Managerial Attention and Worker Performance*

Marina Halac† Andrea Prat‡

May 5, 2016

Abstract

We present a novel theory of the employment relationship. A manager can invest in attention technology to recognize good worker performance. The technology may break and is costly to replace. We show that as time passes without recognition, the worker's belief about the manager's technology worsens and his effort declines. The manager responds by investing, but this investment is insufficient to stop the decline in effort and eventually becomes decreasing. The relationship therefore continues deteriorating, and a return to high performance becomes increasingly unlikely. These deteriorating dynamics do not arise when recognition is of bad performance or independent of effort.

*An earlier version of this paper was circulated under the title "Managerial Attention and Worker Engagement." We thank Dirk Bergemann, Simon Board, Patrick Bolton, Alessandro Bonatti, Sylvain Chassang, Wouter Dessein, Marco Di Maggio, Bob Gibbons, Ricard Gil, Chris Harris, Ben Hermalin, Johannes Hörner, Navin Kartik, Amit Khandelwal, Qingmin Liu, Michael Magill, Jim Malcomson, David Martimort, Niko Matouschek, Meg Meyer, Daniel Rappoport, Paolo Siconolfi, Matt Stephenson, Steve Tadelis, various seminar and conference audiences, and four anonymous referees and the Co-editor for helpful comments. Enrico Zanardo and Weijie Zhong provided excellent research assistance.

†Columbia University and University of Warwick. Email: mhalac@columbia.edu.

‡Columbia University. Email: andrea.prat@columbia.edu.

What motivates employees to work hard? A large literature in economics has been devoted to this question, focusing for the most part on the optimal design of (explicit or implicit) incentive contracts and on how workers respond to different forms of compensation.[1] In this paper, we provide a novel theory of the employment relationship: workers' effort depends not only on compensation, but also on their beliefs about whether management is paying attention to their behavior. Only when paying attention can management recognize (good or bad) worker behavior. Because attention is costly and not directly observable, the moral hazard problem that arises inside the firm is two-sided: workers must be incentivized to exert effort; managers must be incentivized to invest in attention.

The idea that workers care about whether they are being "watched" is related to the widely studied Hawthorne effect, namely the improvement in workers' performance possibly caused by the "feeling that they are being accorded some attention" (Oxford English Dictionary). Interpretations of the original Hawthorne experiments and why workers' behavior may change with their awareness of being observed vary.[2] Our focus is on workers who care about managerial attention because this attention can produce some form of recognition of their performance. Workers value recognition for either psychological or financial reasons, or both. More broadly, our theory is in line with the psychology literature that examines the determinants of worker productivity. This literature finds that employee job satisfaction and workers' perceptions about management matter for both productivity and profits (Judge et al. 2001; Ostroff 1992; Harter, Schmidt and Hayes 2002). In particular, whether workers believe that they are given "recognition or praise for doing good work" affects their engagement and performance (Harter et al., 2002, p. 269), and these beliefs can have a significant impact on an organization's bottom line (Harter et al., 2010).[3]

We model managerial attention as a technology that recognizes worker performance. For example, consider a chief operating officer (COO) and a division

[1] For an excellent survey, see Prendergast (1999).

[2] The Hawthorne effect originated in a set of experiments conducted in a Western Electric factory in the 1920s, where workers' productivity was shown to increase each time a change in lighting was made. For a recent reassessment, see Levitt and List (2011).

[3] Saunderson (2004) finds that managers in the public sector know and believe in the importance of employee recognition, yet many fail to have effective recognition programs.

head or director in a firm. These parties meet regularly to discuss how the director is handling business, yet the relevant cost of monitoring for the COO is not attending the meetings but rather learning and thinking about the tradeoffs, challenges, constraints, and opportunities in the director's division. The acquired understanding allows the COO to recognize good ideas and decisions by the director for some time. Eventually, however, the COO's attention is needed in other areas and she may lose grasp of the division's issues—something the director cannot directly observe. We capture these ideas by introducing an *attention technology*, an intangible asset that is subject to depreciation and in which a manager can invest. At any point in time, the manager can recognize a worker's performance only if she has a high attention technology in place.[4]

Can managers be induced to invest in attention technology? How is workers' effort affected by their perceptions of managerial attention? We provide a model where these two variables are interlinked and explore their dynamic interaction. We show that relationship dynamics depend on the monitoring structure. When recognition is of *good* performance, dynamics are deteriorating: absent recognition, worker effort and eventually managerial investment decrease, and a return to high productivity becomes less likely over time. These dynamics contrast with those that arise when recognition is of *bad* performance or independent of effort.

Our model is cast in continuous time. At each moment, a myopic agent privately chooses effort which generates output for a principal. Output is unobservable; instead, the parties observe a verifiable signal—"recognition"—whose instantaneous arrival rate depends on the principal's attention technology and the agent's effort. If the principal's technology is high, the arrival rate is proportional to effort; if the technology is low, recognition cannot arrive. Recognition yields the agent a fixed reward, which may be purely psychological or also entail a monetary bonus. The principal's attention technology follows a stochastic process akin to those used for productive assets in the industrial organization literature (e.g., Besanko and Doraszelski, 2004): a high technology can "break" at

[4] Another illustrative example of managerial attention is in continuous process improvement, pioneered by Toyota and imitated by scores of manufacturing firms (Gibbons and Henderson, 2013): workers exert effort to identify performance-enhancing changes to production, and while these innovations always benefit the firm, workers can be recognized only if an effective system of management practices is in place to monitor how they engage with the productive process.

any point and become low, and the principal can invest at some cost to instantly "fix" it. The agent observes neither the principal's investment nor her attention technology, which we thus call the principal's *type*. Naturally, the agent's incentive to exert effort depends on his belief that the principal's type is high.

There are several important features of managerial attention that our model tries to capture. First, workers cannot perfectly assess the quality of the attention technology. A manager's ability to identify and document the contributions made by workers depends on many different and interrelated aspects of management, which are themselves difficult to observe.[5] Second, the signals produced by the attention technology are (at least in part) verifiable, typically because they describe the details of contributions made by workers in a specific context familiar to them. As the COO in the example above, a manager cannot learn those details unless a high technology is in place; hence, she cannot simply "fake" recognition at random times. Third, as noted, attention is an asset, although our analysis imposes no restrictions on how likely the technology is to break at any point.[6] Lastly, unlike with productive assets, the manager does not care about the attention technology directly; attention is valuable only insofar as it incentivizes the agent to exert effort.

We focus on continuous equilibria, where the agent's belief about the principal's type is continuous in the absence of observable events. This belief is a function of recognition (or lack thereof) and the agent's belief about the principal's investment. Because recognition reveals that the principal's type is high, the belief jumps up to one when the agent is recognized. Without investment, the agent's belief, and thus his effort, would then decrease continuously as time passes without recognition, due to Bayesian updating and the possibility that the attention technology breaks down. But a principal whose technology breaks could invest to fix it, with certainty or with a high probability, and if the agent

[5] One reason for this is that management practices display synergies with other practices and attributes of the firm, as stressed by Milgrom and Roberts (1990, 1995), Ichniowski, Shaw and Prennushi (1997), Bartling, Fehr and Schmidt (2012), and Brynjolfsson and Milgrom (2013).

[6] Our analysis is valid even when the depreciation rate is arbitrarily high and the investment cost arbitrarily low, so that attention approaches an instantaneous monitoring cost. Our view of attention as an asset is in line with Bloom, Sadun and Van Reenen (2012), where management is a form of technology, as well as with the new approach to growth discussed in Corrado and Hulten (2010), where expenditures on intangibles are treated as capital.

expected that to be the case, his belief about the principal's type would stop decreasing. We show however that this does not occur in equilibrium: the principal invests in attention technology as the agent's belief declines, but this investment is insufficient and in fact becomes decreasing when the agent gets pessimistic enough. Relationship dynamics as a result feature continuous deterioration: absent recognition, effort and eventually investment go down, and the chances of obtaining recognition and reverting to high performance decline.

We contrast the dynamics of this model in which the principal can recognize good performance with those that arise when she can also recognize bad performance. Suppose that if a high attention technology is in place, a verifiable bad signal arrives at a rate that is decreasing in the agent's effort. The agent incurs a (psychological or monetary) penalty when a bad signal arrives. We show that if recognition is primarily of bad performance, or symmetric and thus independent of effort, the model is essentially static: the agent's belief and effort, as well as the principal's investment, remain constant absent recognition once the principal starts investing. The relationship therefore does not fall into deterioration.

Our results show that the presence in an organization of the two-sided moral hazard problem we study has important implications for the dynamics of the employment relationship. When a manager must invest in attention to recognize a worker's behavior, the worker's belief about attention affects both his incentives to work and the manager's incentives to invest, with non-obvious consequences that depend on how recognition relates to worker effort. Our findings illuminate a distinction that had not been identified: the literature on firm reputation (reviewed below) assumes that learning is exogenous, hiding the implications of different forms of endogenous learning.

Related literature. This paper fits into the literature on firm reputation; see Cripps (2006) and Bar-Isaac and Tadelis (2008) for surveys. Most closely related are Board and Meyer-ter-Vehn (2013, 2014) and Dilmé (2014), where a firm can invest in product quality and consumers learn about quality through Poisson signals.[7] We apply the theory of reputation to study dynamics inside the firm:

[7] Board and Meyer-ter-Vehn (2013) compare good news and bad news learning about firm quality (see also Abreu, Milgrom and Pearce, 1991). We note that learning is always good news

rather than analyzing how a firm's reputation for quality affects sales, we examine how a firm's reputation for attention affects worker productivity. More importantly, we depart from the literature by endogenizing the learning process: whereas in these models of firm reputation consumers observe exogenous signals of firm quality, in our model the rate at which information arrives depends on the agent's action. Marinovic, Skrzypacz and Varas (2015) study firm reputation when information is endogenously generated by the firm via voluntary certification. We instead focus on the dynamics generated by the complementarity between the principal's and agent's actions.[8]

Our work is also related to an extensive literature on monitoring, which studies the problem of monitoring or auditing an agent when the monitor cannot commit to a strategy. Early contributions such as Graetz, Reinganum and Wilde (1986), Khalil (1997), and Strausz (1997) analyze static settings. More recently, Dilmé and Garrett (2014) study a dynamic model where in each period an inspector can either wait or inspect a short-lived potential offender, incurring a cost for changing her action. While our focus is on recognition of good behavior, monitoring in this literature is of bad behavior, as in the case that we study in Section 3.

Often motivated by the persistent performance differences across seemingly similar firms that Gibbons and Henderson (2013) document, a series of recent papers emphasize path dependence in equilibrium dynamics. For example, Callander and Matouschek (2014) study a search model in which managers learn about the quality of managerial practices by trial and error, and show that if practices are complementary, quality is persistent over time. Chassang (2010) finds that efficient equilibria can be path dependent in a repeated game in which a party learns to predict her partner's cost of cooperating over time. Li and Matouschek (2012) show that bad shocks can have persistent effects in a relational contracts model in which a principal's cost of making payments to an agent is privately observed. Our paper is related in that it also generates persistent relationship

about the principal's type in our model; we distinguish between good and bad signals about the (uninformed) agent's performance.

[8] Unlike in our model, moral hazard is only one-sided in Board and Meyer-ter-Vehn (2013, 2014), Dilmé (2014), and Marinovic et al. (2015). This is also the case in Ely and Välimäki (2003) where information depends on both players' actions.

dynamics. Unlike these articles, however, we consider a model of reputation, in which workers' beliefs about management's attention play a central role.

Finally, by studying managerial attention, our paper relates to Geanakoplos and Milgrom (1991) and other work on organizations under cognitive limits, although we address different issues.[9] This literature is concerned with the coordination of agents without conflicting interests, while we consider how an attention technology interacts with incentive provision.[10] The role of attention is also stressed in empirical work on the time use of managers and firm productivity, including Bandiera et al. (2011) and Bandiera, Prat and Sadun (2013).

1 The model

Setup. Consider a principal and an agent. Time $t \in [0, \infty)$ is continuous and infinite. At each time t, the agent privately chooses effort $a_t \in [0, 1]$ at instantaneous cost $c(a_t) = \frac{1}{2}a_t^2$. The principal receives an output flow equal to a_t. This output however is unobservable; instead, the parties observe a verifiable signal which we call *recognition*.

Recognition arrives via a (non-homogeneous) Poisson process with parameter $\mu\theta_t a_t$, where $\mu > 0$ and $\theta_t \in \{L, H\}$ is the state of the principal's attention technology at time t, with $L = 0$ and $H = 1$. The agent receives a reward $\bar{b} > 0$ each time he is recognized. For most of our analysis, we take this reward to be a purely psychological benefit, so it is exogenously fixed and entails no costs for the principal. This is in line with our motivation and allows us to focus on the problem of attention costs rather than the problem that the principal may want to save on agent compensation. Section 4 considers the case in which the reward is a monetary bonus chosen and paid by the principal.

The evolution of θ_t is determined by an exogenous Poisson process and endogenous investment. Specifically, at any time t, a high attention technology becomes low ("the technology breaks") with instantaneous probability $\gamma > 0$,

[9] See Garicano and Prat (2013) for a survey of this literature.

[10] Dur (2009) and Dur, Non and Roelfsema (2010) study workers who reciprocate managerial attention with effort. Gil and Mondria (2011) consider a multitasking setting where allocating more attention to a task increases the precision of its performance measures.

and the principal can invest at any point by paying a lump sum cost $F > 0$ to instantly transform a low technology into a high one ("fix the technology").[11] The simple form of depreciation is assumed for tractability. The assumption that the principal can instantly fix the technology is not only convenient but also appealing: it removes additional frictions and thus yields a simple benchmark for our model, as we show below.

The agent observes neither the principal's investment nor her attention technology, which we thus call the principal's *type*.

A "heuristic timing" of the game within each instant is as follows: first, the principal decides whether to fix the technology (if low) and the agent chooses effort; next, recognition arrives or not and the agent receives a reward accordingly; finally, the attention technology (if high) breaks or not.

A remark on terminology: the principal's attention technology is a "monitoring technology." However, "monitoring" is typically used in the literature (see our review in the Introduction) as referring to monitoring that is of bad rather than good performance, and a flow rather than an asset. We use the term "attention" to make clear the distinction with our assumptions.

Strategies and payoffs. Let h^{At-} be the agent's private history up to (but not including) time t, consisting of the history of effort choices and recognition (public signal) arrival times up to t. A (pure) strategy for the agent specifies, for each h^{At-}, a choice of effort at t, a_t. This strategy, $a = \{a_t\}_{t \geq 0}$, is progressively measurable with respect to the filtration induced by the histories h^{At-}. The principal's private history is denoted h^{Pt-} and consists of the history of type realizations (equivalently, θ_0, investment decisions, and depreciation shocks) and recognition arrival times up to t. A strategy for the principal specifies, for each time t at which the technology breaks (i.e. the type switches from high to low)

[11] We model attention as a capital asset in dynamic industrial organization models. In Besanko and Doraszelski (2004), for example, an asset is subject to two forces in each period: endogenous investment raises its value and exogenous depreciation lowers it. We consider a continuous-time two-value version of this process. In the reputation models of Board and Meyer-ter-Vehn (2013, 2014) and Dilmé (2014), firm quality also takes one of two values. Board and Meyer-ter-Vehn assume that the firm can increase quality only if a shock occurs when it invests; in our setting, the principal's instantaneous probability of investment corresponds to the transition probability from a low to a high technology. Dilmé assumes that the firm fully controls quality, so there are no exogenous shocks to quality in his model.

and history up to time t, h^{Pt-}, a cumulative distribution $Q(z,t)$ over the time $z \geq t$ to invest in fixing it. This strategy, $Q = \{Q(z,t)\}_{z\geq t, t\geq 0}$, is progressively measurable with respect to the filtration induced by the histories h^{Pt-}. Note that the investment plan chosen by the principal at any time t at which the technology breaks will be sequentially optimal. In particular, since nothing happens until the principal invests—as the type remains low in the absence of investment and no recognition can occur while the type is low—the principal will want to follow the prescribed distribution.

The agent's belief that the principal's type is high at (the beginning of) time t is $x_t \equiv \mathbb{E}^{\widetilde{Q},x_0}[\theta_t | h^{At-}] \in [0,1]$, where $\widetilde{Q} = \{\widetilde{Q}(z,t)\}_{z\geq t, t\geq 0}$ is the agent's belief about the principal's investment strategy and $x_0 \in (0,1]$ is the exogenous and commonly known prior belief.[12] That is, the agent's belief about the principal's type depends on the history of effort and recognition arrival times through the conjectured investment strategy $\widetilde{Q}$ and prior x_0. Since x_t is then a function of h^{At-} only and we assume the agent's effort strategy to be progressively measurable, $x = \{x_t\}_{t\geq 0}$ is also progressively measurable.

Both parties are risk neutral. For tractability and to focus on the dynamic incentives of the principal, we assume that the agent is myopic; Section 4 considers the case of a forward-looking agent. Given belief x_t and effort a_t, the agent's (instantaneous) expected payoff at time $t \geq 0$ is

$$U_t = \mu x_t a_t \bar{b} - \frac{1}{2} a_t^2.$$

The first-order condition for the optimal choice of effort, given belief x_t, is

$$a_t = \mu x_t \bar{b}. \tag{1}$$

To rule out corner solutions, we assume:

Assumption 1. $\mu \bar{b} \leq 1$.

Hence, condition (1) always pins down the agent's optimal level of effort.

[12] $\widetilde{Q}$ could be a non-trivial probability distribution over Q. In this case, $\widetilde{Q}$ is a Borel probability measure over the set of cumulative distribution functions $Q(z,t)$. As we need to define Borel sets over this set, we endow it with the weak*-topology. See Aliprantis and Border (2007).

