Efficient domination through eigenvalues

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Abstract

The paper begins with a new characterization of (k, τ)-regular sets. Then, using this result as well as the theory of star complements, we derive a simplex-like algorithm for determining whether or not a graph contains a $(0, \tau)$-regular set. When $\tau = 1$, this algorithm can be applied to solve the EFFICIENT DOMINATING SET problem which is known to be NP-complete. If -1 is not an eigenvalue of the adjacency matrix of the graph, this particular algorithm runs in polynomial time. However, although it doesn't work in polynomial time in general, we report on its successful application to a vast set of randomly generated graphs.

AMS classification 05C50

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1 Introduction

All graphs in this paper are undirected, without loops and multiple edges. The vertex set of a graph G is denoted $V(G)$ and its edge set $E(G)$. The complementary graph $\bar{G}$ of G is the graph with the same vertex set as G, where any two distinct vertices are adjacent if and only if they are not adjacent in G. The *open neighbourhood* $N_G(v)$ of a vertex $v \in V(G)$ is the set of its neighbours, i.e., the set of vertices adjacent to v, and the *closed neighbourhood* of v is $N[v] = \{v\} \cup N(v)$. The *degree* $d_G(v)$ of v is the number of its neighbours in G,

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i.e., $d_G(v) = |N_G(v)|$ (in general, given a finite set S, the number of elements of S will be denoted by $|S|$). Given $X \subseteq V(G)$, the subgraph of G *induced* by X is the graph whose vertices are the elements of X and whose edges are the edges of G that link vertices of X. If all vertices of G have the same degree r, we say that G is *r-regular* or simply *regular*. A subset $S \subseteq V(G)$ is an *independent set* if no pair of vertices in S is connected by an edge.

Given a vertex v in a graph G, we say that v *dominates* all vertices in $N[v]$. A set S of vertices of G is *dominating* if every vertex of G outside S is adjacent to at least one vertex of S. The *domination number* of a graph G, denoted $\gamma(G)$, is the minimum size of a dominating set of vertices in G. A dominating set S is an *efficient dominating set* (or *independent perfect dominating set*) if each vertex of G is dominated by precisely one vertex of S or, equivalently, if the minimum length of a path between any two vertices of S is at least three. It is not difficult to see that not every graph has an efficient dominating set (take, for example, C_4, i.e., a cycle on four vertices). In [1], Bange, Barkauskas and Slater (see also [21]) showed that $S = \{s_1, s_2, \ldots, s_k\}$ is an efficient dominating set of G if and only if $\{N[s_1], N[s_2], \ldots, N[s_k]\}$ is a partition of $V(G)$. They also showed that, if G has an efficient dominating set then the cardinality of any efficient dominating set equals the domination number $\gamma(G)$ of G. As a consequence, all efficient dominating sets of G have the same cardinality.

The EFFICIENT DOMINATING SET problem (or simply EFFICIENT DOMINATION) is the problem of determining whether a given graph has an efficient dominating set and finding such a set if it exists. There are many applications for efficient domination in coding theory [2, 20], graph embedding [31, 32], facility location on geographical areas [33] and resource allocation in parallel processing systems [23, 24, 28]. It was proved in [1] that the efficient domination problem is NP-complete for general graphs. The same conclusion has been reached for many very restricted graph families such as bipartite graphs [33], chordal graphs [33], chordal bipartite graphs [27], planar graphs of maximum degree three [18], planar bipartite graphs [27] and many other special families, see e.g. [4, 27]. On the other hand, for graphs in several special classes, EFFICIENT DOMINATION can be solved in polynomial-time (for a list of these special classes of graphs see e.g. [4, 14, 22, 27, 28]).

A problem which is closely related to EFFICIENT DOMINATION is that of determining if G has an *efficient edge dominating set*, i.e., a set S of edges such that every edge of G shares a vertex with precisely one edge in S (assuming that an edge shares a vertex with itself). This problem is also NP-complete in general [19] and received considerable attention in the literature under several names, such as EFFICIENT EDGE DOMINATION or DOMINATING INDUCED MATCHING (see e.g. [3, 5, 9, 10, 12, 25, 26]). An instance of EFFICIENT EDGE DOMINATION can be transformed into an instance of EFFICIENT DOMINATION by associating to the input graph G its line graph $L(G)$, in which case the edges of G become the vertices of $L(G)$ with two vertices being adjacent in $L(G)$ if and only if the respective edges of G share a vertex. As a consequence, EFFICIENT DOMINATION is NP-complete for line graphs and also for claw-free graphs (since line graphs are claw-free), a claw being the 4-vertex tree with three vertices of degree 1.

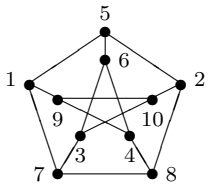

Figure 1: The vertex subsets $S_1 = \{1, 2, 3, 4\}$, $S_2 = \{5, 6, 7, 8, 9, 10\}$ and $S_3 = \{1, 2, 5, 7, 8\}$ are $(0, 2)$-regular, $(1, 3)$-regular and $(2, 1)$-regular, respectively.

An efficient dominating set can also be defined as follows: a set S of vertices of a graph G is an efficient dominating set if S induces in G a regular graph of degree 0 (i.e., S is an independent set) and every vertex of G outside S has precisely one neighbour in S. In the terminology of [7] (see also [8] and [11]), a set S satisfying this property is called a $(0, 1)$-regular set.

