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ON THE DISTRIBUTION OF THE LARGEST REAL EIGENVALUE FOR THE REAL GINIBRE ENSEMBLE

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Let $\sqrt{N}+\lambda_{\max}$ be the largest real eigenvalue of a random $N \times N$ matrix with independent $N(0,1)$ entries (the "real Ginibre matrix"). We study the large deviations behaviour of the limiting $N \to \infty$ distribution $\mathbb{P}[\lambda_{\max} < t]$ of the shifted maximal real eigenvalue $\lambda_{\max}$. In particular, we prove that the right tail of this distribution is Gaussian: for $t > 0$,

$$\mathbb{P}[\lambda_{\max} < t] = 1 - \frac{1}{4}\operatorname{erfc}(t) + O\big(e^{-2t^2}\big).$$

This is a rigorous confirmation of the corresponding result of [*Phys. Rev. Lett.* **99** (2007) 050603]. We also prove that the left tail is exponential, with correct asymptotics up to $O(1)$: for $t < 0$,

$$\mathbb{P}[\lambda_{\max} < t] = e^{\frac{1}{2\sqrt{2\pi}}\zeta(\frac{3}{2})t + O(1)},$$

where ζ is the Riemann zeta-function.

Our results have implications for interacting particle systems. The edge scaling limit of the law of real eigenvalues for the real Ginibre ensemble is a rescaling of a fixed time distribution of annihilating Brownian motions (ABMs) with the step initial condition; see [Garrod, Poplavskyi, Tribe and Zaboronski (2015)]. Therefore, the tail behaviour of the distribution of $X_s^{(\max)}$—the position of the rightmost annihilating particle at fixed time $s > 0$—can be read off from the corresponding answers for $\lambda_{\max}$ using $X_s^{(\max)} \overset{D}{=} \sqrt{4s}\lambda_{\max}$.

1. Introduction and the main result. The laws describing the distribution of eigenvalues of large self-adjoint random matrices near the spectral edge exhibit a large degree of universality. One of the examples is the celebrated family of Tracy–Widom distributions for the largest eigenvalue $F_\beta, \beta = 1, 2, 4$, which has been originally discovered in the context of Gaussian orthogonal, unitary and symplectic ensembles [18, 19]. These distributions also describe the scaling limit of the distribution of the largest eigenvalue for non-Gaussian invariant ensembles [3, 4] as well as noninvariant ensembles of random matrices with independent

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non-Gaussian entries [16]. Moreover, F_β appears in the large number of statistical models not directly related to random matrices, such as random permutations, growth models belonging to the KPZ universality class and related asymmetric exclusion processes; see [1] for a recent review.

There is a growing body of evidence that the extreme statistics of eigenvalues for the real Ginibre ensemble [10] also give rise to a new universality class which is relevant beyond random matrix theory. Recall that the real Ginibre ensemble is a Gaussian measure on the set of random real $N \times N$ matrices such that the matrix elements are independent $N(0, 1)$ random variables. The law of the real eigenvalues in the $N \to \infty$ edge scaling limit is governed by a Pfaffian point process discovered independently in [2, 8] and [15]. Within random matrix theory, this Pfaffian point process turns out to be universal, in the sense that it holds for a large class of non-Gaussian ensembles of real non-symmetric matrices; see [17], Corollary 15.[3] The universality of the extreme eigenvalue statistics can then be shown to follow from the universality of the local correlations functions via the Fredholm Pfaffian representation; see [14]. Outside the random matrix theory, the same Pfaffian point process describes the law of instantaneously annihilating Brownian motions on the line started with half-space initial conditions; see [9].[4] Therefore, the scaling limit of the law of the largest real eigenvalue for the real Ginibre ensemble is the same as the law of the rightmost annihilating particle at a fixed time. In this paper, we will analyse the tails of this distribution and show that

$$\mathbb{P}(\lambda_{\max} < t) \overset{t \to -\infty}{=\!=\!=} e^{\frac{\zeta(3/2)}{2\sqrt{2\pi}} t + O(1)}, \tag{1.1}$$

where $\lambda_{\max}$ is the position of the maximal eigenvalue measured from the right edge of the spectrum for the real Ginibre ensemble or it is the scaled position $X_s^{(\max)}/\sqrt{4s}$ of the rightmost particle for annihilating Brownian motions at time s.

An instance of edge statistics (1.1) outside the random matrix theory *and* Pfaffian point processes is for the symmetric exclusion process on $\mathbb{Z}$ with half-filled initial conditions. That is, all sites to the left of zero are occupied at time $s = 0$, and all sites to the right of zero are empty; at time $s > 0$, particles hop onto empty neighboring sites at rate 1. Let R_s be the position of the rightmost particle at time s. It is shown in [12] that

$$\mathbb{P}(R_s = 0) \overset{s \to \infty}{=\!=\!=} e^{-\frac{\zeta(3/2)}{\sqrt{\pi}} \sqrt{s} + o(\sqrt{s})}, \tag{1.2}$$

which coincides with (1.1) if we identify distance $|t|$ with the diffusive scale $\sqrt{8s}$. To verify that this agreement is not accidental, one must compare $\mathbb{P}(\lambda_{\max} < t)$ and

[3]The cited universality statement covers both the edge and the bulk scaling limits, but only the former is needed for the current investigation.

[4]The bulk scaling limit of the law of the real eigenvalues for the real Ginibre ensemble coincides with the fixed time law for the annihilating Brownian motions corresponding to the maximal entrance law [20].

$\mathbb{P}(R_{t^2/8} = 0)$ for all values of t, not just in the large-$|t|$ limit, but the very possibility that there is a relation between the real Ginibre ensemble and the symmetric exclusion process is intriguing.

Let us stress that formula (1.1) is easy to guess given results on bulk gap probabilities described in Forrester [7]. Building on the paper [5] on gap probability for annihilating Brownian motions and using the relation between the bulk statistics for the real Ginibre ensemble and annihilating Brownian motions with the maximal entrance law initial conditions [20], Forrester argues that the bulk scaling limit of the gap probability for the real Ginibre ensemble satisfies (1.1) if $|t|$ is interpreted as the size of the gap. As the width of the transition region near the edge of the spectrum is of order one for the real Ginibre ensemble, it is only natural to guess that (1.1) stays valid at the edge with edge effects showing only as $O(1)$ terms. The aim of the current paper is to verify these heuristics rigorously.