The principal discounts future payoffs at rate $r > 0$. Denote by $(T_j)_{j=1}^{\infty}$ the (stochastic) instants at which the principal's attention technology breaks. Let $P(z, T_j, T_{j+1}) \equiv \frac{Q(z,T_j)}{Q(T_{j+1},T_j)}$ be the cumulative distribution over the time to invest in fixing the technology that the principal chooses at time T_j given that the technology breaks again at T_{j+1}, which implies that the principal must have invested in between these times (only a high technology can break). Then given $a = \{a_t\}_{t\geq 0}$ and $Q = \{Q(z,t)\}_{z\geq t, t\geq 0}$, the principal's expected payoff at time 0 is

$$\pi_0 = \mathbb{E}^{Q,\theta_0}\left[\int_0^{\infty} e^{-rt} a_t dt - \sum_{j=1}^{\infty} \int_{T_j}^{T_{j+1}} e^{-rz} F dP(z, T_j, T_{j+1})\right].$$

Equilibrium. Each time there is recognition, the agent learns that the principal's type is high, so his belief is reset to $x_t = 1$. The agent receives no information about the principal's type until recognition again occurs. We therefore restrict attention to equilibria in strategies that depend only on what has happened since the last recognition (see, e.g., Board and Meyer-ter-Vehn, 2014). With a slight abuse of notation, we drop the time index in our analysis and write all variables as a function of the time that has passed since recognition, denoted by s.

Analogous to the definitions above, a strategy for the agent is $a = \{a_s\}_{s\geq 0}$ (progressively measurable with respect to the filtration induced by the histories h^{As-}) and a strategy for the principal is $Q = \{Q(z,s)\}_{z\geq s, s\geq 0}$ (progressively measurable with respect to the filtration induced by the histories h^{Ps-}). An *equilibrium* is a quadruple $(a, Q, \widetilde{Q}, x)$ such that: (i) given x, a maximizes the agent's (instantaneous) expected payoff; (ii) given a, Q maximizes the principal's expected payoff; (iii) $\widetilde{Q}$ is correct; and (iv) given a and $\widetilde{Q}$, x is updated by Bayes' rule. A *continuous equilibrium* is an equilibrium in which x_s is continuous for all $s \geq 0$.[13] That is, in a continuous equilibrium, the agent's belief as a function of time t is continuous in the absence of publicly observable events. In any continuous equilibrium, $Q(z,s)$ must be continuous, and we will further assume in this case that it is absolutely continuous, so it admits a density function. We then

[13] Board and Meyer-ter-Vehn (2013) use a similar notion in their model of reputation.

take the principal's strategy in a continuous equilibrium to simply specify an instantaneous probability of investment as a function of the time since recognition and the principal's current type (note that past type realizations are payoff irrelevant for the principal given her current type). More precisely, in a continuous equilibrium, the principal's strategy specifies an instantaneous investment probability q_s for each time $s \geq 0$ conditional on a low type at s, $\theta_s = L$ (and zero investment if $\theta_s = H$). The agent's belief about q_s is denoted $\widetilde{q}_s$.

Observable attention benchmark. Before turning to the equilibrium analysis, consider a benchmark in which the principal's attention technology is observable by the agent. It follows from (1) that the agent's effort at any time $t \geq 0$ is $a_t = \mu\bar{b}$ if $\theta_t = H$ and $a_t = 0$ if $\theta_t = L$. Take now any time t at which the technology breaks. For any $\delta > 0$, the principal prefers to fix the technology at time t and then again at $t+\delta$ if it breaks by then, rather than waiting and fixing the technology at time $t+\delta$, if and only if

$$\int_0^\delta e^{-(\gamma+r)\tau}\mu\bar{b}d\tau - F - \left(1 - e^{-\gamma\delta}\right)e^{-r\delta}F \geq -e^{-r\delta}F.$$

Integrating and simplifying this expression yields that the principal fixes her attention technology whenever it breaks if and only if $\mu\bar{b} \geq (\gamma+r)F$, that is, if and only if the resulting increase in output is larger than the instantaneous rental cost of capital, given by the risk of breakdown plus the interest rate. It is immediate that this condition is also necessary for the principal to invest in attention technology when the technology is unobservable by the agent. Throughout our analysis, we thus assume that parameters are such that this condition is satisfied:

Assumption 2. *$\mu\bar{b} \geq (\gamma + r)F$.*

2 Relationship dynamics

Unlike in the observable attention benchmark described above, in equilibrium the principal cannot fix her attention technology each time it breaks when attention is unobservable: if the principal always invests, the agent's belief that the principal's type is high is $x_s = 1$ at all $s \geq 0$, but then the agent always exerts effort $a_s = \mu\bar{b}$ and the principal has no incentives to invest at cost F.

We construct a continuous equilibrium in which the principal does not invest if the time that has passed since recognition is $s < \bar{s}$, for an endogenous threshold time $\bar{s} \in (0, \infty)$, and she mixes between investing and not investing if $s \geq \bar{s}$. We show that any continuous equilibrium with positive investment must take this form, and one such equilibrium exists if the cost of investment F is small enough.

Consider first the agent's belief about the principal's type, x_s. At $s = 0$ this belief is $x_0 = 1$, since recognition fully reveals that the type is high. Then, given no recognition, the change in x_s is given by three sources: (i) the possibility that a high attention technology breaks, with instantaneous probability γ; (ii) learning about the technology in the absence of recognition, according to Bayes' rule; and (iii) the agent's belief about the principal's investment, which must be correct in equilibrium (i.e. $\widetilde{q}_s = q_s$). If q_s is continuous on a given open interval, then locally in that interval the evolution of x_s is governed by

$$\dot{x}_s = -\gamma x_s - x_s(1 - x_s)\mu a_s + (1 - x_s)q_s. \tag{2}$$

This law of motion is similar to that in Board and Meyer-ter-Vehn (2013), with an important difference: our Bayesian learning term, $x_s(1 - x_s)\mu a_s$, depends on the agent's action a_s, while it is only a function of x_s in their paper (see their equation 2.2). In our setting the learning process is endogenous and depends on the agent's behavior; as shown in Section 3, this difference has important implications for the dynamics of the relationship.

In the equilibrium we are constructing, the principal does not invest before the threshold time $\bar{s}$ is reached. Hence, for $s < \bar{s}$, the law of motion for x_s becomes

$$\dot{x}_s = -\gamma x_s - x_s^2(1 - x_s)\mu^2\bar{b}, \tag{3}$$

where we have substituted $a_s = \mu x_s \bar{b}$. Solving this differential equation with initial condition $x_0 = 1$ uniquely pins down the agent's belief and effort at $s < \bar{s}$. Naturally, x_s and a_s are strictly decreasing over this time period: since the principal is not investing, the agent becomes more pessimistic that her type is high as time passes without recognition, both because of Bayesian updating and because the probability that the principal's technology has broken goes up.

Consider next the principal's incentives to invest. Let π_s^H be the principal's expected payoff at s when her type is $\theta_s = H$ and π_s^L when her type is $\theta_s = L$. The principal is willing to invest at s only if $\pi_s^L \leq \pi_s^H - F$. Since, by construction, at any point in this equilibrium the principal either does not want to invest or is indifferent between investing and not, $\pi_s^L \geq \pi_s^H - F$ for all $s \geq 0$ and we can write the expected payoffs for the two principal types as

$$\pi_s^L = \int_s^\infty e^{-r(\tau-s)} a_\tau d\tau, \tag{4}$$

$$\pi_s^H = \int_s^\infty e^{-(\gamma+r)(\tau-s)-\int_s^\tau \mu a_z dz} \left(a_\tau + \gamma \pi_\tau^L + \mu a_\tau \pi_0^H\right) d\tau. \tag{5}$$

The low type's expected payoff at any point is simply the output given by the agent's effort. To interpret the high type's expected payoff, note that the instantaneous probability that her technology breaks at any time is γ, and the instantaneous probability that recognition occurs given a high type at time z is μa_z. Thus, between times s and τ, the probability that no breakdown occurs is $e^{-\gamma(\tau-s)}$, and the probability that no recognition occurs, given no breakdown, is $e^{-\int_s^\tau \mu a_z dz}$. So long as neither breakdown nor recognition occurs, the high type receives the output flow. If the technology breaks at time τ, her continuation payoff is π_τ^L, whereas if recognition occurs, her continuation payoff is π_0^H.

To convey the economic intuitions more clearly, let $\Psi_s \equiv \pi_0^H - \pi_s^H$ denote the principal's value of recognition at time s and $\Lambda_s \equiv \pi_s^H - \pi_s^L$ her value of investing at s. At each time $s \geq \bar{s}$ in the equilibrium, the principal follows a mixed strategy, so she must be indifferent between investing and not investing:

$$\Lambda_s = F. \tag{6}$$

It follows that $\dot{\Lambda}_s = 0$ for all $s \geq \bar{s}$, which combined with (4)-(6) yields

$$\Psi_s \mu a_s = (\gamma + r) F. \tag{7}$$

The left-hand side is the principal's instantaneous benefit of investment at time

s, given by the value of recognition times the instantaneous probability of recognition conditional on a high type. Equation (7) says that, for the principal to be indifferent at each time $s \geq \overline{s}$, her instantaneous benefit of investment must be equal to the instantaneous rental cost of capital at all such times. The instantaneous benefit of investment must therefore be constant for $s \geq \overline{s}$, which implies that the agent's effort must be decreasing (increasing) at any such time at which the value of recognition is increasing (decreasing). Our main result shows that the value of recognition is in fact strictly increasing, and hence effort strictly decreasing, at all times $s \geq \overline{s}$ in the equilibrium; moreover, effort becomes low enough that the principal's investment must become strictly decreasing as well.

Proposition 1. *Fix any set of parameters $(\gamma, \mu, \overline{b}, r)$. There exist $\overline{F} > 0$ and $\overline{\overline{F}} \in (0, \overline{F})$ such that a continuous equilibrium with positive investment exists if and only if the cost of investment is $F \leq \overline{F}$, and all continuous equilibria have positive investment if $F \leq \overline{\overline{F}}$. In any continuous equilibrium with positive investment, the principal does not invest if the time that has passed since recognition is $s < \overline{s}$, for a threshold time $\overline{s} \in (0, \infty)$; at $s \geq \overline{s}$, the principal invests with a continuous instantaneous probability $q_s \in (0, \infty)$ which is strictly decreasing for $s > \overline{s}$ large enough. The agent's belief x_s and effort a_s, and thus the unconditional instantaneous probability of recognition $\mu x_s a_s$, are continuous and strictly decreasing for all $s \geq 0$. If $r \leq \gamma$, the value of $\overline{s}$, and thus the continuous equilibrium with positive investment, are unique.*

Proof. See Appendix A. *Q.E.D.*

Figure 1 illustrates the equilibrium. When the agent's belief is relatively high, the principal's benefit from being revealed to be a high-attention type is small. Following recognition, there is thus a period of time during which the principal does not invest. As time passes and the agent's belief and effort go down, the principal's value of recognition increases, until at time $\overline{s}$ the principal finds it optimal to start investing. The principal's investment however is insufficient to stop the decline in effort, and at some point it also begins to decline. Hence, as time goes by without recognition, both the agent's effort and the principal's investment decrease, and so does the probability of obtaining recognition.

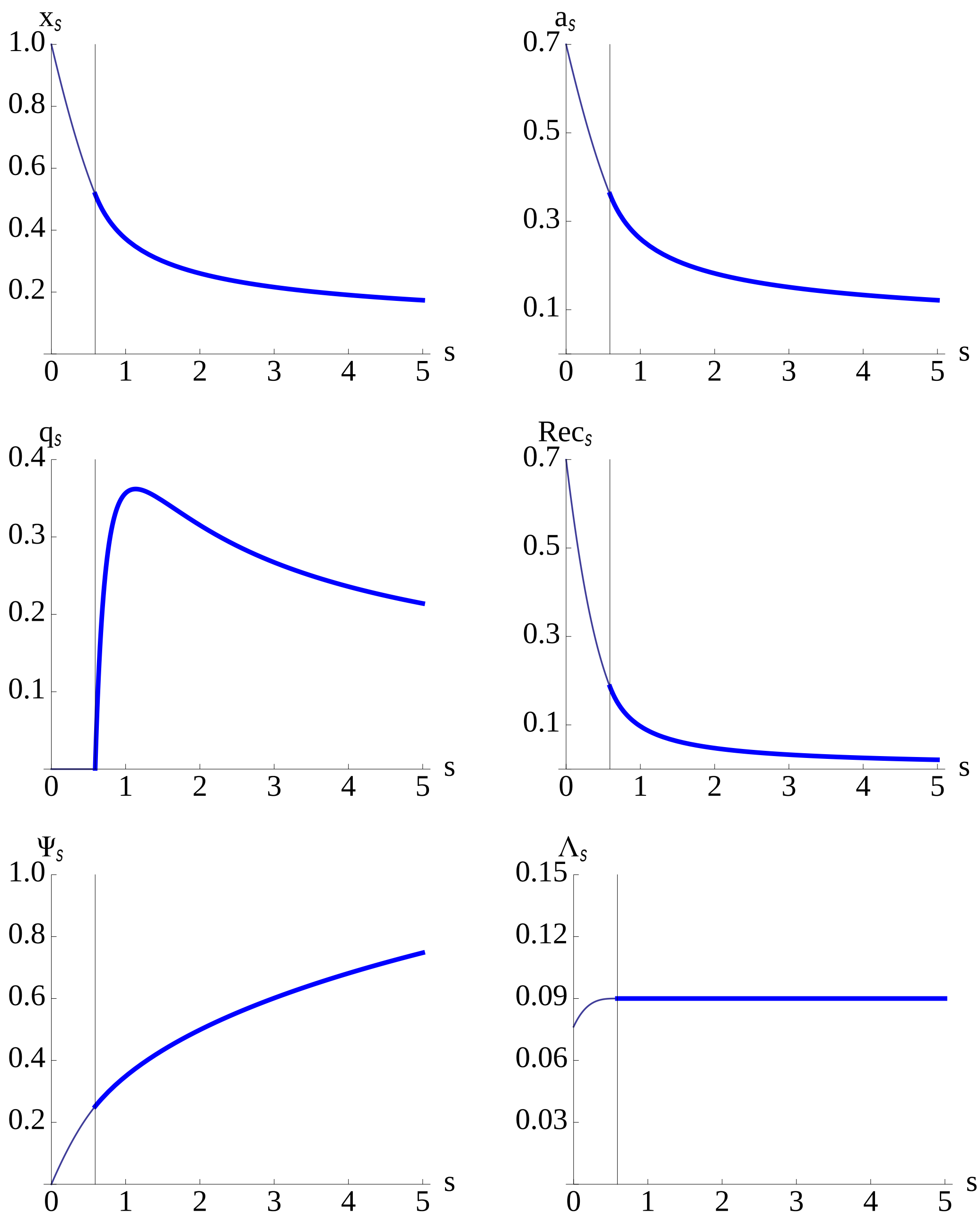

Figure 1: Equilibrium dynamics. Parameters are $\gamma = 1$, $F = 0.09$, $\mu = 1$, $\bar{b} = 0.7$, and $r = 0.01$. Rec_s is the unconditional instantaneous probability of recognition, given by $\mu x_s a_s$. The vertical line indicates the threshold time $\bar{s}$.

As mentioned, the intuition for why the agent's effort decreases prior to $\overline{s}$ is immediate. But why must effort continue going down at $s \geq \overline{s}$ while the principal invests? Recall from (7) that the principal's indifference condition yields $\Psi_s = \frac{(\gamma+r)F}{\mu a_s}$. Moreover, by definition, $\dot{\Psi}_s = \dot{\Lambda}_s - \dot{\pi}_s^L$, and thus for $s \geq \overline{s}$, $\dot{\Psi}_s = -\dot{\pi}_s^L = a_s - r\pi_s^L$; that is, the value of recognition increases (decreases) at s if current effort is above (below) the long-run future average. We can then show that $\dot{\Psi}_s$ and $\dot{a}_s$ cannot change signs at a time $s > \overline{s}$. For a stark intuition, suppose effort were decreasing up to s' and increasing from then on, for $s' > \overline{s}$. Then both the value of recognition Ψ_s and the conditional probability of recognition μa_s would be lower at s' than right before this time. But since the principal is indifferent at s', she would have strict incentives to invest right before, a contradiction. Hence, effort a_s must be a monotonic function over $s \geq \overline{s}$, and in fact an analogous argument shows that it cannot be a strictly increasing function.

It follows that the agent's effort is either constant or strictly decreasing for all $s \geq \overline{s}$. We prove that it must indeed be strictly decreasing in equilibrium by showing that not only the agent's belief is continuous but also the change in the belief must be continuous. Specifically, the equilibrium must satisfy smooth pasting: $\dot{x}_s$ is continuous at $\overline{s}$. Since $\dot{x}_s < 0$ for $s < \overline{s}$, this implies that $\dot{x}_s$, and thus $\dot{a}_s$, are strictly negative in a right neighborhood of $\overline{s}$, and therefore for all $s \geq \overline{s}$. The logic for smooth pasting is similar to that above, namely it is needed to provide the principal the right amount of incentives to invest at each point. Suppose for the purpose of contradiction that $\dot{x}_s$ and thus $\dot{a}_s$ were to jump at $\overline{s}$. Clearly, they can only jump up (recall $q_s = 0$ at $s < \overline{s}$), and since $\dot{\Psi}_s$ is continuous,[14] the change in the principal's instantaneous benefit of investment, $\mu\big(\dot{\Psi}_s a_s + \Psi_s \dot{a}_s\big)$, would then also jump up at $\overline{s}$. However, since $\mu\big(\dot{\Psi}_s a_s + \Psi_s \dot{a}_s\big) = 0$ from $\overline{s}$ on by condition (7), this would imply $\mu\big(\dot{\Psi}_s a_s + \Psi_s \dot{a}_s\big) < 0$ for $s < \overline{s}$ close enough to $\overline{s}$, and therefore (using (7) again) $\Psi_s \mu a_s > (\gamma + r)F$ for such times. The contradiction is then immediate: the principal would have strict incentives to invest in attention technology before reaching the threshold time $\overline{s}$.[15]

[14] Note that $\dot{\Psi}_s = \dot{\Lambda}_s - \dot{\pi}_s^L$, where $\dot{\pi}_s^L = -a_s + r\pi_s^L$ and $\dot{\Lambda}_s = (\gamma+r)\Lambda_s - \mu a_s \Psi_s$ are continuous.

