Limiting Behaviour of the Teichmüller Harmonic Map Flow

by

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Contents

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Declarations

I declare that to the best of my knowledge, the material contained in this thesis is original and my own work except where otherwise indicated. This thesis has not been submitted for a degree at any other university.

Chapter 3 is taken from the paper [13], which is joint work with Melanie Rupflin and my supervisor, Peter Topping. Chapter 4 contains original research which has been obtained in collaboration with Peter Topping.

Abstract

In this thesis we study two problems related to the Teichmüller harmonic map flow, a flow introduced in [21], which aims to deform maps from closed surfaces into closed Riemannian targets of general dimension into branched minimal immersions. It arises as a gradient flow for the energy functional when one varies both the map and the domain metric.

We first consider weak solutions of the flow that exist for all time, with metric degenerating at infinite time. It was shown in [25] that for such solutions one can extract a so-called sequence of almost-minimal maps, which subconverges to a collection of branched minimal immersions (or constant maps). We further improve this compactness theory, in particular showing that no loss of energy can happen, after accounting for all developing bubbles. We also construct an example of a smooth flow where the image of the limit branched minimal immersions is disconnected. These results were obtained in [13], joint with Melanie Rupfling and Peter Topping.

Secondly, we study limits of the coupling constant η, which controls the relative speed of the metric evolution and the map evolution along the flow. We show that when $\eta \downarrow 0$, corresponding to slowing down the metric evolution, one obtains the classical harmonic map flow as a limit of the Teichmüller harmonic map flow when the target N has nonpositive sectional curvature. Finally, we let $\eta \downarrow 0$ and simultaneously rescale time, 'fixing' the speed at which the metric evolves and accelerating the evolution of the map component. We show that in this setting the Teichmüller harmonic map flow converges to a flow through harmonic maps, if one assumes the target N to have strictly negative sectional curvature everywhere and the initial map to not be homotopic to a constant map or a map to a closed geodesic in the target.

Chapter 1

Introduction

A wealth of results on problems of a geometric nature has been obtained through the study of geometric flows. The general idea of such flows is to start with some initial object, e.g. a Riemannian metric or a map between Riemannian manifolds, and deform it into an object which is easier to analyse or has more desirable properties. We are particularly interested in the *harmonic map heat flow* (which we will usually refer to as just the *harmonic map flow*), which was introduced by Eells and Sampson in [8] to study *harmonic maps*. Given a smooth closed orientable surface M, a metric g on M and a smooth closed Riemannian manifold $N = (N, G)$, the *energy* of a map $u : M \to N$ is defined as

$$E(u, g) = \frac{1}{2} \int_M |du|_g^2 dv_g. \tag{1.1}$$

We call u a harmonic map if it is a critical point of E (viewed as a functional on maps). The harmonic map flow is then given by

$$\frac{\partial u}{\partial t} = \tau_g(u), \tag{1.2}$$

where $\tau_g(u) = \operatorname{tr} \nabla du$ is the tension field of the map u. One can view this flow as a gradient flow (with respect to the map) for the energy functional E. The harmonic map flow has been studied extensively, see e.g. [2, 8, 10, 17, 27]. As expected by its gradient flow nature, it aims to transform an initial map into a harmonic map. In general, this is not possible without singularities in the map forming, as some homotopy classes do not contain *any* harmonic maps ([7]). However, assuming nonpositive sectional curvature on N, Eells and Sampson were able to show long-time existence and uniform derivative bounds for this flow in [6] and as a result could show that any homotopy class of maps contains a harmonic representative.

If one also allows the domain metric to vary, a different gradient flow for the functional E can be found. The energy is invariant under conformal changes of the domain metric in two dimensions, so depending on the genus of M one can restrict to flowing through metrics of Gauss curvature $K \in \{-1, 0, 1\}$, corresponding to hyperbolic surfaces, tori and the sphere respectively. This gradient flow was introduced in [21], and is called the *Teichmüller harmonic map flow*. It is given by

$$\begin{aligned} \frac{\partial}{\partial t} u &= \tau_g(u) \\ \frac{\partial}{\partial t} g &= \frac{\eta^2}{4} Re(P_g(\Phi(u,g))) \end{aligned} \tag{1.3}$$

where τ_g again denotes the tension field of u, η a choice of scaling in defining a metric on pairs of maps and metrics, P_g the projection of quadratic differentials onto holomorphic quadratic differentials and $\Phi(u,g)$ is the Hopf differential of u, a quadratic differential that measures how 'close' u is to being conformal. In particular vanishing of $\Phi(u,g)$ implies that u is a weakly conformal map. This flow aims to transform given initial data to branched minimal immersions (or constant maps). In this thesis we study a number of questions concerning the Teichmüller harmonic map flow (1.3).

A priori singularities in this flow might appear both in the metric (i.e. parts of the domain may become arbitrarily 'thin') and the map. The latter type is well-understood for the harmonic map flow, the general idea is that so-called bubbles can be extracted at points and times where energy concentrates ([27]). These bubbles are harmonic maps $S^2 \to N$. Away from such concentration points, the harmonic map flow enjoys higher regularity. This principle carries over to the Teichmüller harmonic map flow ([23]), and allows one to employ a lot of the techniques familiar from the study of the harmonic map flow in the development of the theory for the flow (1.3).

Possible degeneration of the metric on the other hand requires new ideas. When this degeneration only happens at infinite time and M is a hyperbolic surface, it was shown in [25] that the initial map in some sense (sub-)converges to a collection of branched minimal immersions (or constant maps) in the limit. In a joint work with Rupflin and Topping ([13]) we study exactly what properties this convergence has, and some of that material is included in Chapter 3.

When defining the Teichmüller harmonic map flow, there is an arbitrary choice of scaling parameter which determines the relative importance of changes in the map versus changes in the metric. This manifests itself in the coupling constant η in the definition of the flow, (1.3). To remove this arbitrariness, we consider limits as $\eta \downarrow 0$ in Chapter 4.

1.1 Thesis outline

The main body of the thesis consists of three major parts: an introductory Chapter 2 (consisting of known material) and Chapters 3 and 4 presenting new work.

We begin **Chapter 2** by explaining how to arrive at the equations (1.3) as a gradient flow by viewing $E(u, g)$ as a functional of both map and metric, as originally done in [21]. This turns out to require exploiting the symmetry of conformal invariance of the energy functional on surfaces mentioned above, as well as the invariance of the energy functional under pullback by diffeomorphisms. In particular one needs to find a suitable space of pairs of maps and (equivalence classes of) metrics together with an inner product. There is some freedom in choosing this inner product, manifesting itself in the appearance of the coupling constant η in (1.3), which we investigate further in Chapter 4. It turns out that for genus 0 the flow coincides with the classical harmonic map flow, and for genus 1 it simplifies considerably and has been studied in [4]. Therefore from now on we restrict to the case of the domain being of genus ≥ 2, so it admits hyperbolic metrics (i.e. metrics of constant (Gauss-)curvature -1).

Once we have the definition of the flow, we would like to establish certain properties. Two natural questions can be asked immediately:

- Given suitable initial data (u_0, g_0), can we find a solution to (1.3)? Can we characterize the maximal existence time?
- Are solutions unique (in an appropriate class)?

These questions were answered in [23]. The only way for a (weak) solution $(u(t), g(t))$ not to exist for all time was shown to be degeneration of the metric (i.e. $\mathrm{inj}_{g(t)} \to 0$). We give a brief overview of the arguments in [23], and refer to Chapter 4, where we use some related techniques. We also explain what it means for so-called *bubbles* to form under the flow.

Under the assumption that the metric, even at infinite time, does not degenerate (i.e. $\mathrm{inj}_{g(t)} \geq \delta > 0$ for all times t), long time existence and (sub-)convergence to a branched minimal immersion (or a constant map) was obtained in [21]. We present some of the ingredients needed to obtain this result, in particular a Poincaré type inequality for quadratic differentials and the Mumford compactness theorem. In this setting we also first identify a common alternate 'sequential' viewpoint in the study of the Teichmüller harmonic map flow. It turns out that many of the results in the theory do not necessarily rely on the equations (1.3), but only need a sequence (u_i, g_i) of maps and metrics satisfying certain conditions:

Definition 2.2.3 (From [13])**.** Given an oriented closed surface M, a closed Riemannian manifold (N, G), and a pair of sequences $u_i : M \to N$ of smooth maps and g_i of metrics on M with fixed constant curvature and fixed area, we say that (u_i, g_i) is a *sequence of almost-minimal maps* if $E(u_i, g_i)$ is uniformly bounded and

$$\|\tau_{g_i}(u_i)\|_{L^2(M,g_i)} \to 0, \qquad \text{and} \qquad \|P_{g_i}(\Phi(u_i, g_i))\|_{L^2(M,g_i)} \to 0. \tag{1.4}$$

One can then view the results in [21] as a type of compactness statement for such sequences of almost-minimal maps. This can be seen as an analogue to the compactness theory for *sequences of almost-harmonic maps* u_i (see e.g. Lemma 3.1.2), which only satisfy $\|\tau_{g_i}(u_i)\|_{L^2(M,g_i)} \to 0$.

We then consider the results from [25] on metric degeneration at $t = \infty$, and show some of the modifications needed compared to the simpler case of no metric degeneration considered in [21]. This involves understanding the geometry of degenerating hyperbolic surfaces. It turns out that near to points in the domain with small injectivity radius one can view the surface as a long, thin cylinder equipped with a hyperbolic metric which is conformal to the usual euclidean metric (see Lemma A.1), often called a *hyperbolic collar*, or simply *collar*. Furthermore, it is again possible to interpret the theory in [25] as a compactness statement for almost-minimal maps, now from degenerating surfaces. We revisit this theory later in Chapter 3 with the results from [13].

We conclude our introduction to the Teichmüller harmonic map flow by giving a brief overview of a situation in which the flow does always necessarily exist smoothly for all time, namely in the setting of the target N having nonpositive sectional curvature, as developed in [20]. In some sense this can be thought of as a parallel to the classical theory for the harmonic map flow in [8], however the techniques used are very different. In particular we highlight some of the estimates for the projection operator P_g developed in [20], which are important for later applications.

To conclude Chapter 2 we switch focus to some classical results about harmonic maps, in particular situations in which one can find a unique harmonic map in a homotopy class (for some given metric) as studied in [10]. We also recall how this harmonic map changes smoothly under smooth changes of the underlying metric on the domain, as proved in [6].

In **Chapter 3** we present joint work with Rupflin and Topping from [13]. We have the following compactness theorem from [25], which we alluded to in the overview of Chapter 2.

Theorem 2.2.8 (Content from [25])**.** *Suppose we have an oriented closed surface M of genus $\gamma \geq 2$, a closed Riemannian manifold (N, G), and a sequence (u_i, g_i) of almost-minimal maps in the sense of Definition 2.2.3, for which $\lim_{i\to\infty} \ell(g_i) = 0$.*

Then after passing to a subsequence, there exist an integer $1 \leq k \leq 3(\gamma - 1)$ and a hyperbolic punctured surface (Σ, h, c) with $2k$ punctures (i.e. a closed Riemann surface $\hat{\Sigma}$ with complex structure $\hat{c}$, possibly disconnected, that is then punctured $2k$ times to give a Riemann surface Σ with c the restricted complex structure, which is then equipped with a conformal complete hyperbolic metric h) such that the following holds.

1. *The surfaces (M, g_i, c_i) converge to the surface (Σ, h, c) by collapsing k simple closed geodesics σ_i^j in the sense of Proposition A.2 from the appendix; in particular there is a sequence of diffeomorphisms $f_i : \Sigma \to M \setminus \cup_{j=1}^k \sigma_i^j$ such that*
$$f_i^* g_i \to h \text{ and } f_i^* c_i \to c \text{ smoothly locally,}$$
where c_i denotes the complex structure of (M, g_i).

2. *The maps $U_i := u_i \circ f_i$ converge to a limit u_∞ weakly in $W^{1,2}_{loc}(\Sigma)$ and weakly in $W^{2,2}_{loc}(\Sigma \setminus S)$ as well as strongly in $W^{1,p}_{loc}(\Sigma \setminus S)$, $p \in [1, \infty)$, away from a finite set of points $S \subset \Sigma$ at which energy concentrates.*

3. *The limit $u_\infty : \Sigma \to N$ extends to a smooth branched minimal immersion (or constant map) on each component of the compactification $(\hat{\Sigma}, \hat{c})$ of (Σ, c) obtained by filling in each of the $2k$ punctures.*

In the setting of this theorem, we now fix some $j \in \{1, 2, \ldots, k\}$ and consider the sequence of collapsing geodesics $\sigma_i = \sigma_i^j$. As mentioned above we can find a hyperbolic collar $\mathcal{C}_i$ around each geodesic, and it turns out that its length tends to infinity as $i \to \infty$. In particular, in the 'centre' of such a collar the above theorem does not provide us with any information on the behaviour of the maps u_i, as it is purely a local convergence statement. Therefore we consider a number of questions.

- Can energy be 'lost' along such collars? More precisely, after accounting for possible bubbles developing along the sequence of collars (i.e. areas of concentrated energy rescaled to give harmonic maps $S^2 \to N$), does the energy of the limit map equal the limit of the energies?
- What properties do the restrictions of the maps u_i to the collars $\mathcal{C}_i$ have? In particular, do the images of these 'collar maps' become close to curves in the target?
- The limiting domain Σ may be disconnected (e.g. a surface of genus 2 might split into two once punctured tori). Can the image of the limit branched minimal immersion obtained by filling in the punctures as in the above theorem also be disconnected? This would imply that the collar maps do not become close to constant maps.

In contrast to harmonic maps from degenerating domains, where it is indeed possible for energy to be lost as shown in [17, 34], we prove that no energy can be lost and that the images of the collar maps become close to curves. To do this, we first analyse almost-harmonic maps (i.e. maps u with small tension $\tau_g(u)$ in L^2) from long cylinders. This is done as the collar maps in particular satisfy these conditions, because the maps u_i form an almost-minimal sequence, and therefore have small tension in L^2. We can show that the angular energy of such maps decays exponentially away from the ends of the cylinders if they also have locally small energy (in a sense we make precise).

To obtain estimates on the full energy we require some additional geometric information, provided in the form of the condition $\|P_{g_i}(\Phi(u_i, g_i))\|_{L^2(M,g_i)} \to 0$, which turns out to imply L^1 control on the Hopf differential $\Phi(u_i, g_i)$. Using this, we find that the full energy also decays exponentially towards the centre of the cylinder under the earlier assumptions.

Armed with this, we consider sequences of almost-harmonic maps u_i, that also satisfy $\|\Phi(u_i)\|_{L^1} \to 0$, from cylinders $\mathscr{C}_{X_i} := [-X_i, X_i] \times S^1$ with $X_i \to \infty$ as $i \to \infty$, and show that there is no loss of energy (after accounting for all the bubbles). Note that in particular the collar maps corresponding to a sequence of almost-minimal maps can be viewed as such sequences.

We prove this by taking a sequence of maps from longer and longer cylinders and identifying a (finite) number of points where energy concentrates. For regions sufficiently far away from these points of concentrated energy we can then apply our earlier analysis to deduce that the energy of the maps on such regions converges to 0. Similarly the angular energy becomes small, which implies that the image of the maps connecting such regions becomes close to a curve. The regions where energy does concentrate (which are now finite in size) are then analysed using standard methods from the study of almost-harmonic maps (e.g. Lemma 3.1.2).

We then apply this statement for maps from cylinders to the collar maps around each degenerating geodesic obtained by analysing a sequence of almost-minimal maps using Theorem 2.2.8, also establishing that no energy is lost here (as usual, modulo bubbles) and that the 'connecting curves' between regions of concentrated energy are mapped close to curves. Together with analysing the remainder of the maps away from points where the domain degenerates by the aforementioned compactness properties of almost-harmonic maps we then prove the following theorem.

Theorem 3.1.1. *In the setting of Theorem 2.2.8, there exist two finite collections of nonconstant harmonic maps $\{\omega_k\}$ and $\{\Omega_j\}$ mapping $S^2 \to N$, such that after passing to a subsequence we have*

$$\lim_{\delta \downarrow 0} \lim_{i \to \infty} E\left(u_i, g_i; \delta\text{-}thick\,(M, g_i)\right) = E(u_\infty, h) + \sum_k E(\omega_k),$$

and

$$\lim_{i \to \infty} E(u_i, g_i) = E(u_\infty, h) + \sum_k E(\omega_k) + \sum_j E(\Omega_j).$$

Here we denote by δ-thick (M, g_i) all the points $x \in (M, g_i)$ such that $\mathrm{inj}_{g_i}(x) \geq \delta$.

To conclude Chapter 3, we return to again considering solutions to the flow (1.3), and show that the image of the limit branched minimal immersion can indeed be disconnected by constructing an example.

In **Chapter 4** we study limits of the coupling constant η in the flow equations (1.3). We first study the behaviour of the flow as $\eta \downarrow 0$ on compact time intervals, and prove convergence to the classical harmonic map flow (1.2). We do this in the setting of the target N having nonpositive curvature.

Theorem 4.1.1. *Let M be a smooth closed oriented surface of genus $\gamma \geq 2$ and*

$g_0 \in \mathcal{M}_{-1}$, (N, G) a nonpositively curved smooth closed Riemannian manifold and $u_0 : M \to N$ a smooth map. Then on each fixed time interval $[0, T]$ the flows $(u_\eta(t), g_\eta(t))$ satisfying (2.21) with initial condition (u_0, g_0) converge smoothly to the harmonic map flow $u(t)$ satisfying (2.4) (with respect to g_0), starting at the same initial condition $u(0) = u_0$, in the following sense as $\eta \downarrow 0$:

1. *The metrics $g_\eta(t)$ converge to the initial metric g_0 in $C^k(M, g_0)$ uniformly in t for each $k \in \mathbb{N}$.*
2. *The maps $u_\eta(t)$ converge to $u(t)$ smoothly on $M \times [0, T]$.*

The restriction to compact time intervals is motivated by the fact that one can find smooth initial data (u_0, g_0) such that any solution $(u(t), g(t))$ to (1.3) necessarily develops a metric singularity (at infinite time), no matter what $\eta > 0$ is used, hence one cannot expect uniform convergence to the harmonic map flow for all times.

We remark here that it is possible that Theorem 4.1.1 could be obtained by viewing the flow (1.3) as a perturbation to the classical harmonic map flow for small η. We instead opt for a more 'direct' approach, which has the advantage of leading to a number of estimates useful in the latter part of this thesis.

The proof of Theorem 4.1.1 is carried out by first controlling the metric component and then using that control to also obtain good estimates for the map. The metric convergence can be deduced as a consequence of the energy identity

$$\frac{dE}{dt} = -\int_M \left[|\tau_g(u)|^2 + \left(\frac{\eta}{4}\right)^2 |Re(P_g(\Phi(u, g)))|^2 \right] dv_g, \tag{1.5}$$

valid for solutions of the flow (1.3) together with estimates from [23].

To control the map we first establish that the degree of energy concentration along the flows $(u_\eta(t), g_\eta(t))$ from Theorem 4.1.1 can be controlled uniformly in η. We do this by establishing an estimate for the evolution of local energy $E(u_\eta, B_r(x))$ (very similar to [23]), which implies that energy concentration at some time leads to concentrated energy at nearby times. We further prove a bound for the energy concentration of maps in terms of their tension, exploiting the curvature hypothesis on N. In particular, very concentrated energy necessarily leads to $\|\tau_{g_\eta}(u_\eta)\|_{L^2(M, g_\eta)}$ being large. But as $\|\tau_{g_\eta}(u_\eta)\|_{L^2(M, g_\eta)}$ is integrable over $[0, T]$ by (1.5) we find that energy can not become too concentrated.

This allows us to apply standard interpolation estimates to the maps u_η, in particular leading to an a priori $W^{1,4}$-estimate. With this estimate we then consider the evolution equation for the difference $w = u_\eta - u$ of the maps in the Teichmüller harmonic map flow and the classical harmonic map flow. We obtain an integral bound from this equation that shows that indeed $\|w\|_{L^2}$ becomes small as $\eta \downarrow 0$. Higher regularity can then be deduced by interpolation and the results in [23].

Finally we consider a rescaled limit of (1.3) as $\eta \downarrow 0$. In particular, we can slow down time as we decrease η so that in some sense the 'speed' at which the metric moves stays constant, by letting $\bar{t} = \frac{\eta^2}{4}t$. This leads to the rescaled equations

$$\begin{aligned} \frac{\partial}{\partial t}u &= k\tau_g(u) \\ \frac{\partial}{\partial t}g &= Re(P_g(\Phi(u,g))) \end{aligned} \tag{1.6}$$

where we define $k := \frac{4}{\eta^2}$ (hence $\eta \downarrow 0$ now corresponds to $k \to \infty$). One might expect that large k corresponds to the map rapidly becoming harmonic, and in the limit leading to a flow through harmonic maps, instantaneously changing the initial map into a harmonic map. We first establish that the metric component of the rescaled flow (1.6) does not degenerate, at least for small times.

Lemma 4.2.3. *Let M be a smooth oriented closed surface of genus $\gamma \geq 2$ and N be a smooth closed Riemannian manifold. Assume that for some $T > 0$, $k > 0$ we have a smooth solution $(u_k(t), g_k(t))$ to (1.6) on $[0, T]$ with rescaled coupling constant k for some given initial data (u_0, g_0). Then there exists $T_0 = T_0(M, g_0, u_0) > 0$ (in particular* independent *of k) such that the injectivity radius inj_{g_k} is bounded away from 0 up to time $t = \min\{T_0, T\}$.*

We can then use a variety of useful estimates for the metric component, enabled by this control of the injectivity radius, from [22]. In particular we find that (for integers $s \geq 0$) the C^s-norm of the metric tensors satisfies a $C^{0,\frac{1}{2}}$ Hölder condition in time.

In particular, if one assumes that N has *strictly* negative sectional curvature and considers a homotopy class H of maps $M \to N$ that contains no constant maps or maps to closed geodesics in the target, then for each metric g on M there is a unique harmonic representative in H ([10], see also Chapter 2). Hence given such a homotopy class and given a curve $g(t)$ of metrics on $[0, T]$ one can find a corresponding unique curve $u(t)$ of harmonic maps defined such that $u(t)$ is harmonic with respect to $g(t)$. In this context we show that one can indeed find a limit flow for (1.6) as $k \to \infty$.

Theorem 4.2.21. *Let M be a smooth closed oriented surface of genus $\gamma \geq 2$ and N be a smooth closed Riemannian manifold. Given smooth initial data (u_0, g_0) for (4.62), take $0 < T \leq T_0$ with T_0 from Lemma 4.2.3, and consider the sequence $(u_k(t), g_k(t))_{k=1}^{\infty}$ of solutions to (4.62) with rescaled coupling constant k on the fixed time interval $[0, T]$, which we further assume to be smooth up to $t = T$. Then the following is true:*

1. *There exists a limit curve of hyperbolic metrics g (i.e. each $g(t)$ has Gauss curvature $K = -1$) on $[0, T]$, continuous in time and smooth in space in the sense that for all $s \in \mathbb{N}$, g is an element of $C^0([0, T], C^s(\mathrm{Sym}^2(T^*M), g_0))$. After possibly selecting a subsequence in k the curves g_k converge to g in $C^0([0, T], C^s(\mathrm{Sym}^2(T^*M), g_0))$ (i.e. uniformly in time in $C^s(M, g_0)$), again for all $s \in \mathbb{N}$.*

2. *Further assume that N has strictly negative sectional curvature and that the homotopy class of u_0 does not contain maps to closed geodesics in the target or constant maps. Let $u(t) : M \times (0, T] \to N$ be the unique curve of harmonic maps homotopic to u_0 corresponding to $g(t)$, then the limit curve of metrics g is differentiable in time at each point $x \in M$ away from $t = 0$, with derivative given by $\frac{\mathrm{d}}{\mathrm{d}t} g(t)(x) = Re(\Phi(u, g))(x)$, where $\Phi(u, g)$ as usual denotes the Hopf differential. Finally, the maps $u_k(t)$ also converge to $u(t)$ uniformly in t in $C^s(M, g_0)$ away from 0 for all $s \in \mathbb{N}$.*

The existence of a limit curve of metrics can be seen by the metric estimates mentioned earlier. Showing that the maps converge to the corresponding harmonic limit curve and that the limit metric curve is differentiable in time is more involved.

We work in the above setting of the homotopy class of u_0 containing no maps to closed geodesics in the target or constant maps, and N having strictly negative sectional curvature. We then show that small L^2-norm of the tension $\tau_g(u)$ corresponds to being close to a harmonic map in C^0. For the harmonic map flow the L^2-norm of the tension $\mathcal{T}(u(t), g(t)) = \left\| \tau_{g(t)}(u(t)) \right\|_{L^2(M, g(t))}$ is a monotonically decreasing quantity in t when N has nonpositive sectional curvature, and thus has to be small for large times (as it is also integrable). We prove an adapted statement for the evolution of $\mathcal{T}$ under Teichmüller harmonic map flow.

Lemma 4.2.8. *Assume M as usual to be a smooth closed oriented surface of genus*

$\gamma \geq 2$ and N to be a smooth closed Riemannian manifold, which we now also assume to have nonpositive curvature. Let $\eta > 0$ and take $(u(t), g(t))$ to be the (smooth) solution to (1.3) with coupling constant η, starting at initial data (u_0, g_0). Assume that there exists some $\delta > 0$ such that $\mathrm{inj}_{g(t)} \geq \delta > 0$, and denote as usual $E_0 = E(u(0), g(0))$, then

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{T}(t) \leq C E_0^3 \delta^{-2} \eta^4 \tag{1.7}$$

where $\mathcal{T}(t) := \mathcal{T}(u(t), g(t))$ and $C < \infty$ only depends on the genus γ of M.

We thus find that (away from $t = 0$) $\mathcal{T}(t)$ becomes small for solutions of the rescaled flow (1.6) when k is large, and consequently the maps $u_k(t)$ from Theorem 4.2.21 become C^0-close to being harmonic. We next establish higher order convergence. To do this, we apply parabolic regularity to the equation (1.6), using the small tension, to find a priori bounds for u_k and elliptic regularity to their associated harmonic maps $\overline{u}_k$ (which are unique under our assumptions). Note that the bounds on the harmonic maps are standard, and already appear in [8]. By interpolation we then see that $u_k - \overline{u}_k$ becomes small (e.g. in any C^s-norm).

Using the smooth dependence of harmonic maps on the domain metric ([6], again see Chapter 2) together with the convergence of the metric we then find that the maps $u_k(t)$ converge to (the harmonic limit curve) $u(t)$. We finally show that the limit curve of metrics $g(t)$ is differentiable away from 0 by proving that $\frac{\mathrm{d}}{\mathrm{d}t} g_k(t)$ converges uniformly to $Re(\Phi(u, g))$, where we denote as usual by $\Phi(u, g)$ the Hopf differential of the limit. This uses the already established convergence of the maps, the metrics, and properties of the projection P_g from [20] (see also Chapter 2).

Chapter 2

The Teichmüller harmonic map flow

2.1 Construction

2.1.1 The harmonic map flow

Given smooth closed Riemannian manifolds $M = (M, g)$ and $N = (N, G)$, with M assumed to be oriented, and a smooth map $u : M \to N$, we can define the *energy* (sometimes also called the *harmonic map energy* or *Dirichlet energy*) $E(u)$ as

$$E(u) = \frac{1}{2}\int_M |du|_g^2 dv_g, \tag{2.1}$$

where we denote by dv_g the volume element on (M, g). We then define *harmonic maps* to be the critical points of E. Let $u(t)$ be a smooth variation of u (i.e. satisfying $u(0) = u$ and $u(x,t) : M \times [-1,1] \to N$ smooth) with $\frac{\partial u}{\partial t}|_{t=0} = v$ then we find, using partial integration,

$$\frac{\mathrm{d}}{\mathrm{d}t}E(u(t))|_{t=0} = -\int_M \langle v, \tau_g(u)\rangle dv_g \tag{2.2}$$

where $\tau_g(u) = \operatorname{tr}\nabla du$ is the *tension field* of u (e.g. [8]). If we assume the target N to be isometrically embedded into $\mathbb{R}^n$, we can also write

$$\tau_g(u) = \Delta_g u + A_g(u)(\nabla u, \nabla u) \tag{2.3}$$

where Δ_g is the Laplace-Beltrami operator on M and A the second fundamental form of $N \hookrightarrow \mathbb{R}^n$. In particular we can see from (2.2) that (smooth) harmonic maps satisfy $\tau_g(u) = 0$.

To try and find such harmonic maps, the L^2 gradient flow associated with E was considered in [8]. It is given as the *harmonic map flow*

$$\frac{\partial u}{\partial t} = \tau_g(u), \tag{2.4}$$

where $u(x,t) : M \times I \to N$ is now evolving. The hope is that this transforms some initial map $u_0 = u(0)$ into a harmonic map. Indeed, we have the following theorem.

Theorem 2.1.1 ([8, 10])**.** *Let M, N be as before and additionally assume that N has nonpositive sectional curvature. Given a smooth map $u_0 : M \to N$ the harmonic map flow (2.4) with initial condition $u(0) = u_0$ has a smooth solution existing for all times. Furthermore there exists a harmonic map $u_\infty : M \to N$ such that $u(t) \to u_\infty$ smoothly as $t \to \infty$.*

This in particular solves the *homotopy problem* of finding a harmonic map homotopic to some given map when N has nonpositive sectional curvature.

2.1.2 A gradient flow to find branched minimal immersions

From now on we restrict to the domain M being a smooth closed oriented surface. Instead of thinking of the energy E as just a functional of maps, we can consider it as a functional on both maps and metrics $E = E(u,g)$. As before, we can calculate the first variation of the energy, now also considering a variation $g(t)$ of the metric with $\frac{\partial g}{\partial t}|_{t=0} = h$. We find (e.g. [21])

$$\frac{\mathrm{d}}{\mathrm{d}t} E(u(t), g(t))|_{t=0} = -\int_M \langle v, \tau_g(u)\rangle + \frac{1}{4}\langle Re(\Phi(u,g)), \frac{\partial g}{\partial t}\rangle dv_g \tag{2.5}$$

where $\Phi(u,g)$ denotes the *Hopf differential*. In complex notation it is given as $\Phi(u,g) = 4(u^*G)^{(2,0)}$, and writing $\Phi(u,g) = \phi dz^2$ with a local complex coordinate $z = x + iy$ we have the formula

$$\phi = |u_x|^2 - |u_y|^2 - 2i\langle u_x, u_y\rangle. \tag{2.6}$$

More explicitly, one finds $Re(\Phi(u,g)) = 2u^*G - 2e(u,g)g$ where $e(u,g) = \frac{1}{2}|du|_g^2$ is the *energy density* ([23]).

We can read off from (2.5) that a critical point u of E, with respect to both map and metric, needs to be harmonic and to satisfy $Re(\Phi(u,g)) = 0$. Thus $u^*G = e(u,g)g$, implying that u is weakly conformal. Conformal harmonic maps from surfaces are necessarily minimal immersions (i.e. critical points of the area functional). Similarly, it turns out that (non-constant) weakly conformal harmonic maps can also be viewed more geometrically as *branched minimal immersions*. Indeed, away from finitely many points where the differential du vanishes, they are minimal immersions (see [9]). For the purposes of this thesis, we simply use the following definition.

Definition 2.1.2. A branched minimal immersion is a non-constant weakly conformal harmonic map.

Thus, to find branched minimal immersions, one can study critical points of $E(u,g)$. Motivated by the harmonic map flow, one could define the corresponding gradient flow to (2.5) (see [4]):

$$\begin{aligned} \frac{\partial}{\partial t}u &= \tau_g(u) \\ \frac{\partial}{\partial t}g &= Re(\Phi(u,g)). \end{aligned} \tag{2.7}$$

However, as noted already in [4], the equation for the metric in this flow is not expected to have good analytical properties, as the right hand side moves in an infinite dimensional space. Thus, a different approach was taken in [21] to construct a flow designed to find branched minimal immersions. The main idea is to exploit symmetries of the energy functional. This first such symmetry is the invariance of the energy E under conformal changes of the metric g when M is a surface. To this end, we have the following consequence of the classical Uniformisation Theorem for surfaces.

Theorem 2.1.3. *Let M be a smooth closed oriented surface equipped with a Riemannian metric g. Consider a smooth conformal factor $\lambda : M \to (0,\infty)$. If we denote the genus of M by γ, then the following holds:*

- $\gamma = 0$*: M is a sphere and we can find λ such that $K_{\lambda g} = 1$.*
- $\gamma = 1$*: M is a torus and we can find λ such that $K_{\lambda g} = 0$*
- $\gamma \geq 2$*: M is a hyperbolic surface and we can find λ such that $K_{\lambda g} = -1$.*

Here we denote by $K_{\lambda g}$ the Gauss curvature of the surface $(M, \lambda g)$.

