Overconvergent Modular Symbols over Number Fields

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Thesis

submitted to the University of Warwick
for the degree of

Doctor of Philosophy

Warwick Mathematical Institute

June 2016

THE UNIVERSITY OF
WARWICK

To my Grandpa, John Bailey, for whom no mountain was too high

Abstract

The theory of overconvergent modular symbols, developed by Rob Pollack and Glenn Stevens, gives a beautiful and effective construction of the p-adic L-function of a modular form. In this thesis, we develop the theory of overconvergent modular symbols over a completely general number field and use it to construct p-adic L-functions for automorphic forms for GL_2. In particular, we prove control theorems that say that the natural specialisation map from overconvergent to classical modular symbols is an isomorphism on the small slope subspaces, hence attaching a unique overconvergent modular symbol to a small slope cuspidal automorphic eigenform Φ. From this overconvergent symbol we then obtain a p-adic distribution that interpolates certain critical L-values of Φ.

The text is comprised of two largely independent parts. In the first, we develop the theory in concrete detail over imaginary quadratic fields, and in the process present a constructive definition of the p-adic L-function in this setting. In the second, which was joint work with Daniel Barrera Salazar (Université de Montréal), we provide an analogous theory over general number fields, though not in the same explicit detail.

Contents

Acknowledgements

Firstly, I would like to thank my PhD supervisor David Loeffler for his superb guidance throughout my doctoral studies. In addition to teaching me a huge amount of interesting mathematics with unfailing enthusiasm, he has been a constant source of insight, advice and inspiration for the past four years, and I could not have asked for a better supervisor.

I would also like to thank my colleagues in the Warwick number theory group. I am grateful to John Cremona, Haluk Sengun, Vladimir Dokchitser and Marc Masdeu for many conversations that have directly influenced my PhD thesis, and to all of the other staff and PhD students that have continually made it such a pleasure to work at Warwick. Thanks also go to my PhD examiners, Samir Siksek and Tobias Berger, for the comments and corrections they provided for the final version.

The second half of this thesis is joint work with Daniel Barrera Salazar, with whom I have immensely enjoyed working. I would also like to thank Adrian Iovita and the Centre de Recherches Mathématiques for the generous financial support they provided to enable me to visit Montreal in March 2015 to work with Daniel.

On a personal level, I am indebted to my family and friends – including (but certainly not limited to) my flatmates and my hockey and cricket teams – for helping to make my time at Warwick so enjoyable. Special thanks, though, are reserved for Ally, for her encouragement, support and love, which has given me such great happiness during the past two years. Finally, I would like to thank my parents and Rachel; without their support during the times when I needed it most, this thesis would quite simply never have been written.

Declaration

Part I of this thesis is expositional, and none of the results contained in that section are new. The results in Part II appear in the paper [Wil15]. Part III of this thesis was completed in joint work with Daniel Barrera Salazar (Université de Montreal), and the results contained in that section appear in the paper [BSW16]. I declare that unless otherwise indicated, and to the best of my knowledge, the material contained in this thesis is my own original work.

This thesis is submitted to the University of Warwick for the degree of Doctor of Philosophy. No part of this thesis has been submitted towards any other degree.

Introduction

I.1. Background

I.1.1. Motivation

Many of the most famous problems in modern number theory relate to the theory of *L-functions*, which have been a powerful tool in the field for almost two centuries. In particular, the 'special' (or 'critical') values of L-functions have been shown or conjectured to contain important arithmetic information in a huge variety of situations. As a prominent example, the Birch and Swinnerton-Dyer conjecture predicts that important arithmetic data attached to an elliptic curve – such as its rank, the size of its Tate-Shafarevich group, and the order of its torsion group – are related to the value of its L-function at $s = 1$. Work towards this problem, or generalisations such as the Bloch-Kato conjecture or Beilinson's conjecture, could be considered to be the backbone of much current research in number theory.

In recent years, much of the study of L-functions has come through working with *p-adic L-functions*. A p-adic L-function is (loosely) a p-adic analytic object that interpolates the special values of a classical L-function. Since Kubota and Leopoldt's seminal paper [KL64], in which they constructed the first example of such an object, the p-adic Riemann zeta function, p-adic L-functions have been constructed for a wide variety of arithmetic objects, including Dirichlet characters, elliptic curves and modular forms. Where they exist, they have had important consequences; perhaps most strikingly, in [MTT86] Mazur, Tate and Teitelbaum formulated a p-adic analogue of the classical Birch and Swinnerton-Dyer conjecture that has actually been proved in a large number of cases. It is conjecturally equivalent to the classical formulation. Given their applications, it is evidently desirable to construct p-adic L-functions in wide generality. In this thesis, we construct p-adic L-functions for a large class of *(cohomological) automorphic forms for* GL_2 *over number fields.*

Under a wide-ranging series of conjectures known as the *Langlands program*, the theory of automorphic forms provides an attempt at a 'unifying theory' of mathematics. The program can be vaguely summarised as saying that any reasonably 'nice' L-function should come from an automorphic form. A celebrated result in this direction is the *modularity theorem* (or *Shimura–*

Taniyama conjecture), which says that for every elliptic curve $E/\mathbb{Q}$, there is a modular form f_E such that the L-functions of E and f_E are equal. This formed the major component of Wiles' famous proof of Fermat's last theorem.

Modular forms – or automorphic forms for GL_2 over $\mathbb{Q}$ – are arguably the most studied examples of automorphic forms. The first constructions of p-adic L-functions for modular forms were given by Mazur and Swinnerton-Dyer in [MSD74], and have since been followed by a number of other constructions. In particular, in [PS11], Pollack and Stevens gave an alternative construction using the theory of *overconvergent modular symbols*. Until recently, however, p-adic L-functions of automorphic forms for GL_2 over more general number fields had been constructed only in isolated cases. For the most general results previously known, see [Har87a], where such p-adic L-functions are constructed for weight 2 (also known as parallel weight 0) forms that are ordinary at p.

Pollack and Stevens' construction of p-adic L-functions for *small slope* classical modular forms is both beautiful and computationally effective. In this thesis, we generalise their method to construct p-adic L-functions for small slope automorphic forms over a general number field. This was essentially done in three separate papers; the totally real case was completed by Barrera in [BS13], the imaginary quadratic case by the author in [Wil15], and the completely general case in a joint paper between the two in [BSW16]. This thesis contains the accounts given in the second and third of these papers.

I.1.2. An introduction to p-adic L-functions

Since the main goal of this thesis is the construction of p-adic L-functions of automorphic forms, we start by giving a short introduction to the theory. First, consider the case of classical (complex) L-functions. As an example, let $f = \sum_{n\geq 1} a_n q^n \in S_k(\Gamma_0(N))$ be a modular form, with associated L-function

$$L(f,s) := \sum_{n\geq 1} a_n n^{-s}, \qquad s \in \mathbb{C}.$$

For any Hecke character[1] φ, we can define the *twist of* $L(f,s)$ *by* φ to be

$$L(f,\varphi,s) := \sum_{n\geq 1} a_n \varphi(n) n^{-s}, \qquad s \in \mathbb{C}.$$

[1] A rational *Hecke character* is a continuous character $\varphi : \mathbb{Q}^\times\backslash\mathbb{A}_{\mathbb{Q}}^\times \to \mathbb{C}^\times$, where $\mathbb{A}_{\mathbb{Q}}$ is the adele ring of $\mathbb{Q}$. Every rational Hecke character is of the form $\chi|\cdot|^s$, for $s \in \mathbb{C}$, where χ is a finite order character (giving rise to a Dirichlet character).

We can reformulate the above in a nicer way. In his thesis, Tate realised that if we define a function on Hecke characters by

$$\begin{aligned} L(f,*) : \{\text{rational Hecke characters}\} &\longrightarrow \mathbb{C}, \\ \varphi &\longmapsto L(f,\varphi,1), \end{aligned}$$

then we obtain an object that packages the data of the L-function and *all* of its twists into a single complex-valued function.

In this formulation, a complex L-function is a $\mathbb{C}$-valued function on complex Hecke characters. A *p-adic* L-function is a $\mathbb{C}_p$-valued function on 'p-adic Hecke characters'.

We make this more precise. Let $\varphi = \chi|\cdot|^s$ be a Hecke character, where χ has finite order. Suppose that

(i) φ is arithmetic, that is, $s = n \in \mathbb{Z}$, and

(ii) the conductor of χ is a power of p.

Then φ naturally gives rise to a character

$$\begin{aligned} \varphi_{p-\mathrm{fin}} : \mathbb{Z}_p^\times &\longrightarrow \mathbb{C}_p, \\ z &\longmapsto \chi_p(z)z^n, \end{aligned}$$

where χ_p is the restriction of χ to $\mathbb{Z}_p^\times$. The p-adic L-function should then be a function

$$L_p(f,*) : \{\text{analytic functions } \mathbb{Z}_p^\times \to \mathbb{C}_p\} \longrightarrow \mathbb{C}_p$$

satisfying the property that for φ as above, with $0 \leq n \leq k-2$, we have

$$L_p(f,\varphi_{p-\mathrm{fin}}) = (*)L(f,\varphi)$$

for some explicit factor $(*)$. [2]

Denote the space of (locally) analytic functions $\mathbb{Z}_p^\times \to \mathbb{C}_p$ by $\mathcal{A}(\mathbb{Z}_p^\times)$. We have seen that the p-adic L-function should be an element of the dual space

$$\mathcal{D}(\mathbb{Z}_p^\times) := \mathrm{Hom}_{\mathrm{cts}}(\mathcal{A}(\mathbb{Z}_p^\times), \mathbb{C}_p).$$

Elements of $\mathcal{D}(\mathbb{Z}_p^\times)$ are called *distributions*, and will be studied at length later in this thesis.

[2] Note that $L(f,\varphi)$ is a complex number, whilst $L_p(f,\varphi_{p-\mathrm{fin}})$ is p-adic. To make this comparison work, the factor $(*)$ will contain a *complex period* $\Omega_f^\pm$ such that $L(f,\varphi)/\Omega_f^\pm$ is algebraic, where the sign depends on φ.

Since we are interested in constructing p-adic L-functions over a general number field F, rather than just over $\mathbb{Q}$, it is pertinent to discuss how the theory generalises to this case. In particular, rather than studying distributions on $\mathbb{Z}_p^\times$, we study distributions on the narrow ray class group

$$\mathrm{Cl}_F^+(p^\infty) := F^\times\backslash\mathbb{A}_F^\times/F_\infty^+U(p^\infty),$$

where F_∞^+ is the set of totally positive infinite ideles and $U(p^\infty)$ is the set of finite ideles whose components at primes above p are all 1. To see why this is a natural concept, let φ be a Hecke character of p-power conductor. We want to write down a 'p-adic' analogue of φ. By 'moving' the infinite order part from the archimedean places to the places at p, we can write down a p-adic character

$$\varphi_{p-\mathrm{fin}} : \mathbb{A}_F^\times \longrightarrow \mathbb{C}_p^\times$$

in such a way that $\varphi_{p-\mathrm{fin}}$ is invariant under $F^\times F_\infty^+U(p^\infty)$. By the invariance property, this function descends to the space $\mathrm{Cl}_F^+(p^\infty)$. The p-adic L-function should then be a distribution L_p on $\mathrm{Cl}_F^+(p^\infty)$ such that

$$L_p(\varphi_{p-\mathrm{fin}}) = (*)L(\Phi,\varphi),$$

for an explicit factor $(*)$. The definition of $\varphi_{p-\mathrm{fin}}$ is covered in detail in Chapter 1.3.2.

I.1.3. The Pollack–Stevens construction

The following is a summary of the Pollack–Stevens construction of p-adic L-functions of classical modular forms. For a more detailed exposition, see [PS11].

Modular symbols

Almost every known method for constructing p-adic L-functions of modular forms uses *modular symbols*. These are algebraic objects attached to automorphic forms that retain data about the action of the Hecke operators. Since automorphic forms are inherently *analytic* objects, and modular symbols are purely algebraic, they are often much easier to study; we have 'discarded' much of the analytic information. Because of this, they are very powerful computational tools; as an example, Cremona's tables of elliptic curves at the *'L-functions and modular forms database'* (or *LMFDB*) are compiled using a modular symbol algorithm.

We give a brief account of the theory over $\mathbb{Q}$ in weight 2. Let $f \in S_2(\Gamma_0(N))$ be a classical modular form, and consider the function

$$\phi_f : \{\text{paths between cusps}\} \longrightarrow \mathbb{C}$$

given by

$$\phi_f(\{r\}-\{s\}) = \int_r^s f(z)dz,$$

where $\{r\} - \{s\}$ denotes any path between the cusps $r, s \in \mathbb{P}^1(\mathbb{Q})$ in the upper half-plane. We write $\Delta_0 := \mathrm{Div}^0(\mathbb{P}^1(\mathbb{Q}))$ for the free abelian group generated by such paths, and extend ϕ_f to Δ_0 linearly. Note then that we have

$$\phi_f(\gamma D) = \phi_f(D)$$

for any $D \in \Delta_0$ and $\gamma \in \Gamma_0(N)$, where γ acts on cusps by fractional linear transformations (and on Δ_0 by extending linearly). We write $\mathrm{Symb}_{\Gamma_0(N)}(\mathbb{C})$ for the subspace of $\mathrm{Hom}(\Delta_0, \mathbb{C})$ satisfying this invariance property. The remarkable thing now is that not only is this space finite dimensional, but the *Eichler-Shimura isomorphism* says that

$$\mathrm{Symb}_{\Gamma_0(N)}(\mathbb{C}) \cong M_2(\Gamma_0(N)) \oplus S_2(\Gamma_0(N)),$$

so that the study of spaces of modular forms of weight 2 can essentially be reduced to the study of this algebraic symbol space. Even better, there is a natural Hecke action on $\mathrm{Symb}_{\Gamma_0(N)}(\mathbb{C})$, and the isomorphism is equivariant with respect to the Hecke action on both sides. Accordingly, all systems of Hecke eigenvalues in the space of weight 2 modular forms are captured in the symbol space. In particular, in the process of passing from a modular form to a modular symbol – discarding all of the analytic information attached to f – we have *retained* all of the data regarding the Hecke action.

Since the L-function of a modular form f is built out of its Hecke eigenvalues, and the modular symbol attached to f retains this data, it is perhaps not surprising that this symbol also contains data about its L-function. As a toy example, note that

$$\phi_f(\{0\} - \{\infty\}) = \int_0^\infty f(z)dz = 2\pi i L(f, 1),$$

so that ϕ_f sees the critical value of $L(f, s)$ at $s = 1$. Similar formulae exist for all twists of $L(f, s)$ by Dirichlet characters at $s = 1$.

For higher weight, there are slight adjustments to be made. Let $f \in S_{k+2}(\Gamma_0(N))$ be a modular form of weight $k + 2$. Then f naturally gives rise to a function ϕ_f in $\mathrm{Symb}_{\Gamma_0(N)}(V_k(\mathbb{C})) := \mathrm{Hom}_{\Gamma_0(N)}(\Delta_0, V_k(\mathbb{C}))$, where $V_k(\mathbb{C})$ is the space of polynomials over $\mathbb{C}$ of degree at most k.[3] Here, the action of $\Gamma_0(N)$ on $\mathrm{Hom}(\Delta_0, V_k(\mathbb{C}))$ is by

$$\gamma \cdot \phi(D) = \phi(\gamma D)|\gamma,$$

[3] Or, equivalently, $V_k(\mathbb{C})$ is defined as the space of homogeneous polynomials in two variables of degree k over $\mathbb{C}$.

for a suitable right action of $\Gamma_0(N)$ on $V_k(\mathbb{C})$. Again, we get a Hecke equivariant isomorphism

$$\mathrm{Symb}_{\Gamma_0(N)}(V_k(\mathbb{C})) \cong M_{k+2}(\Gamma_0(N)) \oplus S_{k+2}(\Gamma_0(N)).$$

As in the weight 2 case, there are explicit formulae linking the modular symbol ϕ_f with critical values of its L-function.

Overconvergent modular forms and symbols

In this thesis, we will be more interested in the *theoretical* uses of modular symbols and, in particular, with a wonderful idea of Glenn Stevens.

Stevens' theory of *overconvergent modular symbols* has its roots in the study of p-adic variation of modular forms. If $f \in S_k(\Gamma_0(p))$ is a normalised eigenform, then its q-expansion will have algebraic coefficients, and it is natural to ask whether there are congruences between f and other forms. In particular, one might ask:

Question: *Let $m \in \mathbb{N}$. Does there exist a normalised eigenform $g = \sum_n a_n(g)q^n \in S_{k'}(\Gamma_0(p))$, not equal to f, such that*

$$f(z) \equiv g(z) \pmod{p^m},$$

in the sense that $a_n(f) \equiv a_n(g) \pmod{p^m}$ for all n?

This question – and questions similar to it – led to Serre's theory of *p-adic modular forms.* A related question is to ask how *spaces* of modular forms vary as the weight varies p-adically. If one tries to vary spaces of classical modular forms p-adically, then one immediately hits a problem: the dimension of $M_{k+p^n}(\Gamma_0(p))$ is unbounded as n increases, so there is no way that these spaces can be used to 'p-adically approximate' the space $M_k(\Gamma_0(p))$. There have been several clever approaches to circumvent this problem.

Hida considered this question in the case where f is *ordinary*, that is, when the eigenvalue at p is a p-adic unit. His results were remarkable; he showed that the space $S_k^{\mathrm{ord}}(\Gamma_0(p))$ of ordinary cusp forms has dimension that depends only on the weight modulo $p-1$, and succeeded in varying these spaces p-adically. For more general forms, similar results were proved by Coleman. Instead of passing to a subspace, however, he instead passed to a much larger (indeed, infinite dimensional) space, that is, the space of *overconvergent modular forms.* He defined a space $M_k^{\dagger}(\Gamma_0(p))$, containing $M_k(\Gamma_0(p))$, and showed that these spaces also varied nicely p-adically.

The definition of overconvergent modular forms is inherently analytic, relying on rigid p-adic geometry. In [Ste94], Stevens managed to emulate the theory in the world of modular symbols in a purely algebraic manner. His key idea was to replace the coefficient space in the definition of classical modular symbols; instead of considering symbols taking values in polynomials, he considered symbols taking values in (infinite-dimensional) spaces of p-adic distributions. He was then able to show that the spaces of *overconvergent modular symbols* also varied nicely as the weight varied p-adically.

The theory of p-adic variation in number theory is vast and hugely interesting, and the comments above provide but the smallest scratch on its surface, included for motivation only. Whilst going into any further detail here is impractical, there have been huge amounts written on this beautiful topic; for survey articles, see [Maz12] (for the work of Hida) and [Buz04] (for the work of Coleman).

Control theorems

One of the most important aspects of Coleman's theory is his *control theorem*, which says that a 'small slope' overconvergent modular form is in fact classical. There is a natural Hecke action on the space $M_k^\dagger(\Gamma_0(p))$, and if f is an overconvergent eigenform, then the *slope* of f is $v_p(a_p)$, where a_p is the eigenvalue at p. Then in [Col96], in a result known as his *small slope classicality theorem*, Coleman proved:

Theorem (Coleman)**.** *Let $f \in M_k^\dagger(\Gamma_0(p))$ be an overconvergent eigenform of slope $< k-1$. Then f is classical, that is, $f \in M_k(\Gamma_0(p))$.*

Stevens proved the analogous result for modular symbols. Here, however, there is a fundamental difference; whilst the space of classical modular forms is a *subspace* of the space of overconvergent modular forms, the space of classical modular symbols is a *quotient* of the space of overconvergent modular symbols. Stevens' result was then:

Theorem (Stevens)**.** *Denote the space of overconvergent modular symbols by $\mathrm{Symb}_{\Gamma_0(p)}(\mathcal{D}_k)$. There is a natural Hecke-equivariant and surjective specialisation map*

$$\rho_k : \mathrm{Symb}_{\Gamma_0(p)}(\mathcal{D}_k) \longrightarrow \mathrm{Symb}_{\Gamma_0(p)}(V_k)$$

which becomes an isomorphism upon restriction to the slope $< k+1$ subspaces.

Note here the 'shift by 2'; since a weight $k+2$ modular form corresponds to an element of $\mathrm{Symb}_{\Gamma_0(p)}(V_k)$, the slope in this setting is 'small' if it is $< k+1$ rather than $< k-1$.

The construction of the p-adic L-function

Let $f \in S_{k+2}(\Gamma_0(p))$ be an eigenform with $v_p(a_p) < k+1$, where a_p is the eigenvalue at p. To f we can associate a modular symbol $\phi_f \in \mathrm{Symb}_{\Gamma_0(p)}(V_k(\mathbb{C}))$, and in fact, it's possible to renormalise so that the symbol takes values in the p-adic space $V_k(L)$, for L an extension of $\mathbb{Q}_p$. Then, by the control theorem, there is a unique small-slope overconvergent eigensymbol Ψ_f such that $\rho_k(\Psi_f) = \phi_f$.

We stated that the classical modular symbol ϕ_f 'sees' critical L-values of the modular form. In particular, there are formulae relating ϕ_f and the values $L(f,\varphi)$, for $\varphi = \chi|\cdot|^n$, where χ is a finite order character of p-power conductor and $0 \leq n \leq k$. Accordingly, the same information is carried by the overconvergent symbol Ψ_f. Indeed, Pollack and Stevens made the following observation in [PS11]:

Theorem (Pollack–Stevens)**.** *Define μ_p to be the distribution $\Psi_f(\{0\} - \{\infty\})|_{\mathbb{Z}_p^\times}$. Then μ_p satisfies the following interpolation property: let $\varphi = \chi|\cdot|^n$ be a Hecke character, where χ has p-power conductor and $0 \leq n \leq k$; then*

$$\mu_p(\varphi_{p-\mathrm{fin}}) = (*)L(f,\varphi),$$

for an explicit factor $(*)$. The distribution μ_p also satisfies a growth property making it uniquely determined by this interpolation condition. Thus μ_p is the p-adic L-function of f.

It is precisely this theorem that we generalise to automorphic forms over a general number field.

I.2. P-adic L-functions of Bianchi modular forms

The first major work in this thesis concentrates on generalising the results of Pollack and Stevens to the setting of automorphic forms over imaginary quadratic fields. In particular, we construct p-adic L-functions of *Bianchi modular forms*, that is, automorphic forms for GL_2 over imaginary quadratic fields.

I.2.1. Bianchi modular forms

Bianchi modular forms have been increasingly studied in recent years, and the literature regarding them is widespread; in particular, an account of the general theory over arbitrary number fields is given in André Weil's book [Wei71], whilst accounts in the imaginary quadratic case for weight 2 are given by John Cremona and two of his students in [Cre81], [CW94] and [Byg98]. There seem to be a number of different conventions in the various treatments of the theory, and as such, in Part I of the text, we give a largely self-contained introduction to Bianchi modular forms, drawing from the existing literature (and in particular from [Byg98] and [Gha99]) and comparing the various approaches whenever they differ. In the process, we fix the conventions and notation we'll use in the sequel.

The definitions that arise in the theory can seem unnatural to a new reader (indeed, they certainly seemed unnatural to the author at first sight), and it's rarely obvious why they should provide a suitable generalisation of the theory of classical modular forms. Throughout, every attempt is made to motivate each step, often in the case of weight 2 (also known as parallel weight (0,0)) Bianchi modular forms, where the literature is considerably more extensive, owing to the connection between weight 2 Bianchi modular forms for K and elliptic curves defined over K. A conscious effort has been made to write in the greatest possible clarity, even though this inevitably involves labouring certain points.

Let K be an imaginary quadratic field. Broadly speaking, a *Bianchi modular form over* K is a function

$$\Phi : \mathrm{GL}_2(\mathbb{A}_K) \longrightarrow V_{2k+2}(\mathbb{C})$$

that is left-invariant under $\mathrm{GL}_2(K)$, right-invariant under a compact level group $\Omega_1(\mathfrak{n}) \subset \mathrm{GL}_2(\mathcal{O}_K \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}})$ and transforms suitably under the standard irreducible representation $\rho : \mathrm{SU}_2(\mathbb{C}) \rightarrow \mathrm{GL}(V_{2k+2}(\mathbb{C}))$. We say Φ has *weight* (k,k) *and level* $\Omega_1(\mathfrak{n})$. Here $V_{2k+2}(\mathbb{C})$ is the space of homogeneous polynomials in two variables of degree $2k+2$. Such a form corresponds to a collection of automorphic functions $F^1, ..., F^h : \mathrm{GL}_2(\mathbb{C}) \rightarrow V_{2k+2}(\mathbb{C})$, where h is the class number of K. We can descend further to functions $\mathcal{F}^1, ..., \mathcal{F}^h$ on the *upper half-space* $\mathcal{H}_3 := \mathbb{C} \times \mathbb{R}_{>0}$, the analogue of the upper half-plane in this setting. Such a Bianchi modular form has a Fourier expansion, and the Fourier coefficients can be built into an L-function that converges absolutely on a right half-plane. All of this is covered in extensive detail, and with motivation, in Part I.

I.2.2. Bianchi modular symbols

Part II of this thesis is dedicated to the study of (classical and overconvergent) Bianchi modular symbols. Let $\Gamma \subset \mathrm{SL}_2(K)$ be an arithmetic subgroup, and define $V_{k,k}(\mathbb{C}) := V_k(\mathbb{C}) \otimes_{\mathbb{C}} V_k(\mathbb{C})$; then the space of Bianchi modular symbols is the space

$$\mathrm{Symb}_\Gamma(V_{k,k}(\mathbb{C})) := \mathrm{Hom}_\Gamma(\mathrm{Div}^0(\mathbb{P}^1(K)), V_{k,k}(\mathbb{C})^*),$$

analogously to the rational case. Here the superscript $()^*$ denotes the dual space[4]. To a cuspidal Bianchi modular form Φ of weight (k,k) and level $\Omega_1(\mathfrak{n})$, we associate a collection $\mathcal{F}^1, ..., \mathcal{F}^h$ of functions on $\mathcal{H}_3$, as above, each satisfying an automorphy condition for some discrete subgroup $\Gamma_1^i(\mathfrak{n})$ of $\mathrm{SL}_2(K)$, and to each of these $\mathcal{F}^i$, we associate a classical Bianchi modular symbol $\phi_{\mathcal{F}^i} \in \mathrm{Symb}_{\Gamma_1^i(\mathfrak{n})}(V_{k,k}(\mathbb{C})^*)$. The collection $(\phi_{\mathcal{F}^1}, ..., \phi_{\mathcal{F}^h})$ is the *modular symbol attached to* Φ. In fact, we can renormalise this symbol so that it is defined over a sufficiently large number field, and in this way, can define a *p-adic* modular symbol attached to Φ. This construction is

[4] We use the dual space as it is, in many ways, more natural. There is in fact an isomorphism $V_{k,k}(\mathbb{C}) \cong V_{k,k}(\mathbb{C})^*$ that is equivariant with respect to the various actions we consider, so it doesn't really matter which we use.

outlined in Chapter 5.

By studying the L-function of Φ, we can exhibit an explicit link between values of this symbol and critical values of the part of the L-function corresponding to $\mathcal{F}^i$. The calculations involved here are significantly more involved than in the rational case; the definition of the modular symbol is more complicated, and the Fourier expansion is more difficult to work with. The calculations involved here comprise Chapter 6.

We then move onto p-adic methods. We define the space of overconvergent Bianchi modular symbols to be the space of modular symbols taking values in some p-adic distribution space; precisely, we fix a finite extension $L/\mathbb{Q}_p$, and denoting by $\mathbb{A}_k(L)$ the space of rigid analytic functions on the unit disc defined over L, our distribution space is $\mathbb{D}_{k,k}(L) := \mathrm{Hom}(\mathbb{A}_k(L)\hat{\otimes}_L\mathbb{A}_k(L), L)$. We have a specialisation map from overconvergent to classical modular symbols by dualising the inclusion $V_k(L)\otimes_L V_k(L)\hookrightarrow \mathbb{A}_k(L)\hat{\otimes}_L\mathbb{A}_k(L)$, much like in the rational case. We prove the following analogue of Stevens' control theorem; in the case of p inert, it is proved in Corollary 8.2.1, in the case of p ramified it is proved in Corollary 8.2.1 combined with Lemma 9.1.9, and in the case p split it is proved in Theorem 9.1.10.

Theorem. *Let p be a rational prime with $p\mathcal{O}_K = \prod_{\mathfrak{p}|p}\mathfrak{p}^{e_\mathfrak{p}}$. For each prime $\mathfrak{p}|p$, let $\lambda_\mathfrak{p}\in L$. Then, when $v_p(\lambda_\mathfrak{p}) < (k+1)/e_\mathfrak{p}$ for all $\mathfrak{p}|p$, the restriction of the map*

$$\rho : \bigoplus_{i=1}^{h}\mathrm{Symb}_{\Gamma_1^i(\mathfrak{n})}(\mathbb{D}_{k,k}(L))^{\{U_\mathfrak{p}=\lambda_\mathfrak{p}:\mathfrak{p}|p\}} \longrightarrow \bigoplus_{i=1}^{h}\mathrm{Symb}_{\Gamma_1^i(\mathfrak{n})}(V_{k,k}(L)^*)^{\{U_\mathfrak{p}=\lambda_\mathfrak{p}:\mathfrak{p}|p\}}$$

to the simultaneous $\lambda_\mathfrak{p}$-eigenspaces of the $U_\mathfrak{p}$-operators is an isomorphism.

The proof draws from work of Matthew Greenberg in [Gre07], in that we define a series of *finite approximation modules*, and lift compatibly through this system to obtain a overconvergent symbol from a classical one. The proof is constructive.

It is worth remarking that whilst in the rational case, the control theorem gives an analogue of Coleman's small slope classicality theorem, no such theory of 'overconvergent Bianchi modular forms' yet exists. Indeed, the definition of overconvergent modular forms over the rationals relies inherently on geometry, making the conventional construction impossible over an imaginary quadratic field. In this setting, the locally symmetric spaces involved share little of the desirable properties of modular curves over $\mathbb{Q}$ (for example, there is neither a complex structure nor the structure of an algebraic variety on such a space). By passing to modular symbols, and 'forgetting' the analytic structure, it *is* possible to see overconvergent objects in this setting, since we are working purely algebraically.

I.2.3. Distributions and interpolation

In the remainder of Part II, the values of an overconvergent eigenlift $\Psi_{\mathcal{F}^i}$ are studied; namely, we prove that such a symbol takes values in some space of locally analytic distributions, and that it is *admissible* (or *tempered*). We also exploit the link between classical modular symbols and critical L-values to prove an interpolation property. As a formal corollary, we see that we have the following way of constructing the p-adic L-function of Φ (see Theorem 10.4.1):

Theorem. *Suppose we are in the set-up of the control theorem, and let Φ be a small slope cuspidal Bianchi eigenform of weight (k,k) and level $\Omega_1(\mathfrak{n})$. Then to Φ we can associate a small slope eigensymbol $(\phi_1, ..., \phi_h)$ in a direct sum of symbol spaces, which we can lift uniquely to an overconvergent symbol $(\Psi_1, ..., \Psi_h)$ using the control theorem. Then there is a way of patching together the distributions $\Psi_i(\{0\} - \{\infty\})$ to a function μ_p on the ray class group $\mathrm{Cl}_K(p^\infty)$ such that μ_p is the p-adic L-function of Φ.*

Such a result is the natural analogue of the results of Pollack and Stevens in the rational case.

I.3. P-adic L-functions for GL_2

In Part III, in joint work with Daniel Barrera Salazar, we generalise this further to arbitrary number fields. There is a good theory of modular symbols in the more general setting, but this theory can't be made as explicit as in the rational and imaginary quadratic cases; indeed, these symbols live in higher compactly supported cohomology groups. A discussion motivating the general definition is given in Chapter 11.1.

I.3.1. Classical cohomology and L-values

We give a brief overview of the definition of modular symbols in general. Throughout Part III, we take Φ to be a cuspidal automorphic eigenform of weight λ and level $\Omega_1(\mathfrak{n})$ over a number field F, where λ and $\Omega_1(\mathfrak{n})$ are defined as in Chapter 2.3.1 and equation (2.3) respectively. We write $d = r_1 + 2r_2$ for the degree of F, where r_1 (resp. r_2) denotes the number of real (resp. complex) places of F. Let $q = r_1 + r_2$. The space of *modular symbols* of level $\Omega_1(\mathfrak{n})$ and weight λ is the compactly supported cohomology space $\mathrm{H}^q_c(Y_1(\mathfrak{n}), \mathcal{V}_\lambda)$, where $Y_1(\mathfrak{n})$ is the locally symmetric space associated to $\Omega_1(\mathfrak{n})$ and $\mathcal{V}_\lambda$ is a suitable sheaf of polynomials on $Y_1(\mathfrak{n})$ depending on the weight. The *Eichler-Shimura isomorphism* gives a Hecke-equivariant isomorphism between this cohomology group and the direct sum of certain spaces of automorphic forms, mirroring the analogous theorems over $\mathbb{Q}$ and imaginary quadratic fields. All of this is discussed in Chapter 11.

Since we cannot work in the same explicit setting as before, we are forced to use more abstract cohomological methods to study these symbols. In particular, we use the theory of *automorphic* *cycles* as developed by Dimitrov. Using *evaluation maps*, which were described initially by Dimitrov for totally real fields in [Dim13] and which we have generalised to the case of arbitrary number fields, we relate the modular symbol attached to an automorphic form to critical values of its L-function. We show that these results have an algebraic analogue; that is, we can pass to a cohomology class with coefficients in a sufficiently large number field, and then relate this to the algebraic part of the critical L-values of Φ. In particular, via a long and technical argument, we prove the following result (see Theorem 12.2.7 in the paper for a more precise formulation):

Theorem. *For each Hecke character φ of F with infinity type in a range depending on λ, there is a map*

$$\mathrm{Ev}_\varphi : \mathrm{H}^q_c(Y_1(\mathfrak{n}), \mathcal{V}_\lambda(A)) \longrightarrow A$$

such that if Φ is a cuspidal automorphic form of weight λ, with associated A-valued modular symbol ϕ_A (for A either $\mathbb{C}$ or a sufficiently large number field), then

$$\mathrm{Ev}_\varphi(\phi_A) = (*)L(\Phi, \varphi),$$

where $L(\Phi, \cdot)$ is the L-function attached to Φ and $(*)$ is an explicit factor.

I.3.2. Overconvergent cohomology

All of these results are rather classical in nature, and make explicit results that are, in theory, 'well-known' (although the authors could not find the results in the generality they require in the existing literature). At this point, we start using new p-adic methods. Henceforth, assume that $(p)|\mathfrak{n}$, and take L to be a (sufficiently large) finite extension of $\mathbb{Q}_p$. We define the space of *overconvergent modular symbols of level* $\Omega_1(\mathfrak{n})$ *and weight* λ to be the compactly supported cohomology of $Y_1(\mathfrak{n})$ with coefficients in an (infinite-dimensional) space of p-adic distributions equipped with an action of $\Omega_1(\mathfrak{n})$ that depends on λ.

For each prime $\mathfrak{p}|p$ in F, we have the Hecke operator $U_\mathfrak{p}$ at $\mathfrak{p}$ on both automorphic forms and (classical and overconvergent) modular symbols, induced from the action of the matrix $\left(\begin{smallmatrix} 1 & 0 \\ 0 & \pi_\mathfrak{p} \end{smallmatrix}\right)$, where $\pi_\mathfrak{p} \in L$ is a fixed uniformiser at $\mathfrak{p}$.

In Chapter 13.2, we prove that for any $h_\mathfrak{p} \in \mathbb{Q}$, the space of overconvergent modular symbols admits a slope $\leq h_\mathfrak{p}$ decomposition with respect to the $U_\mathfrak{p}$ operator.

Definition. Let M be an L-vector space with an action of the Hecke operators $U_\mathfrak{p}$ for $\mathfrak{p}|p$. Where it exists, we denote the slope $\leq h_\mathfrak{p}$ subspace with respect to the $U_\mathfrak{p}$ operator by $M^{\leq h_\mathfrak{p}, U_\mathfrak{p}}$. If h $:= (h_\mathfrak{p})_{\mathfrak{p}|p}$ is a collection of rationals indexed by the primes above p, we define

$$M^{\leq \mathbf{h}} := \bigcap_{\mathfrak{p}|p} M^{\leq h_\mathfrak{p}, U_\mathfrak{p}}$$

to be the slope $\leq \mathbf{h}$-subspace at p.

Definition. Let $p\mathcal{O}_F = \prod \mathfrak{p}^{e_\mathfrak{p}}$ be the decomposition of p in F, and for each $\mathfrak{p}|p$ let $h_\mathfrak{p} \in \mathbb{Q}$. Let Σ be the set of all infinite places of F, and write the weight λ as $\lambda = ((k_\sigma), (v_\sigma)) \in \mathbb{Z}[\Sigma]^2$. For each $\sigma \in \Sigma$, there is a unique prime $\mathfrak{p}(\sigma)|p$ corresponding to σ, and to denote this we write $\sigma \sim \mathfrak{p}$. Define $k_\mathfrak{p}^0 := \min\{k_\sigma : \sigma \sim \mathfrak{p}\}$ and $\omega_\mathfrak{p}(\lambda) := \sum_{\sigma\sim\mathfrak{p}} v_\sigma$.

We say that the slope $\mathbf{h} := (h_\mathfrak{p})_{\mathfrak{p}|p}$ is *small* if $h_\mathfrak{p} < (k_\mathfrak{p}^0 + \omega_\mathfrak{p}(\lambda) + 1)/e_\mathfrak{p}$ for each $\mathfrak{p}|p$.

For each fixed weight, there is a surjective Hecke-equivariant specialisation map ρ from the space of overconvergent modular symbols to the space of classical modular symbols. In Chapter 13.3, we prove the following *control theorem*:

Theorem. *Let $\mathbf{h} \in \mathbb{Q}^{\{\mathfrak{p}|p\}}$ be a small slope. Then the restriction of the specialisation map ρ to the slope $\leq \mathbf{h}$ subspaces of the spaces of modular symbols is an isomorphism.*

In particular, to a small slope cuspidal eigenform – that is, an eigenform whose associated modular symbol lives in some small-slope subspace of the space of classical modular symbols – one can attach a *unique* small-slope overconvergent eigenlift of its associated modular symbol.

I.3.3. Overconvergent evaluation maps and p-adic L-functions

Let Ψ be an overconvergent eigensymbol. We can use a slightly different version of the evaluation maps from previously to construct a distribution μ_Ψ on the narrow ray class group $\mathrm{Cl}_F^+(p^\infty)$ attached to Ψ, closely following the work of the first author in [BS13]. We prove that the distribution we define is completely canonical. Via compatibility between classical and overconvergent evaluation maps, this distribution then interpolates the critical values of the L-function of Φ, and we hence define the p-adic L-function to be this distribution. To summarise, the main result of Part III is:

Theorem. *Let Φ be a small slope cuspidal eigenform over F. Let ϕ_Φ be the (p-adic) classical modular symbol attached to Φ, and let Ψ_Φ be its (unique) small-slope overconvergent eigenlift. Let μ_Φ be the distribution on $\mathrm{Cl}_F^+(p^\infty)$ attached to Ψ_Φ.*

If φ is a critical Hecke character, then we can define a canonical locally algebraic character $\varphi_{p-\mathrm{fin}}$ on $\mathrm{Cl}_F^+(p^\infty)$ associated to φ. Then

$$\mu_\Phi(\varphi_{p-\mathrm{fin}}) = (*)L(\Phi, \varphi),$$

where $(*)$ is an explicit factor.

Definition. We define the *p-adic L-function of* Φ to be the distribution μ_Φ on $\mathrm{Cl}_F^+(p^\infty)$.

For a precise notion of the range of interpolation and the factor $(*)$, see Theorem 14.3.1.

In the case that F is totally real or imaginary quadratic, given slightly tighter conditions on the slope one can prove that the distribution we obtain is *admissible*, that is, it satisfies a growth property that then determines the distribution uniquely. In the general situation, it is rather more difficult to define the correct notion of admissibility; we discuss this further in Chapter 14.4. We instead settle for proving that our construction is canonical, so that it is indeed reasonable to *define* the p-adic L-function in this manner.

I.4. Structure of the text

The text is split into three major parts. The first is entirely expositional, developing the theory of automorphic forms in the setting we need, and may be entirely skipped by the reader who is comfortable with the theory of automorphic forms for GL_1 and GL_2. In Chapter 1, we focus on Hecke characters, or automorphic forms for GL_1. In Chapter 2, we present motivation and definitions for automorphy conditions for GL_2. In Chapter 3, we complete the definition of automorphic forms by discussing harmonic differential forms and boundedness conditions. Finally, we conclude Part I with a discussion of L-functions in Chapter 4.

Parts II and III are largely independent, and may be read as such. Part II, which contains the results of the paper [Wil15], focuses on the imaginary quadratic setting. In Chapter 5, we develop the theory of Bianchi modular symbols, whilst in Chapter 6 we prove the connection between modular symbols and L-values of automorphic forms. In Chapter 7, we define spaces of distributions and overconvergent Bianchi modular symbols, and then in Chapter 8, we prove the control theorem in this setting and examine overconvergent lifts. In Chapter 9, we refine these results in the case p splits in the imaginary quadratic field. Finally, we conclude Part II in Chapter 10 by using our results to construct the p-adic L-function of an automorphic form.

Part III contains the results of the paper [BSW16], written jointly with Daniel Barrera. In Chapter 11, we discuss the theory of classical modular symbols over number fields and how this generalises the rational and imaginary quadratic cases. In Chapter 12, we use automorphic cycles to define evaluation maps and prove an integral formula for the L-function of an automorphic form. In Chapter 13, we define overconvergent modular symbols, and prove the control theorem for cohomology. Finally, in Chapter 14, we show how to use overconvergent evaluations to define a canonical ray class distribution attached to a small slope automorphic form, and show that it interpolates critical values of its L-function. We define its p-adic L-function to be this distribution.

I.5. Comparison to relevant literature

There are a number of people who have worked on similar things in the recent past. In the Bianchi case, perhaps of most relevance is Mak Trifkovic, who in [Tri06] performed computations with overconvergent Bianchi modular symbols. He proved a lifting theorem in the case of weight 2 ordinary eigenforms over an imaginary quadratic field of class number 1, using similar explicit methods to [Gre07]. The lifting results in Part II are a significant generalisation of his theorem, though the author has not made any efforts to repeat the computational aspects of Trifkovic's work in this more general setting. Trifkovic's work highlights a wider range of applications for overconvergent modular symbols; indeed, they have been used in the efficient computation of *Stark–Heegner points* on elliptic curves. For further details on such computations, see, for example, the work of Darmon and Pollack in [DP06] or Guitart and Masdeu in [GM14].

Notation

Whilst we will usually introduce notation as it is required, this section is intended as an index for the main notation used throughout the text. In particular, we have used the following convention:

Convention: Throughout the text, we will use K to denote an imaginary quadratic field whilst F denotes an arbitrary number field. Since the first half of this thesis will concentrate on the former case and the second half the latter, and both halves are essentially self-contained, the author hopes that this will help the reader distinguish between the two halves more easily. Here, we give the definitions for general F.

Basic objects

Let p be a prime, and fix – once and for all – an embedding $\mathrm{inc}_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$. Let F be a number field of degree $d = r_1 + 2r_2$, where r_1 is the number of real embeddings and r_2 the number of pairs of complex embeddings of F. Write $q = r_1 + r_2$. We write Σ for the set of all infinite embeddings of F. Let $\Sigma(\mathbb{R})$ denote the set of real places of F and let $\Sigma(\mathbb{C})$ be the set containing a (henceforth fixed) choice of embedding from each pair of complex embeddings, so that

$$\Sigma = \Sigma(\mathbb{R}) \cup \Sigma(\mathbb{C}) \cup c\Sigma(\mathbb{C}),$$

where c denotes complex conjugation. When we want to make the field explicit in this notation, we will write Σ_F instead of Σ.