[15] More formally: observe that, as noted in fn. 14, $\dot{\Lambda}_s = (\gamma+r)\Lambda_s - \mu a_s \Psi_s$, and the equilibrium requires $\Lambda_s < F$ for $s < \overline{s}$ and $\Lambda_s = F$ for $s \geq \overline{s}$. But then $\Psi_s \mu a_s > (\gamma + r)F$ for $s < \overline{s}$ close to $\overline{s}$ implies $\dot{\Lambda}_s < 0$ at such times, a contradiction.

A principal who delays investment is therefore punished with continuous deterioration of the relationship. The agent becomes more pessimistic and his effort declines over time, so that even if the principal then decides to invest, it is harder to obtain recognition and return to high performance. In fact, we can show that in the limit for an infinitely patient principal, the equilibrium gives rise to a trap as the moral hazard problem becomes more severe. Consider the limit as the discount rate r goes to zero and let the principal's cost of investment be $F < \overline{F}$ so that the equilibrium of Proposition 1 exists. Appendix B shows that as F approaches $\overline{F}$, $\lim_{s\to\infty} x_s$ and $\lim_{s\to\infty} a_s$ vanish: the probability of obtaining recognition and reverting to high performance goes to zero as time passes without recognition.[16]

Proposition 1 also shows that the equilibrium investment path is hump-shaped. The logic is elaborate because the principal uses a mixed strategy, but to see the main idea, take the aforementioned case in which the agent's belief and effort go to zero absent recognition. At one extreme, it is clear that the principal will not invest when the agent's belief is close to one, as the value of recognition is then close to zero. At the other extreme, it is also clear that as the agent's belief approaches zero, the principal's investment must go to zero: as shown by (2), if $x_s = 0$, $\dot{x}_s$ is determined by (the agent's correct belief over) q_s, so x_s, and hence a_s, would increase in equilibrium if $q_s > 0$. More generally, we show that punishing the principal for not investing requires effort to become low enough over time that investment must eventually become decreasing for effort to continue declining.

Finally, about uniqueness: two arguments are used to show that any continuous equilibrium with positive investment must be as characterized in Proposition 1. First, we show that any such equilibrium where, at each time $s \geq 0$, the principal either is indifferent or does not have incentives to invest, must take this form. Second, we show that a continuous equilibrium where the principal has strict incentives to invest over some time interval does not exist. Intuitively, in any such equilibrium, the principal should strictly want to invest at $s = 0$;

[16] Put differently, for any $\delta > 0$ and $\varepsilon > 0$, there exist $r > 0$, $F < \overline{F}$, and a time since recognition $s > 0$ such that the probability of obtaining recognition in $(s, s+\delta)$ is strictly less than ε.

otherwise, if she has strict incentives to invest at $s' > 0$ but not before, x_s must either increase continuously toward one as s approaches s' or jump to one at s', neither of which can occur. However, since $\Psi_0 = 0$, the principal will not want to invest at $s = 0$. As for the result that any continuous equilibrium must be as described in Proposition 1 if F is small enough, this follows from the fact that a no-investment equilibrium does not exist in that case: without investment, effort decreases as time passes without recognition, but then for F sufficiently small the principal eventually has strict incentives to invest to obtain recognition.[17]

3 Recognition of good and bad performance

We have considered a principal who can recognize *good* performance by the agent. What happens if she can also recognize *bad* performance? Recognition is more likely to be of good performance in jobs based on innovation, where the verifiable event is the presence of something positive like a breakthrough. Recognition of bad performance, on the other hand, may arise in jobs where employees perform well-defined tasks, like maintenance or quality control, and the verifiable event is the presence of something negative like a fault.

Take the model of Section 1 but assume now that there are two types of (verifiable) signals: good-performance signals and bad-performance signals. A good signal arrives via a Poisson process with parameter $\mu\theta_t a_t$, whereas a bad signal arrives via a Poisson process with parameter $\nu\theta_t(1 - a_t)$, where $\mu, \nu \geq 0$. The agent receives a reward $\bar{b} > 0$ when a good signal arrives and incurs a penalty $\underline{b} < 0$ when a bad signal arrives. Analogous to (1) and Assumption 1, for any $t \geq 0$, the agent's effort is $a_t = x_t(\mu\bar{b} - \nu\underline{b})$, where we assume $\mu\bar{b} - \nu\underline{b} \leq 1$.

The benchmark case where the principal's attention technology is observable is qualitatively the same as that in Section 1. In this setting with good- and bad-performance signals, the principal invests in attention if and only if $\mu\bar{b} - \nu\underline{b} \geq (\gamma + r)F$; analogous to Assumption 2, we assume that this condition holds.

Now suppose that the principal's attention technology is unobservable by the

[17] Note that the agent's belief x_s and effort a_s are strictly positive for all finite s. An equilibrium with no effort and no investment cannot be sustained unless the agent's prior belief that the principal's type is high is assumed to be zero.

agent. Each time recognition—of either good or bad performance—occurs, the agent learns that the principal's type is high and thus his belief is reset to one. As in Section 2, we consider equilibria in strategies that depend only on what has happened since recognition last occurred. Let s be the time since recognition. We characterize a continuous equilibrium (i.e., an equilibrium in which the agent's belief x_s is continuous) in which the principal does not invest in attention technology if $s < \widehat{s}$, for an endogenous threshold time $\widehat{s} \in (0, \infty)$, and she mixes between investing and not if $s \geq \widehat{s}$.

Consider the law of motion for the agent's belief x_s. At $s = 0$, the belief is $x_0 = 1$. Then, given no recognition, the evolution of the belief on any open interval over which q_s is continuous is governed by

$$\dot{x}_s = -\gamma x_s - x_s(1 - x_s)[\mu a_s + \nu(1 - a_s)] + (1 - x_s)q_s. \tag{8}$$

Before time $\widehat{s}$ is reached, the principal does not invest. Substituting $a_s = x_s(\mu\bar{b} - \nu\underline{b})$ and $q_s = 0$ into (8), the law of motion for $s < \widehat{s}$ is

$$\dot{x}_s = -\gamma x_s - x_s(1 - x_s)[(\mu - \nu)x_s(\mu\bar{b} - \nu\underline{b}) + \nu]. \tag{9}$$

Solving this differential equation with initial condition $x_0 = 1$ pins down the agent's belief and effort at $s < \widehat{s}$.

Consider now $s \geq \widehat{s}$. The principal must be indifferent between investing and not investing at these times. Analogous to (7), indifference yields

$$\Psi_s[\mu a_s + \nu(1 - a_s)] = (\gamma + r)F. \tag{10}$$

Condition (10) shows that the sign of $\mu - \nu$ is key in determining the qualitative properties of the solution. If $\mu - \nu > 0$, the solution is qualitatively the same as that in Section 2. Suppose instead that $\mu - \nu \leq 0$. Then (10) shows that the agent's effort a_s and the principal's value of recognition Ψ_s must move in the same direction for the principal's instantaneous benefit of investment to be constant for $s \geq \widehat{s}$. Now since $\dot{\Psi}_s = a_s - r\pi^L_s$, it follows that effort must be strictly increasing (decreasing) at any time $s \geq \widehat{s}$ at which it is strictly above (below) its long-run future average, which implies that effort cannot be different from the long-run average at any such time. Therefore, when $\mu - \nu \leq 0$, a_s and Ψ_s must be constant for $s \geq \widehat{s}$.

Proposition 2. *Consider a setting with recognition of good and bad performance. If recognition is primarily of good performance (i.e. $\mu > \nu$), the continuous equilibria are as characterized in Proposition 1. Suppose instead that recognition is primarily of bad performance or symmetric (i.e. $\mu \leq \nu$). Then in any continuous equilibrium with positive investment, the principal does not invest if the time that has passed since recognition is $s < \widehat{s}$, for a threshold time $\widehat{s} \in (0, \infty)$, and she invests with a constant instantaneous probability $\widehat{q} \in (0, \infty)$ for $s \geq \widehat{s}$. The agent's belief x_s and effort a_s are decreasing for $s < \widehat{s}$ and constant for $s \geq \widehat{s}$. The unconditional instantaneous probability of recognition, $x_s[\mu a_s + \nu(1 - a_s)]$, is constant for $s \geq \widehat{s}$.*

Proof. See Appendix C. *Q.E.D.*

Figure 2 illustrates the equilibria. When recognition is primarily of bad performance or symmetric, the equilibrium is essentially static: the principal's investment is zero initially and constant after it jumps up at the threshold time $\widehat{s}$; the agent's belief and effort and the unconditional instantaneous probability of recognition are also constant for $s \geq \widehat{s}$.[18] Hence, unlike when recognition is primarily of good performance, the relationship does not continue deteriorating over time.

The intuition for these results stems from the principal's incentives to invest in attention technology. The principal's instantaneous benefit of investment is the product of the instantaneous probability of recognition conditional on a high attention technology and the value of recognition. When recognition is of bad performance or symmetric, a decline in the agent's effort has a direct effect of (weakly) increasing the principal's incentive to invest because it (weakly) increases the probability of obtaining recognition. However, when recognition is of good performance, a decline in effort has a negative direct effect, as the probability of recognition goes down. Consequently, incentivizing the principal to invest

[18] Appendix C shows that a continuous equilibrium with positive investment exists if and only if the cost of investment F is small enough. Moreover, the value of $\widehat{s}$ is unique.

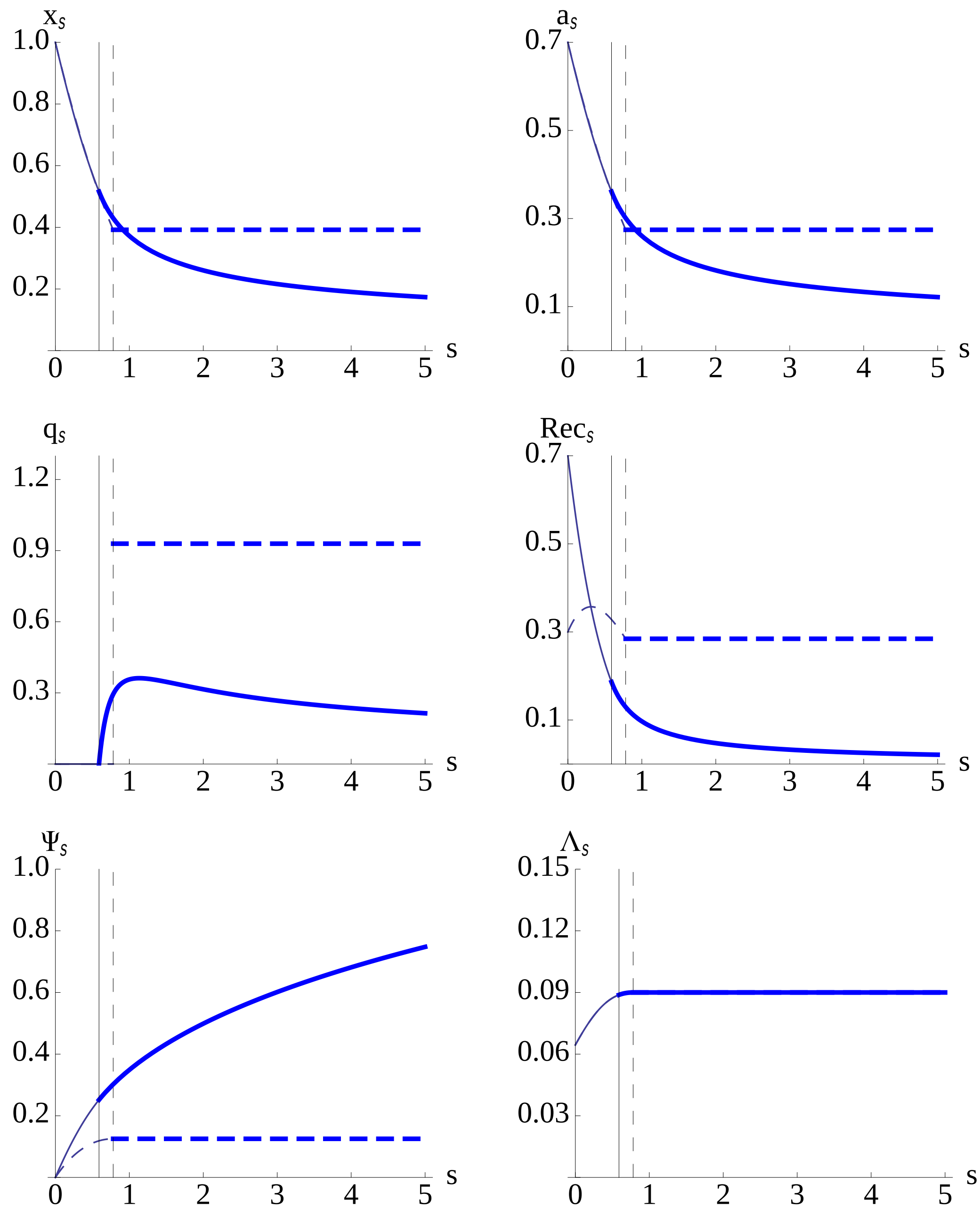

Figure 2: Equilibrium dynamics when recognition is of good performance (solid lines) and when recognition is of bad performance (dashed lines). We set $\mu = 1$ and $\nu = 0$ in the former case and $\mu = 0$ and $\nu = 1$ in the latter; all other parameters are the same as in Figure 1. Rec_s is the unconditional instantaneous probability of recognition, given by $x_s[\mu a_s + \nu(1-a_s)]$. The vertical lines indicate the threshold times $\overline{s}$ and $\widehat{s}$.

in this case requires that her value of recognition increase; that is, she must be threatened with continuously decreasing effort over time.

Our results have implications for the study of firm reputation. Note that when recognition is symmetric ($\mu = \nu$), the instantaneous probability with which a signal arrives is independent of the agent's effort. This case corresponds to the one typically studied in models of firm reputation. For example, Board and Meyer-ter-Vehn (2013) consider a setting in which consumers observe public signals of the quality of a firm's product, but the arrival rate of these signals is independent of consumers' actions (which are not explicitly modeled). The reputational dynamics they obtain are as we characterize for the case of $\mu \leq \nu$.[19]

In reality, however, consumers are more likely to learn about the quality of a product when the volume of sales is larger, both because more consumers experience with the product directly and because more consumers are likely to learn from the experience of others.[20] To map our model into this problem, take θ_s to be firm quality, x_s consumers' expectation of firm quality, and a_s the volume of sales at time s. The case studied in Board and Meyer-ter-Vehn (2013) is one in which the firm sells a single unit at each time and consumers compete in a Bertrand fashion, so the product price equals x_s and a_s is fixed. But another possibility is for the price to be fixed and quantity to adjust, so that the volume of sales a_s is an increasing function of perceived quality x_s, analogous to our model in Section 2. Here sales will change with perceived quality and in turn affect the rate at which information about quality is generated. As shown in Proposition 1 and Proposition 2, this effect leads to qualitatively different reputational dynamics.

4 Discussion

Forward-looking agent. For tractability and to focus on the principal's dynamic incentives, we assumed throughout that the agent is myopic. The presence of a forward-looking player and a myopic one is in line with the literature on firm

[19] Specifically, see the case of a convergent cutoff in their perfect good news setting.

[20] Rob and Fishman (2005) study a model in which information about firm quality spreads in the market through word of mouth.

reputation. In practice, however, workers are not fully myopic, and they can benefit from experimenting: unlike a myopic agent, they value having information in the future about whether managerial attention is high or low.

While we cannot solve the model with a forward-looking agent analytically, we can numerically construct an equilibrium analogous to the one in Proposition 1 and show that it yields qualitatively the same relationship dynamics as with a myopic agent. Consider the setting of Section 1, in which recognition is of good performance, but assume now that the agent is forward-looking and discounts the future at the same rate $r > 0$ as the principal. The agent's expected payoff following recognition is

$$U_0 = \int_0^\infty e^{-\int_0^s (r+\mu x_\tau a_\tau)d\tau} \left[\mu x_s a_s(\overline{b} + U_0) - \frac{1}{2}a_s^2\right] ds.$$

For intuition, note that the instantaneous probability assigned by the agent to recognition occurring at a time τ is $\mu x_\tau a_\tau$, and hence his belief that recognition will not occur by time s is $e^{-\int_0^s \mu x_\tau a_\tau d\tau}$. When recognition occurs, the agent receives the reward $\overline{b}$ plus an expected continuation payoff U_0.

The agent chooses an effort plan $\{a_s\}_{s\geq 0}$ to maximize U_0 subject to the law of motion for x_s given in equation (2), where q_s is taken as given. Appendix D sets up the Hamiltonian for the agent's problem and derives the first-order conditions.[21] We then solve for a continuous equilibrium that parallels the one we constructed in Section 2: for a threshold time $\overline{s} \in (0, \infty)$, the principal does not invest if the time that has passed since recognition is $s < \overline{s}$ and she mixes between investing and not investing if $s \geq \overline{s}$.

Figure 3 provides a graphical illustration. The figure shows that the equilibrium dynamics are qualitatively the same with a forward-looking agent and with a myopic agent. As expected, though, there are quantitative differences. We find that in the forward-looking agent case, the agent's effort and the principal's investment are higher. The intuition is related to the value of experimentation mentioned above: because a forward-looking agent benefits from knowing in the

[21]Note that given the principal's investment strategy, the agent chooses a sequence of effort to maximize his expected utility taking into account how his belief evolves as a function of his effort choices and the principal's investment (and recognition). This computation ensures that no deviation (including double deviations) is profitable for the agent.