More generally, a subset $S \subseteq V(G)$ is a (k, τ)-*regular set* in G if it induces a k-regular subgraph in G and every vertex outside S has exactly τ neighbours in S. We will assume that k is a nonnegative integer and that τ is always a positive integer with the following exception: when a graph G is r-regular, $V(G)$ is considered by convention a $(r, 0)$-regular set. For example, in the Petersen graph depicted in Figure 1, the set $S_1 = \{1, 2, 3, 4\}$ is $(0, 2)$-regular, the set $S_2 = \{5, 6, 7, 8, 9, 10\}$ is $(1, 3)$-regular and the set $S_3 = \{1, 2, 5, 7, 8\}$ is $(2, 1)$-regular. Also, the vertex set of the Petersen graph is a $(3, 0)$-regular set by convention. Some results on the existence of (k, τ)-regular sets have been obtained by means of spectral graph theory in [7, 8, 11].

Since an efficient dominating set can be viewed as a $(0, 1)$-regular set, EFFICIENT DOMINATION is a particular case of the general problem of determining whether a graph contains a $(0, \tau)$-regular set. In this paper, using some spectral results on (k, τ)-regular sets as well as the theory of star complements, we present a simplex-like algorithm for detecting a $(0, \tau)$-regular set in an arbitrary graph. This general approach will subsequently be applied to EFFICIENT DOMINATION. Although this particular algorithm can be used to find an efficient dominating set in any given graph or to conclude that such a set doesn't exist, it is not polynomial in general. However, if -1 is not an eigenvalue of the adjacency matrix of the graph, it works in polynomial time.

The remaining sections of this paper are organized as follows. In Section 2, additional notation is given and some preparatory results on (k, τ)-regular sets needed in the sequel are presented. A new characterization of (k, τ)-regular sets is proposed. Next, in Section 3, we recall some facts from the theory of star complements and use them to describe a simplex-like algorithm for detecting $(0, \tau)$-regular sets. In Section 4, considering $\tau = 1$, the referred simplex-like algorithm is applied to EFFICIENT DOMINATION. Finally, in Section 5, we report on the successful application of this last algorithm to a vast set of randomly generated graphs, namely a set of graphs with eigenvalue -1 containing at least

two efficient dominating sets and also 3600 randomly generated bipartite graphs.

2 Preparatory results

We first introduce some more notation that will be needed below.

The adjacency matrix $A(G) = [a_{ij}]$ of a graph G of order n is the $n \times n$ matrix defined by $a_{ij} = 1$ if $(v_i, v_j) \in E(G)$ and $a_{ij} = 0$ otherwise. This matrix is real and symmetric and all its n eigenvalues $\lambda_1 \geq \ldots \geq \lambda_n$ are real numbers. The eigenvalues of $A(G)$ are also called the eigenvalues of G. The spectrum of $A(G)$ (also called the spectrum of G) is represented by $\sigma(G)$. The eigenspace of an eigenvalue $\lambda \in \sigma(G)$ is denoted by $\mathcal{E}_G(\lambda) = \ker(A(G) - \lambda I_n)$ (in general, given a natural number n, I_n represents the identity matrix of order n; we also use $\ker(M)$ to denote the kernel or null space of a matrix M). The dimension of $\mathcal{E}_G(\lambda_i)$ coincides with the multiplicity of λ_i, which will be denoted by $m_G(\lambda_i)$. Recall also that if G has at least one edge, then $A(G)$ has a negative eigenvalue not greater than -1 and a positive eigenvalue not less than the average degree in G [16].

Throughout the paper, all the vectors considered are column vectors. They will be represented in boldface as for example $\mathbf{x} = (x_1, \ldots, x_n)^T$, that denotes a vector of $\mathbb{R}^n$ (T stands for the transposition operation), $\mathbf{j}$ that will represent the all-one vector or $\mathbf{0}$ that denotes the null vector.

We turn now to the presentation of the results on (k, τ)-regular sets needed in the sequel. The following criterion for the existence of a (k, τ)-regular set is a slight variation of Proposition 2.2. of [7]. The proof is given for the sake of completeness.

Proposition 2.1 *A graph G of order n has a (k, τ)-regular set S if and only if the system*

$$(A(G) - (k - \tau)I_n)\mathbf{x} = \tau\mathbf{j}, \tag{1}$$

has a 0-1 solution. Furthermore, such a solution $\mathbf{x} = (x_1, \ldots, x_n)^T$ is the characteristic vector of S (i.e., $x_i = 1$ if $i \in S$ and $x_i = 0$ otherwise).

Proof. Let us assume that G has a (k, τ)-regular set. Then by Proposition 2.2 of [7], its characteristic vector is a solution of (1) and hence there is a 0-1 solution. Conversely, assume that the system (1) has a 0-1 solution $\mathbf{x}$ and define the vertex subset $S = \{v_i : x_i = 1\}$. Since, from (1),

$$\sum_{j \in N_G(i) \cap S} x_j - (k - \tau)x_i = \tau \Leftrightarrow |N_G(i) \cap S| = (k - \tau)x_i + \tau, \text{ for } i = 1, \ldots, n,$$

we may conclude that

$$|N_G(i) \cap S| = \begin{cases} k & \text{if } v_i \in S \\ \tau & \text{otherwise} \end{cases},$$

and therefore S is a (k, τ)-regular set. ■

The characteristic vectors of (k,τ)-regular sets were studied in Proposition 4.1 and Corollary 4.3 of [11]. Based on different tools, namely the minimal least squares solution of a linear system, we next give a result that subsumes and is equivalent to the above cited statements of [11]. To facilitate the reading, we present the proof of this equivalence in the appendix.