We write $\mathfrak{D}$ for the different of F and D for the discriminant of F. For each finite place v in F, fix (once and for all) a uniformiser π_v in the completion F_v.

Let $\mathbb{A}_F = F_\infty \times \mathbb{A}_F^f$ denote the adele ring of F, with infinite adeles $F_\infty \cong F \otimes_{\mathbb{Q}} \mathbb{R}$ and finite adeles $\mathbb{A}_F^f$. Let $\widehat{\mathcal{O}}_F \cong \widehat{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathcal{O}_F$ denote the integral (finite) adeles. Let $F_\infty^+ \cong \mathbb{R}_{>0}^{r_1} \times (\mathbb{C}^\times)^{r_2}$ be the connected component of the identity in $F_\infty^\times$.

Ray class groups

For an ideal $\mathfrak{f} \subset \mathcal{O}_F$, we define $U(\mathfrak{f})$ to be the set of elements of $\widehat{\mathcal{O}_F}$ that are congruent to $1 \pmod{\mathfrak{f}}$, and denote the narrow ray class group modulo $\mathfrak{f}$ by

$$\mathrm{Cl}_F^+(\mathfrak{f}) := F^\times \backslash \mathbb{A}_F^\times / U(\mathfrak{f}) F_\infty^+.$$

This is equivalent to the usual (ideal-theoretic) formulation. When $\mathfrak{f} = \mathcal{O}_F$, we write simply Cl_F^+ (the narrow class group of F). Write h for the narrow class number of F. Fix a level $\mathfrak{n} \subset \mathcal{O}_F$ with $(p)|\mathfrak{n}$, and choose fixed (ideal) representatives $I_1, ..., I_h$ of the narrow class group, coprime to $\mathfrak{n}$, represented by ideles $a_1, ..., a_h$, with $(a_i)_v = 1$ for all $v|\mathfrak{n}\infty$. (Note that in Part II, where we treat the imaginary quadratic case, the narrow class group is nothing other than the usual class group).

Rings of polynomials

Let R be a ring and $k \geq 0$ a non-negative integer. We write $V_k(R)$ for the space of homogeneous polynomials of degree k in two variables over R. If $\mathbf{k} = (k_i)_i \in \mathbb{Z}^d$, write

$$V_{\mathbf{k}}(R) := \bigotimes_{i=1}^{d} V_{k_i}.$$

We equip this with a left action of $\mathrm{GL}_2(R)^d$, induced from the action of $\mathrm{GL}_2(R)$ on a single component by

$$\gamma \cdot P(X,Y) = P(dX + bY, cX + aY), \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

For a weight $\lambda = (\mathbf{k}, \mathbf{v}) \in \mathbb{Z}^d \times \mathbb{Z}^d$, we will write $V_\lambda(R)$ for the space $V_{\mathbf{k}}(R)$ with the action of $(\gamma_i)_i \in \mathrm{GL}_2(R)^d$ twisted by $\det^{\mathbf{v}} = \prod_{i=1}^d \det(\gamma_i)^{v_i}$. In Part II, we will only consider weights of the form $\lambda = [(k,k),(0,0)]$, in which case we will write $V_{k,k}(R)$.

Automorphic forms and modular symbols

We will typically use Φ to denote an automorphic form for GL_2, whose level will always be denoted by $\Omega_1(\mathfrak{n})$, as in equation (2.3). We will use ϕ to denote a classical modular symbol and Ψ an overconvergent modular symbol, with appropriate subscripts where necessary.

Part I

Automorphic Forms

In this (entirely expository) section, we develop the standard theory of automorphic forms and describe some of their properties. We build the theory from the ground up, although, to keep this thesis at a manageable length, we assume as prerequisites that the reader is familiar with both the theory of classical modular forms and the theory of adeles and ideles. After an introductory chapter on Hecke characters and their properties, we give – with motivation – the definition of an adelic automorphy condition of GL_2 *over general number fields. In Chapter 3, we discuss harmonic differential forms, allowing us to complete the definition of automorphic forms in the generality we require. Finally, we discuss the theory of Hecke operators and Fourier expansions, using them to define L-functions of automorphic forms.*

None of this section is original, and is intended to be an introduction to the theory of automorphic forms. The reader who is comfortable with this material may safely skip to Part II.

Chapter 1

Hecke Characters

Hecke characters can be seen as automorphic forms for GL_1*, and provide a generalisation of the theory of Dirichlet characters. We will extensively refer to them in the sequel, and here we recap some of the basic theory, in the process fixing the notation we shall use throughout this thesis. In particular, we will show that each Hecke character has a well-defined conductor and gives rise to a function on ideals that are coprime to this conductor, before describing the theory of Gauss sums for Hecke characters. We conclude by associating a character of the narrow ray class group* $\mathrm{Cl}_F^+(p^\infty)$ *to a Hecke character of suitable conductor.*

1.1. Motivation and definitions

1.1.1. Dirichlet characters and Hecke characters over $\mathbb{Q}$

Recall that for a positive integer N, a *Dirichlet character* is a homomorphism

$$\chi : (\mathbb{Z}/N\mathbb{Z})^\times \longrightarrow \mathbb{C}^\times.$$

We can see this as a character on the ideles $\mathbb{A}_\mathbb{Q}^\times$ in a natural way. Indeed, note that

$$(\mathbb{Z}/N\mathbb{Z})^\times \cong \prod_p (\mathbb{Z}/p^r\mathbb{Z})^\times,$$

where the product is over all primes and r is such that p^r exactly divides N (so that r is zero almost everywhere), and recall that there is a decomposition

$$\mathbb{Z}_p^\times \cong (\mathbb{Z}/p^r\mathbb{Z})^\times \times (1 + p^r\mathbb{Z}_p).$$

Thus we can see χ as a continuous character on $\widehat{\mathbb{Z}} \coloneqq \prod_p \mathbb{Z}_p^\times$ with suitable conditions; namely, if χ_p is the restriction of χ to $\mathbb{Z}_p^\times$, we have $\chi_p(1 + p^r\mathbb{Z}_p) = 1$ for all p and r as above.

Recall that the ideles decompose as

$$\mathbb{A}_\mathbb{Q}^\times \cong \mathbb{Q}^\times \cdot [\mathbb{R}_{>0} \times \widehat{\mathbb{Z}}],$$

and accordingly that any Dirichlet character χ defines a unique continuous homomorphism

$$\chi : \mathbb{Q}^\times \backslash \mathbb{A}_\mathbb{Q}^\times \longrightarrow \mathbb{C}^\times$$

whose restriction to $\mathbb{R}_{>0}$ is trivial.

A *Hecke character* over $\mathbb{Q}$ is a continuous character

$$\varphi : \mathbb{Q}^\times \backslash \mathbb{A}_\mathbb{Q}^\times \longrightarrow \mathbb{C}^\times.$$

Thus a Dirichlet character determines a Hecke character. Note that φ is allowed to be non-trivial on $\mathbb{R}_{>0}$, and in general, any Hecke character over $\mathbb{Q}$ takes the form $|\cdot|^k\chi$, where χ comes from a Dirichlet character, $|\cdot|$ is the idelic norm map and k is an integer. (This does not remain true for a general number field).

1.1.2. General Hecke characters

We now develop the theory by giving definitions and properties in generality.

Definition 1.1.1. Let F be a number field. A *Hecke character for F* is a continuous homomorphism

$$\varphi : F^\times \backslash \mathbb{A}_F^\times \longrightarrow \mathbb{C}^\times.$$

By restriction, for each place v of F, we obtain a character $\varphi_v : F_v^\times \to \mathbb{C}^\times$, where F_v denotes the completion of F at v.

Definition 1.1.2. The *finite part* of φ is

$$\varphi_f := \prod_{v \nmid \infty} \varphi_v.$$

The *infinite part* of φ is

$$\varphi_\infty := \varphi|_{(F \otimes_\mathbb{Q} \mathbb{R})^\times},$$

so that $\varphi = \varphi_f \varphi_\infty$.

1.2. Basic properties

We now give some of the basic properties that Hecke characters satisfy. In particular, we'll give more details on the possible forms a Hecke character can take on both the finite and infinite ideles.

1.2.1. The finite part and the conductor

First, we show that each Hecke character gives rise to a Dirichlet character over F in a natural way.

Proposition 1.2.1. *Suppose v corresponds to a finite prime $\mathfrak{p}$ of F, and write $\varphi_{\mathfrak{p}}$ for the restriction of φ_v to $\mathcal{O}_{\mathfrak{p}}^\times$, the ring of integers in the completion $F_{\mathfrak{p}} = F_v$.*

(i) There is a non-negative integer $e_{\mathfrak{p}}$ such that $\varphi_{\mathfrak{p}}(1 + \mathfrak{p}^{e_{\mathfrak{p}}}) = 1$ and $e_{\mathfrak{p}}$ is minimal with this property.

(ii) For almost all primes $\mathfrak{p}$ of $\mathcal{O}_F$, we have $e_{\mathfrak{p}} = 0$, that is, $\varphi_{\mathfrak{p}}(\mathcal{O}_{\mathfrak{p}}^\times) = 1$.

Proof. Observe first that $\mathcal{O}_{\mathfrak{p}}^\times$ is a profinite group, with

$$\mathcal{O}_{\mathfrak{p}}^\times = \varprojlim_n \mathcal{O}_{\mathfrak{p}}^\times/(1 + \mathfrak{p}^n).$$

Now, there exists some neighbourhood V of 1 in $\mathbb{C}$ containing no non-trivial subgroup, and by continuity, the inverse image of V in $\mathcal{O}_{\mathfrak{p}}^\times$ is open. By the nature of the profinite topology, any non-empty open set must contain a subgroup of the form $1 + \mathfrak{p}^{e_{\mathfrak{p}}}$ for some non-negative integer $e_{\mathfrak{p}}$. Now, we have

$$\varphi_{\mathfrak{p}}(1 + \mathfrak{p}^{e_{\mathfrak{p}}}) \subset V,$$

but it must also be a subgroup; since we picked V to contain no non-trivial subgroup, we therefore have

$$\varphi_{\mathfrak{p}}(1 + \mathfrak{p}^{e_{\mathfrak{p}}}) = 1,$$

as required. (Note that we can always pick a minimal such $e_{\mathfrak{p}}$ by the well-ordering principle).

Now, the kernel of a continuous homomorphism from a profinite group is open, and hence has finite index (as any open subgroup of a profinite group has finite index). But $\prod_{\mathfrak{p}} \mathcal{O}_{\mathfrak{p}}^\times$ is itself profinite; indeed, it is the inverse limit

$$\prod_{\mathfrak{p}} \mathcal{O}_{\mathfrak{p}}^\times = \varprojlim_I (\mathcal{O}_K/I)^\times,$$

where the limit is taken over all non-zero ideals in $\mathcal{O}_F$ with the usual ordering. Thus the kernel of $\prod_{\mathfrak{p}} \varphi_{\mathfrak{p}}$ has finite index, and this forces all but finitely many of the $e_{\mathfrak{p}}$ to be 0, as required. $\square$

Definition 1.2.2. Let φ be a Hecke character of F. Define the *conductor* of φ to be the ideal $\mathfrak{f} := \prod_{\mathfrak{p}} \mathfrak{p}^{e_{\mathfrak{p}}}$, where the $e_{\mathfrak{p}}$ are as defined in Proposition 1.2.1.

In particular, φ naturally gives rise to a character of $\widehat{\mathcal{O}}_F^\times/(1+\mathfrak{f}\widehat{\mathcal{O}}_F) \cong (\mathcal{O}_F/\mathfrak{f})^\times \to \mathbb{C}^\times$, a Dirichlet character over F with conductor $\mathfrak{f}$. We've shown that all of the information determining this

Dirichlet character is encoded by the finite part φ_f, and in particular at the primes dividing $\mathfrak{f}$, which motivates:

Definition 1.2.3. For (any) ideal $I \subset \mathcal{O}_F$, write

$$\varphi_I := \prod_{v|I} \varphi_v.$$

So $\varphi_{\mathfrak{f}}$ determines the Dirichlet character associated to φ.

1.2.2. The infinite part

Whilst every Dirichlet character of F will arise from a Hecke character in the manner explained above, the association is very far from bijective. Indeed, there will be many distinct Hecke characters that determine the same Dirichlet character. This is because we've said nothing about the behaviour at infinite places.

Recall that $F_\infty := F \otimes_{\mathbb{Q}} \mathbb{R} = \prod_{v|\infty} F_v$. Let F_∞^+ be the connected component of the identity in $F_\infty^\times$, that is,

$$F_\infty^+ \cong \prod_{v|\infty} F_v^+,$$

where

$$F_v^+ = \begin{cases} \mathbb{R}_{>0} & : v \in \Sigma(\mathbb{R}), \\ \mathbb{C}^\times & : v \in \Sigma(\mathbb{C}). \end{cases}$$

There is a canonical decomposition $F_\infty^\times = \{\pm 1\}^{\Sigma(\mathbb{R})} \times F_\infty^+$, and thus any Hecke character φ gives a character φ_∞^+ on F_∞^+ by restriction. We say such a character of F_∞^+ is *arithmetic* (sometimes referred to in the literature as *algebraic*) if it takes the form

$$\mathbf{z} = (z_v)_{v|\infty} \longmapsto \mathbf{z}^{\mathbf{r}} = \prod_{v|\infty} z_v^{r_v}$$

for some $\mathbf{r} \in \mathbb{Z}[\Sigma]$, and we say $\mathbf{r}$ is the *infinity-type* of φ. Henceforth, all Hecke characters will be assumed to be arithmetic.

Not all elements of $\mathbb{Z}[\Sigma]$ can be realised as the infinity type of a Hecke character. The following description of the 'admissible' infinity types is taken from [Hid94], Chapter 3.

Definition 1.2.4. Let F_{CM} be the maximal CM subfield of F (or the maximal totally real subfield if no such CM field exists), and denote its set of infinite places by Σ_{CM}. There is a natural inflation map

$$\begin{aligned}\mathrm{Inf} : \mathbb{Z}[\Sigma_{\mathrm{CM}}] &\longrightarrow \mathbb{Z}[\Sigma],\\ \sum_{\tau\in\Sigma_{\mathrm{CM}}} n_\tau \tau &\longmapsto \sum_{\substack{\sigma\in\Sigma\\ \sigma|_{F_{\mathrm{CM}}}=\tau}} n_\tau \sigma.\end{aligned}$$

Let $\Xi_{\mathrm{CM}} := \{\mathbf{j} \in \mathbb{Z}[\Sigma_{\mathrm{CM}}] : \mathbf{j} + c\mathbf{j} \in \mathbf{t}_{\mathrm{CM}}\mathbb{Z}\}$, where $\mathbf{t}_{\mathrm{CM}} = (1, 1, ..., 1)$. We define the set of *admissible infinity types* to be

$$\Xi = \mathrm{Inf}(\Xi_{\mathrm{CM}}).$$

In more concrete terms, a necessary (but not sufficient) condition for $\mathbf{r} \in \Xi$ is that $\mathbf{r} + c\mathbf{r}$ is parallel. This motivates the following piece of notation, which we'll require in the sequel:

Definition 1.2.5. Let $\mathbf{r} \in \mathbb{Z}[\Sigma]$ be admissible, that is, let $\mathbf{r} \in \Xi$. Then define $[\mathbf{r}] \in \mathbb{R}$ to be the unique number such that

$$\mathbf{r} + c\mathbf{r} = 2[\mathbf{r}]\mathbf{t}.$$

Note that, in particular, for any $\zeta \in F^\times$, we have $N((\zeta))^{[\mathbf{r}]} = \zeta^{\mathbf{r}}$, which we'll use later.

In [Wei56], Weil then shows that:

Proposition 1.2.6. *An element $\mathbf{r} \in \mathbb{Z}[\Sigma]$ can be realised as the infinity type of a Hecke character of F if and only if $\mathbf{r} \in \Xi$, that is, $\mathbf{r}$ is admissible.*

This restriction on the possible infinity types comes from the condition that any Hecke character φ is trivial on $F^\times$. In particular, an element $\mathbf{r} \in \mathbb{Z}[\Sigma]$ is admissible if and only if there exists an integer n such that $\epsilon^{n\mathbf{r}} = \prod_{v\in\Sigma} \epsilon^{nr_v} = 1$ for all $\epsilon \in \mathcal{O}_F^\times$.

Examples:

(i) Suppose F is totally real. As the unit group is as 'big' as it can be relative to the degree of F, this condition is very restrictive, and indeed the only admissible infinity types are parallel.

(ii) Suppose F is imaginary quadratic. Then the unit group is finite, and if we take n to be its order, we see that *any* element of $\mathbb{Z}[\Sigma]$ can be an infinity type.

We also define a character ε_φ of the *Weyl group* $\{\pm 1\}^{\Sigma(\mathbb{R})}$ attached to φ. Rather than defining ε_φ simply by restriction, we do this more subtly. In particular, we can consider $\iota \in \{\pm 1\}^{\Sigma(\mathbb{R})}$ as an infinite idele by setting its entries at non-real places to be 1; then we define

$$\varepsilon_\varphi(\iota) := \varphi_\infty(\iota)\iota^{\mathbf{r}}.$$

In the sequel, we will (in an abuse of notation) write ε_φ for both this character of $\{\pm 1\}^{\Sigma(\mathbb{R})}$ and for the character of the ideles given by $\varepsilon_\varphi(x) = \varepsilon_\varphi((\mathrm{sign}(x_v))_{v\in\Sigma(\mathbb{R})})$. Note then that $\varphi_\infty\varepsilon_\varphi$ is the unique algebraic character of $F_\infty^\times$ that restricts to φ_∞^+ on F_∞^+; namely, it is the character of $F_\infty^\times$ given by $\mathbf{z} \mapsto \mathbf{z}^{\mathbf{r}}$.

Remark: Note that if $F = \mathbb{Q}$ and $\varphi = |\cdot|$ is the norm character on $\mathbb{A}_\mathbb{Q}^\times$, then $\varepsilon_\varphi(-1) = -1$, even though φ itself takes only positive values.

1.2.3. Hecke characters as functions on ideals

We have introduced Hecke characters via their idelic formulation. They were initially defined in a slightly less clean way as functions on ideals, with the conductor built into the definition. There is a close connection between ideles and ideals; indeed, given an idele $x \in \mathbb{A}_F^\times$, we can define its associated fractional ideal by

$$I(x) := \prod_{\mathfrak{p}} \mathfrak{p}^{v_\mathfrak{p}(x_\mathfrak{p})},$$

where the product is over all finite primes of F. This is well-defined since $v_\mathfrak{p}(x_\mathfrak{p}) = 0$ for almost all $\mathfrak{p}$. Any fractional ideal I of F can be written in the form $I(x)$ for some idele x.

Proposition 1.2.7. *Let φ be a Hecke character of conductor $\mathfrak{f}$, and let $J(\mathfrak{f})$ be the group of fractional ideals of F that are coprime to $\mathfrak{f}$. Then there is a well-defined character*

$$\varphi : J(\mathfrak{f}) \longrightarrow \mathbb{C}^\times,$$

defined by

$$\varphi(I) = \varphi_f(x),$$

where x is any idele such that $I = I(x)$ with $x_\mathfrak{p} = 1$ for all primes $\mathfrak{p}|\mathfrak{f}$.

Proof. The only part that is not clear is that this map is well-defined. Suppose we choose a different idele x' representing I that has trivial components at $\mathfrak{f}$; then we know that $v_\mathfrak{p}(x'x^{-1}) = 0$ for all primes $\mathfrak{p}$ not dividing $\mathfrak{f}$. Thus, for such primes, we have $x'x^{-1} \in \mathcal{O}_\mathfrak{p}^\times$, so that $\varphi_\mathfrak{p}((x'x^{-1})_\mathfrak{p}) = 1$ by definition of the conductor. Since x and x' both have trivial components at $\mathfrak{f}$, it follows that $\varphi_f(x'x^{-1}) = 1$, so φ is well-defined as a function on ideals. $\square$

Whilst we have abused notation to write φ for both the idelic Hecke character and the function it determines on ideals, it will always be clear from context which formulation we mean.

Definition 1.2.8. We extend φ to a function on *all* fractional ideals of F by setting $\varphi(I) = 0$ if I is not coprime to $\mathfrak{f}$.

At this stage, it is useful to fix, for each ideal I, a systematic choice of idele x_I such that $I = I(x_I)$. Our choice will make sense for *all* fractional ideals I; indeed, in Part II, we will need to use the value $\varphi(x_{\mathfrak{f}})$, which visibly depends on the choice of idele. As long as we are consistent in our choices, this will not matter.

We define x_I as follows. If $I = (\alpha)$ is principal, define $x_{(\alpha)}$ at each place by

$$(x_{(\alpha)})_v = \begin{cases} \alpha & : v = \mathfrak{p} \text{ finite with } \mathfrak{p} | (\alpha), \\ 1 & : \text{otherwise.} \end{cases}$$

Let $I_1, ..., I_h$ be (fixed) ideals that form a complete set of representatives of the narrow class group of F, and choose (fixed) idelic representatives a_i representing each I_i. Then any fractional ideal I of F has form $(\alpha)I_i$ for some i, and we can define

$$x_I ::= a_i x_{(\alpha)}.$$

Note by definition that when I is coprime to $\mathfrak{f}$, we have

$$\varphi(I) = \varphi(x_I).$$

We conclude this section by giving a very simple result linking the three functions $\varphi_{\mathfrak{f}}$, φ_∞ and φ as a function on ideals.

Proposition 1.2.9. *Let φ be a Hecke character with conductor $\mathfrak{f}$, and let $\alpha \in F^\times$ be such that (α) is coprime to $\mathfrak{f}$. Then*

$$\varphi_\infty(\alpha)\varphi_{\mathfrak{f}}(\alpha)\varphi((\alpha)) = 1.$$

Proof. We have

$$\begin{aligned} \varphi_\infty(\alpha)\varphi_{\mathfrak{f}}(\alpha)\varphi((\alpha)) &= \prod_{v|\infty} \varphi_v(\alpha) \prod_{v|\mathfrak{f}} \varphi_v(\alpha) \prod_{v|(\alpha)} \varphi_v(\alpha) \\ &= \prod_{v|\mathfrak{f}(\alpha)\infty} \varphi_v(\alpha) \\ &= \prod_{v\nmid\mathfrak{f}(\alpha)\infty} \varphi_v(\alpha)^{-1} = 1, \end{aligned}$$

since $\alpha \in \mathcal{O}_v^\times$ for all $v \nmid \mathfrak{f}(\alpha)\infty$ and φ_v is trivial on $\mathcal{O}_v^\times$ for all such v. (For the penultimate equality, we have used the fact that φ is trivial on the diagonal embedding of $F^\times$ in $\mathbb{A}_F^\times$). $\square$

Corollary 1.2.10. *Suppose that the ideals $I_1, ..., I_h$ are all coprime to $\mathfrak{f}$, and let $I = (\alpha)I_i$ be* *a fractional ideal of F. Then*

$$\varphi(I) = \begin{cases} \varphi(a_i)\varphi_{\mathfrak{f}}(\alpha)^{-1}\varphi_\infty(\alpha)^{-1} & : I \in J(\mathfrak{f}), \\ 0 & : \textit{otherwise}. \end{cases}$$

1.3. Further topics

We conclude this chapter by mentioning two further topics that we'll need in the sequel related to Hecke characters; namely, the generalisation of Gauss sums of Dirichlet characters and ways of seeing Hecke characters p-adically.

1.3.1. Gauss sums

For a rational Dirichlet character χ modulo N, we define the *Gauss sum* to be the quantity

$$\tau(\chi) := \sum_{a=0}^{N-1} \overline{\chi}(a)e^{2\pi i a/N}.$$

For primitive χ, this has the nice property that, for $b \in \mathbb{Z}$, we have

$$\sum_{a=0}^{N-1} \overline{\chi}(a)e^{2\pi i ab/N} = \chi(b)\tau(\chi)$$

(see, for example, [DS05], Section 4.3). We will need an object satisfying a similar property for Hecke characters over arbitrary number fields; however, the generalisation of this is again non-obvious.

We first introduce a more general exponential map on the adeles of F.

Definition 1.3.1. Let e_F be the unique function

$$e_F : \mathbb{A}_F/F \longrightarrow \mathbb{C}^\times$$

that satisfies

$$x_\infty \longmapsto e^{2\pi i \mathrm{Tr}_{F/\mathbb{Q}}(x_\infty)},$$

where x_∞ is an infinite adele. We can describe e_F explicitly as

$$e_F(\mathbf{x}) = \prod_{v\in\Sigma(\mathbb{C})} e^{2\pi i \mathrm{Tr}_{\mathbb{C}/\mathbb{R}}(x_v)} \prod_{v\in\Sigma(\mathbb{R})} e^{2\pi i x_v} \prod_{\lambda|\ell \text{ finite}} e_\ell(-\mathrm{Tr}_{F_\lambda/\mathbb{Q}_\ell}(x_\lambda)),$$

where

$$e_\ell\left(\sum_j c_j \ell^j\right) = e^{2\pi i \sum_{j<0} c_j\ell^j}.$$

Now let φ be a Hecke character of conductor $\mathfrak{f}$. Let d be a (finite) idele representing the different $\mathfrak{D}$, and for each finite prime w of F, fix a uniformiser π_w in $\mathcal{O}_w$.

Definition 1.3.2. Define the *Gauss sum attached to* φ to be

$$\tau(\varphi) := \varphi(d^{-1}) \sum_{b \in (\mathcal{O}_F/\mathfrak{f})^\times} \varphi_{\mathfrak{f}}(b) e_F(bd^{-1}(\mathfrak{f}^{-1})_{v|\mathfrak{f}}),$$

where $(\mathfrak{f}^{-1})_{v|\mathfrak{f}}$ is the adele given by

$$((\mathfrak{f}^{-1})_{v|\mathfrak{f}})_w := \begin{cases} \pi_w^{-v_w(\mathfrak{f})} & : w|\mathfrak{f} \\ 0 & : \text{otherwise}. \end{cases}$$

Remarks: (i) This definition is independent of the choice of d.

(ii) This definition is a natural one; in fact, it is the product of the ϵ-factors over $v|\mathfrak{f}$, as defined by Deligne in [Del72]. For this particular iteration of the definition, we've followed [Hid94], page 480 (though we have phrased the definition slightly differently by choosing more explicit representatives).

Proposition 1.3.3. *Let ζ be a non-zero element of $\mathcal{O}_F$. Then we have*

$$\varphi(d^{-1}) \sum_{b \in (\mathcal{O}_F/\mathfrak{f})^\times} \varphi_{\mathfrak{f}}(b) e_F(\zeta bd^{-1}(\mathfrak{f}^{-1})_{v|\mathfrak{f}}) = \begin{cases} \varphi_{\mathfrak{f}}(\zeta)^{-1}\tau(\varphi) & : ((\zeta), \mathfrak{f}) = 1 \\ 0 & : \textit{otherwise} \end{cases},$$

where the notation $((\zeta), \mathfrak{f}) = 1$ means that the two ideals are coprime.

Proof. See [Del72], or, for an English translation, [Tat79]. There is also an account of Gauss sums and their properties in [Nar04]. $\square$

Remark: In Chapter 6.1.2, we'll give an equivalent and more concrete definition of the Gauss sum of a Hecke character defined over an imaginary quadratic field.

1.3.2. Hecke characters on ray class groups

We conclude this chapter by describing a p-adic character associated to a classical Hecke character of suitable conductor. This will be crucial in later sections when describing p-adic L-functions.

Definition 1.3.4. Define the *ray class group of level* p^∞ to be the p-adic analytic group

$$\mathrm{Cl}_F^+(p^\infty) := F^\times \backslash \mathbb{A}_F^\times / U(p^\infty) F_\infty^+,$$

where $U(p^\infty)$ is the group of elements of $\widehat{\mathcal{O}}_F$ that are congruent to $1 \pmod{p^n}$ for all integers n (that is, elements of $\widehat{\mathcal{O}}_F$ such that their components at primes above p are all equal to 1).

Remark: By class field theory, $\mathrm{Cl}_F^+(p^\infty)$ is isomorphic to the Galois group of the maximal abelian extension of F unramified outside p and ∞.

The p-adic L-function of an automorphic form over F should be a distribution on this space in a sense that will be made clear in later sections, and to this end we discuss the structure of this space in the sequel.

Throughout, fix an isomorphism $\mathbb{C} \cong \mathbb{C}_p$ that is compatible with our earlier embedding $\mathrm{inc}_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$. To a Hecke character φ of conductor $\mathfrak{f}|p^\infty$, we can associate a character $\varphi_{p-\mathrm{fin}}$ on $\mathrm{Cl}_F^+(p^\infty)$. Indeed, let φ be a Hecke character with infinity type $\mathbf{r}$ and associated character ε_φ on $\{\pm 1\}^{\Sigma(\mathbb{R})}$, as above. Then there is a unique algebraic homomorphism $w^{\mathbf{r}} : F^\times \longrightarrow \overline{\mathbb{C}}^\times$ given by

$$w^{\mathbf{r}}(\gamma) = \prod_{v \in \Sigma} \sigma_v(\gamma)^{r_v},$$

where σ_v is the complex embedding corresponding to the infinite place v. This then induces maps $w_\infty^{\mathbf{r}} : (F \otimes_{\mathbb{Q}} \mathbb{R})^\times \to \mathbb{C}^\times$ and $w_p^{\mathbf{r}} : (F \otimes_{\mathbb{Q}} \mathbb{Q}_p)^\times \to \overline{\mathbb{Q}}_p^\times \subset \mathbb{C}_p^\times$. Note that $w_\infty^{\mathbf{r}}$ is equal to $\varepsilon_\varphi \varphi_\infty$, the unique algebraic character of $F_\infty^\times$ that agrees with φ_∞ on F_∞^+.

Then under the isomorphism $\mathbb{C} \cong \mathbb{C}_p$, we can see $w_p^{\mathbf{r}}$ and φ as having values in $\mathbb{C}_p$.

Definition 1.3.5. We define $\varphi_{p-\mathrm{fin}}$ to be the function

$$\begin{aligned} \varphi_{p-\mathrm{fin}} : \mathbb{A}_F^\times &\longrightarrow \mathbb{C}_p^\times \\ x &\longmapsto w_\infty^{\mathbf{r}}(x_\infty)^{-1} w_p^{\mathbf{r}}(x_p)\varphi(x) = \varepsilon_\varphi \varphi_f(x) w_p^{\mathbf{r}}(x_p). \end{aligned}$$

Proposition 1.3.6. *Let φ be a Hecke character of conductor $\mathfrak{f}|(p^\infty)$. Then the function $\varphi_{p-\mathrm{fin}}$ gives a well-defined function on the narrow ray class group $\mathrm{Cl}_F^+(p^\infty)$.*

Proof. By definition, $\varphi_{p-\mathrm{fin}}$ is trivial on F_∞^+. As $w_\infty^{\mathbf{r}}$ and $w_p^{\mathbf{r}}$ are both induced from the same function on F, we see that $\varphi_{p-\mathrm{fin}}$ is also trivial on $F^\times$. As φ has conductor $\mathfrak{f}$, it is trivial on $U(\mathfrak{f})$, and hence on $U(p^\infty)$. Finally, if $x \in U(p^\infty)$, then $x_p = x_\infty = 1$, so that $w_p^{\mathbf{r}}(x_p) = w_\infty^{\mathbf{r}}(x_\infty) = 1$. This completes the proof. $\square$

Chapter 2

Automorphy Conditions

In a first introduction to the theory of modular forms, they are defined as holomorphic functions on the upper half-plane that satisfy transformation properties under suitable arithmetic subgroups of $\mathrm{SL}_2(\mathbb{Z})$*. The advantage of this approach is that it is conceptually easy to understand, but it is highly non-obvious how one should generalise the theory to different settings. With this in mind, this chapter starts by focusing on developing a theory of rational automorphic functions that more easily generalises to arbitrary number fields. It goes on to define and study automorphic functions over imaginary quadratic fields as a precursor to eventually treating the completely general case.*

2.1. Classical modular forms via adeles

We start by giving motivation for the *adelic* definition of modular forms which, at first sight, appears to bear little relation to the more familiar definition. This section is heavily based upon Weil's book [Wei71]. To introduce the adelic formulation of modular forms, we keep to level 1 initially.

2.1.1. Level 1

We can consider the upper half-plane as a set B of matrices by defining

$$B := \left\{ \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} : x \in \mathbb{R}, y \in \mathbb{R}_{>0} \right\},$$

and identifying $\left(\begin{smallmatrix} y & x \\ 0 & 1 \end{smallmatrix}\right)$ with $x+iy \in \mathcal{H}$. We then obtain a description of $\mathcal{H}$ as a symmetric space; it is easily seen that

$$\mathcal{H} \cong B \cong \mathrm{GL}_2(\mathbb{R})/\mathbb{R}^\times \cdot \mathrm{O}_2(\mathbb{R}),$$

where

$$\mathbb{R}^\times \cong Z(\mathrm{GL}_2(\mathbb{R})) \subset \mathrm{GL}_2(\mathbb{R})$$

can as the centre of $\mathrm{GL}_2(\mathbb{R})$ via its embedding as scalar matrices. This can be reworked as

$$\mathrm{GL}_2^+(\mathbb{R}) = \mathbb{R}_{>0} \cdot B \cdot \mathrm{SO}_2(\mathbb{R}), \tag{2.1}$$

where now we just consider matrices with positive determinant (that is, the connected component of the identity in $\mathrm{GL}_2(\mathbb{R})$). Thus, if $f : \mathcal{H} \to \mathbb{C}$ is a function, then we can extend f - considered as a function on B - to a function F on $\mathrm{GL}_2^+(\mathbb{R})$ by stipulating that F satisfies suitable conditions under translations by $\mathbb{R}_{>0}$ and $\mathrm{SO}_2(\mathbb{R})$. These we state without motivation, for now; we ask that, for all $g \in \mathrm{GL}_2^+(\mathbb{R})$:

(i) $F(zg) = F(g)$ for all $z \in \mathbb{R}_{>0}$, and

(ii) $F(gr(\theta)) = F(g)e^{ik\theta}$, where $r(\theta) = \left(\begin{smallmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{smallmatrix}\right)$.

Note that the two conditions combine to show that k must be an even integer by considering $\theta = \pi$. Indeed, as the notation suggests, this k will be the weight of our modular form.

To obtain a modularity condition, we'll pass to an even bigger space, namely the group $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$, where $\mathbb{A}_\mathbb{Q} = \mathbb{R} \times \mathbb{A}_\mathbb{Q}^f$ is the rational adele ring, with $\mathbb{A}_\mathbb{Q}^f$ the finite adeles. Let

$$\Omega := \prod_{p\,\mathrm{prime}} \mathrm{GL}_2(\mathbb{Z}_p) \cong \mathrm{GL}_2\left(\widehat{\mathbb{Z}}\right),$$

an open compact subgroup of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q}^f)$, where $\widehat{\mathbb{Z}}$ is the space of finite integral adeles. Now extend our function F to a function $\mathcal{F}$ on $\mathrm{GL}_2^+(\mathbb{R}) \times \Omega$ by setting

$$\mathcal{F}(g_\infty \tau) = \mathcal{F}(g_\infty), \quad g_\infty \in \mathrm{GL}_2^+(\mathbb{R}), \tau \in \prod_{p\,\mathrm{prime}} \mathrm{GL}_2(\mathbb{Z}_p).$$

Proposition 2.1.1. *Let f be a function $B \cong \mathcal{H} \to \mathbb{C}$. If we set $\tau = x + iy$, then $\phi(\tau) := y^{-k/2} f\left(\begin{smallmatrix} y & x \\ 0 & 1 \end{smallmatrix}\right)$ behaves like a modular form of weight k under the action of $\mathrm{SL}_2(\mathbb{Z})$ if and only if f extends to a function $\mathcal{F} : \mathrm{GL}_2^+(\mathbb{R}) \times \Omega \to \mathbb{C}$ such that:*

(i) $\mathcal{F}$ is left-invariant under $Z^+ \cong \mathbb{R}_{>0}$,

(ii) $\mathcal{F}$ transforms as $\mathcal{F}(gr(\theta)) = \mathcal{F}(g)e^{ik\theta}$ for $r(\theta) \in \mathrm{SO}_2(\mathbb{R})$,

(iii) $\mathcal{F}$ is right-invariant under all the groups $\mathrm{GL}_2(\mathbb{Z}_p)$ for prime p,

(iv) and $\mathcal{F}$ is left-invariant under $\mathrm{SL}_2(\mathbb{Z})$.

Proof. Suppose we can extend f as above, and take an element $g \in \mathrm{GL}_2^+(\mathbb{R}) \times \Omega$. Then we can write the component at infinity as

$$g_\infty = z \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} r(\theta)$$

using equation (2.1). Then

$$\mathcal{F}(g) = f\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} e^{ik\theta}.$$

If now σ is any element of $\mathrm{SL}_2(\mathbb{Z})$, then σg is also an element of $\mathrm{GL}_2^+(\mathbb{R})\times\Omega$, since $\det(\sigma g_\infty) > 0$, and $\sigma g_p \in \mathrm{GL}_2(\mathbb{Z}_p)$ for each p. Thus we can write $(\sigma g)_\infty$ in the form

$$(\sigma g)_\infty = z' \begin{pmatrix} y' & x' \\ 0 & 1 \end{pmatrix} r(\theta'),$$

and since by condition (iv) we must have $\mathcal{F}(\sigma g) = \mathcal{F}(g)$, we thus have

$$f\begin{pmatrix} y' & x' \\ 0 & 1 \end{pmatrix} = f\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} e^{ik(\theta-\theta')}. \tag{2.2}$$

Let $\tau = x + iy$, and $\tau' = x' + iy'$, both points in $\mathcal{H}$. After some calculation, we see that $\tau' = \sigma(\tau)$ under the usual action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathcal{H}$.

Define a function $\phi : \mathcal{H} \to \mathbb{C}$ by

$$\phi(\tau) = y^{-k/2} f\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}.$$

We write $f(\tau) = f(x+iy) = f\left(\begin{smallmatrix} y & x \\ 0 & 1 \end{smallmatrix}\right)$ as an abuse of notation. Then equation (2.2) gives

$$\begin{aligned} \phi(\sigma(\tau)) &= \phi(\tau') = \mathrm{Im}(\tau')^{-k/2} f(\tau') \\ &= |c\tau + d|^k \mathrm{Im}(\tau)^{-k/2} f(\tau) e^{ik(\theta-\theta')} \\ &= (c\tau + d)^k \phi(\tau), \end{aligned}$$

so that ϕ behaves like a modular form of weight k, as required.

The same argument in reverse shows that if ϕ behaves like a modular form of weight k, then f extends uniquely to a function $\mathcal{F}$ satisfying conditions (i) to (iv), which completes the proof. $\square$

Now, note that

$$\mathrm{SL}_2(\mathbb{Z}) = \Omega \cap \mathrm{GL}_2(\mathbb{Q}).$$

Furthermore, an approximation theorem gives

$$\mathrm{GL}_2(\mathbb{A}_\mathbb{Q}) = \mathrm{GL}_2(\mathbb{Q}) \cdot [\mathrm{GL}_2^+(\mathbb{R}) \times \Omega].$$

It follows immediately that any function $\mathcal{F}$ on $\mathrm{GL}_2^+(\mathbb{R}) \times \Omega$ that is left-invariant under $\mathrm{SL}_2(\mathbb{Z})$ extends uniquely to a function Φ on $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ that is left-invariant under $\mathrm{GL}_2(\mathbb{Q})$. Thus we have the following:

Theorem 2.1.2. *Let $f : \mathcal{H} \to \mathbb{C}$ be a function. Then f transforms like a modular form of weight k and level $\mathrm{SL}_2(\mathbb{Z})$ if and only if $\mathrm{Im}(\tau)^{k/2} f(\tau)$ can be extended uniquely to give a function $\Phi : \mathrm{GL}_2(\mathbb{A}_\mathbb{Q}) \to \mathbb{C}$ satisfying:*

(i) Φ is invariant under translation by $Z(\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})) \cong \mathbb{A}_\mathbb{Q}^\times$,

(ii) Φ transforms as $\Phi(gr(\theta)) = \Phi(g)e^{ik\theta}$ for $r(\theta) \in \mathrm{SO}_2(\mathbb{R})$,

(iii) Φ is right-invariant under all the groups $\mathrm{GL}_2(\mathbb{Z}_p)$ for prime p,

(iv) and Φ is left-invariant under $\mathrm{GL}_2(\mathbb{Q})$.

In the classical theory, we also stipulate that f should be holomorphic on the extended upper half-plane. In the adelic setting, the holomorphicity on $\mathcal{H}$ corresponds to prescribing a differential equation for Φ. Holomorphicity at the cusps corresponds to a bound for the order of magnitude of Φ in a fundamental domain in $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ for $\mathrm{GL}_2(\mathbb{Q})$. This is discussed in more detail in Chapter 3.1.

Remark: In passing from f to Φ, we multiplied by $\mathrm{Im}(\tau)^{k/2}$. As a result, condition (iii) above is really an invariance condition for Φ under the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathcal{H}$, whereas the usual modularity condition, which scales as $(c\tau + d)^k$, is genuinely different. This will become important later in studying the differences in the literature for imaginary quadratic fields.

2.1.2. Higher levels

To introduce adelic modular forms of level $\Gamma_0(N)$, we make a change in the definition of Ω. The key point of the above was that $\Omega \cap \mathrm{GL}_2(\mathbb{Q}) = \mathrm{SL}_2(\mathbb{Z})$, so we instead look for some group $\Omega_0(N)$ such that

$$\Omega_0(N) \cap \mathrm{GL}_2(\mathbb{Q}) = \Gamma_0(N).$$

We can define such a group in the 'obvious' way; namely, for each prime p, define

$$\Gamma_0(N)_p = \left\{ \begin{pmatrix} a & b \\ Nc & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Q}_p) : a, b, c, d \in \mathbb{Z}_p \right\}.$$

Indeed, if $p \nmid N$, then this is none other than $\mathrm{GL}_2(\mathbb{Z}_p)$, and more generally, if $p^m || N$, we have

$$\Gamma_0(N)_p = \left\{ \begin{pmatrix} a & b \\ p^m c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}_p \right\}.$$

Now we set

$$\Omega_0(N) = \prod_{p \,\mathrm{prime}} \Gamma_0(N)_p.$$

Remark: Note that if we define $\widehat{\mathbb{Z}} = \prod_p \mathbb{Z}_p$ to be the integral finite ideles, we have

$$\Omega_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2\left(\widehat{\mathbb{Z}}\right) : c \in N\widehat{\mathbb{Z}} \right\}.$$

We'll use this formulation in future.