The starting point of our investigation is the following Fredholm determinant representation of $\mathbb{P}[\lambda_{\max} < \alpha]$ due to Rider and Sinclair [14].[5]

THEOREM 1.1 (Rider and Sinclair [14]). *Introduce the integral operator T with kernel*

$$T(x, y) = \frac{1}{\pi} \int_0^\infty e^{-(x+u)^2} e^{-(y+u)^2}\, du. \tag{1.3}$$

Let χ_t be the indicator of (t, ∞). Then

$$\mathbb{P}[\lambda_{\max} < t] = \sqrt{\det(I - T\chi_t)\Gamma_t}, \tag{1.4}$$

where Γ_t is defined as follows. Set $g(x) = \frac{1}{\sqrt{\pi}} e^{-x^2}$, $G(x) = \int_{-\infty}^x g(y)\, dy$ and denote by $R(\cdot, \cdot)$ the kernel of the operator $(I - T\chi_t)^{-1} - I$. Then

$$\begin{aligned}\Gamma_t = {} & \left(1 - \frac{1}{2} a_t\right)\left(1 - \frac{1}{2}\int_{-\infty}^t R(x, t+)\, dx\right) \\ & + \frac{1}{4}(1 - b_t)\left(\int_{-\infty}^t (I - T\chi_t)^{-1} g(x)\, dx - 1\right)\end{aligned}$$

for $a_t = \int_t^\infty G(x)(I - T\chi_t)^{-1} g(x)\, dx$ and $b_t = (I - T\chi_t)^{-1} G(t)$.

The reader is referred to the original paper for precise definitions of the quantities entering the theorem. We mention briefly only that $\det(I - T\chi_t)$ should be understood as the Fredholm determinant of the operator T acting on $L^2(t, \infty)$ and that T is the square of a Hilbert–Schmidt operator, which implies that T is positive and trace class.

[5] The paper [14] contains some minor computational errors, which led to an incorrect expression for $\mathbb{P}(\lambda_{\max} < \alpha)$. Here, we present the corrected version. An interested reader is referred to Appendix B for a discussion of the errors and their correction.

The apparently complicated expression for the factor Γ_t in the statement of the Rider–Sinclair theorem is actually rather simple, and cancellations occur that imply that $\Gamma_t = (1 - a_t)$. Therefore, we have the following.

LEMMA 1.1.

$$\mathbb{P}(\lambda_{\max} < t) = \sqrt{\det(I - T\chi_t)(1 - a_t)}, \tag{1.5}$$

where the operator T and the function a_t are defined in the statement of Theorem 1.1.

In what follows we are going to use the modified version (1.5) of the Rider–Sinclair result.

As it turns out, the asymptotic analysis of (1.5) can be greatly simplified by assigning a probabilistic interpretation to all the factors. In particular, the interpretation allowed us to control the asymptotic expansion beyond the leading order, which is hard to do using purely operator theoretic methods. Let $(B_n, n \geq 0)$ be the discrete time random walk with Gaussian $N(0, 1/2)$ increments started at zero. Let

$$\begin{cases} \tau_t = \inf_{n>0}\{2n - 1 : B_{2n-1} \leq t\}, \\ \tau_0 = \inf_{n>0}\{2n : B_{2n} \geq 0\}. \end{cases} \tag{1.6}$$

In words: τ_0 is the smallest *even* time such that $B_{\tau_0} \geq 0$, τ_t is the smallest *odd* time such that $B_{\tau_t} \leq t$. Also, let

$$I_{\tau_0} = \inf\{B_s : s \text{ is odd}, s \leq \tau_0\} \tag{1.7}$$

be the infimum of the random walk $(B_s)_{\geq 0}$ taken over all odd times not exceeding the exit time τ_0. Then we have the following probabilistic restatement of Theorem 1.1.

THEOREM 1.2.

$$\mathbb{P}(\lambda_{\max} < t) = \sqrt{\mathbb{P}(\tau_t < \tau_0)}e^{-\frac{1}{2}\mathbb{E}((I_{\tau_0} - t)_+ \delta_0(B_{\tau_0}))}, \tag{1.8}$$

where $x_+ := \max(x, 0)$ is the positive part of a real number.

REMARK. We use the expression $\mathbb{E}(X\delta_y(Y))$ to mean a continuous Lebesgue density for the measure $\mathbb{E}(X\mathbb{1}(Y \in dy))$ evaluated at y. If $y = 0$, we sometimes re-write $\mathbb{E}(X\delta_0(Y))$ as $\mathbb{E}(X\mathbb{1}(Y \in d0))$.

Note that (1.8) can be rewritten in a more useful way when $t < 0$ as

$$\mathbb{P}(\lambda_{\max} < t) = \sqrt{\mathbb{P}(\tau_t < \tau_0)}e^{\frac{t}{2}\mathbb{E}(\delta_0(B_{\tau_0}))}e^{-\frac{1}{2}\mathbb{E}(\max(t, I_{\tau_0})\delta_0(B_{\tau_0}))}. \tag{1.9}$$

Our main result below will follow from the above probabilistic representation via an application of some general results for random walks. In particular, it can be

easily shown that $\mathbb{E}(\delta_0(B_{\tau_0})) = \zeta(3/2)/\sqrt{2\pi}$ (see Lemma 3.2), which then controls the first term in the asymptotics for $t < 0$. The following theorem details the asymptotics up to $O(1)$.

THEOREM 1.3. *For $t > 0$,*

$$\mathbb{P}(\lambda_{\max} < t) = 1 - \frac{1}{4}\operatorname{erfc}(t) + O\big(e^{-2t^2}\big). \tag{1.10}$$

For $t < 0$,

$$\mathbb{P}(\lambda_{\max} < t) = \exp\Big(-\frac{\zeta(3/2)}{2\sqrt{2\pi}}|t| + O(1)\Big). \tag{1.11}$$

For $t < 0$, the above statement provides a rigorous justification of (1.1). The right tail asymptotic (1.10) is actually well known [8] since the probability of finding an eigenvalue very far to the right of the spectral edge is approximately equal to the level density. This part of Theorem 1.3 should be considered as a test of the Rider–Sinclair answer complemented by careful bounding of error terms.

As has been already mentioned, the edge scaling limit of the law of real eigenvalues for the real Ginibre ensemble coincides up to a Brownian rescaling with the law of annihilating Brownian motions on the real line started with the step initial condition [9]. The step initial condition corresponds to the maximal entrance law restricted to $x < 0$ and zero density of particles for $x > 0$. The maximal entrance law can be constructed as the limit of the homogeneous Poisson point process initial conditions with intensity diverging to infinity; see [9] for precise definitions and details. Therefore, Theorem 1.3 yields a simple corollary.

