is true for $\delta' = \tilde{\delta}$.

Now we use the compactness of $\mathcal{I}$ to prove (3.60). Therefore fix $\xi \in [0,1]$ and let (ξ, s, δ) be the triple of values such that (3.61) holds. Since $\mathcal{N}_g$ is compact the cover of this zero set with the intervals

$$((\xi - s, \xi + s))_{\{(\xi,s,\delta)|\xi\in(0,1)\}}$$

has a finite subcover $((\xi_i - s_i, \xi_i + s_i))_{i=\{1,2,...,k\}}$. Let $\tilde{\delta}$ be the minimum of all $(\delta_i)_{i=\{1,2,...,k\}}$. Since $\mathcal{N}_g$ is a subset of all $\mathcal{I}(\delta)$ there is a $\bar{\delta}$ such that the cover $\cup_{i=\{1,2,...,k\}}(\xi_i - s_i, \xi_i + s_i)$ has $\mathcal{I}(\bar{\delta})$ as a subset. Let δ' be the minimum of $\bar{\delta}$ and $\tilde{\delta}$, then (3.60) is satisfied.

□

Lemma 3.13. *1. We have*

$$\lim_{N\to\infty} \tfrac{1}{N} \log\Big(\frac{\hat{\mathcal{Z}}^{\mathbf{r}}_N}{\mathcal{Z}^{\mathbf{0}}_N}\Big) \leq \tau(J) \tag{3.64}$$

2. The family $(\hat{\gamma}_N)_{N\in\mathbb{N}}$ is exponentially tight.

Proof. First we prove inequality (3.64). Therefore note that by (2.25) we have the inequality $\mathcal{Z}^{\psi}_{\mathcal{S}} \leq \mathcal{Z}^{\mathbf{0}}_{\mathcal{S}}$ for all subsets $\mathcal{S}$ of $\{1, 2, \dots, N-1\}$. So by the two stage interpretation (1.6), we have $\hat{\mathcal{Z}}^{\mathbf{r}}_N \leq \hat{\mathcal{Z}}^{\mathbf{0}}_N$. So by definition of $\tau(J)$ the above limit is finite.

Now we consider the sequence $(\hat{\gamma}_N)_{N\in\mathbb{N}}$. It is sufficient to show that $(K_\kappa)_{\kappa\in\mathbb{R}}$, where

$$K_\kappa := \{f \in C((0,1)) \mid \|\dot{f}\|_\infty \leq \kappa\},$$

is a family of precompact sets that satisfies

$$\lim_{\kappa\to\infty} \limsup_{N\to\infty} N^{-1} \log \hat{\gamma}_N(K^c_\kappa) = -\infty. \tag{3.65}$$

The precompactness of K_κ follows by Arzelà-Ascoli: for all $f \in K_\kappa$, by the mean value theorem we have $|f(x) - f(y)|_\infty \leq \kappa|x - y|$ and K_κ is bounded and equicontinuous.

The sets $(K_\kappa)_{\kappa\in\mathbb{R}}$ are convenient because they allow us to work with the process $(\phi(i+1)-\phi(i))_{i\in\{1,2,\dots,N-1\}}$: Since

$$\tfrac{\mathrm{d}}{\mathrm{d}\xi}[h_N(\phi)] = N^{-1}[\phi(i+1)-\phi(i)],$$

we have

$$h_N(\phi)\in K_\kappa \Leftrightarrow \max_{i\in\{1,2,\dots,N-1\}} N^{-1}|\phi(i+1)-\phi(i)|\le\kappa.$$

We use the two stage interpretation of $\hat{\gamma}_N$. Assume the result of the first stage is $\mathcal{P}$. In the second stage the process $(\phi(i+1)-\phi(i))_{i\in\{1,2,\dots,N-1\}}$ is a Gaussian process. We use this to show that there is a constant C so that

$$\gamma_{\mathcal{P}^c}(K_\kappa^c)\le e^{-C\kappa^2N+o(N)} \quad \text{for all} \quad \mathcal{P}\subset\{1,2,\dots,N-1\} \quad \text{as} \quad N\to\infty. \tag{3.66}$$

As

$$\gamma_{\mathcal{P}^c}(K_\kappa^c)\le \sum_{i\in\{1,2,\dots,N-1\}} \gamma_{\mathcal{P}^c}\big(\tfrac{|\phi(i+1)-\phi(i)|}{N}>\kappa\big), \tag{3.67}$$

we employ Lemma B.1 in this proof. Therefore note, that since the conditioning only reduces the variance, the variances of the random variables $(\frac{1}{N}(\phi(i+1)-\phi(i)))_{i\in\{1,2,\dots,N-1\}}$ are the largest if the result of the first experiment was $\mathcal{P}=\emptyset$; so the variances are of order $O(\frac{1}{N})$. Furthermore, independent of $\mathcal{P}$, the expectations of $(\frac{1}{N}(\phi(i+1)-\phi(i)))_{i\in\{1,2,\dots,N-1\}}$ are bounded by a constant that does not depend on N. So for κ large enough, there is a C such that

$$\begin{aligned}\gamma_{\mathcal{P}^c}\big(\tfrac{|\phi(i+1)-\phi(i)|}{N}>\kappa\big) &\le e^{-C(\frac{\kappa}{1/\sqrt{N}})^2+o(N)}\\ &= e^{-CN\kappa^2+o(N)} \quad \text{for all } \mathcal{P}\subset\{1,2,\dots,N-1\} \text{ as } N\to\infty.\end{aligned} \tag{3.68}$$

Since the sum on the right hand side of (3.67) has only $N-1$ terms, (3.68) implies (3.66).

The upper bound (3.66) is independent of $\mathcal{P}$ and hence

$$\hat{\gamma}_N(K_\kappa^c)\le e^{-C\kappa^2N+o(N)}. \tag{3.69}$$

After applying $\frac{1}{N}\log(\cdot)$ to both sides of (3.69), the left hand side is, at least asymptotically for large N, bounded by $-C\kappa^2+o(1)$; so we obtain (3.65) by first letting $N\to\infty$ and then letting $\kappa\to\infty$. □

Lemma 3.14. *The very-well-behaved functions are dense in $H_{\mathbf{r}}$ with respect to the*

$\|\cdot\|_\infty$ *norm.*

Proof. Since, by Lemma 3.4, the well-behaved functions are dense in $H_{\mathbf{r}}$, we prove that for all $r > 0$ and all well-behaved functions g there is a very-well-behaved function f such that $f \in B(g, r)$.

We only treat the case $\mathcal{N}_g = \{0\}$ to illustrate the construction. For general well-behaved g the same construction should be performed near every interval (or isolated point) where g is zero.

Let

$$\tilde{\xi} := \inf\{t > 0 \mid |g(t)| = \tfrac{r}{2}\}.$$

The value $\tilde{\xi}$ is strictly positive because g is continuous.

Let

$$f(\xi) := \begin{cases} g(\xi) & , \text{for } \xi \geq \tilde{\xi}, \\ h(\xi) & , \text{otherwise}, \end{cases}$$

where h is a smooth monotone function such that $h(0) = 0$, $h(\tilde{\xi}) = g(\tilde{\xi})$ and $\dot{h}(\tilde{\xi}) = \dot{g}(\tilde{\xi})$ (such a h exists because by definition of $\tilde{\xi}$ we have $|\dot{g}(\tilde{\xi})| \geq 0$).

We show that f is very-well-behaved. Therefore let

$$\delta = \inf\{|g(\xi)| \mid \xi \in [\tilde{\xi}, 1]\}.$$

Since g has no zero in $(0, 1]$ we have $\delta > 0$. By definition of δ we have

$$\{\xi \mid |f(\xi)| \leq \tfrac{\delta}{2}\} \cap (\tilde{\xi}, 1] = \emptyset; \tag{3.70}$$

and since h is monotone the set

$$\{\xi \mid |f(\xi)| \leq \tfrac{\delta}{2}\} \cap [0, \tilde{\xi}] \tag{3.71}$$

is an interval. Combining (3.70) and (3.71), we see that f is a very-well-behaved function.

It remains to show that $f \in B(g, r)$. Therefore we use that in $[0, \tilde{\xi}]$ the functions $|g|$ and $|h|$ are bounded by $\frac{1}{2}r$ and the triangle inequality:

$$|g(\xi) - f(\xi)| \leq |g(\xi)| + |f(\xi)| < r\,, \text{for } \xi \leq \tilde{\xi}.$$

□

Chapter 4

Minimiser of the rate function of the model with pinning interaction

For large N, we want to know which empirical profiles are typical for the Laplacian model with pinning. Therefore we determine for each $\mathcal{Q} \subset L_\infty(0,1)$ the limit

$$\lim_{N\to\infty} \hat{\gamma}_N(\mathcal{Q}). \tag{4.1}$$

By the large deviation principle of $(\hat{\gamma}_N)_{N\in\mathbb{N}}$ (Theorem 1.3), the limit of $\hat{\gamma}_N(\mathcal{Q})$ is necessarily 1 if $\mathcal{Q}$ is an open set that contains all minimisers of the rate function $\Sigma^{\mathbf{r},J}$ (for a definition of this rate function see Theorem 1.3). So we determine the set of minimisers $\mathcal{M}^*_{\mathbf{r}}$ of the rate function $\Sigma^{\mathbf{r},J}$.

The minimisers of $\Sigma^{\mathbf{r},J}$ coincide with the minimiser of

$$\mathcal{E}^J(f) = \frac{1}{2}\int_0^1 (\ddot{f}(\xi))^2 \,\mathrm{d}\,\xi - \tau(J)|\mathcal{N}_f| \tag{4.2}$$

in $H_{\mathbf{r}}$, because by (1.17) the two functions $\Sigma^{\mathbf{r},J}$ and $\mathcal{E}^J$ differ only by an additive constant. So we study the minimisers of $\mathcal{E}^J$ in $H_{\mathbf{r}}$. For this study it is convenient to also study the minimisers of $\mathcal{E}^J$ in $H_{\mathbf{a}}$. We let $\mathcal{M}^*_{\mathbf{a}}$ be the set of minimisers of $\mathcal{E}^J$ in $H_{\mathbf{a}}$.

We are mainly interested in the zero sets $\mathcal{N}_h$ of the elements h of $\mathcal{M}^* = \mathcal{M}^*_{\mathbf{r}}, \mathcal{M}^*_{\mathbf{a}}$. The proposition in Section 4.1 shows that these zero sets are unions of at most three disjoint sets, where at most one of them is a closed interval (we interpret an isolated point as a closed interval) and the others are isolated points.

In the following sections we determine, for each kind of zero set, for which boundary conditions there are minimisers of this kind. In Section 4.2 we do this for $\mathcal{M}^*_{\mathbf{a}}$ and in Section 4.3 we do this for $\mathcal{M}^*_{\mathbf{r}}$.

In this chapter we omit the index J and write $\mathcal{E}^J = \mathcal{E}$.

4.1 Superset of the set of minimisers

We present a superset of the set of minimisers.

Proposition 4.1. *1. The set $\mathcal{M}^*_{\mathbf{r}}$ is a subset of*

$$\{h_{l,r} \mid l + r \leq 1\} \cup \{h_{\mathbf{r}}\}, \tag{4.3}$$

where for $l + r \leq 1$

$$h_{l,r}(\xi) := \begin{cases} h^{(0,l)}_{(\mathbf{a},\mathbf{0})}(\xi) & , \textit{for} \quad \xi \in [0, l), \\ 0 & , \textit{for} \quad \xi \in [l, 1 - r], \\ h^{(1-r,1)}_{(\mathbf{0},\mathbf{b})}(\xi) & , \textit{for} \quad \xi \in (1 - r, 1], \end{cases} \tag{4.4}$$

and where, for all intervals $I \subset \mathbb{R}$ and all $\mathbf{r} \in \mathbb{R}^4$, $h^I_{\mathbf{r}}$ is the minimiser of $h \to Q_I(h) := \frac{1}{2} \int_I (\ddot{h}(\xi))^2 \,\mathrm{d}\xi$ in $H_{\mathbf{r}}(I)$.

2. The set $\mathcal{M}^*_{\mathbf{a}}$ is a subset of

$$\{h_{\mathbf{a}}\} \cup \{h_l \mid l \leq 1\}, \tag{4.5}$$

where

$$h_{\mathbf{a}}(\xi) = a + \alpha\xi \quad \textit{and} \quad h_l(\xi) := \begin{cases} h^{(0,l)}_{(\mathbf{a},\mathbf{0})}(\xi) & , \textit{for} \quad \xi < l, \\ 0 & , \textit{for} \quad \xi \geq l. \end{cases}$$

Proof. We only give the proof for $\mathcal{M}^*_{\mathbf{r}}$ because we obtain a proof for $\mathcal{M}^*_{\mathbf{a}}$ by applying the same methods. Let h be an element of the complement of (4.3). We show that h is not a minimiser. Therefore we show that in (4.3) there is at least one function h^* such that

$$\mathcal{E}(h^*) < \mathcal{E}(h). \tag{4.6}$$

If $\mathcal{E}(h) = \infty$, then (4.6) is satisfied for all h^* in (4.3). If $\mathcal{E}(h) < \infty$, we distinguish two cases:

Case 1: $|\mathcal{N}_h| = 0$.

Since $|\mathcal{N}_h| = |\mathcal{N}_{h_\mathbf{r}}| = 0$, the definitions of $\mathcal{E}$ and $h_\mathbf{r}$ imply that for $h^* = h_\mathbf{r}$ inequality (4.6) is satisfied: In fact, we have

$$\mathcal{E}(h) = Q(h) = \tfrac{1}{2}\int_0^1 (\ddot{h}(\xi))^2 \,\mathrm{d}\,\xi > \tfrac{1}{2}\int_0^1 (\ddot{h}_\mathbf{r}(\xi))^2 \,\mathrm{d}\,\xi = Q(h_\mathbf{r}) = \mathcal{E}(h_\mathbf{r}), \tag{4.7}$$

where the strict inequality holds because $h_\mathbf{r}$ is the unique minimiser of Q in $H_\mathbf{r}$.

Case 2: $|\mathcal{N}_h| > 0$.

Let l be the infimum and $1-r$ be the supremum of the accumulation points of $\mathcal{N}_h$, we show that for $h^* = h_{l,r}$ inequality (4.6) is satisfied. But first of all, note that the points l and r actually exist, because a set with positive Lebesgue measure has at least two accumulation points. The definition of (l,r) has two important consequences:

1. Since $|\mathcal{N}_h \cap [l,r]^c| = 0$, we have

$$\begin{aligned}\mathcal{E}(h) = {} & \frac{1}{2}\int_0^l (\ddot{h}(\xi))^2 \,\mathrm{d}\,\xi \\ & + \frac{1}{2}\int_l^{1-r} (\ddot{h}(\xi))^2 \,\mathrm{d}\,\xi - \tau|\{\xi \in (l, 1-r) \mid h(\xi) = 0\}| \\ & + \frac{1}{2}\int_{1-r}^1 (\ddot{h}(\xi))^2 \,\mathrm{d}\,\xi. \end{aligned} \tag{4.8}$$

2. As l and r are themselves accumulation points of $\mathcal{N}_h$ (the set of accumulation points is closed), Lemma 3.7 yields

$$\dot{h}(l) = \dot{h}(1-r) = 0. \tag{4.9}$$

To check this for $h^* = h_{l,r}$ inequality (4.6) is satisfied, we combine (4.8) and (4.9): By (4.9), the restrictions of h and $h_{l,r}$ to the intervals $(0,l)$, $(l,1-r)$, and $(1-r,1)$ are elements of $H_{(\mathbf{a},\mathbf{0})}(0,l)$, $H_{(\mathbf{0},\mathbf{0})}(l,1-r)$, and $H_{(\mathbf{0},\mathbf{b})}(1-r,1)$, respectively. By (4.8), the optimality of $h^{(0,l)}_{(\mathbf{a},\mathbf{0})}$, $h^{(1-r,1)}_{(\mathbf{0},\mathbf{b})}$ and the fact that $h^{(l,1-r)}_{(\mathbf{0},\mathbf{0})}(\xi) = 0$ for $\xi \in [l,r]$ imply (4.6); where the inequality is strict because, by the uniqueness of the minimisers $h^{(0,l)}_{(\mathbf{a},\mathbf{0})}$ and $h^{(1-r,1)}_{(\mathbf{0},\mathbf{b})}$, we have $\mathcal{E}(h) = \mathcal{E}(h_{l,r})$ if and only if $h = h_{l,r}$. □

Now we are using Proposition 4.1 to study the zero sets of the minimisers and to ultimately determine the set $\mathcal{M}^*$. We study the cases with and without terminal condition separately. First we study the case without terminal condition.

4.2 Without terminal condition

We consider the zero sets of all functions in (4.5). By definition the zero set of h_l is the union of the interval $[l, 1]$ and the zero sets of $h^{(0,l)}_{(\mathbf{a},\mathbf{0})}$. Furthermore, since we have

$$h^{(0,l)}_{(\mathbf{a},\mathbf{0})}(\xi) = a h^{(0,1)}_{(1,sw^{-1}l,\mathbf{0})}(\tfrac{\xi}{l}) \qquad , \text{ for } \quad a \neq 0 \quad \text{and all} \quad \xi \in [0, l],$$

and

$$h^{(0,l)}_{(\mathbf{a},\mathbf{0})}(\xi) = h^{(0,1)}_{(0,\alpha l,\mathbf{0})}(\tfrac{\xi}{l}) \qquad , \text{ for } \quad a = 0 \quad \text{and all} \quad \xi \in [0, l], \tag{4.10}$$

where

$$s := \operatorname{sgn}(a\alpha) \quad \text{and} \quad w := \tfrac{|a|}{|\alpha|},$$

we only need to consider the zero sets of the following functions,

$$(h^{(0,1)}_{(1,l,\mathbf{0})})_{l>0}, \quad (h^{(0,1)}_{(1,-l,\mathbf{0})})_{l>0}, \quad \text{and} \quad (h^{(0,1)}_{(0,l,\mathbf{0})})_{l\in\mathbb{R}\setminus\{0\}}. \tag{4.11}$$

Note that (4.10) is true because the left and right hand sides of (4.10) are polynomials of degree 3 that have the same value and the same first derivative at $\xi = 0$ and $\xi = l$. Now we study the zero sets of the functions in (4.11).

Proposition 4.2.

1. *The functions* $(h^{(0,1)}_{(1,l,\mathbf{0})})_{l>0}$, $(h^{(0,1)}_{(0,l,\mathbf{0})})_{l\in\mathbb{R}\setminus\{0\}}$, *and* $(h^{(0,1)}_{(1,-l,\mathbf{0})})_{0<l\leq 3}$ *have no zeroes in* $(0, 1)$*;*
2. *whereas the functions* $(h^{(0,1)}_{(1,-l,\mathbf{0})})_{l>3}$ *have exactly one zero in* $(0, 1)$.

We give the proof of Proposition 4.2 at the end of this section. In Figure 4.1 we plot the functions $h^{(0,l)}_{1,s}$ for some choices of the parameters l and s.

Propositions 4.2 and 4.1 show that a zero set of a minimiser is the union of at most two disjoint set, where at most one of them is an interval and the other one is an isolated point. Where by (4.10) and by Proposition 4.2, the function h_l has an isolated zero if and only if $s = -1$ and $w^{-1}l < 3$. This isolated zero is element of $(0, l)$. For the following study we reformulate this result as follows: If h is a

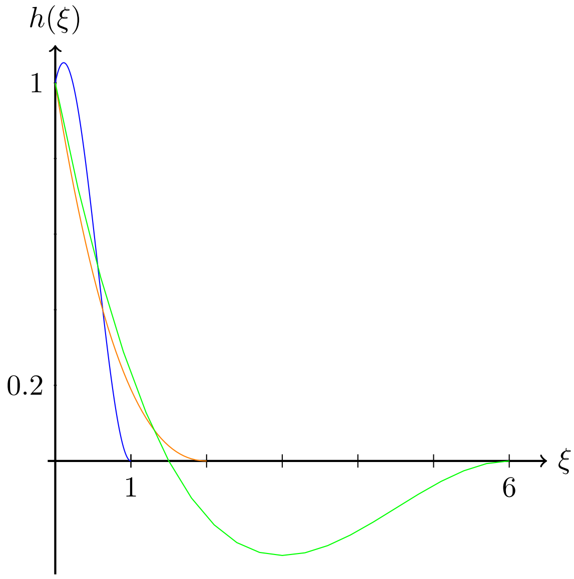

Figure 4.1: These are the plots for $h = h^{(0,1)}_{(1,1,0,0)}$ (blue), $h = h^{(0,l_1)}_{(1,-1,0,0)}$ (orange), and $h = h^{(0,l_2)}_{(1,-1,0,0)}$ (green) where $l_1 = 2$ and $l_2 = 6$. The formula for $h^{(0,1)}_{(1,1,0,0)}$ is given in (D.3), where $\mathbf{r} = (1, 1, 0, 0)$. To obtain the formula for $h^{(0,l)}_{(1,-1,0,0)}$, we plug (D.3) into the right hand side of (4.10).

minimiser of $\mathcal{E}$ in $H_{\mathbf{a}}$, then h is an element one of the following sets:

$$\mathcal{M}_1 := \{h_{\mathbf{a}} \mid \mathbf{a} \in \mathbb{R}^2\},$$
$$\mathcal{M}_2 := \bigcup_{\mathbf{a}\in\mathbb{R}^2} \{h_l \mid l \leq 1, h_l \text{ has a zero in } (0, l)\},$$
$$\mathcal{M}_3 := \bigcup_{\mathbf{a}\in\mathbb{R}^2} \{h_l \mid l \leq 1, h_l \text{ has no zero in } (0, l)\}.$$

We study the sets

$$S_i := \{(\mathbf{a}, J) \in \mathbb{R}^3 \mid \text{there is a } h \in \mathcal{M}_i \text{ that minimises } \mathcal{E} \text{ in } H_{\mathbf{a}}\}, \tag{4.12}$$

for $i \in \{1, 2, 3\}$. Therefore we use the fact that

$$\mathcal{E}(h_l) = 6a^2 l^{-3} + 6a\alpha l^{-2} + 2\alpha^2 l^{-1} - \tau(1 - l),$$

where we have applied equation (D.4) and that by (4.10) we have

$$\int_0^l (\ddot{h}^{(0,l)}_{(\mathbf{a},\mathbf{0})}(\xi))^2 \,\mathrm{d}\xi = \tfrac{1}{l^3} \int_0^1 (\ddot{h}^{(0,1)}_{(a,l\alpha,\mathbf{0})}(\xi))^2 \,\mathrm{d}\xi.$$

For the following it is convenient to introduce $\tilde{E}_{\mathbf{a}}^{\tau} : \mathbb{R}_{>0} \to \mathbb{R}$ given by

$$\tilde{E}_{\mathbf{a}}^{\tau}(l) := 6a^2 l^{-3} + 6a\alpha l^{-2} + 2\alpha^2 l^{-1} + \tau l. \tag{4.13}$$

For the moment we only consider the intersections of $(S_i)_{i\in\{1,2,3\}}$ with the set where $w \in (0, \infty)$. In Corollary 4.8 below, we present the result of this study. The functions that appear in Corollary 4.8 will be defined in the following. After proving Corollary 4.8, we consider the intersections of $(S_i)_{i\in\{1,2,3\}}$ with the set where $w = 0$ or $w = \infty$. We present our results of these latter cases in Proposition 4.10 below.

We are going to transform our current minimisation problem with the parameters $(\mathbf{a}, J)$ into a dual minimisation problem with three new parameters (s, w, K), where one of them, s, assumes only the values 1 and -1.

Proposition 4.3. *Fix $(\mathbf{a}, J) \in \mathbb{R}^3$ such that $w \in (0, \infty)$. Let $K := \tau(J)(\frac{w}{\alpha})^2$, $s := \operatorname{sgn}(a\alpha)$ and let $\mathbf{s} := (1, s)$. We have:*

A function h is a minimiser of $\mathcal{E}$ in $H_{\mathbf{a}}$ if and only if $g : [0, 1] \to \mathbb{R}$ defined by

$$g(\cdot) = \tfrac{1}{a} h(w\cdot), \tag{4.14}$$

is a minimiser of $\mathcal{E}_w^K$ in $H_{\mathbf{s}}$, where $\mathcal{E}_w^K : H_{\mathbf{s}} \to \mathbb{R}$ is given by

$$\mathcal{E}_w^K(g) = \tfrac{1}{2} \int_0^{w^{-1}} (\ddot{g}(\xi))^2 \,\mathrm{d}\xi - K|\mathcal{N}_g \cap (0, w^{-1})|. \tag{4.15}$$

Proof. As $\mathcal{E}(h) = \frac{\alpha^2}{w} \mathcal{E}_w^K(\frac{1}{a} h(w\xi))$ holds for all $h \in H_{\mathbf{a}}$, for every minimiser h of $\mathcal{E}$ in $H_{\mathbf{a}}$ the function g given by (4.14) is a minimiser of $\mathcal{E}_w^K$ and vice versa. □

To make use of Proposition 4.3, we will define sets $(\tilde{S}_i)_{i\in\{1,2,3\}}$ such that $(\mathbf{a}, J) \in S_i$ if and only if $(s, w, K) \in \tilde{S}_i$. The problem is that the map T given by $T(\mathbf{a}, J) = (s, w, K)$ is not injective. Under the map T, the points (a, α, J) and $(-a, -\alpha, J)$ have the the same image. But by (4.14) the minimisers for (a, α, J) and $(-a, -\alpha, J)$ are reflected along the x-axis. So the sets

$$\tilde{S}_i := T(S_i) \quad , \text{for} \quad i \in \{1, 2, 3\} \tag{4.16}$$

actually satisfy that $(\mathbf{a}, J) \in S_i$ if and only if $(s, w, K) \in \tilde{S}_i$. Furthermore, treating the points (a, α, J) and $(-a, -\alpha, J)$ as members of the same equivalence class, the map T is injective.