A simple check shows that $\Omega_0(N)$ satisfies the intersection property we desire, and that we still have $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q}) = \mathrm{GL}_2(\mathbb{Q}) \cdot [\mathrm{GL}_2^+(\mathbb{R}) \times \Omega_0(N)]$. Then an argument almost identical to that in the proof of Proposition 2.1.1 gives:

Theorem 2.1.3. *Let $f : \mathcal{H} \to \mathbb{C}$ be a function. Then f transforms like a modular form of weight $k \in 2\mathbb{Z}$ and level $\Gamma_0(N)$ if and only if $\mathrm{Im}(\tau)^{k/2} f(\tau)$ can be extended uniquely to give a function $\Phi : \mathrm{GL}_2(\mathbb{A}_\mathbb{Q}) \to \mathbb{C}$ satisfying:*

(i) *Φ is invariant under translation by $Z(\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})) \cong \mathbb{A}_\mathbb{Q}^\times$,*

(ii) *Φ transforms as $\Phi(gr(\theta)) = \Phi(g)e^{ik\theta}$ for $r(\theta) \in \mathrm{SO}_2(\mathbb{R})$,*

(iii) *Φ is right-invariant under all the groups $\Gamma_0(N)_p$ for prime p,*

(iv) *and Φ is left-invariant under $\mathrm{GL}_2(\mathbb{Q})$.*

We'd also like a way of describing the character of a modular form of level $\Gamma_0(N)$. Let χ' be a Dirichlet character with conductor dividing N, and recall that in Chapter 1 that this determines a finite order Hecke character χ over $\mathbb{Q}$. We get characters χ_p of $\mathbb{Z}_p^\times$ by restriction. Then we have:

Theorem 2.1.4. *Let $f : \mathcal{H} \to \mathbb{C}$ be a function. Then f transforms like a modular form of weight k, level $\Gamma_0(N)$ and character $(\chi')^{-1}$ if and only if $\mathrm{Im}(\tau)^{k/2} f(\tau)$ can be extended uniquely to give a function $\Phi : \mathrm{GL}_2(\mathbb{A}_\mathbb{Q}) \to \mathbb{C}$ satisfying:*

(i) *$\Phi(gz) = \Phi(g)\chi(z)$ for $z \in Z(\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})) \cong \mathbb{A}_\mathbb{Q}^\times$,*

(ii) *Φ transforms as $\Phi(gr(\theta)) = \Phi(g)e^{ik\theta}$ for $r(\theta) \in \mathrm{SO}_2(\mathbb{R})$,*

(iii) *Φ is right-invariant under all the groups $\mathrm{GL}_2(\mathbb{Z}_p)$ for prime $p \nmid N$,*

(iv) *when $p^m || N$, with $m > 0$, and for $\gamma = \left(\begin{smallmatrix} a & b \\ Nc & d \end{smallmatrix}\right) \in \Gamma_0(N)_p$, we have*

$$\Phi(g\gamma) = \Phi(g)\chi_p(d),$$

(iv) *and Φ is left-invariant under $\mathrm{GL}_2(\mathbb{Q})$.*

Note here that k can be odd, but since $-I \in Z(\mathrm{GL}_2(\mathbb{R})) \cap \mathrm{SO}_2(\mathbb{R})$, this occurs if and only if χ' is odd (that is, if $\chi'(-1) = -1$).

We've chosen χ to be a finite order Hecke character to illustrate the relation to the classical theory of modular forms. There is, however, no reason not to consider a Hecke character with non-trivial infinity type instead. Indeed, in the sequel, we'll actually work with level $\Gamma_1(N)$, allowing us to suppress any mention of the character at finite places (since it will be automatically encoded in the definition), but we'll prescribe the character to have infinity type depending on the weight.

With the above results and remarks taken into account, we're now well placed to make a definition of automorphy conditions in the adelic setting that is more compatible with the existing literature (for example, [Hid94]) in the more general setting. In this definition, $\Omega_1(N)$ is defined in the obvious way, that is, by

$$\Omega_1(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2\left(\widehat{\mathbb{Z}}\right) : c \in N\widehat{\mathbb{Z}}, d \equiv 1 \pmod{N} \right\}.$$

Definition 2.1.5. Let k and N be positive integers, and define a representation

$$\begin{aligned} \rho : \mathrm{SO}_2(\mathbb{R}) \times \mathbb{R}^\times &\longrightarrow \mathbb{C}^\times \\ (r(\theta), x) &\longmapsto e^{ik\theta} x^{-k}. \end{aligned}$$

A function $\Phi : \mathrm{GL}_2(\mathbb{A}_\mathbb{Q}) \longrightarrow \mathbb{C}$ is said to be *automorphic of weight k and level $\Omega_1(N)$* if it satisfies:

(i) (Automorphy) $\Phi(gr(\theta)x) = \Phi(g)\rho(r(\theta), x)$ for $r(\theta) \in \mathrm{SO}_2(\mathbb{R})$ and $x \in \mathbb{Z}(\mathrm{GL}_2(\mathbb{R})) \cong \mathbb{R}^\times$,

(ii) (Level) Φ is right-invariant under translation by elements of $\Omega_1(N)$, and

(iii) Φ is left-invariant under translation by elements of $\mathrm{GL}_2(\mathbb{Q})$.

A function that is automorphic of weight k and level $\Omega_1(N)$ will be automorphic of weight k, level $\Omega_0(N)$ and character χ for some Hecke character χ of infinity type $-k$. We prescribe this infinity type to allow better compatibility with cohomology in the sequel.

Remark: In the above, we've said nothing about holomorphicity; in the general case, this is rather more tricky to get a handle on, and will be mentioned in more detail in later chapters.

2.2. Imaginary quadratic fields

The reformulation above now generalises with a little effort to give a whole plethora of areas of study; we can, for example, replace the algebraic group GL_2 with a different reductive group, such as the symplectic group GSp_4, or, as is more pertinent to our interests, replace $\mathbb{Q}$ with a different number field. In this section, we describe the theory when we replace $\mathbb{Q}$ with an imaginary quadratic field. We isolate this case for clarity; not only is it the major focus of the first half of this thesis, but if one understands the rational and imaginary quadratic cases well, then it is easy to see how the theory should look for general number fields. We start by defining adelic automorphy conditions before discussing explicit descriptions of the theory that are more in line with the classical theory of modular forms.

2.2.1. Adelic automorphy conditions over imaginary quadratic fields

Over $\mathbb{Q}$, the weight is an integer that induces conditions at the infinite place, so over an imaginary quadratic field, we'd expect the weight to be a pair (k, ℓ) of integers – one for each complex embedding of the field. Perhaps the only thing that remains unclear is what the target space of our function should be; in the rational case, we only have functions into the 1-dimensional space $\mathbb{C}$. This is because all the irreducible representations of $\mathrm{SO}_2(\mathbb{R})$ (that is, the circle group) are 1-dimensional, and in fact they behave as in condition (ii) of the above theorems. When we work with an imaginary quadratic field K, we instead look at irreducible representations of the natural analogue $\mathrm{SU}_2(\mathbb{C})$[1]. We then have the following:

Proposition 2.2.1. *Let $n \geq 0$ be an integer, and $V_n(\mathbb{C})$ be the space of homogeneous polynomials in two variables X and Y of degree n with complex coefficients. There is a right action of $\mathrm{SU}_2(\mathbb{C})$ on this space by*

$$(P|u)\begin{pmatrix} X \\ Y \end{pmatrix} = P\left(u\begin{pmatrix} X \\ Y \end{pmatrix}\right),$$

an irreducible representation that induces a map $\rho_n : \mathrm{SU}_2(\mathbb{C}) \to \mathrm{GL}(V_n(\mathbb{C}))$. Moreover, every irreducible representation of $\mathrm{SU}_2(\mathbb{C})$ arises in this way.

Proof. See [Sim91], VIII.4. □

From this, we might expect a weight (k, ℓ) form to take values in $V_{k+\ell}(\mathbb{C})$. In fact, there is a shift by 2, and the correct thing to consider is $V_{k+\ell+2}(\mathbb{C})$. This shift is found liberally in the theory - one can push it around, but not remove it entirely, and for convenience we introduce it now[2]. We define natural analogues of the rational objects we considered above.

Notation: (i) Let K be an imaginary quadratic field with ring of integers $\mathcal{O}_K$, adele ring $\mathbb{A}_K$ and finite integral adeles $\widehat{\mathcal{O}}_K$. Let $\mathfrak{n}$ be an ideal of $\mathcal{O}_K$ and define

$$\Omega_1(\mathfrak{n}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2\left(\widehat{\mathcal{O}}_K\right) : c \in \mathfrak{n}\widehat{\mathcal{O}}_K, d \equiv 1 \pmod{\mathfrak{n}} \right\}. \tag{2.3}$$

[1] We shall later give some motivation for this in Chapter 3.1.3.

[2] In later sections, when we discuss cohomology classes attached to automorphic forms, this will become clearer; it allows us to define differential forms attached to such automorphic forms.

(ii) Define a map

$$\begin{aligned}\rho : \mathrm{SU}_2(\mathbb{C}) \times \mathbb{C}^\times &\longrightarrow \mathrm{GL}(V_{2k+2}(\mathbb{C})),\\ (u,z) &\longmapsto \rho_{k+\ell+2}(u)|z|^{-k}.\end{aligned}$$

Definition 2.2.2. A function $\Phi : \mathrm{GL}_2(\mathbb{A}_K) \longrightarrow V_{k+\ell+2}(\mathbb{C})$ is said to be *automorphic of weight* (k,ℓ) *and level* $\Omega_1(\mathfrak{n})$ if it satisfies:

(i) (Automorphy) $\Phi(guz) = \Phi(g)\rho(u,z)$ for $u \in \mathrm{SU}_2(\mathbb{C})$ and $z \in \mathbb{Z}(\mathrm{GL}_2(\mathbb{C})) \cong \mathbb{C}^\times$,

(ii) (Level) Φ is right-invariant under translation by elements of $\Omega_1(\mathfrak{n})$, and

(iii) Φ is left-invariant under translation by elements of $\mathrm{GL}_2(K)$.

Remark: In fact, we can simplify this somewhat by considering only the case of *parallel* weight, that is, weights with $k = \ell$. Indeed, Harder showed in [Har87b] that non-zero cusp forms (to be defined in the sequel) exist only at parallel weights. In some accounts, such a cusp form is said to have weight $k+2$; thus parallel weight $(0,0)$ corresponds to weight 2, and there is often a correspondence between weight 2 automorphic forms and elliptic curves over K, as we'd hope. From now on we'll deal exclusively with parallel weight (k,k), as our focus will be on cusp forms.

2.2.2. Explicit description for class number one

Now suppose K has class number one. Given a function $\Phi : \mathrm{GL}_2(\mathbb{A}_K) \to V_{2k+2}(\mathbb{C})$ that is automorphic of weight (k,k) and level $\Omega_1(\mathfrak{n})$, condition (iii) says that Φ descends to a function on $\mathrm{GL}_2(K)\backslash\mathrm{GL}_2(\mathbb{A}_K)$, and then conditions (i) and (iii) say that Φ is determined by its values on a set of representatives for the double coset

$$\mathrm{GL}_2(K)\backslash\mathrm{GL}_2(\mathbb{A}_K)/\left[Z(\mathrm{GL}_2(\mathbb{C}))\cdot\Omega_1(\mathfrak{n})\right]$$

(recalling the definition of $\Omega_1(\mathfrak{n})$ in equation (2.3)). Since we are assuming that the class number was one, an approximation theorem similar to the rational case gives that

$$\mathrm{GL}_2(\mathbb{A}_K) = \mathrm{GL}_2(K)\cdot\left[\mathrm{GL}_2(\mathbb{C})\times\Omega_1(\mathfrak{n})\right], \tag{2.4}$$

from which we see that Φ is determined by its values on $\mathrm{GL}_2(\mathbb{C})$. We can go further still, though; condition (ii) means we need only consider the values of Φ on a set of representatives for

$$\mathrm{GL}_2(K)\backslash\mathrm{GL}_2(\mathbb{A}_K)/\left[Z(\mathrm{GL}_2(\mathbb{C}))\cdot\mathrm{SU}_2(\mathbb{C})\times\Omega_1(\mathfrak{n})\right] \cong \mathrm{GL}_2(\mathbb{C})/\mathbb{C}^\times\mathrm{SU}_2(\mathbb{C}).$$

An explicit calculation shows that the set

$$B := \left\{ \begin{pmatrix} t & z \\ 0 & 1 \end{pmatrix} : z \in \mathbb{C}, r \in \mathbb{R}_{>0} \right\} \tag{2.5}$$

gives a complete set of coset representatives for this latter quotient. This motivates:

Definition 2.2.3. Define *hyperbolic 3-space* (or *upper half-space*) to be the 3-dimensional real manifold

$$\mathcal{H}_3 := \{(z,t) : z \in \mathbb{C}, t \in \mathbb{R}_{>0}\}.$$

The above discussion means that over imaginary quadratic fields of class number one, instead of studying functions Φ on $\mathrm{GL}_2(\mathbb{A}_K)$, we can consider functions

$$\mathcal{F} : \mathcal{H}_3 \longrightarrow V_{2k+2}(\mathbb{C}),$$

with suitable additional conditions. In the sequel, we give these explicitly by studying the geometry of $\mathcal{H}_3$.

2.2.3. Higher class number and strong approximation

The case of more general class number remains. In this case, equation (2.4) no longer applies in the form given above; instead, $\mathrm{GL}_2(\mathbb{A}_K)$ decomposes as the disjoint union of h sets, where h is the class number. To describe this, let $I_1, ..., I_h$ be a complete set of representatives for the class group, and let $a_1, ..., a_h$ to be ideles representing these ideals. Then define

$$g_i = \begin{pmatrix} a_i & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{A}_K).$$

The approximation theorem now becomes:

Theorem 2.2.4 (Strong Approximation)**.** *There is a decomposition*

$$\mathrm{GL}_2(\mathbb{A}_K) = \coprod_{i=1}^{h} \mathrm{GL}_2(K) \cdot g_i \cdot [\mathrm{GL}_2(\mathbb{C}) \times \Omega_1(\mathfrak{n})].$$

As motivation for this formula, we give a brief sketch of the proof of this in the level 1 case. The case of level $\mathfrak{n}$ requires a bit more work; the details can be found in [Byg98], Chapter 5.2. The reader who is happy to take this result on faith may skip ahead to the end of the proof below.

Firstly, we quote a theorem of Weil, which describes when a set

$$\{L_{\mathfrak{p}} \subset K_{\mathfrak{p}}^2 \text{ lattice } : \mathfrak{p} \text{ prime in } K\}$$

of local lattices gives rise to a global lattice L in K such that $L_{\mathfrak{p}}$ is the completion of L in $K_{\mathfrak{p}}$ for each prime. The theorem says that such a global lattice exists if and only if $L_{\mathfrak{p}} = \mathcal{O}_{\mathfrak{p}}^2$ for almost all $\mathfrak{p}$, and in this case, we have

$$L = K^2 \cap \bigcap_{\mathfrak{p} \text{ prime}} L_{\mathfrak{p}}.$$

As a result of this, we have an action of $\mathrm{GL}_2(\mathbb{A}_K)$ on the set of lattices in K^2. Let $L \subset K^2$ be a lattice, and $c \in \mathrm{GL}_2(\mathbb{A}_K)$. Defining a lattice $L_{\mathfrak{p}}$ to be the completion of L in $K_{\mathfrak{p}}$ for each prime $\mathfrak{p}$, the theorem says that almost all of the $L_{\mathfrak{p}}$ are equal to $\mathcal{O}_{\mathfrak{p}}^2$. As $c_{\mathfrak{p}} \in \mathrm{GL}_2(\mathcal{O}_{\mathfrak{p}})$ for almost all $\mathfrak{p}$, we see that $c_{\mathfrak{p}} L_{\mathfrak{p}} = \mathcal{O}_{\mathfrak{p}}^2$ for almost all $\mathfrak{p}$. We thus define cL to be the lattice corresponding to the local lattices $c_{\mathfrak{p}} L_{\mathfrak{p}}$.

Now, by a structure theorem for finitely generated torsion-free modules over a Dedekind domain, any lattice $L \subset K^2$ is isomorphic (as an $\mathcal{O}_K$-module) to $\mathcal{O}_K \oplus I$ for some integral ideal I of $\mathcal{O}_K$. The ideal class of I is known as the *Steinitz class* of L. Given another lattice $L' \cong \mathcal{O}_K \oplus I'$, we have that $L \cong L'$ as $\mathcal{O}_K$-modules if and only if they have the same Steinitz class, that is, $[I] = [I']$ in the ideal class group.

Note also that if two lattices L and L' in K^2 are isomorphic as $\mathcal{O}_K$-modules, we obtain an element $\gamma \in \mathrm{GL}_2(K)$ with $\gamma L = L'$ by tensoring an explicit isomorphism $L \to L'$ with K, obtaining an isomorphism $K \to K$ (that is, an element of $\mathrm{GL}_2(K)$).

Finally, to complete the preliminaries before giving the proof in the level one case, note that $\Omega_\infty := \mathrm{GL}_2(\mathbb{C}) \times \prod_{\mathfrak{p}} \mathrm{GL}_2(\mathcal{O}_{\mathfrak{p}})$ is the stabiliser in $\mathrm{GL}_2(\mathbb{A}_K)$ of the lattice $\mathcal{O}_K^2$.

Proof. (Theorem 2.2.4, Level one case). Let $c \in \mathrm{GL}_2(\mathbb{A}_K)$. Recall that $I_1, ..., I_h$ were defined above to be a complete set of (integral and prime) representatives for the class group of K; it thus follows from the above remarks that there is some i such that

$$c\mathcal{O}_K^2 \cong \mathcal{O}_K \oplus I_i.$$

Then, with a_i and g_i as defined above, we have $c\mathcal{O}_K^2 \cong g_i \mathcal{O}_K^2$, which means there is some $\gamma \in \mathrm{GL}_2(K)$ such that

$$\gamma c \mathcal{O}_K^2 = g_i \mathcal{O}_K^2 \quad \Rightarrow \quad g_i^{-1} \gamma c \mathcal{O}_K^2 = \mathcal{O}_K^2.$$

Thus $g_i^{-1}\gamma c \in \Omega_\infty$, which shows that $c \in \mathrm{GL}_2(K) g_i \Omega_\infty$. It's clear that there is only one i for which this holds, and hence we have the disjoint union as claimed. □

The result of this approximation theorem is that we now end up considering functions F from h copies of $\mathrm{GL}_2(\mathbb{C})$ into $V_{2k+2}(\mathbb{C})$, and ultimately $\mathcal{F}$ from h copies of $\mathcal{H}_3$ into $V_{2k+2}(\mathbb{C})$, as

$$\mathcal{F} : g_1\mathcal{H}_3 \sqcup \cdots \sqcup g_h\mathcal{H}_3 \longrightarrow V_{2k+2}(\mathbb{C}).$$

Here g_i acts on $\mathcal{H}_3$ by left multiplication when we consider it as a quotient of $\mathrm{GL}_2(\mathbb{C})$. We can (and will subsequently) consider such functions as h-tuples $(\mathcal{F}^1, ..., \mathcal{F}^h)$ of functions from $\mathcal{H}_3$ to $V_{2k+2}(\mathbb{C})$, with each $\mathcal{F}^i$ satisfying suitable transformation properties depending on g_i.

Remarks: (i) The above discussion highlights the importance of the adelic definition; we'd naturally prefer to categorise things by a single object rather than a collection of many that are in some sense compatible. In this instance, it is natural to use the adelic definition to derive and draw together functions on $\mathcal{H}_3$, which are then sometimes easier to work with.

(ii) Note that the collection $(\mathcal{F}^1, ..., \mathcal{F}^h)$ is *not* canonical, as it very much depends on the choice of idelic class group representatives a_i.

To give these h-tuples explicitly, we define

$$\begin{aligned} F^i : \mathrm{GL}_2(\mathbb{C}) &\longrightarrow V_{2k+2}(\mathbb{C}), \\ g &\longmapsto \Phi(g_i g). \end{aligned}$$

These functions F^i then satisfy automorphy properties under the groups

$$\Gamma_1^i(\mathfrak{n}) := \mathrm{SL}_2(K) \cap g_i\Omega_1(\mathfrak{n})\mathrm{GL}_2(\mathbb{C})g_i^{-1},$$

which are discrete subgroups of $\mathrm{SL}_2(K)$. It turns out that $\Gamma_1^i(\mathfrak{n}) = \Gamma_{I_i}$, where Γ_I is defined as follows:

Definition 2.2.5. Let I be an integral ideal of K that is coprime to $\mathfrak{n}$. Then we define the *twist of* $\Gamma_1(\mathfrak{n})$ *by* I to be

$$\Gamma_I := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(K) : a, d \in \mathcal{O}_K, c \in \mathfrak{n}I^{-1}, b \in I, d \equiv 1 \pmod{\mathfrak{n}} \right\}.$$

Later, we will discuss automorphy properties with regards to these twists, which are not necessarily congruence subgroups. Note however that if we take $I_1 = \mathcal{O}_K$, then $\Gamma_1^1(\mathfrak{n}) = \Gamma_{\mathcal{O}_K} = \Gamma_1(\mathfrak{n})$, hence the terminology.

2.2.4. Hyperbolic 3-Space

The following is a brief sketch of some of the relevant properties of $\mathcal{H}_3$. For a more detailed approach, see [EGM98].

There is an obvious action of $\mathrm{GL}_2(\mathbb{C})$ on $\mathcal{H}_3$ when we consider it as a quotient $\mathrm{GL}_2(\mathbb{C})/\mathbb{C}^\times \mathrm{SU}_2(\mathbb{C})$ as above. There are two further ways to describe this action; firstly, note that we can embed $\mathcal{H}_3$ in the quaternions

$$\mathbb{H} = \{a + bi + cj + dk : a, b, c, d \in \mathbb{C}, i^2 = j^2 = k^2 = -1, ij = k = -ji\}$$

via the map $(z, t) \mapsto z + tj$. The action can then be described as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot x = (ax + b)(cx + d)^{-1}.$$

Alternatively, the action can be worked out painfully and explicitly to give

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (z, t) = \left(\frac{(az + b)\overline{(cz + d)} + a\bar{c}|t|^2}{|cz + d|^2 + |ct|^2}, \frac{(ad - bc)t}{|cz + d|^2 + |ct|^2} \right). \tag{2.6}$$

Hyperbolic 3-space has the obvious structure of a real differentiable 3-manifold. The metric $ds^2 = (dzd\bar{z} + dt^2)/t^2$ further endows it with the structure of a Riemannian manifold.

We endow $\mathcal{H}_3$ with the subspace topology induced from $\mathbb{R}^3$. It can be compactified in a manner analogous to the real case by the addition of cusps; for a fixed imaginary quadratic field K, the set of *cusps of* K is defined to be $\mathbb{P}^1(K)$. The *extended upper half-space* is the space

$$\mathcal{H}_3^* := \mathcal{H}_3 \cup \{(s, 0) : s \in K\} \cup \{\infty\}$$

obtained by adjoining the cusps to $\mathcal{H}_3$. Viewing $\mathcal{H}_3$ as a subspace of the quaternions, and treating a cusp $(s, 0)$ as the quaternion $s + 0j$, we see that the action of $\mathrm{GL}_2(K)$ on $\mathbb{P}^1(K)$ by fractional linear transformations extends the action on $\mathcal{H}_3$, and thus we obtain an action on $\mathcal{H}_3^*$.

2.2.5. Passing from GL_2 to $\mathcal{H}_3$: comparing the literature

It remains to describe exactly how to pass from functions F on $\mathrm{GL}_2(\mathbb{C})$ to functions $\mathcal{F}$ on $\mathcal{H}_3$. Recall that we've defined some function $\Phi : \mathrm{GL}_2(\mathbb{A}_K) \to V_{2k+2}(\mathbb{C})$, and stated that from it we obtain a function $F : \mathrm{GL}_2(\mathbb{C}) \to V_{2k+2}(\mathbb{C})$ by passing to the quotient. We've also stated that any such F is totally defined by its values on the subspace $B \cong \mathcal{H}_3$ (as defined in equation (2.5)), but there are two genuinely different approaches to defining the actual value $\mathcal{F}(z, t)$

given in the literature. Here we give a brief account of both viewpoints.

Recall that in the rational case, in Proposition 2.1.1, we passed from a function F on $\mathrm{GL}_2(\mathbb{C})$ to $x^{-k/2}F(x,y)$ to restrict to $\mathcal{H}$, that is, we scaled the restriction by the 'height' of a point $z \in \mathcal{H}$ relative to the weight k. The result was that we obtained a function that behaved exactly like a modular form of weight k on $\mathcal{H}$. In [Gha99], Ghate takes an approach that specialises in weight 2 to this viewpoint. In particular, he defines his automorphic forms to have level $\Omega_0(\mathfrak{n})$ and character χ (for some Hecke character χ), and translates this scaling to the imaginary quadratic scenario by setting, for a function of weight $k+2$,

$$\mathcal{F}(z,t) = t^{-1-k}F\left[\frac{1}{\sqrt{t}}\begin{pmatrix} t & z \\ 0 & 1\end{pmatrix}\right] = t^{-1-k}\chi_\infty\left(\frac{1}{\sqrt{t}}\right)F\left[\begin{pmatrix} t & z \\ 0 & 1\end{pmatrix}\right],$$

where as above F takes values in $V_{2k+2}(\mathbb{C})$. Specialising to weight 2, if $\Gamma \leq \mathrm{SL}_2(\mathcal{O}_K)$ is some congruence subgroup for K, the automorphy condition we require such functions to satisfy is described as

$$\mathcal{F}(\gamma(z,t)) = \mathcal{F}(z,t)\rho(\gamma,(z,t)),$$

where

$$\rho(\gamma,(z,t)) = \begin{pmatrix} \bar{r}^2 & -\overline{rs} & \bar{s}^2 \\ 2\bar{r}s & |r|^2-|s|^2 & -2r\bar{s} \\ s^2 & rs & r^2\end{pmatrix}, \quad \gamma = \begin{pmatrix} a & b \\ c & d\end{pmatrix} \in \Gamma, r = \overline{cz+d}, s = \bar{c}t. \tag{2.7}$$

In accounts such as [Cre81], [CW94] and [Byg98], all of which deal predominantly with the weight two case, such a function $\mathcal{F}'$ is defined by simply restricting to B. This associates to F the natural analogue over K of the function $\mathrm{Im}(\tau)^{k/2}f(\tau)$ over $\mathbb{Q}$, where f is the associated modular form as described in Proposition 2.1.1. In weight 2, if χ_∞ is trivial[3], this gives $\mathcal{F}' = t\mathcal{F}$. The automorphy condition that they give is

$$\mathcal{F}'(\gamma(z,t)) = \frac{\mathcal{F}'(z,t)\rho(\gamma,(z,t))}{|cz+d|^2+|ct|^2}, \tag{2.8}$$

that is, the same as above up to a scaling factor. It is apparent that conditions (2.7) and (2.8) are compatible by observing the action described in equation (2.6) above; we've introduced a scaling factor of t in $\mathcal{F}$, and for $\gamma \in \Gamma$, we have $ad-bc=1$, showing that t scales as $|cz+d|^2+|ct|^2 = |r|^2+|s|^2$.

In each of these approaches, the authors go on to define a condition for a function $\mathcal{F} : \mathcal{H}_3 \to \mathbb{C}^3$ to be a 'cusp form of weight 2' to be either (2.7) or (2.8), so the spaces $S_2(\Gamma)$ of such forms are not, in a strict sense, equal from paper to paper. They are of course isomorphic, since there is a

[3]In weight 2, this is a natural thing to assume; it says that the infinity type of χ is $(0,0)$, and weight 2 corresponds to parallel weight $(0,0)$. See Section 4.2 for further details.

clear bijection between the sets of functions that arise. For our purposes, it is more convenient to adopt Ghate's convention, since we wish to work with arbitrary weight. Since we work with level $\Omega_1(\mathfrak{n})$ and suppress mention of the character, the following definition – which we'll use for the remainder of this thesis – is thus equivalent to Ghate's:

Definition 2.2.6. Let Φ be an automorphic form of weight (k,k) and level $\Omega_1(\mathfrak{n})$ over K. Let F^i be the function on $\mathrm{GL}_2(\mathbb{C})$ corresponding to the idelic class group representative a_i. Then define

$$\begin{aligned} \mathcal{F}^i : \mathcal{H}_3 &\longrightarrow V_{2k+2}(\mathbb{C}), \\ (z,t) &\longmapsto t^{-1} F^i \begin{pmatrix} t & z \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

Remark: For completeness, we state the automorphy condition for functions on $\mathcal{H}_3$ for general weights. Let

$$\mathcal{F} : \mathcal{H}_3 \longrightarrow V_{2k+2}(\mathbb{C})$$

be a function, and suppose Γ is a congruence subgroup of $\mathrm{SL}_2(\mathcal{O}_K)$. Then let

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma, r = \overline{cz+d}, s = \bar{c}t,$$

and

$$j(\gamma; z, t) = \begin{pmatrix} \bar{r} & s \\ -\bar{s} & r \end{pmatrix} = \begin{pmatrix} c & 0 \\ 0 & \bar{c} \end{pmatrix} \begin{pmatrix} z & -t \\ t & \bar{z} \end{pmatrix} + \begin{pmatrix} d & 0 \\ 0 & \bar{d} \end{pmatrix}.$$

We say $\mathcal{F}$ is *automorphic of level* Γ *and weight* (k,k) if

$$\mathcal{F}\left[\gamma \cdot (z,t); \begin{pmatrix} X \\ Y \end{pmatrix}\right] = \mathcal{F}\left[(z,t); j(\gamma; z, t) \begin{pmatrix} X \\ Y \end{pmatrix}\right]$$

for all $\gamma \in \Gamma$, where now the weight is implicitly defined by the space in which $\mathcal{F}$ takes values. This specialises to the condition given using Ghate's convention above in weight $(0,0)$, a result which the reader is encouraged to take on faith.

2.3. General number fields

Now let F be a general number field of degree $d = r_1 + 2r_s$, where r_1 and r_2 are the number of real and complex places of F respectively. The theory of automorphic functions over F is analogous to the work we've done previously, and the conditions at infinity are essentially a 'product' of the conditions for $\mathbb{Q}$ and imaginary quadratic fields, in a sense that should be clear from the definition below.

We fix $\mathcal{O}_F$, $\mathbb{A}_F$, $\widehat{\mathcal{O}}_F$, $\mathfrak{n}$ and $\Omega_1(\mathfrak{n})$ to be the analogues for general F of the corresponding objects for imaginary quadratic K. Further, we define

$$K_\infty^+ := \mathrm{SO}_2(\mathbb{R})^{r_1} \times \mathrm{SU}_2(\mathbb{C})^{r_2}.$$

The group K_∞^+ is a subgroup of the standard maximal compact subgroup $\mathrm{O}_2(\mathbb{R})^{r_1} \times \mathrm{U}_2(\mathbb{C})^{r_2}$ of $\mathrm{GL}_2(F \otimes_{\mathbb{Q}} \mathbb{R})$.

2.3.1. Weights

The notion of a *weight* is slightly more tricky to define in the general case; indeed, there are subtleties in this case that can be ignored in a first treatment over $\mathbb{Q}$ or imaginary quadratic fields. Over $\mathbb{Q}$ the weight is an integer $k \in \mathbb{Z}[\Sigma_{\mathbb{Q}}]$, and over an imaginary quadratic K it is a pair $(k, \ell) \in \mathbb{Z}[\Sigma_K]$. In reality, though, we define weights by using the reductive group GL_2. In particular, consider the case over $\mathbb{Q}$; we take $G = \mathrm{GL}_2$, and define T to be its maximal torus (corresponding to diagonal matrices). A *weight* should then be an algebraic character of the torus, which will be a *pair* of integers (k, v) such that the character has form

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \longmapsto a^{k+v}b^v = \det{}^v \cdot a^k.$$

Over $\mathbb{Q}$, it is harmless to assume $v = 0$ and suppress it from the definition. The case of an imaginary quadratic field K gives rise to a similar story; indeed, in this case, take $G := \mathrm{Res}_{K/\mathbb{Q}}\mathrm{GL}_2$ and T its maximal torus. Then, for example,

$$T(\mathbb{R}) \cong \mathbb{C}^\times \times \mathbb{C}^\times,$$

and the algebraic characters of this can be parametrised by pairs $[(k_1, k_2), (v_1, v_2)] \in \mathbb{Z}[\Sigma_K]^2$, with

$$[(k_1, k_2), (v_1, v_2)] : \begin{pmatrix} y & 0 \\ 0 & z \end{pmatrix} \longmapsto y^{k_1+v_1} z^{v_1} \cdot \overline{y}^{k_2+v_2} \overline{z}^{v_2}.$$

(This might seem unconventional, but we really do want a change in v to correspond to a change by a factor of the determinant). We restricted to the case where $k_1 = k_2$, and then could take $v_1 = v_2 = 0$. In the general case, however, it is a serious restriction to only consider weights where $v = 0$. With this in mind, we define:

Definition 2.3.1. A *weight* for F is an algebraic character of the maximal torus T of $\mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}_2$. Any weight can be identified with an element

$$\lambda := (\mathbf{k}, \mathbf{v}) \in \mathbb{Z}[\Sigma_F]^2.$$

We are actually interested in a smaller class of weights that lead to automorphic forms that are *cohomological*. For this, we recall the definition of *admissible* elements of $\mathbb{Z}[\Sigma_F]$ from Definition 1.2.4.

Definition 2.3.2. We say a weight $\lambda = (\mathbf{k}, \mathbf{v}) \in \mathbb{Z}[\Sigma_F]^2$ is *admissible* if $\mathbf{k} = c\mathbf{k}$, where c denotes complex conjugation, and $\mathbf{k} + 2\mathbf{v}$ is parallel and admissible as an element of $\mathbb{Z}[\Sigma_F]$.

We do not technically require the condition that $\mathbf{k} + 2\mathbf{v}$ is parallel, but it does not impose any serious restrictions, and we do so for simplicity. Likewise, we impose the condition that $\mathbf{k} = c\mathbf{k}$ for simplicity since non-zero cuspidal automorphic forms can exist only at weights satisfying this condition (see [Har87b]).

Remark: The major reason we introduce more general weights is to provide more general coefficient spaces; indeed, $\mathbf{v}$ represents a twist by the determinant. To give a toy example that readily generalises, over $\mathbb{Q}$ a weight is a pair (k, v), and the irreducible algebraic representations of $\mathrm{GL}_2(\mathbb{C})$ are indexed by such pairs that are *dominant*, that is, where $k \geq 0$ and $k \geq v$. In this case, the corresponding representation is $V_k(\mathbb{C})$ with the natural $\mathrm{GL}_2(\mathbb{C})$ action twisted by $\det^v$. This will be made more clear in the sequel.

2.3.2. Definition of the automorphy condition

Let $\lambda := (\mathbf{k}, \mathbf{v})$ be an admissible weight. We define $\mathbf{k}^* \in \mathbb{Z}[\Sigma_F]$ by

$$k_v^* := \begin{cases} 2k_v + 2 & : v \text{ complex}, \\ 0 & : v \text{ real}. \end{cases}$$

Then define

$$V_{\mathbf{k}^*}(\mathbb{C}) := \bigotimes_{v \in \Sigma_F(\mathbb{C})} V_{k_v^*}(\mathbb{C}).$$

We define a representation

$$\rho : K_\infty^+ \times (F \otimes_{\mathbb{Q}} \mathbb{R})^\times \longrightarrow \mathrm{GL}(V_{\mathbf{k}^*}(\mathbb{C}))$$

individually at each infinite place as follows:

- Suppose $v \in \Sigma(\mathbb{R})$. Then define

$$\rho_v : \mathrm{SO}_2(\mathbb{R}) \times \mathbb{R}^\times \longrightarrow \mathbb{C}^\times,$$
$$(r(\theta), x) \longmapsto e^{ik_v\theta} x^{-k_v - 2v_v}.$$

- Suppose $v \in \Sigma(\mathbb{C})$. Then define

$$\rho_{v,k_v^*} : \mathrm{SU}_2(\mathbb{C}) \longrightarrow \mathrm{GL}(V_{k_v^*}(\mathbb{C}))$$

to be the usual map, and then define

$$\begin{aligned}\rho_v : \mathrm{SU}_2(\mathbb{C}) \times \mathbb{C}^\times &\longrightarrow \mathrm{GL}(V_{k_v^*}(\mathbb{C})),\\ (u,z) &\longmapsto \rho_{v,k_v^*}(u)|z|^{-k_v-2v_v}.\end{aligned}$$

Then define

$$\rho := \bigotimes_{v\in\Sigma(\mathbb{R})\cup\Sigma(\mathbb{C})} \rho_v.$$

Definition 2.3.3. We say a function $\Phi : \mathrm{GL}_2(\mathbb{A}_F) \to V_{\mathbf{k}^*}(\mathbb{C})$ is *automorphic of weight* λ *and level* $\Omega_1(\mathfrak{n})$ if it satisfies:

(i) (Automorphy condition) $\Phi(guz) = \Phi(g)\rho(u,z)$ for $u \in K_\infty^+$ and $z \in Z(\mathrm{GL}_2(F \otimes_\mathbb{Q} \mathbb{R})) = (F \otimes_\mathbb{Q} \mathbb{R})^\times$,

(ii) (Level condition) Φ is right-invariant under translation by $\Omega_1(\mathfrak{n})$,

(iii) and Φ is left-invariant under $\mathrm{GL}_2(F)$.

Remark: The condition that $\mathbf{k}+2\mathbf{v}$ is admissible is crucial for this to be well-defined. Indeed, it means that conditions (i) and (iii) are compatible where they overlap. If we hadn't introduced $\mathbf{v}$, there would only be a narrow range of weights for which these conditions would be compatible and the definition would make sense. We could have modified the definition in a different way to remove this problem, but this would be less natural and indeed would cause problems when we consider cohomology classes associated to automorphic forms.

2.3.3. Explicit descriptions

As in the rational and imaginary quadratic cases, one can write down more explicit descriptions of functions satisfying these conditions. In particular, in much the same way, suppose that F has (narrow)[4] class number h, and recall that we took $a_1, ..., a_h$ to form a complete set of idelic representatives for the narrow class group Cl_F^+. Define $g_i := \left(\begin{smallmatrix} a_i & 0 \\ 0 & 1\end{smallmatrix}\right)$. Then we have strong approximation over F:

$$\mathrm{GL}_2(\mathbb{A}_F) = \bigsqcup_{i=1}^{h} \mathrm{GL}_2(F) \cdot g_i \cdot [\mathrm{GL}_2^+(F \otimes_\mathbb{Q} \mathbb{R}) \times \Omega_1(\mathfrak{n})]. \tag{2.9}$$

Hence, as over K, a function Φ on $\mathrm{GL}_2(\mathbb{A}_F)$ satisfying the above conditions descends to a collection of h functions

$$F^1, ..., F^h : \mathrm{GL}_2^+(F \otimes_\mathbb{Q} \mathbb{R}) \longrightarrow \mathcal{V}_{\mathbf{k}^*}(\mathbb{C}),$$

[4] In the general case, it is necessary to consider the *narrow* class group. Over an imaginary quadratic field, the narrow class group and usual class group coincide.

and then further to a collection of functions

$$\mathcal{F}^1, ..., \mathcal{F}^h : \mathcal{H}_F \longrightarrow \mathcal{V}_{\mathbf{k}^*}(\mathbb{C}),$$

where

$$\mathcal{H}_F := \mathcal{H}^{\Sigma(\mathbb{R})} \times \mathcal{H}_3^{\Sigma(\mathbb{C})}. \tag{2.10}$$

Furthermore, these functions satisfy automorphy conditions under the groups

$$\Gamma_1^i(\mathfrak{n}) := \mathrm{SL}_2(F) \cap g_i \Omega_1(\mathfrak{n}) \mathrm{GL}_2^+(F \otimes_{\mathbb{Q}} \mathbb{R}) g_i^{-1}. \tag{2.11}$$

Remark: Again, note that this identification is not canonical, as it depends on the choice of class group representatives.

Chapter 3

Automorphic Forms

In previous sections, we have defined adelic analogues of the transformation property that classical modular forms satisfy. In this chapter, we complete the definition of an automorphic form by sketching the analogue of the holomorphicity condition in the classical case. We will keep this discussion brief; indeed, clear and detailed expositions of the theory exist in the literature (see, for example, [Wei71] and [Byg98]). We end by giving a complete definition of automorphic forms in the generality we require.

3.1. Harmonic differential forms

In the theory of classical modular forms, we require that modular forms are holomorphic functions on the upper half-plane. This fundamentally relies on the fact that we can exploit the corresponding complex structure on the upper half-plane. When we consider the theory over a general number field F, we have such a complex structure on the analogous space $\mathcal{H}_F = \mathcal{H}^{r_1} \times \mathcal{H}_3^{r_2}$ if and only if F is totally real, that is, if and only if $r_2 = 0$. In this case, we have a natural notion of holomorphicity at each infinite (real) place v, and we say an automorphic function $\Phi : \mathrm{GL}_2(\mathbb{A}_F) \longrightarrow V_{\mathbf{k}^*}$ is *holomorphic* if the h functions it induces on $\mathcal{H}_F$ are holomorphic at each real place.

When F has complex places, the situation becomes more difficult. There is no complex structure on the upper half-space $\mathcal{H}_3$, which means that a whole array of nice properties satisfied over totally real fields have no known analogue in the general case. In particular, to say something is 'holomorphic' becomes meaningless. Instead, we use *harmonicity*, a property that can be readily defined without complex structure and provides a generalisation of holomorphicity in the case where there *is* complex structure.

In this brief section, we discuss harmonic differential forms and show that harmonicity does indeed generalise holomorphicity. We also give some remarks on the definition of the automorphy condition from the previous chapter, which was stated without motivation. The account given here is a short summary of Chapter 4 of [Byg98].

3.1.1. Definitions

If V is a real differential manifold, then the *tangent bundle* is the vector bundle with fibre at z equal to the tangent space T_zV of V at z. We then define another vector bundle on V by setting the fibre at z to be T_zV^*, that is, the *cotangent bundle.* A *differential 1-form* on V is a smooth section of the cotangent bundle. In concrete terms, it means that for each point $z \in V$, we pick a linear map $\omega_z = \omega(z) : T_zV \to \mathbb{R}$ in such a way that the association varies smoothly as we vary z. We denote the space of differential 1-forms by $\Omega^1(V)$. If we have a group G acting on the left of V, then we obtain a left-action on $\Omega^1(V)$ by setting $(\gamma{\cdot}\omega)_z = \omega_{\gamma^{-1}z}$.

More generally, if $E \to V$ is a smooth vector bundle, then an *E-valued differential 1-form* is a smooth section of the tensor product bundle of E and the cotangent bundle; we denote the space of E-valued differential 1-forms by $\Omega^1(V, E)$. Again, less abstractly, we can consider an E-valued differential 1-form as a collection of maps $\omega_z : T_zV \to E_z$. Most pertinently, we will be interested in differentials with values in polynomial spaces, that is, when E corresponds to a local system arising from a tensor product of copies of $V_n(\mathbb{C})$.

Differential r-forms are defined in a very similar fashion; an *r-form* is a smoothly varying choice of r-linear alternating forms on the tangent spaces, that is, a compatible collection of r-antilinear maps $\phi_z : (T_zV)^r \to \mathbb{R}$, one for each $z \in V$. The space of r-forms on V is denoted $\Omega^r(V)$. We can define an E-valued r-form in precisely the same manner as above, and denote the corresponding space by $\Omega^r(V, E)$.

3.1.2. The Hodge star operator and harmonicity

Let V be a Riemannian manifold of dimension m as above. There is an operator

$$* : \Omega^r(V, \mathbb{C}) \longrightarrow \Omega^{m-r}(V, \mathbb{C})$$

on differential forms with the following properties: for r-forms α and β,

(i) $* * \alpha = (-1)^{r(m+1)}\alpha$,

(ii) $\alpha \wedge *\beta = \beta \wedge *\alpha$, and

(iii) $\alpha \wedge *\alpha = f dx_1 \wedge \cdots \wedge dx_m$, where $f \geq 0$ and $\{dx_1, ..., dx_m\}$ is a positively orientated orthonormal basis for $\Omega^m(V, \mathbb{C})$.

See [Byg98], Chapter 4.1.3 for further details.

Definition 3.1.1. Define a Hermitian inner product on $\Omega^r(V, \mathbb{C})$ by $(\alpha, \beta) := \int_V \alpha \wedge *\beta$.

There is also a standard differentiation operator

$$d : \Omega^r(V,\mathbb{C}) \longrightarrow \Omega^{r+1}(V,\mathbb{C}),$$

called the *exterior derivative*. This allows us to define a map

$$\delta : \Omega^r(V,\mathbb{C}) \longrightarrow \Omega^{r-1}(V,\mathbb{C})$$

by setting

$$\delta := (-1)^{m(r+1)+1} * d * .$$

A simple check shows that δ is the adjoint of d under the Hermitian form defined above.

Definition 3.1.2. The *Laplace operator* is defined to be

$$\Delta = \delta d + d\delta.$$

Definition 3.1.3. A differential form ω is said to be *harmonic* if $\Delta\omega = 0$.