COROLLARY 1.1. *Consider the system of instantaneously annihilating Brownian motions on the real line started with the step initial condition. Let $X_s^{(\max)}$ be the position of the rightmost particle at a fixed time $s > 0$. Then*

$$\mathbb{P}\big(X_s^{(\max)} < x\big) = 1 - \frac{1}{4}\operatorname{erfc}\Big(\frac{x}{\sqrt{4s}}\Big) + O\big(e^{-\frac{x^2}{2s}}\big)$$

for $x/\sqrt{s} \to \infty$, while for $x/\sqrt{s} \to -\infty$,

$$\mathbb{P}\big(X_s^{(\max)} < x\big) = e^{\frac{1}{2\sqrt{2\pi}}\zeta(\frac{3}{2})\frac{x}{\sqrt{4s}} + O(1)}.$$

The above result complements the recent study [11] of annihilating Brownian motions near the edge of the distribution, where the average number of particles in the positive half-line $(0, \infty)$ has been calculated.

The correspondence between the real eigenvalues and annihilating particles can also be used for an intuitive explanation of Theorem 1.3: the right tail of $\mathbb{P}(\lambda_{\max} < t)$ is Gaussian as it corresponds to the probability that a Brownian particle travels distance t during the time interval $[0, 1/4]$. The left tail cannot be

thinner than exponential, as the probability of the event $\{\lambda_{\max} < t\}$ for a negative t of large magnitude can be bounded below by the intersection of the following $O(|t|)$ independent events: particles stay within each of $|t|$ boxes of size 1 and completely annihilate each other by the time $s = 1/4$; particles with initial positions to the left of t do not enter the interval (t, ∞) before time $1/4$. Unfortunately, we could not extract the exact rate of the exponential decay from arguments like this.

The rest of the paper is organised as follows. In Section 2, we will prove Lemma 1.1 and Theorem 1.2. The asymptotic analysis of $\mathbb{P}(\lambda_{\max} < s)$ is carried out in Section 3. All technical lemmas needed for the proof of our main result are proved in Appendix A. In Appendix B, we explain how to fix the minor errors we found in the original statement of the Rider–Sinclair theorem.

2. The proof of Lemma 1.1 and Theorem 1.2. Let us examine the Γ-factor defined in the statement of the Rider–Sinclair Theorem 1.1. Throughout the paper, we will use the following basic property of the integral operator T.

LEMMA 2.1. *For any $t \in \mathbb{R}$, the operator T acting on $L^2(t, \infty)$ is positive definite. Its spectral radius ρ_t is bounded away from* 1: *for any $t > 0$,*

$$\rho_t \le \frac{1}{8} e^{-2t^2}. \tag{2.1}$$

There exists $C_0 > 0$ such that for any $t < 0$,

$$\rho_t \le \exp\left[-\frac{C_0}{1+t^2}\right]. \tag{2.2}$$

Therefore, for any $t \in \mathbb{R}$, the resolvent of the operator $T\chi_t$ can be expanded into an absolutely convergent power series,

$$(I - T\chi_t)^{-1} = \sum_{n=0}^{\infty} (T\chi_t)^n.$$

Using the explicit definitions of the integral operator T and functions G, g from Theorem 1.1,

$$a_t := \int_t^{\infty} G(x)(I - T\chi_t)^{-1} g(x)\, dx = \sum_{m=0}^{\infty} p_m, \tag{2.3}$$

where

$$\begin{aligned} p_m = \int_{\mathbb{R}^{2m+2}} \prod_{k=0}^{m} \frac{dy_k\, du_k}{\pi} e^{-y_m^2 - \sum_{n=1}^{m}((u_n - y_n)^2 + (y_{n-1} - u_n)^2) - (y_0 - u_0)^2} \\ \times \left(\prod_{p=0}^{m} \chi(y_p \ge t) \prod_{q=1}^{m} \chi(u_q \le 0) \chi(u_0 \ge 0) \right), \end{aligned} \tag{2.4}$$

where $\chi(\cdot \geq a)$, $\chi(\cdot \leq a)$ are the characteristic functions of $[a, \infty)$ and $(-\infty, a]$ correspondingly. The expression for p_m quoted above arises from some simple changes of variables designed to re-write in terms of the density of a segment of the Gaussian random walk introduced in Section 1. The integrand in (2.4) can be identified with the density for the initial segment of the walk of length $2m+2$ $(B_1, B_2, \dots, B_{2m+2})$ at the point $(y_m, u_m, y_{m-1}, u_{m-1}, \dots, y_0, u_0)$. This reveals that

$$p_m = \mathbb{P}[B_{2k+1} \geq t, k = 0, 1, \dots, m; B_{2l} \leq 0, l = 1, 2, \dots, m; B_{2m+2} \geq 0].$$

Equivalently, in terms of exit times τ_t, τ_0 defined in (1.6),

$$p_m = \mathbb{P}[\tau_0 = 2m+2; \tau_t > \tau_0], \qquad m \geq 0.$$

Substituting this formula into (2.3) and summing over m's, we find that

$$a_t = \mathbb{P}[\tau_t > \tau_0]. \tag{2.5}$$

Note that the above expression is well defined, as the exit time $\tau := \tau_0 \wedge \tau_t$ is a finite random variable, $\mathbb{P}[\tau < \infty] = 1$.

Next,

$$b_t := (I - T\chi_t)^{-1} G(t) = \sum_{m=0}^{\infty} q_m, \tag{2.6}$$

where

$$\begin{aligned} q_m = \int_{\mathbb{R}^{2m+1}} \prod_{k=1}^{m} \frac{dy_k\, du_k}{\pi} \frac{du_{m+1}}{\sqrt{\pi}} e^{-u_1^2 - \sum_{n=1}^{m}((y_n-u_n)^2 + (u_{n+1}-y_n)^2)} \\ \times \left(\prod_{p=1}^{m} \chi(y_p \leq 0) \chi(u_p \geq t) \chi(u_{m+1} \leq t) \right). \end{aligned} \tag{2.7}$$

Consider the density for segment $(B_1, B_2, \dots, B_{2m+1})$ of the random walk at the point $(u_1, y_1, \dots, u_m, y_m, u_{2m+1})$. Re-writing (2.6) as an expectation over this segment, we find that

$$q_m = \mathbb{P}[B_{2k+1} \geq t, k = 0, 1, \dots, m-1; B_{2l} \leq 0, l = 1, 2, \dots, m; B_{2m+1} \leq t].$$

