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Polarized Calabi–Yau 3-folds in codimension 4

Gavin Brown and Konstantinos Georgiadis

Abstract

We construct Calabi–Yau 3-fold orbifolds embedded in weighted projective space in codimension 4. Each Hilbert series we consider is realised by at least two deformation families of Calabi–Yau 3-folds, distinguished by their topology, echoing a similar phenomenon for Fano 3-folds in high codimension.

1 Introduction

Throughout this paper, a *Calabi–Yau 3-fold* is a normal projective variety X of dimension 3, with $K_X = 0$ and $h^1(X, \mathcal{O}_X) = 0$ and with at worst canonical singularities. In practice, in Theorems 1.1 and 1.2 and §5 below, we construct varieties whose singularities are cyclic quotients, so X is an orbifold, and, furthermore, the singularities we consider admit crepant resolutions, and every Calabi–Yau 3-fold we discuss has a crepant resolution by a Calabi–Yau manifold.

1.1 Calabi–Yau 3-folds and Fano 4-folds

The classification of Calabi–Yau 3-folds is a central hard problem in birational geometry of primary importance in its own right. While much attention is given to Calabi–Yau manifolds, more general Calabi–Yau 3-folds are also of great interest. The central motivation for us lies in their potential role as anticanonical divisors in Fano 4-folds.

In one dimension down, the relationship between Fano 3-folds V and polarised K3 surfaces (S, A) , where $S \in |-K_V|$ and $A = -K_{V|S}$, works directly with orbifold K3 surfaces in their own right, rather than their crepant resolutions; see [ABR02, §5]. Although K3 surfaces are unobstructed, those that extend to Fano 3-folds in codimension 4 form obstructed subfamilies, as explained in [BKR12]. Thus, from the perspective of Fano 4-fold geometry, it is important to construct Calabi–Yau 3-fold orbifolds in a systematic way. Ultimately it will be possible to construct many Fano 4-folds in codimension 4 based on knowledge of these polarised Calabi–Yau orbifolds, and the fact that the Calabi–Yau 3-folds are already obstructed promises new phenomena when constructing Fano 4-folds.

We treat Calabi–Yau 3-folds as polarised varieties, (X, A) , including a choice of ample \mathbb{Q} -Cartier divisor A . With this choice, we may regard X as embedded in weighted projective

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space by choosing homogeneous generators for the graded ring

$$R(X, A) = \bigoplus_{m \in \mathbb{N}} H^0(X, mA)$$

and writing $X = \text{Proj } R(X, A)$. The degrees a_1, \dots, a_N of a set of minimal generators of $R(X, A)$ are well defined, and so we write $X \subset \mathbb{P}^{N-1}(a_1, \dots, a_N)$. The case $N = 5$ of weighted hypersurfaces is classified by Kreuzer–Skarke [KS00] into 184,026 distinct deformation families, of which 7,555 have orbifolds as their general member. There are lists (but not complete classifications) of families in codimension ≤ 3 , that is $N \leq 7$, in [BKZ14, Table 1], [GRDB], and an encyclopaedic list of known nonsingular Calabi–Yau 3-folds with Picard rank 1, often described as weighted models, in [K15, Appendix 1].

Our interest here is in Calabi–Yau 3-folds embedded in codimension 4, the case $N = 8$. We first prove the following as a model result.

Theorem 1.1. *There are at least two deformation families of quasismooth, projectively normal Calabi–Yau 3-folds $X \subset \mathbb{P}^7(1, 1, 1, 1, 1, 1, 3, 3)$, nonsingular away from two quotient singularities, $2 \times \frac{1}{3}(1, 1, 1)$, and with Hilbert series*

$$P_X(t) = \frac{1 - 2t^3 - 6t^4 - t^6 + 6t^5 + 4t^6 + 6t^7 - t^6 - 6t^8 - 2t^9 + t^{12}}{(1-t)^6(1-t^3)^2}.$$

The Hilbert series, expressed in this way, faithfully reports that the general such X is defined by 9 equations (two of degree 3, six of degree 4 and one of degree 6) related by 16 syzygies (six of degree 5, etc.); that is, $R(X, A)$ has a 9×16 first syzygy matrix, as is common for Gorenstein codimension 4; see [Rei15], for example. The two families of this theorem are distinguished by their Euler characteristic (or, equivalently in this case, by their Hodge number $h^{2,1}$), and so in particular members of different families cannot be birational (Corollary 2.6 below). This phenomenon of multiple deformation families in codimension 4 echoes the analogous result for Fano 3-folds [BKR12]. In the theory of Calabi–Yau 3-folds, however, it is a well-known phenomenon, already visible in Gross’ obstructedness result for Calabi–Yau [Gro97a], for example, and made explicit in Lee–Oguiso [LO09].

This is one of a large series of theorems, each one constructing general members of two or more deformation families of orbifold Calabi–Yau 3-folds embedding in codimension 4. Theorem 1.2 in §1.2 below is a typical example. The crucial point is the (quasi)smoothability of varieties in a Pfaffian format, and §5 recalls a salutary example where this fails. The rest of the introduction gives a geometric explanation of Theorem 1.1 and the techniques we use; §§2–4 give proofs of the results with additional comments in §5.

To reach codimension 4, we use double Gorenstein unprojection, with the novelty that unprojection divisors cannot be chosen to be coordinate planes (see §4), since where planes meet along lines, new divisors and new singularities arise on the unprojection (in some cases ready for further ‘serial’ unprojection to codimension 5). This complements existing approaches in the literature. In a series of papers Kapustka and Kapustka [KK09, K09, KK10, K11, K15] unproject anticanonically polarised del Pezzo surfaces inside Calabi–Yau 3-folds; the resulting contraction is to a Gorenstein singularity, which they then prove can be smoothed. Neves and Papadakis [NP13] apply ‘parallel’ unprojection, contracting large configurations of subvarieties to build high-dimensional varieties, inside which they describe Calabi–Yau 3-folds (also with $\frac{1}{3}(1, 1, 1)$ orbifold points) as complete intersections.

1.2 The web of Calabi–Yaus

To find other codimension 4 embeddings, in §1.5 we begin an audit of rational functions which have the same coarse numerical properties as those of Hilbert series of polarised Calabi–Yau 3-folds; in §3–4 we construct Calabi–Yau 3-folds that realise each case. Already the fairly crude Hilbert series analysis allows us to predict the codimension of the expected general member; our considerations in §1.5 are largely heuristic, but they are enough to provide a line of argument for the eventual existence proof. This approach is sometimes called the ‘graded rings method’; see [ABR02, Bro07] or [Rei00, §3.2], for example.

Applying this method to find X, A having $h^0(X, A) = 3$, $h^0(X, 2A) = 6$ and quotient singularities $\mathcal{B} = \{n \times \frac{1}{3}(1, 1, 1), m \times \frac{1}{5}(1, 1, 3)\}$, for various values of $n, m \geq 0$, we have candidates for embedded varieties as shown in Table 1 (see §1.5 for details).

$n \setminus m$	0	1	2	3
0	×	$X_{16} \subset \mathbb{P}(1^3, 5, 8)$	$X_{6,10} \subset \mathbb{P}(1^3, 3, 5^2)$	$X_{6^3, 8^3, 10^3} \subset \mathbb{P}(1^3, 3^2, 5^3)$
1	$X_{18} \subset \mathbb{P}(1^3, 6, 9)$	$X_{11} \subset \mathbb{P}(1^3, 3, 5)$	$X_{6^2, 8^2, 10} \subset \mathbb{P}(1^3, 3^2, 5^2)$	
2	$X_{12} \subset \mathbb{P}(1^3, 3, 6)$	$X_{6,8} \subset \mathbb{P}(1^3, 3^2, 5)$	$X_{6^4, 8^4, 10} \subset \mathbb{P}(1^3, 3^3, 5^2)$	
3	$X_9 \subset \mathbb{P}(1^3, 3^2)$	$X_{6^3, 8^2} \subset \mathbb{P}(1^3, 3^3, 5)$		
4	$X_{6^2} \subset \mathbb{P}(1^3, 3^3)$	$X_{6^6, 8^3} \subset \mathbb{P}(1^3, 3^4, 5)$		
5	$X_{6^5} \subset \mathbb{P}(1^3, 3^4)$	Subscript of X_{6^5} indicates 5 defining equations of degree 6,		
6	$X_{6^9} \subset \mathbb{P}(1^3, 3^5)$	and $\mathbb{P}^7(1, 1, 1, 3, 3, 3, 3, 3)$ is abbreviated as $\mathbb{P}(1^3, 3^5)$, etc.		

