# ON THE ASYMPTOTIC GROWTH OF
# BLOCH–KATO–SHAFAREVICH–TATE GROUPS OF MODULAR
# FORMS OVER CYCLOTOMIC EXTENSIONS

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ABSTRACT. We study the asymptotic behaviour of the Bloch–Kato–Shafarevich–Tate group of a modular form $f$ over the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$ under the assumption that $f$ is non-ordinary at $p$. In particular, we give upper bounds of these groups in terms of Iwasawa invariants of Selmer groups defined using $p$-adic Hodge Theory. These bounds have the same form as the formulae of Kobayashi, Kurihara and Sprung for supersingular elliptic curves.

## 1. Introduction

Let $p$ be an odd prime and $f$ a normalised new cuspidal modular eigenform of weight $k \geq 2$, and $p$ an odd prime which does not divide the level of $f$. For notational simplicity, we assume in this introduction that all the Fourier coefficients of $f$ lie in $\mathbb{Z}$. We let $V_f$ be the *cohomological* $p$-adic Galois representation attached to $f$ (so the determinant of $V_f$ is $\chi^{1-k}$ times a finite-order character). Then $V_f$ has Hodge–Tate weights $\{0, 1-k\}$, where our convention[1] is that the Hodge–Tate weight of the cyclotomic character is 1. Let $T_f$ be the canonical $G_\mathbb{Q}$-stable $\mathbb{Z}_p$-lattice in $V_f$ defined by Kato [Kat04, 8.3].

Let $K_\infty$ be the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$ and write $K_n$ for the unique subextension of degree $p^n$. Our aim is to study the asymptotic behaviour of the Bloch–Kato–Shafarevich–Tate groups $\Sha(K_n, T_f(j))$ (with $j \in [1, k-1]$), whose definition we shall recall below.

When $k = 2$, the form $f$ corresponds to an isogeny class of elliptic curves, and we may choose a curve $\mathcal{E}$ in this isogeny class such that $T_f(1) = T_p(\mathcal{E})$, where the latter is the $p$-adic Tate module of $\mathcal{E}$. In this case it can be shown that the group $\Sha(K_n, T_f(1))$ is the quotient of the classical $p$-primary Shafarevich–Tate group $\Sha_p(K_n, \mathcal{E})$ by its maximal divisible subgroup; hence if the latter group is finite (which is a well-known conjecture), the two groups are equal.

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[1]This is usual in $p$-adic Hodge theory, but the opposite convention appears to be common in papers on modularity lifting.

**The ordinary case.** The behaviour of the Selmer and Shafarevich–Tate groups over the cyclotomic extension depends sharply on whether $\mathcal{E}$ has ordinary or supersingular reduction at $p$. If $\mathcal{E}$ is ordinary, then the $p$-Selmer group

$$\mathrm{Sel}_p(K_\infty, \mathcal{E}) = \varinjlim_n \mathrm{Sel}_p(K_n, \mathcal{E})$$

of $A$ over $K_\infty$ is cotorsion over the Iwasawa algebra $\mathbb{Z}_p[\![\mathrm{Gal}(K_\infty/\mathbb{Q})]\!]$, by a theorem of Kato [Kat04, Theorem 17.4]. By Mazur's control theorem [Maz72], this implies that **if** the groups $\mathrm{III}_p(K_n, \mathcal{E})$ are finite for all $n$, then we must have

$$\mathrm{len}_{\mathbb{Z}_p} \mathrm{III}_p(K_n, \mathcal{E}) = \mu p^n + \lambda n + O(1),$$

for some Iwasawa invariants $\mu$ and $\lambda$ associated to $\mathrm{Sel}_p(\mathcal{E}/K_\infty)$.

**The supersingular case.** The case of supersingular elliptic curves with $a_p(\mathcal{E}) = 0$ has been studied by Kurihara [Kur02] and Kobayashi [Kob03]. Suppose that $\mathrm{III}_p(K_n, \mathcal{E})$ is finite for all $n$ and write $s_n(\mathcal{E}) = \mathrm{len}_{\mathbb{Z}_p} \mathrm{III}_p(K_n, \mathcal{E})$. They showed that for $n$ sufficiently large,

$$s_n(\mathcal{E}) - s_{n-1}(\mathcal{E}) = q_n + \lambda_\pm + \mu_\pm(p^n - p^{n-1}) - r_\infty(\mathcal{E}),$$

where $q_n$ is an explicit sum of powers of $p$, $r_\infty(\mathcal{E})$ is the rank of $\mathcal{E}$ over $K_\infty$, $\lambda_\pm$ and $\mu_\pm$ are the Iwasawa invariants of some cotorsion signed Selmer groups, and the sign $\pm$ depends on the parity of $n$.

For supersingular elliptic curves with $a_p(\mathcal{E}) \neq 0$ (which can only occur when $p = 2$ or $3$), Sprung [Spr13] proved a similar formula:

$$s_n(\mathcal{E}) - s_{n-1}(\mathcal{E}) = q_n^\star + \lambda_\star + \mu_\star(p^n - p^{n-1}) - r_\infty(\mathcal{E}),$$

for $n \gg 0$, where $q_n^\star$ is again an explicit sum of powers of $p$, $\star \in \{\#, \flat\}$, $\lambda_\star$ and $\mu_\star$ are Iwasawa invariants of some cotorsion Selmer groups defined in [Spr12] and the choice of $\star$ depends on the "modesty algorithm". An analytic version of this formula has been generalised to arbitrary weight 2 modular forms in [Spr15].

**Higher weights.** The main result of the present article is that a similar formula for modular forms of higher weight would give us an upper bound on the growth of the Bloch–Kato–Shafarevich–Tate groups. Suppose that $\mathrm{ord}_p(a_p(f)) > \frac{k-1}{2p}$ and $3 \leq k \leq p$, where $a_p(f)$ is the $p$-th Fourier coefficient of the modular form $f$. We shall see below that the Selmer coranks

$$r_n(f) = \mathrm{corank}_{\mathbb{Z}_p} \mathrm{Sel}(K_n, T_f(j))$$

stabilise for $n \gg 0$, and we define $r_\infty(f)$ to be the limiting value (see Proposition 5.4). We define

$$s_n(f) = \mathrm{len}_{\mathbb{Z}_p} \mathrm{III}_p(K_n, T_f(j))$$

(which is finite by definition). We prove the inequality (see Theorem 5.5 for the precise statement)

$$s_n(f) - s_{n-1}(f) \leq q_n^\star + \lambda_\star + \mu_\star(p^n - p^{n-1}) + \kappa - r_\infty(f),$$

for $n \gg 0$, where $q_n^\star$ is once again a sum of powers of $p$ that depends on $k$ and the parity of $n$, $\lambda_\star$ and $\mu_\star$ are the Iwasawa invariants of the Selmer groups defined in [LLZ10] for some choice of basis of the Wach module of $T_f$, $\kappa$ is some integer that depends on the image of some Coleman maps that we shall review in §3 of this article and the choice of $\star$ is given by an explicit algorithm (similar to the "modesty algorithm" of Sprung).

The fact that we have an inequality is a result of the growth of the logarithmic matrix contributed from the twists of $T_f(i)$ for $i \neq j$. In the appendix to this paper, we relate the defect of this inequality to the Tamagamwa numbers of $T_f(j)$ using the method developed by Perrin-Riou in [PR03].

## 2. BACKGROUND FROM $p$-ADIC HODGE THEORY

We recall the necessary notation and definitions from $p$-adic analysis and $p$-adic Hodge theory. For more details see [LLZ11, §1.3]. We fix (for the duration of this article) a finite extension $E/\mathbb{Q}_p$ with ring of integers $\mathcal{O}$, which will be the coefficient field for all the representations we shall consider.

### 2.1. Iwasawa algebras and distribution algebras.
Let $\Gamma = \mathrm{Gal}(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q})$. This group is isomorphic to a direct product $\Delta \times \Gamma_1$, where $\Delta$ is a finite group of order $p-1$ and $\Gamma_1 = \mathrm{Gal}(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q}(\mu_p))$. We choose a topological generator $\gamma$ of $\Gamma_1$, which determines an isomorphism $\Gamma_1 \cong \mathbb{Z}_p$. We also fix a finite extension $E$ of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}$ which will be our field of coefficients (i.e. we will consider representations of Galois groups on $E$-vector spaces).

We write $\Lambda = \mathcal{O}[\![\Gamma]\!]$, the Iwasawa algebra of $\Gamma$. The subalgebra $\mathcal{O}[\![\Gamma_1]\!]$ can be identified with the formal power series ring $\mathcal{O}[\![X]\!]$, via the isomorphism sending $\gamma_1$ to $1+X$; this extends to an isomorphism

$$(2.1) \qquad \Lambda = \mathcal{O}[\Delta][\![X]\!].$$

For a character $\eta$ of $\Delta$ and a $\Lambda$-module $M$, $M^\eta$ denotes its $\eta$-isotypic component, which is regarded as an $\mathcal{O}[\![X]\!]$-module. For $n \geq 1$, we write $\Gamma_n$ for the subgroup $\mathrm{Gal}(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q}(\mu_{p^n}))$ and $\Lambda_n = \mathcal{O}[\Gamma/\Gamma_n]$. Note that

$$\Lambda_n = \mathcal{O}[\Delta][\![X]\!]/(\omega_{n-1}(X)),$$

where $\omega_{n-1}(X)$ denotes the polynomial $(1+X)^{p^{n-1}} - 1$.

We may consider $\Lambda$ as a subring of the ring $\mathcal{H}$ of locally analytic $E$-valued distributions on $\Gamma$. The isomorphism (2.1) extends to an identification between $\mathcal{H}$ and the subring of power series $F \in E[\Delta][\![X]\!]$ which converge on the open unit disc $|X| < 1$.

### 2.2. Power series rings.
Let $\mathbb{A}^+_{\mathbb{Q}_p} = \mathcal{O}[\![\pi]\!]$, where $\pi$ is a formal variable. We equip this ring with a $\mathcal{O}$-linear *Frobenius endomorphism* $\varphi$, defined by $\pi \mapsto (1+\pi)^p - 1$, and with an $\mathcal{O}$-linear action of $\Gamma$ defined by $\pi \mapsto (1+\pi)^{\chi(\sigma)} - 1$ for $\sigma \in \Gamma$, where $\chi$ denotes the $p$-adic cyclotomic character.

The Frobenius $\varphi$ has a left inverse $\psi$, satisfying

$$(\varphi \circ \psi)(f)(\pi) = \tfrac{1}{p} \sum_{\zeta:\zeta^p=1} f\left(\zeta(1+\pi) - 1\right).$$

The map $\psi$ is not a morphism of rings, but it is $\mathcal{O}$-linear, and commutes with the action of $\Gamma$.

We write $\mathbb{B}^+_{\mathbb{Q}_p} = \mathbb{A}^+_{\mathbb{Q}_p}[1/p] \subset E[\![\pi]\!]$, and

$$\mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p} = \left\{ F(\pi) \in E[\![\pi]\!] : F \text{ converges on the open unit disc} \right\},$$

so there are natural inclusions

$$\mathbb{A}^+_{\mathbb{Q}_p} \hookrightarrow \mathbb{B}^+_{\mathbb{Q}_p} \hookrightarrow \mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p}.$$

The actions of $\varphi$, $\psi$, and $\Gamma$ extend to these larger rings (via the same formulae as before). We shall write $q = \varphi(\pi)/\pi \in \mathbb{A}^+_{\mathbb{Q}_p}$, and $t = \log(1+\pi) \in \mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p}$.

2.3. **The Mellin transform.** The action of $\Gamma$ on $1 + \pi \in (\mathbb{A}^+_{\mathbb{Q}_p})^{\psi=0}$ extends to an isomorphism of $\Lambda$-modules

$$\mathfrak{M} : \Lambda \xrightarrow{\cong} (\mathbb{A}^+_{\mathbb{Q}_p})^{\psi=0}$$
$$1 \longmapsto 1 + \pi,$$

called the *Mellin transform*. This can be further extended to an isomorphism of $\mathcal{H}$-modules

$$\mathcal{H} \xrightarrow{\cong} (\mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p})^{\psi=0}$$

which we denote by the same symbol.

**Theorem 2.1.** *For every $n \geq 1$, the Mellin transform induces an isomorphism of $\Lambda$-modules*

$$\Lambda_n \cong (\mathbb{A}^+_{\mathbb{Q}_p})^{\psi=0}/\varphi^n(\pi)(\mathbb{A}^+_{\mathbb{Q}_p})^{\psi=0}.$$

*Proof.* If $\mu \in \omega_{n-1}(X)\Lambda$, then $\mathfrak{M}(\mu) \in \varphi^n(\pi)(\mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p})^{\psi=0}$, by [LLZ10, Theorem 5.4]. However, $\varphi^n(\pi)$ is a monic polynomial in $\pi$, so if an element of $\mathbb{A}^+_{\mathbb{Q}_p}$ is divisible by $\varphi^n(\pi)$ in $\mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p}$, it is divisible by the same element in $\mathbb{A}^+_{\mathbb{Q}_p}$. Hence the Mellin transform induces a map $\Lambda_n \to (\mathbb{A}^+_{\mathbb{Q}_p})^{\psi=0}/\varphi^n(\pi)(\mathbb{A}^+_{\mathbb{Q}_p})^{\psi=0}$; and this map is surjective, because the Mellin transform itself is surjective. Since both sides are free $\mathcal{O}$-modules of the same rank, namely $(p-1)p^n$, it follows that the map must in fact be an isomorphism. $\square$

We write $\partial$ for the differential operator $(1+\pi)\frac{\mathrm{d}}{\mathrm{d}\pi}$ on $\mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p}$, and $\mathrm{Tw}$ for the ring automorphism of $\mathcal{H}$ defined by $\sigma \mapsto \chi(\sigma)\sigma$ for $\sigma \in \Gamma$. Then one has the compatibility relation

$$\mathfrak{M} \circ \mathrm{Tw} = \partial \circ \mathfrak{M}.$$

Let $u = \chi(\gamma)$ be the image of our topological generator $\gamma$ under the cyclotomic character, so that $\mathrm{Tw}$ maps $X$ to $u(1+X) - 1$. If $m \geq 0$ is an integer, we define $\omega_{n,m}(X) = \omega_n(u^{-m}(1+X) - 1)$ and $\tilde{\omega}_{n,m} = \prod_{i=0}^m \omega_{n,i}$. By exactly the same argument as Theorem 2.1, this gives the following isomorphism of $\Lambda$-modules

(2.2)  $$\Lambda_{n,m} := \Lambda/\tilde{\omega}_{n-1,m}\Lambda \cong (\mathbb{A}^+_{\mathbb{Q}_p})^{\psi=0}/\varphi^n(\pi^{m+1})(\mathbb{A}^+_{\mathbb{Q}_p})^{\psi=0}.$$

We will need below the following technical result, regarding the interaction between Mellin transforms and the Iwasawa invariants of power series. We recall the *Weierstrass preparation theorem*, which states that any $F \in \mathcal{O}[\![X]\!]$ can be factorized uniquely as

$$F(X) = \varpi^{\mu(F)} \cdot (X^{\lambda(F)} + \varpi G(X)) \cdot u(X),$$

where $\varpi$ is a uniformizer of $\mathcal{O}$, $\lambda(F)$ and $\mu(F)$ are non-negative integers, $G \in \mathcal{O}[X]$ is a polynomial of degree $< \lambda(F)$, and $u \in \mathcal{O}[\![X]\!]^\times$. The quantities $\lambda(F)$ and $\mu(F)$ are called the *Iwasawa invariants* of $F$.

