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Abstract. Given an expanding map of the interval we can associate an absolutely continuous measure. Given an Anosov transformation on a two torus we can associate a Sinai–Ruelle–Bowen measure. In this note we consider first and second derivatives of the change in the average of a reference function. We present an explicit convergent series for these derivatives. In particular, this gives a relatively simple method of computation.

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1. Introduction

Linear response can often be used to describe how physically relevant quantities respond to external stimuli. We recall the following informal description of Ruelle [14]: *"Linear response theory deals with the way a physical system reacts to a small change in the applied forces or the control parameters. The system starts in an equilibrium or a steady state ρ, and is subjected to a small perturbation x, which may depend on time. In first approximation, the change $\Delta\rho$ of ρ is assumed to be linear in the perturbation x"*. A more mathematical formulation is the following. Let $f : M \to M$ be a smooth discrete time dynamical system (on a compact Riemann manifold M) admitting a unique SRB measure μ. Assume that $\lambda \mapsto f_\lambda$ is a smooth path through $f = f_0$ and that there exists a large enough set Λ, containing 0 as an accumulation point, so that f_λ admits an SRB measure μ_λ for each $\lambda \in \Lambda$. One asks how smooth the map $\lambda \mapsto \mu_\lambda$ at 0, in particular whether it is differentiable (see [2]).

Ruelle presented explicit formulae for the first derivative (using the susceptibility function) in [15]. For example, if we associate a vector field X so that $f_\lambda = f_0 + \lambda X \circ f + o(\lambda)$ then one can hope to write

$$\frac{\partial}{\partial\lambda}\left(\int g d\mu_\lambda\right)\Big|_{\lambda=0} = \sum_{n=0}^{\infty} \int \langle X, \mathrm{grad}(g \circ f_0^n)\rangle d\mu. \qquad (1.1) \quad \{\texttt{susf:eq}\}$$

However, to make sense of this expression one needs, for example, that the right hand side of (1.1) converges in a suitable sense.

An approach suggested by Ruelle, was to consider the susceptibility function

$$\Psi(z) = \sum_{n=0}^{\infty} z^n \int \langle X, \mathrm{grad}(g \circ f_0^n)\rangle d\mu, \qquad (1.2)$$

which reduces to (1.1) when $z = 1$.

In an incomplete manuscript of Sondergaard and Cvitanović, the authors propose the idea of studying this problem using a different approach involving a complex function defined using periodic orbits [8]. We want to develop further these ideas in the context of C^ω expanding maps and Anosov diffeomorphisms, a key point being the use of a somewhat different complex function. (A related problem was posed by Baladi in §5 in her survey [2], where she asked about the relationship of periodic points and linear response.) In particular, we present an alternative convergent series for the Right Hand Side of (1.1), which has the merit of being easily computed.

The problem of computing the first derivative of the integral was studied by Bahsoun and Galatolo [10]. Their proof takes a functional analytic approach by rewriting the Fréchet derivative of the measure using transfer operators. Using approximation of the transfer operator by finite (although very large) rank operators, they obtain estimates on the derivative to any prescribed level of accuracy. This method has the distinct advantage that it applies to expanding maps of finite differentiability (for example, C^3). By contrast, we require the stronger hypothesis that the map is real analytic, but then our proof involves deriving an alternative expression for the derivative in (1.1) as a series each of whose terms are defined in terms of weights on periodic points for the map. We require the strong analyticity hypothesis in order to ensure that this series converges and, moreover, sufficiently rapidly to give very good numerical approximations. In particular, this approach provides both good and effective estimates on the error, see Subsection 4.2. The advantage of this approach is numerical stability due to simplicity of the calculations involved. In particular, the algorithm presented in Section 4.1 can be realised on any pocket-sized calculator.

We consider the problem in the settings of expanding maps of the circle and Anosov diffeomorphisms of the torus. Our results are as follows.

1.1. Expanding maps of the circle

The simplest possible setting in which to study these problems is expanding maps of the circle.

Let us take a family of C^ω expanding maps $T_\lambda : \mathbb{T}^1 \to \mathbb{T}^1$, $\lambda \in (-\varepsilon, \varepsilon)$, on the unit circle $K = \mathbb{R}/\mathbb{Z}$. We denote by

$$d\mu_\lambda = \rho_\lambda(x)dx$$

the associated absolutely continuous invariant measure, with density $\rho \in C^\omega(\mathbb{T}^1)$ [7]. Given a C^ω function $g : \mathbb{T}^1 \to \mathbb{C}$ we can consider the average $\int g d\mu_\lambda$, which has an analytic dependence on $\lambda \in (-\varepsilon, \varepsilon)$, and find an expression in terms of periodic orbits for the derivatives.

$$A = \frac{\partial}{\partial \lambda}\left(\int g d\mu_\lambda\right)\Big|_{\lambda=0} \text{ and } B = \frac{\partial^2}{\partial \lambda^2}\left(\int g d\mu_\lambda\right)\Big|_{\lambda=0} \quad (1.3) \quad \{\texttt{ABdef:eq}\}$$

In particular, this would lead to a series expansion

$$\int g d\mu_\lambda = \int g d\mu_0 + \lambda A + \lambda^2 \frac{B}{2} + o(\lambda^2). \qquad (1.4)$$ {avdif:eq}

The following theorem is our main result on expanding maps and gives both the desired expression for the coefficients in (1.3) in terms of periodic points and convergence estimates which will be useful later for computations.

{main1:thm}

Theorem 1.1. *Let T_λ be a family of C^ω expanding maps of the circle, let μ_λ be the absolutely continuous invariant probability measure and let g be a C^ω test function. Then*

(i) The first and the second coefficients A and B may be written as explicit convergent series $A = \sum_{n=0}^{\infty} A_n$ and $B = \sum_{n=0}^{\infty} B_n$;

(ii) The kth term of the series is defined in terms of periodic points of period $\leq k$;

(iii) The partial sums $S_n(A) = \sum_{k=1}^{n} A_k$ and $S_n(B) = \sum_{k=1}^{n} B_k$ of the first n terms in each series converge faster than any exponential to A and B, respectively, i.e., $A_n \leq \alpha \exp(-\beta n^2)$ and $|B_n| \leq \alpha \exp(-\beta n^2)$ for some $\alpha, \beta > 0$.

For clarity of exposition, we have formulated the result for expanding maps of the circle, but an extension to expanding maps in higher dimensions also holds.

Contributions to the study of this and related problems have been made by Ruelle [15] and [16], Baladi and Smania [3], [4], [5]; Dolgopyat [9], Liverani and Butterley [6].

The connection with periodic points is not unfamiliar, we recall

Lemma 1.2. *The average of a test function is related to periodic orbits by the formula*

$$\int g d\mu_\lambda = \lim_{n \to +\infty} \frac{\sum_{T_\lambda^n x_\lambda = x_\lambda} g(x) \cdot |T'_\lambda(x_\lambda)|^{-1}}{\sum_{T_\lambda^n x_\lambda = x_\lambda} |T'_\lambda(x_\lambda)|^{-1}}.$$

The rate of convergence in this lemma is typically only exponential, which is the same as the growth rate of the number of the periodic points needed to compute the terms.

However, our goal is to give an explicit convergent power series in λ, where the coefficients can be efficiently computed in terms of periodic points, as in Theorem 1.1.

1.2. Anosov diffeomorphisms

We recall that a diffeomorphism $f : M \to M$ on a compact manifold is Anosov if

(i) there exists a continuous (in the manifold) Df-invariant splitting $TM = E^s \oplus E^u$ and constants $C > 0$ and $0 < \lambda < 1$ such that $\|Df^n|_{E^s}\| \leq C\lambda^n$ and $\|Df^{-n}|_{E^u}\| \leq C\lambda^n$ for $n \geq 0$.

(ii) f is transitive, i.e. there exists a dense orbit.

Let us consider a family of C^ω Anosov diffeomorphisms $f_\lambda : M \to M$, $\lambda \in (-\varepsilon, \varepsilon)$. Let μ_λ be the associated Sinai–Ruelle–Bowen measures.