Table 1: Candidate Calabi–Yau 3-folds with $\mathcal{B} = \{n \times \frac{1}{3}(1, 1, 1), m \times \frac{1}{5}(1, 1, 3)\}$ for various values of $n, m \geq 0$. (The case $n = m = 0$ does not arise. Higher values of n, m lead to candidates in codimension ≥ 5 .) Those in codimension 3 are shaded light grey, those in codimension 4 in darker grey.

Assuming for a moment that families of Calabi–Yau 3-folds exist as in Table 1, we may treat the table as a small portion of the *web of Calabi–Yaus*, as described in [CGGK96, GH88, Gro97b] following Reid’s fantasy conjecture in [Rei87]. Indeed, denoting a deformation family in Table 1 by $\mathcal{F}^{(n,m)}$ and general member of $\mathcal{F}^{(n,m)}$ by $X^{(n,m)}$, making the *Gorenstein projection* of [PR04] from an orbifold $\frac{1}{3}(1, 1, 1)$ point $P \in X^{(n,m)}$ factorises as

$$\begin{array}{ccc} & \tilde{X} & \\ & \swarrow \quad \searrow & \\ X^{(n,m)} & & \overline{X}^{(n-1,m)} \end{array},$$

where $X^{(n,m)} \longleftarrow \tilde{X}$ is the crepant resolution of $P \in X^{(n,m)}$ and $\tilde{X} \longrightarrow \overline{X}^{(n-1,m)}$ is the contraction of finitely many rational curves to ordinary double points on a special member

of $\mathcal{F}^{(n-1,m)}$. We check that $\overline{X}^{(n-1,m)}$ admits a deformation to an orbifold member of $\mathcal{F}^{(n-1,m)}$ (see Lemma 2.5 below), and, following [Gro97b], we think of this as a connection between the two families which we denote symbolically by $\mathcal{F}^{(n,m)} \rightarrow \mathcal{F}^{(n-1,m)}$. (In the cited papers, the ‘web of Calabi–Yaus’ is a graph whose vertices correspond to deformation families of *nonsingular* Calabi–Yaus; but nonsingularity is a minor point in our context, since $X^{(n,m)}$ is \mathbb{Q} -factorial with unique crepant resolution that factorises $\widetilde{X} \rightarrow X^{(n,m)}$.)

Similarly, projection from an orbifold $\frac{1}{5}(1, 1, 3)$ point of $X^{(n,m)}$ describes an analogous connection $\mathcal{F}^{(n,m)} \rightarrow \mathcal{F}^{(n+1,m-1)}$. Therefore the graph of connections between the families $\mathcal{F}^{(n,m)}$ of Table 1 is connected, in accordance with [Gro97b, Conjecture 0.1].

In fact, we prove that these all entries of Table 1 exist as proposed, and that in codimension 4 they arise in multiple deformation families. (This does not jeopardise connectedness, since the families in codimension ≤ 3 are irreducible.)

Theorem 1.2. *The candidates in Table 1 all exist as quasismooth, projectively normal Calabi–Yau 3-folds in the proposed ambient spaces, nonsingular away from quotient singularities $\mathcal{B} = \{n \times \frac{1}{3}(1, 1, 1), m \times \frac{1}{5}(1, 1, 3)\}$. Moreover, each of those in codimension 4 admit more than one deformation component as follows: there are at least*

- (a) 2 deformation families of $X_{6^9} \subset \mathbb{P}^7(1^3, 3^5)$, $\mathcal{B} = \{6 \times \frac{1}{3}(1, 1, 1)\}$.
- (b) 2 deformation families of $X_{6^6, 8^3} \subset \mathbb{P}^7(1^3, 3^4, 5)$, $\mathcal{B} = \{4 \times \frac{1}{3}(1, 1, 1), \frac{1}{5}(1, 1, 3)\}$.
- (c) 3 deformation families of $X_{6^4, 8^4, 10} \subset \mathbb{P}^7(1^3, 3^3, 5^2)$, $\mathcal{B} = \{2 \times \frac{1}{3}(1, 1, 1), 2 \times \frac{1}{5}(1, 1, 3)\}$.
- (d) 2 deformation families of $X_{6^3, 8^3, 10^3} \subset \mathbb{P}^7(1^3, 3^2, 5^3)$, $\mathcal{B} = \{3 \times \frac{1}{5}(1, 1, 3)\}$.

The Hilbert series in case (d) of Theorem 1.2 can be realised by at least one more deformation family of Calabi–Yau orbifolds $X \subset \mathbb{P}(1^3, 3^2, 5^3)$, but whose general member has quotient singularities $\{\frac{1}{3}(1, 1, 1), \frac{1}{3}(2, 2, 2), 3 \times \frac{1}{5}(1, 1, 3)\}$; we do not consider this family here, and its connection into the web does not work in the same way, but see [BCG] for this construction.

Our approach is by the Tom and Jerry ansatz of [BKR12, §2], which we review in §1.4. We give an elementary model example in §1.3, sketching the proof of Theorem 1.1, which is given in detail in §2. In §§3–4 we construct Calabi–Yau 3-folds in codimensions 3 and 4. Nonsingularity analysis is the crucial central point. Bertini’s theorem does not apply directly, and in any case in the presence of nontrivial weightings linear systems tend to have large base loci; compare Examples 3.1–3 in [BKZ14]. One approach would be to write down random examples and test them by computer, and when done on a large scale this may be the most effective way; compare [BKR12], where a few hundred cases are settled systematically by computer. The significant early examples of Calabi–Yau 3-folds in codimensions 3 and 4 given by Tonoli [Ton04] and Bertin [Ber09] all use computer algebra as a crucial tool for checking nonsingularity. Although we too used a great deal of computer experimentation to identify the families we construct, in the end our arguments do not use computer algebra at all.

1.3 A Hilbert scheme with two Calabi–Yau components

To prove Theorem 1.1, we construct two polarised orbifolds X, A and X', A' with the same Hilbert series but different topological invariants. These serve as general members of two

different components of a common Hilbert scheme. They arise as follows; §2 gives the details, working with explicit equations.

In each case we construct a degree $(3, 3)$ complete intersection $Z = Z_{3,3} \subset \mathbb{P}^5$ that contains the union $\mathbb{D} \cup \mathbb{E}$ of two linearly-embedded copies of $\mathbb{P}^2 \subset \mathbb{P}^5$. The two constructions differ only in that \mathbb{D} and \mathbb{E} are chosen to be disjoint in one case and to meet transversely in a single point in the other. In either case, the general such Z has 12 nodes on each of \mathbb{D} and \mathbb{E} and is nonsingular elsewhere; when \mathbb{D} and \mathbb{E} intersect, one of the nodes of Z lies at the intersection point, so there are only 23 nodes in total on Z in this case, rather than 24.

Blowing up each of $-\mathbb{D}$ and $-\mathbb{E}$ in Z gives a small resolution of singularities $\tilde{Z} \rightarrow Z$, and the birational transforms of the planes $\mathbb{D} \cup \mathbb{E} \subset \tilde{Z}$, now disjoint in either case, can be contracted to a pair of quotient singularities $\frac{1}{3}(1, 1, 1)$ on a Calabi–Yau orbifold X . The polarising divisor on Z lifts to one on \tilde{Z} , still meeting each of \mathbb{D} and \mathbb{E} in a line, and then descends to X .

The general nonsingular $Z_{3,3}$ has Euler characteristic $\chi_{\text{top}} = c_3 = -16 \times 9 = -144$. In either situation, \tilde{Z} arises from Z as a conifold transition, passing through the degeneration to nodes and then their small resolution, so has Euler characteristic -144 plus twice the number of nodes appearing on $\mathbb{D} \cup \mathbb{E}$; thus the Euler characteristic is either -96 or -98 , depending on the case. This distinguishes the topology of the resulting orbifolds X , and so they lie in different deformation families.

In §2, we show that the polarisation on X embeds it in weighted projective space $X \subset \mathbb{P}(1^6, 3^2)$ in either case. We calculate the two different cases separately, but first recall Tom and Jerry from [BKR12].