Therefore instead of searching for branched minimal immersions in the space of all metrics on M, we consider the space $\mathcal{M}_c := \{$Smooth metrics g on M of constant Gauss curvature $c\}$, with c depending on the genus of M as in the above theorem. In the case of $c = 0$ we also additionally restrict to unit area tori.

The second symmetry considered in [21] is pullback by diffeomorphisms $f : M \to M$, as indeed $E(u, g) = E(u \circ f, f^*g)$. Thus we define an equivalence relation on pairs (u, g) of maps and metrics by $(u_1, g_1) \sim (u_2, g_2)$ if there exists a smooth diffeomorphism $f : M \to M$ homotopic to the identity such that $u_2 = u_1 \circ f$ and $g_2 = f^*g_1$ and consider the quotient space (again, from [21])

$$\mathcal{A} = \{C^\infty(M, N) \times \mathcal{M}_c\}/\sim \tag{2.8}$$

of equivalence classes $[(u, g)]$.

Remark 2.1.4. As noted in [21], $\mathcal{A}$ is only a set, but it can be modified to be a smooth (infinite-dimensional) manifold, together with a natural tangent bundle and metric, by considering metrics and functions of sufficiently high Sobolev regularity. This is done using ideas also used in the development of Teichmüller theory (see [21, Appendix B] for more details, [33] for an introduction to Teichmüller theory). However, to motivate defining the Teichmüller harmonic map flow this construction is not required.

We next try to find a 'good' definition (working formally) of a metric structure on $\mathcal{A}$. We first describe tangent vectors of $\mathcal{M}_c$.

Lemma 2.1.5 (Theorem 2.4.1 in [33])**.** *Let M be a smooth closed surface and consider $g(t) \in \mathcal{M}_c$ a smooth curve of metrics with $h = \partial_t g|_{t=0}$, then there exists a tensor $h_0 \in \Gamma(\mathrm{Sym}^2\, T^*M)$ satisfying $\delta_{g(0)} h_0 = 0$ and $\mathrm{tr}_{g(0)}\, h_0 = 0$, such that*

$$h = h_0 + \mathcal{L}_X g(0) \tag{2.9}$$

where X is some (smooth) vector field on M.

Sketch of proof. One first proves that such a decomposition $h = h_0 + \mathcal{L}_X g(0)$ satisfying the divergence free condition exists (indeed, this does not even require M to be a surface and works for arbitrary $h \in \Gamma(\mathrm{Sym}^2\, T^*M)$), see [33, Theorem 1.4.2]) by solving an elliptic equation. We then find that the linearization $DR(g(0))$ of the scalar curvature necessarily vanishes when evaluated at $\mathcal{L}_X g(0)$ and h (as the scalar curvature $R = 2K$ is constant in both those directions). Thus also $DR(g(0))h_0 = 0$. But we have the general formula (e.g. [31, Proposition 2.3.9])

$$DR(g(0))v = -\Delta_{g(0)}(\operatorname{tr}_{g(0)} v) + \delta_{g(0)}\delta_{g(0)}v - \langle \mathrm{Ric}, v\rangle, \tag{2.10}$$

valid for any $v \in \Gamma(\mathrm{Sym}^2 T^*M)$). Using this with $v = h_0$ implies $-\Delta_{g(0)}(\operatorname{tr}_{g(0)} h_0) - K \operatorname{tr}_{g(0)} h_0 = 0$. Multiplying this by $\operatorname{tr}_{g(0)} h_0$ and integrating over M yields

$$\int_M |\nabla \operatorname{tr}_{g(0)} h_0|^2 - K(\operatorname{tr}_{g(0)} h_0)^2 dv_{g(0)} = 0. \tag{2.11}$$

Note that for $K = c = -1$ we are finished. We use arguments from [32] to handle the remaining cases. For $K = 0$ we deduce that $\operatorname{tr}_{g(0)} h_0$ is constant, which implies $\operatorname{tr}_{g(0)} h_0 = 0$ as the area of M was assumed fixed in this case (note that $\partial_t dv_g = \frac{1}{2} \operatorname{tr} h_0 dv_g$, e.g. [31, Proposition 2.3.12]). If $K = 1$ then $(M, g(0))$ is necessarily the (round) sphere, and we can deduce $\operatorname{tr}_{g(0)} h_0 = 0$ as 1 is not an eigenvalue of the Laplacian. □

Tensors h satisfying $\delta_{g(0)}h_0 = 0$ and $\operatorname{tr}_{g(0)} h_0 = 0$ are also called *transverse traceless*, and they can be identified with the space of so-called holomorphic quadratic differentials as follows.

Lemma 2.1.6 (e.g. [33, Chapter 2])**.** *Let (M, g) be a smooth closed surface, then there is a bijection between the space $S_2^{TT}(M, g)$ of transverse traceless symmetric 2-tensors h and the space $\mathcal{H}(M, g)$ of* holomorphic quadratic differentials*, i.e. quadratic differentials that can be written as θdz^2, where $z = x + iy$ is a local complex coordinate for (M, g), $dz^2 = dz \otimes dz$ and θ is a holomorphic function.*

Proof. We first note that any transverse traceless tensor h is the real part of a holomorphic quadratic differential by an argument from [33], computing the derivatives $\partial_k h_{ij}$ explicitly in isothermal coordinates (x, y) (i.e. $g = \rho^2(dx^2 + dy^2)$).

We let $h = h_{11}dx^2 + 2h_{12}dxdy + h_{22}dy^2$, then $\operatorname{tr} h = \rho^2(h_{11} + h_{22}) = 0$, so $h_{11} = -h_{22}$. We find that $0 = (\delta h)_i = \frac{1}{\rho^2}\partial_k h_{ik}$, and setting $i = 1$ and $i = 2$ we obtain

$$\frac{\partial}{\partial x}h_{11} + \frac{\partial}{\partial y}h_{12} = 0 \tag{2.12}$$

$$\frac{\partial}{\partial x}h_{12} - \frac{\partial}{\partial y}h_{11} = 0, \tag{2.13}$$

using $h_{22} = -h_{11}$. Therefore $\theta(z) = h_{11}(x,y) - ih_{12}(x,y)$ (with $z = x + iy$) is a holomorphic function (as h_{11}, $-h_{12}$ satisfy the Cauchy-Riemann equations). Hence $h = Re(\theta(z)(dx + idy)^2) = Re(\theta dz^2)$ is the real part of a holomorphic quadratic differential. We remark here that one could also carry out this computation from a complex viewpoint (e.g. [32]).

We next prove that $\mathcal{H} \cong Re(\mathcal{H})$, as the map $\theta dz^2 \to Re(\theta dz^2)$ is a bijection. To see this, we again compute in local coordinates $z = x + iy$

$$\theta dz^2 = (a + ib)dz^2 = (a + ib)(dx^2 + 2idxdy - dy^2) \tag{2.14}$$

$$Re(\theta dz^2) = a(dx^2 - dy^2) - b(2dxdy) \tag{2.15}$$

where a and b are some (locally defined) real functions on M. We see that $Re(\theta dz^2)$ can only vanish if $a = b = 0$. Additionally, we observe that real parts of quadratic differentials are traceless.

To finish the proof we can simply reverse the argument: if we start with $h = Re(\theta dz^2)$ for some holomorphic quadratic differential θdz^2 (which implies that h is tracefree), we can compute the Cauchy-Riemann equations for θdz^2, and find that the corresponding tensor h is transverse traceless, i.e. also divergence free. $\square$

Therefore we have the formal decomposition $T\mathcal{M}_c = Re(\mathcal{H}) \oplus \mathcal{L}_X g$ (this could be made precise by equipping $\mathcal{M}_c$ with an appropriate manifold structure, see [33]). Often this is referred to as a splitting into a *horizontal* part (consisting of real parts of holomorphic quadratic differentials) and a *vertical* part (corresponding to modifying by diffeomorphisms). Thus given a curve $(u(t), g(t)) \in C^\infty(M,N) \times \mathcal{M}_c$ we can represent the tangent vector of the metric component at $t = 0$ by

$$\partial_t g(0) = Re(\Psi) + \mathcal{L}_X g(0) \tag{2.16}$$

where $\Psi \in \mathcal{H}$. Recall that in the definition of $\mathcal{A}$ we identified pairs (u, g) under pullback by diffeomorphisms. We can in particular modify any curve $(u(t), g(t))$ by a diffeomorphism such that $\partial_t g(0)$ is horizontal (i.e. $\partial_t g(0) = Re(\Psi)$). This can be done by considering the family of diffeomorphisms generated by $-X$ (see [21]).

But now at any given $[(u_0, g_0)] \in A$ we simply take such a path $(u(t), g(t)) \in C^\infty(M,N) \times \mathcal{M}_c$ that starts at $(u(0), g(0)) = (u_0, g_0)$, modified as above so $\partial_t g(0)$ is horizontal. For the tangent vector V at $[(u_0, g_0)]$ corresponding to $(\partial_t g(0), \partial_t u(0))$ a norm is then defined in [21] by

$$\|V\|^2 := \|\partial_t u\|_{L^2}^2 + \eta^{-2}\,\|Re(\Psi)\|_{L^2}^2\,, \tag{2.17}$$

where $\eta > 0$ is the *coupling constant*, a choice of the relative importantce of the map directions and the metric directions. We note that the decomposition into horizontal and vertical parts is canonical, and in particular the choice of holomorphic quadratic differential $Re(\Psi)$ is unique. We then have an inner product on tangent vectors of $\mathcal{A}$ induced by this norm.

Using (2.5) we can express the differential of E at $[(u,g)] \in \mathcal{A}$ evaluated at some tangent vector given by $(\partial_t u, \partial_t g) = (\partial_t u, Re(\Psi))$ via

$$DE(\partial_t u, Re(\Psi)) = -\int_M \langle \partial_t u, \tau_g(u)\rangle dv_g + \frac{1}{4}\langle Re(\Phi(u,g)), Re(\Psi)\rangle dv_g, \tag{2.18}$$

see also [21] for this formula.

We denote by $\mathcal{Q}(M,g)$ the space of quadratic differentials on (M,g), then we have $\Phi(u,g) \in \mathcal{Q}(M,g)$. However, generally $\Phi(u,g) = \phi dz^2$ is not a *holomorphic* quadratic differential, and indeed ϕ is holomorphic precisely when u is a harmonic map. We thus introduce the L^2-orthogonal projection $P_g : \mathcal{Q}(M,g) \to \mathcal{H}(M,g)$ (using the canonical L^2 Hermitian inner product induced by g on $\mathcal{Q}(M,g)$). This allows us to further calculate, as in [21],

$$\begin{aligned} DE(\partial_t u, Re(\Psi)) &= -\int_M \langle \partial_t u, \tau_g(u)\rangle + \frac{1}{4}\langle Re(P_g(\Phi(u,g))), Re(\Psi)\rangle dv_g && (2.19)\\ &= -\langle (\tau_g(u), \frac{\eta^2}{4}(P_g(\Phi(u,g)))), (\partial_t u, Re(\Psi))\rangle_{\mathcal{A}}. && (2.20)\end{aligned}$$

We used that $\langle Re(\Phi(u,g)), Re(\Psi)\rangle_{L^2} = \langle Re(P_g(\Phi(u,g))), Re(\Psi)\rangle_{L^2}$ (as Ψ is a holomorphic quadratic differential and orthogonality of quadratic differentials implies that their real parts are orthogonal, in both cases using the L^2 Hermitian inner product). We can now finally give the flow equations of the *Teichmüller harmonic map flow* by writing down the gradient flow of E corresponding to the inner product defined on $\mathcal{A}$ as some $[(u(t), g(t))]$ satisfying

$$\begin{aligned} \frac{\partial}{\partial t}u &= \tau_g(u) \\ \frac{\partial}{\partial t}g &= \frac{\eta^2}{4}Re(P_g(\Phi(u,g))). \end{aligned} \tag{2.21}$$

From now on we will always think of the Teichmüller harmonic map flow as flowing pairs of maps and metrics (rather than equivalence classes $[(u(t), g(t))]$).

Remark 2.1.7. Starting at some $g_0 \in \mathcal{M}_c$ any solution metric $g(t)$ will necessarily stay in $\mathcal{M}_c$ (by Lemma 2.1.6 $\frac{\partial}{\partial t}g$ is transverse traceless, and the derivative of the scalar curvature vanishes in transverse traceless directions by the formula used in the proof of Lemma 2.1.5). We further have that the volume form $dv_{g(t)}$ is constant along the flow, as $\frac{\partial}{\partial t}g$ is traceless (e.g. [31, Proposition 2.3.12]).

We also observe here that (smooth solutions) of the flow satisfy the following energy identity, by (2.5):

$$\frac{dE}{dt} = -\int_M \left[|\tau_g(u)|^2 + \left(\frac{\eta}{4}\right)^2 |Re(P_g(\Phi(u,g)))|^2\right] dv_g. \tag{2.22}$$

We will later see how this allows us to obtain information about the behaviour of the flow (2.21).

The nature of the flow depends strongly on the genus γ of M. Briefly (see [21] for some more detail):

$\gamma = 0$ The space $\mathcal{H}$ only consists of the 0 element, and thus (2.21) coincides with the harmonic map flow (2.4).

$\gamma = 1$ $\mathcal{H}$ now has real dimension 2, and the flow (2.21) is somewhat easier to analyse than for $\gamma > 1$. In particular, it possible to view the metric as moving in a two dimensional submanifold of $\mathcal{M}_0$ ([21, Section A.1]), and the flow was shown to be equivalent in [21] to a flow studied in [4].

$\gamma \geq 2$ One can find that $\mathcal{H}$ has real dimension $6(\gamma - 1)$, however it is no longer possible to view the metric (globally) as moving in a finite dimensional subspace of $\mathcal{M}_{-1}$. In particular, the flow (2.21) may deform the metric $g(t)$ by pulling back with a diffeomorphism.

For the rest of this thesis we will only be working with $\gamma \geq 2$, i.e. M a hyperbolic surface.

2.2 Properties of the Teichmüller harmonic map flow

2.2.1 Existence and uniqueness

The existence and uniqueness theory for the Teichmüller harmonic map flow (2.21) has been investigated in [23]. We give a definition ([21]) for weak solutions of (2.21).

Definition 2.2.1 (From [21, Definition 1.2]). We call $(u, g) \in H^1_{loc}(M \times [0, T), N) \times C^0([0, T), \mathcal{M}_{-1})$ a weak solution of (2.21) on $[0, T), T \leq \infty$, provided u solves the first equation of (2.21) in the sense of distributions and g is piecewise C^1 (viewed as map from $[0, T)$ into the space of symmetric $(0, 2)$ tensors equipped with any C^k metric, $k \in \mathbb{N}$) and satisfies the second equation of (2.21) away from times where it is not differentiable.

Weak solutions in this sense were shown to exist until the metric degenerates in [23]. The following theorem, which we give here in an abbreviated form, was proved. We use the notion of 'bubbles' in this statement, which we make precise later.

Theorem 2.2.2 (From [23, Theorem 1.1]). *Let M be a smooth closed oriented surface and N a smooth closed Riemannian manifold, then for any given initial data $(u_0, g_0) \in C^\infty(M, N) \times \mathcal{M}_{-1}$ there exists a weak solution (u, g) of (2.21) defined on a maximal interval $[0, T), T \leq \infty$, that satisfies the following properties*

1. *The solution (u, g) is smooth away from at most finitely many singular times $T_i \in (0, T)$ at which bubbles develop at a finite set of points $S(T_i)$.*

2. *For each i there exists a limit map $u(T_i) \in H^1(M, N)$ such that $u(t) \rightharpoonup u(T_i)$ weakly in H^1 and $u(t) \to u(T_i)$ smoothly away from $S(T_i)$, both times as $t \to T_i$. We further have $g(T_i) \in \mathcal{M}_{-1}$ such that $g(t) \to g(T_i)$ smoothly as $t \to T_i$.*

3. *The energy $E(u(t), g(t))$ is non-increasing in t.*

4. *The solution exists as long as the metrics do not degenerate in moduli space. Hence, if we denote the length of shortest closed geodesic in $(M, g(t))$ by $\ell(g(t))$, we either have $T = \infty$ or $\liminf_{t \uparrow T} \ell(g(t)) = 0$.*

Furthermore, the solution is uniquely determined by its initial data in the class of all weak solutions with non-increasing energy.

Note that the same theorem holds true for genus $\gamma = 1$, where it follows by work in [4] as mentioned above, and for $\gamma = 0$, where the flow reduces to the harmonic map flow (2.4) and global existence was established in [27].

The prove this theorem, first certain estimates for the evolution of the metric tensor are established in [23]. In particular, it is shown that the metric $g(t)$ is a Lipschitz-curve with respect to any C^k-norm on the space $\mathcal{M}_{-1}$, as long as one controls $\ell(g(t))$. We give such an estimate in Chapter 4, Lemma 4.1.6. Thus, one can obtain short time existence by an iteration argument.

If $T < \infty$ one finds that either degeneration of the metric occurs in the sense that

$$\liminf_{t \uparrow T} \ell(g(t)) = 0 \tag{2.23}$$

or the metric stays controlled and the map becomes singular as $t \uparrow T$. Finally, if only the map becomes singular it is shown that an analysis similar to [27] can be employed to find that such singularities are necessarily caused by bubbles developing (which we will describe in more detail soon), and that one can flow past such map singularities using the strong control on the metric. The main idea to carry out this analysis is that on a short time interval, the metric is nearly constant, thus estimates from the harmonic map flow can be translated into estimates for Teichmüller harmonic map flow with an additional error term due to the evolving metric. We use this technique and some of the estimates obtained in Chapter 4 of this thesis, in particular Lemma 4.1.13.

We note that the class of solutions considered is natural, as for the harmonic map flow uniqueness no longer holds when one drops the non-increasing assumption on the energy, as shown in [28].

Bubbling at finite times

We now give a short overview of the bubbling phenomenon mentioned above. It was first observed in constructing a global weak solution to the harmonic map flow by Struwe [27].

In our setting for a weak solution (u, g) as given by Theorem 2.2.2, we say that a *bubble* develops at a singular time $T_0 \in (0, T)$ and point $x_0 \in M$ if

1. The energy concentrates at (x_0, T_0) in the sense that for any neighbourhood Ω of x_0 we have an $\epsilon > 0$ with
$$\limsup_{t\uparrow T_0} E(u(t), g(t), \Omega) \geq \epsilon. \tag{2.24}$$

2. Choose local isothermal coordinates centred at the point x_0 (with respect to the metric $g(T_0)$) and view the map u as a map in these coordinates. Then we can find a sequence of times $t_i \uparrow T$ such that there exists a non-constant harmonic map $\omega : S^2 \to N$, called a bubble (which we view as a map from $\mathbb{R}^2 \cup \infty \to N$ via stereographic projection), and sequences $a_i \to 0 \in \mathbb{R}^2$, $\lambda_i \downarrow 0$ as $i \to \infty$ such that
$$u(a_i + \lambda_i x, t_i) \to \omega \text{ in } W^{2,2}_{loc}(\mathbb{R}^2, N) \tag{2.25}$$
as $i \to \infty$.

We remark that much more is known about this bubbling behaviour. It is possible that one can extract more than one bubble at any given singular point by choosing different sequences a_i and λ_i in the above construction. For the harmonic map flow it has been shown ([5]) that if one extracts all the bubbles, forming a so-called *bubble tree*, one in particular finds that no energy is lost along the flow in the sense that
$$\lim_{t\uparrow T_0} E(u(t)) = E(u(T_0)) + \sum_i E(\omega_i) \tag{2.26}$$
where T_0 again denotes some singular time, $u(t)$ is the weak solution to (2.4) from [27] and the ω_i are the bubbles.

This energy analysis carries over easily to the Teichmüller harmonic map flow if the metric is assumed not to degenerate ([23]), however if metric degeneration occurs, new ideas are needed. We revisit this question in Chapter 3 of this thesis (see also [13]) in the context of infinite time degeneration (see also Lemma 3.1.2).

2.2.2 No metric degeneration

Recall that $\ell(g(t))$ denotes the length of the shortest closed geodesic in $(M, g(t))$. If we assume that $\ell(g(t)) \geq \epsilon > 0$ for all times t we see that the weak solution $(u(t), g(t))$ provided by Theorem 2.2.2 exists for all time. The motivation for defining the Teichmüller harmonic map flow (2.21) was to transform maps into branched minimal immersions, thus we would like to establish that (in some appropriate sense) the flow converges to some limit branched minimal immersion (u, g) (or a constant map) as $t \to \infty$. This was done in [21]. Note that we cannot expect this to be (global) smooth convergence in general, as in particular certain homotopy classes of maps do not contain any harmonic maps, and thus do not contain any branched minimal immersions (i.e. non-constant weakly harmonic maps) either (e.g. [7]). As already described in Theorem 2.2.2, finite time map singularities in the form of bubbles developing may form, and similarly bubbles can develop at infinite time.

To analyse limits of the flow (2.21) it is convenient to make the following definition. This is taken from [13].

Definition 2.2.3. Given an oriented closed surface M, a closed Riemannian manifold (N, G), and a pair of sequences $u_i : M \to N$ of smooth maps and g_i of metrics on M with fixed constant curvature and fixed area, we say that (u_i, g_i) is a *sequence of almost-minimal maps* if $E(u_i, g_i)$ is uniformly bounded and

$$\|\tau_{g_i}(u_i)\|_{L^2(M,g_i)} \to 0, \qquad \text{and} \qquad \|P_{g_i}(\Phi(u_i, g_i))\|_{L^2(M,g_i)} \to 0. \tag{2.27}$$

We will apply this when M is a hyperbolic surface and the g_i lie in $\mathcal{M}_{-1}$.

To motivate this definition, recall that we can view E as a functional on the space of maps and metrics modulo diffeomorphisms isotopic to the identity, and compute the gradient as in (2.19). We then see that the gradient of E converges to 0 along sequences of almost-minimal maps, and we classified critical points of E as branched minimal immersions (or constant maps, see also [21]). The assumption of bounded energy is added to enable us to prove certain compactness statements for sequences of almost-minimal maps.

As a consequence of the energy identity (2.22) we can extract such sequences from any globally defined Teichmüller harmonic map flow.

Proposition 2.2.4. *Consider a solution (in the sense of Theorem 2.2.2) $(u(t), g(t))$ to (2.21) which is defined up to $T = \infty$. Then we can find a sequence $t_i \to \infty$ such that $(u_i := u(t_i), g_i := g(t_i))$ is a sequence of almost-minimal maps.*

Proof. This argument is from [21]. By integrating the energy identity (2.22) we find

$$\int_0^\infty \int_M |\tau_g(u)|^2 + \left(\frac{\eta}{4}\right)^2 |Re(P_g(\Phi(u,g)))|^2 dv_g dt \le E(u(0)) - \lim_{t\to\infty} E(u(t)) < \infty \tag{2.28}$$

where we used that E is non-increasing (by (2.19)). Note that the flow $(u(t), g(t))$ is smooth away from at most finitely many times, so we can necessarily extract a (smooth) sequence of almost-minimal maps, as otherwise the integral on the left would not converge. Here we used that (e.g. [13])

$$\|P_{g_i}(\Phi(u_i,g_i))\|^2_{L^2(M,g_i)} = 2\|Re(P_{g_i}(\Phi(u_i,g_i)))\|^2_{L^2(M,g_i)}.$$

□

Having defined such sequences of almost-minimal maps we can give the following compactness statement from [21], where we assume that the metric stays uniformly controlled.

Theorem 2.2.5 (Content from [21, Theorem 1.4])**.** *Let M be a smooth closed oriented surface of genus $\gamma \ge 2$, (N,G) be a smooth closed Riemannian manifold, and consider a sequence (u_i, g_i) of almost-minimal maps in the sense of Definition 2.2.3, for which there exists $\epsilon > 0$ such that $\ell(g_i) \ge \epsilon$ for all i.*

Then, after passing to a subsequence in i, there exists a sequence of orientation-preserving diffeomorphisms $f_i : M \to M$, a hyperbolic metric $\bar{g}$ on M, a weakly conformal harmonic map $\bar{u} : (M,\bar{g}) \to N$ and a finite set of points $S \subset M$ such that

1. *$f_i^*(g_i) \to \bar{g}$ smoothly;*
2. *$u_i \circ f_i \rightharpoonup \bar{u}$ weakly in $H^1(M)$;*
3. *$u_i \circ f_i \to \bar{u}$ strongly in $W^{1,p}_{loc}(M \setminus S)$ for any $p \in [1,\infty)$, and thus also in $C^0_{loc}(M \setminus S)$;*
4. *the map $\bar{u}$ has the same action on $\pi_1(M)$ as u_0.*

At each point in the singular set S a bubble develops, in the sense that energy concentrates and we can extract a harmonic map $\omega : S^2 \to N$, as described in Section 2.2.1.

Thus, by Proposition 2.2.4 we indeed find that the Teichmüller harmonic map flow transforms an initial map into a branched minimal immersion (or constant map) if no degeneration of the metric is present.

We highlight some of the ideas involved in the proof of Theorem 2.2.5. The first main point is that the hyperbolic metrics $\{g_i\}$ of the almost-minimal sequence in the above theorem form a compact set in the moduli space of metrics, as we assumed $\ell(g_i) \geq \epsilon$, by the Mumford compactness theorem.

Theorem 2.2.6 (Mumford compactness, e.g. [33, Appendix C])**.** *Let $\epsilon > 0$ and $g_i \in \mathcal{M}_{-1}$ be such that $\ell(g_i) \geq \epsilon$, then, after passing to a subsequence in i, there exists a sequence of orientation-preserving diffeomorphisms $f_i : M \to M$ and $\bar{g} \in \mathcal{M}_{-1}$ such that $f_i^* g_i \to \bar{g}$ smoothly.*

The other key point is the following so-called elliptic-Poincaré inequality for a quadratic differential Ψ on a closed oriented surface (M, g) ([21, Lemma 2.1]), which is in particular valid in our setting of M having genus $\gamma \geq 2$ and $g \in \mathcal{M}_{-1}$:

$$\|\Psi - P_g(\Psi)\|_{L^1(M,g)} \leq C \left\|\bar{\partial}(\Psi)\right\|_{L^1(M,g)}. \tag{2.29}$$

Here the constant C depends on the genus of the surface M, and on $\ell(g)$ - although it turns out that for $\gamma \geq 2$, this dependency can be removed, as we will discuss in the next section.

This allows us to deduce $\|\Phi(u_i, g_i)\|_{L^1(M,g_i)} \to 0$ for a sequence (u_i, g_i) of almost-minimal maps by applying the above inequality with $\Psi = \Phi(u_i, g_i)$ and noting that (e.g. [21, Section 3])

$$\left\|\bar{\partial}(\Phi(u_i, g_i))\right\|_{L^1(M,g_i)} \leq \sqrt{2} \left\|\tau_{g_i}(u_i)\right\|_{L^2(M,g_i)} E(u_i, g_i)^{\frac{1}{2}}. \tag{2.30}$$

The authors of [21] then carry out a ‘bubbling’ argument, modified to allow for the changing metric, which is essentially proving a compactness statement for maps with tension converging to 0 in L^2 (see also Lemma 3.1.2 for a corresponding result with respect to a fixed metric). This way one obtains a limit harmonic map, and using the elliptic-Poincaré inequality one finds that it is also weakly conformal.

2.2.3 Metric degeneration at infinity

We now consider the behaviour of the flow (2.21) when we allow degeneration of the metric, but only at infinite time, i.e. $\liminf_{t\to\infty} \ell(g(t)) = 0$. This was studied in [25].

In addition to Proposition 2.2.4 we can now extract a sequence of almost-minimal maps (u_i, g_i) such that also $\lim_{i\to\infty} \ell(g_i) = 0$.

Proposition 2.2.7 ([13, Proposition 1.2])**.** *Given an oriented closed surface M of genus $\gamma \geq 2$, a closed Riemannian manifold (N, G), and a smooth flow (u, g) solving (2.21) for which $\liminf_{t\to\infty} \ell(g(t)) = 0$, there exists a sequence $t_i \to \infty$ such that $\lim_{i\to\infty} \ell(g(t_i)) = 0$ and $(u(t_i), g(t_i))$ is a sequence of almost-minimal maps.*

Proof from [13]. As in Proposition 2.2.4, we can extract a sequence of times $t_i \to \infty$ at which $(u(t_i), g(t_i))$ is a sequence of almost-minimal maps. We have to adjust the times t_i to ensure that $\ell(g(t_i)) \to 0$ as $i \to \infty$ (by virtue of the hypothesis $\liminf_{t\to\infty} \ell(g(t)) = 0$) and thus must argue that it is impossible for $\ell(g(t))$ to spend almost all of the time away from zero, but drop quickly and occasionally down near zero. To do this, we pick times $\tilde{t}_i \to \infty$ with $\tilde{t}_{i+1} \geq \tilde{t}_i + 1$ such that $\ell(g(\tilde{t}_i)) \leq 1/i$ for sufficiently large i, and claim that $\ell(g(t)) \leq C/i$ for $t \in [\tilde{t}_i, \tilde{t}_i + 1/i]$, with C depending only on the genus γ, the coupling constant η and an upper bound E_0 for the energy. If this claim were true then we would be able to pick our sequence of times t_i from the set $\cup_i[\tilde{t}_i, \tilde{t}_i + 1/i]$, which has infinite measure, in the usual way.

When the genus of M is at least 2, the claim follows from [20, Lemma 2.3], which implies in particular the Lipschitz bound

$$\left| \frac{d}{dt} \ell(g(t)) \right| \leq C(\gamma, \eta, E_0),$$

whenever $\ell < 2 \operatorname{arsinh}(1)$. See also the proof of Lemma 4.2.3 where we explicitly calculate the η-dependence of this bound. $\square$

We remark that we may assume the flow to be smooth in this proposition: given any weak solution as defined in 2.2.2 it will necessarily be smooth for sufficiently large times as there can only be a finite number of singular times.

To obtain a compactness statement for such sequences of almost-minimal maps in this setting of degenerating metrics one has to allow more general *hyperbolic punctured surfaces* (see the theorem below for a description) as the limit domain. The following theorem was then proved in [25].

Theorem 2.2.8 (Content from [25])**.** *Suppose we have an oriented closed surface M of genus $\gamma \geq 2$, a closed Riemannian manifold (N, G), and a sequence (u_i, g_i) of almost-minimal maps in the sense of Definition 2.2.3, for which $\lim_{i\to\infty} \ell(g_i) = 0$.*

Then after passing to a subsequence, there exist an integer $1 \leq k \leq 3(\gamma - 1)$ and a hyperbolic punctured surface (Σ, h, c) with $2k$ punctures (i.e. a closed Riemann surface $\hat{\Sigma}$ with complex structure $\hat{c}$, possibly disconnected, that is then punctured $2k$ times to give a Riemann surface Σ with c the restricted complex structure, which is then equipped with a conformal complete hyperbolic metric h) such that the following holds.

1. *The surfaces (M, g_i, c_i) converge to the surface (Σ, h, c) by collapsing k simple closed geodesics σ_i^j in the sense of Proposition A.2 from the appendix; in particular there is a sequence of diffeomorphisms $f_i : \Sigma \to M \setminus \cup_{j=1}^k \sigma_i^j$ such that*
$$f_i^* g_i \to h \text{ and } f_i^* c_i \to c \text{ smoothly locally,}$$
where c_i denotes the complex structure of (M, g_i).

2. *The maps $U_i := u_i \circ f_i$ converge to a limit u_∞ weakly in $W^{1,2}_{loc}(\Sigma)$ and weakly in $W^{2,2}_{loc}(\Sigma \setminus S)$ as well as strongly in $W^{1,p}_{loc}(\Sigma \setminus S)$, $p \in [1, \infty)$, away from a finite set of points $S \subset \Sigma$ at which energy concentrates.*

3. *The limit $u_\infty : \Sigma \to N$ extends to a smooth branched minimal immersion (or constant map) on each component of the compactification $(\hat{\Sigma}, \hat{c})$ of (Σ, c) obtained by filling in each of the $2k$ punctures.*

Similarly to the proof of Theorem 2.2.5, the first step in proving Theorem 2.2.8 in [25] is a compactness statement for hyperbolic metrics of Deligne-Mumford type, now allowing for degeneration of the metrics, see Lemma A.2 for the precise statement. The main point is that as in the above theorem, a collection of simple closed geodesics is allowed to collapse, thus forming a punctured surface in the limit.

The authors of [25] then carry out a similar bubbling analysis to [21] to find a harmonic limit map from a punctured surface Σ. To deduce that this limit map is conformal, one could then use a uniform version of the Poincaré estimate (2.29), established in [24], where the constant C in particular was shown to only depend on the genus γ. This relies on a careful analysis of the structure of the space of holomorphic quadratic differentials on degenerating surfaces.