Given a C^ω function $g : M \to \mathbb{R}$ we can consider the average $\int g d\mu_\lambda$, which has an analytic dependence on $\lambda \in (-\varepsilon, \varepsilon)$ and find an expression for its derivatives in terms of periodic orbits.

$$A = \frac{\partial}{\partial\lambda}\left(\int g d\mu_\lambda\right)\Big|_{\lambda=0} \text{ and } B = \frac{\partial^2}{\partial\lambda^2}\left(\int g d\mu_\lambda\right)\Big|_{\lambda=0}$$

In particular, this would lead to a series expansion, similar to (1.4)

$$\int g d\mu_\lambda = \int g d\mu_0 + \lambda A + \lambda^2 \frac{B}{2} + o(\lambda^2). \tag{1.5}$$ {ava:dif}

The following theorem is our main result on Anosov diffeomorphisms and gives both the desired expression for the coefficients in (1.5) in terms of periodic points and convergence estimates which will be useful later for computations. Let us restrict to the case of Anosov diffeomorphisms of the two torus $\mathbb{T}^2$.

{main2:thm}

Theorem 1.3. *Let T_λ be a C^ω family of Anosov diffeomorphisms of $\mathbb{T}^2$, let μ_λ be the SRB measures and let g be a C^ω test function. Then*

(i) The first and the second coefficients A and B may be written as explicit convergent series $A = \sum_{n=0}^{\infty} A_n$ and $B = \sum_{n=0}^{\infty} B_n$;

(ii) The k'th term of the series is defined in terms of periodic points of period $\leq k$;

(iii) The partial sums $S_n(A) \stackrel{\text{def}}{=} \sum_{k=1}^{n} A_k$ and $S_n(B) \stackrel{\text{def}}{=} \sum_{k=1}^{n} B_k$ of the first k terms in each series converge faster than any exponential to A and B, respectively. In particular, there exist constants $\alpha, \beta > 0$, which can be explicitely estimated, such that $|A_n| \leq \alpha \exp(-\beta n^2)$ and $|B_n| \leq \alpha \exp(-\beta n^2)$.

For clarity of exposition, we have formulated the result for Anosov diffeomorphisms of $\mathbb{T}^2$, but an extension to higher dimensional Anosov diffeomorphisms also holds.

We present the explicit formulae for the $S_n(A)$ and $S_n(B)$ in a later section.

The connection with periodic points is again well known:

Lemma 1.4. *The average of a test function is related to periodic orbits by the formula*

$$\int g d\mu_\lambda = \lim_{n\to+\infty} \frac{\sum_{T_\lambda^n x_\lambda = x_\lambda} g(x) \cdot |\det(DT_\lambda|_{E^u})(x_\lambda)|^{-1}}{\sum_{T_\lambda^n x_\lambda = x_\lambda} |\det(DT_\lambda|_{E^u})(x_\lambda)|^{-1}}$$

The rate of convergence in this lemma is typically only exponential, which is the same as the growth of number of the periodic points needed to compute the terms. Thus Theorem 1.3 provides a faster and more efficient means of approximation.

1.3. Examples

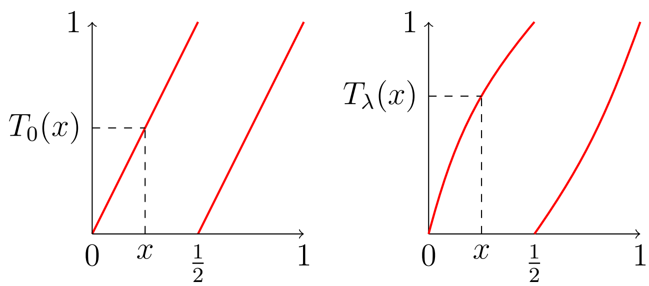
{numeric:se

To illustrate the efficiency of the approach to numerical computation we can consider several simple examples. We say that an estimate A_N is accurate to k-decimal places we mean that the sequence of approximations A_N have the property that consecutive approximations A_N and A_{N+1} agree to k-places. Moreover, our approach does give explcit bounds, but they are not as sharp as the numerics suggest would be possible.

1.3.1. Expanding maps of the circle

Example 1.5. *Let $T_0 : [0,1] \to [0,1]$ be the doubling map defined by*

$$T_0(x) = \begin{cases} 2x & \text{if } 0 \le x \le \frac{1}{2} \\ 2x - 1 & \text{if } \frac{1}{2} < x \le 1. \end{cases}$$



n	$S_n(A)$	$S_n(B)$
7	-0.00189764	7.847249099
8	$2.503 \cdot 10^{-5}$	7.655670510
9	$-1.73 \cdot 10^{-7}$	7.658058404
10	$6.24 \cdot 10^{-10}$	7.658050630
11	$-1.15 \cdot 10^{-12}$	7.658050565
12	$1.42 \cdot 10^{-13}$	7.658050566

Figure 1. (a) The doubling map T_0; (b) The small perturbation T_λ; (c) Approximations to the first derivative and to the second derivatives. {conv:tab}

Let $T_\lambda : [0,1] \to [0,1]$ $(-\frac{1}{2\pi} < \lambda < \frac{1}{2\pi})$ be the map defined by

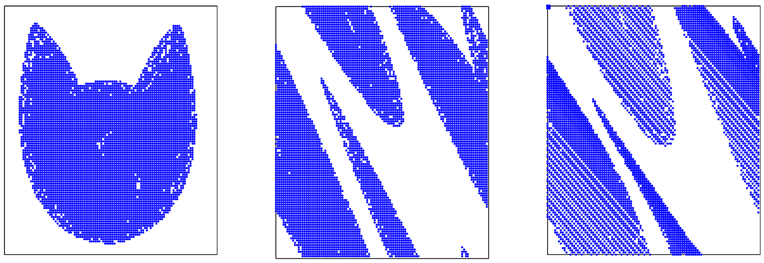
$$T_\lambda(x) = \begin{cases} 2x + \lambda \sin(2\pi x) & \text{if } 0 \le x \le \frac{1}{2} \\ 2x - 1 + \lambda \sin(2\pi x) & \text{if } \frac{1}{2} < x \le 1. \end{cases}$$

Let $g(x) = \cos(2\pi x)$. In this case the value $A = 0$ can be obtained as a part of general statement, outlined in the Appendix 4.3 p. 17, and this leads to a useful check on the numerics. In particular, using only ≈ 2000 periodic points with period ≤ 10 we see from Table 1 (a) accuracy to 9 decimal places. Similarly, we have a method for finding numerical approximations B_k to $B = \frac{\partial^2}{\partial \lambda^2} \int g d\mu_{T_\lambda}|_{\lambda=0}$ using periodic orbits of period $\le k$. For instance, using only ≈ 8000 periodic points with period ≤ 12 we get accuracy to 9 decimal places.

1.3.2. Anosov diffeomorphisms

Example 1.6. *We can consider the Arnol'd CAT map $T_0 : \mathbb{T}^2 \to \mathbb{T}^2$ given by*

$$T_0(x, y) = (2x + y, x + y) \bmod 1;$$



n	$S_n(A)$
4	8.103482255823
5	3.191273209097
6	-5.060637127313
7	-1.072969824595
8	0.002785871255
9	0.002790864776
10	0.002790864709

Figure 2. The case of Anosov diffeomorphisms. (1) Original domain (2) the image under T_0. (3) The image under T_λ. (4) Approximations to the first derivative $\frac{\partial}{\partial\lambda}\int g d\mu_\lambda$.

{convan:tab

and define a small perturbation

$$T_\lambda(x, y) = (2x + y + \lambda\cos(2\pi x), x + y) \bmod 1.$$

The number of periodic points of T_0 *grows exponentially like* $\left(\frac{3+\sqrt{5}}{2}\right)^n$*, and they are equidistributed. We need to choose test function changing rapidly in order to reduce computational error. For example, one can consider* $g(x, y) = \sin(19\sin(2\pi x) + 41\cos(2\pi y))$*. We obtain* $A = 0.00279\ldots$ *with* ≈ 6000 *periodic points of period* 9 *with accuracy to* 10 *decimal places.*

2. Approach of thermodynamics

We will present the argument in a simple case of expanding maps and explain afterwards the changes needed in the case of invertible Anosov diffeomorphisms.

