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INFINITE GAMES AND σ-POROSITY

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ABSTRACT

We show a new game characterizing various types of σ-porosity for Souslin sets in terms of winning strategies. We use the game to prove and reprove some new and older inscribing theorems for σ-ideals of σ-porous type in locally compact metric spaces.

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1. Introduction

The theory of porous and σ-porous sets forms an important part of real analysis and Banach space theory for more than forty years. It originated in 1967 when E. P. Dolženko used for the first time the nomenclature 'porous set' and proved that some sets of his interest are σ-porous ([3]). Since then the porosity as well as many variants of this notion (see Section 4) have been used many times. The interested reader can consult the survey papers of L. Zajíček ([9, 11]) on porous and σ-porous sets.

Here we are interested in structural properties of σ-ideals of σ-porous type. More precisely, the main question we will consider in this work is the following one.

QUESTION: *Let A be a Souslin subset of a metric space X and $\mathcal{I}$ be a σ-ideal of subsets of X. Suppose that $A \notin \mathcal{I}$. Does there exist a closed set $F \subset A$ which is not in $\mathcal{I}$?*

This question was posed by L. Zajíček in [9] (for a Borel set A) for classical Dolženko σ-porosity. An affirmative answer was given independently by J. Pelant (for any topologically complete metric space X) and M. Zelený (for any compact metric space X). Their results are demonstrated in a joint paper [12] which combines the original idea of J. Pelant (giving an explicit construction of the set F) and techniques developed by M. Zelený. The case of some other types of porosity (including the ordinary one in a locally compact metric space X but also $\langle g \rangle$-porosity in a locally compact metric space X and symmetrical porosity in $\mathbb{R}$) was solved (also affirmatively) by M. Zelený and L. Zajíček in [13]. They offer a less complicated method of construction of F using so called 'porosity-like' relations and their proof uses tools from Descriptive Set Theory. However, the authors admitted that their method cannot be applied to strong porosity and so the Question for strong porosity still remained open (even in a compact metric space X).

Later on, I. Farah and J. Zapletal introduced a new powerful tool to describe σ-porous sets. This was an infinite game which can be used to characterize σ-porous sets in 2^ω considered with respect to a certain metric compatible with the product topology. This game is used to reprove the positive answer to the Question in this particular case ([6, Example 4.20]). The only attempt to answer the Question for strong porosity (and ordinary porosity once again) was made by D. Rojas-Rebolledo, who generalized in [8] the ideas from [6]. He managed to give an affirmative answer to the Question in any zero-dimensional compact metric space X. Further, M. Doležal ([2]) showed a characterization of σ-$\mathbf{P}$-porous sets for any porosity-like relation $\mathbf{P}$ via an infinite game.

Our aim is to generalize results of [6, 8, 13] in two directions. We give an affirmative answer to the Question in (locally) compact spaces which generalizes [6, 8] (although [12] deals even with topologically complete spaces) and also for σ-ideals of σ-porous type which are not included in [13].

Let us look at the content of this work a little closer. The basic notation is presented in Section 2. In Section 3, we prove the main result of the paper (Theorem 3.4). The complete formulation is a little bit technical so let us formulate the result in an informal way.

Let X be a compact metric space and $\mathbf{P}$ be a porosity-like relation on X satisfying some additional conditions. Then every Souslin non-σ-$\mathbf{P}$-porous subset A of X contains a compact non-σ-$\mathbf{P}$-porous subset.

To prove this we proceed as follows. We introduce a modification of the infinite games from [6, 2] for two players played with a set A. We prove that the first player has a winning strategy whenever $A \subset X$ is a Souslin non-σ-$\mathbf{P}$-porous set, while the second player has a winning strategy whenever $A \subset X$ is any σ-$\mathbf{P}$-porous set. Now consider non-σ-$\mathbf{P}$-porous Souslin subset A of X, so that the first player has a winning strategy. Using this winning strategy, we find a compact subset K of A such that the first player still has a winning strategy in the game played with K. This means that the second player does not have a winning strategy and so K is not σ-$\mathbf{P}$-porous.

In Section 4, we apply the last result to specific porosities and obtain an (affirmative) answer to several different variants of the Question. Namely, we deal with ordinary porosity, strong porosity, strong right porosity, and 1-symmetrical porosity. As it is described earlier, the first result has been already known but the method used in our work (based on an infinite game) aspires to be more elegant and easier than the known proofs. The other results are new. Finally, we show that there exists a closed set in $\mathbb{R}$ which is σ-$(1-\varepsilon)$-symmetrically porous for every $\varepsilon \in (0,1)$ but which is not σ-1-symmetrically porous. This answers a question posed by M. J. Evans and P. D. Humke in [5].

2. Preliminaries

Let (X,d) be a metric space. An open ball with center $x \in X$ and radius $r > 0$ is denoted by $B(x,r)$. Since an open ball (considered as a set) does not uniquely determine its center and radius, we will identify every open ball with the pair (center, radius) throughout this work. Therefore two different open balls (i.e., two different pairs (center, radius)) can still determine the same subset of X. Let $A \subset X$ and $x \in X$. Then $d(x,A)$ denotes the distance of x from the set A. If A is empty then we set $d(x,A) = \infty$.