We will revisit Theorem 2.2.8 in the next chapter, where we provide more information on the asymptotics of the convergence to the limit map, in particular proving that no energy can be lost in the limit after accounting for all possible bubbles developing.

2.2.4 Targets with nonpositive sectional curvature

We now turn our attention to targets $N = (N, G)$ with nonpositive sectional curvature, which were studied in [20]. For the harmonic map flow (2.21) it was shown in [8] that this leads to smooth long-time existence, i.e. Theorem 2.1.1. The curvature condition for N in particular stops bubbles from developing, in the sense that there cannot be any (non-constant) harmonic maps $\omega : S^2 \to N$ (e.g. [20]). One thus sees from Theorem 2.2.2 that the Teichmüller harmonic map flow will necessarily stay smooth, as long as the metric does not degenerate.

The difficulty is showing that the metric cannot degenerate (i.e. $\ell(g(t))$ is bounded away from 0 for all finite times t). This was proved in [20], leading to the following theorem, analogous to Theorem 2.1.1.

Theorem 2.2.9 ([20, Theorem 1.1])**.** *Let M be a smooth closed oriented surface of genus $\gamma \geq 2$ and $N = (N, G)$ a smooth closed Riemannian manifold with nonpositive sectional curvature. Then for any initial data $(u_0, g_0) \in C^\infty(M, N) \times \mathcal{M}_{-1}$, there exists a smooth solution $(u(t), g(t))$ to (2.21), for $t \in [0, \infty)$.*

We remark that it is actually sufficient to assume that N *supports no bubbles*, i.e. that there are no nonconstant harmonic maps $\omega : S^2 \to N$ (see [20]).

To prove Theorem 2.2.9 the evolution of $\ell(g(t))$ along the flow is controlled in [20]. The starting point is a decomposition of hyperbolic surfaces into so-called *δ-thick* and *δ-thin* parts, referring to regions of injectivity radius $\mathrm{inj}_g \geq \delta$ and $\mathrm{inj}_g < \delta$ respectively. It turns out that for sufficiently small δ, the δ-thin part of M is given by a collection of *hyperbolic collars*. Each such collar $\mathcal{C}(l)$ is a cylindrical region around a simple closed geodesic of length l, which can be viewed as a cylinder $\mathscr{C}(-X(l), X(l)) = [-X(l), X(l)] \times S^1$ equipped with a metric conformal to the usual euclidean metric. The conformal factor, as well as the length $X(l)$ of the collar, can be computed explicitly in terms of l, with formulas given by the 'Collar lemma' A.1.

Given a solution $(u(t), g(t))$ to the flow (2.21) and a collar $\mathcal{C}(l)$ at some time t_0, an estimate for $\frac{\mathrm{d}}{\mathrm{d}t} l(t)|_{t=t_0}$ is proved in [20]. We give this result in Chapter 4 (see Lemma 4.2.1), where we in particular analyse how it depends on the coupling constant η. Using this estimate and controlling a so-called 'weighted energy', degeneration of the metric is ruled out, leading to Theorem 2.2.9.

The full proof is somewhat complex, and we only highlight one particular estimate obtained in [20] which we independently apply later on. Recall that the projection operator $P_g : \mathcal{Q}(M,g) \to \mathcal{H}(M,g)$ was defined as the L^2-orthogonal projection from the space of quadratic differentials (e.g. including the Hopf differential) onto the space of holomorphic quadratic differentials. It is therefore clearly bounded as an $L^2 - L^2$ operator. However, in [20] it was shown that one further has the $L^1 - L^1$ bound

$$\|P_g(\Psi)\|_{L^1(M,g)} \leq C \|\Psi\|_{L^1(M,g)}, \tag{2.31}$$

for any quadratic differential $\Psi \in \mathcal{Q}(M,g)$ with a constant $C < \infty$ *only* depending on the genus γ of M. This is a very useful estimate when working with the flow (2.21), as with $\Psi = \Phi(u,g)$ one has the estimate $\|\Phi(u,g)\|_{L^1(M,g)} \leq CE(u(0),g(0))$. Hence in particular one finds that

$$\|\partial_t g\|_{L^1(M,g)} = \left\| \frac{\eta^2}{4} Re(P_g(\Phi(u,g))) \right\|_{L^1(M,g)} \leq C\frac{\eta^2}{4} E(u(0),g(0)). \tag{2.32}$$

But indeed even more is true: as a consequence of the fact that the C^k-norms of a holomorphic function are controlled by its L^1-norm, we even have

$$\|\partial_t g\|_{C^k(M,g)} \leq C \|\partial_t g\|_{L^1(M,g)} \leq C\frac{\eta^2}{4} E(u(0),g(0)) \tag{2.33}$$

although C now also depends on $\ell(g)$ (see e.g. [25, Lemma A.9]). We will apply this several times in chapter 4, in particular in the proof of Lemma 4.2.8.

Even more refined estimates for the evolution of the metric under the flow (2.21) (and more generally for horizontal curves of metrics, i.e. metrics moving in the direction of a holomorphic quadratic differential) have been obtained in [22].

2.3 Some properties of harmonic maps

Let (M,g) and (N,G) be smooth closed oriented Riemannian manifolds, with N having nonpositive sectional curvature. We require some properties of harmonic maps $u : M \to N$ for applications in Chapter 4. Note that the results of this section are valid for higher-dimensional domains M, but we will only need them in the case of M being a surface later on.

2.3.1 Uniqueness in a homotopy class

By Theorem 2.1.1 we know that any smooth map $u_0 : M \to N$ can be smoothly deformed into a harmonic map $u : M \to N$ (with respect to the metric g) by evolving it under the harmonic map flow, and hence in particular any homotopy class of maps $M \to N$ contains (at least) one harmonic representative. We are interested in situations when u_0 is the unique harmonic map homotopic to u (again, with respect to g). We note that there are two obvious obstructions to uniqueness:

- u_0 is a constant map;
- u_0 maps to a closed geodesic σ, as we can 'rotate' each point in the image by a fixed amount on σ.

Without any further assumptions on N, a slightly more general phenomenon can happen. Indeed, imagine N to be a two-dimensional torus, which we view as $N = S^1 \times S^1$. Now take u_0 to map to one of the S^1 (at unit speed) and rotate it around the other S^1. This will keep the map harmonic, as the image stays a geodesic. Hence it possible to 'translate' u_0 through a family of harmonic maps.

As shown by Hartman in [10] this is the only way that uniqueness can fail.

Theorem 2.3.1 ([10])**.** *Let M, N be as above. If u_0, u_1 are homotopic harmonic maps $M \to N$, then there exists a C^∞ homotopy $u(x,s) : M \times [0,1] \to N$ with $u(\cdot,0) = u_0$ and $u(\cdot,1) = u_1$ such that:*

1. *For all fixed s, $u(\cdot,s) : M \to N$ is a harmonic map;*
2. *For fixed x, the arc $u(x,s)$ is a geodesic arc, with length independent of x, and u proportional to arc length.*

If one further assumes that N has strictly negative sectional curvature, it was shown in [10] that this simplifies to the two initial obstructions (i.e. mapping to a constant or closed geodesic).

Theorem 2.3.2 ([10])**.** *Let M as before and additionally assume that N has strictly negative sectional curvature. If u_0, u_1 are homotopic harmonic maps $M \to N$, then either $u_0 \equiv u_1$ or one the following is true:*

1. *The images $u_0(M)$ and $u_1(M)$ are both points.*
2. *The images $u_0(M)$ and $u_1(M)$ are both equal to some closed geodesic $\sigma \subset N$.*

Remark 2.3.3. We will use this setting of maps u_0 not homotopic to constant maps or maps to closed geodesics in Chapter 4 to construct certain limits of the flow (2.21). In particular, it allows us to take some given curve of metrics $g(t) : [0, T] \to \mathcal{M}_{-1}$ and some initial map $u_0 : M \to N$ and find a curve of harmonic maps $u(t) : [0, T] \to N$ such that each $u(t) : (M, g(t)) \to N$ is the unique harmonic map (with respect to $g(t)$) homotopic to u_0 by Theorem 2.3.2. In the next section we study how the harmonic maps $u(t)$ change as we change the underlying metric.

2.3.2 Dependence on the domain metric

Consider M and N as above and additionally assume again that N has strictly negative sectional curvature. Given a smooth map $u : M \to N$ (satisfying the topology assumptions from Remark 2.3.3) it induces a map from metrics to harmonic maps, as described in the remark. The problem of the dependence of harmonic maps on their domain metrics was considered in [6] (the authors also study deformations of the target metric, but we do not need that here). Let $k \in \mathbb{N}$, $\alpha \in (0, 1)$ and consider the space $\mathcal{M}^{k,\alpha}$ of $C^{k,\alpha}$-metrics on M, which is in particular a subset of $C^{k,\alpha}(\mathrm{Sym}^2 T^*M)$. We note that given any metric g on M we can define a norm on $C^{k,\alpha}(\mathrm{Sym}^2 T^*M)$ with respect to g, which we will denote by $\|\cdot\|_{C^{k,\alpha}(M,g)}$, which also induces a norm on $\mathcal{M}^{k,\alpha}$. Similarly we consider the space $C^{k,\alpha}(M, N)$ of maps $u : M \to N$ with derivatives of order $\leq k$ being α-Hölder continuous, which again we can equip with a norm with respect to any metric g on M. In this setting we can state the following theorem from [6].

Theorem 2.3.4 (Special case of [6, Theorem 3.1])**.** *Let (M, g_0) and (N, h) be smooth closed Riemannian manifolds, with N having strictly negative sectional curvature. Consider a smooth harmonic map $u_0 : (M, g_0) \to (N, h)$ such that $u_0(M)$ is not a point or a geodesic in N. Then for integers $0 \leq k < \infty$ there is a neighbourhood V of $g_0 \in \mathcal{M}^{k+1,\alpha}$ and a unique C^1-map $S : V \to C^{k+2,\alpha}(M, N)$ such that $S(g_0) = u_0$ and $S(g) : (M, g) \to (N, h)$ is a harmonic map for each $g \in V$.*

Remark 2.3.5. By C^1 we mean that S is a continuously differentiable map between Banach spaces, in particular between $C^{k+1,\alpha}(\mathrm{Sym}^2 T^*M)$ and $C^{k+2,\alpha}(M, N)$. Note that more is true: indeed, for any $r \in \mathbb{N}$ one can find such a neighbourhood V with the map S being C^r, but C^1 is sufficient for our purposes (again, see [6]).

As a consequence of this Theorem we obtain the following simplified corollary, which will be the statement needed in Chapter 4.

Corollary 2.3.6. *In the situation of Theorem 2.3.4, the map S is Lipschitz, and for any metric $g \in V$ we find*

$$\|S(g_1) - S(g_2)\|_{C^{k+1}} \leq C \, \|g_1 - g_2\|_{C^{k+1}} \tag{2.34}$$

with some constant $C < \infty$, depending on u_0, g_0, M, N and k.

Proof. This follows by simply replacing the Hölder norms with appropriate C^k-norms through standard embeddings ($C^{k+1}(M,N) \hookrightarrow C^{k,\alpha}(M,N) \hookrightarrow C^k(M,N)$ etc., as M and N are smooth). $\square$

Chapter 3

Refined asymptotics

This chapter is taken from [13], and is joint work with Melanie Rupflin and Peter Topping.

3.1 Overview

Let M be a smooth closed oriented surface of genus $\gamma \geq 2$, and consider sequences of almost-minimal maps (as defined in 2.2.3) (u_i, g_i). If $\liminf_{i\to\infty} \ell(g_i) = 0$ then the theory of [25], as outlined in the last section, provides us with a compactness statement, Theorem 2.2.8, for such sequences.

In this chapter, we take this analysis of the asymptotics of (u_i, g_i) and we refine it in several ways. First, after passing to a further subsequence, we extract all bubbles that can develop. What is well understood is that we can extract bubbles at each of the points in S (where possibly multiple bubbles can develop). In what follows we will call these bubbles $\{\omega_k\}$. Our first task is to isolate a new set of bubbles, called $\{\Omega_j\}$ below, that are disappearing into the $2k$ punctures found in Theorem 2.2.8, or equivalently (as we describe below), being lost down the one or more collars that degenerate in the domain (M, g_i) as $i \to \infty$.

Having extracted the complete set of bubbles, we show that the chosen subsequence enjoys a no-loss-of-energy property in which the limit $\lim_{i\to\infty} E(u_i, g_i)$ is precisely equal to the sum of the energies (or equivalently areas) of the branched minimal immersions found in Theorem 2.2.8 and the new branched minimal immersions obtained as bubbles. A special case of what we prove below in Theorem 3.1.8, combined with existing theory, is the following result. (Recall that δ-thick(M,g) consists of all points in M at which the injectivity radius is at least δ. Its complement is δ-thin(M,g).)

Theorem 3.1.1. *In the setting of Theorem 2.2.8, there exist two finite collections of nonconstant harmonic maps $\{\omega_k\}$ and $\{\Omega_j\}$ mapping $S^2 \to N$, such that after passing to a subsequence we have*

$$\lim_{\delta\downarrow 0}\lim_{i\to\infty} E\left(u_i, g_i; \delta\text{-}thick\,(M, g_i)\right) = E(u_\infty, h) + \sum_k E(\omega_k),$$

and

$$\lim_{i\to\infty} E(u_i, g_i) = E(u_\infty, h) + \sum_k E(\omega_k) + \sum_j E(\Omega_j).$$

Showing that no loss of energy occurs in intermediate regions around the bubbles developing at points in S is standard, following in particular the work of Ding and Tian [5] we describe in a moment, although one could also use energy decay estimates of the form we prove in this chapter. However, showing that no energy is lost near the $2k$ punctures, away from where the bubbles develop, is different, and a key ingredient is the Poincaré estimate for quadratic differentials discovered in [24], which is applied globally, not locally where the energy is being controlled. In this step we exploit the smallness of $P_g(\Phi(u,g))$ that holds for almost-minimal maps. That this is essential is demonstrated by the work of T. Parker [17] and M. Zhu [34], which established that energy *can* be lost along 'degenerating collars' in general sequences of harmonic maps from degenerating domains.

The following is the foundational compactness result when the domain is fixed, cf. [27, 5, 18, 15, 29, 30].

Theorem 3.1.2. *Suppose (Υ, g_0) is a fixed surface, possibly noncompact, possibly incomplete, and let u_i be a sequence of smooth maps into (N, G) from either (Υ, g_0), or more generally from a sequence of subsets $\Upsilon_i \subset \Upsilon$ that exhaust Υ. Suppose that $E(u_i) \leq E_0$ and that $\|\tau_{g_0}(u_i)\|_{L^2} \to 0$ as $i \to \infty$. Then there is a subsequence for which the following holds true.*

There exist a smooth harmonic map $u_\infty : (\Upsilon, g_0) \to (N, G)$ (possibly constant) and a finite set of points $S \subset \Upsilon$ such that we have, as $i \to \infty$,

$$u_i \to u_\infty \quad in\ W^{2,2}_{loc}(\Upsilon\backslash S, N),\ \ and$$

$$u_i \rightharpoonup u_\infty \quad \textit{weakly in } W^{1,2}_{loc}(\Upsilon, N).$$

At each point in S, a bubble tree develops in the following sense. After picking local isothermal coordinates centred at the given point in S, there exist a finite number of nonconstant harmonic maps $\omega_j : S^2 \to (N, G)$, for $j \in \{1, \ldots, J\}$, $J \in \mathbb{N}$ (so-called bubbles) which we view as maps from $\mathbb{R}^2 \cup \{\infty\}$ via stereographic projection, and sequences of numbers $\lambda_i^j \downarrow 0$ and coordinates $a_i^j \to 0 \in \mathbb{R}^2$, such that

$$u_i\left(a_i^j + \lambda_i^j x\right) \rightharpoonup \omega_j \qquad \textit{weakly in } W^{1,2}_{loc}(\mathbb{R}^2, N).$$

Moreover, we do not count bubbles more than once in the sense that

$$\frac{\lambda_i^j}{\lambda_i^k} + \frac{\lambda_i^k}{\lambda_i^j} + \frac{|a_i^j - a_i^k|^2}{\lambda_i^j \lambda_i^k} \to \infty, \tag{3.1}$$

for each $j, k \in \{1, \ldots, J\}$ with $j \neq k$.

The bubbling has no energy loss in the sense that for each point $x_0 \in S$ analysed as above, and each neighbourhood $U \subset\subset \Upsilon$ of x_0 such that $\overline{U} \cap S = \{x_0\}$ only, we have

$$\lim_{i\to\infty} E\left(u_i; U\right) = E(u_\infty; U) + \sum_j E(\omega_j).$$

Moreover, the bubbling enjoys the no-necks property

$$u_i(x) - \sum_j \left(\omega_j\left(\frac{x - a_i^j}{\lambda_i^j}\right) - \omega_j(\infty)\right) \to u_\infty(x) \tag{3.2}$$

in $L^\infty(U)$ and $W^{1,2}(U)$ as $i \to \infty$.

Remark 3.1.3. We note that the proof of the first part of Theorem 3.1.1 (virtually) immediately follows from Theorems 2.2.8 and 3.1.2: Away from S we can combine the strong $W^{1,2}$- convergence of the maps with the convergence of the metrics. To analyse the maps $U_i = u_i \circ f_i$ near points in S we then apply Theorem 3.1.2 on small geodesic balls $B_r^{f_i^* g_i}(p) \subset (\Sigma, f_i^* g_i)$, which are of course isometric to one another provided $r > 0$ is chosen sufficiently small as the metrics g_i are all hyperbolic. Finally, the convergence of the metrics allows us to relate the δ-thick part of $(\Sigma, f_i^* g_i)$ to the δ-thick part of (Σ, h), compare [25, Lemma A.7], as well as the geodesic balls $B_r^{f_i^* g_i}(p)$ in $(\Sigma, f_i^* g_i)$ to geodesic balls in (Σ, h). This completes the proof of the first part of Theorem 3.1.1.

Remark 3.1.4. To do more, we must recall more about the structure of sequences of degenerating hyperbolic metrics, and in particular we need the precise description of the metrics g_i near to the geodesics σ_i^j of Theorem 2.2.8 given by the Collar Lemma A.1 in the appendix. In particular, for $\delta \in (0, \operatorname{arsinh}(1))$ sufficiently small, the δ-thin part of (M, g_i) is isometric to a finite disjoint union of cylinders $\mathcal{C}_i^{\delta,j} := (-X_\delta(\ell_i^j), X_\delta(\ell_i^j)) \times S^1$ with the metric from Lemma A.1; each cylinder has a geodesic σ_i^j at the centre, with length $\ell_i^j \to 0$ as $i \to \infty$. These initial observations motivate us to analyse in detail almost-harmonic maps from cylinders.

Definition 3.1.5. When we apply Theorem 3.1.2 in the case that $(\Upsilon, g_0) = \mathbb{R} \times S^1$ is the cylinder with its standard flat metric, then we say that the maps u_i *converge to a bubble branch*, and extract bubbles $\{\Omega_j\}$ as follows. First we add all the bubbles $\{\omega_j\}$ to the list $\{\Omega_j\}$. In the case that $u_\infty : \mathbb{R} \times S^1 \to N$ is nonconstant, we view it (via a conformal map of the domain) as a harmonic map from the twice punctured 2-sphere, remove the two singularities (using the Sacks-Uhlenbeck removable singularity theorem [26]) to give a smooth nonconstant harmonic map from S^2, and add it to the list $\{\Omega_j\}$. We say that u_i converges to a *nontrivial* bubble branch if the collection $\{\Omega_j\}$ is nonempty.

We use the term 'bubble branch' alone to informally refer to the collection of bubbles together with the limit u_∞.

In this chapter we prove a refinement of the above convergence to a bubble branch. To state this result, we shall use the following notations: For $a < b$, define $\mathscr{C}(a, b) := (a, b) \times S^1$ to be the finite cylinder which will be equipped with the standard flat metric $g_0 = ds^2 + d\theta^2$ unless specified otherwise. For $\Lambda > 0$ we write for short $\mathscr{C}_\Lambda = \mathscr{C}(-\Lambda, \Lambda)$. Furthermore, given sequences a_i and b_i of real numbers we write $a_i \ll b_i$ if $a_i < b_i$ for all $i \in \mathbb{N}$ and $b_i - a_i \to \infty$ as $i \to \infty$.

Theorem 3.1.6. *Let $X_i \to \infty$ and let $u_i : \mathscr{C}_{X_i} \to N$ be a sequence of smooth maps with uniformly bounded energy, $E(u_i; \mathscr{C}_{X_i}) \leq E_0 < \infty$, which are almost harmonic in the sense that*

$$\|\tau_{g_0}(u_i)\|_{L^2(\mathscr{C}_{X_i})} \to 0. \tag{3.3}$$

Then after passing to a subsequence in i, there exist a finite number of sequences s_i^m (for $m \in \{0, \ldots, \bar{m}\}$, $\bar{m} \in \mathbb{N}$) with $-X_i =: s_i^0 \ll s_i^1 \ll \cdots \ll s_i^{\bar{m}} := X_i$ such that the following holds true.

1. *For each $m \in \{1, \ldots, \bar{m} - 1\}$ (if nonempty) the translated maps $u_i^m(s, \theta) := u_i(s + s_i^m, \theta)$ converge to a nontrival bubble branch in the sense of Definition*

3.1.5.

2. *The* connecting cylinders $\mathscr{C}(s_i^{m-1}+\lambda, s_i^m-\lambda)$, *$\lambda$ large, are mapped near curves in the sense that*

$$\lim_{\lambda\to\infty}\limsup_{i\to\infty}\sup_{s\in(s_i^{m-1}+\lambda,s_i^m-\lambda)}\operatorname{osc}(u_i;\{s\}\times S^1)=0, \tag{3.4}$$

for each $m\in\{1,\dots,\bar{m}\}$.

3. *If we suppose in addition that the Hopf-differentials tend to zero*

$$\|\Phi(u_i)\|_{L^1(\mathscr{C}_{X_i})}\to 0 \tag{3.5}$$

then there is no loss of energy on the connecting cylinders $\mathscr{C}(s_i^{m-1},s_i^m)$ in the sense that for each $m\in\{1,\dots,\bar{m}\}$ we have

$$\lim_{\lambda\to\infty}\limsup_{i\to\infty}E(u_i;\mathscr{C}(s_i^{m-1}+\lambda,s_i^m-\lambda))=0. \tag{3.6}$$

Definition 3.1.7. In the setting of Theorem 3.1.6, we abbreviate the conclusions of parts 1 and 2 by saying that the maps u_i *converge to a full bubble branch.* In the case that (3.6) also holds (i.e. the conclusion of part 3) we say that the maps u_i *converge to a full bubble branch with no loss of energy.*

Returning to the observations of Remark 3.1.4, we note that the length of each of the cylinders $\mathcal{C}_i^{\delta,j}$ is converging to infinity, and that any fixed length portion of either end of any of these cylinders will lie within the $\hat{\delta}$-thick part of (M,g_i) for some small $\hat{\delta}\in(0,\delta)$, and thus be captured by the limit u_∞ from Theorem 2.2.8. Our main no-loss-of-energy result can therefore be stated as the following result about the limiting behaviour on the middle of the collars, which constitutes our main theorem.

Theorem 3.1.8. *In the setting of Theorem 2.2.8, we fix $j\in\{1,\dots,k\}$ in order to analyse the j^{th} collar surrounding the geodesic σ_i^j. Now that j is fixed, we drop it as a label for simplicity. Thus we consider the collar $\mathcal{C}(\ell_i)=(-X(\ell_i),X(\ell_i))\times S^1$, with its hyperbolic metric, where $X(\ell_i)\to\infty$.*

Then after passing to a subsequence, the restrictions of the maps u_i to the collars $\mathcal{C}(\ell_i)$ converge to a full bubble branch with no loss of energy in the sense of Definition 3.1.7.

Remark 3.1.9. Similar results have been obtained when the domain M is a torus, details can be found in the paper [13].

Theorem 3.1.8 indirectly describe the map u_i on 'connecting cylinders' as being close to an i-dependent curve, thanks to (3.4). We are not claiming that this curve has zero length in the limit, as is the case in some similar situations, e.g. for necks in harmonic maps [17] and the harmonic map flow from fixed domains [18]. We are also not claiming that in some limit the curve should satisfy an equation, for example that it might always be a geodesic as would be the case for sequences of harmonic maps from degenerating surfaces, see [3]. The following construction can be used to show that these claims would be false in general.

Proposition 3.1.10. *Given a closed Riemannian manifold (N, G), a C^2 unit-speed curve $\alpha : [-L/2, L/2] \to N$, and any sequence of degenerating hyperbolic collars $\mathscr{C}_{X_i}$, $X_i \to \infty$, equipped with their collar metrics g_i as in the Collar Lemma A.1, the maps $u_i : \mathscr{C}_{X_i} \to N$ defined by*

$$u_i(s, \theta) = \alpha\left(\frac{Ls}{2X_i}\right)$$

satisfy

$$E(u_i; \mathscr{C}_{X_i}) \leq \frac{CL^2}{X_i} \to 0,$$

$$\|\tau_{g_i}(u_i)\|^2_{L^2(\mathscr{C}_{X_i}, g_i)} \leq C\frac{L^4}{X_i} \to 0$$

and

$$\|\Phi(u_i, g_i)\|^2_{L^2(\mathscr{C}_{X_i}, g_i)} \leq C\frac{L^4}{X_i} \to 0.$$

We give the computations in Section 3.4. The proposition can be used to construct a sequence of almost-minimal maps with nontrivial connecting curves. For example, one can take any curve α as above, and any sequence of hyperbolic metrics g_i with a separating collar degenerating, and then take the maps u_i to be essentially constant on either side of this one degenerating collar where the map is modelled on that constructed in the proposition. A slight variation of the construction would show that the i-dependent connecting curve need not have a reasonable limit as $i \to \infty$ in general, whichever subsequence we take, and indeed that its length can converge to infinity as $i \to \infty$.

Finally, we consider the more specific question of what the connecting curves can look like in the case that we are considering the flow (2.21) and we have applied Proposition 2.2.7 to get a sequence of almost-minimal maps $u(t_i) : (M, g(t_i)) \to N$.

One consequence of Theorem 3.1.6 is that for large λ and very large i, the restriction of $u(t_i)$ to the connecting cylinders $\mathscr{C}(s_i^{m-1}+\lambda, s_i^m-\lambda)$ is close in C^0 to curves $\gamma_i(s)$ connecting the end points

$$p_+^{m-1} := \lim_{\lambda\to\infty}\lim_{i\to\infty} u(t_i)(s_i^{m-1}+\lambda, \theta=0) \tag{3.7}$$

and

$$p_-^{m} := \lim_{\lambda\to\infty}\lim_{i\to\infty} u(t_i)(s_i^{m}-\lambda, \theta=0) \tag{3.8}$$

in the images of branched minimal immersions that we have already found. (Note that it is not important to take $\theta = 0$ in these limits. Any sequence θ_i would give the same limits.)

Now that we have restricted to the particular case in which our theory is applied to the Teichmüller harmonic map flow, one might hope to rule out or restrict necks from developing. However, these necks do exist, and we do not have to have $p_+^{m-1} = p_-^m$, as we now explain.

Theorem 3.1.11. *On any oriented closed surface M of genus at least two, there exists a smooth solution of the Teichmüller harmonic map flow into S^1 that develops a nontrivial neck as $t \to \infty$. More precisely, if we extract a sequence of almost-minimal maps $(u(t_i), g(t_i))$ as in Proposition 2.2.7, then we can analyse it with Theorem 3.1.8, and after passing to a further subsequence we obtain*

$$\lim_{\lambda\to\infty}\liminf_{i\to\infty} \operatorname{osc}(u(t_i); \mathscr{C}(s_i^{m-1}+\lambda, s_i^m-\lambda)) > 0 \tag{3.9}$$

for some degenerating collar and some $m \in \{1, \ldots, \bar{m}\}$. Moreover, there exist examples for which

$$p_+^{m-1} \neq p_-^m, \tag{3.10}$$

i.e. at least one neck connects distinct points.

The simplest way of constructing an example as required in the theorem is to arrange that there can be no nonconstant branched minimal immersions in the limit, while preventing the flow from being homotopic to a constant map. The flow then forces a collar to degenerate in the limit $t \to \infty$, and maps it to a curve in the target as we describe in Theorem 3.1.8. The precise construction will be given in Section 3.4. A key ingredient is the regularity theory for flows into nonpositively curved targets developed in [20].

Remark 3.1.12. It would be interesting to prove that in a large class of situations the connecting curves of Theorem 3.1.8, when applied to the Teichmüller harmonic map flow, will have a limit, and that that limit will necessarily be a geodesic. In the case that $M = T^2$, and under the assumption that the total energy converges to zero as $t \to \infty$, Ding-Li-Liu [4] proved that the image of the torus indeed converges to a closed geodesic.

The conclusion of the theory outlined above is a much more refined description of how the flow decomposes an arbitrary map into a collection of branched minimal immersions from lower genus surfaces.

The rest of this chapter is organised as follows: In the next section we derive bounds on the angular part of the energy of almost harmonic maps on long Euclidean cylinders. The main results about almost-minimal maps are then established in Section 3.3, where we first prove Theorem 3.1.6, which then allows us to show Theorem 3.1.8, and as a consequence to complete the proof of Theorem 3.1.1. In Section 3.4 we prove the results on the images of the connecting cylinders stated in Proposition 3.1.10 and Theorem 3.1.11. In the appendix we include the statements of two well-known results for hyperbolic surfaces, the Collar lemma and the Deligne-Mumford compactness theorem, the statements and notations of which are used throughout this chapter.

3.2 Angular energy decay along cylinders for almost-harmonic maps

Throughout this section we consider smooth maps $u : \mathscr{C}_\Lambda \to N \hookrightarrow \mathbb{R}^{N_0}$, $\Lambda > 0$, where $N = (N, G)$ is a compact Riemannian manifold that we isometrically embed in $\mathbb{R}^{N_0}$ and the cylinder $\mathscr{C}_\Lambda$ is equipped with the flat metric $(ds^2 + d\theta^2)$. The tension τ of u is given by

$$\tau := u_{\theta\theta} + u_{ss} + A(u)(u_s, u_s) + A(u)(u_\theta, u_\theta),$$

where $A(u)$ denotes the second fundamental form of the target $N \hookrightarrow \mathbb{R}^{N_0}$.

Our goal is to prove a decay result for almost-harmonic maps from cylinders, forcing the angular energy to be very small on the middle of the cylinder $\mathscr{C}_\Lambda$ when we apply it in the setting of Theorem 3.1.8. This will be done by first controlling the angular energy, defined in terms of the angular energy on circles

$$\vartheta(s) = \vartheta(u, s) := \int_{\{s\}\times S^1} |u_\theta|^2.$$

The proof of the following lemma is very similar to [29, Lemma 2.13], which in turn optimised [15]. Energy decay in such situations arose earlier in [18], and such results for harmonic functions are classical. More sophisticated decay results were required in [20].

Lemma 3.2.1. *For a smooth map $u : \mathscr{C}_\Lambda \to N$ with $E(u; \mathscr{C}_\Lambda) \leq E_0$, there exist $\delta > 0$ and $C \in (0, \infty)$ depending only on N and E_0, such that if*

$$E(u; \mathscr{C}(s-1, s+2)) < \delta \text{ for every } s \text{ such that } \mathscr{C}(s-1, s+2) \subset \mathscr{C}_\Lambda$$

and

$$\|\tau\|^2_{L^2(\mathscr{C}_\Lambda)} < \delta,$$

then for any $s \in (-\Lambda + 1, \Lambda - 1)$ we have

$$\vartheta(s) \leq Ce^{|s|-\Lambda} + \int_{-\Lambda}^{\Lambda} e^{-|s-q|}\mathcal{T}(q)dq \tag{3.11}$$

where

$$\mathcal{T}(s) := \int_{\{s\}\times S^1} |\tau|^2.$$

Furthermore when $1 < \lambda < \Lambda$, we have the angular energy estimate

$$\int_{-\Lambda+\lambda}^{\Lambda-\lambda} \vartheta(s)ds \leq Ce^{-\lambda} + 2\|\tau\|^2_{L^2(\mathscr{C}_\Lambda)}, \tag{3.12}$$

and thus

$$E(u; \mathscr{C}_{\Lambda-\lambda}) \leq Ce^{-\lambda} + 2\|\tau\|^2_{L^2(\mathscr{C}_\Lambda)} + \frac{1}{4}\|\Phi\|_{L^1(\mathscr{C}_{\Lambda-\lambda})}. \tag{3.13}$$

We require a standard 'small-energy' estimate, very similar to e.g. [5, Lemma 2.1] or [29, Lemma 2.9].