Proposition 2.2 *Let G be a graph of order n with at least one (k,τ)-regular set and denote by $\mathbf{x}^+$ the minimal least squares solution of system (1).*

Then a subset $S \subseteq V(G)$ is a (k,τ)-regular set in G if and only if its characteristic vector $\mathbf{x}$ is such that

$$\mathbf{x} = \mathbf{x}^+ + \mathbf{q}, \tag{2}$$

where $\mathbf{q} = \mathbf{0}$ if $(k-\tau) \notin \sigma(G)$ and $\mathbf{q} \in \mathcal{E}_G(k-\tau)$ otherwise. Moreover,

$$\mathbf{x}^+ = \begin{cases} \left(A(G) - (k-\tau)I_n\right)^{-1}(\tau\mathbf{j}) & \text{if } (k-\tau) \notin \sigma(G) \\ \displaystyle\sum_{i=1}^{n-t} \tau \frac{\mathbf{j}^T\mathbf{u}_i}{\lambda_i - (k-\tau)}\mathbf{u}_i & \text{if } (k-\tau) \in \sigma(G) \end{cases}, \tag{3}$$

where $t = \dim \mathcal{E}_G(k-\tau)$, $\lambda_1, \ldots, \lambda_{n-t}$ are the eigenvalues of $A(G)$ different from $k-\tau$ and $\mathbf{u}_1, \ldots, \mathbf{u}_{n-t}$ are corresponding mutually orthonormal eigenvectors.

Proof. First, notice that $\mathbf{x}^+$ is a solution of the system (1) and from Proposition 2.1 $S \subseteq V(G)$ is (k,τ)-regular if and only if its characteristic vector $\mathbf{x}$ is also a solution of the system (1), that is, if and only if $\exists \mathbf{q} \in \ker(A(G) - (k-\tau)I_n)$ such that $\mathbf{x} = \mathbf{x}^+ + \mathbf{q}$. Saying that $\mathbf{q}$ belongs to the null space of the matrix $A(G) - (k-\tau)I_n$ is equivalent to saying that $\mathbf{q}$ is an eigenvector of $A(G)$ associated to the eigenvalue $(k-\tau)$ when $(k-\tau) \in \sigma(G)$ and $\mathbf{q} = \mathbf{0}$ otherwise. Therefore, the first part of the proposition is proven.

It remains to prove that $\mathbf{x}^+ = \sum_{i=1}^{n-t} \tau \frac{\mathbf{j}^T\mathbf{u}_i}{\lambda_i - (k-\tau)}\mathbf{u}_i$ when $(k-\tau) \in \sigma(G)$. Let U_{n-t} be the matrix whose columns are the eigenvectors $\mathbf{u}_1, \ldots, \mathbf{u}_{n-t}$ and let D_{n-t} be the diagonal matrix whose diagonal entries are $\lambda_i - (k-\tau)$, $i = 1, \ldots, n-t$. Since

$$A(G) - (k-\tau)\, I = U_{n-t} D_{n-t} U_{n-t}^T,$$

considering the Moore-Penrose generalized inverse of $A(G) - (k-\tau)\, I$ (see for instance [6]) which is usually denoted by $(A(G) - (k-\tau)\, I)^\dagger$ it follows that

$$\begin{aligned} (A(G) - (k-\tau)\, I)^\dagger &= \left(U_{n-t} D_{n-t} U_{n-t}^T\right)^\dagger \\ &= U_{n-t} D_{n-t}^{-1} U_{n-t}^T = \sum_{i=1}^{n-t} \frac{1}{\lambda_i - (k-\tau)} \mathbf{u}_i \mathbf{u}_i^T. \end{aligned}$$

Therefore, $\mathbf{x}^+ = (A(G) - (k-\tau)\, I)^\dagger \ (\tau\mathbf{j}) = \sum_{i=1}^{n-t} \tau \frac{\mathbf{j}^T\mathbf{u}_i}{\lambda_i - (k-\tau)}\mathbf{u}_i$, as required. ■

We complete this section by noting that if G has a (k,τ)-regular set, its size can easily be computed as it is stated in the following corollary of Proposition 2.2.

Corollary 2.2.1 *If a graph G has a (k,τ)-regular set $S \subseteq V(G)$, then $|S| = \mathbf{j}^T\mathbf{x}^+$.*

Proof. If $S \subseteq V(G)$ is (k,τ)-regular set in G, then its characteristic vector $\mathbf{x}$ satisfies $\mathbf{x} = \mathbf{x}^+ + \mathbf{q}$. If $\mathbf{q} = \mathbf{0}$, $|S| = \mathbf{j}^T\mathbf{x} = \mathbf{j}^T\mathbf{x}^+$ and the corollary follows. Otherwise, $(A(G) - (k-\tau)I_n)\,\mathbf{x} = \tau\mathbf{j}$ implies that $\mathbf{j}^T\mathbf{q} = 0$, i.e., $\mathbf{q}$ is orthogonal to the all-one vector $\mathbf{j}$. Therefore, $|S| = \mathbf{j}^T\mathbf{x} = \mathbf{j}^T\mathbf{x}^+ + \mathbf{j}^T\mathbf{q} = \mathbf{j}^T\mathbf{x}^+$. ■

This corollary provides an easy sufficient condition for the non-existence of a (k,τ)-regular set.

