The Dynamics of Bidding Markets with Financial Constraints*

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November 20, 2014

Abstract

We develop a model of bidding markets with financial constraints à la Che and Gale (1998b) in which two firms choose their budgets optimally and we extend it to a dynamic setting over an infinite horizon. We provide three main results for the case in which the exogenous cash-flow is not too large and the opportunity cost of budgets is positive but arbitrarily low. First, firms keep small budgets and markups are high most of the time. Second, the dispersion of markups and "money left on the table" across procurement auctions hinges on differences, both endogenous and exogenous, in the availability of financial resources rather than on significant private information. Third, we explain why the empirical analysis of the size of markups based on the standard auction model may have a bias, downwards or upwards, positively correlated with the availability of financial resources. A numerical example illustrates that our model is able to generate a rich set of values for markups, bid dispersion and concentration.

JEL Classification Numbers: L13, D43, D44. Keywords: bidding markets, financial constraints, markups, money left on the table, industry dynamics, all pay auctions.

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*We are grateful to Luis Cabral, Dan Kovenock, Gilat Levy and Matthew Shum for their comments. We also thank an anonymous referee for very detailed comments that greatly improved the paper. We thank Fundación Ramón Areces and the Spanish Ministry of Economics and Competitiveness (project ECO2012-38863) for their financial support.

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1 Introduction

An implicit assumption of the standard model of bidding is that the size of the project is relatively small compared to the financial resources of the firm. That this assumption is key to derive the main predictions of the standard model is known since the analysis of Che and Gale (1998b). In their model, the extent to which a firm is financially constrained depends on its budget, *working capital* hereafter, which is assumed exogenous. In our paper, as it happens in reality, the firm's working capital is not exogenous but chosen out of the firm's internal financial resources, the *cash* hereafter, which in turn depends on the past performance of the firm.

Our first main result, stated in Theorem 1, challenges the view that "auctions [still] work well if raising cash for bids is easy" (Aghion, Hart, and Moore (1992, p. 527)).[1] Although the standard model arises in our infinite horizon setup when working capitals are sufficiently abundant, firms tend to keep too little of it and markups are high if the exogenous cash-flow is not too large, in a sense we formalise later, and the opportunity cost of working capital is positive but arbitrarily low

Besides, our model displays sensible features regarding the behaviour of markups, "money left on the table" and market shares that suggests that we should be more cautious in the empirical analysis of bidding markets. Our second main result, see Corollaries 3 and 5, provides a new explanation for the dispersion of markups and "money left on the table"[2] observed across procurement auctions. Interestingly, this explanation, discussed below the aforementioned corollaries, does not hinge on significant private information about working capitals and costs, but on differences in the availability of financial resources across auctions in a sense that we formalise later. This casts doubts about the usual interpretation for the dispersion of markups and "money left on the table" observed in procurement as indicative of incomplete information and large heterogeneity in production cost.[3] Our third main result, see Corollaries 4 and 6, explains why the empirical analysis of

[1]This conjecture has been recently questioned by Rhodes-Kropf and Viswanathan (2005) under the assumption that firms finance their bids by borrowing in a competitive financial market.

[2]"Money left on the table" is the difference between the two lowest bids in procurement auctions.

[3]Indeed, as Weber (1981) pointed out: " Some authors have cited the substantial uncertainty concerning the extractable resources present on a tract, as a factor which makes large bid spreads [i.e. 'money left on the table'] unavoidable." More recently, Krasnokutskaya (2011) noted that "The magnitude of the 'money

the size of markups may be biased downwards or upwards with a bias positively correlated with the availability of financial resources when the researcher assumes that the data are generated by the standard model. We also use a numerical example to illustrate that the model is able to generate a rich set of values for key variables like markups, bid dispersion and concentration.

We are interested in markets in which only bids that have secured financing can be submitted, i.e. are *acceptable*,[4] as when surety bonds are required.[5] We also follow Che and Gale's (1998b) insight that the set of acceptable bids increases with the working capital. This feature is present in a number of settings in which firms have limited access to external financial resources. One example is an auction in which the price must be paid upfront, and hence the maximum acceptable bid increases with the firm's working capital. Another example is a procurement contest in which the firm must be able to finance the difference between its working capital and the cost of production. If the external funds that are available to the firm increase with its bid or its profitability, it follows that the firm's minimum acceptable bid decreases in the firm's working capital. The latter property arises when the sponsor pays in advance a fraction of the price,[6] a feature of the common practice of progress payments, or when the amount banks are willing to lend depends on the profitability of the project, as it is usually the case.[7]

A representative example of the institutional details of the bidding markets we are interested in is highway maintenance procurement. As Hong and Shum (2002) point out "many of the contractors in these auctions bid on many contracts over time, and likely

left on the table' variable [...] indicates that cost uncertainty may be substantial."

[4]Alternatively, we could have assumed that it was costly for the firm to default on a submitted bid, e.g. the firm may bear a direct cost in case of default.

[5]In the U.S., the Miller Act and "Little Miller Acts" regulate the provision of surety bonds for federal and state construction projects, respectively. A surety bond plays two roles: first, it certifies that the proposed bid is not jeopardized by the technological and financial conditions of the firm, and second, it insures against the losses in case of non-compliance. Indeed, the Surety Information Office highlights that "Before issuing a bond the surety company must be fully satisfied that the contractor has [...] the financial strength to support the desired work program." See `http://suretyinfo.org/?wpfb_dl=149`.

[6]A numerical illustration can be found in Beker and Hernando-Veciana (2011).

[7]We show in Section S4 of the supplementary material that this is also the theoretical prediction of a model inspired by the observation of Tirole (2006), page 114, that "The borrower must [...] keep a sufficient stake in the outcome of the project in order to have an incentive not to waste the money."

derive a large part of their revenues from doing contract work for the state." Besides, Porter and Zona (1993) explain that "The set of firms submitting bids on large projects was small and fairly stable[...] There may have been significant barriers to entry, and there was little entry in a growing market."[8]

Motivated by these observations, we build a *static* model in which two firms endowed with some cash choose working capitals to compete in a first price auction for a procurement contract. The cost of complying is known and identical across firms, the minimum acceptable bid increases with the firm's working capital and only cash is publicly observable.[9] Since using cash as working capital means postponing consumption, it is costly.[10] Firms choose their working capitals and bids optimally. The *static* model provides a simple setting with a unique equilibrium that illustrates the strategic forces that shape our results. The *dynamic* model consists of the infinite repetition of the static model. The cash at the beginning of each period is equal to the last period unspent working capital plus the earnings in previous procurement contract and some exogenous cash-flow.

In our static model, to carry more working capital than strictly necessary to make the bid acceptable is strictly dominated because of its cost. Thus, the firm that carries more working capital wins the contract[11] and both firms incur the cost of their working capital.

The strategic considerations that shape the equilibrium working capitals are the same as in the all pay auction with complete information.[12] Not surprisingly, in a version of our game with unlimited cash, there is a unique symmetric equilibrium in which firms randomize in a bounded interval with an atomless distribution. This is also the unique equilibrium in our game when the firms' cash is larger than the upper bound of the support of the equilibrium randomization. We call the scenario *symmetric* if this is the case, and *laggard-leader* otherwise. In this latter case, firms also randomize in a bounded interval,

[8] Moreover, it can be shown that in a model with many firms and entry the natural extension of the equilibrium we study has the feature that only two firms with the most cash enter the market.

[9] Our first main result and the part of our second main result regarding markups also hold in a version of our model with observable working capital, see Beker and Hernando-Veciana (2011).

[10] Any other motivation for the cost of working capital would deliver similar results.

[11] This feature seems realistic in many procurement contracts: "It is thought that Siemens' superior financial firepower was a significant factor in it beating Canada's Bombardier to preferred bidder status on Thameslink," in *Minister blocks...*, The Guardian, 11/Dec/2011.

[12] It resembles Che and Gale's (1998a) model of an all pay auction with caps in that working capitals are bounded by cash. Our model is more general in that they assume exogenous caps common to all agents.

though the firm with less cash, the *laggard* hereafter, puts an atom at zero and the other firm, the *leader*, at the laggard's cash.

In our dynamic model, we characterize a class of equilibria that contains the limit of the sequence of the unique equilibrium of models with an increasing number of periods. Remarkably, the marginal continuation value of cash is equal to its marginal consumption value under a mild assumption about the minimum acceptable bid. Thus, as in the static model, firms do not carry more working capital than strictly necessary to make the bid acceptable and the strategic interaction each period is, again, similar to an all pay auction.

On the equilibrium path, the frequency of each scenario is determined both by the exogenous cash-flow and by the minimum acceptable bid as a function of the working capital. If one keeps the latter fixed, the following cases arise. If the exogenous cash-flow is sufficiently small, the laggard-leader scenario occurs most of the time as the cost of working capital becomes negligible. This insight implies our first main result (Theorem 1). Another consequence is that one of the firms tends to win consecutive procurement contracts.[13] If the exogenous cash-flow is sufficiently large, the symmetric scenario occurs each period. In this case, the probability that a given firm wins the contract is constant across periods.

To understand the second main result (Corollaries 3 and 5), note that the dispersion of markups and "money left on the table" is due to heterogeneity across auctions in the availability of financial resources. Financial resources in the form of cash and minimum acceptable bids affect the equilibrium working capitals which determine the bids, and hence the markups and "money left on the table". To understand the third main result (Corollaries 4 and 6), note that biases in the structural estimation of markups can also arise if, as it is often the case, the researcher does not observe costs. Imagine bid data from several auctions with identical financial conditions and suppose the data are generated by our static model. On the one hand, if the laggard has little cash, there are large markups and little "money left on the table". However, a researcher who assumed the standard model would conclude that there is little cost heterogeneity and, consequently, small markups, i.e. the estimation would be biased downwards. On the other hand, if

[13] To the extent that joint profits are larger in the laggard-leader scenario than in the symmetric scenario, our result is related to the literature on increasing dominance due to efficiency effects (see Budd, Harris, and Vickers (1993), Cabral and Riordan (1994) and Athey and Schmutzler (2001)

the laggard has relatively large cash, but not too large, there is sizable "money left on the table" and relatively low markups. However, a researcher who assumed the standard model would conclude that there is large cost heterogeneity and, as a consequence, large markups, i.e. the estimation would be biased upwards.

Che and Gale (1998b) and Zheng (2001) show that the dispersion of markups can reflect heterogeneity of working capital if it is sufficiently scarce.[14] We show that scarcity is the typical situation if firms choose their working capital. Whereas they assume that the distribution of working capitals is constant across firms, our results show that this distribution is seldom constant across firms. This difference is important because the lack of asymmetries in the distribution of working capitals precludes the possibility of large expected money left on the table when private information is small.

Firms also choose working capitals in Galenianos and Kircher's (2008) model of monetary policy and in Burkett's (2014) principal-agent model of bidding. Whereas the all pay auction structure only arises in the former, the laggard-leader scenario does not occur because working capital is not bounded by cash.

Our paper contributes to a recent literature that explains how asymmetries in market shares arise and persist in otherwise symmetric models. In particular, Besanko and Doraszelski (2004), and Besanko, Doraszelski, Kryukov, and Satterthwaite (2010) show that firm-specific shocks can give rise to a dynamic of market shares similar to ours. The difference, though, is that the dynamic in our model arises because firms randomize their working capital due to the all pay auction structure.

Our characterization of the dynamics resembles that of Kandori, Mailath, and Rob (1993) in that we study a Markov process in which two persistent scenarios occur infinitely often and we analyse their frequencies as the randomness vanishes. While the transition function of their process is exogenous, ours stems from the equilibrium strategies.

Section 2 explains how we model financial constraints. Sections 3 and 4 analyse the static and the dynamic model, respectively. Section 5 concludes. All the proofs are relegated to the Appendix and the supplementary material.

[14] See also Che and Gale (1996, 2000), and DeMarzo, Kremer, and Skrzypacz (2005). Pitchik and Schotter (1988), Maskin (2000), Benoit and Krishna (2001) and Pitchik (2009) study how bidders distribute a fixed budget in a sequence of auctions. This is not an issue in our setup.

2 A Reduced Form Model of Procurement with Financial Constraints

In this section, we describe a model of procurement that we later embed in the models of Sections 3 and 4. Two firms[15] compete for a procurement contract of common and known cost c in a first price auction: each firm submits a bid, and the firm who submits the lowest bid gets the contract at a price equal to its bid.[16] Only bids in a restricted set, the *acceptable* bids, are allowed. In particular, we assume that the minimum acceptable bid of a firm with working capital $w \geq 0$ is given by[17]

$$b^*(w) \equiv \pi(w) + c, \tag{1}$$

where π is strictly decreasing, satisfies $\pi(0) > 0$ and $\lim_{w\to\infty} \pi(w) < 0$ and is continuously differentiable.

As we discuss in the Introduction, our assumption that firms can submit only acceptable bids captures a wide range of institutional arrangements whose aim is to preclude firms from submitting unsustainable bids such as bids that cannot be financed. [18] Alternatively, the sponsor may provide incentives to guarantee that firms submit only acceptable bids by making them bear some of the cost of default. The monotonicity of the set of acceptable bids arises naturally in markets in which firms have limited access to external financial resources, as we discuss in the Introduction and Section S4 of the supplementary material.

For any given bids b_1 and b_2, we use *markup* to denote $\frac{\min\{b_1,b_2\}-c}{c}$ and we use "*money left on the table*" to denote $\frac{|b_1-b_2|}{c}$.

[15]As in all pay auctions, see Baye, Kovenock, and de Vries (1996), if there are more than two firms then there are multiple equilibria. One such equilibrium is that in which two firms choose the equilibrium strategies of the two-firm model and the other firms choose zero working capital.

[16]A sale auction of a good with common and known value v can be easily encompassed in our analysis assuming that $c = -v < 0$ and bids are negative numbers.

[17]Thus, the model of auctions with budget constraints analysed by Che and Gale (1998b) in Section 3.2 corresponds in our framework with $b^*(w) = -w$ and $\pi(w) = v - w$, and the interpretation in Footnote 16.

[18]For instance, Meaney (2012) says that "As well as considering the financial aspects of bids, the DfT [the sponsor] assesses the deliverability and quality of the bidders' proposals so as to be confident that the successful bidder is able to deliver on the commitments made in the bidding process."

Definition 1. θ is the working capital for which the minimum acceptable bid is equal to the cost of the procurement contract c so that $\pi(\theta) = 0$ or, equivalently, $\theta = \pi^{-1}(0)$.

Our assumptions on π imply that there exists a unique $\theta \in (0, \infty)$.

3 The Static Model

Each firm $i \in \{1, 2\}$ starts with some cash $m_i \geq 0$. We assume the firm's cash to be publicly observable. Each firm i chooses simultaneously and independently (I) how much of its cash to keep as working capital $w_i \in [0, m_i]$ and (II) an acceptable bid $b_i \geq b^*(w_i)$ for a market as described in Section 2. A pure strategy is thus denoted by the vector $(b_i, w_i) \in \{(b, w) : b \geq b^*(w), w \in [0, m_i]\}$. Firm i's expected[19] profit in the market against another firm with cash m_j that bids b_j is equal to:

$$V(b_i, b_j, m_i, m_j) \equiv \begin{cases} b_i - c & \text{if } b_i = b_j \text{ and } m_i > m_j \text{ or if } b_i < b_j, \\ \frac{1}{2}(b_i - c) & \text{if } b_i = b_j \text{ and } m_i = m_j, \\ 0 & \text{otherwise,} \end{cases} \tag{2}$$

where we are applying the usual uniformly random tie breaking rule except in the case in which one firm has strictly more cash than the other. In this case, we assume that the firm with strictly more cash wins.[20] We assume that the firm maximises

$$m_i - w_i + \beta(w_i + V(b_i, b_j, m_i, m_j)), \tag{3}$$

that is, $m_i - w_i$, its *consumption* hereafter, plus the discounted sum, at rate $\beta \in (0, 1)$, of the working capital and the expected profit in the market. Note that a unit increase in working capital is costly in the sense that it reduces the current utility in one unit and increases the future utility in β. Thus, the cost of working capital becomes negligible when β increases to 1.

We start by simplifying the strategy space. First, any strategy (b, w) in which $b > b^*(w)$ is strictly dominated by the strategy $(b, \tilde{w})$ where $\tilde{w}$ satisfies $b = b^*(\tilde{w})$ so that it is never

[19] We take expectations with respect to the tie breaking rule in the case $b_i = b_j$ and $m_i = m_j$.

[20] We deviate from the more natural uniformly random tie-breaking rule that is usual in Bertrand games and all pay auctions in order to guarantee the existence of an equilibrium. In our game, a sufficiently fine discretisation of the action space would overcome the existence problem and yield our results with the usual uniformly random tie-breaking rule at the cost of a more cumbersome notation.

optimal to carry more working capital than is strictly necessary.[21] Thus, we restrict to the set of pure strategies $\{(b, w) : b = b^*(w), w \in [0, m]\}$ where m denotes the firm's cash.

In our second simplification of the strategy space, we use the following definition:

Definition 2. $\overline{\nu}^\beta \in [0, \theta)$ is the unique solution[22] in w to

$$\beta\pi(w) = (1 - \beta)w. \tag{4}$$

Since $w = \theta$ solves (4) for $\beta = 1$, the Implicit Function Theorem implies:

$$\lim_{\beta \uparrow 1} \overline{\nu}^\beta = \theta \tag{5}$$

Thus, $\overline{\nu}^\beta$ denotes the working capital for which $\beta\pi(\overline{\nu}^\beta)$, the discounted procurement profits associated with the minimum acceptable bid corresponding to working capital $\overline{\nu}^\beta$, equals $(1 - \beta)\overline{\nu}^\beta$, the implicit costs of selecting working capital $\overline{\nu}^\beta$ that are associated with postponing consumption. Any pure strategy $(b^*(w), w)$ in which $w > \overline{\nu}^\beta$ is strictly dominated by $(b^*(\overline{\nu}^\beta), \overline{\nu}^\beta)$. As a consequence, we further restrict the set of pure strategies to $\{(b, w) : b = b^*(w), w \in [0, \min\{m, \overline{\nu}^\beta\}]\}$ where m denotes the firm's cash.

Once we eliminate the above strictly dominated strategies, the resulting reduced game has a unidimensional strategy space as an all pay auction. Each firm chooses a working capital and its corresponding minimum acceptable bid. The firm with the higher working capital wins the procurement contract and carrying working capital is costly for each firm. As in all pay auctions, there is no pure strategy equilibrium. This can be easily understood when each of the two firms' cash is weakly larger than $\overline{\nu}^\beta$. If both firms choose different working capitals, the one with more working capital has a strictly profitable deviation: to decrease marginally its working capital.[23] If both firms choose the same working capital w, there is also a strictly profitable deviation: to increase marginally its working capital if $w < \overline{\nu}^\beta$, and to choose zero working capital if $w = \overline{\nu}^\beta$.[24]

[21]The probability that a firm wins the contract is unaffected but the cost of working capital increases.

[22]Note that this equation is equivalent to $m - w + \beta w + \beta\pi(w) = m$.

[23]It saves on the cost of working capital without affecting to the cases in which the firm wins and increases the profits from the procurement contract because it increases the price.

[24]In the former case, the deviation is profitable because winning the procurement contract at $w < \overline{\nu}^\beta$ gives strictly positive profits and the deviation breaks the tie in favor of the deviating firm with an arbitrarily small increase in the cost of working capital and an arbitrarily small decrease in the profits from the procurement contract. In the latter case, $w = \overline{\nu}^\beta$ implies that one of the firms is winning with a probability strictly less than one, and hence the definition of $\overline{\nu}^\beta$, see Footnote 22, means that this firm makes strictly lower expected payoffs than with zero working capital.