Lemma 3.2.2. *There exist constants $\delta_0 \in (0, 1]$ and $C \in (0, \infty)$ depending only on N such that any map $u \in W^{2,2}(\mathscr{C}(-1, 2), N))$ which satisfies $E(u; \mathscr{C}(-1, 2)) < \delta_0$ must obey the inequality*

$$\|u - \bar{u}\|_{W^{2,2}(\mathscr{C}(0,1))} \leq C\left(\|\nabla u\|_{L^2(\mathscr{C}(-1,2))} + \|\tau\|_{L^2(\mathscr{C}(-1,2))}\right)$$

where $\bar{u}$ is the average value of u over $\mathscr{C}(-1,2)$.

Applying the Sobolev Trace Theorem gives the following (cf. [29]):

Corollary 3.2.3. *For any map $u \in C^\infty(\mathscr{C}(-1,2), N)$ satisfying $E(u; \mathscr{C}(-1,2)) < \delta_0$ (where δ_0 originates in Lemma 3.2.2) and for any $s \in (0,1)$, there holds the estimate*

$$\int\limits_{\{s\}\times S^1} (|u_\theta|^2 + |u_s|^2) \leq C \left(\|\nabla u\|_{L^2(\mathscr{C}(-1,2))} + \|\tau\|_{L^2(\mathscr{C}(-1,2))} \right)^2$$

with some constant C, again only depending on N.

We now establish a differential inequality for $\vartheta(s)$. This is similar to [15, Lemma 2.1], but without requiring a bound on $\sup |\nabla u|$. It is proved analogously to [29, Lemma 2.13], working on cylinders instead of annuli and considering a general target N.

Lemma 3.2.4. *There exists a constant $\delta > 0$ depending on N such that for $u \in C^\infty(\mathscr{C}(-1,2), N)$ satisfying $E(u; \mathscr{C}(-1,2)) < \delta$ and $\|\tau\|^2_{L^2(\mathscr{C}(-1,2))} < \delta$, and for any $s \in (0,1)$, we have the differential inequality*

$$\vartheta''(s) \geq \vartheta(s) - 2 \int\limits_{\{s\}\times S^1} |\tau|^2.$$

Proof of Lemma 3.2.4. From the proof of [20, Lemma 3.7] we have the expression

$$\vartheta''(s) = 2 \int_{\{s\}\times S^1} |u_{s\theta}|^2 + |u_{\theta\theta}|^2 - u_{\theta\theta} \cdot \tau + u_{\theta\theta} \left[A(u)(u_s, u_s) + A(u)(u_\theta, u_\theta) \right]. \quad (3.14)$$

We can estimate the penultimate term as in [20] using integration by parts and Young's inequality:

$$\begin{aligned} \left| 2 \int u_{\theta\theta} \cdot [A(u)(u_s, u_s)] \right| &\leq C \int |u_\theta|^2 |u_s|^2 + |u_{s\theta}||u_s||u_\theta| \\ &\leq \int |u_{s\theta}|^2 + C \int |u_\theta|^2 |u_s|^2, \end{aligned}$$

while the final term of (3.14) requires just Young's inequality:

$$\left| 2 \int u_{\theta\theta} \cdot [A(u)(u_\theta, u_\theta)] \right| \leq C \int |u_{\theta\theta}||u_\theta|^2 \leq \frac{1}{4} \int |u_{\theta\theta}|^2 + C \int |u_\theta|^4$$

where C is a constant only depending on N, that is revised at each step. Summing gives

$$2\left|\int_{\{s\}\times S^1} u_{\theta\theta}\cdot[A(u)(u_s,u_s)+A(u)(u_\theta,u_\theta)]\right| \leq C\int_{\{s\}\times S^1}|u_\theta|^2(|u_s|^2+|u_\theta|^2) + \int_{\{s\}\times S^1}|u_{s\theta}|^2+\frac{1}{4}|u_{\theta\theta}|^2. \tag{3.15}$$

To apply Corollary 3.2.3, we can ask that $\delta<\delta_0$, and thus handle the first term on the right-hand side as follows:

$$\begin{aligned}\int_{\{s\}\times S^1}&|u_\theta|^2(|u_s|^2+|u_\theta|^2)\\ &\leq C\sup_{\{s\}\times S^1}|u_\theta|^2\left(\|\nabla u\|_{L^2(\mathscr{C}(-1,2))}+\|\tau\|_{L^2(\mathscr{C}(-1,2))}\right)^2\\ &\leq C\left(\int_{\{s\}\times S^1}|u_{\theta\theta}|^2\right)\delta\end{aligned}$$

and thus for δ sufficiently small, depending on N, we can improve (3.15) to

$$2\left|\int_{\{s\}\times S^1} u_{\theta\theta}\cdot[A(u)(u_s,u_s)+A(u)(u_\theta,u_\theta)]\right| \leq \int_{\{s\}\times S^1}|u_{s\theta}|^2+\frac{1}{2}|u_{\theta\theta}|^2. \tag{3.16}$$

It remains to estimate the inner product of $u_{\theta\theta}$ with the tension in (3.14). By Young's inequality, we have

$$\left|2\int_{\{s\}\times S^1}u_{\theta\theta}\cdot\tau\right|\leq\frac{1}{2}\int_{\{s\}\times S^1}|u_{\theta\theta}|^2+2\int_{\{s\}\times S^1}|\tau|^2, \tag{3.17}$$

and so combining (3.16) and (3.17) with (3.14) gives the estimate

$$
\begin{aligned}
\vartheta''(s) \geq\ & 2\int_{\{s\}\times S^1} |u_{s\theta}|^2 + |u_{\theta\theta}|^2 \\
& - \left(\frac{1}{2}\int_{\{s\}\times S^1} |u_{\theta\theta}|^2 + 2\int_{\{s\}\times S^1} |\tau|^2 + \int_{\{s\}\times S^1} |u_{s\theta}|^2 + \frac{1}{2}|u_{\theta\theta}|^2 \right) \\
\geq\ & \int_{\{s\}\times S^1} |u_{\theta\theta}|^2 - 2\int_{\{s\}\times S^1} |\tau|^2 \\
\geq\ & \int_{\{s\}\times S^1} |u_{\theta}|^2 - 2\int_{\{s\}\times S^1} |\tau|^2
\end{aligned}
$$

by Wirtinger's inequality. $\square$

Lemma 3.2.4 can be applied all along a long cylinder $\mathscr{C}_\Lambda$ as arising in Lemma 3.2.1, and we can analyse the resulting differential inequality as in the next lemma to deduce bounds on ϑ.

Lemma 3.2.5. *Consider a smooth function $f : [S_1, S_2] \to \mathbb{R}$ satisfying the inequality*

$$f''(s) - f(s) \geq -2\mathcal{T}(s), \tag{3.18}$$

with given boundary values $f(S_1)$, $f(S_2) \in [0, 2E_0]$, and $\mathcal{T} : [S_1, S_2] \to [0, \infty)$ smooth. Then

$$f(s) \leq 2E_0 \left(e^{s-S_2} + e^{S_1-s}\right) + \int_{S_1}^{S_2} e^{-|s-q|}\mathcal{T}(q)dq$$

for $s \in (S_1, S_2)$.

Proof. Recall that in the equality case for (3.18) a solution $\tilde{f}$ can be written explicitly as

$$\tilde{f}(s) := Ae^s + Be^{-s} + \int_{S_1}^{S_2} e^{-|s-q|}\mathcal{T}(q)dq, \qquad A, B \in \mathbb{R}.$$

We then select $A = 2E_0 e^{-S_2}$ and $B = 2E_0 e^{S_1}$ to obtain such a solution for which $\tilde{f}(S_1) \geq 2E_0 \geq f(S_1)$ and $\tilde{f}(S_2) \geq 2E_0 \geq f(S_2)$. The maximum principle implies $\tilde{f} \geq f$ and thus the claim. $\square$

We now apply the estimate from Lemma 3.2.5 to establish decay of angular energy.

Proof of Lemma 3.2.1. First note that we may assume that $\Lambda \geq 1$, otherwise the lemma is vacuous. By definition of ϑ, and the upper bound on the total energy, we have

$$\int_{-\Lambda}^{\Lambda} \vartheta(q)dq \leq 2E(u;\mathscr{C}_\Lambda) \leq 2E_0.$$

We choose $\delta > 0$ smaller than both the δ of Lemma 3.2.4 and the δ_0 in Corollary 3.2.3.

From the above we obtain that there must exist $S_1 \in [-\Lambda, -\Lambda+1)$ and $S_2 \in (\Lambda-1, \Lambda]$ such that $\vartheta(S_1) \leq 2E_0$ and $\vartheta(S_2) \leq 2E_0$. As before, we write

$$\mathcal{T}(s) := \int_{\{s\}\times S^1} |\tau|^2.$$

Then by Lemma 3.2.4, ϑ satisfies $\vartheta'' - \vartheta \geq -2\mathcal{T}$ on $[S_1, S_2]$. Applying Lemma 3.2.5 then gives the first conclusion (3.11) of Lemma 3.2.1.

To prove the energy estimate (3.12) we integrate (3.11) and obtain

$$\int_{-\Lambda+\lambda}^{\Lambda-\lambda} \vartheta(s)ds \leq C\int_{-\Lambda+\lambda}^{\Lambda-\lambda} e^{|s|-\Lambda}ds + \int_{-\Lambda+\lambda}^{\Lambda-\lambda}\int_{-\Lambda}^{\Lambda} e^{-|s-q|}\mathcal{T}(q)dqds. \tag{3.19}$$

We can calculate the first integral on the right-hand side explicitly:

$$\int_{-\Lambda+\lambda}^{\Lambda-\lambda} e^{|s|-\Lambda}ds = 2\left(e^{-\lambda} - e^{-\Lambda}\right) \leq 2e^{-\lambda}. \tag{3.20}$$

In the second integral we change the order of integration

$$\int_{-\Lambda+\lambda}^{\Lambda-\lambda}\int_{-\Lambda}^{\Lambda} e^{-|s-q|}\mathcal{T}(q)dqds = \int_{-\Lambda}^{\Lambda}\mathcal{T}(q)\int_{-\Lambda+\lambda}^{\Lambda-\lambda} e^{-|s-q|}dsdq,$$

and estimate

$$\int_{-\Lambda+\lambda}^{\Lambda-\lambda} e^{-|s-q|}ds \leq \int_{-\infty}^{\infty} e^{-|s-q|}ds = 2,$$

to find that

$$\int_{-\Lambda+\lambda}^{\Lambda-\lambda}\int_{-\Lambda}^{\Lambda} e^{-|s-q|}\mathcal{T}(q)dqds \leq 2\int_{-\Lambda}^{\Lambda}\mathcal{T}(q)dq \leq 2\left\|\tau\right\|^2_{L^2(\mathscr{C}_\Lambda)}.$$

Together with (3.19) and (3.20) this implies claim (3.12). To prove (3.13) we compute

$$E(u;\mathscr{C}_{\Lambda-\lambda}) = \frac{1}{2}\int_{\mathscr{C}_{\Lambda-\lambda}} (|u_\theta|^2 + |u_s|^2)\, d\theta ds = \frac{1}{2}\int_{\mathscr{C}_{\Lambda-\lambda}} (|u_s|^2 - |u_\theta|^2)\, d\theta ds + \int_{\mathscr{C}_{\Lambda-\lambda}} |u_\theta|^2 d\theta ds \tag{3.21}$$

so by (3.12), the definition (2.6) of Φ, and the conformal invariance of the L^1 norm of Φ (see (A.3)), we have

$$E(u;\mathscr{C}_{\Lambda-\lambda}) \leq Ce^{-\lambda} + 2\|\tau\|^2_{L^2(\mathscr{C}_\Lambda)} + \frac{1}{4}\|\Phi\|_{L^1(\mathscr{C}_{\Lambda-\lambda})}.$$

□

3.3 Proofs of the main theorems; convergence to full bubble branches

Our main initial objective in this section is to prove Theorem 3.1.6, giving convergence of almost-harmonic maps to full bubbles branches. This will then be combined with the Poincaré estimate for quadratic differentials of [24] to give Theorem 3.1.8.

Proof of Theorem 3.1.6. Let $u_i : \mathscr{C}_{X_i} \to N$ be a sequence of smooth almost harmonic maps as considered in Theorem 3.1.6. The first task is to construct sequences s_i^m as in the statement of the theorem. We would like to apply (3.13) on the regions $\mathscr{C}(s_i^{m-1}+\lambda, s_i^m-\lambda)$ to the maps u_i for large i, so we let $\delta > 0$ be as in Lemma 3.2.1, which will be independent of i, of course.

We proceed to construct auxiliary sequences $\hat{s}_i^m$, where $m \in \{0, \ldots, \hat{m}+1\}$ for some $\hat{m} \geq 0$. For each i, consider the overlapping chunks of length 3 of the form $(k-1, k+2) \times S^1 \subset \mathscr{C}_{X_i}$ for $k \in \mathbb{Z}$, i.e. for integral k such that $-X_i < k-1 < k+2 < X_i$. These chunks cover $\mathscr{C}_{X_i}$ except possibly for cylinders of length no more than 1 at the ends.

For each i, we initially choose the numbers $\hat{s}_i^m$, for $m = 1, 2, \ldots, m_i$, to be the increasing sequence of integers so that $(\hat{s}_i^m - 1, \hat{s}_i^m + 2) \times S^1$ are precisely the chunks above that have energy at least $\frac{\delta}{2}$.

Note that by the bound on the total energy, there is a uniform bound on the number m_i of such chunks, depending only on N and E_0. Finally, we add in $\hat{s}_i^0 = -X_i$ and

$\hat{s}_i^{m_i+1} = X_i$. By passing to a subsequence of the u_i we can assume that for each i, we have the same number of sequence elements $\hat{s}_i^m$, i.e. $m_i = \hat{m}$ for each i. Note also that for any region $(s-1, s+2) \times S^1 \subset \mathscr{C}_{X_i}$ with energy at least δ there is some associated overlapping integer chunk $(k-1, k+2) \times S^1 \subset \mathscr{C}_{X_i}$ of energy at least $\frac{\delta}{2}$ which is assigned a label in the above construction, except possibly for regions very close to the ends of the cylinder in the sense that $s-1 < -X_i + 1$ or $s+2 > X_i - 1$.

From this auxiliary sequence we form s_i^m. Set $s_i^0 = -X_i = \hat{s}_i^0$, and consider the difference $\hat{s}_i^1 - s_i^0$. If this has a subsequence converging to infinity, pass to that subsequence and take $s_i^1 = \hat{s}_i^1$; if not, discard $\hat{s}_i^1$. Proceed iteratively to define s_i^m (i.e. s_i^2 is the next $\hat{s}_i^m$ such that the respective difference $\hat{s}_i^m - s_i^1$ diverges for some subsequence, after having passed to that subsequence). This process will terminate with the selection of $s_i^{\bar{m}}$, for some $\bar{m}$. Whatever sequence $s_i^{\bar{m}}$ was chosen, redefine it as $s_i^{\bar{m}} = X_i$, which can only change it by an amount that is uniformly bounded in i. This finishes the construction.

For each $m \in \{1, 2, \ldots, \bar{m}-1\}$, consider the shifted maps $u_i^m(s,\theta) := u_i(s + s_i^m, \theta)$. These maps have uniformly bounded energy and $\tau(u_i^m) \to 0$ in L^2. Theorem 3.1.2 applied in the case of Definition 3.1.5 gives, for a subsequence, convergence of each sequence u_i^m to a nontrivial bubble branch with associated bubbles $\{\Omega_j\}$. This completes the proof of Part 1 of the theorem.

Next we consider the connecting cylinders $\mathscr{C}(s_i^{m-1} + \lambda, s_i^m - \lambda)$ for $m \in \{1, 2, \ldots, \bar{m}\}$ and large λ. By construction, there exists a constant $K > 0$ such that

$$E(u_i; \mathscr{C}(s-1, s+2)) < \delta \text{ for } s \in (s_i^{m-1} + K + 1, s_i^m - K - 2),$$

for sufficiently large i (otherwise we would not have discarded the respective $\hat{s}_i^m$). Now let

$$\Lambda_i^m = \frac{(s_i^m - K) - (s_i^{m-1} + K)}{2} = \frac{s_i^m - s_i^{m-1} - 2K}{2}.$$

By translation we can consider u_i on $\mathscr{C}_{\Lambda_i^m}$. We denote the shifted maps as

$$\hat{u}_i^m(s,\theta) = u_i\left(s + \frac{s_i^m + s_i^{m-1}}{2}, \theta\right).$$

For each $\lambda > 1$, the estimate (3.13) from Lemma 3.2.1 applies (as in particular we have no concentration of energy) for sufficiently large i, giving

$$\begin{aligned} E(u_i;\mathscr{C}(s_i^{m-1}+K+\lambda, s_i^m-K-\lambda)) &= E(\hat{u}_i^m;\mathscr{C}_{\Lambda_i^m-\lambda}) \\ &\leq Ce^{-\lambda}+2\left\|\tau(\hat{u}_i^m)\right\|^2_{L^2(\mathscr{C}_{\Lambda_i^m})}+\frac{1}{4}\left\|\Phi(\hat{u}_i^m)\right\|_{L^1(\mathscr{C}_{\Lambda_i^m-\lambda})}. \end{aligned}$$

Taking the limit $i\to\infty$, and using the assumption (3.3) we find that

$$\limsup_{i\to\infty} E(u_i;\mathscr{C}(s_i^{m-1}+K+\lambda, s_i^m-K-\lambda)) \leq Ce^{-\lambda}+\frac{1}{4}\limsup_{i\to\infty}\left\|\Phi(u_i)\right\|_{L^1(\mathscr{C}_{X_i})}.$$

Letting $\lambda\to\infty$ proves that the 'no-loss-of-energy' claim (3.6) holds true provided the maps satisfy the additional assumption (3.5), which completes the proof of Part 3 of the theorem. We remark that this last step is the only part of the proof where (3.5) is used.

Finally we consider the quantity

$$\sup_{s\in(s_i^{m-1}+K+\lambda,s_i^m-K-\lambda)} \operatorname{osc}(u_i,\{s\}\times S^1),$$

again for $1<\lambda<\Lambda_i^m$. After applying the same shift as above, this is equivalent to

$$\sup_{s\in(-\Lambda_i^m+\lambda,\Lambda_i^m-\lambda)} \operatorname{osc}(\hat{u}_i^m,\{s\}\times S^1).$$

On each circle $\{s\}\times S^1$ we have a bound on the (shifted) angular energy $\vartheta_i^m(s) := \vartheta(\hat{u}_i^m, s)$ from Lemma 3.2.1 (at least for sufficiently large i) for $s\in(-\Lambda_i^m+\lambda,\Lambda_i^m-\lambda)$ given by

$$\vartheta_i^m(s)\leq Ce^{|s|-\Lambda_i^m}+\int_{-\Lambda_i^m}^{\Lambda_i^m} e^{-|s-q|}\mathcal{T}(q)dq \leq Ce^{|s|-\Lambda_i^m}+\int_{-\Lambda_i^m}^{\Lambda_i^m}\mathcal{T}(q)dq,$$

and thus we have

$$\begin{aligned} \sup_{s\in(-\Lambda_i^m+\lambda,\Lambda_i^m-\lambda)} \vartheta_i^m(s) &\leq Ce^{-\lambda}+\int_{-\Lambda_i^m}^{\Lambda_i^m}\mathcal{T}(q)dq \\ &\leq Ce^{-\lambda}+\left\|\tau\right\|^2_{L^2(\mathscr{C}_{\Lambda_i^m})}. \end{aligned}$$

Taking limits, and using once more (3.3), gives

$$\limsup_{i\to\infty}\sup_{s\in(-\Lambda_i^m+\lambda,\Lambda_i^m-\lambda)} \vartheta_i^m(s)\leq Ce^{-\lambda},$$

and then

$$\lim_{\lambda\to\infty}\limsup_{i\to\infty}\sup_{s\in(-\Lambda_i^m+\lambda,\Lambda_i^m-\lambda)}\vartheta_i^m(s)=0.$$

We conclude by observing that by the fundamental theorem of calculus and Cauchy-Schwarz, we can control the oscillation of $\hat{u}_i^m$ on a circle in terms of the angular energy on that circle, by

$$\left[\operatorname{osc}(\hat{u}_i^m,\{s\}\times S^1)\right]^2\leq 2\pi\vartheta_i^m(s),$$

which implies the oscillation bound for u_i claimed as Part 2 of the theorem. □

The key step needed to derive Theorem 3.1.8 from Theorem 3.1.6 is to use the Poincaré inequality for quadratic differentials to get control on the Hopf differential.

Lemma 3.3.1. *In the setting of Theorem 3.1.8, the Hopf differential decays according to*

$$\|\Phi(u_i,g_i)\|_{L^1(M,g_i)}\to 0,$$

as $i\to\infty$.

Proof. The Poincaré estimate for quadratic differentials [24] states that for any quadratic differential Φ on the domain (M,g), and in particular for the Hopf differential Φ, we have

$$\|\Phi-P_g(\Phi)\|_{L^1}\leq C\|\bar{\partial}\Phi\|_{L^1},\tag{3.22}$$

where C depends only on the genus $\gamma\geq 2$ of M and is thus in particular *independent* of g. By (2.27), as the area of (M,g_i) is fixed, we know that

$$\|P_{g_i}(\Phi(u_i,g_i))\|_{L^1}\to 0,$$

and by direct computation (see e.g. [21, Lemma 3.2]) we know that

$$\|\bar{\partial}\Phi(u_i,g_i)\|_{L^1}\leq CE_0^{\frac{1}{2}}\|\tau_{g_i}(u_i)\|_{L^2},$$

where E_0 is an upper bound on the energies $E(u_i,g_i)$. Therefore, by (2.27) we find that

$$\|\bar{\partial}\Phi(u_i,g_i)\|_{L^1}\to 0,$$

and we conclude from (3.22) that

$$\|\Phi(u_i,g_i)\|_{L^1(M,g_i)}\to 0,$$

as required. $\square$

Based on Lemma 3.3.1 and Theorem 3.1.6 we can now give the

Proof of Theorem 3.1.8. To derive Theorem 3.1.8 from Theorem 3.1.6, we want to view the restriction of the maps u_i to the collars $\mathcal{C}(\ell_i)$ as maps from *Euclidean* cylinders $\mathscr{C}_{X_i}$, $X_i = X(\ell_i)$, which are almost harmonic (with respect to g_0).

We first remark that $E(u_i; \mathscr{C}_{X_i})$ is bounded uniformly thanks to the conformal invariance of the energy and the assumed uniform bound on $E(u_i, g_i)$.

We then note that the conformal factors of the metrics $\rho^2(s)(ds^2 + d\theta^2)$ of the hyperbolic collars $(\mathcal{C}(\ell), \rho^2 g_0)$, $\ell \in (0, 2\operatorname{arsinh}(1))$, described in Lemma A.1 are bounded uniformly by

$$\rho(s) \leq \rho(X(\ell)) = \frac{\ell}{2\pi \tanh\frac{\ell}{2}} \leq \frac{\sqrt{2}\operatorname{arsinh}(1)}{\pi} \leq 1.$$

Given that the norm of the tension scales as

$$\|\tau_g(u)\|_{L^2(\mathscr{C},g)} = \left\|\rho^{-1}\tau(u)\right\|_{L^2(\mathscr{C})} \tag{3.23}$$

under a conformal change of the metric $g = \rho^2 g_0$, where we continue to abbreviate $\tau(u) := \tau_{g_0}(u)$ and equip $\mathscr{C}$ with the flat metric unless specified otherwise, we obtain from (2.27) that

$$\|\tau(u_i)\|_{L^2(\mathscr{C}_{X_i})} \leq \|\tau_{g_i}(u_i)\|_{L^2(\mathcal{C}(\ell_i),g_i)} \to 0.$$

We furthermore note that the L^1-norm of quadratic differentials is invariant under conformal changes of metric, compare (A.3), and that the Hopf-differential depends only on the conformal structure. Lemma 3.3.1 thus yields

$$\|\Phi(u_i)\|_{L^1(\mathscr{C}_{X_i})} = \|\Phi(u_i, g_i)\|_{L^1(\mathcal{C}(\ell_i),g_i)} \to 0.$$

Consequently all assumptions of Theorem 3.1.6, including (3.5), are satisfied and Theorem 3.1.8 follows. $\square$

Proof of Theorem 3.1.1. Continuing on from Remarks 3.1.3 and 3.1.4 it remains to analyse the energy on the degenerating collars $\mathcal{C}(\ell_i^j)$. After passing to a subsequence, Theorem 3.1.8 gives convergence to a full bubble branch without loss of energy on each of these collars so that the energy on δ-thin$(M, g_i) = \bigcup_j \mathcal{C}_i^{\delta,j}$ satisfies

$$\begin{aligned}\lim_{\delta\downarrow 0}\lim_{i\to\infty} E(u_i; \delta\text{-thin}(M, g_i)) &= \lim_{\delta\downarrow 0}\lim_{i\to\infty}\sum_j E(u_i; \mathscr{C}(-X(\ell_i^j) + \lambda_\delta(\ell_i^j), X(\ell_i^j) - \lambda_\delta(\ell_i^j)) \\ &= \sum_k E(\Omega_k)\end{aligned}$$

Here we use that $\lambda_\delta(\ell_i^j) := X(\ell_i^j) - X_\delta(\ell_i^j) \geq \frac{\pi}{\delta} - C \to \infty$ as $\delta \to 0$, compare (A.1) and [24, Prop. A.2], and we denote by $\{\Omega_k\}$ the collection of all bubbles developing on the degenerating collars. As noted in Remark 3.1.3 this concludes the proof of Theorem 3.1.1. □

3.4 Construction of a nontrivial neck

The main purpose of this section is to prove Theorem 3.1.11, but we first record the following elementary computations.

Proof of Proposition 3.1.10. To ease notation, we drop all subscripts i for the following computations. We also simplify matters by embedding (N, G) isometrically in some Euclidean space and composing u with that embedding. The energy is conformally invariant, thus we calculate with respect to the flat metric

$$E(u; \mathscr{C}_X) = \frac{1}{2}\int_{S^1}\int_{-X}^{X} |u_s|^2 ds d\theta \leq \frac{CL^2}{X}.$$

Using (3.23) and the fact that $\rho^{-1} \leq \ell^{-1} \leq CX$, which follows from Lemma A.1, we compute

$$\|\tau_g(u)\|^2_{L^2(\mathscr{C}_X,g)} = \|\rho^{-1}\tau(u)\|^2_{L^2(\mathscr{C}_X)} \leq C\int_{-X}^{X}\rho^{-2}|u_{ss}|^2 ds \leq \frac{CL^4}{X^4}\int_{-X}^{X}\rho^{-2}ds \leq \frac{CL^4}{X}.$$

Finally, we compute the L^2-norm of $\Phi(u, g)$. With $z = s + i\theta$, $\Phi(u, g) = |u_s|^2 dz^2$. Recalling that $|dz^2|_g = 2\rho^{-2}$ (see A.2) we similarly find

$$\|\Phi(u, g)\|^2_{L^2(\mathscr{C}_X,g)} = \int_{\mathscr{C}_X} |u_s|^4 4\rho^{-4}\rho^2 ds\, d\theta \leq C\frac{L^4}{X^4}\int_{-X}^{X}\rho^{-2}ds \leq C\frac{L^4}{X}.$$

□

The remainder of this section is devoted to the proof of Theorem 3.1.11, constructing a flow that develops a nontrivial neck. We opt for a general approach, although essentially explicit constructions are also possible. To this end, consider any closed oriented surface M of genus at least 2, and take the target N to be S^1. Choose a smooth initial map $u_0 : M \to S^1$ that maps some closed loop α on M exactly once around S^1, and take any hyperbolic metric g_0 on M. We claim that the subsequent flow (2.21) develops a nontrivial neck.

The first key point is that since S^1 has nonpositive sectional curvature, the regularity theory from [20, Theorem 1.1, Theorem 1.2] applies, so the flow exists for all time.

The second key point is that because the target is S^1, there do not exist any branched minimal immersions, except if one allows constant maps. In particular, no bubbles can form. If no collar degenerated in this flow, i.e. if there were a uniform positive lower bound for the lengths of all closed geodesics in $(M, g(t))$, then by the results in [21], the map u_0 would be homotopic to the constant map, which is false by hypothesis.

Therefore there are degenerating collars, and we can analyse them with Theorems 2.2.8 and 3.1.8 (using Proposition 2.2.7). We next demonstrate that a neck forms that is nontrivial in the sense that (3.9) holds. If not, then after passing to a subsequence, the maps from each degenerating collar would become C^0 close to constant maps. By [25, Theorem 1.1] this would imply that u_0 would be C^0 close to a constant map, and thus in particular it would be homotopic to a constant map, which again is false by hypothesis.

We have proved that our flow develops a neck in the sense that (3.9) holds for some degenerating collar, and some m. By Theorem 3.1.8 the image of the subcollar $\mathscr{C}(s_i^{m-1} + \lambda, s_i^m - \lambda)$ will be close to a curve for each i. However, the limiting endpoints (3.7) and (3.8) of the curves may not be distinct, i.e. (3.10) might fail in general.

To make a construction in which (3.10) must hold for some collar and some m, it suffices to adjust our construction so that again all extracted branched minimal immersions must be constant, but so that the union of the images is not just one point. By our theory, the connecting cylinders will thus be mapped close to curves connecting these distinct image points so a nontrivial neck with distinct end points must develop.

To achieve this, we will lift the whole flow to a finite cover $\overline{M}$ of M. Given such a cover, we need to check that the lifted flow still satisfies (2.21). Locally, the lifting will not affect the tension field of u, so the lifted flow will satisfy the first equation of (2.21). However, care is required with the second equation since when we pass to the cover, new holomorphic quadratic differentials arise in addition to the lifts of the original holomorphic quadratic differentials. The following lemma will establish that this causes no problems.

Lemma 3.4.1. *Let $q : \overline{M} \to M$ be a smooth orientation-preserving covering map from one oriented closed surface to another. Suppose g is a metric on M, and $\overline{g} := q^*g$. Then for all quadratic differentials Ψ on (M, g), we have*

$$\overline{P}_{\overline{g}}(q^*\Psi) = q^*(P_g\Psi), \tag{3.24}$$

where $\overline{P}_{\overline{g}}$ and P_g are the L^2-orthogonal projections onto the space of holomorphic quadratic differentials on $(\overline{M}, \overline{g})$ and (M, g), respectively.

As a consequence, given a solution (u, g) to (2.21) on M, and a covering map q as in the lemma, the lifted pair $(u \circ q, q^*g)$ will be a solution to (2.21) on $\overline{M}$.

Proof. If q is an n-fold cover, then the linear map from quadratic differentials on (M, g) to quadratic differentials on $(\overline{M}, \overline{g})$ given by

$$\Psi \mapsto \frac{q^*\Psi}{\sqrt{n}} \tag{3.25}$$

is an L^2 isometry onto its image, because

$$\left\| \frac{q^*\Psi}{\sqrt{n}} \right\|^2_{L^2(\overline{M},\overline{g})} = \frac{\langle q^*\Psi, q^*\Psi\rangle}{n} = \langle \Psi, \Psi\rangle = \|\Psi\|^2_{L^2(M,g)}.$$

Thus it is clear that $q^*(P_g\Psi)$ coincides with the $L^2(\overline{M}, \overline{g})$ projection of $q^*\Psi$ onto the space $q^*\mathcal{H}$, where $\mathcal{H}$ is the space of holomorphic quadratic differentials on (M, g). Therefore to establish (3.24), it remains to prove that if Φ is a holomorphic quadratic differential on $(\overline{M}, \overline{g})$, then $\langle \Phi, q^*\Theta\rangle = 0$ for every quadratic differential $\Theta \in \mathcal{H}^\perp$. To see this, we consider the adjoint Υ to the map $\Psi \mapsto q^*\Psi$, which pushes down a quadratic differential on $(\overline{M}, \overline{g})$ to (M, g), adding up the n preimages. In particular, Υ maps *holomorphic* quadratic differentials to *holomorphic* quadratic differentials, and so

$$\langle \Phi, q^*\Theta\rangle = \langle \Upsilon(\Phi), \Theta\rangle = 0,$$

as required. $\square$

Returning to our construction, we may assume that all the branched minimal immersions are mapping to the same limit point p (or we are done already). The aim is now to use a lifting construction, justified by the above, to obtain a lifted flow with the images of the corresponding branched minimal immersions being the two *different* lifts of p in a double cover of the target.

To this end, fix an arbitrary base point x_0 on M. Now consider the index 2 subgroup H of $\pi_1(M, x_0)$ consisting of loops whose images under the initial map u_0 go round the target S^1 an even number of times. We can pass to a (double) cover $q : \overline{M} \to M$ of the domain satisfying $q_* \left(\pi_1 \left(\overline{M}, \overline{x}_0\right)\right) = H$ (see e.g. [11, Prop. 1.36]) and lift u_0 to the map $\overline{u}_0 = u_0 \circ q : \overline{M} \to S^1$. By the choice of H we can further lift $\overline{u}_0$ to a map $\tilde{u}_0$ into a (connected) double cover of the target S^1 (e.g. [11, Prop. 1.33]). By Lemma 3.4.1, the flow (2.21) on $\overline{M}$ starting at $(\tilde{u}_0, q^* g_0)$ covers the original flow on M starting at (u_0, g_0). We can analyse this lifted flow using Theorem 3.1.8 for a subsequence of the times t_i at which we analysed the flow on M.