It is clear that, for $x \in \mathcal{O}_{\mathbf{C}_p}$ with $\mathrm{ord}_p(x) > 0$, we have the lower bound

(2.3)  $$\mathrm{ord}_p F(x) \geq \min\left(\tfrac{\mu+1}{e}, \tfrac{\mu}{e} + \lambda\,\mathrm{ord}_p(x)\right),$$

where $e = 1/\operatorname{ord}_p(\varpi)$ is the absolute ramification degree of $F$. Moreover, if $\operatorname{ord}_p(x)$ is sufficiently small (depending on $F$), this lower bound is an equality (it suffices to take $\operatorname{ord}_p(x) < 1/(e\lambda)$).

**Proposition 2.2.** *Let $f \in \mathbb{A}^+_{\mathbb{Q}_p}$, and let $g$ be the unique element of $\Lambda(\Gamma_1)$ such that $\mathfrak{M}(g) = (1 + \pi)\varphi(f)$. Then the $\lambda$- and $\mu$-invariants of $f$ (as an element of $\mathcal{O}[\![\pi]\!]$) coincide with those of $g$ (as an element of $\mathcal{O}[\![X]\!]$).*

*Proof.* This is a consequence of Proposition 7.2 of [LZ12], which shows that for any $f \in \mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p}$ and $g \in \mathcal{H}$ such that $\mathfrak{M}(g) = (1+\pi)\varphi(f)$, and any real $s$ with $0 < s < 1$, we have $v_s(f) = v_s(g)$, where

$$v_s(f) := \inf\{\operatorname{ord}_p f(x) : \operatorname{ord}_p(x) \geq s\}.$$

When $f \in \mathcal{O}[\![X]\!]$ and $s$ is sufficiently small, $v_s(f)$ is determined by the Iwasawa invariants of $f$: from the inequality (2.3) and the discussion following, we have $v_s(f) = \frac{1}{e}\mu(f) + \lambda(f)s$ for any $s < \frac{1}{e\lambda(f)}$. So the cited proposition implies the equalities $\lambda(f) = \lambda(g)$ and $\mu(f) = \mu(g)$. $\square$

2.4. **Crystalline representations and Wach modules.** Fontaine has defined a certain topological $\mathbb{Q}_p$-algebra $\mathbb{B}_{\mathrm{cris}}$, equipped with an action of $G_{\mathbb{Q}_p}$, a filtration $\mathrm{Fil}^\bullet$, and a Frobenius endomorphism $\varphi$.

For any $p$-adic representation $V$ of $G_{\mathbb{Q}_p}$, we define the *crystalline Dieudonné module* of $V$ by

$$\mathbb{D}_{\mathrm{cris}}(V) = \left(V \otimes_{\mathbb{Q}_p} \mathbb{B}_{\mathrm{cris}}\right)^{G_{\mathbb{Q}_p}}.$$

The space $\mathbb{D}_{\mathrm{cris}}(V)$ inherits a filtration and a Frobenius endomorphism from those of $\mathbb{B}_{\mathrm{cris}}$. It is known that $\dim_{\mathbb{Q}_p} \mathbb{D}_{\mathrm{cris}}(V) \leq \dim_{\mathbb{Q}_p} V$, and we say $V$ is *crystalline* if equality holds. If in fact $V$ is an $E$-linear representation, then $\mathbb{D}_{\mathrm{cris}}(V)$ is naturally an $E$-vector space (and its filtration and Frobenius are $E$-linear).

**Definition 2.3.** *Let $a \leq b$ be integers. A* Wach module *over $\mathbb{B}^+_{\mathbb{Q}_p}$ with weights in $[a, b]$ is a finite free $\mathbb{B}^+_{\mathbb{Q}_p}$-module $N$, equipped with an action of $\Gamma$ and a Frobenius*

$$\varphi : N[1/\pi] \to N[1/\varphi(\pi)]$$

*compatible with those of $\mathbb{B}^+_{\mathbb{Q}_p}$, satisfying the following conditions:*

- *$\Gamma$ acts trivially on $N/\pi N$,*
- *$\varphi(\pi^b N) \subseteq \pi^b N$,*
- *if $\varphi^*(\pi^b N)$ is the $\mathbb{B}^+_{\mathbb{Q}_p}$-submodule of $\pi^b N$ generated by $\varphi(\pi^b N)$, then the quotient $\pi^b N/\varphi^*(\pi^b N)$ is killed by $q^{b-a}$.*

Cf. [Ber04, Definition III.4.1]. In *op.cit.* it is shown how to attach to every crystalline $E$-linear representation $V$ of $G_{\mathbb{Q}_p}$ a Wach module $\mathbb{N}(V)$ over $\mathbb{B}^+_{\mathbb{Q}_p}$, in such a way that there is a canonical isomorphism

$$\mathbb{N}(V) \otimes_{\mathbb{B}^+_{\mathbb{Q}_p}} \mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p}[1/t] \cong \mathbb{D}_{\mathrm{cris}}(V) \otimes_E \mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p}[1/t].$$

Moreover, the definition of Wach modules also makes sense integrally, i.e. over $\mathbb{A}^+_{\mathbb{Q}_p}$; and we may associate to each $\mathcal{O}$-lattice $T$ in $V$ that is stable under $G_{\mathbb{Q}_p}$ an integral Wach module $\mathbb{N}(T) \subset \mathbb{N}(V)$ (Lemme II.1.3 of *op.cit.*).

**Definition 2.4.** *We say $V$ satisfies the* Fontaine–Laffaille condition *if it is crystalline and has Hodge–Tate weights in $[a, a + (p-1)]$ for some $a \in \mathbb{Z}$.*

If $V$ satisfies the Fontaine–Laffaille condition, and $V$ is irreducible of dimension $\geq 2$, then one has a particularly convenient parametrisation of $G_{\mathbb{Q}_p}$-stable lattices in $V$. We say a $\mathcal{O}$-lattice $M \subset \mathbb{D}_{\mathrm{cris}}(V)$ is a *strongly divisible lattice* if the equality

$$\varphi\left(M \cap \mathrm{Fil}^i \mathbb{D}_{\mathrm{cris}}(V)\right) \subset p^i M$$

holds for all $i \in \mathbb{Z}$. Then there is a bijection $T \mapsto \mathbb{D}_{\mathrm{cris}}(T)$ between $G_{\mathbb{Q}_p}$-stable lattices in $V$, and strongly divisible lattices in $\mathbb{D}_{\mathrm{cris}}(V)$, given by defining $\mathbb{D}_{\mathrm{cris}}(T)$ to be the image of $\mathbb{N}(T)$ in $\mathbb{N}(V)/\pi\mathbb{N}(V) \cong \mathbb{D}_{\mathrm{cris}}(V)$; cf. [Ber04, Propositions V.2.1 & V.2.3].

We shall need below the following technical result.

**Theorem 2.5.** *Let $T$ be a $G_{\mathbb{Q}_p}$-stable $\mathcal{O}$-lattice in a crystalline $E$-linear representation $V$. Then $(\varphi^*\mathbb{N}(T))^{\psi=0}$ is a free $\Lambda$-module of rank $d = \dim_E V$. Moreover, if $\{n_1, \ldots, n_d\}$ is an $\mathbb{A}_{\mathbb{Q}_p}^+$-basis of $\mathbb{N}(T)$ which satisfies the condition*

$$(\gamma - 1)n_i \in \pi^2\mathbb{N}(T)$$

*for all $i$, then $\{(1+\pi)\varphi(n_i) : i = 1, \ldots, d\}$ is a $\Lambda$-module basis of $(\varphi^*\mathbb{N}(T))^{\psi=0}$.*

*Proof.* This is shown in the course of the proof of Theorem 3.5 of [LLZ10]. The condition on the basis modulo $\pi^2$ is the conclusion of Lemma 3.9 in *op. cit.* $\qquad\square$

2.5. **Iwasawa cohomology and the Fontaine isomorphism.** If $V$ is an $E$-linear $p$-adic representation of $G_{\mathbb{Q}_p}$, and $T \subset V$ is a $G_{\mathbb{Q}_p}$-stable $\mathcal{O}_E$-lattice, then we define *Iwasawa cohomology* groups by

$$H^i_{\mathrm{Iw}}(\mathbb{Q}_p(\mu_{p^\infty}), T) = \varprojlim_n H^1(\mathbb{Q}_p(\mu_{p^n}), T)$$

(where the inverse limit is with respect to the corestriction maps). These groups are finitely-generated $\Lambda$-modules, zero unless $i \in \{1, 2\}$. If $H^0(\mathbb{Q}_p(\mu_{p^\infty}), T/pT) = 0$, which is the case in our applications below, then $H^2_{\mathrm{Iw}}$ is zero, and $H^1_{\mathrm{Iw}}$ is a free $\Lambda$-module of rank equal to the $\mathcal{O}$-rank of $T$.

The following theorem is the starting-point for our study of Iwasawa cohomology:

**Theorem 2.6** (Fontaine–Berger). *If $V$ is crystalline with all Hodge–Tate weights $\geq 0$, and $V$ has no non-zero quotient on which $G_{\mathbb{Q}_p}$ acts trivially, then there is a canonical $\Lambda$-module isomorphism*

$$h^1_T : \mathbb{N}(T)^{\psi=1} \to H^1_{\mathrm{Iw}}(\mathbb{Q}_p(\mu_{p^\infty}), T).$$

See [CC99, §II.1], where it is shown that (for any $T$) there is an isomorphism $\mathbb{D}(T)^{\psi=1} \to H^1_{\mathrm{Iw}}(\mathbb{Q}_p(\mu_{p^\infty}), T)$ where $\mathbb{D}(T)$ is the $(\varphi, \Gamma)$-module of $T$; and [Ber03, §A], where it is shown that $\mathbb{N}(T)^{\psi=1} = \mathbb{D}(T)^{\psi=1}$ under the above hypotheses.

## 3. Wach modules and Coleman maps

3.1. **Review on the definition of Coleman maps.** Let $f = \sum a_n q^n$ be a normalised new cuspidal modular eigenform of weight $k \geq 3$ (note that the case $k = 2$ can be dealt with using the method of Sprung in [Spr13]), nebentypus $\varepsilon$ and level $N$ with $(p, N) = 1$. We take $E$ to be the completion of the smallest number field containing all the coefficients of $f$ at some fixed prime above $p$. We assume that $f$ is non-ordinary at $p$, and that $k \leq p$. We write $T_f$ for the $\mathcal{O}$-linear representation of

$G_{\mathbb{Q}}$ associated to $f$ as defined by Kato [Kat04, 8.3]. It is crystalline, with Hodge–Tate weights 0 and $1 - k$. We fix an integer $j \in [1, k-1]$ and write $T = T_f(j)$ and $\mathcal{T} = T_f(k-1)$. Note that $T = \mathcal{T}(j - k + 1)$.

The representation $T/\varpi T$ (where $\varpi$ is a uniformiser of $\mathcal{O}$) is irreducible as a representation of $G_{\mathbb{Q}_p}$, so in particular we have

$$H^0(\mathbb{Q}_p(\mu_{p^\infty}), T/\varpi T) = 0.$$

Both $T_f$ and $\mathcal{T}$ are $G_{\mathbb{Q}_p}$-stable $\mathcal{O}_E$-lattices in crystalline representations of $G_{\mathbb{Q}_p}$, so we may consider their Wach modules and Dieudonné modules. By [Ber04, Proposition III.2.1], there are inclusions of $\mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p}$-modules

$$\mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p} \otimes_{\mathbb{A}^+_{\mathbb{Q}_p}} \mathbb{N}(\mathcal{T}) \subset \mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p} \otimes_{\mathcal{O}} \mathbb{D}_{\mathrm{cris}}(\mathcal{T}),$$

$$\mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p} \otimes_{\mathcal{O}} \mathbb{D}_{\mathrm{cris}}(T_f) \subset \mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p} \otimes_{\mathbb{A}^+_{\mathbb{Q}_p}} \mathbb{N}(T_f),$$

where the elementary divisors of the inclusions are given by 1 and $(t/\pi)^{k-1}$ in both cases.