We define the sets $(\tilde{S}_i)_{i\in\{1,2,3\}}$ as images of the sets $(S_i)_{i\in\{1,2,3\}}$ which in turn are defined in terms of properties of the minimisers of $\mathcal{E}$ in $H_{\mathbf{a}}$ (recall (4.12)). Now we apply Proposition 4.3 in order to express $\tilde{S}_i$ directly in terms of properties of the minimisers of $\mathcal{E}_w^K$ in $H_{\mathbf{s}}$. We have

$$h_l \text{ minimises } \mathcal{E} \text{ in } H_{\mathbf{a}} \text{ if and only if } g_{lw^{-1}} \text{ minimises } \mathcal{E}_w^K \text{ in } H_{\mathbf{s}}, \tag{4.17}$$

where for $l \in \mathbb{R}_{>0}$

$$g_l(\xi) := \begin{cases} h^{(0,l)}_{(\mathbf{s},\mathbf{0})}(\xi) & \text{, for } \quad \xi < l, \\ 0 & \text{, otherwise.} \end{cases}$$

To verify (4.17) we first plug $h = h_l$ into (4.14); then for $\xi = 0$ and $\xi = lw^{-1}$ the values and the first derivatives of the left hand side of (4.14) coincide with the corresponding values for $g_{lw^{-1}}$. Since in $[0, lw^{-1}]$ the functions $\xi \mapsto h_l(w\xi)$ and $g_{lw^{-1}}$ are polynomials of degree 3 the left hand side of (4.14) coincides with $g_{lw^{-1}}$ in $[0, lw^{-1}]$. Furthermore in $[lw^{-1}, w^{-1}]$ the left hand side of (4.14) is zero and this is also true for $g_{lw^{-1}}$. So if $h = h_l$, then the left hand side of (4.14) coincides with $g_{lw^{-1}}$. Now plug $g = g_{lw^{-1}}$ into (4.14); then for $\xi = 0$ and $\xi = lw^{-1}$ the values and the first derivatives of the right hand side of (4.14) coincide with the corresponding values for $\xi \mapsto \frac{1}{a} h_l(w\xi)$. Repeating the previous argument, we see that if $g = g_{lw^{-1}}$, then the the right hand side of (4.14) coincides with $\xi \mapsto \frac{1}{a} h_l(w\xi)$.

In the following proposition, we use (4.17) to express $\tilde{S}_i$ in terms of properties of the minimisers of $\mathcal{E}_w^K$ in $H_{\mathbf{s}}$.

Proposition 4.4. *For* $(\mathbf{a}, J) \in \mathbb{R}^3$ *such that* $w \in (0, \infty)$*, we have*

1. $(s, w, K) \in \tilde{S}_1$ *if and only if* $g_{\mathbf{s}}(\xi) := 1 + s\xi$ *is a minimiser of* $\mathcal{E}_w^K$,
2. $(s, w, K) \in \tilde{S}_2$ *if and only if* $s = 1$ *and one of the functions* $(g_l)_{l>0}$ *is a minimiser of* $\mathcal{E}_w^K$*, or if* $s = -1$ *and one of the functions* $(g_l)_{l \leq 3}$ *is a minimiser of* $\mathcal{E}_w^K$,
3. $(s, w, K) \in \tilde{S}_3$ *if and only if* $s = -1$ *and one of the functions* $(g_l)_{l>3}$ *is a minimiser of* $\mathcal{E}_w^K$.

Proof. We shall only prove Statement 3 as the other ones follow in an analogous way. First we prove that if g_l with $l > 3$ is a minimiser of $\mathcal{E}_w^K$ then $(s, w, K) \in \tilde{S}_3$. Assume that g_l with $l > 3$ is a minimiser of $\mathcal{E}_w^K$. Then $h_{lw}(\xi) = a g_l(\xi w^{-1})$ is a minimiser of $\mathcal{E}$ in $H_{\mathbf{a}}$. By Proposition 4.2 the function g_l has a zero in $(0, l)$. So h_{lw} has a zero in $(0, lw)$. By definition of S_3 we have $(\mathbf{a}, J) \in S_3$ and hence, by (4.16), $(s, w, K) \in \tilde{S}_3$.

Now we prove that if $(s, w, K) \in \tilde{S}_3$ then there is a $l > 3$ such that g_l is a minimiser of $\mathcal{E}_w^K$. Assume that $(s, w, K) \in \tilde{S}_3$ then by (4.16) we have $(\mathbf{a}, J) \in S_3$.

By definition of S_3 there is an $l' \leq 1$ such that $h_{l'}$ has a zero in $(0, l')$ and that $h_{l'}$ minimises $\mathcal{E}$ in $H_{\mathbf{a}}$. By (4.17), $g_{l'w^{-1}}$ minimises $\mathcal{E}_w^K$ in $H_{\mathbf{s}}$. By (4.14), the function $g_{l'w^{-1}}$ has a zero in $(0, l'w^{-1})$ because $h_{l'}$ has a zero in $(0, l')$. Let $l = l'w^{-1}$. We have $l > 3$ and $s = -1$ because g_l has a zero in $(0, l)$ if and only if $s = -1$ and $l > 3$ (see Proposition 4.2). □

To make use of Proposition 4.4, we need to know the minimisers of $\mathcal{E}_w^K$. By Proposition 4.1, the minimisers are a subset of

$$\{g_l \mid l \leq w^{-1}\} \cup \{g_{\mathbf{s}}\}, \tag{4.18}$$

where $g_{\mathbf{s}}$ is the linear function with $g_{\mathbf{s}}(0) = 1$ and $\dot{g}_{\mathbf{s}}(0) = s$. We use the fact that

$$\mathcal{E}_w^K(g_l) = \tilde{E}_{\mathbf{s}}^K(l) - Kw^{-1}, \tag{4.19}$$

where we define $\tilde{E}_{\mathbf{s}}^K$ in (4.13). Recall that $l \mapsto \tilde{E}_{\mathbf{s}}^K(l)$ is a differentiable function in $(0, \infty)$. For a local minimiser l^* of $\tilde{E}_{\mathbf{s}}^K$ we show the following: The function g_{l^*} minimises $\mathcal{E}_w^K$ in $H_{\mathbf{s}}$ if and only if

$$l^* \leq w^{-1}, \tag{4.20}$$

$$\tilde{E}_{\mathbf{s}}^K(l^*) \leq Kw^{-1}, \tag{4.21}$$

and

$$\tilde{E}_{\mathbf{s}}^K(l^*) \leq \tilde{E}_{\mathbf{s}}^K(l), \text{ for all local minimiser } l \text{ of } \tilde{E}_{\mathbf{s}}^K \text{ that satisfy (4.20) and (4.21).} \tag{4.22}$$

Condition (4.20) is necessary because if $l > w^{-1}$ then g_l is not in the set (4.18). Condition (4.21) is necessary because it corresponds to the condition $\mathcal{E}_w^K(g_l) \leq \mathcal{E}_w^K(g_{\mathbf{s}})$. To verify this first note that we have $\mathcal{E}_w^K(g_{\mathbf{s}}) = 0$, because for the linear function $g_{\mathbf{s}}$ both terms on the right hand side of (4.15) are zero. Combining this with the fact that right hand side of (4.19) is smaller than zero if and only if (4.21) is satisfied, we see that (4.21) is satisfied if and only if $\mathcal{E}_w^K(g_l) \leq \mathcal{E}_w^K(g_{\mathbf{s}})$. Hence if l^* satisfies the two conditions (4.20) and (4.21) then we know that for at least one local minimiser l the function g_l is a minimiser of $\mathcal{E}_w^K$ in $H_{\mathbf{s}}$. If (4.20) and (4.21) are satisfied, the only possible reason why l^* does not correspond to a minimiser of $\mathcal{E}_w^K$ is that there is another local minimiser for which the corresponding minimiser has a smaller rate $\mathcal{E}_w^K$.

Furthermore the function $g_{\mathbf{s}}$ minimises $\mathcal{E}_w^K$ in $H_{\mathbf{s}}$ if and only if no local minimiser satisfies (4.20) and (4.21) or if there is a local minimiser l^* that satisfies (4.20), (4.21) and (4.22) but $\mathcal{E}_w^K(g_{l^*}) = 0$.

In the next proposition we determine, for all $s \in \{1, -1\}$ and all $K \in \mathbb{R}_{>0}$, the set of local minimisers of $l \to \tilde{E}_{\mathbf{s}}^K(l)$.

Proposition 4.5. *Fix $K \in \mathbb{R}_{>0}$.*

1. *For $s = 1$, the function $\tilde{E}_{\mathbf{s}}^K$ has exactly one local minimiser l_1, where*

$$l_1(s, K) := \tfrac{1+\sqrt{1+6\sqrt{2K}}}{\sqrt{2K}}.$$

2. *For $s = -1$, the function $\tilde{E}_{\mathbf{s}}^K$ has up to two local minimiser l_1 and l_2, where*

$$l_1(s, K) := \tfrac{-1+\sqrt{1+6\sqrt{2K}}}{\sqrt{2K}},$$

   and where l_2 exists only for $K \le \frac{1}{72}$ and is given by

$$l_2(s, K) := \tfrac{1+\sqrt{1-6\sqrt{2K}}}{\sqrt{2K}}.$$

Furthermore

$$\begin{aligned} l_1(-1, K) &\in (0, 3) \quad , \textit{for} \quad K \in \mathbb{R}_{>0}, \\ l_2(-1, K) &\in [6, \infty) \quad , \textit{for} \quad K \in (0, \tfrac{1}{72}]. \end{aligned}$$

We give the proof of Proposition 4.5 at the end of this Section. We combine Proposition 4.5 and Proposition 4.4 to obtain the following representations of the sets $(\tilde{S}_i)_{i\in\{2,3\}}$ and $(S_i)_{i\in\{2,3\}}$, respectively.

Corollary 4.6. *We have*

$$\begin{aligned} \tilde{S}_2 &= \{(s, w, K) \in \{1, -1\} \times \mathbb{R}^2_{>0} \mid \textit{for}\, l = l_2(s, K),\ g_l \textit{ is a minimiser of}\, \mathcal{E}_w^K\}, \\ \tilde{S}_3 &= \{(s, w, K) \in \{1, -1\} \times \mathbb{R}^2_{>0} \mid \textit{for}\, l = l_1(s, K),\ g_l \textit{ is a minimiser of}\, \mathcal{E}_w^K\}, \end{aligned}$$

and

$$\begin{aligned} S_2 \cap \mathcal{R} &= \{(\mathbf{a}, J) \in \mathcal{R} \mid \textit{for}\, l = l_2(\mathbf{a}, J),\ h_l \textit{ is a minimiser of}\, \mathcal{E} \textit{ in}\, H_{\mathbf{a}}\}, \\ S_3 \cap \mathcal{R} &= \{(\mathbf{a}, J) \in \mathcal{R} \mid \textit{for}\, l = l_1(\mathbf{a}, J),\ h_l \textit{ is a minimiser of}\, \mathcal{E} \textit{ in}\, H_{\mathbf{a}}\}, \end{aligned}$$

where we use

$$\mathcal{R} := \{(\mathbf{a}, J) \in \mathbb{R}^3 \mid w \in (0, \infty)\},$$
$$l_1(\mathbf{a}, J) := w l_1(s, K)\ ,\ \textit{for}\, (\mathbf{a}, J) \in \mathbb{R}^3,$$
$$l_2(\mathbf{a}, J) := w l_2(s, K)\ ,\ \textit{for}\, s = -1\ \textit{and}\, K \leq \tfrac{1}{72}.$$

After our preceding preparations we are now able to represent the sets $(\tilde{S}_i)_{i\in\{1,2,3\}}$ via functions in the following proposition.

Proposition 4.7.

1. *For $s = -1$, we have*

$$\begin{aligned}
\tilde{S}_1 \cap [\{-1\} \times \mathbb{R}^2_{>0}] &= \{(s, w, K) \in \{-1\} \times \mathbb{R}^2_{>0} \mid K \leq G(w)\},\\
\tilde{S}_2 \cap [\{-1\} \times \mathbb{R}^2_{>0}] &= \{(s, w, K) \in \{-1\} \times (0, \bar{w}] \times \mathbb{R}_{>0} \mid G(w) \leq K \leq \bar{K}\},\\
\tilde{S}_3 \cap [\{-1\} \times \mathbb{R}^2_{>0}] &= \{(s, w, K) \in \{-1\} \times (0, \bar{w}] \times \mathbb{R}_{>0} \mid K \geq \bar{K}\}\\
&\quad \cup \{(s, w, K) \in \{-1\} \times (\bar{w}, \infty] \times \mathbb{R}_{>0} \mid K \geq G(w)\},
\end{aligned}$$

where

- *for $w \in \mathbb{R}_{>0}$ we let*

$$G(w) := \begin{cases} K_{2,*}(w) & ,\ \textit{if} \quad w \leq \bar{w}, \\ K_{1,*}(w) & ,\ \textit{otherwise}, \end{cases}$$

- *for $i \in \{1, 2\}$ and $w \in \mathbb{R}_{>0}$, the value $K_{i,*}(w)$ is the unique value for K such that*

$$\mathcal{E}^K_w(g_{l^*}) = 0, \quad \textit{for} \quad l^* = l_i(K); \tag{4.23}$$

  if there is no such K set $K_{i,*}(w) = \infty$,

- *$\bar{w}$ is the unique value for w such that*

$$K_{1,*}(w) = K_{2,*}(w), \tag{4.24}$$

- *$\bar{K} := K_{1,*}(\bar{w})$.*

2. For $s=1$ we have

$$\tilde{S}_1 \cap [\{1\} \times \mathbb{R}^2_{>0}] = \{(s,w,K) \in \{1\} \times \mathbb{R}^2_{>0} \mid K \leq K_{1,*}(w)\},$$
$$\tilde{S}_2 \cap [\{1\} \times \mathbb{R}^2_{>0}] = \emptyset,$$
$$\tilde{S}_3 \cap [\{1\} \times \mathbb{R}^2_{>0}] = \{(s,w,K) \in \{1\} \times \mathbb{R}^2_{>0} \mid K \geq K_{1,*}(w)\},$$

where $K_{1,*}(w)$ is defined analogously as in the case for $s=-1$.

Before we give a proof of Proposition 4.7 we summarise our findings for the minimisers of the rate function

Corollary 4.8. *Fix $\mathbf{a} \in \mathbb{R}^2$ such that $w = \frac{|a|}{|\alpha|} \in (0,\infty)$. Let $\mathcal{M}^*(J)$ be the set of minimisers of $\mathcal{E}$ in $H_{\mathbf{a}}$.*

1. If $\operatorname{sgn}(a\alpha) = -1$, we have

(a) for $w < \bar{w}$

$$\mathcal{M}^*(J) = \begin{cases} \{h_{\mathbf{a}}\} & \text{, for } \tau(J) < (\frac{\alpha}{w})^2 G(w), \\ \{h_{\mathbf{a}}, h_{l_2}\} & \text{, for } \tau(J) = (\frac{\alpha}{w})^2 G(w), \\ \{h_{l_2}\} & \text{, for } G(w)(\frac{\alpha}{w})^2 < \tau(J) < (\frac{\alpha}{w})^2 \bar{K}, \\ \{h_{l_1}, h_{l_2}\} & \text{, for } \tau(J) = (\frac{\alpha}{w})^2 \bar{K}, \\ \{h_{l_1}\} & \text{, for } \tau(J) > (\frac{\alpha}{w})^2 \bar{K}. \end{cases}$$

(b) for $w = \bar{w}$

$$\mathcal{M}^*(J) = \begin{cases} \{h_{\mathbf{a}}\} & \text{, for } \tau(J) < (\frac{\alpha}{w})^2 G(w), \\ \{h_{\mathbf{a}}, h_{l_1}, h_{l_2}\} & \text{, for } \tau(J) = (\frac{\alpha}{w})^2 G(w) = (\frac{\alpha}{w})^2 \bar{K}, \\ \{h_{l_1}\} & \text{, for } \tau(J) > (\frac{\alpha}{w})^2 \bar{K}. \end{cases}$$

(c) for $w > \bar{w}$

$$\mathcal{M}^*(J) = \begin{cases} \{h_{\mathbf{a}}\} & \text{, for } \tau(J) < (\frac{\alpha}{w})^2 G(w), \\ \{h_{\mathbf{a}}, h_{l_1}\} & \text{, for } \tau(J) = (\frac{\alpha}{w})^2 G(w), \\ \{h_{l_1}\} & \text{, for } \tau(J) > (\frac{\alpha}{w})^2 G(w). \end{cases}$$

2. If $\operatorname{sgn}(a\alpha) = 1$ *we have*

$$\mathcal{M}^*(J) = \begin{cases} \{h_{\mathbf{a}}\} & , \textit{for} \quad \tau(J) < (\frac{\alpha}{w})^2 K_{1,*}(w), \\ \{h_{\mathbf{a}}, h_{l_1}\} & , \textit{for} \quad \tau(J) = (\frac{\alpha}{w})^2 K_{1,*}(w), \\ \{h_{l_1}\} & , \textit{for} \quad \tau(J) > (\frac{\alpha}{w})^2 K_{1,*}(w). \end{cases}$$

Proof of Proposition 4.7. We only give the proof for $s = -1$. Our arguments can easily be adapted to the case $s = 1$; in fact $s = 1$ is simpler because the local minimiser of $\tilde{E}^K_{1,1}$ is unique. We omit the parameter s in the following.

Step 1: We show that the values $(K_{i,*}(w))_{i\in\{1,2\}}$ and $w = \bar{w}$ are well defined. We start with the values $(K_{i,*}(w))_{i\in\{1,2\}}$. Fix $i \in \{1, 2\}$. Since by definition $K_{i,*}(w) = \infty$ if (4.23) has no solution we only have to prove that (4.23) has at most one solution. We do this by contradiction. Assume that (4.23) has two solutions $K_1 < K_2$. This implies

$$\mathcal{E}^{K_1}_w(g_{l_i(K_1)}) = \mathcal{E}^{K_2}_w(g_{l_i(K_2)}). \tag{4.25}$$

But for fixed l, the function $K \to \mathcal{E}^K_w(g_l)$ is strictly decreasing and hence we have

$$\mathcal{E}^{K_1}_w(g_l) > \mathcal{E}^{K_2}_w(g_l) \quad , \text{for} \quad l = l_i(K_1). \tag{4.26}$$

By Proposition 4.5 we have

$$\begin{aligned} \mathcal{E}^{K_2}_w(g_{l_1(K_1)}) &\geq \min_{l\in(0,3)} \mathcal{E}^{K_2}_w(g_l) = \mathcal{E}^{K_2}_w(g_{l_1(K_2)}), \\ \mathcal{E}^{K_2}_w(g_{l_2(K_1)}) &\geq \min_{l\in[6,\infty)} \mathcal{E}^{K_2}_w(g_l) = \mathcal{E}^{K_2}_w(g_{l_2(K_2)}). \end{aligned} \tag{4.27}$$

Combining (4.26) and (4.27) we see that

$$\mathcal{E}^{K_1}_w(g_{l_i(K_1)}) > \mathcal{E}^{K_2}_w(g_{l_i(K_1)}) \geq \mathcal{E}^{K_2}_w(g_{l_i(K_2)}).$$

This is a contradiction to (4.25). So the values $(K_{i,*}(w))_{i\in\{1,2\}}$ are unique.

Now we show that $\bar{w}$ is well defined. We will use that $w \mapsto K_{1,*}(w)$ is invertible for w such that $K_{1,*}(w) < \infty$. So before we prove that $\bar{w}$ is well defined we show that $w \mapsto K_{1,*}(w)$ is strictly increasing. The following argument works also for $w \mapsto K_{2,*}(w)$ and so we prove for $i \in \{1, 2\}$ that $w \mapsto K_{i,*}(w)$ is strictly increasing. Fix $i \in \{1, 2\}$. By definition of $K_{i,*}$ and by (4.19), the inverse function of $K_{i,*}$ is given by

$$\frac{K}{\tilde{E}^K_{\mathbf{s}}(l_i(K))}. \tag{4.28}$$

The function (4.28) is strictly increasing because, by (4.52), its first derivative coincides with

$$\frac{\tilde{E}_{\mathbf{s}}^{K}(l_i(K)) - Kl_i(K)}{[\tilde{E}_{\mathbf{s}}^{K}(l_i(K))]^2}, \tag{4.29}$$

which is, by definition of $\tilde{E}_{\mathbf{s}}^{K}$, strictly positive. So the functions $(K_{i,*})_{i\in\{1,2\}}$ are strictly increasing.

In the following we derive two conditions that a solution of (4.24) necessarily satisfies. We will see that these conditions imply that (4.24) has one and only one solution. Assume that (4.24) has a solution w^* and let $K^* := K_{1,*}(w^*) = K_{2,*}(w^*)$. Then by definition of $K_{1,*}$ and $K_{2,*}$ we have

$$0 = \mathcal{E}_{w^*}^{K^*}(g_{l_1}) - \mathcal{E}_{w^*}^{K^*}(g_{l_2}) = \tilde{E}_{\mathbf{s}}^{K^*}(l_1) - \tilde{E}_{\mathbf{s}}^{K^*}(l_2). \tag{4.30}$$

Since by Proposition 4.11 below, the function

$$K \mapsto \tilde{E}_{\mathbf{s}}^{K}(l_1(K)) - \tilde{E}_{\mathbf{s}}^{K}(l_2(K))$$

has a unique zero, namely K_0, the value w^* has to satisfy

$$K_{1,*}(w^*) = K_0 \,\text{and}\, K_{2,*}(w^*) = K_0. \tag{4.31}$$

Since $K_{1,*}$ and $K_{2,*}$ are invertible functions of w the conditions (4.31) imply that if (4.24) has a solution then this solution is necessarily unique. Note that as the inverse functions $[K_{1,*}]^{-1}$ and $[K_{2,*}]^{-1}$ are given by (4.28) both conditions from (4.31) lead to the condition

$$w^* = \frac{K_0}{\tilde{E}_{\mathbf{s}}^{K_0}(l_1(K_0))} = \frac{K_0}{\tilde{E}_{\mathbf{s}}^{K_0}(l_1(K_0))}.$$

By going backwards through the steps that lead us to the necessary conditions (4.31), we see that the two conditions do not contradict each other and that the solution to (4.24) coincides with the solution to $K_{1,*}(w) = K_0$ and thus we obtain existence of a solution.

Step 2: To determine the set $\tilde{S}_1$ we consider its complement $\tilde{S}_1^c$. We have $(w, K) \in \tilde{S}_1^c$ if and only if $g_{\mathbf{s}}$ is not a minimiser of $\mathcal{E}_w^K$. And $g_{\mathbf{s}}$ is not a minimiser of $\mathcal{E}_w^K$ if and only if at least one g_l satisfies $\mathcal{E}_w^K(g_l) < \mathcal{E}_w^K(g_{\mathbf{s}})$. This is the case if and only if for at least one $i \in \{1, 2\}$ and $l^* = l_i$

- (4.21) is satisfied with strict inequality,
- and (4.20) is satisfied.

For the moment, fix $i \in \{1,2\}$. The area in the (w, K)-plane in which (4.20) is satisfied is

$$\{(w,K) \in \mathbb{R}_{>0} \times \mathbf{D}_i \mid w \leq \frac{1}{l_i(K)}\}, \tag{4.32}$$

where $\mathbf{D}_1 := \mathbb{R}_{>0}$ and $\mathbf{D}_2 := (0, \frac{1}{72}]$; and the area in which (4.21) is satisfied with strict inequality is

$$\{(w,K) \in \mathbb{R}_{>0} \times \mathbf{D}_i \mid w < \frac{K}{\tilde{E}^K_{\mathbf{s}}(l_i(K))}\}. \tag{4.33}$$

The intersection of (4.32) and (4.33) coincides with (4.33) because as we will see

$$\frac{K}{\tilde{E}^K_{\mathbf{s}}(l_i(K))} < \frac{1}{l_i(K)} \quad \text{, for all} \quad K \in \mathbf{D}_i.$$

To check this we use that by the definition of $\tilde{E}^K_{\mathbf{s}}(l)$ (see (4.13)) there is a function $R(l)$ (independent of K) with $R(l) > 0$ such that $\tilde{E}^K_{\mathbf{s}}(l) = R(l) + Kl$

$$\frac{Kl}{\tilde{E}^K_{\mathbf{s}}(l)} = \frac{1}{1 + \frac{R(l)}{Kl}} < 1 \quad \text{, for all} \quad l \in \mathbb{R}_{>0}.$$

So, we have

$$\begin{aligned}
\tilde{S}^c_1 &= \bigcup_{i\in\{1,2\}} \{(w,K) \in \mathbb{R}_{>0} \times \mathbf{D}_i \mid w < \frac{K}{\tilde{E}^K_{\mathbf{s}}(l_i(K))}\} \\
&= \bigcup_{i\in\{1,2\}} \{(w,K) \in \mathbb{R}_{>0} \times (0, \tfrac{1}{72}] \mid w < \frac{K}{\tilde{E}^K_{\mathbf{s}}(l_i(K))}\} \\
&\quad \cup \{(w,K) \in \mathbb{R}_{>0} \times (\tfrac{1}{72}, \infty] \mid w < \frac{K}{\tilde{E}^K_{\mathbf{s}}(l_1(K))}\},
\end{aligned}$$

where

$$\begin{aligned}
&\bigcup_{i\in\{1,2\}} \{(w,K) \in \mathbb{R}_{>0} \times (0, \tfrac{1}{72}] \mid w < \frac{K}{\tilde{E}^K_{\mathbf{s}}(l_i(K))}\} \\
&= \{(w,K) \in \mathbb{R}_{>0} \times (0, \tfrac{1}{72}] \mid w < \max_{i\in\{1,2\}} \frac{K}{\tilde{E}^K_{\mathbf{s}}(l_i(K))}\},
\end{aligned} \tag{4.34}$$

which we show in the following. To check (4.34), note that (w, K) is, by definition of the union of two sets, an element of the left hand side of (4.34) if and only if at least one of the following two conditions is satisfied:

$$w < \frac{K}{\tilde{E}^K_{\mathbf{s}}(l_1(K))} \quad \text{or} \quad w < \frac{K}{\tilde{E}^K_{\mathbf{s}}(l_2(K))}. \tag{4.35}$$

By definition of the maximum, (w, K) satisfies (4.35) if and only if

$$w < \max_{i\in\{1,2\}} \frac{K}{\tilde{E}^K_{\mathbf{s}}(l_i(K))},$$

and this is equivalent to (w, K) being an element of the right hand side of (4.34).