The symbol $\omega^{<\omega}$ stands for the set of all finite sequences of elements of ω including the empty sequence. We denote the concatenation of sequences $s \in \omega^{<\omega}$ and (i), where $i \in \omega$, by $s^\wedge i$. If $t \in \omega^{<\omega} \cup \omega^\omega$, then the symbol $|t|$ denotes the length of t. Given $s \in \omega^{<\omega}$ and $\nu \in \omega^{<\omega} \cup \omega^\omega$, we write $s \preceq \nu$ if ν is an extension of s. If $s,t \in \omega^{<\omega}$, then we say that s,t are compatible if either $s \preceq t$ or $t \preceq s$. If $\nu \in \omega^{<\omega} \cup \omega^\omega$, $n \in \omega$, and $|\nu| \geq n$, then the symbol $\nu|n$ means the finite sequence $(\nu(0), \dots, \nu(n-1))$. We will use the notation $\mathbb{N} = \omega \setminus \{0\}$.

We will prove our results for porosity-like relations satisfying some additional assumptions and then apply it to specific cases. To do this, we need the following definition.

Definition 2.1: Let (X,d) be a metric space and let $\mathbf{P} \subset X \times 2^X$ be a relation between points of X and subsets of X. Then $\mathbf{P}$ is called a *point-set relation on* X. The symbol $\mathbf{P}(x,A)$ where $x \in X$ and $A \subset X$ means that $(x,A) \in \mathbf{P}$. A point-set relation $\mathbf{P}$ on X is called a *porosity-like relation* if the following conditions hold for every $A \subset X$ and $x \in X$:

- (P1) if $B \subset A$ and $\mathbf{P}(x,A)$ then $\mathbf{P}(x,B)$,
- (P2) we have $\mathbf{P}(x,A)$ if and only if there exists $r > 0$ such that $\mathbf{P}\big(x, A \cap B(x,r)\big)$,
- (P3) we have $\mathbf{P}(x,A)$ if and only if $\mathbf{P}(x,\overline{A})$.

If $\mathbf{P}$ is a porosity-like relation on X, $A \subset X$, and $x \in X$, we say that

- A is $\mathbf{P}$-*porous at* x if $\mathbf{P}(x,A)$,
- A is $\mathbf{P}$-*porous* if it is $\mathbf{P}$-porous at each $x \in A$,
- A is σ-$\mathbf{P}$-*porous* if it is a countable union of $\mathbf{P}$-porous sets.

If $\mathbf{P}$ is a porosity-like relation on X and $A \subset X$, we denote

$$\mathrm{por}_{\mathbf{P}}(A) = \{x \in A \colon \mathbf{P}(x,A)\} \quad \text{and} \quad N_{\mathbf{P}}(A) = \{x \in A \colon \neg\mathbf{P}(x,A)\}.$$

We also need to recall the definition of a Foran–Zajíček scheme and one related proposition from [1].

Definition 2.2 (cf. [1, Definition 3.4]): Let X be a complete metric space and $\mathbf{P}$ be a porosity-like relation on X. Let $\mathrm{F} = \{F(t)\colon t \in \omega^{<\omega}\}$ be a system of nonempty subsets of X such that for each $t \in \omega^{<\omega}$ and each $k \in \omega$ we have

(i) $\bigcup_{j\in\omega} F(t^\wedge j)$ is a dense subset of $F(t)$,
(ii) $F(t)$ is $\mathbf{P}$-porous at no point of $F(t^\wedge k)$,
(iii) for any $\nu \in \omega^\omega$ and any sequence $\{G_n\}_{n\in\omega}$ of open sets satisfying:
  (a) $\lim_{n\to\infty} \operatorname{diam}(G_n) = 0$,
  (b) $\overline{G_{n+1}} \subset G_n$ for every $n \in \omega$,
  (c) $F(\nu|n) \cap G_n \neq \emptyset$ for every $n \in \omega$,

  we have
$$\bigcap_{n\in\omega} \big(F(\nu|n) \cap G_n\big) \neq \emptyset.$$

Then we say that F is a $\mathbf{P}$-*Foran–Zajíček scheme in* X.

Proposition 2.3 ([1, Proposition 3.11]): *Let X be a complete metric space and $\mathbf{P}$ be a porosity-like relation on X. Suppose that $N_{\mathbf{P}}(A)$ is a Souslin set whenever $A \subset X$ is Souslin. If $S \subset X$ is a Souslin non-σ-$\mathbf{P}$-porous set, then there exists a $\mathbf{P}$-Foran–Zajíček scheme F in X such that each element of F is a subset of S.*

Remark 2.4: The above definition and proposition are formulated in [1] in a slightly more general form.

3. Main result

3.1. The class $\mathcal{P}_X$. For a metric space X and a porosity-like relation $\mathbf{P}$ on X, we consider the following two conditions:

(α) there is a nondecreasing function $\psi\colon [0,\infty) \to [0,\infty)$ with $\psi(0) = 0$ and $0 < \psi(r) \le r$ for every $r > 0$, such that whenever $x \in \operatorname{por}_{\mathbf{P}}(C)$ for some $x \in X$ and $C \subset X$, then also
$$x \in \operatorname{por}_{\mathbf{P}}\big(\{y \in X\colon d(y,C) \le \psi\big(d(y,x)\big)\}\big),$$
(β) $N_{\mathbf{P}}(A)$ is a Souslin set whenever $A \subset X$ is a Souslin set.

We denote by $\mathcal{P}_X$ the class of all porosity-like relations $\mathbf{P}$ on X which satisfy both conditions (α) and (β).