In this section we introduce determinants, which are complex functions whose zeros can be expressed in terms of a suitable thermodynamical pressure function. We will also recall that the integral of the test function which we study can be expressed in terms of a suitable derivative of this pressure. We are interested in the derivative of this integral as given in (1.1). So by the Implicit Function Theorem we will see that this can be expressed in terms of the derivatives of the determinant. This is the key to our approach.

2.1. Expanding maps of the circle

We will begin by reviewing thermodynamic formalism for expanding maps of the circle. This then allows us to describe the zeros of the complex determinant function we need to introduce. Finally, we explain how the determinant function can be used to study the linear response problem for expanding maps of the circle.

2.1.1. Thermodynamic formalism Let $T\colon \mathbb{T}^1 \to \mathbb{T}^1$ be an expanding map on the unit circle. We can consider the C^ω function $F : \mathbb{T}^1 \to \mathbb{R}$ defined by

$$F(x) = -\log |T'(x)|.$$

Definition 2.1. *Let $h(m)$ be the entropy of the measure m. We define the pressure function $P : C(\mathbb{T}^1, \mathbb{R}) \to \mathbb{R}$ by*

$$P(F) := \sup_m \left\{ h(m) + \int F dm \right\}$$

where the supremum is over all T-invariant probability measures. Let μ_F be the Gibbs measure associated to F, i.e.,

$$P(F) = h(\mu_F) + \int F d\mu_F$$

Let us consider an analytic family $F_\lambda\colon \mathbb{T}^1 \to \mathbb{T}^1$ of expanding maps on the circle with parameter $\lambda \in (\varepsilon, \varepsilon)$ and denote $\mu_\lambda := \mu_{F_\lambda}$.

The following result is well known [17].

{difP:lem}

Lemma 2.2. *Let $g : \mathbb{T}^1 \mapsto \mathbb{R}$ be a real analytic function. Then the function $t \mapsto P(F_\lambda + tg)$ is analytic and we can write*

$$\frac{\partial P(F_\lambda + tg)}{\partial t}\Big|_{t=0} = \int g d\mu_\lambda$$

2.1.2. Determinant for the expanding maps We now introduce a complex-valued function of three variables, associated to the family $F_\lambda\colon \mathbb{T}^1 \to \mathbb{T}^1$ and a test function $g\colon \mathbb{T}^1 \to \mathbb{C}$

Definition 2.3. *The determinant $d\colon \mathbb{C} \times \mathbb{R} \times (-\varepsilon, \varepsilon) \to \mathbb{C}$, is a formally defined function*

$$d(z, u, \lambda) = \exp\left(-\sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{T_\lambda^n x_\lambda = x_\lambda} \frac{\exp(-ug^n(x_\lambda))}{|(T_\lambda^n)'(x_\lambda)| - 1} \right) \tag{2.1}$$

{det:eq}

where the second summation is over periodic points x_λ for T_λ of period n and we write $g^n(x_\lambda) = \sum_{k=0}^{n-1} g(T_\lambda^k x_\lambda)$.

It is relatively classical to show the following.

{ancov:lem}

Lemma 2.4. *For $z \in \mathbb{C}$, $\lambda \in (-\varepsilon, \varepsilon)$ and $u \in \mathbb{R}$ we have that:*

(i) $d(z, u, \lambda)$ converges to an analytic function for $|z| < \exp(-P(F_\lambda - ug))$;

(ii) $d(z, u, \lambda)$ has an analytic extension in $z \in \mathbb{C}$ to the entire complex plane $\mathbb{C}$;

(iii) $z \mapsto d(z, u, \lambda)$ has a simple zero at $z(u, \lambda) = \exp(-P(F_\lambda - ug))$.

These results can be easily deducted from the paper of Ruelle [18] and his book [17], but we briefly recall the idea of the argument.

Let us treat the circle $\mathbb{T}^1$ as the unit interval $[0,1]$. Let $[0,1] \subset U \subset \mathbb{C}$ be its complex neighbourhood. We let B be the Banach space of bounded analytic functions $f : U \to \mathbb{C}$ with the supremum norm $\|\cdot\|_\infty$.

{trop:def}

Definition 2.5. *To a family of maps $F_\lambda \in B$ and a test function $g \in B$ we associate the transfer operator $\mathcal{L}_{u,\lambda} : B \to B$:*

$$(\mathcal{L}_{u,\lambda})f(x) = \sum_k \exp\big((F_\lambda - ug)(T_k x)\big) f(T_k x)$$

where $T_k : U \to U$ are C^ω contractions with $\overline{T_k(U)} \subset U$, and $T_\lambda \circ T_k$ is the identity map.

Providing that $F_\lambda : U \to \mathbb{C}$ and $u : U \to \mathbb{C}$ are analytic, the operators $\mathcal{L}_{u,\lambda}$ are nuclear. In particular, the determinant

$$\det(I - z\mathcal{L}_{u,\lambda}) = \exp\left(-\sum_{n=1}^{\infty} \frac{z^n}{n} \mathrm{trace}(\mathcal{L}^n_{u,\lambda})\right)$$

is an entire function in z. The previous statements come easily from results of Ruelle [18], after Grothendieck [11]:

{acoefs:lem

Lemma 2.6 (Grothendieck–Ruelle)**.** *We can expand the determinant in a power series $\det(I - \mathcal{L}_{u,\lambda}) = 1 + \sum_{n=1}^{\infty} a_n(u,\lambda) z^n$, where the coefficients a_n satisfy: there exists $\alpha > 0$ and $0 < \theta < 1$ such that $|a_n(u,\lambda)| \le \alpha \theta^{n^2}$.*

In particular, we see the following

{z0:cor}

Corollary 2.7. *Let $z = z(u,\lambda)$ be the real zero for $d(z,u,\lambda)$, i.e. $d(z(u,\lambda),u,\lambda) = 0$. Then $z(0,\lambda) = 1$ for all $\lambda \in (-\varepsilon,\varepsilon)$.*

Proof. By Rohlin's equality we have that $P(F_\lambda) = 0$ for all $\lambda \in (-\varepsilon,\varepsilon)$. □

Using Lemma 2.2 we can observe

$$\frac{\partial}{\partial\lambda} z(u,\lambda) = \frac{\partial}{\partial\lambda} \exp(-P(F_\lambda - ug)) = -z(u,\lambda) \frac{\partial}{\partial\lambda} P(F_\lambda - ug)$$

2.1.3. Analytic dependence of the average on measure Implicit to our analysis is that the function $\lambda \mapsto \int g d\mu_\lambda$ is analytic in λ, from which we can then turn to the problem of solving the derivatives. This is part of a general result whereby we consider analyticity of the determinant $d(z,u,\lambda)$, defined by (2.1). We may introduce an analytic function $\eta\colon \mathbb{C} \times (-\varepsilon,\varepsilon) \to \mathbb{C}$ by

$$\begin{aligned}\eta(z,\lambda) := \frac{\partial \log d(z,u,\lambda)}{\partial u}\Big|_{u=0} &= \frac{1}{d(z,u,\lambda)} \frac{\partial d(z,u,\lambda)}{\partial u}\Big|_{u=0} \\ &= \sum_{n=1}^{\infty} z^n \sum_{T^n_\lambda x_\lambda = x_\lambda} \frac{g^n(x)}{n} \frac{1}{|(T^n_\lambda)'(x_\lambda)|} \qquad (2.2)\end{aligned}$$

Lemma 2.8. *The function $\eta(z,\lambda)$ has a simple pole at $s=1$ with residue $\int g d\mu_\lambda$.*

For each individual periodic point $T_\lambda^n(x_\lambda) = x_\lambda$ we have a C^ω function $(-\varepsilon,\varepsilon) \ni \lambda \mapsto x_\lambda$. Moreover, we can find a common neighbourhood $(-\varepsilon,\varepsilon) \subset U$ such that $(-\varepsilon,\varepsilon) \ni \lambda \mapsto x_\lambda$ has an analytic extension to U.