Corollary 2.2.2 *If $\mathbf{j}^T\mathbf{x}^+$ is not a natural number, then G has no (k,τ)-regular set.*

3 Detecting $(0,\tau)$-regular sets

According to Proposition 2.1, detecting a $(0,\tau)$-regular set in a graph G of order n is equivalent to searching for a 0-1 solution of $(A(G)+\tau I_n)\mathbf{x} = \tau\mathbf{j}$. Note that, if any solution $\mathbf{x}$ of this system is nonnegative, then $\mathbf{x} \leq \mathbf{j}$. In fact, this system can be written in the form

$$\tau x_i + \sum_{k \in N_G(i)} x_k = \tau, \quad i = 1, \ldots, n;$$

thus, if $\mathbf{x} \geq 0$, we have, for all i, $\tau x_i \leq \tau$, and then $x_i \leq 1$ since $\tau > 0$. Therefore, the set of solutions of the system

$$\left\{ \begin{array}{rcl} (A(G)+\tau I_n)\mathbf{x} & = & \tau\mathbf{j} \\ \mathbf{x} & \geq & 0 \end{array} \right. \tag{4}$$

is included in the hypercube $[0,1]^n$. Then, obtaining a $(0,\tau)$-regular set in G is equivalent to determining an extreme vertex of the convex polyhedron $[0,1]^n$. The theory of star complements allows us to link the 0-1 solutions of (4) and, more generally, its basic feasible solutions with the star set concept (see below). We begin this section by recalling some basic facts of that theory.

3.1 Star sets and star complements

Given a graph G, let λ be an eigenvalue of G with multiplicity $k > 0$ and $\mathcal{E}_G(\lambda)$ its eigenspace. Also, if X is a subset of $V(G)$, $\bar{X} = V(G)\backslash X$ is its complementary set and $G - X$ is the subgraph of G induced by $\bar{X}$. We say that X is a *star set* for λ in G (or simply a λ-*star set*) if $|X| = k$ and λ is not an eigenvalue of $G - X$. In addition, $\bar{X} = V(G)\backslash X$ is called a λ-*co-star set*, while $G - X$ is a *star complement* for λ in G.

The next proposition follows directly from the proof of Theorem 5.1.7 in [17] which is known as the Reconstruction Theorem.

Proposition 3.1 *Given a graph G of orden n, let $X \subset V(G)$, $\bar{X} = V(G)\backslash X$ and $A(G) = \begin{bmatrix} A_X & N^T \\ N & C_{\bar{X}} \end{bmatrix}$, where A_X and $C_{\bar{X}}$ are the adjacency matrices of the subgraphs induced by X and $\bar{X}$, respectively. Then X is a λ-star set of G iff λ is not an eigenvalue of $C_{\bar{X}}$ and the rows of $\begin{bmatrix} N & C_{\bar{X}} - \lambda I_{|\bar{X}|} \end{bmatrix}$ form a basis of the row space of $A(G) - \lambda I_n$. Furthermore, $\mathcal{E}_G(\lambda)$ is spanned by the vectors*

$$\begin{bmatrix} \mathbf{y} \\ -\left(C_{\bar{X}} - \lambda I_{|\bar{X}|}\right)^{-1} N\mathbf{y} \end{bmatrix}, \tag{5}$$

where $\mathbf{y} \in \mathbb{R}^{|X|}$.

It should be noted that, from the definition of a star set, we may conclude that for every graph G and every eigenvalue λ of G there is at least one λ-star set.

We recall another basic result from the theory of star complements also given in [17].

Proposition 3.2 *Let G be a graph, λ one of its eigenvalues, X a λ-star set and $\bar{X} = V(G)\backslash X$ the corresponding co-star set. If $\lambda \neq 0$, then $\bar{X}$ is a dominating set for G.*

3.2 A simplex-like approach

In this subsection we describe a simplex-like algorithm for the detection of $(0, \tau)$-regular sets in a graph G when $\lambda = -\tau \in \sigma(G)$. Note that if $\lambda = -\tau \notin \sigma(G)$, obtaining the unique solution of $(A(G) + \tau I_n)\mathbf{x} = \tau\mathbf{j}$ is sufficient to decide if G has or not a $(0, \tau)$-regular set.

From Proposition 3.1, assuming X is a star set for $\lambda = -\tau \in \sigma(G)$ in G we conclude that (4) is equivalent to the system:

$$\begin{cases} \begin{bmatrix} N & C_{\bar{X}} + \tau I_{|\bar{X}|} \end{bmatrix} \mathbf{x} = \tau\mathbf{j}_{\bar{X}} \\ \mathbf{x} \geq 0 \end{cases}, \tag{6}$$

where $\mathbf{j}_{\bar{X}}$ is the all-one vector in $\mathbb{R}^{|\bar{X}|}$. Notice that (6) is a subsystem of (4) just formed by the equation associated to the indices in $\bar{X}$. Let us call $\mathbf{x}$ a *star solution* of (6) if $\mathbf{x}$ is a solution of this system and there exists a star set X for $\lambda = -\tau$ in G such that $x_i = 0$ if $i \in X$. We now state a result that slightly generalizes Theorem 14 of [13].