A mixed strategy over the set of strictly undominated strategies is described by a distribution function with support[25] contained in the set $\{(b,w) : b = b^*(w), w \in [0, \min\{m, \overline{\nu}^\beta\}]\}$ where m denotes the firm's cash. This randomization can be described by the marginal distribution over working capitals F. With a slight abuse of notation, we denote by (b^*, F) the mixed strategy where the firm randomises its working capital w according to F and submits a bid $b^*(w)$. If a firm uses (b^*, F) where F is differentiable and has support $[\underline{w}, \overline{w}]$, then the expected payoff to the other firm with cash $m \geq \overline{w}$ from choosing $w \in (\underline{w}, \overline{w})$ is

$$m - w + \beta w + \beta \pi(w) F(w) \tag{6}$$

so that indifference across the support results only if F satisfies the differential equation

$$1 - \beta = \beta F'(w)\pi(w) + F(w)\beta\pi'(w) \tag{7}$$

for any $w \in (\underline{w}, \overline{w})$. Thus, $(1-\beta)$, the increase in the cost of working capital $w(1-\beta)$, must equal $\beta F'(w)\pi(w) + F(w)\beta\pi'(w)$, the change in the expected discounted profits $\beta\pi(w)F(w)$. There is both a positive effect and a negative effect of an increase in w on the change in expected discounted profits. The former arises due to the higher probability of winning a contract and the latter due to the lower profits associated with a win.

We distinguish two scenarios:

Definition 3. Let $m_l \equiv \min\{m_1, m_2\}$. The *symmetric scenario* denotes the case in which $m_l \geq \overline{\nu}^\beta$. The *laggard-leader scenario* denotes the complementary case.

Let χ_y denote the degenerate distribution that puts weight 1 on $y \in \mathbb{R}$.

Proposition 1. If $m_l \geq \overline{\nu}^\beta$, then the unique equilibrium is symmetric and denoted by the single (mixed) strategy $\left(b^*, F^\beta\right)$ where b^* is defined in (1) and

$$F^\beta(w) \equiv \frac{(1-\beta)w}{\beta\pi(w)} \tag{8}$$

with support $[0, \overline{\nu}^\beta]$ solves the differential equation (7) with initial condition $F(0) = 0$. Besides: (i) the equilibrium probability of winning the contract is common across firms; (ii) the equilibrium is unaffected by any change in cash that leaves $m_l \geq \overline{\nu}^\beta$; and (iii) for any $m_l \geq \theta$, as β increases to 1, $F^\beta(w)$ converges to $\chi_\theta(w)$.

[25] We use the definition of support of a probability measure in Stokey and Lucas (1999). According to their definition, the support is the smallest closed set with probability one.

This equilibrium satisfies the usual property of all pay auctions that bidders without competitive advantage get their outside opportunity, i.e. the payoff of carrying zero working capital and losing the procurement contract.

Besides, one can deduce the following corollary from (8) using (5).

Corollary 1. If $m_l \geq \theta$, then in equilibrium, $|\pi(w_1) - \pi(w_2)|$ and $\pi(\max\{w_1, w_2\})$ converge in distribution to χ_0 as β increases to 1.

In the standard auction model, cost heterogeneity vanishes as the distribution of costs converges to the degenerate distribution that puts all the weight on one value. As cost heterogeneity vanishes, the markup and "money left on the table" vanish (Krishna (2002), Chapter 2). Corollary 1 says that this limit outcome also arises as β increases to 1 in the symmetric scenario, see Definition 3, since the markup $\frac{\min\{b_1,b_2\}-c}{c}$ is equal to $\frac{\pi(\max\{w_1,w_2\})}{c}$ and "money left on the table" $\frac{|b_1-b_2|}{c}$ is equal to $\frac{|\pi(w_1)-\pi(w_2)|}{c}$. In this sense, financial constraints become irrelevant as β increases to 1.

We next consider the laggard-leader scenario, see Definition 3. In what follows, the *leader* refers to the firm that starts with more cash and the *laggard* to the other firm.

Proposition 2. If $m_l < \overline{\nu}^\beta$ and $m_1 \neq m_2$, then in the unique equilibrium,[26] the laggard's strategy is $\left(b^*, F_l^\beta\right)$ and the leader's strategy is $\left(b^*, F_L^\beta\right)$ where b^* is defined in (1) and each of the distributions

$$F_l^\beta(w) \equiv F^\beta(w) + \frac{\beta\pi(m_l) - (1-\beta)m_l}{\beta\pi(w)} \text{ if } w \in [0, m_l], \tag{9}$$

$$F_L^\beta(w) \equiv \begin{cases} F^\beta(w) & \text{if } w \in [0, m_l), \\ 1 & \text{if } w = m_l, \end{cases} \tag{10}$$

has support $[0, m_l]$ and solves the differential equation (7) with separate boundary conditions so that $F_l^\beta(m_l) = 1$ and F_l^β has an atom at 0 while $F_L^\beta(0) = 0$ and F_L^β has an atom at 1.

One can deduce the following corollary from Proposition 2 using (5).

Corollary 2. If $m_l < \overline{\nu}^\beta$ and $m_1 \neq m_2$, then (i) the leader is more likely to win the

[26] Interestingly, this equilibrium has similar qualitative features as the equilibrium of an all pay auction in which both agents have the same cap but the tie-breaking rule allocates to one of the agents only. The latter model has been studied in an independent and simultaneous work by Szec (2010).

contract and (ii) for any $m_l < \theta$, as β increases to 1, $F_L^\beta(w)$ converges to $\chi_{m_l}(w)$ and the equilibrium probability that the winner is the leader converges to 1

Since each firm is indifferent among all points in its support, the laggard receives a payoff equal to the symmetric payoff (when it chooses its atom 0) and the leader receives a premium over the symmetric payoff (when it chooses its atom m_l). This difference occurs because, unlike the symmetric case, the leader is not only able but also willing to undercut any acceptable bid of the laggard.

Corollary 3. If $m_l < \overline{\nu}^\beta$ and $m_1 \neq m_2$, (i) an increase in m_l for which $m_l < \overline{\nu}^\beta$ increases (in the sense of first order stochastic dominance) both equilibrium distributions of working capitals and hence, decreases the equilibrium expectation of $\pi(\max\{w_1, w_2\})$, and (ii) the equilibrium probability that each firm chooses its atom simultaneously is:

$$\frac{\pi(m_l)}{\pi(0)} \left(1 - \frac{(1-\beta)m_l}{\beta\pi(m_l)}\right)^2. \tag{11}$$

Corollary 3 is direct from (9) and (10) and it is the starting point for our second main result. Point (i) shows that the dispersion of markups, $\frac{\min\{b_1,b_2\}-c}{c} = \frac{\pi(\max\{w_1,w_2\})}{c}$ observed across auctions can be explained by variations in the laggard's cash and it suggests that the same can apply to the dispersion of "money left on the table", $\frac{|b_1-b_2|}{c} = \frac{|\pi(w_1)-\pi(w_2)|}{c}$. Note that a similar argument also applies with respect to changes in π. Point (ii) also casts doubts about the usual interpretation of "money left on the table" as indicative of incomplete information. To see why, consider the linear example[27] $\pi(w) = \theta - w$. In this case, as θ increases to infinity, the probability that each firm chooses its atom simultaneously tends to 1 so that the "money left on the table" tends to $\frac{\pi(0)-\pi(m_l)}{c} = \frac{m_l}{c}$. Thus, a sufficiently large θ implies almost no uncertainty together with sizable "money left on the table." Note that the implications about "money left on the table" that are only suggested by Corollary 3, are proved in Corollary 5 for the dynamic model under the assumptions that the exogenous cashflow (defined in Section 4.1) is not too large, in a sense we formalise later, and β is sufficiently close to 1.

[27]If $\pi(w) = \theta - w$, the equilibrium probability that each firm chooses its atom simultaneously is:

$$\frac{\pi(m_l)}{\pi(0)} \left(1 - \frac{(1-\beta)m_l}{\beta\pi(m_l)}\right)^2 = \left(\frac{\theta - m_l}{\theta}\right) \left(1 - \frac{(1-\beta)m_l}{\beta(\theta - m_l)}\right)^2.$$

Here, the laggard's cash is exogenous but in the model of Section 4 we show in a numerical example that the endogenous distribution of the laggard's cash has sufficient variability to generate significant dispersion of markups and "money left on the table" across otherwise identical auctions. Interestingly, these results are provided for parameter values for which there is little uncertainty.

Corollary 4. If $m_l < \theta$ and $m_1 \neq m_2$, then as β increases to 1: (i) in equilibrium, $\pi(\max\{w_1, w_2\})$ converges in distribution to $\chi_{\pi(m_l)}$, and (ii) the equilibrium expectation of $|\pi(w_1) - \pi(w_2)|$ converges to[28] $\pi(m_l)\left(\ln\left(\frac{\pi(0)}{\pi(m_l)}\right)\right)$.

The corollary follows by inspection of (5), 9) and (10). Intuitively, (i) can be explained because the leader increases its probability of winning by shifting all its probability mass to m_l as β increases to 1. Since working capital is costless in the limit, the laggard's randomization guarantees the indifference of the leader by balancing the positive and negative effects of an increase in working capital on the expected discounted profits, which explains (ii).

Corollary 4 implies that when β is close to 1 and $m_l < \hat{m}$, where $\hat{m} \equiv \pi^{-1}(\frac{\pi(0)}{e})$ and e denotes the Euler constant 2.718..., the markup, $\frac{\min\{b_1,b_2\}-c}{c} = \frac{\pi(\max\{w_1,w_2\})}{c}$, decreases[29] and the expected "money left on the table", $\frac{|b_1-b_2|}{c} = \frac{|\pi(w_1)-\pi(w_2)|}{c}$, increases as the laggard's cash m_l increases. This is the basis for our third main result. Suppose that β is close to 1 and that the bid data from several auctions with identical financial constraints are generated by the model with constant procurement cost c. If $m_l < \overline{\nu}^{\beta}$, then Corollary 4 states that the average "money left on the table" will be small and there will be large markups when m_l is close to zero but the average "money left on the table" will be substantial and markups small when $m_l = \hat{m}$. In what follows we assume that $m_l < \overline{\nu}^{\beta}$. The bid data reveals the "money left on the table" but costs and, therefore, markups are not observable. If the average "money left on the table" were small, as would happen if m_l is

[28]Proving (ii) requires some non-trivial computations. F_l^{β} converges to a distribution with an atom of probability $\frac{\pi(m_l)}{\pi(0)}$ at zero and density $-\pi'(w)\frac{\pi(m_l)}{\pi(w)^2}$ in $(0, m_l]$. This together with the convergence of $F_L^{\beta}(w)$ to $\chi_{m_l}(w)$ implies that the expectation of $|b_1 - b_2| = \pi(\min\{w_1, w_2\}) - \pi(\max\{w_1, w_2\})$ converges to:

$$\pi(0)\frac{\pi(m_l)}{\pi(0)} + \int_0^{m_l} \pi(w)\left(-\pi'(w)\frac{\pi(m_l)}{\pi(w)^2}\right)dw - \pi(m_l) = \pi(m_l)\int_0^{m_l}\left(\frac{-\pi'(w)}{\pi(w)}\right)dw = \pi(m_l)\left(\ln\left(\frac{\pi(0)}{\pi(m_l)}\right)\right).$$

[29]Since $\frac{\partial}{\partial m}\left(\pi(m)\left(\ln\left(\frac{\pi(0)}{\pi(m)}\right)\right)\right) > 0$ if $m < \hat{m}$.

close to zero, an interpretation of the bid data using the standard model would conclude that there was little cost heterogeneity and small markups even though there were large markups in the generated data. That is, the results would be biased downward. If the average "money left on the table" were substantial, as would happen if $m_l = \hat{m}$, then an interpretation of the bid data using the standard model would conclude that there was large cost heterogeneity and therefore large markups even though there were small markups in the generated data. That is, the results on markups would be biased upwards.

Finally, in Proposition 3 we describe the equilibrium strategies when each firm has cash $m < \nu^\beta$. We use $\xi^\beta \in (0, \theta)$ to denote the function implicitly defined as the unique solution in m to:

$$\frac{\beta}{2}\pi(m) - (1-\beta)m = 0. \tag{12}$$

By (1), (2) and (3), the left hand side of (12) is equal to the difference in a firm's expected payoffs between choosing working capital m and zero working capital when the other firm chooses working capital m. If $m \in (\xi^\beta, \overline{\nu}^\beta)$, we let $\lambda(m) \in [0, m]$ be implicitly defined by:[30]

$$\left(F^\beta(\lambda(m)) + \frac{1 - F^\beta(\lambda(m))}{2}\right)\beta\pi(m) - (1-\beta)m = 0, \tag{13}$$

where F^β is defined in (8). By (1), (2) and (3), the left hand side of (13) is equal to the difference in a firm's expected payoffs between choosing working capital m and zero working capital when the other firm chooses a working capital in $(0, \lambda(m))$ with probability $F^\beta(\lambda(m))$ and a working capital equal to m with probability $1 - F^\beta(\lambda(m))$.

Proposition 3. If $m_1 = m_2 = m$ then the unique equilibrium is symmetric and denoted by (b^*, χ_m) if $m \in (0, \xi^\beta]$; and by (b^*, F^{**}) if $m \in (\xi^\beta, \overline{\nu}^\beta)$ where b^* is defined in (1),

$$F^{**}(w) \equiv \begin{cases} F^\beta(w) & \text{if } w \in [0, \lambda(m)] \\ F^\beta(\lambda(m)) & \text{if } w \in (\lambda(m), m) \\ 1 & \text{if } w \geq m, \end{cases}$$

and F^β is defined in (8).

[30] Existence and uniqueness of the solution follow from the properties of the left hand side of the equation. This is increasing in $\lambda(m)$, it is negative at $\lambda(m) = 0$ and it is strictly positive at $\lambda(m) = m$. The first one is direct, the second can be deduced from (12) using that $m > \xi^\beta$, and the third from the definition of $\overline{\nu}^\beta$, in (4), using that $m < \overline{\nu}^\beta$, and the definition of F^β in (8).

The equilibrium in the first case is explained by the fact that $m \leq \xi^{\beta}$ implies that the left hand side of (12) is weakly positive and hence the best response to χ_m is χ_m. This is not the case when $m > \xi^{\beta}$ as the left hand side of (12) is strictly negative. Instead, the equilibrium in this case is constructed by shifting probability away from the common amount of cash and placing it at the bottom of the space of working capitals according to a distribution that solves the differential equation (7).

We shall not discuss the implications of Proposition 3 as in our dynamic model the case in which both firms cash is less than θ does not arise along the game tree. See our discussion after introducing Assumption 1.

4 The Dynamic Model

In this section, we endogenise the distribution of cash by assuming that it is derived from the past market outcomes. This approach provides a natural framework to analyse the conventional wisdom in economics that "auctions [still] work well if raising cash for bids is easy." In Theorem 1, we provide conditions under which the laggard-leader scenario occurs most of the time. This is the basis for our first main result. Besides, we provide formal results in Corollaries 5 and 6 and a numerical example that, on the one hand, complement the previous section analysis of the second and third main results and, on the other hand, shed some light on the concentration and asymmetries of market shares.

4.1 The Game

We consider the infinite horizon dynamic version of the game in the last section. We assume that both firms have the same amount of cash in the first period. Afterwards each firm's cash is equal to its working capital in the previous period plus the profits in the procurement contract and some exogenous cash flow[31] $\underline{m} > 0$. We assume that $\underline{m}$ is constant across time and firms, and interpret it as derived from other activities of the firm. Hence, in any period t in which firms start with cash $(m_{1,t}, m_{2,t})$, choose working capitals $(w_{1,t}, w_{2,t})$ and bids $(b_{1,t}, b_{2,t})$, and Firm 1 wins the procurement contract with

[31] All our results also hold true for the case $\underline{m} = 0$. However, the analysis in Section 4.3 differs, as explained in Footnote 42.

profits $b_{1,t} - c$, the next period distribution of cash is equal to:

$$(m_{1,t+1}, m_{2,t+1}) = (w_{1,t} + b_{1,t} - c + \underline{m}, w_{2,t} + \underline{m}). \tag{14}$$

Firm $i \in \{1, 2\}$ wins in period t with probability one if $b_{i,t} < b_{j,t}$ or if $b_{i,t} = b_{j,t}$ and $m_{i,t} > m_{j,t}$, with probability $1/2$ if $b_{i,t} = b_{j,t}$ and $m_{i,t} = m_{j,t}$, and loses otherwise. The payoff in period t of a firm with cash m_t that chooses working capital w_t is equal to its consumption $m_t - w_t$. The firm's lifetime payoff in a subgame beginning at period τ is:

$$\sum_{t=\tau}^{\infty} \beta^{t-\tau}(m_t - w_t),$$

where (m_t, w_t) denotes its cash and working capital holdings in period t. We assume that the firm maximises its expected lifetime payoff at any period τ.

The following assumption[32] is used in the proof of Proposition 4.

Assumption 1. $\pi(w) \geq \theta - \underline{m} - w$ for any $w \in [0, \infty)$.

Since $\pi(w)$ is the minimum profit that a firm with working capital w can make when it wins the procurement contract, (14) and Assumption 1 imply that the firm that wins the procurement contract one period, starts next period with cash at least θ. As we explain after Proposition 4, this assumption guarantees that firms do not want to carry more working capital than strictly necessary to make the bid acceptable. Assumption 1 also implies that θ must be less than any common amount of cash held by the firms in any information set after the first period. We show in Proposition 3, for the case of the static model, that a tedious case differentiation is necessary if one allows firms to have identical cash less than θ. For the same reason, we assume that both firms start in the first period with cash greater than θ.[33]

We denote by Ω the set of cash vectors that may arise in the information sets of the game tree. A Markov mixed strategy consists of a randomization over the set of working capitals and acceptable bids for each point (m, m') in Ω, where m denotes the firm's cash and m' the rival's. We shall restrict to equilibria in Markov mixed strategies with support contained in the set $\{(b, w) : b = \tilde{b}(w|m, m'), w \in [0, m]\}$ for some function

[32] A large class of functions satisfy this assumption, for instance the linear function $\pi(w) = \theta - w$.

[33] In this sense, our result that firms carry too little cash in the long term arises even when firms start with sufficiently large amounts of cash.

$\tilde{b}(\cdot|m,m') : [0,m] \to \mathbb{R}$ that satisfies that $\tilde{b}(w|m,m') \geq \pi(w)+c$ for any $w \in [0,m]$. This Markov mixed strategy can be described by its marginal distribution function $\sigma(\cdot|m,m')$ over working capitals and the bid function $\tilde{b}\,(\cdot|m,m')$.

We let $W(m,m')$ denote the lifetime expected payoff of a firm that has cash m when its rival has m'. In Definition 4 below, we denote the expected continuation payoff of a firm who bids b with working capital w, cash m and face a rival who bids b', has working capital w' and cash m' by $\tilde{W}\,(b,w,m,b',w',m')$ which is equal to:

$$\rho(b,m,b',m')W\big(w+\underline{m}+b-c, w'+\underline{m}\big)+(1-\rho(b,m,b',m'))W\big(w+\underline{m}, w'+\underline{m}+b'-c\big)\,, \tag{15}$$

where:

$$\rho(b,m,b',m') = \begin{cases} 1 & \text{if either } b<b'\text{, or if } b=b' \text{ and } m>m' \\ 0 & \text{if either } b>b'\text{, or if } b=b' \text{ and } m<m' \\ \frac{1}{2} & \text{if } b=b' \text{ and } m=m'. \end{cases}$$

describes the allocation rule of the procurement contract.