**Lemma 3.1.** *There exists an $\mathcal{O}$-basis $\mathfrak{v}_1, \mathfrak{v}_2$ of $\mathbb{D}_{\mathrm{cris}}(\mathcal{T})$ such that $\mathfrak{v}_1 \in \mathrm{Fil}^0 \mathbb{D}_{\mathrm{cris}}(\mathcal{T})$ and $\mathfrak{v}_2 = \varphi(\mathfrak{v}_1)$, where $\varphi$ is the Frobenius action on $\mathbb{D}_{\mathrm{cris}}(\mathcal{T})$.*

*Proof.* The Fontaine–Laffaille condition of [FL82] implies that for all integers $i$

(a) $\mathrm{Fil}^i \mathbb{D}_{\mathrm{cris}}(\mathcal{T})$ is a direct summand of $\mathbb{D}_{\mathrm{cris}}(\mathcal{T})$;
(b) $\varphi(\mathrm{Fil}^i \mathbb{D}_{\mathrm{cris}}(\mathcal{T})) \subset p^i \mathbb{D}_{\mathrm{cris}}(\mathcal{T})$;
(c) $\mathbb{D}_{\mathrm{cris}}(\mathcal{T}) = \sum_i p^{-i} \varphi(\mathrm{Fil}^i \mathbb{D}_{\mathrm{cris}}(\mathcal{T}))$.

The Hodge–Tate weights of $\mathcal{T}$ are 0 and $k-1$, so $\mathrm{Fil}^0 \mathbb{D}_{\mathrm{cris}}(\mathcal{T})$ is of rank 1, say $\mathrm{Fil}^0 \mathbb{D}_{\mathrm{cris}}(\mathcal{T}) = \mathcal{O} \cdot \mathfrak{v}_1$ and (b) says that $\mathfrak{v}_2 := \varphi(\mathfrak{v}_1) \in \mathbb{D}_{\mathrm{cris}}(\mathcal{T})$. Furthermore, (a) tells us that there exists some $\mathfrak{v}' \in \mathbb{D}_{\mathrm{cris}}(\mathcal{T})$ such that

$$\mathbb{D}_{\mathrm{cris}}(\mathcal{T}) = \mathcal{O} \cdot \mathfrak{v}_1 \oplus \mathcal{O} \cdot \mathfrak{v}'.$$

By (c), we have

$$\mathbb{D}_{\mathrm{cris}}(\mathcal{T}) = \mathcal{O} \cdot \varphi(\mathfrak{v}_1) + p^{k-1} \varphi(\mathbb{D}_{\mathrm{cris}}(\mathcal{T})).$$

Combing the last two equations gives

(3.1) $$\mathbb{D}_{\mathrm{cris}}(\mathcal{T}) = \mathcal{O} \cdot \varphi(\mathfrak{v}_1) \oplus \mathcal{O} \cdot p^{k-1} \varphi(\mathfrak{v}').$$

Let $D$ be the $\mathcal{O}$-lattice generated by $\mathfrak{v}_1$ and $\mathfrak{v}_2$. Note that (3.1) implies that

(3.2) $$\mathfrak{v}' \in D + \mathcal{O} \cdot p^{k-1} \varphi(\mathfrak{v}').$$

As $\mathfrak{v}_2 = \varphi(\mathfrak{v}_1)$ and

$$\varphi^2 - \frac{a_p}{p^{k-1}} \varphi + \frac{\varepsilon(p)}{p^{k-1}} = 0$$

on $\mathbb{D}_{\mathrm{cris}}(\mathcal{T})$, we have $p^{k-1} \varphi(\mathfrak{v}_2) = a_p \mathfrak{v}_2 - \varepsilon(p) \mathfrak{v}_1$. In particular, this implies that $p^{k-1} \varphi(D) \subset D$. Hence, we may iterate the inclusion (3.2) to deduce that

$$\mathfrak{v}' \in D + \mathcal{O} \cdot (p^{k-1} \varphi)^n (\mathfrak{v}')$$

for all $n \geq 0$. However, as $f$ is non-ordinary at $p$, $p^{k-1} \varphi$ is an $\mathcal{O}$-operator on $\mathbb{D}_{\mathrm{cris}}(\mathcal{T})$ with strictly positive slope. This implies that $(p^{k-1} \varphi)^n \to 0$ as $n \to \infty$, which forces that $\mathfrak{v}' \in D$. Hence, $D = \mathbb{D}_{\mathrm{cris}}(\mathcal{T})$ as required. $\square$

We fix an $\mathcal{O}$-basis $\mathfrak{v}_1, \mathfrak{v}_2$ of $\mathbb{D}_{\mathrm{cris}}(\mathcal{T})$, as given by Lemma 3.1. Since $\mathbb{D}_{\mathrm{cris}}(\mathcal{T}) = \mathbb{N}(\mathcal{T})/\pi\mathbb{N}(\mathcal{T})$, this basis can be lifted to a basis $\mathfrak{n}_1, \mathfrak{n}_2$ of $\mathbb{N}(\mathcal{T})$ as an $\mathbb{A}^+_{\mathbb{Q}_p}$-module. There is a change of basis matrix $M \in M_{2\times 2}(\mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p})$ such that

$$(3.3) \qquad\qquad \begin{pmatrix} \mathfrak{n}_1 & \mathfrak{n}_2 \end{pmatrix} = \begin{pmatrix} \mathfrak{v}_1 & \mathfrak{v}_2 \end{pmatrix} M$$

and $M \equiv I_2 \mod \pi$, where $I_2$ is the $2 \times 2$ identity matrix. We write $v_i = \mathfrak{v}_i \cdot t^{k-j-1} e_{-k+j+1}$, $n_i = \mathfrak{n}_i \cdot \pi^{k-j-1} e_{-k+j+1}$, $v_{f,i} = \mathfrak{v}_i \cdot t^{k-1} e_{1-k}$ and $n_{f,i} = \mathfrak{n}_i \cdot \pi^{k-1} e_{1-k}$ for the corresponding bases of $\mathbb{D}_{\mathrm{cris}}(T)$, $\mathbb{N}(T)$, $\mathbb{D}_{\mathrm{cris}}(T_f)$ and $\mathbb{N}(T_f)$ respectively. Here $e_r$ denotes a basis of the Tate motive $\mathcal{O}(\chi^r)$ for $r \in \mathbb{Z}$. By [Ber04, proof of Proposition V.2.3] and [Lei15, Proposition 4.2], we may choose our bases so that

$$(3.4) \qquad\qquad M \equiv I_2 \mod \pi^{k-1}$$

and that the matrices of $\varphi$ with respect to $v_{1,f}, v_{2,f}$ and $n_{1,f}, n_{2,f}$ are given by

$$\begin{pmatrix} 0 & -\varepsilon(p) \\ p^{k-1} & a_p \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & -\varepsilon(p) \\ (\delta q)^{k-1} & a_p \end{pmatrix}$$

respectively, where $\delta = p/(q - \pi^{p-1}) \in (\mathbb{A}^+_{\mathbb{Q}_p})^\times$. Then, the matrices of $\varphi$ with respect to $\mathfrak{v}_1, \mathfrak{v}_2$ and $\mathfrak{n}_1, \mathfrak{n}_2$ are given by

$$A = \begin{pmatrix} 0 & -\frac{\varepsilon(p)}{p^{k-1}} \\ 1 & \frac{a_p}{p^{k-1}} \end{pmatrix} \quad \text{and} \quad P = \begin{pmatrix} 0 & -\frac{\varepsilon(p)}{q^{k-1}} \\ \delta^{k-1} & \frac{a_p}{q^{k-1}} \end{pmatrix}.$$

**Definition 3.2.** *We define the* logarithmic matrix $M_{\log}$ *(with respect to the chosen bases) to be* $\mathfrak{M}^{-1}\left((1+\pi)A\varphi(M)\right)$.

**Theorem 3.3.** *Let* $\mathfrak{n}_1, \mathfrak{n}_2$ *be the basis of* $\mathbb{N}(\mathcal{T})$ *chosen above. Then,* $(1+\pi)\varphi(\mathfrak{n}_1), (1+\pi)\varphi(\mathfrak{n}_2)$ *form a* $\Lambda$*-basis of* $(\varphi^*\mathbb{N}(\mathcal{T}))^{\psi=0}$.

*Proof.* Let $\gamma \in \Gamma$ be a topological generator. Then, (3.3) tells us that

$$\begin{pmatrix} \gamma \cdot \mathfrak{n}_1 & \gamma \cdot \mathfrak{n}_2 \end{pmatrix} = \begin{pmatrix} \mathfrak{v}_1 & \mathfrak{v}_2 \end{pmatrix} \gamma(M).$$

This gives the equation

$$\begin{pmatrix} \gamma \cdot \mathfrak{n}_1 & \gamma \cdot \mathfrak{n}_2 \end{pmatrix} = \begin{pmatrix} \mathfrak{n}_1 & \mathfrak{n}_2 \end{pmatrix} M^{-1} \cdot \gamma(M).$$

Hence, for both $i = 1, 2$, we have

$$(1 - \gamma)\mathfrak{n}_i \in \pi^{k-1}\mathbb{N}(\mathcal{T})$$

thanks to (3.4). As we assume that $k \geq 3$, we have in particular

$$(1 - \gamma)\mathfrak{n}_i \in \pi^2\mathbb{N}(\mathcal{T}),$$

which is the condition required in Theorem 2.5[2]. Therefore, our result follows. $\square$

Recall from [LLZ10, Remark 3.4] that for all $z \in \mathbb{N}(\mathcal{T})^{\psi=1}$, we have $(1 - \varphi)z \in (\varphi^*\mathbb{N}(\mathcal{T}))^{\psi=0}$. The latter is free of rank 2 over $\Lambda$, with basis $(1 + \pi)\varphi(\mathfrak{n}_1), (1 + \pi)\varphi(\mathfrak{n}_2)$ as given by Theorem 3.3. This allows us to define the Coleman maps (again, with respect to our chosen bases) as follows.

---

[2]This is the only place where we use the assumption that $k \geq 3$.

**Definition 3.4.** *For $i \in \{1, 2\}$, we define the $\Lambda$-homomorphisms $\mathrm{Col}_i : \mathbb{N}(\mathcal{T})^{\psi=1} \to \Lambda$ given by the relation*

$$(1 - \varphi)z = \sum_{i=1}^{2} \mathrm{Col}_i(z) \cdot (1 + \pi)\varphi(\mathfrak{n}_i) = \begin{pmatrix} \mathfrak{v}_1 & \mathfrak{v}_2 \end{pmatrix} \cdot M_{\log} \cdot \begin{pmatrix} \mathrm{Col}_1(z) \\ \mathrm{Col}_2(z) \end{pmatrix}.$$

Let $h_{\mathcal{T}}^1 : \mathbb{N}(\mathcal{T})^{\psi=1} \to H_{\mathrm{Iw}}^1(\mathbb{Q}_p(\mu_{p^\infty}), \mathcal{T})$ be the $\Lambda$-isomorphism given by Theorem 2.6. By an abuse of notation, we shall write $\mathrm{Col}_1, \mathrm{Col}_2$ for the compositions $\mathrm{Col}_1 \circ (h_{\mathcal{T}}^1)^{-1}$ and $\mathrm{Col}_2 \circ (h_{\mathcal{T}}^1)^{-1}$ as well.

3.2. **A finite projection of the Coleman maps.**

**Definition 3.5.** *For each $n \geq 1$, we define $H_n = \varphi^{n-1}(P^{-1}) \cdots \varphi(P^{-1})$ and $\mathscr{H}_n = \mathfrak{M}^{-1}((1 + \pi)H_n)$.*

**Remark 3.6.** *Note that $H_n \in \mathbb{A}_{\mathbb{Q}_p}^+$, and $\mathscr{H}_n \in \Lambda$; and $H_1 = \mathscr{H}_1 = 1$.*

**Lemma 3.7.** *We have the congruence*

$$M_{\log} \equiv A^n \cdot \mathscr{H}_n \mod \tilde{\omega}_{n-1,k-2}(X)\mathcal{H}.$$

*Proof.* From (3.3), we have the relation

$$MP = A\varphi(M),$$

which we may rewrite as $M = A\varphi(M)P^{-1}$. On iteration, we have

$$M = A^{n-1}\varphi^{n-1}(M)\varphi^{n-2}(P^{-1}) \cdots \varphi(P^{-1})P^{-1}.$$

By (3.4), we have $\varphi^{n-1}(M) = 1 \mod \varphi^{n-1}(\pi^{k-1})$, so this implies that

$$M \equiv A^{n-1}\varphi^{n-2}(P^{-1}) \cdots \varphi(P^{-1})P^{-1} \mod \varphi^{n-1}(\pi^{k-1}).$$

This implies that

$$\varphi(M) \equiv A^{n-1} \cdot H_n \mod \varphi^n(\pi^{k-1}).$$

Hence the result by (2.2). $\square$

**Lemma 3.8.** *For all $n \geq 1$ and $z \in \mathbb{N}(\mathcal{T})^{\psi=1}$, $(1 \otimes \varphi^{-n}) \circ (1 - \varphi)z$ is congruent to an element in $\Lambda_{n,k-2} \otimes \mathbb{D}_{\mathrm{cris}}(\mathcal{T})$ modulo $\tilde{\omega}_{n-1,k-2}(X)\mathcal{H} \otimes \mathbb{D}_{\mathrm{cris}}(\mathcal{T})$.*

*Proof.* By Lemma 3.7 and the equation in Definition 3.4, we have the congruence

$$(1 - \varphi)z \equiv \begin{pmatrix} \mathfrak{v}_1 & \mathfrak{v}_2 \end{pmatrix} \cdot A^n \cdot \mathscr{H}_n \cdot \begin{pmatrix} \mathrm{Col}_1(z) \\ \mathrm{Col}_2(z) \end{pmatrix} \mod \tilde{\omega}_{n-1,k-2}(X)\mathcal{H} \otimes \mathbb{D}_{\mathrm{cris}}(\mathcal{T}).$$

If we apply $(1 \otimes \varphi^{-n})$ to both sides, we obtain

$$\begin{pmatrix} 1 \otimes \varphi^{-n} \end{pmatrix} \circ (1 - \varphi)z \equiv \begin{pmatrix} \mathfrak{v}_1 & \mathfrak{v}_2 \end{pmatrix} \cdot \mathscr{H}_n \cdot \begin{pmatrix} \mathrm{Col}_1(z) \\ \mathrm{Col}_2(z) \end{pmatrix} \mod \tilde{\omega}_{n-1,k-2}(X)\mathcal{H} \otimes \mathbb{D}_{\mathrm{cris}}(\mathcal{T}).$$

As $\mathscr{H}_n$, $\mathrm{Col}_1(z)$ and $\mathrm{Col}_2(z)$ are all defined over $\Lambda$, we see that $(1 \otimes \varphi^{-n}) \circ (1 - \varphi)z$ is indeed congruent to an element in $\Lambda_{n,p-2} \otimes \mathbb{D}_{\mathrm{cris}}(\mathcal{T})$. $\square$

This allows us to give the following definition.