Proposition 3.1.4. *A differential form ω is harmonic if and only if it is closed and co-closed, that is, if $d\omega = d(*\omega) = 0$.*

Proof. Using the fact that d and δ are adjoint to each other under the Hermitian inner product, we see that

$$(\Delta\omega, \omega) = (d\omega, d\omega) + (\delta\omega, \delta\omega) = (\omega, \Delta\omega).$$

The result follows, since $\delta\omega = 0$ if and only if $d(*\omega) = 0$. $\square$

Remarks 3.1.5: Having given the definition, it is worthwhile pointing out why harmonic differential forms are so useful in practice. They satisfy two major nice properties, of which the first has more relevance to our situation.

(i) A harmonic differential form can be integrated in a path-independent manner; that is, if we take two points $x, y \in V$ and any path γ between them, then the integral $\int_x^y \omega := \int_\gamma \omega$ is well-defined. This will be important when attaching a modular symbol to a cusp form.

(ii) The second important feature of harmonic forms is their relation to de Rham cohomology groups. The *de Rham cohomology* of a Riemannian manifold V is the homology of the cochain complex given by the abelian groups $\Omega^r(V)$ together with the maps $d : \Omega^r(V) \to \Omega^{r+1}(V)$ (since it can be shown that $d^2 = 0$). The relation mentioned above is that each de Rham cohomology class contains a unique harmonic representative.

3.1.3. Aside: irreducible representations of $\mathrm{SU}_2(\mathbb{C})$

In our earlier definition of automorphic functions over a number field F, we made – without motivation – the stipulation that such a function Φ should satisfy

$$\Phi(gu) = \Phi(g)\rho(u), \quad g \in \mathrm{GL}_2(\mathbb{A}_F), u \in K_\infty^+,$$

where ρ is an irreducible representation of K_∞^+. In this section, we give some motivation for such a condition in the case that $F = K$ is imaginary quadratic and the weight is $(0,0)$.

Consider first the rational case. At first introduction, the automorphy condition defining classical rational modular forms appears quite unnatural, yet it turns out to be closely linked to differentials, in the sense that the 1-form dz satisfies the property that $d(\gamma z) = (cz+d)^{-2}dz$ for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$. This bears direct comparison with the transformation condition for weight 2 classical modular forms. To motivate the imaginary quadratic case, we'll again look at certain differentials and how they translate under suitable groups.

Let G be an arbitrary Lie group that has the structure of a real Riemannian manifold. Left translation by an element $g \in G$, denoted L_g, induces a pull-back action L_g^* on (real- or complex-valued) differentials. We say a differential $\omega \in \Omega^r(G,\mathbb{C})$ is *left-invariant* if $L_g^*\omega = \omega$ for all $g \in G$. The space of left-invariant differentials has the desirable property that we can choose a basis; if, for example, we choose a set of complex differentials (β_i) such that the evaluations $((\beta_i)_I)$ at the identity span the space $(T_I G)^*$, then we can write any left-invariant 1-form ω uniquely as

$$\omega = \sum_i \alpha_i \beta_i, \quad \alpha_i \in \mathbb{C}.$$

Definition 3.1.6. We choose a basis $\beta = (\beta_0, \beta_1, \beta_2)^T$ for the left-invariant 1-forms on $B = \mathcal{H}_3$ by setting

$$\beta_0 = \frac{dz}{t}, \quad \beta_1 = -\frac{dt}{t}, \quad \text{and } \beta_2 = -\frac{d\overline{z}}{t}.$$

The projection $\pi : \mathrm{GL}_2(\mathbb{C}) \to \mathcal{H}_3$ gives differentials $\pi^*\beta_i$ on $\mathrm{GL}_2(\mathbb{C})$. These will certainly be left-invariant under B, since the β_i are, but they need not be left-invariant under the whole group $\mathrm{GL}_2(\mathbb{C})$. The question of measuring how far from left-invariant the $\pi^*\beta_i$ are now dictates how an automorphic form should behave under translation by $B\backslash\mathrm{GL}_2(\mathbb{C}) = \mathbb{C}^\times \cdot \mathrm{SU}_2(\mathbb{C})$.

We've written $\beta = \beta_z$ (for $z \in \mathcal{H}_3$) as a column vector. Define, for $g \in \mathrm{GL}_2(\mathbb{C})$ and $z \in \mathcal{H}_3$, the Jacobian matrix $J(g;z)$ by

$$\beta_{gz} = J(g;z)\beta_z.$$

As a function, J satisfies the cocycle relation

$$J(g_1g_2, z) = J(g_1, g_2z)J(g_2, z);$$

left-invariance under B also gives $J(b, z) = 1$ for all $b \in B$, and combined with the cocycle relation we have

$$J(g; z) = J(\pi(g)^{-1}g; z),$$

where $\pi(g)^{-1}g \in \mathbb{C}^\times \cdot \mathrm{SU}_2(\mathbb{C})$. Now, we have $\pi \circ L_g = L_g \circ \pi$, and pulling this equality back gives the relation

$$(\pi^*\beta)(gh) = J(g; \pi(h))(\pi^*\beta)(h),$$

which measures the failure of left-invariance, as desired. Note that we have $\pi(h) = h \cdot \pi(1)$, that is, $\pi(1)$ with h acting on the left. Using this to expand further, the cocycle relation yields

$$\begin{aligned} J(g, \pi(h)) &= J(gh; \pi(1))J(h; \pi(1))^{-1} \\ &= J(\pi(gh)^{-1}gh; \pi(1))J(\pi(h)^{-1}h; \pi(1)), \end{aligned}$$

so that J is entirely determined by the values $J(u; \pi(1))$ for $u \in \mathbb{C}^\times \cdot \mathrm{SU}_2(\mathbb{C})$. Define a function

$$\rho : \mathbb{C}^\times \cdot \mathrm{SU}_2(\mathbb{C}) \to \mathbb{C}$$

by $\rho(u) = J(u; \pi(1))$. The cocycle relation now says that

$$\rho(uv) = J(uv; \pi(1)) = J(u; v\pi(1))J(v; \pi(1)) = \rho(u)\rho(v),$$

since $v\pi(1) = \pi(1)$ for $v \in \mathbb{C}^\times \cdot \mathrm{SU}_2(\mathbb{C})$. So ρ is a representation of $\mathbb{C}^\times \cdot \mathrm{SU}_2(\mathbb{C})$, and the transformation properties of left-invariant differentials are thus given by such representations, giving the desired motivation.

3.1.4. Harmonicity generalising holomorphicity

We earlier claimed that harmonicity was the appropriate generalisation of holomorphicity in the rational case. The following is a sketch of the proof of this statement.

Recall that a differential form ω is harmonic if and only if $d\omega = d(*\omega) = 0$. In the rational case, the space of left-invariant complex valued differentials on $\mathcal{H}$ has dimension 2; we choose the basis $(\beta_1, \beta_2) = (dz/y, -d\overline{z}/y)$, where $z = x + iy$. Then a (not necessarily left-invariant) differential 1-form of the form $\omega = f_1\beta_1 + f_2\beta_2$, for any functions $f_i : \mathcal{H} \to \mathbb{C}$, has

$$*(f_1\beta_1 + f_2\beta_2) = -i(\overline{f_1}\beta_2 - \overline{f_2}\beta_1)$$

(see [Byg98], Lemma 60). After taking the complex conjugate, we see further that $*(f_1\beta_1+f_2\beta_2)$ is closed if and only if $f_1\beta_1 - f_2\beta_2$ is. This then shows that $f_1\beta_1 + f_2\beta_2$ is harmonic if and only if $f_1\beta_1$ and $f_2\beta_2$ are both closed, which is if and only if

$$\frac{f_1}{y}dz \text{ and } \frac{\overline{f_2}}{y}dz$$

are closed.

Now, a differential of form $h(z)dz$ is closed if and only if $\partial h/\partial \overline{z} = 0$. But we have the identity

$$\frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right),$$

so that if $h = u + iv$ for real valued functions u and v, then $\partial h/\partial \overline{z} = 0$ if and only if u and v satisfy the Cauchy-Riemann equations, that is, if h is holomorphic. To conclude, this shows that

Proposition 3.1.7. *The differential $f_1\beta_1 + f_2\beta_2$ is harmonic if and only if $f_1(z)/y$ and $\overline{f_2}/y$ are holomorphic functions of $z = x + iy$.*

How does this relate to modular forms? The above argument - again for simplicity of exposition - only treats the weight 2 case. If we take a rational modular form f of weight 2, then we define $f_1(z) = \mathrm{Im}(z)f(z)$ and $f_2(z) = f_1(\overline{z})$, and obtain a differential form

$$\omega_f := f_1(z)\beta_1 + f_2(z)\beta_2 = f(z)dz - f(\overline{z})d\overline{z}.$$

Then ω_f is harmonic if and only if f is holomorphic.

Remark: Here we also have a $d\overline{z}$ term. In the more typical rational theory, we'd only consider differentials of the form $f(z)dz$. If, in the definition of automorphic functions, we worked with $\mathrm{GL}_2(\mathbb{R})$ rather than $\mathrm{GL}_2^+(\mathbb{R})$, we'd have two terms, yielding two functions – a 'holomorphic' part, f, and an 'anti-holomorphic' part, $f(\overline{z})$. Since in this case the anti-holomorphic part depends entirely on the holomorphic part, we can – and do – usually safely ignore it.

3.2. Differential forms attached to automorphic functions

In this section, we attach differential forms to the automorphic functions defined in the previous chapter. We'll start by describing the modules these differential forms take values in, before giving the construction in the rational and imaginary quadratic cases, from which the construction in the general case follows.

First, we need a definition:

Definition 3.2.1. Let F be a number field, and let $\mathbf{k} \in \mathbb{Z}[\Sigma_F]$. We define

$$V_{\mathbf{k}}(\mathbb{C}) := \bigotimes_{\sigma\in\Sigma} V_{k_\sigma}(\mathbb{C}).$$

This is naturally a (right) $\mathrm{SU}_2(\mathbb{C})$-module via the action on the components.

Recall that we wrote the degree of F as $d = r_1 + 2r_2$, for r_1 and r_2 the number of real and complex places of F respectively. Define $q := r_1 + r_2$. The main result we require is the following:

Theorem 3.2.2. *Let $\Phi : \mathrm{GL}_2(\mathbb{A}_F) \longrightarrow V_{\mathbf{k}^*}(\mathbb{C})$ be an automorphic function of weight $\lambda = (\mathbf{k}, \mathbf{v})$ and level $\Omega_1(\mathfrak{n})$ (in the sense of Definition 2.3.3), descending to a collection of h functions $\mathcal{F}^1, ..., \mathcal{F}^h$ on $\mathcal{H}_F$. For each i, there is a $V_{\mathbf{k}}(\mathbb{C})$-valued differential q-form $\omega_{\mathcal{F}^i}$ on $\mathcal{H}_F$ attached to $\mathcal{F}^i$.*

If one knows the construction of this differential in the cases that F is $\mathbb{Q}$ or imaginary quadratic, then one can obtain the construction in the general case by essentially taking a tensor product over the infinite places. Given this, we will only treat these two cases here, and refer the reader to [Hid94], Section 2 for the situation over a general number field.

Suppose $F = \mathbb{Q}$, and that $f : \mathcal{H} \longrightarrow \mathbb{C}$ is a modular function of level $\Gamma_1(N)$ and weight k.[1] We define ω_f to be the differential

$$\omega_f(z) := f(z)(zX + Y)^k dz$$

on $\mathcal{H}$.

Now suppose $F = K$ is imaginary quadratic. In the sequel, we will treat this case in detail, so here we include only a sketch proof for completeness. Suppose that $\Phi : \mathrm{GL}_2(\mathbb{A}_K) \longrightarrow V_{2k+2}(\mathbb{C})$ is an automorphic function of weight (k, k) and level $\Omega_1(\mathfrak{n})$, giving rise to functions $\mathcal{F}^1, ..., \mathcal{F}^h$ as in the theorem. Fix an i, and let F^i be the corresponding function on $\mathrm{GL}_2(\mathbb{C})$. Note that by the Clebsch-Gordon map, there is an injection

$$V_{2k+2} \hookrightarrow V_k(\mathbb{C}) \otimes V_k(\mathbb{C}) \otimes V_2(\mathbb{C}),$$

of $\mathrm{SU}_2(\mathbb{C})$-modules, and that after restricting to $\mathrm{SL}_2(\mathbb{C})$ and composing with this map, we can

[1] It is very important to note here that a 'modular function of weight k' in the sense of Definition 2.3.3 actually corresponds to a modular form of weight $k + 2$. In particular, a weight 2 modular form corresponds to a weight 0 automorphic function in this sense. This is another case of the 'shift by 2' that we can push around but not eliminate.

consider F^i as a function

$$F^i : \mathrm{SL}_2(\mathbb{C}) \longrightarrow V_k(\mathbb{C}) \otimes V_k(\mathbb{C}) \otimes V_2(\mathbb{C}). \tag{3.1}$$

One can identify $V_2(\mathbb{C})$ with the space of differentials spanned by the basis $(\beta_0, \beta_1, \beta_2)$ considered in Definition 3.1.6. Via this, we can see the map F^i in (3.1) as determining an element of $V_k(\mathbb{C}) \otimes V_k(\mathbb{C}) \otimes \Omega^1(\mathrm{SL}_2(\mathbb{C}), \mathbb{C}) \cong \Omega^1(\mathrm{SL}_2(\mathbb{C}), V_k(\mathbb{C}) \otimes V_k(\mathbb{C}))$. After a small modification, this differential descends to the quotient $\mathcal{H}_3 = \mathrm{SL}_2(\mathbb{C})/\mathrm{SU}_2(\mathbb{C})$, giving the required differential on $\mathcal{H}_3$.

For a detailed proof in this case, see Chapter 5.4.3.

3.3. B-moderacy

The one remaining condition that we're yet to mention in any form is the analogue of 'holomorphicity at the cusps'. This relates to a condition known as *B-moderacy*, which prescribes 'boundedness' as a function approaches a cusp. We continue to assume that K is imaginary quadratic, though the ideas generalise easily. Let $||\cdot||$ denote any norm on the complex vector space $V_{2k+2}(\mathbb{C})$, and let

$$B := \left\{ \begin{pmatrix} t & z \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{A}_K) \right\}.$$

Definition 3.3.1. Let $\Phi : \mathrm{GL}_2(\mathbb{A}_K) \to V_{2k+2}(\mathbb{C})$ be an automorphic function of weight (k, k) and level $\Omega_1(\mathfrak{n})$. We say that Φ is *B-moderate* if there exists an $N \geq 0$ such that for every compact subset S of B,

$$\left|\left| \Phi\left[\begin{pmatrix} t & z \\ 0 & 1 \end{pmatrix}\right] \right|\right| = O(|t|^N + |t|^{-N}), \quad t \in \mathbb{A}_K^\times, z \in \mathbb{A}_K$$

uniformly over $\left(\begin{smallmatrix} t & z \\ 0 & 1 \end{smallmatrix}\right) \in S$.

3.4. Definition of automorphic forms

We now come to the definition of automorphic forms.

Notation: Recall the notation: we take F to be a number field, $\lambda = (\mathbf{k}, \mathbf{v}) \in \mathbb{Z}[\Sigma_F]^2$ to be an admissible weight, and $\Omega_1(\mathfrak{n})$ to be the (adelic) level group defined in equation (2.3).

Definition 3.4.1. Let $\Phi : \mathrm{GL}_2(\mathbb{A}_F) \longrightarrow V_{\mathbf{k}^*}(\mathbb{C})$ be a function. We say that Φ is an *automorphic form of weight λ and level* $\Omega_1(\mathfrak{n})$ if:

(i) Φ is an automorphic function of weight λ and level $\Omega_1(\mathfrak{n})$ (see Definition 2.3.3),

(ii) Φ is B-moderate, and

(iii) if we write $\mathcal{F}^1, ..., \mathcal{F}^h$ for the functions on $\mathcal{H}_F$ corresponding to Φ for some fixed choice of representatives for the narrow class group, then the differential q-forms $\omega_{\mathcal{F}^i}$ are harmonic for $i = 1, ..., h$.

To compare to the classical case, condition (ii) is the analogue of being holomorphic at the cusps, whilst condition (iii) is the analogue of being holomorphic on the upper half-plane.

Remark: Condition (iii) is independent of the choice of class group representatives. Indeed, it can be stated in the following intrinsic form:

(iii′) write Φ_∞ for the restriction of Φ to $\mathrm{GL}_2(F_\infty^+)$, where F_∞^+ is the connected component of the identity in F_∞. Then Φ_∞ is an eigenfunction of the operators δ_v for all infinite places v, with

$$\delta_v(\Phi_\infty) = \left(\frac{k_v^2}{2} + k_v\right)\Phi_\infty,$$

where δ_v is a component of the Casimir operator in the Lie algebra $\mathfrak{sl}_2(\mathbb{C}) \otimes_{\mathbb{R}} \mathbb{F}_v$ (see [Hid93], Section 1.3).

Indeed, this form of the condition is necessary to define automorphic forms in even greater generality.

3.5. Cusp Forms

We are primarily interested in cusp forms, and whilst we will not be in a position to explain the cuspidality condition until the next section, it is straightforward to state.

Definition 3.5.1. Let $\Phi : \mathrm{GL}_2(\mathbb{A}_F) \to V_{\mathbf{k}^*}(\mathbb{C})$ be an automorphic form of weight λ and level $\Omega_1(\mathfrak{n})$. Then Φ is a *cusp form* if it satisfies the additional condition

(iv) For all $g \in \mathrm{GL}_2(\mathbb{A}_F)$, we have

$$\int_{F\backslash\mathbb{A}_F} \Phi(ug)du = 0,$$

where we consider $\mathbb{A}_F$ to be embedded inside $\mathrm{GL}_2(\mathbb{A}_F)$ by the map sending u to $\left(\begin{smallmatrix} 1 & u \\ 0 & 1 \end{smallmatrix}\right)$, and du is the Lebesgue measure on $\mathbb{A}_F$.

Remarks: (i) The cuspidal condition is a natural one; the value of the integral for a fixed g corresponds to a constant Fourier coefficient. We will give some motivation for this statement in the sequel in the case where F is imaginary quadratic and the weight is $(0, 0)$.

(ii) The general definitions given above are already slightly tailored to work with cusp forms, in that we've restricted to weights satisfying $\mathbf{k} = c\mathbf{k}$, where c denotes complex conjugation; in [Har87b], it is shown that all cusp forms have such weights. More generally, it is possible to define general automorphic forms of weight $\mathbf{k}$ where $\mathbf{k} \neq c\mathbf{k}$. We make no further mention of this here, however.

Definition 3.5.2. We write $S_\lambda(\Omega_1(\mathfrak{n}))$ for the space of adelic cusp forms of weight λ and level $\Omega_1(\mathfrak{n})$.

Chapter 4

L-Functions

In this chapter, we attach classical complex L-functions to the automorphic forms defined above. There are two equivalent ways of doing this: either through the theory of Hecke operators or Fourier expansions. For the former, we take an eigenform for the Hecke operators and build a Dirichlet series out of the Hecke eigenvalues, and for the latter, we do so by using the Fourier coefficients. We start by describing the theory of Hecke operators, then by writing down the Fourier expansion of an automorphic form, before finally defining the L-function.

4.1. Hecke operators

As in the classical case, there is a rich theory of Hecke operators for the automorphic forms defined in the previous chapter. We will define them here.

Let $\mathfrak{p}$ be a prime of F, and consider the double coset

$$\Omega_1(\mathfrak{n})\gamma_{\mathfrak{p}}\Omega_1(\mathfrak{n}),$$

where we define $\gamma_{\mathfrak{p}} \in \mathrm{GL}_2(\mathbb{A}_F^f)$ by

$$(\gamma_{\mathfrak{p}})_v = \begin{cases} \begin{pmatrix} \pi_{\mathfrak{p}} & 0 \\ 0 & 1 \end{pmatrix} & : v \text{ corresponds to the finite prime } \mathfrak{p}, \\ \mathrm{I} & : \text{otherwise}, \end{cases}$$

where $\pi_{\mathfrak{p}}$ is a (fixed) uniformiser of $\mathcal{O}_{\mathfrak{p}}$. Then the $T_{\mathfrak{p}}$ Hecke operator is given by the double coset operator $[\Omega_1(\mathfrak{n})\gamma_{\mathfrak{p}}\Omega_1(\mathfrak{n})]$. More explicitly, we have a decomposition

$$\Omega_1(\mathfrak{n})\gamma_{\mathfrak{p}}\Omega_1(\mathfrak{n}) = \bigsqcup_{i\in I} \Omega_1(\mathfrak{n})\gamma_i,$$

for some finite set I and $\gamma_i \in \mathrm{GL}_2(\mathbb{A}_F^f)$. Then, for an automorphic form Φ of weight λ and level $\Omega_1(\mathfrak{n})$, we set

$$\Phi|T_{\mathfrak{p}}(g) = \sum_{i\in I} \Phi(g\gamma_i).$$

When $\mathfrak{p}$ divides $\mathfrak{n}$, we denote the operator instead by $U_{\mathfrak{p}}$.

Remark: This is independent of the choice of representatives γ_i using the invariance of Φ under $\Omega_1(\mathfrak{n})$.

We can similarly define Hecke operators for each integral ideal I of F. Indeed, let $\gamma_I := \prod_{\mathfrak{p}} \gamma_{\mathfrak{p}}^r$, where $\mathfrak{p}^r$ exactly divides I; then the Hecke operator at I is given by the double coset operator $[\Omega_1(\mathfrak{n})\gamma_I\Omega_1(\mathfrak{n})]$. Such Hecke operators are totally determined by the Hecke operators at primes, as in the rational case. For ideals I that are coprime to $\mathfrak{n}$, we write T_I for the Hecke operator at I, and for others, we write U_I.

Remarks: Many of the nice properties that classical modular forms enjoy with respect to Hecke operators also carry over to this situation. In particular:

(i) If Φ is an eigenform for the Hecke operators, then the Hecke eigenvalues are algebraic, and for a suitable choice of $\mathbf{v}$ in the weight, they also satisfy integrality conditions (see [Hid94], Section 6).

(ii) We have *multiplicity one* theorems, as over $\mathbb{Q}$.

4.2. Fourier expansions

Here, we give a brief introduction to Fourier expansions of automorphic forms with a view to defining their L-functions. In the sequel, we will return to this to give accounts that are more tailored to our purposes; in particular, in Part II, we shall describe Fourier expansions of the individual components $\mathcal{F}^i$ of an automorphic form over an imaginary quadratic field (removing references to adeles).

4.2.1. Statement of the expansion

The following is taken almost verbatim from [Hid94], Section 6.

We need the following definitions:

Definition 4.2.1. Throughout, let F be an arbitrary number field with set of real places $\Sigma(\mathbb{R})$ and complex places $\Sigma(\mathbb{C})$. Let $\lambda = (\mathbf{k}, \mathbf{v})$ be an admissible weight.

(i) Let n be an integer, and let $K_n(x)$ be the *modified Bessel function*, that is, the unique solution to

$$\frac{d^2K_n}{dx^2} + \frac{1}{x}\frac{dK_n}{dx} - \left(1 + \frac{n^2}{x^2}\right)K_n = 0,$$

with asymptotic behaviour

$$K_n(x) \sim \sqrt{\frac{\pi}{2x}}e^{-x}$$

as $x \to \infty$. Note that, in particular, $K_{-n} = K_n$.

(ii) For $v \in \Sigma(\mathbb{R})$, define the *Whittaker function at v* to be

$$\begin{aligned} W_{\lambda,v} : \mathbb{R}^\times &\longrightarrow \mathbb{C}, \\ y &\longmapsto |y|^{-v_v} e^{-2\pi|y|}. \end{aligned}$$

(iii) For $v \in \Sigma(\mathbb{C})$, define the Whittaker function at v to be

$$\begin{aligned} W_{\lambda,v} : \mathbb{C}^\times &\longrightarrow V_{2k_v+2}(\mathbb{C}), \\ y &\longmapsto \sum_{n=0}^{k_v+2} \binom{2k_v+2}{n} |y|^{-2v_v} \left(\frac{y}{i|y|}\right)^{k_v+1-n} K_{n-k_v-1}(4\pi|y|) S_v^{2k_v+2-n} T_v^n. \end{aligned}$$

(iv) Define

$$\begin{aligned} W_\lambda : (F \otimes_{\mathbb{Q}} \mathbb{R})^\times &\longrightarrow V_{\mathbf{k}^*}(\mathbb{C}), \\ \mathbf{y} &\longmapsto \prod_{v \in \Sigma(\mathbb{R}) \cup \Sigma(\mathbb{C})} W_{\lambda,v}(y_v). \end{aligned}$$

(v) Define $F_+^\times$ to be the set of totally positive elements of $F^\times$, that is, the elements of F that are positive under every real embedding.

(vi) Write $\mathfrak{D}$ for the different of F.

Recall the definition of e_F from Definition 1.3.1. Then we have:

Theorem 4.2.2. *Let Φ be an automorphic form of weight $\lambda = (\mathbf{k}, \mathbf{v})$ and level $\Omega_1(\mathfrak{n})$. Then there is a Fourier expansion*

$$\Phi\left(\begin{pmatrix} \mathbf{y} & \mathbf{x} \\ 0 & 1 \end{pmatrix}\right) = |\mathbf{y}|_{\mathbb{A}_F} \sum_{\zeta \in F_+^\times} c(\zeta\mathbf{y}\mathfrak{D}, \Phi) W_\lambda(\zeta y_\infty) e_F(\zeta\mathbf{x}), \qquad \mathbf{x} \in \mathbb{A}_F, \mathbf{y} \in \mathbb{A}_F^\times,$$

where here $\zeta\mathbf{y}\mathfrak{D}$ is the fractional ideal $(\zeta)I(\mathbf{y})\mathfrak{D}$ of F. Moreover, the function $c(\cdot, \Phi)$ gives a well-defined function on fractional ideals that is supported at the integral ideals of F.

Proof. See [Hid94], Theorem 6.1. □

4.2.2. Motivation for weight $(0, 0)$ over imaginary quadratic fields

At first glance, it is entirely non-obvious where the above Fourier expansion comes from. To give some motivation, in this section, we sketch the derivation for the case that $F = K$ is imaginary quadratic of class number 1 and the weight is $(0, 0)$, which has received substantially more treatment in the literature. For a full derivation in this case, see [Cre81], Chapter 3, on which this account is based.

The general idea is to study *harmonic functions* $h : \mathcal{H}_3 \to \mathbb{C}^3$, that is, functions h with suitably bounded growth such that $h \cdot \beta$ is a harmonic differential. The harmonicity condition gives rise to a series of partial differential equations which turn out to be precisely those defining K-Bessel functions. With an appropriate extra condition (which we label as the function being *standard*), this leads to an expression for h in terms of the K-Bessel functions K_0 and K_1.

On a different tack, the automorphy condition for $\mathcal{F}$ gives invariance under the subgroup $\{\left(\begin{smallmatrix} 1 & \alpha \\ 0 & 1 \end{smallmatrix}\right) : \alpha \in \mathcal{O}_K\}$, which translates into a periodicity condition

$$\mathcal{F}(z + \alpha, t) = \mathcal{F}(z, t),$$

and hence $\mathcal{F}$ has a Fourier expansion with regard to the additive characters of $\mathbb{C}$ which are trivial on $\mathcal{O}_K$. In [Tat50], it is proved that that if we take an additive character

$$\Psi(z) := e^{-2\pi i(z+\bar{z})}$$

of $\mathbb{C}$, then every additive character of $\mathbb{C}$ that is trivial on $\mathcal{O}_K$ has the form

$$\Psi_\alpha(z) = \Psi(\alpha z), \quad \alpha \in \mathcal{O}_K;$$

hence $\mathcal{F}$ has a Fourier expansion of the form

$$\mathcal{F}(z, t) = \sum_{\alpha \in K} c'(\alpha, t)\Psi(\alpha z).$$

Some calculations using the automorphy condition allow us to show that, as a function on K, the function $c'(\cdot, t)$ is supported only on elements α that belong to the inverse different $\mathfrak{D}^{-1}$, and so replacing α with $\alpha\delta^{-1}$, where δ is a generator of the different, allows us to write the sum over $\alpha \in \mathcal{O}_K$ instead.

The Fourier coefficients are given by the usual integral formula

$$c'(\alpha, t) = \int_{\mathcal{O}_K \backslash \mathbb{C}} \mathcal{F}(z, t)\Psi(\alpha z)dz,$$

and cusp forms correspond to the zeroth Fourier coefficient being 0, that is,

$$c'(0, t) = \int_{\mathcal{O}_K \backslash \mathbb{C}} \mathcal{F}(z, t)dz = 0.$$

This gives some slight motivation for the (much more general) integral cuspidal condition given for forms on $\mathrm{GL}_2(\mathbb{A}_K)$ in the previous section.

If we define $c''(\alpha, t) = \Psi(\alpha)c'(\alpha, t)$, then a simple calculation shows that the $c''(\alpha, t)$ are standard harmonic functions of t. This means that they have the form $c(\alpha)H(t)$, where $H(t)$ is a vector-valued function depending on K-Bessel functions. Substituting this into the above gives the required form of the Fourier expansion.

4.3. Defining the L-function

We now have two alternative methods of defining the L-function. The following proposition shows that they are equivalent.

Proposition 4.3.1. *Let Φ be an automorphic form of weight λ and level $\Omega_1(\mathfrak{n})$ that is an eigenform for all of the Hecke operators. Let λ_I be the eigenvalue of Φ at T_I. Then*

$$\lambda_I = c(I, \Phi),$$

where $c(I, \Phi)$ is the Fourier coefficient of Φ at I.

Proof. See [Hid94], Corollary 6.2. □

Thus we define:

Definition 4.3.2. In the set-up of above, for $s \in \mathbb{C}$ define the *L-function of* Φ to be the sum

$$L(\Phi, s) := \sum_{0 \neq I \subset \mathcal{O}_F} c(I, \Phi)N(I)^{-s} = \sum_{0 \neq I \subset \mathcal{O}_F} \lambda_I N(I)^{-s}.$$

Of course, we can define the L-function for an automorphic form that is *not* an eigenform by using the Fourier coefficients, but such L-functions don't satisfy as many nice properties, and we'll only be concerned with the case where we *do* have an eigenform.

Definition 4.3.3. Let φ be a Hecke character of F, and recall that this naturally gives rise to a function on fractional ideals of F that we also denote by φ. We define the *twist of* $L(\Phi, s)$ *by* φ to be

$$L(\Phi, \varphi, s) := \sum_{0 \neq I \subset \mathcal{O}_F} c(I, \Phi)\varphi(I)N(I)^{-s}.$$

Proposition 4.3.4. *The sum $L(\Phi, \varphi, s)$ converges absolutely in a right half-plane for Re(s) sufficiently large.*

Proof. See [Wei71], Chapter II. □

It is often more convenient to parcel the L-function and all of its twists together into a single function on characters in the style of Tate's thesis. With that in mind, we make the following definition:

Definition 4.3.5. Let Φ be an automorphic form as above, and let φ be a Hecke character. Then define

$$L(\Phi, \varphi) := L(\Phi, \varphi, 1).$$

Note that this makes sense for an arbitrary Hecke character, not just one that is arithmetic. In particular, for a complex number s, we have a Hecke character $|\cdot|^s$, where $|\cdot|$ denotes the adelic norm character, and then we have

$$L(\Phi, \varphi, s) = L(\Phi, \varphi|\cdot|^{s-1}).$$

We make one more definition. We've shown that the L-function is related to the Hecke eigenvalues, so that in a sense, the L-function is built from local data at the finite primes (much like in the classical case, where the L-function of a Hecke eigenform has an Euler product). We *complete* the L-function by adding the appropriate factors at infinity.

Definition 4.3.6. Let φ be a Hecke character of infinity type $\mathbf{j} + \mathbf{v}$. Define

$$\Lambda(\Phi, \varphi) := \left[\prod_{v \in \Sigma} \frac{\Gamma(j_v + 1)}{(-2\pi i)^{j_v+1}}\right] L(\Phi, \varphi),$$

where Γ is the usual Gamma function. This is the L-function renormalised by Deligne's Γ-factors at infinity.

4.4. Periods and algebraicity

We record one further very important result here. The following theorem facilitates the study of p-adic L-functions of automorphic forms for GL_2, and thus is crucial to the rest of this thesis. It was proved by Hida in [Hid94], Theorem 8.1. Earlier, Shimura proved this result over $\mathbb{Q}$ in [Shi77] and later over totally real fields in [Shi78].

Theorem 4.4.1. *Let Φ be a cuspidal eigenform over F of weight $\lambda = (\mathbf{k}, \mathbf{v})$ and level $\Omega_1(\mathfrak{n})$, with associated L-function $L(\Phi, \cdot)$. Let φ be an arithmetic Hecke character of infinity type $\mathbf{j}+\mathbf{v}$, where $0 \leq \mathbf{j} \leq \mathbf{k}$, and let $\varepsilon = \varepsilon_\varphi$ be its associated character on $\{\pm 1\}^{\Sigma(\mathbb{R})}$ (as in Chapter 1.2.2). Let E be a number field containing the normal closure of F and the Hecke eigenvalues of Φ. Then there is a period*

$$\Omega_\Phi^\varepsilon \in \mathbb{C}^\times,$$

depending only on Φ *and* ε*, such that*

$$\frac{\Lambda(\Phi,\varphi)}{\Omega_\Phi^\varepsilon \tau(\varphi)} \in E(\varphi),$$

where $E(\varphi)$ *is the number field generated over* E *by adjoining the values of* φ*.*

Remarks: (i) We are assuming that all Hecke characters are arithmetic; if we dropped this assumption, then $E(\varphi)$ need not be finite over E (see [Hid94], Section 8).

(ii) This results bears comparison with a conjecture of Deligne in [Del79]. Deligne's conjecture is stated for *motives*, and says that the L-function of the (conjectural) motive attached to an automorphic form as above should satisfy a similar algebraicity result after renormalising by an (explicit) transcendental period.

(iii) There are many choices of such a period, differing by elements of $E^\times$. Throughout the rest of the paper, we shall assume that we fix a period for each character ε.

(iv) Note that the period depends on the character $\varepsilon_\varphi(\iota) := \varphi|_{\{\pm 1\}^{\Sigma(\mathbb{R})}}(\iota)\iota^{\mathbf{j}+\mathbf{v}}$ of the Weyl group, and *not* the character $\varphi|_{\{\pm 1\}^{\Sigma(\mathbb{R})}}$. This latter version is incompatible with Shimura's results over $\mathbb{Q}$.

Thus we have a collection of 2^{r_1} periods attached to Φ, and each corresponds to a different collection of L-values, depending on the parity of the corresponding Hecke characters.

Part II

Imaginary Quadratic Fields

A Bianchi modular form is an automorphic form over an imaginary quadratic field K. In this part, we use a modular symbol method to construct p-adic L-functions for Bianchi modular forms. We do so via very explicit methods; in particular, in this setting, we have a very concrete description of the spaces of modular symbols, and all of our proofs are constructive.

We start by describing the classical theory, attaching classical modular symbols to Bianchi modular forms and then linking them with special values of their L-functions. We then develop the theory of overconvergent modular symbols in the Bianchi setting in explicit detail, showing that the natural specialisation map from overconvergent to classical modular symbols is an isomorphism on the small slope eigenspaces of the Hecke operators at p. Using this, to a small slope classical Bianchi eigenform, one can attach a canonical small slope overconvergent eigensymbol; in the remainder of this part, we examine the distributions that arise as values of this symbol, showing that they have good growth properties and interpolate special L-values of the original eigenform. In particular, we use it to construct the p-adic L-function of the eigenform.

The results that appear here are contained in the paper "P-adic L-functions of Bianchi modular forms" *(see [Wil15]).*

Chapter 5
Classical Bianchi Modular Symbols

Let K be an imaginary quadratic field. In this chapter, we discuss the theory of automorphic forms over K, reformulating some of the above theory in a more concrete setting; typically, we will work with functions defined on $\mathcal{H}_3$ rather than on $\mathrm{GL}_2(\mathbb{A}_K)$. We then define modular symbols over imaginary quadratic fields; firstly, we do this abstractly with an arbitrary module of values, and secondly, with a specific module of polynomials. After defining Hecke operators on the space of modular symbols, we show how to attach a Bianchi modular symbol to a Bianchi modular form in a Hecke-equivariant manner. We conclude by refining the construction to construct a modular symbol with p-adic (rather than complex) coefficients.

5.1. Notation and recap

Throughout Part II, K will denote an imaginary quadratic field of class number h.

Definition 5.1.1. A *Bianchi modular form* is an automorphic form of some weight and level over K.

Notation: We use the following conventions for the weight and level throughout the rest of this part.

(i) We will always take the weight to be $\lambda = [(k,k),(0,0)]$, that is, we'll set $\mathbf{v} = 0$. We'll write this simply as *weight* (k,k).

(ii) All (adelic) automorphic forms will have level $\Omega_1(\mathfrak{n})$, as defined in Definition 2.3, for $\mathfrak{n}$ some integral ideal of K.

(iii) We write $S_{k,k}(\Omega_1(\mathfrak{n}))$ for the space of Bianchi modular forms of weight (k,k) and level $\Omega_1(\mathfrak{n})$.

Let $\Phi : \mathrm{GL}_2(\mathbb{A}_K) \longrightarrow V_{2k+2}(\mathbb{C})$ be a Bianchi modular form of weight (k,k) and level $\Omega_1(\mathfrak{n})$. Recall that to Φ, we associate a (non-canonical) collection of h functions $\mathcal{F}^1, ..., \mathcal{F}^h : \mathcal{H}_3 \rightarrow V_{2k+2}(\mathbb{C})$ defined in the following manner: fix a choice of representatives $I_1, ..., I_h$ for the class group of K that are all pairwise coprime to $\mathfrak{n}$, and choose ideles $a_1, ..., a_h$ representing these ideals. Then define

$$\begin{aligned} F^i : \mathrm{GL}_2(\mathbb{C}) &\longrightarrow V_{2k+2}(\mathbb{C}), \\ g &\longmapsto \Phi\left(\begin{pmatrix} a_i & 0 \\ 0 & 1 \end{pmatrix} g\right). \end{aligned}$$

We then define

$$\begin{aligned} \mathcal{F}^i : \mathcal{H}_3 &\longrightarrow V_{2k+2}(\mathbb{C}), \\ (z,t) &\longmapsto t^{-1} F^i \begin{pmatrix} t & z \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

These functions then satisfy automorphy conditions under the discrete subgroups

$$\Gamma_1^i(\mathfrak{n}) := \mathrm{SL}_2(K) \cap \begin{pmatrix} a_i & 0 \\ 0 & 1 \end{pmatrix} \Omega_1(\mathfrak{n}) \mathrm{GL}_2(\mathbb{C}) \begin{pmatrix} a_i^{-1} & 0 \\ 0 & 1 \end{pmatrix}$$

of $\mathrm{SL}_2(K)$. (For further details of this construction, see Chapter 2.2.5).

With this construction in mind, it is also useful to define the notion of Bianchi modular forms on $\mathcal{H}_3$. To this end, let Φ be a cuspidal automorphic form of weight (k,k) and level $\Omega_1(\mathfrak{n})$; then for any idele a, we obtain a function F_a on $\mathrm{GL}_2(\mathbb{C})$ by

$$F_a(g) := \Phi\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g\right),$$

which naturally gives a function $\mathcal{F}_a$ on $\mathcal{H}_3$. We say that $\mathcal{F}_a$ has weight (k,k) and level

$$\Gamma_a := \mathrm{SL}_2(K) \cap \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \Omega_1(\mathfrak{n}) \mathrm{GL}_2(\mathbb{C}) \begin{pmatrix} a^{-1} & 0 \\ 0 & 1 \end{pmatrix}.$$

Definition 5.1.2. Let $\mathcal{F} : \mathcal{H}_3 \longrightarrow V_{k,k}(\mathbb{C})$ be a function.

(i) We say that $\mathcal{F}$ is a *Bianchi modular form of weight* (k,k) *and level* Γ if there exists an idele a, with trivial components at infinity and $\mathfrak{n}$, such that $\mathcal{F} = \mathcal{F}_a$ and $\Gamma = \Gamma_a \leq \mathrm{SL}_2(K)$ for some automorphic form Φ over K of weight (k,k) and level $\Omega_1(\mathfrak{n})$.

(ii) We say that $\mathcal{F}$ is a *cusp form* if the automorphic form Φ is cuspidal.

In other words, a function on $\mathcal{H}_3$ is a cusp form if and only if it comes from an (adelic) automorphic form in the natural way. (When we talk about cusp forms, it should be clear whether they are functions on $\mathcal{H}_3$ or $\mathrm{GL}_2(\mathbb{A}_K)$ by context).

Remark: We could define cusp forms on $\mathcal{H}_3$ more directly by writing down an automorphy condition under the group Γ (see, for example, equation (2.7) above). We've chosen to define them in this more abstract way to emphasise the fact that we really want to see such functions as coming from adelic automorphic forms. In particular, we want to see each one as part of a collection of h forms coming from one automorphic form, and that whilst we can consider such forms individually, we get far nicer structures and properties by considering them as one part of a collection.

As a simple example of this, note that if $\mathfrak{p}$ is a prime of K that is not principal, then there is no notion of a $T_{\mathfrak{p}}$ operator on cusp forms on $\mathcal{H}_3$. For this, we really need to see a Bianchi modular form as a collection of h cusp forms on $\mathcal{H}_3$, and then the $T_{\mathfrak{p}}$ operator permutes the components in addition to acting on each individually. This will all be made clearer in the sequel.

5.2. Modular symbols and differentials

We now come to the definition of modular symbols over imaginary quadratic fields. In this section, we discuss the basic theory as a precursor to the sequel, where we attach such a modular symbol to a cusp form on $\mathcal{H}_3$.

5.2.1. Abstract modular symbols over K

We begin this section by giving the definition of modular symbols over K with values in a completely general module V. To this end, let $\Gamma \leq \mathrm{SL}_2(K)$ be a discrete subgroup, and let V be a right Γ-module.

Definition 5.2.1. Define

$$\Delta_0 := \mathrm{Div}^0(\mathbb{P}^1(K)),$$

and note that Δ_0 has a left action of Γ (and indeed, of $\mathrm{SL}_2(K)$) given by fractional linear transformations. Concretely, this is the action induced linearly by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot r := \frac{ar+b}{cr+d}, \quad a,b,c,d \in K, r \in \mathbb{P}^1(K),$$

where (as usual) this means that $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \cdot r = \infty$ when $cr + d = 0$ and $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \cdot \infty = a/c$.

Remarks: (i) We define the *completed upper half-space* $\mathcal{H}_3^* := \mathcal{H}_3 \cup \mathbb{P}^1(K)$. This is the imaginary quadratic analogue of 'adding in the cusps' over the rationals; in particular, we say that $\mathbb{P}^1(K)$ is the set of *cusps* of $\mathcal{H}_3^*$ and given some $r \in K$, we see r as an element $(r, 0)$ on the boundary of $\mathcal{H}_3 = \mathbb{C} \times \mathbb{R}_{>0}$.

(ii) Note that Δ_0 is spanned by elements of the form $\{r\} - \{s\}$ for $r, s \in \mathbb{P}^1(K)$. One should view this element as representing a path between the cusps r and s.

Definition 5.2.2. (i) We say that a map $\phi \in \mathrm{Hom}(\Delta_0, V)$ is Γ-*invariant* if we have

$$\phi(\gamma \cdot D)|\gamma = \phi(D)$$

for all $D \in \Delta_0$ and $\gamma \in \Gamma$.

(ii) Write

$$\mathrm{Symb}_\Gamma(V) := \mathrm{Hom}_\Gamma(\Delta_0, V)$$

for the space of Γ-invariant maps from Δ_0 to V. We call this the space of V*-valued modular symbols of level* Γ.

For suitable modules V – to be defined below – there is a close link between V-valued modular symbols of level Γ and Bianchi modular forms on $\mathcal{H}_3$ of level Γ.