Theorem 3.3 *Let G be a graph, $\lambda = -\tau \in \sigma(G)$ and X a star set for $\lambda = -\tau$ in G. Then, $\mathbf{x}$ is a star solution of (6) if and only if $\mathbf{x}$ is a basic feasible solution of this system.*

Proof. Theorem 14 of [13] proves the same assertion for system (6) when $\lambda = -\tau$ is the least eigenvalue of G. It happens that the same proof holds if instead we consider any other eigenvalue $\lambda = -\tau \in \sigma(G)$. ■

In other words, the last theorem asserts that every vertex subset $\bar{X}' \subset V(G)$ is a co-star set for the eigenvalue $\lambda = -\tau$ if and only if the columns of the matrix $\left[\begin{array}{cc} N & C_{\bar{X}} + \tau I_{|\bar{X}|} \end{array} \right]$ whose indices are in $\bar{X}'$ define a basic submatrix of (6). The next result guarantees that the search for a 0-1 solution of (6) can be limited to its star solutions, i.e., to its basic feasible solutions (this amounts to saying that it suffices to search on some of the co-star sets for $\lambda = -\tau$).

Proposition 3.4 *Every 0-1 solution of the system* (6) *is a basic feasible solution.*

Proof. The polytope defined by (6) is included in the hypercube $[0,1]^n$. Therefore, each 0-1 solution is an extreme vertex of this hypercube and thus a basic feasible solution. ■

Based on the above results, we may apply a simplex technique to system (6) for deciding whether this system has or not a 0-1 solution (i.e., for deciding if G has or has not a $(0,\tau)$-regular set). We may start from the simplex tableau

$$\begin{array}{c|c|c} & x_N & \\ \hline x_B & \left(C_{\bar{X}'} + \tau I_{\bar{X}'}\right)^{-1} N & \tau \left(C_{\bar{X}'} + \tau I_{\bar{X}'}\right)^{-1} \mathbf{j}_{\bar{X}'} \\ & & \end{array} \tag{7}$$

where X' is some star set for $-\tau$ in G (which defines the indices of the nonbasic variables x_N) and $\bar{X}' = V(G) \setminus X$ defines the indices of the basic variables x_B. It should be noted that the initial star solution (x_B, x_N) and the corresponding star set X' can be obtained by first computing a feasible solution of (6) and subsequently applying the nullifying procedure described in [13] and analyzed in Theorem 12 of that paper. Thus, if the right-hand side of (7) is nonnegative but not integer, we may apply the fractional dual algorithm for Integer Linear Programming (ILP) with Gomory cuts (described for example in [29]) until a 0-1 star solution is determined or the conclusion that such a solution does not exist is obtained.

4 An algorithm for efficient domination

As mentioned in Section 1, an efficient dominating set can be viewed as a $(0,1)$-regular set. Considering $\tau = 1$, we now apply to EFFICIENT DOMINATION the simplex-like approach for detecting $(0,\tau)$-regular sets given in Section 3. Before proceeding we introduce an upper bound for the size of an efficient dominating set using the star complement theory.

Proposition 4.1 *Let G be a graph of order n. If G has an efficient dominating set S, then*

$$|S| \leq n - \max\{m_G(\lambda) : \lambda \in \sigma(G) \backslash \{0\}\}.$$

Proof. Recalling what has been said in Section 1, $|S| = \gamma(G)$. By Proposition 3.2, any co-star set $\bar{X}$ for any eigenvalue $\lambda \neq 0$ is a dominating set for G. As $\left|\bar{X}\right| = n - m_G(\lambda)$, the result follows. ■

Now, applying the results of Section 2 specifically to efficient dominating sets we obtain:

Proposition 4.2 *Let G be a graph of order n. Consider $k = 0$, $\tau = 1$ and $\mathbf{x}^+$ given by (3). Then:*

1. *If G has an efficient dominating set $S \subseteq V(G)$, then $|S| = \mathbf{j}^T\mathbf{x}^+$.*
2. *If $\mathbf{j}^T\mathbf{x}^+$ is not a natural number, then G has no efficient dominating set.*
3. *If $-1 \notin \sigma(G)$, G has an efficient dominating set if and only if the components of $\mathbf{x}^+$ are 0-1. Furthermore, if $\mathbf{x}^+$ is a 0-1 vector then it is the characteristic vector of the unique efficient dominating set of G.*

Proof. Proposition 2.2 and its corollaries imply facts 1, 2 and the first part of 3. To prove the last part of fact 3, assume that there are two distinct $(0,1)$-regular sets S and S' with characteristic vectors $\mathbf{x}$ and $\mathbf{x}'$, respectively; then $\mathbf{q} = \mathbf{x} - \mathbf{x}' \in \ker(A(G) + I_n) \backslash \{\mathbf{0}\}$, i.e., $-1 \in \sigma(G)$ which is a contradiction. ■

Based on this proposition and taking into account Subsection 3.2, we give below Algorithm 1 for the EFFICIENT DOMINATION problem.

Algorithm 1 (for EFFICIENT DOMINATION).

Require: The adjacency matrix of a graph G of order n.