It suffices to show that the images of the branched minimal immersions we can construct from the lifted flow consist of *both* of the lifts of p, not just one. These distinct points can then only be connected by nontrivial necks.

To see this, note that from the analysis of the original flow on M with Theorem 1.1 from [25] (i.e. with Proposition 2.2.7 and Theorem 2.2.8), we can find some $\delta > 0$ sufficiently small such that for sufficiently large i, the δ-thin part of $(M, g(t_i))$ will consist of a (disjoint) union of (sub)collars that eventually degenerate. For large enough i, the image of the δ-thick part of $(M, g(t_i))$ will be contained in a small neighbourhood of p. For each such large i, we pick a point y_i in the δ-thick part, and deform α to pass through y_i. We view α then as a path that starts and ends at y_i, and by assumption, the composition $u_0 \circ \alpha$ takes us exactly once round the target S^1. In particular, as we pass once round the lift of α, we move from one lift of y_i to the other, and the flow map moves from being close to one lift of p to being close to the other lift.

In particular, the branched minimal immersions in the lifted picture map to both lifts of p as required.

Chapter 4

Coupling constant limits

Recall that in the construction of the Teichmüller harmonic map flow (2.21) as outlined in Section 2.1.2 a choice of the relative weight of map and metric 'directions' is made, resulting in the appearance of a coupling constant $\eta > 0$ in the flow equations (2.21). In this chapter we investigate the limiting behaviour of the flow (2.21) as $\eta \downarrow 0$. In the first section we establish smooth convergence to the harmonic map flow on compact time intervals, assuming the target $N = (N, G)$ has nonpositive sectional curvature. In the second section we show convergence to a limit flow 'through harmonic maps' when one combines $\eta \downarrow 0$ with a rescaling of time.

4.1 Convergence to the harmonic map flow as $\eta \downarrow 0$ on compact time intervals

For the rest of this section fix some smooth closed oriented surface M of genus $\gamma \geq 2$ and a smooth closed target manifold (N, G). Then for given smooth initial data (u_0, g_0) (with $g_0 \in \mathcal{M}_{-1}$), any $\eta > 0$ corresponds to a smooth solution (u_η, g_η) of (2.21) with coupling constant η, at least up to some small time, by [21] and in particular Theorem 2.2.2. Under the additional assumption of the target (N, G) having nonpositive sectional curvature we know from [20] that these flows in fact exist smoothly for all times, see Theorem 2.2.9. Hence we can consider the sequence (u_η, g_η) as $\eta \downarrow 0$ of smooth global solutions to (2.21).

We are interested in conditions ensuring convergence to the classical harmonic map flow. From e.g. Section 3.4 we know that we can choose a specific configuration of initial data (u_0, g_0), domain M and target N such that the resulting flows (u_η, g_η) necessarily develop a metric singularity (at infinite time), no matter what $\eta > 0$ is used. Therefore we certainly will not have any kind of uniform convergence to the harmonic map flow for all time, hence we restrict our attention to compact time intervals.

We state our main theorem of this section in this setting, which essentially says that the limit flow for $\eta \downarrow 0$ (on such compact time intervals) is given by the classical harmonic map flow.

Theorem 4.1.1. *Let M be a smooth closed oriented surface of genus $\gamma \geq 2$ and $g_0 \in \mathcal{M}_{-1}$, (N, G) a nonpositively curved smooth closed Riemannian manifold and $u_0 : M \to N$ a smooth map. Then on each fixed time interval $[0, T]$ the flows $(u_\eta(t), g_\eta(t))$ satisfying (2.21) with initial condition (u_0, g_0) converge smoothly to the harmonic map flow $u(t)$ satisfying (2.4) (with respect to g_0), starting at the same initial condition $u(0) = u_0$, in the following sense as $\eta \downarrow 0$:*

1. *The metrics $g_\eta(t)$ converge to the initial metric g_0 in $C^k(M, g_0)$ uniformly in t for each $k \in \mathbb{N}$.*

2. *The maps $u_\eta(t)$ converge to $u(t)$ smoothly on $M \times [0, T]$.*

Here we define the $C^k(M, g_0)$-norm of a tensor g by

$$\|g\|_{C^k(M,g_0)} := \sup_{x \in M} \sum_{l=0}^{k} |\nabla^l g|_{g_0}(x). \tag{4.1}$$

Remark 4.1.2. This theorem is unlikely to be true if one drops the curvature assumption on the target N. For the classical harmonic map flow it is possible to construct flows with finite time singularities, where a small variation of the initial map leads to a smooth flow (which in particular rules out uniform convergence as in the above theorem), so it seems probable that one can achieve the same effect by perturbing the metric.

We also note that it is likely that one can prove the above theorem using more abstract tools, e.g. the implicit function theorem. However, an advantage of the more direct route taken in this section is that we obtain a number of useful estimates which we can apply in the next section.

Notation for this section: All the flows (2.21) are assumed to start at smooth initial data (u_0, g_0), where we always take $u_0 : M \to N$, $g_0 \in \mathcal{M}_{-1}$. We further always use

N to mean (N, G).

The proof of Theorem 4.1.1 will be carried out in two steps. We first obtain the claimed estimates on the metric, which are then used in estimating the map component.

4.1.1 Metric evolution for small η

We proceed with estimating the evolution of the metric g_η for small η. As a consequence of the energy identity (2.22) for the flow (see also [21]) we obtain the following lemma.

Lemma 4.1.3. *Let $T > 0$, $\eta > 0$ and consider the curve of metrics $g_\eta(t)$ defined on $[0, T]$ as in Theorem 4.1.1. Define the L^2-length of $g_\eta(t)$ on $[0, T]$ by*

$$L(g_\eta, [0, T]) = \int_0^T \|\partial_t g_\eta\|_{L^2(M, g_\eta(t))} \, dt.$$

We then have the estimate $L(g_\eta, [0, T]) \leq \eta\sqrt{TE(u_0, g_0)}$.

Proof. Simply observe that by integrating (2.22) in t and the monotonicity of energy we have

$$\int_0^T \int_M \left(\frac{\eta}{4}\right)^2 |Re(P_{g_\eta}(\Phi(u_\eta, g_\eta)))|^2 dv_{g_\eta} dt \leq E_0.$$

for $E_0 = E(u_0, g_0)$ denoting the initial energy. Hence

$$\int_0^T \int_M |\partial_t g_\eta|^2 \, dv_{g_\eta} dt \leq \eta^2 E_0,$$

which allows us to estimate L via Hölder's inequality as

$$L(g_\eta, [0, T]) = \int_0^T \|\partial_t g_\eta\|_{L^2(M, g_\eta(t))} \, dt \leq \eta\sqrt{TE_0}. \tag{4.2}$$

□

Remark 4.1.4. Note that after projecting a curve of metrics $g(t)$ down to a path $[g(t)]$ in Teichmüller space the L^2-length of $g(t)$ defined in the above lemma corresponds to the length of $[g(t)]$, computed with respect to the classical Weil-Petersson metric (up to a constant) .

Thus if we consider the family of curves $g_\eta(t)$ starting at g_0 on a fixed interval $[0,T]$, we see that their L^2-length becomes arbitrarily small as η decreases.

We are now in a position to state an estimate from [23] for so-called *horizontal curves* of metrics with small L^2-length.

Definition 4.1.5 (See also [22])**.** In our setting a *horizontal curve* is a curve of metrics $g(t) \in C^1([t_1,t_2],\mathcal{M}_{-1})$ such that for all $t \in [t_1,t_2]$ we have $\frac{\mathrm{d}}{\mathrm{d}t}g(t) = Re(\Psi(t))$, where $\Psi(t)$ is a holomorphic quadratic differential on $(M, g(t))$.

Note that in particular solutions $g_\eta(t)$ to (2.21) are horizontal curves.

Proposition 4.1.6 (Proposition 2.2 in [23])**.** *For every $\epsilon > 0$ and every $s > 3$ there exists a number $\theta = \theta(\epsilon, s) > 0$ such that the following holds true. Let $g_0 \in \mathcal{M}^s_{-1}$ be any hyperbolic metric of class H^s for which the length $\ell(g_0)$ of the shortest closed geodesic in (M, g_0) is no less than ϵ. Then there is a number $C = C(g_0, s) < \infty$ such that for any horizontal curve $g(t)$ with $g(0) = g_0$ and $L(g,[0,T]) \leq \theta$ we have*

$$\left\|\frac{\mathrm{d}}{\mathrm{d}t}g(t)\right\|_{H^s} \leq C\left\|\frac{\mathrm{d}}{\mathrm{d}t}g(t)\right\|_{L^2(M,g(t))} \quad \text{for every } t \in [0,T). \tag{4.3}$$

We also have the following result from [23] controlling the projection P_g.

Lemma 4.1.7 (Lemma 2.9 in [23])**.** *For any $g_0 \in \mathcal{M}_{-1}$ and any $s > 3$ there exists a neighbourhood W of g_0 in $\mathcal{M}^s_{-1}$ and a constant $C = C(g_0, s)$ such that for all $g \in W$ and $k \in \Gamma(\mathrm{Sym}^2(T^*M))$ we have*

$$\|P_g k\|_{H^s} \leq C\,\|k\|_{L^1(M,g_0)}\,. \tag{4.4}$$

The H^s-norm in both these results is to be understood with respect to some fixed set of local coordinate charts on M.

Remark 4.1.8. We can apply Proposition 4.1.6 to $g_\eta(t)$ on a fixed time interval $[0,T]$ by Lemma 4.1.3 for all sufficiently small η. By integrating (4.3) and estimating as before we obtain

$$\|g_\eta(t) - g_0\|_{C^k(M,g_0)} \leq C\eta\sqrt{t} \leq C\eta. \tag{4.5}$$

Here C depends on T, k, g_0 and $E_0 = E(u_0, g_0)$ and we used the fact that H^s embeds continuously into C^k for sufficiently large s.

We claim that we can find $\eta_0 = \eta_0(T, g_0, E_0, N) \leq 1$ such that for $\eta \leq \eta_0$ and $t \in [0, T]$ the following properties hold:

1. For any vector field $X \in \Gamma(TM)$ we have $\frac{1}{2}|X|_{g_0} \leq |X|_{g_\eta(t)} \leq 2|X|_{g_0}$.

2. For any smooth map $u : M \to N$ we have $\frac{1}{2}|du|_{g_0} \leq |du|_{g_\eta(t)} \leq 2|du|_{g_0}$.

3. For any $(0, 2)$-tensor $h \in \Gamma(\mathrm{Sym}^2 T^*M)$ we have $\frac{1}{2}|h|_{g_0} \leq |h|_{g_\eta(t)} \leq 2|h|_{g_0}$.

4. The metrics $g_\eta(t)$ lie in the neighbourhood W from Lemma 4.1.7.

5. The injectivity radius satisfies $\frac{1}{2}\,\mathrm{inj}_{g_0} \leq \mathrm{inj}_{g_\eta(t)} \leq 2\,\mathrm{inj}_{g_0}$.

6. For all $x \in M$ and $r > 0$ metric balls satisfy $B^{g_0}_{\frac{r}{2}}(x) \subset B^{g_\eta(t)}_r(x) \subset B^{g_0}_{2r}(x)$.

7. The metrics $g_\eta(t)$ and g_0 satisfy the assumptions of Lemma C.1 (i.e. $\|g_0 - g_\eta(t)\|_{C^0(M,g_0)} \leq C_1$ with C_1 from Lemma C.1).

Claims 1,2 and 3 can be seen directly by working in local coordinates as in Lemma C.1. Claim 4 follows from estimate (4.3). We further obtain Claim 5 and Claim 6 as a consequence of Claim 1 (as in particular the length of any curve measured with respect to g_0 will be comparable to the length of the same curve measured with respect to $g_\eta(t)$). Finally Claim 7 follows immediately from estimate (4.5).

Applying Lemma 4.1.7 when $\eta \leq \eta_0$ we then find

$$\left\| \frac{\mathrm{d}}{\mathrm{d}t} g_\eta \right\|_{C^k(M,g_0)} \leq C\eta^2 \left\| \Phi(u_\eta, g_\eta) \right\|_{L^1(M,g_0)} \leq C\eta^2 \tag{4.6}$$

where C depends on k, g_0 and E_0, in particular showing that $\left\|\frac{\mathrm{d}}{\mathrm{d}t} g_\eta\right\|_{C^k(M,g_0)}$ is uniformly bounded for all times $t \in [0, T]$ and $\eta \leq \eta_0$.

Corollary 4.1.9. *Under the assumptions of Theorem 4.1.1, we have uniform convergence in t of $g_\eta(t)$ to g_0 on $[0, T]$ in $C^k(M, g_0)$ for any $k \in \mathbb{N}$ as $\eta \downarrow 0$.*

Proof. This follows from the estimate (4.5), which by Lemma 4.1.3 we can apply for all sufficiently small η. □

4.1.2 Map evolution for small η

We now investigate how the map component of the flow behaves as η becomes small. As before, let $u(t)$ be the solution to the harmonic map flow equation (2.4) on (M, g_0) with $u(0) = u_0$, and let $u_\eta(t)$ be as usual the solution to (2.21) with coupling constant η and initial conditions (u_0, g_0). It remains to see why $u_\eta(t)$ becomes close to $u(t)$. Our strategy is to obtain integral bounds for the maps $u_\eta(t)$, uniformly in η. These bounds will later allows us to show that $u - u_\eta(t)$ becomes small in L^2 as $\eta \downarrow 0$.

We require an interpolation estimate for higher norms in terms of the tension away from points of concentrated energy. We first state a standard local version of this estimate, for maps from flat disks D_r.

Lemma 4.1.10 (e.g. [20, Lemma 3.3])**.** *Given a smooth closed target N there exists $\epsilon_0 = \epsilon_0(N) > 0$ and $C = C(N) < \infty$, such that for each $r > 0$ and each smooth map $u : D_r \to N$ with $E(u, D_r) \leq \epsilon_0$, we have the estimate*

$$\int_{D_{\frac{r}{2}}} |\nabla u|^4 + |\nabla^2 u|^2 \leq C \left(\frac{E(u, D_r)}{r^2} + \|\tau(u)\|^2_{L^2(D_r)} \right). \tag{4.7}$$

From this estimate we can obtain a global version for maps $u : (M, g) \to N$. We will need to understand how to translate local estimates defined with respect to the flat metric into estimates using the hyperbolic metric.

Remark 4.1.11. Consider (M, g) a closed hyperbolic surface, with injectivity radius bounded below by $r_0 < \mathrm{inj}_g$, for some $r_0 \leq 1$, and let $x \in M$. Let $r < r_0$, then we can choose particular local isothermal coordinates on the geodesic ball $B_r(x)$, which allow us to view $B_r(x)$ as a disk $(D_{r'}, g_H)$ for some $r' < 1$ only depending on r, where g_H denotes the Poincaré metric. This follows immediately by considering a local isometry from (M, g) to the Poincaré disk, centred at x. We will also refer to these coordinates as *hyperbolic isothermal coordinates.* Hence we can write $g_H = \lambda^2 g_{eucl}$ where $\lambda : D_{r'} \to [2, K]$ denotes the conformal factor, with an upper bound given by a universal constant $K < \infty$ (as we considered disks of hyperbolic radius $r_0 \leq 1$, and thus the disk $D_{r'}$ stays away from the boundary of the Poincaré disk).

We will now see why Lemma 4.1.10 is still true when taken with respect to the hyperbolic metric on $D_{r'}$ (after adjusting the constant C). We observe the flat metric and the hyperbolic metric on $D_{r'}$ are equivalent in the sense that there are universal constants C_1, C_2 with

$$C_1 g_{eucl} \leq g_H \leq C_2 g_{eucl}. \tag{4.8}$$

Denote the connection induced through g_H by $\tilde{\nabla}$ and the (flat) connection with respect to g_{eucl} by ∇. Note of course that ∇ and $\tilde{\nabla}$ agree on functions. We now consider a function $u : D_{r'} \to N \hookrightarrow \mathbb{R}^n$. As a direct consequence of the equivalence of the metrics, we again have universal constants C_1, C_2 such that

$$C_1 |\nabla u|_{g_{eucl}} \leq |\tilde{\nabla} u|_{g_H} \leq C_2 |\nabla u|_{g_{eucl}}. \tag{4.9}$$

Finally, we consider how $|\nabla^2 u|$ changes. In local coordinates we have

$$\tilde{\nabla}^2 u_{ij} = \partial_i \partial_j u - \tilde{\Gamma}_{ij}^k \partial_k u, \tag{4.10}$$

where $\tilde{\Gamma}$ denotes the Christoffel symbols of the connection $\tilde{\nabla}$. We can explicitly compute these Christoffel symbols in terms of the conformal factor λ, and find that they are uniformly bounded (as λ is a continuously differentiable function in the region we considered). Hence we have

$$|\tilde{\nabla}^2 u|_{g_H} \leq C |\tilde{\nabla}^2 u|_{g_{eucl}} \leq C |\nabla^2 u|_{g_{eucl}} + C |\nabla u|_{g_{eucl}}, \tag{4.11}$$

with C again a universal constant. We finally recall that the tension scales via $\|\tau_{g_H}(u)\|^2_{L^2(D_r, g_H)} = \left\|\lambda^{-2} \tau_{g_{eucl}}(u)\right\|^2_{L^2(D_r, g_{eucl})}$ ((3.23)), and the volume form as $dv_H = \lambda^2 dv_{eucl}$. Combining these, we find

$$\begin{aligned}
\int_{D_{\frac{r'}{2}}} |\tilde{\nabla} u|^4_{g_H} + |\tilde{\nabla}^2 u|^2_{g_H} dv_H &\leq C \int_{D_{\frac{r'}{2}}} |\nabla u|^4_{g_{eucl}} + |\nabla^2 u|^2_{g_{eucl}} + |\nabla u|^2_{g_{eucl}} dv_{eucl} && (4.12)\\
&\leq C \left(\frac{E(u, D_{r'})}{r'^2} + \|\tau_{g_{eucl}}(u)\|^2_{L^2(D_{r'}, g_{eucl})} \right) && (4.13)\\
&\leq C \left(\frac{E(u, D_{r'})}{r'^2} + \|\tau_{g_H}(u)\|^2_{L^2(D_{r'}, g_H)} \right) && (4.14)
\end{aligned}$$

where we absorbed the additional $|\nabla u|^2_{g_{eucl}}$ term into the energy and the constant C is now allowed to depend on N as in Lemma 4.1.10. Note that the energy is conformally invariant, so it does not matter whether we compute it with respect to g_H or g_{eucl} on $D_{r'}$.

Lemma 4.1.12. *Take N to be a smooth closed manifold and M a smooth closed oriented surface of genus $\gamma \geq 2$. Let $\epsilon_0 > 0$ be the constant from Lemma 4.1.10 and take $g \in \mathcal{M}_{-1}$. Consider a smooth map $u : (M, g) \to N$ with energy bounded by $E(u, g) \leq E_0$. Assume there is some $r > 0$ such that we have a bound on the local energy for all $x \in M$ of*

$$E(u, B_r(x)) \leq \epsilon_0, \tag{4.15}$$

then there exists $C < \infty$ (depending on r, E_0, (M, g) and N) such that

$$\int_M |\nabla u|^4 + |\nabla^2 u|^2 dv_g \leq C \left(1 + \int |\tau_g(u)|^2 dv_g\right). \tag{4.16}$$

Proof. Let $r_0 < \min\{\mathrm{inj}_g(M), r, 1\}$, then we can cover M with geodesic balls $B_{\frac{r_0}{2}}(x_i)$. The total number of such balls will only depend on (M, g) and r. We then consider the cover $\{B_{r_0}(x_i)\}$. As explained in Remark 4.1.11, we can view each $B_{r_0}(x_i)$ to be some disk $D_{r'}$ equipped with the Poincaré metric. Viewing those disks as *flat* disks allows us to apply Lemma 4.1.10 (note that the energy is conformally invariant, and hence the necessary assumption is satisfied). As in Remark 4.1.11 we can replace flat metric in Lemma 4.1.10 with the hyperbolic metric on each disk, after modifying the constant C. Adding up all the individual estimates yields the claim after again adjusting the constant. □

Note that in the setting of Theorem 4.1.1 this lemma applies to all maps $u_\eta(t)$, as by the curvature assumption these maps are actually even smooth at each time. In particular, we can find sufficiently small $r > 0$ for any fixed flow $u_\eta(t)$ such that (4.15) is satisfied on some compact time interval. However, we want to deduce uniform (in η) bounds on u_η by integrating (4.16) in time. This requires us to control the amount of energy that can concentrate (again, uniformly in η).

Uniform control of energy concentration

We start with an estimate for the evolution of local energy.

Lemma 4.1.13. *Variant of [23, Lemma 3.3], see also [27]. In the setting of Theorem 4.1.1, let $T > 0$, $\eta_0(T) \geq \eta > 0$, with η_0 from Remark 4.1.8, and consider the associated curve of maps $u_\eta(t)$ corresponding to the solution of the flow (2.21) with coupling constant η on $[0, T]$, starting at initial data (u_0, g_0). Then for any radii $0 < r < r + r' < \mathrm{inj}_{g_0}$ and any point $x \in M$ we have*

$$E(u_\eta(t), B_r(x)) \leq 2E(u_0, B_{r+r'}(x)) + C\frac{t}{r'^2}, \tag{4.17}$$

where $t \in [0, T]$ and we take both the energies and geodesic balls to be defined with respect to the initial metric g_0. The constant C depends on (M, g_0), N and $E_0 = E(u_0, g_0)$.

Proof. This can be seen by the same argument as in [23] after modifying the cut-off function used in the proof. For the sake of completeness we give some more details on the calculations involved. Take $\phi \in C_0^\infty(B_{r+r'}(x), [0,1])$, satisfying $\phi \equiv 1$ on $B_r(x)$ and $|\nabla\phi|_{g_0} \leq \frac{C}{r'}$. We can then multiply equation (2.21) for u_η by $\phi^2\partial_t u_\eta$ and integrate over M with respect to g_0 (exactly as in [23]) to arrive at

$$0 = \int \phi^2 |\partial_t u_\eta|^2 dv_{g_0} - \int \phi^2 \partial_t u_\eta \Delta_{g_\eta} u_\eta dv_{g_0}, \tag{4.18}$$

where we view the target as isometrically embedded $N \hookrightarrow \mathbb{R}^n$. Using integration by parts we find

$$-\int \phi^2 \partial_t u_\eta \Delta_{g_\eta} u_\eta dv_{g_0} = \int \langle d(\phi^2 \partial_t u_\eta), du_\eta \rangle_{g_\eta} dv_{g_0} \tag{4.19}$$

$$= \int \partial_t u_\eta \langle d(\phi^2), du_\eta \rangle_{g_\eta} dv_{g_0} + \int \phi^2 \langle d(\partial_t u_\eta), du_\eta \rangle_{g_\eta} dv_{g_0}. \tag{4.20}$$

We then note that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \int \phi^2 \langle du_\eta, du_\eta \rangle_{g_\eta} dv_{g_0} = \int \phi^2 \langle d(\partial_t u_\eta), du_\eta \rangle_{g_\eta} dv_{g_0} + R(u_\eta, g_\eta) \tag{4.21}$$

with an error term given by

$$R(u_\eta, g_\eta) = -\int \phi^2 \langle \frac{\mathrm{d}}{\mathrm{d}t} g_\eta, du \otimes du \rangle_{g_\eta} dv_{g_0} \tag{4.22}$$

which we can estimate (using $\eta \leq \eta_0$ and Remark 4.1.8) as

$$|R(u_\eta, g_\eta)| \leq C \left\| \frac{\mathrm{d}}{\mathrm{d}t} g_\eta \right\|_{C^0(g_\eta)} E(u_\eta(t), g_\eta(t)) \leq C \left\| \frac{\mathrm{d}}{\mathrm{d}t} g_\eta \right\|_{C^0(g_0)} E_0. \tag{4.23}$$

We further estimate

$$\begin{aligned} \left| \int \partial_t u_\eta \langle d(\phi^2), du_\eta \rangle_{g_\eta} dv_{g_0} \right| &\leq C \int |\phi| |\partial_t u_\eta| |d\phi|_{g_\eta} |du_\eta|_{g_\eta} dv_{g_0} \\ &\leq \int \phi^2 |\partial_t u_\eta|^2 dv_{g_0} + C \int |d\phi|_{g_\eta}^2 |du_\eta|_{g_\eta}^2 dv_{g_0}. \end{aligned} \tag{4.24}$$

As $\eta \leq \eta_0$ we see from Remark 4.1.8 that we also have the bound $|d\phi|_{g_\eta(t)} \leq \frac{C}{r'}$ (as in particular $g_\eta \leq 2g_0$). Thus, combining (4.24), (4.21) and (4.20) with (4.18) we find

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int \phi^2|du_\eta|^2_{g_\eta}dv_{g_0} \leq \left(\frac{C}{r'^2} + C\left\|\frac{\mathrm{d}}{\mathrm{d}t}g_\eta\right\|_{C^0(g_0)}\right)E_0. \tag{4.25}$$

Again from Remark 4.1.8 we see that $\|\frac{\mathrm{d}}{\mathrm{d}t}g_\eta\|_{C^0(g_0)}$ is uniformly bounded.

Note that $r' < \mathrm{inj}_{g_0}$, hence r' is bounded from above in terms of only the genus of M, and we can simplify the above estimate to

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int \phi^2|du_\eta|^2_{g_\eta}dv_{g_0} \leq \frac{C}{r'^2} \tag{4.26}$$

after adjusting the constant C (now depending on (M, g_0), N and E_0).

As $\eta \leq \eta_0$ we have $|du_\eta|_{g_0} \leq 2|du_\eta|_{g_\eta}$ and we can integrate inequality (4.25) to find

$$E(u_\eta(t), B_r(x)) \leq \int \phi^2|du_\eta|^2_{g_0}dv_{g_0} \leq 2\int \phi^2|du_\eta|^2_{g_\eta}dv_{g_0} \leq 2E(u_0, B_{r+r'}(x)) + C\frac{t}{r'^2} \tag{4.27}$$

which establishes the claim. □

Using this, we now show that it is possible to choose a uniform (in η) $r > 0$ in (4.15). To this end, consider some domain Ω, which for us will be either contained in (M, g) (e.g. a geodesic ball or the whole of M) or a flat disk. Given a metric g on Ω and a smooth map $u : \Omega \to N$, we can then define the concentration radius

$$r_\epsilon(u, \Omega, g) = \inf\{r > 0 : \exists x \in \Omega, E(u, B_r(x), g) \geq \epsilon\}. \tag{4.28}$$

If the set is empty (i.e. $E < \epsilon$) we agree to let this equal the diameter of Ω, and if Ω is the whole domain of u we write $r_\epsilon(u)$ for simplicity. We first prove a local result, relating the concentration radius to the size of the tension on flat disks. This is similar to [20, Lemma 3.2] and exploits the nonpositive curvature of the target.

Lemma 4.1.14. *Let (N, G) be a nonpositively curved smooth closed Riemannian manifold, and let further $r > 0$ and take as usual D_r to be the flat disk of radius r. Then for smooth maps $u : D_r \to N$ with energy bounded by $E(u, D_r) \leq E_0$, and $\epsilon > 0$, there exists a constant $2 \leq K < \infty$, only depending on N, E_0 and ϵ, such that*

$$r_\epsilon(u, D_{\frac{r}{2}}, g_{eucl}) \geq \frac{r}{K(1 + r\|\tau(u)\|_{L^2(D_r)})}. \tag{4.29}$$

Proof. Observe that we can restrict to the case $r = 1$ and deduce the remaining cases by scaling. We argue by contradiction. Assume the lemma is false for some $\epsilon > 0$, $E_0 > 0$, then we obtain a sequence $r_i = \frac{1}{i(1+\|\tau(u_i)\|_{L^2(D_1)})} \in (0, \frac{1}{2})$ for $i \geq 2$ together with maps $u_i : D_1 \to N$ with $E(u_i, D_1) \leq E_0$ and points $x_i \in D_{\frac{1}{2}}$ with $E(u_i, D_{r_i}(x_i)) \geq \epsilon$. Note that $r_i \to 0$. We can therefore consider the restrictions of the maps u_i to $D_{\frac{r}{2}}(x_i) \subset D_1$, and after shifting x_i to the origin these form a sequence of maps (still labeled as u_i for convenience) $u_i : D_{\frac{1}{2}} \to N$ with $E(u_i, D_{r_i}) > \epsilon$. We now rescale these maps by $\frac{1}{r_i}$ to obtain maps from larger and larger disks $\tilde{u}_i : D_{\frac{1}{2r_i}} \to N$, with $E(\tilde{u}_i, D_1) \geq \epsilon$. By the rescaling, the tension now satisfies

$$\|\tau(\tilde{u}_i)\|_{L^2} = r_i \, \|\tau(u_i)\|_{L^2(D_{\frac{1}{2}})} \leq \frac{1}{i} \frac{\|\tau(u_i)\|_{L^2(D_1)}}{1 + \|\tau(u_i)\|_{L^2(D_1)}} \to 0. \tag{4.30}$$

Therefore a standard bubbling argument, see e.g. Lemma 3.1.2, allows us to extract a nonconstant harmonic map $\tilde{u}_\infty : \mathbb{R}^2 \to N$, which can be extended to a (also non-constant) harmonic map from $S^2 \to N$, which contradicts the curvature assumption on N (see e.g. [20, Lemma 2.1]). □

We now deduce a global version of this for smooth maps $u : (M, g) \to N$.

Lemma 4.1.15. *Let again (N, G) be a nonpositively curved smooth closed Riemannian manifold and further take M to be a smooth closed oriented surface of genus $\gamma \geq 2$. Given $r_0 > 0$ and $g \in \mathcal{M}_{-1}$ with $\mathrm{inj}_{g_0} \geq r_0$, a smooth map $u : (M, g) \to N$ with bounded energy $E(u, g) \leq E_0$ and an $\epsilon > 0$, there exists a constant $K < \infty$, only depending on ϵ, r_0, N and E_0, such that*

$$r_\epsilon(u, g) \geq \frac{1}{K(1 + \|\tau(u)\|_{L^2(M,g)})}. \tag{4.31}$$

Proof. Locally around each point on M we can find geodesic balls of radius $\frac{r_0}{2}$. We then apply Lemma 4.1.14 at each point $x \in M$ on the corresponding flat disk $D_{r'}$ (obtained by taking hyperbolic isothermal coordinates, see Remark 4.1.11), which will give a lower bound of the form (4.29). Initially, this bound on the concentration radius will be with respect to the euclidean metric, but it also implies a bound for the hyperbolic metric on $D_{r'}$ by the conformal equivalence of the two metrics. By the same argument we find that $\|\tau(u)\|_{L^2(D_{r'}, g_{eucl})} \leq C \, \|\tau(u)\|_{L^2(M,g)}$, which finishes the proof. □

We aim to use this estimate to show uniform (in η) control for the energy concentration along solutions (u_η, g_η) of the flow (2.21). Note however that we do not know in advance that the tension $\tau_{g_\eta}(u_\eta)$ is small in L^2 at each time, but only that $\left\|\tau_{g_\eta}(u_\eta)\right\|^2_{L^2(M,g_\eta)}$ is controlled in $L^1[0,T]$ (by (2.22)). Hence we combine Lemma 4.1.15 with Lemma 4.1.13, which implies that very concentrated energy at some time also leads to concentrated energy at *nearby* times, allowing us to use the L^1 bound to control the radius of concentration.

Lemma 4.1.16. *Again, take (N,G) to be a nonpositively curved smooth closed Riemannian manifold and M a smooth oriented closed surface of genus $\gamma \geq 2$. Let $T > 0$, $\eta_0 \geq \eta > 0$, with η_0 from Remark 4.1.8, and consider the solution $(u_\eta(t), g_\eta(t))$ with coupling constant η to (2.21) defined on $[0,T]$ with initial data (u_0, g_0). Then for any $\epsilon > 0$ there exists an $R = R((M,g_0), u_0, \epsilon, N) > 0$ such that $E(u_\eta(t), B_R(x)) \leq \epsilon$ for all $t \in [0,T]$ and $x \in M$, where both the energies and the geodesic balls are defined with respect to the initial metric g_0.*

Proof. As u_0 is smooth we can find $0 < R_0(u_0) \leq \frac{1}{4}\,\mathrm{inj}_{g_0}$ such that $r_{\frac{\epsilon}{4}}(u_0, g_0) \geq 2R_0$. Hence we can apply Lemma 4.1.13 with $r = r' = R_0$ and find (as $2R_0 \leq \frac{1}{2}\,\mathrm{inj}_{g_0}$) at any point $x \in M$

$$E(u_\eta(t), B_{R_0}(x)) \leq 2E(u_0, B_{2R_0}(x)) + C\frac{t}{R_0^2} \leq \frac{1}{2}\epsilon + C\frac{t}{R_0^2}. \tag{4.32}$$

We see that with $\tau = \epsilon\frac{R_0^2}{C}$ (which is in particular *independent* of η) we have $E(u_\eta(t), B_{R_0}(x)) \leq \epsilon$ for all $t \in [0,\tau]$, and thus close to 0 we can take $R = R_0 > 0$ as required.