Lemma 2.9. *In a neighbourhood $1 \in V \subset \mathbb{C}$ we have that $V \ni z \mapsto \eta(z,\lambda)^{-1}$ is analytic. Moreover, $U \ni \lambda \mapsto \eta(z,\lambda)^{-1} \in C^\omega(V,\mathbb{C})$ is also analytic.*

Recall Corollary 2.7. We can use the residue theorem to deduce that

$$U \ni \lambda \mapsto \frac{1}{\eta(z,\lambda)} \mapsto \int g d\mu_\lambda$$

is analytic.

2.2. Anosov diffeomorphisms

Let $T\colon \mathbb{T}^2 \to \mathbb{T}^2$ be an Anosov diffeomorphism of the torus, i.e. we assume that there exists a DT-invariant splitting $\mathbb{T}^2 = E^s \oplus E^u$, and $C,\rho > 0$ such that $\left\| DT^n|_{E^s} \right\| \le C\rho^n$ and $\left\| DT^{-1}|_{E^u} \right\| \le C\rho^n$. We also assume that the map T has a dense orbit.

We will begin by reviewing thermodynamic formalism for Anosov maps of the torus. This then allows us to describe the zeros of the complex determinant function we need to introduce. We also include a brief description of the Banach space and operators (due to Rugh) that we use.

2.2.1. Thermodynamic formalism We can consider the Hölder function $\varphi^u\colon \mathbb{T}^2 \to \mathbb{R}$ defined by

$$\varphi^u(x) = -\log|\det(D_x T|_{E^u})|$$

and the C^ω function $\varphi\colon \mathbb{T}^2 \to \mathbb{R}$ given by

$$\varphi(x) = -\log|\det(I - DT)|$$

Definition 2.10. *We define the pressure function $P: C(\mathbb{T}^2,\mathbb{R}) \to \mathbb{R}$ by*

$$P(T) := \sup_m \left\{ h(m) + \int T dm_T \right\}$$

where the supremum is over all T-invariant probability measures. Let μ_T be the Gibbs measure associated to T, i.e., the unique T-invariant probability measure such that

$$P(T) = h(\mu_T) + \int T d\mu_T.$$

Let $T_\lambda\colon \mathbb{T}^2 \to \mathbb{T}^2$ be a family of Anosov diffeomorphisms. Let φ_λ^u and φ_λ^s be the associated functions. The following result is well known [17].

{difP2:lem}

Lemma 2.11. *Let $w : \mathbb{T}^2 \mapsto \mathbb{R}$ be a real analytic function. The function $t \mapsto P(-\varphi_\lambda + tw)$ is analytic and we can write*

$$\frac{\partial P(\varphi_\lambda + tw)}{\partial t}\Big|_{t=0} = \int w d\mu_\lambda,$$

where μ_λ is the SRB measure and $P(\varphi^u_\lambda) = 0$.

2.2.2. Determinant for Anosov diffeomorphisms We recall the result of Rugh from [19]. For a real analytic Anosov diffeomorphism $T : \mathbb{T}^2 \to \mathbb{T}^2$ and a positive real analytic function $g : \mathbb{T}^2 \to \mathbb{R}^+$ given by $g(z) = \exp(w(z))$ we can associate the function

$$d(z) \stackrel{\text{def}}{=} \exp\left(-\sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{T^n x = x} \frac{\prod_{k=0}^{n-1} g(T^k x)}{\det(DT^n(x) - I)}\right).$$

which converges for $|z| < \exp(P(-\phi^u + w))$. In particular, we observe that

$$\lim_{n\to+\infty} \frac{\exp\left(\sum\limits_{k=0}^{n-1} \phi^u(T^k x)\right)}{\det(DT^n(x) - I)} = 1. \tag{7.1}$$

We have the following interpretation.

Proposition 2.12 (Rugh). *The function $d(z)$ has an analytic extension to $\mathbb{C}$ with a simple zero at $z = \exp(P(-\phi^u + w))$.*

However, examining the proof we see that there is an additional analytic dependence. We therefore define

$$d(z, s, \lambda) := \exp\left(-\sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{T^n_\lambda x_\lambda = x_\lambda} \frac{\exp\left(s \sum_{k=0}^{n-1} w(T^k_\lambda x)\right)}{\det(DT^n_\lambda(x) - I)}\right). \tag{2.3}$$ {det2:eq}

Lemma 2.13 (Ruelle–Grothendieck–Rugh)**.** *The function $d\colon \mathbb{C} \times \mathbb{R} \times (-\varepsilon, \varepsilon) \to \mathbb{C}$, given by (2.3) is analytic. Furthermore, we can write*

$$d(z, s, \lambda) = 1 + \sum_{n=1}^{\infty} a_n(s, \lambda) z^n$$

where there exists $0 < \theta < 1$ such that such that $|a_n(s, \lambda)| = O(\theta^{n^2})$.

Thus the truncations

$$d^{(N)}(z, s, \lambda) = 1 + \sum_{n=1}^{N} a_n(s, \lambda) z^n$$

are efficient approximations to $d(z, s, \lambda)$ and lead to approximations to $\int w d\mu_\lambda$ via the implicit function theorem. As in the case of the expanding maps, one additional ingredient is the expansion $x_\lambda = x_0 + \lambda x^{(1)} + \cdots$ and replacing the family of fixed points x_λ by $x_0 + \lambda x^{(1)}$, after solving for $x^{(1)}$.

Example 2.14. *We can consider the Arnol'd CAT map $T_0 : \mathbb{T}^2 \to \mathbb{T}^2$ defined by $T_0(x, y) = (2x + y, x + y)$ (mod 1). We can then define $T_\lambda(x, y) = (2x + y + \lambda \sin(2\pi x), x + y)$. The periodic points for T_0 correspond to $\left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right) = (A^n - I)^{-1} \left(\begin{smallmatrix} n \\ m \end{smallmatrix}\right)$ where $n, m \in \mathbb{Z}$, and $A = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$.*

2.2.3. The Banach spaces and transfer operators of Rugh For completeness, we briefly recall the approach by Rugh.

The spaces constructed by Rugh in his paper [19] were the forerunners of the modern theory of Anisotropic Banach spaces. For our purposes, the most important feature is that it retains the property of being a nuclear space.

One associates to the Anosov map a Markov partition $\mathcal{P} = \{P_1, \cdots, P_k\}$. Each piece of the partition can be written in the form $[U_i, S_i]$ where $U_i \subset W^u(z_i)$ and $S_i \subset W^u(z_i)$, for some $z_i \in P_i$, and we write $[x, y] = W^s(x, \varepsilon) \cap W^s(y, \varepsilon)$ for sufficiently small $\varepsilon > 0$, depending only on T. Following the original work of Adler and Weiss [1], and Sinai [20], we can model $T : \mathbb{T}^2 \to \mathbb{T}^2$ by a subshift of finite type $\sigma : \Sigma_A \to \Sigma_A$ with transition matrix A.

On each piece P_i of the partition one can consider the natural coordinates associated to the stable and unstable manifolds (i.e., we can identify points in P_i with $U_i \times S_i$ using the above. As is well known, these coordinates are typically only C^1. In order to recover analytic coordinates we need to use an approach introduced by Rugh.

Assume that $z_0 \in P_{i_0}$, $Tz_1 \in P_{i_1}$. In particular, writing $z_0 = (x_0, y_0)$ and $z_1 = (x_1, y_1)$ we see that for each

(i) $y_0 \in U_{i_0}$ the map $x_0(\cdot, y_0) : S_{i_1} \to S_{i_0}$ is an analytic contraction.
(ii) $x_1 \in S_{i_1}$ the map $y_1(x_1, \cdot) : U_{i_0} \to U_{i_1}$ is an analytic expansion.

Here contraction and expansion are understood in terms of the modulus of derivative being smaller, or larger, than 1 respectively.