Ensure: The characteristic vector of an efficient dominating set of G or the conclusion that such a vertex subset does not exists.

1: Determine the $\mathbf{x}^+$ vector given in (3).
2: **if** $\mathbf{j^T x^+} \notin \mathbb{N}$ **then STOP** (G has no efficient dominating set) **end if**.
3: **if** $-1 \notin \sigma(G)$ **then** return the output obtained from fact 3 of Proposition 4.2 and **STOP end if**.
4: Determine a co-star set for the eigenvalue -1 and the associated simplex tableau (7).
5: **while** no conclusion about the existence of a 0-1 solution is obtained from the simplex tableau **do**
6: Apply the fractional dual algorithm for ILP with Gomory cuts.
7: **end while**
8: **if** the fractional dual algorithm stopped with a 0-1 solution **then** return such solution as the characteristic vector of an efficient dominating set **else** return the conclusion that G has no efficient dominating set **end if**.

Notice that the step 3 of Algorithm 1 guarantees that it stops if $-1 \notin \sigma(G)$. Clearly in this case the algorithm works in polynomial time. Otherwise, although we have no guarantee of polynomiality, the Algorithm 1 is finite since the fractional dual algorithm for ILP with Gomory cuts (steps 5–7) is finite

too. Since, by Proposition 3.4, all 0-1 solutions are basic, we can assert the correctness of the output produced by the algorithm in step 8.

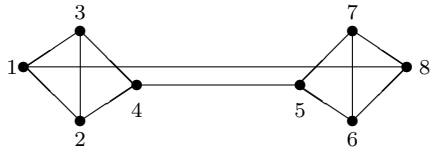

Figure 2: Graph G with $\sigma(G) = \{3, \sqrt{5}, 1, -1, -\sqrt{5}\}$, where $m_G(-1) = 4$

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Example 4.1 *Let us apply the Algorithm 1 to the graph G depicted in the Figure 2 to decide if G has or not a $(0,1)$-regular set.*

1: Since the graph G is 3-regular, it is immediate that $\mathbf{x}^+ = \frac{1}{4}\mathbf{j}_8$, where $\mathbf{j}_8$ denotes an all-one vector with eight entries.

2: Since $\mathbf{j}^T\mathbf{x}^+ = 2 \in \mathbb{N}$ we proceed to the next step.

3: Since $m_G(-1) = 4$ we proceed to the next step;

4: Since the vertex subset $\bar{X} = \{4, 5, 7, 8\}$ is a co-star set for the eigenvalue -1, then the matrix $\begin{bmatrix} N & C_{\bar{X}} + \tau I_{|\bar{X}|} \end{bmatrix}$ associated to this co-star set takes the form

$$\begin{array}{c} \\ \boldsymbol{4} \\ \boldsymbol{5} \\ \boldsymbol{7} \\ \boldsymbol{8} \end{array} \begin{array}{c} \begin{array}{cccccccc} 1 & 2 & 3 & 6 & \boldsymbol{4} & \boldsymbol{5} & \boldsymbol{7} & \boldsymbol{8} \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \end{pmatrix} \end{array},$$

and the corresponding simplex tableau (7) is

	x_1	x_2	x_3	x_6	
x_4	1	1	1	0	1
x_5	-1	0	0	0	0
x_7	0	-1	-1	1	0
x_8	1	1	1	0	1

6: The solution obtained is feasible and 0-1;

8: Therefore, $S = \{4, 8\}$ is an efficient dominating set for G.

Notice that there are other efficient dominating sets which can be obtained from the above simplex tableau by pivoting operations.

5 Computational experiments

We have tested Algorithm 1 on two groups of randomly generated graphs, namely a series of bipartite graphs and a set of graphs with eigenvalue -1 containing at least two efficient dominating sets (recall that this case is the interesting one since, for graphs without eigenvalue -1, the efficient domination problem is easily solved by Algorithm 1).

The last set of graphs was generated taking into account Theorem 2.5 of [16]: *If the spectrum of the graph G contains an eigenvalue λ_0 with multiplicity $p > 4$, then the spectrum of the complementary graph $\bar{G}$ contains an eigenvalue $-\lambda_0 - 1$ with multiplicity q, where $p - 1 \leq q \leq p + 1$*. Thus, starting with a null graph G_0 with a predefined number of vertices as well as with a predefined cardinality, two different efficient dominating sets with this cardinality were randomly generated and implanted in G_0 giving rise to graph G_1; then, for a predefined density, a random graph G_2 was generated (see for instance [15, p. 376]) on the vertices of G_1 that don't belong to any of the two implanted efficient dominating sets. Next, 5 of these last vertices were duplicated in $\bar{G}_2$ allowing us to obtain a graph, denoted by G_3, which necessarily has the eigenvalue $\lambda_0 = 0$ with multiplicity at least 5; the above cited Theorem 2.5 of [16] finally guarantees that the complementary graph $\bar{G}_3$ has -1 as a eigenvalue with multiplicity at least 4. Algorithm 1 was applied to the graphs $\bar{G}_3$ generated as described. Table 1 summarizes some of the results obtained. Its first column, denoted by "n ", lists the twelve different graph orders considered, ranging from $n = 25$ until $n = 4005$ vertices. The second column, denoted by "$|S|$", presents three different cardinalities of the efficient dominating sets S generated for each considered value of n. The next three columns report on the densities of the generated graphs; it should be noted that, for each n and $|S|$, thirty instances were generated according to three predefined density levels (namely the 0.25, 0.5 and 0.75 densities); the density columns of the table show the minimum, median and maximum of the set of final densities reached by each of the thirty instances. Finally, the last column gives the average time (in seconds) spent by the algorithm on each set of thirty instances.