5.2.2. Classical Bianchi modular symbols

Definition 5.2.3. As above we define, for a non-negative integer k, the space $V_k(\mathbb{C})$ to be the space of homogeneous polynomials of degree k in two variables over $\mathbb{C}$. Furthermore, for notational convenience, we define $V_{k,k}(\mathbb{C}) := V_k(\mathbb{C}) \otimes_{\mathbb{C}} V_k(\mathbb{C})$.

Note that we can identify $V_{k,k}(\mathbb{C})$ with the space of polynomials that are homogeneous of degree k in two variables X, Y and homogeneous of degree k in two further variables $\overline{X}, \overline{Y}$. Furthermore, recall that $V_k(\mathbb{C})$ is an irreducible $\mathrm{SU}_2(\mathbb{C})$-module from Proposition 2.2.1, with $\mathrm{SU}_2(\mathbb{C})$ acting on the right by

$$(P|\gamma)\begin{pmatrix} X \\ Y \end{pmatrix} = P\left(\gamma\begin{pmatrix} X \\ Y \end{pmatrix}\right).$$

The following defines a different action on this space, with a view to obtaining a 'nice' action on the dual space of $V_{k,k}(\mathbb{C})$.

Definition 5.2.4. We have a left-action of $\mathrm{SL}_2(\mathbb{C})$ on $V_k(\mathbb{C})$ defined by

$$(\gamma \cdot P)\begin{pmatrix} X \\ Y \end{pmatrix} = P\begin{pmatrix} dX + bY \\ cX + aY \end{pmatrix}, \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

We then obtain a left-action of $\mathrm{SL}_2(\mathbb{C})$ on $V_{k,k}(\mathbb{C})$ by

$$(\gamma \cdot P)\left[\begin{pmatrix} X \\ Y \end{pmatrix}, \begin{pmatrix} \overline{X} \\ \overline{Y} \end{pmatrix}\right] = P\left[\begin{pmatrix} dX + bY \\ cX + aY \end{pmatrix}, \begin{pmatrix} \bar{d}\overline{X} + \bar{b}\overline{Y} \\ \bar{c}\overline{X} + \bar{a}\overline{Y} \end{pmatrix}\right].$$

Remark: This action, whilst appearing unconventional, is chosen so that it is compatible with an action on a space of locally analytic functions on some p-adic space. This compatibility simplifies matters considerably when considering specialisation maps from overconvergent to classical modular symbols.

The left action of $\mathrm{SL}_2(\mathbb{C})$ described above translates into a *right*-action on the dual space $V_{k,k}(\mathbb{C})^* := \mathrm{Hom}(V_{k,k}(\mathbb{C}), \mathbb{C})$. For $\mu \in V_{k,k}(\mathbb{C})^*$, we set

$$(\mu|\gamma)(P) = \mu(\gamma \cdot P).$$

Definition 5.2.5. (i) For a discrete subgroup Γ of $\mathrm{SL}_2(K)$, the space of *Bianchi modular symbols of weight* (k,k) *and level* Γ is defined to be the space $\mathrm{Symb}_\Gamma(V_{k,k}(\mathbb{C})^*)$ of $V_{k,k}(\mathbb{C})^*$-valued modular symbols of level Γ.

(ii) For a fixed choice of class group representatives $I_1, ..., I_h$, the space of *Bianchi modular symbols of weight* (k,k) *and level* $\Omega_1(\mathfrak{n})$ is defined to be the space

$$\mathrm{Symb}_{\Omega_1(\mathfrak{n})}(V_{k,k}(\mathbb{C})^*) := \bigoplus_{i=1}^{h} \mathrm{Symb}_{\Gamma_1^i(\mathfrak{n})}(V_{k,k}(\mathbb{C})^*).$$

In the sequel, we'll attach a modular symbol of level $\Omega_1(\mathfrak{n})$ to a full adelic cuspidal Bianchi modular form by attaching a modular symbol of level $\Gamma_1^i(\mathfrak{n})$ to each cusp form $\mathcal{F}^i$ on $\mathcal{H}_3$. It is often easier to study these objects component by component at the level of $\mathcal{H}_3$, and where possible, we do so.

Remark: We can equip each of these spaces with a Hecke action, and then there are natural Hecke-equivariant isomorphisms between the symbol spaces defined above and certain compactly supported cohomology groups. In particular, the space $\mathrm{Symb}_{\Omega_1(\mathfrak{n})}(V_{k,k}(\mathbb{C})^*)$ can be defined independently of the choice of class group representatives for K. For more details of this approach, see Part III.

5.2.3. Differentials on $\mathcal{H}_3$

We studied differential forms in Chapter 3.1.1. In this section, we specialise to study differentials on hyperbolic 3-space $\mathcal{H}_3$, which naturally has the structure of a real differentiable 3-manifold. Accordingly, at each point of $x \in \mathcal{H}_3$, the tangent space $T_x\mathcal{H}_3$ is a 3-dimensional real vector space, which then gives the following:

Proposition 5.2.6. *The space* $\Omega^1(\mathcal{H}_3, \mathbb{C})$ *is a 3-dimensional* $C^\infty(\mathcal{H}_3)$*-module spanned by the elements* dz, dt *and* $d\overline{z}$.

Recall that in Chapter 2.2.4 we defined an action of $\mathrm{SL}_2(\mathbb{C})$ on $\mathcal{H}_3$. By pulling back this action to the space of differentials, we obtain:

Proposition 5.2.7. *The expression*

$$\gamma \cdot \begin{pmatrix} dz \\ -dt \\ -d\overline{z} \end{pmatrix} = \rho(\gamma, (z,t))^{-1} \begin{pmatrix} dz \\ -dt \\ -d\overline{z} \end{pmatrix}, \tag{5.1}$$

for ρ as defined in equation (2.7), defines a left action of $\mathrm{SL}_2(\mathbb{C})$ on $\Omega^1(\mathcal{H}_3, \mathbb{C})$.

We can obtain a more concrete description of this action by passing to $V_2(\mathbb{C})$, the space of homogeneous polynomials of degree two in the variables A, B. Previously, we have defined a right-action of $\mathrm{SU}_2(\mathbb{C})$ on $V_2(\mathbb{C})$. We can translate between right and left actions by setting

$$(u \cdot P)\begin{pmatrix} A \\ B \end{pmatrix} := (P|u^{-1})\begin{pmatrix} A \\ B \end{pmatrix} = P\left(u^{-1}\begin{pmatrix} A \\ B \end{pmatrix}\right), \quad P \in V_2(\mathbb{C}), u \in \mathrm{SU}_2(\mathbb{C}).$$

To emphasise that this space is now equipped with a *left*-action, we denote it $V_2^{\ell}(\mathbb{C})$ (and likewise $V_2^{r}(\mathbb{C})$ when we use the right-action).

Proposition 5.2.8. *Let $\Omega_0^1(\mathcal{H}, \mathbb{C})$ be the $\mathbb{C}$-vector space spanned by dz, dt and $d\overline{z}$. There is an isomorphism $\Omega_0^1(\mathcal{H}_3, \mathbb{C}) \xrightarrow{\sim} V_2^{\ell}(\mathbb{C})$ of $\mathrm{SU}_2(\mathbb{C})$-modules given by the map sending*

$$dz \mapsto A^2, \quad dt \mapsto -AB, \quad d\overline{z} \mapsto -B^2.$$

Proof. The proof, whilst not stated as an explicit proposition, is contained in [Gha99], Section 2.2, and uses the theory of Lie algebras. The map factors through an isomorphism $T_{(0,1)}\mathcal{H}_3^* \otimes_{\mathbb{R}} \mathbb{C} \cong V_2^{\ell}(\mathbb{C})$, with the map $\Omega^1(\mathcal{H}_3, \mathbb{C}) \to T_{(0,1)}\mathcal{H}_3^* \otimes_{\mathbb{R}} \mathbb{C}$ induced from

$$\begin{aligned} \Omega^1(\mathcal{H}_3, \mathbb{R}) &\longrightarrow T_{(0,1)}\mathcal{H}_3^*, \\ \omega &\longmapsto \omega_{(0,1)} \end{aligned}$$

by tensoring with $\mathbb{C}$. $\square$

Using this isomorphism, we can define a left-action of $\mathrm{SL}_2(\mathbb{C})$ on $V_2^{\ell}(\mathbb{C})$ corresponding to the pull-back of the action on differentials. An explicit check shows this to be given by

$$\gamma \cdot P\begin{pmatrix} A \\ B \end{pmatrix} = P\left[\frac{1}{|a|^2 + |c|^2}\begin{pmatrix} \overline{a} & \overline{c} \\ -c & a \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix}\right], \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{5.2}$$

We'll later use the space $V_2^\ell(\mathbb{C})$ and this concrete definition of the action to obtain an element of $V_{k,k}^\ell(\mathbb{C}) \otimes_{\mathbb{C}} V_2^\ell(\mathbb{C})$ associated to a Bianchi modular form $\mathcal{F}$, and then use Proposition 5.2.8 to turn this into a $V_{k,k}^\ell(\mathbb{C})$-valued modular symbol.

5.3. Hecke operators

In the classical theory, the Hecke operators allow us to endow spaces of automorphic forms with additional structure. We can also define Hecke operators on the space of modular symbols. In this section, we'll give the definition, in the process providing motivation for considering the *full* 'adelic' space $\mathrm{Symb}_{\Omega_1(\mathfrak{n})}(V_{k,k}(\mathbb{C})^*)$ of modular symbols.

5.3.1. Hecke operators at principal ideals

Let I be a principal ideal of K, and pick a generator β of I. Let Γ be a discrete subgroup of $\mathrm{SL}_2(K)$.

Definition 5.3.1. The T_I operator on $\mathrm{Symb}_\Gamma(V_{k,k}(\mathbb{C})^*)$ is defined to be the double coset operator

$$T_I = \left[\Gamma \begin{pmatrix} 1 & 0 \\ 0 & \beta \end{pmatrix} \Gamma\right].$$

Concretely, define

$$\Gamma_\beta := \begin{pmatrix} 1 & 0 \\ 0 & \beta^{-1} \end{pmatrix} \Gamma \begin{pmatrix} 1 & 0 \\ 0 & \beta \end{pmatrix} \cap \Gamma,$$

and choose representatives $\gamma_1', ..., \gamma_n'$ for the quotient Γ/Γ_β. Then, defining

$$\gamma_i = \begin{pmatrix} 1 & 0 \\ 0 & \beta \end{pmatrix} \gamma_i',$$

we have

$$(\phi|T_I)(D) = \sum_{i=1}^{n} (\phi|\gamma_i)(D)|\gamma_i$$

for $\phi \in \mathrm{Symb}_\Gamma(V_{k,k}(\mathbb{C})^*)$ and $D \in \Delta_0$. (Note here that we've extended the action of $\mathrm{SL}_2(\mathbb{C})$ to $\mathrm{GL}_2(\mathbb{C})$ without modification). The operator T_I is independent of both the choice of β and the choice of representatives.

Hence: for a *principal* ideal, we can define a Hecke operator T_I on each component of a full modular symbol.

5.3.2. Hecke operators at non-principal ideals

Now let I be a *non*-principal prime of K. The approach outlined above no longer works in this setting; we can't choose a generator to define the double coset operator. In fact, there is no natural definition of a T_I operator on the components of a modular symbol; we really do need to use the *full* space.

As ever, consider our fixed representatives $I_1, ..., I_h$ for the class group. For each $i \in \{1, ..., h\}$ there is a unique $j_i \in \{1, ..., h\}$ such that

$$II_i = (\alpha_i)I_{j_i},$$

for $\alpha_i \in K$.

Definition 5.3.2. The T_I operator is defined on $\mathrm{Symb}_{\Omega_1(\mathfrak{n})}(V_{k,k}(\mathbb{C})^*)$ by

$$(\phi_1, ..., \phi_h)|T_I := \left(\phi_{j_1}\middle|\left[\Gamma_{j_1}\begin{pmatrix}1 & 0\\ 0 & \alpha_1\end{pmatrix}\Gamma_1\right], ..., \phi_{j_h}\middle|\left[\Gamma_{j_h}\begin{pmatrix}1 & 0\\ 0 & \alpha_h\end{pmatrix}\Gamma_h\right]\right).$$

Again, we can compute this concretely in the same manner as above by writing down explicit representatives of the double coset. Note that the T_I operator really does permute the individual components, depending on the class of I in the class group; indeed, this permutation corresponds to multiplication by $[I]$ in the class group.

If $\mathfrak{p}$ is a prime ideal, note that if n is an integer such that $\mathfrak{p}^n = (\sigma)$ is principal, then this action becomes significantly simpler; namely, we just act on each component via the double coset $[\Gamma \left(\begin{smallmatrix}1 & 0\\ 0 & \sigma\end{smallmatrix}\right) \Gamma]$. When working with Hecke operators in the sequel, we will use this approach, as it allows us to work with individual components.

5.3.3. Hecke operators at p

We make one further observation. When I is not coprime to the level $\mathfrak{n}$, we write U_I instead of T_I. In the sequel, we will always take $\mathfrak{n}$ to be divisible by (p), so that if $\mathfrak{p}$ is a prime above p, we have an even simpler description of the Hecke operator $U_{\mathfrak{p}}$. Let n be an integer such that $\mathfrak{p}^n = (\sigma)$ is principal; then $U_{\mathfrak{p}^n} = U_{\mathfrak{p}}^n$ acts on each component by

$$\phi|U_{\mathfrak{p}}^n(D) = \sum_{a \pmod{\mathfrak{p}^n}} \left(\phi\middle|\begin{pmatrix}1 & a\\ 0 & \sigma\end{pmatrix}\right)(D)\middle|\begin{pmatrix}1 & a\\ 0 & \sigma\end{pmatrix},$$

where $\phi \in \mathrm{Symb}_\Gamma(V_{k,k}(\mathbb{C})^*)$ and $D \in \Delta_0$.

Remark: Note that we have $U_{\mathfrak{p}^n} = U_{\mathfrak{p}}^n$ only because $\mathfrak{p}$ divides the level. In general, the Hecke operators are *not* multiplicative.

5.4. The modular symbol attached to a Bianchi modular form

Let $\mathcal{F} : \mathcal{H}_3 \longrightarrow V_{2k+2}(\mathbb{C})$ be a Bianchi cusp form on $\mathcal{H}_3$ of weight (k, k) and level Γ. In this section, we describe a way of attaching an element of $\mathrm{Symb}_\Gamma(V_{k,k}(\mathbb{C})^*)$ to $\mathcal{F}$. We'll start by giving some motivation for the construction; firstly, a (very brief) description of the rational case, before tackling the case of a weight $(0, 0)$ form over an imaginary quadratic field of class number 1, for which the literature is rather more broad and the construction is much simpler. Finally, we'll describe the *Eichler-Shimura-Harder isomorphism*, which describes the construction in the general case for imaginary quadratic fields.

5.4.1. Modular symbols over $\mathbb{Q}$

Let $\Gamma \leq \mathrm{SL}_2(\mathbb{Z})$ be a congruence subgroup. The space of *modular symbols of weight* $k + 2$ *and level* Γ is the space

$$\mathrm{Symb}_\Gamma(V_k(\mathbb{C})) := \mathrm{Hom}_\Gamma(\mathrm{Div}^0(\mathbb{P}^1(\mathbb{Q}), V_k(\mathbb{C})),$$

where these objects are all defined in a manner analogous to before. If $f \in S_2(\Gamma)$ is a classical weight 2 cusp form, we define its associated modular symbol to be defined on paths between cusps – that is, on generators of $\mathrm{Div}^0(\mathbb{P}^1(\mathbb{Q}))$ – by

$$\phi_f(\{r\} - \{s\}) = \int_r^s f(z)dz.$$

In other words, we define a (harmonic) differential on $\mathcal{H}_3$ associated to f with values in $\mathbb{C} \cong V_0(\mathbb{C})$, and then integrate it over a path between two cusps (noting that, by harmonicity, the integral is independent of the choice of path). For weight $k + 2$, then, we look for a suitable harmonic differential on $\mathcal{H}$ with values in $V_k(\mathbb{C})$, and then integrate it over such paths to give the modular symbol. Such a differential is given by $f(z)(zX + Y)^k dz$; it is a simple check to show that the function defined on pairs of cusps by

$$\phi_f(\{r\} - \{s\}) = \int_r^s f(z)(zX + Y)^k dz$$

induces a modular symbol.

Remarks: (i) We met this differential in earlier sections; in particular, it is the differential introduced in Theorem 3.2.2.

(ii) For further details on this construction, see the papers of Pollack and Stevens ([PS11] and [PS12]).

(iii) In some ways, it is more natural to consider modular symbols over $\mathbb{Q}$ to have values in $V_k(\mathbb{C})^*$, in line with the Bianchi case. This certainly gives better compatibility with the theory of overconvergent modular symbols. The two formulations are equivalent via an isomorphism $V_k(\mathbb{C}) \cong V_k(\mathbb{C})^*$ of $\mathrm{SL}_2(\mathbb{C})$-modules (see Chapter 5.4.4).

The strategy over imaginary quadratic fields will be similar. In particular, we look to associate a $V_{k,k}(\mathbb{C})^*$-valued harmonic differential to a Bianchi modular form – as in Theorem 3.2.2 – and then integrate it over cusps.

5.4.2. In weight (0,0) and class number 1

A cusp form of weight (0,0) and level $\Gamma_1(\mathfrak{n})$ over an imaginary quadratic field of class number 1 can be described simply as a function

$$\mathcal{F} = (\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2) : \mathcal{H}_3 \longrightarrow V_2(\mathbb{C}) \cong \mathbb{C}^3.$$

To such a cusp form we associate a differential

$$\omega_{\mathcal{F}} := \mathcal{F}_0(z,t)dz - \mathcal{F}_1(z,t)dt - \mathcal{F}_2(z,t)d\overline{z}.$$

Remarks 5.4.1: (i) This can be realised more elegantly as

$$\omega_{\mathcal{F}} = t\mathcal{F} \cdot \beta,$$

where we recall that

$$\beta = \left(\frac{dz}{t}, \frac{-dt}{t}, \frac{-d\overline{z}}{t}\right)$$

is a basis for the left-invariant differentials on $\mathcal{H}_3$. Note that this differs slightly (by the factor of t) from that defined in [Cre81], [CW94] and [Byg98], as explained in Section 2.2.5 above.

(ii) An explicit calculation shows that $\omega_{\mathcal{F}}$ is $\Gamma_1(\mathfrak{n})$ invariant; in Section 2.2.5 we showed that, for $\gamma \in \Gamma_1(\mathfrak{n})$, we have

$$\mathcal{F}(\gamma(z,t)) = \mathcal{F}(z,t)\rho(\gamma,(z,t)),$$

where the action of $\Gamma_1(\mathfrak{n})$ on $\mathcal{H}_3$ was defined in equation (2.6) and where ρ is as defined in equation (2.7). We also have an explicit description of the action of $\mathrm{SL}_2(\mathbb{C})$ on $\Omega^1(\mathcal{H}_3, \mathbb{C})$ as

$$\gamma \cdot \begin{pmatrix} dz \\ -dt \\ -d\overline{z} \end{pmatrix} = \rho(\gamma,(z,t))^{-1} \begin{pmatrix} dz \\ -dt \\ -d\overline{z} \end{pmatrix}$$

(see equation (5.1)). These combine easily to give invariance, as stated, and shows that in fact this differential is defined on the quotient $Y_1(\mathfrak{n}) := \Gamma_1(\mathfrak{n})\backslash\mathcal{H}_3$, that is,

$$\omega_{\mathcal{F}} \in \mathrm{H}^1\left(Y_1(\mathfrak{n}), \widetilde{\mathbb{C}}\right),$$

where $\widetilde{\mathbb{C}}$ is the constant sheaf on the real 3-manifold $Y_1(\mathfrak{n})$ associated to $V_{0,0}(\mathbb{C}) = \mathbb{C}$.

This differential brings us to the desired definition:

Definition 5.4.2. Let $\mathcal{F} : \mathcal{H}_3 \to \mathbb{C}^3$ be a cusp form of weight (0,0) and level $\Gamma_1(\mathfrak{n})$ over an imaginary quadratic field of class number 1. The *modular symbol attached to* $\mathcal{F}$ is defined pointwise by

$$\phi_{\mathcal{F}}(\{r\} - \{s\}) := \int_r^s \omega_{\mathcal{F}}.$$

5.4.3. The Eichler-Shimura-Harder isomorphism

Now consider the case of general weight (k,k), with arbitrary class number h. Let Φ be a cusp form on $\mathrm{GL}_2(\mathbb{A}_K)$ of weight (k,k) and level $\Omega_1(\mathfrak{n})$, and write $\mathcal{F}^1, ..., \mathcal{F}^h$ for the associated cusp forms on $\mathcal{H}_3$ (under our fixed set of class group representatives). We pick one of these functions $\mathcal{F}^i$, henceforth denoting it simply by $\mathcal{F}$, and describe how to attach a modular symbol $\phi_{\mathcal{F}}$ to it. Throughout, we write $\Gamma = \Gamma_1^i(\mathfrak{n})$ for the level of $\mathcal{F}$. The construction described here appears in [Gha99], Section 5.1.

Above, we considered a right-action of $\mathrm{SU}_2(\mathbb{C})$ on $V_k(\mathbb{C})$. Here, we must pass to the corresponding *left*-action, defined by $\gamma \cdot P := P|\gamma^{-1}$ (see Section 5.4.4 below for more details on this change). When talking about this space with a left-action, as before we write $V_k^\ell(\mathbb{C})$ (and similarly $V_k^r(\mathbb{C})$ when we talk about the right-action). We extend this notation to the tensor product, and write $V_{k,k}^\ell(\mathbb{C})$ and $V_{k,k}^r(\mathbb{C})$ for the tensor products considered with the left- and right-action respectively.

We recap some basic representation theory. The *Clebsch-Gordan formula* says that, for $k \geq \ell$,

$$V_k(\mathbb{C}) \otimes_{\mathbb{C}} V_\ell(\mathbb{C}) = V_{k+\ell}(\mathbb{C}) \oplus V_{k+\ell-2}(\mathbb{C}) \oplus \cdots \oplus V_{k-\ell}(\mathbb{C})$$

as (left *or* right) $\mathrm{SU}_2(\mathbb{C})$-modules. This says that, as $V_{k,k}(\mathbb{C}) \cong V_k(\mathbb{C}) \otimes_{\mathbb{C}} V_k(\mathbb{C})$, we have

$$V_{k,k}(\mathbb{C}) \otimes_{\mathbb{C}} V_2(\mathbb{C}) = V_{2k+2}(\mathbb{C}) \oplus V_{2k}(\mathbb{C})^2 \oplus \cdots \oplus V_0(\mathbb{C})$$

as right $\mathrm{SU}_2(\mathbb{C})$-modules, and hence that there is an injection of (left) $\mathrm{SU}_2(\mathbb{C})$-modules

$$\sigma : V_{2k+2}^\ell(\mathbb{C}) \longhookrightarrow V_{k,k}^\ell(\mathbb{C}) \otimes_{\mathbb{C}} V_2^\ell(\mathbb{C}).$$

A conceptual construction

We echo Ghate's approach by first describing a conceptual method of obtaining a differential and then computing the result more explicitly. We start by constructing this differential on $\mathrm{SL}_2(\mathbb{C})$; let F be the function $\mathrm{GL}_2(\mathbb{C}) \to V_{2k+2}^\ell(\mathbb{C})$ corresponding to $\mathcal{F}$, where here we have passed to a left-action of $\mathrm{SU}_2(\mathbb{C})$ rather than the conventional right-action. From now on we will only consider the restriction of F to $\mathrm{SL}_2(\mathbb{C})$, and, in an abuse of notation, we will also write F for this restriction. We compose F with the map σ defined above to give

$$\sigma \circ F : \mathrm{SL}_2(\mathbb{C}) \longrightarrow V_{k,k}^{\ell}(\mathbb{C}) \otimes V_2^{\ell}(\mathbb{C}).$$

This associates to F a polynomial that is homogeneous of degree k in variables X and Y, homogeneous of degree k in variables $\overline{X}$ and $\overline{Y}$ and homogeneous of degree 2 in variables A and B. We then use Proposition 5.2.8 to pass from $V_2(\mathbb{C})$ to differentials; namely, we replace A^2 with dz, AB with $-dt$ and B^2 with $-d\overline{z}$ to obtain a differential 1-form on $\mathrm{SL}_2(\mathbb{C})$ with values in $V_{k,k}^{\ell}(\mathbb{C})$.

At the moment, this procedure is not well-defined on $\mathcal{H}_3 = \mathrm{SL}_2(\mathbb{C})/\mathrm{SU}_2(\mathbb{C})$. To this end, we scale by the action of $\mathrm{SL}_2(\mathbb{C})$.

Definition 5.4.3. Given $F : \mathrm{SL}_2(\mathbb{C}) \to V_{2k+2}^{\ell}(\mathbb{C})$ as above, define a differential ω_F on $\mathrm{SL}_2(\mathbb{C})$ by

$$\omega_F(g) = g \cdot (\sigma \circ F(g)), \quad g \in \mathrm{SL}_2(\mathbb{C}).$$

Here $\mathrm{SL}_2(\mathbb{C})$ acts on the left of $V_{k,k}^{\ell}(\mathbb{C})$ in exactly the same manner as $\mathrm{SU}_2(\mathbb{C})$, whilst the action on $V_2^{\ell}(\mathbb{C})$ is described by equation (5.2) above. Thus, concretely, the action on $V_{k,k}^{\ell}(\mathbb{C}) \otimes_{\mathbb{C}} V_2^{\ell}(\mathbb{C})$ is

$$\gamma \cdot P\left[\begin{pmatrix} X \\ Y \end{pmatrix}, \begin{pmatrix} \overline{X} \\ \overline{Y} \end{pmatrix}, \begin{pmatrix} A \\ B \end{pmatrix}\right] = P\left[\gamma^{-1}\begin{pmatrix} X \\ Y \end{pmatrix}, \overline{\gamma}^{-1}\begin{pmatrix} \overline{X} \\ \overline{Y} \end{pmatrix}, \frac{1}{|a|^2 + |c|^2}\begin{pmatrix} \overline{a} & \overline{c} \\ -c & a \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix}\right], \tag{5.3}$$

where here $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$.

Proposition 5.4.4. *This differential is invariant under right multiplication by* $\mathrm{SU}_2(\mathbb{C})$, *that is,*

$$\omega_F(gu) = \omega_F(g), \quad u \in \mathrm{SU}_2(\mathbb{C}), g \in \mathrm{SL}_2(\mathbb{C})$$

so ω_F gives a well-defined differential on $\mathcal{H}_3 = \mathrm{SL}_2(\mathbb{C})/\mathrm{SU}_2(\mathbb{C})$.

Proof. The map σ is $\mathrm{SU}_2(\mathbb{C})$-equivariant. Now,

$$\begin{aligned}\omega_F(gu) = gu \cdot (\sigma \circ F(gu)) &= gu \cdot u^{-1}(\sigma(F(g)) \\ &= g \cdot (\sigma \circ F(g)) = \omega_F(g),\end{aligned}$$

as required. $\square$

The Eichler-Shimura-Harder construction

To make this construction more concrete, we describe the map σ in detail. Firstly, define

$$\mathbf{Q} := \left(V^{2k+2},\ -(2k+2)V^{2k+1}U,\ \dots,\ (-1)^j \begin{pmatrix} 2k+2 \\ j \end{pmatrix} V^{2k+2-j}U^j,\ \dots,\ U^{2k+2} \right),$$

a vector of homogeneous monomials in two (new) variables U, V. Note that the components are precisely the monomials that appear in $(V-U)^{2k+2}$. Now define a vector $\Psi \in [V_{k,k}^{\ell}(\mathbb{C}) \otimes_{\mathbb{C}} V_2^{\ell}(\mathbb{C})]^{2k+3}$ by

$$(XV - YU)^k(\overline{X}U + \overline{Y}V)^k(AV - BU)^2 = \mathbf{Q} \cdot \Psi.$$

To emphasise the dependence on the polynomial variables, we will sometimes write

$$\Psi = \Psi(\mathbf{x}, \overline{\mathbf{x}}, \mathbf{a}) = \Psi\left(\begin{pmatrix} X \\ Y \end{pmatrix}, \begin{pmatrix} \overline{X} \\ \overline{Y} \end{pmatrix}, \begin{pmatrix} A \\ B \end{pmatrix} \right).$$

An explicit calculation shows that, for $u \in \mathrm{SU}_2(\mathbb{C})$, we have the relation

$$u \cdot \Psi(\mathbf{x}, \overline{\mathbf{x}}, \mathbf{a}) = \Psi(u^{-1}\mathbf{x}, \overline{u}^{-1}\overline{\mathbf{x}}, u^{-1}\mathbf{a}) = \rho_{2k+2}(u)^{-1}\Psi(\mathbf{x}, \overline{\mathbf{x}}, \mathbf{a}),$$

which is used below.

Define the components F_n of F to be functions $F_n : \mathrm{GL}_2(\mathbb{C}) \to \mathbb{C}$ in such a way that

$$F(g) = \sum_{j=0}^{2k+2} F_n(g) X^{2k+2-n} Y^n.$$

Writing F as a vector $(F_0, ..., F_{2k+2})$, is thus meaningful to take the dot product of $F(g)$ and Ψ to obtain a map

$$F \cdot \Psi : \mathrm{SL}_2(\mathbb{C}) \longrightarrow V_{k,k}^{\ell}(\mathbb{C}) \otimes_{\mathbb{C}} V_2^{\ell}(\mathbb{C}).$$

This is the promised explicit description of the map $\sigma \circ F$. It remains to make this well-defined on $\mathcal{H}_3$; as above, define a function

$$\delta(g, \mathbf{x}, \overline{\mathbf{x}}, \mathbf{a}) := g \cdot (F(g) \cdot \Psi(\mathbf{x}, \overline{\mathbf{x}}, \mathbf{a})).$$

Then, for $u \in \mathrm{SU}_2(\mathbb{C})$,

$$\begin{aligned} \delta(gu, \mathbf{x}, \overline{\mathbf{x}}, \mathbf{a}) &= gu \cdot (F(gu) \cdot \Psi(\mathbf{x}, \overline{\mathbf{x}}, \mathbf{a})) \\ &= g \cdot (F(gu) \cdot (u \cdot \Psi(\mathbf{x}, \overline{\mathbf{x}}, \mathbf{a}))) \\ &= g \cdot (F(g)\rho_{2k+2}(u) \cdot \rho_{2k+2}(u)^{-1}\Psi(\mathbf{x}, \overline{\mathbf{x}}, \mathbf{a})) \\ &= \delta(g, \mathbf{x}, \overline{\mathbf{x}}, \mathbf{a}), \end{aligned}$$

so that δ is well-defined on $\mathrm{SU}_2(\mathbb{C})$, as required. Now, under the isomorphism of Proposition 5.2.8, this construction gives a $V_{k,k}^{\ell}(\mathbb{C})$-valued differential $\omega_{\mathcal{F}} = \omega_F$ on $\mathcal{H}_3$. A lengthy calculation using the modularity of F and the action of $\mathrm{SL}_2(\mathbb{C})$ on $\Omega^1(\mathcal{H}_3)$ shows that in fact, this differential is invariant under the action of Γ, so that we get a well-defined differential on the quotient, that is, an element

$$\omega_{\mathcal{F}} \in \mathrm{H}^1\left(\Gamma\backslash\mathcal{H}_3, \mathcal{L}(V_{k,k}^{\ell}(\mathbb{C}))\right), \tag{5.4}$$

where $\mathcal{L}(V_{k,k}^{\ell}(\mathbb{C}))$ is the local system on $\Gamma\backslash\mathcal{H}_3$ corresponding to the Γ-module $V_{k,k}^{\ell}(\mathbb{C})$ (see Chapter 11.3 for further details). This differential is harmonic from the definition of automorphic forms, and hence we can integrate it between cusps of $\mathcal{H}_3$ in a path-independent manner. A simple check then shows:

Proposition 5.4.5. *Let $\mathcal{F} : \mathcal{H}_3 \longrightarrow V_{2k+2}(\mathbb{C})$ be a cusp form of weight (k,k) and level Γ. Then the map $\phi'_{\mathcal{F}} : \Delta_0 \to V_{k,k}^{\ell}(\mathbb{C})$ given by*

$$\phi'_{\mathcal{F}}(\{r\} - \{s\}) := \int_r^s \omega_{\mathcal{F}}$$

defines a modular symbol $\phi'_{\mathcal{F}} \in \mathrm{Symb}_{\Gamma}(V_{k,k}^{\ell}(\mathbb{C}))$.

Proof. This is an elementary consequence of the work done above. □

Remark: This is not quite the modular symbol attached to $\mathcal{F}$. In particular, note that this symbol takes values in $V_{k,k}^{\ell}(\mathbb{C})$, not $V_{k,k}(\mathbb{C})^*$. This discrepancy will be addressed in Chapter 5.4.4.

The connection to cohomology

In equation (5.4), we stated a relation between modular forms and cohomology spaces. We really want to say that this identification gives an isomorphism between the space of cusp forms of weight (k,k) and level $\Gamma_1^i(\mathfrak{n})$ and this cohomology group. Indeed, this is the case, and it is equivariant with respect to the Hecke operators at *principal* primes. However, as we've seen, the Hecke operators at *non*-principal primes are not defined on either of these spaces. To state the Eichler-Shimura-Harder isomorphism in its full form, we need some new notation to allow us to define an 'adelic' cohomology group.

Definition 5.4.6. Define the *locally symmetric space* $Y_1(\mathfrak{n})$ *of level* $\Omega_1(\mathfrak{n})$ to be the quotient

$$Y_1(\mathfrak{n}) := \mathrm{GL}_2(K)\backslash\mathrm{GL}_2(\mathbb{A}_K)/\Omega_1(\mathfrak{n})\mathrm{SU}_2(\mathbb{C})Z(\mathrm{GL}_2(\mathbb{C})),$$

where $Z(\mathrm{GL}_2(\mathbb{C})) \cong \mathbb{C}^\times$ is the centre of $\mathrm{GL}_2(\mathbb{C})$. This is the equivalent of the modular curve in this setting.

Lemma 5.4.7. *For a fixed choice of representatives $I_1, ..., I_h$ of the class group, there is an isomorphism*

$$\mathrm{H}^1_{\mathrm{cusp}}\left(Y_1(\mathfrak{n}), \mathcal{L}(V^\ell_{k,k}(\mathbb{C}))\right) \cong \bigoplus_{i=1}^{h} \mathrm{H}^1_{\mathrm{cusp}}\left(\Gamma^i_1(\mathfrak{n})\backslash\mathcal{H}_3, \mathcal{L}(V^\ell_{k,k}(\mathbb{C}))\right).$$

Proof. This follows immediately from strong approximation (see Theorem 2.2.4). □

There is a good theory of Hecke operators – given by the usual double coset operators – on this larger cohomology space. Given this, we then have:

Theorem 5.4.8 (Eichler-Shimura-Harder)**.** *Let $\Phi \in S_{k,k}(\Omega_1(\mathfrak{n}))$ be an (adelic) cusp form, and let $\mathcal{F}^1, ..., \mathcal{F}^h$ be the cusp forms on $\mathcal{H}_3$ associated to Φ. The association*

$$\Phi \longmapsto (\mathcal{F}^1, ..., \mathcal{F}^h) \longmapsto (\omega_{\mathcal{F}^1}, ..., \omega_{\mathcal{F}^h}) \in \bigoplus_{i=1}^{h} \mathrm{H}^1_{\mathrm{cusp}}\left(\Gamma^i_1(\mathfrak{n})\backslash\mathcal{H}_3, \mathcal{L}(V^\ell_{k,k}(\mathbb{C}))\right)$$

defines a Hecke-equivariant isomorphism

$$S_{k,k}(\Omega_1(\mathfrak{n})) \cong \mathrm{H}^1_{\mathrm{cusp}}\left(Y_1(\mathfrak{n}), \mathcal{L}(V^\ell_{k,k}(\mathbb{C}))\right).$$

Proof. This is [Hid94], Proposition 3.1, where the result is given in general. Hida in turn cites [Har87b], Section 3. □

5.4.4. Acting up: remarks on action conventions

The theory we are using here switches almost wilfully between right- and left-actions as well as to dual representations. Often this is for convenience, but sometimes it is a necessity.

For example, above we've defined a *left*-action of $\mathrm{SU}_2(\mathbb{C})$ on the space $\Omega^1(\mathcal{H}_3, \mathbb{C})$ of differentials, whilst also using the space $V^\ell_{k,k}(\mathbb{C})$, to define the vector

$$\Psi \in [V^\ell_{k,k}(\mathbb{C}) \otimes_{\mathbb{C}} V^\ell_2(\mathbb{C})]^{2k+3}.$$

This means that, when we look at the differential $\delta(g) \coloneqq g(F(g)\cdot\Psi) = F(g)\cdot(g\Psi)$ on $\mathrm{SL}_2(\mathbb{C})$, where g acts on Ψ component-wise, we have $\mathrm{SU}_2(\mathbb{C})$-invariance, and thus a well-defined differential on $\mathcal{H}_3$. This argument is simplified by using a left-action, since

$$\delta(gu) = g(F(gu)\cdot(u\Psi)) = g[F(g)\rho_{2k+2}(u)\rho_{2k+2}(u)^{-1}\cdot\Psi] = \delta(g);$$

if it were a *right*-action, then we do not get this cancellation in the middle.

This poses a slight technical problem. The differential we've defined takes values in $V_{k,k}^{\ell}(\mathbb{C})$ (which is the same as $V_{k,k}^{r}(\mathbb{C})$, when we switch to the corresponding right action). A Bianchi modular symbol, as defined above, takes values in $V_{k,k}(\mathbb{C})^*$, where here this is the dual of $V_{k,k}(\mathbb{C})$ equipped with a *different* left-action. For the purposes of p-adic interpolation, however, we cannot use the control theorem as proved to lift such a symbol. To this author, the technicalities of passing between symbols with values in $V_{k,k}^{r}(\mathbb{C})$ and $V_{k,k}(\mathbb{C})^*$ appeared somewhat frustrating. We record the following lemma.

Lemma 5.4.9. *(i) There is an* $\mathrm{SL}_2(\mathbb{C})$*-equivariant isomorphism*

$$V_k^r(\mathbb{C}) \xrightarrow{\sim} V_k(\mathbb{C})^*$$

given on monomials by

$$X^rY^{k-r} \longmapsto \binom{k}{r}^{-1} \mathcal{X}^{k-r}\mathcal{Y}^r,$$

where $\mathcal{X}^{k-r}\mathcal{Y}^r$ *is the element of the dual basis defined by*

$$\mathcal{X}^{k-r}\mathcal{Y}^r(X^{k-s}Y^s) = \begin{cases} 1 & : r = s, \\ 0 & : \textit{otherwise.} \end{cases}$$

(ii) Similarly, there is an $\mathrm{SL}_2(\mathbb{C})$*-equivariant isomorphism*

$$V_{k,k}^r(\mathbb{C}) \xrightarrow{\sim} V_{k,k}^*(\mathbb{C})$$

given by

$$X^rY^{k-r}\overline{X}^s\overline{Y}^{k-s} \mapsto \binom{k}{r}^{-1}\binom{k}{s}^{-1} \mathcal{X}^{k-r}\mathcal{Y}^r\overline{\mathcal{X}}^{k-s}\overline{\mathcal{Y}}^s.$$

Proof. An explicit check confirms the isomorphism of part (i), using the (easily verified) identity

$$\binom{k}{j}\binom{j}{n}\binom{k-j}{r-n} = \binom{r}{n}\binom{k-r}{j-n}\binom{k}{r}$$

of binomial coefficients. Part (ii) follows easily from part (i). □

Let $\mathcal{F} : \mathcal{H}_3 \to V_{2k+2}(\mathbb{C})$ be a cusp form of weight (k,k) and level Γ. Recall that in the previous section, we defined an element $\phi'_{\mathcal{F}} \in \mathrm{Symb}_\Gamma(V_{k,k}^{\ell}(\mathbb{C}))$ attached to $\mathcal{F}$. We consider this symbol as taking values in $V_{k,k}^r(\mathbb{C})$ via the usual compatibility.

Definition 5.4.10. The *modular symbol attached to* $\mathcal{F}$ is the element

$$\phi_{\mathcal{F}} \in \mathrm{Symb}_{\Gamma}(V_{k,k}(\mathbb{C})^*)$$

given by composing $\phi'_{\mathcal{F}}$ with the map $V^r_{k,k}(\mathbb{C}) \longrightarrow V_{k,k}(\mathbb{C})^*$ of the lemma.

Now let $\Phi \in S_{k,k}(\Omega_1(\mathfrak{n}))$ be a full Bianchi modular form on $\mathrm{GL}_2(\mathbb{A}_K)$, corresponding to a collection $\mathcal{F}^1, ..., \mathcal{F}^h$ of cusp forms on $\mathcal{H}_3$.

Definition 5.4.11. The *modular symbol attached to* Φ is the element

$$\phi_{\Phi} \in \mathrm{Symb}_{\Omega_1(\mathfrak{n})}(V_{k,k}(\mathbb{C})^*) = \bigoplus_{i=1}^{h} \mathrm{Symb}_{\Gamma_1^i(\mathfrak{n})}(V_{k,k}(\mathbb{C})^*)$$

given by the tuple $(\phi_{\mathcal{F}^1}, ..., \phi_{\mathcal{F}^h})$.

Remark: Using Lemma 5.4.7, this determines an element of $\mathrm{H}^1_{\mathrm{cusp}}(Y_1(\mathfrak{n}), \widetilde{V_{k,k}(\mathbb{C})^*})$. Whilst both the isomorphism of this lemma and the tuple $(\phi_{\mathcal{F}^1}, ..., \phi_{\mathcal{F}^h})$ both depend on the choice of class group representatives, this cohomology class is independent of choices. Indeed, in [Har87b], Harder states the Eichler-Shimura-Harder isomorphism independently of such representatives, and this class is the image of the automorphic form under this isomorphism.

5.5. Algebraic and p-adic modular symbols

Thus far, we have worked exclusively with complex coefficients. Our ultimate aim is to p-adically interpolate spaces of modular symbols, and to do so, we'll need to work with *algebraic* coefficients. That we can indeed do so relies on a result known as *multiplicity one*, which says that systems of Hecke eigenvalues occur in the space of modular symbols in one-dimensional eigenspaces. We obtain the following algebraicity results.

Lemma 5.5.1. *Let* $\Phi \in S_{k,k}(\Omega_1(\mathfrak{n}))$ *be a Bianchi modular form that is an eigenform for all the Hecke operators, and let* $K(\Phi)$ *be the field extension obtained by adjoining all of the Hecke eigenvalues to* K*. Then* $K(\Phi)$ *is a number field.*

Proof. See [Hid94], Chapter 6. ☐

Theorem 5.5.2. *Let* $\Phi \in S_{k,k}(\Omega_1(\mathfrak{n}))$ *be a Bianchi modular form that is an eigenform for all the Hecke operators, and let* $\phi_\Phi \in \mathrm{Symb}_{\Omega_1(\mathfrak{n})}(V_{k,k}(\mathbb{C})^*)$ *be its associated modular symbol. Then there is a finite extension* $F/K(\Phi)$ *and a complex period* $\Omega_\Phi \in \mathbb{C}^\times$ *such that we have*

$$\phi_\Phi/\Omega_\Phi \in \mathrm{Symb}_{\Omega_1(\mathfrak{n})}(V_{k,k}(F)^*).$$

Proof. (Sketch). A similar result is proved in [Hid94], Chapter 8. As mentioned above, the key step is to show that for suitable coefficient spaces – including the fields F and $\mathbb{C}$ – the space where the Hecke operators act with the same system of eigenvalues as Φ is one dimensional. □

So, to a cusp form $\mathcal{F}$ on $\mathcal{H}_3$, we can renormalise the corresponding modular symbol to have coefficients in some sufficiently large number field. This, in turn, can be embedded into a sufficiently large finite extension L of $\mathbb{Q}_p$. Accordingly, we obtain a p-adic modular symbol attached to $\mathcal{F}$.