We now combine Lemma 4.1.15 with Lemma 4.1.13 as mentioned above to extend this control of the energy concentration to larger times. Let $T \geq t_0 > \tau$, and $r_0 = r_\epsilon(u_\eta(t_0), g_0)$ be as defined above.

We consider two cases.

Case 1: $r_0 \geq \frac{1}{2}\mathrm{inj}_{g_0}$, we set $R_1 = \frac{1}{2}\mathrm{inj}_{g_0}$.

Case 2: $r_0 < \frac{1}{2}\mathrm{inj}_{g_0}$. We proceed to show an a priori lower bound for r_0. By the interior energy estimate (4.17), we see that for any point $x \in M$ and $0 < \delta < t_0$

$$E(u_\eta(t_0), B_{r_0}(x)) \leq 2E(u_\eta(t_0 - \delta), B_{r_0+r'}(x)) + \frac{C\delta}{r'^2}. \tag{4.33}$$

This will hold for any r' satisfying $r' + r_0 < \mathrm{inj}_{g_0}$. After rearranging, this gives us a lower bound on the energy concentration at intermediate times

$$E(u_\eta(t_0-\delta), B_{r_0+r'}(x)) \geq \frac{1}{2}E(u_\eta(t_0), B_{r_0}(x)) - \frac{C_1\delta}{r'^2}. \tag{4.34}$$

By construction, we can choose some $x_0 \in M$ such that $E(u_\eta(t_0), B_{r_0}(x_0)) \geq \epsilon$. Therefore with $r(t) := r_0 + \sqrt{\frac{4C_1(t_0-t)}{\epsilon}}$, we have $E(u_\eta(t), B_{r(t)}(x_0)) \geq \frac{\epsilon}{4}$ for $t \in (0, t_0)$ such that $r(t) < \mathrm{inj}_{g_0}$. In particular, we can find δ_1 (depending on ϵ, C_1 and inj_{g_0}) such that $r(t) < \mathrm{inj}_{g_0}$ on $(t_0-\delta_1, t_0)$. To ensure we only consider positive times we then set $\delta_2 = \min\{\delta_1, \tau\}$.

By Remark 4.1.8 we find that $\mathrm{inj}_{g_\eta(t)} \geq \frac{1}{2}\mathrm{inj}_{g_0}$ for $t \in [0, T]$ and we can apply (4.31) with respect to $g_\eta(t)$ to find

$$r_{\frac{\epsilon}{8}}(u_\eta(t), g_\eta(t)) \geq \frac{1}{K_{\frac{\epsilon}{8}}(1 + \left\|\tau_{g_\eta(t)}(u_\eta(t))\right\|_{L^2(M, g_\eta(t))})} \tag{4.35}$$

on the interval $t \in (t_0-\delta_2, t_0) \subset [0, t_0]$. Crucially, the constant $K_{\frac{\epsilon}{8}}$ does *not* depend on t or η (as we have a uniform lower bound for the injectivity radius of $g_\eta(t)$). We now translate this bound for concentration with respect to the changing metric $g_\eta(t)$ into a bound with respect to the initial metric g_0. We have

$$\begin{aligned} \frac{\epsilon}{4} \leq E(u_\eta(t), B_{r(t)}(x_0)) &= E(u_\eta(t), B^{g_0}_{r(t)}(x_0); g_0) && (4.36)\\ &\leq 2E(u_\eta(t), B^{g_0}_{r(t)}(x_0); g_\eta(t)) && (4.37)\\ &\leq 2E(u_\eta(t), B^{g_\eta(t)}_{2r(t)}(x_0); g_\eta(t)), && (4.38) \end{aligned}$$

where we used Remark 4.1.8. Hence $r_{\frac{\epsilon}{8}}(u_\eta(t), g_\eta(t)) \leq 2r(t)$. Combining this with (4.35) yields

$$2r(t) \geq \frac{1}{K_{\frac{\epsilon}{8}}(1 + \left\|\tau_{g_\eta(t)}(u_\eta(t))\right\|_{L^2(M, g_\eta(t))})}, \tag{4.39}$$

which after squaring and some manipulation leads to

$$C(1 + \left\|\tau_{g_\eta(t)}(u_\eta(t))\right\|^2_{L^2(M,g)}) \geq \frac{1}{r_0^2 + C(t_0-t)}, \tag{4.40}$$

where the constant C is now allowed to depend on $K_{\frac{\epsilon}{8}}$, C_1 and ϵ. We can integrate this inequality over $(t_0-\delta_2, t_0)$ (using the energy identity (2.22) to integrate the tension) and see that the right hand side diverges as $r_0 \to 0$ which implies a lower bound $R_1 > 0$ on r_0 in terms of (M, g_0), ϵ, u_0 and N.

Hence in either case we have found $R_1 > 0$, only depending on the initial data

(M, g_0), u_0, N and on ϵ, such that $E(u_\eta(t_0), B_r(x)) \leq \epsilon$ for all $r \leq R_1$, $x \in M$. As R_1 does not depend on t, we can now set $R = \min\{R_0, R_1\}$ and obtain the claim for all times $t \in [0, T]$. $\square$

L^2-bound for the map component

We now use the established uniform bound on the concentration of local energy to deduce a priori integral bounds on the flows u_η on some time interval $[0, T]$. The main idea is to integrate the bound (4.16). The following result is similar to [23, Section 4].

Lemma 4.1.17. *As before, take (N, G) to be a nonpositively curved smooth closed Riemannian manifold and M a smooth oriented closed surface of genus $\gamma \geq 2$. Let $T > 0$, then there exists $\eta_0 \geq \eta_1 > 0$, with η_0 from Remark 4.1.8, such that the following holds for all $\eta \leq \eta_1$. Consider the solution $(u_\eta(t), g_\eta(t))$ with coupling constant η to (2.21) defined on $[0, T]$ with initial data (u_0, g_0) as well as the harmonic map flow $u(t)$ solving (2.4) with the same initial data. We define $|\nabla V| := \max\{|\nabla u|_{g_0}, |\nabla u_\eta|_{g_0}\}$, then*

$$\int_0^T \int_M |\nabla V|_{g_0}^4 dv_{g_0} \leq C \tag{4.41}$$

for some constant C only depending on (M, g_0), u_0, N and T. Furthermore $|\nabla V|_{g_0}$ is in $L^2(M, g_0)$ on each time slice (with a bound only depending on $E_0 = E(u_0, g_0)$).

Proof. We first check the claims of the lemma for $|\nabla u_\eta|_{g_0}$. We will initially assume $\eta \leq \eta_0$, and choose η_1 later. We apply the bound (4.16) at each time $t \in [0, T]$ to the map $u_\eta(t)$ with respect to the initial metric $g = g_0$ and radius $r = R$ as obtained from Lemma 4.1.16 (with $\epsilon = \epsilon_0$ from Lemma 4.1.12). We find

$$\int_M |\nabla u_\eta|_{g_0}^4 dv_{g_0} + |\nabla^2 u_\eta|_{g_0}^2 dv_{g_0} dt \leq C(1 + \int |\tau_{g_0}(u_\eta)|^2) dv_{g_0}, \tag{4.42}$$

with a constant C depending on (M, g_0), u_0 and N, where we take ∇ to denote the connection induced by g_0.

We would like to use the energy identity (2.22) to integrate this inequality and further estimate the right hand side. Thus, as in [23], we want to replace $\tau_{g_0}(u_\eta)$

with $\tau_{g_\eta}(u_\eta)$ in (4.42). To do this, we estimate $|\tau_{g_\eta}(u_\eta) - \tau_{g_0}(u_\eta)|$ pointwise

$$|\tau_{g_\eta}(u_\eta) - \tau_{g_0}(u_\eta)| \leq C \left\| g_\eta(t) - g_0 \right\|_{C^0(M,g_0)} (|\nabla u_\eta|^2_{g_0} + |\nabla u_\eta|_{g_0} + |\nabla^2 u_\eta|_{g_0}), \quad (4.43)$$

see Lemma C.2 in the Appendix for a proof of this estimate. This allows us to estimate

$$\int |\tau_{g_0}(u_\eta)|^2 dv_{g_0} \leq C \int |\tau_{g_\eta}(u_\eta)|^2 dv_{g_0} \\ + C \left\| g_\eta(t) - g_0 \right\|_{C^0(M,g_0)} \int |\nabla u_\eta|^4_{g_0} + |\nabla u_\eta|^2_{g_0} + |\nabla^2 u_\eta|^2_{g_0} dv_{g_0}, \quad (4.44)$$

which we can use together with estimate (4.42) to find

$$\int_M |\nabla u_\eta|^4_{g_0} dv_{g_0} + |\nabla^2 u_\eta|^2_{g_0} dv_{g_0} dt \leq C + C \int |\tau_{g_\eta}(u_\eta)|^2 dv_{g_0} \\ + C \left\| g_\eta(t) - g_0 \right\|_{C^0(M,g_0)} \int |\nabla u_\eta|^4_{g_0} + |\nabla^2 u_\eta|^2_{g_0} dv_{g_0}, \quad (4.45)$$

where we absorbed the $|\nabla u_\eta|^2_{g_0}$ integral into the constant C by estimating it via e.g. $2E_0$ (using $\eta \leq \eta_0$). Hence we can pick $0 < \eta_1 \leq \eta_0$, only depending on (M, g_0), u_0, T and N, so that we can also absorb the other terms on the right when $\eta \leq \eta_1$ and obtain

$$\int_M |\nabla u_\eta|^4_{g_0} dv_{g_0} + |\nabla^2 u_\eta|^2_{g_0} dv_{g_0} dt \leq C(1 + \int |\tau_{g_\eta}(u_\eta)|^2) dv_{g_0} \quad (4.46)$$

with a constant C also depending on (M, g_0), u_0 and N. We now drop the $|\nabla^2 u_\eta|^2_{g_0}$ term (it was only required to control the error introduced by switching the metric of the tension) and integrate (4.46) over $[0, T]$ to find (4.41) for $|\nabla u_\eta|_{g_0}$, after using the energy identity (2.22) to estimate the tension term. We see that $|\nabla u_\eta|_{g_0}$ is in $L^2(M, g_0)$ on each time slice by Remark 4.1.8 combined with the monotonicity of the energy $E(u_\eta(t), g_\eta(t))$.

Finally we also have estimates of the same type for the harmonic map flow $u(t)$ (we could either replicate the previous work to deduce an a priori bound on the concentration of energy in the harmonic map flow, or simply allow C to depend on the flow itself, which is fixed), finishing the proof. □

We are now ready to prove the main closeness result for the map component.

Lemma 4.1.18. *Take M to be a smooth closed oriented surface of genus $\gamma \geq 2$ and*

N to be a smooth closed Riemannian manifold with nonpositive sectional curvature. Let $T > 0$, $\eta_1 \geq \eta > 0$, with η_1 from the previous lemma, and as before denote by $u(t)$, $u_\eta(t)$ the solutions (with respect to initial data (u_0, g_0)) to the usual harmonic map flow (2.4), respectively Teichmüller harmonic map flow (2.21) on $[0, T]$. Then given any $\epsilon > 0$, we can find $\eta_1 \geq \eta_\epsilon > 0$, depending on ϵ, T, u_0, (M, g_0) and N, such that for all $\eta_\epsilon \geq \eta > 0$ we have

$$\|u_\eta(t) - u(t)\|_{L^2(M,g_0)} < \epsilon \tag{4.47}$$

for $t \in [0, T]$.

Proof. We now use techniques from [23, Section 3], based on [27]. In particular the difference $w = u - u_\eta$ satisfies the evolution equation

$$\frac{\partial}{\partial t} w - \Delta_{g_0}(w) = (\Delta_{g_0} - \Delta_{g_\eta})(u_\eta) + A_{g_0}(u)(\nabla u, \nabla u) - A_{g_\eta}(u_\eta)(\nabla u_\eta, \nabla u_\eta), \tag{4.48}$$

where A denotes the second fundamental form of the target $N \hookrightarrow \mathbb{R}^n$ and we write $A_g(u)(\nabla u, \nabla u) = g^{ij} A(u)(\partial_i u, \partial_j u)$.

Multiplying this equation by w and integrating over (M, g_0) together with partial integration (with respect to g_0) we obtain

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \|w\|^2_{L^2(M,g_0)} + \|\nabla w\|^2_{L^2(M,g_0)} = &\int_M (\Delta_{g_0} - \Delta_{g_\eta})(u_\eta) w dv_{g_0} \\ &+ \int_M A_{g_0}(u)(\nabla u, \nabla u) w dv_{g_0} \\ &- \int_M A_{g_\eta}(u_\eta)(\nabla u_\eta, \nabla u_\eta) w dv_{g_0}. \end{aligned} \tag{4.49}$$

We now estimate the terms on the right hand side, and all norms in the following, as well as the volume form dv, are taken to be defined with respect to g_0. For the first term we find

$$|\int_M (\Delta_{g_0} - \Delta_{g_\eta})(u_\eta) w dv| \leq C \|g_0 - g_\eta\|_{C^0} \|\nabla w\|_{L^2} \|\nabla u\|_{L^2} \leq C \|g_0 - g_\eta\|_{C^0} \|\nabla w\|_{L^2} \tag{4.50}$$

where we used integration by parts (with respect to both g_0 and g_η, as the volume forms are the same by 2.1.7) and $\|\nabla u\|^2_{L^2} \leq C E_0 \leq C$, as well as Lemma C.1 to estimate the difference of the inverse metric tensors of g_0 and g_η. Here and in the following we let C denote positive constants, which may depend on (M, g_0), N and the initial energy $E(u_0, g_0)$. We proceed to estimate the second fundamental form terms. Note that

$$A_{g_0}(u)(\nabla u, \nabla u) - A_{g_\eta}(u_\eta)(\nabla u_\eta, \nabla u_\eta) = A_{g_0}(u)(\nabla u, \nabla u) - A_{g_0}(u_\eta)(\nabla u_\eta, \nabla u_\eta) + A_{g_0}(u_\eta)(\nabla u_\eta, \nabla u_\eta) - A_{g_\eta}(u_\eta)(\nabla u_\eta, \nabla u_\eta), \tag{4.51}$$

which we estimate pointwise

$$|A_{g_0}(u)(\nabla u, \nabla u) - A_{g_0}(u_\eta)(\nabla u_\eta, \nabla u_\eta)| \leq C(|\nabla w||\nabla V| + |w||\nabla V|^2) \tag{4.52}$$

$$|A_{g_0}(u_\eta)(\nabla u_\eta, \nabla u_\eta) - A_{g_\eta}(u_\eta)(\nabla u_\eta, \nabla u_\eta)| \leq C \|g_0 - g_\eta\|_{C^0} |\nabla V|^2. \tag{4.53}$$

The first inequality follows by an application of the mean value theorem to A (which is a smooth function on N) (see also [27] for this estimate), for the second we can again apply Lemma C.1, see also the proof of Lemma C.2 in the Appendix. Multiplying (4.51) by w, integrating and applying Hölder's inequality gives

$$\int_M A_{g_0}(u)(\nabla u, \nabla u)w - A_{g_\eta}(u_\eta)(\nabla u_\eta, \nabla u_\eta)w dv \leq C \|g_0 - g_\eta\|_{C^0} \|\nabla V\|^2_{L^4} \|w\|_{L^2} + C \|\nabla w\|_{L^2} \|\nabla V\|_{L^4} \|w\|_{L^4} + C \|\nabla V\|^2_{L^4} \|w\|^2_{L^4}. \tag{4.54}$$

Combining (4.50) and (4.54) with (4.49) we arrive at our main estimate (see also [23, Section 4] for a similar estimate)

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \|w\|^2_{L^2} + \|\nabla w\|^2_{L^2} \leq C \|g_0 - g_\eta\|_{C^0} \left(\|\nabla w\|_{L^2} + \|w\|_{L^2} \|\nabla V\|^2_{L^4}\right) + C \|\nabla w\|_{L^2} \|\nabla V\|_{L^4} \|w\|_{L^4} + C \|\nabla V\|^2_{L^4} \|w\|^2_{L^4}. \tag{4.55}$$

The strategy is now to derive an estimate for $\frac{\mathrm{d}}{\mathrm{d}t} \|w(t)\|_{L^2}$ that allows us to apply Gronwall's lemma to deduce our desired smallness. In particular we need to control all the terms in (4.55) through quantities integrable in time (e.g. $\|\nabla V\|_{L^4}$), $\|\nabla w\|^2_{L^2}$ and $\|w(t)\|^2_{L^2}$.

To this end, recall the following consequence of Sobolev's inequality (see [23]):

$$\|w\|^2_{L^4} \leq C \|w\|_{L^2} (\|w\|_{L^2} + \|\nabla w\|_{L^2}). \tag{4.56}$$

Using Young's inequality we further find

$$C\,\|\nabla w\|_{L^2}\,\|\nabla V\|_{L^4}\,\|w\|_{L^4}+C\,\|\nabla V\|_{L^4}^2\,\|w\|_{L^4}^2\le\frac{1}{4}\,\|\nabla w\|_{L^2}^2+C\,\|\nabla V\|_{L^4}^2\,\|w\|_{L^4}^2\,.$$

Together with (4.56) and Young's inequality this implies

$$C\,\|\nabla w\|_{L^2}\,\|\nabla V\|_{L^4}\,\|w\|_{L^4}+C\,\|\nabla V\|_{L^4}^2\,\|w\|_{L^4}^2\le\frac{1}{2}\,\|\nabla w\|_{L^2}^2+C(1+\|\nabla V\|_{L^4}^4)\,\|w\|_{L^2}^2\,, \tag{4.57}$$

which is of the desired form.

Similarly we can estimate the first term in the right hand side of (4.55) via Young's inequality to obtain

$$\begin{aligned}C\,\|g_0-g_\eta\|_{C^0}\left(\|\nabla w\|_{L^2}+\|w\|_{L^2}\,\|\nabla V\|_{L^4}^2\right)&\\ \le C\,\|g_0-g_\eta\|_{C^0}^2+\frac{1}{4}\,\|\nabla w\|_{L^2}^2+C\,\|w\|_{L^2}^2\,\|\nabla V\|_{L^4}^4\,.&\end{aligned} \tag{4.58}$$

Putting these together and using the notation $\psi(t)=1+\|\nabla V\|_{L^4}^4\in L^1([0,T])$ from [23] we arrive at

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\,\|w\|_{L^2}^2\le C\,\|g_0-g_\eta\|_{C^0}^2+C\,\|w\|_{L^2}^2\,\psi(t). \tag{4.59}$$

We can apply Gronwall's lemma to this inequality for the function $\|w(t)\|_{L^2}^2$ and finally get

$$\|w(t)\|_{L^2}^2\le C\int_0^t\|g_0-g_\eta\|_{C^0}^2\,e^{C\int_0^t\psi(s)ds}\le C\int_0^t\|g_0-g_\eta\|_{C^0}^2\,e^{C(t+1)}. \tag{4.60}$$

From (4.5) we see that we can indeed choose η_ϵ small such that

$$\|u(t)-u_\eta(t)\|_{L^2(M,g_0)}<\epsilon \tag{4.61}$$

for all $t\in[0,T]$ and $\eta\le\eta_\epsilon$. Explicitly we can choose η_ϵ such that

$$C\int_0^T\|g_0-g_{\eta_\epsilon}\|_{C^0}^2\le\epsilon\,e^{-C(T+1)}.$$

□

Higher order bounds

We now improve this L^2-bound using a simple interpolation argument, exploiting the fact that the Teichmüller harmonic map flow enjoys good a priori estimates.

This requires an $R(\epsilon_0) > 0$ such that we have control on the local energy $E(u_\eta, B_R) < \epsilon_0$ for all sufficiently small η, which is provided by Lemma 4.1.16. Note that the precise ϵ_0 required comes as an ingredient from [23].

Lemma 4.1.19. *In the setting of Lemma 4.1.18, let $T > 0$, then there exists $\eta_3 > 0$ such that the Hölder-norms (in space and time, and up to arbitrary order) of the difference w on the interval $[0, T]$ stay bounded for $\eta \leq \eta_3$, with (uniform in η) bounds depending only on the initial data (u_0, g_0), T, M and N.*

Proof. We initially set $\eta_3 = \eta_0$, with η_0 from Remark 4.1.8. This allows us to use Lemma 4.1.16 to establish uniform control of the concentration radius of u_η. We then also demand η_3 to be small enough so that the theory in [23, Section 3] applies, in particular we require the estimate $\|g_0 - g_\eta(t)\|_{H^s} \leq \epsilon_1$ with $\epsilon_1 = \epsilon_1(g_0, s) > 0$ as defined in [23, Equation (3.3)]. The claim is then a direct consequence of [23, Lemma 3.5, Remark 3.6, Remark 3.7], together with the smoothness of the harmonic map flow starting at u_0.

To obtain estimates for the Teichmüller harmonic map flow, a very similar technique to the harmonic map flow case ([27]) was used in [23]. The basic idea is to bound $\int |\tau_{g_\eta}(t)|^2$ using the control of the energy concentration and equation (2.21) for the map. Once that is done, the estimate from lemma 4.1.12 provides a $W^{2,2}$-bound on the map, which implies control in (any) $W^{1,p}$-norm by the Sobolev embedding theorem. One can then use parabolic regularity applied to the equation $\partial_t u = \tau_g(u)$ viewed as the inhomogeneous heat equation $\partial_t u - \Delta_g(u) = A_g(u)(\nabla u, \nabla u)$ to obtain bounds in the parabolic Sobolev space $W_p^{2,1}$. A bootstrapping argument then gives higher order control. $\square$

By a standard interpolation argument, this allows us to extend the results of Lemma 4.1.18 to higher norms.

Corollary 4.1.20. *Under the assumptions of the above lemma, the Hölder-norms (in space and time, and up to arbitrary order) of the difference w on $[0, T]$ converge to 0 as $\eta \downarrow 0$.*

Proof. Using interpolation (e.g. Ehrling's lemma), for any $\epsilon > 0$ we can find some $C(\epsilon) > 0$ such that

$$\|w\|_{C^{k,\alpha}} \leq \epsilon \|w\|_{C^{k+1,\alpha}} + C \|w\|_{L^2(M,g_0)},$$

which implies the claim by Lemma 4.1.18 and Lemma 4.1.19. Here we mean $C^{k,\alpha}$ to be a generic parabolic Hölder space of order k. $\square$

We now obtain the main Theorem 4.1.1 from Corollaries 4.1.9 and 4.1.20.

4.2 A rescaled limit as $\eta \downarrow 0$

We can study limits of (2.21) for varying η under different time scales. One that turns out to be particularly interesting is the scaling $\bar{t} = \frac{\eta^2}{4}t$. This in some sense 'fixes' the speed at which the metric evolves. The equations for the flow become

$$\begin{aligned} \frac{\partial}{\partial t}u &= k\tau_g(u) \\ \frac{\partial}{\partial t}g &= Re(P_g(\Phi(u,g))) \end{aligned} \tag{4.62}$$

where we defined $k = \frac{4}{\eta^2}$, so $\eta \to 0$ corresponds to $k \to \infty$.

Setting for this section: We again take (u_0, g_0) to be smooth initial data for all flows considered, with $u_0 : M \to N$, $g_0 \in \mathcal{M}_{-1}$. We further take $N = (N, G)$ to be a smooth closed Riemannian manifold, which we will consider to be isometrically embedded as $N \hookrightarrow \mathbb{R}^n$. Initially, we will not assume N to be nonpositively curved (as opposed to the last section), but instead assume that we have a smooth solution of (4.62) up to some time T.

4.2.1 Metric control on large time intervals for small η

Analysing the flow (4.62) on a time interval $[0, T]$ corresponds to studying the original flow (2.21) on $[0, kT]$. In particular, for fixed T we consider longer and longer time intervals as $\eta \to 0$, as opposed to the last section. We first show that the injectivity radius of the metric does not degenerate in this setting, independent of k, as long as $T > 0$ is chosen sufficiently small. We then use the controlled injectivity radius to establish estimates on the metric.

Denote the length of shortest closed geodesic on $(M, g(t))$ by $\ell(t)$, then it is a standard result that $\text{inj}_{g(t)} = \frac{1}{2}\ell(t)$ when M is a compact hyperbolic surface. We will proceed by bounding $\ell(t)$ from below. It is convenient to do this in the setting of the original flow (2.21), i.e. without rescaling time. We use the theory in [20]. We will analyse how the lower bounds for $\ell(t)$ derived therein depend on η.

Recall that a 'collar' on a hyperbolic surface is a specific region around a central closed geodesic, which is isometric to a cylinder equipped with a metric conformal to the flat metric (see Section 2.2.4 and Lemma A.1). The problem of bounding $\ell(t)$ from below is then equivalent to controlling the lengths of the central geodesics on such collars (see e.g. [25]). We have the following result from [20] to control the evolution of collars.

Lemma 4.2.1 (Slight variant of [20, Lemma 2.3])**.** *Let M be a closed oriented smooth surface of genus $\gamma \geq 2$ and N be a smooth closed Riemannian manifold. Assume that we have some $T > 0$, $\eta > 0$ such that the flow (u_η, g_η) is a smooth solution to (2.21) on $[0, T]$ starting at initial data (u_0, g_0) with coupling constant η. Then given a collar $\mathcal{C}$ in $(M, g_\eta(t_0))$ for some time $t_0 \in [0, T]$, with central geodesic σ of length $\ell(t_0) < 2\,\text{arsinh}(1)$, there holds*

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\log \ell(t_0)\right| \leq C\eta^2 \ell(t_0)[I + E_0], \tag{4.63}$$

with a constant $C < \infty$ depending only on the genus γ of M, where $E_0 = E(u_0, g_0)$ and I is a weighted energy given by

$$I := \int_{\mathcal{C}} e(u_\eta, g_\eta)\rho^{-2} dv_{g_\eta}, \tag{4.64}$$

where $e(u_\eta, g_\eta)$ denotes the energy density (e.g. $e(u_\eta, g_\eta) = \frac{1}{2}|du_\eta|^2_{g_\eta}$) and ρ is the conformal factor on the collar (as defined in Lemma A.1).

Proof. This follows directly from equation (2.1) in [20]. $\square$

We use this lemma to deduce a (short-time) bound on inj_{g_η}, which improves as $\eta \downarrow 0$.

Proposition 4.2.2. *Let M, N and $(u_\eta(t), g_\eta(t))$ be as in the previous lemma. Denote the shortest closed geodesic on $(M, g_\eta(t))$ for $t \in [0, T]$ by $\ell(t)$, then we can estimate ℓ by*

$$\ell(t) \geq \min\{\text{arsinh}\, 1, \ell(g_0)\} - CT\eta^2 \tag{4.65}$$

where $C < \infty$ now depends on $E_0 = E(u_0, g_0)$ and the genus γ of M.

Proof. If $\ell(t) \geq 2\,\mathrm{arsinh}(1)$ at all times, we take that as our lower bound. Otherwise, consider some time $t_0 \in [0, T]$ with $\ell(t_0) < 2\,\mathrm{arsinh}(1)$. Hence we can find some collar $\mathcal{C}$ on $(M, g(t_0))$ with central geodesic of length $\ell(t_0) < 2\,\mathrm{arsinh}(1)$. We can therefore analyse this collar using Lemma 4.2.1. At t_0 we find

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\log \ell(t_0)\right| \leq C\eta^2 \ell(t_0)[I + E_0]. \tag{4.66}$$

We now note that by its definition the weighted energy satisfies

$$I(t_0) \leq CE(u_\eta(t_0), g_\eta(t_0))\ell(t_0)^{-2} \tag{4.67}$$

(as the conformal factor ρ is always bounded from below in terms of ℓ, see e.g. [20]). Using this we find the bound

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\ell(t_0)\right| \leq C\eta^2 E_0 \tag{4.68}$$

with C only depending on the genus γ of M. Hence in particular, if $\ell(t) < 2\,\mathrm{arsinh}(1)$ for all $t \in [0, t_0]$ we find that

$$\ell(t_0) \geq \ell(g_0) - Ct_0\eta^2 E_0 \geq \ell(g_0) - CT\eta^2 \tag{4.69}$$

where C now also depends on E_0. Otherwise we can find some time $0 < t_1 < t_0$ such that $\mathrm{arsinh}(1) \leq \ell(t_1) < 2\,\mathrm{arsinh}(1)$ and apply the same argument over $[t_1, t_0]$ to find that

$$\ell(t_0) \geq \mathrm{arsinh}(1) - CT\eta^2. \tag{4.70}$$

In either case we have $\ell(t_0) \geq \min\{\mathrm{arsinh}\,1, \ell(g_0)\} - CT\eta^2$, finishing the proof. $\square$

Lemma 4.2.3. *Let M be a smooth oriented closed surface of genus $\gamma \geq 2$ and N be a smooth closed Riemannian manifold. Assume that for some $T > 0$, $k > 0$ we have a smooth solution $(u_k(t), g_k(t))$ to (4.62) on $[0, T]$ with rescaled coupling constant k for some given initial data (u_0, g_0). Then there exists $T_0 = T_0(M, g_0, u_0) > 0$ (in particular* independent *of k) such that the injectivity radius inj_{g_k} is bounded away from 0 up to time $t = \min\{T_0, T\}$.*

Proof. This is a consequence of the bound derived in the previous proposition. Explicitly, we can choose $T_0 = \frac{1}{8C}\min\{\mathrm{arsinh}\,1, \ell(g_0)\}$ (with the C from Proposition

4.2.2) to achieve $\mathrm{inj}_{g_k} \geq \frac{1}{2}\ell(g_k) \geq \frac{1}{4}\min\{\operatorname{arsinh} 1, \ell(g_0)\}$ on $[0, \min\{T_0, T\}]$. $\square$

We now assume an injectivity radius bound for solutions of 2.21. This allows us to state the following C^k-estimate from [22] for horizontal curves (as defined in 4.1.5).

Lemma 4.2.4 (Special case of [22, Lemma 3.2])**.** *Let M be a smooth oriented closed surface of genus $\gamma \geq 2$. Let $\epsilon > 0$ and consider a horizontal curve of metrics $g(t)$ on M defined on the interval $[0, T]$, with uniformly bounded injectivity radius $\mathrm{inj}_{g(t)} \geq \epsilon$. Then there exists a $\delta > 0$, depending only on γ and ϵ, such that if the L^2-length (as defined in Lemma 4.1.3) satisfies $L(g, [s, t]) < \delta$ for some $[s, t] \subset [0, T]$, then we have some $C_1 > 0$ only depending on γ such that for any $t_1, t_2 \in [s, t]$ there holds*

$$g(t_1) \leq C_1 g(t_2). \tag{4.71}$$

We further have some constant C_2 only depending on M, ϵ and k such that

$$|g(t_1) - g(t_2)|_{C^k(g(t_0))}(x) \leq C_2 L(g, [t_1, t_2]), \tag{4.72}$$

for any $s \leq t_1 \leq t_2 \leq t$, $t_0 \in [s, t]$.

Remark 4.2.5. Note that this is strictly stronger than Proposition 4.1.6 cited in the last section, as the constant C_2 here does not depend on the initial metric of the considered horizontal curve.

We can now apply this lemma along horizontal curves $g(t)$ arising from solutions to (4.62). We first show that the metrics (assuming an injectivity radius bound) stay equivalent.

Corollary 4.2.6. *As usual, take M to be a smooth closed oriented surface of genus $\gamma \geq 2$ and N to be a smooth closed Riemannian manifold. Let $T > 0$, $k > 0$ and consider a smooth solution $(u_k(t), g_k(t))$ to (4.62) on the time interval $[0, T]$ with rescaled coupling constant k starting at the initial data (u_0, g_0). Assume that there exists $\delta > 0$ such that $\mathrm{inj}_{g_k(t)} \geq \delta > 0$ for $t \in [0, T]$. Then we can find a constant $C > 0$, only depending on δ, M, $E(u_0, g_0)$ and T such that for any $s, t \in [0, T]$ we have*

$$g_k(s) \leq C g_k(t). \tag{4.73}$$

Proof. By Lemma 4.1.3, we obtain a bound of the form

$$L(g_k, [s, t]) \leq \sqrt{(t - s) E(u_0, g_0)}. \tag{4.74}$$

Now consider two arbitrary times $s, t \in [0, T]$, assume $s < t$. We can then find a sufficiently large integer $K = K(\delta, M, E(u_0, g_0), T) > 0$ such that $g(t)$ restricted to time intervals of length at most $\Delta = \frac{t-s}{K}$ satisfies the L^2-length condition required for Lemma 4.2.4. Then simply apply estimate (4.71) from the above lemma on the intervals $[s, s+\Delta], \ldots, [t-\Delta, t]$. After combining the resulting inequalities we obtain the claim. $\square$

Hence the metric along solutions of the flow (4.62) stays uniformly equivalent to e.g. $g(0) = g_0$, as long as the injectivity radius is controlled, in particular up to the time T_0 from Lemma 4.2.3. We now observe that certain norms defined with respect to the changing metric also stay controlled. This problem was already considered in [22, Section 3], and the same methods directly apply in our situation.