By virtue of real analyticity, we can fix small neighbourhoods $\mathcal{S}_{i_0} \supset S_{i_0}$ and $\mathcal{U}_{i_1} \supset U_{i_1}$ with smooth boundaries corresponding to complexifications of these pieces of unstable and stable maps such that:

(i) for any $y_0 \in \mathcal{U}_{i_0}$ the map $x_0(\cdot, y_0) : \mathcal{S}_{i_0} \to \mathcal{S}_{i_1}$ is an analytic contraction and, in particular, $\overline{x_0(\mathcal{S}_{i_0}, y)} \subset \mathcal{S}_{i_1}$.
(ii) for any $x_1 \in \mathcal{S}_{i_1}$ the maps $y_1(x_1, \cdot) : \mathcal{U}_{i_1} \to \mathcal{U}_{i_0}$ is an analytic expansion and, in particular, $\overline{y_1(x_1, \mathcal{U}_{i_1})} \subset \mathcal{U}_{i_0}$.
(iii) We can solve $y_j(\xi_0, \phi_s(\xi_0, \eta_1)) = \eta_1$ to get a family of contractions $\phi_s(\xi_0, \cdot) : \mathcal{U}_{i_1} \to \mathcal{U}_{i_0}$ (indexed by ξ_0).
(iv) We define a family of contractions $\phi_u(\cdot, \eta_1) : \mathcal{S}_{i_0} \to \mathcal{S}_{i_1}$ by $\phi_u(\xi_0, \eta_1) = y_1(\phi_s(\xi_0, \eta_1))$ (indexed by η_1).

We can consider the space of functions $B := \oplus_i C^\omega(\mathcal{S}_i \times (\widehat{\mathbb{C}} - \mathcal{U}_i))$ consisting of bounded analytic functions $f : \coprod_i \mathcal{U}_i \to \mathbb{C}$ with the supremum norm. We can then define a transfer operator $\mathcal{L} : B \to B$ by

$$\mathcal{L}f(x_1, y_1) = -\sum_{A(i_0,i_1)=1} \frac{1}{4\pi^2} \int_{\partial \mathcal{S}_i} \int_{\partial \mathcal{U}_i} \frac{f(x_0, y_0) \cdot G(x_1, y_0)}{(x_0 - \varphi_u(x_0, y_1))(y_1 - \varphi_s(x_0, y_1))} dx_0 dy_0$$

where A is the transition matrix, $(x_0, y_0) \in \mathcal{S}_{i_0} \times (\widehat{\mathbb{C}} - \mathcal{U}_{i_0})$ $(x_1, y_1) \in \mathcal{S}_j \times (\widehat{\mathbb{C}} - \mathcal{U}_j)$, and $G(x_0, y_1) = \partial_2 \phi_s(x_0, y_1)$ is a weight function associated with the change of variables (cf. Rugh [13]).

3. Determinant and test function

The coefficients A and B, defined by (1.3), can be written in terms of the determinant. They give linear and quadratic approximations to the derivative of the average (1.4). We keep the notation introduced in the previous section.

The first coefficient A may be written in a relatively easy closed form.

{difD:lem}

Lemma 3.1 (Linear approximation)**.**

$$A = \frac{\partial}{\partial \lambda} \left(\int g d\mu_\lambda \right) \Big|_{\lambda=0} = -\left(\frac{\frac{\partial^2 d(1,u,\lambda)}{\partial u \partial \lambda}\big|_{u=0,\lambda=0}}{\frac{\partial d(z,0,0)}{\partial z}\big|_{z=1}} \right) + \left(\frac{\frac{\partial^2 d(z,0,\lambda)}{\partial z \partial \lambda}\big|_{z=1,\lambda=0} \left(\frac{\partial d(1,u,0)}{\partial u}\big|_{u=0} \right)}{\left(\frac{\partial d(z,0,0)}{\partial z}\big|_{z=1} \right)^2} \right).$$

Proof. By the implicit function theorem applied to $d(z(u, \lambda), u, \lambda) = 0$ we can write

$$-\frac{\frac{\partial d(z(0,\lambda),u,\lambda)}{\partial u}\big|_{u=0}}{\frac{\partial d(z,0,\lambda)}{\partial z}\big|_{z=z(0,\lambda)}} = \frac{\partial z(u, \lambda)}{\partial u}\Big|_{u=0} = z(0, \lambda) \frac{\partial P(F_\lambda - ug)}{\partial u}\Big|_{u=0} = \frac{\partial P(F_\lambda - ug)}{\partial u}\Big|_{u=0}, \quad (3.1)$$ {difD1:eq}

using the corollary 2.7 to see that $z(0, \lambda) = 1$, and by Lemma 2.2

$$\frac{\partial P(F_\lambda - ug)}{\partial u}\Big|_{u=0} = -\int g d\mu_\lambda. \quad (3.2)$$ {difD2:eq}

We thus see from the two identities (3.1) and (3.2) that

$$\int g d\mu_\lambda = -\frac{\frac{\partial d(z(0,\lambda),u,\lambda)}{\partial u}\big|_{u=0}}{\frac{\partial d(z,0,\lambda)}{\partial z}\big|_{z=z(0,\lambda)}}.$$

Differentiating with respect to λ and taking into account that $\frac{\partial z(0,\lambda)}{\partial \lambda}\big|_{\lambda=0} = 0$, we get the result. ☐

The expression for the second coefficient $B = \frac{\partial^2}{\partial \lambda^2} \int g d\mu_\lambda \big|_{\lambda=0}$ involves third-order derivatives of the determinant.

{B:lem}

Lemma 3.2 (Quadratic approximation)**.**

$$B = \left(\frac{\partial d}{\partial z} \right)^{-1} \left(\frac{\partial^3 d}{\partial u \partial^2 \lambda} - \frac{\partial^3 d}{\partial u \partial \lambda^2} - \frac{\partial^3 d}{\partial z \partial \lambda^2} \cdot \int g d\mu_0 - 2\frac{\partial^2 d}{\partial z \partial \lambda} \cdot A - \frac{\partial^2 d}{\partial z^2} \cdot A \cdot \int g d\mu_0 \right) \Big|_{u=0,\lambda=0,z=1},$$

where $A = \frac{\partial}{\partial \lambda} \int g d\mu_\lambda \big|_{\lambda=0}$.

Proof. To estimate the value B we differentiate the determinant twice, and calculate $\frac{\partial^2}{\partial\lambda^2}\left(\frac{\partial}{\partial u} d(z(u,\lambda),u,\lambda))|_{u=0}\right)|_{\lambda=0}$ using the identities $z(0,\lambda)\equiv 0$ and $\frac{\partial z(0,\lambda}{\partial\lambda}\Big|_{\lambda=0} = 0$. □

It is clear therefore that in order to estimate the coefficients A and B, it is sufficient to be able to compute efficiently derivatives of the determinant. Below we provide theoretical background and outline computational method.

3.1. Derivatives of $d(z,u,\lambda)$

Since the determinant is an analytic function, we can expand it in a power series.

$$d(z,u,\lambda) = 1 + \sum_{n=1}^{\infty} a_n(u,\lambda) z^n. \tag{3.3}$$ {dets:eq}

Comparing the terms in the expansion for $d(z,u,\lambda)$ given by (2.1) we get the following.

{anest:lem}

Lemma 3.3. *Let $g : \mathbb{T}^1 \to \mathbb{R}$ be real analytic, and let $T_\lambda \colon \mathbb{T}^1 \to \mathbb{T}^1$ be a family of the expanding maps of the circle. Then each $a_n(u,\lambda)$ depends only on periodic points of period n, i.e.,*

$$a_n(u,\lambda) = \sum_{n_1+\cdots+n_r=n} \frac{1}{r!}\prod_{j=1}^{r}\left(\frac{1}{n_j}\sum_{T^{n_j}x_\lambda = x_\lambda}\frac{\exp(-ug^{n_j}(x))}{(T_\lambda^{n_j})'(x_\lambda)-1}\right)$$

Moreover, in the case of the doubling map on the circle, we can take any $\frac{1}{2} < \theta < 1$ and then α can be explicitly estimated in the upper bound $|a_n| \le \alpha\theta^{n^2}$.

{anap:lem}

Lemma 3.4. *The derivatives of the determinant (2.1) can be approximated by the sums of derivatives of coefficients a_n. Moreover, an upper bound for the approximation error can be explicitly calculated.*

4. Numerical results

We begin with an outline of the algorithm we use for computing the first and second derivatives of the integrals. We then illustrate this, firstly, for expanding maps of the circle and then, secondly, for Anosov diffeomorphisms of the torus.