3.2. THE INFINITE GAME. For the rest of this section, let us fix a complete metric space (X, d), a porosity-like relation $\mathbf{P}$ on X, and sequences $(R_n)_{n=1}^{\infty}$ and $(r_n)_{n=1}^{\infty}$ of positive real numbers such that

$$\frac{R_n}{n} > r_n > R_{n+1}, \qquad n \in \mathbb{N}. \tag{1}$$

For every $A \subset X$, we define a game $H(A)$ for two players named I and II. The game is played as follows:

I	x_1		x_2		x_3	
						$\dots$
II		(S_1^1)		(S_2^1, S_2^2)		(S_3^1, S_3^2, S_3^3)

On the first move, I plays $x_1 \in X$ and II plays an open set $S_1^1 \subset B(x_1, R_1)$. On the second move, I plays $x_2 \in B(x_1, R_1 - r_1)$ and II plays two open sets S_2^1 and S_2^2 such that $S_2^1 \cup S_2^2 \subset B(x_2, R_2)$. On the $(n+1)$th move, $n \in \mathbb{N}$, I plays $x_{n+1} \in B(x_n, R_n - r_n)$ and II replies by playing open sets $S_{n+1}^1, S_{n+1}^2, \dots, S_{n+1}^{n+1}$ such that $\bigcup_{j=1}^{n+1} S_{n+1}^j \subset B(x_{n+1}, R_{n+1})$.

CLAIM 3.1: *We have* $\lim_{n\to\infty} R_n = 0$ *and for every* $n \in \mathbb{N}$ *we have* $\overline{B(x_{n+1}, r_{n+1})} \subset B(x_n, r_n)$*. Consequently, the limit* $\lim_{n\to\infty} x_n$ *exists in* X *and is equal to a unique element of* $\bigcap_{n=1}^{\infty} B(x_n, R_n)$.

Proof. The equality $\lim_{n\to\infty} R_n = 0$ follows from (1). Let $n \in \mathbb{N}$ and $y \in \overline{B(x_{n+1}, R_{n+1})}$. Then we have

$$d(y, x_n) \le d(y, x_{n+1}) + d(x_{n+1}, x_n) < R_{n+1} + (R_n - r_n) \overset{(1)}{<} r_n + (R_n - r_n) = R_n,$$

and so $\overline{B(x_{n+1}, R_{n+1})} \subset B(x_n, R_n)$. ∎

We call the limit point $x = \lim_{n\to\infty} x_n$ the *outcome* of the game. Player II wins the game $H(A)$ if

- either $x \notin A$,
- or there is $m \in \mathbb{N}$ (called a *witness of II's victory* then) such that

$$x \in \operatorname{por}_{\mathbf{P}}\Big(X \setminus \bigcup_{n=m}^{\infty} \big(S_n^m \cap B(x, r_n)\big)\Big).$$

Now suppose that $\mathcal{S} = (\mathcal{S}_n)_{n=1}^{\infty}$ is a sequence of collections of open subsets of X. Then we define a game $H_{\mathcal{S}}(A)$ between player I and player II in the same way as the game $H(A)$, but now player II has to choose the sets S_n^m, $m \le n$, from the collection $\mathcal{S}_n$ in his nth move, whenever it is possible. Note that in applications below player II will always be able to choose the sets S_n^m, $m \le n$, from $\mathcal{S}_n$ in his nth move.

3.3. The main theorem. Let $A \subset X$ be σ-$\mathbf{P}$-porous. We have $A = \bigcup_{n=1}^{\infty} A_n$, where A_n is $\mathbf{P}$-porous for every $n \in \mathbb{N}$. Then player II can win the game $H(A)$ by playing $S_n^m = B(x_n, R_n) \setminus \overline{A_m}$, $m \le n$, $n \in \mathbb{N}$. Indeed, for every $m \in \mathbb{N}$ we then have

$$X \setminus \bigcup_{n=m}^{\infty} \big(S_n^m \cap B(x, r_n)\big) = \overline{A_m} \cup \left(X \setminus \bigcup_{n=m}^{\infty} \big(B(x_n, R_n) \cap B(x, r_n)\big)\right),$$

and so if the outcome x of a run of the game is in A_m for some $m \in \mathbb{N}$, then m is a witness of II's victory (by (P2) and (P3)). In the next lemma, we prove that for certain porosity-like relations $\mathbf{P}$ on X and certain sequences $\mathcal{S}$ of collections of open subsets of X, player II can win even the game $H_{\mathcal{S}}(A)$.

Lemma 3.2: *Let X be a complete metric space and $\mathbf{P}$ be a porosity-like relation on X satisfying condition (α) witnessed by a function ψ. Let $(t_n)_{n=1}^{\infty}$ be a nonincreasing sequence of real numbers such that for every $n \in \mathbb{N}$ we have*

$$0 < t_n \le \frac{r_n - R_{n+1}}{2}. \tag{2}$$

Let $\mathcal{S} = (\mathcal{S}_n)_{n=1}^{\infty}$ be a sequence of collections of open subsets of X such that for every $n \in \mathbb{N}$ and for every set $D \subset X$, there is $S \in \mathcal{S}_n$ such that

$$D \subset S \subset \{y \in X : d(y, D) < \psi(t_{n+1})\}.$$

Suppose that $A \subset X$ is σ-$\mathbf{P}$-porous. Then II has a winning strategy in the game $H_{\mathcal{S}}(A)$.

Proof. Let $A = \bigcup_{m=1}^{\infty} A_m$ where A_m is a $\mathbf{P}$-porous set for every $m \in \mathbb{N}$. Suppose that in his nth move, player I played x_n. Let $m \in \{1, \dots, n\}$. Let us denote

$$D_n = \{y \in B(x_n, R_n - t_{n+1}) : d(y, A_m) > \psi(t_{n+1})\}. \tag{3}$$

By the assumption on the collection $\mathcal{S}_n$, there is $S_n^m \in \mathcal{S}_n$ such that

$$D_n \subset S_n^m \subset \{y \in X : d(y, D_n) < \psi(t_{n+1})\}. \tag{4}$$

We show that

$$S_n^m \subset B(x_n, R_n) \setminus A_m. \tag{5}$$

Suppose that $y \in S_n^m$. By (4) there is $z \in D_n$ such that $d(y,z) < \psi(t_{n+1})$. Then we have

$$d(y,x_n) \le d(y,z) + d(z,x_n) \overset{(3)}{<} \psi(t_{n+1}) + (R_n - t_{n+1}) \le R_n$$

and

$$d(y,A_m) \ge d(z,A_m) - d(y,z) \overset{(3)}{>} \psi(t_{n+1}) - d(y,z) > 0.$$

Thus we have verified (5).