To evaluate the maximum on the right hand side of (4.34), we identify the subset of $(0, \frac{1}{72}]$ in which the function given by

$$\frac{K}{\tilde{E}^K_{\mathbf{s}}(l_2(K))} - \frac{K}{\tilde{E}^K_{\mathbf{s}}(l_1(K))} = K\frac{\tilde{E}^K_{\mathbf{s}}(l_1(K)) - \tilde{E}^K_{\mathbf{s}}(l_2(K))}{\tilde{E}^K_{\mathbf{s}}(l_1(K))\tilde{E}^K_{\mathbf{s}}(l_2(K))} \tag{4.36}$$

is positive. Since, by Proposition 4.11 below, $\tilde{E}^K_{\mathbf{s}}(l_1(K)) - \tilde{E}^K_{\mathbf{s}}(l_2(K))$ is strictly negative if and only if $K > \bar{K}$, we have

$$\max_{i\in\{1,2\}} \frac{K}{\tilde{E}^K_{\mathbf{s}}(l_i(K))} = \begin{cases} \frac{K}{\tilde{E}^K_{\mathbf{s}}(l_2(K))} & \text{, for} \quad K \le \bar{K}, \\ \frac{K}{\tilde{E}^K_{\mathbf{s}}(l_1(K))} & \text{, otherwise.} \end{cases}$$

So,

$$\begin{aligned} \tilde{S}^c_1 = \{(w, K) \in \mathbb{R}_{>0} \times (0, \bar{K}] \mid w < \frac{K}{\tilde{E}^K_{\mathbf{s}}(l_2(K))}\} \\ \cup \{(w, K) \in \mathbb{R}_{>0} \times (\bar{K}, \infty) \mid w < \frac{K}{\tilde{E}^K_{\mathbf{s}}(l_1(K))}\}. \end{aligned} \tag{4.37}$$

By definition and the monotonicity of $K_{i,*}$, equation (4.37) turns into

$$\begin{aligned} \tilde{S}^c_1 &= \{(w, K) \in (0, \bar{w}] \times \mathbb{R}_{>0}] \mid K_{2,*}(w) < K\} \\ &\cup \{(w, K) \in (\bar{w}, \infty) \times \mathbb{R}_{>0} \mid K_{1,*}(w) < K\} \\ &= \{(w, K) \in \mathbb{R}^2_{>0} \mid K > G(w)\}. \end{aligned}$$

We plotted $K_{1,*}$ and $K_{2,*}$ in Figure 4.2.

Step 3: Now we consider $\tilde{S}_2$ and $\tilde{S}_3$. Therefore note that we have

$$\tilde{S}_2 \cup \tilde{S}_3 = \tilde{S}^c_1 \cup \{(w, G(w)) \mid w \in \mathbb{R}\}, \tag{4.38}$$

because, by the definitions of $(S_i)_{i\in\{1,2,3\}}$ and (4.17),

- $(w, K) \in \tilde{S}_2 \cup \tilde{S}_3$ if and only if at least one of the functions $\{g_l \mid l \le 1\}$ is a minimiser of $\mathcal{E}^K_w$
- $(w, K) \in \tilde{S}^c_1$ if and only if $g_{\mathbf{s}}$ is not a minimiser of $\mathcal{E}^K_w$,

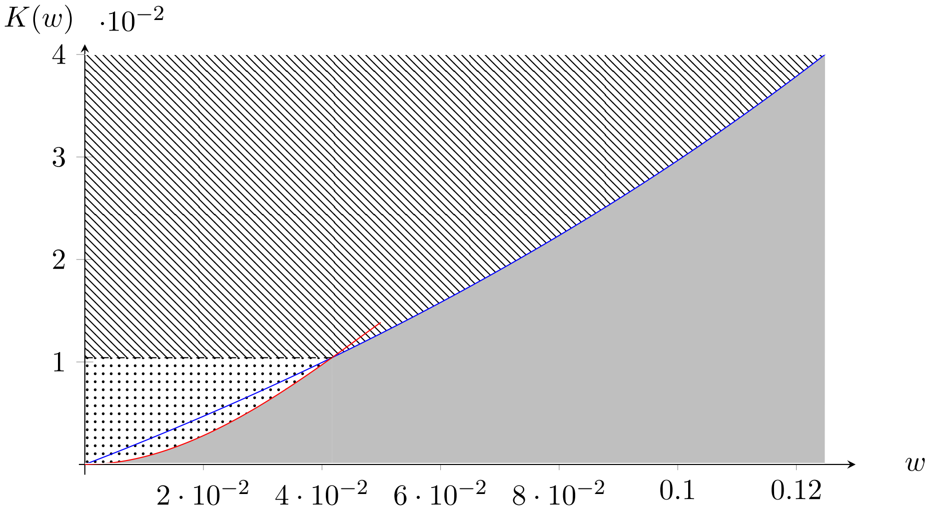

Figure 4.2: This figure illustrates Proposition 4.7: It is a graph of the functions $K = K_{1,*}$ (blue) and $K = K_{2,*}$ (red) and of the sets $\tilde{S}_1$, $\tilde{S}_2$, $\tilde{S}_3$ in the (w, K)-plane for $s = -1$. The grey area corresponds to the set $\tilde{S}_1$ where the minimiser is the straight line, the area filled with lines going from north to west coincides with the set $\tilde{S}_3$ where the minimiser does not cross the zero line (like the orange minimiser from Figure 4.1), the area filled with dots corresponds to $\tilde{S}_2$ where the minimiser crosses zero once (like the green minimiser from Figure 4.1), $\tilde{S}_4$ and $\tilde{S}_5$ coincide and are the intersection of $\tilde{S}_2$ and $\tilde{S}_3$. (Minimisers like the blue one from Figure 4.1 do not appear, because we assume $s = -1$ and hence the gradient of the minimiser at $\xi = 0$ has to be negative.)

- and for $(w, K) \in \{(w, G(w)) \mid w \in \mathbb{R}\}$, there is a $l \in \mathbb{R}$ such that both g_l and $g_{\mathbf{s}}$ are minimisers of $\mathcal{E}_w^K$: $\mathcal{E}_w^K(g_l) = \mathcal{E}_w^K(g_{\mathbf{s}}) = 0$.

To distinguish $\tilde{S}_2$ and $\tilde{S}_3$, we use condition (4.22): Since $\bar{K} = K_0$, we have

$$\begin{aligned}\tilde{S}_2 &= \{(w, K) \in \mathbb{R}_{>0} \times (0, \tfrac{1}{72}] \mid K \leq \bar{K}\} \cap [\tilde{S}_1^c \cup \{(w, G(w) \mid w \in \mathbb{R}_{>0}\}] \\ &= \{(w, K) \in \mathbb{R}_{>0} \times (0, \tfrac{1}{72}] \mid K \in [G(w), \bar{K}]\},\end{aligned}$$

and

$$\begin{aligned}\tilde{S}_3 &= \{(w, K) \in \mathbb{R}_{>0} \times \mathbb{R}_{>0} \mid K \geq \bar{K}\} \cap [\tilde{S}_1^c \cup \{(w, G(w) \mid w \in \mathbb{R}_{>0}\}] \\ &= \{(w, K) \in \mathbb{R}_{>0} \times \mathbb{R}_{>0} \mid K \geq \max(\bar{K}, G(w))\}.\end{aligned}$$

□

We summarise the minimisers of the original function $\tilde{E}_{\mathbf{a}}^\tau$ using Proposition 4.5.

Proposition 4.9. *Let* $\mathbf{a} \in \mathbb{R}^2$ *such that* $w \in \mathbb{R}_{>0}$ *and* $s = \operatorname{sgn}(a\alpha)$*.*

1. *For* $\tau \in \mathbb{R}_{>0}$*, the set of local minima of* $\tilde{E}_{\mathbf{a}}^\tau$ *depends on* $\mathbf{a}$*.*
   (a) *For* $s = 1$ *the function* $\tilde{E}_{\mathbf{a}}^\tau$ *has only one local minimum in*
   $$l_1(\mathbf{a}, \tau) = \frac{|\alpha| + \sqrt{\alpha^2 + 6|a|\sqrt{2\tau}}}{\sqrt{2\tau}}. \tag{4.39}$$
   (b) *For* $s = -1$ *the set of local minima of* $\tilde{E}_{\mathbf{a}}^\tau$ *has up to two elements*
   $$l_1(\mathbf{a}, \tau) = \frac{-|\alpha| + \sqrt{\alpha^2 + 6|a|\sqrt{2\tau}}}{\sqrt{2\tau}} \tag{4.40}$$
   and
   $$l_2(\mathbf{a}, \tau) = \frac{|\alpha| + \sqrt{\alpha^2 - 6|a|\sqrt{2\tau}}}{\sqrt{2\tau}}, \tag{4.41}$$
   where l_1 *is always a local minimum and* l_2 *is only a local minimum if* $\tau \leq \frac{\alpha^4}{72a^2}$*.*
2. *For* $\tau = 0$ *the function* $\tilde{E}_{\mathbf{a}}^\tau$ *has no local minimum but* $\lim_{l\to\infty} \tilde{E}_{\mathbf{a}}^\tau(l) = 0$*.*
3. *The function* $\tilde{E}_{(a,0)}^\tau$ *has only one local minimum attained at*
   $$l_1(\tau, a, 0) := \sqrt{a}(\tfrac{18}{\tau})^{\frac{1}{4}}. \tag{4.42}$$

4. The function $\tilde{E}^{\tau}_{(0,\alpha)}$ has only one local minimum attained at

$$l_1(\tau, 0, \alpha) := \alpha\sqrt{\tfrac{2}{\tau}}.$$

We give the proof of Proposition 4.9 at the end of this section. Now we study the sets intersections of $(S_i)_{i\in\{1,2,3\}}$ with the set where $w = 0$ or $w = \infty$.

Proposition 4.10. *Fix $\mathbf{a} \in \mathbb{R}^2$ such that $w \in \mathbb{R}_{>0}$.*

1. Let $\mathcal{M}^*(J)$ be the set of minimisers of $\mathcal{E}$ in $H_{(a,0)}$. There is a G such that

$$\mathcal{M}^*(J) = \begin{cases} \{h_{(a,0)}\} & \text{, for} \quad \tau(J) < G, \\ \{h_{(a,0)}, h_{l_1}\} & \text{, for} \quad \tau(J) = G, \\ \{h_{l_1}\} & \text{, for} \quad \tau(J) > G. \end{cases}$$

2. Let $\mathcal{M}^*(J)$ be the set of minimisers of $\mathcal{E}$ in $H_{(0,\alpha)}$. There is a G such that

$$\mathcal{M}^*(J) = \begin{cases} \{h_{(0,\alpha)}\} & \text{, for} \quad \tau(J) < G, \\ \{h_{(0,\alpha)}, h_{l_1}\} & \text{, for} \quad \tau(J) = G, \\ \{h_{l_1}\} & \text{, for} \quad \tau(J) > G. \end{cases}$$

Proof. The existence of G follows since for fixed l the function $J \to \mathcal{E}(h_l)$ is monotone decreasing. □

Collection of remaining proofs

Proof of Proposition 4.2. We consider the functions $h^{(0,1)}_{(1,sl,\mathbf{0})}$ for $s \in \{1, -1\}$ and $l > 0$. Fix $s \in \{1, -1\}$ and $l > 0$. The function $h^{(0,1)}_{(1,sl,\mathbf{0})}(\xi)$ has a zero in $(0, 1)$ if and only if it has a local minimum at which it has a negative value. Its derivative has maximal one zero in $(0, 1)$ because by Proposition (D.1), the derivative

$$\dot{h}^{(0,1)}_{(1,sl,\mathbf{0})}(\xi) = ls + 2[-3 - 2ls]\xi + 3[2 + ls]\xi^2$$

is a quadratic function and, the value at $\xi = 1$ is already zero by assumption. Hence the function has maximal one local extrema; for our discussion of the first derivative we distinguish three cases:

1. For $s = 1$, the local extrema is a maximum because, as the function value at $\xi = 0$ is greater then the function value at $\xi = 1$, the first derivative changes sign from positive to negative.

2. For $s = -1$ and $l \leq 3$ there is no local extrema or the local extrema is a maximum. To show this we use that the first derivative is a quadratic function. Since the second derivative at $\xi = 1$ is given by

$$\ddot{h}^{(0,1)}_{(1,sl,\mathbf{0})}(1) = -2l + 6$$

the gradient of the first derivative at $\xi = 1$ is positive, and since the first derivative is strictly smaller then zero at $\xi = 0$ and is equal to zero at $\xi = 1$, the first derivative stays below zero.

3. For $s = -1$ and $l > 3$ there is a local minimum. To show this we use again that the first derivative is a quadratic function. In this case the second derivative is strictly negative at $\xi = 1$ and the first derivative is strictly smaller than zero at $\xi = 0$ and equal to zero at $\xi = 1$. So the first derivative has a zero at which it changes sign from negative to positive. The function value at the local minimiser has to be negative because otherwise it would not be a minimiser.

That $h_{0,l,\mathbf{0}}$ has no zero follows by definition. □

Proof of Proposition 4.5. We have to find the positive local minima of $\tilde{E}^K_{\mathbf{s}}$ for all $K \in \mathbb{R}_{>0}$ and $s \in \{-1, 1\}$. Fix $K \in \mathbb{R}_{>0}$ and $s \in \{-1, 1\}$. Since $l \to \tilde{E}^K_{\mathbf{s}}(l)$ is a two times continuously differentiable function from $\mathbb{R} \setminus \{0\} \to \mathbb{R}$, its local minima are those zeros of

$$\tfrac{\mathrm{d}}{\mathrm{d}l}\tilde{E}^K_{\mathbf{s}}(l) = -\tfrac{1}{l^4}[2(9 + 6sl + l^2) - \tau l^4] = -\tfrac{1}{2l^4}[2(l + 3s) - \sqrt{2K}l^2][2(l + 3s) + \sqrt{2K}l^2] \tag{4.43}$$

at which

$$\tfrac{\mathrm{d}^2}{\mathrm{d}l\mathrm{d}l}\tilde{E}^K_{\mathbf{s}}(l) = 4l^{-5}(l + 6s)(l + 3s) \tag{4.44}$$

is strictly positive. The zeros of the first derivative are

$$z_1 = \sqrt{(2K)}^{-1}(-1 - \sqrt{1 - 6s\sqrt{2K}}), \quad z_2 = \sqrt{(2K)}^{-1}(-1 + \sqrt{1 - 6s\sqrt{2K}}),$$
$$z_3 = \sqrt{(2K)}^{-1}(1 - \sqrt{1 + 6s\sqrt{2K}}), \quad z_4 = \sqrt{(2K)}^{-1}(1 + \sqrt{1 + 6s\sqrt{2K}}).$$

Depending on s and K not all of this zeros are positive real values.

Case $s = 1$**:** Only z_4 is an element of $\mathbb{R}_{>0}$. Since the second derivative is strictly positive for all $l > 0$, the zero z_4 is the only local minimum of $\tilde{E}^K_{(1,1)}$. As $l_1(1, K) = z_4(K)$, $l_1(1, K)$ is the unique local minimum of $\tilde{E}^K_{(1,1)}$ in $\mathbb{R}_{>0}$.
Case $s = -1$**:** As the second derivative is strictly positive if and only if $l \in (0, 3) \cup (6, \infty)$ and as $l_1(-1, K) = z_2(K)$ and $l_2(-1, K) = z_4(K)$ it is sufficient to check that, for $K \in \mathbb{R}_{>0}$,

$$\begin{aligned} &z_1(K) \in (-\infty, 0), \qquad z_2(K) \in (0, 3) \\ &z_3(K) \in (3, 6] \cup \mathbb{C} \setminus \mathbb{R}, \quad z_4(K) \in [6, \infty) \cup \mathbb{C} \setminus \mathbb{R}. \end{aligned} \tag{4.45}$$

A direct way to prove (4.45) is to study the behaviour of all four maps $(K \to z_i(K))_{i\in\{1,2,3,4\}}$. For the way that we use it is sufficient to study the behaviour of the function $F(l) : \mathbb{R}_{>0} \to \mathbb{R}$ given by

$$F(l) = \frac{2(9 - 6l + l^2)}{l^4} = \frac{2(l - 3)^2}{l^4}. \tag{4.46}$$

By (4.43), the zeros $(z_i)_{i\in\{1,2,3,4\}}$ that are elements of $\mathbb{R}$ are the l coordinates of the points of intersection between the graphs of $F(l)$ (see Figure 4.3) and the straight line at level K:

$$\{(l, k) \in \mathbb{R}^2 \mid k = F(l)\} \cap \{(l, k) \in \mathbb{R}^2 \mid k = K\} = \bigcup_{\substack{i\in\{1,2,3,4\} \\ \text{s.t } z_i\in\mathbb{R}}} \{(z_i, K)\}. \tag{4.47}$$

To determine the left hand side of (4.47), we need to obtain the solutions of the equation $F(l) = K$. We do this by discussing the monotonicity of F. Since the first derivative of F coincides with (4.44), the function F is

- strictly increasing for $l \in (-\infty, 0)$ and $\lim_{l\to-\infty} F(l) = 0$, $\lim_{l\to0} F(l) = \infty$
- strictly decreasing for $l \in (0, 3)$ and $\lim_{l\to0} F(l) = \infty$, $\lim_{l\to3} F(l) = 0$
- strictly increasing for $l \in (3, 6)$ and $\lim_{l\to3} F(l) = 0$, $\lim_{l\to6} F(l) = \frac{1}{72}$
- strictly decreasing for $l \in (6, \infty)$ and $\lim_{l\to6} F(l) = \frac{1}{72}$,$\lim_{l\to\infty} F(l) = 0$.

By monotonicity $F(l) = K$ has maximal one solution in any of the intervals $(-\infty, 0)$, $(0, 3)$, $(3, 6)$ and $[6, \infty)$. Since $F(l) \leq \frac{1}{72}$ for $l \geq 3$, the equation $F(l) = K$ has solutions $l \in [3, \infty)$ if and only if $K \leq \frac{1}{72}$.

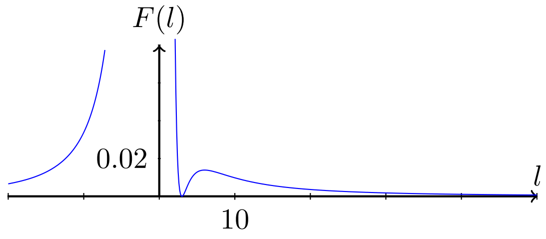

Figure 4.3: This is a plot of the function $F(l)$ (see (4.46)). We use $F(l)$ to determine the local minima of $\tilde{E}_{\mathbf{s}}^K$.

Case $K \leq \frac{1}{72}$: The equation $F(l) = K$ has four solutions

$$\begin{aligned} &s_1 \in (-\infty, 0), \qquad s_2 \in (0,3) \\ &s_3 \in (3,6] \cup \mathbb{C} \setminus \mathbb{R}, \quad s_4 \in [6,\infty) \cup \mathbb{C} \setminus \mathbb{R}. \end{aligned} \tag{4.48}$$

Since, by definition of $(z_i)_{i \in \{1,2,3,4\}}$,

$$z_1 < z_2 < z_3 \leq z_4,$$

and since by (4.47), $\{z_1, z_2, z_3, z_4\} = \{s_1, s_2, s_3, s_4\}$ we see that (4.45) is satisfied for $K \leq \frac{1}{72}$.

Case $K > \frac{1}{72}$: The equation $F(l) = K$ has two solutions

$$s_1 \in (-\infty, 0), \qquad s_2 \in (0,3),$$

and only the zeros z_1 and z_2 are in $\mathbb{R}$. Since $z_1 < z_2$ we have $z_1 = s_1$ and $z_2 = s_2$. So (4.45) is also satisfied for $K > \frac{1}{72}$. □

Proof of Proposition 4.9. 1.)By Corollary 4.6, we just need to check that the right hand sides of (4.39), (4.40) and (4.41) coincide with $wl_i(s, K)$ where $w = \frac{|a|}{|\alpha|}$, $K = (\frac{w}{\alpha})^2 \tau(J)$ and where $i \in \{1,2\}$, $s \in \{-1,1\}$ respectively.

2.) By (4.43), the function $l \to \tilde{E}_{\mathbf{s}}^0$ is strictly decreasing and as $\tilde{E}_{\mathbf{s}}^0(l) = O(\frac{1}{l})$ for $l \to \infty$ the function converges to zero.

3.) and 4.): We have:

$$\begin{aligned} &\text{the local minimum of} \quad \tilde{E}_{1,0}^K \quad \text{is} \quad (\tfrac{18}{K})^{\frac{1}{4}}, \\ &\text{the local minimum of} \quad \tilde{E}_{0,1}^K \quad \text{is} \quad \sqrt{\tfrac{2}{K}}. \end{aligned} \tag{4.49}$$

As

$$\tilde{E}^{\tau}_{a,0}(l) = a^2 \tilde{E}^{Ca^{-2}}_{1,0}(l), \quad \text{and} \quad \tilde{E}^{\tau}_{0,\alpha}(l) = \alpha^2 \tilde{E}^{C\alpha^{-2}}_{0,1}(l),$$

we have the following statements

$$\begin{aligned} &\text{if } l \text{ is a local minimum of } \tilde{E}^{\tau}_{1,0}, \text{ then is } l \text{ also a minimiser of } \tilde{E}^{K\alpha^2}_{(a,0)}, \\ &\text{if } l \text{ is a local minimum of } \tilde{E}^{K}_{0,1}, \text{ then is } l \text{ also a minimiser of } \tilde{E}^{Ka^2}_{(0,\alpha)}. \end{aligned} \tag{4.50}$$

Combining (4.49) and (4.50), we see that

$$\begin{aligned} &\sqrt{a}(\tfrac{18}{C})^{\frac{1}{4}} \quad \text{is the local minimiser of} \quad \tilde{E}^{\tau}_{(a,0)} \text{ and that} \\ &\alpha\sqrt{\tfrac{2}{C}} \quad \text{is the local minimiser of} \quad \tilde{E}^{\tau}_{(0,\alpha)}. \end{aligned}$$

□

Proposition 4.11. *The function from* $(0, \frac{1}{72}]$ *to* $\mathbb{R}$ *given by*

$$\tilde{E}^{K}_{\mathbf{s}}(l_1(K)) - \tilde{E}^{K}_{\mathbf{s}}(l_2(K)) \tag{4.51}$$

has a unique zero called K_0*, is strictly decreasing and strictly positive for* $K < K_0$*.*

Proof. By evaluating the function for $K \to 0$ and $K = \frac{1}{72}$, we see that the continuous function changes its sign, so it must have a zero. (For $K = \frac{3}{4}\frac{1}{72}$ we have the following approximations $\tilde{E}^{K}_{\mathbf{s}}(l_1(K)) \approx \tilde{E}^{K}_{\mathbf{s}}(l_2(K)) \approx 0.25$) For uniqueness we show that the function (4.51) is strictly decreasing. Therefore we consider its first derivative. Since for a fixed K we have

$$\tfrac{\mathrm{d}}{\mathrm{d}l}\tilde{E}^{K}_{\mathbf{s}}(l) = 0 \quad , \text{for } l \in \{l_1(K), l_2(K)\},$$

we obtain

$$\tfrac{\mathrm{d}}{\mathrm{d}K}\tilde{E}^{K}_{\mathbf{s}}(l_i(K)) = l_i(K) \quad , \text{for} \quad i \in \{1,2\} \quad \text{and} \quad K \in (0, \tfrac{1}{72}]. \tag{4.52}$$

By Proposition 4.5, $l_1(K) < l_2(K)$ for all $K \in (0, \frac{1}{72}]$ and hence the first derivative of (4.51) is negative for all $K \in (0, \frac{1}{72}]$. □

4.3 With terminal condition

We consider the zero sets of all functions in (4.3). By definition, the zero set of $h_{l,r}$ is the union of the interval $[l, r]$ and the zero sets of the minimisers $h^{(0,l)}_{(\mathbf{a},\mathbf{0})}$ and $h^{(1-r,1)}_{(\mathbf{0},\mathbf{b})}$. To obtain the zero set of $h^{(1-r,1)}_{(\mathbf{0},\mathbf{b})}$ we use that, by definition of $h^{(1-r,1)}_{(b,-\beta,\mathbf{0})}$, we have

$$h^{(1-r,1)}_{(\mathbf{0},\mathbf{b})}(\xi) = h^{(1-r,1)}_{(b,-\beta,\mathbf{0})}(2 - r - \xi).$$

So we only need to consider the zero sets of $h^{(0,r)}_{(\mathbf{a},\mathbf{0})}$ for $\mathbf{a} = (b, -\beta)$ and $r \in \mathbb{R}_{>0}$. Shifting this zero set back into the interval $(1 - r, 1)$ we get the zero set of $h^{(1-r,1)}_{(\mathbf{0},\mathbf{b})}$. The zero set of $h^{(0,r)}_{(\mathbf{a},\mathbf{0})}$ for $\mathbf{a} \in \mathbb{R}$ has already been studied in Proposition 4.2.