4.1. Outline of the algorithm

{ss:algorit

Our expression for the first derivative of the integral from Lemma 3.1 together with Lemma 3.4 provides the basis for an efficient algorithm for estimating the numerical value of the derivative (1.1).

To present the algorithm used, we will need the following simple technical result.

{recc:lem}

Lemma 4.1. *In notation introduced above, consider the values*

$$b_n(u,\lambda) \stackrel{\text{def}}{=} \sum_{T_\lambda^n(x_\lambda)=x_\lambda} \frac{\exp(-ug^n(x_\lambda))}{|(T_\lambda^n)'(x_\lambda)|-1}, \quad 1 \le n \le N \tag{4.1}$$ {bdef:eq}

Then the coefficients of the series (3.3) satsify the reccurent relation

$$a_n(u,\lambda) = -\frac{1}{n}\sum_{j=0}^{n-1} a_j(u,\lambda) b_{n-j}(u,\lambda) \tag{4.2}$$ {mainrec:ec

where $a_0 = 1$.

Proof. We recall the determinant identity, that follows from (2.1) and (2.3)

$$\exp\left(-\sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{T_\lambda^n x_\lambda = x_\lambda} \frac{\exp(-u g^n(x_\lambda))}{|(T_\lambda^n)'(x_\lambda)| - 1}\right) = 1 + \sum_{n=1}^{\infty} z^n a_n(u,\lambda);$$

With notation introduced, it can be rewritten as

$$\exp\Big(-\sum_{n=1}^{\infty} \frac{z^n}{n} b_n\Big) = 1 + \sum_{n=1}^{\infty} z^n a_n(u,\lambda);$$

The Lemma follows by induction in n. Differentiating n times both sides of the latter equation in z and evaluating the result at $z = 0$ we obtain the required relation. □

Our algorithm is the following.

Step 1 Fix N. We can compute the periodic points of the map T_0 (for example, using iterations of the inverse transformation) up to period N and associate the values $b_n(u,\lambda)$ defined by (4.1) as well as partial derivatives $\frac{\partial b_n}{\partial u}$, $\frac{\partial b_n}{\partial \lambda}$ and $\frac{\partial^2 b_n}{\partial u \partial \lambda}$. (We would like to stress out that in order to avoid round-off errors, we calculate the derivatives analytically for each combination of perturbation and test function g.)

Step 2 We can derive the expressions for a_n $(1 \le n \le N)$ in terms of b_n $(1 \le n \le N)$ using reccurent relation (4.2).

Step 3 We can define approximations

$$d_N(z,u,\lambda) = \sum_{n=1}^{N} z^n a_n(u,\lambda)$$

to $d_N(z,u,\lambda)$. The derivatives of $d(z,u\lambda)$ that appear in the formula in (3.1) can be approximated by the derivatives of $d_N(z,u,\lambda)$ which take an explicit form:

$$\frac{\partial^2 d^N(z,u,\lambda)}{\partial u \partial \lambda} = \sum_{n=1}^{N} z^n \frac{\partial^2 a_n(u,\lambda)}{\partial u \partial \lambda}$$

$$\frac{\partial d^N(z,u,\lambda)}{\partial \lambda \partial z} = \sum_{n=1}^{N} n z^{n-1} \frac{\partial a_n(u,\lambda)}{\partial \lambda}$$

$$\frac{\partial d^N(z,u,\lambda)}{\partial u} = \sum_{n=1}^{N} z^n \frac{\partial a_n(u,\lambda)}{\partial u}$$

$$\frac{\partial d^N(z,u,\lambda)}{\partial z} = \sum_{n=1}^{N} n z^{n-1} a_n(u,\lambda)$$

Step 4 We obtain partial derivatives of a_n involved in the formulae above using reccurent relation (4.2), using derivatives of b_n obtained in Step 1.

Step 5 We define subsequent approximations

$$A_N := -\left(\frac{\frac{\partial^2 d^N(z,u,\lambda)}{\partial u \partial \lambda}}{\frac{\partial d^N(z,u,\lambda)}{\partial z}}\right) + \left(\frac{\frac{\partial d^N(z,u,\lambda)}{\partial \lambda \partial z}\frac{\partial d^N(z,u,\lambda)}{\partial u}}{\frac{\partial d^N(z,u,\lambda)}{\partial z}}\right)^2$$

In particular, we need only sum expressions involving the derivatives of the coefficients $a_n(u,\lambda)$ constructed in Step 4. It follows from Lemma 3.1 that

$$A_N \longrightarrow \frac{\partial}{\partial \lambda}\left(\int g d\mu_\lambda\right)\Big|_{\lambda=0} \text{ as } N \to \infty.$$

In the next subsection we provide an estimate on the rate of convergence.

4.2. Convergence estimates

{ss:convest

The rate at which A_N converges to A us controlled by the size of the discarded tail (from N to infinity) of the series.

To illustrate the approach, consider the case of real analytic expanding map $T_\lambda : [0,1] \to [0,1]$ with $\lambda \in (-\varepsilon, \varepsilon)$. We denote the inverse branches by $T_{\lambda,j} : [0,1] \to [0,1]$, with $j = 1, \cdots, k$. Let us assume that each $T_{\lambda,j}$ has an analytic extension to a neighbourhood $B(x,r) \supset [0,1]$ for $\lambda \in V$, a bounded complex neighbourhood of x such that

$$\overline{\cup_j T_{\lambda,j} B(x,r)} \subset B(x, \theta^{\frac{1}{2}} r)$$

for some $0 < \theta^{\frac{1}{2}} < 1$. Let us assume that $u \in U$, a bounded complex neighbourhood of x. We can then bound a_n using the approach in the proof of Lemma 2.6 given in [18] (see also [12])

$$|a_n(u,\lambda)| \le \|\mathcal{L}_{u,\lambda}\|_\infty^n n^{n/2} \theta^{\frac{n}{2}}, \qquad n \ge 0,$$

where we use the supremum norm $\|\mathcal{L}_{u,\lambda}\|_\infty$ for the operator $\mathcal{L}_{u,\lambda}$ acting on bounded analytic functions on $B(x, \theta^{\frac{1}{2}} r)$ with respect to the supremum norm.

The additional analytic dependence on λ and u is important to us in order to use Cauchy's theorem to bound the derivatives of $a_n(u,\lambda)$. In particular, we can write

$$\frac{\partial a_n(u,\lambda)}{\partial u}\Big|_{u=0} = \frac{1}{2\pi i}\int_{|\xi-1|=\rho_0} \frac{a_n(\xi,\lambda)d\xi}{(\xi-1)^2}$$

for any $\rho_0 > 0$ such that $B(0,\rho_0) \subset U$. Similarly,

$$\frac{\partial a_n(u,\lambda)}{\partial \lambda}\Big|_{\lambda=0} = \frac{1}{2\pi i}\int_{|\xi|=\rho_1} \frac{a_n(u,\xi)d\xi}{(\xi-1)^2}$$

for any $\rho_1 > 0$ such that $B(0, \rho_1) \subset V$. Finally, we can also write

$$\frac{\partial^2 a_n(u,\lambda)}{\partial u \partial \lambda}\Big|_{u=0,\lambda=0} = -\frac{1}{(2\pi)^2}\int_{|\xi-1|=\rho_0}\int_{|\eta|=\rho_1} \frac{a_n(\xi,\eta)d\xi d\eta}{(\xi-1)^2\eta^2}.$$

In particular, we have

$$\Big|\frac{\partial a_n(u,\lambda)}{\partial u}\Big|_{u=0}\Big| \leq \frac{1}{\rho_0^2}\|\mathcal{L}_{u,\lambda}\|^n n^{n/2}\theta^{\frac{n}{2}}$$

$$\Big|\frac{\partial a_n(u,\lambda)}{\partial \lambda}\Big|_{\lambda=0}\Big| \leq \frac{1}{\rho_1^2}\|\mathcal{L}_{u,\lambda}\|^n n^{n/2}\theta^{\frac{n}{2}}$$

$$\Big|\frac{\partial^2 a_n(u,\lambda)}{\partial u \partial \lambda}\Big|_{u=0,\lambda=0}\Big| \leq \frac{1}{\rho_0\rho_1}\|\mathcal{L}_{u,\lambda}\|^n n^{n/2}\theta^{\frac{n}{2}}$$

Bounds on error in approximation for the doubling map. We would like to give explicite estimates in the cases we studied in Examples 4.3 and 4.4. Assume that $T_\lambda(x) = 2x + \lambda\cos(2\pi x) \pmod 1$ and $g(x) = \sin(2\pi x)$.