Let s be a non-negative integer here (to avoid confusion with the rescaled coupling constant k). Recall that we defined the $C^s(M, g)$-norm of (in particular) tensors $h \in \mathrm{Sym}^2(T^*M)$ via

$$\|g\|_{C^s(M,g)} := \sup_{x\in M} \sum_{l=0}^{s} |\nabla^l g|_g(x), \tag{4.75}$$

where ∇ refers to the Levi-Civita connection on (M, g) and its extensions. We can similarly define a norm $C^s(M, g)$ for maps $u : M \to N \hookrightarrow \mathbb{R}^n$ by

$$\|u\|_{C^s(M,g)} := \sup_{x\in M} \sum_{l=0}^{s} |\nabla^l u|_g(x). \tag{4.76}$$

Lemma 4.2.7 (From Section 3 of [22])**.** *Let $T > 0, k > 0$ and take M, N and (u_k, g_k) as in the above lemma. Assume again that there is some $\epsilon > 0$ such that $\mathrm{inj}_{g_k} \geq \epsilon$ on $[0, T]$. Then the $C^s(M, g(t))$ norms as defined above are uniformly equivalent on $[0, T]$, in the sense that there exists some $C = C(s, \epsilon, M, E(u_0, g_0), T) > 0$ such that for any $t_1, t_2 \in [0, T]$ we have*

$$\|u\|_{C^s(M,g(t_1))} \leq C \|u\|_{C^s(M,g(t_2))}.$$

Note that u here can be either a tensor or a map $u : M \to N \hookrightarrow \mathbb{R}^n$.

Proof. This can be seen as in the proof of [22, Lemma 3.2], combined with standard estimates for $|\partial_t g(t)|_{C^s(M,g(t))}$ (e.g. as mentioned in Section 2.2.4), when $u \in \mathrm{Sym}^2(T^*M)$. For maps $u : M \to N \hookrightarrow \mathbb{R}^n$ Lemma 3.2 in [22] extends, in particular for $s \geq 1$ we can find an inequality of the form

$$\frac{\partial}{\partial t} |\nabla^s u|_{g(t)}(x) \leq C |u|_{C^s(g(t))}(x) |\partial_t g|_{C^{s-1}(g(t))}(x), \tag{4.77}$$

which holds for almost all times t and all $x \in M$, and can then be integrated. Note that for $s = 0$ there is nothing to prove as the C^0-norm of a map does not depend on the underlying metric. □

4.2.2 Evolution of the tension

The previous section motivates us to further study solutions to (2.21) with injectivity radius bounded from below. A quantity that proved very important in the study of the harmonic map flow is the L^2-norm of the tension $\mathcal{T}(u, g) = \|\tau_g(u)\|^2_{L^2(M,g)}$. For the classical harmonic map flow into nonpositively curved targets this turns out to be monotonically decreasing in time. This monotonicity can be seen by computing the second variation of the energy along solutions of the harmonic map flow, which is a well-known calculation (see e.g. [8]). We show that for solutions to (2.21) (assuming the metric does not degenerate) with a target of nonpositive curvature we have a bound on how fast the tension can increase instead, which improves for small η. The curvature hypothesis on N also again means that any smooth initial data (u_0, g_0) together with a choice of η now leads to a smooth solution to (2.21) that exists for all times t.

Lemma 4.2.8. *Assume M as usual to be a smooth closed oriented surface of genus $\gamma \geq 2$ and N to be a smooth closed Riemannian manifold, which we now also assume to have nonpositive curvature. Let $\eta > 0$ and take $(u(t), g(t))$ to be the (smooth) solution to (2.21) with coupling constant η, starting at initial data (u_0, g_0). Assume that there exists some $\delta > 0$ such that $\mathrm{inj}_{g(t)} \geq \delta > 0$, and denote as usual $E_0 = E(u(0), g(0))$, then*

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{T}(t) \leq C E_0^3 \delta^{-2} \eta^4 \tag{4.78}$$

where $\mathcal{T}(t) := \mathcal{T}(u(t), g(t))$ and $C < \infty$ only depends on the genus γ of M.

Proof. In the following we use the formalism of the induced covariant derivative ∇_t on $M \times [0, T]$, which agrees with $\frac{\partial}{\partial t}$ for time-dependent functions, as explained in e.g. [16, p. 86ff]. We further consider $N \hookrightarrow \mathbb{R}^n$ to be isometrically embedded. We have $\mathcal{T}(u,g) = \int_M g^{ij}\nabla_i\nabla_j u^k g^{mn}\nabla_m\nabla_n u^k dv_g$. Hence

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{T} = 2\int_M \nabla_t(g^{ij}\nabla_i\nabla_j u^k)g^{mn}\nabla_m\nabla_n u^k dv_g, \tag{4.79}$$

where we used that the flow (2.21) leaves the induced volume form invariant (see Remark 2.1.7), and thus no additional term involving the metric appears upon differentiating the integral. We now evaluate the first factor and drop the volume form and the superscript on the map for simplicity, and adopt the shorthand $h := \frac{\partial}{\partial t}g = \frac{\eta^2}{4}Re(P_g(\Phi(u,g)))$:

$$\nabla_t(g^{ij}\nabla_i\nabla_j u) = -h^{ij}\nabla_i\nabla_j u + g^{ij}\nabla_t\nabla_i\nabla_j u. \tag{4.80}$$

We switch derivatives in the second term and obtain (for a derivation, see e.g. [16, A.14, p. 86ff])

$$g^{ij}\nabla_t\nabla_i\nabla_j u = g^{ij}\left(\nabla_i\nabla_j\frac{\partial}{\partial t}u + \mathrm{Rm}^N(\frac{\partial}{\partial t}u, \nabla_i u)\nabla_j u - (\frac{\partial}{\partial t}\Gamma^k_{ij})\nabla_k u\right). \tag{4.81}$$

From standard formulas we have the evolution of the Christoffel symbols given by

$$\frac{\partial}{\partial t}\Gamma^k_{ij} = \frac{1}{2}g^{kq}(\nabla_i h_{jq} + \nabla_j h_{iq} - \nabla_q h_{ij}). \tag{4.82}$$

Note that after tracing this with the metric g (in i,j) it vanishes as $\delta h = \mathrm{tr}h = 0$ by Remark 2.1.7. We now simplify (4.80) further, using the convention of repeated indices denoting traces (as we carried out the time derivatives now):

$$\nabla_t(g^{ij}\nabla_i\nabla_j u) = -h^{ij}\nabla_i\nabla_j u + \nabla_i\nabla_i\nabla_j\nabla_j u + \mathrm{Rm}^N(\nabla_k\nabla_k u, \nabla_i u)\nabla_i u. \tag{4.83}$$

Putting this back into (4.79) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{T} = 2\int_M \left(-h^{ij}\nabla_i\nabla_j u^k + \nabla_i\nabla_i\nabla_j\nabla_j u^k + (\mathrm{Rm}^N(\nabla_p\nabla_p u, \nabla_i u)\nabla_i u)^k\right)\nabla_m\nabla_m u^k. \tag{4.84}$$

Inspecting the terms we have L^2-inner products (in the bundle $u^*(TN)$), we now integrate the first (using $\delta h = 0$) and second term by parts to get

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{T} = 2\int_M h^{ij}\nabla_j u^k\nabla_i\nabla_m\nabla_m u^k - 2\int_M \nabla_i\nabla_j\nabla_j u^k\nabla_i\nabla_m\nabla_m u^k + 2\int_M \langle \mathrm{Rm}^N(\tau_g(u), du(e_i))du(e_i), \tau_g(u))\rangle. \tag{4.85}$$

Using inner product notation we finally arrive at

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{T} = 2\int_M \langle h_{ij}, \langle \nabla_i u, \nabla_j \tau_g(u)\rangle_{u^*(TN)}\rangle - 2\int_M \langle \nabla\tau_g(u), \nabla\tau_g(u)\rangle + 2\int_M \langle \mathrm{Rm}^N(du(e_i), \tau_g(u))\tau_g(u), du(e_i)\rangle. \tag{4.86}$$

We now proceed to estimate the first term in (4.86). Recall that given a lower bound δ on inj_g we can estimate

$$\|Re(\theta)\|_{L^\infty(M,g)} \leq \|\theta\|_{L^\infty(M,g)} \leq C\delta^{-1}\|\theta\|_{L^1(M,g)} \tag{4.87}$$

for any holomorphic quadratic differential θ with a constant $C < \infty$ depending only on γ ([22, Section 2]). We further know from [20, Proposition 4.10] that P_g is a bounded operator from L^1 to L^1, i.e.

$$\|P_g(\phi)\|_{L^1(M,g)} \leq C\|\phi\|_{L^1(M,g)} \tag{4.88}$$

for any quadratic differential ϕ where $C < \infty$ again only depends on γ. Together with the uniform bound $\|\Phi(u,g)\|_{L^1(M,g)} \leq CE(u(t), g(t)) \leq CE_0$ we see that

$$\left\|Re(P_{g(t)}(\Phi(u(t), g(t))))\right\|_{L^\infty(M,g)} \leq C\delta^{-1}\left\|P_{g(t)}(\Phi(u(t), g(t))\right\|_{L^1(M,g)} \leq C\delta^{-1}E_0. \tag{4.89}$$

Using this we estimate the first integral in (4.86) as

$$\left|\int_M \langle \frac{\eta^2}{4} Re(P_{g(t)}(\Phi(u(t), g(t)))_{ij}, \langle \nabla_i \tau_{g(t)}(u(t)), du(e_j)\rangle\rangle\right| \tag{4.90}$$

$$\leq \int_M C\delta^{-1}E_0\frac{\eta^2}{4}|\langle\nabla\tau_{g(t)}(u(t)), du\rangle| \tag{4.91}$$

$$\leq \int_M |\nabla\tau_{g(t)}|^2 + C\delta^{-2}E_0^2\eta^4\int_M |du|^2 \tag{4.92}$$

$$\leq \int_M |\nabla\tau_{g(t)}|^2 + C\delta^{-2}E_0^3\eta^4. \tag{4.93}$$

Here we used Young's inequality in the second inequality. From (4.86), using the nonpositive sectional curvature of the target, we therefore obtain the claim

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{T}(t) \leq CE_0^3\delta^{-2}\eta^4. \tag{4.94}$$

□

We use this bound to see that for large k the flow (4.62) very quickly has small tension. Note that the limiting $\eta = 0$ (or $k = \infty$) case corresponds to the classical harmonic map flow, which satisfies $\mathcal{T}(t) \to 0$ as $t \to \infty$. Our estimate could be considered a quantitative version of this statement, allowing the metric to move slightly.

Corollary 4.2.9. *With M, N, $(u(t), g(t))$ and δ as in the previous lemma, we have for any $\varepsilon > 0$, and $k = \frac{4}{\eta^2} \geq 1$*

$$\mathcal{T}(t) \leq C(\varepsilon)k^{-1} \ , \ t \geq \varepsilon k, \tag{4.95}$$

where $C(\varepsilon) \to \infty$ for $\varepsilon \to 0$ and $C(\varepsilon)$ also depends on E_0, in addition to δ and the genus γ of M. In particular, after carrying out the rescaling $\bar{t} = \frac{1}{k}t$, this gives $\mathcal{T}(\bar{t}) \leq C(\varepsilon)k^{-1}$ for $\bar{t} > \varepsilon$.

Proof. We note that Lemma 4.2.8 provides us with a linear estimate on $\mathcal{T}(t)$. Together with the L^1-bound $\int_0^T \mathcal{T}(t) \leq E(u_0, g_0)$ this implies the claimed point-wise bound, by simply comparing with an appropriate linear function and calculating the respective L^1-norm.

In particular, consider some time $t_0 \geq \varepsilon k$ and to simplify notation set $A := CE_0^3\delta^{-2}\eta^4 = Ck^{-2}$ to be the derivative bound from the previous lemma and $h := \varepsilon k$. We want to show an upper bound for $\mathcal{T}(t_0)$. If $\mathcal{T}(t_0) \leq Ah = C\varepsilon k^{-1}$, we take Ah as our upper bound. Otherwise, define a (positive) linear function $f(t)$ by $f'(t) \equiv A$ and $f(t_0) = \mathcal{T}(t_0)$ on $[t_0 - h, t_0]$. We find that $\frac{\mathrm{d}}{\mathrm{d}t}(\mathcal{T}(t) - f(t)) \leq 0$, hence $f(t) \leq \mathcal{T}(t)$ on $[t_0 - h, t_0]$. Thus the L^1-norm of f is bounded from above by the L^1-norm of $\mathcal{T}(t)$ (on $[t_0 - h, t_0]$), and in particular by E_0. We compute the L^1-norm of f as

$$\|f\|_{L^1(t_0-h,t_0)} = h(\mathcal{T}(t_0) - Ah) + \frac{1}{2}Ah^2 = h(\mathcal{T}(t_0) - \frac{1}{2}Ah). \tag{4.96}$$

Therefore we have

$$h(\mathcal{T}(t_0) - \frac{1}{2}Ah) \leq E_0 \tag{4.97}$$

$$\mathcal{T}(t_0) \leq \frac{1}{h}E_0 + \frac{1}{2}Ah \tag{4.98}$$

$$\mathcal{T}(t_0) \leq k^{-1}(E_0\varepsilon^{-1} + C\varepsilon) = k^{-1}C(\varepsilon). \tag{4.99}$$

Thus $\mathcal{T}(t) \leq k^{-1}C(\varepsilon)$ in either case, proving the claim. □

4.2.3 A priori estimates assuming small tension

We now proceed to establish (long-term) a priori estimates for the flow (2.21) assuming an L^2-bound on the tension and a lower injectivity radius bound $\mathrm{inj}_g > r_0$ on the metric, aiming to then apply these estimates to the rescaled flow (4.62).

To this end, we work on geodesic balls $B_r(x)$ of radius comparable to r_0 with respect to a metric g. As explained in Remark 4.1.11, we can take hyperbolic isothermal coordinates on these balls, which allows us to view them as hyperbolic disks $(D_{r'}, g_H)$, which are in particular conformally equivalent to the euclidean disks $(D_{r'}, g_{eucl})$. The conformal factor only depends on an upper bound of the size of the disks considered (i.e. r). In the rest of this section we will always use $B_r(x)$ to refer to this particular choice of coordinates, and we will carry out all the regularity theory below on the corresponding euclidean disks $(D_{r'}, g_{eucl})$, unless indicated otherwise.

We first establish a local bound for maps with small tension, using the control of the energy concentration established in the last section.

Lemma 4.2.10. *Let $B_r(x)$ for some $x \in M$ be as above with radius $r < r_0 < \mathrm{inj}_g$ where (M, g) is a smooth closed oriented hyperbolic surface as usual, N has nonpositive sectional curvature, and $u : (M, g) \to N$ is a smooth map with energy bounded by $E(u, g) \leq E_0$ and tension bounded by $\|\tau_g(u)\|_{L^2(M,g)} \leq K$. Then we have $\|\nabla^2 u\|^2_{L^2(B_{\frac{r}{2}})} \leq C(\|\tau_g(u)\|^2_{L^2(M,g)} + 1)$, for some C only depending on r_0, E_0, K and N.*

Proof. Recall Lemma 4.1.15, this implies that with $\|\tau_g(u)\|_{L^2(M,g)} \leq K$ we have some $0 < r_1 \leq r_0$, only depending on r_0, E_0, K and N, such that for any $y \in M$ we have $E(u, B_{r_1}(y)) < \epsilon_0$, where $\epsilon_0 > 0$ is the constant from Lemma 4.1.10. We can then use the bound from Lemma 4.1.10 on disks of radius r_1 covering $B_r(x)$ if $r_1 < r$, otherwise we directly apply it on $B_r(x)$. The total number of disks required to cover $B_r(x)$ is bounded in terms of a constant only depending on r_0 and r_1 (which depended on E_0, K and N). After summing up we see that

$$\|\nabla^2 u\|^2_{L^2(B_{\frac{r}{2}})} \leq C E_0 \frac{1}{r_1^2} + C \|\tau_g(u)\|^2_{L^2(B_r)} . \tag{4.100}$$

We can then estimate $\|\tau_g(u)\|^2_{L^2(B_r)} \leq C \|\tau_g(u)\|^2_{L^2(M,g)}$ (using the scaling of the flat tension, see Remark 4.1.11), which implies the claim after we absorb the additional factors occurring in (4.100) into the constant C. $\square$

Using this H^2-bound we then apply parabolic theory to the equation for the map.

Lemma 4.2.11. *Given a smooth closed oriented surface M of genus $\gamma \geq 2$ and N a smooth closed Riemannian manifold with nonpositive sectional curvature, let $(u(t), g(t))$ be a solution to (2.21) on $[0,T]$ with coupling constant η, starting at initial data (u_0, g_0), with energy bounded by $E(u_0, g_0) \leq E_0$ and such that there exists $r_0 > 0$ with $\mathrm{inj}_{g(t)} \geq r_0$. Further assume that the tension field satisfies $\|\tau_{g(t)}(u(t))\|_{L^2(M,g(t))} < K$ for all t in some time interval $[T_1, T_2] \subset [0,T]$ of length at least 2. Choose some time $t_0 \in [T_1+1, T_2-1]$, then on any hyperbolic isothermal chart $U = B_r(x)$ for $r < \frac{r_0}{4}$ defined with respect to $g(t_0)$, we obtain a bound in the parabolic Sobolev space $W_p^{2,1}(B_{\frac{r}{2}}(x) \times [t_0 - \frac{1}{2}, t_0+1])$ for all $p < \infty$ of the form*

$$\|u(t)\|_{W_p^{2,1}} \leq C \tag{4.101}$$

where C depends only on r_0, E_0, p, K and N, assuming $\eta \leq \eta_0$ with $\eta_0 > 0$ only depending on E_0, r_0 and γ.

Proof. To see this, consider some given time $t_0 \in [T_1+1, T_2-1]$ and study the equation $u_t - \Delta_g(u) = A_g(u)(\nabla u, \nabla u)$ in local coordinates (chosen as in Remark 4.1.11) as an inhomogeneous linear parabolic equation on some cylinder $B_r(x) \times [t_0 - 1, t_0+1]$. Note that the coefficients satisfy the necessary assumptions (i.e uniform parabolicity, continuity and boundedness) to apply Theorem B.1, with bounds in particular independent of the particular metric $g(t_0)$, but only depending on r_0. To see this, note that using Lemma 4.2.4 together with Lemma 4.1.3 for sufficiently small η_0 we have $g(t) \geq C g(t_0)$ for $t \in [t_0 - 1, t_0 + 1]$ by (4.71), implying that $g(t)$ is also uniformly parabolic on $B_r(x)$ (or in other words, the corresponding bilinear form is coercive). We also have the C^k-bounds from Lemma 4.2.4, thus all (spatial) derivatives of the metric coefficients are bounded (as well as Christoffel symbols etc.).

To proceed we use a standard interior regularity argument. More precisely, consider $\phi : B_r(x) \times [t_0 - 1, t_0 + 1] \to [0,1]$ a smooth function in time and space, which is compactly supported in space and has initial values $\phi(\cdot, t_0 - 1) = 0$, such that $\phi(x,t) \equiv 1$ for $x \in B_{\frac{r}{2}}$ and $t \in [t_0 - \frac{1}{2}, t_0 + 1]$. Then $u\phi$ solves a parabolic equation with zero initial and boundary data on $B_r \times [t_0 - 1, t_0 + 1]$. This equation is of the form $(\phi u)_t - \Delta_g(\phi u) = F(u, \nabla u, \phi) + \phi A_g(u)(\nabla u, \nabla u)$, where the error term $F(u, \nabla u, \phi)$ is some linear combination of u and ∇u, with coefficients bounded in any space-time L^p (depending on derivative bounds for ϕ and bounds on the metric coefficients). Thus $F(u, \nabla u, \phi)$ is bounded in $L^p(B_r \times [t_0 - 1, t_0 + 1])$.

By Lemma 4.2.10, combined with the embedding $W^{2,2}(B_r) \hookrightarrow W^{1,p}(B_r)$ for any $p < \infty$, we can see that also $A_g(u)(\nabla u, \nabla u)$ is bounded in this space. To this end, we apply Lemma 4.2.10 and find a $W^{1,p}$-bound for u on flat disks corresponding to geodesic balls defined with respect to the changing metric $g(t)$. However, in particular we have $B_{\frac{r_0}{4}}(x) \subset B^{g(t)}_{\frac{r_0}{2}}(x)$ for $t \in [t_0 - 1, t_0 + 1]$ (after possibly choosing an even smaller η_0), hence we also get the bound on our fixed chart $B_r(x) \subset B_{\frac{r_0}{4}}(x)$.

Hence Theorem B.1 applies (to each component of u) over the time interval $[t_0 - 1, t_0 + 1]$, and we deduce the claimed bound using that $\phi \equiv 1$ on the domain $B_{\frac{r}{2}}(x) \times [t_0 - \frac{1}{2}, t_0 + 1]$. $\square$

Corollary 4.2.12. *Let s be a nonnegative integer and $\alpha \in (0, 1)$. Under the assumptions of the previous lemma, we have the spatial Hölder norms $C^{s,\alpha}$ of $u(t_0)$ bounded (on some slightly smaller ball $B_{r'}(x)$), with bounds only depending on s, α, K, r_0, E_0, r' and N.*

Proof. This is a standard bootstrapping argument, see also the proof of [23, Theorem 3.8]. We can apply the previous lemma to see that $u \in W^{2,1}_p$ (for any $p < \infty$). We now differentiate the equation satisfied by u (i.e. $u_t - \Delta_g(u) = A_g(u)(\nabla u, \nabla u)$) in space, and obtain an equation for $\partial_i u$, which we can again consider as an inhomogeneous heat equation. As $u \in W^{2,1}_p$ we have $\partial_i A_g(u)(\nabla u, \nabla u) \in L^{\frac{p}{2}}$, which together with the interior regularity argument of the last lemma will lead us to see that in fact $\partial_i u \in W^{2,1}_{\frac{p}{2}}$. Note that the C^k-bounds for the metric following from Lemma 4.2.4 are critical for this argument, as they allow us to estimate the spatial derivatives of metric coefficients which appear after differentiating (2.21). As p was arbitrary, we can repeat this process to get $W^{2,1}_p$-bounds on any order (spatial) derivative of u over a slightly smaller time interval and domain (e.g. $B_{r'} \times [t_0 - \frac{1}{4}, t_0 + 1]$), but in particular including t_0. Finally, by standard embedding theorems (see e.g. [14, Chapter II, Lemma 3.3]), we obtain the claimed spatial Hölder regularity at $t = t_0$. $\square$

Remark 4.2.13. Note that we could also get control on higher order time derivatives of the map by considering an explicit formula for the projection operator P_g, which would lead to bounding (higher order) time derivatives of the metric coefficients and allow us to carry out an iteration argument similar to the above. This was done in [23]. However, as we only apply these estimates to the rescaled flow (4.62) (where the time regularity would degenerate) we do not do this here.

We also observe that by elliptic regularity the restriction of any *harmonic* map

$u : (M, g(t)) \to N$ to a chart $B_r(x)$ enjoys good estimates. This is standard in our setting (as we assumed N to have nonpositive sectional curvature, see e.g. [8]), but we give an outline of a proof for completeness.

Lemma 4.2.14. *Consider again a smooth closed oriented surface M of genus $\gamma \geq 2$ and N a smooth closed Riemannian manifold with nonpositive sectional curvature. Let $u : (M, g) \to N$ be a harmonic map with energy bounded by $E(u, g) \leq E_0$, and assume that $\mathrm{inj}_g > r_0 > 0$. Take $B_r(x)$ as before with $r < r_0$, then for any nonnegative integer s and $\alpha \in (0, 1)$ the Hölder-norms $C^{s,\alpha}$ of u on $B_r(x)$ are controlled in terms of s, α, E_0, r_0, r and N.*

Proof. This is very similar to the previous argument. We now consider the equation $\Delta_g u = A_g(u)(\nabla u, \nabla u)$ and use elliptic theory, initially on $B_{r_0}(x)$. In particular as in Lemma 4.2.10 we can obtain an H^2-bound on u, implying an L^p-bound for $A_g(u)(\nabla u, \nabla u)$, which together with elliptic regularity gives us $u \in W^{2,p}$ (or more precisely, interior elliptic regularity) and a standard bootstrapping argument gives higher order bounds. $\square$

So far in this section we have only obtained *local* bounds, on flat disks, for harmonic maps and solutions to (4.62) with a priori bounded tension and injectivity radius. In fact, using estimates established earlier, we can actually obtain bounds with respect to the fixed norm $\|\cdot\|_{C^s(M,g_0)}$ away from $t = 0$ for any solution to (4.62).

Proposition 4.2.15. *Take M to be a smooth closed oriented surface of genus $\gamma \geq 2$ and N to be a smooth closed Riemannian manifold with nonpositive sectional curvature. Given initial data (u_0, g_0), let $0 < T \leq T_0$ with T_0 from Lemma 4.2.3, $k \geq 1$ large enough so that $\eta \leq \eta_0$ with η_0 from Lemma 4.2.11, and consider the associated solution $(u_k(t), g_k(t))$ of (4.62) defined on $[0, T]$ with rescaled coupling constant k, starting at (u_0, g_0). Let further s be a nonnegative integer and $\frac{1}{k} < \epsilon$, then for $0 < \epsilon \leq t \leq T$, we can obtain bounds on $\|u_k(t)\|_{C^s(M,g_0)}$, in terms of s, ϵ, $E(u_0, g_0)$, (M, g_0) and N. Additionally, if $\bar{u}(t)$ is a harmonic map with respect to $g_k(t)$, we can also bound $\|\bar{u}(t)\|_{C^s(M,g_0)}$ in terms of the same quantities.*

Proof. We begin by considering $(u_k(t), g_k(t))$. Note that we are now working with the rescaled equations (4.62). Observe that we have a uniform lower bound r_0 on the injectivity radius $\mathrm{inj}_{g_k(t)}$, in terms of $E(u_0, g_0)$ and (M, g_0) by Lemma 4.2.3. We also have a uniform upper bound on $\|\tau_{g_k}(u_k)\|_{L^2(g_k(t))}$ for $t \geq \epsilon$ in terms of ϵ, $E(u_0, g_0)$, M and r_0 by Lemma 4.2.9.

Hence for any fixed time $t \geq \epsilon$, we can apply Corollary 4.2.12 to obtain estimates on $\|u_k(t)\|_{C^s(D_{r'},g_{eucl})}$, for any disk $D_{r'}$ corresponding (by taking a hyperbolic isothermal coordinate chart, as in Remark 4.1.11) to a geodesic ball $B_r(x)$, defined with respect to $g_k(t)$ for e.g. $r = \frac{r_0}{4}$. Note that this directly implies a bound on $\|u_k(t)\|_{C^s(B_r(x))}$, computed with respect to the hyperbolic metric on $D_{r'}$, as these norms are equivalent

$$\|u_k(t)\|_{C^s(B_r(x))} \leq C \|u_k(t)\|_{C^s(D_{r'},g_{eucl})}, \tag{4.102}$$

with a constant C only depending on s and γ (as we can find an upper bound for r_0 in terms of γ) which can be seen by a direct calculation as in Remark 4.1.8. We can then bound $\|u_k\|_{C^s(M,g_k(t))}$, as we can estimate it by simply taking the supremum of $\|u_k(t)\|_{C^s(B_r(x))}$ over all $x \in M$, and we obtain

$$\|u_k(t)\|_{C^s(M,g_k(t))} \leq C(s, \epsilon, E(u_0, g_0), N, (M, g_0)). \tag{4.103}$$

But now the result follows by Lemma 4.2.7, as the norms $\|\cdot\|_{C^s(M,g_k(t))}$ for $t \in [0,T]$ are uniformly equivalent to $\|\cdot\|_{C^s(M,g_0)}$.

The corresponding claim for harmonic maps can be seen in the same way, now using Lemma 4.2.14 instead to obtain the local bounds. □

4.2.4 C^s-closeness to harmonic maps using small tension

We now exploit the small tension to see that the flow (4.62) becomes C^0-close to a harmonic map at each (positive) time for sufficiently large k, under appropriate topological assumptions on the initial map u_0. Specifically, we will assume that the homotopy class of u_0 contains no constant maps nor maps to closed geodesics in the target, which ensures that for any metric on M there exists a unique harmonic map homotopic to u_0 (see Theorem 2.3.2).

Lemma 4.2.16. *Let M be a smooth closed oriented surface of genus $\gamma \geq 2$ and N be a smooth closed Riemannian manifold, which we now additionally assume to have strictly negative sectional curvature. Fix a homotopy class H of maps $u : M \to N$ that contains no constant maps nor maps to closed geodesics in the target. Consider any smooth map $u \in H$ and $g \in \mathcal{M}_{-1}$ with injectivity radius $\mathrm{inj}_g \geq r_0 > 0$ and energy $E(u, g) \leq E_0$. Then given any $\epsilon > 0$ there exists some $\delta = \delta(\epsilon, r_0, E_0, N, H) > 0$ such that $\|\tau_g(u)\|_{L^2(M,g)} < \delta$ implies $|u - \bar{u}|_{C^0} < \epsilon$ where $\bar{u}$ denotes the unique harmonic map with respect to g in H.*

Proof. Assume the claim was false, then there exists an $\epsilon > 0$ such that we can find a sequence (u_i, g_i) satisfying the assumptions of the lemma with $\|\tau_{g_i}(u_i)\|_{L^2(M,g_i)} \to 0$ and $|u_i - \bar{u}_i|_{C^0} \geq \epsilon$ for the (unique in H) g_i-harmonic map $\bar{u}_i$.

As $\mathrm{inj}_{g_i} \geq r_0 > 0$ we can apply Mumford compactness (see Theorem 2.2.6) and obtain a subsequence (u_i, g_i) together with diffeomorphisms $f_i : M \to M$ such that $f_i^*(g_i) \to h$ smoothly for some limit metric $h \in \mathcal{M}_{-1}$. We modify the maps u_i with the same diffeomorphisms and denote $v_i = u_i \circ f_i$, $h_i = f_i^* g_i$. Then $\|\tau_{h_i}(v_i)\|_{L^2(M,h_i)} \to 0$ and $|v_i - \bar{v}_i|_{C^0} \geq \epsilon$, for any harmonic map (homotopic to v_i) $\bar{v}_i$ with respect to h_i. To see this, first note that this map is necessarily unique, as the homotopy class of v_i also contains no constant maps nor maps to closed geodesics in the target. Hence it is given by $\bar{v}_i = \bar{u}_i \circ f_i$, and the C^0 distance considered is invariant under diffeomorphisms.

We can now see from the modified bubbling analysis carried out in [21, Lemma 3.2] that there exists a harmonic map $\bar{v} : (M, h) \to N$ such that $v_i \to \bar{v}$ strongly in $W^{1,p}(M \setminus S)$ for all $p \in [0, \infty)$ where

$$S := \{x \in M : \text{for any neighbourhood } \Omega \text{ of } x, \limsup_{i\to\infty} E(u_i, g_i, \Omega) \geq \epsilon_0\}. \tag{4.104}$$

In particular, the set S of points where energy concentrates is empty in our case as a consequence of the curvature assumption on N. Hence $v_i \to \bar{v}$ in C^0 (as $W^{1,p} \hookrightarrow C^0$ for $p > 2$). It remains to see that $\bar{v}_i$ and $\bar{v}$ necessarily become close (in C^0) to deduce a contradiction to $|v_i - \bar{v}_i|_{C^0} \geq \epsilon$. By the C^0-convergence of $v_i \to \bar{v}$ we see that $\bar{v}$ is homotopic to v_i for all sufficiently large i. Even though the f_i are not necessarily homotopic to the identity, it therefore follows that $\bar{v}$ is not a constant map nor maps to a closed geodesic in the target. We finally obtain that $\bar{v}_i \to \bar{v}$ (in C^0) using the continuous dependence of the harmonic map on the metric in our setting (see Corollary 2.3.6). In particular, this requires the strictly negative sectional curvature on N in addition to the topological condition satisfied by $\bar{v}$.

□

We finally obtain that the flow becomes close to a harmonic map at all (positive) times for large k in $C^s(M, g_0)$ by interpolation.