Definition 4. A (symmetric) Bidding and Investment (BI) equilibrium[34] is a value function W, a working capital distribution σ and a bid function b such that for every $(m,m') \in \Omega$, W is the value function and $\sigma\,(\cdot|\,m,m')$ and $b(\cdot|m,m')$ are the optimisers of the right hand side of the following Bellman equation:

$$W(m,m') = \max_{\substack{\tilde{\sigma}(w)\,\in\,\Delta(m)\\ \tilde{b}(w)\,\geq\,\pi(w)+c}} \iint \Big[m-w+\beta\tilde{W}(\tilde{b}(w),w,m,b(w'|m',m),w',m')\Big]\sigma\big(dw'|m',m\big)\tilde{\sigma}\,(dw)\,,$$

where $\Delta(m)$ denotes the set of distributions with support in $[0,m]$ and $\tilde{W}$ is defined by (15).

4.2 The Equilibrium Strategies

In what follows, we define a value function, a bid function and a working capital distribution and show that they are a BI equilibrium. Our proposed strategies generalize the equilibrium strategies in Section 3. The bid function is, as in the static model, the minimum acceptable bid (with a slight abuse of notation):

[34] In a version of our model with finitely many periods studied in the supplementary material there is a unique equilibrium that is symmetric. We also show that as the horizon increases to infinity, the limit of that equilibrium is a BI equilibrium.

$$b^*(w|m,m') \equiv \pi(w) + c. \tag{16}$$

We find our equilibrium distribution of working capital by setting up a fixed point problem over a set of functions and then use the solution of this problem to describe the equilibrium distribution. We set up the fixed point problem as follows. We start with a non-empty, closed, bounded and convex subset $\mathcal{P}^{\beta}$ of the space of all bounded continuous functions. For each function Ψ in this class $\mathcal{P}^{\beta}$ we set up a differential equation that depends on Ψ. We then consider the unique continuous solution, F_m^{Ψ}, to this differential equation with initial condition $F(m) = 1$. Lastly, we seek Ψ in $\mathcal{P}^{\beta}$ that is a fixed point of an operator $T : \mathcal{P}^{\beta} \to \mathcal{P}^{\beta}$ where $T(\Psi)$ is described in terms of F_m^{Ψ}. Once we have this fixed point, Ψ^{β} say, we then use $F_m^{\Psi^{\beta}}$ to define the equilibrium distribution of working capital and it turns out that Ψ^{β} determines the equilibrium premium earned by a leader (see (25)).

Let $\mathcal{P}^{\beta}$ be defined as:

$$\left\{ \Psi : [0,\infty) \to \left[0, \frac{\beta}{1-\beta}\pi(0)\right] \text{ is continuous, decreasing and } \Psi(m) = 0 \, \forall m \geq \theta \right\}. \tag{17}$$

Definition 5. For any $\Psi \in \mathcal{P}^{\beta}$ and $m \in [0, \theta)$, we denote by $F_m^{\Psi} : [0, m] \to \mathbb{R}$ the unique continuous solution to the first order differential equation:[35]

$$1 - \beta = \beta F'(w) \left(\pi(w) + \Psi(w + \underline{m})\right) + F(w)\beta\pi'(w) \text{ and } F(m) = 1. \tag{18}$$

The functional form of F_m^{Ψ} can be found in (A4) in the Appendix.[36] Note that (18) is analogous to (7) and that (18) is identical to (7) when Ψ is the zero function.

Definition 6. We denote by $\hat{\nu}^{\Psi}$ the unique value of $m \in [0, \theta)$ for which $F_m^{\Psi}(0) = 0$.

By (8) and Definitions 2 and 6, we see that $\overline{\nu}^{\beta} = \left(F^{\beta}\right)^{-1}(1)$ and $\hat{\nu}^{\Psi} = \overline{\nu}^{\beta}$ when Ψ is the zero function.

We underscore that, for any $m \leq \hat{\nu}^{\Psi}$, $F_m^{\Psi}(w)$ is a distribution of w (given the pair (Ψ, m)) with support in $[0, m]$ that is continuous for $w \in (0, m)$ but it has an atom of size $F_m^{\Psi}(0)$ at $w = 0$ when $m < \hat{\nu}^{\Psi}$. Recall that $F_{\hat{\nu}^{\Psi}}^{\Psi}(0) = 0$ by Definition 6.

Consider the following functional equation:

$$T(\Psi) = \Psi, \tag{19}$$

[35]The uniqueness of the solution follows from Theorem 7.1 in Coddington and Levinson (1984), pag. 22.

[36]We thank an anonymous referee for pointing out that (18) has an explicit solution.

where $T : \mathcal{P}^\beta \to \mathcal{P}^\beta$ is defined as:[37]

$$T(\Psi)(m) \equiv \begin{cases} \beta F_m^\Psi(0)\left(\pi(0)+\Psi(\underline{m})\right) & \text{if } 0 \leq m \leq \hat{\nu}^\Psi, \\ 0 & \text{if } m > \hat{\nu}^\Psi. \end{cases} \tag{20}$$

Definition 7. For any $\beta \in (0,1)$, we denote by $\widehat{\mathcal{P}}^\beta \subset \mathcal{P}^\beta$ the set of fixed points of T, by Ψ^β an element of $\widehat{\mathcal{P}}^\beta$ and by $\nu^\beta \equiv \hat{\nu}^{\Psi^\beta}$ the upper end of the support of the distribution $F_{\nu^\beta}^{\Psi^\beta}$.

Lemma S2 in the supplementary material shows that the set of fixed points $\widehat{\mathcal{P}}^\beta$ is not empty. Let:

$$F_{l,m}^\beta(w) = F_{L,m}^\beta(w) \;=\; F_{\nu^\beta}^{\Psi^\beta}(w) \qquad \text{if } w \leq \nu^\beta \leq m, \tag{21}$$

$$F_{l,m}^\beta(w) \;=\; F_m^{\Psi^\beta}(w) \qquad \text{if } w \leq m < \nu^\beta, \tag{22}$$

$$F_{L,m}^\beta(w) \equiv \begin{cases} F_{\nu^\beta}^{\Psi^\beta}(w) & \text{if } w < m < \nu^\beta, \\ 1 & \text{if } w = m < \nu^\beta. \end{cases} \tag{23}$$

For any $(m, m') \in \Omega$, let:

$$\sigma^*(w|m,m') \equiv \begin{cases} F_{l,m}^\beta(w) & \text{if } m \leq m', \\ F_{L,m'}^\beta(w) & \text{if } m > m', \end{cases} \tag{24}$$

and:

$$W^*\left(m,m'\right) \equiv \begin{cases} m + \frac{\beta}{1-\beta}\underline{m} & \text{if } m \leq m', \\ m + \frac{\beta}{1-\beta}\underline{m} + \Psi^\beta\left(m'\right) & \text{if } m > m'. \end{cases} \tag{25}$$

Thus, $\Psi^\beta(m')$ is an additive premium associated to being leader.

Note that Assumption 1 implies that the case in which both firms have the same cash $m = m'$ and $(m, m') \in \Omega$ can only arise if $m = m' \geq \theta$. By Definitions 6 and 7, $\nu^\beta < \theta$. Thus, (21) implies that $F_{l,m}^\beta = F_{L,m'}^\beta = F_{\nu^\beta}^{\Psi^\beta}$, and (19) and (20) imply that $\Psi^\beta\left(m'\right) = 0$. Thus, neither σ^* nor W^* change discontinuously at any of these points.

Proposition 4. For each $\Psi^\beta \in \widehat{\mathcal{P}}^\beta$, (W^*, σ^*, b^*) is a BI equilibrium where W^*, σ^* are defined by (21)-(25) and b^* by (16).[38]

[37] That $T(\Psi) \in \mathcal{P}^\beta$ follows from checking the conditions in (17). Since F_m^Ψ decreases in m, by (A4) in the Appendix, (20) implies that $T(\Psi)(m)$ decreases continuously from $\beta F_0^\Psi(0)(\pi(0)+\Psi(\underline{m}))$ to $\beta F_{\hat{\nu}^\Psi}^\Psi(0)(\pi(0)+\Psi(\underline{m}))$ as m increases from 0 to $\hat{\nu}^\Psi$, and it is then equal to zero. Besides, $T(\Psi)(m) = 0$ for $m \geq \theta$ since $\theta > \hat{\nu}^\Psi$, by Definition 6, $\beta F_{\hat{\nu}^\Psi}^\Psi(0)(\pi(0) + \Psi(\underline{m})) = 0$ since $F_{\hat{\nu}^\Psi}^\Psi(0) = 0$, by Definition 6, and $\beta F_0^\Psi(0)(\pi(0) + \Psi(\underline{m})) \leq \frac{\beta\pi(0)}{(1-\beta)}$ since $F_0^\Psi(0) = 1$, by Definition 5, and $\Psi(\underline{m}) \leq \frac{\beta\pi(0)}{(1-\beta)}$ since $\Psi \in \mathcal{P}^\beta$.

[38] The limit of the unique equilibrium of the finite horizon model is one of the equilibria described in Proposition 4, see the supplementary material.

The intuition behind the proposition is based on our results in the static model. There, we use the property that the game has the all pay auction structure: after deleting strictly dominated strategies, the firm that carries more working capital wins but carrying working capital is costly for both firms. This argument also applies here because this property is inherited from one period to the previous one in the following sense: if the payoffs of the reduced game in period t satisfy the property, so do the payoffs of the reduced game in period $t-1$. To see why, note that the usual result of all pay auctions that bidders without competitive advantage get their outside opportunity implies here that the laggard's equilibrium payoffs in the reduced game of period t are equal to the payoffs of consuming all its cash and starting period $t+1$ as a laggard with cash $\underline{m}$. The leader's equilibrium payoffs in the reduced game in period t have an additive premium which is a consequence of the leader's ability to carry sufficient working capital to undercut any acceptable bid of the laggard. This ability is independent of the amount of cash the leader has and so it is the premium. Consequently, the value of a marginal increase in the cash with which the firm starts period t is equal to its consumption value plus the value of switching from laggard to leader. The value of switching from laggard to leader is zero because a marginal increase in cash switches the leadership only when the cash is common and no less than θ (by (14) and Assumption 1) so that the premium is zero because none of the firms is constrained by cash to bid above cost. We can thus conclude that, in period $t-1$, a unit increase in working capital, keeping constant the bid, is costly in the sense that it reduces the current consumption in one unit but only increases the future utility in its discounted value β. This means, as in the static model, that it is not profitable to carry more working capital than necessary to make the bid acceptable. Thus, in period $t-1$, after deleting strictly dominated strategies, the firm that carries more working capital wins but carrying working capital is costly for both firms.[39]

We can also distinguish here between the symmetric and laggard-leader scenarios and it may be shown that an analogous version of points (i)-(iii) in Proposition 1 and properly adapted versions of Corollaries 1-4 hold true as well.

[39]Note that the property that firms do not want to carry more working capital than strictly necessary to make the bid acceptable is also a property of the unique equilibrium of the finite version of our model. This is because the recursive argument in the previous paragraph can be applied starting from the last period since the last period is the same game as the static model. See the supplementary material.

4.3 The Equilibrium Dynamics

To study the frequency of the symmetric and the laggard-leader scenarios, we study the stochastic process of the laggard's cash induced by our equilibrium. Its state space is equal to $[\underline{m}, \nu^\beta + \underline{m}]$ because the procurement profits are non negative and none of the firms' working capitals is larger than ν^β. In period $t+1$, the pair of cash holdings $(m_{1,t+1}, m_{2,t+1})$ (see (14)) and, therefore, the laggard's cash in period $t+1$, denoted by $m_{t+1} \equiv min\{m_{1,t+1}, m_{2,t+1}\}$, are determined by the distribution over working capitals $(w_{1,t}, w_{2,t})$ and bids $(b_{1,t}, b_{2,t})$ in period t which is completely determined by the laggard's cash m_t in period t. Thus, the laggard's cash follows a Markov process. Let $\mathcal{B}$ denote the Borel sets of $[\underline{m}, \nu^\beta + \underline{m}]$. The probability that m_{t+1} lies in a Borel set given that $m_t = m$ is given by a transition function $Q^\beta : [\underline{m}, \nu^\beta + \underline{m}] \times \mathcal{B} \to [0,1]$ that can be easily deduced from the equilibrium. In particular, it is defined by:[40]

$$Q^\beta (m, [\underline{m}, x]) = \begin{cases} 1 - \left(1 - F^\beta_{l,m}(x - \underline{m})\right)\left(1 - F^\beta_{L,m}(x - \underline{m})\right) & \text{if } x - \underline{m} < m, \nu^\beta, \\ 1 & \text{o.w.} \end{cases} \tag{26}$$

This expression is equal to 1 minus the probability that both the laggard's and the leader's working capitals are strictly larger than $x - \underline{m}$.

Definition 8. A distribution $\mu : \mathcal{B} \to [0,1]$ is invariant if it satisfies:

$$\mu(\mathcal{M}) = \int Q^\beta(m, \mathcal{M})\,\mu(dm) \qquad \text{for all } \mathcal{M} \in \mathcal{B}. \tag{27}$$

Standard arguments[41] can be used to show that there exists a unique invariant distribution (which we denote by μ^β), and that μ^β is globally stable and has support[42] $[\underline{m}, \nu^\beta + \underline{m}]$. A suitable law of large numbers can be applied to show that the fraction of time that the Markov process spends on any set $\mathcal{M} \in \mathcal{B}$ converges (almost surely) to $\mu(\mathcal{M})$.

Typically, the frequency of each scenario depends on a non trivial way on the transition probabilities. An exception is when the transition probabilities do not depend on the

[40] As a convention, we denote by $[\underline{m}, \underline{m}]$ the singleton $\{\underline{m}\}$.

[41] See Hopenhayn and Prescott (1992).

[42] Here is where the assumption $\underline{m} > 0$ makes a difference as the support of the invariant distribution would be equal to $\{0\}$ if $\underline{m} = 0$. This is because zero becomes an absorbing state of the dynamics of the laggard's cash when $\underline{m} = 0$. To see why, note that a feature of the equilibrium is that a laggard that chooses zero working capital in any given period loses with probability one in the auction of that period. Thus, the laggard starts next period with zero cash if $\underline{m} = 0$ and its only feasible working capital is zero.

state.[43] By (21) and (26), this independence occurs in our model when the exogenous cashflow $\underline{m} \geq \nu^\beta$ so that only the symmetric scenario can occur. Since (20) implies that $\Psi^\beta(m) = 0$ for any $m \geq \underline{m} \geq \nu^\beta$, we obtain that $\underline{m} \geq \nu^\beta$, (8), (21) and (A4) in the Appendix imply $F_{\nu^\beta}^{\Psi^\beta} = F_{L,m}^\beta = F_{l,m}^\beta = F^\beta$ for $m \geq \underline{m}$ so that the induced equilibrium in each period is the symmetric scenario of the static model. Thus, the following proposition is an immediate consequence of (26), (27), Proposition 1, Corollary 1 and the fact that $\theta > \nu^\beta$ by Definitions 6 and 7 so no proof is provided.

Proposition 5. If $\frac{\theta}{\underline{m}} < 1$, then: (i) the equilibrium probability of winning the contract at any date t is common across firms, (ii) $\lim_{\beta\uparrow 1} \mu^\beta\left(\{\theta + \underline{m}\}\right) = 1$ and (iii) both (a) the fraction of time that both firms choose working capital structure arbitrarily close to θ, and (b) $\pi(\max\{w_{1,t}, w_{2,t}\})$, and $|\pi(w_{1,t}) - \pi(w_{2,t})|$ are arbitrarily close to 0 converges (almost surely) to 1 as β increases to 1.

The ratio $\frac{\theta}{\underline{m}}$ decreases in the cash flow $\underline{m}$ and increases in the working capital θ needed to push the bid down to c. Proposition 5 illustrates the conventional wisdom that "auctions [still] work well if raising cash for bids is easy" as in our model, for a fixed $\underline{m}$, it is easy to raise cash for bids from internal or external resources if β is close to 1 or if θ is small, respectively. Next, Theorem 1 and Corollary 5(i), which are the basis for our first main result, show that the ease to raise cash from internal resources is not sufficient for auctions to work well.

Theorem 1. If $\frac{\theta}{\underline{m}} > 4$ and $\pi(2\underline{m}) + \pi(\underline{m}) > \pi(0)$, then $\lim_{\beta\uparrow 1} \mu^\beta(\{\underline{m}\}) = 1$.

The first hypothesis requires that $\underline{m}$ be sufficiently high relative to θ. Since π is continuous and decreasing, the second hypothesis in Theorem 1 requires that $\underline{m}$ be sufficiently small given π. When $\underline{m}$ is sufficiently small relative to θ and π, then as β increases to 1, the laggard's cash equals $\underline{m}$.

Corollary 5. If $\frac{\theta}{\underline{m}} > 4$ and $\pi(2\underline{m}) + \pi(\underline{m}) > \pi(0)$, the fraction of time the following

[43] In the more difficult case in which the transition probabilities depend on the state, the invariant distribution associated to the limit transition probabilities as β increases to 1 has an easy characterization. This is because the transition probabilities become degenerate and concentrate its probability in one point only, either $\underline{m}$ or $\theta + \underline{m}$, and thus any distribution with support in $\{\underline{m}, \theta + \underline{m}\}$ is an invariant distribution. Since there are multiple invariant distributions, we cannot apply a continuity argument to characterize what happens when the cost of working capital is small.

properties hold in equilibrium converges to 1 (almost surely) as β increases to 1: (i) $\pi(\max\{w_{1,t}, w_{2,t}\})$ is arbitrarily close to $\pi(\underline{m})$, and (ii) $|\pi(w_{1,t}) - \pi(w_{2,t})|$ is arbitrarily close to $\pi(0) - \pi(\underline{m})$.

Corollary 5 follows since as β increases to 1: $\mu^{\beta}(\{\underline{m}\})$ increases to 1, by Theorem 1, and the laggard and the leader play with probability arbitrarily close to 1 at their atoms when the laggard's cash is $\underline{m}$, by Lemma A5 in the Appendix, the assumption that $\frac{\theta}{\underline{m}} > 4$ and $\pi(2\underline{m}) + \pi(\underline{m}) > \pi(0)$, the first line of (A13) and Lemma A8(ii) in the Appendix. Thus, if $\underline{m}$ is sufficiently small given π, then, as β tends to 1, the leader wins the procurement contract most of the time so that large concentrations and asymmetries in market shares occur. The next section provides a quantitative example.

Corollary 5 is the basis for extending our second main result to the dynamic model: the dispersion of markups $\frac{\min\{b_1,b_2\}-c}{c} = \frac{\pi(\max\{w_1,w_2\})}{c}$, and "money left on the table", $\frac{|b_1-b_2|}{c} = \frac{|\pi(w_1)-\pi(w_2)|}{c}$, across auctions arises only due to differences in the exogenous cash flow $\underline{m}$ and the function π. Thus, Corollary 5 gives a general setting that goes beyond the linear example and sufficiently large θ discussed after Corollary 3. Each case shows that it is incorrect to infer, as is typically done, that the dispersion of markups and "money left on the table" indicates incomplete information. That is, the usual interpretation is incorrect in this setting.

Corollary 6. If $\frac{\theta}{\underline{m}} > 4$ and $\pi(2\underline{m}) + \pi(\underline{m}) > \pi(0)$, the following properties hold in equilibrium (almost surely) as β increases to 1: (i) $\lim_{\tau\to\infty} \frac{1}{\tau}\sum_{t=1}^{\tau} \pi(\max\{w_{1,t}, w_{2,t}\})$ is arbitrarily close to $\pi(\underline{m})$, and (ii) $\lim_{\tau\to\infty} \frac{1}{\tau}\sum_{t=1}^{\tau} |\pi(w_{1,t}) - \pi(w_{2,t})|$ is arbitrarily close to $\pi(0) - \pi(\underline{m})$.

For those $\underline{m}$ satisfying the assumptions of Corollary 6, an argument analogous to the one we made after Corollary 4 let us extend our third main result to the dynamic model.[44]

4.4 A Numerical Example

In this section, we use a numerical example to shed some light on the intermediate case (not covered in Section 4.3) in which $\frac{\theta}{\underline{m}}$ lies in $(1, 4)$.