1.4 Review of Tom and Jerry

The unprojection theorem [PR04, Theorem 1.5] takes as input a Gorenstein scheme Y and a codimension 1 Gorenstein subscheme $D \subset Y$. It returns a new Gorenstein scheme X , the *unprojection of D in Y* , whose coordinate ring is an extension of that of Y by a single new generator. The theorem is stated locally, but [PR04, §2.4] gives the graded version we use here. The relationship between Y and X depends on the precise situation, but here, as in [BKR12] and [CPR00], it has a straightforward birational description; see the proof of Proposition 2.2 below.

Section 2.3 of [BKR12] defines two ways to set up the unprojection data $D \subset Y$ in the case that $Y \subset w\mathbb{P}^N$ is a codimension 3 scheme minimally defined by the maximal Pfaffians of an antisymmetric 5×5 matrix $M = (m_{ab})$ of forms on $w\mathbb{P}^N$ and $D \subset w\mathbb{P}^N$ is a codimension 4 complete intersection inside $w\mathbb{P}^N$. These two ways are called *Tom* and *Jerry*, and, for fixed i and j , are respectively denoted and defined by:

Tom _{i} for any $1 \leq i \leq 5$: the entry m_{ab} of M must lie in I_D whenever neither a nor b equals i . In other words, all entries of M lie in I_D except possibly those in row and column i .

Jer _{ij} for any $1 \leq i < j \leq 5$: the entry m_{ab} of M must lie in I_D whenever either a or b equals i or j . In other words, all entries of both rows and columns i and j of M lie in I_D , while other entries are unconstrained.

Papadakis [Pap04] gives formulas for the unprojection X , but we do not need these here.

In practice, it is often easy to show that some Tom or Jerry setup cannot work: the degrees of entries in the matrix, combined with the row–column symmetries one can use to simplify entries, often mean that without loss of generality some entries can be assumed to be zero; that often implies that Y is so singular that X could not be quasismooth. We do not repeat this analysis here, since it follows [BKR12, §5] closely.

The harder part is to prove that some cases of $D \subset Y$ do work to give quasismooth X . We use unprojection arguments from codimension 2, where Bertini’s theorem applies, for this part. So in practice our approach is in three steps:

Step 1: Analyse the numerics of Hilbert series to generate candidates for Calabi–Yau 3-folds in codimensions up to 4. (See §1.5.)

Step 2: Use the Tom and Jerry ansatz for candidates in codimension 4 to identify those cases that may plausibly exist as unprojections of some $D \subset Y$ in codimension 3.

Step 3: For each case $D \subset Y$ proposed in Step 2, construct a variety Z in codimension 2 that contains two divisors $D \cup E \subset Z$ for which the unprojection of $E \subset Z$ realises $D \subset Y$.

The statements of our results come from the heuristic arguments of Steps 1 and 2, and so are largely invisible in the discussion, while the proofs are complete and rigorous applications of Step 3, with various configurations of $D \cup E \subset Z$ introduced out of the blue because they work.

1.5 Review of Hilbert series heuristics

A polarised variety X, A has a Hilbert series $P_X = P_{X,A}(t)$ defined by

$$P_X = \sum_{m \in \mathbb{N}} h^0(X, mA) t^m,$$

the Hilbert series of its homogeneous coordinate ring $R(X, A)$. The orbifold Riemann–Roch formula computes P_X from initial data $P_1 = h^0(X, A)$, $P_2 = h^0(X, 2A)$ and the basket \mathcal{B} of singularities. We use the formula of Buckley–Reid–Zhou [BRZ13, Theorem 1.3], which applies in any case where K_X is a multiple of A (in our case the zero multiple) and \mathcal{B} is a collection of isolated cyclic quotient singularities.

It is convenient to be able to recognise when a series $P(t)$ is plausibly the Hilbert series of some X, A in codimensions 3 or 4. We mainly encounter the 5×5 Pfaffian case in codimension 3 and the 9×16 resolution shape in codimension 4. When $X \subset \mathbb{P}(a_0, \dots, a_n)$ is a variety in codimension 3 defined by the Pfaffians of a skew-symmetric 5×5 -matrix M with homogeneous polynomials as entries, its Hilbert series has the form

$$P_X(t) = \frac{1 - t^{d_1} - \dots - t^{d_5} + t^{e_5} + \dots + t^{e_1} - t^k}{\prod_{i=0}^n (1 - t^{a_i})}$$

where the $d_i = \deg \text{Pf}_i(M)$ are the degrees of the Pfaffians, and $k = 2 \sum \deg M_{ii}$ is the adjunction number; see [CR02]. The exponents $e_i = k - d_i$ are the degrees of the syzygies.

If $X \subset \mathbb{P}(a_0, \dots, a_n)$ is in codimension 4 with 9×16 resolution shape, then we have

$$P_X(t) = \frac{1 - t^{d_1} - \dots - t^{d_9} + t^{e_1} + \dots + t^{e_{16}} - t^{c_1} - \dots - t^{c_9} + t^k}{\prod_{i=0}^n (1 - t^{a_i})}$$

where again the d_i are the degrees of the equations, the e_i are the degrees of first syzygies, c_i the degrees of second syzygies and k the adjunction number. Typically of course the e_i tend to be bigger than d_j , and c_i tend to be bigger than e_j , so quite generally, as a rough guide, we interpret the number of sign changes in the coefficients of the numerator of P_X as the codimension of X in $\mathbb{P}(a_0, \dots, a_n)$; but note the example of Theorem 1.2(c), where

$$P_X(t) = \frac{1 - 4t^6 - 4t^8 + 4t^9 - t^{10} + 8t^{11} - t^{12} + 4t^{13} - 4t^{14} - 4t^{16} + t^{22}}{(1-t)^3(1-t^3)^3(1-t^5)^2}$$

requires care to notice the overlapping syzygy degrees in this rule of thumb.

In our search for varieties, then, our starting point is to fix two integers P_1 and P_2 (which will be $h^0(X, A)$ and $h^0(X, 2A)$ if we are successful in constructing X, A) and the basket \mathcal{B} of singularities (which we hope will be the polarised singularities of X, A , if successful). Applying Riemann–Roch with these values gives a rational function which we attempt to express in a form, such as those above, to suggest an embedding of X into some weighted projective space. We illustrate this standard technique with an example.

Example 1.3. Let $P_1 = 3$, $P_2 = 6$, and $\mathcal{B} = \{4 \times \frac{1}{3}(1, 1, 1), \frac{1}{5}(1, 1, 3)\}$. Orbifold Riemann–Roch [BRZ13] computes a function that we denote by P :

$$P = \frac{1 - t - t^3 + t^4}{(1-t)^4} + 4 \times \frac{t^3}{(1-t)^3(1-t)} + \frac{t^5 + t^3}{(1-t)^3(1-t)}$$

To express P in a recognisable ‘free resolution form’ as above, we first expand as a power series,

$$P = 1 + 3t + 6t^2 + 14t^3 + 27t^4 + 46t^5 + \dots,$$

and then multiply by factors $1 - t^{a_i}$ for weights a_i that we guess inductively. In this case, $P_1 = 3$, the first nontrivial coefficient of a power of t , so we expect three variables of weight 1, and correspondingly multiply both sides of the above equation by $(1-t)^3$ to get

$$(1-t)^3 P = 1 + 4t^3 + t^5 + 4t^6 + t^8 + 4t^9 + \dots.$$

The next nontrivial power of t is $4t^3$, so we multiply by $(1-t^3)^4$ to eliminate this term:

$$(1-t)^3(1-t^3)^4 P = 1 + t^5 - 6t^6 - 3t^8 + \dots.$$

Continuing, we multiply by $(1-t^5)$ to get

$$(1-t)^3(1-t^3)^4(1-t^5)P = 1 - 6t^6 - 3t^8 + 8t^9 + 8t^{11} - 3t^{12} - 6t^{14} + t^{20}$$

and conclude that

$$P = \frac{1 - 6t^6 - 3t^8 + 8t^9 + 8t^{11} - 3t^{12} - 6t^{14} + t^{20}}{(1-t)^3(1-t^3)^4(1-t^5)}.$$

(Comparing with the original expression shows that the tail of the power series cancels.) The number of sign changes in the numerator of P is four, which suggests a codimension four 3-fold $X_{6^6, 8^3} \subset \mathbb{P}(1^3, 3^4, 5)$ with Hilbert series $P_X = P$; this is case (b) of Theorem 1.2. In §4 we prove that there are in fact two deformation families of such a quasismooth X .