**Definition 3.9.** *For $n \geq 1$, define*

$$\underline{\mathrm{Col}}_n : H_{\mathrm{Iw}}^1(\mathbb{Q}_p(\mu_{p^\infty}), \mathcal{T}) \to \Lambda_{n,k-2} \otimes \mathbb{D}_{\mathrm{cris}}(\mathcal{T})$$

$$z \mapsto \begin{pmatrix} 1 \otimes \varphi^{-n} \end{pmatrix} \circ (1 - \varphi) \circ (h_{\mathcal{T}}^1)^{-1}(z) \mod \tilde{\omega}_{n-1,k-2}(X).$$

We recall that $h_{\mathcal{T}}^1$ is an isomorphism by Theorem 2.6. Therefore, Lemma 3.8 tells us that the map $\underline{\mathrm{Col}}_n$ is well-defined.

For an integer $m$, we define the twisting map

$$\mathrm{Tw}_m \coloneqq \mathrm{Tw}^{-m} \otimes t^{-m} e_m : \mathcal{H} \otimes \mathbb{D}_{\mathrm{cris}}(\mathcal{T}) \to \mathcal{H} \otimes \mathbb{D}_{\mathrm{cris}}(\mathcal{T}(m)).$$

Consider the twisting map $\mathrm{Tw}^{k-j-1} : \sigma \mapsto \chi^{k-j-1}(\sigma)\sigma$ on $\Lambda$. Since $k-j-1 \leq k-1$, $\mathrm{Tw}^{k-j-1}(\tilde{\omega}_{n-1,k-2}(X))$ is divisible by $\omega_{n-1}(X)$. Hence, $\mathrm{Tw}^{k-j-1}$ induces a natural map $\Lambda_{n,k-2} \to \Lambda_n$. Therefore, we may define

$$\underline{\mathrm{Col}}_{T,n} : H_{\mathrm{Iw}}^1(\mathbb{Q}_p(\mu_{p^\infty}), T) \to \Lambda_n \otimes \mathbb{D}_{\mathrm{cris}}(T)$$

$$z \mapsto \mathrm{Tw}_{-k+j+1} \circ \underline{\mathrm{Col}}_n(z \cdot e_{k-j-1}) \mod \omega_{n-1}(X),$$

on identifying $H_{\mathrm{Iw}}^1(\mathbb{Q}_p(\mu_{p^\infty}), T) \cdot e_{k-j-1}$ with $H_{\mathrm{Iw}}^1(\mathbb{Q}_p(\mu_{p^\infty}), \mathcal{T})$.

**Lemma 3.10.** *The map* $\underline{\mathrm{Col}}_{T,n}$ *defines a* $\Lambda_n$*-homomorphism from* $H^1(\mathbb{Q}_p(\mu_{p^n}), T)$ *to* $\Lambda_n \otimes \mathbb{D}_{\mathrm{cris}}(T)$.

*Proof.* We note that $\underline{\mathrm{Col}}_{T,n}$ is a $\Lambda$-homomorphism since both $\underline{\mathrm{Col}}_n$ and $x \mapsto \mathrm{Tw}^{k-j-1} \circ (x \cdot e_{k-j-1})$ are $\Lambda$-linear. The fact that $\underline{\mathrm{Col}}_{T,n}$ factors through $H^1(\mathbb{Q}_p(\mu_{p^n}), T)$ follows from the equation $H_{\mathrm{Iw}}^1(\mathbb{Q}_p(\mu_{p^\infty}), T)_{\Gamma_n} = H^1(\mathbb{Q}_p(\mu_{p^n}), T)$ (because of the vanishing of $H_{\mathrm{Iw}}^2(\mathbb{Q}_p(\mu_{p^\infty}), T)$). $\square$

We have the explicit formula

$$(3.5) \quad \underline{\mathrm{Col}}_{T,n}(z) \equiv \begin{pmatrix} v_1 & v_2 \end{pmatrix} \cdot \mathrm{Tw}^{k-1-j} \left( \mathscr{H}_n \cdot \begin{pmatrix} \mathrm{Col}_1(z \cdot e_{k-1-j}) \\ \mathrm{Col}_2(z \cdot e_{k-1-j}) \end{pmatrix} \right)$$
$$\mod \omega_{n-1}(X)\Lambda \otimes \mathbb{D}_{\mathrm{cris}}(T),$$

by Lemma 3.7 and the expansion of $1 - \varphi$ as given in Definition 3.4.

We now modify the definition of $\underline{\mathrm{Col}}_{T,n}$ to define a map that lands in $\Lambda_n$. For any $u \in \mathbb{Z}_p^\times$, we define $\underline{\mathrm{Col}}_{T,n,u} : H_{\mathrm{Iw}}^1(\mathbb{Q}_p(\mu_{p^\infty}), T) \to \Lambda_n$ to be the composition of $\underline{\mathrm{Col}}_{T,n}$ and the linear functional on $\Lambda_n \otimes \mathbb{D}_{\mathrm{cris}}(T) \to \Lambda_n$ given by $a \cdot v_1 + b \cdot v_2 \mapsto a + ub$. More explicitly, (3.5) tells us that $\underline{\mathrm{Col}}_{T,n,u}$ is given by
(3.6)

$$\underline{\mathrm{Col}}_{T,n,u}(z) \equiv \begin{pmatrix} 1 & u \end{pmatrix} \cdot \mathrm{Tw}^{k-1-j} \left( \mathscr{H}_n \cdot \begin{pmatrix} \mathrm{Col}_1(z \cdot e_{k-1-j}) \\ \mathrm{Col}_2(z \cdot e_{k-1-j}) \end{pmatrix} \right) \mod \omega_{n-1}(X)\Lambda.$$

Note that Lemma 3.10 tells us that $\underline{\mathrm{Col}}_{T,n,u}$ is $\Lambda_n$-linear.

### 3.3. Analysis of Bloch–Kato subgroups via Coleman maps.

If $F$ is a finite extension of $\mathbb{Q}_p$, we write $H_f^1(F, T) \subset H^1(F, T)$ for the usual Bloch–Kato subgroup from [BK90] and $H_{/f}^1(F, T)$ denotes the quotient $H^1(F, T)/H_f^1(F, T)$. The goal of this section is to study $H_{/f}^1(\mathbb{Q}_p(\mu_{p^n}), T)$ via the map $\underline{\mathrm{Col}}_{T,n,u}$.

Let $\mathcal{T}^*$ be the $\mathcal{O}$-linear dual of $\mathcal{T}$. For each $n \geq 1$, we define the pairing

$$\langle \sim, \sim \rangle_n : H^1(\mathbb{Q}_p(\mu_{p^n}), \mathcal{T}) \times H^1(\mathbb{Q}_p(\mu_{p^n}), \mathcal{T}^*(1)) \to \Lambda_n$$

$$(x, y) \mapsto \sum_{\sigma \in \Gamma/\Gamma_n} [x, y^\sigma]_n \sigma,$$

where $[\sim, \sim]_n$ is the standard cup-product pairing

$$H^1(\mathbb{Q}_p(\mu_{p^n}), \mathcal{T}) \times H^1(\mathbb{Q}_p(\mu_{p^n}), \mathcal{T}^*(1)) \to \mathcal{O}.$$

On taking inverse limits, this induces a pairing

$$\langle \sim, \sim \rangle : H^1_{\mathrm{Iw}}(\mathbb{Q}_p(\mu_{p^\infty}), \mathcal{T}) \times H^1_{\mathrm{Iw}}(\mathbb{Q}_p(\mu_{p^\infty}), \mathcal{T}^*(1)) \to \Lambda.$$

It is semi-linear over $\Lambda$ with respect to the involution on $\Lambda$ (which we denote by $\tilde{\iota}$) in the following sense:

$$\langle \sigma x, y \rangle = \sigma \langle x, y \rangle, \quad \langle x, \sigma y \rangle = \sigma^{\tilde{\iota}} \langle x, y \rangle$$

We may extend the pairing $\langle \sim, \sim \rangle$ by semi-linearity to

$$\left( \mathcal{H} \otimes_{\mathcal{O}} H^1_{\mathrm{Iw}}(\mathbb{Q}_p(\mu_{p^\infty}), \mathcal{T}) \right) \times \left( \mathcal{H} \otimes_{\mathcal{O}} H^1_{\mathrm{Iw}}(\mathbb{Q}_p(\mu_{p^\infty}), \mathcal{T}^*(1)) \right) \to \mathcal{H},$$

which is again denoted by $\langle \sim, \sim \rangle$ by an abuse of notation.

Recall that in [PR94], Perrin-Riou defined the big exponential map

$$\Omega_{\mathcal{T}^*(1),1} : (\mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p})^{\psi=0} \otimes \mathbb{D}_{\mathrm{cris}}(\mathcal{T}^*(1)) \to \mathcal{H} \otimes H^1_{\mathrm{Iw}}(\mathbb{Q}_p(\mu_{p^\infty}), \mathcal{T}^*(1)).$$

By [LLZ11, proof of Proposition 4.8], for all $z \in H^1_{\mathrm{Iw}}(\mathbb{Q}_p(\mu_{p^\infty}), \mathcal{T})$,

$$(\mathfrak{M}^{-1} \otimes 1)(1 - \varphi)z = \sum_{i=1}^{2} \langle z, \Omega_{\mathcal{T}^*(1),1}((1 + \pi) \otimes \mathfrak{v}'_i) \rangle \mathfrak{v}_i$$

where $\mathfrak{v}'_1, \mathfrak{v}'_2$ is the dual basis of $\mathbb{D}_{\mathrm{cris}}(\mathcal{T}^*(1))$ to $\mathfrak{v}_1, \mathfrak{v}_2$ with respect to the natural pairing

$$[\sim, \sim] : \mathbb{D}_{\mathrm{cris}}(\mathcal{T}) \times \mathbb{D}_{\mathrm{cris}}(\mathcal{T}^*(1)) \to \mathcal{O}.$$

Therefore,

$$\underline{\mathrm{Col}}_n(z) = \sum_{i=1}^{2} \langle z, \Omega_{\mathcal{T}^*(1),1}((1 + \pi) \otimes \mathfrak{v}'_i) \rangle \varphi^{-n}(\mathfrak{v}_i) \mod \tilde{\omega}_{n-1,k-2}$$

$$= \sum_{i=1}^{2} \langle z, \Omega_{\mathcal{T}^*(1),1}((1 + \pi) \otimes (p\varphi)^n(\mathfrak{v}'_i)) \rangle \mathfrak{v}_i \mod \tilde{\omega}_{n-1,k-2}$$

as the dual of $\varphi^{-1}$ with respect to $[\sim, \sim]$ is $p\varphi$. This description allows us to make the following choice of $u$ to describe the kernel of $\underline{\mathrm{Col}}_{T,n,u}$.

**Proposition 3.11.** *There exists $u \in \mathbb{Z}_p^\times$ such that $\ker(\underline{\mathrm{Col}}_{T,n,u}) = H^1_f(\mathbb{Q}_p(\mu_{p^n}), T)$.*

*Proof.* Write $v' = (\mathfrak{v}'_1 + u\mathfrak{v}'_2) \cdot t^{-k+j+1} e_{k-j-1} \in \mathbb{D}_{\mathrm{cris}}(\mathcal{T}^*(1))$ and let $z \in H^1(\mathbb{Q}_p(\mu_{p^n}), T)$. If $\theta$ is a Dirichlet character of conductor $p^m > 1$, we have the interpolation formula of Perrin-Riou [PR94, §3.2.3] (see also [Lei11, §3.2])

$$(3.7) \qquad \frac{\theta\left(\underline{\mathrm{Col}}_{T,n,u}(z)\right)}{(-1)^{k-j-1}(k-j-1)!} = \sum_{\sigma \in \Gamma/\Gamma_m} \frac{\theta^{-1}(\sigma)}{\tau(\theta)} [\exp^*_{T,m}(z^\sigma), p^n \varphi^{n-m}(v')],$$

where $\exp^*_{T,m} : H^1(\mathbb{Q}_p(\mu_{p^m}), T) \to \mathbb{Q}_p(\mu_{p^m}) \otimes \mathrm{Fil}^0 \mathbb{D}_{\mathrm{cris}}(T)$ is the Bloch–Kato dual exponential map and $\tau(\theta)$ is the Gauss sum of $\theta$. There is a similar formula when $\theta$ is the trivial character on replacing $\varphi^{-m}$ by $\left(1 - \frac{\varphi^{-1}}{p}\right)(1 - \varphi)^{-1}$. We note that here $\exp^*_{T,m}(z)$ is the shorthand for $\exp^*_{T,m} \circ \mathrm{cor}_{n/m}(z)$, where $\mathrm{cor}_{n/m}$ denotes the the corestriction map $H^1(\mathbb{Q}_p(\mu_{p^n}), T) \to H^1(\mathbb{Q}(\mu_{p^m}), T)$. Recall that $\exp^*_{T,n}$ factors through $H^1_f(\mathbb{Q}_p(\mu_{p^n}), T)$. Therefore, we see that $H^1_f(\mathbb{Q}_p(\mu_{p^n}), T)$ is contained in $\ker(\underline{\mathrm{Col}}_{T,n,u})$.