Our example illustrates the following properties that are useful for empirical work:

[44] Indeed, one can replace m_l by $\underline{m}$ and consider two arbitrary values for $\underline{m}$

(I) the endogenous distribution of the laggard's cash has sufficient variability to generate significant dispersion of markups and bids across otherwise identical auctions; (II) changes in $\frac{\theta}{\underline{m}}$ give rise to a rich set of values for bid dispersion and concentration; and (III) large concentration and asymmetries in market shares arise for large values of $\frac{\theta}{\underline{m}}$.

We use empirically grounded values for the parameters: a year consists of four periods[45] and we assume[46] $\beta = 0.9602$, $\pi(w) = \theta - w$ and[47] $\frac{\theta}{c} = 1$. We compute a solution to the functional equation (19) by iterating the function T (see (20)) from the initial condition $\Psi = 0$ to obtain a fixed point[48] Ψ^{β}. Afterwards, we use Proposition 4 to construct a BI equilibrium (W^*, σ^*, b^*). Finally, we compute μ^{β}, the invariant measure (over the laggard's cash) associated with the BI equilibrium (W^*, σ^*, b^*).

The left panel of Figure 1 illustrates (I).[49] It shows that, for a given ratio $\frac{\theta}{\underline{m}}$, markups and "money left on the table" may have significant volatility across auctions due to the endogenous volatility of the firm's working capital and cash.

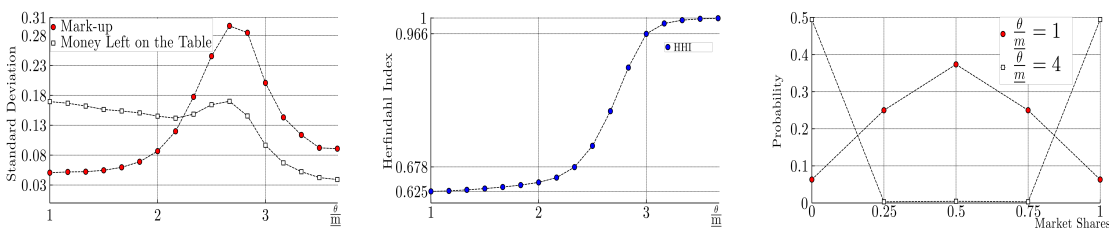

Figure 1: $\pi(w) = \theta - w$, $\beta = 0.9602$ and $\frac{\theta}{c} = 1$. Left and central panels (resp.): St. dev. of $\frac{\min\{b_{1,t}, b_{2,t}\} - c}{c}$ and $\frac{|b_{1,t} - b_{2,t}|}{c}$, and HHI as a function of $\frac{\theta}{\underline{m}}$. Right panel: Invariant distribution of annual market share of Firm 1 (1 yr = 4 periods) for $\frac{\theta}{\underline{m}} = 1$ and $\frac{\theta}{\underline{m}} \approx 4$.

[45] In the data of Hong and Shum (2002) firms bid on average in 4 contracts per year: "[in] a data set of bids submitted in procurement contract auctions conducted by the NJDOT in the years 1989-1997,[...] firms which are awarded at least one contract bid in an average of 29.43 auctions."

[46] This assumption implies an annual discount rate of 0.85, slightly higher than the 0.80 used in Jofre-Bonet and Pesendorfer (2003), and an expected cost of working capital of 0.15.

[47] Since π and, hence, μ, are independent of c, any measure of markups or money left on the table is arbitrary unless we provide a relationship between c and the other parameters. We explain in Footnote 49 how to generate the graph in the right panel of Figure 1 for different values of the ratio θ/c.

[48] Lemma S3 in the supplementary material shows that the generated sequence converges to a fixed point of Equation (19).

[49] In Figure 1 we keep $\theta = c = 1$ and vary $\underline{m}$ between 0.25 and 1. Interestingly, the graph remains the same for any combination of $\underline{m}$ and θ for which the ratio $\theta/\underline{m}$ varies between 1 and 4 while keeping $c = \theta$. One can obtain the graph for other values of $\frac{\theta}{c}$ simply multiplying the values in the vertical axis by $\frac{\theta}{c}$.

Regarding (II), as $\frac{\theta}{m}$ increases from 1 to 4, the left panel of Figure 1 shows that the standard deviation of the "money left on the table" varies from 0.17 to 0.04 whereas the central and right panels show, respectively, that the Herfindahl-Hirschman Index (HHI) varies from 0.625 to almost 1 and that the distribution of Firm 1 market share shifts.[50] The fact that HHI is almost 1 and that the distribution of Firm 1 market share becomes concentrated on 0 and 1 for $\frac{\theta}{m} \approx 4$ illustrate (III). To the extent that there is a direct relationship between the size of the ratio $\frac{\theta}{m}$ to the project's cost, our model predicts that concentration is greater for larger projects than for smaller ones.[51]

5 Conclusion

We have studied a model of bidding markets with financial constraints. A key element of our analysis is that the stage at which firms choose their working capitals resembles an all pay auction with caps. This feature, and thus our results, seems pertinent for other models of investing under winner-take-all competition, like patent races. The introduction of private information about cost is a natural extension that nests both the standard model and our model and provides a framework to test between these two models. Existing results for all pay auctions and general contests[52] suggest these may be fruitful lines of future research. Furthermore, our analysis points out a tractable way to incorporate the dynamics of liquidity in Galenianos and Kircher's (2008) analysis of monetary policy. Although the main focus of our paper is positive, it also offers interesting normative insights for markets in the absence of surety bonds (which implies firms' bids are unconstrained). It is well known that the possibility of bankruptcy creates distortions in these markets (see Calveras, Ganuza, and Hauk (2004), and Zheng (2001)). Our paper shows that using surety bonds to insure against bankruptcy could also have dramatic consequences for markups and concentration.

[50] The same firm wins all the contracts 98.92% of the years if $\frac{\theta}{m} \approx 4$, and only 13% of the years if $\frac{\theta}{m} = 1$.

[51] Porter and Zona (1993) explain that "the market for large jobs [in procurement of highway maintenance] was highly concentrated. Only 22 firms submitted bids on jobs over $1 million. On the 25 largest jobs, 45 percent of the 76 bids were submitted by the four largest firms."

[52] Amann and Leininger (1996) study the relationship between the equilibrium of the all pay auction with and without private information and Alcalde and Dahm (2010) study the similarities between the equilibrium outcome in an all pay auction and in some other models of contests.

Appendix: Proofs

The proofs of the next lemmae are available in the supplementary material. We start with some auxiliary results that are used in the proofs of Propositions 1, 2 and 3. First, recall that we can restrict to mixed strategies (b^*, F_i) in which $F_i\colon \mathbb{R} \to [0,1]$ has support in $[0, \min\{\overline{\nu}^\beta, m_i\}]$ and b^* is as in (1). In Lemmas A1-A3 and Propositions 1-3, we study the equilibrium choices of working capital assuming that the firm bids according to b^*.

Lemma A1. Suppose an equilibrium $((b^*, F_1), (b^*, F_2))$. F_j puts strictly positive probability on $[w-\epsilon, w]$ for any $\epsilon > 0$, if $w \in (0, \min\{\overline{\nu}^\beta, m_i\}]$ belongs to the support of F_i, for $\{i,j\} = \{1,2\}$.

Lemma A2. Suppose an equilibrium $((b^*, F_1), (b^*, F_2))$. F_i is continuous at $w \in [0, \min\{\overline{\nu}^\beta, m_j\})$ if F_j puts strictly positive probability on $[w-\epsilon, w]$ for any $\epsilon > 0$ and $\{i,j\} = \{1,2\}$.

Lemma A3. Suppose an equilibrium $((b^*, F_1), (b^*, F_2))$. For $\{i,j\} = \{1,2\}$:

(i) If the support of F_i contains $w \neq 0$, then the support of F_j also contains w.

(ii) If $w \in (0, \min\{\overline{\nu}^\beta, m_l\})$, then F_i is continuous at w.

(iii) If $m_l < \overline{\nu}^\beta$ and $m_i < m_j$, then F_i is continuous at m_i.

(iv) If F_j has an atom at 0 then F_i is continuous at 0.

(v) If the support of F_i contains $w \in (0, \min\{\overline{\nu}^\beta, m_l\})$, then it also contains $[0, w]$. Besides, when $m_l < \overline{\nu}^\beta$ and $m_i \neq m_j$, the claim also holds true for $w = m_l$.

(vi) If F_i is continuous in $(0, \nu)$ and $(0, \nu)$ belongs to the support of F_j then:

$$F_i(w) = F^\beta(w) + \frac{\pi(0)}{\pi(w)} F_i(0) \, \forall w \in [0, \nu). \tag{A1}$$

Proof of Proposition 1

Proof. To see why the proposed strategy is an equilibrium note that the expected payoff of Firm i with cash m_i when it chooses working capital w and the other firm randomizes its working capital according to F^β, see (6), is equal to:

$$m_i - (1-\beta)w + \beta\pi(w)F^\beta(w),$$

which, by definition of F^{β} in (8), is equal to m_i if $w \leq \overline{\nu}^{\beta}$, and strictly less than m_i otherwise. Thus, deviations are not profitable, as required.

We now prove that the equilibrium is unique. The maximum of the support of F_i, $i = 1, 2$ is common by Lemma A3(i), strictly positive by Lemma A3(iv), and weakly less than $\overline{\nu}^{\beta}$ by the restriction to strictly undominated strategies. These results and Lemma A3(v) imply that F_1 and F_2 (a) each have an atom at $\overline{\nu}^{\beta}$ or (b) each have support equal to $[0, \nu]$ for some $\nu \in (0, \overline{\nu}^{\beta}]$. (a) cannot occur because (2) and Definition 2 imply that at least one firm earns less than m_i when each chooses ν^{β} and so that firm can strictly improve its payoff by choosing zero working capital. In Case (b), Lemma A3(ii) implies that F_1 and F_2 are continuous in $(0, \nu)$. Thus, if $\nu = \overline{\nu}^{\beta}$, Lemma A3(vi) and $F^{\beta}(\overline{\nu}^{\beta}) = 1$ imply that $F_1(0) = F_2(0) = 0$, and hence $F_1 = F_2 = F^{\beta}$, as desired. To finish the proof we show that $\nu = \overline{\nu}^{\beta}$. Lemma A3(iv) implies that $F_i(0) = 0$ for some $i \in \{1, 2\}$. Hence Lemma A3(vi) implies that $F_i(w) = F^{\beta}(w)$ for $w \in [0, \nu)$. To get a contradiction, suppose $\nu < \overline{\nu}^{\beta}$. Then, $F_i(\nu) = F^{\beta}(\nu)$ because Lemma A3(ii) implies that F_i is continuous at ν. Thus, $F^{\beta}(\nu) < 1$, by Definition 2 and (8), which contradicts that F_i has support $[0, \nu]$.

Properties (i) and (ii) are straightforward and (iii) follows from (5) and (8). ■

Proof of Proposition 2

Proof. We first show that the proposed candidate is an equilibrium. By (6) (replacing F with F_L^{β} and m with m_l), (7), (8) and (10), the laggard's expected payoff from $w \in [0, m_l)$ is constant and equal to m_l. The tie-breaking rule guarantees that this payoff is continuous at $w = m_l$ so that the laggard has no incentive to deviate. By (6) (replacing F with F_l^{β} and m with m_L), (7), (8) and (9), the leader's expected payoff from $w \in [0, m_l]$ is constant and equal to $m_L - (1-\beta)m_l + \beta\pi(m_l) \geq m_L - (1-\beta)w' + \beta\pi(w')$ for any $w' \in (m_l, m_L]$ so that the leader has no incentive to deviate.

To prove uniqueness, we use the fact that $m_l < \nu^{\beta}$ along with Lemma A3 ((i) and (iv)), to infer that the supports of the equilibrium distributions must have a common maximum that is weakly less than m_l. Since $m_l < \nu^{\beta}$, and $m_l \neq m_L$, Lemma A3(v) can then be used to imply that each support equals $[0, \nu]$ for some $\nu \in (0, m_l]$ and Lemma A3(ii) implies that both distributions must be continuous on $(0, \nu)$. Since $\nu \leq m_l < \nu^{\beta}$, (2) and (8) imply that $F^{\beta}(w) < 1$ for any $w \in [0, \nu]$, so that each distribution must have an atom either at 0 or at ν. Lemma A3 ((ii) and (iii)) implies that the laggard's atom is at 0 and the laggard's payoff must equal m_l. Lemma A3(iv) then implies that the leader's atom is at ν. If $\nu < m_l$, then the laggard can obtain a payoff higher than m_l by choosing $w \in (\nu, m_l)$. Thus, $\nu = m_l$ so that (9) and (10) define the unique equilibrium distributions. ■

Proof of Proposition 3

Proof. We first show that our proposal is an equilibrium. If $m \in (0, \xi^\beta)$, then the payoff to each firm is $\beta\left(m + \frac{\pi(m)}{2}\right)$ and the payoff to a deviation to $w < m$ is $(m - (1-\beta)w)$. Deviations are unprofitable since $w > 0$, π decreases and ξ^β satisfies (12) implies

$$\begin{aligned}\beta\left(m + \frac{\pi(m)}{2}\right) - (m - (1-\beta)w) &\geq \beta\frac{\pi(m)}{2} - (1-\beta)m \\ &\geq \beta\frac{\pi(\xi^\beta)}{2} - (1-\beta)\xi^\beta = 0. \qquad \text{(A2)}\end{aligned}$$

If $m \in (\xi^\beta, \overline{\nu}^\beta)$, then, by construction, using (6), (8), (12), and (13), the payoff to each firm is constant and equal to m on $[0, \lambda(m)] \cup \{m\}$, the support of F^{**}. If a firm uses $w \in (\lambda(m), m)$, then the payoff equals $m - w + \beta\left(w + \pi(w)F^\beta(\lambda(m))\right) < m - \lambda(m) + \beta(\lambda(m) + \pi(\lambda(m))F^\beta(\lambda(m))) = m$ since the payoff decreases in w in this range and the payoff is continuous on $[0, m]$ and therefore equals m at the boundary.

We now show uniqueness. If $m \in (0, \xi^\beta)$, Lemma A3 ((i), (iv) and (v)) imply that either (a) the support of F_i equals $\{0, m\}$ and the support of F_j equals $\{m\}$ for $\{i, j\} = \{1, 2\}$; or for $i = 1, 2$ (b) the support of F_i is $\{m\}$; (c) the support of F_i is $[0, \nu] \cup \{m\}$ for some $\nu \in (0, m]$; or (d) the support of F_i is $[0, \nu]$ for some $\nu \in (0, m)$. In case (a) i's payoff at $w = 0$ (i.e., m) must equal its expected payoff at $w = m$ (i.e., $\beta m + \frac{\beta}{2}\pi(m)$) which holds by (12) only if $m = \xi^\beta$. If $m = \xi^\beta$, j's expected payoff at ξ^β is $\beta\xi^\beta + \left(F_i(0) + \frac{1-F_i(0)}{2}\right)\beta\pi(\xi^\beta)$ and the limit, as $w \downarrow 0$, of j's expected payoff at w is $\xi^\beta + F_i(0)\beta\pi(0)$ which is greater than its payoff at ξ^β since $\pi(w) > \pi(\xi^\beta) > 0$ and (12) imply

$$\begin{aligned}\xi^\beta + F_i(0)\beta\pi(0) &> \xi^\beta + F_i(0)\beta\pi(\xi^\beta) = \xi^\beta + F_i(0)\beta\pi(\xi^\beta) + \frac{\beta}{2}\pi(\xi^\beta) - (1-\beta)\xi^\beta \\ &= \beta\xi^\beta + \left(F_i(0) + \frac{1}{2}\right)\beta\pi(\xi^\beta) \\ &> \beta\xi^\beta + \left(F_i(0) + \frac{1-F_i(0)}{2}\right)\beta\pi(\xi^\beta)\end{aligned}$$

which violates equilibrium requirements. In case (b) we know that equilibrium results by construction when $m \in [0, \xi^{\beta}]$. If $m \notin [0, \xi^{\beta}]$, then equilibrium cannot result since the expected payoff of $\beta\left(m + \frac{\pi(m)}{2}\right) < m$ in this case. In case (c) we know that equilibrium results when $F_i = F^{**}$ for $i = 1, 2$ by construction when $m \in \left(\xi^{\beta}, \overline{\nu}^{\beta}\right)$. That it is the only equilibrium follows since the expected payoffs at 0 and at m must be equal so that

$$m = \beta\left(m + \left(\lim_{w \uparrow \nu} F_j(w) + \frac{1 - \lim_{w\uparrow\nu} F_j(w)}{2}\right)\pi(m)\right) \tag{A3}$$

which implies that $\lim_{w\uparrow\nu} F_j(w) = \frac{2m(1-\beta)-\beta\pi(m)}{\beta\pi(m)} \in (0, 1)$ for $j = 1, 2$ only if $m \in (\xi^{\beta}, \overline{\nu}^{\beta})$ by (4) and (12). In this case, $F_1(w) = F_2(w)$ is continuous on $(0, \nu)$ (Lemma A3(ii)) and $F_1(0) = F_2(0) = 0$ (Lemma A3(iv)) so that $F_1(w) = F_2(w) = F^{\beta}(w)$ for $w \in (0, \nu)$. In this case, (13) and (A3) imply that $\nu = \lambda(m)$ so that Lemma A3(vi) implies $F_1(w) = F_2(w) = F^{\beta}(w)$ for $w \in [0, \lambda(m)]$ and so $F_1(w) = F_2(w) = F^{**}(w)$.

Finally, in case (d), Lemma A3(ii) implies that F_1 and F_2 are continuous on $(0, \nu)$ and so Lemma A3(vi) implies $F_i(w)$ satisfies (A1) for $i = 1, 2$. Lemma A3(iv) implies that either (i) F_i is continuous on $[0, \nu]$ for $i = 1, 2$, (ii) F_i is continuous on $[0, \nu]$, F_j has an atom at 0 for $\{i, j\} = \{1, 2\}$, or (iii), F_i has an atom at 0, F_j has an atom at ν for $\{i, j\} = \{1, 2\}$. Cases d(i)-(ii) are not possible because (A1) implies that $F_i(\nu) = \frac{(1-\beta)\nu}{\beta\pi(\nu)} < 1$ since $\nu < \overline{\nu}^{\beta}$. In case d(iii), by (A1), $F_i(w)$ is described by the right-hand side of (9) and $F_j(w)$, by that of (10) after replacing m_l with ν in (9) and (10). In this case, the payoff to i is constant and equal to m but the payoff to j is constant and equal to $m - \nu(1 - \beta) + \beta\pi(\nu) > m$ since $\nu < \overline{\nu}^{\beta}$ and so i can do better by deviating to $w = m + \epsilon$ for some small $\epsilon > 0$. ■

Solutions to the Differential Equation in (18)

It can be shown by taking derivatives that:

$$\begin{aligned} F^{\Psi}_{m}(w) &= e^{\int_w^m \frac{\pi'(y)}{\pi(y)+\Psi(y+\underline{m})}dy}\left(1 - \frac{1-\beta}{\beta}\int_w^m \frac{e^{\int_x^m \frac{(-\pi'(y))}{\pi(y)+\Psi(y+\underline{m})}dy}}{\pi(x) + \Psi(x + \underline{m})}dx\right) && \text{(A4)} \\ &= e^{\int_w^m \frac{\pi'(y)}{\pi(y)+\Psi(y+\underline{m})}dy} - \frac{1-\beta}{\beta}\int_w^m \frac{e^{\int_w^x \frac{\pi'(y)}{\pi(y)+\Psi(y+\underline{m})}dy}}{\pi(x) + \Psi(x + \underline{m})}dx, && \text{(A5)} \end{aligned}$$

where we use in the second step that $e^{\int_a^b A(x)dx} \cdot e^{-\int_c^b A(x)dx} = e^{\int_a^c A(x)dx}$. One can also show by taking derivatives that in the the particular case of $F^{\Psi}_{\hat{\nu}^{\Psi}}$, see Definition 6:

$$F^{\Psi}_{\hat{\nu}^{\Psi}}(w) = \frac{1-\beta}{\beta}\int_0^w \frac{e^{\int_x^w \frac{(-\pi'(y))}{\pi(y)+\Psi(y+\underline{m})}dy}}{\pi(x) + \Psi(x + \underline{m})}dx. \tag{A6}$$

Proof of Proposition 4

To show that our bid function b^* solves the right hand side of the firm's Bellman equation in Definition 4, we prove the more general argument that for our continuation value W^*, and for any given bid and working capital of the rival, a working capital w and a bid $\tilde{b} > \pi(w) + c$ does strictly worse than the same bid $\tilde{b}$ and the minimum working capital that makes this bid acceptable, i.e. $\tilde{w}$ such that $\pi(\tilde{w}) + c = \tilde{b}$. The argument is the same as in the static case: reducing today's working capital while keeping constant the bid increases today's utility in the amount of working capital reduced while it decreases tomorrow's continuation value in its discounted value. This is easy to deduce from the functional form of W^*, see (25), when the reduction in today's working capital (keeping constant the bid) does not change the identity of tomorrow's leader. Otherwise, it is a consequence of both firms having the same cash when the identity of the leader changes, the implication of Assumption 1 that at least one firm has cash larger than θ at any information set, and that $\Psi^\beta(m') = 0$ if $m' \geq \theta$, by (17) and Definition 7.