Using this we see that the zero set of a minimiser is the union of at most five disjoint sets. For the following study we reformulate this as follows: If h is a minimiser of $\mathcal{E}$ in $H_{\mathbf{r}}$, then h is an element of one of the following sets:

$$\begin{aligned}
\mathcal{M}_1 &:= \{h_{\mathbf{r}} \mid \mathbf{r} \in \mathbb{R}^4\},\\
\mathcal{M}_2 &:= \bigcup_{\mathbf{r}\in\mathbb{R}^4} \{h_{l,r} \mid l + r \le 1, h_{l,r} \text{ has one zero in } (0, l) \text{ and one in } (1 - r, 1)\},\\
\mathcal{M}_3 &:= \bigcup_{\mathbf{r}\in\mathbb{R}^4} \{h_{l,r} \mid l + r \le 1, h_{l,r} \text{ has no zero in } (0, l) \text{ and no zero in } (1 - r, 1)\},\\
\mathcal{M}_4 &:= \bigcup_{\mathbf{r}\in\mathbb{R}^4} \{h_{l,r} \mid l + r \le 1, h_{l,r} \text{ has one zero in } (0, l) \text{ and no zero in } (1 - r, 1)\},\\
\mathcal{M}_5 &:= \bigcup_{\mathbf{r}\in\mathbb{R}^4} \{h_{l,r} \mid l + r \le 1, h_{l,r} \text{ has no zero in } (0, l) \text{ and one zero in } (1 - r, 1)\}.
\end{aligned}$$

In Figure 4.4 we illustrate one element of each of this sets. We like to study the sets

$$S_i := \{(\mathbf{r}, J) \in \mathbb{R}^5 \mid \text{ there is a } h \in \mathcal{M}_i \text{ that minimises } \mathcal{E} \text{ in } H_{\mathbf{r}}\},$$

for $i \in \{1, 2, \ldots, 5\}$. Therefore we use that

$$\mathcal{E}(h_{l,r}) = \tfrac{2}{l}[3(\tfrac{a}{l})^2 + 3\tfrac{a\alpha}{l} + \alpha^2] - \tau(J)(1 - l - r) + \tfrac{2}{r}[3(\tfrac{b}{r})^2 - 3\tfrac{b\beta}{r} + \beta^2]. \quad (4.53)$$

We denote the right hand side of (4.53), which is defined for all $(l, r) \in \mathbb{R}^2_+$, by $E^\tau(l, r)$ and note that by (4.13) we have

$$E^\tau(l, r) = \tilde{E}^\tau_{(a,\alpha)}(l) + \tilde{E}^\tau_{(b,-\beta)}(r) - \tau. \quad (4.54)$$

The local minimisers of E^τ are the pairs (l, r), where l is a local minimiser of $\tilde{E}^\tau_{(a,\alpha)}$ and r is a local minimiser of $\tilde{E}^\tau_{(b,-\beta)}$.

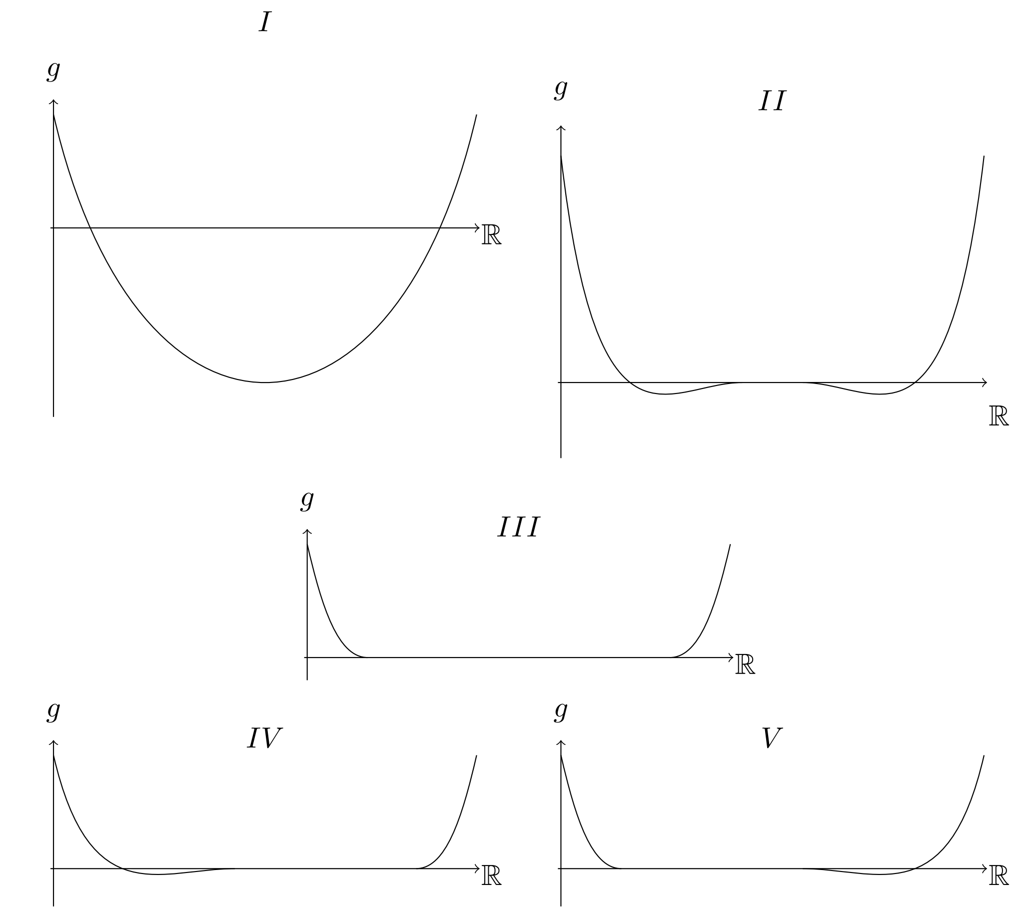

Figure 4.4: These are sketches of possible minimisers of $\mathcal{E}$ in $H_{\mathbf{r}}$ (for a definition of $\mathcal{E}$, see (4.2)): A minimiser either does not pick reward (I), or picks reward and crosses zero twice (II), or picks reward with out crossing the zero line (III), or picks reward and crosses zero only once (IV) or (V).

For a local minimiser (l^*, r^*) of E^τ we show the following: The function $h_{(l^*,r^*)}$ is a minimiser of $\mathcal{E}$ in $H_{\mathbf{a}}$ if and only if the following three conditions are satisfied:

$$l^* + r^* \leq 1, \tag{4.55}$$

$$E^\tau(l^*, r^*) \leq \mathcal{E}(h_{\mathbf{r}}), \tag{4.56}$$

and

$$E^\tau(l^*, r^*) \leq E^\tau(l, r), \text{ for every local minimiser } (l, r) \text{ of } E^\tau \text{ that satisfies (4.55) and (4.56).} \tag{4.57}$$

The argument for this equivalence is analogously to the argument for the three conditions (4.20), (4.21) and (4.22): The three conditions are necessary, because if one of them is not satisfied the function $h_{(l^*,r^*)}$ is not a minimiser. Now we show that the conditions are sufficient. First of all note that by the first two conditions the set $\{h_{l,r} \mid l + r \leq 1\}$ contains a minimiser. Then note that for all l, r such that $l + r = 1$ the functions $h_{l,r}$ satisfy, by (4.7), $\mathcal{E}(h_{l,r}) \geq \mathcal{E}(h_{\mathbf{r}})$. So if a local minimiser (l^*, r^*) satisfies all three conditions the function h_{l^*,r^*} is a minimiser of $\mathcal{E}$ in $H_{\mathbf{a}}$.

Additionally, the function $h_{\mathbf{r}}$ is a minimiser of $\mathcal{E}$ in $H_{\mathbf{r}}$ if and only if there is no local minimiser satisfying (4.55) and (4.56) or if $h_{(l^*,r^*)}$ minimises $\mathcal{E}$ in $H_{\mathbf{r}}$ but $E^\tau(l^*, r^*) = \mathcal{E}(h_{\mathbf{r}})$.

We shall not study the set $\mathcal{M}^*$ in full detail for all possible boundaries $\mathbf{r}$. We restrict ourselves to the following symmetric boundary conditions $\mathbf{r} = (\mathbf{a}, \tilde{\mathbf{a}})$ where $\tilde{\mathbf{a}} = (a, -\alpha)$.

In the next four subsections we consider the following subsets of the parameter space

- $\{(\mathbf{r}, J) \in \mathbb{R}^5 \mid \mathbf{r} \text{ is symmetric with } w \in (0, \infty) \text{ and } s = -1\}$,
- $\{(\mathbf{r}, J) \in \mathbb{R}^5 \mid \mathbf{r} \text{ is symmetric with } w \in (0, \infty) \text{ and } s = 1\}$,
- $\{(\mathbf{r}, J) \in \mathbb{R}^5 \mid \mathbf{r} \text{ is symmetric with } \alpha = 0\}$,
- $\{(\mathbf{r}, J) \in \mathbb{R}^5 \mid \mathbf{r} \text{ is symmetric with } a = 0\}$.

We now discus two results that we use for the first two subsections. The boundary conditions that we consider in these two sections are convenient because they allow us to formulate a dual problem that is directly related to the dual problem that we are using in Section 4.2. We now present this dual problem.

Proposition 4.12. *If $a \neq 0$ and $\alpha \neq 0$, let $w := \frac{|a|}{|\alpha|}$, $K := \tau(J)(\frac{w}{\alpha})^2$ and $s = \operatorname{sgn}(a\alpha)$. We have:*

A function h is a minimiser of $\mathcal{E}$ in $H_{(\mathbf{a},\tilde{\mathbf{a}})}$ if and only if

$$g(\xi) := \tfrac{1}{a} h(w\xi) \tag{4.58}$$

is a minimiser of $\mathcal{E}_w^K$ in $H_{(\mathbf{s},\tilde{\mathbf{s}})}(0, w^{-1})$, where $\tilde{\mathbf{s}} = (1, -s)$ and where $\mathcal{E}_w^K : H_{(\mathbf{s},\tilde{\mathbf{s}})} \to \mathbb{R}$ is given by (4.15)*.*

Proof. Analogously to the proof of Proposition 4.3. □

Now we proceed as in Section 4.2. We consider the sets

$$\tilde{S}_i := T(S_i) \quad \text{, for} \quad i \in \{1, 2, \ldots, 5\},$$

where T is the map such that $T(\mathbf{a}, J) = (s, w, K)$. Analogously to (4.17) we have that $h_{l,r}$ minimises $\mathcal{E}$ in $H_{(\mathbf{a},\tilde{\mathbf{a}})}$ if and only if $g_{lw^{-1},rw^{-1}}$ minimises $\mathcal{E}_w^K$ where, for all (l, r), such that $l + r \leq w^{-1}$,

$$g_{l,r}(\xi) := \begin{cases} h_{(\mathbf{s},\mathbf{0})}^{(0,l)}(\xi) & \text{, for} \quad \xi < l, \\ h_{(0,0,\tilde{\mathbf{s}})}^{(w^{-1}-r,w^{-1})}(\xi) & \text{, for} \quad \xi > 1 - r, \\ 0 & \text{, otherwise.} \end{cases}$$

Since

$$\mathcal{E}_w^K(g_{l,r}) = \tilde{E}_{\mathbf{s}}^K(l) + \tilde{E}_{\mathbf{s}}^K(r) - Kw^{-1}, \tag{4.59}$$

we obtain, analogously to Corollary 4.6, the following corollary.

Corollary 4.13. *We have*

$$\tilde{S}_2 = \{(s, w, K) \in \{1, -1\} \times \mathbb{R}^2_{>0} \mid \textit{for}\, l = l_2(s, K), r = l_2(s, K),\ g_{l,r} \textit{ minimises}\, \mathcal{E}_w^K\},$$
$$\tilde{S}_3 = \{(s, w, K) \in \{1, -1\} \times \mathbb{R}^2_{>0} \mid \textit{for}\, l = l_1(s, K), r = l_1(s, K),\ g_{l,r} \textit{ minimises}\, \mathcal{E}_w^K\},$$
$$\tilde{S}_4 = \{(s, w, K) \in \{1, -1\} \times \mathbb{R}^2_{>0} \mid \textit{for}\, l = l_2(s, K), r = l_1(s, K),\ g_{l,r} \textit{ minimises}\, \mathcal{E}_w^K\},$$
$$\tilde{S}_5 = \{(s, w, K) \in \{1, -1\} \times \mathbb{R}^2_{>0} \mid \textit{for}\, l = l_1(s, K), r = l_2(s, K),\ g_{l,r} \textit{ minimises}\, \mathcal{E}_w^K\},$$

and

$$S_2 \cap \mathcal{R} = \{(\mathbf{a}, J) \in \mathcal{R} \mid \text{for } l = l_2(\mathbf{a}, J), r = l_2(\mathbf{a}, J),\ h_{l,r} \text{ is a minimiser of } \mathcal{E} \text{ in } H_{\mathbf{r}}\},$$
$$S_3 \cap \mathcal{R} = \{(\mathbf{a}, J) \in \mathcal{R} \mid \text{for } l = l_1(\mathbf{a}, J), r = l_1(\mathbf{a}, J),\ h_{l,r} \text{ is a minimiser of } \mathcal{E} \text{ in } H_{\mathbf{r}}\},$$
$$S_4 \cap \mathcal{R} = \{(\mathbf{a}, J) \in \mathcal{R} \mid \text{for } l = l_2(\mathbf{a}, J), r = l_1(\mathbf{a}, J),\ h_{l,r} \text{ is a minimiser of } \mathcal{E} \text{ in } H_{\mathbf{r}}\},$$
$$S_5 \cap \mathcal{R} = \{(\mathbf{a}, J) \in \mathcal{R} \mid \text{for } l = l_1(\mathbf{a}, J), r = l_2(\mathbf{a}, J),\ h_{l,r} \text{ is a minimiser of } \mathcal{E} \text{ in } H_{\mathbf{r}}\},$$

where we use

$$\mathcal{R} := \{(\mathbf{a}, J) \in \mathbb{R}^3 \mid w \in (0, \infty)\}.$$

4.3.1 Symmetric boundary condition where $a \neq 0$ and $a\alpha < 0$

Since $s = -1$ in this subsection we identify $(-1, w, K)$ with (w, K).

Proposition 4.14. *We have (see Figure 4.5)*

$$\begin{aligned}
\tilde{S}_1 &= \{(w, K) \in \mathbb{R}^2_{>0} \mid K \leq G(w)\}, \\
\tilde{S}_2 &= \{(w, K) \in (0, \hat{w}] \times \mathbb{R}_{>0} \mid G(w) \leq K \leq \underline{K}\}, \\
\tilde{S}_3 &= \{(w, K) \in (0, \hat{w}] \times \mathbb{R}_{>0} \mid K \geq \underline{K}\} \\
&\quad \cup \{(w, K) \in (\hat{w}, \infty) \times \mathbb{R}_{>0} \mid K \geq G(w)\}, \\
\tilde{S}_4 = \tilde{S}_5 &= \{(w, K) \in (0, \hat{w}] \times \mathbb{R}_{>0} \mid K = \underline{K}\},
\end{aligned}$$

where

- *for* $w \in \mathbb{R}^+$,
$$G(w) := \begin{cases} \hat{K}_{2,*}(w) & , \text{for} \quad w \in (0, \hat{w}], \\ \hat{K}_{1,*}(w) & , \text{for} \quad w \in (\hat{w}, \infty), \end{cases}$$
- *for* $w \in \mathbb{R}^+$ *and* $i \in \{1, 2\}$, $\hat{K}_{i,*}(w)$ *is the unique value for* K *such that for* $l^* = l_i(K)$
$$\mathcal{E}^K_w(g_{l^*,l^*}) = \mathcal{E}^K_w(g_{\mathbf{s}}), \tag{4.60}$$
if there is no such K *set* $\hat{K}_{i,*}(w) = \infty$.
- $\hat{w}$ *is the unique value for* w *such that*
$$\hat{K}_{1,*}(w) = \hat{K}_{2,*}(w), \tag{4.61}$$
- $\underline{K} := \hat{K}_{1,*}(\hat{w})$.

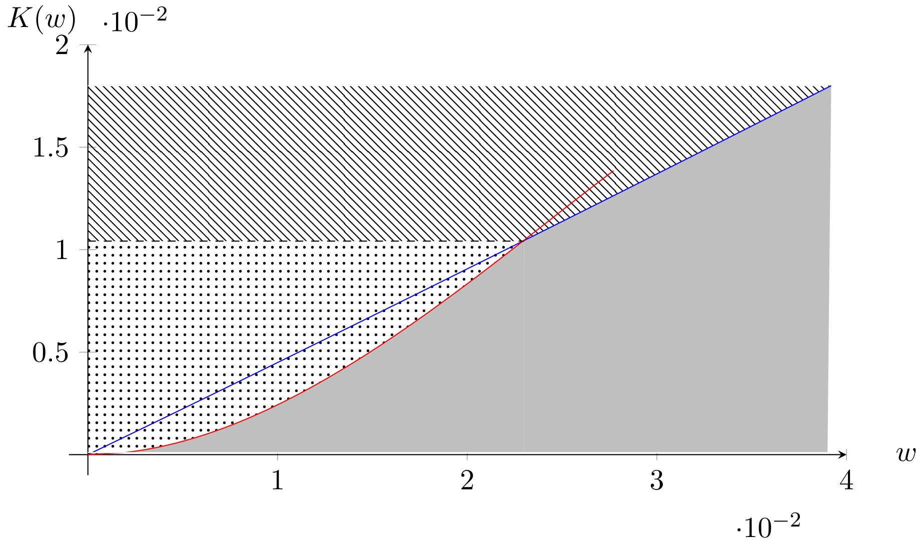

Figure 4.5: This figure illustrates Proposition 4.14: It is a graph of the functions $K = \hat{K}_{1,*}$ (blue) and $K = \hat{K}_{2,*}$ (red) and of the sets $\tilde{S}_1, \ldots, \tilde{S}_5$ in the (w, K)-plane for $s = -1$. The grey area corresponds to the set $\tilde{S}_1$ where the minimiser does not pick reward (in this set the minimisers have a form like the sketch (I) in Figure 4.4), the area filled with lines going from north to west (including the dashed line) corresponds to the set $\tilde{S}_3$ (in this set the minimisers have a form like the sketch (III) in Figure 4.4), the area filled with dots (including the dashed line) corresponds to the set $\tilde{S}_2$ (in this set the minimisers have a form like the sketch (II) in Figure 4.4), $\tilde{S}_4$ and $\tilde{S}_5$ coincide and are the intersection of $\tilde{S}_2$ and $\tilde{S}_3$, i.e. the dashed line (in these sets minimisers appear that have forms like the sketches (IV) and (V) in Figure 4.4).

We prove Proposition 4.14 after the following corollary.

Corollary 4.15. *Fix* $(a, \alpha) \in \mathbb{R}^2$ *such that* $a \neq 0, \alpha \neq 0$ *and* $\operatorname{sgn}(a\alpha) = -1$. *Let* $\mathbf{r} = (a, \alpha, a, -\alpha)$, $\mathcal{M}^*(J)$ *be the set of minimisers of* $\mathcal{E}$ *in* $H_{\mathbf{r}}$ *and* $w = \frac{|a|}{|\alpha|}$. *We have,*

1. *for* $w < \hat{w}$

$$\mathcal{M}^*(J) = \begin{cases} \{h_{\mathbf{r}}\} & , \textit{for} \quad \tau(J) < (\frac{\alpha}{w})^2 G(w), \\ \{h_{\mathbf{r}}, h_{l_2,l_2}\} & , \textit{for} \quad \tau(J) = (\frac{\alpha}{w})^2 G(w), \\ \{h_{l_2,l_2}\} & , \textit{for} \quad (\frac{\alpha}{w})^2 G(w) < \tau(J) < (\frac{\alpha}{w})^2 \underline{K}, \\ \{h_{l_1,l_1}, h_{l_1,l_2}, h_{l_2,l_1}, h_{l_2,l_2}\} & , \textit{for} \quad \tau(J) = (\frac{\alpha}{w})^2 \underline{K}, \\ \{h_{l_1,l_1}\} & , \textit{for} \quad \tau(J) > (\frac{\alpha}{w})^2 \underline{K}, \end{cases}$$

2. for $w = \hat{w}$

$$\mathcal{M}^*(J) = \begin{cases} \{h_{\mathbf{r}}\} & \text{, for } \tau(J) < (\frac{\alpha}{w})^2 G(w), \\ \{h_{\mathbf{r}}, h_{l_1,l_1}, h_{l_1,l_2}, h_{l_2,l_1}, h_{l_2,l_2}\} & \text{, for } \tau(J) = (\frac{\alpha}{w})^2 \underline{K} = (\frac{\alpha}{w})^2 G(\hat{w}), \\ \{h_{l_1,l_1}\} & \text{, for } \tau(J) > (\frac{\alpha}{w})^2 \underline{K}, \end{cases}$$

3. for $w > \hat{w}$

$$\mathcal{M}^*(J) = \begin{cases} \{h_{\mathbf{r}}\} & \text{, for } \tau(J) < (\frac{\alpha}{w})^2 G(w), \\ \{h_{\mathbf{r}}, h_{l_1,l_1}\} & \text{, for } \tau(J) = (\frac{\alpha}{w})^2 G(w), \\ \{h_{l_1,l_1}\} & \text{, for } \tau(J) > (\frac{\alpha}{w})^2 G(w). \end{cases}$$

Proof. We obtain this corollary by combining Proposition 4.14 with Corollary 4.13. □

Proof of Proposition 4.14.
Step 1: The values $(\hat{K}_{i,*}(w))_{i\in\{1,2\}}$ and $\hat{w}$ are well defined. This follows analogously to Step 1 in the proof of Proposition 4.7. Note that we also have $\underline{K} = K_0$.
Step 2: To determine $(\tilde{S}_i)_{i\in\{1,2,\dots,5\}}$, we need to decide whether $g_{l,r}$ is a minimiser of $\mathcal{E}_w^K$. Therefore we transform the conditions (4.55), (4.56) and (4.57) (these are necessary and sufficient conditions for $h_{l,r}$ to be a minimiser of $\mathcal{E}$ in $H_{\mathbf{a}}$) into conditions for $g_{lw^{-1},rw^{-1}}$.

- Condition (4.55) turns into

$$l^* + r^* \le w^{-1}. \tag{4.62}$$

- Condition (4.56) turns into

$$\tilde{E}_{\mathbf{s}}^K(l^*, r^*) - Kw^{-1} \le \mathcal{E}_w^K(g_{\mathbf{s},\tilde{\mathbf{s}}}), \tag{4.63}$$

  where

$$\tilde{E}_{\mathbf{s}}^K(l^*, r^*) := \tilde{E}_{\mathbf{s}}^K(l^*) + \tilde{E}_{\mathbf{s}}^K(r^*). \tag{4.64}$$

As

$$
\begin{aligned}
\mathcal{E}_w^K(g_{\mathbf{s},\tilde{\mathbf{s}}}) &= \tfrac{1}{2}\int_0^{w^{-1}} (\ddot{g}_{(\mathbf{s},\tilde{\mathbf{s}})}(\xi))^2 \mathrm{d}\xi \\
&= w^3 \tfrac{1}{2}\int_0^1 (\ddot{h}^{(0,1)}_{(1,sw^{-1},1,-sw^{-1})}(\xi))^2 \mathrm{d}\xi \\
&= 2w,
\end{aligned}
$$

condition (4.63) is equivalent to

$$
\tilde{E}_{\mathbf{s}}^K(l^*, r^*) - Kw^{-1} \leq 2w, \tag{4.65}
$$

- condition (4.57) turns into

$$
\tilde{E}_{\mathbf{s}}^K(l^*, r^*) \leq \tilde{E}_{\mathbf{s}}^K(l, r), \text{ for all local minimiser} \quad (l,r) \quad \text{of} \quad \tilde{E}_{\mathbf{s}}^K \\
\text{that satisfy (4.63) and (4.65).} \tag{4.66}
$$

Hence if (l^*, r^*) is a local minimiser of $\tilde{E}_{\mathbf{s}}^K$, the function g_{l^*,r^*} is a minimiser of $\mathcal{E}_w^K$ if and only if (l^*, r^*) satisfies (4.62), (4.65), and (4.66).

Step 3: We consider $\tilde{S}_1$. As in Section 4.2, it is more convenient to study $\tilde{S}_1^c$ because $(w, K) \in \tilde{S}_1^c$ if and only if $g_{\mathbf{s},\tilde{\mathbf{s}}}$ is not a minimiser of $\mathcal{E}_w^K$; and $g_{\mathbf{s},\tilde{\mathbf{s}}}$ is not a minimiser of $\mathcal{E}_w^K$ if and only if there is a $g_{l,r}$ that satisfies $\mathcal{E}_w^K(g_{l,r}) < \mathcal{E}_w^K(g_{\mathbf{s},\tilde{\mathbf{s}}})$. So we have $(w, K) \in \tilde{S}_1^c$ if and only if for at least one pair $(i, j) \in \{1, 2\} \times \{1, 2\}$ the following two conditions are satisfied for $(l^*, r^*) = (l_i, l_j)$

- condition (4.62) and
- condition (4.65) with strict inequality.