We define the strategy for II by choosing this S_n^m, $m \in \{1,\dots,n\}$, in his nth move.

Let x be the outcome of the game where II followed this strategy. We may suppose that $x \in A_m$ for some $m \in \mathbb{N}$ (otherwise $x \notin A$ and II wins) and we will show that m is a witness for II's victory then. By (5) we have $A_m \subset X \setminus \bigcup_{n=m}^\infty S_n^m$, and so

$$x \in A_m \subset X \setminus \bigcup_{n=m}^{\infty} S_n^m \subset X \setminus \bigcup_{n=m}^{\infty} \bigl(S_n^m \cap B(x,r_n)\bigr).$$

So it remains to prove that $\mathbf{P}\bigl(x, X \setminus \bigcup_{n=m}^\infty (S_n^m \cap B(x,r_n))\bigr)$. Since A_m is $\mathbf{P}$-porous and $x \in A_m$, we have by condition (α) that

$$\mathbf{P}\bigl(x, \bigl\{y \in X \colon d(y,A_m) \le \psi(d(y,x))\bigr\}\bigr).$$

So by the definition of a porosity-like relation (namely by (P1) and (P2)) it suffices to show that

$$\Bigl(X \setminus \bigcup_{n=m}^{\infty} \bigl(S_n^m \cap B(x,r_n)\bigr)\Bigr) \cap B(x,t_m) \subset \bigl\{y \in X \colon d(y,A_m) \le \psi(d(y,x))\bigr\}.$$

So let $y \in B(x,t_m)$ be arbitrary and suppose that $d(y,A_m) > \psi(d(y,x))$. By (1) and (2), it easily follows that $\lim_{n\to\infty} t_n = 0$, and so there is $n_0 \ge m$ such that $t_{n_0+1} \le d(y,x) < t_{n_0}$. By the monotonicity of ψ, we then have

$$d(y,A_m) > \psi(t_{n_0+1}). \tag{6}$$

We also have

$$\begin{aligned} d(y,x_{n_0}) &\le d(y,x) + d(x,x_{n_0+1}) + d(x_{n_0+1},x_{n_0}) < t_{n_0} + R_{n_0+1} + (R_{n_0} - r_{n_0}) \\ &\overset{(2)}{\le} \frac{r_{n_0} - R_{n_0+1}}{2} + R_{n_0+1} + R_{n_0} - r_{n_0} \\ &= \frac{R_{n_0+1} - r_{n_0}}{2} + R_{n_0} \overset{(2)}{\le} R_{n_0} - t_{n_0} \le R_{n_0} - t_{n_0+1}. \end{aligned}$$

By this estimate, (3) and (6), it follows that $y \in D_{n_0}$, and so (by (4)) $y \in S^m_{n_0}$. But we also have $y \in B(x, t_{n_0})$ and so (by (2)) $y \in B(x, r_{n_0})$. By the last two facts, we conclude that $y \in \bigcup_{n=m}^{\infty} \left(S^m_n \cap B(x, r_n)\right)$ and the proof is finished. ∎

Proposition 3.3: *Let X be a complete metric space, $A \subset X$, $\mathbf{P}$ be a porosity-like relation on X. Suppose that there exists a $\mathbf{P}$-Foran–Zajíček scheme $\mathrm{F} = \{F(s) \colon s \in \omega^{<\omega}\}$ such that each element of F is contained in A. Then player I has a winning strategy in the game $H(A)$.*

Proof. In the nth move of the game $H(A)$, let player I choose, in addition to x_n, also an auxiliary object $s_n \in \omega^{<\omega}$ such that

(i) $|s_n| \le n$,
(ii) s_n and s_{n+1} are compatible,
(iii) $x_n \in F(s_n)$.

In the sequel, we always denote $p_n = |s_n|$, $n \in \mathbb{N}$. The strategy for player I is the following. He starts by $s_1 = \emptyset$ and by choosing arbitrary $x_1 \in F(\emptyset)$. Now suppose that I is in his $(n+1)$th move, and so x_j, s_j, and S^m_j for $m \le j \le n$ are already known. For $m \le n$ denote $G^m_n = \bigcup_{i=m}^{n} S^m_i$. Player I distinguishes the following cases. (Observe that by (i) and (1) we have $R_n - pr_n > 0$ for every $p \le n$.)

(A) If there is $1 \le p < p_n$ such that $B(x_n, R_n) \not\subset G^p_n$ and
$$G^p_n \cap B(x_n, R_n - pr_n) \cap F(s_n|p) \neq \emptyset,$$
pick the least such p, let x_{n+1} be an arbitrary element of
$$G^p_n \cap B(x_n, R_n - pr_n) \cap F(s_n|p),$$
and set $s_{n+1} = s_n|p$.

(B) If there is no p as in (A) and if $x_n \in G^{p_n}_n$ and $B(x_n, R_n) \not\subset G^{p_n}_n$, set $x_{n+1} = x_n$ and $s_{n+1} = s_n$.