Note that Table 1 reports on results of applying Algorithm 1 for a total of 1080 randomly generated graphs. The tests were carried out on a computer using an Intel(R) Core(TM) i7-3770K/3.50GHz processor with 16.0 Gb RAM and Windows 7 (64 bits) as the operating system. The overall procedure was implemented in MATLAB (version 7.6), where the built-in functions `randperm` and `rand` were respectively called to randomly generate the implanted efficient dominating sets and the graph induced by the remaining vertices.

As a first comment to the results presented in Table 1 we have observed that the graph density is apparently uncorrelated with the time spent by the algorithm to solve the efficient domination problem (the computed correlation coefficient is close to zero). Instead, the time spent by the algorithm is heavily dependent on the graph order as expected. It should also be noted that we have run tests with similar results considering several other values of $|S|$ and of the densities and with graphs where the eigenvalue -1 has high multiplicity.

On the other hand, in all the tested cases, the algorithm used no Gomory cuts; in fact, the determination of a co-star set associated to the eigenvalue -1 immediately yielded a 0-1 solution and consequently an efficient dominating set. Although this perhaps explains the low running times observed, it can also be a motivation for future work to try to understand the reasons of this behaviour. However, as a conclusion, we can say that Algorithm 1 is very suitable for solving the efficient domination problem in large graphs generated according to the foregoing procedure.

		Density			Average
n	$\|S\|$	*min*	*median*	*max*	time (s)
25	2	0.18	0.40	0.86	0.001
25	4	0.17	0.31	0.66	0.002
25	6	0.15	0.25	0.51	0.001
45	4	0.19	0.37	0.76	0.002
45	8	0.15	0.28	0.52	0.003
45	12	0.11	0.20	0.39	0.002
65	6	0.19	0.37	0.71	0.004
65	12	0.13	0.26	0.50	0.004
65	18	0.09	0.17	0.33	0.004
85	8	0.18	0.35	0.70	0.005
85	16	0.12	0.25	0.51	0.006
85	24	0.08	0.16	0.32	0.006
105	10	0.17	0.35	0.70	0.009
105	20	0.12	0.24	0.48	0.008
105	30	0.08	0.15	0.31	0.007
305	30	0.17	0.34	0.67	0.06
305	60	0.11	0.22	0.44	0.06
305	90	0.07	0.14	0.29	0.06

		Density			Average
n	$\|S\|$	*min*	*median*	*max*	time (s)
605	60	0.17	0.34	0.67	0.3
605	120	0.11	0.21	0.43	0.3
605	180	0.07	0.13	0.26	0.3
905	90	0.17	0.34	0.66	0.8
905	180	0.11	0.21	0.42	0.7
905	270	0.06	0.13	0.26	0.7
1005	50	0.21	0.41	0.82	0.9
1005	100	0.17	0.34	0.67	0.9
1005	300	0.07	0.13	0.27	0.9
2005	100	0.21	0.41	0.82	8.7
2005	200	0.17	0.33	0.66	8.7
2005	600	0.06	0.13	0.26	8.5
3005	150	0.21	0.41	0.81	29.3
3005	300	0.17	0.33	0.66	29.1
3005	900	0.06	0.13	0.25	28.0
4005	200	0.21	0.41	0.81	68.1
4005	400	0.17	0.33	0.66	67.8
4005	1200	0.06	0.13	0.25	65.9

Table 1: Some computational results for randomly generated graphs with at least two efficient domination sets

Using the same computer and Matlab environment as above, we have also tested Algorithm 1 on some randomly generated bipartite graphs. As noted in Section 1, the efficient domination problem is NP-complete for bipartite graphs, as proved in [33]. However, our tests reveal that this negative result does not prevent the successful use of Algorithm 1 for bipartite graphs. Table 2 reports on the computational results obtained for 3600 randomly generated bipartite graphs. The first column, denoted by "n", gives the graph considered orders. Each row presents the results for 600 bipartite graphs of corresponding order. It should be noted that the bipartition of each vertex set of these graphs was also randomly generated. In addition, three density levels (200 graphs for each level) were considered in the generation of graphs corresponding to each row; the respective minimum, median and maximum of the final observed densities are shown in Table 2. The remaining columns present, for each n, the number of graphs without and with efficient dominating set (eds) followed by the number of those with eigenvalue -1, respectively. Finally, the average time (in seconds)

spent by the algorithm for each group of 600 bipartite graphs is reported.

Some similarities were observed between the tests with bipartite graphs and those first described in this section. In fact, in the present case, the graph density also seems to be uncorrelated with the time spent by the algorithm, which continues to be heavily dependent of graph order. In addition, the determination of a co-star set associated to the eigenvalue -1 immediately yielded an efficient dominating set, preventing the use of Gomory cuts. Finally, the low running times observed grants to the Algorithm 1 a promising practical value.