Analogous calculations with different $\mathcal{B} = \{n \times \frac{1}{3}(1, 1, 1), m \times \frac{1}{5}(1, 1, 3)\}$ give the candidates of Table 1. There are a couple of tricks we employ beyond the basic game to end up with candidates that can be realised in practice. The singularities of \mathcal{B} should arise as restrictions of those of the ambient weighted projective space; for example, if \mathcal{B} includes $\frac{1}{5}(1, 1, 3)$, then the weights a_i should include some that are divisible by 5, and at least one that is congruent to 3 modulo 5. And a $\frac{1}{5}(1, 1, 3)$ point would lead to a $\frac{1}{3}(1, 1, 1)$ point after projection, so unless the degree $A^3 \leq 1/15$ (so there could be no projection) there should be some weights divisible by 3 to accommodate this. If the power series process does not provide such weights, we must experiment a little by hand. Although this is somewhat heuristic, it is justified in the end by the results of §§3–4.

2 The proof of Theorem 1.1

2.1 Disjoint planes, parallel unprojection and Jerry

Consider \mathbb{P}^5 with homogeneous coordinates x, y, z, u, v, w . Let $\mathbb{D} = (u = v = w = 0)$ and $\mathbb{E} = (x = y = z = 0)$. Any 3-fold $Z = (f = g = 0)$ containing $\mathbb{D} \cup \mathbb{E}$ is defined by two polynomials

$$\begin{aligned} f &= X_2u + Y_2v + Z_2w + U_2x + V_2y + W_2z \\ g &= X'_2u + Y'_2v + Z'_2w + U'_2x + V'_2y + W'_2z, \end{aligned}$$

where the forms $X_2 = X_2(x, y, z), \dots, Z'_2(x, y, z)$ are quadrics in x, y, z and the forms $U_2 = U_2(u, v, w), \dots, W'_2(u, v, w)$ are quadrics in u, v, w . Since all these forms can be chosen freely, the general such $Z = (f = g = 0)$ is nonsingular away from the base locus $\mathbb{D} \cup \mathbb{E}$ by Bertini's theorem.

Lemma 2.1. *In the notation above, the singular locus $\text{Sing}(Z)$ of Z consists of exactly 12 points on \mathbb{D} and 12 points on \mathbb{E} , and all the singular points are ordinary nodes.*

Proof. Consider first the singularities that lie on \mathbb{E} . We rewrite the equations of Z as

$$\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} A & B & C \\ D & E & F \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad (2.1)$$

where A, \dots, F are quadric forms in the ideal $I_{\mathbb{D}} = (u, v, w)$. In this notation, the singularities of Z on \mathbb{D} occur where the 2×3 matrix in (2.1) drops rank. If we choose both A and E to be the zero form, this rank condition is equivalent to

$$BD = CD = BF = 0.$$

With this choice, the singular locus of Z on \mathbb{E} is

$$\text{Sing}(Z) \cap \mathbb{E} = (B = C = 0) \cup (B = D = 0) \cup (D = F = 0). \quad (2.2)$$

Each of the factors in this union is the intersection of two conics in $\mathbb{E} = \mathbb{P}^2$, and so for general B, C, D, F this singular locus is 12 distinct reduced points, which are therefore

necessarily nodes. Since the singularities are nodes for this choice of A, \dots, F , they are also nodes for the general member Z of this family.

The analogous argument for \mathbb{D} shows that the general member of the family also has exactly 12 nodes on \mathbb{D} , which completes the proof of the lemma. \square

We refer to this choice of A, \dots, F in the proof of the lemma as a *standard choice*; we use it throughout to show that a Jacobian ideal is reduced in general, and so the singularities are nodes.

Next we make the \mathbb{E} -ample small blowup of Z followed by contraction of \mathbb{E} in one step using the Kustin–Miller unprojection of [PR04, Theorem 1.5]; we set up the algebra of unprojection first before proving this geometric claim in Proposition 2.2.

Unprojecting $\mathbb{E} \subset Z$ introduces a new variable s , the *unprojection variable*. With the equations of Z written as in (2.1) the unprojection calculus realises s as the rational form

$$s = \frac{BF - CE}{x} = \frac{AF - CD}{-y} = \frac{AE - BD}{z}. \quad (2.3)$$

The five equations—namely f, g and with the implicit equations for s above, $sx = BF - CE$ and so on—can be mounted as the maximal Pfaffians of an anti-symmetric 5×5 matrix (we display only the strict upper triangle)

$$\begin{pmatrix} z & y & A & D \\ & x & B & E \\ & & C & F \\ & & & s \end{pmatrix}. \quad (2.4)$$

In the language of §1.4, the matrix (2.4) is in Jerry format for the ideal $I_{\mathbb{D}}$; more precisely, it is Jer_{45} , since each entry of the fourth and fifth rows and columns lies in $I_{\mathbb{D}}$.

These five equations define a 3-fold $Y \subset \mathbb{P}(1^6, 3)$, in coordinates x, y, z, u, v, w, s , which contains the birational transform of \mathbb{D} , also denoted $\mathbb{D} \subset Y$, defined by ($u = v = w = s = 0$).

Proposition 2.2. *The point $P_s = (0 : 0 : 0 : 0 : 0 : 0 : 1) \in Y$ is quasismooth, and is a quotient singularity $\frac{1}{3}(1, 1, 1)$. Away from P_s , Y is smooth outside \mathbb{D} , and has exactly 12 isolated singularities on \mathbb{D} all of which are nodes.*

Proof. The three unprojection equations ($sx = BF - CE$, etc., of (2.3)) eliminate x, y, z near P_s , so Y is quasismooth there, and has an orbifold chart (analytically) locally isomorphic to a neighbourhood of the origin in the index 3 orbifold chart of the ambient weighted projective space; locally it is just the origin in the quotient of \mathbb{C}^3 with coordinates u, v, w by $\mathbb{Z}/3$ acting by multiplication, which is the claim.

The unprojection map $\varphi: Z \dashrightarrow Y$ is defined by

$$(x, y, z, u, v, w) \mapsto (x, y, z, u, v, w, s)$$

where s is the rational function defined in (2.3). Multiplying through by the denominators of s gives polynomial expressions for φ ,

$$\begin{aligned} (x, y, z, u, v, w) &\mapsto (x^2, xy, xz, xu, xv, xw, BF - CE) \\ &= (xy, y^2, yz, yu, yv, yw, CD - AF) \\ &= (xz, yz, z^2, zu, zv, zw, AE - BD), \end{aligned}$$

from which it follows immediately that φ is defined everywhere on Z except possibly at the locus

$$x = y = z = AE - BD = CD - AF = BF - CE = 0.$$

This locus is precisely the set of nodes $\{Q_1, \dots, Q_{12}\}$ on \mathbb{E} . The birational inverse $\psi: Y \dashrightarrow Z$ is simply the elimination of s , and so is defined everywhere on Y except at the point P_s .