(C) Suppose that neither (A) nor (B) occur. The set $\bigcup_{i\in\omega} F(s_n{}^\wedge i)$ is dense in $F(s_n)$ and $F(s_n) \cap B(x_n, R_n - p_n r_n) \neq \emptyset$ by the induction hypothesis (namely by (iii)). Thus there exists $i \in \omega$ such that
$$F(s_n{}^\wedge i) \cap B(x_n, R_n - p_n r_n) \neq \emptyset.$$
Set $s_{n+1} = s_n{}^\wedge i$ and choose any $x_{n+1} \in F(s_{n+1}) \cap B(x_n, R_n - p_n r_n)$.

It follows from the definition of the strategy that conditions (i)–(iii) are satisfied.

An intuitive description of I's strategy is the following. If, during his first n moves, II does not 'try very hard to win' by a witness smaller than p_n, i.e., if for every $1 \le p < p_n$, the set G_n^p is too small to intersect $F(s_n|p)$ 'close' to x_n, then player I plays x_{n+1} 'close' to x_n. However, if G_n^p intersects $F(s_n|p)$ 'close' to x_n for some $1 \le p < p_n$ and if it is still possible for II to win by such p (i.e., if $B(x_n, R_n) \not\subset G_n^p$), then during a finite number of turns, player I tries to ruin any possible II's chances of having p as a witness (by ensuring $B(x_j, R_j) \subset G_j^p$ for some $j > n$) for the least such p. But during these finitely many turns of the game, II may try to win by an even smaller witness p'. Then I immediately changes his plan and focuses on this smaller p'.

We show that this strategy is winning for player I. Suppose that we have a run of the game where player I followed the above strategy. Let s_n, x_n, S_n^m be the corresponding objects constructed during the run.

Let us fix $p \in \mathbb{N}$ for a while. Whenever I applies (A) to choose s_{n+1} of the length p (which means that II threatens to win by some witness smaller than p_n and p is the least such witness), he decides to continue by a finite (possibly empty) chain of applications of (B) until, for some $j > n$, $B(x_j, R_j) \subset G_j^p$. It is possible that he will have to apply (A) for some $1 \le p' < p$ before he fulfills this intention. But then again, he changes his plan and decides to continue by a finite chain of applications of (B) until, for some $j > n$, $B(x_j, R_j) \subset G_j^{p'}$, and so on. It follows that whenever I applies (A) for some p, he always achieves, after a finite number of turns (during which he only applies (A) or (B)), the validity of the inclusion $B(x_j, R_j) \subset G_j^{p'}$ for some $j > n$ and for some $1 \le p' \le p$, for which the condition (A) was also used during the game (at least once). Such a p' cannot be used for applying (A) anymore. It follows that for every $p \in \mathbb{N}$, the condition (A) can be applied only finitely many times during the game (at most p times, to be precise). By analogous considerations, for every $p \in \mathbb{N}$, it can happen also only finitely many times that II applies condition (B) to choose s_{n+1} of the length p. And since applying (C) gives $p_{n+1} = p_n + 1$, we easily conclude that

(iv) $\lim_{n\to\infty} p_n = \infty$.

Using (ii) and (iv), there exists a unique $\nu \in \omega^\omega$ such that

$$\forall n \in \omega\ \exists k_0 \in \mathbb{N}\ \forall k \ge k_0 \colon s_k|n = \nu|n. \tag{7}$$

Let x be the outcome of the run. By (1) we have $\lim_{n\to\infty} \operatorname{diam} B(x_n, R_n) = 0$. By Claim 3.1 we have $\overline{B(x_{n+1}, R_{n+1})} \subset B(x_n, R_n)$ for every $n \in \mathbb{N}$. Further by (iii) and (7), for each $n \in \mathbb{N}$ there is $k \geq n$ large enough such that $x_k \in F(s_k) \subset F(s_k|n) = F(\nu|n)$, and so $F(\nu|n) \cap B(x_n, R_n) \neq \emptyset$. By the definition of a **P**-Foran–Zajíček scheme, we conclude that $\bigcap_{n\in\mathbb{N}}(F(\nu|n) \cap B(x_n, R_n)) \neq \emptyset$. But the only element of $\bigcap_{n\in\mathbb{N}}^{\infty} B(x_n, R_n)$ is x, and so $x \in \bigcap_{n\in\mathbb{N}} F(\nu|n)$. Using the fact that each element of F is contained in A we get $x \in A$.

Let $m \in \mathbb{N}$ be arbitrary. We show that m is not a witness of II's victory. If $x \in \bigcup_{n=m}^{\infty} S_n^m$, then

$$x \notin \operatorname{por}_{\mathbf{P}}\Big(X \setminus \bigcup_{n=m}^{\infty} \big(S_n^m \cap B(x, r_n)\big)\Big).$$

Suppose that $x \notin \bigcup_{n=m}^{\infty} S_n^m$. There is $q \geq m$ such that for every $n \geq q$, we have $p_n \geq m + 2$ and $s_n|m = \nu|m$. So for every $n \geq q$, condition (A) was not applied with $p = m$ which means that either $B(x_n, R_n) \subset G_n^m$ or $G_n^m \cap B(x_n, R_n - mr_n) \cap F(s_n|m) = \emptyset$. But the former case does not hold since $B(x_n, R_n)$ contains x and $x \notin \bigcup_{n=m}^{\infty} S_n^m = \bigcup_{n=m}^{\infty} G_n^m$. So for every $n \geq q$ we have

$$G_n^m \cap B(x_n, R_n - mr_n) \cap F(s_n|m) = \emptyset. \tag{8}$$

Since $p_n \geq m + 2$ for $n \geq q$, it also immediately follows from the described strategy for I that

$$x_{n+1} \in B(x_n, R_n - (m+2)r_n), \quad n \geq q. \tag{9}$$

We then have

$$\begin{aligned} d(x, x_q) &\leq d(x, x_{q+1}) + d(x_{q+1}, x_q) \overset{(9)}{<} R_{q+1} + (R_q - (m+2)r_q) \\ &\overset{(1)}{<} r_q + (R_q - (m+2)r_q) = R_q - (m+1)r_q < R_q - mr_q, \end{aligned}$$

and so there is $r > 0$ such that

$$B(x, r) \subset B(x_q, R_q - mr_q). \tag{10}$$

Now we show that

$$F(s_q|m) \cap B(x, r) \cap \bigcup_{n=m}^{\infty} (B(x, r_n) \cap S_n^m) = \emptyset. \tag{11}$$