In what follows, we assume that both firms use the bid function b^* and write down the expected payoff to a firm with cash m that chooses a working capital $w \in [0, m]$ when the opponent with cash m' chooses working capital according to the equilibrium distribution. We consider different cases depending on the relationship between m and m' and show that in each case the expected payoff equals $W^*(m, m')$.

If $m, m' \geq \nu^\beta$ then the opponent's distribution of working capital is the atomless distribution $F^{\Psi^\beta}_{\nu^\beta}$ with support equal to $[0, \nu^\beta]$, see Definition 7 and (21) and (24). Using that $b^*(w|m, m') \geq b^*(w'|m', m)$ if and only if $w \leq w'$, the definition of W^* in (25) and some algebra, we obtain that the expected payoff is

$$m - (1-\beta)w + \frac{\beta}{1-\beta}\underline{m} + \beta \int_0^{\min\{w,\nu^\beta\}} (\pi(w) + \Psi^\beta(\tilde{w} + \underline{m}))(F^{\Psi^\beta}_{\nu^\beta})'(\tilde{w})d\tilde{w}. \tag{A7}$$

The derivative of Equation (A7) with respect to w is 0 for $w \in [0, \nu^\beta]$ because $F^{\Psi^\beta}_{\nu^\beta}$ solves (18) and it is negative for $w > \nu^\beta$. Thus the firm is indifferent among all $w \in [0, \nu^\beta]$ and strictly prefers these levels to anything strictly greater than ν^β. The expected payoff to the firm with cash m equals the expected payoff in Equation (A7) when $w = 0$ which equals $W^*(m, m')$ for $m, m' \geq \nu^\beta$ as required.

If $m < m'$ and $m \in [0, \nu^\beta)$, our firm is the laggard and the other firm the leader. The leader's distribution of working capital is $F^{\beta}_{L,m}$ with support $[0, m]$ and an atom at m, see (23). Using that $b^*(w|m, m') \geq b^*(w'|m', m)$ if and only if $w \leq w'$, the definition of W^* in (25) and some algebra, we obtain that the laggard's expected payoff is

$$m - (1-\beta)w + \frac{\beta}{1-\beta}\underline{m} + \beta \int_0^w (\pi(w) + \Psi^{\beta}(\tilde{w} + \underline{m}))(F^{\beta}_{L,m})'(\tilde{w})d\tilde{w}, \tag{A8}$$

if $w \in [0, m)$. The derivative of (A8) with respect to w, for $w \in [0, m)$, is 0 because $F^{\beta}_{L,m}$ solves (18), see (23). Our tie-breaking rule ensures that the laggard's expected payoff is continuous at $w = m$. Thus, the laggard is indifferent among all $w \in [0, m)$ and its expected payoff is equal to the expected payoff in (A8) when $w = 0$ which equals $W^*(m, m')$ for $m < m'$ and $m \in [0, \nu^{\beta}]$ as required.

If $m > m'$ and $m' \in [0, \nu^{\beta}]$, our firm is the leader and the other is the laggard. The laggard's distribution of working capital is $F^{\beta}_{l,m'} \equiv F^{\Psi^{\beta}}_{m'}$ with support $[0, m']$ and an atom at 0, see (22). Using that $b^*(w|m, m') \geq b^*(w'|m', m)$ if and only if $w \leq w'$, the definition of W^* in (25) and some algebra, the leader's expected payoff is

$$\begin{aligned} m - (1-\beta)w + \frac{\beta}{1-\beta}\underline{m} + \beta F^{\beta}_{l,m}(0)(\pi(w) + \Psi^{\beta}(\underline{m}))+ \\ \beta \int_0^{\min\{w,m'\}} (\pi(w) + \Psi^{\beta}(\tilde{w} + \underline{m}))(F^{\beta}_{l,m})'(\tilde{w})d\tilde{w}, \end{aligned} \tag{A9}$$

The derivative of Equation (A9) with respect to w for $w \in [0, m]$ is 0 because $F^{\beta}_{l,m}$ defined in (21) solves (18) in $[0, m]$. The leader's expected payoff is given by (A9) evaluated at $w = 0$ and it equals $W^*(m, m')$ for $m > m'$ and $m' \in [0, \nu^{\beta}]$ as required, as can be deduced using that Ψ^{β} is a fixed point of the operator T on $\mathcal{P}^{\beta}$. ■

Proof of Theorem 1

To prove Theorem 1 we show a more general result that we state as Theorem A1 below. The lemma and definition that follows are used, respectively, in the proof of Lemma A5 and the statement of Theorem A1.

Lemma A4. $\lim_{\beta \uparrow 1} \nu^{\beta} = \theta$.

Definition A1. Let $\Lambda \equiv \left\{(\pi, \underline{m}) : \lim_{\beta \uparrow 1}\left(\inf\left\{x : x = \Psi^{\beta}(\underline{m}) \text{ for some } \Psi^{\beta} \in \widehat{\mathcal{P}}^{\beta}\right\}\right) = \infty\right\}$.

Λ consists of the $(\pi, \underline{m})$ such that every selection of fixed points of T diverges as $\beta \uparrow 1$.

Theorem A1. If $\frac{\theta}{\underline{m}} > 4$ and $(\pi, \underline{m}) \in \Lambda$, then $\lim_{\beta\uparrow 1} \mu^\beta(\{\underline{m}\}) = 1$.

The following lemma together with Theorem A1 imply Theorem 1.

Lemma A5. If $\pi(2\underline{m}) + \pi(\underline{m}) > \pi(0)$, then $(\pi, \underline{m}) \in \Lambda$.

Auxiliary Results Used in the Proof of Theorem A1

Lemma A6. $F_{l,m}$ and $F_{L,m}$ decrease in m for $m < \nu^\beta$ and are constant in m for $m \geq \nu^\beta$.

In the proof of Theorem A1, we use the implication of (26) that:

$$Q^\beta(m, [\underline{m}, x]) = F^\beta_{l,m}(x-\underline{m}) + F^\beta_{L,m}(x-\underline{m}) - F^\beta_{l,m}(x-\underline{m})F^\beta_{L,m}(x-\underline{m}), \tag{A10}$$

for $x - \underline{m} < m, \nu^\beta$, which implies that:

$$Q^\beta(m, [\underline{m}, x]) = (2 - F^\beta_{L,\theta}(x-\underline{m}))F^\beta_{L,\theta}(x-\underline{m}) = Q^\beta(\theta, [\underline{m}, x]), \tag{A11}$$

for $m \geq \theta$, by (21)-(23), because $\nu^\beta < \theta$ by Definitions 6 and 7.

Finally, note that for $m'' < m'$:

$$Q^\beta(m, (m'', m']) = Q^\beta(m, [\underline{m}, m']) - Q^\beta(m, [\underline{m}, m'']), \tag{A12}$$

which is equal to zero when $m'' - \underline{m} \geq m$ by (26).

Lemma A7. If $m' - \underline{m} < \nu^\beta$ and $\mathcal{M} \subset (\underline{m}, m']$ then $Q^\beta(m, \mathcal{M}) \leq 2(1 - F^\beta_{l,m'-\underline{m}}(0))$.

Lemma A8.

(i) Suppose $(\pi, \underline{m}) \in \Lambda$ and $\theta > 2\underline{m}$ then:

$$\lim_{\beta\to 1} F^\beta_{l,m}(w) = \begin{cases} 1 & \text{if } m < \theta - \underline{m} \text{ and } w \in [0, m], \\ \frac{\pi(m)}{\pi(w)} & \text{if } m \in [\theta - \underline{m}, \theta) \text{ and } w \in [\theta - \underline{m}, m], \\ \frac{\pi(m)}{\pi(\theta - \underline{m})} & \text{if } m \in [\theta - \underline{m}, \theta) \text{ and } w \in [0, \theta - \underline{m}), \\ 0 & \text{if } m \geq \theta \text{ and } w \in [0, \theta), \end{cases} \tag{A13}$$

(ii) $\lim_{\beta\to 1} F^\beta_{L,m}(w) = 0$ if $w < \min\{\theta, m\}$.

(iii) Suppose $(\pi, \underline{m}) \in \Lambda$ and $\theta \geq 3\underline{m}$ then:

$$\lim_{\beta\to 1}(1-\beta)\Psi^\beta(m) = \begin{cases} \pi(\underline{m}) & \text{if } m < \theta - \underline{m}, \\ \pi(\underline{m})\frac{\pi(m)}{\pi(\theta - \underline{m})} & \text{if } m \in [\theta - \underline{m}, \theta). \end{cases} \tag{A14}$$

Lemma A9. Suppose $(\pi, \underline{m}) \in \Lambda$.

(i) $\lim_{\beta\uparrow 1} \frac{F^{\beta}_{L,m}(w)}{1-\beta} = 0$ if $w < \min\{\theta - \underline{m}, m\}$.

(ii) $\lim_{\beta\uparrow 1} \frac{F^{\beta}_{L,\theta}(\theta-2\underline{m}-\epsilon)}{(1-\beta)^2} = \int_0^{\theta-2\underline{m}-\epsilon} \frac{1}{\pi(\underline{m})} dz > 0$, if $\theta \geq 3\underline{m}$.

(iii) $\lim_{\beta\uparrow 1} \frac{1-F^{\beta}_{l,\theta-\underline{m}-\epsilon}(0)}{1-\beta} = \int_0^{\theta-\underline{m}-\epsilon} \frac{(-\pi'(z))\min\left\{\frac{\pi(m)}{\pi(\theta-\underline{m})},1\right\}}{\pi(\underline{m})\min\left\{\frac{\pi(z+\underline{m})}{\pi(\theta-\underline{m})},1\right\}} dz > 0$ for any $\epsilon > 0$, if $\theta \geq 3\underline{m}$.

Lemma A10. If $(\pi, \underline{m}) \in \Lambda$ and $\theta > 2\underline{m}$, then $\lim_{\beta\uparrow 1} \mu^{\beta}((\underline{m}, \theta]) = 0$.

Proof of Theorem A1

For $\epsilon \in (0, \theta - 4\underline{m})$, we define the following sets $A \equiv \{\underline{m}\}$, $B \equiv (\underline{m}, \theta - 2\underline{m} - \epsilon]$, $C = (\theta - 2\underline{m} - \epsilon, \theta - \underline{m} - \epsilon]$, $D \equiv (\theta - \underline{m} - \epsilon, \theta - \epsilon]$, $E \equiv (\theta - \epsilon, \theta + \underline{m}]$. We also let $\hat{B} \equiv (\underline{m}, 2\underline{m})$. The definition of ϵ implies that $\hat{B} \subset B$.

By Lemma A10, it is sufficient to show that $\lim_{\beta\uparrow 1} \mu^{\beta}(E) = 0$. We provide an upper bound for $\mu^{\beta}(E)$ for β close to 1 and show that this bound converges to zero.

That $Q(m, E) = 0$ if $m \notin D \cup E$ (which follows from (A12) and (26)) and (27) imply:

$$\begin{aligned}
\mu^{\beta}(E) &= \mu^{\beta}(D) \int_D Q^{\beta}(m, E) \frac{\mu^{\beta}(dm)}{\mu^{\beta}(D)} + \mu^{\beta}(E) \int_E Q^{\beta}(m, E) \frac{\mu^{\beta}(dm)}{\mu^{\beta}(E)} \\
&\leq \mu^{\beta}(D) + \left(\mu^{\beta}(E) \int_E Q^{\beta}(m, E) \frac{\mu^{\beta}(dm)}{\mu^{\beta}(E)} \right). \qquad \text{(A15)}
\end{aligned}$$

That $Q(m, D) = 0$ if $m \notin C \cup D \cup E$ (which follows from (A12) and (26)) and (27) imply:

$$\mu^{\beta}(D) = \mu^{\beta}(C \cup D) \int_{C\cup D} Q^{\beta}(m, D) \frac{\mu^{\beta}(dm)}{\mu^{\beta}(C \cup D)} + \mu^{\beta}(E) \int_E Q^{\beta}(m, D) \frac{\mu^{\beta}(dm)}{\mu^{\beta}(E)}. \qquad \text{(A16)}$$

Substituting (A16) into (A15), using that $1 - Q^{\beta}(m, D) - Q^{\beta}(m, E) = Q^{\beta}(m, A \cup B \cup C)$ and solving for $\mu^{\beta}(E)$, one gets the first inequality below:

$$\begin{aligned}
\mu^{\beta}(E) &\leq \frac{\int_{C\cup D} Q^{\beta}(m, D)\mu^{\beta}(dm)}{\int_E Q^{\beta}(m, A \cup B \cup C) \frac{\mu^{\beta}(dm)}{\mu^{\beta}(E)}} \\
&\leq \frac{\int_{C\cup D} Q^{\beta}(m, D)\mu^{\beta}(dm)}{Q^{\beta}(\theta, A \cup B \cup C)} \\
&\leq \frac{2(1 - F^{\beta}_{l,\theta-\underline{m}-\epsilon}(0))\mu^{\beta}(C \cup D)}{Q^{\beta}(\theta, A \cup B \cup C)} \\
&\leq \frac{2(1 - F^{\beta}_{l,\theta-\underline{m}-\epsilon}(0)) \left(2(1 - F^{\beta}_{l,\theta-\underline{m}-\epsilon}(0))\mu^{\beta}((\underline{m}, \theta]) + Q^{\beta}(\theta, C \cup D)\mu^{\beta}([\theta, \theta + \underline{m}]) \right)}{Q^{\beta}(\theta, A \cup B \cup C)} \\
&= \frac{\frac{2(1-F^{\beta}_{l,\theta-\underline{m}-\epsilon}(0))}{1-\beta} \left(\frac{2(1-F^{\beta}_{l,\theta-\underline{m}-\epsilon}(0))}{1-\beta} \mu^{\beta}((\underline{m}, \theta]) + \frac{Q^{\beta}(\theta,C\cup D)}{1-\beta} \mu^{\beta}([\theta, \theta + \underline{m}]) \right)}{\frac{Q^{\beta}(\theta,A\cup B\cup C)}{(1-\beta)^2}}, \qquad \text{(A17)}
\end{aligned}$$

where the remaining inequalities are explained as follows. The second inequality follows from the property that $Q^\beta(m, A\cup B\cup C) \geq Q^\beta(\theta, A\cup B\cup C)$ for $m \in E$. This property follows from $Q^\beta(m, A\cup B\cup C) = Q^\beta(m, [\underline{m}, \theta - \underline{m} - \epsilon])$ and because the right hand side of (26) is increasing in $F^\beta_{l,m}(x-\underline{m})$ and $F^\beta_{L,m}(x-\underline{m})$ and the fact that $F^\beta_{l,m}(x) \geq F^\beta_{l,\theta}(x)$ and $F^\beta_{L,m}(x) \geq F^\beta_{L,\theta}(x)$ by Lemma A6 since $\nu^\beta < \theta$ by Definitions 6 and 7. The third inequality follows from Lemma A7 for $\mathcal{M} = D$ and $m' = \theta - \epsilon$ which is less than ν^β by Lemma A4. Finally, the last inequality follows from:

$$\begin{aligned}
\mu^\beta(C\cup D) &= \int_{\theta-3\underline{m}-\epsilon}^{\theta} Q^\beta(m, C\cup D)\mu^\beta(dm) + Q^\beta(\theta, C\cup D)\mu^\beta([\theta, \theta+\underline{m}]) && \text{(A18)}\\
&\leq 2(1 - F_{l,\theta-\underline{m}-\epsilon}(0))\mu^\beta((\underline{m}, \theta]) + Q^\beta(\theta, C\cup D)\mu^\beta([\theta, \theta+\underline{m}]). && \text{(A19)}
\end{aligned}$$

The first equality uses (27), $Q^\beta(m, C\cup D) = 0$ if $m < \theta - 3\underline{m} - \epsilon$, see right below (A12), and $Q^\beta(m, C\cup D) = Q^\beta(\theta, C\cup D)$ if $m \geq \theta$ by (A11). The inequality uses Lemma A7 for $\mathcal{M} = C\cup D$ and $m' = \theta - \epsilon < \nu^\beta$ by Lemma A4.

To conclude the proof, we show that the last line of the right hand side of (A17) tends to zero as β tends to 1.

First, note that

$$\begin{aligned}
\lim_{\beta\uparrow 1} \frac{Q^\beta(\theta, A\cup B\cup C)}{(1-\beta)^2} &= \lim_{\beta\uparrow 1}\left((2 - F^\beta_{L,\theta}(\theta - 2\underline{m} - \epsilon))\frac{F^\beta_{L,\theta}(\theta - 2\underline{m} - \epsilon)}{(1-\beta)^2}\right)\\
&= 2\int_0^{\theta-2\underline{m}-\epsilon} \frac{1}{\pi(\underline{m})}dy > 0 && \text{(A20)}
\end{aligned}$$

where the first step uses (A11); the second step uses that $\lim_{\beta\to 1} F^\beta_{L,\theta}(\theta - 2\underline{m} - \epsilon) = 0$ and $\lim_{\beta\uparrow 1} \frac{F^\beta_{L,\theta}(\theta-2\underline{m}-\epsilon)}{(1-\beta)^2} = \int_0^{\theta-2\underline{m}-\epsilon} \frac{1}{\pi(\underline{m})}dy$ by Lemmas A8(i) and A9(ii), respectively, and that the limit of the product equals the product of the limits. Next note that:

$$\lim_{\beta\uparrow 1} \frac{Q^\beta(\theta, C\cup D)}{1-\beta} = \lim_{\beta\uparrow 1}\left(\frac{Q^\beta(\theta, [\underline{m}, \theta-\epsilon])}{1-\beta} - \frac{Q^\beta(\theta, [\underline{m}, \theta - 2\underline{m} - \epsilon])}{1-\beta}\right) = 0, \qquad \text{(A21)}$$

by application of (A12), in the first step, and of (A11) and Lemma A9(i), and the property that the limit of a difference is equal to the difference of the limits, in the second step.

That the right hand side of the last line of (A17) tends to zero as β tends to 1 follows from (A20)-(A21), Lemmas A9(iii) and A10, $\mu^\beta([\theta, \theta+\underline{m}]) \leq 1$ and that the limit of the ratio equals the ratio of the limits when the denominator's limit is not zero. ■

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The Dynamics of Bidding Markets with Financial Constraints: Supplementary Material[53]

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November 20, 2014

This supplementary material consists of four parts. Section S1 contains the proofs of all the Lemmae of the main text. Section S2 provides a proof for the existence of a fixed point to the operator T defined in (20). Section S3 studies a finite horizon version of our model of Section 4. Section S4 analyses a model with moral hazard and limited liability that endogenizes our function π, see (1).