	Density			Without	With	With	With	Average
n	*min*	*median*	*max*	eds	$\lambda = -1$	eds	$\lambda = -1$	time (s)
20	0.03	0.27	0.53	418	8	182	39	0.004
50	0.02	0.25	0.51	390	2	210	144	0.006
100	0.01	0.25	0.51	399	4	201	182	0.013
300	0.01	0.24	0.50	408	4	192	178	0.124
500	0.01	0.24	0.50	424	6	176	148	0.393
1000	0.01	0.25	0.50	586	23	14	4	4.352

Table 2: Some computational results for randomly generated bipartite graphs

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Appendix

We establish in this appendix the equivalence between Proposition 2.2 proved in Section 2 and the statements of Proposition 4.1 and Corollary 4.3 of [11]. To achieve this goal, we need to introduce some more notation and terminology.

Given a graph G of order n, its distinct eigenvalues μ_i, $i = 1, \ldots, p$ ($p \leq n$), are said to be *main* if their associated eigenspaces $\mathcal{E}_G(\mu_i)$ are not orthogonal to the all-one vector $\mathbf{j}$. The remaining distinct eigenvalues are referred to as *non-main*. The concept of main (non-main) eigenvalues was introduced in [16] and further investigated in several papers. An overview was published in [30]. Note that the maximum eigenvalue of the adjacency matrix $A(G)$, also called the *index* of G, is a main eigenvalue.

In [11], the so called (k, τ)-parametric vector $\mathbf{g}$ was defined. If $(k - \tau) \in \sigma(G)$, this vector was characterized in Proposition 4.1 of [11] as follows:

$$\mathbf{g} = \sum_{i=1}^{p} \tau \frac{\mathbf{j}^T \mathbf{x}_i}{\mu_i - (k - \tau)} \mathbf{x}_i.$$

Here, $\mu_1, \ldots, \mu_p$ are the distinct main eigenvalues of G and $\{\mathbf{x}_1, \ldots, \mathbf{x}_p\}$ an orthonormal basis of the so called Main(G) which is the space spanned by the eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_p$, such that $\mathbf{v}_i \in \mathcal{E}_G(\mu_i)$ and $\mathbf{v}_i^T \mathbf{j} \neq 0$ for each $i = 1, \ldots, p$.

It should be noted that, in view of Proposition 2.2, to assert that this statement and the statements of Proposition 4.1 and Corollary 4.3 of [11] are equivalent, it remains to show that, when $(k - \tau) \in \sigma(G)$, the minimal least squares solution $\mathbf{x}^+$ defined in (3) coincides with the parametric vector $\mathbf{g}$ given above. To see this, consider the notations of Proposition 2.2 and assume without loss of generality that the n eigenvalues of G can be grouped as follows: the first $r \geq p$ eigenvalues, say $\lambda_1, \ldots, \lambda_r$, are the main eigenvalues of G where each μ_i appears $m_G(\mu_i) = d_i$ times, $i = 1, \ldots, p$; the following eigenvalues $\lambda_{r+1}, \ldots, \lambda_{n-t}$ are non-main and the last t eigenvalues coincide with $k - \tau$ (which is non-main too). Using this grouping, $\mathbf{x}^+$ can be written as

$$\mathbf{x}^+ = \sum_{i=1}^{r} \tau \frac{\mathbf{j}^T \mathbf{u}_i}{\lambda_i - (k - \tau)} \mathbf{u}_i$$

since $\mathbf{j}^T \mathbf{u}_i = 0$ if $i = r + 1, \ldots, n - t$. Additionally, for each main eigenvalue μ_i, denoting by P_i the orthogonal projection matrix of $\mathbb{R}^n$ onto $\mathcal{E}_G(\mu_i)$ with respect to the canonical orthonormal basis of $\mathbb{R}^n$ (i.e., $P_i = \mathbf{u}_1 \mathbf{u}_1^T + \cdots + \mathbf{u}_{d_i} \mathbf{u}_{d_i}^T$ and $\{\mathbf{u}_1, \ldots, \mathbf{u}_{d_i}\}$ is an orthonormal basis of $\mathcal{E}_G(\mu_i)$), we can write

$$\mathbf{x}^+ = \sum_{i=1}^{r} \tau \frac{\mathbf{j}^T \mathbf{u}_i}{\lambda_i - (k - \tau)} \mathbf{u}_i = \sum_{i=1}^{p} \frac{\tau}{\mu_i - (k - \tau)} P_i \mathbf{j} = \sum_{i=1}^{p} \tau \frac{||P_i \mathbf{j}||}{\mu_i - (k - \tau)} \frac{P_i \mathbf{j}}{||P_i \mathbf{j}||}.$$

Putting $\mathbf{x}_i = \frac{P_i \mathbf{j}}{||P_i \mathbf{j}||}$, $i = 1, \ldots, p$, we have that $\{\mathbf{x}_1, \ldots, \mathbf{x}_p\}$ is an orthonormal basis of Main(G) ([30, p. 463]); since $\mathbf{j}^T \mathbf{x}_i = \frac{\mathbf{j}^T P_i \mathbf{j}}{||P_i \mathbf{j}||} = \frac{||P_i \mathbf{j}||^2}{||P_i \mathbf{j}||} = ||P_i \mathbf{j}||$, it can finally be asserted that

$$\mathbf{x}^+ = \sum_{i=1}^{p} \tau \frac{\mathbf{j}^T \mathbf{x}_i}{\mu_i - (k - \tau)} \mathbf{x}_i = \mathbf{g}.$$