We choose $u$ so that $\varphi^{n-m}(v')$, $1 \le m \le n$ and $\varphi^n \left(1 - \frac{\varphi^{-1}}{p}\right)(1 - \varphi)^{-1}(v')$ are not contained inside $\mathrm{Fil}^0 \mathbb{D}_{\mathrm{cris}}(V)$. We note that such $u$ exists since all maps are

surjective on $\mathbb{D}_{\mathrm{cris}}(V)$ and $\mathrm{Fil}^0 \mathbb{D}_{\mathrm{cris}}(V)$ is of dimension one. Let $v''$ be any $\mathcal{O}$-basis of $\mathbb{D}_{\mathrm{cris}}(T)/\mathrm{Fil}^0 \mathbb{D}_{\mathrm{cris}}(T)$. In particular, for each $m \geq 1$, there exists a non-zero constant $c_m \in \mathcal{O}$ such that $\varphi^{n-m}(v') \equiv c_m v''$ and $\varphi^n \left(1 - \frac{\varphi^{-1}}{p}\right)(1-\varphi)^{-1}(v') \equiv c_0 v''$ modulo $\mathrm{Fil}^0 \mathbb{D}_{\mathrm{cris}}(T)$.

Suppose that $\underline{\mathrm{Col}}_{T,n,u}(z) = 0$. From (3.7), we deduce that

$$\sum_{\sigma \in \Gamma/\Gamma_n} \theta^{-1}(\sigma)[\exp^*_{T,n}(z^\sigma), v''] = 0$$

for all characters $\theta$ on $\Gamma/\Gamma_n$. By the independence of the characters, this implies that $[\exp^*_{T,n}(z^\sigma), v''] = 0$ for all $\sigma$. In particular, $z$ is contained in the kernel of $\exp^*_{T,n}$, which is $H^1_f(\mathbb{Q}_p(\mu_{p^n}), T)$. $\qquad\square$

**Corollary 3.12.** *For any $u \in \mathbb{Z}_p^\times$ that satisfies the condtion of Proposition 3.11, $\underline{\mathrm{Col}}_{T,n,u}$ induces an injection of $\Lambda_n$-modules*

$$H^1_{/f}(\mathbb{Q}_p(\mu_{p^n}), T) \hookrightarrow \Lambda_n,$$

*whose cokernel is finite.*

*Proof.* The injectivity is given by Proposition 3.11. By [BK90, Theorem 4.1], $H^1_f(\mathbb{Q}_p(\mu_{p^n}), V)$ is isomorphic to $\mathbb{D}_{\mathrm{cris}}(V)/\mathrm{Fil}^0 \mathbb{D}_{\mathrm{cris}}(V) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p(\mu_{p^n})$. Hence, by duality $H^1_{/f}(\mathbb{Q}_p(\mu_{p^n}), V)$ is isomorphic to $\mathrm{Fil}^0 \mathbb{D}_{\mathrm{cris}}(V) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p(\mu_{p^n})$. Therefore, the finiteness of the cokernel follows from the fact that the two sides have the same $\mathbb{Z}_p$-rank. $\qquad\square$

We remark that our map $\underline{\mathrm{Col}}_{T,n,u}$ does depend on the choice of $u$. But it does not affect our calculations later, see the proof of Proposition 4.11 below.

## 4. Results on $p$-adic valuations

4.1. **Review of Kobayashi rank.** Given an $\mathcal{O}$-module $N$, we shall write $\mathrm{len}(N)$ for the $\mathcal{O}$-length of $N$. We fix a family of primitive $p^n$-th root of unity $\zeta_{p^n}$ and write $\epsilon_n = \zeta_{p^n} - 1$.

**Definition 4.1.** *Let $N = (N_n)$ be an inverse system of finitely generated $\mathcal{O}$-modules with transition maps $\pi_n : N_n \to N_{n-1}$. If $\pi_n$ has finite kernel and cokernel, the Kobayashi rank $\nabla N_n$ is defined as*

$$\nabla N_n := \mathrm{len}(\ker \pi_n) - \mathrm{len}(\mathrm{coker}\, \pi_n) + \mathrm{rank}_{\mathcal{O}} N_{n-1}.$$

*If $L$ is an $\mathcal{O}[\![X]\!]$-module, we define $\nabla_n L$ to be $\nabla(L/\omega_n(X)L)$, with the connecting map given by the natural projection $L/\omega_n(X)L \to L/\omega_{n-1}(X)L$, if its kernel and cokernel are finite.*

**Lemma 4.2.** *Let $F \in \mathcal{O}[\![X]\!]$ be a non-zero element. Let $N$ be the inverse limit defined by $N_n = \mathcal{O}[\![X]\!]/(F, \omega_n)$, where the the connecting maps are the natural projections.*

  (a) *Suppose that $F(\epsilon_n) \neq 0$, then $\nabla N_n$ is defined and is equal to $\mathrm{ord}_{\epsilon_n} F(\epsilon_n)$.*
  (b) *When $n$ is sufficiently large, then $\nabla N_n$ is defined. Furthermore,*

$$\nabla N_n = e \times \mathrm{ord}_{\epsilon_n} F(\epsilon_n) = e\lambda(F) + (p^n - p^{n-1})\mu(F),$$

  *where $e$ is the ramification index of $E/\mathbb{Q}_p$ and $\lambda(F), \mu(F)$ are the Iwasawa invariants as defined in §2.3 above.*

(c) *If $L$ is a finitely generated torsion $\mathcal{O}[\![X]\!]$-module, then $\nabla_n L$ is defined for $n \gg 0$ and its value is given by*

$$\lambda(L) + (p^n - p^{n-1})\mu(L),$$

*where $\lambda(L)$ and $\mu(L)$ are the $\lambda$- and $\mu$-invariants of a generator of the characteristic ideal of $L$.*

*Proof.* This follows from the same proof as [Kob03, Lemma 10.5]. $\qquad\square$

We write $p^r$ for the size of the residue field of $E$. The following lemma allows us to relate the growth in the size of a tower of finite $\mathcal{O}$-modules and Kobayashi ranks.

**Lemma 4.3.** *Suppose that $N = (N_n)$ is an inverse limit of finite $\mathcal{O}$-modules such that $|N_n| = p^{s_n}$ for some integer $s_n \in r\mathbb{Z}$ for all $n \geq 1$. Then, $r\nabla N_n = s_n - s_{n-1}$.*

*Proof.* Since $N_{n-1}$ is finite, we have

$$
\begin{aligned}
\nabla N_n &= \mathrm{len}(\ker \pi_n) - \mathrm{len}(\mathrm{coker}\, \pi_n) \\
&= (\mathrm{len}(N_n) - \mathrm{len}(\mathrm{Im}\, \pi_n)) - (\mathrm{len}(N_{n-1}) - \mathrm{len}(\mathrm{Im}\, \pi_n)) \\
&= \mathrm{len}(N_n) - \mathrm{len}(N_{n-1}).
\end{aligned}
$$

In general, if $L$ is a finite $\mathcal{O}$-module, then $|L| = p^{r\mathrm{len}(L)}$. Hence the result. $\qquad\square$

Finally, we prove a lemma on $p$-adic valuations that will be needed later.

**Lemma 4.4.** *Let $F \in \mathcal{O}[\![X]\!]$ be non-zero. Then for all sufficiently large integers $n$ we have*

$$\mathrm{ord}_p F(\epsilon_n) = \mathrm{ord}_p \mathfrak{M}(F)(\epsilon_{n+1}).$$

*Moreover, for $n \gg 0$ we also have*

$$\mathrm{ord}_p F(\epsilon_n) = \mathrm{ord}_p \mathrm{Tw}(F)(\epsilon_n).$$

*Proof.* We may write $\mathfrak{M}(F) = (1+\pi)\varphi(G)$ for some $G \in \mathbb{A}_{\mathbb{Q}_p}^+$. By Proposition 2.2, $F$ and $G$ have the same Iwasawa invariants, so $\mathrm{ord}_p F(\epsilon_n) = \mathrm{ord}_p G(\epsilon_n)$ for $n \gg 0$. This implies the first part of the lemma since $(1+\pi)\varphi(G)(\epsilon_{n+1}) = \zeta_{p^{n+1}} G(\epsilon_n)$. The second part of the lemma follows from the fact that $\mathrm{Tw}$ preserves $\mu$- and $\lambda$-invariants. $\qquad\square$

4.2. **Calculations on evaluation matrices.** From now on, we shall write $v = \mathrm{ord}_p(a_p)$, where $a_p$ is the $p$-th Fourier coefficient of $f$. Following [Spr13, §4.1], given any 2×2 matrix $\phi = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ defined over $\overline{\mathbb{Q}_p}$, we write $\mathrm{ord}_p(\phi) = \begin{pmatrix} \mathrm{ord}_p(a) & \mathrm{ord}_p(b) \\ \mathrm{ord}_p(c) & \mathrm{ord}_p(d) \end{pmatrix}$.

**Lemma 4.5.** *Let $1 \leq i \leq n-2$, then*

$$\mathrm{ord}_p\left(\varphi^i(P^{-1})(\epsilon_n)\right) = \begin{pmatrix} v & 0 \\ \frac{k-1}{p^{n-i-1}} & \infty \end{pmatrix}.$$

*Proof.* Recall that

$$P = \begin{pmatrix} 0 & -\frac{\varepsilon(p)}{q^{k-1}} \\ \delta^{k-1} & \frac{a_p}{q^{k-1}} \end{pmatrix},$$

so its inverse is given by

$$P^{-1} = \begin{pmatrix} \frac{a_p}{\delta^{k-1}\varepsilon(p)} & \frac{1}{\delta^{k-1}} \\ -\frac{q^{k-1}}{\varepsilon(p)} & 0 \end{pmatrix}.$$

Therefore, our result follows from the fact that $\delta \in \mathbb{Z}_p^\times$, $\varepsilon(p) \in \mathcal{O}^\times$ and $\varphi^i(q)$ is equal to the $p^{i+1}$-cyclotomic polynomial, so $\varphi^i(q)(\epsilon_n) = \frac{\zeta_{p^{n-i-1}}-1}{\zeta_{p^{n-i}}-1}$ whose $p$-adic valuation is $1/p^{n-i-1}$. $\qquad\square$

**Proposition 4.6.** *Assume that* $2v > \frac{k-1}{p}$. *For all* $n \geq 1$,

$$\operatorname{ord}_p(H_n(\epsilon_n)) = \begin{cases} \begin{pmatrix} v + \sum_{i=1}^{\frac{n-1}{2}} \frac{k-1}{p^{2i-1}} & \sum_{i=1}^{\frac{n-1}{2}} \frac{k-1}{p^{2i}} \\ \infty & \infty \end{pmatrix} & \text{if } n \text{ is odd.} \\[2em] \begin{pmatrix} \sum_{i=1}^{\frac{n}{2}} \frac{k-1}{p^{2i-1}} & v + \sum_{i=1}^{\frac{n}{2}-1} \frac{k-1}{p^{2i}} \\ \infty & \infty \end{pmatrix} & \text{if } n \text{ is even.} \end{cases}$$

*Proof.* By Lemma 4.5, we have

$$\operatorname{ord}_p(H_n(\epsilon_n)) = \begin{pmatrix} v & 0 \\ \infty & \infty \end{pmatrix} \begin{pmatrix} v & 0 \\ \frac{k-1}{p} & \infty \end{pmatrix} \cdots \begin{pmatrix} v & 0 \\ \frac{k-1}{p^{n-1}} & \infty \end{pmatrix}.$$

In particular,

$$(4.1) \qquad \operatorname{ord}_p(H_{n+1}(\epsilon_{n+1})) = \operatorname{ord}_p(H_n(\epsilon_n)) \begin{pmatrix} v & 0 \\ \frac{k-1}{p^n} & \infty \end{pmatrix}.$$

Therefore,

$$\operatorname{ord}_p(H_1(\epsilon_1)) = \begin{pmatrix} v & 0 \\ \infty & \infty \end{pmatrix} \quad \text{and} \quad \operatorname{ord}_p(H_2(\epsilon_2)) = \begin{pmatrix} \frac{k-1}{p} & v \\ \infty & \infty \end{pmatrix}$$

since $2v > \frac{k-1}{p}$ by our assumption.

Suppose that

$$\operatorname{ord}_p(H_{2\ell-1}(\epsilon_{2\ell-1})) = \begin{pmatrix} v + \sum_{i=1}^{\ell-1} \frac{k-1}{p^{2i}} & \sum_{i=1}^{\ell-1} \frac{k-1}{p^{2i-1}} \\ \infty & \infty \end{pmatrix},$$

$$\operatorname{ord}_p(H_{2\ell}(\epsilon_{2\ell})) = \begin{pmatrix} \sum_{i=1}^{\ell} \frac{k-1}{p^{2i-1}} & v + \sum_{i=1}^{\ell-1} \frac{k-1}{p^{2i}} \\ \infty & \infty \end{pmatrix}$$

for some integer $\ell \geq 1$. By (4.1), we have first of all

$$\operatorname{ord}_p(H_{2\ell+1}(\epsilon_{2\ell+1})) = \begin{pmatrix} v + \sum_{i=1}^{\ell} \frac{k-1}{p^{2i}} & \sum_{i=1}^{\ell} \frac{k-1}{p^{2i-1}} \\ \infty & \infty \end{pmatrix}$$

because $\sum_{i=1}^{\ell} \frac{k-1}{p^{2i}} < \sum_{i=1}^{\ell} \frac{k-1}{p^{2i-1}}$. On applying (4.1) again, we have

$$\operatorname{ord}_p(H_{2\ell+2}(\epsilon_{2\ell+2})) = \begin{pmatrix} \sum_{i=1}^{\ell+1} \frac{k-1}{p^{2i-1}} & v + \sum_{i=1}^{\ell} \frac{k-1}{p^{2i}} \\ \infty & \infty \end{pmatrix}$$

thanks to our assumption that $2v > \frac{k-1}{p}$, which implies that

$$2v + \sum_{i=1}^{\ell} \frac{k-1}{p^{2i}} > \sum_{i=1}^{\ell+1} \frac{k-1}{p^{2i-1}}.$$

Therefore, our result follows from induction. $\qquad\square$

For $i = 1, 2$, we fix two elements $F_1, F_2 \in \mathcal{O}[\![X]\!]$ with $\mu_i$ and $\lambda_i$ being its $\mu$- and $\lambda$-invariants.