In terms of exit times, this is

$$q_m = \mathbb{P}[\tau_t = 2m+1, \tau_t < \tau_0], \qquad m \geq 0.$$

Substituting this expression into (2.6) and summing over m, we find that

$$b_t = \mathbb{P}[\tau_t < \tau_0] = 1 - a_t. \tag{2.8}$$

In a very similar fashion, we find that

$$\int_{-\infty}^{t} R(x, t)\, dx = \mathbb{P}[\tau_t > \tau_0] = a_t \tag{2.9}$$

and

$$\int_{-\infty}^{t} (I - T\chi_t)^{-1} g(x)\,dx = \mathbb{P}[\tau_t < \tau_0] = 1 - a_t. \tag{2.10}$$

Substituting (2.5), (2.8), (2.9) and (2.10) into the expression for Γ_t presented in Theorem 1.1 gives

$$\Gamma_t = (1 - a_t/2)^2 - a_t^2/4 = 1 - a_t = \mathbb{P}[\tau_t < \tau_0].$$

Lemma 1.1 and the pre-factor in formula (1.8) of Theorem 1.2 are verified.

To check the exponent in the right-hand side of (1.8), we use Lemma 2.1 to justify the application of the trace-log formula to the Fredholm determinant appearing in the Rider–Sinclair theorem and the subsequent expansion of the logarithm:

$$\log\det(I - T\chi_t) = \operatorname{Tr}\log(I - T\chi_t) = -\sum_{m=1}^{\infty} \frac{1}{m} \operatorname{Tr}(T\chi_t)^m.$$

Explicitly,

$$\log\det(I - T\chi_t) = -\sum_{m=1}^{\infty} \frac{1}{m} r_m(t), \tag{2.11}$$

where

$$\begin{aligned} r_m(t) = \int_{\mathbb{R}^{2m}} \prod_{k=1}^{m} \frac{dy_k\,du_k}{\pi} e^{-\sum_{n=1}^{m}((u_n - y_n)^2 + (y_{n+1} - u_n)^2)} \\ \times \left(\prod_{p=1}^{m} \chi(u_p \le 0)\chi(y_p \ge t) \right), \end{aligned} \tag{2.12}$$

where $y_{m+1} \equiv y_1$. Differentiating r_m with respect to t gives a sum of m terms which are all identical due to the cyclic symmetry of the integrand. Interpreting each of the terms using the Gaussian random walk of length $2m$, we find that

$$\frac{dr_m}{dt}(t) = -m\mathbb{P}[\tau_0 = 2m;\ \tau_t > \tau_0;\ B_{\tau_0} \in d0].$$

Substituting the above into the t-derivative of (2.11), one gets

$$\begin{aligned} \frac{d}{dt}\log\det(I - T\chi_t) &= \mathbb{P}[\tau_t > \tau_0;\ B_{\tau_0} \in d0] \\ &= \mathbb{E}\big(\mathbb{1}(I_{\tau_0} > t)\delta_0(B_{\tau_0})\big), \end{aligned}$$

where we used that $\{\tau_t > \tau_0\} = \{I_{\tau_0} > t\}$. Integrating the above expression over the interval $(t, +\infty)$ and applying the boundary condition $\det(I - T\chi_t)\,|_{t=+\infty} = 1$, we arrive at the claimed expression for the exponent:

$$\log\det(I - T\chi_t) = -\mathbb{E}\big((I_{\tau_0} - t)_+\delta_0(B_{\tau_0})\big). \tag{2.13}$$

Formula (1.8) is proved.

Finally, equation (1.9) follows from (1.8) due to the following elementary identity: for any t, I,

$$t + (I - t)_+ = \max(t, I).$$

Theorem 1.2 is proved.

3. The proof of Theorem 1.3.

3.1. *Asymptotic expansion for $t < 0$: The proof of* (1.11). A natural starting point for our analysis is equation (1.9). First, let us investigate the coefficient of the $O(t)$ term, $\mathbb{E}(\delta_0(B_{\tau_0}))$. The event $\{B_{\tau_0} \in d0\}$ depends only on the position of the random walk at even times. Averaging over positions at odd times, we find that

$$\mathbb{P}[B_{\tau_0} \in d0] = \mathbb{P}[\tilde{B}_{t_0} \in d0],$$

where $(\tilde{B}_s)_{s \geq 0}$ is a Gaussian random walk with $N(0, 1)$-increments and $t_0 = \inf\{s = 1, 2, \dots : \tilde{B}_s > 0\}$. Therefore,

$$\begin{aligned}
\mathbb{E}\big(\delta_0(B_{\tau_0})\big) &= \sum_{n=1}^{\infty} \mathbb{P}[t_0 = n, \tilde{B}_n \in d0] \\
&= \sum_{n=1}^{\infty} \mathbb{P}[\tilde{B}_1 < 0; \tilde{B}_2 < 0; \tilde{B}_{n-1} < 0; \tilde{B}_n \in d0] \\
&= \sum_{n=1}^{\infty} \mathbb{P}[\tilde{B}_1 < 0; \tilde{B}_2 < 0; \tilde{B}_{n-1} < 0 \mid \tilde{B}_n = 0]\mathbb{P}[\tilde{B}_n \in d0].
\end{aligned}$$

The last expression can be evaluated using the following remarkable combinatorial lemma.

LEMMA 3.1.

$$\mathbb{P}[\tilde{B}_1 < 0; \tilde{B}_2 < 0; \tilde{B}_{n-1} < 0 \mid \tilde{B}_n = 0] = \frac{1}{n}. \tag{3.1}$$

This lemma follows from a more general combinatorial result concerning the total time a random walk spends above zero; see [6], Chapter XII, Section 6. For the sake of completeness, a very simple proof of Lemma 3.1 is presented in Appendix A. We conclude that

$$\mathbb{E}\big(\delta_0(B_{\tau_0})\big) = \sum_{n=1}^{\infty} \frac{\mathbb{P}[\tilde{B}_n \in d0]}{n} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{2\pi n^3}} = \frac{\zeta(3/2)}{\sqrt{2\pi}}. \tag{3.2}$$

The leading order asymptotic for $\log \mathbb{P}[\lambda_{\max} < t]$ is derived.