We want to consider the case of a small $\lambda \neq 0$. Let $\widehat{T}_{1,\lambda}, \widehat{T}_{2,\lambda} : \mathbb{C} \to \mathbb{C}$ be defined by $\widehat{T}_{1,\lambda}(x) = 2x + \lambda\cos(2\pi x)$ and $\widehat{T}_{1,\lambda}(x) = 2x + \lambda\cos(2\pi x) + 1$. We can choose $r > \frac{1}{2}$ and then consider the images $\cap_j \widehat{T}_{j,\lambda}\big(B\big(\frac{1}{2}, r\big)\big)$. We also require that $T_{j,\lambda}$ are bijections from $B\big(\frac{1}{2}, r\big)$ onto the image. We would then like to choose $R > r$ and $|\lambda|$ sufficiently small that $\cap_j \widehat{T}_{j,\lambda}\big(B\big(\frac{1}{2}, r\big)\big) \supset B\big(\frac{1}{2}, R\big)$. In particular, we can choose any

$$R < 2r - \frac{1}{2} - |\lambda| \cdot \|\cos(2\pi x)\|_\infty$$

where the supremum is over the disk $B\big(\frac{1}{2}, r\big)$. We can trivially bound this by $\|\cos(2\pi x)\|_\infty \leq \exp(2\pi r)$. We can then choose any

$$\theta^{\frac{1}{2}} = \frac{r}{R} \geq \frac{2r}{4r - 1 - 2\lambda\exp(2\pi r)}$$

For example, if we choose $R = 2$, $r = \frac{3}{2}$, and $\lambda < \exp(2\pi r)\big(2 - \frac{1}{2r}\big)$, then we can choose any $\theta > \frac{9}{16}$.

We next want to bound the norm of the operator $\mathcal{L}_{\lambda,u}$ acting on bounded analytic functions on $B\big(\frac{1}{2}, 2\big)$ with respect to the supremum norm. In addition let us choose $|u| \leq \rho_1 = \frac{1}{100}$. Directly from Definition 2.5 we deduce an upper bound

$$\|\mathcal{L}_{\lambda,u}\| \leq 2\|(T_\lambda)^{-1}\|_\infty \exp\Big(\frac{\sqrt{e^5}}{100}\Big) \leq \frac{200}{200 - \sqrt{e^5}}\exp\Big(\frac{\sqrt{e^5}}{100}\Big),$$

since for $z \in B\big(\frac{1}{2}, 2\big)$ we have bounds $|\sin(2\pi z)| < \sqrt{e^5}$ and $|\cos(2\pi z)| < \sqrt{e^5}$.

Finally, using the above estimates we can explicitly bound the tail of the series for the derivatives, i.e., the difference between the derivatives for $d(z, \lambda, u)$ and $d^N(z, \lambda, u)$, which leads to a bound on the difference of the value A and its approximation A_N.

4.3. Some rigorous values

{ap2}

In the case of expanding maps of the circle, there is a nice criteria for estimating a linear approximation to the average $\int g d\mu$, which is of independent interest.

{Aexpr:thm}

Theorem 4.2. *Assume that T_λ and g are chosen so that there exist a constant C_0 and a polynomial P_0 such that for any n*

$$\frac{1}{|(T_0^n)'|-1}\cdot\Big|\sum_{T_\lambda^n(x_\lambda)=x_\lambda}\frac{\partial}{\partial\lambda}(T_\lambda^n)'(x_\lambda)\Big|_{\lambda=0}\Big|\le P_0(n) \tag{4.3}$$ {hyp1:eq}

$$\Big|\frac{1}{n}\sum_{T_0^n(x)=x} g^n(x)\Big|\le C_0 \tag{4.4}$$ {hyp2:eq}

then

$$\frac{\partial}{\partial\lambda}\int g d\mu_\lambda\Big|_{\lambda=0}=\lim_{n\to\infty}\frac{1}{n}\frac{\partial^2}{\partial u\partial\lambda}b_n(u,\lambda)\Big|_{u=0,\lambda=0}; \tag{4.5}$$

where $b_n(u,\lambda)$ are sums over periodic orbits given by (4.1), providing the latter limit exists.

The hypothesis of Theorem 4.2 are satisfied; in particular, in the examples we will consider below. The second condition (4.4) holds true for any test function g with zero average $\int g d\mu = 0$.

Proof. The argument is very straightforward. The conditions (4.3) and (4.4), imposed on the diffeomorphism and the test function, allows one to show, relying on the analyticity of the determinant, that $\frac{\partial d(1,u,0)}{\partial u}\Big|_{u=0}=0$ and

$$\frac{\partial^2}{\partial u\partial\lambda}d(1,u,\lambda)\Big|_{u=0,\lambda=0}=\frac{\partial}{\partial z}d(z,u,\lambda)\Big|_{z=1,u=0,\lambda=0}\cdot\lim_{n\to\infty}\frac{1}{n}\frac{\partial^2}{\partial u\partial\lambda}b_n(u,\lambda)\Big|_{u=0,\lambda=0}.$$

Theorem follows from Lemma 3.1. □

4.4. Expanding maps of the circle

Using the method described above, one can calculate partial derivatives of the first 16 coefficients $a_1,\dots,a_{16}$ very rapidly.

{ex:4.3}

Example 4.3 ($T_\lambda(x)=2x+\lambda\cos(2\pi x)$ and $g(x)=\sin(2\pi x)$)**.** *The left graph in Figure 3 shows a plot of sums over periodic orbits, b_n and its derivatives against n in logarithmic scale. We observe that $\log(b_n)=\ln(1+\frac{1}{2^n-1})\approx\frac{1}{2^n-1}$ converges to 0, as it should, and each of partial derivatives are asymptotic to $\exp(-\alpha n)$ for some constant $\alpha>0$. The right graph in Figure 3 shows a plot of the Taylor series coefficients a_n and their derivatives in the logarithmic scale. We observe that the coefficients and their derivatives converge to zero superexponentially $a_n\approx\exp(-\alpha n^2)$ for some $\alpha>0$. The numerical values for partial*

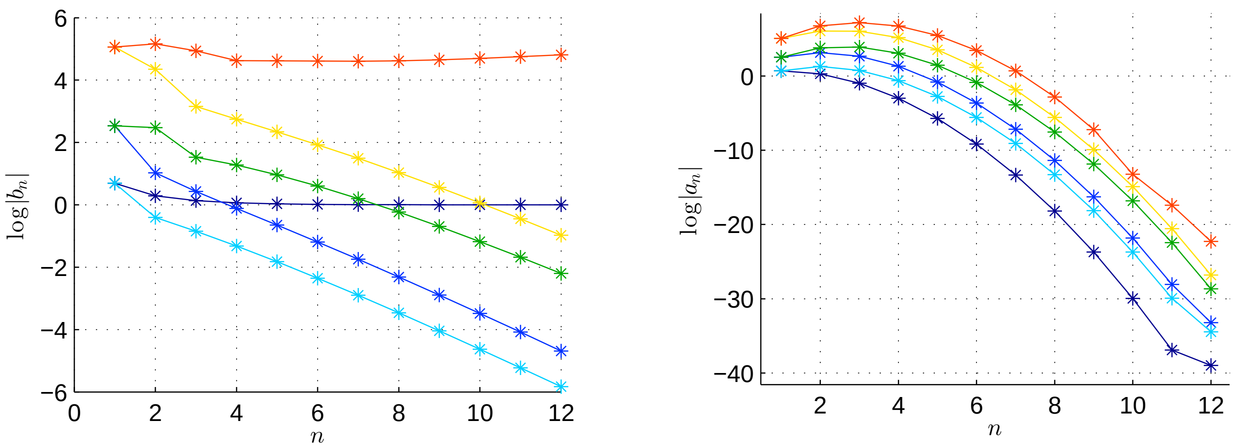


Figure 3. Representative plots. On the left hand side we see the plot of sums $|b_n|$ (dark blue) and partial derivatives $\left|\frac{\partial b_n}{\partial u}\right|$ (blue), $\left|\frac{\partial b_n}{\partial \lambda}\right|$ (light blue), $\left|\frac{\partial^2 b_n}{\partial u \partial \lambda}\right|$ (green), $\left|\frac{\partial^2 b_n}{\partial \lambda^2}\right|$ (yellow), and $\left|\frac{\partial^3 b_n}{\partial u \partial^2 \lambda}\right|$ (red) in logarithmic scale. On the right, the corresponding derivatives of a_n are shown. All derivatives are evaluated at $\lambda = 0$, $u = 0$. {sin2cos2:f