Fix $(i, j) \in \{1, 2\} \times \{1, 2\}$. The area in the (w, K)-plane in which (4.62) is satisfied is given as

$$
\{(w, K) \in \mathbb{R}_{>0} \times \mathbf{D}_{i,j} \mid w \leq \frac{1}{l_i(K) + l_j(K)}\}, \tag{4.67}
$$

where $\mathbf{D}_{i,j} := \mathbf{D}_i \cap \mathbf{D}_j$; and the area in which (4.65) is satisfied with strict inequality is the set

$$
\{(w, K) \in \mathbb{R}_{>0} \times \mathbf{D}_{i,j} \mid 2w^2 - \tilde{E}_{\mathbf{s}}^K(l_i, l_j)w + K > 0\}. \tag{4.68}
$$

For fixed K, the quadratic function

$$
w \mapsto 2w^2 - \tilde{E}_{\mathbf{s}}^K(l_i, l_j)w + K \tag{4.69}
$$

has two zeros, namely

$$w_1^{(i,j)} = \frac{\tilde{E}_{\mathbf{s}}^K(l_i,l_j) - \sqrt{[\tilde{E}_{\mathbf{s}}^K(l_i,l_j)]^2 - 8K}}{4}$$

and

$$w_2^{(i,j)} = \frac{\tilde{E}_{\mathbf{s}}^K(l_i,l_j) + \sqrt{[\tilde{E}_{\mathbf{s}}^K(l_i,l_j)]^2 - 8K}}{4}$$

Since the function in (4.69) is strictly positive if and only if $w \in (0, w_1^{(i,j)}) \cup (w_2^{(i,j)}, \infty)$, the set in (4.68) coincides with the set

$$\{(w,K) \in \mathbb{R}_{>0} \times \mathbf{D}_{i,j} \mid w \in (0, w_1^{(i,j)}(K)) \cup (w_2^{(i,j)}(K), \infty)\}. \tag{4.70}$$

In Proposition 4.20 below we show that $w_1^{(i,j)}(K)$ and $w_2^{(i,j)}(K)$ are elements of $\mathbb{R}$ for all $K \in \mathbf{D}_{i,j}$.

The intersection of (4.67) and (4.70) is given by

$$\begin{aligned} &\{(w,K) \in \mathbb{R}_{>0} \times \mathbf{D}_{i,j} \mid w \leq \tfrac{1}{l_i(K)+l_j(K)}, w < w_1^{(i,j)}(K)\} \\ \cup&\{(w,K) \in \mathbb{R}_{>0} \times \mathbf{D}_{i,j} \mid w_2^{(i,j)}(K) < w \leq \tfrac{1}{l_i(K)+l_j(K)}\}, \end{aligned} \tag{4.71}$$

where the second set is empty if $w_2^{(i,j)}(K) > \frac{1}{l_i(K)+l_j(K)}$ for all $K \in \mathbf{D}_{i,j}$. Since we have by Proposition 4.21 that

$$w_1^{(i,j)}(K) \leq \tfrac{1}{l_i(K)+l_j(K)} \leq w_2^{(i,j)}(K) \quad \text{, for} \quad K \in \mathbf{D}_{i,j},$$

the set (4.71) coincides with the set

$$\{(w,K) \in \mathbb{R}_{>0} \times \mathbf{D}_{i,j} \mid w < w_1^{(i,j)}(K)\}. \tag{4.72}$$

By using a result that is analogous to (4.34), we obtain that

$$\begin{aligned} \tilde{S}_1^c &= \bigcup_{(i,j)\in\{1,2\}\times\{1,2\}} \{(w,K) \in \mathbb{R}_{>0} \times \mathbf{D}_{i,j} \mid w < w_1^{(i,j)}(K)\} \\ &= \{(w,K) \in \mathbb{R}_{>0} \times (0, \tfrac{1}{72}] \mid w < \max_{(i,j)\in\{1,2\}\times\{1,2\}} w_1^{(i,j)}(K)\} \\ &\cup \{(w,K) \in \mathbb{R}_{>0} \times (\tfrac{1}{72}, \infty) \mid w < w_1^{(1,1)}(K)\}. \end{aligned} \tag{4.73}$$

By Proposition 4.23 below, we further get

$$\max_{(i,j)\in\{1,2\}\times\{1,2\}} w_1^{(i,j)}(K) = w_1^{(2,2)}(K) \quad \text{, for} \quad K \in (0, \underline{K}],$$
$$\max_{(i,j)\in\{1,2\}\times\{1,2\}} w_1^{(i,j)}(K) = w_1^{(1,1)}(K) \quad \text{, for} \quad K \in (\underline{K}, \tfrac{1}{72}],$$

and henceforth

$$\begin{aligned}\tilde{S}_1^c &= \{(w,K) \in \mathbb{R}_{>0} \times (0, \underline{K}] \mid w < w_1^{(2,2)}(K)\} \\ &\cup \{(w,K) \in \mathbb{R}_{>0} \times (\underline{K}, \infty) \mid w < w_1^{(1,1)}(K)\}. \end{aligned} \tag{4.74}$$

Now we use the inverse function of $K \mapsto w_1^{(i,i)}(K)$ for $K \in \mathbf{D}_{i,i}$. Therefore we show that $w_1^{(i,i)}(K)$ coincides with the inverse of $\hat{K}_{i,*}$. Fix $i \in \{1,2\}$ and $K \in \mathbf{D}_{i,i}$, we seek a value $w \in \mathbb{R}_{>0}$ such that $K = \hat{K}_{i,*}(w)$. By definition of $w \mapsto \hat{K}_{i,*}(w)$ the value K solves (4.60). So it is necessary that $w^{-1} \geq 2l_i(K)$ because otherwise the function g_{l^*,l^*} in (4.60) is not well defined. Furthermore since for $l^* + r^* \leq w^{-1}$ we have

$$\mathcal{E}_w^K(g_{l^*,r^*}) = \tilde{E}_{\mathbf{s}}^K(l^*, r^*) - Kw^{-1}.$$

the value w has to be a zero of (4.69) (recall that the value of the function (4.69) is the value for K such that (4.63) is satisfied with equality). Now we know that (4.69) has two solutions $w_1^{(i,i)}(K)$ and $w_2^{(i,i)}(K)$. But by Proposition 4.21, $w_2^{(i,i)}(K) \geq \frac{1}{2l_i(K)}$ and $w_2^{(i,i)}(K) = \frac{1}{2l_i(K)}$ if and only if $w_1^{(i,i)}(K) \geq \frac{1}{2l_i(K)}$. So the only w that solves $K = \hat{K}_{i,*}(w)$ is $w_1^{(i,i)}(K)$.

As $w_1^{(i,i)}(K)$ is continuous and invertible, it is either strictly increasing or decreasing. By evaluating $w_1^{(i,i)}(K)$ for two values for K, we see that $w_1^{(i,i)}(K)$ is strictly increasing. So $\hat{K}_{i,*}$ is also strictly increasing.

Applying that the increasing function $\hat{K}_{i,*}$ is the inverse function of $w_1^{(i,i)}(K)$ to (4.74), we immediately get that

$$\begin{aligned}\tilde{S}_1^c &= \{(w,K) \in (0, \hat{w}] \times \mathbb{R}_{>0} \mid K > \hat{K}_{2,*}(w)\} \\ &\cup \{(w,K) \in (\hat{w}, \infty) \times \mathbb{R}_{>0} \mid K > \hat{K}_{1,*}(w)\} \\ &= \{(w,K) \in \mathbb{R}^2_{>0} \mid K > G(w)\}. \end{aligned} \tag{4.75}$$

Step 4: We consider the remaining sets $(\tilde{S}_i)_{i\in\{2,3,4,5\}}$, and we shall first consider their union. By the definitions of $(\tilde{S}_i)_{i\in\{2,3,4,5\}}$, this is the union of all (w,K) such that $\{g_{l,r} \mid l + r \leq 1\}$ contains a minimiser of $\mathcal{E}_w^K$. By definition of $G(w)$, we have that

$$\bigcup_{i\in\{2,3,4,5\}} \tilde{S}_i = \tilde{S}_1^c \cup \{(w,K) \in \mathbb{R}^2_{>0} \mid K = G(w)\}.$$

To distinguish between the sets $(\tilde{S}_i)_{i\in\{2,3,4,5\}}$, we use, as in Section 4.2, that $\underline{K} = K_0$. We outline our arguments only for the set $\tilde{S}_4$. Since $2l_i(K) < w^{-1}$, we have $(w,K) \in \tilde{S}_4$ if and only if $(w,K) \in \bigcup_{i\in\{2,3,4,5\}} \tilde{S}_i$ and

$$\tilde{E}^K_{\mathbf{s}}(l_1(K)) = \tilde{E}^K_{\mathbf{s}}(l_2(K)).$$

But this is only possible if $K = K_0$ and hence we obtain that

$$\begin{aligned}\tilde{S}_4 &= \{(w,K) \in \mathbb{R}^2 \mid K = K_0\} \cap \bigcup_{i\in\{2,3,4,5\}} \tilde{S}_i \\ &= \{(w,K) \in (0,\hat{w}] \times \mathbb{R}_{>0} \mid K = \underline{K}\}.\end{aligned}$$

□

4.3.2 Symmetric boundary condition where $a \neq 0$ and $a\alpha > 0$

Since $s = 1$ in this subsection we identify $(1,w,K)$ with (w,K). All proofs in this subsection are analogous to the ones in Section 4.3.1.

Proposition 4.16. *We have*

$$\begin{aligned}\tilde{S}_1 &= \{(w,K) \in \mathbb{R}^2_{>0} \mid K \leq \hat{K}_{1,*}(w)\}, \\ \tilde{S}_3 &= \{(w,K) \in \mathbb{R}^2_{>0} \mid K \geq \hat{K}_{1,*}(w)\},\end{aligned}$$

where, for $w \in \mathbb{R}_{>0}$, the value $\hat{K}_{1,*}(w)$ is the unique value for K such that for $l^* = l_1(1,K)$ we have (4.60).

Corollary 4.17. *Fix $(a,\alpha) \in \mathbb{R}^2$ such that $a \neq 0, \alpha \neq 0$ and $\operatorname{sgn}(a\alpha) = 1$. Let $\mathbf{r} = (a,\alpha,a,-\alpha)$, $\mathcal{M}^*(J)$ be the set of minimisers of $\mathcal{E}$ in $H_{\mathbf{r}}$ and $w = \frac{|a|}{|\alpha|}$. We have,*

$$\mathcal{M}^*(J) = \begin{cases} \{h_{\mathbf{r}}\} & , \textit{for} \quad \tau(J) < (\frac{\alpha}{w})^2 \hat{K}_{1,*}(w), \\ \{h_{\mathbf{r}}, h_{l_1,l_1}\} & , \textit{for} \quad \tau(J) = (\frac{\alpha}{w})^2 \hat{K}_{1,*}(w), \\ \{h_{l_1,l_1}\} & , \textit{for} \quad \tau(J) > (\frac{\alpha}{w})^2 \hat{K}_{1,*}(w). \end{cases}$$

4.3.3 Symmetric boundary condition with $\alpha = 0$

We consider symmetric boundary conditions such that $a \neq 0$ and $\alpha = 0$. Fix $J \in \mathbb{R}$. Since, by Proposition 4.9, $\tilde{E}_{\mathbf{a}}^{\tau(J)}$ has at most one local minimiser, namely $l_1(\tau(J), \mathbf{a}) = \sqrt{a}(\frac{18}{\tau(J)})^{\frac{1}{4}}$, the set of possible minimisers in (4.3) turns out to be the set

$$\mathcal{M} = \{h_{l,l} \mid 2l \leq 1\} \cup \{h_{\mathbf{r}}\}.$$

By (4.2), all functions $h_{l,l}$ are elements of $\mathcal{M}_3$. So all parameters $(\mathbf{a}, J)$ that we consider in this section are either in S_1 or S_3. The parameters $(\mathbf{a}, J)$ are in S_3 if and only if

$$\begin{aligned} 2l_1 &\leq 1 \\ 2\tilde{E}_{\mathbf{a}}^{\tau(J)}(l_1) &\leq \tau(J);. \end{aligned} \tag{4.76}$$

this conditions are equivalent to the conditions

$$\begin{aligned} \tau(J) &\geq 288a^2 \\ \tau(J) &\geq \tfrac{32(1+3a)^4}{9a^2}. \end{aligned}$$

We have thus shown the following proposition.

Proposition 4.18. *We have*

$$S_3 \cap \mathcal{X} = \{(\mathbf{a}, J) \in \mathcal{X} \mid \tau(J) \geq \tfrac{32(1+3a)^4}{9a^2}\},$$

and

$$S_1 \cap \mathcal{X} = \{(\mathbf{a}, J) \in \mathcal{X} \mid \tau(J) \leq \tfrac{32(1+3a)^4}{9a^2}\},$$

where we use

$$\mathcal{X} := \{(\mathbf{r}, J) \in \mathbb{R}^5 \mid \mathbf{r} \textit{ is symmetric with } \alpha = 0\}.$$

4.3.4 Symmetric boundary condition with $a = 0$

We consider symmetric boundary conditions such that $a = 0$ and $\alpha \neq 0$. Let $\tilde{\mathcal{X}}$ be the set of this boundary conditions.

Proposition 4.19. *We have*

$$S_3 \cap \tilde{\mathcal{X}} = \{(\mathbf{a}, J) \in \tilde{\mathcal{X}} \mid \tau(J) \geq 32\alpha^2\},$$

and

$$S_1 \cap \tilde{\mathcal{X}} = \{(\mathbf{a}, J) \in \tilde{\mathcal{X}} \mid \tau(J) \leq 32\alpha^2\}.$$

Proof. This proof is analogous to the one of Proposition 4.18. □

4.3.5 Collection of remaining proofs

We collect in this subsection remaining proofs for auxiliary statements used in the previous subsections.

Proposition 4.20. *For all $(i,j) \in \{1,2\} \times \{1,2\}$, the values $w_1^{(i,j)}(K)$ and $w_2^{(i,j)}(K)$ are in $\mathbb{R}$ for all $K \in \mathbf{D}_{i,j}$.*

Proof. Fix $(i,j) \in \{1,2\} \times \{1,2\}$. The values $w_1^{(i,j)}(K)$ and $w_2^{(i,j)}(K)$ are in $\mathbb{R}$ for all $K \in \mathbf{D}_{i,j}$ if and only if the function given by $K \to [\tilde{E}_{\mathbf{s}}^K(l_i, l_j)]^2 - 8K$ is positive in $\mathbf{D}_{i,j}$. This function is positive if and only if

$$\frac{1}{2}\tilde{E}_{\mathbf{s}}^K(l_i, l_j) - \sqrt{2K} \tag{4.77}$$

is positive. To see that the function given by (4.77) is positive we consider its first derivative,

$$\frac{l_i(K) + l_j(K)}{2} - \frac{1}{\sqrt{2K}}. \tag{4.78}$$

We have

- for $i = j = 1$, the derivative (4.78) has exactly one zero in $\mathbb{R}_{>0}$ namely $K = \frac{1}{8}$ (to check this note $l_1(\frac{1}{8}) = 2$); it is positive for $K < \frac{1}{8}$ and it is negative for $K > \frac{1}{8}$;
- for $i = j = 2$ the derivative (4.78) has exactly one zero in $(0, \frac{1}{72}]$ namely $K = \frac{1}{72}$ and it is positive for $K < \frac{1}{72}$;
- for $i \neq j$ the (4.78) is positive for for $K \leq \frac{1}{72}$ because $2l_1 < l_1 + l_2$ for $K \leq \frac{1}{72}$.

These properties of the derivative have the following consequences on the properties of the function given by (4.77):

- for $i = j = 1$, the function has a local minimum at $K = \frac{1}{8}$ and hence, since (4.77) has a zero at $K = \frac{1}{8}$, the function given by (4.77) is positive for all $K \in \mathbb{R}_{>0}$;
- for $i = j = 2$, the function is strictly increasing and hence, since the expression (4.77) converges to zero for $K \to 0$, the function given by (4.77) is positive in $(0, \frac{1}{72}]$.

- for $i \neq j$, the function is strictly increasing and hence, since the expression (4.77) has a positive limit for $K \to 0$, the function given by (4.77) is positive on $(0, \frac{1}{72}]$.

□

Proposition 4.21. *We have:*

1. *For* $(i,j) \neq (1,1)$*:*

$$w_1^{(i,j)}(K) < \tfrac{1}{l_i(K)+l_j(K)} < w_2^{(i,j)}(K) \quad , \textit{for} \quad K \in (0, \tfrac{1}{72}]. \tag{4.79}$$

2. *For* $(i,j) = (1,1)$*:*

$$w_1^{(1,1)}(K) < \tfrac{1}{2l_1(K)} < w_2^{(1,1)}(K) \quad , \textit{for} \quad K \in (0, \tfrac{1}{8}), \tag{4.80}$$

$$w_1^{(1,1)}(K) = \tfrac{1}{2l_1(K)} = w_2^{(1,1)}(K) \quad , \textit{for} \quad K = \tfrac{1}{8}, \tag{4.81}$$

$$w_1^{(1,1)}(K) < \tfrac{1}{2l_1(K)} < w_2^{(1,1)}(K) \quad , \textit{for} \quad K \in (\tfrac{1}{8}, \infty). \tag{4.82}$$

Proof. By the definitions of $w_1^{(i,j)}$ and $w_2^{(i,j)}$ we have $w_1^{(i,j)} \leq w_2^{(i,j)}$. To show the remaining two inequalities we consider the following cases.
Case $(i,j) \neq (1,1)$**:** First of all (4.79) is satisfied for $K = \frac{1}{72}$. To show that (4.79) is satisfied for all other $K \in (0, \frac{1}{72}]$, we show that the graph of the continuous function $\frac{1}{l_i+l_j}$ never intersects with any of the graphs of the continuous functions $w_1^{(i,j)}$ or $w_2^{(i,j)}$. We prove this by contradiction. Fix $v \in \{1,2\}$. Suppose there is a K such that

$$\tfrac{1}{l_i(K)+l_j(K)} = w_v^{(i,j)}(K), \tag{4.83}$$

let $\breve{w} := w_v^{(i,j)}(K)$. Then, by definition of $w_v^{(i,j)}$, we have

$$l_i + l_j = \breve{w}^{-1}, \tag{4.84}$$

$$\mathcal{E}^K_{\breve{w}}(g_{l_i,l_j}) = \mathcal{E}^K_{\breve{w}}(g^{(0,\breve{w}^{-1})}_{(\mathbf{s},\tilde{\mathbf{s}})}). \tag{4.85}$$

By (4.84), the zero set of g_{l_i,l_j} has Lebesgue measure zero and hence by (4.85) we have

$$g_{l_i,l_j} = g^{(0,\breve{w}^{-1})}_{(\mathbf{s},\tilde{\mathbf{s}})}. \tag{4.86}$$

By Proposition (4.22) below, (4.86) is equivalent to $(\breve{w}, l_i, l_j) = (\frac{1}{4}, 2, 2)$. Since either $i = 2$ or $j = 2$, we necessarily have $l_2(K) = 2$. This is a contradiction, because by Proposition 4.5, $l_2(K) \in (6, \infty)$ for all $K \in \mathbb{R}_{>0}$.

Case $(i,j) = (1,1)$**:** We start by showing (4.81). We do this by evaluating the functions; using the fact that $l_1(\frac{1}{8}) = 2$ makes this evaluation handy.

To show the remaining two statements note that for $K = \frac{1}{8^2}$ (4.80) is satisfied, and that for $K = \frac{1}{\sqrt{8}}$ (4.82) is satisfied. To show that the statements are satisfied for all other values of K, namely $K \in (0, \frac{1}{8})$ and $K \in (\frac{1}{8}, \infty)$, we show that the graphs of the three functions $w_1^{(1,1)}(K)$, $w_2^{(1,1)}(K)$ and $\frac{1}{2l_1(K)}$ intersect only in $(w, K) = (\frac{1}{4}, \frac{1}{8})$. We do this again by contradiction. Fix $v \in \{1, 2\}$. Suppose there is a $K \neq \frac{1}{8}$ such that

$$\frac{1}{2l_1(K)} = w_v^{(1,1)}(K), \tag{4.87}$$

let $\breve{w} := w_v^{(1,1)}(K)$. Exactly the same steps that lead us from (4.83) to (4.86) lead us from (4.87) to (4.86) again. By Proposition (4.22) below, (4.86) is equivalent to $(\breve{w}, l_1, l_1) = (\frac{1}{4}, 2, 2)$. But as $K \neq \frac{1}{8}$, $l_1(K) \neq 2$. Hence we have a contradiction.

□

Proposition 4.22. *The equation*

$$g_{l^*,r^*} = g_{1,-1,1,1} \tag{4.88}$$

is satisfied if and only if $(w, l^*, r^*) = (\frac{1}{4}, 2, 2)$.

Proof. We show that $(w, l, r) = (\frac{1}{4}, 2, 2)$ is a necessary condition for (4.88). Since the function $g_{1,-1,1,1}$ has the line at $\xi = \frac{1}{2}w^{-1}$ as axis of symmetry and $g_{l,r}$ has this symmetry only if $l = r = \frac{1}{2}w^{-1}$ the condition

$$l = r = \tfrac{1}{2}w^{-1}, \tag{4.89}$$

is necessary. Since $g_{\frac{1}{2}w^{-1},\frac{1}{2}w^{-1}}$ has a zero at $\frac{1}{2}w^{-1}$, while $g_{1,-1,1,1}$, that satisfies by definition

$$g_{1,-1,1,1}(\xi) = h^{(0,1)}_{1,-w^{-1},1,w^{-1}}(\xi w),$$

has, a zero in $\frac{1}{2}w^{-1}$ (if and) only if $w = \frac{1}{4}$, we see that $(w, l, r) = (\frac{1}{4}, 2, 2)$ is a necessary condition for (4.88).

Clearly $(w, l, r) = (\frac{1}{4}, 2, 2)$ is a sufficient condition for (4.88). □

Proposition 4.23. *We have*

$$w_1^{(1,1)} < w_1^{(1,2)} < w_1^{(2,2)} \quad \text{, for} \quad K < K_0, \tag{4.90}$$

$$w_1^{(1,1)} = w_1^{(1,2)} = w_1^{(2,2)} \quad \text{, for} \quad K = K_0, \tag{4.91}$$

$$w_1^{(1,1)} > w_1^{(1,2)} > w_1^{(2,2)} \quad \text{, for} \quad K > K_0. \tag{4.92}$$

Proof. Fix $K \in \mathbb{R}_{>0}$. In order to apply Proposition 4.11, we use the function given by

$$E \to \frac{E - \sqrt{E^2 - 8K}}{4}. \tag{4.93}$$

First of all note that, for all $(i,j) \in \{1,2\} \times \{1,2\}$, $w_1^{(i,j)}$ coincides with the image of the sum $\tilde{E}^K(l_i) + \tilde{E}^K(l_j)$ of the function (4.93). Since the function (4.93) is strictly decreasing in $E > \sqrt{8K}$(check that the first derivative is negative for all E), we apply Proposition 4.11 in the following way:

- for $K < K_0$ we have $\tilde{E}^K(l_1) > \tilde{E}^K(l_2)$ and hence

  $$\tilde{E}^K(l_1, l_1) > \tilde{E}^K(l_2, l_1) > \tilde{E}^K(l_2, l_2),$$

  so by the monotonicity of (4.93) we get (4.90).

- for $K = K_0$ we have $\tilde{E}^K(l_1) = \tilde{E}^K(l_2)$ and hence

  $$\tilde{E}^K(l_1, l_1) = \tilde{E}^K(l_2, l_1) = \tilde{E}^K(l_2, l_2);$$

  so by the monotonicity of (4.93) we get (4.91).

- for $K > K_0$ we have $\tilde{E}^K(l_1) < \tilde{E}^K(l_2)$ and hence

  $$\tilde{E}^K(l_1, l_1) < \tilde{E}^K(l_2, l_1) < \tilde{E}^K(l_2, l_2),$$

  so by the monotonicity of (4.93) we get (4.92).

□

4.3.6 Remarks on how to deal with general boundary conditions

To describe the behaviour of the interface for non symmetric boundary conditions we define a list of critical values.

Definition 4.24. Fix $\mathbf{r}$. Let

$C_{1,1}(\mathbf{r})$ be the unique value for C such that $l_1(C, \mathbf{a}) + l_1(C, \mathbf{b}) = 1$,
$C_{1,2}(\mathbf{r})$ be the unique value for C such that $l_1(C, \mathbf{a}) + l_2(C, \mathbf{b}) = 1$,
$C_{2,1}(\mathbf{r})$ be the unique value for C such that $l_2(C, \mathbf{a}) + l_1(C, \mathbf{b}) = 1$,
$C_{2,2}(\mathbf{r})$ be the unique value for C such that $l_2(C, \mathbf{a}) + l_2(C, \mathbf{b}) = 1$,

and let

$$C_{1,1,*}(\mathbf{r}) \text{ be the value } C \geq C_{1,1}(\mathbf{r}) \text{ such that} \tilde{E}^C_{\mathbf{a}}(l_1(C,\mathbf{a})) + \tilde{E}^C_{\mathbf{b}}(l_1(C,\mathbf{b})) - C = E_{\mathbf{r}},$$
$$C_{1,2,*}(\mathbf{r}) \text{ be the value } C \geq C_{1,2}(\mathbf{r}) \text{ such that} \tilde{E}^C_{\mathbf{a}}(l_1(C,\mathbf{a})) + \tilde{E}^C_{\mathbf{b}}(l_2(C,\mathbf{b})) - C = E_{\mathbf{r}},$$
$$C_{2,1,*}(\mathbf{r}) \text{ be the value } C \geq C_{2,1}(\mathbf{r}) \text{ such that} \tilde{E}^C_{\mathbf{a}}(l_2(C,\mathbf{a})) + \tilde{E}^C_{\mathbf{b}}(l_1(C,\mathbf{b})) - C = E_{\mathbf{r}},$$
$$C_{2,2,*}(\mathbf{r}) \text{ be the value } C \geq C_{2,2}(\mathbf{r}) \text{ such that} \tilde{E}^C_{\mathbf{a}}(l_2(C,\mathbf{a})) + \tilde{E}^C_{\mathbf{b}}(l_2(C,\mathbf{b})) - C = E_{\mathbf{r}}.$$

The values $C_{i,j}(\mathbf{r})$ are the critical values for h_{l_i,l_j} being defined at all. The values $C_{i,j,*}(\mathbf{r})$ are the critical values such that for $C \geq C_{i,j,*}(\mathbf{r})$ the minimiser has a smaller rate than $h_{\mathbf{r}}$. The behaviour of the interface depends on the ordering of these 8 critical values and additional on the solutions of

$$\tilde{E}^C_{\mathbf{a}}(l_i(C,\mathbf{a})) + \tilde{E}^C_{\mathbf{b}}(l_j(C,\mathbf{b})) = \tilde{E}^C_{\mathbf{a}}(l_m(C,\mathbf{a})) + \tilde{E}^C_{\mathbf{b}}(l_n(C,\mathbf{b})) \tag{4.94}$$

for all $(i,j,m,n) \in \{1,2\}^4$. The solutions of (4.94) are the critical values at which two local minimisers have the same rate.