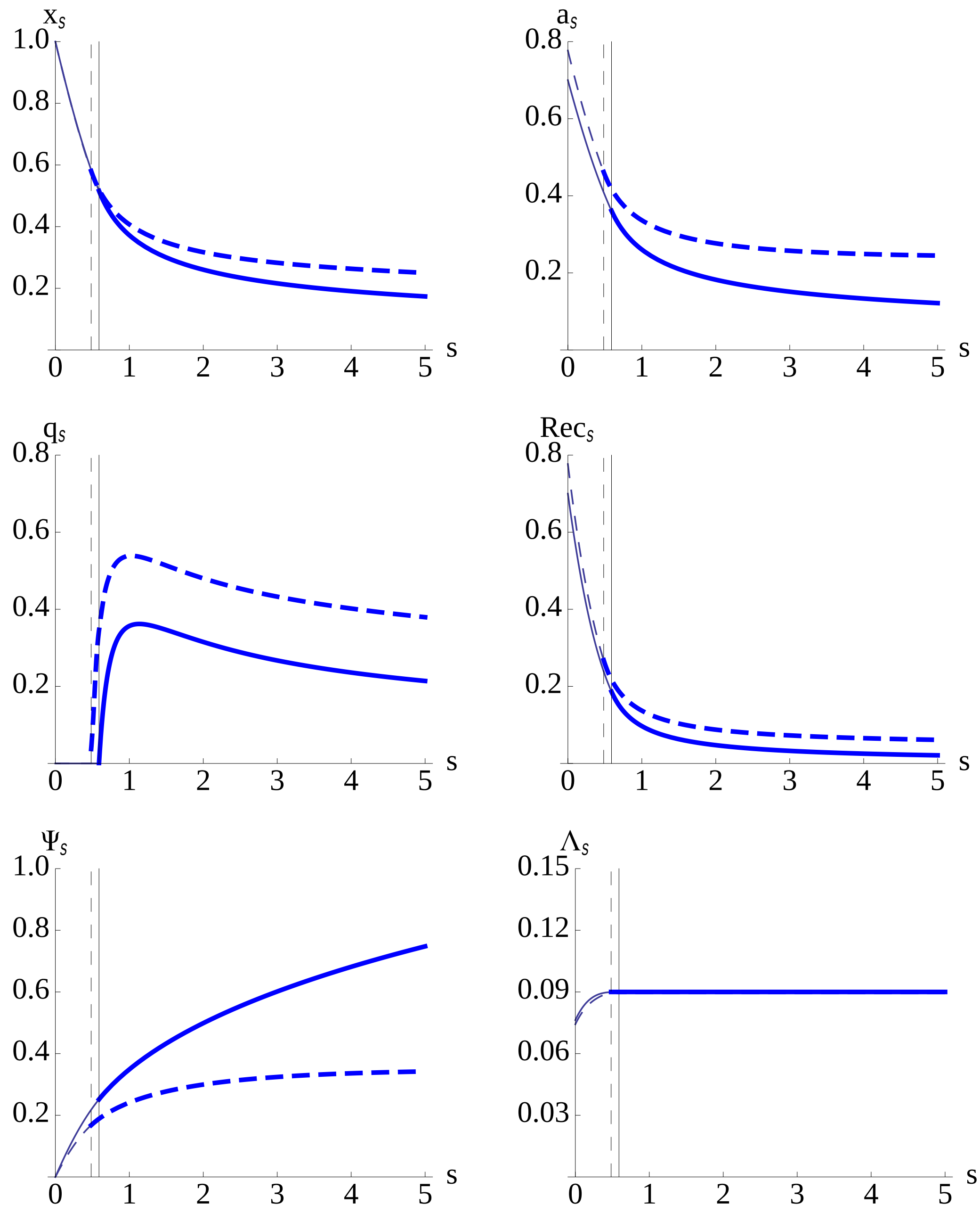

Figure 3: Equilibrium dynamics with a myopic agent (solid lines) and with a forward-looking agent (dashed lines). Parameters are the same as in Figure 1. Rec_s is the unconditional instantaneous probability of recognition, given by $\mu x_s a_s$. The vertical lines indicate the threshold times $\overline{s}$ in each equilibrium.

future whether the principal's type is high or low, for any given belief about the principal's current type, his incentive to exert effort is higher than that of a myopic agent. Given the complementarity between effort and investment, the principal in turn invests more when the agent is forward-looking.

Recognition reward. In our model, the recognition reward $\bar{b}$ entails no costs to the principal and has a fixed exogenous value. This formulation is appealing if the reward is taken to be purely psychological. Suppose we instead take $\bar{b}$ to be (partly) a monetary bonus. Then our model has assumed that this bonus is paid not by the principal but by some external, unmodeled party, and that its value is set exogenously. The first assumption is convenient to focus on the moral hazard problem due to the principal's cost of investment and abstract from another source of moral hazard: if the principal incurs the cost of the bonus directly, she may want to decrease her investment in attention technology to save on this cost. The second assumption is a natural consequence of the first.

Our qualitative results are unchanged if we remove the first assumption while keeping the second one. That is, suppose $\bar{b}$ is an exogenously set bonus but the principal bears the cost of bonus payments. We can incorporate this by simply re-defining the principal's payoff following recognition as $\overline{\pi}_0^H \equiv \pi_0^H - \bar{b}$; our analysis can then be performed without change. The dynamics of the relationship are qualitatively the same as in our main model; quantitatively, of course, the principal's incentives to invest will now be lower.

Allowing for an endogenous (and time-varying) bonus, on the other hand, can lead to different dynamics, as the principal may increase the bonus over time to boost the agent's incentives. While a full solution to this case is beyond the scope of this paper, we highlight here a negative result: endogenizing the bonus does not eliminate the inefficiency in effort. To see why, suppose by contradiction that the agent's effort is always at the efficient level. Then the principal does not invest, as she receives the largest possible payoff when her attention technology is low and she bears no investment nor bonus costs. It follows that in equilibrium the agent's belief about the principal's type must go down as time passes without recognition,[22] and inducing efficient effort requires the bonus to become

[22]Note that the principal cannot signal her type through the bonus offer: the low type can

arbitrarily high over time. However, the high type is not willing to offer such a high bonus: the gain is no larger than the present value of future efficient effort, while the cost is proportional to the bonus as the high type has to pay the agent if recognition occurs before her technology breaks.

5 Conclusion

This paper has studied a dynamic two-sided moral hazard problem in which a worker chooses effort and the manager chooses whether to invest in an attention technology to recognize worker performance. We showed that when recognition is of good performance, the relationship falls into deterioration: absent recognition, worker effort and eventually managerial investment decrease, and a return to high productivity becomes less likely as time passes. These deteriorating dynamics do not arise when recognition is of bad performance or independent of effort.

Our work highlights the role of workers' beliefs about managerial attention. These beliefs have important implications for the dynamics of the employment relationship, particularly in jobs such as those based on innovation, where workers are rewarded for good contributions rather than punished for bad outcomes. We find that, as workers get pessimistic about the presence of a monitoring system that can recognize their contributions, they reduce their effort, and even if management then improves its monitoring system, it will find it difficult to restore its reputation. More broadly, our paper contributes to the theory of reputation by endogenizing the learning process and uncovering the effects of different forms of endogenous learning.

Our analysis restricted attention to continuous equilibria. There also exist discontinuous equilibria of our game, in which the worker's belief jumps in the absence of recognition. Discontinuous equilibria can in principle take many arbitrary forms. In Appendix E, we study a simple class of stationary discontinuous equilibria: as a function of the time since recognition, the manager invests only in countably many points, where the time in between these points is fixed and the manager invests with the same mass probability at each of them. We show that

always mimic the high type at no cost.

the manager prefers a continuous equilibrium, as characterized in Proposition 1, to any discontinuous equilibrium in this class. A characterization of the whole set of equilibria and their properties is left for future work.

References

Abreu, Dilip, Paul Milgrom, and David Pearce, "Information and Timing in Repeated Partnerships," *Econometrica*, 1991, *59* (6), 1713–1733.

Aliprantis, Charalambos and Kim Border, *Infinite Dimensional Analysis: A Hitchhiker's Guide*, Berlin: Springer-Verlag, 3rd ed., 2007.

Bandiera, Oriana, Andrea Prat, and Raffaella Sadun, "Managing the Family Firm: Evidence from CEOs at Work," 2013. NBER Working paper No. 19722.

_ , Luigi Guiso, Andrea Prat, and Raffaella Sadun, "What Do CEOs Do?," 2011. CEPR Working paper No. 8235.

Bar-Isaac, Heski and Steven Tadelis, "Seller Reputation," *Foundations and Trends in Microeconomics*, 2008, *4* (4), 273–351.

Bartling, Björn, Ernst Fehr, and Klaus Schmidt, "Screening, Competition, and Job Design: Economic Origins of Good Jobs," *American Economic Review*, 2012, *102* (2), 834–864.

Besanko, David and Ulrich Doraszelski, "Capacity Dynamics and Endogenous Asymmetries in Firm Size," *RAND Journal of Economics*, 2004, *35* (1), 23–49.

Bloom, Nicholas, Raffaella Sadun, and John Van Reenen, "Management as a Technology?," 2012. Working paper.

Board, Simon and Moritz Meyer-ter-Vehn, "Reputation for Quality," *Econometrica*, 2013, *81* (6), 2381–2462.

_ and _ , "A Reputational Theory of Firm Dynamics," 2014. Working paper.

Brynjolfsson, Erik and Paul Milgrom, "Complementarity in Organizations," in Robert Gibbons and John Roberts, eds., *The Handbook for Organization Economics*, Princeton: Princeton University Press, 2013.

Callander, Steven and Niko Matouschek, "Managing on Rugged Landscapes," 2014. Working paper.

Chassang, Sylvain, "Building Routines: Learning, Cooperation and the Dynamics of Incomplete Relational Contracts," *American Economic Review*, 2010, *100*, 448–465.

Corrado, Carol and Charles Hulten, "How Do You Measure a "Technological Revolution"?," *American Economic Review*, 2010, *100* (2), 99–104.

Cripps, Martin, "Reputation," *New Palgrave Dictionary of Economics*, 2006, *Second edition.*

Dilmé, Francesc, "Reputation Building through Costly Adjustment," 2014. Working paper.

_ **and Daniel Garrett**, "Residual Deterrence," 2014. Working paper.

Dur, Robert, "Gift Exchange in the Workplace: Money or Attention?," *Journal of the European Economic Association*, 2009, *7* ((2-3)), 550–560.

_ **, Arjan Non, and Hein Roelfsema**, "Reciprocity and Incentive Pay in the Workplace," *Journal of Economic Psychology*, 2010, *31*, 676–686.

Ely, Jeffrey and Juuso Välimäki, "Bad Reputation," *Quarterly Journal of Economics*, 2003, *118*, 785–812.

Garicano, Luis and Andrea Prat, "Organizational Economics with Cognitive Costs," in D. Acemoglu, M. Arellano, and E. Dekel, eds., *Advances in Economics and Econometrics, Tenth World Congress of the Econometric Society*, Cambridge: Cambridge University Press, 2013.

Geanakoplos, John and Paul Milgrom, "A Theory of Hierarchies Based on Limited Managerial Attention," *Journal of the Japanese and International Economies*, 1991, *5*, 205–225.

Gibbons, Robert and Rebecca Henderson, "What Do Managers Do? Exploring Persistent Performance Differences among Seemingly Similar Enterprises," in R. Gibbons and J. Roberts, eds., *Handbook of Organizational Economics*, Princeton: Princeton University Press, 2013.

Gil, Ricard and Jordi Mondria, "Introducing Managerial Attention Allocation in Incentive Contracts," *SERIEs – Journal of the Spanish Economic Association*, 2011, *2* (3), 335–358.

Graetz, Michael, Jennifer Reinganum, and Louis Wilde, "The Tax Compliance Game: Toward an Interactive Theory of Law Enforcement," *Journal of Law, Economics, and Organization*, 1986, *2* (1), 1–32.

Harter, James, Frank Schmidt, and Theodore Hayes, "Business-Unit-Level Relationship Between Employee Satisfaction, Employee Engagement, and Business Outcomes: A Meta-Analysis," *Journal of Applied Psychology*, 2002, *87* (2), 268–279.

_ , _ , James Asplund, Emily Killham, and Sangeeta Agrawal, "Causal Impact of Employee Work Perceptions on the Bottom Line of Organizations," *Perspectives on Psychological Science*, 2010, *5* (4), 378–389.

Hartman, Philip, *Ordinary Differential Equations*, Boston: Birkhäuser, 2nd ed., 1982.

Ichniowski, Casey, Kathryn Shaw, and Giovanna Prennushi, "The Effects of Human Resource Management Practices on Productivity: A Study of Steel Finishing Lines," *American Economic Review*, 1997, *87* (3), 291–313.

Judge, Timothy, Carl. Thoresen, Joyce Bono, and Gregory Patton, "The Job Satisfaction–Job Performance Relationship: A Qualitative and Quantitative Review," *Psychological Bulletin*, 2001, *127* (3), 376–407.

Khalil, Fahad, "Auditing without Commitment," *RAND Journal of Economics*, 1997, *28* (4), 629–640.

Levitt, Steven and John List, "Was There Really a Hawthorne Effect at the Hawthorne Plant? An Analysis of the Original Illumination Experiments," *American Economic Journal: Applied Economics*, 2011, *3* (1), 224–238.

Li, Jin and Niko Matouschek, "Managing Conflicts in Relational Contracts," *American Economic Review*, 2012, *103* (6), 2328–2351.

Marinovic, Iván, Andrzej Skrzypacz, and Felipe Varas, "Dynamic Certification and Reputation for Quality," 2015. Working paper.

Milgrom, Paul and John Roberts, "The Economics of Modern Manufacturing: Technology," *American Economic Review*, 1990, *80* (3), 511–528.

_ **and** _ , "Complementarities and Fit: Strategy, Structure, and Organizational Change in Manufacturing," *Journal of Accounting and Economics*, 1995, *19* (2-3), 179–208.

Ostroff, Cheri, "The Relationship Between Satisfaction, Attitudes, and Performance: An Organizational Level Analysis," *Journal of Applied Psychology*, 1992, *77* (6), 963–974.

Prendergast, Canice, "The Provision of Incentives in Firms," *Journal of Economic Literature*, 1999, *37* (1), 7–63.

Rob, Rafael and Arthur Fishman, "Is Bigger Better? Customer Base Expansion through Word-of-Mouth Reputation," *Journal of Political Economy*, 2005, *113* (5), 1146–1162.

Saunderson, Roy, "Survey Findings of the Effectiveness of Employee Recognition in the Public Sector," *Public Personnel Management*, 2004, *33* (3), 255–275.

Strausz, Roland, "Delegation of Monitoring in a Principal-Agent Relationship," *Review of Economic Studies*, 1997, *64* (3), 337–357.

A Appendix: Proof of Proposition 1

This proof is divided into six steps. Steps 1-2 solve for the equilibrium dynamics; we proceed backwards by first solving for the dynamics at $s \geq \overline{s}$ and then considering $s < \overline{s}$. Step 3 proves smooth pasting. Step 4 shows existence. Steps 5-6 deal with the uniqueness results.

Step 1: Dynamics at $\mathbf{s \geq \overline{s}}$. At each time $s \geq \overline{s}$, the principal must be indifferent between investing and not investing. Using (4)-(6), the evolution of Λ_s, Ψ_s, and π_s^L at $s \geq \overline{s}$ is given by

$$\dot{\Lambda}_s = 0, \tag{11}$$
$$\dot{\Psi}_s = -\dot{\pi}_s^L, \tag{12}$$
$$\dot{\pi}_s^L = -a_s + r\pi_s^L, \tag{13}$$

with initial conditions $\Lambda_{\overline{s}} = F$, $\Psi_{\overline{s}} = \overline{\Psi}$, and $\pi_{\overline{s}}^L = \overline{\pi}^L$, where $\overline{\Psi}$ and $\overline{\pi}^L$ are derived subsequently. To solve, note that as shown in the text, (4)-(6) and (11) imply that condition (7) holds for $s \geq \overline{s}$; that is, $\Psi_s \mu a_s = (\gamma + r)F$ at each such time.[23] Combining these equations we obtain that the evolution of Ψ_s for $s \geq \overline{s}$ is given by[24]

$$\dot{\Psi}_s = \frac{(\gamma + r)F}{\mu \Psi_s} + r\Psi_s - r(\overline{\Psi} + \overline{\pi}^L). \tag{14}$$

Condition (7) implies that Ψ_s is bounded away from zero for all $s \geq \overline{s}$. Hence, the right-hand side of equation (14) is uniformly Lipshitz continuous and, by the Picard-Lindelöf theorem (Hartman, 1982), the initial value problem given by (14) and the initial condition $\Psi_{\overline{s}} = \overline{\Psi} > 0$ has a unique solution on the whole interval $[\overline{s}, \infty)$. Let Ψ_s^* denote this unique solution given initial value $\overline{\Psi}$. Then using (7) and the fact that $a_s = \mu \bar{b} x_s$ for all $s \geq 0$, we can express the equilibrium belief

[23] To derive this equation, note that, differentiating (4) and (5), we have

$$\dot{\Lambda}_s = \dot{\pi}_s^H - \dot{\pi}_s^L = -a_s - \gamma\pi_s^L - \mu a_s \pi_0^H + (r + \gamma + \mu a_s)\pi_s^H + a_s - r\pi_s^L.$$

Canceling terms, substituting $F = \pi_s^H - \pi_s^L$ and $\Psi_s = \pi_0^H - \pi_s^H$, and setting $\dot{\Lambda}_s = 0$ yields the equation.