Denote the (closure of the) preimage of a node Q_i on \mathbb{D} under ψ by Γ_i . Obviously, $\Gamma_i \cong \mathbb{P}^1$, since it is the cone on Q_i over P_s . The restriction of φ on $Z \setminus \mathbb{E}$ gives an isomorphism

$$Z \setminus \mathbb{E} \longrightarrow Y \setminus \Gamma, \quad \text{where } \Gamma = \bigcup_{i=1}^{12} \Gamma_i. \quad (2.5)$$

Note that $\mathbb{D} \subset Z \setminus \mathbb{E}$, so a neighbourhood of \mathbb{D} is mapped isomorphically, and its 12 nodes survive on Y . Since Z is smooth away from \mathbb{D} and \mathbb{E} the isomorphism (2.5) shows that Y is smooth away from \mathbb{D} and Γ . Furthermore, the equations defining φ show that it induces a contraction

$$\mathbb{E} \setminus \text{Sing}(Z) \longrightarrow \{P_s\}.$$

It remains only prove that Y is quasismooth on $\Gamma \setminus \{P_s\}$. This is elementary: the locus Γ is defined by the same equations (2.2) that define the nodes on \mathbb{E} , and using a *standard choice* of forms A, \dots, F , the Jacobian matrix of Y restricted to any of the three components of the union in (2.2) has rank three along that part of Γ (c.f. a similar elementary calculation for a harder case in §3 below). \square

Next we unproject $\mathbb{D} \subset Y$. By [PR04, Theorem 1.5], this introduces another unprojection variable of degree 3, and constructs some $X \subset \mathbb{P}(1^6, 3^2)$, embedded in codimension 4. The whole picture is this:

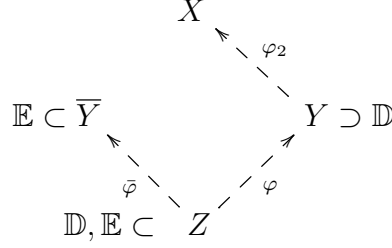
$$\begin{array}{c} X \subset \mathbb{P}(1^6, 3^2) \\ \uparrow \\ \mathbb{D} \subset Y \subset \mathbb{P}(1^6, 3) \\ \uparrow \\ \mathbb{D} \cup \mathbb{E} \subset Z \subset \mathbb{P}^5 \end{array}$$

and we prove next that X is an orbifold birational to Z .

Theorem 2.3. *In the notation above, X is a quasismooth 3-fold with exactly two $\frac{1}{3}(1, 1, 1)$ quotient singularities.*

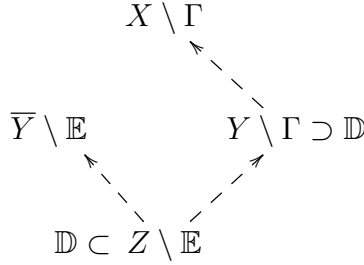
Proof. The variety X is defined to be the unprojection of $\mathbb{D} \subset Y$. The crucial point is to show that X is quasismooth. As well as the two unprojections used so far, we consider the unprojection of $\mathbb{D} \subset Z$ to give another variety \overline{Y} in codimension 3 that contains an

isomorphic copy $\mathbb{E} \subset \bar{Y}$.



By Proposition 2.2, φ is an isomorphism in a neighborhood of \mathbb{D} , and the analogous argument shows that $\bar{\varphi}$ is an isomorphism in a neighborhood of \mathbb{E} and $\bar{Y} \setminus \mathbb{E}$ is quasismooth.

By (2.5) there is an isomorphism $Z \setminus \mathbb{E} \cong Y \setminus \Gamma$, where $\Gamma = \bigcup_{i=1}^{12} \Gamma_i$ is the union of exceptional rational curves blown up from the nodes lying on \mathbb{E} . Therefore, the unprojection of $\mathbb{D} \subset Z \setminus \mathbb{E}$ is isomorphic to the unprojection of $\mathbb{D} \subset Y \setminus \Gamma$.



(We have identified Γ with its isomorphic image in X in the diagram.) Consequently, $X \setminus \Gamma \cong \bar{Y} \setminus \mathbb{E}$, and so $X \setminus \Gamma$ is quasismooth. Since φ_2 is an isomorphism in a neighborhood of Γ it follows that X is quasismooth. Moreover, \mathbb{D} and $\mathbb{E} \subset Z$ are contracted to two $\frac{1}{3}(1, 1, 1)$ quotient singularities on X , since they are on \bar{Y} and Y , respectively. \square

2.2 Intersecting planes, serial unprojection and Tom

For the second calculation, consider instead a pair of planes $\mathbb{D} = (x = u = v = 0)$ and $\mathbb{E} = (x = y = z = 0)$ which meet transversely in the coordinate point $P_w = (0 : 0 : 0 : 0 : 0 : 1) \in \mathbb{P}^5$. As before, the general complete intersection $Z' = Z'_{3,3} = (f = g = 0)$ containing $\mathbb{D} \cup \mathbb{E}$ is also nonsingular outside 12 nodes on each of \mathbb{D} and \mathbb{E} , although this time the point $P_w \in Z'$ is necessarily one of those nodes, and so there are only 23 in total.

Unprojecting $\mathbb{E} \subset Z'$ first, as before, the equations f, g can be presented as in (2.1), but this time under the conditions that $B, C, E, F \in (u, v)$, so that both f and g lie in $I_{\mathbb{D} \cup \mathbb{E}} = (x, yu, yv, zu, zv)$. The unprojection variable s satisfies the same equations (2.3) as before, and the five equations of the unprojection $Y' \subset \mathbb{P}(1^6, 3)$ can again be presented as the maximal Pfaffians of the matrix (2.4).

But now, in contrast to §2.1, the matrix (2.4) for Y' is in Tom format for the ideal $I_{\mathbb{D}}$, in fact Tom_1 , since each entry outside the first row and column lies in $I_{\mathbb{D}}$.

Using the same arguments as Lemma 2.1 and Proposition 2.2, general choices for the forms A, \dots, F defining Z' , subject to the condition $B, C, E, F \in (u, v)$, results in a 3-fold Y' containing an isomorphic copy $\mathbb{D} \subset Y'$ that is quasismooth away from \mathbb{D} and has only nodes on \mathbb{D} itself. This unprojection of $\mathbb{E} \subset Z$ resolves all 12 nodes lying on \mathbb{E} , including the one that also lies on \mathbb{D} , and so Y' has only 11 nodes.

Again, unprojecting $\mathbb{D} \subset Y'$ constructs a quasismooth variety $X' \subset \mathbb{P}(1^6, 3^2)$ with two $\frac{1}{3}(1, 1, 1)$ quotient singularities.

Thus we have constructed two 3-fold orbifolds $X, X' \subset \mathbb{P}(1^6, 3^2)$.

Theorem 2.4. *The varieties X and X' are quasismooth Calabi–Yau 3-folds with the same Hilbert series but with different topological Euler characteristic. In particular, X and X' lie in different components of Calabi–Yau 3-fold orbifolds in the same Hilbert scheme.*

Proof. By construction, the two varieties have the same quotient singularities $2 \times \frac{1}{3}(1, 1, 1)$. The hypersurfaces Z and Z' have the same Hilbert series, and by [PR04] their Hilbert series are related to that by

$$P_X = P_Z + \frac{2t^3}{(1-t)^3(1-t^3)} = P_{X'}.$$

The initial variety $Z \subset \mathbb{P}^5$ is a Calabi–Yau 3-fold. Unprojection of the planes preserves this property. Indeed all singularities are rational, so $h^1(X, \mathcal{O}_X) = h^1(X', \mathcal{O}_{X'}) = 0$. (Alternatively one can use the unprojection theorem [PR04, Theorem 5.1] directly: it proves that the homogeneous coordinate ring of each unprojection is Gorenstein, and so a theorem of Goto–Watanabe [GW78, 5.1.9–11] implies all intermediate cohomology vanishes; in particular $h^1(X, \mathcal{O}_X) = 0$, and similarly for X' .) By the proof of Proposition 2.2, geometrically each unprojection factorises as the crepant resolution of $\frac{1}{3}(1, 1, 1)$ and small resolution of nodes (which is necessarily crepant), so $K_X = 0$ and $K_{X'} = 0$. Thus X and X' are both Calabi–Yau 3-folds.

It remains to check the claim about Euler characteristics. We consider the final unprojection to X and X' from codimension 3 Calabi–Yau 3-folds Y and Y' . For X , we unproject a nodal variety Y containing a divisor \mathbb{D} in Jer_{45} format; for X' , we unproject a nodal variety Y' with a divisor, also denoted \mathbb{D} , in Tom_1 format. In both cases, the unprojection φ factorizes as the \mathbb{D} -ample small resolution of nodes, $\tilde{Y} \rightarrow Y$ and $\tilde{Y}' \rightarrow Y'$, followed by the contraction, $\tilde{Y} \rightarrow X$ and $\tilde{Y}' \rightarrow X'$, of \mathbb{D} .

Both Y and Y' are defined by the maximal Pfaffians of skew 5×5 matrices of forms that have degrees

$$\begin{pmatrix} 1 & 1 & 2 & 2 \\ & 1 & 2 & 2 \\ & & 2 & 2 \\ & & & 3 \end{pmatrix}. \quad (2.6)$$

Lemma 2.5. *Any 3-fold $Y_{\text{gen}} \subset \mathbb{P}(1^6, 3)$ defined by the maximal Pfaffians of a matrix of general forms with degrees as indicated by (2.6) is quasismooth.*

This lemma follows at once by applying the proof of Proposition 2.2 to the case of general $Z_{3,3} \subset \mathbb{P}^5$ which contains \mathbb{E} but is not constrained to contain \mathbb{D} .