[53] In this supplementary material we use cross references to equations, definitions, lemmas and propositions in the original paper. The numbering of footnotes is consecutive to the original paper.

S1 Proofs of Lemmae

Proof of Lemma A1

Proof. To get a contradiction, suppose that $w \in (0, \min\{\overline{\nu}^{\beta}, m_i\}]$ belongs to the support of F_i and F_j puts zero probability on $[w - \epsilon, w]$ for some $\epsilon > 0$. We shall argue that Firm i has a profitable deviation when Firm j plays (b^*, F_j). The contradiction hypothesis has two implications. (a) $w - \epsilon$ gives Firm i strictly greater expected payoffs than w since the former saves on the cost of working capital and increases the profit when winning without affecting the probability of winning. (b) Firm i's expected payoffs are continuous in its working capital at w since F_j does not put an atom at w. (a) and (b) mean that there exists an $\epsilon' \in (0, \epsilon)$ such that any working capital in $(w - \epsilon', w + \epsilon')$ gives strictly less expected payoffs than a working capital $w - \epsilon$. The fact that w belongs to the support of F_i means that F_i puts strictly positive probability in $(w - \epsilon', w + \epsilon')$ and thus Firm i has a profitable deviation: shift the probability mass in $(w - \epsilon', w + \epsilon')$ to $w - \epsilon$. ■

Proof of Lemma A2

Proof. To get a contradiction, suppose there exists $w \in [0, \min\{\overline{\nu}^{\beta}, m_j\})$ for which F_j puts strictly positive probability on $[w - \epsilon, w]$ for all $\epsilon > 0$ and F_i has an atom at w. Suppose, first, that $m_j > m_i$ and F_j puts zero probability on $[w - \varepsilon', w)$ for some $\varepsilon' > 0$. Our tie breaking rule in (2) implies that Firm i loses with probability one conditional on both firms choosing w. Thus, Firm i can profitably deviate by shifting the probability that F_i puts in w to $w - \varepsilon'$. Since F_j puts zero probability in $[w - \varepsilon', w)$, this deviation does not affect the probability that i wins the contract but it saves the cost of working capital ε'. We conclude that either (i) $m_j > m_i$ and F_j puts strictly positive probability on $[w - \varepsilon, w)$ for all $\varepsilon > 0$ or (ii) $m_j \leq m_i$ and F_j puts strictly positive probability on $[w - \varepsilon, w)$ for all $\varepsilon > 0$. In each case one can find $\varepsilon' > 0$ small enough such that Firm j can improve by moving the probability that F_j puts in $[w - \epsilon', w)$ in case (i) and in $[w - \epsilon', w]$ in case (ii), to a point slightly above w. Each deviation affects marginally Firm j's cost of working capital and profits conditional on winning but allows the firm to increase from at most $\frac{1}{2}$ to 1 the probability of winning the procurement contract at a strictly positive profit if Firm i plays the atom at w. ■

Proof of Lemma A3

Proof. Lemma A1 and the definition of support implies (i). (ii) and (iii) follow from putting Lemmas A1 and A2 together. (iv) follows from Lemma A2 for $w = 0$. To prove (v), note that Lemma A1 and (i) (with the roles of i and j interchanged) imply that it is sufficient to show that F_j is continuous at w. That follows from (ii) (applied to F_j instead of F_i). The case $w = m_l$ is similar, the only differences are that we use (iii) instead of (ii) and we take $m_i < m_j$ without loss of generality by (i). To prove (vi), note that for $\{i, j\} = \{1, 2\}$ the continuity of F_i in $(0, \nu)$ implies that j's equilibrium expected payoffs of choosing $w \in (0, \nu)$ are equal to (6) for $F = F_i$. Since $(0, \nu)$ belongs to the support of F_j, the usual indifference condition of a mixed strategy equilibrium implies that F_i is a continuous solution of (7) in $(0, \nu)$. Since F_i is a distribution and thus it is right-continuous at 0, the uniqueness result in Theorem 7.1 in Coddington and Levinson (1984), pag. 22, implies that the solutions we seek are characterized by (A1) as can be proved by taking derivatives. ■

Proof of Lemma A4

Proof. It follows from (5) and $\nu^\beta \in [\overline{\nu}^\beta, \theta)$. That $\nu^\beta < \theta$ follows from Definitions 6 and 7. To show that $\nu^\beta \geq \overline{\nu}^\beta$ we use that (A4) and Definition 6 imply:

$$1 - \frac{1-\beta}{\beta} \int_0^{\hat{\nu}^\Psi} \frac{e^{\int_x^{\hat{\nu}^\Psi} \frac{(-\pi'(y))}{\pi(y)+\Psi(y+\underline{m})} dy}}{\pi(x) + \Psi(x + \underline{m})} dx = 0. \tag{S1}$$

Thus, $\nu^\beta \geq \overline{\nu}^\beta$ follows from the facts that the left hand side of (S1) increases in Ψ and decreases in $\hat{\nu}^\Psi$ and that, by Definitions 2 and 7, two solutions to (S1) are $(\Psi, \hat{\nu}^\Psi) = (0, \overline{\nu}^\beta)$ and $(\Psi, \hat{\nu}^\Psi) = (\Psi^\beta, \nu^\beta)$ ■

Proof of Lemma A5

Proof. In this lemma, we use five auxiliary results. First, we use the implication of $\pi(2\underline{m}) > \pi(0) - \pi(\underline{m})$ and π strictly decreasing that $\pi(2\underline{m}) > 0$, and hence $2\underline{m} < \theta$, by Definition 1, which together with Lemma A4 means that for β close to 1,

$$2\underline{m} < \nu^\beta. \tag{S2}$$

Second, note that for $m \leq \nu^\beta$, (22) implies that

$$F^\beta_{l,m}(0) = F^{\Psi^\beta}_m(0) \geq \frac{\pi(m)}{\pi(0)}\left(1 - \frac{1-\beta}{\beta}\frac{m}{\pi(m)}\right), \tag{S3}$$

where the inequality follows from the fact that its right hand side is equal to the right hand side of (A4) for Ψ equal to the zero function and $w = 0$, and that the right hand side of (A4) decreases if the function Ψ shifts downwards.

Third, we use the following implication of (19)-(20) for $m, \underline{m} < \nu^\beta$:

$$\frac{\Psi^\beta(m)}{\Psi^\beta(\underline{m})} = \frac{F^{\Psi^\beta}_m(0)}{F^{\Psi^\beta}_{\underline{m}}(0)}. \tag{S4}$$

Fourth, we use that $F^\beta_{l,m}$ is a solution to (18) in $[w', w''] \subset [0, \min\{m, \nu^\beta\}]$, by (21)-(23), and hence satisfies:

$$F^\beta_{l,m}(w'') - F^\beta_{l,m}(w') = \int_{w'}^{w''} \frac{\frac{1-\beta}{\beta} + (-\pi'(y))F^\beta_{l,m}(y)}{\pi(y) + \Psi(y+\underline{m})}dy. \tag{S5}$$

Fifth, we use that $\underline{m} \leq \nu^\beta$ means that,

$$\begin{aligned}
1 - F^\beta_{l,\underline{m}}(0) &= F^\beta_{l,\underline{m}}(\underline{m}) - F^\beta_{l,\underline{m}}(0) \\
&= \int_0^{\underline{m}} \frac{\frac{1-\beta}{\beta} + (-\pi'(y))F^\beta_{l,\underline{m}}(y)}{\pi(y) + \Psi^\beta(y+\underline{m})}dy \\
&\leq \int_0^{\underline{m}} \frac{\frac{1-\beta}{\beta} + (-\pi'(y))}{\Psi^\beta(2\underline{m})}dy \\
&= \frac{\frac{1-\beta}{\beta}\underline{m} + \pi(0) - \pi(\underline{m})}{\Psi^\beta(2\underline{m})},
\end{aligned} \tag{S6}$$

where the first step follows from $F^\beta_{l,\underline{m}}(\underline{m}) = F^{\Psi^\beta}_{\underline{m}}(\underline{m})$, by (22), and $F^{\Psi^\beta}_{\underline{m}}(\underline{m}) = 1$, by Definition 5, the second step from (S5), the third step from $F^\beta_{l,\underline{m}}(y) \leq 1$, $\pi(y) \geq 0$ and $\Psi^\beta(y+\underline{m}) \geq \Psi^\beta(2\underline{m})$, and the last step from standard algebra.

The lemma is proved using that $\pi(2\underline{m}) > 0$, as explained above, $\lim_{\beta\uparrow 1} F^\beta_{l,\underline{m}}(0) > 0$, by (S3), and the following equalities and inequalities, for β close to 1:

$$\begin{aligned}
(1-\beta)\Psi^\beta(\underline{m}) &= \beta F^\beta_{l,\underline{m}}(0)\left(\pi(0) + \Psi^\beta(\underline{m})\right) - \beta\Psi^\beta(\underline{m}) \\
&= \beta F^\beta_{l,\underline{m}}(0)\left(\pi(0) - \frac{1 - F^\beta_{l,\underline{m}}(0)}{F^\beta_{l,\underline{m}}(0)}\Psi^\beta(\underline{m})\right) \\
&\geq \beta F^\beta_{l,\underline{m}}(0)\left(\pi(0) - \frac{\frac{1-\beta}{\beta}\underline{m} + \pi(0) - \pi(\underline{m})}{F^\beta_{l,2\underline{m}}(0)}\right) \\
&\geq \beta F^\beta_{l,\underline{m}}(0)\pi(0)\frac{\pi(2\underline{m}) + \pi(\underline{m}) - \pi(0) - \frac{1-\beta}{\beta}3\underline{m}}{\pi(2\underline{m}) - \frac{1-\beta}{\beta}2\underline{m}},
\end{aligned}$$

where we use in the first equality (19) and (20); in the second equality, an algebraic transformation; in the first inequality, (S2), (S4) for $m = 2\underline{m}$ and (S6); and in the second inequality, (S2), (S3) for $m = 2\underline{m}$ and some algebra. ■

Proof of Lemma A6

Proof. Direct from (21), (22), (23), and (A5). ■

Proof of Lemma A7

Proof. It is sufficient to show that:

$$Q^{\beta}(m, [\underline{m}, m']) - F^{\beta}_{l,m}(0) \leq 2(1 - F^{\beta}_{l,m'-\underline{m}}(0)). \tag{S7}$$

This can be deduced from $Q^{\beta}(m, \mathcal{M}) \leq Q^{\beta}(m, (\underline{m}, m'])$, (A12) for $m'' = \underline{m}$ and $Q^{\beta}(m, \{\underline{m}\}) = F^{\beta}_{l,m}(0)$. That $Q^{\beta}(m, \{\underline{m}\}) = F^{\beta}_{l,m}(0)$ can be deduced from (26) since $F^{\beta}_{L,m}(0) = 0$ by (21) and (23) and Definition 6.

If $m' - \underline{m} \geq m$, (S7) follows from $Q^{\beta}(m, [\underline{m}, m']) = 1$, by (26), and $F^{\beta}_{l,m}(0) \geq F^{\beta}_{l,m'-\underline{m}}(0)$, by Lemma A6. If $m' - \underline{m} < m$, (S7) follows from:

$$\begin{aligned}
& Q^{\beta}(m, [\underline{m}, m']) - F^{\beta}_{l,m}(0) \\
&\quad = F^{\beta}_{l,m}(m' - \underline{m}) + F^{\beta}_{L,m}(m' - \underline{m})(1 - F^{\beta}_{l,m}(m' - \underline{m})) - F^{\beta}_{l,m}(0) \\
&\quad = F^{\beta}_{l,m}(m' - \underline{m}) - F^{\beta}_{l,m}(0) + (F^{\beta}_{L,m}(m' - \underline{m}) - F^{\beta}_{L,m}(0))(1 - F^{\beta}_{l,m}(m' - \underline{m})) \\
&\quad \leq F^{\beta}_{l,m}(m' - \underline{m}) - F^{\beta}_{l,m}(0) + (F^{\beta}_{L,m}(m' - \underline{m}) - F^{\beta}_{L,m}(0)) \\
&\quad \leq 2(F^{\beta}_{l,m}(m' - \underline{m}) - F^{\beta}_{l,m}(0)) \\
&\quad \leq 2(F^{\beta}_{l,m'-\underline{m}}(m' - \underline{m}) - F^{\beta}_{l,m'-\underline{m}}(0)) \\
&\quad = 2(1 - F^{\beta}_{l,m'-\underline{m}}(0)),
\end{aligned}$$

where the first step uses (A10) which applies by $m' - \underline{m} < m$ and the Lemma's hypothesis; the second step uses again that $F^{\beta}_{L,m}(0) = 0$; the third step uses that $F^{\beta}_{l,m} \leq 1$ and that

$F^{\beta}_{L,m}(\cdot)$ is increasing and $m' - \underline{m} \geq 0$; and the fourth step uses that:

$$\begin{aligned} F^{\beta}_{L,m}(m' - \underline{m}) - F^{\beta}_{L,m}(0) &= \int_0^{m'-\underline{m}} \frac{\frac{1-\beta}{\beta} + (-\pi'(y))F^{\beta}_{L,m}(y)}{\pi(y) + \Psi(y+\underline{m})} dy \\ &\leq \int_0^{m'-\underline{m}} \frac{\frac{1-\beta}{\beta} + (-\pi'(y))F^{\beta}_{l,m}(y)}{\pi(y) + \Psi(y+\underline{m})} dy \\ &= F^{\beta}_{l,m}(m' - \underline{m}) - F^{\beta}_{l,m}(0), \end{aligned}$$

where the equalities use $m' - \underline{m} < m$ and the Lemma's hypothesis to apply (S5) and a similar equation in which $F^{\beta}_{L,m}$ replaces $F^{\beta}_{l,m}$, and the inequality follows from Lemma A6 and (21)-(23); in the fifth step, we use the Lemma's hypothesis, $m' - \underline{m} < m$, (S5) for $w'' = m' - \underline{m}$ and $w' = 0$ and Lemma A6; and in the sixth step, that $F^{\beta}_{l,m'-\underline{m}}(m' - \underline{m}) = 1$ by (21) and (22). ■

Proof of Lemma A8

Proof. We use the following implication of $(\pi, \underline{m}) \in \Lambda$, first line, and $\Psi^{\beta} \in \mathcal{P}^{\beta}$, second line:

$$\lim_{\beta \to 1} \Psi^{\beta}(m) = \begin{cases} \infty & \text{if } m < \theta, \\ 0 & \text{if } m \geq \theta. \end{cases} \tag{S8}$$

The definition of $\mathcal{P}^{\beta}$ in (17) implies the second line. The first line can be deduced using Lemma A4, (S4), $\lim_{\beta \to 1} \Psi^{\beta}(\underline{m}) = \infty$, $F^{\Psi^{\beta}}_{\underline{m}}(0) \leq 1$ and (S3).

To prove (i) and (ii), we use that $\lim_{\beta \uparrow 1} \nu^{\beta} = \theta$, by Lemma A4, $\nu^{\beta} < \theta$, by Definitions 6 and 7, and (21)-(23) imply that for β close to 1:

$$F^{\beta}_{l,m}(w) = \begin{cases} F^{\Psi^{\beta}}_{m}(w) & \text{if } m < \theta \\ F^{\Psi^{\beta}}_{\nu^{\beta}}(w) & \text{if } m \geq \theta \end{cases}, \tag{S9}$$

and,

$$F^{\beta}_{L,m}(w) = \; F^{\Psi^{\beta}}_{\nu^{\beta}}(w) \quad \text{if } w \in [0, \min\{m, \theta\}). \tag{S10}$$

The limit, as β goes to 1, of the first line of the right hand side of (S9) is equal to the corresponding expressions in the first three lines of the right of (A13). This is because: (a) the two terms of the product in the right hand side of (A4) evaluated at Ψ^{β} have the necessary finite limits as β goes to 1; and (b) the product of (finite) limits equals the limit of the product. Result (a) can be shown by using the the bounded convergence theorem

(Royden (1988), page 81), denoted BCT hereafter,[54] and (S8). That the limit, as β goes to 1, of the second line of the right of (S9) is equal to the right of the fourth line of (A13) is a consequence of the integral in the last line of the right hand side of (A6) being bounded for $\Psi = \Psi^\beta$. Since this argument does not require neither $(\pi, \underline{m}) \in \Lambda$ nor $\theta > 2\underline{m}$, it also implies the limit in (ii).

To prove (iii), we start with the case $m = \underline{m}$. We use that for β close to 1, $F^\beta_{l,\underline{m}}(0) = F^{\Psi^\beta}_{\underline{m}}(0)$ by (22) and $\underline{m} < \nu^\beta$, a consequence of $3\underline{m} < \theta$ and Lemma A4. Thus, Definition 7 means that:

$$\begin{aligned}(1-\beta)\Psi^\beta(\underline{m}) &= \beta F^\beta_{l,\underline{m}}(0)(\pi(0) + \Psi^\beta(\underline{m})) - \beta\Psi^\beta(\underline{m}) \\ &= \beta F^\beta_{l,\underline{m}}(0)\pi(0) - \beta\Psi^\beta(\underline{m})(1 - F^\beta_{l,\underline{m}}(0)),\end{aligned} \tag{S11}$$

where the first term on the second line of the right hand side of (S11) tends to $\pi(0)$ since $\lim_{\beta\uparrow 1} F^\beta_{l,\underline{m}}(0) = 1$ by the first line of the right of (A13) and $3\underline{m} > \theta$. We use that $F^\beta_{l,\underline{m}}(\underline{m}) = 1$, by (22) and Definition 5, and (S5) to rewrite the last term of (S11) as

$$\beta\Psi^\beta(\underline{m})\int_0^{\underline{m}} \frac{\frac{1-\beta}{\beta} + (-\pi'(z))F^\beta_{l,\underline{m}}(z)}{\pi(z) + \Psi^\beta(z+\underline{m})}dz = \beta\int_0^{\underline{m}} \frac{\frac{1-\beta}{\beta} + (-\pi'(z))F^\beta_{l,\underline{m}}(z)}{\frac{\pi(z)}{\Psi^\beta(\underline{m})} + \frac{F^\beta_{l,z+\underline{m}}(0)}{F^\beta_{l,\underline{m}}(0)}}dz. \tag{S12}$$

where the equality in (S12) follows from (S4). By application of the BCT, the limit of this last term when β tends to 1 is equal to $\pi(0) - \pi(\underline{m})$, as required, since $\Psi^\beta(\underline{m})$ diverges to infinity and $F^\beta_{l,m}(w)$ tends to 1 for $w \leq \underline{m}$ and $m \in [\underline{m}, 2\underline{m}]$. The former follows from the assumption that $(\pi, \underline{m}) \in \Lambda$ and Definition A1, and the latter follows from the first line of the right hand side of (A13) since we assume that $3\underline{m} < \theta$ which means that $2\underline{m} < \theta - \underline{m}$.

The result for a general m can be deduced from the result for $m = \underline{m}$ and (S4) and the limit results for $F^\beta_{l,m}$ in part (i) of this lemma. ■

Proof of Lemma A9

Proof. To show (i), note that (21) and (23) and Lemma A4 imply that $F_{L,m}(w) = F^{\Psi^\beta}_{\nu^\beta}(w)$ for $w < \min\{\theta, m\}$ and β close to 1. Thus, we can deduce (i) from (A6), (S8), and the BCT.

[54] The BCT applies because each integrand in (A4) is uniformly bounded along the sequence since Ψ being non negative implies that $|\frac{-\pi'(y)}{\pi(y)+\Psi(y+\underline{m})}| \leq \frac{\pi'(y)}{\pi(y)}$ and $0 \leq \frac{1}{\pi(x)+\Psi(x+\underline{m})} \leq \frac{1}{\pi(x)}$.