To this end, let $n \geq m$ be arbitrary and we will show that

$$F(s_q|m) \cap B(x, r) \cap B(x, r_n) \cap S_n^m = \emptyset.$$

Suppose first that $n \le q$. Then we have

$$\begin{aligned} F(s_q|m) \cap B(x,r) \cap B(x,r_n) \cap S_n^m &\subset F(s_q|m) \cap B(x,r) \cap S_n^m \\ &\overset{(10)}{\subset} F(s_q|m) \cap B(x_q, R_q - mr_q) \cap G_q^m \overset{(8)}{=} \emptyset. \end{aligned}$$

Now suppose that $n > q$. Then for any $y \in B(x,r_n)$, we have

$$\begin{aligned} d(y,x_n) &\le d(y,x) + d(x,x_{n+1}) + d(x_{n+1},x_n) \overset{(9)}{<} r_n + R_{n+1} + (R_n - (m+2)r_n) \\ &\overset{(1)}{<} r_n + r_n + (R_n - (m+2)r_n) = R_n - mr_n, \end{aligned}$$

and so $B(x,r_n) \subset B(x_n, R_n - mr_n)$. So we have

$$\begin{aligned} F(s_q|m) \cap B(x,r) \cap B(x,r_n) \cap S_n^m &\subset F(s_q|m) \cap B(x,r_n) \cap S_n^m \\ &\subset F(s_q|m) \cap B(x_n, R_n - mr_n) \cap G_n^m \\ &= F(s_n|m) \cap B(x_n, R_n - mr_n) \cap G_n^m \overset{(8)}{=} \emptyset. \end{aligned}$$

Thus we have verified (11).

It follows that

$$F(s_q|m) \cap B(x,r) \subset \Big(X \setminus \bigcup_{n=m}^{\infty} \big(S_n^m \cap B(x,r_n)\big)\Big) \cap B(x,r). \tag{12}$$

By the definition of a **P**-Foran–Zajíček scheme, the set $F(s_q|m) = F(\nu|m)$ is **P**-porous at no point of $F(\nu|(m+1))$, so in particular it is not **P**-porous at x. By this fact, (P1), (P2), and (12), we easily conclude that

$$x \notin \operatorname{por}_{\mathbf{P}}\Big(X \setminus \bigcup_{n=m}^{\infty} \big(S_n^m \cap B(x,r_n)\big)\Big).$$

This finishes the proof. ∎

Theorem 3.4: *Let X be a compact metric space, $\mathbf{P}$ be a porosity-like relation from the class $\mathcal{P}_X$, and $A \subset X$ be a non-σ-$\mathbf{P}$-porous Souslin set. Then there exists a compact set $K \subset A$ which is not σ-$\mathbf{P}$-porous.*

Proof. Let ψ be a function witnessing that $\mathbf{P}$ satisfies condition (α) and let $(t_n)_{n=1}^{\infty}$ be a nonincreasing sequence of real numbers such that $0 < t_n \le \frac{1}{2}(r_n - R_{n+1})$, $n \in \mathbb{N}$. For every $n \in \mathbb{N}$, let D_n be a finite subset of X such that $X = \bigcup_{y \in D_n} B\big(y, \frac{1}{2}\psi(t_{n+1})\big)$ and let $\mathcal{S}_n$ be the collection of all subsets of X of the form $\bigcup_{y \in E} B\big(y, \frac{1}{2}\psi(t_{n+1})\big)$ where $E \subset D_n$. It is easy to verify that $\mathcal{S}_n$,

$n \in \mathbb{N}$, are finite collections of open subsets of X such that for every $D \subset X$ and $n \in \mathbb{N}$, there is $S \in \mathcal{S}_n$ such that

$$D \subset S \subset \{y \in X : d(y, D) < \psi(t_{n+1})\}.$$

Let us denote $\mathcal{S} = (\mathcal{S}_n)_{n=1}^{\infty}$. By Proposition 2.3, there exists a **P**-Foran–Zajíček scheme $\mathrm{F} = \{F(s) : s \in \omega^{<\omega}\}$ in X such that each element of F is a subset of A. By Proposition 3.3, player I has a winning strategy ρ in the game $H(A)$. Thus I has a winning strategy also in the game $H_{\mathcal{S}}(A)$. Let $K \subset X$ be the set of all possible outcomes of the game $H_{\mathcal{S}}(A)$ where player I follows the strategy ρ. Since the strategy ρ is winning we have $K \subset A$.

Let $\mathcal{T}$ be the tree of all legal positions of the game $H_{\mathcal{S}}(A)$ where I follows the strategy ρ. Let us equip the body $[\mathcal{T}]$ of the tree $\mathcal{T}$ by the product topology. Then $[\mathcal{T}]$ is a compact topological space since $\mathcal{T}$ is a finitely branching tree (because the collections $\mathcal{S}_n$, $n \in \mathbb{N}$, are finite). Moreover, the mapping $u : [\mathcal{T}] \to X$ which assigns to every run of the game $H_{\mathcal{S}}(A)$ its outcome is obviously continuous. It follows that $K = u\big([\mathcal{T}]\big)$ is a compact set.