Remark: Note that this modular symbol is not canonical. Indeed, the choice of Ω_Φ is well-defined only up to multiplication by elements of $F^\times$.

Henceforth, we fix a period Ω_Φ and in an abuse of notation write $\phi_{\mathcal{F}} \in \mathrm{Symb}_\Gamma(V_{k,k}(L)^*)$ for the *p-adic* modular symbol attached to a cusp form $\mathcal{F}$ on $\mathcal{H}_3$.

5.6. Summary of construction

We quickly summarise the construction above. From a cuspidal Bianchi modular form Φ, and a fixed choice of representatives for the class group of K, we obtained a collection $\mathcal{F}^1, ..., \mathcal{F}^h$ of cusp forms on $\mathcal{H}_3$. For each fixed i, we associated to $\mathcal{F}^i$ a harmonic differential $\omega_{\mathcal{F}^i}$ on $\mathcal{H}_3$ with values in $V_{k,k}^\ell(\mathbb{C})$, and from this, we obtained an element

$$\phi'_{\mathcal{F}^i} \in \mathrm{Symb}_{\Gamma_1^i(\mathfrak{n})}(V_{k,k}^\ell(\mathbb{C}))$$

by integrating $\omega_{\mathcal{F}^i}$ over paths between cusps. We viewed this as an element of $\mathrm{Symb}_{\Gamma_1^i(\mathfrak{n})}(V_{k,k}^r(\mathbb{C}))$ by passing to the corresponding right-action on $V_{k,k}(\mathbb{C})$, and then to an element

$$\phi_{\mathcal{F}^i} \in \mathrm{Symb}_{\Gamma_1^i(\mathfrak{n})}(V_{k,k}(\mathbb{C})^*)$$

by using the isomorphism of Lemma 5.4.9. We then renormalised by an appropriate fixed choice of period to obtain a symbol defined over a sufficiently large number field, which then allowed us to define our symbol over a sufficiently large finite extension L of $\mathbb{Q}_p$. In an abuse of notation, we have written

$$\phi_{\mathcal{F}^i} \in \mathrm{Symb}_{\Gamma_1^i(\mathfrak{n})}(V_{k,k}(L)^*)$$

for this symbol.

Finally, we define a p-adic modular symbol attached to the full Bianchi modular form Φ by collecting these symbols together into a tuple

$$(\phi_{\mathcal{F}^1}, ..., \phi_{\mathcal{F}^h}) \in \mathrm{Symb}_{\Omega_1(\mathfrak{n})}(V_{k,k}(L)^*).$$

We need to consider this collection to be able to define Hecke operators at non-principal primes, since such operators permute the components.

Chapter 6

L-values via Modular Symbols

In this chapter, we give an explicit link between the modular symbol attached to a Bianchi modular form Φ and critical values of its L-function. We start by recalling definitions that are relevant in the study of this L-function, before deriving an integral formula for it. We explicitly compute the modular symbol attached to Φ and show that, using this integral formula, we can link its values to a certain range of special L-values. After this, we refine this statement into progressively neater forms by first renormalising the L-function and then considering it as a function on characters.

6.1. The L-function of a Bianchi modular form

6.1.1. Definitions and Fourier expansions revisited

Recall that we defined the L-function of a general automorphic form Φ using either its Fourier expansion or Hecke eigenvalues (in the case where the form is an eigenform of all the Hecke operators). In particular, writing the Fourier coefficients as $c(I, \Phi)$ and letting φ be a Hecke character, we defined

$$L(\Phi, \varphi, s) := \sum_{0 \neq I \subset \mathcal{O}_K} c(I, \Phi)\varphi(I)N(I)^{-s}.$$

It is convenient to also define the following 'partial' L-functions, each corresponding to an element of the class group.

Definition 6.1.1. Let $w = |\mathcal{O}_K^\times|$ be the size of the unit group of K (noting that this is finite by Dirichlet's unit theorem). Then define

$$L^j(\Phi, \varphi, s) = L(\mathcal{F}^j, \varphi, s) := w^{-1} \sum_{\zeta \in K^\times} c(\zeta I_j, \Phi)\varphi(\zeta I_j)N(\zeta I_j)^{-s}.$$

Note here that whilst $\mathcal{F}^j$ is dependent on the class group representative a_j, the partial L-function $L^j(\Phi, \varphi, s)$ is not. We also have

$$L(\Phi, \varphi, s) = L^1(\Phi, \varphi, s) + \cdots + L^h(\Phi, \varphi, s),$$

where here we scale by w^{-1} as when we sum over elements of $K^\times$, we include each ideal w times (that is, once for each unit). As before, each of the partial L-functions converges absolutely on some right-half plane, that is, for $\mathrm{Re}(s)$ sufficiently large.

Remark: We can describe these partial L-functions in terms of a Fourier expansion solely for $\mathcal{F}^j$. Indeed, such a Fourier expansion can be worked out to be

$$\mathcal{F}^j\left(z,t;\begin{pmatrix}X\\Y\end{pmatrix}\right) = |a_j|t\sum_{n=0}^{2k+2}\begin{pmatrix}2k+2\\n\end{pmatrix}\sum_{\zeta\in K^\times}\left[c(\zeta\delta I_j,\Phi)\left(\frac{\zeta}{i|\zeta|}\right)^{k+1-n}\times\right. \tag{6.1}$$
$$\left.\zeta^{-v_1}\overline{\zeta}^{-v_2}K_{n-k-1}(4\pi|\zeta|t)e^{2\pi i(\zeta z+\overline{\zeta z})}\right]X^{2k+2-n}Y^n.$$

Here we've written $(z,t)\in\mathcal{H}_3\cong\mathbb{C}^\times\times\mathbb{R}_{>0}$. This version of the Fourier expansion can be obtained naturally from the version stated previously. Then the coefficients appearing in the definition of L^j are precisely those appearing in the expansion of $\mathcal{F}^j$.

We make one further definition.

Definition 6.1.2. Let $\mathcal{F}^j_n:\mathcal{H}_3\longrightarrow\mathbb{C}$ be the functions determined by the expression

$$\mathcal{F}^j\left(z,t;\begin{pmatrix}X\\Y\end{pmatrix}\right) = \sum_{n=0}^{2k+2}\mathcal{F}^j_n(z,t)X^{2k+2-n}Y^n.$$

We'll use this terminology when computing the integral formula later in this section.

6.1.2. Gauss sums revisited

In Chapter 1.3.1, we defined Gauss sums attached to Hecke characters over a general number field. In keeping with the more concrete treatment of the theory over imaginary quadratic fields, here we give a different formulation referring to adeles only where necessary. We use the exposition in [Nem93], which itself draws on papers [Hec20] and [Hec23] of Hecke (that are written in German). Another way of defining these objects in terms of local ε-factors is described in [Del72], Section 3 (and is translated into English in [Tat79], Section 3). Hecke's Gauss sums are shown to be a product of local Gauss sums that agree with Deligne's ε-factors in [Nar04], Proposition 6.14.

Definition 6.1.3. Let K be an imaginary quadratic field with different $\mathfrak{D}=\delta\mathcal{O}_K$, and let φ be a Hecke character for K with conductor $\mathfrak{f}$. The *Gauss sum* for φ is defined to be

$$\tau(\varphi) := \sum_{\substack{[a]\in\mathfrak{f}^{-1}/\mathcal{O}_K\\(a\mathfrak{f},\mathfrak{f})=1}}\varphi(a\mathfrak{f})\varphi_\infty\left(\frac{a}{\delta}\right)e^{2\pi i\mathrm{Tr}_{K/\mathbb{Q}}(a/\delta)},$$

where here the notation $(I,J)=1$ for ideals I and J mean that I and J are coprime.

Remarks: (i) The coprimality condition is essential, as otherwise $\varphi(a\mathfrak{f})$ is not well-defined as the value of φ at the ideal $a\mathfrak{f}$. Excluding these terms from the sum corresponds to setting $\chi(a) = 0$ for χ a rational Dirichlet character and a an integer not coprime to the conductor.

(ii) This isn't quite as Nemchenok defines a Gaussian sum in [Nem93]; rather, he defines the sum to be over $\mathfrak{f}^{-1}\mathfrak{D}^{-1}/\mathfrak{D}^{-1}$. We've used the fact that δ generates $\mathfrak{D}$ and rescaled to get the version above.

We also have the following reformulation of Proposition 1.3.3:

Proposition 6.1.4. *(i) For all $b \in \mathcal{O}_K$, we have*

$$\sum_{\substack{[a]\in\mathfrak{f}^{-1}/\mathcal{O}_K \\ (a\mathfrak{f},\mathfrak{f})=1}} \varphi(a\mathfrak{f})\varphi_\infty\left(\frac{a}{\delta}\right) e^{2\pi i \mathrm{Tr}_{K/\mathbb{Q}}(ab/\delta)} = \tau(\varphi)\varphi_{\mathfrak{f}}(b).$$

(ii) By replacing φ with φ^{-1}, we have

$$\frac{1}{\tau(\varphi^{-1})} \sum_{\substack{[a]\in\mathfrak{f}^{-1}/\mathcal{O}_K \\ (a\mathfrak{f},\mathfrak{f})=1}} \varphi(a\mathfrak{f})^{-1}\varphi_\infty\left(\frac{a}{\delta}\right)^{-1} e^{2\pi i \mathrm{Tr}_{K/\mathbb{Q}}(ab/\delta)} = \begin{cases} \varphi_{\mathfrak{f}}(b)^{-1} & : ((b),\mathfrak{f}) = 1, \\ 0 & : \textit{otherwise.} \end{cases}$$

Proof. We give only the briefest details of how to prove this statement. All of the ingredients required are contained in [Nem93], though the result is not stated explicitly. To combine the results stated therein: from the Gauss sum defined above (which is Nemchenok's normalised Gauss sum, that is, a Gauss sum of parameter 1 and auxiliary ideal $\mathfrak{f}^{-1}\mathfrak{D}^{-1}$), we can define a Gauss sum for $\chi = \varphi_{\mathfrak{f}}$ with parameter 1. The sum in the proposition corresponds to a Gaussian sum for φ of parameter b and auxiliary ideal $\mathfrak{f}^{-1}\mathfrak{D}^{-1}$, which then corresponds to a Gauss sum for χ with parameter b. Nemchenok's Proposition 4 (part (6)) gives the relation between Gauss sums of parameters 1 and b to be multiplication by $\overline{\chi}(b)$, which then translates into the required result. □

6.1.3. An integral formula for the L-function

We want to write $L^j(\Phi, \varphi, s)$ in an integral form, much like in the rational case (see [DS05], Chapter 5.10). To do so, we will use Gauss sums to deal with coefficients at ideals that are not coprime to the conductor. Throughout this section, φ denotes a Hecke character of infinity type $(-u, -v)$ and conductor $\mathfrak{f}$.

Ultimately, the result we will prove is:

Theorem 6.1.5. *With the notation as above, for* $(u,v) = (\frac{k+1-n}{2}, -\frac{k+1-n}{2})$*, we have*

$$L^j(\Phi,\varphi,s) = A(j,n,\varphi,s) \sum_{[a]\in\mathfrak{f}^{-1}/\mathcal{O}_K} \varphi(a\mathfrak{f})^{-1} a^u \overline{a}^v \int_0^\infty t^{2s-2}\mathcal{F}_n^j(a,t)dt,$$

where $\mathcal{F}_n^j$ *is as defined in Definition 6.1.2 and*

$$A(j,n,\varphi,s) = \varphi(a_j)|a_j|_f^{s-1} \cdot \frac{4\cdot(2\pi)^{2s} i^{k+1-n}\binom{2k+2}{n}^{-1}}{|D|^s\Gamma\left(s+\frac{n-k-1}{2}\right)\Gamma\left(s-\frac{n-k-1}{2}\right)w\tau(\varphi^{-1})}.$$

is an explicit function of s*. (Here* $D = N(\mathfrak{D})$ *is the discriminant of* K*, whilst* $|\cdot|$ *denotes the usual norm on* $K^\times$ *and* $|\cdot|_f = \prod_{v\nmid\infty}|\cdot|_v$ *denotes the finite adelic norm).*

The proof is contained in the rest of this section.

We can write

$$L^j(\Phi,\varphi,s) = w^{-1}\varphi(\delta I_j)N(\delta I_j)^{-s} \sum_{\substack{\zeta\in K^\times \\ ((\zeta\delta),\mathfrak{f})=1}} c(\zeta\delta I_j,\Phi)\varphi_\infty(\zeta)^{-1}\varphi_{\mathfrak{f}}(\zeta)^{-1}|\zeta|^{-2s}$$

$$= \frac{\varphi(a_j)|a_j|_f^s}{w|D|^s} \sum_{\substack{\zeta\in K^\times, \\ ((\zeta\delta),\mathfrak{f})=1}} c(\zeta\delta I_j,\Phi)\varphi_\infty(\zeta\delta)^{-1}\varphi_{\mathfrak{f}}(\zeta\delta)^{-1}|\zeta|^{-2s},$$

using that $N(\delta I_j)^{-s} = |D|^{-s}|a_j|_f^s$ (recalling that $D = N((\delta))$ is the discriminant of K). Using Proposition 6.1.4, this is

$$= \frac{\varphi(a_j)|a_j|_f^s}{w|D|^s\tau(\varphi^{-1})} \sum_{\zeta\in K^\times} c(\zeta\delta I_j,\Phi)\varphi_\infty(\zeta\delta)^{-1} \times \left[\sum_{\substack{[a]\in\mathfrak{f}^{-1}/\mathcal{O}_K \\ (a\mathfrak{f},\mathfrak{f})=1}} \varphi(a\mathfrak{f})^{-1}\varphi_\infty\left(\frac{a}{\delta}\right)^{-1} e^{2\pi i \mathrm{Tr}_{K/\mathbb{Q}}(a\zeta)}\right]|\zeta|^{-2s},$$

where we've substituted our Gauss sum for $\varphi_{\mathfrak{f}}(\zeta\delta)^{-1}$ and accordingly eliminated the condition $((\zeta\delta),\mathfrak{f}) = 1$ from the sum over $K^\times$. Continuing, this is

$$= \frac{\varphi(a_j)|a_j|_f^s}{w\tau(\varphi^{-1})|D|^s} \sum_{\substack{[a]\in\mathfrak{f}^{-1}/\mathcal{O}_K \\ (a\mathfrak{f},\mathfrak{f})=1}} \varphi(a\mathfrak{f})^{-1}a^u\overline{a}^v \sum_{\zeta\in K^\times} c(\zeta\delta I_j,\Phi)\zeta^u\overline{\zeta^v}e^{2\pi i\mathrm{Tr}_{K/\mathbb{Q}}(a\zeta)}|\zeta|^{-2s},$$

via some cancellation and using that $\varphi_\infty(x) = x^u\overline{x}^v$.

To get our integral formula, we will use the standard integral (see [Hid94], section 7)

$$\int_0^\infty t^{\ell-1}K_{n-k-1}(at)dt = a^{-\ell}2^{\ell-2}\Gamma\left(\frac{\ell+n-k-1}{2}\right)\Gamma\left(\frac{\ell-n+k+1}{2}\right).$$

Setting $a = 4\pi|\zeta|$, and $\ell = 2s$, this translates into

$$|\zeta|^{-2s} = \frac{4 \cdot (2\pi)^{2s}}{\Gamma\left(s + \frac{n-k-1}{2}\right) \Gamma\left(s - \frac{n-k-1}{2}\right)} \int_0^\infty t^{2s-1} K_{n-k-1}(4\pi|\zeta|t)dt.$$

To get this into the form of the Fourier expansion for $\mathcal{F}^j$, we fix $(u,v) = (\frac{k+1-n}{2}, -\frac{k+1-n}{2})$, which means we now have

$$L^j(\Phi,\varphi,s) = \frac{\varphi(a_j)|a_j|_f^s}{w\tau(\varphi^{-1})|D|^s} \sum_{\substack{[a]\in\mathfrak{f}^{-1}/\mathcal{O}_K \\ (a\mathfrak{f},\mathfrak{f})=1}} \varphi(a\mathfrak{f})^{-1} a^u \overline{a}^v \sum_{\zeta\in K^\times} \left[c(\zeta\delta I_j,\Phi) \left[\frac{\zeta}{|\zeta|}\right]^{k+1-n} \times \right.$$
$$\left. e^{2\pi i \mathrm{Tr}_{K/\mathbb{Q}}(a\zeta)} |\zeta|^{-2s} \right].$$

Substituting the expression for $|\zeta|^{-2s}$ into this gives

$$L^j(\Phi,\varphi,s) = A(j,n,\varphi,s) \sum_{\substack{[a]\in\mathfrak{f}^{-1}/\mathcal{O}_K \\ (a\mathfrak{f},\mathfrak{f})=1}} \varphi(a\mathfrak{f})^{-1} a^u \overline{a}^v \sum_{\zeta\in K^\times} \left[c(\zeta\delta I_j,\Phi) \left[\frac{\zeta}{i|\zeta|}\right]^{k+1-n} \times \right.$$
$$\left. e^{2\pi i \mathrm{Tr}_{K/\mathbb{Q}}(a\zeta)} \int_0^\infty t^{2s-1} K_{n-k-1}(4\pi|\zeta|t)dt \right]$$

$$= A(j,n,\varphi,s) \sum_{\substack{[a]\in\mathfrak{f}^{-1}/\mathcal{O}_K \\ (a\mathfrak{f},\mathfrak{f})=1}} \varphi(a\mathfrak{f})^{-1} a^u \overline{a}^v \int_0^\infty \sum_{\zeta\in K^\times} \left[c(\zeta\delta I_j,\Phi) \left(\frac{\zeta}{i|\zeta|}\right)^{k+1-n} \times \right.$$
$$\left. e^{2\pi i \mathrm{Tr}_{K/\mathbb{Q}}(a\zeta)} t^{2s-1} K_{n-k-1}(4\pi|\zeta|t) \right] dt$$

$$= A(j,n,\varphi,s) \sum_{\substack{[a]\in\mathfrak{f}^{-1}/\mathcal{O}_K \\ (a\mathfrak{f},\mathfrak{f})=1}} \varphi(a\mathfrak{f})^{-1} a^u \overline{a}^v \int_0^\infty t^{2s-2} \mathcal{F}_n^j(a,t)dt,$$

where

$$A(j,n,\varphi,s) = \varphi(a_j)|a_j|_f^{s-1} \cdot \frac{4 \cdot (2\pi)^{2s} i^{k+1-n} \binom{2k+2}{n}^{-1}}{|D|^s \Gamma\left(s + \frac{n-k-1}{2}\right) \Gamma\left(s - \frac{n-k-1}{2}\right) w\tau(\varphi^{-1})}.$$

Remark: We can swap the integral and summation in the calculation above in some right half-plane due to absolute convergence. Via meromorphic continuation, this integral then gives a definition of the L-function on all of $\mathbb{C}$, whereas previously we'd shown only that it was well-defined on a right half-plane. In fact, a little more work shows that this function is an *analytic* continuation and the L-function is holomorphic on the whole complex plane. This is a special case of a well-known result of Langlands on the analytic continuation of L-functions of automorphic representations for GL_n.

6.2. Explicitly linking the modular symbol and L-values

There is an important link between modular symbols and critical L-values. In this section, we derive this link by explicitly computing the differential defined previously. First, though, we give the link in the case of weight $(0,0)$ and class number 1, where the arguments are, as usual, much simpler.

6.2.1. L-values in weight $(0,0)$

Suppose K has class number 1, and recall the definition of the modular symbol attached to a Bianchi cusp form $\mathcal{F} = (\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2)$ of weight $(0,0)$ over K from Definition 5.4.2. In particular, we defined

$$\omega_{\mathcal{F}} := \mathcal{F}_0(z,t)dz - \mathcal{F}_1(z,t)dt - \mathcal{F}_2(z,t)d\overline{z},$$

where the $\mathcal{F}_i$ are the components of $\mathcal{F}$. We then defined

$$\phi_{\mathcal{F}}(\{r\} - \{s\}) = \int_r^s \omega_{\mathcal{F}}.$$

Since the integral we obtain is path independent, we can choose our path from r to s to be composed of the two vertical paths from r to ∞ and then ∞ to s. When we integrate over these paths, there is no change in the z or $\overline{z}$ directions, and hence in these cases the integral collapses to give

$$\phi_{\mathcal{F}}(\{r\} - \{\infty\}) = -\int_0^\infty \mathcal{F}_1(r,t)dt.$$

Now consider the integral formula in this situation. Set $s = 1$, $n = 1$, $k = 0$ and $r = 0$; then in this situation, and with $\varphi = 1$, we see that

$$\phi_{\mathcal{F}}(\{0\} - \{\infty\}) = -\frac{|\delta|^2 w}{8\pi^2} \cdot L(\mathcal{F}, 1).$$

Thus $\phi_{\mathcal{F}}$ sees the (single) special L-value of $\mathcal{F}$. This is precisely the kind of property we want the modular symbol to have; the task is now to make this connection for higher weights and class number and non-trivial characters.

Remark: This expression is proved directly, in this special case, in [CW94]; the proof of the integral formula given previously is a direct generalisation of their proof.

6.2.2. An explicit description of the modular symbol

In the weight (0,0) and class number 1 case, we exhibited a link between the modular symbol $\phi_{\mathcal{F}}$ and the critical L-value $L(\mathcal{F}, 1)$ of $\mathcal{F}$. We'd like to extend this link to general weights and class number. To that end, let $\mathcal{F}^j$ be a cusp form on $\mathcal{H}_3$ of weight (k,k) and level $\Gamma_1^j(\mathfrak{n})$. In

Section 6.1.1, we stated that $\mathcal{F}^j$ has a Fourier expansion of form

$$\mathcal{F}^j\left(z,t;\begin{pmatrix}X\\Y\end{pmatrix}\right)=|a_j|_K t\sum_{n=0}^{2k+2}\binom{2k+2}{n}\sum_{\zeta\in K^\times}\left[c(\zeta\delta I_j,\Phi)\left(\frac{\zeta}{i|\zeta|}\right)^{k+1-n}\times\right.$$
$$\left.K_{n-k-1}(4\pi|\zeta|t)e^{2\pi i(\zeta z+\overline{\zeta z})}\right]X^{2k+2-n}Y^n. \tag{6.2}$$

where a_j is an idele representing I_j, our fixed representative of the element of the class group corresponding to $\mathcal{F}^j$.

To implement this, we calculate Ψ more explicitly following Ghate ([Gha99]). Write $\Psi=(\Psi_0,...,\Psi_{2k+2})$. Then some work shows that

$$\Psi_n(\mathbf{x},\overline{\mathbf{x}},\mathbf{a})=(-1)^n\binom{2k+2}{n}^{-1}[A^2\cdot C_n(\mathbf{x},\overline{\mathbf{x}})-2AB\cdot C_{n-1}(\mathbf{x},\overline{\mathbf{x}})+B^2\cdot C_{n-2}(\mathbf{x},\overline{\mathbf{x}})],$$

where

$$C_n(\mathbf{x},\overline{\mathbf{x}}):=\sum_{\substack{q,r=0\\k-(q-r)=n}}^{k}(-1)^r\binom{k}{q}\binom{k}{r}X^{k-q}Y^q\overline{X}^{k-r}\overline{Y}^r.$$

For $(z,t)\in\mathcal{H}_3$, let

$$g=\frac{1}{\sqrt{t}}\begin{pmatrix}t&z\\0&1\end{pmatrix},$$

an element of $\mathrm{SL}_2(\mathbb{C})$ that represents it. Then, under the left-action of $\mathrm{SL}_2(\mathbb{C})$ defined by equation (5.3), we have

$$\begin{aligned}\delta(\mathcal{F}^j)(z,t)&:=\delta(F^j)(g)\\&=F^j(g)\cdot\Psi\left[\frac{1}{\sqrt{t}}\begin{pmatrix}1&-z\\0&t\end{pmatrix}\begin{pmatrix}X\\Y\end{pmatrix},\frac{1}{\sqrt{t}}\begin{pmatrix}1&-\overline{z}\\0&t\end{pmatrix}\begin{pmatrix}\overline{X}\\\overline{Y}\end{pmatrix},\frac{1}{\sqrt{t}}I_2\begin{pmatrix}A\\B\end{pmatrix}\right].\end{aligned}$$

The convention we are using (see Section 2.2.5) dictates that

$$\mathcal{F}^j(z,t)=t^{-k-1}F^j(g),$$

so that this becomes

$$\begin{aligned}\delta(\mathcal{F}^j)(z,t)&=\sum_{n=0}^{2k+2}\sqrt{t}^{2k+2}\mathcal{F}^j_n(z,t)\Psi_j\left(\frac{X-zY}{\sqrt{t}},Y\sqrt{t},\frac{\overline{X}-\overline{z}\overline{Y}}{\sqrt{t}},\overline{Y}\sqrt{t},\frac{A}{\sqrt{t}},\frac{B}{\sqrt{t}}\right)\\&=\sum_{n=0}^{2k+2}\mathcal{F}^j_n(z,t)\Psi_j\left(X-zY,Yt,\overline{X}-\overline{z}\overline{Y},\overline{Y}t,A,B\right),\end{aligned}$$

with A^2 replaced by dz, AB replaced by $-dt$, and B^2 replaced by $-d\overline{z}$. From hereon in, we begin to simplify matters with our exact goal in mind. We will integrate over a vertical path between a cusp and ∞, so we may as well fix $z = a$ for some $a \in K$. We also only need to look at the AB term of this polynomial, corresponding to dt, as the integrals in the z and $\overline{z}$ direction will vanish. Under these simplifications, the resulting differential $\delta'(\mathcal{F}^j)(a,t)$ that we obtain is

$$\delta'(\mathcal{F}^j)(a,t) = 2\sum_{n=0}^{2k+2}(-1)^n \binom{2k+2}{n}^{-1} \mathcal{F}^j_n(a,t)C_{n-1}(X - aY, tY, \overline{X} - \overline{a}\overline{Y}, t\overline{Y})dt.$$

We can simplify this by using the isomorphism $V^r_{k,k}(\mathbb{C}) \to V_{k,k}(\mathbb{C})^*$ given in Section 5.4.4, which (via some simple calculations) has the effect of replacing $C_{n-1}(X - aY, tY, \overline{X} - \overline{a}\overline{Y}, t\overline{Y})$ with

$$C'_{n-1}(\mathcal{Y} - a\mathcal{X}, t\mathcal{X}, \overline{\mathcal{Y}} - a\overline{\mathcal{X}}, t\overline{\mathcal{X}}) := \sum_{\substack{q,r=0\\ k-(q-r)=n-1}}^{k} (-1)^q t^{q+r}(\mathcal{Y} - a\mathcal{X})^{k-q}\mathcal{X}^q(\overline{\mathcal{Y}} - a\overline{\mathcal{X}})^{k-r}\overline{\mathcal{X}}^r. \tag{6.3}$$

Substituting the expression (6.3) above, integrating over 0 to ∞, eliminating n from the sum, and making some trivial simplifications, we get:

Proposition 6.2.1. *We can explicitly describe the modular symbol attached to $\mathcal{F}^j$ at generating divisors as*

$$\phi_{\mathcal{F}^j}(\{a\} - \{\infty\}) = \sum_{q,r=0}^{k} c^j_{q,r}(a)(\mathcal{Y} - a\mathcal{X})^{k-q}\mathcal{X}^q(\overline{\mathcal{Y}} - \overline{a}\overline{\mathcal{X}})^{k-r}\overline{\mathcal{X}}^r,$$

for $a \in K$, where

$$c^j_{q,r}(a) := 2\binom{2k+2}{k-q+r+1}^{-1}(-1)^{k+r+1}\int_0^\infty t^{q+r}\mathcal{F}^j_{k-q+r+1}(a,t)dt.$$

6.2.3. Linking modular symbols and L-values

We now refer back to Section 6.1.3, and, in particular, the integral formula we obtained for the L-function $L^j(\Phi,\varphi,s)$, where φ is a Hecke character with infinity type

$$(-u,-v) = \left(-\frac{q-r}{2}, \frac{q-r}{2}\right)$$

and conductor $\mathfrak{f}$. We want to set $2s-2 = q+r$, that is, $s = \frac{q+r+2}{2}$. Again writing $n = k-q+r+1$ for ease of notation, we obtain

$$\sum_{\substack{[a]\in\mathfrak{f}^{-1}/\mathcal{O}_K\\ (a\mathfrak{f},\mathfrak{f})=1}} \varphi(a\mathfrak{f})^{-1}a^u\overline{a}^v c^j_{q,r}(a) = (-1)^{k+r+1}2\left(\begin{smallmatrix}2k+2\\ n\end{smallmatrix}\right)^{-1} A(j,n,\varphi,\tfrac{q+r+2}{2})^{-1}L^j(\Phi,\varphi,\tfrac{q+r+2}{2}).$$

Some cancellation using the explicit form for $A(j,n,\varphi,\frac{q+r+2}{2})$ now gives

$$\sum_{\substack{[a]\in\mathfrak{f}^{-1}/\mathcal{O}_K\\(a\mathfrak{f},\mathfrak{f})=1}}\varphi(a\mathfrak{f})^{-1}a^u\overline{a}^v c^j_{q,r}(a)$$
$$=\left[\varphi(a_j)|a_j|_f^{\frac{q+r}{2}}\cdot\frac{(-1)^{q+r+k}2(2\pi i)^{q+r+2}}{|\delta|^{q+r+2}\Gamma(q+1)\Gamma(r+1)w\tau(\varphi^{-1})}\right]^{-1}L^j(\Phi,\varphi,\tfrac{q+r+2}{2}).$$

Here we've used that $|D|=|\delta|^2$. This gives us a link between modular symbols and L-values. Indeed, combining the results above for each j, we have the following theorem:

Theorem 6.2.2. *We have:*

$$L(\Phi,\varphi,\tfrac{q+r+2}{2})=$$
$$\left[\frac{(-1)^k2(-2\pi i)^{q+r+2}}{|\delta|^{q+r+2}q!r!w\tau(\varphi^{-1})}\right]\sum_{j=1}^h\left[\varphi(a_j)|a_j|_f^{\frac{q+r}{2}}\sum_{\substack{[a]\in\mathfrak{f}^{-1}/\mathcal{O}_K\\(a\mathfrak{f},\mathfrak{f})=1}}\varphi(a\mathfrak{f})^{-1}a^u\overline{a}^v c^j_{q,r}(a)\right].$$

Here:

- Φ *is a Bianchi cusp form of weight* (k,k) *and level* $\Omega_1(\mathfrak{n})$*,*
- w *is the size of the unit group of* $\mathcal{O}_K$*,*
- δ *is a generator of the different,*
- h *is the class number,*
- a_j *is an idele corresponding to the* j*th representative* I_j *of the class group, which is coprime to* $\mathfrak{n}$*,*
- *For an ideal* $I\subset\mathcal{O}_K$*,* x_I *is the fixed idele corresponding to* I *defined in Chapter 1.2.3,*
- φ *is a Hecke character of infinity type* $(-u,-v)=(-\frac{q-r}{2},\frac{q-r}{2})$ *and conductor* $\mathfrak{f}$*, where* $0\le q,r\le k$*,*
- $c^j_{q,r}(a)$ *is the coefficient of* $(\mathcal{Y}-a\mathcal{X})^{k-q}\mathcal{X}^q(\overline{\mathcal{Y}}-\bar{a}\overline{\mathcal{X}})^{k-r}\overline{\mathcal{X}}^r$ *in* $\phi_{\mathcal{F}^j}(\{a\}-\{\infty\})$*, where*
- $\phi_{\mathcal{F}^j}$ *is the modular symbol attached to the function* $\mathcal{F}^j$ *on* $\mathcal{H}_3$ *induced by* Φ*.*

6.3. L-functions as functions on characters

Recall the definition of L-functions as functions on Hecke characters from previously. For a Hecke character φ, we defined

$$L(\Phi,\varphi)=L(\Phi,\varphi,1).$$

This relates to more general values via

$$L(\Phi,\varphi|\cdot|_{\mathbb{A}_K}^{s-1})=L(\Phi,\varphi,s).$$

With this in mind, let $\varphi' := \varphi|\cdot|_{\mathbb{A}_K}^{\frac{q+r}{2}}$. This is now a Hecke character of conductor $\mathfrak{f}$ and infinity type

$$\left(\frac{q-r}{2}+\frac{q+r}{2}, -\frac{q-r}{2}+\frac{q+r}{2}\right) = (q,r),$$

where $0 \leq q, r \leq k$. This is the 'critical square' predicted by Deligne.

Proposition 6.3.1. *If φ is a Hecke character of conductor $\mathfrak{f}$ and $\varphi' := \varphi|\cdot|_{\mathbb{A}_K}^{\frac{q+r}{2}}$, then there is a relation*

$$\tau(\varphi') = |x_{\mathfrak{f}}|_{\mathbb{A}_K}^{\frac{q+r}{2}}|\delta|^{-(q+r)}\tau(\varphi)$$

of Gauss sums, where the idele $x_{\mathfrak{f}}$ associated to $\mathfrak{f}$ is as defined in Section 1.2.3.

Proof. We have

$$|x_{a\mathfrak{f}}|_{\mathbb{A}_K} = |x_{a\mathfrak{f}}|_f = |a|_f|x_{\mathfrak{f}}|_f = |a|^{-2}|x_{\mathfrak{f}}|_f,$$

where the first equality follows from the definition of $x_{a\mathfrak{f}}$ (as the infinite components are trivial). We also know that $|a|_{\mathbb{A}_K} = 1$, so that $|a|_f = |a|^{-2}$ (where $|\cdot|$ is the standard norm on $\mathbb{C}$). Now, computing with the Gauss sums,

$$\begin{aligned}
\tau(\varphi') &= \sum_{\substack{[a]\in\mathfrak{f}^{-1}/\mathcal{O}_K\\(a\mathfrak{f},\mathfrak{f})=1}} \varphi'(a\mathfrak{f})\varphi'_\infty\left(\frac{a}{\delta}\right)e^{2\pi i \mathrm{Tr}_{K/\mathbb{Q}}(a/\delta)}\\
&= \sum_{\substack{[a]\in\mathfrak{f}^{-1}/\mathcal{O}_K\\(a\mathfrak{f},\mathfrak{f})=1}} \varphi(a\mathfrak{f})|x_{a\mathfrak{f}}|_{\mathbb{A}_K}^{\frac{q+r}{2}}\varphi_\infty\left(\frac{a}{\delta}\right)\left|\frac{a}{\delta}\right|^{q+r}e^{2\pi i \mathrm{Tr}_{K/\mathbb{Q}}(a/\delta)}\\
&= \sum_{\substack{[a]\in\mathfrak{f}^{-1}/\mathcal{O}_K\\(a\mathfrak{f},\mathfrak{f})=1}} \varphi(a\mathfrak{f})|x_{\mathfrak{f}}|_f^{\frac{q+r}{2}}\varphi_\infty\left(\frac{a}{\delta}\right)|\delta|^{-(q+r)}e^{2\pi i \mathrm{Tr}_{K/\mathbb{Q}}(a/\delta)}\\
&= |x_{\mathfrak{f}}|_{\mathbb{A}_K}^{\frac{q+r}{2}}|\delta|^{-(q+r)}\tau(\varphi),
\end{aligned}$$

as required. ☐

Passing to inverses, we find that

$$\tau((\varphi')^{-1}) = |\delta|^{q+r}|x_{\mathfrak{f}}|_f^{-\frac{q+r}{2}}\tau(\varphi^{-1}).$$

Accordingly, we see that

$$
\begin{aligned}
L(\Phi,\varphi') &= \left[\frac{(-1)^k 2(-2\pi i)^{q+r+2}}{|\delta|^{q+r+2} q!r!w\tau(\varphi^{-1})}\right]\sum_{j=1}^{h}\left[\varphi(a_j)|a_j|_f^{\frac{q+r}{2}}\sum_{\substack{[a]\in\mathfrak{f}^{-1}/\mathcal{O}_K\\(a\mathfrak{f},\mathfrak{f})=1}}\varphi(a\mathfrak{f})^{-1}a^u\overline{a}^v c^j_{q,r}(a)\right]\\
&= \left[\frac{(-1)^k 2(-2\pi i)^{q+r+2}}{|\delta|^2|x_{\mathfrak{f}}|_f^{\frac{q+r}{2}} q!r!w\tau((\varphi')^{-1})}\right]\sum_{j=1}^{h}\left[\varphi'(a_j)\sum_{\substack{[a]\in\mathfrak{f}^{-1}/\mathcal{O}_K\\((a)\mathfrak{f},\mathfrak{f})=1}}\varphi'(a\mathfrak{f})^{-1}|a\mathfrak{f}|^{\frac{q+r}{2}}a^u\overline{a}^v c^j_{q,r}(a)\right]\\
&= \left[\frac{(-1)^k 2(-2\pi i)^{q+r+2}}{|D|q!r!w\tau((\varphi')^{-1})}\right]\sum_{j=1}^{h}\left[\varphi'(a_j)\sum_{\substack{[a]\in\mathfrak{f}^{-1}/\mathcal{O}_K\\((a)\mathfrak{f},\mathfrak{f})=1}}\varphi'(a\mathfrak{f})^{-1}(\varphi'_\infty)^{-1}(a)c^j_{q,r}(a)\right].
\end{aligned}
$$

This simplifies further; indeed, for any Hecke character φ, an explicit check shows that:

Lemma 6.3.2. *Let φ be a Hecke character of conductor $\mathfrak{f}$, and let $a\in\mathfrak{f}^{-1}$ such that $(a\mathfrak{f},\mathfrak{f})=1$. Then we have*

$$\varphi(a\mathfrak{f})^{-1}\varphi_\infty(a)^{-1}=\varphi(x_{\mathfrak{f}})^{-1}\varphi_{\mathfrak{f}}(ax_{\mathfrak{f}}).$$

Proof. Write $a\mathfrak{f}=abI_j$ for $b\in K^\times$ and some j. Then

$$
\begin{aligned}
\varphi(a\mathfrak{f})=\varphi((ab))\varphi(a_j)&=\varphi(a_j)\prod_{\mathfrak{p}:v_{\mathfrak{p}}(ab)\neq 0}\varphi_{\mathfrak{p}}(ab)\\
&=\varphi(a_j)\prod_{\mathfrak{p}\nmid\mathfrak{f}}\varphi_{\mathfrak{p}}(ab),
\end{aligned}
$$

since as $a\mathfrak{f}$ and I_j are coprime to $\mathfrak{f}$, we also see that (ab) is coprime to $\mathfrak{f}$. Extending the product to all finite primes, we get

$$=\varphi(a_j)\prod_{\mathfrak{p}}\varphi_{\mathfrak{p}}(ab)\prod_{\mathfrak{p}|\mathfrak{f}}\varphi_{\mathfrak{p}}(ab)^{-1}=\varphi(a_j)\varphi_\infty(ab)^{-1}\varphi_{\mathfrak{f}}(ab)^{-1}.$$

Combining this gives

$$\varphi(a\mathfrak{f})^{-1}\varphi_\infty(a)^{-1}=\varphi(a_j)^{-1}\varphi_\infty(b)\varphi_{\mathfrak{f}}(ab).$$

Now, as

$$(ax_{\mathfrak{f}})_v=\begin{cases} ab & : v|\mathfrak{f},\\ ab\pi_{I_j} & : v=I_j,\\ a & : \text{ otherwise}\end{cases}$$

we see that $\varphi_{\mathfrak{f}}(ab) = \varphi_{\mathfrak{f}}(ax_{\mathfrak{f}})$. Furthermore,

$$\begin{aligned}
\varphi(x_{\mathfrak{f}}) &= \varphi(a_j) \prod_{\mathfrak{p}: v_{\mathfrak{p}}(b) \neq 0} \varphi_{\mathfrak{p}}(b) \\
&= \varphi(a_j)\varphi_{I_j}(b) \prod_{\mathfrak{p}|\mathfrak{f}} \varphi_{\mathfrak{p}}(b) \\
&= \varphi(a_j) \prod_{\mathfrak{p}} \varphi_{\mathfrak{p}}(b) = \varphi(a_j)\varphi_\infty(b)^{-1},
\end{aligned}$$

as $\varphi_{\mathfrak{p}}(b) = 1$ for all primes not dividing I_j or $\mathfrak{f}$. We conclude that

$$\varphi(a\mathfrak{f})^{-1}\varphi_\infty(a)^{-1} = \varphi(x_{\mathfrak{f}})^{-1}\varphi_{\mathfrak{f}}(ax_{\mathfrak{f}}),$$

as required. □

Using the Lemma, our equation becomes

$$L(\Phi, \varphi') = \left[\frac{(-1)^k 2(-2\pi i)^{q+r+2}}{\varphi'(x_{\mathfrak{f}})|\delta|^2 q! r! w \tau((\varphi')^{-1})}\right] \sum_{j=1}^{h} \left[\varphi'(a_j) \sum_{\substack{[a]\in\mathfrak{f}^{-1}/\mathcal{O}_K \\ ((a)\mathfrak{f},\mathfrak{f})=1}} \varphi'_{\mathfrak{f}}(ax_{\mathfrak{f}}) c^j_{q,r}(a)\right],$$

where φ' is a Hecke character of conductor $\mathfrak{f}$ and infinity type (q, r) with $0 \leq q, r \leq k$. Henceforth, we simply write φ for this character, dropping the prime from the notation.

We make one further change with the aim of massaging this formula into something a little nicer; namely, we renormalise, using the Deligne Γ-factor at infinity. As before, define

$$\Lambda(\Phi, \varphi, t) := \frac{\Gamma(q+t)\Gamma(r+t)}{(2\pi i)^{q+t}(2\pi i)^{r+t}} L(\Phi, \varphi, t).$$

The upshot of all of these calculations is the following:

Theorem 6.3.3. *Let Φ be a Bianchi modular form of weight (k, k) and level $\Omega_1(\mathfrak{n})$, and let $\Lambda(\Phi, *)$ denote its normalised L-function (as a function on Hecke characters). Let φ be a Hecke character of conductor $\mathfrak{f}$ and infinity type (q, r), where $0 \leq q, r \leq k$. Then we have*

$$\Lambda(\Phi, \varphi) = \left[\frac{(-1)^{k+q+r} 2\varphi_{\mathfrak{f}}(x_{\mathfrak{f}})}{\varphi(x_{\mathfrak{f}})|\delta|^2 w \tau(\varphi^{-1})}\right] \sum_{j=1}^{h} \left[\varphi(a_j) \sum_{\substack{[a]\in\mathfrak{f}^{-1}/\mathcal{O}_K \\ ((a)\mathfrak{f},\mathfrak{f})=1}} \varphi_{\mathfrak{f}}(a) c^j_{q,r}(a)\right], \tag{6.4}$$

where all other notation is as defined in Theorem 6.2.2.

Chapter 7
Overconvergent Bianchi Modular Symbols

In this chapter, we develop the theory of overconvergent modular symbols over imaginary quadratic fields, generalising an idea of Stevens to this setting. To do so, we consider modular symbols with values in a space of p-adic distributions, a much larger space equipped with a natural surjective map to the space of polynomials used in the definition of classical modular symbols. After defining these distributions, we endow them with an action of a suitable semi-group, allowing us to use them as modules of values for modular symbols. Following this, we prove that it is possible to work with integral coefficients. We conclude by writing down suitable filtrations and submodules of this space that will be essential in future chapters.

7.1. A conceptual description

Overconvergent modular symbols are modular symbols that take values in a space of p-adic distributions. In a similar style to the rest of Part II of this thesis, we begin this section by defining these spaces in the most natural way before passing to a more workable and explicit description. We will keep this discussion brief, since all of the objects mentioned will be defined (in the more explicit setting) later in the chapter. The reader who has not met the theory of p-adic distributions before is encouraged to skip straight to Chapter 7.2 on a first reading.