To show (ii), note that (21) and Lemma A4 imply that $F^{\beta}_{L,\theta}(\theta - 2\underline{m} - \epsilon) = F^{\Psi^\beta}_{\nu^\beta}(\theta - 2\underline{m} - \epsilon)$ for β close to 1. Thus, we can deduce from (A6) and Definition 7 that:

$$\frac{F^{\beta}_{L,\theta}(\theta - 2\underline{m} - \epsilon)}{(1-\beta)^2} = \frac{1}{\beta}\int_0^{\theta - 2\underline{m}-\epsilon} \frac{e^{\int_x^{\theta-2\underline{m}-\epsilon} \frac{(-\pi'(y))}{\pi(y)+\Psi^\beta(y+\underline{m})}dy}}{(1-\beta)\pi(x) + (1-\beta)\Psi^\beta(x+\underline{m})}dx.$$

We can deduce (ii) from this equation, applying the limits in (S8), the first line of the right of (A14) and the BCT. The BCT can be applied to the integral in x because, first, its integrand is non negative and bounded above by:

$$\frac{e^{\int_0^{\theta-2\underline{m}-\epsilon} \frac{(-\pi'(y))}{\pi(y)}dy}}{(1-\beta)\Psi^\beta(\theta - \underline{m} - \epsilon)}, \tag{S13}$$

since $\pi(x)$ is non-negative for $x \in [0, \theta]$ and $\Psi^\beta(z + \underline{m})$ is decreasing and non-negative in x; and second, expression (S13) is uniformly bounded above for any β in $(1/2, 1)$ as a consequence of the first line of the right of (A14).

To show (iii), note that for β close to 1:

$$\begin{aligned}
\lim_{\beta\uparrow 1}\frac{1 - F^{\beta}_{l,\theta-\underline{m}-\epsilon}(0)}{1-\beta} &= \lim_{\beta\uparrow 1}\frac{F^{\beta}_{l,\theta-\underline{m}-\epsilon}(\theta - \underline{m} - \epsilon) - F^{\beta}_{l,\theta-\underline{m}-\epsilon}(0)}{1-\beta} \\
&= \lim_{\beta\uparrow 1}\int_0^{\theta-\underline{m}-\epsilon}\frac{\frac{1-\beta}{\beta} + (-\pi'(z))F^{\beta}_{l,\theta-\underline{m}-\epsilon}(z)}{(1-\beta)\pi(z) + (1-\beta)\Psi^\beta(z+\underline{m})}dz \\
&= \int_0^{\theta-\underline{m}-\epsilon}\lim_{\beta\uparrow 1}\left(\frac{\frac{1-\beta}{\beta} + (-\pi'(z))F^{\beta}_{l,\theta-\underline{m}-\epsilon}(z)}{(1-\beta)\pi(z) + (1-\beta)\Psi^\beta(z+\underline{m})}\right)dz \\
&= \int_0^{\theta-\underline{m}-\epsilon}\frac{(-\pi'(z))\min\left\{\frac{\pi(m)}{\pi(\theta-\underline{m})},1\right\}}{\pi(\underline{m})\min\left\{\frac{\pi(z+\underline{m})}{\pi(\theta-\underline{m})},1\right\}}dz,
\end{aligned}$$

where first equality follows from Lemma A4, (22), and $F^{\Psi^\beta}_{m}(m) = 1$; the second equality follows by Lemma A4 and (S5); in the third equality we apply, once again, the BCT using that the integrand is nonnegative and bounded above by:

$$\frac{\frac{1-\beta}{\beta} + \gamma}{(1-\beta)\pi(z) + (1-\beta)\Psi^\beta(z+\underline{m})},$$

which is uniformly bounded above in β for $\beta \in (1/2, 1)$ and $z \in [0, \theta - \underline{m} - z]$ as can be shown by applying Lemma A8(iii) to the denominator; and finally, in the fourth equality we use the following three properties: (a) the numerator converges to $(-\pi'(w))\min\left\{\frac{\pi(m)}{\pi(\theta-\underline{m})},1\right\}$ since the first and third line of the right hand side of (A13) implies that $\lim_{\beta\uparrow 1} F^{\beta}_{l,m}(z) =$

$\min\left\{\frac{\pi(m)}{\pi(\theta-\underline{m})},1\right\}$ because $z\leq\theta-\underline{m}-\epsilon$, (b) the denominator converges to $\pi(\underline{m})\min\left\{\frac{\pi(w+\underline{m})}{\pi(\theta-\underline{m}-\epsilon)},1\right\}>0$ since Lemma A8(iii) implies that $\lim_{\beta\uparrow 1}(1-\beta)\Psi^{\beta}(z+\underline{m})=\pi(\underline{m})\min\left\{\frac{\pi(z+\underline{m})}{\pi(\theta-\underline{m})},1\right\}$ because $z+\underline{m}<\theta$ and (c) the limit of the ratio equals the ratio of the limits when the denominator has a positive limit. ■

Proof of Lemma A10

Proof. For $\mathcal{M}=(\underline{m},\theta]$, (27) implies that $\mu^{\beta}(\mathcal{M})=\int Q^{\beta}(m,\mathcal{M})\mu^{\beta}(dm)$, which is less than $2(1-F^{\beta}_{l,\theta-\underline{m}}(0))$ for β close to 1 since Lemma A7 for $m'=\theta$ can be applied by Lemma A4 and since μ^{β} is a probability measure. Thus, the lemma follows from the third line on the right hand side of (A13). ■

S2 Existence of Fixed Points of the Operator T

To simplify the notation, we adopt the convention that

$$(a)^{+}\equiv\max\{a,0\}. \tag{S14}$$

We also find convenient to compute a bound for $\hat{\nu}^{\Psi}$ for any $\Psi\in\mathcal{P}^{\beta}$.

Definition S1. $\overline{x}^{\beta}$ is the unique $w\in[(\theta-\underline{m})^{+},\theta]$ that solves:

$$\beta\pi(w)=(1-\beta)(w-(\theta-\underline{m})^{+}).$$

Note the similarity between the equation that defines $\overline{x}^{\beta}$ and (4). Indeed, as next lemma shows, $\overline{x}^{\beta}$ is an upper bound to the support of the randomizations that firms play over working capitals in the models of Section 4 and Section S3. In this sense, it is an analogue of $\overline{\nu}^{\beta}$ in the static model.

Lemma S1. $\hat{\nu}^{\Psi}\leq\overline{x}^{\beta}$ for any $\Psi\in\mathcal{P}^{\beta}$.

Proof. By Definition 5, $F^{\Psi}_{\hat{\nu}^{\Psi}}(\hat{\nu}^{\Psi})=1$. Besides, the right hand side of (A6) is increasing in w for $w<\theta$. Thus, it is sufficient to show the right hand side of (A6) is greater than 1 at

$w = \overline{x}^{\beta}$. This expression is equal to:

$$
\begin{aligned}
\frac{1-\beta}{\beta}\int_0^{\overline{x}^{\beta}} \frac{e^{\int_x^{\overline{x}^{\beta}} \frac{(-\pi'(y))}{\pi(y)+\Psi(y+\underline{m})}dy}}{\pi(x)+\Psi(x+\underline{m})}dx &\geq \frac{1-\beta}{\beta}\int_{(\theta-\underline{m})^+}^{\overline{x}^{\beta}} \frac{e^{\int_x^{\overline{x}^{\beta}} \frac{(-\pi'(y))}{\pi(y)+\Psi(y+\underline{m})}dy}}{\pi(x)+\Psi(x+\underline{m})}dx \\
&= \frac{1-\beta}{\beta}\int_{(\theta-\underline{m})^+}^{\overline{x}^{\beta}} \frac{e^{\int_x^{\overline{x}^{\beta}} \frac{(-\pi'(y))}{\pi(y)}dy}}{\pi(x)}dx \\
&= \frac{1-\beta}{\beta}\cdot\frac{\overline{x}^{\beta}-(\theta-\underline{m})^+}{\pi(\overline{x}^{\beta})} \\
&= 1,
\end{aligned}
$$

where we use that the integrand is positive in the first step, that $\Psi(m) = 0$ for $m \geq \theta$ in the second step, standard algebra in the third step and Definition S1 in the last step. ∎

We prove that T has a fixed point applying Schauder Fixed-Point Theorem. For this purpose, we restrict the domain of Ψ to $[0, \theta+\underline{m}]$. Define $\tilde{\mathcal{P}}^{\beta}$ as:

$$
\left\{\Psi\colon [0,\theta+\underline{m}] \to \left[0, \frac{\beta}{1-\beta}\pi(0)\right] \text{ is continuous, decreasing and } \Psi(m) = 0\,\forall m \geq \theta\right\}. \tag{S15}
$$

For any $\tilde{\Psi} \in \tilde{\mathcal{P}}^{\beta}$, we define its (unique) extension to $\mathcal{P}^{\beta}$ by the function Ψ that satisfies $\Psi(m) = \tilde{\Psi}(m)$ if $m \in [0, \theta+\underline{m}]$ and $\Psi(m) = 0$ if $m \in (\theta+\underline{m}, \infty)$. For any $\tilde{\Psi} \in \tilde{\mathcal{P}}^{\beta}$, let $F_m^{\tilde{\Psi}} \equiv F_m^{\Psi}$ where Ψ is the extension of $\tilde{\Psi}$ to $\mathcal{P}^{\beta}$. For any $\tilde{\Psi} \in \tilde{\mathcal{P}}^{\beta}$, let $\tilde{T}(\tilde{\Psi})(m) \equiv T(\Psi)(m)$ for $m \in [0, \theta+\underline{m}]$, where T is defined in (20) and Ψ is the extension of $\tilde{\Psi}$ to $\mathcal{P}^{\beta}$.

Lemma S2. The operator $\tilde{T} : \tilde{\mathcal{P}}^{\beta} \to \tilde{\mathcal{P}}^{\beta}$ has a fixed point. Moreover, the extension of any fixed point of $\tilde{T}$ is a fixed point of the operator $T : \mathcal{P}^{\beta} \to \mathcal{P}^{\beta}$ and any fixed point of T is the extension of some fixed point of $\tilde{T}$.

Proof. We endow $\tilde{\mathcal{P}}^{\beta}$ with the sup-norm, that we denote by $||\cdot||$, and check that the operator $\tilde{T}$ on $\tilde{\mathcal{P}}^{\beta}$ satisfies all the conditions of Schauder Fixed-Point Theorem, see Stokey and Lucas (1999), Theorem 17.4, page 520.

$\tilde{\mathcal{P}}^{\beta}$ is a nonempty, closed, bounded and convex subset of the set of continuous functions on $[0, \theta+\underline{m}]$. Furthermore, it is easy to see that $T : \mathcal{P}^{\beta} \to \mathcal{P}^{\beta}$, see Footnote 37, implies that $\tilde{T} : \tilde{\mathcal{P}}^{\beta} \to \tilde{\mathcal{P}}^{\beta}$. We show below that $\tilde{T}$ is continuous and that the family $\tilde{T}(\tilde{\mathcal{P}}^{\beta})$ is equicontinuous, as desired.

Claim 1: $\tilde{T}$ is continuous.

We assume an arbitrary sequence $\{\Psi_n\} \to \Psi$ in $\tilde{\mathcal{P}}^\beta$ and show that $\tilde{T}(\Psi_n) \to \tilde{T}(\Psi)$. Note that $||\tilde{T}(\Psi_n) - \tilde{T}(\Psi)||$ is equal to:

$$\begin{aligned}
& \sup_{m\in[0,\overline{x}^\beta]} \left|\beta(F_m^{\Psi_n}(0))^+(\pi(0)+\Psi_n(\underline{m})) - \beta(F_m^{\Psi}(0))^+(\pi(0)+\Psi(\underline{m}))\right| \\
= & \sup_{m\in[0,\overline{x}^\beta]} \left|((F_m^{\Psi_n}(0))^+ - (F_m^{\Psi}(0))^+)\beta(\pi(0)+\Psi(\underline{m})) + \beta(F_m^{\Psi_n}(0))^+(\Psi_n(\underline{m}) - \Psi(\underline{m}))\right| \\
\leq & \sup_{m\in[0,\overline{x}^\beta]} \left\{\left|(F_m^{\Psi_n}(0))^+ - (F_m^{\Psi}(0))^+\right| \beta(\pi(0)+\Psi(\underline{m})) + \beta(F_m^{\Psi_n}(0))^+ \left|\Psi_n(\underline{m}) - \Psi(\underline{m})\right|\right\} \\
\leq & \sup_{m\in[0,\overline{x}^\beta]} \left\{\left|(F_m^{\Psi_n}(0))^+ - (F_m^{\Psi}(0))^+\right| \frac{\beta}{1-\beta}\pi(0) + \beta\left|\Psi_n(\underline{m}) - \Psi(\underline{m})\right|\right\} \\
\leq & \frac{\beta}{1-\beta}\pi(0) \sup_{m\in[0,\overline{x}^\beta]} \left|F_m^{\Psi_n}(0) - F_m^{\Psi}(0)\right| + \beta \sup_{m\in[0,\overline{x}^\beta]} \left|\Psi_n(\underline{m}) - \Psi(\underline{m})\right|,
\end{aligned}$$

where the first step follows from (20), Lemma S1 and because (S14) implies that $(F_m^\Psi(0))^+ = F_m^\Psi(0)$ if $m \leq \hat{\nu}^\Psi$ and[55] $(F_m^\Psi(0))^+ = 0$ if $m > \hat{\nu}^\Psi$; in the second step we add and subtract $\beta\Psi(\underline{m})(F_m^{\Psi_n}(0))^+$; the third step follows from $|A+B| \leq |A| + |B|$; the fourth step because $\Psi(m) \leq \frac{\beta}{1-\beta}\pi(0)$ and $(F_m^{\Psi_n}(0))^+ \leq 1$; and the fifth step from the property that $\sup_x\{a(x)+b(x)\} \leq \sup_x a(x) + \sup_x b(x)$, and that $|(A)^+ - (B)^+| \leq |A - B|$.

Definition S1 implies that $\overline{x}^\beta < \theta + \underline{m}$ and so,

$$\sup_{m\in[0,\overline{x}^\beta]} |\Psi_n(\underline{m}) - \Psi(\underline{m})| \leq \sup_{m\in[0,\theta+\underline{m}]} |\Psi_n(m) - \Psi(m)| = ||\Psi_n - \Psi||.$$

Since we assume that $||\Psi_n - \Psi||$ converges to zero, it only remains to be shown that:

$$\lim_{n\to\infty} \sup_{m\in[0,\overline{x}^\beta]} \left|F_m^{\Psi_n}(0) - F_m^{\Psi}(0)\right| = 0. \tag{S16}$$

Let $\underline{\Psi}_n(w) \equiv (\Psi(w) - \epsilon_n)^+$ and $\overline{\Psi}_n(w) \equiv \Psi(w) + \epsilon_n$ for $\epsilon_n \equiv \sup_{\tilde{n}\geq n} ||\Psi_{\tilde{n}} - \Psi||$. With a slight abuse of notation, we denote by $F_m^{\underline{\Psi}_n(w)}(w)$ and $F_m^{\overline{\Psi}_n(w)}(w)$ the right hand side of (A4) at $\Psi = \underline{\Psi}_n$ and $\Psi = \overline{\Psi}_n$, respectively. Thus, that $\Psi(w)$ and $\Psi_n(w)$ belong to $[\underline{\Psi}_n(w), \overline{\Psi}_n(w)]$ and that $F_m^\Psi \geq F_m^{\tilde{\Psi}}$ if $\Psi(w) \geq \tilde{\Psi}(w) \geq 0$ for any w, by (A4), imply that $F_m^{\Psi(w)}(0)$ and $F_m^{\Psi_n(w)}(0)$ belong to $[F_m^{\underline{\Psi}_n(w)}(0), F_m^{\overline{\Psi}_n(w)}(0)]$. Thus:

$$|F_m^{\Psi_n}(0) - F_m^{\Psi}(0)| \leq \left|F_m^{\overline{\Psi}_n}(0)) - F_m^{\underline{\Psi}_n}(0)\right|. \tag{S17}$$

Note that,

$$\lim_{n\to\infty} \sup_{m\in[0,\overline{x}^\beta]} \left|F_m^{\overline{\Psi}_n}(0) - F_m^{\underline{\Psi}_n}(0)\right| = 0, \tag{S18}$$

[55] Since $F_m^\Psi(0)$ is decreasing in m, see (A4), and $F_{\hat{\nu}^\Psi}^\Psi(0) = 0$, see Definition 6.

by an application of Theorem 7.13 in Rudin (1976), pag. 150. We can apply this theorem, because (A4) implies that: (a) $F_m^{\overline{\Psi}_n}(0)$ and $F_m^{\underline{\Psi}_n}(0)$ are continuous in $m \in [0, \overline{x}^\beta]$, (b) $\{F_m^{\overline{\Psi}_n}(0) - F_m^{\underline{\Psi}_n}(0)\}_n$ is a decreasing sequence since $\{\overline{\Psi}_n\}_n$ and $\{\underline{\Psi}_n\}_n$ are respectively decreasing and increasing sequences, and (c) $F_m^{\overline{\Psi}_n}(0) - F_m^{\underline{\Psi}_n}(0)$ converges to zero pointwise in $m \in [0, \overline{x}^\beta]$ since $\lim_{n\to\infty} ||\overline{\Psi}_n - \underline{\Psi}_n|| = 0$ and the BCT[56] applies.

Equations (S17) and (S18) imply (S16) as desired.

Claim 2: the family $\tilde{T}(\tilde{\mathcal{P}}^\beta)$ is equicontinuous.

(20), $\Psi(\underline{m}) \in \left[0, \frac{\beta}{1-\beta}\pi(0)\right]$ for $\Psi \in \tilde{\mathcal{P}}^\beta$ and Lemma S1 means that it is sufficient to show that there exists a finite κ such that:

$$\left|\frac{\partial F_m^\Psi(0)}{\partial m}\right| \leq \kappa \text{ for any } \Psi \in \tilde{\mathcal{P}}^\beta \text{ and } m \in [0, \overline{x}^\beta]. \tag{S19}$$

Equation (A5) evaluated at $w = 0$ implies that:

$$\left|\frac{\partial F_m^\Psi(0)}{\partial m}\right| = \frac{e^{\int_0^m \frac{\pi'(y)}{\pi(y)+\Psi(y+\underline{m})}dy}}{\pi(m) + \Psi(m + \underline{m})} \left|\pi'(m) - \frac{1-\beta}{\beta}\right| \text{ for } m \in [0, \overline{x}^\beta]. \tag{S20}$$

Note that $e^{\int_0^m \frac{\pi'(y)}{\pi(y)+\Psi(y+\underline{m})}dy} \leq 1$ because $\frac{\pi'(y)}{\pi(y)+\Psi(y+\underline{m})} < 0$ and that $\pi(m) + \Psi(m + \underline{m}) \geq \pi(\overline{x}^\beta)$ because π is a decreasing function and $\Psi(m + \underline{m}) \geq 0$. Besides, since $-\pi'$ is continuous, there exists a finite $\gamma \geq 0$ such that:

$$-\pi'(z) \leq \gamma, \text{ for all } z \in [0, \theta]. \tag{S21}$$

These arguments imply the condition in (S19) for

$$\kappa = \frac{1}{\pi(\overline{x}^\beta)} \left|\gamma + \frac{1-\beta}{\beta}\right|.$$

■

Lemma S2 implies the following proposition (and no proof is required).

Proposition S1. The operator T defined in (20) has a fixed point in $\mathcal{P}^\beta$.

[56]The same arguments as in Footnote 54 can be used to show that each integrand in (A4) is uniformly bounded in β.