Each of Y and Y' is a (flat) degeneration of Y_{gen} : varying entries in the skew matrix deforms the entire free resolution of the defining ideal, and Y and Y' are made by varying particular Tom or Jerry entries to lie in the ideal $I_{\mathbb{E}}$. All this fits in a picture:

$$\begin{array}{ccccc} X' & \longleftarrow & \tilde{Y}' & & \tilde{Y} & \longrightarrow & X \\ & \nearrow \varphi & \downarrow & & \downarrow & \nearrow \varphi & \\ & & Y' & \longleftarrow Y_{\text{gen}} & \longrightarrow & Y & \end{array}$$

where “ \rightsquigarrow ” denotes degeneration. The passage from Y_{gen} to \tilde{Y} or \tilde{Y}' is a conifold transition. By Clemens’ calculation [Cle83, Rei87], the Euler characteristic of \tilde{Y} is related to that of Y_{gen} by

$$\chi(Y_{\text{gen}}) = \chi(\tilde{Y}) + 2 \cdot 12,$$

where 12 is the number of nodes on \mathbb{D} in the Jer_{45} case. Similarly $\chi(Y_{\text{gen}}) = \chi(\tilde{Y}') + 2 \cdot 11$, where 11 is the number of nodes on \mathbb{E} in the Tom_1 case. So

$$\chi(\tilde{Y}') = \chi(\tilde{Y}) + 2.$$

The maps $\tilde{Y} \rightarrow X$ and $\tilde{Y}' \rightarrow X'$ are crepant blowups of the same singularity $\frac{1}{3}(1, 1, 1)$, so the change in the Euler characteristic from \tilde{Y} to X and from \tilde{Y}' to X' is the same in either case. Therefore, $\chi(X') = \chi(X) + 2$, and X and X' have different Euler characteristics. \square

The Hilbert series $P_X = P_{X'}$ can be expanded as

$$P_X = P_{X'} = \frac{1 - 2t^3 - 6t^4 - t^6 + 6t^5 + 4t^6 + 6t^7 - t^6 - 6t^8 - 2t^9 + t^{12}}{(1-t)^6(1-t^3)^2},$$

presenting the numerator to emphasise the 9 equations and 16 syzygies, or 9×16 resolution, even though one could cancel some t^6 terms. The two varieties X and X' that we construct are both defined minimally by 2 cubics, 6 quadrics and a single sextic in the variables of $\mathbb{P}(1^6, 3^2)$. Our approach uses the unprojection theorem [PR04, Theorem 1.5] merely to show that X exists in the ambient space we claim; it also determines the degrees of the equations, which is how we see the degree 6 equation that is otherwise masked in the Hilbert numerator. We could instead apply [Pap04, §5.7] to write down the equations of X , but we do not need them here.

Corollary 2.6. *X and X' are not birational.*

This follows from Batyrev’s theorem [Bat99, Theorem 1.1], and also [Bri02, Theorem 1.1]: smooth Calabi–Yau 3-folds that are birational have the same Betti numbers. Indeed, if X and X' were birational, then, in the notation of the proof, \tilde{Y} and \tilde{Y}' would be birational too. But these are smooth Calabi–Yau 3-folds, and the proof showed that their Euler characteristics differ, and so their Betti numbers cannot be equal.

3 Calabi–Yau 3-folds in codimension 3

We proceed with the proof of Theorem 1.2. The complete intersection Calabi–Yau 3-folds are all well known and their existence follows at once from Bertini’s theorem. We show that the general 5×5 codimension 3 variety $Y \subset \mathbb{P}(1^3, 3^3, 5)$ defined with syzygy degrees

$$\begin{pmatrix} 1 & 3 & 3 & 3 \\ & 3 & 3 & 3 \\ & & 5 & 5 \\ & & & 5 \end{pmatrix}$$

is quasismooth; such Y has quotient singularities $\{3 \times \frac{1}{3}(1, 1, 1), \frac{1}{5}(1, 1, 3)\}$, which is the case $n = 3, m = 1$ of Table 1.

The starting point is the general codimension 2 complete intersection

$$\mathbb{E} \subset Z_{6,6} \subset \mathbb{P}(1^3, 3^3),$$

where $\mathbb{E} = (z = u = v = 0) = \mathbb{P}(1, 1, 3)$, in coordinates x, y, z, u, v, w on $\mathbb{P}(1^3, 3^3)$.

Lemma 3.1. *In the notation above, the non-quasismooth locus of Z consists of exactly 13 points, and these are all ordinary nodes lying on \mathbb{E} .*

The proof follows that of Lemma 2.1. We write the equations of Z as

$$\begin{pmatrix} A_5 & B_3 & C_3 \\ D_5 & E_3 & F_3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

for general forms A, \dots, F of the indicated degrees. Bertini's theorem shows that Z is quasismooth away from \mathbb{E} . The non-quasismooth locus of Z along \mathbb{E} , denoted Σ , occurs where the 2×3 matrix of forms drops rank, so has Hilbert series

$$\begin{aligned} P_{\Sigma|\mathbb{E}} &= \frac{1 - t^6 - 2t^8 + 2t^{11}}{(1 - t)^2(1 - t^3)} \\ &= \frac{1 + t + t^2 + 2t^3 + 2t^4 + 2t^5 + 2t^6 + 2t^7}{1 - t}, \end{aligned}$$

which is that of a 0-dimensional scheme of length 13. The standard choice $A = E = 0$ and B, C, D, F general gives cases with

$$\Sigma = (B = C = 0) \cup (B = D = 0) \cup (D = F = 0) \subset \mathbb{E} = \mathbb{P}(1, 1, 3) \quad (3.1)$$

reduced and disjoint from the orbifold locus of Z ; so the singularities of such Z are ordinary nodes, and so the same holds for general Z and the lemma is proved.

Unprojecting \mathbb{E} gives a new 3-fold Y :

$$\begin{array}{c} Y \subset \mathbb{P}(1^3, 3^3, 5) \\ \downarrow \varphi \\ \mathbb{E} \subset Z \subset \mathbb{P}(1^3, 3^3) \end{array}$$

with unprojection variable s of degree 5. This variety Y is defined by three unprojection equations and the defining equations f, g of Z :

$$sz - BF + CE = su + AF - DC = sv - AE + BD = f = g = 0.$$

We follow the proof of Proposition 2.2. The preimage in Y of the nodes of Z is a bouquet of rational curves $\Gamma = \bigcup_{i=1}^{13} \Gamma_i$, the cone over Σ with vertex the coordinate point P_s . The restriction of φ on $Z \setminus \mathbb{E}$ is an isomorphism

$$Z \setminus \mathbb{E} \longrightarrow Y \setminus \Gamma,$$

so Y is quasismooth outside Γ , and furthermore the equations of φ show at once that it gives a contraction $\mathbb{E} \setminus \text{Sing}(Y) \longrightarrow \{P_s\}$. It remains to show that Y is quasismooth on Γ . Using the *standard choice* $A = E = 0$, we calculate the Jacobian of the five defining equations (3.1) restricted to the three pieces of the union in turn. First, consider the piece ($z = u = v = B = C = 0$). The Jacobian restricted to the rational curves Γ_i of this piece is

$$\begin{pmatrix} -B_x F & -B_y F & s - B_z F & -B_u F & -B_v F & -B_w F & 0 \\ -DC_x & -DC_y & -DC_z & s - DC_u & -DC_v & -DC_w & 0 \\ B_x D & B_y D & B_z D & B_u D & s + B_v D & B_w D & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & D & 0 & F & 0 & 0 \end{pmatrix}.$$

Consider the submatrix

$$\begin{pmatrix} -DC_x & -DC_y & -DC_v & -DC_w \\ B_x D & B_y D & s + B_v D & B_w D \\ 0 & 0 & F & 0 \end{pmatrix}.$$

with entries in x, y, w and s . In general, $(B = C = 0)$ defines 3 distinct points in $\mathbb{E} = \mathbb{P}(1, 1, 3)$ —a nonsingular zero-dimensional subscheme of \mathbb{E} . The polynomial D is not zero there, so columns 1, 2, 4 of the top two rows of this submatrix has rank 2. The general F is not zero there either, so the submatrix, and so in particular the Jacobian matrix itself, has rank 3 on this piece of the union that makes up Γ . The argument is the similar for the other two pieces.