We work with the space $\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{Z}_p$. Note that this space embeds naturally in $\overline{\mathbb{Q}}_p^2$; let $L/\mathbb{Q}_p$ be a finite extension such that the image of this embedding lies in L^2. Now, an element of $V_{k,k}(L)$ can be seen as a function on L^2 that is polynomial of degree at most k in each variable, and accordingly, we can see such an element as a function on $\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{Z}_p$ in a natural way. In particular, the following definition gives an alternative definition of the space $V_{k,k}(L)$ from earlier chapters.

Definition 7.1.1. Define $V_{k,k}(\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{Z}_p, L)$ to be the space of functions on $\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{Z}_p$ that are polynomial of degree k in each variable with coefficients in L, with a left action of $\mathrm{GL}_2(\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{Z}_p)$

given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot P(x) = (a+cx)^k P\left(\frac{b+dx}{a+cx}\right). \tag{7.1}$$

The polynomial spaces $V_{k,k}(\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{Z}_p, L)$ are naturally subspaces of the space $\mathbb{A}(\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{Z}_p, L)$ of rigid analytic functions on $\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{Z}_p$. We then find that, by dualising the inclusion $V_{k,k}(\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{Z}_p, L) \hookrightarrow \mathbb{A}(\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{Z}_p, L)$, we have a surjection from the space $\mathbb{D}(\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{Z}_p, L)$ of rigid analytic distributions on $\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{Z}_p$ to the dual space $V_{k,k}(\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{Z}_p, L)^*$, giving a surjective specialisation map. To see that modular symbols with values in $\mathbb{D}(\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{Z}_p, L)$ make sense, we consider the semigroup

$$\Sigma(\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{Z}_p) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{Z}_p) : c \in p\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{Z}_p, a \in (\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{Z}_p)^\times, ad - bc \neq 0 \right\}.$$

This semigroup acts on $\mathbb{A}(\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{Z}_p, L)$ in the manner defined in equation (7.1), and using this we obtain an action of suitable discrete subgroups of $\mathrm{SL}_2(\mathcal{O}_K)$ on the distribution space, allowing us to use it as a value space. We also get a Hecke action on the resulting modular symbols. All of this directly generalises the work of Pollack and Stevens over $\mathbb{Q}$, and with suitable small adjustments generalises further to the case of arbitrary number fields (see Part III).

We've already shown that we can see $V_{k,k}$ as either a tensor product of polynomial spaces or as a space of polynomials on $\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{Z}_p$. Similarly, we can describe these distribution spaces more explicitly via tensor products. In particular, the space $\mathbb{A}(\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{Z}_p)$ can be thought of as just being the space of rigid analytic functions on $\mathbb{Z}_p^2$, or, in the weak topology, the completed tensor product $\mathbb{A}(\mathbb{Z}_p)\widehat{\otimes}_{\mathbb{Z}_p}\mathbb{A}(\mathbb{Z}_p)$ of two copies of the space of rigid analytic functions on one copy of $\mathbb{Z}_p$. The group $\Sigma(\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{Z}_p)$ becomes the direct product of two copies of the group $\Sigma_0(p)$ as written down by Pollack and Stevens. When p is not split, we are carrying around some redundant information with this approach; in particular, the useful information given by the action of $\Sigma_0(p)^2$ is entirely determined by the action of one of the components. There are significant advantages to using this more explicit approach, however. The spaces have nice descriptions that are easy to work with and allow us to generalise the filtration proof of Greenberg ([Gre07]) to the imaginary quadratic case. In the remainder of this section, we work with this explicit approach exclusively.

7.2. Distribution spaces

In this section, we give a concrete description of the spaces of distributions we'll use to define overconvergent modular symbols.

7.2.1. Notation and preliminaries

Notation 7.2.1: Throughout, as before, $K/\mathbb{Q}$ denotes an imaginary quadratic field. Let p be a rational prime with $p\mathcal{O}_K = \prod \mathfrak{p}^{e_\mathfrak{p}}$, and define $f_\mathfrak{p}$ to be the residue index of $\mathfrak{p}$. Note that $\sum e_\mathfrak{p} f_\mathfrak{p} = 2$. Fix an embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$; then for each prime $\mathfrak{p}|p$, we have $e_\mathfrak{p} f_\mathfrak{p}$ embeddings $K_\mathfrak{p} \hookrightarrow \overline{\mathbb{Q}}_p$, and combining these for each prime, we get an embedding $\sigma = (\sigma_1, \sigma_2) : K \hookrightarrow \overline{\mathbb{Q}}_p \times \overline{\mathbb{Q}}_p$. Let L be a finite extension of $\mathbb{Q}_p$ containing the image of $K_\mathfrak{p}$ under all of the possible embeddings into $\overline{\mathbb{Q}}_p$ for all primes $\mathfrak{p}$ above p. We equip L with a valuation v, normalised so that $v(p) = 1$, and denote the ring of integers in L by $\mathcal{O}_L$, with uniformiser π_L. Then in fact, for each integral ideal I of K coprime to (p), we have

$$\sigma : I^{-1} \longrightarrow \mathcal{O}_L \times \mathcal{O}_L. \tag{7.2}$$

In the obvious way, we then have an embedding

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in I^{-1}, ad - bc \neq 0 \right\} \longrightarrow \mathrm{GL}_2(\mathcal{O}_L) \times \mathrm{GL}_2(\mathcal{O}_L).$$

In particular, for $\mathfrak{n}$ an integral ideal with $(p)|\mathfrak{n}$, we have embeddings of the groups $\Gamma_1^i(\mathfrak{n})$, where each $\Gamma_1^i(\mathfrak{n}) = \Gamma_{I_i}$ is a twist of $\Gamma_1(\mathfrak{n})$ as defined in Definition 2.2.5.

These embeddings will be used in the sequel to define the action of suitable discrete subgroups of $\mathrm{SL}_2(K)$, as well a Hecke action on suitable modular symbol spaces. We'll define some monoid $\Sigma_0(p) \leq \mathrm{GL}_2(\mathcal{O}_\mathfrak{p})$ and an action of $\Sigma_0(p) \times \Sigma_0(p)$. Every matrix whose action we study will have image in $\Sigma_0(p)^2$ under each of the embeddings above. Thus in proving facts about the action of $\Sigma_0(p) \times \Sigma_0(p)$, we'll encapsulate everything we'll later need *regardless* of the splitting behaviour of p in $\mathcal{O}_K$.

Remark: For our purposes, we may need to take L to be larger than this. Let Φ be a cuspidal Bianchi eigenform with Fourier coefficients $c(I, \Phi)$, normalised so that $c(\mathcal{O}_K, \Phi) = 1$. Then the Fourier coefficients are algebraic. In particular, when studying the action of the U_p operator we may need to consider eigenvalues living in the number field $F := K(\{c(\mathfrak{p}, \Phi) : \mathfrak{p}|p\})$. As we can easily enlarge L to contain all possible embeddings of the completions of this field into $\overline{\mathbb{Q}_p}$, we will henceforth assume that these eigenvalues live in L.

7.2.2. Rigid analytic functions and distributions

To define overconvergent modular symbols, Stevens used spaces from p-adic analysis. For more details on the results here, including p-adic function and distribution spaces as well as the completed tensor product, see [Col10].

Definition 7.2.2 (Modules of Values for Overconvergent Modular Symbols)**.** Let R be either a p-adic field or the ring of integers in a finite extension of $\mathbb{Q}_p$, and let $\mathbb{A}(R)$ be the ring of *rigid analytic functions on the closed unit disc defined over R,* that is, the ring

$$\mathbb{A}(R) = \left\{ \sum_{n \geq 0} a_n x^n : a_n \in R, a_n \to 0 \text{ as } n \to \infty \right\}.$$

For $R = L$, this is a L-Banach space with the sup norm. We write $\mathbb{A}_2(R)$ for the completed tensor product $\mathbb{A}(R)\hat{\otimes}_R\mathbb{A}(R)$. We define the space of *rigid analytic distributions* to be the topological dual $\mathbb{D}(R) = \mathrm{Hom}_{\mathrm{cts}}(\mathbb{A}(R), R)$, and analogously we define $\mathbb{D}_2(R) = \mathrm{Hom}_{\mathrm{cts}}(\mathbb{A}_2(R), R)$ be the topological dual of $\mathbb{A}_2(R)$.

Having defined these spaces, our primary spaces of interest, we immediately give two alternate descriptions that are easier to work with.

Definition 7.2.3. (i) Let $\mu \in \mathbb{D}$ be a distribution. Define the *moments* of μ to be the values $(\mu(x^i))_{i \geq 0}$, noting that these values totally determine the distribution since the span of the x^i is dense in $\mathbb{A}$.

(ii) Let $\mu \in \mathbb{D}_2$ be a two variable distribution. Define the *moments* of μ analogously to be the values $(\mu(x^i y^j))_{i,j \geq 0}$.

Proposition 7.2.4. *(i) As L-Banach spaces, the space $\mathbb{A}_2(L)$ is isomorphic to the space $\mathbb{A}(\mathbb{Z}_p^2, L)$ of rigid analytic functions in two variables on the closed unit disc defined over L.*

(ii) We can identify $\mathbb{D}(L)$ with the set of bounded sequences in L and $\mathbb{D}_2(L)$ with the set of doubly indexed bounded sequences in L.

Proof. (i) Note that we have

$$\mathbb{A}(\mathbb{Z}_p^2, L) := \left\{ \sum_{m,n \geq 0} a_{mn} x^m y^n : a_{mn} \in L, a_{mn} \to 0 \text{ as } m+n \to \infty \right\},$$

where the condition on a_{mn} can be stated equivalently and more formally as 'a_{mn} tends to 0 in the filter of cofinite sets.' Thus $\mathbb{A}(\mathbb{Z}_p^2, L)$ has a Banach basis given by the functions $\{x^m y^n : m, n \geq 0\}$. Now, $\mathbb{A}(L)$ has a Banach basis $\{x^i : i \geq 0\}$, so $\mathbb{A}_2(L)$ has a Banach basis $\{x^i y^j : i, j \geq 0\}$, and thus it follows that the spaces are isomorphic, as claimed.

(ii) Suppose $\mu \in \mathbb{D}$ is a distribution. Then we claim that the set of moments, as defined above, is a bounded sequence in L. Indeed, suppose that the sequence (b_m) we obtain is not bounded; then there is some subsequence (b_{m_k}) with strictly decreasing valuations (and hence strictly increasing p-adic absolute value). Thus the power series $f = \sum x^{m_k}/b_{m_k}$

defines an element of $\mathbb{A}(L)$. However, $\mu(f)$ doesn't converge, so we obtain a contradiction and the sequence must be bounded. Furthermore, since the coefficients of any power series in $\mathbb{A}(L)$ tend to zero, any such bounded sequence gives rise to a unique distribution.

A near identical argument shows that there is a one-to-one correspondence between elements of $\mathbb{D}_2(L)$ and bounded doubly indexed sequences in L via the map that sends a distribution to its moments. $\square$

Remarks 7.2.5: (i) This identification with bounded sequences means that we have $\mathbb{D}_2(L) \cong \mathbb{D}_2(\mathcal{O}_L) \otimes_{\mathcal{O}_L} L$, where we see that $\mathbb{D}_2(\mathcal{O}_L)$ is simply the subspace of $\mathbb{D}_2(K)$ consisting of distributions with integral moments.

(ii) Note that the spaces $\mathbb{D}_2(L)$ and $\mathbb{D}(L)\hat{\otimes}_L\mathbb{D}(L)$ are *not* canonically isomorphic as L-Banach spaces when we endow $\mathbb{D}(L)$ with the strong topology (the topology arising from the p-adic valuation). Indeed, there is a norm preserving injection $\mathbb{D}(L)\hat{\otimes}_L\mathbb{D}(L) \hookrightarrow \mathbb{D}_2(L)$ induced by taking a pair of bounded sequences $(a_m), (b_n)$ to the bounded doubly indexed sequence $(a_m b_n)$, but this map is not surjective. If instead we equip $\mathbb{D}(L)$ with the weak topology (that is, the topology of pointwise convergence), then the spaces $\mathbb{D}(L)\hat{\otimes}_L\mathbb{D}(L)$ and $\mathbb{D}_2(L)$ *are* canonically isomorphic (indeed, we still have the injection above, and since we need check only pointwise convergence, we have surjectivity). We continue to use the space $\mathbb{D}_2(L)$ to avoid thinking about this.

We want to equip our spaces with an action of suitable congruence subgroups and a Hecke action. To this end, define

$$\Sigma_0(p) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathcal{O}_L) : p \mid c,\ (a,p) = 1, ad - bc \neq 0 \right\}.$$

Suppose R is a ring containing $\mathcal{O}_L$. Then there is a *weight k left action* of $\Sigma_0(p)$ on $\mathbb{A}(R)$ defined by the rule

$$(\gamma \cdot_k f)(x) = (cx+a)^k f\left(\frac{dx+b}{cx+a}\right), \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

giving rise to a weight k right action on $\mathbb{D}(R)$ given by

$$(\mu|_k\gamma)(f) = \mu(\gamma \cdot_k f).$$

When we talk about these spaces with equipped with this weight k action, we denote them by $\mathbb{A}_k$ and $\mathbb{D}_k$ respectively. Such an action generalises immediately to the two variable case.

Define the weight (k, ℓ) left action of $\Sigma_0(p) \times \Sigma_0(p)$ on $\mathbb{A}_2(R)$ by

$$((\gamma_1, \gamma_2) \cdot_{(k,\ell)} f)(x, y) = (c_1 x + a_1)^k (c_2 y + a_2)^\ell f\left(\frac{d_1 x + b_1}{c_1 x + a_1}, \frac{d_2 y + b_2}{c_2 y + a_2}\right),$$

where $\gamma_i = \left(\begin{smallmatrix} a_i & b_i \\ c_i & d_i \end{smallmatrix}\right)$, again giving rise to a weight (k, ℓ) right action on $\mathbb{D}_2(R)$. When talking about these spaces equipped with the weight (k, ℓ) action, we denote them by $\mathbb{A}_{k,\ell}(R)$ and $\mathbb{D}_{k,\ell}(R)$ respectively.

Remark 7.2.6: Note that the subspace $V_{k,\ell}(R)$ of $\mathbb{A}_2(R)$ is stable under the action of $\Sigma_0(p)^2$, and hence it inherits a left action of $\Sigma_0(p)^2$. This is the action we defined earlier in Definition 5.2.4.

7.3. Overconvergent modular symbols

7.3.1. Definition

The action of $\Sigma_0(p)$ above allows us to define modular symbols with values in distributions. In particular, recall that we defined $\Gamma := \Gamma_1(\mathfrak{n})$, and note that for any ideal I coprime to $\mathfrak{n}$, we have a right action of Γ_I (as in Definition 2.2.5) on the space $\mathbb{D}_{k,\ell}(L)$ via the embedding (7.2).

Definition 7.3.1. (i) Define the space of *overconvergent Bianchi modular symbols for K of weights (k, ℓ) and level Γ_I with coefficients in L* to be

$$\mathrm{Symb}_{\Gamma_I}(\mathbb{D}_{k,\ell}(L)) := \mathrm{Hom}_{\Gamma_I}(\Delta_0, \mathbb{D}_{k,\ell}(L)).$$

(ii) Recall the definition of $\Omega_1(\mathfrak{n})$ from equation (2.3). Define the space of *overconvergent Bianchi modular symbols for K of weights (k, ℓ) and level $\Omega_1(\mathfrak{n})$ with coefficients in L* to be

$$\mathrm{Symb}_{\Omega_1(\mathfrak{n})}(\mathbb{D}_{k,\ell}(L)) := \bigoplus_{i=1}^{h} \mathrm{Symb}_{\Gamma_1^i(\mathfrak{n})}(\mathbb{D}_{k,\ell}(L)).$$

Remark: Note that, for this to make sense, we *need* (p) to divide the level. For a matrix $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ to act on $\mathbb{A}_2(L)$, we need $p|c$, or the action will not be well-defined (as 'dividing by $cx + a$' is not in general a well-defined concept on p-adic power series unless $p|c$). In particular, we will not have an action of $\Gamma_1(\mathfrak{n})$ on distributions unless $(p)|\mathfrak{n}$.

7.3.2. The action of Hecke operators

For a right $\Sigma_0(p)^2$-module V, we define a Hecke action on the adelic space

$$\mathrm{Symb}_{\Omega_1(\mathfrak{n})}(V) := \bigoplus_{i=1}^{h} \mathrm{Symb}_{\Gamma_1^i(\mathfrak{n})}(V)$$

in much the same way as in Chapter 5.3. Since the Hecke operators at p are of critical importance in the sequel, we recall them here. For a prime $\mathfrak{p}$ dividing (p), for each $i \in \{1, ..., h\}$ there is a unique $j_i \in \{1, ..., h\}$ such that

$$\mathfrak{p}I_i = (\alpha_i)I_{j_i},$$

for $\alpha_i \in K$. Then the $U_\mathfrak{p}$ operator is

$$(\phi_1, ..., \phi_h)|U_\mathfrak{p} := \left(\phi_{j_1}\middle|\left[\Gamma_{j_1}\begin{pmatrix}1 & 0\\ 0 & \alpha_1\end{pmatrix}\Gamma_1\right], ..., \phi_{j_h}\middle|\left[\Gamma_{j_h}\begin{pmatrix}1 & 0\\ 0 & \alpha_h\end{pmatrix}\Gamma_h\right]\right).$$

We can work out the double coset operators explicitly to be given by

$$\left[\Gamma_{j_i}\begin{pmatrix}1 & 0\\ 0 & \alpha_i\end{pmatrix}\Gamma_1^i(\mathfrak{n})\right] = \sum_{a \pmod{\mathfrak{p}}}\begin{pmatrix}1 & a\\ 0 & \alpha_i\end{pmatrix}$$

using the usual methods. Note that if n is an integer such that $\mathfrak{p}^n = (\sigma)$ is principal, then this action becomes significantly simpler; namely, we just act on each component, with no permuting, via the sum $\sum_{a \pmod{\mathfrak{p}^n}}\left(\begin{smallmatrix}1 & a\\ 0 & \sigma\end{smallmatrix}\right)$. Because of this much simpler description, in the sequel we much prefer to use a principal power of $\mathfrak{p}$ and prove results using just one component at a time.

7.3.3. Integral overconvergent modular symbols

Ideally, we'd prefer to work with integral distributions. The following results, via Remark 7.2.5(i), allow us to do just that.

Lemma 7.3.2. *Let F be (any) number field and I be an ideal coprime to $\mathfrak{n}$. Then* $\mathrm{Div}^0(\mathbb{P}^1(F))$ *is a finitely generated* $\mathbb{Z}[\Gamma_I]$*-module.*

Proof. Recall that Γ_I is a finitely generated group (see, for example, [Swa71]). Fix generators $\gamma_1, ..., \gamma_r$, say. The set of orbits of the action of Γ_I on $\mathbb{P}^1(F)$ is finite, and in fact has order equal to the class number h of F. Write these orbits as $[c_1], ..., [c_h]$ for any fixed choice of representatives in $\mathbb{P}^1(F)$. Now take any element

$$d_i - d_j \in \mathrm{Div}^0(\mathbb{P}^1(F)),$$

with $d_i = gc_i$ and $d_j = g'c_j$, some g and g' in Γ_I. Then

$$d_i - d_j = d_i - c_i + c_i - c_j + c_j - d_j = (g-1)c_i + (c_i - c_j) + (1 - g')c_j,$$

and it immediately follows that $\mathrm{Div}^0(\mathbb{P}^1(F))$ is generated as a $\mathbb{Z}[\Gamma_I]$-module by the set $\{(1 -$

$g)c_i : g \in \Gamma_I, 1 \leq i \leq h\} \cup \{c_i - c_j : 1 \leq i < j \leq h\}$. But we also have

$$(1 - g_1 g_2)c_i = (1 - g_1)c_i - g_1(1 - g_2)c_i,$$

so that in fact $\mathrm{Div}^0(\mathbb{P}^1(F))$ is generated as a $\mathbb{Z}[\Gamma_I]$-module by the finite set

$$\{(1 - \gamma_i)c_j : 1 \leq i \leq r, 1 \leq j \leq h\} \cup \{c_i - c_j : 1 \leq i < j \leq h\}.$$

In particular, it is finitely generated, as required. $\square$

Proposition 7.3.3. *Let Γ_I be as above, and let D have the structure of both a $\mathcal{O}_L$-module and a right Γ_I-module. Then we have*

$$\mathrm{Symb}_{\Gamma_I}(D \otimes_{\mathcal{O}_L} L) \cong \mathrm{Symb}_{\Gamma_I}(D) \otimes_{\mathcal{O}_L} L.$$

Proof. Let $\phi \in \mathrm{Symb}_{\Gamma_I}(D \otimes_{\mathcal{O}_L} L)$. Using Lemma 7.3.2, take a finite set of generators $\alpha_1, ..., \alpha_n$ for Δ_0 as a $\mathbb{Z}[\Gamma_I]$-module. We can find some element $c \in \mathcal{O}_L$ such that $c\phi(\alpha_i) \in D$ for each i. But then it follows immediately that $c\phi \in \mathrm{Symb}_{\Gamma_I}(D)$, and the result follows. $\square$

Remark: In particular, with Remark 7.2.5(i), this shows that

$$\mathrm{Symb}_{\Gamma_I}(\mathbb{D}_{k,\ell}(L)) \cong \mathrm{Symb}_{\Gamma_I}(\mathbb{D}_{k,\ell}(\mathcal{O}_L)) \otimes_{\mathcal{O}_L} L.$$

With this structure in place, we can now work with the space $\mathbb{D}_{k,\ell}(\mathcal{O}_L)$.

7.4. Filtrations and submodules

In the last section of this chapter, we write down filtrations and submodules of the space of distributions that will allow us to prove a *control theorem* in the next chapter. In particular, this control theorem says that the natural map from overconvergent to classical modular symbols is an isomorphism when we restrict to suitable 'small slope' eigenspaces of the U_p operator. The modules defined in this section are crucial in the proof of surjectivity.

7.4.1. Finite approximation modules

Remark: From now on, we work exclusively with parallel weights, i.e. we consider only $k = \ell$ and use the space $\mathbb{D}_{k,k}(R)$. There are no classical cuspidal Bianchi modular forms at non-parallel weights, so in proving a control theorem in the spirit of Stevens' work, it suffices to exclude the case $k \neq \ell$. We'll also focus on looking at the space of Bianchi modular symbols one component at a time. Henceforth, to this end, Γ will denote one of the $\Gamma_1^i(\mathfrak{n})$ for $i \in \{1, ..., h\}$.

In the one variable case, Matthew Greenberg ([Gre07]) gave an alternative proof of Stevens' control theorem using *finite approximation modules*, defining a $\Sigma_0(p)$-stable filtration of $\mathbb{D}_k(L)$ and then lifting modular symbols through this filtration. We aim to mimic this. First we recap Greenberg's work, recasting it slightly to make it more favourable for our generalisation. He defines:

Definition 7.4.1. Define:

(i) $\mathcal{F}^N\mathbb{D}_k(\mathcal{O}_L) := \{\mu \in \mathbb{D}_k(\mathcal{O}_L) : \mu(x^i) \in \pi_L^{N-i}\mathcal{O}_L\}$, and

(ii) $\mathbb{D}_k^0(\mathcal{O}_L) = \{\mu \in \mathbb{D}_k(\mathcal{O}_L) : \mu(x^i) = 0 \text{ for } 0 \leq i \leq k\}$. Note that this is the kernel of the natural map $\mathbb{D}_k(\mathcal{O}_L) \to V_k(\mathcal{O}_L)^*$ obtained by dualising the inclusion $V_k(\mathcal{O}_L) \hookrightarrow \mathbb{A}_k(\mathcal{O}_L)$.

Greenberg shows in Lemma 2 of [Gre07] that the $\mathcal{F}^N\mathbb{D}_k(\mathcal{O}_L)$ and $\mathbb{D}_k^0(\mathcal{O}_L)$ are both $\Sigma_0(p)$-stable by considering the action of matrices of form $\left(\begin{smallmatrix} 1 & 0 \\ c & 1\end{smallmatrix}\right)$, where $p|c$, and $\left(\begin{smallmatrix} a & b \\ 0 & d\end{smallmatrix}\right)$, where a is a p-adic unit, since such matrices generate $\Sigma_0(p)$.

To define our own filtration, we take a similar route, imposing suitable conditions on the moments of distributions.

Definition 7.4.2. Define:

(i) $\mathcal{F}^N\mathbb{D}_{k,k}(\mathcal{O}_L) := \{\mu \in \mathbb{D}_{k,k}(\mathcal{O}_L) : \mu(x^iy^j) \in \pi_L^{N-i-j}\mathcal{O}_L\}$.

(ii) $\mathbb{D}_{k,k}^0(\mathcal{O}_L) := \{\mu \in \mathbb{D}_{k,k}(\mathcal{O}_L) : \mu(x^iy^j) = 0 \text{ for } 0 \leq i,j \leq k\}$.

(iii) $F^N\mathbb{D}_{k,k}(\mathcal{O}_L) := \mathcal{F}^N\mathbb{D}_{k,k}(\mathcal{O}_L) \cap \mathbb{D}_{k,k}^0(\mathcal{O}_L)$.

Viewing a doubly indexed sequence (a_{ij}) as an infinite matrix, these conditions demand first that the top left $(k+1)\times(k+1)$ entries are all zero, and then that each moment is suitably divisible by some power of π depending on $i+j$ and N.

Proposition 7.4.3. *This filtration is $\Sigma_0(p)^2$-stable.*

Proof. There is an obvious switching map $s : \mathbb{D}_{k,\ell}(\mathcal{O}_L) \to \mathbb{D}_{\ell,k}(\mathcal{O}_L)$. Thus it suffices to prove the result for elements of form (γ, I_2) for $\gamma \in \Sigma_0(p)$, as the action of a more general element can be described as

$$\mu\Big|_k(\gamma_1,\gamma_2) = s^{-1}\left[s\left(\mu\Big|_k(\gamma_1, I_2)\right)\Big|_k(\gamma_2, I_2)\right].$$

To each two-variable distribution $\mu \in \mathbb{D}_{k,k}(\mathcal{O}_L)$, associate a family of distributions $\{\mu_j \in \mathbb{D}_k(\mathcal{O}_L)\}$ by defining the moments of μ_j to be

$$\mu_j(x^i) = \mu(x^iy^j).$$

Then note that we have

$$\mu \in F^N\mathbb{D}_{k,k}(\mathcal{O}_L) \iff \mu_j \in \begin{cases} \mathcal{F}^{N-j}\mathbb{D}_k(\mathcal{O}_L) \cap \mathbb{D}_k^0(\mathcal{O}_L) & : 0 \leq j \leq k \\ \mathcal{F}^{N-j}\mathbb{D}_k(\mathcal{O}_L) & : j > k \end{cases},$$

where the condition must hold for all $j \geq 0$. The result we require then follows from the observation that

$$\mu\Big|_k (\gamma, I_2)(x^i y^j) \;=\; \mu_j\Big|_k \gamma(x^i)$$

combined with the stability (in the one variable case) of each of the modules $\mathbb{D}_k^0(\mathcal{O}_L)$ and $\mathcal{F}^{N-j}\mathbb{D}_k(\mathcal{O}_L)$ under the action of $\Sigma_0(p)$. $\square$

In particular, this result shows that we have a collection of $\Sigma_0(p)^2$-modules

$$A^N\mathbb{D}_{k,k}(\mathcal{O}_L) := \frac{\mathbb{D}_{k,k}(\mathcal{O}_L)}{F^N\mathbb{D}_{k,k}(\mathcal{O}_L)},$$

with action inherited from $\mathbb{D}_{k,k}(\mathcal{O}_L)$, and where this is well-defined since the $F^N\mathbb{D}_{k,k}(\mathcal{O}_L)$ are $\Sigma_0(p)^2$-stable. Furthermore, we see that

$$\begin{aligned} A^N\mathbb{D}_{k,k}(\mathcal{O}_L) &\cong \mathcal{O}_L^{(k+1)^2} \times T \\ &\cong V_{k,k}(\mathcal{O}_L)^* \times T, \end{aligned}$$

where T is some finite product of copies of $\mathbb{Z}/p, \mathbb{Z}/p^2$, and so on up to $\mathbb{Z}/p^{N-k-1}$. In particular, we also have $A^0\mathbb{D}_{k,k}(\mathcal{O}_L) \cong \cdots \cong A^{k+1}\mathbb{D}_{k,k}(\mathcal{O}_L) \cong V_{k,k}(\mathcal{O}_L)^*$. We also have $\Sigma_0(p)^2$-equivariant projection maps π^N from $\mathbb{D}_{k,k}(\mathcal{O}_L)$ to $A^N\mathbb{D}_{k,k}(\mathcal{O}_L)$, and we see that the map

$$\pi^0 : \mathbb{D}_{k,k}(\mathcal{O}_L) \longrightarrow V_{k,k}(\mathcal{O}_L)^*$$

gives rise to a ($\Sigma_0(p)^2$-equivariant) specialisation map

$$\rho^0 : \mathrm{Symb}_\Gamma(\mathbb{D}_{k,k}(\mathcal{O}_L)) \longrightarrow \mathrm{Symb}_\Gamma(V_{k,k}(\mathcal{O}_L)^*).$$

Further to this, for each $M \geq N$, we have a $\Sigma_0(p)^2$-equivariant map

$$\pi^{M,N} : A^M\mathbb{D}_{k,k}(\mathcal{O}_L) \to A^N\mathbb{D}_{k,k}(\mathcal{O}_L)$$

given by projection (and hence also maps $\rho^{M,N}$). The projection maps are all compatible in the obvious ways. Thus we get an inverse system of modules, and it's straightforward to see that

$$\mathbb{D}_{k,k}(\mathcal{O}_L) \cong \varprojlim A^N\mathbb{D}_{k,k}(\mathcal{O}_L). \tag{7.3}$$

To emulate Greenberg's proof of surjectivity in the control theorem in one variable, we take a suitable Bianchi modular eigensymbol, and aim to lift it through each of the $\mathrm{Symb}_\Gamma(A^N\mathbb{D}_{k,k}(\mathcal{O}_L))$ in a compatible way. This gives an element of the inverse limit that corresponds to an overconvergent Bianchi modular symbol, since it follows from equation (7.3) that

$$\mathrm{Symb}_\Gamma(\mathbb{D}_{k,k}(\mathcal{O}_L)) \cong \varprojlim \mathrm{Symb}_\Gamma(A^N\mathbb{D}_{k,k}(\mathcal{O}_L)). \tag{7.4}$$

7.4.2. Moments of functionals on polynomials

Given an element $f \in V_{k,k}(\mathcal{O}_L)^*$, define the *moments of* f to be the quantities $f(x^my^n)$ for $0 \leq m, n \leq k$, analogously to the moments of more general distributions from previously. Any such f is entirely determined by its moments. We'll now define a subspace of $V_{k,k}(\mathcal{O}_L)^*$ with moments that satisfy suitable properties, and then in the proof of the control theorem, take a modular symbol with values in this space and exhibit an explicit lift to $\mathbb{D}_{k,k}(\mathcal{O}_L)$. Tensoring with L will then give the full theorem.

Definition 7.4.4. Let $\lambda \in L^\times$. Define

$$V_{k,k}^\lambda(\mathcal{O}_L) := \left\{ f \in V_{k,k}(\mathcal{O}_L)^* : f(x^iy^j) \in \lambda p^{-(i+j)}\mathcal{O}_L, 0 \leq i+j \leq \lfloor v(\lambda) \rfloor \right\}.$$

Proposition 7.4.5. *$V_{k,k}^\lambda(\mathcal{O}_L)$ is $\Sigma_0(p)^2$-stable.*

Proof. Consider first the one variable situation, as studied by Greenberg in [Gre07]. In particular, he defines $V_k^\lambda(\mathcal{O}_L) = \{f \in V_k(\mathcal{O}_L)^* : f(x^i) \in \lambda p^{-i}\mathcal{O}_L\}$, and states (without proof) that $V_k(\mathcal{O}_L)^*$ is $\Sigma_0(p)$ stable. For completeness, we provide a simple proof. It suffices to show this for matrices of the form $\left(\begin{smallmatrix} 1 & 0 \\ c & 1 \end{smallmatrix}\right)$, where $p|c$, and $\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)$, where $a \in \mathbb{Z}_p^*$, since any element of $\Sigma_0(p)$ factorises as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{c}{a} & 1 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & \frac{ad-bc}{a} \end{pmatrix}.$$

Consider first the case $\gamma = \left(\begin{smallmatrix} 1 & 0 \\ c & 1 \end{smallmatrix}\right)$. We have

$$f\Big|\gamma(x^i) = f((1+cx)^{k-i}x^i) = \sum_{m=0}^{i} \begin{pmatrix} i \\ m \end{pmatrix} c^m f(x^{m+i}).$$

Since $f(x^{m+i}) \in \lambda p^{-m-i}\mathcal{O}_L$, $c^m f(x^{m+i}) \in p^{-i}\lambda\mathcal{O}_L$, and hence the claim follows. The case $\gamma = \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)$ is similar (and, in fact, more straightforward).

Moving back to the two variable case, as in Proposition 7.4.3, it suffices to check the result for elements of form (γ, I_2) of $\Sigma_0(p)^2$. To each $f \in V_{k,k}(\mathcal{O}_L)^*$, associate a family of functionals $\{f_j \in V_k(\mathcal{O}_L)^* : 0 \leq j \leq k\}$ by setting the moments to be $f_j(x^i) = f(x^iy^j)$. Observe that

$$f\Big|(\gamma, I_2)(x^iy^j) = f_j\Big|\gamma(x^i),$$

and that we have

$$f \in V_{k,k}^{\lambda}(\mathcal{O}_L) \iff f_j \in V_k^{\lambda p^{-j}}(\mathcal{O}_L) \text{ for } 0 \leq j \leq k.$$

As $V_k^{\lambda p^{-j}}(\mathcal{O}_L)$ is $\Sigma_0(p)$-stable for each j, the result follows. $\square$

The next result is a technical lemma describing the action of certain matrices in $\Sigma_0(p)^2$. It gives us nice properties of the U_p operator.

Lemma 7.4.6. *(i) Let $\mu \in \mathbb{D}_{k,k}(\mathcal{O}_L)$ be such that $\pi^0(\mu) \in V_{k,k}^{\lambda}(\mathcal{O}_L)$. Then, for $a_i \in \mathcal{O}_K$, we have*

$$\mu\Big|_k \left[\begin{pmatrix} 1 & a_1 \\ 0 & p \end{pmatrix}, \begin{pmatrix} 1 & a_2 \\ 0 & p \end{pmatrix}\right] \in \lambda\mathbb{D}_{k,k}(\mathcal{O}_L).$$

(ii) Let $\mu \in F^N\mathbb{D}_{k,k}(\mathcal{O}_L)$, and suppose $v(\lambda) < k+1$. Then

$$\mu\Big|_k \left[\begin{pmatrix} 1 & a_1 \\ 0 & p \end{pmatrix}, \begin{pmatrix} 1 & a_2 \\ 0 & p \end{pmatrix}\right] \in \lambda F^{N+1}\mathbb{D}_{k,k}(\mathcal{O}_L).$$

Proof. Take some $\mu \in \mathbb{D}_{k,k}(\mathcal{O}_L)$. Then

$$\begin{aligned}\mu\Big|_k \left(\begin{pmatrix} 1 & a_1 \\ 0 & p \end{pmatrix}, \begin{pmatrix} 1 & a_2 \\ 0 & p \end{pmatrix}\right)(x^m y^n) &= \mu((a_1+px)^m(a_2+py)^n) \\ &= \sum_{i=0}^{m}\sum_{j=0}^{n}\begin{pmatrix} m \\ i \end{pmatrix}\begin{pmatrix} n \\ j \end{pmatrix} a_1^{m-i}a_2^{n-j}p^{i+j}\mu(x^i y^j).\end{aligned}$$

(i) Suppose that $\pi^0(\mu)$ lies in $V_{k,k}^{\lambda}(\mathcal{O}_L)$, so that $\mu(x^i y^j) \in \lambda p^{-(i+j)}\mathcal{O}_L$ for any $i+j \leq \lfloor v(\lambda)\rfloor$. It follows that each term of the sum above lies in $\lambda\mathcal{O}_L$, and hence the result follows. If instead $i+j > v(\lambda)$, then the result follows as $\mu(x^i y^j) \in \mathcal{O}_L$.

(ii) Now suppose $\mu \in F^N\mathbb{D}_{k,k}(\mathcal{O}_L)$. Again considering the sum above, the terms where $i,j \leq k$ vanish. If $i+j > k$, then

$$i+j \geq k+1 > v(\lambda),$$

since λ has p-adic valuation $< k+1$. As p^{i+j} and λ are divisible by integral powers of π_L, it follows that $p^{i+j} \in \pi_L\lambda\mathcal{O}_L$. Hence, as $\mu(x^i y^j) \in \pi_L^{N-i-j}\mathcal{O}_L$, it follows that

$$p^{i+j}\mu(x^i y^j) \in \lambda\pi_L^{(N+1)-i-j}\mathcal{O}_L,$$

which completes the proof. $\square$

Recall the definition of Hecke operators. Formally endow the set of maps from Δ_0 to $\mathbb{D}_{k,k}(\mathcal{O}_L)$

with the action of an operator U_p defined as

$$U_p := \sum_{[a]\in\mathcal{O}_K/(p)} \begin{pmatrix} 1 & a \\ 0 & p \end{pmatrix}.$$

Here $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ acts as

$$f|\gamma(D) = f(\gamma D)|\gamma,$$

and we consider such matrices as acting via the embedding of $\mathrm{GL}_2(\mathcal{O}_K) \hookrightarrow \mathrm{GL}_2(\mathcal{O}_L)\times\mathrm{GL}_2(\mathcal{O}_L)$, arising from equation (7.2). Note that the image of $\left(\begin{smallmatrix} 1 & a \\ 0 & p \end{smallmatrix}\right)$ under this embedding has the form as described in Lemma 7.4.6.

Remark: Note that we've made some choice of orbit representatives for the action of Γ on the double coset $\Gamma\left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right)\Gamma$. If we consider U_p as a double coset operator on *set-theoretic* maps, this is not well-defined up to choice of such representatives. However, as long as we are consistent in our choice, it shall not matter; hence we simply define U_p in this very specific way and ignore where it comes from.

7.5. Summary

Before embarking on the proof of a suitable lifting theorem, we first take stock of the work of this chapter. We have a monoid $\Sigma_0(p)$ acting on an $\mathcal{O}_L$-module $\mathbb{D}_{k,k}(\mathcal{O}_L)$, with a $\Sigma_0(p)$-stable filtration $F^N\mathbb{D}_{k,k}(\mathcal{O}_L)$ of $\mathbb{D}_{k,k}(\mathcal{O}_L)$ leading to an inverse system $(A^N\mathbb{D}_{k,k}(\mathcal{O}_L))$ of $\mathcal{O}_L$-modules satisfying

$$\varprojlim A^N\mathbb{D}_{k,k}(\mathcal{O}_L) = \mathbb{D}_{k,k}(\mathcal{O}_L).$$

Furthermore, $A^0\mathbb{D}_{k,k}(\mathcal{O}_L) \cong V_{k,k}(\mathcal{O}_L)^*$. Take $\lambda \in L^\times$ with $v(\lambda) < k+1$, and define

$$D = \{\mu \in \mathbb{D}_{k,k}(\mathcal{O}_L) : \rho^0(\mu) \in V_{k,k}^{\lambda}(\mathcal{O}_L)\}.$$

Suppose γ is a summand of the U_p operator defined above; then Lemma 7.4.6(i) tells us that if $\mu \in D$, then $\mu|\gamma \in \lambda\mathbb{D}_{k,k}(\mathcal{O}_L)$, while Lemma 7.4.6(ii) says that if $\mu \in F^N\mathbb{D}_{k,k}(\mathcal{O}_L)$ then $\mu|\gamma \in \lambda F^{N+1}\mathbb{D}_{k,k}(\mathcal{O}_L)$.

This then gives us the exact situation described in the next section.

Chapter 8

Lifting Small Slope Eigensymbols

In this chapter, we prove that the space of overconvergent modular symbols gives a p-adic deformation of the spaces of classical modular symbols in a very 'controlled' manner. In particular, we prove an analogue of Stevens' control theorem over imaginary quadratic fields. This says that the restriction of the natural map from overconvergent to classical modular symbols is an isomorphism when restricted to the small slope eigenspaces of the U_p operator. This bears comparison with Coleman's small slope classicality theorem for overconvergent modular forms.

The chapter starts by proving an abstract control theorem, then applies it to our particular case using the set-up from the previous chapter. After this, we take a small slope classical eigensymbol, and examine its overconvergent lift. In particular, we show that this symbol actually takes values in a much smaller space of locally analytic *distributions, before showing that it is* admissible, *that is, it satisfies good growth properties. These properties will allow us to define the p-adic L-function uniquely in the sequel.*

8.1. An abstract lifting theorem

We start by proving the control theorem in a completely general setting, since we will use essentially the same ideas multiple times. To this end, the following is an abstraction of some of the elements of Greenberg's work in [Gre07]. The notation given is suggestive, and throughout, the reader should imagine the objects of (i) to (vii) below to be the obvious analogues from Chapter 7.5.

Suppose $L/\mathbb{Q}_p$ is a finite extension, and that we have:

(i) a monoid Σ,

(ii) a $\mathcal{O}_L$-module D that has a right action of Σ,

(iii) a Σ-stable filtration of D, $D \supset \mathcal{F}^0 D \supset \mathcal{F}^1 D \supset \cdots$, where if we define $\mathcal{A}^N D := D/\mathcal{F}^N D$, then we have

$$\varprojlim \mathcal{A}^N D = D,$$

and where the $\mathcal{F}^N D$ have trivial intersection,

(iv) a right Σ-stable submodule A of $\mathcal{A}^0 D$, denoting $D_A := \{\mu \in D : \mu \pmod{\mathcal{F}^0 D} \in A\}$,

(v) an operator $U = \sum_{i=0}^{r} \gamma_i$, where $\gamma_i \in \Sigma$,

(vi) a subgroup $\Gamma \leq \Sigma$ such that for each j, we have

$$\Gamma\gamma_j\Gamma = \coprod_{i=0}^{r} \Gamma\gamma_i,$$

(vii) and a (countable) left $\mathbb{Z}[\Gamma]$-module Δ.

For a right $\mathbb{Z}[\Sigma]$-module $\mathbb{D}$, endow the space of homomorphisms from Δ to $\mathbb{D}$ with a right Σ-action by

$$(\phi|\gamma)(E) = \phi(\gamma \cdot E)|\gamma.$$

For such $\mathbb{D}$, write $\mathrm{Symb}_\Gamma(\mathbb{D}) = \mathrm{Hom}_\Gamma(\Delta, \mathbb{D})$ for the space of Γ-equivariant homomorphisms. Note that U acts on this space.

Theorem 8.1.1. *Suppose that λ is a non-zero element of $\mathcal{O}_L$, that D_A and A have trivial λ-torsion, and that for each γ_i appearing in the U operator, we have:*

(a) if $\mu \in D_A$, then $\mu|\gamma_i \in \lambda D$, and

(b) if $\mu \in \mathcal{F}^N D$, then $\mu|\gamma_i \in \lambda\mathcal{F}^{N+1} D$.

Then the restriction of the natural map $\rho^0 : \mathrm{Symb}_\Gamma(D_A) \to \mathrm{Symb}_\Gamma(A)$ to the λ-eigenspaces of the U operator is an isomorphism.