**Corollary 4.7.** *Under the condition that* $2v > \frac{k-1}{p}$, *for* $n \gg 0$ *we have the formulae*

$$
\mathrm{ord}_{\epsilon_n}\left((\mathscr{H}_{n+1})_{1,1} \cdot F_1(\epsilon_n)\right) = 
\begin{cases}
\lambda_1 + (p^n - p^{n-1})\left(\frac{\mu_1}{e} + v + \sum_{i=1}^{\frac{n-1}{2}} \frac{k-1}{p^{2i-1}}\right) & n \ \text{odd}, \\
\lambda_1 + (p^n - p^{n-1})\left(\frac{\mu_1}{e} + \sum_{i=1}^{\frac{n}{2}} \frac{k-1}{p^{2i-1}}\right) & n \ \text{even},
\end{cases}
$$

$$
\mathrm{ord}_{\epsilon_n}\left((\mathscr{H}_{n+1})_{1,2} \cdot F_2(\epsilon_n)\right) = 
\begin{cases}
\lambda_2 + (p^n - p^{n-1})\left(\frac{\mu_2}{e} + \sum_{i=1}^{\frac{n-1}{2}} \frac{k-1}{p^{2i}}\right) & n \ \text{odd}, \\
\lambda_2 + (p^n - p^{n-1})\left(\frac{\mu_2}{e} + v + \sum_{i=1}^{\frac{n}{2}-1} \frac{k-1}{p^{2i}}\right) & n \ \text{even}.
\end{cases}
$$

*Proof.* By Lemma 4.4, $\mathrm{ord}_p \mathscr{H}_{n+1}(\epsilon_n) = \mathrm{ord}_p H_n(\epsilon_n)$. Hence, our result follows from combining Proposition 4.6 with Lemma 4.2(b). $\square$

**Corollary 4.8.** *Suppose that* $2v > \frac{k-1}{p}$. *For* $n \gg 0$ *and* $n$ *odd, we have*

$$
\mathrm{ord}_{\epsilon_n}\left((\mathscr{H}_{n+1})_{1,1} \cdot F_1(\epsilon_n)\right) < \mathrm{ord}_{\epsilon_n}\left((\mathscr{H}_{n+1})_{1,2} \cdot F_2(\epsilon_n)\right) \quad \text{if } \frac{\mu_1}{e} + v + \frac{k-1}{p+1} \leq \frac{\mu_2}{e}
$$

$$
\mathrm{ord}_{\epsilon_n}\left((\mathscr{H}_{n+1})_{1,1} \cdot F_1(\epsilon_n)\right) > \mathrm{ord}_{\epsilon_n}\left((\mathscr{H}_{n+1})_{1,2} \cdot F_2)(\epsilon_n)\right) \quad \text{if } \frac{\mu_1}{e} + v + \frac{k-1}{p+1} > \frac{\mu_2}{e}.
$$

*For* $n \gg 0$ *and* $n$ *even, we have*

$$
\mathrm{ord}_{\epsilon_n}\left((\mathscr{H}_{n+1})_{1,1} \cdot F_1(\epsilon_n)\right) < \mathrm{ord}_{\epsilon_n}\left((\mathscr{H}_{n+1})_{1,2} \cdot F_2(\epsilon_n)\right) \quad \text{if } \frac{\mu_1}{e} < \frac{\mu_2}{e} + v + \frac{k-1}{p+1}
$$

$$
\mathrm{ord}_{\epsilon_n}\left((\mathscr{H}_{n+1})_{1,1} \cdot F_1(\epsilon_n)\right) > \mathrm{ord}_{\epsilon_n}\left((\mathscr{H}_{n+1})_{1,2} \cdot F_2(\epsilon_n)\right) \quad \text{if } \frac{\mu_1}{e} \geq \frac{\mu_2}{e} + v + \frac{k-1}{p+1}.
$$

*Proof.* Note that

$$
\sum_{i=1}^{\frac{n-1}{2}} \frac{k-1}{p^{2i-1}} - \sum_{i=1}^{\frac{n-1}{2}} \frac{k-1}{p^{2i}} > 0 \quad \text{and} \quad \sum_{i=1}^{\frac{n}{2}} \frac{k-1}{p^{2i-1}} - \sum_{i=1}^{\frac{n}{2}-1} \frac{k-1}{p^{2i}} > 0
$$

and that both sequences are strictly increasing and tend to $\frac{k-1}{p+1}$ as $n \to \infty$. Hence the result. $\square$

4.3. **Some global Iwasawa modules.** For $n \geq 0$ let us write $K_n = \mathbb{Q}(\mu_{p^n})$.

**Definition 4.9** (cf. [Kat04, §12.2]). *For* $m \geq 0$, *we define*

$$
\mathbb{H}^m(T) := \varprojlim_n H^m_{\text{ét}}\left(\mathrm{Spec}\ \mathcal{O}_{K_n}[1/p], j_* T\right),
$$

*where the inverse limit is respect to the corestriction maps, and* $j$ *is the inclusion map* $\mathrm{Spec}\ K_n \hookrightarrow \mathrm{Spec}\ \mathcal{O}_{K_n}[1/p]$.

By [Kat04, 12.4(3)], the modules $\mathbb{H}^m(T)$ are finitely-generated over $\Lambda$, and are zero unless $m \in \{1, 2\}$; and $\mathbb{H}^1(T)$ is free of rank 1 over $\Lambda$. We fix an element $\mathbf{z} \in \mathbb{H}^1(T)$ so that $\mathbb{H}^1(T) = \Lambda \cdot \mathbf{z}$. Tensoring with the basis vector $e_{k-1-j}$ of $\mathcal{O}(k-1-j)$ gives a bijection

$$
\mathbb{H}^1(T) \cong \mathbb{H}^1(\mathcal{T}),
$$

and (in a slight abuse of notation) we shall write $\mathrm{Col}_i(\mathbf{z})$ for the image of $\mathbf{z} \cdot e_{k-1-j}$ under $\mathrm{Col}_i$ composed with the localization map $\mathbb{H}^1(\mathcal{T}) \to H^1_{\mathrm{Iw}}(\mathbb{Q}_p(\mu_{p^\infty}), \mathcal{T})$.

**Definition 4.10.** *For $i = 1, 2$ and $\eta$ a Dirichlet character modulo $p$. Let $\mu_i^\eta$ be the $\mu$-invariant of $\mathrm{Col}_i(\mathbf{z})^\eta$. For each $n \geq 1$, we define an integer $\tau(n, \eta) \in \{1, 2\}$ by*

$$\begin{cases} 1 & \text{if } \frac{\mu_1^\eta}{e} + v + \frac{k-1}{p+1} \leq \frac{\mu_2^\eta}{e} \text{ and } n \text{ odd or } \frac{\mu_1^\eta}{e} < \frac{\mu_2^\eta}{e} + v + \frac{k-1}{p+1} \text{ and } n \text{ even,} \\ 2 & \text{otherwise.} \end{cases}$$

*Furthermore, we write $q_n^* = \mathrm{ord}_{\epsilon_n}\left( (\mathscr{H}_{n+1})_{1,\tau(n,\eta)}(\epsilon_n) \right)$.*

Note in particular that $q_n^*$ is a sum of some powers of $p$, together with possibly $v$, as given by Proposition 4.6. Furthermore, Corollary 4.8 tells us that

$$(4.2) \qquad \mathrm{ord}_{\epsilon_n}\left( \sum_{i=1}^{2} (\mathscr{H}_{n+1})_{1,i} \cdot \mathrm{Col}_i(\mathbf{z})^\eta(\epsilon_n) \right) = q_n^* + \mathrm{ord}_{\epsilon_n} \mathrm{Col}_{\tau(n,\eta)}(\mathbf{z})^\eta(\epsilon_n).$$

### 4.4. Analysis of some local Iwasawa modules.

For $n \geq 1$, we define

$$\mathcal{X}_{\mathrm{loc}}(\mathbb{Q}(\mu_{p^n})) = \mathrm{coker}\left( \mathbb{H}^1(T)_{\Gamma_n} \to H^1_{/f}(\mathbb{Q}_p(\mu_{p^n}), T) \right),$$

which gives an inverse limit with the connecting maps given by the corestriction maps. We would like to study $\nabla \mathcal{X}_{\mathrm{loc}}(\mathbb{Q}(\mu_{p^{n+1}}))^\eta$ for a fixed Dirichlet character $\eta$ modulo $p$.

**Proposition 4.11.** *Suppose that $\underline{\mathrm{Col}}_1(\mathbf{z})^\eta$ and $\underline{\mathrm{Col}}_2(\mathbf{z})^\eta$ are non-zero. For $n \gg 0$, $\nabla \mathcal{X}_{\mathrm{loc}}(\mathbb{Q}(\mu_{p^{n+1}}))^\eta$ is defined, and its value is bounded above by*

$$\nabla_n \mathcal{X}_{\mathrm{loc}}^\eta \leq e q_n^* + \nabla_n(\mathcal{O}[\![X]\!]/\mathrm{Col}_{\tau(n,\eta)}(\mathbf{z})^\eta).$$

*Proof.* Recall from Corollary 3.12, we have the injection

$$\underline{\mathrm{Col}}_{T,n+1,u} : H^1_{/f}(\mathbb{Q}_p(\mu_{p^{n+1}}), T) \hookrightarrow \Lambda_{n+1}.$$

On taking $\Gamma_n$-coinvariants, the *same* map (*not* $\underline{\mathrm{Col}}_{T,n,u}$) induces an injection

$$\underline{\mathrm{Col}}_{T,n+1,u} : H^1_{/f}(\mathbb{Q}_p(\mu_{p^n}), T) \hookrightarrow \Lambda_n,$$

which admits the same description as (3.6). We write $\mathrm{coker}_{n+1}$ and $\mathrm{coker}_n$ for the cokernels of these two maps respectively. Then, we have the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^1_{/f}(\mathbb{Q}_p(\mu_{p^{n+1}}), T) & \xrightarrow{\underline{\mathrm{Col}}_{T,n+1,u}} & \Lambda_{n+1} & \longrightarrow & \mathrm{coker}_{n+1} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \pi \downarrow & & \\ 0 & \longrightarrow & H^1_{/f}(\mathbb{Q}_p(\mu_{p^n}), T) & \xrightarrow{\underline{\mathrm{Col}}_{T,n+1,u}} & \Lambda_n & \longrightarrow & \mathrm{coker}_n & \longrightarrow & 0, \end{array}$$

where the vertical maps are all natural projections. This gives

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{X}_{\mathrm{loc}}(\mathbb{Q}_p(\mu_{p^{n+1}})) & \longrightarrow & \Lambda_{n+1}/(\underline{\mathrm{Col}}_{T,n+1,u}(\mathbf{z})) & \longrightarrow & \mathrm{coker}_{n+1} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow \pi & & \\ 0 & \longrightarrow & \mathcal{X}_{\mathrm{loc}}(\mathbb{Q}_p(\mu_{p^n})) & \longrightarrow & \Lambda_n/(\underline{\mathrm{Col}}_{T,n+1,u}(\mathbf{z})) & \longrightarrow & \mathrm{coker}_n & \longrightarrow & 0. \end{array}$$

Recall from Corollary 3.12 that $\mathrm{coker}_{n+1}$ is finite (in particular, $\mathrm{coker}_n$ too). Hence, on taking $\eta$-isotypic components, $\nabla \mathrm{coker}_{n+1}^\eta$ (with respect to $\pi$) is defined. In fact, it is given by $\mathrm{len}(\ker \pi^\eta)$, which is $\geq 0$.

Furthermore, recall that we assume $\mathrm{Col}_i(\mathbf{z})^\eta \neq 0$ for $i = 1, 2$. Proposition 4.6 tells us that the second row of $\mathscr{H}_{n+1}(\epsilon_n)$ is 0. So, the formulae (3.6) and (4.2) imply

that $\underline{\mathrm{Col}}_{T,n+1,u}(\mathbf{z})(\epsilon_n) \neq 0$ when $n \gg 0$. Hence, $\nabla\left(\Lambda_{n+1}/(\underline{\mathrm{Col}}_{T,n+1,u}(\mathbf{z}))\right)^\eta = \nabla_n\left(\mathcal{O}[\![X]\!]/\underline{\mathrm{Col}}_{T,n+1,u}(\mathbf{z})^\eta\right)$ is defined. Its value is given by

$$eq_n^* + \nabla_n(\mathcal{O}[\![X]\!]/\mathrm{Col}_{\tau(n,\eta)}(\mathbf{z})^\eta),$$

thanks to Lemma 4.2.

Therefore, the fact that the Kobayashi rank $\nabla$ respects short exact sequences ([Kob03, Lemma 10.4]) tells us that $\nabla\mathcal{X}_{\mathrm{loc}}(\mathbb{Q}_p(\mu_{p^{n+1}}))^\eta$ is defined and its value is equal to

$$\nabla_n\left(\mathcal{O}[\![X]\!]/\underline{\mathrm{Col}}_{T,n+1,u}(\mathbf{z})^\eta\right) - \mathrm{len}(\ker \pi^\eta).$$

Hence the result. $\qquad\square$

This can be considered as a weakened version of the modesty proposition [Spr13, Proposition 3.10]. In the $k = 2$ case, equality holds because the projection $\pi$ turns out to be an injection (see [Kob03, Lemma 10.7] and [Spr13, Lemma 4.12]).