To finish the derivation of equation (1.11) from (1.9), we have to show that

$$\log \mathbb{P}[\tau_t < \tau_0] - \mathbb{E}\big(\max(t, I_{\tau_0})\delta_0(B_{\tau_0})\big) = O(1). \tag{3.3}$$

By a Brownian motion analogue, it is clear that each of the terms on the left-hand side of (3.3) is $O(\log|t|)$. The main challenge is to show that the logarithms cancel. This cancellation follows from the following two results.

LEMMA 3.2. *There exists a positive constant C such that for a sufficiently large $|t|$,*

$$\frac{1}{\sqrt{2}|t|} \geq \mathbb{P}[\tau_t < \tau_0] \geq \frac{1}{\sqrt{2}|t|}\big(1 - C|t|^{-1/2}\big). \tag{3.4}$$

LEMMA 3.3. *Consider a Gaussian random walk with $N(0,1)$ increments. There exists a positive constant μ such that for $y < 0$,*

$$\mathbb{E}_y\big(\delta_0(B_{\tau_0})\big) = \sqrt{2} + O\big(e^{-|y|\mu}\big). \tag{3.5}$$

Here, the subscript y means that the random walk is started from y and $\tau_0 = \inf(n : B_n > 0)$.

Lemma 3.2 can be easily proved using martingale methods; see Appendix A. In contrast, Lemma 3.3 is a corollary of Theorem 4 of [13], which is a result of an intricate asymptotic analysis of Wiener–Hopf equations associated with exit problems for Gaussian random walks.

It follows from Lemma 3.2 that

$$\log \mathbb{P}[\tau_t < \tau_0] = -\log|t| + O(1). \tag{3.6}$$

The second term on the left-hand side of (3.3) can be simplified using integration by parts. The result is

$$\mathbb{E}\big(\max(t, I_{\tau_0})\delta_0(B_{\tau_0})\big) = -\int_t^0 dy\mathbb{E}\big(\mathbb{1}_{I_{\tau_0}<y}\delta_0(B_{\tau_0})\big) + \int_0^\infty dy\mathbb{E}\big(\mathbb{1}_{I_{\tau_0}>y}\delta_0(B_{\tau_0})\big).$$

The second term on the right-hand side of the above identity is t-independent and is finite as the the right tail of the distribution of I_{τ_0} conditioned on $B_{\tau_0} = 0$ can be shown to have Gaussian decay. The first term can be evaluated using Lemma 3.3: let $L < 0$ be a t-independent constant. Then

$$\begin{aligned}
\int_t^0 dy\mathbb{E}\big(\mathbb{1}_{I_{\tau_0}<y}\delta_0(B_{\tau_0})\big) &= \left(\int_t^L + \int_L^0\right) dy\mathbb{E}\big(\mathbb{1}_{I_{\tau_0}<y}\delta_0(B_{\tau_0})\big) \\
&= \int_t^L dy\mathbb{E}\big(\mathbb{1}_{I_{\tau_0}<y}\delta_0(B_{\tau_0})\big) + O(1) \\
&= \int_t^L dy\mathbb{E}\big(\mathbb{1}_{I_{\tau_0}<y}\mathbb{E}_{B_{\tau_y}}\big(\delta_0(B_{\tau_0})\big)\big) + O(1)
\end{aligned}$$

$$\overset{\text{Lemma 3.3}}{=\!=\!=} \int_t^L dy \mathbb{E}\big(\mathbb{1}_{I_{\tau_0}<y}\big(\sqrt{2}+O\big(e^{-\mu|B_{\tau_y}|}\big)\big)\big)+O(1)$$

$$\overset{\text{Lemma 3.2}}{=\!=\!=} \log\big(|t|\big)+O(1).$$

We conclude that

$$\mathbb{E}\big(\max(t, I_{\tau_0})\delta_0(B_{\tau_0})\big) = -\log\big(|t|\big)+O(1). \tag{3.7}$$

Substituting (3.6) and (3.7) into the left-hand side of (3.3), we observe that the logarithms cancel and we are left with terms of order 1, as claimed. Theorem 1.3 is proved for $t<0$.

3.2. *Asymptotic expansion for $t>0$: The proof of* (1.10). The $t>0$ case is best tackled starting directly from the Rider–Sinclair theorem with the pre-factor given by Lemma 1.1. Recall that the operator $T\chi_t$ is a positive trace class operator on $L^2(t,\infty)$ with spectral radius strictly less than 1; see Lemma 2.1. Therefore,

$$\mathbb{P}[\lambda_{\max}<t] = \sqrt{(1-a_t)}e^{-\frac{1}{2}Q_t},$$

where $Q_t = \sum_{m=1}^{\infty}\frac{1}{m}\operatorname{Tr}(T\chi_t)^m$. An explicit calculation [see (A.2)] shows that

$$\operatorname{Tr}(T\chi_t) \le \frac{1}{8}e^{-2t^2}.$$

This leads to the following estimate:

$$\begin{aligned} 0 \le Q_t &= \sum_{m=1}^{\infty}\frac{1}{m}\operatorname{Tr}(T\chi_t)^m \le \sum_{m=1}^{\infty}\frac{1}{m}\big(\operatorname{Tr}(T\chi_t)\big)^m \\ &= -\log\big(1-\operatorname{Tr}(T\chi_t)\big) \le -\log\Big(1-\frac{1}{8}e^{-2t^2}\Big). \end{aligned}$$

Therefore, we conclude that

$$\mathbb{P}[\lambda_{\max}<t] = \sqrt{(1-a_t)}\big(1+O\big(e^{-2t^2}\big)\big). \tag{3.8}$$

To calculate the pre-factor, notice that

$$T\chi_t g(y) = g(y)\int_t^{\infty}dx\int_0^{\infty}\frac{dz}{\pi}e^{-2(x^2+z^2+xz+yz)} \le \frac{1}{8\sqrt{2\pi}t}e^{-2t^2}g(y),$$

so,

$$a_t = \int_t^{\infty}G(x)\sum_{k=0}^{\infty}(T\chi_t)^k g(x)\,dx = \int_t^{\infty}G(x)g(x)\,dx+\rho_t,$$

where for suitably large t

$$0 \le \rho_t \le \int_t^\infty G(x)g(x)\,dx \sum_{m=1}^\infty \left(\frac{1}{8\sqrt{2\pi}\,t} e^{-2t^2}\right)^m$$
$$= \int_t^\infty G(x)g(x)\,dx \cdot \frac{\frac{1}{8\sqrt{2\pi}\,t} e^{-2t^2}}{1 - \frac{1}{8\sqrt{2\pi}\,t} e^{-2t^2}}.$$

Also notice that

$$\int_t^\infty G(x)g(x)\,dx = \int_t^\infty G(x)G'(x)\,dx$$
$$= \frac{1}{2}\big(1 - G(t)^2\big)$$
$$= \frac{\operatorname{erfc}(t)}{2} - \frac{\operatorname{erfc}^2(t)}{8}.$$

Therefore,

$$a_t = \frac{1}{2}\operatorname{erfc}(t)\big(1 + O\big(e^{-t^2}\big)\big). \tag{3.9}$$

Substituting (3.9) into (3.8), we find that for $t > 0$,

$$\mathbb{P}[\lambda_{\max} < t] = 1 - \frac{1}{4}\operatorname{erfc}(t) + O\big(e^{-2t^2}\big).$$

Formula (1.10) is proved.