sums $S_n(A)$ and $S_n(B)$, approximating the coefficients A and B, respectively, were given in Table 1. In this example we obtain

$$\frac{\partial}{\partial \lambda} \int g d\mu_\lambda \Big|_{\lambda=0} = 0; \quad \text{and} \quad \frac{\partial^2}{\partial^2 \lambda} \int g d\mu_\lambda \Big|_{\lambda=0} = 7.6505\ldots$$

{ex:4.4}

Example 4.4 ($T_\lambda(x) = 2x + \lambda \cos(4\pi x)$ and $g(x) = \sin(4\pi x)$)**.** *Increasing the frequency of perturbation and test function, we observe that for the second order partial derivative* $\log\left|\frac{\partial^2 b_n}{\partial u \partial \lambda}\right|_{u=0,\lambda=0} \not\to 0$, *and, consequently, we get* $\frac{\partial}{\partial \lambda} \int g d\mu_\lambda \big|_{\lambda=0} = 1.570796326\ldots$; *which corresponds to the value* $\frac{\pi}{2}$ *from Theorem 4.2 up to an error* 10^{-12}.

These estimates took only 7 seconds on a modern Desktop computer.

Example 4.5 ($T_\lambda(x) = 2x + \lambda \cos(2\pi x)$ and $g(x) = \cos(2\pi x)$)**.** *In this example we consider synchronised perturbation and test function. As a result, we observe that one of the derivatives* $\frac{\partial a_n}{\partial \lambda}\big|_{\lambda=0,u=0}$ *vanishes, but* $\log\left|\frac{\partial^2 b_n}{\partial u \partial \lambda}\right|_{u=0,\lambda=0} \not\to 0$, *and we obtain* $\frac{\partial}{\partial \lambda} \int g d\mu_\lambda \big|_{\lambda=0} = 1.570796326\ldots$; *which corresponds to the value* $\frac{\pi}{2}$ *from Theorem 4.2 up to an error* 10^{-14}.

4.5. Anosov diffeomorphisms of the torus

It is well known that for an Anosov diffeomorphism A the total number of periodic points of period n is equal to $\det(A^n - I)$, therefore we see that $b_n(0,0) \equiv 1$ for all n, and $d(z,0,0) = 1 - z$, i.e. $a_0(0,0) = 1$, $a_1(0,0) = -1$, and $a_n(0,0) = 0$ for all $n \neq 1, 2$. Using a similar method with obvious adjustments, we calculate partial derivatives of the first 10 coefficients $a_1, \ldots, a_{10}$ of the Taylor series expansion of the determinant (2.3), evaluated at $\lambda = 0$, $u = 0$. The Figure 4 shows the plots of sums over the orbits b_n and the coefficients a_n in logarithmic scale. We see a very rapid convergence.

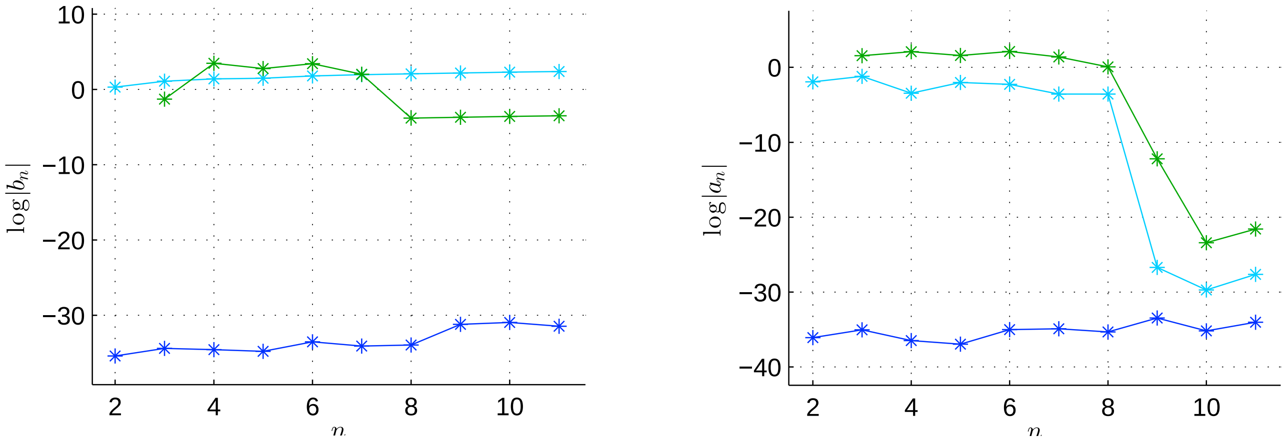


Figure 4. Representative plots. On the left hand side we see the plot of partial derivatives $\left|\frac{\partial b_n}{\partial u}\right|$ (blue), $\left|\frac{\partial b_n}{\partial \lambda}\right|$ (light blue), and $\left|\frac{\partial^2 b_n}{\partial u \partial \lambda}\right|$ (green) in the logarithmic scale. On the right, the corresponding derivatives of a_n are shown. All derivatives are evaluated at $\lambda = 0$, $u = 0$. {anosov:fig

5. Generalizations

Finally, we formulate generalizations of Theorem 1.1 and Theorem 1.3 which can be proved with the same basic method.

We begin by considering the generalization of Theorem 1.1 to expanding maps on d-dimensional compact manifolds.

Theorem 5.1. *Let T_t be a C^ω family of expanding maps on a d-dimensional compact manifold, let μ_{T_t} be the absolutely continuous invariant probability measure and let g be a C^ω test function. Then*

(i) The first and the second coefficients A and B may be written as explicit convergent series $A = \sum_{n=0}^{\infty} A_n$ and $B = \sum_{n=0}^{\infty} B_n$;

(ii) The kth term of the series is defined in terms of periodic points of period $\leq k$;

(iii) The partial sums $S_n(A) = \sum_{k=1}^{n} A_k$ and $S_n(B) = \sum_{k=1}^{n} B_k$ of the first k terms in each series converge faster than any exponential to A and B, respectively, i.e., $|A_n| \leq \alpha e^{-\beta n^{1+1/d}}$ and $|B_n| \leq C e^{-B n^{1+1/d}}$ for some $\alpha, \beta > 0$.

Finally, we have generalization of Theorem 1.3 to Anosov diffeomorphisms on d-dimensional compact manifolds.

Theorem 5.2. *Let T_t be a C^ω family of Anosov diffeomorphisms, let μ_{f_t} be the SRB measures and let g be a C^ω test function. Then*

(i) There are expressions for $A = \sum_{n=0}^{\infty} A_n$ and $B = \sum_{n=0}^{\infty} B_n$ in terms of explicit convergent series;

(ii) *The kth term of the series is defined in terms of periodic points of period $\leq k$;*

(iii) *The partial sums A_k and B_k of the first k terms in each series converge faster than any exponential to A and B, respectively.*

Remark 5.3. The method we have described might also be be applied to C^ω expanding semi-flows and Anosov flows, by using Markov sections.

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