To demonstrate how the phase transition is determined using these critical values we pick the arbitrary boundary condition $\mathbf{r} = (0.1, -40, 1, 40)$.

Example 4.25. For $\mathbf{r} = (0.1, -40, 1, -40)$ we have

$$\mathcal{M}^*(J) = \begin{cases} \{h_{\mathbf{r}}\} & \text{, for } \tau(J) < C_{2,1,*}(\mathbf{r}), \\ \{h_{\mathbf{r}}, h_{l_2(C,\mathbf{a}),l_1(C,\mathbf{b})}\} & \text{, for } \tau(J) = C_{2,1,*}(\mathbf{r}), \\ \{h_{l_2(C,\mathbf{a}),l_1(C,\mathbf{b})}\} & \text{, for } C_{2,1,*}(\mathbf{r}) < \tau(J) < \bar{C}, \\ \{h_{l_2(C,\mathbf{a}),l_1(C,\mathbf{b})}, h_{l_1(C,\mathbf{a}),l_1(C,\mathbf{b})}\} & \text{, for } \tau(J) = \bar{C}, \\ \{h_{l_1(C,\mathbf{a}),l_1(C,\mathbf{b})}\} & \text{, for } \tau(J) > \bar{C}, \end{cases} \tag{4.95}$$

where $\bar{C}$ is the value for C such that (4.94) holds for $(i,j,m,n) = (2,1,1,1)$.

Statement (4.95) is true because

$$\begin{aligned} &C_{1,1} \approx 3.8 \times 10^3, && C_{1,1,*} \approx 2.5 \times 10^6, \\ &C_{2,1} \approx 1.4 \times 10^4, && C_{2,1,*} \approx 3.3 \times 10^4, \\ &\bar{C} \approx 2.66 \times 10^6. \end{aligned}$$

and hence

$$C_{2,1,*} < C_{1,1,*} < \bar{C}.$$

Chapter 5

Outlook and conclusion

We give possible approaches for dealing with the concentration problem (see Section 5.1) and describe some of the problems related to the wetting model (see Section 5.2). Then we give a conclusion to this thesis and mention possible future projects.

5.1 Concentration

A topic to which the results of this thesis can contribute is the study of the limits

$$\lim_{N\to\infty} \hat{\gamma}_N^{\mathbf{r}}(B_\delta), \tag{5.1}$$

where B_δ is a small ball around a minimiser of the rate function $\Sigma_{\mathbf{r}}^J$ of the model with pinning (see Theorem 1.3). If the minimiser is unique, then the **LDP** from Theorem 1.3 implies that this limit is one. But if the minimiser is not unique the **LDP** is not sufficient to determine the limit.

If the rate function has $M \in \mathbb{N}$ minimisers let $(B_i)_{i\in\{1,2,\dots,M\}}$ be M sufficiently small balls around these minimisers. We say that there is a coexistence of empirical profiles if for at least one ball B_i the limit (5.1) with $B_\delta = B_i$ is in $(0, 1)$. If all but one limit are zero we say the empirical profile concentrates on the minimiser in the ball B_i for which the limit is non zero. For the gradient model the concentration problem has been studied in [5].

For the Laplacian model, we consider the following problem: Fix $a \in \mathbb{R}\setminus\{0\}$ and let $\mathbf{r} = (a, 0, a, 0)$, furthermore let J be such that there are two minimisers $\{\bar{h}, \hat{h}\}$, where $\bar{h} \equiv a$ and $\hat{h}$ picks reward in $[l_1, 1 - l_1]$. Such a reward J exists by Proposition 4.18.

We claim that the limit (5.1) is zero if the ball B_δ is around the minimiser $\hat{h}$

and one if the ball is around $\bar{h}$. Our approach to prove this is similar to the approach of Bolthausen, Funaki and Otobe [5]. We consider the limit of the following quotient

$$\frac{\hat{\gamma}_N^{\mathbf{r}}(\hat{B})}{\hat{\gamma}_N^{\mathbf{r}}(\bar{B})} = \frac{\frac{\hat{\mathcal{Z}}_N^{\mathbf{r}}}{\mathcal{Z}_N^{\mathbf{r}}}\hat{\gamma}_N^{\mathbf{r}}(\hat{B})}{\frac{\hat{\mathcal{Z}}_N^{\mathbf{r}}}{\mathcal{Z}_N^{\mathbf{r}}}\hat{\gamma}_N^{\mathbf{r}}(\bar{B})}, \tag{5.2}$$

where $\hat{B}$ is a ball around $\hat{h}$ and $\bar{B}$ is a ball around $\bar{h}$. Considering this quotient is sufficient because by the **LDP** we get

$$\lim_{N\to\infty}(\hat{\gamma}_N^{\mathbf{r}}(\hat{B}) + \hat{\gamma}_N^{\mathbf{r}}(\bar{B})) = 1.$$

Hence the limit of the quotient (5.2) is zero if and only if the limit of $(\hat{\gamma}_N^{\mathbf{r}}(\hat{B}))_{N\in\mathbb{N}}$ is zero.

In the following we consider the right hand side of (5.2), because by the expansion (1.6) we have

$$\frac{\hat{\mathcal{Z}}_N^{\mathbf{r}}}{\mathcal{Z}_N^{\mathbf{r}}}\hat{\gamma}_N^{\mathbf{r}}(B_\delta) = \frac{1}{\mathcal{Z}_N^{\mathbf{r}}}\sum_{\mathcal{P}\subset\Lambda_N} e^{J|\mathcal{P}|} Z_{\mathcal{P}^c}^{\psi}\gamma_{\mathcal{P}^c}^{\psi}(B_\delta), \tag{5.3}$$

where $\psi = \psi^{\mathbf{r},N}$ (for the definition of $\psi^{\mathbf{r},N}$ see (1.13)). Note that for the boundary condition $\mathbf{r} = (a, 0, a, 0)$ we have $\mathcal{Z}_N^{\mathbf{r}} = \mathcal{Z}_N^{\mathbf{0}}$.

Using (5.3) we see that the limit of the denominator of the quotient on the right hand side of (5.2) is one: For $B_\delta = \bar{B}$ the only non zero term on the right hand side of (5.3) is the one for $\mathcal{P} = \emptyset$. This term coincides with $\gamma_N^{\psi}(\bar{B}) = \gamma_N^{\mathbf{r}}(\bar{B})$ and hence by the **LDP** for $(\gamma_N^{\mathbf{r}})_{N\in\mathbb{N}}$ the limit of the denominator of the quotient on the right hand side of (5.2) is 1.

Thus we only consider the numerator on the right hand side of (5.2). We claim that for $B_\delta = \hat{B}$ we can neglect some of the terms on the right hand side of (5.3) without having an effect on the limit. The idea is that a typical empirical profile in $\hat{B}$ should pick reward in an interval $[t_1, 1-t_2]$ such that t_1 and t_2 are near to l_1 (recall that l_1 is the infimum of the zero set of $\hat{h}$). We expect these profiles to be typical because the rate function $\Sigma_{\mathbf{r}}^J$ for such profiles is close to zero while the rate function for other profiles in $\hat{B}$ is strictly positive.

Based on this idea we conjecture that the typical contact set $\mathcal{P}$ of an interface in $\hat{B}$ should be such that $p_* = \min\mathcal{P}$ is close to Nl_1 and such that $p^* = \max\mathcal{P}$ is close to $N - Nl_1$. Furthermore the set $\mathcal{P}$ should have a cardinality of order N. For the empirical profile, we also expect that the gradients where the profile enters and leaves its zero set coincide with zero. So a typical set $\mathcal{P}$ should contain at least two more points, namely one in a small distance from p_* and one near to p^*. It is plausible to assume that the next point after p_* is within a distance of order N^κ where $\kappa < 1$, because then the slope of the empirical profile at $\frac{p_*}{N}$ is zero for large N.

To formally describe the typical profiles we define sets $D_{(t_1,s_1,t_2,s_2)}$ for vectors (t_1, s_1, t_2, s_2) such that (t_1N, s_1N, t_2N, s_2N) is a vector in $\mathbb{Z}^4$. We let $\mathcal{P} \in D_{(t_1,s_1,t_2,s_2)}$ if $p_* = Nt_1$, $p^* = N - Nt_2$ and $\min \mathcal{P}\setminus\{p_*\} = N(t_1+s_1)$, $\max \mathcal{P}\setminus\{p^*\} = N(1 - t_2 - s_2)$. Fix $\kappa_1 < 0$ and $\kappa_2 < 0$. A sets $\mathcal{P}$ is typical if it is an elements of $D_{(t_1,s_1,t_2,s_2)}$ for (t_1, s_1, t_2, s_2) such that (t_1N, s_1N, t_2N, s_2N) is a vector in $\mathbb{Z}^4$ and such that

$$\begin{aligned} &|t_1 - l_1| < N^{\kappa_1}, 0 \le s_1 < N^{\kappa_2},\\ &|t_2 - l_1| < N^{\kappa_1}, 0 \le s_2 < N^{\kappa_2}. \end{aligned} \tag{5.4}$$

Let W be the set of vectors (t_1, s_1, t_2, s_2) such that $N(t_1, s_1, t_2, s_2) \in \mathbb{Z}^4$ and (5.4) are satisfied.

We claim that there are values for κ_1 and κ_2 such that we can drop the terms that correspond to the not typical $\mathcal{P}$ from the right hand side of (5.3) and still obtain the same limit. In other words we claim that we can neglect the terms that do not correspond to one $\mathcal{P} \in D_{(t_1,s_1,t_2,s_2)}$ for $(t_1, s_1, t_2, s_2) \in W$.

To simplify the sum over the terms that we do not neglect, note that

$$\tfrac{1}{\mathcal{Z}_N^{\mathrm{r}}} \sum_{\mathcal{P}\in D_{(t_1,s_1,t_2,s_2)}} e^{J|\mathcal{P}|} Z^{\psi}_{\mathcal{P}^c} \gamma^{\psi}_{\mathcal{P}^c}(B_\delta) = \tfrac{1}{\mathcal{Z}_N^{\mathrm{r}}} \int_{B_\delta} e^{-\mathcal{H}_N} \mathrm{d}\, m_{(t_1,s_1,t_2,s_2)}, \tag{5.5}$$

where

$$\mathrm{d}\, m_{(t_1,s_1,t_2,s_2)} := e^{4J} \prod_{i\in F} (\mathrm{d}\,\phi_i + e^J \delta_0(\mathrm{d}\,\phi_i)) \prod_{i\in F^c\setminus E} \mathrm{d}\,\phi_i \prod_{i\in\{-1,0\}\cup E} \delta_{\psi_i}(\phi_i),$$

and where

$$\begin{aligned} F &:= \{N(t_1+s_1)+1, N(t_1+s_1)+2, \dots, N(1-t_2-s_2)-1\},\\ E &:= \{Nt_1, N(t_1+s_1), N(1-t_2), N(1-t_2-s_2)\}. \end{aligned}$$

Now we apply the splitting property with lag 2 to the right hand side of (5.5). Therefore we consider the sites $U := \{Nt_1 + 1, N(t_1 + s_1) - 1, N(1 - t_2) - 1, N(1 - t_2 - s_2) + 1\}$. Similar to the proof of the upper bound in Section 3.2.2, we apply a generalisation of the law of total expectation to the right hand side of (5.5): The right hand side of (5.5) coincides with

$$\int_{B_1}\int_{B_2}\int_{B_3}\int_{B_4} \mathcal{Z}_L^{\psi^v}\mathcal{Z}_{S_1}^{\psi^v}\hat{\mathcal{Z}}_M^{\psi^v}\mathcal{Z}_{S_2}^{\psi^v}\mathcal{Z}_R^{\psi^v} \times \gamma_L^{\psi^v}(\hat{B})\gamma_{S_1}^{\psi^v}(\hat{B})\hat{\gamma}_M^{\psi^v}(\hat{B})\gamma_{S_2}^{\psi^v}(\hat{B})\gamma_R^{\psi^v}(\hat{B})\mathrm{d}\, v_1\mathrm{d}\, v_2\mathrm{d}\, v_3\mathrm{d}\, v_4, \tag{5.6}$$

where

$$\begin{aligned}
B_1 &:= [N^2\hat{h}(\tfrac{Nt_1+1}{N}) - \delta N^2, N^2\hat{h}(\tfrac{Nt_1+1}{N}) + \delta N^2],\\
B_2 &:= [N^2\hat{h}(\tfrac{N(t_1+s_1)-1}{N}) - \delta N^2, N^2\hat{h}(\tfrac{N(t_1+s_1)-1}{N}) + \delta N^2],\\
B_3 &:= [N^2\hat{h}(\tfrac{N(1-t_2)-1}{N}) - \delta N^2, N^2\hat{h}(\tfrac{N(1-t_2)-1}{N}) + \delta N^2],\\
B_4 &:= [N^2\hat{h}(\tfrac{N(1-t_2-s_2)+1}{N}) - \delta N^2, N^2\hat{h}(\tfrac{N(1-t_2-s_2)+1}{N}) + \delta N^2],
\end{aligned}$$

where

$$\begin{aligned}
L &:= \{1, 2, \ldots, t_1N - 1\},\\
S_1 &:= \{t_1N + 2, Nt_1 + 3, \ldots, (t_1 + s_1)N - 2\},\\
M &:= \{(t_1 + s_1)N + 1, (t_1 + s_1)N + 2, \ldots, (1 - t_2 - s_2)N - 1\},\\
S_2 &:= \{(1 - t_2 - s_2)N + 2, (1 - t_2 - s_2)N + 3, \ldots, N(1 - t_2) - 2\},\\
R &:= \{N(1 - t_2) + 1, N(1 - t_2) + 2, \ldots, N - 1\},
\end{aligned}$$

and where

$$\psi^v(i) := \begin{cases} \psi^{N,\mathbf{r}}(i) & \text{, for } i \notin U,\\ v_1 & \text{, for } i = Nt_1 + 1,\\ v_2 & \text{, for } i = N(t_1 + s_1) - 1,\\ v_3 & \text{, for } i = N(1 - t_2 - s_2) + 1,\\ v_4 & \text{, for } i = N(1 - t_2) - 1. \end{cases}$$

We use the trivial upper bound 1 for the probabilities in (5.6) to obtain the following upper bound

$$\int_{B_1}\int_{B_2}\int_{B_3}\int_{B_4} \mathcal{Z}_L^{\psi^v}\mathcal{Z}_{S_1}^{\psi^v}\hat{\mathcal{Z}}_M^{\psi^v}\mathcal{Z}_{S_2}^{\psi^v}\mathcal{Z}_R^{\psi^v}\mathrm{d}\, v_1\mathrm{d}\, v_2\mathrm{d}\, v_3\mathrm{d}\, v_4 \tag{5.7}$$

To deal with the partition functions, recall that in (2.25) we show that

$$\mathcal{Z}_N^{\mathbf{r}} = e^{-\mathcal{H}_N(\phi_{\mathbf{r}})}\mathcal{Z}_N^{\mathbf{0}},$$

where $\phi_{\mathbf{r}} = \phi_{\mathbf{r},N}$ is the minimiser of the Hamiltonian in the set $H_{\mathbf{r}}^N$. Furthermore recall that by Proposition C.1 we have

$$\mathcal{Z}_N^{\mathbf{0}} = \frac{\sqrt{2\pi}^{N-1}}{\sqrt{p(N)}},$$

where p is a polynomial of degree 4.

Now we claim that the mass of the integral (5.7) lies in a subset of the domain of integration. Therefore fix a x such that $1 + \kappa_2/2 < x < 1$. We claim that for $v = (v_1, v_2, v_3, v_4)$ with $\|v\|_\infty > N^x$ the expression under the integral decays exponentially, with an exponent of order $-N^y$ for a $y > 1$.

To justify this claim we study the scaling of the minimal value of the Hamiltonian $\mathcal{H}_N(\phi_{\mathbf{r},N})$; therefore we use the notation $\mathcal{H}_N(N^2 a, N\alpha, bN^2, \beta N) = \mathcal{H}_N(\phi_{\mathbf{r},N})$. By Lemma D.3, we approximately have

$$\mathcal{H}_N(N^2 a, N\alpha, bN^2, \beta N) \approx NQ(a, \alpha, b, \beta),$$

where $Q(a, \alpha, b, \beta) = Q(h_{\mathbf{r}})$ is the minimiser of Q in $H_{\mathbf{r}}$. Since $(\alpha, \beta) \mapsto Q(0, \alpha, 0, \beta)$ is quadratic we have

$$\mathcal{H}_{N^\kappa}(0, N^p\alpha, 0, N^p\beta) \approx N^{-\kappa+2p} Q(0, \alpha, 0, \beta) \tag{5.8}$$

where we used that $N^p = N^\kappa N^{p-\kappa}$. Since s_1 and s_1 are smaller than N^{κ_2}, the partition functions for the sets S_1 and S_2 dominate the behaviour of the integral if $\|v\|_\infty > N^x$: To see this note that by our approximation (5.8) this partition functions decay exponentially with an exponent of order $-N^{-\kappa+2p}$, where $\kappa = 1+\kappa_2$ (because $|S_1| \approx Ns_1 < N^{1+\kappa_2}$) and $p = x$. Since $1 + \kappa_2/2 < x < 1$ the exponent is of order $-N^y$ for some $y > 1$.

Restricting the domain of integration in (5.7) to $\|v\|_\infty < N^x$, substituting (v_1, v_2, v_3, v_v) by $(v_1\sqrt{Ns_1}, v_2\sqrt{Ns_1}, v_3\sqrt{Ns_2}, v_4\sqrt{Ns_2})$ and applying Proposition C.1 and Lemma 1.1, we expect that (5.7) divided by $\mathcal{Z}_N^{\mathbf{0}}$ is approximately equal to the following expression

$$N^2 s_1 s_2 \sqrt{\frac{p(N)}{p(t_1N)p(s_1N)p(t_2N)p(s_2N)}} e^{-NE(t_1,t_2)} \int\int\int\int e^{-\mathbf{Q}(v)} \mathrm{d}\, v_1 \mathrm{d}\, v_2 \mathrm{d}\, v_3 \mathrm{d}\, v_4, \tag{5.9}$$

where

- the factor $N^2 s_1 s_2$ stems from the substitution,

- the square roots of the polynomials p are the ones form the partition functions (see Proposition C.1),
- the function $\mathbf{Q}$ is a quadratic function that stems form the limit of the Hamiltonians in the partition functions for the sets S_1 and S_2 (this term is of order 1 because of the substitution),
- and where the function $E(t_1, t_2)$ is a function that has a local minimum at (l_1, l_1) (and that is approximately equal to a quadratic function in a small neighbourhood of (l_1, l_1)) it corresponds to the Hamiltonians in the partition functions for the sets L, R, M.

Since $\mathbf{Q}$ is quadratic, the integral in the above approximation is finite.

Now we return to the study of the sum over all terms of the form (5.5) such that $(t_1, s_1, t_2, s_2) \in W$. Recall that these are the terms that we claim to be the only ones that we can not neglect in the sum on the right hand side of (5.3) for $B_\delta = \hat{B}$. Note that since p is a polynomial of degree 4 the limit of

$$N^2 \sqrt{\frac{p(N)}{p(t_1 N) p(t_2 N)}}$$

is a constant.

To get an upper bound for the sum over all terms of the form (5.5) such that $(t_1, s_1, t_2, s_2) \in W$, we take the sum over the approximations in (5.9). We write this sum as a product of a sum over the terms that depend on (t_1, t_2) and a sum over the terms that depend on (s_1, s_2).

We first consider the sum over the terms in (5.9) that depend on (t_1, t_2). To obtain an approximation of order 1 we divide the sum by N. Using that $E(t_1, t_2)$ has a local minimum at (l_1, l_1) and the Taylor approximation we approximately have

$$\begin{aligned} \tfrac{1}{N} \sum \sum_{t_1, t_2 \in W} e^{-NE(t_1,t_2)} &\approx \tfrac{1}{N} \sum \sum_{t_1, t_2 \in W} e^{-E(\sqrt{N} t_1, \sqrt{N} t_2)} \\ &\approx \int \int e^{-c_1 t_1^2 - c_2 t_2^2} \mathrm{d}\, t_1 \mathrm{d}\, t_2, \end{aligned}$$

where c_i is the second derivative of E in the ith direction at the point l_1; note that for this Taylor approximation we are using the grid points $\mathbb{Z}^2/\sqrt{N}$.

Now we consider the sum over the terms in (5.9) that depend on (s_1, s_2). Since we divide the previous sum by N we multiply this sum with N: Since p is a polynomial of degree 4, there is a constants C such that

$$N\sum\sum_{s_1,s_2\in W}\frac{s_1s_2}{\sqrt{p(s_1N)p(s_2N)}}\approx CN\sum\sum_{s_1,s_2\in W}\frac{s_1s_2}{(s_1N)^2(s_2N)^2}$$
$$=C\sum\sum_{s_1,s_2\in W}\frac{1}{N^3s_1s_2}$$
$$=C[\frac{1}{\sqrt{N}}\sum_{s_1\in W}\frac{1}{Ns_1}][\frac{1}{\sqrt{N}}\sum_{s_2\in W}\frac{1}{Ns_2}]. \tag{5.10}$$

Since $s_1\in W$ only if $s_1N\in\mathbb{Z}$, the fact that the sum

$$\sum_{n=1}^{N}\frac{1}{n}$$

is of order $\log(N)$ implies that the right hand side of (5.10) converges to zero.

Our preceding discussion needs to be made rigorous, in particular the validity of the approximation (5.9). Furthermore we need to prove that the sum of the terms that we ignore actually converges to zero.

We make the following conjecture.

Conjecture 5.1. *If the minimiser of the rate function of the Laplacian model is not unique, the model concentrates on the minimiser that does not pick up pinning reward.*

5.2 Wetting

The model for the wetting interaction coincides with the model for pinning interactions with the field being conditioned on having positive values. For the gradient model Bolthausen, Funaki and Otobe [5] prove an **LDP** for the model with wetting interaction. Analogously to the model with pinning interaction they prove the **LDP** for the model with wetting interaction via the **LDP** for the models with $J=-\infty$.

For the Laplacian model a study of the model with wetting interaction and $J=-\infty$ is the first step. This model is given by

$$\gamma_N^{\mathbf{r},w}(\cdot):=\gamma_N^{\mathbf{r}}(\cdot\mid H_{\mathbf{r}}^+).$$

For the Laplacian model we expect that the rate function of the **LDP** for the model with wetting interaction and $J = -\infty$ is given by

$$\Sigma^{\mathbf{r},w}(f) := \begin{cases} Q(f) - Q(H_{\mathbf{r}}^+) & \text{, for } f \in H_{\mathbf{r}}^+, \\ \infty & \text{, otherwise,} \end{cases} \tag{5.11}$$

where $H_{\mathbf{r}}^+ := \{f \in H_{\mathbf{r}} \mid f(\xi) \geq 0, \text{ for } \xi \in [0,1]\}$. Recall that Q is given by

$$Q(f) = \frac{1}{2}\int_0^1 (\ddot{f}(\xi))^2 \,\mathrm{d}\,\xi.$$

We give a proof for boundary conditions $\mathbf{r} = (a, \alpha, b, \beta)$ with $a > 0$ and $b > 0$. The key tool is that since $\gamma_N^{\mathbf{r},w}$ coincides with the measure $\gamma_N^{\mathbf{r}}$ conditioned on having positive heights, we have

$$\gamma_N^{\mathbf{r},w}(A) = \gamma_N^{\mathbf{r}}(A \mid H_{\mathbf{r}}^+) = \tfrac{\gamma_N^{\mathbf{r}}(A \cap H_{\mathbf{r}}^+)}{\gamma_N^{\mathbf{r}}(H_{\mathbf{r}}^+)}. \tag{5.12}$$

Step 1: We prove

$$\lim_{N\to\infty} \tfrac{1}{N} \log \gamma_N^{\mathbf{r}}(H_{\mathbf{r}}^+) = -(Q(H_{\mathbf{r}}^+) - Q(H_{\mathbf{r}})). \tag{5.13}$$

We show that $-(Q(H_{\mathbf{r}}^+) - Q(H_{\mathbf{r}}))$ is an upper bound and a lower bound of this limit. We start with the upper bound. Since $H_{\mathbf{r}}^+$ is a closed set, the **LDP** upper bound of the model without pinning (see Theorem 1.2) yields the upper bound

$$\limsup_{N\to\infty} \tfrac{1}{N} \log \gamma_N^{\mathbf{r}}(H_{\mathbf{r}}^+) \leq -\Sigma^{\mathbf{r}}(H_{\mathbf{r}}^+) = -(Q^{\mathbf{r}}(H_{\mathbf{r}}^+) - Q(H_{\mathbf{r}})).$$

Now we show the lower bound. To apply the **LDP** lower bound of the model without pinning we construct a suitable open set in $H_{\mathbf{r}}^+$. If the boundary condition $\mathbf{r}$ is such that the minimiser $h_{\mathbf{r}}^+$ of Q in $H_{\mathbf{r}}^+$ has no zero, then we consider an open ball $B(h_{\mathbf{r}}^+, \delta)$ around this minimiser. We fix a radius $\delta > 0$ such that $B(h_{\mathbf{r}}^+, \delta) \subset H_{\mathbf{r}}^+$. In this case the **LDP** lower bound of the model without pinning and the fact that $Q(h_{\mathbf{r}}^+) = Q(H_{\mathbf{r}}^+)$ yields the lower bound

$$\begin{aligned}\liminf_{N\to\infty} \tfrac{1}{N} \log \gamma_N^{\mathbf{r}}(H_{\mathbf{r}}^+) &\geq \liminf_{N\to\infty} \tfrac{1}{N} \log \gamma_N^{\mathbf{r}}(B(h_{\mathbf{r}}^+, \delta)) \\ &\geq -\Sigma^{\mathbf{r}}(h_{\mathbf{r}}^+) = -(Q(H_{\mathbf{r}}^+) - Q(H_{\mathbf{r}})).\end{aligned}$$

If the minimiser $h_{\mathbf{r}}^+$ has a zero, then we show that for each $\epsilon > 0$ there is a h_ϵ that has no zero and that satisfies $Q(h_\epsilon) \leq Q(h_{\mathbf{r}}^+) + \epsilon$. Therefore fix a function $g \in H_{\mathbf{0}}^+$ that has no zero in $(0,1)$. Since $a > 0$ and $b > 0$ we see that for each

$\lambda > 0$ the sum $h_{\mathbf{r}}^{+} + \lambda g$ has no zero in $[0,1]$. Furthermore there is a λ such that $Q(h_{\mathbf{r}}^{+} + \lambda g) \le Q(h_{\mathbf{r}}^{+}) + \epsilon$. So for each $\epsilon > 0$ there is a $\lambda > 0$ such that $h_\epsilon = h_{\mathbf{r}}^{+} + \lambda g$ is a valid choice. Since h_ϵ has no zero in $[0,1]$, there is a $\delta > 0$ such that $B(h_\epsilon, \delta) \subset H_{\mathbf{r}}^{+}$. As above the **LDP** lower bound yields the lower bound

$$\begin{aligned}\liminf_{N\to\infty} \tfrac{1}{N} \log \gamma_N^{\mathbf{r}}(H_{\mathbf{r}}^{+}) &\ge \liminf_{N\to\infty} \tfrac{1}{N} \log \gamma_N^{\mathbf{r}}(B(h_\epsilon, \delta)) \\ &\ge -\Sigma^{\mathbf{r}}(h_\epsilon) = -(Q(H_{\mathbf{r}}^{+}) - Q(H_{\mathbf{r}}) + \epsilon).\end{aligned}$$

Since $\epsilon > 0$ was arbitrary we see that even if $h_{\mathbf{r}}^{+}$ has a zero, we have the lower bound $-(Q(H_{\mathbf{r}}^{+}) - Q(H_{\mathbf{r}}))$. Note that for the existence of δ we use that $a > 0$ and $b > 0$.