APPENDIX A: THE PROOF OF TECHNICAL LEMMAS

A.1. Lemma 2.1. Positive definiteness: for any $\phi \in L^2(t, \infty)$,

$$(\phi, T\phi) = \int_0^\infty \frac{dz}{\pi}\left(\int_t^\infty dx e^{-(z+x)^2}\phi(x)\right)^2 \ge 0.$$

Therefore, we have the following estimate for the spectral radius: for any $m = 1, 2, 3, \dots,$

$$\rho_t \le \big(\operatorname{Tr} T^m\big)^{\frac{1}{m}}. \tag{A.1}$$

For $t > 0$, we can set $m = 1$. An explicit calculation gives

$$\begin{aligned} \rho_t \le \operatorname{Tr} T &= \frac{1}{\pi}\int_0^\infty dx \int_t^\infty dy e^{-2(x+y)^2} \\ &\le \frac{1}{\pi}\int_0^\infty dx e^{-2x^2}\int_t^\infty dy e^{-2y^2} \le \frac{1}{8}e^{-2t^2}. \end{aligned} \tag{A.2}$$

Bound (2.1) is proved.

For $t < 0$, let us re-write $\operatorname{Tr} T^m$ in terms of the Gaussian random walk:

$$\operatorname{Tr} T^m = \int_{-\infty}^{-t} dy \mathbb{P}[y + B_{2m} \in dy;\, y + B_{2k+1} \geq 0,$$
$$k = 0, 1, \dots, m-1;\, y + B_{2k} \leq -t,\, 1 \leq k \leq m-1].$$

Clearly,

$$\operatorname{Tr} T^m \leq \int_{-\infty}^{-t} dy \mathbb{P}[y + B_{2m} \in dy;\, y + B_1 \geq 0].$$

A simple bound on the double integral in the right-hand side gives

$$\operatorname{Tr} T^m \leq \frac{1}{\sqrt{2\pi m}} \left(-t + \frac{1}{2\sqrt{2\pi}} e^{-t^2} \right).$$

Substituting this bound into (A.1) and optimising with respect to m, we arrive at (2.2).

A.2. Lemma 3.1. Let $X_1, X_2, \dots, X_n$ be the Gaussian increments of a random walk $\tilde{B}_k = \sum_{m=1}^{k} X_m$, $k = 1, 2, \dots, n$, conditioned to have $\tilde{B}_n = 0$. In total, there are $n!$ associated random walks corresponding to all permutations of the increments. All these permutations are equiprobable. Let us say that two random walks are equivalent if one can be obtained from the other by a cyclic permutation of the increments. Each equivalence class contains n random walks. Almost surely, precisely one representative in each of the classes will stay negative at all times between 1 and $n-1$. To construct such a random walk, let us pick an arbitrary representative

$$\tilde{B} = (X_1, X_1 + X_2, \dots, X_1 + X_2 + \cdots + X_{n-1}, 0)$$

of the equivalence class. Let k be the point of global maximum of $\tilde{B}$. Then the following cyclic permutation of $\tilde{B}$ is a random walk which stays below the origin:

$$(X_{k+1}, X_{k+1} + X_{k+2}, \dots, X_{k+1} + X_{k+2} + \cdots + X_n + X_1 + \cdots + X_{k-1}, 0).$$

All other cyclic permutations of $\tilde{B}$ will have at least one point above the origin.

We conclude that the fraction of random walks conditioned to finish at zero which stay below the origin at all times between 1 and $n-1$ is equal to $(n!/n)/n! = 1/n$. Therefore,

$$\mathbb{P}[\tilde{B}_1 < 0, \dots, \tilde{B}_{n-1} < 0 \mid \tilde{B}_n = 0] = \frac{1}{n}.$$

A.3. Lemma 3.2. Set $\tau = \tau_0 \wedge \tau_t$. By Wald's identity [6] (or the optional stopping theorem for random walks),

$$\mathbb{E}(B_\tau) = \mathbb{E}(B_0) = 0. \tag{A.3}$$

Therefore,

$$\mathbb{E}(B_\tau \mathbb{1}_{\tau_0 > \tau_t}) + \mathbb{E}(B_\tau \mathbb{1}_{\tau_0 < \tau_t}) = 0.$$

Let

$$L_{\tau_0} = B_{\tau_0}, \qquad L_{\tau_t} = t - B_{\tau_t}$$

be the positive overlaps (or "ladder heights") of the random walk at the exit times τ_0 and τ_t correspondingly. It follows from the Wald identity that

$$\mathbb{E}(\mathbb{1}_{\tau_0 > \tau_t}) = \frac{1}{|t|}\big(\mathbb{E}(L_{\tau_0}) - \mathbb{E}\big((L_{\tau_0} + L_{\tau_t})\mathbb{1}_{\tau_0 > \tau_t}\big)\big).$$

Applying Spitzer's theorem (see, e.g., [6], Chapter XVIII.5) to the Gaussian random walk, we find that

$$\mathbb{E}(B_{\tau_0}) = \frac{1}{\sqrt{2}},$$

which implies that

$$\mathbb{P}[\tau_0 > \tau_t] = \mathbb{E}(\mathbb{1}_{\tau_0 > \tau_t}) = \frac{1}{|t|}\left(\frac{1}{\sqrt{2}} - \mathbb{E}\big((L_{\tau_0} + L_{\tau_t})\mathbb{1}_{\tau_0 > \tau_t}\big)\right).$$

As the second term on the right-hand side of the above formula is non-negative, the upper bound on $\mathbb{P}[\tau_0 > \tau_t]$ claimed in Lemma 3.2 is proved.