## 5. Selmer groups and Shafarevich–Tate groups

5.1. **Signed Selmer groups.** Let $T^\vee$ be the Pontryagin dual of $T$. As in [LLZ10], the Coleman maps allow us to define the Selmer groups

$$\mathrm{Sel}_i(T^\vee/\mathbb{Q}(\mu_{p^\infty})) = \ker\left(\mathrm{Sel}(T^\vee/\mathbb{Q}(\mu_{p^\infty})) \to \frac{H^1(\mathbb{Q}_p(\mu_{p^\infty}), T^\vee)}{\ker(\mathrm{Col}_i)^\perp}\right)$$

for $i = 1, 2$. Here $\mathrm{Sel}(T^\vee/\mathbb{Q}(\mu_{p^\infty}))$ is the Bloch–Kato Selmer group from [BK90]. We shall write $\mathcal{X}(\mathbb{Q}(\mu_{p^n})) = \mathrm{Sel}(T^\vee/\mathbb{Q}(\mu_{p^n}))^\vee$ for $n \geq 1$.

Let $\mathcal{X}_i$ be the Pontryagin dual of $\mathrm{Sel}_i(T^\vee/\mathbb{Q}(\mu_{p^\infty}))$. We subsequently assume that for any Dirichlet character $\eta$ that factors through $\mathrm{Gal}(\mathbb{Q}(\mu_p)/\mathbb{Q})$, both $\mathcal{X}_1^\eta$ and $\mathcal{X}_2^\eta$ are $\mathcal{O}[\![X]\!]$-torsion. Note that this is the case if either $k \geq 3$ or $a_p = 0$ by [LLZ10, Theorem 6.5]. In particular, $\nabla_n\mathcal{X}_i^\eta$ are defined for $n \gg 0$ by Lemma 4.2(c).

We have the Poitou-Tate exact sequence (see for example [LLZ10, (61)])

$$(5.1) \qquad \mathbb{H}^1(T) \to \mathrm{Im}\,\mathrm{Col}_i \to \mathcal{X}_i \to \mathcal{X}_0 \to 0,$$

where $\mathcal{X}_0$ is $\mathbb{H}^2(T)$ and can be realized as the Pontryagin dual of the zero Selmer group $\mathrm{Sel}_0(T^\vee/\mathbb{Q}(\mu_{p^\infty}))$, which is defined to be

$$\ker\left(H^1(\mathbb{Q}(\mu_{p^\infty}), T^\vee) \to \prod_v H^1(\mathbb{Q}(\mu_{p^\infty})_v, T^\vee)\right),$$

where $v$ runs through all places of $\mathbb{Q}(\mu_{p^\infty})$. Note that $\mathcal{X}_0$ is a torsion $\Lambda$-module by [Kat04, Theorem 12.4] and hence $\nabla_n\mathcal{X}_0^\eta$ is defined for $n \gg 0$ by Lemma 4.2(c). Note that (5.1) gives the short exact sequence

$$0 \to \frac{\mathrm{Im}\,\mathrm{Col}_i}{(\mathrm{Col}_i(\mathbf{z}))} \to \mathcal{X}_i \to \mathcal{X}_0 \to 0.$$

Hence, our assumption that $\mathcal{X}_i^\eta$ be torsion implies that $\mathrm{Col}_i(\mathbf{z})^\eta \neq 0$. In particular, Proposition 4.11 applies.

Recall from [LLZ11, §5] that $\mathrm{Im}\,\mathrm{Col}_i^\eta$ is pseudo-isomorphic to $\prod_m(X - \chi(\gamma)^m + 1)\mathcal{O}[\![X]\!]$, where $m$ runs through some subset of $\{0, 1, \ldots, k-2\}$ depending on $i$ and $\eta$. Let us write $\kappa_i(\eta)$ for the cardinality of this subset and write $\kappa(n, \eta) = \kappa_{\tau(n,\eta)}(\eta)$. We have the following generalization of [Spr13, Proposition 3.11].

**Proposition 5.1.** *For $i = 1, 2$, $\eta$ any Dirichlet character modulo $p$ and $n \gg 0$,*

$$\nabla_n \mathcal{X}_i^\eta = \nabla_n(\Lambda/\mathrm{Col}_i(\mathbf{z}))^\eta + \nabla_n \mathcal{X}_0^\eta - e\kappa_i(\eta).$$

*Proof.* The following sequence

$$0 \to \left(\frac{\mathrm{Im}(\mathrm{Col}_i)}{\mathrm{Col}_i(\mathbf{z})}\right)^\eta \to \left(\frac{\Lambda}{\mathrm{Col}_i(\mathbf{z})}\right)^\eta \to \frac{\mathcal{O}[\![X]\!]}{\prod_m (X - \chi(\gamma)^m + 1)\mathcal{O}[\![X]\!]} \to G \to 0$$

is exact, where $G$ is some finite subgroup. In particular, $\nabla_n G = 0$ for $n \gg 0$. We may work out the Kobayashi rank of the second last term using Lemma 4.2(b). Recall from [Kob03, Lemma 10.4] that Kobayashi ranks respect exact sequences, therefore,

$$\nabla_n \left(\frac{\mathrm{Im}(\mathrm{Col}_i)}{\mathrm{Col}_i(\mathbf{z})}\right)^\eta + e\kappa_i(\eta) = \nabla_n \left(\frac{\Lambda}{\mathrm{Col}_i(\mathbf{z})}\right)^\eta.$$

From (5.1), we have furthermore the following exact sequence

$$0 \to \frac{\mathrm{Im}(\mathrm{Col}_i)}{\mathrm{Col}_i(\mathbf{z})} \to \mathcal{X}_i \to \mathcal{X}_0 \to 0,$$

which implies that

$$\nabla_n \left(\frac{\mathrm{Im}(\mathrm{Col}_i)}{\mathrm{Col}_i(\mathbf{z})}\right)^\eta + \nabla_n \mathcal{X}_0^\eta = \nabla_n \mathcal{X}_i^\eta.$$

Combing the two equations gives our result. $\qquad\square$

**Remark 5.2.** Let $\mu_0^\eta$ be the $\mu$-invariant of $\mathcal{X}_0^\eta$. For $i = 1, 2$, let $\tilde{\mu}_i^\eta$ be the $\mu$-invariant of $\mathcal{X}_i^\eta$. Then, Proposition 5.1 implies that $\tilde{\mu}_i^\eta = \mu_i^\eta - \mu_0^\eta$. In particular, $\mu_1^\eta - \mu_2^\eta = \tilde{\mu}_1^\eta - \tilde{\mu}_2^\eta$. Therefore, we may replace $\mu_1^\eta$ and $\mu_2^\eta$ by $\tilde{\mu}_1^\eta$ and $\tilde{\mu}_2^\eta$ respectively in Definition 4.10. In other words, we may define $\tau(n, \eta)$ using the $\mu$-invariants of the dual Selmer groups $\mathcal{X}_i$, instead of $\mathrm{Col}_i(\mathbf{z})$.

**Corollary 5.3.** *For $n \gg 0$, $\nabla \mathcal{X}(\mathbb{Q}(\mu_{p^{n+1}}))^\eta$ is defined. Furthermore, its value is bounded above by*

$$eq_n^* + \nabla_n \mathcal{X}_{\tau(n,\eta)}^\eta + e\kappa(n, \eta).$$

*Proof.* Let $\mathcal{Y}(\mathbb{Q}(\mu_{p^n})) = \mathrm{coker}(H^1(G_{n,S}, T) \to H^1_{/f}(\mathbb{Q}_p(\mu_{p^n}), T))$ and $\mathcal{X}_0(\mathbb{Q}(\mu_{p^n})) = \mathrm{Sel}_0(T^\vee/\mathbb{Q}(\mu_{p^n}))^\vee$. As a consequence of the Poitou–Tate exact sequence, we have the short exact sequence

$$0 \to \mathcal{Y}(\mathbb{Q}(\mu_{p^n})) \to \mathcal{X}(\mathbb{Q}(\mu_{p^n})) \to \mathcal{X}_0(\mathbb{Q}(\mu_{p^n})) \to 0$$

(c.f. [Kob03, (10.35)]). But Proposition 10.6 in *op. cit.* says that

- $\nabla \mathcal{Y}(\mathbb{Q}(\mu_{p^{n+1}}))^\eta$ is defined for $n \gg 0$ and is equal to $\nabla \mathcal{X}_{\mathrm{loc}}(\mathbb{Q}(\mu_{p^{n+1}}))^\eta$;
- $\nabla \mathcal{X}_0(\mathbb{Q}(\mu_{p^{n+1}}))^\eta = \nabla_n \mathcal{X}_0^\eta$.

Therefore,

$$\nabla \mathcal{X}(\mathbb{Q}(\mu_{p^{n+1}}))^\eta = \nabla \mathcal{X}_{\mathrm{loc}}(\mathbb{Q}(\mu_{p^{n+1}}))^\eta + \nabla_n \mathcal{X}_0^\eta$$

and our result follows from Propositions 4.11 and 5.1. $\qquad\square$

5.2. **Bloch–Kato–Shafarevich–Tate groups.** Let $L$ be a number field. We recall that the Bloch–Kato–Shafarevich–Tate group of $T^\vee$ over $L$ is defined to be

$$
(5.2) \qquad\qquad \mathrm{III}(L, T^\vee) = \frac{\mathrm{Sel}(T^\vee/L)}{\mathrm{Sel}(T^\vee/L)_{\mathrm{div}}},
$$

where $(\star)_{\mathrm{div}}$ denotes the maximal divisible subgroup of $\star$. (See e.g. [BK90, Remark 5.15.2]). If $f$ corresponds to an elliptic curve $\mathcal{E}$ and the $p$-primary part of the classical Shafarevich–Tate group $\mathcal{E}$ is finite, then the two definitions of ($p$-primary) Shafarevich–Tate groups agree.

**Proposition 5.4.** *There exists integers $n_0^\eta, r_\infty^\eta \geq 0$ such that*

$$
\mathrm{corank}_{\mathcal{O}}\mathrm{Sel}(T^\vee/\mathbb{Q}(\mu_{p^{n+1}}))^\eta = r_\infty^\eta
$$

*for all $n \geq n_0^\eta$.*

*Proof.* By Corollary 5.3, $\nabla\mathcal{X}(\mathbb{Q}(\mu_{p^{n+1}}))^\eta$ is defined for $n \gg 0$. In particular, the kernel and cokernel of the connecting map

$$
\mathrm{Sel}(T^\vee/\mathbb{Q}(\mu_{p^{n+1}}))^\vee \to \mathrm{Sel}(T^\vee/\mathbb{Q}(\mu_{p^n}))^\vee
$$

are finite for $n \gg 0$. In particular, $\mathrm{Sel}(T^\vee/\mathbb{Q}(\mu_{p^{n+1}}))$ and $\mathrm{Sel}(T^\vee/\mathbb{Q}(\mu_{p^n}))$ must have the same $\mathbb{Z}_p$-corank. $\qquad\square$

This implies that $\mathrm{Sel}(T^\vee/\mathbb{Q}(\mu_{p^{n+1}}))_{\mathrm{div}}^\eta \cong (E/\mathcal{O})^{\oplus r_\infty^\eta}$ (as $\mathbb{Z}_p$-modules) for $n \gg 0$. Combined this with (5.2), we obtain the following short exact sequence of $\mathbb{Z}_p$-modules

$$
0 \to (E/\mathcal{O})^{\oplus r_\infty^\eta} \to \mathrm{Sel}(T^\vee/\mathbb{Q}(\mu_{p^{n+1}}))^\eta \to \mathrm{III}(\mathbb{Q}(\mu_{p^{n+1}}), T^\vee)^\eta \to 0.
$$

Therefore, on taking Pontryagin duals, we deduce that

$$
\nabla\mathcal{X}(\mathbb{Q}(\mu_{p^{n+1}}))^\eta = r_\infty^\eta + \nabla\mathrm{III}(\mathbb{Q}(\mu_{p^{n+1}}), T^\vee)^\eta.
$$

From Corollary 5.3, we deduce that

$$
\nabla\mathrm{III}(\mathbb{Q}(\mu_{p^{n+1}}), T^\vee)^\eta \leq eq_n^* + \nabla_n\mathcal{X}_{\tau(n,\eta)}^\eta + e\kappa(n,\eta) - r_\infty^\eta.
$$

Therefore, we obtain the following theorem on applying Lemma 4.3.

**Theorem 5.5.** *Let $\#\mathrm{III}(\mathbb{Q}(\mu_{p^n}), T^\vee)^\eta = p^{s_n^\eta}$. For $n \gg 0$,*

$$
s_{n+1}^\eta - s_n^\eta \leq r\left(eq_n^* + \nabla_n\mathcal{X}_{\tau(n,\eta)}^\eta + e\kappa(n,\eta) - r_\infty^\eta\right),
$$

*where $r$ is the integer so that the residue field of $E$ has cardinality $p^r$.*

Using Lemma 4.2, we may rewrite this formula as

$$
s_{n+1}^\eta - s_n^\eta \leq d\left(q_n^* + \lambda_{\tau(n,\eta)} + (p^n - p^{n-1})\frac{\mu_{\tau(n,\eta)}}{e} + \kappa(n,\eta) - \frac{r_\infty^\eta}{e}\right),
$$

where $d = [E : \mathbb{Q}_p]$.

## Appendix A. Growth of Tamagawa numbers over cyclotomic extensions

We let $T = T_f(j)$ and $\mathcal{T} = T_f(k-1)$ be the representations studied in the main part of the article. In particular, we assume all the previous hypotheses on $T$ and $\mathcal{T}$ are satisfied throughout. Furthermore, we shall assume that the eigenvalues of $\varphi$ on $\mathbb{D}_{\mathrm{cris}}(\mathcal{T})$ are not integral powers of $p$. For notational simplicity, we shall assume that the coefficient field $E$ is $\mathbb{Q}_p$ throughout.