Corollary 4.2.17. *Take M and N to be as in the previous lemma. Assume the homotopy class of u_0 does not contain maps to closed geodesics in the target or constant maps. Given initial data (u_0, g_0) let $0 < T \leq T_0$ with T_0 from Lemma 4.2.3* *and consider the sequence $(u_k, g_k)_{k=1}^{\infty}$ of solutions to (4.62) with rescaled coupling constant k, starting at (u_0, g_0). Let $\epsilon > 0$ and s be a nonnegative integer, then for any $t \geq \epsilon$ we have $\|u_k(t) - \bar{u}_k(t)\|_{C^s(M,g_0)} \to 0$ as $k \to \infty$, where $\bar{u}_k(t)$ denotes the unique harmonic map with respect to $g_k(t)$ in the homotopy class of u_0. Furthermore, this convergence is uniform on the interval $[\epsilon, T]$.*

Proof. Note that we have bounds (independent of k, as long as k is large enough so that $\eta \leq \eta_0$ with η_0 from Lemma 4.2.11) on both $u_k(t)$ and $\bar{u}_k(t)$ in $C^s(M, g_0)$ by Proposition 4.2.15 applied on $[\epsilon, T]$. But we also have $|u_k(t) - \bar{u}_k(t)|_0 \to 0$ as $k \to \infty$ by Lemma 4.2.16 for $t \in [\epsilon, T]$ (using the tension bound from Corollary 4.2.9 and the injectivity radius bound from Lemma 4.2.3). Therefore the claim follows by interpolation, using e.g. Ehrling's lemma as in 4.1.20.

□

4.2.5 Constructing a limit flow

Using the C^s-bounds established for the metric in Lemma 4.2.4 we can show existence of a limit flow for the flows (4.62) as $k \to \infty$. To this end, we consider time-dependent Hölder functions valued in some Banach space.

Definition 4.2.18. Let X be a Banach space, then we can define $C^0([0,T], X)$ to be the Banach space of bounded continuous functions valued in X on $[0,T]$ equipped with the norm

$$\|f\|_{C^0_X} = \sup_{t\in[0,T]} \|f(t)\|_X = \|f\|_0 \,. \tag{4.105}$$

Similarly for $\alpha \in (0,1)$ we denote by $C^{0,\alpha}([0,T], X)$ the Banach space of functions $f : [0,T] \to X$ which are Hölder continuous with exponent α, with the canonical norm

$$\begin{aligned} \|f\|_{C^{0,\alpha}_X} =& \sup_{t_1,t_2\in[0,T],t_1\neq t_2} \frac{\|f(t_1) - f(t_2)\|_X}{|t_1 - t_2|^{\alpha}} + \sup_{t\in[0,T]} \|f(t)\|_X & (4.106)\\ =&[f]_\alpha + \|f\|_0 \,. & (4.107) \end{aligned}$$

In particular, if we let $X = C^s(\mathrm{Sym}^2(T^*M), g_0)$ for any nonnegative integer s, we see that the metric $g(t)$ of a solution to (4.62) lies in $C^{0,\frac{1}{2}}([0,T], X)$, as a consequence of Lemmas 4.2.4, 4.2.7 and 4.1.3. We will need a compactness statement for these time-dependent Hölder spaces.

Lemma 4.2.19. *Assume X, Y are Banach spaces such that X compactly embeds into Y. Let $\alpha \in (0,1)$, then the embedding $C^{0,\alpha}([0,T],X) \hookrightarrow C^0([0,T],Y)$ is compact.*

Proof. This is a variant of the Arzela-Ascoli theorem. For completeness we include a proof in Appendix C. $\square$

Lemma 4.2.20. *For s a nonnegative integer the embedding $C^{s+1}(\mathrm{Sym}^2(T^*M), g_0) \hookrightarrow C^s(\mathrm{Sym}^2(T^*M), g_0)$ is compact.*

Proof. This is standard (and in fact works for arbitrary vector bundles, not just $\mathrm{Sym}^2(T^*M)$), and follows from the usual Arzela-Ascoli theorem, see e.g. [1, Corollary 9.14]. $\square$

Using this compactness we now obtain a limit flow for (4.62) as $k \to \infty$.

Theorem 4.2.21. *Let M be a smooth closed oriented surface of genus $\gamma \geq 2$ and N be a smooth closed Riemannian manifold. Given smooth initial data (u_0, g_0) for (4.62), take $0 < T \leq T_0$ with T_0 from Lemma 4.2.3, and consider the sequence $(u_k(t), g_k(t))_{k=1}^{\infty}$ of solutions to (4.62) with rescaled coupling constant k on the fixed time interval $[0,T]$, which we further assume to be smooth up to $t = T$. Then the following is true:*

1. *There exists a limit curve of hyperbolic metrics g (i.e. each $g(t)$ has Gauss curvature $K = -1$) on $[0,T]$, continuous in time and smooth in space in the sense that for all $s \in \mathbb{N}$, g is an element of $C^0([0,T], C^s(\mathrm{Sym}^2(T^*M), g_0))$. After possibly selecting a subsequence in k the curves g_k converge to g in $C^0([0,T], C^s(\mathrm{Sym}^2(T^*M), g_0))$ (i.e. uniformly in time in $C^s(M, g_0)$), again for all $s \in \mathbb{N}$.*

2. *Further assume that N has strictly negative sectional curvature and that the homotopy class of u_0 does not contain maps to closed geodesics in the target or constant maps. Let $u(t) : M \times (0,T] \to N$ be the unique curve of harmonic maps homotopic to u_0 corresponding to $g(t)$, then the limit curve of metrics g is differentiable in time at each point $x \in M$ away from $t = 0$, with derivative given by $\frac{\mathrm{d}}{\mathrm{d}t} g(t)(x) = Re(\Phi(u,g))(x)$, where $\Phi(u,g)$ as usual denotes the Hopf differential. Finally, the maps $u_k(t)$ also converge to $u(t)$ uniformly in t in $C^s(M, g_0)$ away from 0 for all $s \in \mathbb{N}$.*

Proof. We will use the shorthand notation C^s to refer to the space $C^s(\mathrm{Sym}^2(T^*M), g_0)$. Let $s \in \mathbb{N}$, then each g_k is an element of $C^{0,\frac{1}{2}}([0,T], C^{s+1})$, as observed above. We denote the norm on this space by $\|\cdot\|_{C^{0,\frac{1}{2}}_{s+1}}$. Also observe that we have a uniform (independent of k) bound on $\|g_k\|_{C^{0,\frac{1}{2}}_{s+1}}$ by Lemmas 4.2.4, 4.2.7 and 4.1.3 together with the bound on the injectivity radius 4.2.3. Hence the g_k form a bounded sequence in $C^{0,\frac{1}{2}}([0,T], C^{s+1})$, and therefore we can find a convergent subsequence in $C^0([0,T], C^s)$, which we again denote by g_k, by Lemmas 4.2.19 and 4.2.20, which converges to a limit g in $C^0([0,T], C^s)$. By repeating this subsequence argument we see that the limit g lies in $C^0([0,T], C^s)$ for all $s \in \mathbb{N}$, and we may assume that g_k converges in $C^0([0,T], C^s)$ (again, for all $s \in \mathbb{N}$).

Note that $g(t)$ is necessarily a metric for all t by the uniform equivalence of the metrics $g_k(t)$ from Corollary 4.2.6, and we have $g(t) \in \mathcal{M}_{-1}$ as in particular the curvatures $R(g_k(t))$ converge (by taking $s \geq 2$). This proves Claim 1.

To see the next claim, fix some $\epsilon > 0$ and some $s \in \mathbb{N}$. We first show that $\frac{\mathrm{d}}{\mathrm{d}t} g_k$ converges to a limit in the space $C^0([\epsilon, T], C^s)$. We will denote the norm on this space by $\|\cdot\|_{C^0_s}$ (see Definition 4.2.18). Set $\Psi(t) = \Phi(u(t), g(t))$ with u denoting the curve of harmonic maps associated to the limit curve of metrics g defined as before. We claim that $\frac{\mathrm{d}}{\mathrm{d}t} g_k \to Re(\Psi)$ in $C^0([\epsilon, T], C^s)$. Let $\Psi_k(t) = \Phi(u_k(t), g_k(t))$, then we can estimate

$$\begin{aligned} \left\| \frac{\mathrm{d}}{\mathrm{d}t} g_k - Re(\Psi) \right\|_{C^0_s} &= \|Re(P_{g_k}(\Psi_k)) - Re(\Psi)\|_{C^0_s} \\ &\leq \|Re(P_{g_k}(\Psi_k)) - Re(\Psi_k)\|_{C^0_s} + \|Re(\Psi_k) - Re(\Psi)\|_{C^0_s}. \end{aligned} \tag{4.108}$$

Note that indeed $\frac{\mathrm{d}}{\mathrm{d}t} g_k = Re(P_{g_k}(\Psi_k))$ is an element of $C^0([\epsilon, T], C^s)$ for each k, as each $(u_k(t), g_k(t))$ is a smooth flow. Thus it is sufficient to show that the right hand side of (4.108) converges to 0 as $k \to \infty$ (in particular, that will also show that $Re(\Psi) \in C^0([\epsilon, T], C^s)$).

We start by estimating $\|Re(P_{g_k}(\Psi_k)) - Re(\Psi_k)\|_{C^0_s}$. This requires us to bound $\left\|Re(P_{g_k(t)}(\Psi_k(t))) - Re(\Psi_k(t))\right\|_{C^s}$ at each time $t \in [\epsilon, T]$ uniformly in t. We first bound the L^1-norm of this tensor using an elliptic Poincaré estimate for quadratic differentials from [24]. This tells us

$$\left\|Re(P_{g_k(t)}(\Psi_k(t))) - Re(\Psi_k(t))\right\|_{L^1(M,g_k(t))} \leq \left\|P_{g_k(t)}(\Psi_k(t)) - \Psi_k(t)\right\|_{L^1(M,g_k(t))} \tag{4.109}$$

$$\leq C\left\|\bar{\partial}\Psi_k(t)\right\|_{L^1(M,g_k(t))}, \tag{4.110}$$

where $C > 0$ is some constant only depending on the genus γ of M. By a standard calculation, see e.g. [21, Lemma 3.1], we can estimate

$$\left\|\bar{\partial}\Psi_k(t)\right\|_{L^1(M,g_k(t))} \leq \sqrt{2}\left\|\tau_{g_k(t)}(u_k(t))\right\|_{L^2(M,g_k(t))} E(u_k(t), g_k(t))^{\frac{1}{2}}. \tag{4.111}$$

From Corollary 4.2.9 we obtain that $\left\|\tau_{g_k(t)}(u_k(t))\right\|_{L^2(M,g_k(t))} \to 0$ uniformly in t (for $t \in [\epsilon, T]$), and as usual we can bound $E(u_k(t), g_k(t)) \leq E(u_0, g_0)$, hence

$$\left\|Re(P_{g_k(t)}(\Psi_k(t))) - Re(\Psi_k(t))\right\|_{L^1(M,g_k(t))} \to 0 \tag{4.112}$$

uniformly in t (again for $t \in [\epsilon, T]$). To obtain convergence in C^s, we bound the C^{s+1}-norm and can then argue by interpolation as usual. We observe that we can write the real part of the Hopf differential explicitly (see [23]) as

$$Re(\Psi_k(t)) = 2u_k(t)^*G - 2e(u_k(t), g_k(t))g_k, \tag{4.113}$$

which is therefore bounded in C^{s+1}, again uniformly in t, as a consequence of Proposition 4.2.15, together with $g_k(t) \to g(t)$ in C^{s+1} (by construction of g). To bound $Re(P_{g_k(t)}(\Psi_k(t)))$, we can estimate

$$\left\|Re(P_{g_k(t)}(\Psi_k(t)))\right\|_{C^{s+1}} \leq C\left\|P_{g_k(t)}(\Psi_k(t))\right\|_{L^1(M,g_k(t))} \leq C\left\|\Psi_k(t)\right\|_{L^1(M,g_k(t))}, \tag{4.114}$$

with a constant C only depending on a lower bound for $\mathrm{inj}_{g_k(t)}$ and γ. This is a consequence of the fact that the C^s-norms of holomorphic functions are controlled by their L^1-norm (see [22]) combined with the $L^1 - L^1$-boundedness of the projection operator P_g (as already used in Lemma 4.2.8, from [20, Proposition 4.10]). Thus $\|Re(P_{g_k}(\Psi_k)) - Re(\Psi_k)\|_{C^0_s} \to 0$ as $k \to \infty$.

We proceed to estimate the second term $\|Re(\Psi_k) - Re(\Psi)\|_{C^0_s}$. We again do this by bounding $\|Re(\Psi_k(t)) - Re(\Psi(t))\|_{C^s}$ uniformly in t for $t \in [\epsilon, T]$. Take $\bar{u}_k(t)$ as usual to be the unique $g_k(t)$-harmonic map homotopic to u_0, and set $\bar{\Psi}_k(t) =$

$\Phi(\bar{u}_k(t), g_k(t))$. We find

$$\|Re(\Psi_k(t)) - Re(\Psi(t))\|_{C^s} \leq \left\|Re(\Psi_k(t)) - Re(\bar{\Psi}_k(t))\right\|_{C^s} + \left\|Re(\bar{\Psi}_k(t)) - Re(\Psi(t))\right\|_{C^s}. \tag{4.115}$$

As a consequence of Corollary 4.2.17 we have $u_k(t) \to \bar{u}_k(t)$ in C^s uniformly in t for $t \in [\epsilon, T]$, and hence $\left\|Re(\Psi_k(t)) - Re(\bar{\Psi}_k(t))\right\|_{C^s}$ converges to 0 uniformly in t (again for $t \in [\epsilon, T]$) by the explicit formula (4.113).

Finally, it remains to estimate $\left\|Re(\bar{\Psi}_k(t)) - Re(\Psi(t))\right\|_{C^s}$. We already know that $g_k(t)$ converges to $g(t)$ in C^s as $k \to \infty$ uniformly in t (by construction of g), so by (4.113) we only need to check that the same holds for $\bar{u}_k(t)$ and $u(t)$ (which will also prove the last statement of Claim 2, as we already know $u_k(t) \to \bar{u}_k(t)$ in C^s). Note that the conditions to apply the results from [6], in particular Corollary 2.3.6, are satisfied: N has strictly negative sectional curvature and the homotopy class of u_0 does not contain maps to closed geodesics in the target or constant maps. Thus $\|\bar{u}_k(t) - u(t)\|_{C_s} \to 0$ can be deduced for each $t \in [\epsilon, T]$ by applying Corollary 2.3.6 on neighbourhoods covering the limit curve g. As $g(t) : [0, T] \to C^s(\mathrm{Sym}^2(T^*M))$ is a continuous function, a finite such cover can be found, which implies that the convergence is uniform in t as claimed. Therefore we also have $\|Re(\Psi_k) - Re(\Psi)\|_{C_s^0} \to 0$ as $k \to \infty$.

This establishes $\left\|\frac{\mathrm{d}}{\mathrm{d}t} g_k(t) - Re(\Psi(t))\right\|_{C_s^0} \to 0$ as $k \to \infty$. As a consequence (e.g. using the fundamental theorem of calculus, or more abstractly viewing the sequence g_k as a convergent sequence in the Banach space $C^1([0, T], C^s)$, with limit necessarily equal to the curve g), we see that $g(t)$ is differentiable in t on $[\epsilon, T]$, with derivative given by $\frac{\mathrm{d}}{\mathrm{d}t} g(t) = Re(\Psi(t))$.

Thus Claim 2 is proved. □

Remark 4.2.22. One finds that the injectivity radius of a solution to the Teichmüller harmonic map flow (2.21) is bounded away from 0 when the initial map is *incompressible*, which we take to mean that $u_0 : M \to N$ is homotopically nontrivial and its action on the fundamental group of M has trivial kernel (hence any simple closed homotopically nontrivial curve is mapped to another homotopically nontrivial curve in the target) (see [21]).

Thus the proof of the above theorem can be adapted (assuming u_0 is incompressible) to see that the limit flow exists for all time. We can then calculate the evolution of the energy along this limit flow $(u(t), g(t))$ using (2.5) and find

$$\frac{\mathrm{d}}{\mathrm{d}t} E(u(t), g(t)) = -\int_M \frac{1}{4} |Re(\Phi(u, g))|^2 dv_g, \tag{4.116}$$

allowing us to find a subsequence of times $t_i \to \infty$ with $\Phi(u(t_i), g(t_i)) \to 0$ (this is basically the same argument as in [21]). We can then continue arguing as in [21], in particular applying Mumford compactness (see Theorem 2.2.6) to find a limit metric $\bar{g}$ and a limit weakly conformal harmonic map $\bar{u}$, i.e. constant map or branched minimal immersions (after possibly adjusting by diffeomorphisms).

Hence, assuming the target N has strictly negative sectional curvature and the initial map u_0 is incompressible, and satisfies the homotopy class assumptions of Theorem 4.2.21, the rescaled flow (4.62) converges to a limit flow which deforms the initial map into a branched minimal immersion (or constant map) through homotopic harmonic maps.

Appendix A

Hyperbolic geometry

We will need Keen's 'Collar lemma' several times in this thesis.

Lemma A.1 ([19]). *Let (M, g) be a smooth closed oriented hyperbolic surface and let σ be a simple closed geodesic of length ℓ. Then there is a neighbourhood around σ, a so-called collar, which is isometric to the cylinder $\mathcal{C}(\ell) := (-X(\ell), X(\ell)) \times S^1$ equipped with the metric $\rho^2(s)(ds^2 + d\theta^2)$ where*

$$\rho(s) = \frac{\ell}{2\pi \cos(\frac{\ell s}{2\pi})} \qquad \text{and} \qquad X(\ell) = \frac{2\pi}{\ell}\left(\frac{\pi}{2} - \arctan\left(\sinh\left(\frac{\ell}{2}\right)\right)\right).$$

The geodesic σ corresponds to the circle $\{s = 0\} \subset \mathcal{C}(\ell)$.

For $\delta \in (0, \operatorname{arsinh}(1))$, the δ-thin part of a collar is given by the subcylinder

$$(-X_\delta(\ell), X_\delta(\ell)) \times S^1 \subseteq \mathcal{C}(\ell), \qquad \text{where} \qquad X_\delta(\ell) = \frac{2\pi}{\ell}\left(\frac{\pi}{2} - \arcsin\left(\frac{\sinh(\frac{\ell}{2})}{\sinh\delta}\right)\right) \tag{A.1}$$

for $\delta \geq \ell/2$, respectively zero for smaller values of δ.

To analyse sequences of degenerating hyperbolic surfaces we make repeatedly use of the differential geometric version of the Deligne-Mumford compactness theorem.

Proposition A.2. (Deligne-Mumford compactness, cf. [12] and [13]) *Let (M, g_i, c_i) be a sequence of closed hyperbolic Riemann surfaces of genus $\gamma \geq 2$ that degenerate in the sense that $\liminf_{i\to\infty} \operatorname{inj}_{g_i} M = 0$. Then, after selection of a subsequence, (M, g_i, c_i) converges to a complete hyperbolic punctured Riemann surface (Σ, h, c), where Σ is obtained from M by removing a collection $\mathscr{E} = \{\sigma^j, j = 1, ..., k\}$ of*

k pairwise disjoint, homotopically nontrivial, simple closed curves on M and the convergence is as follows:

For each i there exists a collection $\mathscr{E}_i = \{\sigma_i^j, j = 1, ..., k\}$ of pairwise disjoint simple closed geodesics on (M, g_i, c_i) of length $\ell(\sigma_i^j) =: \ell_i^j \to 0$ as $i \to \infty$, and a diffeomorphism $F_i : M \to M$ mapping σ^j onto σ_i^j, such that the restriction $f_i = F_i|_\Sigma : \Sigma \to M \setminus \cup_{j=1}^k \sigma_i^j$ satisfies

$$(f_i)^* g_i \to h \text{ and } (f_i)^* c_i \to c \text{ in } C^\infty_{loc} \text{ on } \Sigma.$$

Finally, for metrics of the form $g = \xi^2 g_0$, $\xi : \mathscr{C} \to \mathbb{R}^+$ any function on a cylinder $\mathscr{C} = \mathscr{C}(s_1, s_2)$, we have

$$|dz^2|_g = \xi^{-2}|dz^2|_{g_0} = 2\xi^{-2}. \tag{A.2}$$

Thus the L^1 norm of a quadratic differential $\Psi = \psi dz^2$ is independent of the conformal factor:

$$\|\Psi\|_{L^1(\mathscr{C},g)} = \|\Psi\|_{L^1(\mathscr{C},g_0)} = 2\int_{\mathscr{C}} |\psi| ds d\theta. \tag{A.3}$$

Appendix B

Parabolic regularity

Let $D_r \subset \mathbb{R}^n$ be a euclidean ball of radius r, and let $S = \partial D_r$. Denote the cylinder $D_r \times [0, T]$ by Q_T, and consider functions $u(x,t) : Q_T \to \mathbb{R}$. Then we define the standard parabolic Sobolev space $W_p^{2,1}(Q_T)$ for $p \in [1, \infty]$ as

$$W_p^{2,1}(Q_T) = \{u : Q_T \to \mathbb{R} : D_t^\alpha D_x^\beta u \in L^p(Q_T), 2|\alpha| + |\beta| \leq 2\} \tag{B.1}$$

where D_t denotes weak derivatives in the 'time'-direction t, and similarly D_x refers to the 'space'-directions x_i, and as usual α and β are multi-indices. We define a norm on this space by

$$\|u\|_{W_p^{2,1}(Q_T)} = \left\|D_x^2 u\right\|_{L^p(Q_T)} + \|D_x u\|_{L^p(Q_T)} + \|u_t\|_{L^p(Q_T)}\,. \tag{B.2}$$

The idea behind defining these spaces is that they are a good setting to study parabolic equations, as they 'weight' the time derivative appropriately.

Define the linear parabolic operator L by

$$Lu := \partial_t u - a_{ij}\partial_i\partial_j u + b_i\partial_i u + c, \tag{B.3}$$

then we have the following regularity result for the initial-boundary value problem with zero data in $W_p^{2,1}(Q_T)$.

Theorem B.1 (Special case of [14, Chapter IV, Theorem 9.1])**.** *Let $p > 1$, $f \in$*

$L^p(Q_T)$. Consider the problem

$$Lu = f \tag{B.4}$$

$$u_{|t=0} = 0 \tag{B.5}$$

$$u_{|S} = 0 \tag{B.6}$$

where the coefficients of L satisfy

1. *a_{ij}, b_i, c are bounded continuous functions on Q_T.*

2. *L is uniformly parabolic: There exists $\lambda > 0$ such that for $\xi \in \mathbb{R}^n$, we have $a_{ij}\xi_i\xi_j \geq \lambda|\xi|^2$.*

Then this problem has a unique solution $u \in W_p^{2,1}(Q_T)$. This solution satisfies the estimate

$$\|u\|_{W_p^{2,1}(Q_T)} \leq C \|f\|_{L^p(Q_T)} \tag{B.7}$$

where $C < \infty$ is a constant, allowed to depend on $|a|_0$, $|b|_0$, $|c|$, λ, r, n, T and p.

Remark B.2. Note that we state a very simplified version of [14, Chapter IV, Theorem 9.1], adapted to our situation. In particular, it is possible to weaken the assumptions on b and c: they only need to satisfy certain L^p conditions. Further one can allow non-zero initial and boundary data and obtain a corresponding estimate, as well as considering general domains instead of just balls.

Appendix C

Some technical facts and proofs

Given a domain M with a Riemannian metric g we can consider a perturbed metric $\tilde{g}$, such that $g-\tilde{g}$ is small in $C^0(M,g)$. Locally, the coefficients of $\tilde{g}$ are then close to the coefficients of g, and similarly their inverse coefficients are close. Analogously, when we have a perturbation that is small in $C^1(M,g)$ we can also control the difference of the Christoffel symbols (i.e. first derivatives of the metric coefficients). We check this in the following lemma for the benefit of any reader unfamiliar with these ideas.

Lemma C.1 (Controlling the difference of metric tensors)**.** *Let M be a smooth closed surface and consider metrics g, $\tilde{g}$ on M. Then there exist universal constants $C_1 > 0$ and $C_2 > 0$ such that around any point $p \in M$ we can find local coordinates $\{x_i\}$ for which the following is true.*

1. *The metrics g, $\tilde{g}$ are diagonal at p and we have $g_{ij}(p) = \delta_{ij}$, furthermore the partial derivatives $\partial_k g_{ij}$ vanish at p.*
2. *If $\|g-\tilde{g}\|_{C^0(M,g)} \leq C_1$, we have (at p) $|g_{ij}-\tilde{g}_{ij}| \leq \|g-\tilde{g}\|_{C^0(M,g)}$ as well as $|g^{ij}-\tilde{g}^{ij}| \leq C_2 \|g-\tilde{g}\|_{C^0(M,g)}$.*
3. *We further have (again at p) $|\tilde{\Gamma}^k_{ij}| \leq C_2 \|g-\tilde{g}\|_{C^1(M,g)}$.*

Proof. Let $p \in M$ be given. We can choose normal coordinates with respect to g at p. The coefficients of $\tilde{g}$ in these coordinates are not necessarily diagonal, however the matrix $\tilde{g}_{ij}(p)$ is symmetric, and hence we can find an orthogonal linear transformation to diagonalise $\tilde{g}$ (at p). The coordinates obtained this way will satisfy Claim 1.

We now write $h = g - \tilde{g}$. Any components of tensors in this proof are computed at p.

We note that the first part of Claim 2 is immediate from the definition of $\|\cdot\|_{C^0(M,g)}$:

$$\|h\|_{C^0(M,g)}^2 \geq |h(p)|_g^2 = h_{ij}h_{ij}, \tag{C.1}$$

where we used $g_{ij} = \delta_{ij}$. We now control the inverse coefficients $g^{ij} - \tilde{g}^{ij}$. As $\tilde{g}_{ij}$ is diagonal, so is $\tilde{g}^{ij}$, and we find $\tilde{g}^{ii} = (1 - h_{ii})^{-1}$. Thus

$$g^{ii} - \tilde{g}^{ii} = 1 - (1 - h_{ii})^{-1} = -\frac{h_{ii}}{1 - h_{ii}}, \tag{C.2}$$

assuming e.g. $|h_{ii}| < 1$. Hence choosing e.g. $C_1 = \frac{1}{2}$ (which implies $|h_{ii}| \leq \|g - \tilde{g}\|_{C^0(M,g)} \leq \frac{1}{2}$ by (C.1)) allows us to estimate

$$|g^{ii} - \tilde{g}^{ii}| \leq 2|h_{ii}| \leq 2\,\|g - \tilde{g}\|_{C^0(M,g)}\,, \tag{C.3}$$

establishing the rest of Claim 2 (with e.g. $C_1 = \frac{1}{2}, C_2 = 2$).

The final claim is very similar. Note that the Christoffel symbols of g vanish at p. We have

$$\|h\|_{C^1(M,g)} \geq |h(p)|_g + |\nabla h(p)|_g, \tag{C.4}$$

and can calculate $\nabla_k h_{ij} = \partial_k h_{ij}$, thus $\|h\|_{C^1(M,g)}$ controls the first derivatives of h, in the sense that $|\partial_k h_{ij}| \leq \|h\|_{C^1(M,g)}$. Explicitly calculating $\tilde{\Gamma}_{ij}^k$ yields

$$\tilde{\Gamma}_{ij}^k = \frac{1}{2}\tilde{g}^{kk}(\partial_j \tilde{g}_{ki} + \partial_i \tilde{g}_{kj} - \partial_k \tilde{g}_{ij}), \tag{C.5}$$

where we used that $\tilde{g}$ is diagonal at p. We note that all the derivatives $\partial_k g_{ij}$ vanish at p, and hence we can rewrite this as

$$\tilde{\Gamma}_{ij}^k = \frac{1}{2}\tilde{g}^{kk}(\partial_j h_{ki} + \partial_i h_{kj} - \partial_k h_{ij}). \tag{C.6}$$

We bound $\tilde{g}^{kk} = (1 - h_{kk})^{-1} \leq 2$ using (C.1), for $C_1 \leq \frac{1}{2}$ as before. Thus

$$|\tilde{\Gamma}_{ij}^k| \leq 3\,\|h\|_{C^1(M,g)}\,, \tag{C.7}$$

finishing the proof of the second claim. $\square$

Lemma C.2. *Let M be a smooth closed surface and N a smooth closed Riemannian manifold, which we view as isometrically embedded $N \hookrightarrow \mathbb{R}^n$. Let g, $\tilde{g}$ be*

Riemannian metrics on M and $v : M \to N$ be a smooth map, viewed as a map $v : M \to N \hookrightarrow \mathbb{R}^n$ using the isometric embedding. Then there exists a universal constant $C_1 > 0$ and a constant $C_2 = C_2(N) > 0$ such that for $\|g - \tilde{g}\|_{C^0(M,g)} \leq C_1$ we have the pointwise inequality

$$|\tau_g(v) - \tau_{\tilde{g}}(v)| \leq C_2 \|g - \tilde{g}\|_{C^1(M,g)} (|\nabla v|_g^2 + |\nabla v|_g + |\nabla^2 v|_g). \tag{C.8}$$

Proof. We choose the constant C_1 from Lemma C.1, and take the choice of coordinates provided by Lemma C.1 with respect to g at some point $x \in M$. In the remainder of this proof we work at this particular point x. In these local coordinates we find

$$\tau_g(v) - \tau_{\tilde{g}}(v) = (g^{ij} - \tilde{g}^{ij})\partial_i\partial_j v + \tilde{g}^{ij}\tilde{\Gamma}_{ij}^k \partial_k v + (g^{ij} - \tilde{g}^{ij})A(v)(\partial_i v, \partial_j v), \tag{C.9}$$

where A denotes the second fundamental form of N, and we used that the Christoffel symbols of g vanish at x and denote the Christoffel symbols of $\tilde{g}$ by $\tilde{\Gamma}$. The first two terms on the right can now be estimated using Lemma C.1 to find

$$|(g^{ij} - \tilde{g}^{ij})\partial_i\partial_j v + \tilde{g}^{ij}\tilde{\Gamma}_{ij}^k \partial_k v| \leq C \|g - \tilde{g}\|_{C^1(M,g)} (|\nabla v|_g + |\nabla^2 v|_g). \tag{C.10}$$

To estimate the last term, we note

$$|(g^{ij} - \tilde{g}^{ij})A(v)(\partial_i v, \partial_j v)| \leq C(N) \|g - \tilde{g}\|_{C^0(M,g)} |\nabla v|_g^2 \leq C(N) \|g - \tilde{g}\|_{C^1(M,g)} |\nabla v|_g^2, \tag{C.11}$$

using again Lemma C.1, as well as compactness of N to bound $A(v)$ and the Cauchy-Schwarz inequality to bound $|\partial_i v \partial_j v|$ by $C|\nabla v|_g^2$, which finishes the proof. □

Arzela-Ascoli

Proof of Lemma 4.2.19. For a definition of the spaces used, see Definition 4.2.18. We can prove this analogously to the usual Arzela-Ascoli theorem. Assume we are given a bounded sequence u_j in $C^{0,\alpha}([0,T], X)$, we aim to show that we can extract some convergent subsequence in $C^0([0,T], Y)$. First take an enumeration $\{t_i\}$ of the rationals in $[0,T]$. At each t_i we then see that the sequence $u_j(t_i)$ is a bounded sequence in X, hence we can select a subsequence of u_j such that $u_j(t_i)$ converges in Y by assumption. This way we construct sequences $\{u_{1,i}\}, \{u_{2,i}\}, \ldots$ for $t_1, t_2, \ldots$. We then form the diagonal subsequence $v_i := u_{i,i}$ which by definition converges at each rational time in $[0,T]$. We now claim that v_i is a Cauchy sequence in $C^0([0,T],Y)$. Note that the u_i are equicontinuous on $[0,T]$ with respect to the Y-norm (as a consequence of the Hölder condition they satisfy), thus given $\epsilon > 0$ we can find $\delta > 0$ such that $|s-t| < \delta$ implies $\|u_i(t) - u_i(s)\|_Y < \epsilon$ for all indices i and $s, t \in [0,T]$.

In particular we can choose finitely many intervals I_k of length $|I_k| < \delta$ that cover $[0,T]$, each containing a rational time $t_j \in I_k$. Given some $t \in I_k \subset [0,T]$ we can then estimate

$$\|v_m(t) - v_n(t)\|_Y \leq \|v_m(t_j) - v_n(t_j)\|_Y + \|v_m(t_j) - v_m(t)\|_Y + \|v_n(t_j) - v_n(t)\|_Y \leq 3\epsilon \tag{C.12}$$

for m, n sufficiently large. This proves the claim (as we can apply this argument at each $t \in [0,T]$). $\square$

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