To establish the lower bound, we apply Cauchy–Schwarz inequality to bound $\mathbb{E}((L_{\tau_0} + L_{\tau_t})\mathbb{1}_{\tau_0 > \tau_t})$. Solving the resulting inequality with respect to $\mathbb{E}(\mathbb{1}_{\tau_0 > \tau_t})$, we find that

$$\mathbb{E}(\mathbb{1}_{\tau_0 > \tau_t}) \geq \frac{1}{\sqrt{2}|t|}\Big(1 - \sqrt{2\sqrt{2}\mathbb{E}\big(L_{\tau_0}^2 + L_{\tau_t}^2\big)}|t|^{-1/2}\Big).$$

The upper bound of Lemma 3.2 will be proved with $C = \sqrt{2\sqrt{2}\mathbb{E}(L_{\tau_0}^2 + L_{\tau_t}^2)}$ if we can show that the moments $\mathbb{E}(L_{\tau_0}^2)$ and $\mathbb{E}(L_{\tau_t}^2)$ exist. The existence of moments is an immediate consequence of the following observation: let $W_s \sim N(0, s)$. Then, for any positive function f on $\mathbb{R}$,

$$\begin{aligned} \mathbb{E}\big(f(W_1)\mathbb{1}_{W_1 > 0}\big) &\leq \big\{\mathbb{E}\big(f(B_{\tau_0})\big), \mathbb{E}\big(f(B_{\tau_t})\big)\big\} \\ &\leq 2 \sup_{s \in (0,1)} \mathbb{E}\big(f(W_s)\mathbb{1}_{W_s > 0}\big). \end{aligned} \tag{A.4}$$

The desired result follows from the finiteness of the second moment of a Gaussian distribution.

To prove the lower bound in (A.4), notice first that

$$\mathbb{E}\big(f(B_{\tau_0})\big) = \mathbb{E}\big(f(\tilde{B}_{t_0})\big),$$

where $(\tilde{B}_n)_{n\geq 0}$ is the Gaussian random walk with $N(0,1)$ increments and t_0 is the exit time through 0. Then

$$\mathbb{E}\big(f(B_{\tau_0})\big) = \sum_{m=1}^{\infty} \mathbb{E}\big(f(\tilde{B}_m)\mathbb{1}_{\tau_0=m}\big) \geq \mathbb{E}\big(f(\tilde{B}_1)\mathbb{1}_{\tau_0=1}\big) = \mathbb{E}\big(f(W_1)\mathbb{1}_{W_1>0}\big),$$

which proves the lower bound in (A.4).

To derive the upper bound, let us consider the standard Brownian motion $(W_s)_{s\geq 0}$ coupled to $(\tilde{B}_n)_{n\geq 0}$ in such a way that $W_s = \tilde{B}_s$ for $s = 1, 2, \dots$. Let $(F_s^W)_{s\geq 0}$ be its natural filtration. Let

$$\sigma_m = \inf\{s > m-1 : W_s = 0\}, \qquad m = 1, 2, \dots.$$

Then applying the strong Markov property,

$$\begin{aligned}
&\mathbb{E}\big(f(B_{\tau_0})\big) \\
&\quad = \sum_{m=1}^{\infty} \mathbb{E}\big(f(\tilde{B}_m)\mathbb{1}_{\tau_0=m}\big) \\
&\quad = \sum_{m=1}^{\infty} \mathbb{E}\big(\big(f(\tilde{B}_m)\mathbb{1}_{\tilde{B}_m>0}\big)\mathbb{1}_{\tau_0>m-1}\big) \\
&\quad = \sum_{m=1}^{\infty} \mathbb{E}\big(\big(f(\tilde{B}_m)\mathbb{1}_{\tilde{B}_m>0}\big)\mathbb{1}_{\tau_0>m-1}\mathbb{1}_{\sigma_m\in(m-1,m)}\big) \\
&\quad = \sum_{m=1}^{\infty} \mathbb{E}\big(\mathbb{E}\big(\big(f(\tilde{B}_m)\mathbb{1}_{\tilde{B}_m>0}\big)\mathbb{1}_{\tau_0>m-1}\mathbb{1}_{\sigma_m\in(m-1,m)} \mid F_{\sigma_m}^W\big)\big) \\
&\quad = \sum_{m=1}^{\infty} \mathbb{E}\big(\mathbb{E}\big(\big(f(\tilde{B}_m)\mathbb{1}_{\tilde{B}_m>0}\big) \mid F_{\sigma_m}^W\big)\mathbb{1}_{\tau_0>m-1}\mathbb{1}_{\sigma_m\in(m-1,m)}\big) \\
&\quad = \sum_{m=1}^{\infty} \mathbb{E}\big(\mathbb{E}\big(\big(f(W_{m-\sigma_m})\mathbb{1}_{W_{m-\sigma_m}>0}\big) \mid F_{\sigma_m}^W\big)\mathbb{1}_{\tau_0>m-1}\mathbb{1}_{\sigma_m\in(m-1,m)}\big) \\
&\quad \leq \sup_{\tau\in(0,1)} \mathbb{E}\big(f(W_\tau)\mathbb{1}_{W_\tau>0}\big) \sum_{m=1}^{\infty} \mathbb{E}(\mathbb{1}_{\tau_0>m-1}\mathbb{1}_{\sigma_m\in(m-1,m)}).
\end{aligned} \tag{A.5}$$

By the reflection principle,

$$\mathbb{E}(\mathbb{1}_{\tau_0=m}\mathbb{1}_{\sigma_m\in(m-1,m)}) = \frac{1}{2}\mathbb{E}(\mathbb{1}_{\tau_0>m-1}\mathbb{1}_{\sigma_m\in(m-1,m)}).$$

Substituting this result into (A.5) and summing over m's, we find that

$$\mathbb{E}\big(f(B_{\tau_0})\big) \le 2 \sup_{\tau\in(0,1)} \mathbb{E}\big(f(W_\tau)\mathbb{1}_{W_\tau>0}\big).$$

The proof of the upper bound in (A.4) for $\mathbb{E}(f(B_{\tau_0}))$ is complete. The derivation of bounds for $\mathbb{E}(f(B_{\tau_t}))$ is a carbon copy of the above proof. Lemma 3.2 is proved.

APPENDIX B: A NOTE ON THE RIDER–SINCLAIR THEOREM

For all the definitions, notation and numberings used in this Appendix, we refer readers to the original paper [14]. Here, we deal only with the case of even sized matrices, therefore, we fix the problems happening in paragraph 4.1 of the paper.