Combining this lower bound with the upper bound from above we see that the claim (5.13) is true if $a > 0$ and $b > 0$.

Step 2: To prove that (5.11) is the rate function of the **LDP** for the model with wetting interaction, we verify that the **LDP** lower and upper bounds are satisfied. So let $\mathcal{C}$ be a closed set in $H_{\mathbf{r}}$. Since $\mathcal{C} \cap H_{\mathbf{r}}^{+}$ is a closed set the **LDP** for the model without pinning interaction and (5.12) yield that

$$\begin{aligned}\limsup_{N\to\infty} \tfrac{1}{N} \log \gamma_N^{\mathbf{r},w}(\mathcal{C}) &= \limsup_{N\to\infty} \tfrac{1}{N} [\log \gamma_N^{\mathbf{r}}(\mathcal{C} \cap H_{\mathbf{r}}^{+}) - \log \gamma_N^{\mathbf{r}}(H_{\mathbf{r}}^{+})] \\ &\le \limsup_{N\to\infty} \tfrac{1}{N} \log \gamma_N^{\mathbf{r}}(\mathcal{C} \cap H_{\mathbf{r}}^{+}) - \liminf_{N\to\infty} \tfrac{1}{N} \log \gamma_N^{\mathbf{r}}(H_{\mathbf{r}}^{+}) \\ &\le -\Sigma^{\mathbf{r}}(\mathcal{C} \cap H_{\mathbf{r}}^{+}) + Q(H_{\mathbf{r}}^{+}) - Q(H_{\mathbf{r}}) \\ &= -\Sigma^{\mathbf{r},w}(\mathcal{C}).\end{aligned}$$

To make a similar argument for the lower bound we define $H_{\mathbf{r}}^{o} := \{f \in H_{\mathbf{r}} \mid f(\xi) > 0, \text{for}\, \xi \in [0,1]\}$. For open sets $\mathcal{O}$ the set $\mathcal{O} \cap H_{\mathbf{r}}^{o}$ is also open. So by the **LDP** for the model with pinning interaction and by (5.12) we have

$$\begin{aligned}\liminf_{N\to\infty} \tfrac{1}{N} \log \gamma_N^{\mathbf{r},w}(\mathcal{O}) &\ge \liminf_{N\to\infty} \tfrac{1}{N} \log \gamma_N^{\mathbf{r},w}(\mathcal{O} \cap H_{\mathbf{r}}^{o}) \\ &= \liminf_{N\to\infty} \tfrac{1}{N} \log \gamma_N^{\mathbf{r}}(\mathcal{O} \cap H_{\mathbf{r}}^{o}) - \limsup_{N\to\infty} \tfrac{1}{N} \log \gamma_N^{\mathbf{r}}(H_{\mathbf{r}}^{+}) \\ &\ge -\Sigma^{\mathbf{r}}(\mathcal{O} \cap H_{\mathbf{r}}^{o}) + Q(H_{\mathbf{r}}^{+}) - Q(H_{\mathbf{r}}) \\ &\ge -\Sigma^{\mathbf{r},w}(\mathcal{O}).\end{aligned}$$

So for $a > 0$ and $b > 0$ we see that (5.11) is the **LDP** rate function for the model with wetting and $J = -\infty$. The generalisation of this approach to boundary conditions that do not satisfy $a > 0$ and $b > 0$ is an open problem.

5.3 Conclusion

In this thesis we investigate for which pinning reward the pinning has an effect on the empirical profile of the Laplacian model. Therefore we prove in Chapter 2 an **LDP** for the models without pinning. First we show an **LDP** for the integrated random walk without terminal boundary conditions. For this **LDP** we use a more general assumption than the random walk representation being Gaussian. Then we prove an **LDP** for the integrated random walk with terminal conditions. Our main tool is Mogulskii's theorem. We use the formula for Gaussian bridges to extend the **LDP** for an integrated random walk to the **LDP** for an integrated random walk conditioned on its terminal values.

In Chapter 3 we apply our results from Chapter 2 to obtain an **LDP** for the model with pinning interaction. We see that it is sufficient to prove the lower bound for open balls around a dense subset of $H_{\mathbf{r}}$, the so called well-behaved functions. Well-behaved functions are more typical than not well-behaved functions in the sense that in every neighbourhood of a not well-behaved function there is a well-behaved function with smaller rate. For the upper bound we show that the measures are exponentially tight. Hence we see that it is sufficient to prove the **LDP** upper bound for compact sets. The main obstacle in the proof of the **LDP** upper bound for the Laplacian model is that this model does not satisfy a splitting property with lag 1. We use a generalisation of the law of total expectation to deal with the fact that the Laplacian model satisfies the splitting property only with lag 2.

In Section 4 we study the minimisers of the rate function of the model with pinning interaction. Therefore we determine a superset for the set of minimisers. Then we transform the problem finding the minimiser into a dual problem. We discuss the minimiser of the dual problem and transform them back into minimisers of the original problem. Studying the minimisers of the **LDP** we see that for small pinning reward and non zero boundary conditions the minimiser of the rate function for the model with pinning interaction and the one for the model with no pinning coincide, even if the pinning free energy is strictly positive. So for a small pinning reward the pinning has no effect on the empirical profile. Furthermore for each boundary condition there is a critical reward after which the empirical profile behaves different than the one for the model without pinning. Depending on the boundary condition the rate function can have up to five different minimisers (see Figure 4.4).

Further projects

In the case that the minimiser of the rate function for the model with pinning is not unique, it is interesting whether there is a coexistence of the empirical profiles or whether the model concentrates on one minimiser. We already discussed an approach for analysing the concentration problem if the rate function has two minimisers, see Section 5.1. We expect that the results from this thesis could be used to make the arguments from Section 5.1 rigorous. Then we study the case where the rate function has more minimisers. We conjecture that also for this case the model concentrates on the minimiser that does not pick reward (see Section 5.1).

A harder problem is the model with wetting interaction. Therefore we need to study the logarithmic rate of decay of the probability that the heights are positive. In Section 5.2 we give the arguments for boundary conditions with $a > 0$ and $b > 0$. These arguments have to be extended to the case where $a = 0$ or $b = 0$.

A further project is to study models with Hamiltonians such that the random variables $X_1, X_2, \ldots$ from the random walk representation are non Gaussian. For the gradient model Funaki and Otobe [18] prove the **LDP** for Hamiltonians where the random variable X_1 is such that the log moment generating function and its Fenchel-Legendre transform are finite and such that the Fenchel-Legendre transform is at least three times differentiable. For these models they can not use the representation of the Gaussian bridge process, instead they use a change of measure argument: To do so they fix a piecewise linear function h. Then they perform a change of measure such that the empirical profile of the new measures concentrates on h. To justify that the new measure (see [18, Proposition 5.3]) actually concentrates on h they apply [23, Theorem 3.4]. Applying [23, Theorem 3.4] is possible here, because [23, Theorem 3.4] is a result for the empirical mean of i.i.d. random variables and for the gradient case the height differences of the field (without terminal boundary conditions) are i.i.d. random variables. For the Laplacian model [23, Theorem 3.4] has to be generalised such that both the height and the velocity difference are considered.

After all these problems are understood we can proceed to study the empirical profiles for the Laplacian model in $(d+1)$-dimensions. For these models no integrated random walk representation exists. Hence we can not use Mogulskii's theorem to prove the **LDP** for the model without pinning. Finally we can study the $(d+s)$-dimensional model.

Appendix A

Large deviation theory

We briefly summarise the results from the theory of large deviations that we apply in this thesis. The main reference for the results presented in this chapter is the book [14]. Large deviation theory studies sequences of probability measures $(\mu_n)_{n\in\mathbb{N}}$. It aims to characterise the logarithmic decay, that is the limit of

$$\tfrac{1}{N}\log\mu_N(\mathcal{Q}), \text{ as } \quad N\to\infty$$

by a so called rate function.

Definition A.1. Let $\mathcal{X}$ be a metric space and let $(\mu_n)_{n\in\mathbb{N}}$ be a sequence of probability measures on $(\mathcal{X},\mathcal{B})$, where $\mathcal{B}$ is the Borel-σ algebra for $\mathcal{X}$. The sequence $(\mu_n)_{n\in\mathbb{N}}$ satisfies an **LDP** in $\mathcal{X}$ with rate function $I:\mathcal{X}\to\mathbb{R}$ and speed n if

1. for all open sets $G\subset\mathcal{X}$,

$$\liminf_{n\to\infty}\tfrac{1}{n}\log\mu_n(G)\geq-\inf_{x\in G}I(x), \tag{A.1}$$

2. for all closed sets $F\subset\mathcal{X}$,

$$\limsup_{n\to\infty}\tfrac{1}{n}\log\mu_n(F)\leq-\inf_{x\in F}I(x). \tag{A.2}$$

The rate function I is called good rate function if the level sets of I, these are the sets

$$\{x\in\mathcal{X}\mid I(x)\leq\kappa\},\ \text{for}\ \kappa\in\mathbb{R},$$

are compact.

For the lower bound it suffices to establish (A.1) only for open balls. But for the upper bound it is in general not enough to prove (A.2) only for the closed balls. For example consider the sequence $(\delta_n)_{n\in\mathbb{N}}$ of delta measures at $n \in \mathbb{N}$. This sequence eventually assigns the measure zero to any closed ball (with finite radius) and hence the logarithmic rate of decay for these balls is ∞; while for the closed set $\mathbb{R}$ the rate of decay is clearly zero. By only considering closed balls we ignore non compact sets, from the previous example we see that this leads to an error if mass vanishes at infinity. For sequences that posses the property given in the next proposition this problem does not occur.

Definition A.2. A family of probability measures $(\mu_n)_{n\in\mathbb{N}}$ on $\mathcal{X}$ is exponentially tight if for every $\kappa < \infty$, there exists a precompact set $K_\kappa \subset \mathcal{X}$ such that

$$\limsup_{n\to\infty} \tfrac{1}{n} \log \mu_n(K_\kappa) < -\kappa.$$

The most important result from large deviation theory for our work is Mogulskii's Theorem.

Theorem A.3. *Let $X_1, X_2, \dots$ be a sequence of i.i.d. real valued random variables such that $\Lambda(\lambda) := \log \mathbf{E}[e^{\lambda X_1}] < \infty$ for all $\lambda \in \mathbb{R}$. Let*

$$Z_n(t) := \tfrac{1}{n} \sum_{i=1}^{\lfloor nt \rfloor} X_i,\ 0 \le t \le 1,$$

*and let μ_n be the law of Z_n in $L^\infty(0,1)$. The sequence $(\mu_n)_{n\in\mathbb{N}}$ satisfies in $L^\infty(0,1)$ the **LDP** with speed n and good rate function*

$$I^M(f) := \begin{cases} \int_0^1 \Lambda^*(\dot{f}(\xi))\,\mathrm{d}\xi & \text{, for } f \in \mathcal{AC}, \\ \infty & \text{, otherwise,} \end{cases} \tag{A.3}$$

where $\mathcal{AC}$ are the absolutely continuous functions.

Proof. For a proof see [14, Theorem 5.1.2]. □

The following result is known as contraction principle.

Theorem A.4. *Let $(\mu_n)_{n\in\mathbb{N}}$ be a sequence of probability measures on $\mathcal{X}$ that satisfies an **LDP** with rate function I. Furthermore let $\Phi : \mathcal{X} \to \mathcal{Y}$ be a continuous map, where $\mathcal{Y}$ is a metric space. Then the sequence $(\mu_n \circ \Phi^{-1})_{n\in\mathbb{N}}$ satisfies an **LDP** with rate function*

$$y \mapsto \inf_{x\in\mathcal{S}_y} I(x),$$

where

$$\mathcal{S}_y := \{x \in \mathcal{X} \mid \Phi(x) = y\},$$

and where $\inf_{x\in\mathcal{S}_y} I(x) = \infty$ *if* $\mathcal{S}_y = \emptyset$.

Proof. For a proof see [14, Theorem 4.2.1]. □

The following tool is used to obtain large deviation principles for sequences that are exponentially equivalent. The authors of [14] define exponential equivalence as follows.

Definition A.5. Let $(\mathcal{Y}, d)$ be a metric space. The sequences of probability measures $(\mu_n)_{n\in\mathbb{N}}$ and $(\tilde{\mu}_n)_{n\in\mathbb{N}}$ are exponentially equivalent if there exist probability spaces $((\Omega, \mathcal{B}_n, P_n))_{n\in\mathbb{N}}$ and two families of $\mathcal{Y}$-valued random variables $(Z_n)_{n\in\mathbb{N}}$ and $(\tilde{Z}_n)_{n\in\mathbb{N}}$ with joint laws $(P_n)_{n\in\mathbb{N}}$ and marginals $(\mu_n)_{n\in\mathbb{N}}$ and $(\tilde{\mu}_n)_{n\in\mathbb{N}}$, respectively, such that the following condition is satisfied:

For each $\delta > 0$, the set $\{w \mid (\tilde{Z}_n(w), Z_n(w)) \in \Gamma_\delta\}$ is $\mathcal{B}_n$ measurable, and

$$\limsup_{n\to\infty} \tfrac{1}{n} \log P_n(\Gamma_\delta) = -\infty,$$

where

$$\Gamma_\delta := \{(y, \tilde{y}) \mid d(y, \tilde{y}) > \delta\} \subset \mathcal{Y} \times \mathcal{Y}.$$

Example A.6. Let $(P_n)_{n\in\mathbb{N}}$ be a family of probability measures on a probability space $(\Omega, \mathcal{B})$ and let $(f_n)_{n\in\mathbb{N}}$ and $(h_n)_{n\in\mathbb{N}}$ be two families of measurable maps $\Omega \to \mathcal{Y}$ such that

$$\sup_{w\in\Omega} d(f_n(w), h_n(w)) \to 0 \text{ , for } n \to \infty.$$

Then the sequences $(\mu_n)_{n\in\mathbb{N}}$ and $(\tilde{\mu}_n)_{n\in\mathbb{N}}$, given by $\mu_n = P_n \circ f_n^{-1}$ and $\tilde{\mu}_n = P_n \circ h_n^{-1}$, are exponentially equivalent.

Theorem A.7. *Let* $(\mu_n)_{n\in\mathbb{N}}$ *and* $(\tilde{\mu}_n)_{n\in\mathbb{N}}$ *be two exponentially equivalent sequences. If* $(\mu_n)_{n\in\mathbb{N}}$ *satisfies an **LDP** with rate* I *then* $(\tilde{\mu}_n)_{n\in\mathbb{N}}$ *satisfies an **LDP** with rate* I *as well.*

Proof. See [14, Theorem 4.2.13]. □

Appendix B

Gaussian measure

Let $X = (X_1, X_2, \dots, X_N)$ be $N \in \mathbb{N}$ normally distributed random variables with expectations zero and covariance matrix $C := (C_{i,j})_{(i,j)\in\{1,2,\dots,N\}\times\{1,2,\dots,N\}}$. The law of X is a Gaussian measure. We partition X into $(X_a, X_b) := (X_{1,2,\dots,n}, X_{n+1,\dots,N})$ and in line with that we partition C into

$$C = \begin{bmatrix} C_{a,a} & C_{a,b} \\ C_{b,a} & C_{b,b} \end{bmatrix}.$$

If $C_{b,b}$ is invertible, then the distribution of X_a conditioned on $X_b = 0$ is a centered Gaussian measure with covariance matrix

$$C_{a,a} - C_{a,b}C_{b,b}^{-1}C_{b,a};$$

see [13]. A random vector that has this distribution is

$$X_a - C_{a,b}C_{b,b}^{-1}X_b, \tag{B.1}$$

where X_a has a Gaussian distribution with expectation 0 and covariance $C_{a,a}$ and X_b has a Gaussian distribution with expectation 0 and covariance $C_{b,b}$.

The random vectors $X_a - C_{a,b}C_{b,b}^{-1}X_b$ and $C_{a,b}C_{b,b}^{-1}X_b$ are uncorrelated. Since this random vectors are Gaussian they are independent. An important consequence of this is that for any $i \in \{1, 2, \dots N\}$ the variance of X_i given $X_b = 0$ is smaller than the one of the unconditioned random variable X_i. To see this, we use that the variance of the sum of two independent random variables is the sum of the variances of these two random variables and that

$$X_a = [X_a - C_{a,b}C_{b,b}^{-1}X_b] + [C_{a,b}C_{b,b}^{-1}X_b]. \tag{B.2}$$

The map $B : \mathbb{R}^N \to \mathbb{R}^N$ given by

$$B(x_a, x_b) = (X_a - C_{a,b}C_{b,b}^{-1}X_b, 0)$$

is a projection of $\mathbb{R}^N$ onto $H_0 := \{x \in \mathbb{R}^N \mid x_b = 0\}$ because its range is H_0 and because $B(x) = x$ for $x \in H_0$. Furthermore the map B is the orthogonal projection of $(\mathbb{R}^N, < \cdot, C^{-1}\cdot >)$ to H_0: Let $y \in H_0$ and let $U := C^{-1}$ where we partition U analogous to C , then

$$< (I - B)x, C^{-1}y >= x_b^{\mathrm{t}}[C_{a,b}C_{b,b}^{-1}]^{\mathrm{t}}U_{a,a}y_a + x_b^{\mathrm{t}}U_{b,a}y_a,$$

where we use $(I - B)x = (C_{a,b}C_{b,b}^{-1}X_b, x_b)$. By [13, Equation (4)] we have $U_{b,a} = -[C_{a,b}C_{b,b}^{-1}]^{\mathrm{t}}U_{a,a}$ and hence $< (I - B)x, C^{-1}y >= 0$ for all $x \in \mathbb{R}^N$ and all $y \in H_0$.

Another property of the Gaussian measure that we use is the following upper bound to its tails.

Lemma B.1. *If X has a normal distribution with expectation 0 and variance σ^2, then*

$$P(X \geq x) \leq \tfrac{1}{\sqrt{2\pi}}\tfrac{\sigma}{x}e^{-\frac{1}{2}(\frac{x}{\sigma})^2}.$$

Proof. Partial integration. □

Appendix C

The partition function

We derive a formula for the partition functions $(\mathcal{Z}_N^{\mathbf{0}})_{N\in\mathbb{N}}$. The following result has also been derived in [6, Remark 1.2]. In [6] this result is a corollary to a more general problem.

Proposition C.1. *For all* $N \in \mathbb{N}$

$$\mathcal{Z}_N^{\mathbf{0}} = \frac{\sqrt{2\pi}^{N-1}}{\sqrt{p(N)}},$$

where $p : \mathbb{N} \to \mathbb{R}$ *is a polynomial of degree* 4 *given by*

$$p(N) = \tfrac{1}{12}N^4 + \tfrac{1}{3}N^3 + \tfrac{5}{12}N^2 + \tfrac{1}{6}N.$$

Since the Hamiltonian $\mathcal{H}_N$ defines a $N-1$ dimensional Gaussian distribution, we just have to check its precision matrix, that is the inverse of the covariance matrix, has the determinant $p(N)$.

By (D.9), the precision matrix, $\mathbf{R}_N$, is the element of $\mathbb{R}^{\Lambda_N\times\Lambda_N}$ given by

$$\mathbf{R}_N(i,j) = \begin{cases} 1, \text{ for } j = i-2, \\ -4, \text{ for } j = i-1, \\ 6, \text{ for } j = i, \\ -4, \text{ for } j = i+1, \\ 1, \text{ for } j = i+2, \\ 0, \text{ otherwise.} \end{cases}$$

We show that the sequence of determinants $(\det \mathbf{R}_N)_{N\in\mathbb{N}}$ is a discrete polynomial of degree 4. Therefore we use that a sequence $(D_N)_{N\in\mathbb{N}}$ of real numbers is a discrete polynomial of degree 4 if and only if

$$(1-S)^5 D_N = 0, \tag{C.1}$$

where the operator $S : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$ shifts the sequence $D = (D_N)_{N\in\mathbb{N}}$ such that $S(D)_N = D_{N+1}$ (this equivalence is true for polynomials of all degrees, see [2, Theorem 6.3.2]).

To make use of the criterion (C.1), we need to express the determinant of $\mathbf{R}_N$ by the determinants of its four predecessors. We achieve this recursive expression of $\det \mathbf{R}_N$ by repeatedly applying the expansion by minors formula.

For the expansion by minors we only use certain properties of the sequence $(\mathbf{R}_N)_{N\in\mathbb{N}}$: we use that the matrices are symmetric pentadiagonal matrices such that any two elements of the same diagonal have the same value, furthermore we use that for two different matrices from the sequence the entries of two corresponding diagonals are the same especially that the entries in the first and last diagonal are always 1; in other words we use that $\mathbf{R}_{N+1}$ is an element of $\mathbb{R}^{N\times N}$ of the form

$$M_N = M_N(a,b) = \begin{bmatrix} a & b & 1 & & & & \\ b & a & b & \ddots & & & \\ 1 & \ddots & \ddots & \ddots & & & \\ & & & & & & 1 \\ & & & & & \ddots & b \\ & & & & 1 & b & a \end{bmatrix}.$$

Expansion by minors is a suitable tool to get a recursive formula for $(\det M_N)_{N\in\mathbb{N}}$ since $(M_N)_{N\in\mathbb{N}}$ satisfies the recursion

$$M_1 = a, \quad M_2 = \begin{bmatrix} a & b \\ b & a \end{bmatrix}, \quad M_3 = \begin{bmatrix} a & b & c \\ b & a & b \\ c & b & a \end{bmatrix} = \begin{bmatrix} a & b & c \\ b & & \\ c & & M_2 \end{bmatrix},$$
$$M_4 = \begin{bmatrix} a & b & c & 0 \\ b & & & \\ c & M_3 & & \\ 0 & & & \end{bmatrix}, \quad \ldots \quad M_N = \begin{bmatrix} a & b & c & 0 & \ldots & 0 \\ b & & & & & \\ c & & & & & \\ 0 & M_{N-1} & & & & \\ \vdots & & & & & \\ 0 & & & & & \end{bmatrix}. \tag{C.2}$$

We present the recursive formula for $(\det M_N)_{N\in\mathbb{N}}$ in the next lemma. An obvious consequence of this lemma is that (C.1) is true (just plugin $a = 6$ and $b = -4$).

Lemma C.2. *For $a \neq 0, b \neq 0$ and $N \in \mathbb{N}$ let $D_N = \det M_N(a, b)$ then*

$$D_N = (a-1)D_{N-1} - (b^2-a)D_{N-2} + (b^2-a)D_{N-3} - (a-1)D_{N-4} + D_{N-5}, \quad \text{(C.3)}$$

with starting values $D_1 = a, D_2 = \det M_2(a, b), \ldots, D_5 = \det M_5(a, b)$.

Before we give the proof of the lemma, note that it implies that $(\det \mathbf{R}_N)_{N\in\mathbb{N}}$ satisfies the criterion (C.1). Applying With (C.3) for $a = 6$, $b = -4$ and for $N = M + 5$ we get

$$0 = -1D_{M+5} + 5D_{M+4} - 10D_{M+3} + 10D_{M+2} - 5D_{M+1} + D_M = -(1-S)^5 D_M,$$

where we used that $D_{M+x} = S^x(D)_M$ and that the coefficients are the binomial coefficients for the power 5.