S3 A Finite Horizon Model

In this section, we show that the unique equilibrium of a finite period version of the model in Section 4 converges as the number of periods tend to infinity to one of the equilibria that we analyse in Section 4. As Athey and Schmutzler (2001) have argued:

> If there are multiple equilibria, one equilibrium of particular interest (if it exists) is an equilibrium attained by taking the limit of first-period strategies as the horizon T approaches infinity.

We consider a $\bar{t}+1$-period model with periods denoted by $t \in \{1, 2, \ldots, \bar{t}+1\}$. All the periods but the last one are as in the model of Section 4. In the last period, all the firm's cash is consumed. The dynamic link between periods and the firms' objective functions, adapted to the finite time horizon, are also as in Section 4. Both firms start with identical cash larger than θ and we assume Assumption 1. We study the subgame perfect equilibria of the game.

Definition S2. We let Ψ_t, $t = 2, \ldots, \bar{t}+1$, be defined recursively (starting from $\bar{t}+1$) by $\Psi_{\bar{t}+1} \equiv \mathbf{0}$, and $\Psi_t \equiv T(\Psi_{t+1})$, where T is the operator defined in (20). We let ν_t, $t = 1, \ldots, \bar{t}$, denote $\hat{\nu}^{\Psi_{t+1}}$.

For $t = 1, \ldots, \bar{t}$, let,

$$F_{t,l,m}(w) = F_{t,L,m}(w) = F_{\nu_t}^{\Psi_{t+1}}(w) \qquad \text{if } w \leq \nu_t \leq m \tag{S22}$$

$$F_{t,l,m}(w) = F_{m}^{\Psi_{t+1}}(w) \qquad \text{if } w \leq m < \nu_t \tag{S23}$$

$$F_{t,L,m}(w) \equiv \begin{cases} F_{\nu_t}^{\Psi_{t+1}}(w) & \text{if } w < m < \nu_t \\ 1 & \text{if } w = m < \nu_t \end{cases} \tag{S24}$$

Let Ω denote (as in Section 4) the cash vectors that can arise along the game tree. For any $t \in \{1, 2, ..., \bar{t}\}$ and $(m, m') \in \Omega$, let:

$$\sigma_t^*(w|m, m') \equiv \begin{cases} F_{t,l,m}(w) & \text{if } m \leq m' \\ F_{t,L,m'}(w) & \text{if } m > m', \end{cases} \tag{S25}$$

and,

$$W_t(m, m') \equiv \begin{cases} m + \underline{m} \sum_{\tau=1}^{\bar{t}+1-t} \beta^\tau & \text{if } m \leq m' \\ m + \underline{m} \sum_{\tau=1}^{\bar{t}+1-t} \beta^\tau + \Psi_t(m') & \text{if } m > m'. \end{cases} \tag{S26}$$

Note that W_t is continuous in Ω because the only conflicting point is when $m = m'$ and Assumption 1 implies that in this case $m' \geq \theta$ and hence $\Psi_t(m') = 0$ by (17). We also let $W_{\bar{t}+1}(m, m') \equiv m$, for any $(m, m') \in \Omega$.

Proposition S2. There is a unique subgame perfect equilibrium of the game. In this equilibrium, and at any period $t \in \{1, 2, ..., \bar{t}\}$, both firms randomize their working capital according to σ_t^*, bid according to b^* and have W_t expected continuation payoffs at the beginning of period t.

Proof: We prove the proposition using backward induction. It is trivial that the continuation payoffs at the beginning of period $\bar{t}+1$ are described by $W_{\bar{t}+1}$ in period $\bar{t}+1$. We can then apply recursively the following claim:

Claim: *There is a unique equilibrium in the reduced game defined by period t and the continuation payoffs βW_{t+1}. In this equilibrium both firms use the strategy (b^*, σ_t^*), and get expected equilibrium payoffs described by W_t.*

To prove the claim, note that the argument in the first paragraph of the proof of Proposition 4 also applies here using W_{t+1}, Ψ_{t+1}, (S26) and Definition S2 instead of W^*, Ψ^β, (25) and Definition 7, respectively. Thus, in equilibrium both firms use b^*. Under this assumption, the expected payoffs of a firm with cash m and working capital $w \geq \theta - \underline{m}$ that faces a rival that randomizes according to F are equal to:

$$\begin{aligned} m - w + \beta \int_0^w W_t(w + \pi(w) + \underline{m}, \tilde{w} + \underline{m}) F(d\tilde{w}) + \beta \int_w^\infty W_t(w + \underline{m}, \tilde{w} + \pi(\tilde{w}) + \underline{m}) F(d\tilde{w}) \\ = m - (1-\beta)w + \underline{m} \sum_{\tau=1}^{\bar{t}+1-t} \beta^\tau + \beta \pi(w) F(w) + \beta \int_0^{\min\{w, \overline{x}^\beta - \underline{m}\}} \Psi_{t+1}(\tilde{w} + \underline{m}) F(d\tilde{w}) \end{aligned} \tag{S27}$$

since $\Psi_{t+1}(\tilde{w} + \underline{m}) = 0$ for $\tilde{w} + \underline{m} \geq \overline{x}^\beta$ by (20) and Lemma S1.

We use (S27) to show, first, that σ_t^* is an equilibrium with equilibrium expected payoffs given by W_t and, second, that there is no other equilibrium.

To show that σ_t^* is optimal when the other firm uses σ_t^* we distinguish whether the laggard's cash is larger than ν_t. If this is the case, the proof follows by an adaptation of the proof of the case $m, m' \geq \nu^\beta$ in Proposition 4, but using ν_t, $F_{\nu_t}^{\Psi_{t+1}}(\cdot)$, Ψ_{t+1}, W_{t+1}, (S22) and Definition S2 instead of ν^β, $F^{\Psi^\beta}_{\nu^\beta}(\cdot)$, Ψ^β, W^*, (21) and Definition 7, respectively. Otherwise, the proof follows by an adaptation of the proof of the case $m < m'$ and $m \in [0, \nu^\beta)$ and the case $m > m'$ and $m' \in [0, \nu^\beta)$ in Proposition 4, but using ν_t, $F_{t,L,m}$, $F_{t,l,m}$, Ψ_{t+1}, W_{t+1} and (S22), (S23) and (S26) instead of ν^β, $F^\beta_{L,m}$, $F^\beta_{l,m}$, Ψ^β, W^*, and (21), (22) and (25), respectively.

Similarly, one can apply the argument in the proof of Proposition 4 that the value function of the Bellman equation is equal to W^* to show that the expected equilibrium payoffs are equal to W_t.

To prove uniqueness of our equilibrium we explain how to adapt the proof of Propositions 1 and 2. First, we show that one can restrict to working capitals in $[0, \overline{x}^\beta]$ because for the expected payoff function in (S27), a working capital $w \geq \overline{x}^\beta$ is strictly dominated by a working capital of $(\theta - \underline{m})^+$. To see why, note that one can deduce from (S27) and the definition of $\overline{x}^\beta$ in Definition S1 that the difference in expected payoffs between working capital $w > \overline{x}^\beta$ and working capital $(\theta - \underline{m})^+$ is equal to:

$$\begin{aligned} \beta\pi(w)F(w) - \beta\pi((\theta - \underline{m})^+)F((\theta - \underline{m})^+) \\ -(1-\beta)(w - (\theta - \underline{m})^+) &\leq \beta\pi(w) - (1-\beta)(w - (\theta - \underline{m})^+) \\ &< \beta\pi(\overline{x}^\beta) - (1-\beta)(\overline{x}^\beta - (\theta - \underline{m})^+) \\ &= 0, \end{aligned}$$

as desired.

We can show that a version of Lemmas A1, A2 and A3 in which $\overline{\nu}^\beta$ is replaced by $\overline{x}^\beta$ and (A1) by:

$$F_i(w) = e^{\int_0^w \frac{-\pi'(y)}{\pi(y)+\Psi_{t+1}(y+\underline{m})}dy} F_i(0) + \frac{1-\beta}{\beta} \int_0^w \frac{e^{\int_x^w \frac{(-\pi'(y))}{\pi(y)+\Psi_{t+1}(y+\underline{m})}dy}}{\pi(x) + \Psi_{t+1}(x+\underline{m})} \, \forall w \in [0, \nu). \tag{S28}$$

is satisfied by the game defined by the expected payoffs in (S27). We can proceed as in the proof of Propositions 1 and 2 to prove uniqueness when the laggard's cash is greater than ν_t, and when the laggard's cash is less than ν_t, respectively. The only difference is that we use $\overline{\nu}_t$ and $\overline{x}^\beta$ instead of $\overline{\nu}^\beta$, Definition S1 instead of Definition 2, (18) for $\Psi = \Psi_{t+1}$ instead of (7), $F^{\Psi_{t+1}}_{\nu_t}$ instead of F^β, $F_{t,l,m}$ instead of $F_{l,m}$, and $F_{t,L,m}$ instead of $F_{L,m}$, and thus (S22)-(S24) instead of (8)-(10). ■

Finally, we show that the limit of the equilibrium of this finite game as the number of periods goes to infinity is equal to the equilibrium of the model in Section 4. To get this result, we abuse a little bit of the notation and denote by $\sigma^*_{t,\bar{t}}$, $W_{t,\bar{t}}$, $\Psi_{t,\bar{t}}$ and $\nu_{t,\bar{t}}$ the functions σ^*_t, W_t, Ψ_t and ν_t, respectively, to make the dependence in the length of the time horizon of the game explicit.

Lemma S3. $\{\Psi_{t,\bar{t}}\}_{\bar{t}=t+1}^{\infty}$ is an increasing sequence in $\mathcal{P}^\beta$ with limit $\Psi_\infty \in \mathcal{P}^\beta$. Besides, Ψ_∞ is a fixed point of T, where T is defined in (20).

Proof. We first note that by Definition S2, $\Psi_{t,\bar{t}} = T^{\bar{t}-t}(\Psi)$ for Ψ the zero function in $\mathcal{P}^\beta$, and $T^n : \mathcal{P}^\beta \to \mathcal{P}^\beta$ an operator defined recursively by $T^1 = T$ and $T^n = T^{n-1}$ for $n > 1$.

The operator T is monotone in the sense that $\Psi \geq \Psi'$ implies that $T(\Psi) \geq T(\Psi')$. This is a consequence of (20) because (A4) implies that $F_m^{\Psi}(0) \geq F_m^{\Psi'}(0)$. Furthermore, the operator T is continuous and the set $T(\mathcal{P}^\beta)$ is equicontinuous by analogous arguments to Claim 1 and Claim 2, respectively, in the proof of Lemma S2. Thus, $\{\Psi_{t,\bar{t}}\}_{\bar{t}=t+1}^{\infty}$ is an increasing sequence in an equicontinuous set $T(\mathcal{P}^\beta)$. Consequently, $\{\Psi_{t,\bar{t}}\}_{\bar{t}=t+1}^{\infty}$ has a limit in $\mathcal{P}^\beta$ that we denote by Ψ_∞. By continuity of T and the definitions of Ψ_∞ and $\Psi_{t,\bar{t}}$:

$$T(\Psi_\infty) = T(\lim_{\bar{t}\to\infty} \Psi_{t,\bar{t}}) = \lim_{\bar{t}\to\infty} T(\Psi_{t,\bar{t}}) = \lim_{\bar{t}\to\infty} \Psi_{t-1,\bar{t}} = \Psi_\infty.$$

Thus Ψ_∞ is a fixed point of T as desired. ∎

Denote by σ^* and W^* the equilibrium of Proposition 4 that corresponds to the function $\Psi^\beta = \Psi_\infty$, where Ψ_∞ is defined in Lemma S3.

Proposition S3. For any $(m, m') \in \Omega$, $\sigma^*_{t,\bar{t}}(\cdot|m, m')$ converges weakly to $\sigma^*(\cdot|m, m')$ and $W_{t,\bar{t}}(m, m')$ converges uniformly to $W^*(m, m')$ as $\bar{t}$ goes to infinity.

Proof. The uniform convergence of $W_{t,\bar{t}}(m, m')$ to $W^*(m, m')$ is a straightforward consequence of Lemma S3 and the definitions of $W^*(m, m')$ and $W_{t,\bar{t}}(m, m')$ in (25) and (S26).

The convergence of $\sigma^*_{t,\bar{t}}$ follows the definitions in (21)-(24) and in (S22)-(S25) and the limits:

$$F_m^{\Psi_{t,\bar{t}}}(w) \xrightarrow{\bar{t}\to\infty} F_m^{\Psi^\beta}(w) \tag{S29}$$

$$F_{\nu_{t,\bar{t}}}^{\Psi_{t,\bar{t}}}(w) \xrightarrow{\bar{t}\to\infty} F_{\nu^\beta}^{\Psi^\beta}(w) \tag{S30}$$

$$\nu_{t,\bar{t}} \xrightarrow{\bar{t}\to\infty} \nu^\beta, \tag{S31}$$

for $\nu_{t,\bar{t}} = \hat{\nu}^{\Psi_{t,\bar{t}}}$, $\Psi^\beta = \Psi_\infty$ and $\nu^\beta = \hat{\nu}^{\Psi_\infty}$.

Equation (S29) follows from Lemma S3 and the application of the property that the limit of the product is equal to the product of the limits if finite and the BCT to (A4) for $\Psi = \Psi_{t,\bar{t}}$. BCT can applied because the same arguments as in Footnote 54 mean here that the integrands of the corresponding limits are uniformly bounded in $\bar{t}$. Equation (S30) follows from the application of Lemma S3 and the BCT to (A6) for $\Psi = \Psi_{t,\bar{t}}$. Equation (S31) follows from the application of the property that the sequence of unique solutions $x_{\bar{t}}$ to the sequence of equations $\Upsilon_{\bar{t}}(x_{\bar{t}}) = 1$ converges to the unique solution x of the limit equation $\lim_{\bar{t}\to\infty} \Upsilon_{\bar{t}}(x) = 1$ when $\Upsilon_{\bar{t}}$ is a strictly increasing function and $\{\Upsilon_{\bar{t}}\}$ converges to a strictly increasing function. This property applies to $x_{\bar{t}} = \nu_{t,\bar{t}}$ and $x = \nu^\beta$ because (a) $\nu_{t,\bar{t}}$ and ν^β are the unique solutions in w to 1 equal to the last line of the right hand side of (A6) for $\Psi = \Psi_{t,\bar{t}}$ and $\Psi = \Psi_\infty$, respectively; (b) the last line of the right hand side of (A6) is strictly increasing in w; and (c) the last line of the right hand side of (A6) evaluated at $\Psi = \Psi_{t,\bar{t}}$ converges to the last line of the right hand side of (A6) evaluated at $\Psi = \Psi_\infty$ by application of the BCT to the two integrals and Lemma S3. ■

S4 A Model of Financial Constraints

In this section, we endogenize the function π in a model in which moral hazard and limited liability restrict the set of acceptable bids. In this model, the firm who wins the procurement contract can divert some funds at the cost of jeopardizing the success of the procurement contract. The main implication is that the minimum acceptable bid for a firm with working capital w is given by an endogenous function π which under natural assumptions is strictly decreasing.

We endogenize the set of acceptable bids in the model[57] of Section 4 by assuming that a bid b is acceptable if and only if the firm has incentives to comply with the procurement contract in case of winning. We begin by formalizing this incentive compatibility constraint, and later we discuss institutional frameworks that enforce it.

Suppose the same game tree as in Section 4 with an additional stage each period after a firm wins the procurement contract and before the firm complies with the contract. To

[57]It is straightforward how to adapt this variation to the static model of Section 3.

describe this new stage, we use a differentiable function $\alpha : [0,\infty)^2 \to [0,\infty)$. As before, the total funds of the firm that wins the procurement contract are equal to its working capital w. We assume that either the working capital is sufficient to pay for the cost c, i.e. $w \geq c$, or the firm gets a loan $d = c - w$ at zero interest rate.[58] In both cases, the firm can choose between complying with the procurement contract and the loan, if any, or defaulting.[59] If the firm complies with both, it starts next period with cash equal to $w+d$, minus the production cost c, plus the procurement price b, minus the loan repayment d and plus the exogenous cash flow $\underline{m}$, i.e.[60]

$$w + b - c + \underline{m}. \tag{S32}$$

If the same firm defaults, it starts next period with cash equal to $\alpha(d, w) \in [0, d+w]$ plus the exogenous cash flow $\underline{m}$, i.e.

$$\alpha(d, w) + \underline{m}, \tag{S33}$$

Thus, $\alpha(d, w)$ denotes the funds of the firm that cannot be expropriated after default.

This model is realistic. For instance, it is a common practice that an entrepreneur who participates in a procurement contest uses a limited liability company (LLC) that can be liquidated in case of default. The entrepreneur could divert the funds from the LLC to its personal account before defaulting. $\alpha(d, w)$ represents the diverted funds that cannot be expropriated even after litigation and $w + d - \alpha(d, w)$ represents the funds that the entrepreneur cannot keep after default because they are either used to pay a compensation to the sponsor, spent on litigation costs or recouped by the lender. In this case, the limited liability status of the LLC implies that neither the lender nor the sponsor can seize any future revenue of the entrepreneur. Finally, default may not restrict the entrepreneur's future ability to borrow if there are other lenders who are willing to lend to the entrepreneur. Formally, this would happen in a model in which firms and lenders are matched only once and lenders do not observe the outcome of past matches.

The comparison of (S32) and (S33) means that a firm that borrows $d = \max\{c - w, 0\}$

[58] A zero interest rate is consistent with a competitive banking sector, no discounting between the moment the money is transferred to the firm and when it pays back and the fact that acceptable bids are risk free.

[59] For simplicity, we do not allow for default only in the loan or only in the procurement contract.

[60] Formally, we identify the case $w \geq c$ and no loan with the case of borrowing $d = 0$.

prefers to comply rather than default if and only if:

$$w + b - c \geq \alpha(\max\{c - w, 0\}, w). \tag{S34}$$

Thus, and no proof is necessary:

Proposition S4. If the firm's continuation value W is increasing in its own cash, only bids greater than $\alpha(\max\{c-w,0\},w)+c-w$ are acceptable for a firm with working capital $w \in [0,\infty)$. This implies that the set of acceptable bids is characterized by a function $\pi(w) = \alpha(\max\{c - w, 0\}, w) - w$.

The endogenous function π is strictly decreasing under natural assumptions. For instance, the derivative of π at[61] $w < c$ is negative if substituting one unit of working capital for one unit of debt increases the amount of non-expropriable funds by no more than one unit. Furthermore, one can find meaningful economic conditions under which the linear case, $\pi(w) = \theta - w$, arises. For instance, if $\frac{\partial\alpha(d,w)}{\partial d} = \frac{\partial\alpha(d,w)}{\partial w}$, for $d = c - w$, i.e. if the amount of non-expropriable funds does not change when debt is substituted by working capital.

In reality, the incentive compatibility constraint that we analyse above is usually enforced by different institutional frameworks. A natural example in procurement is the requirement of a surety bond. Indeed, the role of sureties[62] is to guarantee that the firm will comply with the procurement contract. An alternative explanation for the particular case $w < c$ is that the sponsor requires proof of the availability of the required external financing and banks are happy to provide it only if they do not expect the firm to default.

[61] The relevant case is $w < c$ because of two reasons. First, $\theta \leq c$ since θ solves $\pi(\theta) = \alpha(\max\{c - w, 0\}, \theta) - \theta = 0$ and $\pi(c) = \alpha((0, c) - c \leq 0$ because $\alpha(0, w) \leq w$. Second, the analysis in Section 4 only requires characterizing π in the domain $[0, \theta)$, since firms have no incentive to carry more working capital than θ. To see why, recall that a firm with working capital w bids $b^*(w|m, m')$ and note that $w > \theta$ implies that $\pi(w) < 0$, and hence $b^*(w|m, m') < c$. If continuation values are increasing in the firm's cash, a firm does not have incentives to bid below c if the rival does not do it, as in our proposed equilibrium. Bidding less than c does not increase the cases in which the firm wins but reduces the profits in case of winning.

[62] See Footnote 5.