Notice that the only difference between the original statement of the Rider–Sinclair theorem and Theorem 1.1 is in the factor Γ_t. The origin of this factor is a finite rank perturbation of the operator $T_n\chi$ (see [14], (4.9)). In summary, we will correct the following two errors:

(i) the limits of the functions $\tilde\phi_n$, $\tilde\psi_n$ (see [14], (4.15)) were calculated incorrectly, and this leads to new definitions for $G(x)$ and $g(x)$ and some changes to the constants contained in the third term of the Γ_t expression as well;

(ii) it was noted in [14], page 1644, last paragraph, that $\int_{-\infty}^t \tilde\psi_n(x)\,dx$ converges to $G(t)$ which is true only up to an additive constant and yields corrections to the fourth term of the Γ_t expression.

We start with the correct derivation of the limits $\lim_{n\to\infty}\tilde\phi_n(x)$ and $\lim_{n\to\infty}\tilde\psi_n(x)$.

$$\begin{aligned}\tilde\phi_n(x) &= \kappa_n \int_0^{x+\sqrt{n}} u^{n-2}e^{-u^2/2}\,du \\ &= \kappa_n 2^{\frac{n-3}{2}}\Gamma\left(\frac{n-1}{2}\right)P\left(\frac{n-1}{2},\frac{(x+\sqrt{n})^2}{2}\right),\end{aligned}$$

where $P(a,x)=\frac{\Gamma(a,x)}{\Gamma(a)}$ is the incomplete regularized Gamma function. It is easy to check by using the duplication formula for Gamma functions that the pre-factor in front of P is asymptotically equal to $2^{-1/2}$. For the last factor, we use the well-known asymptotic formula (see, e.g., [2], (9.16))

$$P(a, a+\sqrt{2a}x) \sim \frac{1}{2}\operatorname{erfc}(-x), \qquad a\to\infty.$$

Together with the above identity, we get

$$G(x) = \lim_{n\to\infty}\tilde\phi_n(x) = \frac{1}{2\sqrt{2}}\operatorname{erfc}(-x) = \int_{-\infty}^x \frac{e^{-t^2}}{\sqrt{2\pi}}\,dt. \tag{B.1}$$

For $\tilde\psi_n$, we have

$$\tilde\psi_n(x) = \kappa_n'(\sqrt{n}+x)^{n-1}e^{-(x+\sqrt{n})^2/2}$$

$$= \kappa_n' n^{\frac{n-1}{2}} e^{-n/2} e^{(n-1)\log(1+\frac{x}{\sqrt{n}}) - x\sqrt{n} - x^2/2}$$
$$= \kappa_n' n^{\frac{n-1}{2}} e^{-n/2} e^{-x^2 + O(n^{-1/2})}.$$

The pre-factor can be shown asymptotically equal to $\frac{1}{\sqrt{2\pi}}$ and, therefore,

$$g(x) = \lim_{n\to\infty} \tilde{\psi}_n(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2}. \tag{B.2}$$

REMARK B.1. Notice that the the correct expression for the product

$$g(x)G(y) = \frac{1}{4\sqrt{\pi}} e^{-x^2} \operatorname{erfc}(-y)$$

is consistent with the scaling limit of the kernel $S_n(x, y)$ from [2].

REMARK B.2. Throughout this Appendix, we use the definitions (B.1), (B.2) for the functions g and G which differ from the conventions adopted in Theorem 1.1. This is done to highlight the changes needed to calculate the correct scaling limit of the 2×2 determinant $\det(1 - (\alpha_i, \beta_j))_{1\le i,j\le 2}$ (see [14], pages 1641–1644).

The first matrix element (α_1, β_1) converges to

$$\int_t^\infty G(x)(I - T\chi_t)^{-1} g(x)\, dx,$$

with operators T and χ_t defined as in Theorem 1.1. The second term (α_1, β_2) converges to

$$(I - T\chi_t)^{-1} G(t) - G(\infty) = (I - T\chi_t)^{-1} G(t) - \frac{1}{\sqrt{2}}.$$

To calculate the remaining two terms (see [14], (4.12)) we follow [14], page 1644, last paragraph, to arrive at

$$(\alpha_2, \beta_1) = \frac{1}{2}\int_{-\infty}^t (1 - \tilde{T}_n \chi_t)^{-1} \tilde{\psi}_n(x)\, dx$$
$$= \frac{1}{2}\int_{-\infty}^t \tilde{T}_n \chi_t (1 - \chi_t \tilde{T}_n \chi_t)^{-1} \tilde{\psi}_n(x)\, dx + \frac{1}{2}\int_{-\infty}^t \tilde{\psi}_n(x)\, dx.$$

The first summand on the right-hand side converges to its formal limit

$$\frac{1}{2}\int_{-\infty}^t T\chi(1 - \chi_t T_n \chi_t)^{-1} g(x)\, dx.$$

The answer for the second summand depends on the parity of n:

$$\int_{-\infty}^t \tilde{\psi}_n(x)\, dx \approx -\frac{1}{2\sqrt{2}} \operatorname{erfc}(t) + \frac{1 - (-1)^n}{\sqrt{2}} = G(t) - \frac{(-1)^n}{\sqrt{2}}.$$

Considering only the even sized matrices we conclude that (α_2, β_1) converges to

$$\frac{1}{2}\int_{-\infty}^{t}(I - T\chi_t)^{-1}g(x)\,dx - \frac{1}{2\sqrt{2}}.$$

Therefore, the limit of (α_2, β_2) coincides with the answer stated in the original paper and is equal to

$$\frac{1}{2}\int_{-\infty}^{t}R(x, t+)\,dx.$$

Gathering the answers derived above we find that

$$\begin{aligned}\Gamma_t = {} & \left(1 - \int_t^{\infty} G(x)(I - T\chi_t)^{-1}g(x)\,dx\right)\left(1 - \frac{1}{2}\int_{-\infty}^{t}R(x, t+)\,dx\right) \\ & + \frac{1}{2}\left(\frac{1}{\sqrt{2}} - (I - T\chi_t)^{-1}G(t)\right)\left(\int_{-\infty}^{t}(I - T\chi_t)^{-1}g(x)\,dx - \frac{1}{\sqrt{2}}\right),\end{aligned}$$

which is exactly the statement of Theorem 1.1 after the re-definition $g, G \to \frac{1}{\sqrt{2}}g, \frac{1}{\sqrt{2}}G$.

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