4 Calabi–Yau 3-folds in codimension 4

We prove Theorem 1.2(b); that is, we construct members of two deformation families of Calabi–Yau 3-folds

$$X_{6^6, 8^3} \subset \mathbb{P}(1^3, 3^4, 5) \quad \text{with quotient singularities } 4 \times \frac{1}{3}(1, 1, 1), \frac{1}{5}(1, 1, 3).$$

The starting point is a $(6, 6)$ complete intersection $Z_{6,6} \subset \mathbb{P}^5(1^3, 3^3)$ containing divisors $\mathbb{D} \cong \mathbb{P}^2$ and $\mathbb{E} \cong \mathbb{P}(1, 1, 3)$. As in the model case, we build a variety $\mathbb{D} \subset Y$ by unprojecting \mathbb{E} in Z , and then construct X by unprojecting \mathbb{D} in Y , but unlike the model case these divisors cannot both be chosen to be coordinate hyperplanes. The choice of \mathbb{E} is delicate: after unprojection we want a variety Y that contains \mathbb{D} in Tom_1 format or, later on, in Jer_{45} format, and this constrains \mathbb{E} .

4.1 Construction of Tom_1

Fix homogeneous coordinates x, y, z, t, u, v on $\mathbb{P}(1^3, 3^3)$ and define $\mathbb{D} = \mathbb{P}^2$ by $I_{\mathbb{D}} = (t, u, v)$. Define \mathbb{E} by setting $\alpha = x$, $\beta = yz(y - z) + u$, and $\varepsilon = t$ and taking $I_{\mathbb{E}} = (\alpha, \beta, \varepsilon)$. It follows that $I_{\mathbb{D} \cup \mathbb{E}} = (t, ux, u\beta, vx, v\beta)$ and

$$\mathbb{D} \cap \mathbb{E} = \{(0 : 1 : 0 : 0 : 0 : 0), (0 : 0 : 1 : 0 : 0 : 0), (0 : 1 : 1 : 0 : 0 : 0)\}.$$

Let $Z = Z_{6,6}$ be a general $(6, 6)$ complete intersection that contains $\mathbb{D} \cup \mathbb{E}$.

Lemma 4.1. *In the notation above, the singular locus $\text{Sing}(Z)$ of Z consists of exactly 27 points on \mathbb{D} and 13 points on \mathbb{E} , and all singular points are ordinary nodes. In each case, 3 of the nodes are at $\mathbb{D} \cap \mathbb{E}$, so that Z has 37 nodes in total.*

We follow the proof of Lemma 2.1, using the same standard choice.

Proof. First write the equations of Z , from the point of view of $\mathbb{E} \subset Z$, as

$$\begin{pmatrix} A_5 & B_3 & C_3 \\ D_5 & E_3 & F_3 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \varepsilon \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad (4.1)$$

where A, \dots, F are forms of the indicated degrees and $A, B, D, E \in I_{\mathbb{D}}$. The singular locus of Z on \mathbb{E} is defined by the 2×2 minors of the matrix in (4.1). Making the standard choice that both A and E are the zero form, we have

$$\text{Sing}(Z) \cap \mathbb{E} = (B_3 = C_3 = 0) \cup (B_3 = D_5 = 0) \cup (D_5 = F_3 = 0). \quad (4.2)$$

We may choose further $B' = v$ and $F' = v + h_3$, for a general cubic $h = h(x, y, z)$, so that each factor in this disjoint union is a transverse intersection away from the index 3 point of $\mathbb{E} \cong \mathbb{P}(1, 1, 3)$, of 3, 5 and 5 reduced points respectively. (Since $B, D \in I_{\mathbb{D}}$, the three points of $\mathbb{D} \cap \mathbb{E}$ fall into the second factor of the union; the other factors do not meet this intersection since C and F are not constrained.) Thus the general Z has 13 nodes on \mathbb{E} .

Now re-expressing the equations of Z from the point of view of $\mathbb{D} \subset Z$ as

$$\begin{pmatrix} A' & B' & C' \\ D' & E' & F' \end{pmatrix} \begin{pmatrix} t \\ u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad (4.3)$$

where A', \dots, F' are cubic forms and $B', C', E', F' \in I_{\mathbb{E}}$. The singular locus of Z on \mathbb{D} is defined by the 2×2 minors of the matrix in (4.3). Making the standard choice $A' = E' = 0$, we have

$$\text{Sing}(Z) \cap \mathbb{D} = (B' = C' = 0) \cup (B' = D' = 0) \cup (D' = F' = 0). \quad (4.4)$$

Since $B', C' \in I_{\mathbb{E}}$, the first factor in this union consists of 9 reduced points in $\mathbb{D} = \mathbb{P}^2$, of which 3 are $\mathbb{D} \cap \mathbb{E}$. The cubic D' is not constrained, so the remaining two factors provide a further 9 disjoint reduced points each, and so for general B', C', D', F' this singular locus is 27 distinct reduced points, all necessarily nodes. Since the singularities are nodes for this choice of A', \dots, F' , they are also nodes for the general member Z of this family. \square

Unprojection of \mathbb{E} At this point, we have a general $Z_{6,6} \in \mathbb{P}(1^3, 3^3)$ containing two divisors. Unprojecting \mathbb{E} gives a 3-fold $Y \subset \mathbb{P}(1^3, 3^3, 5)$ in codimension 3:

$$\begin{array}{ccc} \mathbb{D} & \subset & Y \subset \mathbb{P}(1^3, 3^3, 5) \\ & & \uparrow \\ & & \mathbb{E}, \mathbb{D} \subset Z_{6,6} \subset \mathbb{P}(1^3, 3^3) \end{array}$$

The unprojection variable, denoted by w , has weight 5. In the notation of (4.1), the equations of Y are those of $Z_{6,6}$ together with three new equations:

$$w\alpha = BF - CE, \quad w\beta = -AF + CD, \quad w\varepsilon = AE - BD.$$

As usual, these five equations can be realised as the maximal Pfaffians of a matrix,

$$\begin{pmatrix} \alpha & \beta & C & F \\ & \varepsilon & -B & -E \\ & & A & D \\ & & & w \end{pmatrix}.$$

By design, these are in Tom_1 format for $I_{\mathbb{D}} = (t, u, v, w)$. The polynomials A, \dots, F do not contain w , so $P_w = \frac{1}{5}(1, 1, 3) \in Y$ is a quasismooth point: the coefficients of w on the left-hand side of the three unprojection equations contain independent linear terms.

Proposition 4.2. *Y is quasismooth outside \mathbb{D} , and has in total 24 isolated singularities on \mathbb{D} , all of which are nodes.*

Proof. Also here, we follow the same pattern as in the proof of Proposition 2.2, and use analogous definitions and notation. The restriction of φ on $Z \setminus \mathbb{E}$ gives an isomorphism

$$Z \setminus \mathbb{E} \longrightarrow Y \setminus \bigcup_{i=1}^{13} \Gamma_i. \quad (4.5)$$

Note that $(\mathbb{D} \setminus (\mathbb{D} \cap \mathbb{E})) \subset Z \setminus \mathbb{E}$. Since $Z_{6,6}$ is quasismooth away from $\mathbb{D} \cup \mathbb{E}$ the isomorphism (4.5) shows that Y is quasismooth away from \mathbb{D} and $\bigcup_{i=1}^{13} \Gamma_i$ and that \mathbb{D} contains 24 nodes of Y . Furthermore, φ induces a contraction $\mathbb{E} \setminus \text{Sing}(Z) \longrightarrow \{P_w\}$. As before, the unprojection factorizes as the \mathbb{E} -ample small resolution followed by the contraction of \mathbb{E} , and Y is quasismooth along the Γ_i by the Jacobian argument of §3. \square

Unprojection of \mathbb{D} Unprojecting the divisor \mathbb{D} in Y gives a new 3-fold $X \subset \mathbb{P}(1^3, 3^4, 5)$:

$$\begin{array}{c} X \subset \mathbb{P}(1, 1, 1, 3, 3, 3, 3, 5) \\ \uparrow \\ \mathbb{D} \subset Y \subset \mathbb{P}(1, 1, 1, 3, 3, 3, 5) \\ \uparrow \\ \mathbb{E}, \mathbb{D} \subset Z_{6,6} \subset \mathbb{P}(1, 1, 1, 3, 3, 3) \end{array}$$

The unprojection variable, denoted s , has weight 3. Note that $P_s = \frac{1}{3}(1, 1, 1) \in X'$. The proof of Theorem 2.3 now applies to show the following.