[24] To obtain this equation, substitute (13) into (12) and use (7) to substitute for a_s. To substitute for π_s^L, note that by definition, $\pi_0^H = \Psi_s + \Lambda_s + \pi_s^L$ for any $s \geq 0$; hence, given $\overline{\Psi}$ and $\overline{\pi}^L$, we have $\pi_0^H = \overline{\Psi} + F + \overline{\pi}^L$, and using (6), we obtain $\pi_s^L = \overline{\Psi} + \overline{\pi}^L - \Psi_s$ for all $s \geq \overline{s}$.

and effort at $s \geq \overline{s}$ in terms of Ψ_s^*:

$$x_s^* = \frac{(\gamma+r)F}{\Psi_s^* \mu^2 \overline{b}}, \tag{15}$$

$$a_s^* = \frac{(\gamma+r)F}{\Psi_s^* \mu}. \tag{16}$$

Furthermore, note that q_s must be continuous for $s > \overline{s}$. This follows from the fact that, by condition (7), $x_s = \frac{(\gamma+r)F}{\mu^2 \overline{b} \Psi_s}$, which is a C^∞ function of Ψ_s, which is C^1; hence, x_s is also C^1 and in particular $\dot{x}_s$, and thus q_s, are continuous for $s > \overline{s}$. Using (2), the equilibrium investment for $s > \overline{s}$ is then given by

$$q_s^* = \frac{\dot{x}_s^* + \gamma x_s^*}{(1 - x_s^*)} + x_s^* \mu a_s^*. \tag{17}$$

Remark 1. *As shown in Step 4 below, the equilibrium conditions imply that q_s^* given in (17) is positive for all $s \geq \overline{s}$.*

Observe that if $\dot{\Psi}_{\overline{s}} > 0$, the solution to (14) has $\dot{\Psi}_s^* > 0$ for all $s > \overline{s}$ (since the solution is unique, no stationary point can be reached in finite time). We show below that $\dot{\Psi}_{\overline{s}} > 0$ must indeed hold. Using (15) and (16), this implies that the agent's belief x_s^* and the agent's effort a_s^* are strictly decreasing for all $s \geq \overline{s}$.

Finally, we show that the principal's investment q_s^* is decreasing for $s > \overline{s}$ large enough. We write the proof assuming that x_s is twice differentiable and hence q_s differentiable once. However, a similar argument can be used if x_s is differentiable only once, in which case $\ddot{x}_s$ must be replaced with $(\dot{x}_{s+\delta} - \dot{x}_s)/\delta$ and $\dot{q}_s$ with $(q_{s+\delta} - q_s)/\delta$. Since the main point of the argument involves taking limits on $s \to \infty$ leaving δ fixed, the result obtains for the case of x_s only C^1.

Differentiating (17) (and omitting the symbol * below to ease the exposition), $\dot{q}_s$ for $s > \overline{s}$ is given by

$$\dot{q}_s = \frac{\ddot{x}_s(1 - x_s) + \gamma \dot{x}_s + \dot{x}_s^2 + 2\mu^2 \overline{b} x_s (1 - x_s)^2 \dot{x}_s}{(1 - x_s)^2}. \tag{18}$$

Rearranging terms,

$$(1 - x_s)\frac{\dot{q}_s}{\dot{x}_s} = \frac{\ddot{x}_s}{\dot{x}_s} + \frac{\gamma + \dot{x}_s + 2\mu^2\bar{b}x_s(1-x_s)^2}{(1-x_s)}. \tag{19}$$

We will show that $\lim_{s\to\infty}(1-x_s)\frac{\dot{q}_s}{\dot{x}_s} > 0$. Since $\dot{x}_s < 0$ and $1 - x_s > 0$ for all $s \in [\bar{s}, \infty)$, this implies $\dot{q}_s < 0$ for s large enough. Using (14) and (15), note that

$$\begin{aligned}\frac{\ddot{x}_s}{\dot{x}_s} &= -\frac{2\dot{\Psi}_s}{\Psi_s} + \frac{\ddot{\Psi}_s}{\dot{\Psi}_s} \\ &= -\frac{2\dot{\Psi}_s}{\Psi_s} - \frac{\mu\bar{b}x_s}{\Psi_s} + r.\end{aligned}$$

As $s \to \infty$, $\dot{x}_s \to 0$ and $\dot{\Psi}_s \to 0$. Thus, substituting in (19) and denoting $\underline{x} \equiv \lim_{s\to\infty} x_s$ and $\underline{\Psi} \equiv \lim_{s\to\infty} \Psi_s$,

$$\lim_{s\to\infty}(1-x_s)\frac{\dot{q}_s}{\dot{x}_s} = -\frac{\mu\bar{b}\underline{x}}{\underline{\Psi}} + r + \frac{\gamma + 2\mu^2\bar{b}\underline{x}(1-\underline{x})^2}{(1-\underline{x})}. \tag{20}$$

It follows that $\lim_{s\to\infty}(1-x_s)\frac{\dot{q}_s}{\dot{x}_s} > 0$ if and only if

$$\begin{aligned}0 &< -\mu\bar{b}\underline{x}(1-\underline{x}) + r\underline{\Psi}(1-\underline{x}) + \gamma\underline{\Psi} + 2\mu^2\bar{b}\underline{x}(1-\underline{x})^2\underline{\Psi} \\ &= r(\underline{\Psi} - \underline{\pi}^L)(1-\underline{x}) + \gamma\underline{\Psi} + 2\mu^2\bar{b}\underline{x}(1-\underline{x})^2\underline{\Psi}, \end{aligned}\tag{21}$$

where $\underline{\pi}^L \equiv \lim_{s\to\infty} \pi_s^L$ and we have used the fact that $-\mu\bar{b}\underline{x} + r\underline{\pi}^L = \lim_{s\to\infty} \dot{\pi}_s^L = 0$. Observe that for all $s > 0$,

$$\begin{aligned}r(\Psi_s - \pi_s^L) &= r(\pi_0^H - \pi_s^H) - r\pi_s^L \\ &> r\mu\bar{b}\int_0^s e^{-r\tau}x_\tau d\tau - \mu\bar{b}x_s \\ &> \mu\bar{b}x_s\left(1 - e^{-rs}\right) - \mu\bar{b}x_s,\end{aligned}$$

where we have used the fact that $x_\tau > x_s$ for all $\tau \in [0, s)$. Therefore, we obtain $\lim_{s\to\infty} r(\Psi_s - \pi_s^L) = r(\underline{\Psi} - \underline{\pi}^L) > 0$, implying that the right-hand side of (21) is strictly positive.

Step 2: Dynamics at $\mathbf{s < \bar{s}}$. The agent's belief x_s for $s \leq \overline{s}$ is pinned down by the solution to the differential equation (3) with initial condition $x_0 = 1$. Since the right-hand side of (3) is Lipshitz continuous in x, it follows from the Picard-Lindelöf theorem that there exists a solution and it is unique. Note that this solution does not depend on $\overline{s}$. Using this solution and (1), the agent's effort at $s \leq \overline{s}$ is $a_s = \mu \bar{b} x_s$. Note that both x_s and a_s are decreasing for all $s < \overline{s}$. We denote the values at $\overline{s}$ by $x_{\overline{s}} \equiv \overline{x}(\overline{s})$ and $a_{\overline{s}} \equiv \overline{a}(\overline{s})$; when not confusing, we omit the dependence of $\overline{x}$ and $\overline{a}$ on $\overline{s}$. It follows from (7) that

$$\overline{\Psi} = \frac{(\gamma + r)F}{\mu \overline{a}}. \tag{22}$$

Using (4)-(5), the evolution of Ψ_s, Λ_s, and π_s^L at $s \in [0, \overline{s}]$ is given by

$$\dot{\Lambda}_s = (\gamma + r)\Lambda_s - \mu a_s \Psi_s, \tag{23}$$

$$\dot{\Psi}_s = -\dot{\Lambda}_s - \dot{\pi}_s^L, \tag{24}$$

$$\dot{\pi}_s^L = -a_s + r\pi_s^L, \tag{25}$$

where the following boundary conditions must be satisfied: $\Lambda_{\overline{s}} = F$ by definition of $\overline{s}$, $\Psi_0 = 0$ by definition, and $\pi_{\overline{s}}^L = \overline{\pi}^L$ by continuity of π_s^L. To solve, note that given a_s for $s \leq \overline{s}$ and the boundary condition $\pi_{\overline{s}}^L = \overline{\pi}^L$, there is a unique solution to (25). Moreover, by definition, $\Lambda_s = F + \overline{\Psi} + \overline{\pi}^L - \Psi_s - \pi_s^L$. Hence, having the solution for π_s^L, we can obtain Λ_s and Ψ_s for $s < \overline{s}$ by solving[25]

$$\dot{\Psi}_s = -(\gamma + r)(F + \overline{\Psi} + \overline{\pi}^L - \Psi_s - \pi_s^L) + \mu a_s \Psi_s + a_s - r\pi_s^L, \tag{26}$$

with initial condition $\Psi_0 = 0$. This differential equation is linear and thus has a simple closed-form integral solution.

Step 3: Smooth pasting. We next prove smooth pasting, namely that $\dot{x}_s$ must be continuous at $\overline{s}$. This implies $\dot{x}_{\overline{s}} < 0$ for $\overline{s} < \infty$, and hence $\dot{a}_{\overline{s}} < 0$ and $\dot{\Psi}_{\overline{s}} > 0$ as claimed above. Note also that smooth pasting implies q_s continuous at $\overline{s}$ (i.e. $q_{\overline{s}} = 0$) and thus q_s is continuous for all $s \geq 0$ in the equilibrium (see Step 1).

[25] To obtain this equation, substitute (23), (25), and $\Lambda_s = F + \overline{\Psi} + \overline{\pi}^L - \Psi_s - \pi_s^L$ into (24).

To prove smooth pasting, note first that $\lim_{s \to \overline{s}^-} \dot{x}_s \leq \lim_{s \to \overline{s}^+} \dot{x}_s$. This is immediate since $q_s = 0$ for $s < \overline{s}$, so it follows from (1) and (2) that x_s can only jump up at $\overline{s}$. Next, we show that $\lim_{s \to \overline{s}^-} \dot{x}_s \geq \lim_{s \to \overline{s}^+} \dot{x}_s$. Note that

$$\dot{\Lambda}_s = (\gamma + r)\Lambda_s - \mu^2 \overline{b} x_s \Psi_s,$$

so given that Λ_s, x_s, and Ψ_s are continuous, $\dot{\Lambda}_s$ is also continuous, with $\lim_{s \to \overline{s}} \dot{\Lambda}_s = 0$. Moreover, since the principal cannot have incentives to invest before time $\overline{s}$, $\dot{\Lambda}_s \geq 0$ for $s < \overline{s}$, s sufficiently close to $\overline{s}$, and hence we must have $\lim_{s \to \overline{s}^-} \ddot{\Lambda}_s \leq 0$. Noting from (14) and (23)-(25) that $\dot{\Psi}_s$ is also continuous, we can write this as

$$\lim_{s \to \overline{s}^-} \ddot{\Lambda}_s = -\mu^2 \overline{b} x_{\overline{s}} \dot{\Psi}_{\overline{s}} - \mu^2 \overline{b} \Psi_{\overline{s}} \lim_{s \to \overline{s}^-} \dot{x}_s \leq 0,$$

where we have used the fact that $\lim_{s \to \overline{s}} \dot{\Lambda}_s = 0$. Manipulating this inequality,

$$\lim_{s \to \overline{s}^-} \dot{x}_s \geq -\frac{x_{\overline{s}}}{\Psi_{\overline{s}}} \dot{\Psi}_{\overline{s}}.$$

Now consider $\lim_{s \to \overline{s}^+} \dot{x}_s$. Using (15), we have

$$\lim_{s \to \overline{s}^+} \dot{x}_s = -\frac{(\gamma + r)F}{\mu^2 \overline{b} \Psi_{\overline{s}}^2} \dot{\Psi}_{\overline{s}},$$

where we have used again the fact that $\dot{\Psi}_s$ is continuous. Hence, a sufficient condition for the claim to be true is

$$-\frac{x_{\overline{s}}}{\Psi_{\overline{s}}} \dot{\Psi}_{\overline{s}} \geq -\frac{(\gamma + r)F}{\mu^2 \overline{b} \Psi_{\overline{s}}^2} \dot{\Psi}_{\overline{s}},$$

which is equivalent to

$$x_{\overline{s}} \leq \frac{(\gamma + r)F}{\mu^2 \overline{b} \Psi_{\overline{s}}}.$$

By (16) and $a_s = \mu \overline{b} x_s$, this inequality holds with equality. The claim follows.

Finally, note that the result that $\dot{x}_s$ is continuous at $\overline{s}$ implies $q_{\overline{s}} = 0$ and thus

$$-\gamma x_{\overline{s}} - x_{\overline{s}}^2 (1 - x_{\overline{s}}) \mu^2 \overline{b} = -\frac{(\gamma + r)F}{\mu^2 \overline{b} \Psi_{\overline{s}}^2} \dot{\Psi}_{\overline{s}},$$

where by (15) we have $x_{\overline{s}} = \frac{(\gamma+r)F}{\Psi_{\overline{s}}\mu^2\overline{b}}$. Using $\Psi_s = \pi_0^H - \pi_s^L - F$ and (14) to substitute Ψ_s and $\dot{\Psi}_s$ and rearranging terms yields the following condition on $\pi_{\overline{s}}^L$:

$$\pi_{\overline{s}}^L = \frac{1}{r}\left\{\overline{a}(\overline{s}) - [\gamma + (1-\overline{x}(\overline{s}))\mu\overline{a}(\overline{s})]\overline{\Psi}(\overline{s})\right\} \equiv \overline{\pi}^L(\overline{s}). \tag{27}$$

Step 4: Existence. Having the values of $\overline{\Psi}$ and $\overline{\pi}^L$ (given by (22) and (27) respectively), we can solve for the equilibrium dynamics as explained above. As noted in Remark 1, one can verify that the equilibrium conditions ensure that q_s given in (17) is positive for all $s \geq \overline{s}$. To see this, suppose by contradiction that $q_s < 0$ for some $s \geq \overline{s}$. Equation (2) implies $\dot{x}_s < -\gamma x_s - x_s^2(1-x_s)\mu^2\overline{b}$ for such s, which using (15) can be rewritten as

$$\dot{\Psi}_s > \gamma\Psi_s + (\gamma + r)F\left(1 - x_s\right). \tag{28}$$

Now substituting (16) and (27) into equation (14) yields

$$\dot{\Psi}_s = (a_s - \overline{a}) + r(\Psi_s - \overline{\Psi}) + \gamma\overline{\Psi} + (1-\overline{x})(\gamma + r)F, \tag{29}$$

and combining (28) and (29) implies

$$(a_s - \overline{a}) + r(\Psi_s - \overline{\Psi}) + \gamma\overline{\Psi} + (1-\overline{x})(\gamma + r)F > \gamma\Psi_s + (1-x_s)(\gamma + r)F.$$

We reach a contradiction since $\overline{\Psi} \leq \Psi_s$, $(1-\overline{x}) \leq (1-x_s)$, and $(a_s - \overline{a}) + r(\Psi_s - \overline{\Psi}) = (a_s - \overline{a}) + r(\overline{\pi}^L - \pi_s^L) \leq 0$ for any $s \geq \overline{s}$. Note that these inequalities are strict for $s > \overline{s}$ finite; hence, we obtain that q_s is strictly positive for any such s.

An equilibrium as we have characterized therefore exists if and only if there exists a value $\overline{s} \in (0,\infty)$ such that: (i) $\overline{\pi}(\overline{s}) > 0$; and (ii) the solution to (26) with initial condition $\Psi_0 = 0$ satisfies the boundary condition $\Psi_{\overline{s}} = \overline{\Psi}(\overline{s})$. Note that since $\dot{\pi}_s^L < 0$ for all $s < \overline{s}$, (i) implies $\pi_s^L > 0$ for $s < \overline{s}$; moreover, using the solution for $s \geq \overline{s}$, we also obtain $\pi_s^L \geq 0$ for all $s > \overline{s}$.

We show that $\overline{s} \in (0,\infty)$ satisfying (i) and (ii) exists if F is small enough. Consider condition (i). Note from (27) that $\overline{\pi}^L(\overline{s})$ is decreasing in $\overline{s}$ with $\lim_{\overline{s}\to\infty}\overline{\pi}^L(\overline{s}) = -\infty$ and $\lim_{\overline{s}\to 0}\overline{\pi}^L(\overline{s}) = \frac{\mu\bar{b}}{r} - \frac{\gamma(\gamma+r)F}{r\mu^2\overline{b}}$. Hence, to have $\overline{\pi}^L(\overline{s}) > 0$ for some $\overline{s}$, a necessary condition is

$$\frac{\mu\bar{b}}{r} - \frac{\gamma(\gamma + r)F}{r\mu^2\bar{b}} > 0 \tag{30}$$

and $\bar{s} \in (0, s^*)$, where (given (30)) s^* is defined by $\bar{\pi}(s^*) = 0$. Observe that for $F \to 0$, (30) is satisfied and $s^* \to \infty$. That is, as F approaches zero, $\bar{\pi}^L(\bar{s}) > 0$ for any $\bar{s} \geq 0$.