The strategy ρ is a winning strategy for player I also in the game $H_{\mathcal{S}}(K)$ by the definition of the game. Thus there is no winning strategy for player II in the game $H_{\mathcal{S}}(K)$. By Lemma 3.2, the compact set $K \subset A$ is not σ-**P**-porous. ∎

4. Applications to specific porosities.

Using Theorem 3.4 we prove inscribing theorems for σ-porosity, σ-strong porosity, σ-strong right porosity, and σ-1-symmetrical porosity. It will be clear that Theorem 3.4 can be applied to many other types of porosity. First of all we recall definitions of the mentioned porosities.

Let (X, d) be a metric space. Let $M \subset X$, $x \in X$, and $R > 0$. Then we define

$$\begin{aligned}\theta(x, R, M) = \sup\{r > 0 &: \text{there exists an open ball } B(z, r) \\ &\text{such that } d(x, z) < R \text{ and } B(z, r) \cap M = \emptyset\},\end{aligned}$$

$$p(x, M) = \limsup_{R \to 0+} \frac{\theta(x, R, M)}{R}.$$

We say that $M \subset X$ is

- *porous at* $x \in X$ if $p(x, M) > 0$,
- *strongly porous at* $x \in X$ if $p(x, M) \geq 1$.

Let $M \subset \mathbb{R}$, $x \in \mathbb{R}$, and $R > 0$. Then we define

$$\begin{aligned} \theta^+(x,R,M) &= \sup\{r>0\colon \text{ there exists an open ball } B(z,r), z > x, \\ &\qquad \text{such that } |x-z|<R, \text{ and } B(z,r)\cap M = \emptyset\}, \\ p^+(x,M) &= \limsup_{R\to 0+} \frac{\theta^+(x,R,M)}{R}, \\ \theta^s(x,R,M) &= \sup\{r>0\colon \text{ there exists an open ball } B(z,r), \\ &\qquad \text{such that } |x-z|<R, \text{ and } \big(B(z,r)\cup B(2x-z,r)\big)\cap M = \emptyset\}, \\ p^s(x,M) &= \limsup_{R\to 0+} \frac{\theta^s(x,R,M)}{R}. \end{aligned}$$

Let $c > 0$. We say that $M \subset \mathbb{R}$ is

- *right porous at* $x \in \mathbb{R}$ if $p^+(x,M) > 0$,
- *strongly right porous at* $x \in \mathbb{R}$ if $p^+(x,M) \geq 1$,
- *c-symmetrically porous at* $x \in \mathbb{R}$ if $p^s(x,M) \geq c$.

Lemma 4.1: *Let (X,d) be a metric space and $C \subset X$. Let $\psi\colon [0,\infty) \to [0,\infty)$ be a nondecreasing function such that $\psi(0) = 0$ and $0 < \psi(R) \leq R$ for every $R > 0$, and such that $\lim_{R\to 0+} \frac{\psi(R)}{R} = 0$. Let us denote $\tilde{C} = \{y \in X\colon d(y,C) \leq \psi(d(y,x))\}$. Then for every $x \in C$, we have $p(x,\tilde{C}) = p(x,C)$.*

If, moreover, (X,d) is the real line equipped with euclidean metric, then for every $x \in C$, we also have $p^+(x,\tilde{C}) = p^+(x,C)$ and $p^s(x,\tilde{C}) = p^s(x,C)$.

Proof. Let $x \in C$ be arbitrary. Since $C \subset \tilde{C}$, we clearly have $p(x,\tilde{C}) \leq p(x,C)$. So suppose that $p(x,C) > 0$ and let $\varepsilon \in (0, p(x,C))$ and $R_0 > 0$ be arbitrary. There exists $R_1 > 0$ such that

$$\frac{\psi(2R)}{R} < \frac{\varepsilon}{2}, \qquad R \in (0,R_1). \tag{13}$$

There also exists $R \in \big(0, \min(R_0,R_1)\big)$ such that $\frac{\theta(x,R,C)}{R} > p(x,C) - \frac{\varepsilon}{2}$. By the last inequality, there is an open ball $B(z,r)$ such that

$$d(x,z) < R, \qquad B(z,r)\cap C = \emptyset, \qquad \text{and} \qquad \frac{r}{R} > p(x,C) - \frac{\varepsilon}{2}. \tag{14}$$

Note that (14) together with the fact that $x \in C$ imply that $r < R$. Note also that (14) together with the inequality $\varepsilon < p(x,C)$ imply that $r - \frac{1}{2}R\varepsilon > 0$. Let $y \in B(z, r - \frac{1}{2}R\varepsilon)$ be arbitrary. Then

$$d(y,x) \leq d(y,z) + d(z,x) < r + R < 2R \tag{15}$$

and so we have

$$d(y,C) \geq d\big(y, X \setminus B(z,r)\big) > \tfrac{1}{2}R\varepsilon \overset{(13)}{>} \psi(2R) \overset{(15)}{\geq} \psi(d(y,x)).$$

It follows that $B\big(z, r - \frac{1}{2}R\varepsilon\big) \cap \tilde{C} = \emptyset$, and so

$$\theta(x,R,\tilde{C}) \geq \frac{r - \frac{1}{2}R\varepsilon}{R} = \frac{r}{R} - \frac{\varepsilon}{2} \overset{(14)}{>} p(x,C) - \varepsilon.$$

Since $R_0 > 0$ was chosen arbitrarily, we have $p(x,\tilde{C}) \geq p(x,C) - \varepsilon$. And since $\varepsilon \in (0, p(x,C))$ was also chosen arbitrarily, we conclude that $p(x,\tilde{C}) \geq p(x,C)$.