Recall the Perrin-Riou $p$-adic regulator

$$\mathcal{L}_{\mathcal{T}} : H^1_{\mathrm{Iw}}(\mathbb{Q}_p(\mu_{p^\infty}), \mathcal{T}) \to \mathcal{H} \otimes \mathbb{D}_{\mathrm{cris}}(\mathcal{T})$$

defined by $\mathfrak{M}^{-1} \circ (1 - \varphi) \circ (h^1_{\mathcal{T}})^{-1}$, which is the map used to define the Coleman maps in Definition 3.4. We have the following interpolation formula

**Proposition A.1.** *Let $n \geq 1$. For any $z \in H^1_{\mathrm{Iw}}(\mathbb{Q}_p(\mu_{p^\infty}), \mathcal{T})$, $i \geq 0$ and a Dirichlet character $\delta$ of conductor $p^n$, we have*

$$\mathcal{L}_{\mathcal{T}}(z)(\chi^i \delta) = \begin{cases} i!(1-\varphi)(1-p^{-1}\varphi^{-1})^{-1} \left(\exp^*(z_{0,-i})\right) \cdot t^{-i} e_i & \text{if } n = 0, \\ \frac{i! p^n}{\tau(\delta)} \varphi^n \left(\exp^*(\tilde{e}_\delta \cdot z_{n,-i})\right) \cdot t^{-i} e_i & \text{otherwise,} \end{cases}$$

*where $\tau(\delta)$ is the Gauss sum of $\delta$, $z_{n,-i}$ is the projection of $z$ in $H^1(\mathbb{Q}_p(\mu_{p^n}), \mathcal{T}(-i))$ and $\tilde{e}_\delta$ is the element $\sum_{\sigma \in \mathrm{Gal}(\mathbb{Q}_p(\mu_{p^n})/\mathbb{Q}_p)} \delta^{-1}(\sigma)\sigma$.*

*Proof.* This is a slight reformulation of [LZ14, Theorem B.5] since we have the equation

$$\varphi(t^{-i} e_i) = p^{-i} \cdot t^{-i} e_i.$$

$\square$

**Corollary A.2.** *Let $z \in H^1_{\mathrm{Iw}}(\mathbb{Q}_p, \mathcal{T})$. Then, $\mathcal{L}_{\mathcal{T}}(z)(\chi^i \delta) = 0$ if and only if $\tilde{e}_\delta \cdot z_{n,-i} \in \tilde{e}_\delta \cdot H^1_f(\mathbb{Q}_p(\mu_{p^n}), \mathcal{T}(-i))$.*

*Proof.* This is because our assumption on the eigenvalues of $\varphi$ implies that $(1 - \varphi)(1 - p^{-1}\varphi^{-1})^{-1}$ and $\varphi^n$ are both invertible. $\square$

We write $K = \mathbb{Q}(\mu_{p^n})$ and $\Delta_K = \mathrm{Gal}(K/\mathbb{Q})$. For each character $\delta$ on $\Delta_K$, we write $p^{n_\delta}$ for its conductor. Let $K_p$ be the completion of $K$ at the unique place above $p$ (which may be identified with $\mathbb{Q}_p(\mu_{p^n})$). We fix a basis $v$ for $\mathrm{Fil}^0 \mathbb{D}_{\mathrm{cris}}(T)$ and its dual $v'$ in $\mathbb{D}_{\mathrm{cris}}(T^*(1))/\mathrm{Fil}^0 \mathbb{D}_{\mathrm{cris}}(T^*(1))$. We have the definition of the Tamagawa number as defined by Bloch–Kato [BK90]:

$$\mathrm{Tam}(T/K) = [H^1_f(K_p, T) : \mathcal{O}_{K_p} \cdot v] L_p(T, 1),$$

where $L_p(T, 1)$ is the Euler factor of the complex $L$-function $L_p(T, 1)$ at $p$ and we identify $\mathcal{O}_{K_v} v$ with its image under the Bloch–Kato exponential map. We may decompose the Tamagawa number into isotypic components, namely

$$\mathrm{Tam}(T/K) = \prod_\eta \mathrm{Tam}(T/K)^\eta,$$

where the product runs through all the Dirichlet characters modulo $p$ and $\mathrm{Tam}(T/K)^\eta$ is given by

$$[H^1_f(K_p, T)^\eta : \left(\mathcal{O}_{K_p} \cdot v\right)^\eta] L_p(T(\eta), 1),$$

which we may identify with $\mathrm{Tam}(T(\eta)/K^\Delta)$.

**Lemma A.3.** *Let $d_K$ be the discriminant of $K$. Then, we have the formula*
$$\mathrm{Tam}(T/K) = |d_K|_p^{-1}[\mathcal{O}_{K_p} \cdot v : H^1_{/f}(K_p, T)]L_p(T, 1),$$
*where we identify $H^1_{/f}(K_p, T)$ with its image under the Bloch–Kato dual exponential map.*

*Proof.* This follows from the commutative diagram

$$
\begin{array}{ccc}
\left(K_p \otimes \mathrm{Fil}^0 \mathbb{D}_{\mathrm{cris}}(T)\right) \times \left(K_p \otimes \frac{\mathbb{D}_{\mathrm{cris}}(T^*(1))}{\mathrm{Fil}^0 \mathbb{D}_{\mathrm{cris}}(T^*(1))}\right) & \longrightarrow & K_p \\
\uparrow{\scriptstyle\exp^*} \qquad\qquad\qquad\qquad & \downarrow{\scriptstyle\exp} & \downarrow{\scriptstyle\mathrm{Tr}_{K_p/\mathbb{Q}_p}} \\
\left(\mathbb{Q}_p \otimes H^1_{/f}(K_p, T)\right) \times \left(\mathbb{Q}_p \otimes H^1_f(K_p, T^*(1))\right) & \longrightarrow & \mathbb{Q}_p.
\end{array}
$$
□

Take $\mathbf{z}$ to be a $\Lambda$-generator of $\mathbb{H}^1(T)$ as in the main part of the article. This gives a $\Lambda$-basis $\mathbf{z} \cdot e_{k-j-1}$ of $\mathbb{H}^1(\mathcal{T})$. We shall write $\mathcal{L}_T(\mathbf{z})$ for $\mathrm{Tw}_{-k+j+1} \circ \mathcal{L}_{\mathcal{T}}(\mathbf{z} \cdot e_{k-j-1})$ and
$$\tilde{v}_K = \bigotimes_{\delta \in \hat{\Delta}_K} \left(\varphi^{n_\delta}(1 - \delta(p)\varphi)(1 - p^{-1}\bar{\delta}(p)\varphi^{-1})^{-1}v\right).$$

**Theorem A.4.** *Suppose that $\mathcal{L}_T(\mathbf{z})(\delta) \neq 0$ for all $\delta \in \hat{\Delta}_K$. Then,*
$$\bigotimes_{\delta \in \hat{\Delta}_K} \mathcal{L}_T(\mathbf{z})(\delta) \sim_p \frac{\mathrm{Tam}(T/K)}{L_p(T, 1)} \prod_\delta \left[e_\delta H^1_{/f}(K_p, T) : e_\delta \mathbf{z}_K\right] \tilde{v}_K.$$
*Here, we write $a \sim_p b$ if $a$ and $b$ have the same $p$-adic valuation.*

*Proof.* Let $\mathbf{z}_K$ be the projection of $\mathbf{z}$ in $H^1(K_p, T)$. For each character of $\Delta_K$, we write $e_\delta = \sum_{\sigma \in \Delta_K} \delta^{-1}(\sigma)\sigma$ and let $K_\delta$ for the subfield of $K$ defined by the kernel of $\delta$. Our assumption means that $e_\delta \cdot \mathbf{z}_K \notin e_\delta \cdot H^1_f(\mathbb{Q}_p(\mu_{p^n}), T)$ for all $\delta$ by Corollary A.2. Note that $\sum e_\delta = [K : \mathbb{Q}]$. On applying Proposition A.1, we deduce that

$$\bigotimes_{\delta \in \hat{\Delta}_K} \mathcal{L}_T(\mathbf{z})(\delta) \sim_p \prod_\delta \left[\frac{e_\delta}{[K : \mathbb{Q}]}\mathcal{O}[\Delta_K]v : e_\delta \mathcal{O}[\Delta_K]\frac{p^{n_\delta}}{\tau(\delta)}\exp^*(\mathbf{z}_K)\right] \tilde{v}_K$$

$$\sim_p \prod_\delta p^{n_\delta}\left[e_\delta \mathcal{O}[\Delta_K]\frac{\tau(\delta)}{[K : \mathbb{Q}]} : e_\delta \mathcal{O}[\Delta_K]\right]$$
$$\times [e_\delta \mathcal{O}[\Delta_K]v : e_\delta \mathcal{O}[\Delta_K]\exp^*(\mathbf{z}_K)] \tilde{v}_K.$$

Note that the factor $(k - j - 1)!$ does not appear because of the Fontaine–Laffaille condition. Now, [Gil79, Proposition 1] tells us that
$$\left[e_\delta \mathcal{O}[\Delta_K]\frac{\tau(\delta)}{[K : \mathbb{Q}]} : e_\delta \mathcal{O}[\Delta_K]\right] = [K : K_\delta]\left[e_\delta \mathcal{O}[\Delta_K]\frac{\tau(\delta)}{[K_\delta : \mathbb{Q}]} : e_\delta \mathcal{O}[\Delta_K]\right] = 1.$$
Therefore, we deduce from the conductor-discriminant formula that
$$\bigotimes_{\delta \in \hat{\Delta}_K} \mathcal{L}_T(\mathbf{z})(\delta) \sim_p |d_K|_p^{-1} \prod_\delta [e_\delta \mathcal{O}[\Delta_K]v : e_\delta \mathcal{O}[\Delta_K]\exp^*(\mathbf{z}_K)] \tilde{v}_K.$$
Combining this with Lemma A.3 gives us the result.        □

**Remark A.5.** *There is in fact a similar formula without assuming the non-vanishing of $\mathbb{I}_{\mathrm{arith}}(T)(\delta)$. It would involve Perrin-Riou's $p$-adic height. See* [PR03, p.180].

**Corollary A.6.** *Let $\eta$ be a Dirichlet character modulo $p$. Under the conditions of Theorem A.4, we have*

$$\nabla_n \mathcal{X}_{\mathrm{loc}}^\eta + b_{n+1}^\eta - b_n^\eta = q_n^* + \nabla_n(\mathbb{Z}_p[\![X]\!]/\mathrm{Col}_{\tau(n,\eta)}(\mathbf{z})^\eta) + p^{n-1}(p-1)n(k-j-1)$$

*for $n \gg 0$, where $\tau(n,\eta)$ is as defined in Definition 4.10 and $b_i^\eta$ denotes the $p$-adic valuation of $\mathrm{Tam}(T/\mathbb{Q}(\mu_{p^i})^\eta)$ for $i = n, n+1$.*

*Proof.* Let $\Delta_{n+1}$ be the set of Dirichlet characters of conductor $p^{n+1}$ whose $\Delta$-component is $\eta$. Its cardinality is given by $p^{n-1}(p-1)$. By Theorem A.4, we have

$$\bigotimes_{\delta \in \Delta_{n+1}} \mathcal{L}_T(\mathbf{z})(\delta) \sim_p \frac{\mathrm{Tam}(T/\mathbb{Q}(\mu_{p^{n+1}}))^\eta}{\mathrm{Tam}(T/\mathbb{Q}(\mu_{p^n}))^\eta} \prod_{\delta \in \Delta_{n+1}} \left[ e_\delta H^1_{/f}(K_p, T) : e_\delta \mathbf{z}_K \right] \varphi^{n+1}(v)^{\otimes|\Delta_{n+1}|}.$$

This gives
(A.1)
$$\bigotimes_{\delta \in \Delta_{n+1}} \varphi^{-n-1} \circ \mathcal{L}_T(\mathbf{z})(\delta) \sim_p \frac{\mathrm{Tam}(T/\mathbb{Q}(\mu_{p^{n+1}}))^\eta}{\mathrm{Tam}(T/\mathbb{Q}(\mu_{p^n}))^\eta} \prod_{\delta \in \Delta_{n+1}} \left[ e_\delta H^1_{/f}(K_p, T) : e_\delta \mathbf{z}_K \right] v^{\otimes|\Delta_{n+1}|}.$$

Note that $\varphi^{-n-1} \circ \mathrm{Tw}_{-k+j+1} = p^{(n+1)(k-j-1)} \mathrm{Tw}_{-k+j+1} \circ \varphi^{-n-1}$. The terms appearing on the left-hand side are therefore simply $p^{(n+1)(k-j-1)} \underline{\mathrm{Col}}_{T,n+1}(\mathbf{z})(\delta)$. Therefore, the $p$-adic valuation of the left-hand side of (A.1) is given by

$$p^{n-1}(p-1)(n+1)(k-j-1) + q_n^* + \mathrm{ord}_{\epsilon_n} \mathrm{Col}_{\tau(n,\eta)}(\mathbf{z})^\eta(\epsilon_n)$$

thanks to (4.2). Hence the result.                                           $\square$

The proof of our Proposition 4.11 implies that the defect of our inequality in Theorem 5.5 is in fact given by the length of $\ker \pi^\eta$, where $\pi$ is some projection map. We see here that we may in fact relate this defect to the Tamagawa numbers, namely,

$$\mathrm{len}_{\mathbb{Z}_p} \ker \pi^\eta = b_{n+1}^\eta - b_n^\eta - p^{n-1}(p-1)n(k-j-1).$$

Let $t_n^\eta$ be the integer $s_n^\eta + b_n^\eta$, which is the $p$-adic valuation of $\#\mathrm{III}(\mathbb{Q}(\mu_{p^n}), T^\vee)^\eta \times \mathrm{Tam}(T/\mathbb{Q}(\mu_{p^n}))^\eta$. The Bloch–Kato conjecture predicts that this quantity should be related to the leading coefficient of the complex $L$ function of $T$ at 1. Theorem 5.5 tells us that we have the equality

$$t_{n+1}^\eta - t_n^\eta = q_n^* + \nabla_n \mathcal{X}_{\tau(n,\eta)}^\eta + \kappa(n,\eta) - r_\infty^\eta + p^{n-1}(p-1)n(k-j-1).$$

for $n \gg 0$.

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