Consider next condition (ii). Note that the differential equation (25) with boundary condition $\pi^L_{\bar{s}} = \bar{\pi}(\bar{s})$ has a unique solution given by

$$\pi^L_s = \int_s^{\bar{s}} e^{-r(\tau - s)}\mu\bar{b}x_\tau d\tau + e^{-r(\bar{s}-s)}\bar{\pi}(\bar{s}) \tag{31}$$

for $s < \bar{s}$. Given π^L_s, consider now the solution to (24). This is a linear equation in Ψ_s with time dependent coefficients that can be rewritten as

$$\dot{\Psi}_s = f_1(s, \bar{s}) + f_2(s)\Psi_s,$$

where $f_1(s, \bar{s}) = \mu\bar{b}x_s - (\gamma + r)\pi^H_0 + \gamma\pi^L_s$, $f_2(s) = \mu^2\bar{b}x_s + \gamma + r$, and $\pi^H_0 = \overline{\Psi} + F + \bar{\pi}^L$. A closed-form solution given initial condition $\Psi_0 = 0$ is

$$\Psi_s = \int_0^s f_1(\tau, \bar{s})e^{\int_\tau^s f_2(i)di}d\tau.$$

Hence, there exists $\bar{s} > 0$ such that $\Psi_{\bar{s}} = \frac{(\gamma+r)F}{\mu^2\bar{b}\bar{x}}$ if and only if

$$\int_0^{\bar{s}} f_1(\tau, \bar{s})e^{\int_\tau^{\bar{s}} f_2(i)di}d\tau = \frac{(\gamma + r)F}{\mu^2\bar{b}x_{\bar{s}}}. \tag{32}$$

Note that for every $s > 0$, $\frac{(\gamma+r)F}{\mu^2\bar{b}x_s}$ goes to zero as $F \to 0$, since x_s is given by the solution to (3) with initial condition $x_0 = 1$ and is thus independent of F. Furthermore, as $F \to 0$, $f_1(s, \bar{s})$ tends to a positive limit, uniformly in $s \in (0, \bar{s})$. To see this, note that, using (27), $\lim_{F\to 0} \pi^H_0 = \frac{\mu\bar{b}\bar{x}}{r}$, and thus

$$\lim_{F\to 0} f_1(s, \bar{s}) = \mu\bar{b}x_s - (\gamma + r)\frac{\mu\bar{b}\bar{x}}{r} + \gamma\pi^L_s.$$

Since $\pi_s^L > e^{-r(\overline{s}-s)}\pi_{\overline{s}}^L$ for $s < \overline{s}$ (see (31)),

$$\lim_{F\to 0} f_1(s,\overline{s}) > \mu\bar{b}x_s - (\gamma + r)\frac{\mu\bar{b}\overline{x}}{r} + \gamma e^{-r(\overline{s}-s)}\frac{\mu\bar{b}x_s}{r}.$$

Simplifying this expression, we find that $\lim_{F\to 0} f_1(s,\overline{s})$ is positive for all $s \in (0,\overline{s})$ if and only if

$$\frac{x_s}{\overline{x}} - 1 \geq \frac{\gamma}{r}\left(1 - e^{-r(\overline{s}-s)}\right). \tag{33}$$

Both sides of this inequality are zero at $s = \overline{s}$; hence, it suffices to show that for $s \in (0,\overline{s})$, the derivative of the left-hand side is smaller than that of the right-hand side, i.e.

$$\dot{x}_s < -\gamma\overline{x}e^{-r(\overline{s}-s)}. \tag{34}$$

By (3) (and the fact that $x_s > \overline{x}$ and $1 > e^{-r(\overline{s}-s)}$ for $s < \overline{s}$), this condition is satisfied for all $s < \overline{s}$.

The fact that $f_1(s,\overline{s})$ goes to a positive limit uniformly in $s \in (0,\overline{s})$ when F goes to zero (while $f_2(s)$ does not depend on F) implies that the left-hand side of (32) also goes to a positive value, as it is the integral of a uniformly positive function. Therefore, we have obtained that as $F \to 0$, the left-hand side of (32) tends to something positive and the right-hand side to zero. Note also that as $\overline{s} \to 0$, the left-hand side of (32) tends to zero while the right-hand side has a positive limit. By continuity of (32) in $\overline{s}$, we conclude that for F small enough, there exists $\overline{s}$ that satisfies (32).

We have then shown that there exists $\overline{F} > 0$ such that if $F \leq \overline{F}$, both conditions (i) and (ii) are satisfied, and note that Assumption 2 (i.e. $\mu\bar{b} \geq (\gamma + r)F$) also holds. It is immediate that if $F > \overline{F}$, these conditions cannot be satisfied.

Step 5: Uniqueness of $\overline{\mathbf{s}}$. We show that the threshold time $\overline{s} \in (0,\infty)$ is unique if $\gamma > r$. Rewrite (32) as

$$\overline{x}\int_0^{\overline{s}} \left[\mu\bar{b}x_\tau - (\gamma + r)\pi_0^H + \gamma\pi_\tau^L\right] e^{\int_\tau^{\overline{s}}(\mu^2\bar{b}x_i + \gamma + r)di} d\tau = \frac{(\gamma + r)F}{\mu^2\bar{b}}. \tag{35}$$

Note that using (22) and (27), we have

$$-(\gamma+r)\pi_0^H = -\gamma(\overline{\Psi}+F+\overline{\pi}^L) - r(\overline{\Psi}+F) - \mu\bar{b}\overline{x} + \gamma\overline{\Psi} + (1-\overline{x})(\gamma+r)F.$$

Thus, using again (22), we can rewrite (35) as

$$\overline{x}\int_0^{\overline{s}} \left\{ \begin{array}{c} \mu\bar{b}(x_\tau-\overline{x}) + \gamma(\pi_\tau^L-\overline{\pi}^L) \\ -(\gamma+r)F\overline{x} - r\frac{F(\gamma+r)}{\mu^2\bar{b}\overline{x}} \end{array} \right\} e^{\int_\tau^{\overline{s}}(\mu^2\bar{b}x_i+\gamma+r)di} d\tau = \frac{(\gamma+r)F}{\mu^2\bar{b}}. \tag{36}$$

Denote the left-hand side of (36) by $LHS(\overline{s})$ and the right-hand side by RHS. We show that if $\overline{s}'$ is an equilibrium, then $LHS(\overline{s})$ is strictly increasing at $\overline{s}=\overline{s}'$. Since $LHS(\overline{s})$ and RHS are continuous, this implies that the equilibrium threshold time $\overline{s}$ is unique: for any set of parameters, there exists a unique point $\overline{s}'$ where $LHS(\overline{s}')=RHS$. The derivative of $LHS(\overline{s})$ with respect to $\overline{s}$ is

$$\frac{\partial LHS(\overline{s})}{\partial\overline{s}} = \Psi_{\overline{s}}\frac{\partial\overline{x}}{\partial\overline{s}} + \overline{x}\left\{ \begin{array}{c} \int_0^{\overline{s}}\left[\left[-\mu\bar{b}-(\gamma+r)F+r\frac{F(\gamma+r)}{\mu^2\bar{b}\overline{x}^2}\right]\frac{\partial\overline{x}}{\partial\overline{s}} + \gamma\frac{\partial(\pi_\tau^L-\overline{\pi}^L)}{\partial\overline{s}}\right] e^{\int_\tau^{\overline{s}}(\mu^2\bar{b}x_i+\gamma+r)di}d\tau \\ +\Psi_{\overline{s}}(\mu^2\bar{b}\overline{x}+\gamma+r) - (\gamma+r)F\overline{x} - r\frac{F(\gamma+r)}{\mu^2\bar{b}\overline{x}} \end{array} \right\}.$$

Substituting with $\frac{\partial\overline{x}}{\partial\overline{s}} = -\gamma\overline{x}-\overline{x}^2(1-\overline{x})\mu^2\bar{b}$, $\frac{\partial(\pi_\tau^L-\overline{\pi}^L)}{\partial\overline{s}} = e^{r\tau}\mu\bar{b}\overline{x}+\mu\bar{b}\overline{x}(1-e^{-r(\overline{s}-\tau)})$, and $\Psi_{\overline{s}}=\overline{\Psi}$, and canceling and rearranging terms yields

$$\frac{\partial LHS(\overline{s})}{\partial\overline{s}} = \overline{x}\int_0^{\overline{s}} \left\{ \begin{array}{c} \left[\mu\bar{b}+(\gamma+r)F\right]\left[\gamma\overline{x}+\overline{x}^2(1-\overline{x})\mu^2\bar{b}\right] - r\gamma\frac{F(\gamma+r)}{\mu^2\bar{b}\overline{x}} \\ -rF(\gamma+r)(1-\overline{x}) + \gamma\mu\bar{b}\overline{x}\left(e^{r\tau}+1-e^{-r(\overline{s}-\tau)}\right) \end{array} \right\} e^{\int_\tau^{\overline{s}}(\mu^2\bar{b}x_i+\gamma+r)di}d\tau.$$

Note that (27) and $\overline{\pi}^L>0$ imply $\mu\bar{b}\overline{x} > \gamma\frac{F(\gamma+r)}{\mu^2\bar{b}\overline{x}} + F(\gamma+r)(1-\overline{x})$. Hence, a sufficient condition for $\frac{\partial LHS(\overline{s})}{\partial\overline{s}}>0$ is

$$\left[\mu\bar{b}+(\gamma+r)F\right]\left[\gamma\overline{x}+\overline{x}^2(1-\overline{x})\mu^2\bar{b}\right] + \gamma\mu\bar{b}\overline{x}\left(e^{r\tau}+1-e^{-r(\overline{s}-\tau)}\right) > r\mu\bar{b}\overline{x},$$

which is satisfied for r small enough, and in particular if $\gamma\geq r$.

Step 6: Other equilibria. Consider first the claim that any continuous equilibrium with positive investment must be as characterized in Proposition 1. We begin by showing that any continuous equilibrium with positive investment where, at each time $s\geq 0$, the principal either is indifferent or does not have incentives to invest, must be as characterized in the proposition. Suppose by contradiction that this is not the case. Then there exists an equilibrium in which the principal is indifferent (and invests) over an interval of time $[s', s'']$ and she strictly prefers not to invest over (s'', s'''), where $s' \geq 0$ and s'' is finite. Such an equilibrium must have $s' > 0$, as otherwise the principal's indifference would require $\mu\Psi_0 a_0 = (\gamma + r)F$, which cannot be satisfied since $\Psi_0 = 0$ by definition. Note that q_s cannot go to zero continuously as s approaches s'': as shown in Step 1 and Step 4, for any given initial value $s' > 0$, the principal's indifference conditions uniquely determine the evolution of q_s and imply $q_s > 0$ for finite s. Hence, q_s must jump down to zero at s''. It follows that $\dot{x}_s$ and therefore $\mu\big(\dot{\Psi}_s a_s + \Psi_s \dot{a}_s\big)$ also jump down at s''. Now note that the principal's indifference requires $\Lambda_s = F$ for $s \in [s', s'']$, implying $\dot{\Lambda}_s = (\gamma + r)\Lambda_s - \mu a_s \Psi_s = 0$ and $\ddot{\Lambda}_s = (\gamma + r)\dot{\Lambda}_s - \mu\big(\dot{\Psi}_s a_s + \Psi_s \dot{a}_s\big) = 0$ for $s \in (s', s'')$. Therefore, we obtain that $\ddot{\Lambda}$ jumps up to a strictly positive value at s'' (note that $\dot{\Lambda}_s$ is continuous), and as a consequence Λ_s increases strictly above F at that time. This yields a contradiction as the principal's strict incentives not to invest over (s'', s''') require $\Lambda_s < F$ for those times.

We next show that a continuous equilibrium in which the principal has strict incentives to invest over some time interval does not exist. Suppose by contradiction that it does. Let $s' \geq 0$ be the earliest time at which the principal strictly wants to invest in this equilibrium. The agent's belief at s' is $x_{s'} = 1$. Note that the agent's belief cannot increase continuously towards $x_{s'} = 1$: this would require that the principal use a mixed investment strategy, and therefore that she be indifferent between investing and not, while the belief increases; however, the principal's indifference conditions yield a strictly decreasing agent belief (see Step 1). Hence, if $s' > 0$, the agent's belief would jump up at s', which cannot occur in a continuous equilibrium. It follows that $s' = 0$ and the principal has strict incentives to invest at $s \in [0, \Delta)$, for some $\Delta > 0$. This means that for each $s \in [0, \Delta)$, the principal (weakly) prefers to invest at s rather than at $s + \delta$, for any $\delta \in (0, \Delta - s)$. However, for $s = 0$ and δ arbitrarily small, this implies $\mu a_0 \Psi_0 \geq (\gamma + r)F$, which cannot be satisfied since $\Psi_0 = 0$ by definition. This completes the proof that any continuous equilibrium with positive investment must be as characterized in Proposition 1.

Finally, consider the claim that there exists $\overline{\overline{F}} > 0$ such that a continuous equilibrium with no investment does not exist if $F \leq \overline{\overline{F}}$. Suppose that a continuous equilibrium with no investment exists. Then the agent's belief x_s follows the law of motion in (3), and so the agent's effort $a_s = \mu\bar{b}x_s$ is decreasing at all s. But then it is immediate that $\Lambda_s > 0$ for all $s < \infty$ and $\dot{\Lambda}_s > 0$ in a neighborhood of 0 (cf. (23) and $\Psi_0 = 0$). Hence, for $F > 0$ small enough, there exists s such that $\Lambda_s > F$ and $\mu a_s \Psi_s \geq (\gamma + r)F$; that is, the principal has strict incentives to invest at s, yielding a contradiction.

Supplementary Appendix for Online Publication

This Online Appendix contains the proof of our results for the undiscounted limit discussed in Section 2, the proof of Proposition 2, details for the discussion of the forward-looking agent case described in Section 4, and an analysis of discontinuous equilibria.

B Undiscounted limit

Denote $\underline{a} \equiv \lim_{s\to\infty} a_s$. We prove the following result:

Proposition 3. *Fix any set of parameters $(\gamma, \mu, \overline{b})$ and consider the limit as r goes to zero. Let $F < \overline{F}$ so that the equilibrium of Proposition 1 exists. As F approaches $\overline{F}$, $\underline{a}$ vanishes, so the probability of returning to high performance goes to zero as time passes without recognition.*

Proof. Consider $\dot{\Psi}_s$ for $s \leq \overline{s}$, given by equation (26). In the limit as $r \to 0$, we have

$$\dot{\Psi}_s = \mu\bar{b}x_s - \gamma(F + \overline{\Psi} - \Psi_s) + \gamma\mu\bar{b}\int_s^{\overline{s}} x_t dt + \mu^2\bar{b}x_s\Psi_s - \underline{a}, \tag{37}$$

where $\underline{a} = r\pi_s^L$. Using (27) and substituting with (22),

$$\underline{a} = \mu\bar{b}\overline{x} - \frac{\gamma^2 F}{\mu^2\bar{b}\overline{x}} - (1-\overline{x})\gamma F. \tag{38}$$

Substituting (22) and (38) in (37) yields

$$\begin{aligned}
\dot{\Psi}_s &= \mu\bar{b}x_s - \gamma\left(F + \frac{\gamma F}{\mu^2\bar{b}\overline{x}} - \Psi_s\right) + \gamma\mu\bar{b}\int_s^{\overline{s}} x_t dt + \mu^2\bar{b}x_s\Psi_s - \left[\mu\bar{b}\overline{x} - \frac{\gamma^2 F}{\mu^2\bar{b}\overline{x}} - (1-\overline{x})\gamma F\right] \\
&= \left(\mu^2\bar{b}x_s + \gamma\right)\Psi_s + \mu\bar{b}(x_s - \overline{x}) + \gamma\mu\bar{b}\int_s^{\overline{s}} x_t dt - \gamma F\overline{x}.
\end{aligned}$$

Solving this differential equation with initial condition $\Psi_0 = 0$ gives that for $s \leq \overline{s}$,

$$\Psi_s = \int_0^s \left[\mu\bar{b}(x_\tau - \overline{x}) + \gamma\mu\bar{b}\int_\tau^{\overline{s}} x_t dt - \gamma F\overline{x}\right] e^{\int_\tau^s \left(\mu^2\bar{b}x_i + \gamma\right)di} d\tau. \tag{39}$$

Following the same steps as in the proof of Proposition 1, an equilibrium is a value of $\overline{s} \in (0, \infty)$ such that (i) $\underline{a} \geq 0$ and (ii) $\Psi_{\overline{s}} = \overline{\Psi}$. For condition (i), note that the right-hand side of (38) is increasing in $\overline{x}$ and $\frac{\partial \overline{x}}{\partial \overline{s}} = -\gamma\overline{x} - \overline{x}^2(1-\overline{x})\mu^2\overline{b} < 0$, so $\underline{a}$ is decreasing in $\overline{s}$. Note also that the value of $\overline{s}$ that makes $\underline{a} = 0$ is finite. Hence, making the dependence of $\underline{a}$ on $\overline{s}$ explicit, $\underline{a}(\overline{s}) \geq 0$ is equivalent to $\overline{s} \leq \overline{s}^{\max}$ for $\overline{s}^{max}$ defined by $\underline{a}(\overline{s}^{\max}) = 0$. Note that $\overline{s}^{max}$ is a continuous and differentiable function of parameters.