In the case of $X = \mathbb{R}$, the equalities $p^+(x,\tilde{C}) = p^+(x,C)$ and $p^s(x,\tilde{C}) = p^s(x,C)$ can be shown analogously. ∎

Theorem 4.2 (cf. [12, Theorem 3.1]): *Let (X,d) be a locally compact metric space. Let $A \subset X$ be a non-σ-porous Souslin set. Then there exists a non-σ-porous compact set $K \subset A$.*

Proof. Suppose first that the space (X,d) is compact. Let $\mathbf{P}$ be the porosity-like relation on X given by

$$\mathbf{P}(x,B) \Leftrightarrow B \text{ is porous at } x$$

where $x \in X$ and $B \subset X$.

Let $\psi\colon [0,\infty) \to [0,\infty)$ be a nondecreasing function such that $\psi(0) = 0$ and $0 < \psi(R) \leq R$ for every $R > 0$, and such that $\lim_{R\to 0+} \frac{\psi(R)}{R} = 0$. By Lemma 4.1, such ψ witnesses that $\mathbf{P}$ satisfies the condition (α).

Let $A \subset X$ be a Souslin set. Then $N_{\mathbf{P}}(A)$ consists of all $x \in A$ such that for every (rational) $\varepsilon > 0$ there is (rational) $R_0 > 0$ such that for every (rational) $R \in (0, R_0)$, we have $\frac{\theta(x,R,A)}{R} \leq \varepsilon$. The last inequality can be further reformulated such that for every $z \in X$ (or every z from some fixed countable dense subset of X) with $d(x,z) < R$ and every (rational) $r > R\varepsilon$, there exists $y \in A$ such that $d(y,z) < r$. Now it is easy to see that these conditions describe a Souslin set and so $\mathbf{P}$ satisfies the condition (β).

We verified that $\mathbf{P}$ is from the class $\mathcal{P}_X$ and so the statement follows from Theorem 3.4 for any compact metric space (X,d).

Now suppose that (X,d) is an arbitrary locally compact metric space. By [10, Lemma 3], there exists $x \in X$ such that $A \cap B(x,r)$ is a non-σ-porous subset of X for every $r > 0$. Let us take $r_0 > 0$ such that $\overline{B(x,r_0)}$ is compact and denote $A' = A \cap B(x,r_0)$. Since porosity is a local property, every $M \subset B(x,r_0)$ is σ-porous in X if and only if M is σ-porous in the compact metric space $\overline{B(x,r_0)}$.

Therefore, A' is non-σ-porous in $\overline{B(x,r_0)}$. Due to the previous part of the proof, there exists a non-σ-porous (in $\overline{B(x,r_0)}$ and therefore also in X) compact set $K \subset A' \subset A$. ∎

Theorem 4.3: *Let (X,d) be a locally compact metric space. Let $A \subset X$ be a non-σ-strongly porous Souslin set. Then there exists a non-σ-strongly porous compact set $K \subset A$.*

Proof. Similarly as in the proof of the previous theorem, we may assume that X is compact. The porosity-like relation on X given by

$$\mathbf{P}(x,B) \Leftrightarrow B \text{ is strongly porous at } x$$

is from the class $\mathcal{P}_X$ by similar arguments as in the proof of the previous theorem as well. So the statement again follows from Theorem 3.4. ∎

Theorem 4.4: *Let $A \subset \mathbb{R}$ be a non-σ-strongly right porous (non-σ-1-symmetrically porous respectively) Souslin set. Then there exists a non-σ-strongly right porous (non-σ-1-symmetrically porous respectively) compact set $K \subset A$.*

Proof. We show the proof for the strong right porosity, 1-symmetrical porosity being analogous.

Similarly as above, we may assume that $A \subseteq (0,1)$. Let us consider the porosity-like relation on $[0,1]$ given by

$$\mathbf{P}(x,B) \Leftrightarrow B \text{ (considered as a subset of } \mathbb{R}) \text{ is strongly right porous at } x,$$

so that for $B \subseteq [0,1]$, we have

$$B \text{ is } \mathbf{P}\text{-porous in } [0,1] \Leftrightarrow B \text{ is strongly right porous in } \mathbb{R}.$$

Using similar arguments as above it is not difficult to show that $\mathbf{P}$ is from the class $\mathcal{P}_{[0,1]}$. So the statement follows from Theorem 3.4. ∎

Remark 4.5: Theorem 4.4 has been already used in [7].

Finally, we apply Theorem 4.4 to answer a question posed by Evans and Humke in [5]. This is the following question.

Question: *Does there exist an F_σ set in $[0,1]$ which is σ-$(1-\varepsilon)$-symmetrically porous for every $0<\varepsilon<1$ but which is not σ-1-symmetrically porous?*

We answer this question affirmatively by proving the next theorem.

THEOREM 4.6: *There exists a closed set $K \subset [0,1]$ which is σ-$(1-\varepsilon)$-symmetrically porous for every $0 < \varepsilon < 1$ but which is not σ-1-symmetrically porous.*

Proof. There exists a Borel set $A \subset (0,1)$ which is σ-$(1-\varepsilon)$-symmetrically porous for every $0 < \varepsilon < 1$ but which is not σ-1-symmetrically porous ([4]). By Theorem 4.4, there exists a compact non-σ-1-symmetrically porous set $K \subset A$. Since K is a subset of A, it is still σ-$(1-\varepsilon)$-symmetrically porous for every $0 < \varepsilon < 1$. ∎

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