Proof of Lemma C.2. For $N > 2$, expansion by minors gives us

$$\det M_N = a \det M_N^{1,1} - b \det M_N^{1,2} + \det M_N^{1,3}, \quad \text{(C.4)}$$

where $M_N^{x,y}$ is the minor of M_N, that is the $\mathbb{R}^{(N-1)\times(N-1)}$ obtained from M_N by deleting row x and column y. Only if we can express all terms on the right of C.4 by elements of $(D_N)_{N\in\mathbb{N}}$, we can use (C.4) to define a recursion for $(D_N)_{N\in\mathbb{N}}$. While the recursive definition (C.2) of M_N leads directly to $\det M_N^{1,1} = \det M_{N-1} = D_{N-1}$, there is no tool to make the same straight forward argument for the other two terms. Actually, as we will see, $\det M_N^{1,2}$ depends on all D_M with $M < N$. As this blocks the most direct way we have to choose a more complex way: we introduce the helping sequence $(E_N)_{N\in\mathbb{N}}$ where $E_N = \det M_{N+1}^{1,2}$ and apply expansions by minors to the last two determinants on the right hand side of (C.4). We will prove firstly that

$$E_N = bD_{N-1} - E_{N-1} \quad \text{(C.5)}$$

and that

$$\det M_N^{1,3} = bE_{N-2} - aD_{N-3} + D_{N-4}. \quad \text{(C.6)}$$

Before we give the proofs note that by combining (C.4), (C.5) and (C.6) we get a recursion for the helping sequence $(\tilde{E}_N)_{N\in\mathbb{N}}$, where $\tilde{E}_N := \frac{E_N}{b}$, namely

$$\tilde{E}_N = (a-1)\tilde{E}_{N-1} - (b^2-a)\tilde{E}_{N-2} + (b^2-a)\tilde{E}_{N-3} - (a-1)\tilde{E}_{N-4} + \tilde{E}_{N-5}, \quad \text{(C.7)}$$

We use (C.7) to express $\tilde{E}_{N+1}+\tilde{E}_N$ in terms of $(\tilde{E}_{i+1}+\tilde{E}_i)_{i\in\{N-1,N-2,...N-5\}}$. Since, by (C.5), we have $[\tilde{E}_{N+1}+\tilde{E}_N] = D_N$, we found an expression of D_N in terms of $(D_i)_{i\in\{N-1,N-2,...N-5\}}$. This expression coincides with (C.3).

Now we prove (C.5) and (C.6). It is convenient to introduce a notation for matrices which are obtained by only removing a row, lets say the row x. If M_N is the original matrix we denote the matrix with deleted row x by $M_N^{x,-}$ analogously the matrix with deleted column y is denoted by $M_N^{-,y}$.

We prove (C.5) by expansion by minors

$$
\begin{aligned}
E_{N-1} = \det \begin{bmatrix} b & \\ 1 & \\ 0 & M_{n-1}^{-,1} \\ \vdots & \\ 0 & \end{bmatrix} &= \det \begin{bmatrix} b & b \quad 1 \quad 0 \dots \quad 0 \\ 1 & \\ 0 & M_{n-2} \\ \vdots & \\ 0 & \end{bmatrix} \\
&= b \det M_{n-2} - \det \begin{bmatrix} b \quad 1 \quad 0 \dots \quad 0 \\ M_{n-2}^{1,-} \end{bmatrix} \\
&= bD_{N-2} - 1 \det \begin{bmatrix} b & \\ 1 & \\ 0 & M_{n-2}^{-,1} \\ \vdots & \\ 0 & \end{bmatrix}.
\end{aligned}
$$

We prove (C.6) by employing expansion by minors three times

$$
\begin{aligned}
\det M_N^{1,3} = \det \begin{bmatrix} b & \\ 1 & \\ 0 & M_{N-1}^{-,2} \\ \vdots & \\ 0 & \end{bmatrix} &= b \det \left[M_{N-1}^{1,2} \right] - \det \begin{bmatrix} a & 1 \quad 0 \quad \dots \quad 0 \\ 1 & \\ 0 & M_{N-3} \\ \vdots & \\ 0 & \end{bmatrix} \\
&= b \det \left[M_{N-1}^{1,2} \right] - a \det \left[M_{N-3} \right] + \det M_{N-4},
\end{aligned}
$$

for the last equation we used expansion by minors twice. □

Appendix D

Minimisers of the rate function of the model without pinning interaction and of the Hamiltonian

We study the problem to minimise $f \mapsto Q(f) = \frac{1}{2}\int_0^1 (\ddot{f}(\xi))^2 \,\mathrm{d}\,\xi$ in $H_{\mathbf{r}}$ alongside with a discrete approximation to this problem.

D.1 Minimiser of Q

A way to find the minimum of Q in $H_{\mathbf{r}}$ is to derive the Euler-Lagrange equation; but there is also a way that does not need any further theory: we find the minimum by just analysing what it means to be a minimum of Q in $H_{\mathbf{r}}$. In the following we use the second way because this prepares our discussion of the minimiser of the Hamiltonian. Since for any $f \in H_{\mathbf{r}}$ we have $H_{\mathbf{r}} = \{h + f \mid f \in H_{\mathbf{0}}\}$, a function $h_{\mathbf{r}} \in H_{\mathbf{r}}$ is the minimiser of Q in $H_{\mathbf{r}}$ if and only if

$$0 \le Q(h_r + f) - Q(h_{\mathbf{r}}) = \int_0^1 \ddot{f}(\xi)\ddot{h}_{\mathbf{r}}(\xi)\,\mathrm{d}\xi + Q(f)\,, \text{ for all } f \in H_{\mathbf{0}}. \tag{D.1}$$

The following argument shows that (D.1) is equivalent to

$$\int_0^1 \ddot{f}(\xi)\ddot{h}_{\mathbf{r}}(\xi)\,\mathrm{d}\xi = 0, \text{ for all } f \in H_{\mathbf{0}}. \tag{D.2}$$

Clearly (D.2) implies (D.1). To prove the other direction, we first show by contradiction that the left hand side of (D.2) has to be positive or zero for all $f \in H_{\mathbf{0}}$. Therefore, assume there is a f such that this integral is negative and plug $\frac{1}{n}f$ into (D.1). Since $Q(\frac{1}{n}f) = \frac{1}{n^2}Q(f)$, the right hand side of (D.1) is the difference between a positive term of order $O(\frac{1}{n^2})$ and one of order $O(\frac{1}{n})$; hence the inequality (D.1) is violated for n large enough. So the integral in (D.2) has to be positive or zero for all $f \in H_{\mathbf{0}}$; and since $H_{\mathbf{0}}$ always contains both f and $-f$ this is equivalent to (D.2).

Since (D.2) holds for all $f \in C^\infty$, applying integration by parts two times yields that the fourth weak derivative of a minimiser $h_{\mathbf{r}}$ has to be zero, i.e. $h_{\mathbf{r}}$ has to be a polynomial of order three in $H_{\mathbf{r}}$. There is only one such polynomial and clearly it satisfies (D.2) (for all function in $H_{\mathbf{0}}$ not just the ones in C^∞).

We have already given the major step for the proof of the following proposition.

Proposition D.1. *The minimiser of Q in $H_{\mathbf{r}}$ is unique. We denote this minimiser by $h_{\mathbf{r}}$. The function $h_{\mathbf{r}}$ is given by*

$$h_{\mathbf{r}}(\xi) = a + \alpha\xi + k\xi^2 + c\xi^3, \tag{D.3}$$

where

$$k(a,\alpha,b,\beta) := 3(b-a) - 2\alpha - \beta, \qquad c(a,\alpha,b,\beta) := (\alpha+\beta) - 2(b-a).$$

We have

$$\begin{aligned} Q(h_{\mathbf{r}}) &= [2k^2 + 6kc + 6c^2] \\ &= 6a^2 + 6b^2 + 2\alpha^2 + 2\alpha\beta + 2\beta^2 - 6b(\alpha+\beta) + 6a(-2b+\alpha+\beta). \end{aligned} \tag{D.4}$$

Furthermore for all $f \in H_{\mathbf{r}}$ we have

$$Q(f) = Q(h_{\mathbf{r}}) + Q(f - h_{\mathbf{r}}). \tag{D.5}$$

Proof. From the discussion above we know that the minimiser is the polynomial of degree three that satisfies the boundary condition (a, α, b, β). The four boundary conditions yield four equations for the unknown coefficients of the polynomial; solving them we obtain the coefficients of $h_{\mathbf{r}}$.

To prove (D.5) we use that the minimiser has a geometrical interpretation.

Therefore consider the Sobolev space H^2 with semi inner product $< \cdot, \cdot >$ given by

$$\langle f, g\rangle = \int_0^1 \ddot{f}(\xi)\ddot{g}(\xi)\,\mathrm{d}\xi. \tag{D.6}$$

By (D.2), $h_{\mathbf{r}}$ is orthogonal to $H_{\mathbf{0}}$ and since $f - h_{\mathbf{r}} \in H_{\mathbf{0}}$ for all $f \in H_{\mathbf{r}}$ the minimiser is also orthogonal to $f - h_{\mathbf{r}}$. This implies the version of the Pythagorean theorem from (D.5) above. □

D.2 Minimiser of the Hamiltonian

In this section we seek the minimum of the Hamiltonian $\mathcal{H}_N$ in $H^N_{\mathbf{r}}$, where

$H^N_{\mathbf{r}} :=$
$\{\phi \in \mathbb{R}^{\overline{\Lambda_N}} \mid \phi(0) = aN^2, \phi(N) = bN^2, \phi(-1) = aN^2 - \alpha N, \phi(N+1) = bN^2 + \beta N\},$

where $\overline{\Lambda_N} := \{-1, 0, 1, \ldots, N, N+1\}$. We use that

$$\mathcal{H}_N(\phi) = \tfrac{1}{2}\left\langle \Delta\phi, \Delta\phi\right\rangle,$$

where $< \cdot, \cdot >$ is the inner product on $\mathbb{R}^{\{0,1,\ldots,N\}}$ and Δ is the $\{0, 1, \ldots, N\} \times \overline{\Lambda_N}$ matrix such that $\Delta\phi_i = \phi_{i-1} - 2\phi_i + \phi_{i+1}$.

Fix $N \in \mathbb{N}$. Analogously to the previous section, a function $\phi_{\mathbf{r},N} = \phi_{\mathbf{r}}$ is a minimiser of $\mathcal{H}_N$ in $H^N_{\mathbf{r}}$ if and only if

$$0 \le \mathcal{H}_N(\phi_{\mathbf{r}} + \psi) - \mathcal{H}_N(\phi_{\mathbf{r}}) = \langle \Delta\phi_{\mathbf{r}}, \Delta\psi\rangle + \mathcal{H}_N(\psi)\,, \text{ for all}\, \psi \in H^N_{\mathbf{0}}. \tag{D.7}$$

Also analogously this is equivalent to

$$\langle \Delta\phi_{\mathbf{r}}, \Delta\psi\rangle = 0\,, \text{ for all}\, \psi \in H^N_{\mathbf{0}}. \tag{D.8}$$

This implies $\Delta^{\mathrm{t}}\Delta\phi_{\mathbf{r}}(i) = 0$ for all $i \in \Lambda_N$. We have

$$\Delta^{\mathrm{t}}\Delta\phi(i) = \phi(i-2) - 4\phi(i-1) + 6\phi(i) - 4\phi(i+1) + \phi(i+2)\,, \text{ for all}\, i \in \Lambda_N. \tag{D.9}$$

So for $\phi(i) = 1$, $\phi(i) = i$, $\phi(i) = i^2$ or $\phi(i) = i^3$ we have $\Delta^{\mathrm{t}}\Delta\phi(i) = 0$ for all $i \in \Lambda_N$. While for $\phi(i) = i^z$ for $z > 3$ we have $\Delta^{\mathrm{t}}\Delta\phi(2) \neq 0$. In other words the polynomial of order three with domain $\overline{\Lambda_N}$ in $H^N_{\mathbf{r}}$ is the minimiser of $\mathcal{H}_N$.

Proposition D.2. *Fix the boundary condition* $\mathbf{r}$. *The minimiser of* $\mathcal{H}_N$ *in* $H^N_{\mathbf{r}}$ *is the polynomial of order three with domain* $\overline{\Lambda_N}$ *in* $H^N_{\mathbf{r}}$*, we denote this polynomial by*

$\phi_{\mathbf{r},N}$.

Let a_N, α_N, k_N, c_N be the coefficients of $h_{\mathbf{r},N}\colon \xi \mapsto \frac{1}{N^2}\phi_{\mathbf{r},N}(\xi N)$ with domain $\overline{\Lambda_N}/N$, then

$$\begin{aligned}
a_N(a,\alpha,b,\beta) &= a, \\
\alpha_N(a,\alpha,b,\beta) &= \tfrac{2b-a(2+3N)+N(3b+\alpha(N+1)-\beta))}{(N+1)(N+2)}, \\
k_N(a,\alpha,b,\beta) &= N\tfrac{(-\alpha+\beta+N(3(b-a)-2\alpha-\beta))}{(N+1)(N+2)}, \\
c_N(a,\alpha,b,\beta) &= N^2\tfrac{2(a-b)+\alpha+\beta}{(N+1)(N+2)}.
\end{aligned} \tag{D.10}$$

We have

$$\lim_{N\to\infty} a_N(\mathbf{r}) = a, \quad \lim_{N\to\infty} \alpha_N(\mathbf{r}) = \alpha, \quad \lim_{N\to\infty} k_N(\mathbf{r}) = k(\mathbf{r}), \quad \lim_{N\to\infty} c_N(\mathbf{r}) = c(\mathbf{r}).$$

Furthermore, for all $\psi \in H^N_{\mathbf{r}}$ we have

$$\mathcal{H}_N(\psi) = \mathcal{H}_N(\phi_{\mathbf{r},N}) + \mathcal{H}_N(\psi - \phi_{\mathbf{r},N}). \tag{D.11}$$

Proof. First we consider (D.10). Therefore note that the boundary conditions

$$\begin{aligned}
&\tfrac{1}{N^2}h_{\mathbf{r},N}(0) = a, \quad \tfrac{1}{N^2}h_{\mathbf{r},N}(1) = b, \\
&\tfrac{1}{N^2}h_{\mathbf{r},N}(-\tfrac{1}{N}) = a - \tfrac{1}{N}\alpha, \quad \tfrac{1}{N^2}h_{\mathbf{r},N}(1+\tfrac{1}{N}) = b + \tfrac{1}{N}\beta.
\end{aligned}$$

yield four equations that determine $a_N(\mathbf{r}), \alpha_N(\mathbf{r}), k_N(\mathbf{r}), c_N(\mathbf{r})$.

The property (D.11) follows by the analogy between (D.8) and (D.2). As in our argument for (D.5), we use that (D.8) implies that $\phi_{\mathbf{r},N}$ and $\psi - \phi_{\mathbf{r},N}$ are orthogonal. So (D.11) is a version of the Pythagorean theorem. □

Lemma D.3. *We have*

$$|Q(h_{\mathbf{r}}) - \tfrac{1}{N}\,\mathcal{H}_N(\phi_{\mathbf{r},N})| \to 0 \quad \text{, for } N \to \infty$$

Proof. By the triangle inequality we have

$$\begin{aligned}
|Q(h_{\mathbf{r}}) - \tfrac{1}{N}\,\mathcal{H}_N(\phi_{\mathbf{r},N})| \le{}& |Q(h_{\mathbf{r}}) - \tfrac{1}{N}\sum_{i\in\{0,1,\ldots,N\}} (\ddot{h}_{\mathbf{r}}(\tfrac{i}{N}))^2| \\
&+ |\tfrac{1}{N}\sum_{i\in\{0,1,\ldots,N\}} (\ddot{h}_{\mathbf{r}}(\tfrac{i}{N}))^2 - \tfrac{1}{N}\,\mathcal{H}_N(\phi_{\mathbf{r},N})|.
\end{aligned}$$

The first term on the right is of order $O(\frac{1}{N})$, since the Riemann sum approximation is of order $O(\frac{1}{N})$. For the second term note that

$$
\begin{aligned}
&\tfrac{1}{N} \sum_{i\in\{0,1,\ldots,N\}} (\ddot{h}_{\mathbf{r}}(\tfrac{i}{N}))^2 - \tfrac{1}{N}\,\mathcal{H}_N(\phi_{\mathbf{r},N}) \\
&= \tfrac{1}{N} \sum_{i\in\{0,1,\ldots,N\}} [\ddot{h}_{\mathbf{r}}(\tfrac{i}{N} - \Delta\phi_{\mathbf{r},N}(x)][(\ddot{h}_{\mathbf{r}}(\tfrac{i}{N}) + \Delta\phi_{\mathbf{r},N}(x)] \\
&= O(\tfrac{1}{N^2}) \sum_{i\in\{0,1,\ldots,N\}} [(\ddot{h}_{\mathbf{r}}(\tfrac{i}{N}) + \Delta\phi_{\mathbf{r},N}(x)] \\
&= O(\tfrac{1}{N}),
\end{aligned}
$$

□

Nomenclature

$\mathbb{N}$ Natural numbers not including zero: $1, 2, \ldots$

$\mathbb{Z}$ Integers

$\mathbb{R}$ Real numbers

$\mathbb{R}_{>0}$ The strictly positive real numbers

Λ_N $\{1, 2, \ldots N-1\}$, page 2

$\overline{\Lambda_N}$ $\{-1, 0, \ldots, N+1\}$, page 3

(s_*, s^*) $\{s_* + 1, \ldots, s^* - 1\} \subset \mathbb{Z}$, see, page 7

$\mathbb{R}^{\mathbb{Z}}$ Space with the elements $\phi = (\phi_i)_{i \in \mathbb{Z}}$, and analogously for subsets of $\mathbb{Z}$

$\mathbb{Z}/N$ $\{\frac{i}{N} \mid i \in \mathbb{Z}\}$ and analogously for subsets of $\mathbb{Z}$, page 8

$\lceil \cdot \rceil$ for $x \in \mathbb{R}$ the value $\lceil x \rceil$ is the smallest integer larger than or equal to x , see equation (2.30), page 31

$\lfloor \cdot \rfloor$ for $x \in \mathbb{R}$ the value $\lfloor x \rfloor$ is the largest integer smaller than or equal to x, page 8

Function Spaces

$\mathcal{N}_f$ Zero set of f , see equation (1.18), page 11

$\dot{f}, \ddot{f}, f^{(3)}, f^{(4)}, \ldots$ Weak derivatives of f of orders $1, 2, 3, 4, \ldots$

$(C(0,1), \|\cdot\|_\infty)$ Space of continuous functions $[0,1] \to \mathbb{R}$ with norm $\|\cdot\|_\infty$

$(L^1, \|\cdot\|_1) \supset (L^2, \|\cdot\|_2) \supset \ldots \supset L^\infty$ Lebesgue spaces of functions $[0,1] \to \mathbb{R}$

$C(I)$ Continuous function on the interval I

$\mathcal{AC}$ Space of functions $f\colon [0,1] \to \mathbb{R}$ such that $\dot{f} \in L^1$; i.e. the absolutely continuous functions

$H_{\mathbf{a}}$ see, page 10

$H_{\mathbf{r}}$ see, page 10

$H_{\mathbf{r}}(I)$ see, page 32

$O(\cdot), o(\cdot)$ big and little O notation

Boundary conditions

$\mathbf{r} = (\mathbf{a}, \mathbf{b}) = (a, \alpha, b, \beta)$ Boundary condition for the empirical profile: the values a and b are the values at the boundaries and the values α and β are the gradients at the boundaries, page 8

$\psi^{\mathbf{r},N}$ Boundary condition, page 8

$\psi^{\mathbf{r},N,I}$ Boundary condition for models on I, page 31

Random Walk Representation

$(\Omega, \mathcal{E}, P^{\psi})$ Probability Space for **IRW**, page 3

$X_1, X_2, \ldots$ i.i.d. real valued random variables, page 3

$Y_0, Y_1, Y_2, \ldots$ Random walk with increments $X_1, X_2, \ldots$, page 3

$\zeta_0, \zeta_1, \zeta_2, \ldots$ Integrated random walk, page 3

Models and Rates

h_N Empirical profile, page 8

$\gamma_N^{\mathbf{a}}$ Model without pinning and without terminal boundary conditions, page 9

ϑ_N Law of the empirical profile of the IRW, especially if X_1 is not Gaussian, page 18

$\gamma_N^{\mathbf{r}}$ Model without pinning and with terminal boundary conditions, page 9

$\hat{\gamma}_N^{\mathbf{r}}$ Model with pinning and with terminal boundary conditions, page 9

$\gamma_{N,I}^{\mathbf{r}}$ Model without pinning for the interval I, page 31

$\Pi^{\mathbf{a}}$ Rate function of $(\vartheta_N^{\mathbf{a}})_{N\in\mathbb{N}}$, page 19

Q	Not centralised rate function for the model without pinning, page 10
$\Sigma^{\mathbf{a}}, \Sigma^{\mathbf{r}}$	Rate functions for the models without pinning, page 10
$\mathcal{E}^J$	Not centred rate function for the model with pinning, page 11
$\Sigma^{\mathbf{r},J}$	Rate function for the model with pinning, page 11
$\Sigma^{\mathbf{r}}_I$	Rate function of the model without pinning for the interval I, page 31
I^M	Rate function in Mogulskii's theorem, page 19

Bibliography

[1] S. Asmussen, *Applied Probability and Queues*, Springer-Verlag New York, 2003.

[2] R. Beeler, *How to Count*, Springer International Publishing, 2015.

[3] E. Bolthausen, J.-D. Deuschel, and G. Giacomin, *Entropic repulsion and the maximum of the two-dimensional harmonic*, Annals of Probability, 29 (2001), pp. 1670–1692.

[4] E. Bolthausen, J.-D. Deuschel, and O. Zeitouni, *Entropic repulsion of the lattice free field*, Communications in Mathematical Physics, 170 (1995), pp. 417–443.

[5] E. Bolthausen, T. Funaki, and T. Otobe, *Concentration under scaling limits for weakly pinned gaussian random walks*, Probability Theory and Related Fields, 143 (2009), pp. 441–480.

[6] M. Borecki, *Pinning and Wetting Models for Polymers with* $(\nabla + \Delta)$-*Interaction*, PhD thesis, TU Berlin, 2010.

[7] J. Bricmont, A. El Mellouki, and J. Fröhlich, *Random surfaces in statistical mechanics: Roughening, rounding, wetting,...*, Journal of Statistical Physics, 42 (1986), pp. 743–798.

[8] R. Bundschuh, M. Lässig, and R. Lipowsky, *Semi-flexible polymers with attractive interactions*, The European Physical Journal E, 3 (2000), pp. 295–306.

[9] F. Caravenna, F. den Hollander, and N. Pétrélis, *Lectures on random polymers*, in Probability and statistical physics in two and more dimensions, Clay Mathematics Proceedings, American Mathematical Society, 2010, pp. 319–393.

[10] F. Caravenna and J.-D. Deuschel, *Pinning and wetting transition for* $(1+1)$*-dimensional fields with laplacian interaction*, Annals of Probability, 36 (2008), pp. 2388–2433.

[11] ——, *Scaling limits of* $(1+1)$*-dimensional pinning models with laplacian interaction*, Annals of Probability, 37 (2009), pp. 903–945.

[12] T. Colledge, *Free energy and large deviations for 1-dimensional pinning models*, Master's thesis, University of Warwick, 2012.

[13] R. W. Cottle, *Manifestations of the schur complement*, Linear Algebra and its Applications, 8 (1974), pp. 189–211.

[14] A. Dembo and O. Zeitouni, *Large Deviations Techniques and Applications*, Springer-Verlag Berlin Heidelberg, 2010.

[15] L. Dobrushin, Roland, R. Kotecký, and S. B. Shlosman, *Wulff Construction: A Global Shape from Local Interaction*, AMS translation series, 1992.

[16] T. Funaki, *Stochastic interface models*, in Lectures on Probability Theory and Statistics, J. Picard, ed., vol. 1869 of Ecole d'Et de Probabilits de Saint-Flour, Springer-Verlag Berlin Heidelberg, 2003.

[17] ——, *Topics in random interfaces*, in Lecture notes for 3rd NIMS Summer School in Probability, 2015.

[18] T. Funaki and T. Otobe, *Scaling limits for weakly pinned random walks with two large deviation minimizers*, Journal of the Mathematical Society of Japan, 62 (2010), pp. 1005–1041.

[19] T. Funaki and H. Sakagawa, *Large deviations for* $\nabla\varphi$ *interface model and derivation of free boundary problems*, in Stochastic Analysis on Large Scale Interacting Systems, 2002, pp. 173–211.

[20] H.-O. Georgii, *Gibbs Measures and Phase Transitions*, De Gruyter, 2011.

[21] G. Giacomin, *Anharmonic lattices, random walks and random interfaces*, Recent research developments in statistical physics, (2000), pp. 97–118.

[22] ——, *Random Polymer Models*, Imperial College Press, 2007.

[23] M. Z. Guo, G. C. Papanicolaou, and S. R. S. Varadhan, *Nonlinear diffusion limit for a system with nearest neighbor interactions*, Communications in Mathematical Physics, 118 (1988), pp. 31–59.

[24] O. Hryniv and Y. Velenik, *Some rigorous results on semiflexible polymers, I: Free and confined polymers*, Stochastic Processes and their Applications, 119 (2009), pp. 3081–3100.

[25] N. Kurt, *Entropic repulsion for a class of gaussian interface models in high dimensions*, Stochastic Processes and their Applications, 117 (2007), pp. 23–34.

[26] ——, *Maximum and entropic repulsion for a gaussian membrane model in the critical dimension*, Annals of Probability, 37 (2009), pp. 687–725.

[27] H. Sakagawa, *Entropic repulsion for a gaussian lattice field with certain finite range interaction*, Journal of Mathematical Physics, 44 (2003), pp. 2939–2951.

[28] ——, *On the free energy of a gaussian membrane model with external potentials*, Journal of Statistical Physics, 147 (2012), pp. 18–34.

[29] Y. Velenik, *Localization and delocalization of random interfaces*, Probability Surveys, 3 (2006), pp. 112–169.