Using (39), condition (ii) is equivalent to

$$\overline{x} \int_0^{\overline{s}} \left[\mu\overline{b}(x_\tau - \overline{x}) + \gamma\mu\overline{b} \int_\tau^{\overline{s}} x_t dt - \gamma F \overline{x} \right] e^{\int_\tau^{\overline{s}} \left(\mu^2 \overline{b} x_i + \gamma\right) di} d\tau = \frac{\gamma F}{\mu^2 \overline{b}}. \qquad (40)$$

Denote the left-hand side of (40) by $lhs(\overline{s})$ and the right-hand side by rhs. We show that if $\overline{s}'$ is an equilibrium, then $lhs(\overline{s})$ is strictly increasing at $\overline{s} = \overline{s}'$. The derivative of $lhs(\overline{s})$ with respect to $\overline{s}$ is

$$\frac{\partial lhs(\overline{s})}{\partial \overline{s}} = \Psi_{\overline{s}} \frac{\partial \overline{x}}{\partial \overline{s}} + \overline{x} \left\{ \begin{array}{c} \int_0^{\overline{s}} \left[-(\mu\overline{b} + \gamma F) \frac{\partial \overline{x}}{\partial \overline{s}} + \gamma\mu\overline{b}\overline{x} \right] e^{\int_\tau^{\overline{s}} \left(\mu^2 \overline{b} x_i + \gamma\right) di} d\tau \\ + \Psi_{\overline{s}}(\mu^2 \overline{b}\overline{x} + \gamma) - \gamma F \overline{x} \end{array} \right\}.$$

Substituting with $\frac{\partial \overline{x}}{\partial \overline{s}} = -\gamma\overline{x} - \overline{x}^2(1-\overline{x})\mu^2\overline{b}$ and $\Psi_{\overline{s}} = \overline{\Psi}$ and canceling and rearranging terms yields

$$\frac{\partial lhs(\overline{s})}{\partial \overline{s}} = \overline{x} \left\{ (\mu\overline{b} + \gamma F)[\gamma\overline{x} + \overline{x}^2(1-\overline{x})\mu^2\overline{b}] + \gamma\mu\overline{b}\overline{x} \right\} \int_0^{\overline{s}} e^{\int_\tau^{\overline{s}} \left(\mu^2 \overline{b} x_i + \gamma\right) di} d\tau > 0.$$

Since both $lhs(\overline{s})$ and rhs are continuous, this implies that the equilibrium threshold time $\overline{s}$ is unique: there exists a unique point $\overline{s}'$ where $lhs(\overline{s}') = rhs$. Moreover, the fact that the derivative of $lhs(\overline{s})$ is bounded away from zero allows to apply the Implicit Function Theorem and obtain that the equilibrium is continuous (in fact differentiable) in the parameters. Hence, given an original equilibrium $\overline{s}'$ with $\overline{s}' < \overline{s^{\max}}'$, a new equilibrium with $\overline{s}'' < \overline{s^{max}}''$ exists for any local change of parameters.

We next show that increasing F reduces $\underline{a}$. Together with the results above, this implies that starting from any given continuous equilibrium with investment, one can increase F until $\underline{a}$ becomes arbitrarily close to zero in equilibrium. Note that for a fixed $\overline{s}$, rhs increases when F increases whereas $lhs(\overline{s})$ decreases pointwise. Therefore, the point $\overline{s}$ at which $lhs(\overline{s}) = rhs$ increases when F increases. Note that $\overline{x}$ depends on F only through $\overline{s}$, and $\frac{\partial \overline{s}}{\partial F} > 0$ implies $\frac{\partial \overline{x}}{\partial F} < 0$. Hence, using (38),

$$\frac{\partial \underline{a}}{\partial F} = \left(\mu \bar{b} + \frac{\gamma^2 F}{\mu^2 \bar{b} \overline{x}^2} + \gamma F\right) \frac{\partial \overline{x}}{\partial F} - \frac{\gamma^2}{\mu^2 \bar{b} \overline{x}} - (1 - \overline{x})\gamma < 0.$$

Q.E.D.

C Proof of Proposition 2

The case with $\mu > \nu$ is analogous to that studied in Proposition 1 and thus omitted. Consider $\mu \leq \nu$.

The construction of the equilibrium is simple. Given a threshold time $\widehat{s} \in (0, \infty)$, the law of motion for the agent's belief on $[0, \widehat{s}]$ is given by (9). The solution to this differential equation with initial condition $x_0 = 1$ yields the agent's belief x_s and the agent's effort $a_s = (\mu \bar{b} - \nu \underline{b}) x_s$ for $s \in [0, \widehat{s}]$. Since the right-hand side of (9) is Lipshitz continuous in x, it follows from the Picard-Lindelöf theorem that there exists a solution and it is unique. Note that this solution does not depend on $\widehat{s}$, and that both x_s and a_s are decreasing for all $s < \widehat{s}$. Denote the values at $\widehat{s}$ by $x_{\widehat{s}} \equiv \widehat{x}(\widehat{s})$ and $a_{\widehat{s}} \equiv \widehat{a}(\widehat{s})$; we omit the dependence on $\widehat{s}$ in what follows.

As explained in the text, (10) implies that the agent's effort must be constant for all $s \geq \widehat{s}$. Therefore, given continuity of x_s, the agent's belief and effort must be $x_s = \widehat{x}$ and $a_s = \widehat{a}$ for all $s \geq \widehat{s}$. Moreover, given these constant values, the principal's investment is also pinned down: setting $\dot{x}_s = 0$ in (8), we obtain that for all $s \geq \widehat{s}$, q_s must be equal to

$$\widehat{q} = \frac{\gamma \widehat{x}}{1 - \widehat{x}} + \widehat{x}[\mu \widehat{a} + \nu(1 - \widehat{a})].$$

Consider now the claim that any continuous equilibrium with positive investment must take this form. First, note that this equilibrium is the unique continuous equilibrium with positive investment where, at each time $s \geq 0$, the principal either is indifferent or does not have incentives to invest. This follows from (10), which implies that in any continuous equilibrium, the agent's effort and the principal's value of recognition must be constant for all times $s \geq \widetilde{s}$ if the principal is indifferent between investing and not investing at $\widetilde{s}$. Next, consider continuous equilibria in which the principal has strict incentives to invest over some time interval. By the same reasoning as in the proof of Proposition 1, there exists $\Delta > 0$ such that the principal has strict incentives to invest at $s \in [0, \Delta]$. However, this requires $[\mu a_0 + \nu(1 - a_0)]\Psi_0 \geq (\gamma + r)F$, which cannot be satisfied since $\Psi_0 = 0$. Thus, a continuous equilibrium in which the principal has strict incentives to invest does not exist, and the claim follows.

Finally, we prove the claims in fn. 18. As explained above, the solution to (9) (with initial condition $x_0 = 1$) uniquely determines x_s, and thus $a_s = (\mu\bar{b} - \nu\underline{b})x_s$, for $s \leq \widehat{s}$, independently of the value of $\widehat{s}$. Moreover, note that for any given $\widehat{s}$, the values of a_s, π^L_s and π^H_s are pinned down for $s \geq \widehat{s}$—these values are $a_s = \widehat{a}$, $\pi^L_s = \frac{\widehat{a}}{r}$, and $\pi^H_s = \pi^L_s + F$—and as a result the values of π^L_s and π^H_s are also pinned for $s \leq \widehat{s}$:

$$\begin{aligned}
\pi^L_s &= \int_s^{\widehat{s}} e^{-r(\tau - s)} a_\tau d\tau + e^{-r(\widehat{s} - s)} \frac{\widehat{a}}{r}, \\
\pi^H_s &= \int_s^{\widehat{s}} e^{-(\gamma + r)(\tau - s) - \int_s^\tau [\mu a_z + \nu(1 - a_z)]dz} \left\{ a_\tau + \gamma \pi^L_\tau + [\mu a_\tau + \nu(1 - a_\tau)]\pi^H_0 \right\} d\tau \\
&\quad + e^{-(\gamma + r)(\widehat{s} - s) - \int_s^{\widehat{s}} [\mu a_z + \nu(1 - a_z)]dz} \left(\frac{\widehat{a}}{r} + F \right). \qquad (41)
\end{aligned}$$

Using (41), it follows that for any given $\widehat{s}$, $\Psi_{\widehat{s}} = \pi^H_0 - \pi^H_{\widehat{s}}$ is given by

$$\Psi_{\widehat{s}} = \int_0^{\widehat{s}} e^{-(\gamma + r)\tau - \int_0^\tau [\mu a_z + \nu(1 - a_z)]dz} \left\{ \begin{array}{c} (a_\tau - \widehat{a}) + \gamma(\pi^L_\tau - \pi^L_{\widehat{s}}) \\ + [\mu a_\tau + \nu(1 - a_\tau)]\Psi_{\widehat{s}} - (\gamma + r)F \end{array} \right\} d\tau.$$

It is immediate to verify that $\Psi_{\widehat{s}}$ is strictly increasing in $\widehat{s}$ and is thus bounded above by $\lim_{\widehat{s} \to \infty} \Psi_{\widehat{s}}$, which is finite. Note that $\mu\widehat{a} + \nu(1 - \widehat{a})$ is also strictly increasing in $\widehat{s}$ and is bounded above by ν. Therefore, it follows that there exists $\widehat{F} > 0$ such that a time $\widehat{s}$ at which (10) is satisfied (i.e., $\Psi_{\widehat{s}}[\mu a_{\widehat{s}} + \nu(1 - a_{\widehat{s}})] = (\gamma + r)F$) exists if and only if $F \leq \widehat{F}$, and such a time $\widehat{s}$ is unique. Given the construction and claims above, this proves that a continuous equilibrium with positive investment exists if and only if F is small enough, and such an equilibrium is unique.

D Details for Section 4

In this section, we describe how we solve numerically the case of a forward-looking agent discussed in Section 4.

Agent's problem. The agent's expected payoff at $s = 0$ is

$$U_0 = \int_0^\infty e^{-rs}(1 - R_s)\left[\mu x_s a_s \left(\bar{b} + U_0\right) - \frac{1}{2}a_s^2\right] ds, \tag{BC0}$$

where $R_s \equiv 1 - e^{-\int_0^s \mu x_\tau a_\tau d\tau}$. The agent's optimization problem is

$$\max_{a_s} U_0 = \int_0^\infty e^{-rs}(1 - R_s)\left[\mu x_s a_s \left(\bar{b} + U_0\right) - \frac{1}{2}a_s^2\right] ds$$

$$\text{subject to} \quad \dot{x} = -\gamma x_s - x_s(1 - x_s)\mu a_s + (1 - x_s)q_s, \tag{42}$$

$$\dot{R}_s = (1 - R_s)\mu a_s x_s, \tag{43}$$

$$x_0 = 1, \; R_0 = 0. \tag{BC1}$$

Note that given the principal's equilibrium strategy, the agent faces a relatively simple single-agent experimentation problem where the evolution of the underlying state (the principal's type) depends only on recognition, as the principal's investment is only a function of the time that has passed since recognition (and her type). The agent's action affects both the payoff process and the learning process, and the forward-looking agent takes this into account when choosing effort. In particular, we solve for the agent's optimal sequence of effort taking into account how his belief will evolve and affect effort choices depending on the effort he chooses. This computation ensures that no deviation (including double deviations) is profitable for the agent.

For multipliers $\lambda_{1s}, \lambda_{2s}$, the Hamiltonian is:

$$H = \int_0^\infty \left\{ e^{-rs}(1-R_s)\left[\mu x_s a_s\left(\overline{b}+U_0\right) - \frac{1}{2}a_s^2\right] + \lambda_{1s}\dot{x}_s + \lambda_{2s}\dot{R}_s \right\} ds.$$

The first order conditions with respect to a_s, λ_{1s} and λ_{2s} yield

$$\begin{aligned} 0 &= e^{-rs}(1-R_s)\left[\mu x_s\left(\overline{b}+U_0\right) - a_s\right] - \lambda_{1s}x_s(1-x_s)\mu + \lambda_{2s}(1-R_s)\mu x_s, \\ -\dot{\lambda}_{1s} &= e^{-rs}(1-R_s)\mu a_s(\overline{b}+U_0) - \lambda_{1s}\left[\gamma + (1-2x_s)\mu a_s + q_s\right] + \lambda_{2s}(1-R_s)\mu a_s, \\ -\dot{\lambda}_{2s} &= -e^{-rs}\left[\mu x_s a_s(\overline{b}+U_0) - \frac{1}{2}a_s^2\right] - \lambda_{2s}\mu a_s x_s. \end{aligned}$$

Replacing with $m_{1s} = \lambda_{1s}e^{rs}$ and $m_{2s} = \lambda_{2s}e^{rs}$,

$$\begin{aligned} 0 &= (1-R_s)\left[\mu x_s(\overline{b}+U_0) - a_s\right] - m_{1s}x_s(1-x_s)\mu + m_2(1-R_s)\mu x_s, && (44) \\ -\dot{m}_{1s} + rm_{1s} &= (1-R_s)\mu a_s(\overline{b}+U_0) - m_{1s}\left[\gamma + (1-2x_s)\mu a_s + q_s\right] + m_{2s}(1-R_s)\mu a_s, && (45) \\ -\dot{m}_{2s} + rm_{2s} &= -\left[\mu x_s a_s(\overline{b}+U_0) - \frac{1}{2}a_s^2\right] - m_{2s}\mu a_s x_s. && (46) \end{aligned}$$

The transversality condition on m_{1s} and m_{2s} is

$$\lim_{s\to\infty} e^{-rs}m_{1s} = \lim_{s\to\infty} e^{-rs}m_{2s} = 0. \tag{BC2}$$

Given the principal's investment q_s, the agent's effort and belief are determined by equations (42)-(46) and boundary conditions (BC0)-(BC2).

Equilibrium dynamics and smooth pasting. We consider an equilibrium with threshold time $\overline{s} \in (0, \infty)$ so that the principal does not invest at $s < \overline{s}$ and she mixes between investing and not investing at $s \geq \overline{s}$. As in the case of a myopic agent, we have that for $s < \overline{s}$,

$$\begin{aligned} \dot{\Lambda}_s &= (\gamma + r)\Lambda_s - \mu a_s \Psi_s, && (47) \\ \dot{\Psi}_s &= -\dot{\Lambda}_s - \dot{\pi}_s^L, && (48) \\ \dot{\pi}_s^L &= -a_s + r\pi_s^L, && (49) \end{aligned}$$

with boundary conditions

$$\Lambda_{\overline{s}} = F, \;\; \Psi_0 = 0, \;\text{ and } \pi^L_{\overline{s}} = \overline{\pi}^L, \tag{BC3}$$

where $\overline{\Psi} = \frac{(\gamma+r)F}{\mu a_{\overline{s}}}$ and $\overline{\pi}^L$ is derived below.

For $s \geq \overline{s}$, $\Lambda_s = F$, so the system is

$$\begin{aligned}
\dot{\Lambda}_s &= 0, && (50)\\
\dot{\Psi}_s &= -\dot{\pi}^L_s, && (51)\\
\dot{\pi}^L_s &= -a_s + r\pi^L_s, && (52)
\end{aligned}$$

with boundary conditions

$$\Lambda_{\overline{s}} = F, \;\; \Psi_{\overline{s}} = \overline{\Psi}, \;\text{ and } \pi^L_{\overline{s}} = \overline{\pi}^L. \tag{BC4}$$

The value of $\overline{\pi}^L$ is obtained from smooth pasting: we require that a_s and x_s be continuously differentiable at $\overline{s}$. This gives

$$\overline{\pi}^L = \frac{a_{\overline{s}} - \dot{\Psi}_{\overline{s}}}{r} = \frac{1}{r}\left(\frac{(r+\gamma)F}{\mu\Psi_{\overline{s}}} + \frac{\dot{a}_{\overline{s}}\mu\Psi^2_{\overline{s}}}{(\gamma+r)F}\right).$$

Solving for the equilibrium. We find values $(\overline{s}, U_0, m_{10}, m_{20})$ such that equations (42)-(52) and boundary conditions (BC0)-(BC4) are satisfied. Begin by fixing a set of initial values $(\overline{s}, U_0, m_{10}, m_{20})$. We proceed as follows:

1. Solve the agent's problem for $s < \overline{s}$. Given $(\overline{s}, U_0, m_{10}, m_{20})$ and initial conditions (BC1), and setting $q_s = 0$, we can solve (42)-(46) on $[0, \overline{s}]$. We obtain a_s, x_s, R_s, m_{1s}, and m_{2s} for $s < \overline{s}$.

2. Solve the system characterizing equilibrium dynamics for $s \geq \overline{s}$. We solve (50)-(52) given the boundary conditions (BC4). We obtain a_s, π^L_s, Ψ_s, and Λ_s for $s \geq \overline{s}$.

3. Solve for the agent's belief and the principal's investment for $s \geq \overline{s}$. We obtain the belief x_s on $[\overline{s}, \infty)$ by inputting the effort path a_s obtained in step 2 into the agent's problem (42)-(46). Then having solved for x_s and

a_s, we can solve for the investment q_s on $[\bar{s}, \infty)$. We obtain q_s, x_s, R_s, m_{1s}, and m_{2s} for $s \geq \bar{s}$.

4. Solve the system characterizing equilibrium dynamics for $s < \bar{s}$. We solve (47)-(49) given boundary conditions (BC3). Note that the value of Λ_s is unknown here but we can solve the system because a_s is pinned down at this point. We obtain π_s^L, Ψ_s, and Λ_s for $s \leq \bar{s}$.

5. Compare solution to initial values. Having solved for all variables, compute now the resulting values for the value of recognition and the agent's expected payoff at time $s = 0$, which we can denote by $\widetilde{\Psi}_0$ and $\widetilde{U}_0$ respectively, and the limits $\lim_{s\to\infty} e^{-rs} m_{1s}$ and $\lim_{s\to\infty} e^{-rs} m_{2s}$. If given initial values $(\bar{s}, U_0, m_{10}, m_{20})$, we obtain $\widetilde{\Psi}_0 = 0$, $\widetilde{U}_0 = U_0$, $\lim_{s\to\infty} e^{-rs} m_{1s} = 0$, and $\lim_{s\to\infty} e^{-rs} m_{2s} = 0$, then we have found an equilibrium. Otherwise we change the initial values, searching on a grid of $(\bar{s}, U_0, m_{10}, m_{20})$, until these four conditions are satisfied up to some precision target.

E Discontinuous equilibria

Consider the setting of Section 1. Our analysis in the paper restricted attention to equilibria in which the agent's belief as a function of the time since recognition, x_s, is continuous. In this section, we study equilibria in which this belief can jump. Because such equilibria can in principle take many arbitrary forms, we focus on a simple class of discontinuous equilibria that are stationary. We show that the principal prefers the continuous equilibrium characterized in Proposition 1 to any discontinuous equilibrium in this class.