Proposition 4.3. *X is a quasismooth Calabi-Yau 3-fold with two quasismooth singularities each of type $\frac{1}{3}(1, 1, 1)$.*

4.2 Construction of Jer_{45}

We describe the construction in the case, stating the intermediate steps; the proofs follow those of §4.1. Fix coordinates x, y, z, t, u, v on $\mathbb{P}(1^3, 3^3)$ and define $\mathbb{D} = \mathbb{P}^2$ by $I_{\mathbb{D}} = (t, u, v)$. Define \mathbb{E} by $I_{\mathbb{E}} = (x, u + p_3, t + q_3)$ where p_3 and q_3 are general cubics in y, z . Since p_3 and q_3 are general, \mathbb{E} and \mathbb{D} are disjoint.

Let $Z' = Z'_{6,6}$ be a general (6, 6) complete intersection that contains $\mathbb{D} \cup \mathbb{E}$. We may write the equations of Z' as

$$\begin{pmatrix} A_5 & B_3 & C_3 \\ D_5 & E_3 & F_3 \end{pmatrix} \begin{pmatrix} x \\ u + p_3 \\ t + q_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

where A, \dots, F are general polynomials of the indicated degrees that lie in $I_{\mathbb{D}}$.

Lemma 4.4. *In the notation above, the singular locus $\text{Sing}(Z')$ of Z' consists of exactly 27 points on \mathbb{D} and 13 points on \mathbb{E} , and all singular points are ordinary nodes.*

Unprojecting $\mathbb{E} \subset Z'$ gives a 3-fold $Y' \subset \mathbb{P}(1^3, 3^3, 5)$ which contains the divisor $\mathbb{D} = (t = u = v = w = 0)$. The unprojection variable has weight 5 and is denoted by w . The equations of Y' can be realised as the five maximal Pfaffians of the antisymmetric 5×5 matrix

$$\begin{pmatrix} x & u + p_3(y, z) & C_3 & F_3 & \\ & t + q_3(y, z) & -B_3 & -E_3 & \\ & & A_5 & D_5 & \\ & & & & w \end{pmatrix}$$

which is in Jer_{45} format for \mathbb{D} .

Proposition 4.5. *Y' is quasismooth outside \mathbb{D} , and has in total 27 isolated singularities on \mathbb{D} , all of which are nodes.*

Unprojecting $\mathbb{D} \subset Y'$ gives a new 3-fold $X' \subset \mathbb{P}(1^3, 3^4, 5)$. The unprojection variable, denoted s , has weight 3, and $P_s = \frac{1}{3}(1, 1, 1) \in X'$.

Proposition 4.6. *X' is a quasismooth Calabi–Yau 3-fold with two quasismooth singularities each of type $\frac{1}{3}(1, 1, 1)$.*

Theorem 1.2(b) now follows: the two Calabi–Yau 3-folds X and X' have the same Hilbert series (they are calculated by identical orbifold Riemann–Roch formulas), but have different Euler characteristics since the number of nodes on $\mathbb{D} \subset Y$ and $\mathbb{D} \subset Y'$ differ; therefore they are members of two different deformation families.

The remaining cases of Theorem 1.2 in codimensions 3 and 4 are proved by the same method. The key point is the starting configurations $\mathbb{D} \cup \mathbb{E} \subset Z \subset w\mathbb{P}^5$.

In case (a), we start with $\mathbb{D} = (u = v = w = 0)$ and $\mathbb{E} = (u - \alpha = v - \beta = w - \gamma = 0)$ inside an otherwise general $Z_{6,6} \subset \mathbb{P}(1^3, 3^3)$ with coordinates x, y, z, u, v, w , where α, β and γ are cubics in x, y, z . If α, β and γ are all chosen generally, then \mathbb{D} and \mathbb{E} are disjoint; if there is a single linear relation among them, then $\mathbb{D} \cap \mathbb{E}$ is 9 points, the intersection of two cubics in \mathbb{P}^2 . After unprojecting $\mathbb{E} \subset Z$ and mounting the equations of the resulting

$Y \subset \mathbb{P}(1^3, 3^4)$ in an antisymmetric matrix, as in (2.4), these two cases correspond to Jer_{45} and Tom_3 , respectively, and the number of nodes on Y differs by 9 between the two cases, being those nodes lying at $\mathbb{D} \cap \mathbb{E}$, which are resolved by the unprojection of \mathbb{E} .

In case (c), in the same ambient $\mathbb{P}(1^3, 3^3)$, with $\mathbb{D} = (z = v = w = 0)$, the three cases arise from the choice of $\mathbb{E} = (z = u = v - \alpha_3(x, y) = 0)$ or $(y = u = v - \alpha = 0)$ with either $\alpha = 0$ or x^3 . Case (d) is similar to (a) but working with $Z_{6,8} \subset \mathbb{P}(1^3, 3^2, 5)$.

5 Final remarks

Of course this is just the start. The web of Calabi–Yaus is vast: compare with Batyrev–Kreuzer [BK10], who navigate the web using conifold transitions starting from toric hypersurfaces and find over 30,000 cases. With our approach, the initial parameters are P_1, P_2 and the basket of singularities \mathcal{B} (which we have taken to be isolated quotient singularities). For example, varying P_1 and P_2 with these same possible baskets $\mathcal{B} = \{n \times \frac{1}{3}(1, 1, 1), m \times \frac{1}{5}(1, 1, 3)\}$ gives other candidates. In cases where $P_1 \geq 2$ and $m, n \geq 1$ these are as follows:

P_1	P_2	n	m	codim	Calabi–Yau 3-fold
2	5	1	3	4	$X_{8^3, 9^3, 10^3} \subset \mathbb{P}(1, 1, 3, 3, 4, 5, 5, 5)$
3	6				See Theorem 1.2
3	7	1	1	3	$X_{5, 6^2, 7, 8} \subset \mathbb{P}(1, 1, 1, 2, 3, 3, 5)$
3	7	2	1	4	$X_{5^2, 6^4, 7, 8^2} \subset \mathbb{P}(1, 1, 1, 2, 3, 3, 3, 5)$
4	10	1	1	3	$X_{4, 6^3, 8} \subset \mathbb{P}(1, 1, 1, 1, 3, 3, 5)$
4	10	1	1	4	$X_{4^2, 6^5, 8^2} \subset \mathbb{P}(1, 1, 1, 1, 3, 3, 3, 5)$
4	11	1	1	4	$X_{4^2, 5^2, 6^3, 7, 8} \subset \mathbb{P}(1, 1, 1, 1, 2, 3, 3, 5)$
5	15	1	1	4	$X_{4^4, 6^4, 8} \subset \mathbb{P}(1, 1, 1, 1, 1, 3, 3, 5)$

In codimension 3 we construct a single deformation family, while again in codimension 4 we construct at least two deformation families in each case.

It is worth noting, finally, that we go to some trouble in §3 to prove existence and quasismoothness even in codimension 3. Our caution is justified by an example of a Calabi–Yau 3-fold $Y \subset \mathbb{P}(1, 1, 2, 5, 8, 13, 19)$ given in [BKZ14, §4.2.2]. Any general such Y fails to be quasismooth at a single point, where it has a node: Y is not (quasi)smoothable within its Pfaffian format. This is striking, since it is easy to check that Y is quasismooth along all nontrivial orbifold strata of the ambient space, and the common experience is that smoothness elsewhere is then routine.

Applying the methods of §4, this nodal codimension 3 Calabi–Yau Y admits an unprojection to a quasismooth Calabi–Yau $X \subset \mathbb{P}(1, 1, 2, 5, 8, 13, 19, 37)$ in codimension 4 which has a single quotient singularity, $\frac{1}{37}(5, 13, 19)$. But, unlike all cases here, this arises only as a single Jerry unprojection, and so provides only one deformation family of such X .

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