We define a stationary discontinuous equilibrium as an equilibrium in which the principal does not invest except in countably many points $s^1, s^2, \ldots$ such that, for all $n \in \mathbb{N} = \{1, 2, \ldots\}$, (i) $s^{n+1} = s^n + \Delta$ for some $\Delta > 0$, and (ii) the principal invests with a mass probability $\kappa > 0$ at s^n. Denote the set of times at which the principal invests by $J = \{s^1, s^1 + \Delta, s^1 + 2\Delta, \ldots\}$. s^1, Δ, and κ are such that for some $0 \leq x^- < x^+ \leq 1$, the agent's belief that the principal is a high type satisfies $x_{s^-} = x^-$ and $x_{s^+} = x^+$ for all $s \in J$. Let $s^0 \equiv \min\{s : x_s = x^+\}$; note that $s^1 - s^0 = \Delta$.

Figure 4 depicts a discontinuous equilibrium. (While the scale makes it difficult to see, the values of all variables shown in the figure are strictly positive at all $s \geq 0$.) At each time $s \in J$ at which the principal invests, the agent's belief x_s jumps from x^- to x^+, and so effort a_s jumps from $a^- = \mu\bar{b}x^-$ to $a^+ = \mu\bar{b}x^+$. At all other times $s \notin J$, the evolution of x_s is given by the law of motion (3), the same one that describes the agent's belief over $[0, \bar{s}]$ in the continuous equilibrium. Note that since the principal has incentives to invest only at the instants $s \in J$, she must be indifferent between investing and not investing at these times.[26] Also, by construction, the low type and high type's expected payoffs are the same at each $s \in J \cup s^0$, and hence the principal is also indifferent at s^0. Analogous to (6) and (7), it follows that $\Lambda_s = F$ and $\mu a^+ \Psi_s = (\gamma + r)F$ at all $s \in J \cup s^0$.

It is worth noting that in any stationary discontinuous equilibrium, Δ must be bounded from below by a strictly positive value.[27] Although the smooth pasting condition need not be satisfied in a discontinuous equilibrium, roughly speaking the intuition is related to that for smooth pasting in the continuous equilibrium: if Δ is too small, the principal's indifference between investing and not when she invests would imply that she has strict incentives to invest at a previous point. Thus, as discussed in the paper, an equilibrium where the agent's belief is constant from (approximately) the time at which the principal starts investing does not exist.

Comparing with the continuous equilibrium of Proposition 1, we find:[28]

Proposition 4. *The principal's expected payoff at $s = 0$, π_0^H, is higher in the continuous equilibrium of Proposition 1 than in any stationary discontinuous equilibrium.*

[26] If the principal had strict incentives to invest at $s \in J$, she would invest over a time interval $[s - \varepsilon, s + \varepsilon]$ for some $\varepsilon > 0$.

[27] To prove this, we can show that $s^0 \leq \bar{s} \leq s^1$, which implies that if Δ (and thus κ) were to go to zero, then s^0 and s^1 would go to $\bar{s}$. However, in this limit, the discontinuous equilibrium would yield a higher payoff for the principal at $s = 0$, π_0^H, than the continuous equilibrium of Proposition 1, contradicting Proposition 4 below. A formal proof for the claim that $s^0 \leq \bar{s} \leq s^1$ is available from the authors upon request.

[28] A welfare analysis of the agent is uninteresting because the agent is myopic. A myopic agent is indifferent between the continuous and discontinuous equilibria at time $s = 0$; at any other time, he prefers the equilibrium that induces higher effort.

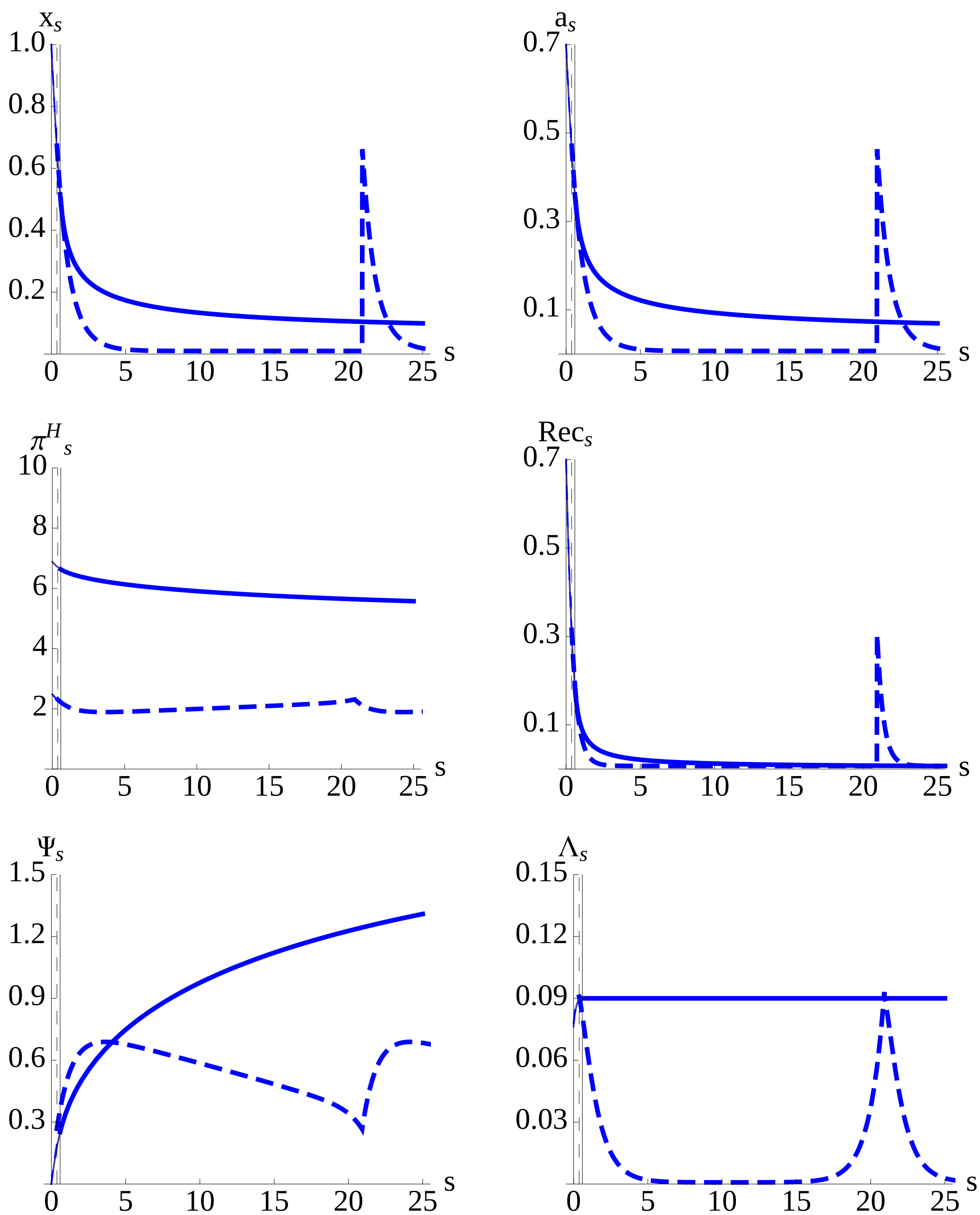

Figure 4: Dynamics in the continuous equilibrium (solid lines) and the discontinuous equilibrium (dashed lines). Parameters are the same as in Figure 1. Rec_s is the unconditional instantaneous probability of recognition, given by $\mu x_s a_s$. The vertical lines indicate the times $\bar{s}$ and s^0.

Proof. Using superscripts d and c to denote variables in the discontinuous equilibrium and the continuous equilibrium respectively, we have $x_s^d = x_s^c$ and $a_s^d = a_s^c$ at all $0 \le s \le \min\{\overline{s}, s^1\}$, and $x_{s+}^d = x_{s^0}^d$ and $a_{s+}^d = a_{s^0}^d$ for all $s \in J$. As for the principal's incentives, as noted, indifference implies $\Lambda_s^d = F$ and $\mu a_{s^0}^d \Psi_s^d = (\gamma + r)F$ at each $s \in J \cup s^0$, and at any $s \notin J \cup s^0$ we must have $\Lambda_s^d < F$. Finally, we will use the fact that in the continuous equilibrium, $\mu a_s^c \Psi_s^c \le (\gamma + r)F$ for all $s \le \overline{s}$. This follows from the proof of Proposition 1, where we show that the equilibrium threshold time $\overline{s}$ is such that the left-hand side of (32) is less than the right-hand side at all $s \le \overline{s}$.

We now proceed by proving two claims.

Claim 1. *If $s^0 > \overline{s}$, then $\pi_0^{Hc} \ge \pi_0^{Hd}$.*

Proof of Claim 1. Suppose by contradiction that $s^0 > \overline{s}$ and $\pi_0^{Hd} > \pi_0^{Hc}$. Note that $s^0 > \overline{s}$ implies

$$\mu a_{s^0}^d \Psi_{s^0}^d = (\gamma + r)F = \mu a_{s^0}^c \Psi_{s^0}^c$$

where $a_{s^0}^d < a_{s^0}^c$. Therefore,

$$\Psi_{s^0}^d = \pi_0^{Hd} - \pi_{s^0}^{Hd} > \pi_0^{Hc} - \pi_{s^0}^{Hc} = \Psi_{s^0}^c. \tag{53}$$

Now note that we can write

$$\begin{aligned}
\pi_0^{Hd} &= \int_0^{s^0} e^{-(\gamma+r)\tau - \int_0^\tau \mu a_{\tilde{\tau}}^d d\tilde{\tau}} \left[a_\tau^d + \gamma \pi_\tau^{Ld} + \mu a_\tau^d \pi_0^{Hd} \right] d\tau + e^{-(\gamma+r)s^0 - \int_0^{s^0} \mu a_{\tilde{\tau}}^d d\tilde{\tau}} \pi_{s^0}^{Hd} \\
&< \int_0^{s^0} e^{-(\gamma+r)\tau - \int_0^\tau \mu a_{\tilde{\tau}}^c d\tilde{\tau}} \left[a_\tau^c + \gamma \left(\int_\tau^{s^0} e^{-r(\tilde{\tau}-\tau)} a_{\tilde{\tau}}^c d\tilde{\tau} + e^{-r(s^0-\tau)} \pi_{s^0}^{Ld} \right) + \mu a_\tau^c \pi_0^{Hd} \right] d\tau \\
&\quad + e^{-(\gamma+r)s^0 - \int_0^{s^0} \mu a_{\tilde{\tau}}^c d\tilde{\tau}} \pi_{s^0}^{Hd},
\end{aligned}$$

where the inequality follows from the fact that $a_s^d = a_s^c$ for $s \in [0, \overline{s}]$, $a_s^d < a_s^c$ for $s \in [\overline{s}, s^0]$, and $\pi_0^{Hd} > \pi_s^{Hd}$ for $s \in (0, s^0]$. It then follows that

$$\begin{aligned}
\pi_0^{Hd} - \pi_0^{Hc} &< \int_0^{s^0} e^{-(\gamma+r)\tau - \int_0^\tau \mu a_{\tilde{\tau}}^c d\tilde{\tau}} \left[\gamma e^{-r(s^0-\tau)} \left(\pi_{s^0}^{Ld} - \pi_{s^0}^{Lc} \right) + \mu a_\tau^c \left(\pi_0^{Hd} - \pi_0^{Hc} \right) \right] d\tau \\
&\quad + e^{-(\gamma+r)s^0 - \int_0^{s^0} \mu a_{\tilde{\tau}}^c d\tilde{\tau}} \left(\pi_{s^0}^{Hd} - \pi_{s^0}^{Hc} \right).
\end{aligned} \tag{54}$$

Note that $\pi_{s^0}^{Ld} = \pi_{s^0}^{Hd} - F$ and $\pi_{s^0}^{Lc} = \pi_{s^0}^{Hc} - F$; hence, substituting,

$$\begin{aligned}\pi_0^{Hd} - \pi_0^{Hc} \quad < \quad & \int_0^{s^0} e^{-(\gamma+r)\tau - \int_0^\tau \mu a_{\tilde{\tau}}^c d\tilde{\tau}} \left[\gamma e^{-r(s^0-\tau)} \left(\pi_{s^0}^{Hd} - \pi_{s^0}^{Hc}\right) + \mu a_\tau^c \left(\pi_0^{Hd} - \pi_0^{Hc}\right)\right] d\tau \\ & + e^{-(\gamma+r)s^0 - \int_0^{s^0} \mu a_{\tilde{\tau}}^c d\tilde{\tau}} \left(\pi_{s^0}^{Hd} - \pi_{s^0}^{Hc}\right). \end{aligned} \tag{55}$$

Recall that by the contradiction assumption, $\pi_0^{Hd} > \pi_0^{Hc}$. But then (55) requires $\pi_0^{Hd} - \pi_0^{Hc} < \pi_{s^0}^{Hd} - \pi_{s^0}^{Hc}$, contradicting (53).[29]‖

Claim 2. *If $s^0 \leq \overline{s}$, then $\pi_0^{Hc} \geq \pi_0^{Hd}$.*

Proof of Claim 2. Suppose by contradiction that $s^0 \leq \overline{s}$ and $\pi_0^{Hd} > \pi_0^{Hc}$. Note that $s^0 \leq \overline{s}$ implies

$$\mu a_{s^0}^d \Psi_{s^0}^d = (\gamma + r)F \geq \mu a_{s^0}^c \Psi_{s^0}^c.$$

Note that $a_s^d = a_s^c$ for $s \in [0, s^0]$. Hence, we obtain

$$\Psi_{s^0}^d = \pi_0^{Hd} - \pi_{s^0}^{Hd} \geq \pi_0^{Hc} - \pi_{s^0}^{Hc} = \Psi_{s^0}^c. \tag{56}$$

Now note that given $a_s^d = a_s^c$ for $s \in [0, s^0]$, we can write

$$\begin{aligned}\pi_0^{Hd} - \pi_0^{Hc} \quad = \quad & \int_0^{s^0} e^{-(\gamma+r)\tau - \int_0^\tau \mu a_{\tilde{\tau}}^c d\tilde{\tau}} \left[\gamma \left(\pi_\tau^{Ld} - \pi_\tau^{Lc}\right) + \mu a_\tau^c \left(\pi_0^{Hd} - \pi_0^{Hc}\right)\right] d\tau \\ & + e^{-(\gamma+r)s^0 - \int_0^{s^0} \mu a_{\tilde{\tau}}^c d\tilde{\tau}} \left(\pi_{s^0}^{Hd} - \pi_{s^0}^{Hc}\right). \end{aligned} \tag{57}$$

[29] To see why (55) requires $\pi_0^{Hd} - \pi_0^{Hc} < \pi_{s^0}^{Hd} - \pi_{s^0}^{Hc}$, divide both sides by $\pi_0^{Hd} - \pi_0^{Hc}$ under the assumption that $\pi_0^{Hd} - \pi_0^{Hc} > 0$:

$$1 < \int_0^{s^0} e^{-(\gamma+r)\tau - \int_0^\tau \mu a_{\tilde{\tau}}^c d\tilde{\tau}} \left[\gamma e^{-r(s^0-\tau)} \left(\frac{\pi_{s^0}^{Hd} - \pi_{s^0}^{Hc}}{\pi_0^{Hd} - \pi_0^{Hc}}\right) + \mu a_\tau^c\right] d\tau + e^{-(\gamma+r)s^0 - \int_0^{s^0} \mu a_{\tilde{\tau}}^c d\tilde{\tau}} \left(\frac{\pi_{s^0}^{Hd} - \pi_{s^0}^{Hc}}{\pi_0^{Hd} - \pi_0^{Hc}}\right).$$

The claim follows from the fact that

$$1 = \int_0^{s^0} e^{-(\gamma+r)\tau - \int_0^\tau \mu a_{\tilde{\tau}}^c d\tilde{\tau}} \left[r + \gamma + \mu a_\tau^c\right] d\tau + e^{-(\gamma+r)s^0 - \int_0^{s^0} \mu a_{\tilde{\tau}}^c d\tilde{\tau}}.$$

For any $\tau \leq s^0$,

$$\pi_\tau^{Ld} - \pi_\tau^{Lc} = e^{-r(s^0-\tau)} \left(\pi_{s^0}^{Ld} - \pi_{s^0}^{Lc}\right) \leq e^{-r(s^0-\tau)} \left(\pi_{s^0}^{Hd} - \pi_{s^0}^{Hc}\right),$$

where the last inequality follows from the fact that $\pi_{s^0}^{Ld} = \pi_{s^0}^{Hd} - F$ whereas $\pi_{s^0}^{Lc} \geq \pi_{s^0}^{Hc} - F$. Hence, substituting $\left(\pi_\tau^{Ld} - \pi_\tau^{Lc}\right)$ in (57), we obtain

$$\begin{aligned} \pi_0^{Hd} - \pi_0^{Hc} \quad \leq \quad & \int_0^{s^0} e^{-(\gamma+r)\tau - \int_0^\tau \mu a_{\widetilde{\tau}}^c d\widetilde{\tau}} \left[\gamma e^{-r(s^0-\tau)} \left(\pi_{s^0}^{Hd} - \pi_{s^0}^{Hc}\right) + \mu a_\tau^c \left(\pi_0^{Hd} - \pi_0^{Hc}\right)\right] d\tau \\ & + e^{-(\gamma+r)s^0 - \int_0^{s^0} \mu a_{\widetilde{\tau}}^c d\widetilde{\tau}} \left(\pi_{s^0}^{Hd} - \pi_{s^0}^{Hc}\right). \end{aligned} \tag{58}$$

Recall that by the contradiction assumption, $\pi_0^{Hd} - \pi_0^{Hc} > 0$. But then (58) requires $\pi_0^{Hd} - \pi_0^{Hc} < \pi_{s^0}^{Hd} - \pi_{s^0}^{Hc}$, contradicting (56).[30]$\|$ *Q.E.D.*

[30]This can be verified following analogous steps to those in fn. 29.