For clarity, the proof will be broken into a series of smaller steps. We have natural Σ-equivariant projection maps

$$\pi^N : D \longrightarrow \mathcal{A}^N D$$

that induce Σ-equivariant maps

$$\rho^N : \mathrm{Symb}_\Gamma(D) \longrightarrow \mathrm{Symb}_\Gamma(\mathcal{A}^N D),$$

(and hence $\rho^0 : \mathrm{Symb}_\Gamma(D_A) \to \mathrm{Symb}_\Gamma(A)$ by restriction) as well as maps $\pi^{M,N} : \mathcal{A}^M D \to \mathcal{A}^N D$ for $M \geq N$ that similarly induce maps $\rho^{M,N}$. Thus we have an inverse system, and also it is straightforward to see that

$$\varprojlim \mathrm{Symb}_\Gamma(\mathcal{A}^N D) = \mathrm{Symb}_\Gamma(D).$$

First we pass to a filtration where the Σ-action is nicer. Define $\mathcal{F}^N D_A = \mathcal{F}^N D \cap D_A$. This is a Σ-stable filtration of D_A, since A is Σ-stable and the projection maps are Σ-equivariant. It's immediate that if γ_i is a summand of the U operator and $\mu \in \mathcal{F}^N D_A$, then $\mu|\gamma_i \in \lambda\mathcal{F}^{N+1}D_A$. Define $\mathcal{A}^N D_A = D_A/\mathcal{F}^N D_A$, so that we have the following (where the vertical maps are injections):

$$\begin{tikzcd} D \arrow[r, "\pi^M"] & \mathcal{A}^M D \arrow[r, "\pi^{M,N}"] & \mathcal{A}^N D \\ D_A \arrow[u] \arrow[r, "\pi^M"] & \mathcal{A}^M D_A \arrow[u] \arrow[r, "\pi^{M,N}"] & \mathcal{A}^N D_A \arrow[u] \end{tikzcd}$$

Again, we see easily that

$$\varprojlim \mathrm{Symb}_\Gamma(\mathcal{A}^N D_A) = \mathrm{Symb}_\Gamma(D_A). \tag{8.1}$$

Firstly, since $\mathcal{A}^N D_A$ may have non-trivial λ-torsion, we should make the statement "U-eigensymbol in $\mathrm{Symb}_\Gamma(\mathcal{A}^N D_A)$" more precise. By condition (b) of 8.1.1, if γ is a summand of U, and $\mu \in D_A$, then $\mu|\gamma \in \lambda D$. Accordingly, given a homomorphism φ from Δ to D_A, we have $(\varphi|\gamma)(E) = \varphi(\gamma E)|\gamma = \lambda x$, for $E \in \Delta$ and some $x \in D$. Define a formal operator

$$V_\gamma : \mathrm{Hom}(\Delta, D_A) \longrightarrow \mathrm{Hom}(\Delta, D)$$

by

$$(\varphi|V_\gamma)(E) = x,$$

so that we have an equality of operators $\lambda V_\gamma = \gamma|_{\mathrm{Hom}(\Delta,D_A)}$. Note that the operator V_γ is well-defined since D_A has trivial λ-torsion.

Remarks: (i) Note that as ρ_0 is Σ-equivariant and A is Σ-stable, D_A is Σ-stable, so $\gamma|_{\mathrm{Hom}(\Delta,D_A)}$ is indeed an operator on $\mathrm{Hom}(\Delta, D_A)$.

(ii) The reason we don't simply just define $V_\gamma = \lambda^{-1}\gamma$ is that 'dividing by λ' is not in general a well-defined notion on D.

Further define

$$V = \sum_{i=0}^{r} V_{\gamma_i},$$

so that we have an equality of operators $\lambda V = U|_{\mathrm{Hom}(\Delta,D_A)}$ (where the right hand side is an operator on $\mathrm{Hom}(\Delta, D_A)$ by the first remark above).

Note that V gives rise to an operator V_N on each $\mathcal{A}^N D_A$ by Σ-equivariance. We say an element $\varphi_N \in \mathrm{Symb}_\Gamma(\mathcal{A}^N D_A)$ is a *U-eigensymbol of eigenvalue* λ if $\varphi_N|V_N = \varphi_N$.

Take a U-eigensymbol $\phi_0 \in \mathrm{Symb}_\Gamma(A) = \mathrm{Symb}_\Gamma(\mathcal{A}^0 D_A)$ with eigenvalue λ. Suppose a lift to a U-eigensymbol $\phi_N \in \mathrm{Symb}_\Gamma(\mathcal{A}^N D_A)$ exists. We can take an arbitrary lift of ϕ_N to some homomorphism

$$\phi : \Delta \longrightarrow D.$$

Such a lift exists, as we can take some $\mathbb{Z}$-basis of Δ (using countability) and define ϕ on this basis, extending $\mathbb{Z}$-linearly (noting that this gives a well-defined lift since ϕ_N is also a homomorphism).

Now, since ϕ_N is a U-eigensymbol with eigenvalue λ, it follows that $\phi|V$ is also a lift of ϕ_N to $\mathrm{Hom}(\Delta, D_A)$. The maps π^N, inducing the maps ρ^N, can be used immediately to extend the definition of ρ^N to the space of homomorphisms from Δ to D_A (rather than just the ones that are Γ-equivariant). Define

$$\phi_{N+1} = \rho^{N+1}\left(\phi|V\right),$$

a homomorphism from Δ to $\mathcal{A}^{N+1}D_A$. Note that since the maps $\rho^N, \rho^{M,N}$ are Σ-equivariant (and hence V-equivariant), we have compatibility relations

$$\rho^{N+1,N}(\phi_{N+1}) = \phi_N,$$

so that the following lemma says that the family $\{\phi_N\}$ we obtain gives an element of the inverse limit given in equation (8.1).

Lemma 8.1.2. *The homomorphism ϕ_{N+1} is a well-defined U-eigensymbol in* $\mathrm{Symb}_\Gamma(\mathcal{A}^{N+1}D_A)$.

We prove this lemma in a series of claims.

Claim 8.1.2.1. *If $\gamma = \gamma_i$ is a summand of the U operator, and ϕ' is another lift of ϕ_N, then*

$$\rho_{N+1}\left(\phi|V_\gamma\right) = \rho_{N+1}\left(\phi'|V_\gamma\right).$$

In particular, ϕ_{N+1} is independent of the choice of ϕ above ϕ_N.

Proof. To say that ϕ and ϕ' are both lifts of ϕ_N is to say that the image of $\phi - \phi'$ under ρ^N is 0 in $\mathcal{A}^N D_A$, that is, $\phi - \phi' \in \mathcal{F}^N D_A$. Thus by condition (b) in Theorem 8.1.1,

$$(\phi - \phi')\Big|\gamma \in \lambda\mathcal{F}^{N+1}D_A,$$

that is,

$$(\phi - \phi')\Big|V_\gamma \in \mathcal{F}^{N+1}D_A.$$

Indeed, the image of $(\phi - \phi')|V_\gamma$ under ρ^{N+1} in $\mathcal{A}^{N+1}D_A$ is 0. But this is precisely the statement that we required. $\square$

Claim 8.1.2.2. *The homomorphism ϕ_{N+1} is Γ-equivariant.*

Proof. Let $\gamma \in \Gamma$. As the map ρ^{N+1} is Σ-equivariant, it follows that

$$\phi_{N+1}\Big|\gamma = \frac{1}{\lambda}\rho^{N+1}\left(\sum_{i=0}^{r}\phi\Big|_k(\gamma_i\gamma)\right),$$

where the division by λ is purely formal and well-defined by the remarks above. By condition (vi) above, we have a double coset decomposition

$$\Gamma\gamma_j\Gamma = \coprod_{i=0}^{r}\Gamma\gamma_i,$$

hence we can find $\eta_j \in \Gamma$ such that

$$\sum_{i=0}^{r}\phi\Big|_k(\gamma_i\gamma) = \sum_{j=0}^{r}\left(\phi\Big|_k\eta_j\right)\Big|_k\gamma_j.$$

Since ϕ_N is Γ-invariant, $\phi|\eta_j$ is a lift of ϕ_N, and hence by Claim 8.1.2.1 it follows that

$$\begin{aligned}\phi_{N+1}\Big|\gamma &= \frac{1}{\lambda}\rho^{N+1}\left(\sum_{j=0}^{r}\left(\phi\Big|_k\eta_j\right)\Big|\gamma_j\right)\\ &= \frac{1}{\lambda}\rho^{N+1}\left(\sum_{j=0}^{r}\phi\Big|\gamma_j\right) = \phi_{N+1},\end{aligned}$$

as required. □

Claim 8.1.2.3. *The homomorphism ϕ_{N+1} is a U-eigensymbol with eigenvalue λ.*

Proof. Recall that when we consider the operator V acting on $\mathcal{A}^N D_A$, we denote it V_N, and that ϕ_N is a U-eigensymbol with eigenvalue λ if ϕ_N is a fixed point of V_N. Note then that $\phi|V$ is also a lift of ϕ_N to $\mathrm{Hom}(\Delta, D_A)$. In particular, $\phi|V$ also lives in $\mathrm{Hom}(\Delta, D_A)$, so we can apply V to it again. Thus, by Σ-equivariance and Claim 8.1.2.1,

$$\begin{aligned}\phi_{N+1}\Big|V_{N+1} &= \rho^{N+1}\left(\phi\Big|V\right)\Big|V_{N+1}\\ &= \rho^{N+1}\left(\phi\Big|V^2\right) = \phi^{N+1},\end{aligned}$$

as required. □

Proof. (*Theorem 8.1.1*). Surjectivity follows from the results above; take an element $\phi_0 \in \mathrm{Symb}_\Gamma(A)$. Then for each N we can construct $\phi_N \in \mathrm{Symb}_\Gamma(\mathcal{A}^N D_A)$, compatibly with the projection maps $\pi^{M,N}$, and thus obtain a well-defined element of the inverse limit, which is the domain. By construction, this element has image ϕ_0 under ρ^0.

To prove injectivity, take some

$$\varphi \in \ker(\rho^0) = \mathrm{Symb}_\Gamma(D_A \cap \mathcal{F}^0 D_A).$$

Applying the operator V recursively gives

$$\varphi = \varphi \Big| V^N \in \mathcal{F}^N D_A$$

by condition (b) in Theorem 8.1.1. Thus φ lies in the intersection of all the $\mathcal{F}^N D_A$, that is, $\varphi = 0$ (by condition (iii)) as required.

Thus the map is a bijection, and thus an isomorphism, as required. □

Remark: This result bears comparison to [PP09], Theorem 3.1, where Pollack and Pollack generalise Greenberg's argument to the (slightly different) setting of group cohomology. In particular, they prove an analogue of this control theorem in the *ordinary* case, that is, for $v(\lambda) = 0$. In [Wil16], this condition on $v(\lambda)$ is removed under some extra hypotheses (which are analogous to conditions (i) and (ii) of Theorem 8.1.1 above).

8.2. The Bianchi control theorem

By considering the result above in combination with the objects in Chapter 7.5, we have the following corollary:

Theorem 8.2.1. *(i) Let $K/\mathbb{Q}$ be an imaginary quadratic field, p a rational prime, and $L/\mathbb{Q}_p$ the finite extension defined in Notation 7.2.1. Let $\Gamma = \Gamma_1^i(\mathfrak{n})$ be a twist of $\Gamma_1(\mathfrak{n}) \leq \mathrm{SL}_2(\mathcal{O}_K)$ with $(p)|\mathfrak{n}$.*

Let $\lambda \in L^\times$. Then, when $v_p(\lambda) < k+1$, the restriction of the specialisation map

$$\rho^0 : \mathrm{Symb}_\Gamma(\mathbb{D}_{k,k}(L))^{U_p=\lambda} \longrightarrow \mathrm{Symb}_\Gamma(V_{k,k}(L)^*)^{U_p=\lambda}$$

(where the superscript $(U_p = \lambda)$ denotes the λ-eigenspace for U_p) is an isomorphism.

(ii) In the same set up, with $\Omega_1(\mathfrak{n})$ defined as in equation (2.3), we have an isomorphism

$$\rho^0 : \mathrm{Symb}_{\Omega_1(\mathfrak{n})}(\mathbb{D}_{k,k}(L))^{U_p=\lambda} \longrightarrow \mathrm{Symb}_{\Omega_1(\mathfrak{n})}(V_{k,k}(L)^*)^{U_p=\lambda}.$$

Proof. To prove (i), recall that in the set up of Section 7.5, Theorem 8.1.1 says that the restriction of ρ^0 to the map

$$\rho^0 : \mathrm{Symb}_\Gamma(D)^{U_p=\lambda} \longrightarrow \mathrm{Symb}_\Gamma(V_{k,k}^\lambda(\mathcal{O}_L))^{U_p=\lambda}$$

is an isomorphism. The result now follows by right-exactness of tensor product and Proposition 7.3.3, since $D \otimes_{\mathcal{O}_L} L \cong \mathbb{D}_{k,k}(L)$ and $V_{k,k}^\lambda \otimes_{\mathcal{O}_L} K \cong V_{k,k}(L)^*$. Part (ii) is a trivial consequence as the U_p operator acts separately on each component. $\square$

8.3. Values of overconvergent lifts

This section will examine the spaces in which overconvergent lifts take values, refining our earlier results. Recall that $\Gamma = \Gamma_1^i(\mathfrak{n})$ is a twist of $\Gamma_1(\mathfrak{n})$.

8.3.1. Locally analytic distributions

We've shown that any classical Bianchi eigensymbol of suitable slope can be lifted to an overconvergent symbol that takes values in a space of rigid analytic distributions. However, the module of values we're truly interested in is a smaller space of distributions. A p-adic L-function should be a function on characters; but a rigid analytic distribution can take as input only functions that can be written as a single convergent power series. However, finite order characters are locally constant, and thus most cannot be written in this form. Instead, we want our lift to take values in the dual of *locally analytic* functions.

Definition 8.3.1. Let $r, s \in \mathbb{R}_{>0}$. Define the $(r,s)-$*ball in* $\mathbb{C}_p$ to be

$$B(\mathcal{O}_\mathfrak{p}, r, s) = \{(x,y) \in \mathbb{C}_p^2 : \exists u \in \mathcal{O}_K \otimes_\mathbb{Z} \mathbb{Z}_p \text{ such that } |x - \sigma_1(u)| \leq r, |y - \sigma_2(u)| \leq s\},$$

where σ_1, σ_2 are the embeddings $K \otimes \mathbb{Q}_p \hookrightarrow L$.

Example: When r and s are both at least 1, $B(\mathcal{O}_\mathfrak{p}, r, s)$ is the cartesian product of the closed discs of radii r and s in $\mathbb{C}_p$. If $r = s = 1/p$, this is the cartesian product of two copies of the disjoint union of closed discs of radius $1/p$ with centres at 0, 1, ..., $p-1$. As r and s tend to zero, $B(\mathcal{O}_\mathfrak{p}, r, s)$ comprises smaller and smaller discs around the points of $\mathcal{O}_K \otimes_\mathbb{Z} \mathbb{Z}_p$.

Definition 8.3.2. Let r and s be as above. Then define the space of *locally analytic functions of radius* (r,s) *over* L, denoted $\mathbb{A}[L,r,s]$, to be the space of rigid analytic functions on $B(\mathcal{O}_\mathfrak{p}, r, s)$ that are defined over L.

Example: The space $\mathbb{A}[L,1,1]$ is just the space $\mathbb{A}_2(L)$ of 'fully analytic' functions described previously. The space $\mathbb{A}[L,1/p,1/p]$ consists of functions that are 'slightly more locally analytic,' in the sense that each element can be written as a collection of p (possibly independent) convergent power series, one for each closed disc around the points $0, 1, ..., p-1$. In particular, this space contains the space $\mathbb{A}[L, 1, 1]$. As r and s get smaller and smaller, the functions become more and more locally analytic in the sense that at each point of $\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{Z}$, a function in $\mathbb{A}[L, r, s]$ can be written as a convergent power series in a disc of progressively smaller radius around that point.

Definition 8.3.3. Define the space of *locally analytic distributions of order (r, s) over L* to be

$$\mathbb{D}[L, r, s] = \mathrm{Hom}_{\mathrm{cts}}(\mathbb{A}[L, r, s], L).$$

We endow $\mathbb{A}[L, r, s]$ with a weight (k, ℓ) action of $\Sigma_0(p)^2$ identical to the action defined earlier on $\mathbb{A}_2(L)$; it is obvious that this action extends immediately to the larger space. It's then clear that by dualising, the action we obtain on $\mathbb{D}[L, r, s]$ is the restriction of the action on $\mathbb{D}_2(L)$. When talking about these spaces equipped with these actions, we denote them $\mathbb{A}_{k,\ell}[L, r, s]$ and $\mathbb{D}_{k,\ell}[L, r, s]$.

For $r \leq r'$ and $s \leq s'$, we have a natural and completely continuous injection $\mathbb{A}[L, r', s'] \hookrightarrow \mathbb{A}_{k,\ell}[L, r, s]$, since $B(\mathcal{O}_{\mathfrak{p}}, r, s) \subset B(\mathcal{O}_{\mathfrak{p}}, r', s')$. Since the polynomials are dense in each of these spaces, the image of this injection is dense. Using this compatibility, we make the following definitions:

Definition 8.3.4. Define the space of *locally analytic functions over L* to be the direct limit

$$\mathcal{A}_{k,\ell}(L) := \varinjlim \mathbb{A}_{k,\ell}[L, r, s] = \bigcup_{r,s} \mathbb{A}_{k,\ell}[L, r, s].$$

Definition 8.3.5. Define the space of *locally analytic distributions over K* to be

$$\mathcal{D}_{k,\ell}(L) := \mathrm{Hom}_{\mathrm{cts}}(\mathcal{A}_{k,\ell}, L).$$

Proposition 8.3.6. *There is a canonical $\Sigma_0(p)^2$-equivariant isomorphism*

$$\mathcal{D}_{k,\ell}(L) \cong \varprojlim \mathbb{D}_{k,\ell}[L, r, s] = \bigcap_{r,s} \mathbb{D}_{k,\ell}[L, r, s].$$

Proof. An element of the left hand side is a functional on a direct limit. Each element $\{\mu_{r,s}\}$ of the right hand side is an element of the direct product of all the $\mathbb{D}_{k,\ell}[L, r, s]$. Given such an element $\{\mu_{r,s}\}$, define an element μ of the left hand side by

$$\mu(f) = \mu_{r_0,s_0}(f_{r_0,s_0})$$

for some choice of r_0, s_0 such that f is represented by $f_{r_0,s_0} \in \mathbb{A}_{k,\ell}[L, r_0, s_0]$ in the direct limit. This is independent of the choice of r_0 and s_0; suppose we choose different values r_0' and s_0'

with $f_{r_0',s_0'}$ representing f, then we can choose some values r_1 and s_1 with $r_1 \leq r_0, r_0'$ and $s_1 \leq s_0, s_0'$. Then

$$\mu_{r_0,s_0}(f_{r_0,s_0}) = \mu_{r_1,s_1}(f_{r_1,s_1}) = \mu_{r_0',s_0'}(f_{r_0',s_0'}),$$

by definition of inverse limit, where we choose $f_{r_1,s_1} \in \mathbb{A}_{k,\ell}[L, r_1, s_1]$ also representing f, which is possible by definition of direct limit.

We define an inverse to the corresponding map

$$\varprojlim \mathbb{D}_{k,\ell}[L, r, s] \longrightarrow \mathcal{D}_{k,\ell}(L)$$

as follows. For an element $\mu \in \mathcal{D}_{k,\ell}(L)$, define an element $\{\mu_{r,s}\}$ by $\mu_{r_0,s_0}(f_{r_0,s_0}) = \mu(f)$, where f_{r_0,s_0} represents f in the direct limit. This does give a family in the inverse limit using properties of μ, and this gives a well-defined inverse.

The $\Sigma_0(p)^2$-equivariance of this map follows from a simple check. □

8.3.2. The action of $\Sigma_0(p)^2$

The action of certain elements of $\Sigma_0(p)^2$ naturally moves us up and down the direct/inverse systems.

Lemma 8.3.7. *(i) Let $g \in \mathbb{A}_{k,\ell}[L, r, s]$, and $a_1, a_2 \in \mathcal{O}_\mathfrak{p}$. Then*

$$[\gamma_1, \gamma_2] \cdot_{(k,\ell)} g, \qquad \gamma_i = \begin{pmatrix} 1 & a_i \\ 0 & p^n \end{pmatrix},$$

naturally extends to $B(\mathcal{O}_\mathfrak{p}, rp^n, sp^n)$ and thus gives an element of $\mathbb{A}_{k,\ell}[L, rp^n, sp^n]$.

(ii) Let $\mu \in \mathbb{D}_{k,\ell}[L, r, s]$, and γ_i as above for $i = 1, 2$. Then $\mu|_{(k,\ell)}[\gamma_1, \gamma_2]$ naturally gives an element of the smaller space $\mathbb{D}_{k,\ell}[L, rp^{-n}, sp^{-n}]$.

Proof. For $x, y \in B(\mathcal{O}_\mathfrak{p}, rp^n, sp^n)$, there exist $b_1, b_2 \in \mathcal{O}_\mathfrak{p}$ such that $|x - b_1| \leq rp^n$ and $|y - b_2| \leq sp^n$. Then $|(a_1 + p^n x) - (a_1 + p^n b_1)| \leq r$, and similarly for y, so that

$$(a_1 + p^n x, a_2 + p^n y) \in B(\mathcal{O}_\mathfrak{p}, r, s).$$

For such x, y, we have

$$[\gamma_1, \gamma_2] \cdot_{k,\ell} g(x, y) = g(a_1 + p^n x, a_2 + p^n y),$$

and since g is defined on $B(\mathcal{O}_\mathfrak{p}, r, s)$, the result follows.

For part (ii), note that the action of $[\gamma_1, \gamma_2]$ gives a map

$$\mathbb{A}_{k,\ell}[L, r, s] \longrightarrow \mathbb{A}_{k,\ell}[L, rp^n, sp^n],$$

and hence dualising, the action gives a map

$$\mathbb{D}_{k,\ell}[L, rp^n, sp^n] \longrightarrow \mathbb{D}_{k,\ell}[L, r, s].$$

This is the required result (though scaled by a factor of p^n). □

Proposition 8.3.8. *Suppose that $\Psi \in \mathrm{Symb}_\Gamma(\mathbb{D}_{k,k}(L))$ is a U_p-eigensymbol with non-zero eigenvalue. Then Ψ is an element of $\mathrm{Symb}_\Gamma(\mathcal{D}_{k,k}(L))$.*

Proof. Firstly, $\mathbb{D}_{k,k}(L) = \mathbb{D}_{k,k}[L, 1, 1]$. Note that U_p acts invertibly on the U_p-eigenspace, so that for each integer n, there exists some eigensymbol Ψ' with $\Psi = \Psi'|U_p^n$. The U_p^n operator can be described explicitly as

$$U_p^n = \sum_{[a] \in \mathcal{O}_K/(p^n)} \begin{pmatrix} 1 & a \\ 0 & p^n \end{pmatrix},$$

so combining with Lemma 8.3.7 shows that Ψ takes values in $\mathbb{D}_{k,k}[L, p^{-n}, p^{-n}]$ for each n, and thus in $\varprojlim \mathbb{D}_{k,k}[L, r, s] \cong \mathcal{D}_{k,k}(L)$, using Proposition 8.3.6. The result follows. □

Corollary 8.3.9. *(i) Let $K/\mathbb{Q}$ be an imaginary quadratic field, p a rational prime, and $L/\mathbb{Q}_p$ the finite extension defined in Notation 7.2.1. Let $\Gamma = \Gamma_1^i(\mathfrak{n})$ be a twist of $\Gamma_1(\mathfrak{n}) \leq \mathrm{SL}_2(\mathcal{O}_K)$ with $(p)|\mathfrak{n}$.*

Let $\lambda \in L^\times$. Then, when $v(\lambda) < k + 1$, the restriction of the natural map

$$\mathrm{Symb}_\Gamma(\mathcal{D}_{k,k}(L))^{U_p=\lambda} \longrightarrow \mathrm{Symb}_\Gamma(V_{k,k}(L)^*)^{U_p=\lambda}$$

is an isomorphism.

(ii) In the set-up of above, and with $\Omega_1(\mathfrak{n})$ as defined in equation (2.3), the restriction of the natural map

$$\mathrm{Symb}_{\Omega_1(\mathfrak{n})}(\mathcal{D}_{k,k}(L))^{U_p=\lambda} \longrightarrow \mathrm{Symb}_{\Omega_1(\mathfrak{n})}(V_{k,k}(L)^*)^{U_p=\lambda}$$

is an isomorphism.

8.3.3. Admissible distributions

In this section, we prove that the values of an overconvergent symbol lifting a classical eigensymbol are *admissible*, that is, they satisfy good growth properties. In the sequel, we will define the p-adic L-function as the value of an overconvergent symbol at the divisor $\{0\} - \{\infty\}$; the admissibility condition then shows that this distribution is *unique*.

We start by defining the admissibility condition. For each pair r, s, the space $\mathbb{D}_{k,k}[L, r, s]$ admits a natural operator norm $||\cdot||_{r,s}$ via

$$||\mu||_{r,s} = \sup_{0 \neq f \in \mathbb{A}_{k,k}[L,r,s]} \frac{|\mu(f)|_p}{|f|_{r,s}},$$

where $|\cdot|_p$ is the usual p-adic absolute value on L and $|\cdot|_{r,s}$ is the sup norm on $\mathbb{A}_{k,k}[L, r, s]$. Note that if $r \leq r', s \leq s'$, then $||\mu||_{r,s} \geq ||\mu||_{r',s'}$ for $\mu \in \mathbb{D}_{k,k}[L, r', s']$.

These norms give rise to a family of norms on the space of locally analytic functions. It is natural to classify locally analytic distributions by growth properties as we vary in this family.

Definition 8.3.10. Let $\mu \in \mathcal{D}_{k,k}(L)$ be a locally analytic distribution. We say μ is *h-admissible* if

$$||\mu||_{r,r} = O(r^{-h})$$

as $r \to 0^+$.

The following lemma is a useful technical result describing the family of norms of a Γ-orbit in $\mathbb{A}_{k,k}[L, r, s]$. It gives universal constants that will be useful in the sequel.

Lemma 8.3.11. *There exist positive constants C and C' such that*

$$C|\gamma \cdot_{(k,k)} f|_{r,s} \leq |f|_{r,s} \leq C'|\gamma \cdot_{(k,k)} f|_{r,s}$$

for every $\gamma \in \Gamma$ and $f \in \mathbb{A}_{k,k}[L, r, s]$.

Proof. The action of γ by

$$\gamma^{-1} \cdot (x, y) = \left(\frac{b + dx}{a + cx}, \frac{b' + d'y}{a' + c'y}\right), \quad \sigma(\gamma^{-1}) = \left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}\right]$$

maps $B(\mathcal{O}_\mathfrak{p}, r, s)$ bijectively to itself. Furthermore, we have

$$|(a + cx)^k(a' + c'y)^k|_p \leq \max\{1, |x|_p^k, |y|_p^k, |xy|_p^k\} \leq \max\{1, r^k, s^k, (rs)^k\} = C^{-1},$$

say, noting that C^{-1} is certainly positive. Thus

$$\begin{aligned}|\gamma \cdot_{(k,k)} f|_{r,s} &= \sup_{(x,y)\in B(\mathcal{O}_\mathfrak{p},r,s)} \left|(a+cx)^k(a'+c'y)^k\right|_p \cdot \left|f\Big(\gamma\cdot(x,y)\Big)\right|_p \\ &\leq C^{-1} \sup_{(x,y)\in B(\mathcal{O}_\mathfrak{p},r,s)} |f(x,y)|_p \\ &= C^{-1}|f|_{r,s},\end{aligned}$$

from which the first inequality follows. The reverse direction follows from symmetry by considering the action of γ^{-1}. □

Definition-Proposition 8.3.12. *Let* $\Psi \in \mathrm{Symb}_\Gamma(\mathcal{D}_{k,k}(L))$*, and* $r,s \leq 1$*. The expression*

$$||\Psi||_{r,s} := \sup_{D\in\Delta_0} ||\Psi(D)||_{r,s}$$

gives a well-defined norm on $\mathrm{Symb}_\Gamma(\mathcal{D}_{k,k}(L))$*.*

Proof. It suffices to show that $||\Psi(D)||_{r,s}$ is bounded. Pick a finite set of generators $D_1,...,D_m$ for Δ_0 as a $\mathbb{Z}[\Gamma]-$module. For any $D\in\Delta_0$, write $D = \alpha_1D_1 + \cdots \alpha_mD_m$ with $\alpha_i \in \mathbb{Z}[\Gamma]$. Then

$$||\Psi(D)||_{r,s} \leq \max_i ||\Psi(\alpha_iD_i)||_{r,s} = ||\Psi(\alpha_1D_1)||_{r,s}$$

(without loss of generality). Write $\alpha_1 = a_1\gamma_1 + \cdots a_\ell\gamma_\ell$, with $a_i \in \mathbb{Z}, \gamma_i \in \Gamma$. Then

$$||\Psi(\alpha_1D_1)||_{r,s} \leq \max_j |a_j|\cdot||\Psi(\gamma_jD_1)||_{r,s} \leq \left|\left|\Psi(D_1)\right|_{(k,k)}\gamma_1^{-1}\right|\right|_{r,s},$$

where we again, without loss of generality, take this max to be at $j=1$. We have

$$\begin{aligned}\left|\left|\Psi(D_1)\right|_{(k,k)}\gamma_i^{-1}\right|\right|_{r,s} &= \sup_{0\neq f\in\mathbb{A}_{k,k}[L,r,s]} \frac{|\Psi(D_1)(\gamma_1^{-1}\cdot_{(k,k)} f)|_p}{|f|_{r,s}} \\ &= \sup_{0\neq f\in\mathbb{A}_{k,k}[L,r,s]} \frac{|\Psi(D_1)(f)|_p}{|\gamma_1\cdot_{(k,k)} f|_{r,s}} \\ &\leq C' \sup_{0\neq f\in\mathbb{A}_{k,k}[L,r,s]} \frac{|\Psi(D_1)(f)|_p}{|f|_{r,s}} \\ &\leq C'||\Psi(D_1)||_{r,s},\end{aligned}$$

for some universal constant C', using Lemma 8.3.11. Combining, this gives

$$||\Psi(D)||_{r,s} \leq C'||\Psi(D_1)||_{r,s},$$

so in particular, it is finite and hence gives a well-defined norm, as required. □

The values of an overconvergent eigensymbol satisfy further conditions of the type above depending on their slope.

Proposition 8.3.13. *Suppose $\Psi \in \mathrm{Symb}_\Gamma(\mathcal{D}_{k,k}(L))$ is a U_p-eigensymbol with eigenvalue λ and slope $h = v(\lambda)$. Then, for every $D \in \Delta_0$, the distribution $\Psi(D)$ is h-admissible.*

Proof. For any r and a positive integer n, we have

$$||\Psi(D)||_{\frac{r}{p^n},\frac{r}{p^n}} = |\lambda|^{-n}\left|\left|\left(\Psi\Big|_{(k,\ell)} U_p^n\right)(D)\right|\right|_{\frac{r}{p^n},\frac{r}{p^n}}$$
$$\leq |\lambda|^{-n} \max_{[a]\in\mathcal{O}_K/(p^n)} \left|\left|\Psi\left(\begin{pmatrix}1 & a\\ 0 & p^n\end{pmatrix} D\right)\Big|_{k,\ell}\sigma\left[\begin{pmatrix}1 & a\\ 0 & p^n\end{pmatrix}\right]\right|\right|_{\frac{r}{p^n},\frac{r}{p^n}},$$

where σ is the embedding $\mathcal{O}_K \hookrightarrow \mathbb{Z}_p \times \mathbb{Z}_p$,

$$\leq |\lambda|^{-n} \max_{[a]\in\mathcal{O}_K/(p^n)} \left|\left|\Psi\left(\begin{pmatrix}1 & a\\ 0 & p^n\end{pmatrix} D\right)\right|\right|_{r,r}$$
$$\leq |\lambda|^{-n}||\Psi||_{r,r},$$

the norm defined in Definition-Proposition 8.3.12. Here the second to last inequality follows since for $\gamma_i = \left(\begin{smallmatrix}1 & a_i\\ 0 & p^n\end{smallmatrix}\right)$, we have, for any $\mu \in \mathbb{D}_{k,\ell}[L,r,s]$,

$$\left|\left|\mu\Big|_{k,\ell}[\gamma_1,\gamma_2]\right|\right|_{\frac{r}{p^n},\frac{s}{p^n}} \leq ||\mu||_{r,s}.$$

This is simply because, for $f \in \mathbb{A}_{k,\ell}[L, rp^{-n}, sp^{-n}]$,

$$||f||_{rp^{-n},sp^{-n}} \geq ||[\gamma_1,\gamma_2]\cdot_{(k,\ell)} f||_{r,s},$$

and hence

$$\left|\left|\mu\Big|_{k,\ell}[\gamma_1,\gamma_2]\right|\right|_{\frac{r}{p^n},\frac{s}{p^n}} = \sup_{f\in\mathbb{A}_{k,\ell}[L,\frac{r}{p^n},\frac{s}{p^n}]} \frac{|\mu([\gamma_1,\gamma_2]\cdot_{(k,\ell)} f)|_p}{||f||_{\frac{r}{p^n},\frac{s}{p^n}}}$$
$$\leq \sup_{f\in\mathbb{A}_{k,\ell}[L,\frac{r}{p^n},\frac{s}{p^n}]} \frac{|\mu([\gamma_1,\gamma_2]\cdot_{(k,\ell)} f)|_p}{||[\gamma_1,\gamma_2]\cdot_{(k,\ell)} f||_{r,s}}$$
$$\leq \sup_{g\in\mathbb{A}_{k,\ell}[L,r,s]} \frac{|\mu(g)|_p}{||g||_{r,s}} = ||\mu||_{r,s}.$$

From the inequality $||\Psi(D)||_{rp^{-n},rp^{-n}} \leq |\lambda|^{-n}||\Psi||_{r,r}$, the result follows. ∎

Remark 8.3.14: To summarise the results of this section: if we take an eigensymbol ϕ associated to a Bianchi cuspidal eigenform of small slope h, and lift it to some overconvergent eigensymbol Ψ using the control theorem, then the components of the distribution $\Psi(\{0\}-\{\infty\})$ are h-admissible locally analytic distributions.

Chapter 9

The Case p Split

The results above hold for the U_p operator regardless of the splitting behaviour of the prime p in K. When p is inert, this is the whole story, and if p is ramified, then there are only slight modifications to make to obtain a finer result for the $U_{\mathfrak{p}}$ operator, where $(p) = \mathfrak{p}^2$ in K. In the case that p splits in K as $\mathfrak{p}\overline{\mathfrak{p}}$, however, we can obtain more subtle results. The crux of this section is that we can prove a control theorem for a 'small slope' condition that encompasses far more possible eigensymbols. For example, suppose $\phi \in \mathrm{Symb}_\Gamma(V_{k,k}(L)^*)$ is a Bianchi eigensymbol with slope $(k+1)/2$ at $\mathfrak{p}$ and $(k+1)/2$ at $\overline{\mathfrak{p}}$. This will have slope $k+1$ under the U_p operator, and hence we cannot lift it using the control theorem proved above. The results below will allow us to lift even symbols such as this.

Many of the results and proofs closely mirror those of previous sections. We first prove a more refined control theorem, again by writing down a suitable filtration and then using Theorem 8.1.1, and then go on to prove an admissibility result for such lifts.

Throughout, we assume that p splits as $\mathfrak{p}\overline{\mathfrak{p}}$. Whilst ultimately we want to prove a control theorem for the full space of modular symbols for $\Omega_1(\mathfrak{n})$, it is simpler to instead work with a principal power of $\mathfrak{p}$ and look at each individual component of the direct sum separately, combining together at the end.

9.1. Lifting simultaneous eigensymbols of $U_{\mathfrak{p}}$ and $U_{\overline{\mathfrak{p}}}$

The following results will show that it is possible to lift a classical Bianchi eigensymbol to a space of Bianchi modular symbols that are overconvergent in one variable, and then again from this space to the space of fully overconvergent Bianchi modular symbols we considered previously.

To do so, consider the space

$$[\mathbb{D}_k \otimes V_k^*](R) = \mathrm{Hom}_{\mathrm{cts}}(\mathbb{A}_k(R) \otimes_R V_k(R), R),$$

with the appropriate action of $\Sigma_0(p)^2$ (where this makes sense) induced from the action on $\mathbb{A}_{k,k}$. This gives us

$$V_{k,k}(R) \subset [\mathbb{D}_k \otimes V_k^*](R) \subset \mathbb{D}_{k,k}(R).$$

Now put $R = \mathcal{O}_L$, and recall the filtration in the one variable case from Definition 7.4.1. We now define new filtrations to reflect lifting by one variable at a time.

Definition 9.1.1. Define

(i) $\mathcal{F}^N[\mathbb{D}_k \otimes V_k^*](\mathcal{O}_L) = \mathcal{F}^N\mathbb{D}_k(\mathcal{O}_L) \otimes_{\mathcal{O}_L} V_k(\mathcal{O}_L)^*$
$$= \{\mu \in [\mathbb{D}_k \otimes V_k^*](\mathcal{O}_L) : \mu(x^i y^j) \in \pi_L^{N-i}\mathcal{O}_L \text{ for all } j\},$$

(ii) $[\mathbb{D}_k \otimes V_k^*]^0(\mathcal{O}_L) = \ker([\mathbb{D}_k \otimes V_k^*](\mathcal{O}_L) \to V_{k,k}(\mathcal{O}_L)^*)$
$$= \{\mu \in [\mathbb{D}_k \otimes V_k^*](\mathcal{O}_L) : \mu(x^i y^j) = 0 \text{ for all } 0 \leq i \leq k\},$$

(iii) and $F^N[\mathbb{D}_k \otimes V_k^*](\mathcal{O}_L) = \mathcal{F}^N[\mathbb{D}_k \otimes V_k^*](\mathcal{O}_L) \cap [\mathbb{D}_k \otimes V_k^*]^0(\mathcal{O}_L)$.

Definition 9.1.2. Define

(i) $\mathcal{F}_{\mathfrak{p}}^N\mathbb{D}_{k,k}(\mathcal{O}_L) = \mathbb{D}_k(\mathcal{O}_L)\hat{\otimes}_{\mathcal{O}_L}\mathcal{F}^N(\mathcal{O}_L)$
$$= \{\mu \in \mathbb{D}_{k,k}(\mathcal{O}_L) : \mu(x^i y^j) \in \pi_L^{N-j}\mathcal{O}_L \text{ for all } i\},$$

(ii) $\mathbb{D}_{k,k,\mathfrak{p}}^0(\mathcal{O}_L) = \ker(\mathbb{D}_{k,k}(\mathcal{O}_L) \to [\mathbb{D}_k \otimes V_k^*](\mathcal{O}_L))$
$$= \{\mu \in \mathbb{D}_{k,k}(\mathcal{O}_L) : \mu(x^i y^j) = 0 \text{ for all } j \geq 0\},$$

(iii) and $F_{\mathfrak{p}}^N\mathbb{D}_{k,k}(\mathcal{O}_L) = \mathcal{F}_{\mathfrak{p}}^N\mathbb{D}_{k,k}(\mathcal{O}_L) \cap \mathbb{D}_{k,k,\mathfrak{p}}^0(\mathcal{O}_L)$.

Further define

$$A^N[\mathbb{D}_k \otimes V_k^*](\mathcal{O}_L) = [\mathbb{D}_k \otimes V_k^*](\mathcal{O}_L)/F^N[\mathbb{D}_k \otimes V_k^*](\mathcal{O}_L),$$

$$A_{\mathfrak{p}}^N\mathbb{D}_{k,k}(\mathcal{O}_L) = \mathbb{D}_{k,k}(\mathcal{O}_L)/F_{\mathfrak{p}}^N\mathbb{D}_{k,k}(\mathcal{O}_L).$$

Hence we now have filtrations

$$\mathcal{F}^0[\mathbb{D}_k \otimes V_k^*](\mathcal{O}_L) \subset \cdots \subset \mathcal{F}^M[\mathbb{D}_k \otimes V_k^*](\mathcal{O}_L) \subset \cdots \subset [\mathbb{D}_k \otimes V_k^*](\mathcal{O}_L) \subset$$
$$\cdots \subset \mathcal{F}_{\mathfrak{p}}^N\mathbb{D}_{k,k}(\mathcal{O}_L) \subset \cdots \subset \mathbb{D}_{k,k}(\mathcal{O}_L).$$

Proposition 9.1.3. *These filtrations are $\Sigma_0(p)^2$-stable.*

Proof. This follows from the one variable case, as these filtrations are defined to be a tensor product of $\Sigma_0(p)$-stable spaces, and in the two variable case, $\Sigma_0(p)^2$ acts separately on each component. □

These filtrations lead to $\Sigma_0(p)^2$-equivariant projection maps

$$\pi_1^N : [\mathbb{D}_k \otimes V_k^*](\mathcal{O}_L) \longrightarrow A^N[\mathbb{D}_k \otimes V_k^*](\mathcal{O}_L)$$

and

$$\pi_2^N : \mathbb{D}_{k,k}(\mathcal{O}_L) \longrightarrow A_{\mathfrak{p}}^N \mathbb{D}_{k,k}(\mathcal{O}_L),$$

which again give maps ρ_1^N and ρ_2^N on the corresponding symbol spaces.

Having defined two $\Sigma_0(p)^2$-stable filtrations, the next pieces we need are $\Sigma_0(p)^2$-stable submodules of $V_{k,k}(\mathcal{O}_L)^*$ and $[\mathbb{D}_k \otimes V_k^*](\mathcal{O}_L)$ to play the role of the module A in Theorem 8.1.1.

Definition 9.1.4. Let $\lambda \in K^*$. Define

(i) $V_{k,k,\mathfrak{p}}^{\lambda}(\mathcal{O}_L) = V_k^{\lambda}(\mathcal{O}_L) \otimes_{\mathcal{O}_L} V_k(\mathcal{O}_L)^*$
$$= \{f \in V_{k,k}(\mathcal{O}_L)^* : f(x^i y^j) \in \lambda p^{-i}\mathcal{O}_L \text{ for } 0 \leq i \leq \lfloor v(\lambda) \rfloor\},$$

(ii) and $[\mathbb{D}_k \otimes V_k^*]^{\lambda}(\mathcal{O}_L) = \mathbb{D}_k(\mathcal{O}_L) \otimes_{\mathcal{O}_L} V_k^{\lambda}(\mathcal{O}_L)$
$$= \{f \in [\mathbb{D}_k \otimes V_k^*](\mathcal{O}_L) : f(x^i y^j) \in \lambda p^{-j}\mathcal{O}_L \text{ for } 0 \leq j \leq \lfloor v(\lambda) \rfloor\}.$$

This gives the following situation:

$$\mathbb{D}_{k,k}(\mathcal{O}_L) \xrightarrow{\pi_2^0} [\mathbb{D}_k \otimes V_k^*](\mathcal{O}_L) \supset [\mathbb{D}_k \otimes V_k^*]^{\lambda}(\mathcal{O}_L),$$

$$[\mathbb{D}_k \otimes V_k^*](\mathcal{O}_L) \xrightarrow{\pi_1^0} V_{k,k}(\mathcal{O}_L)^* \supset V_{k,k,\mathfrak{p}}^{\lambda}(\mathcal{O}_L).$$

Proposition 9.1.5. *These modules are $\Sigma_0(p)^2$-stable.*

Proof. As in Proposition 9.1.3, this follows from the one variable case, as these are nothing but a tensor product of $\Sigma_0(p)$-stable spaces. $\square$

We want an analogue of Lemma 7.4.6 for this setting. Choose n such that $\mathfrak{p}^n = (\beta)$ is principal (noting that this also forces $\overline{\mathfrak{p}}^n$ to be principal). Then we define $U_{\mathfrak{p}^n} = U_{\mathfrak{p}}^n$ as

$$\sum_{a \pmod{\mathfrak{p}^n}} \begin{pmatrix} 1 & a \\ 0 & \beta \end{pmatrix}.$$

We will prove control theorems for eigenspaces of the operators $U_{\mathfrak{p}}^n$ for $\mathfrak{p}|p$, which will give us the theorem for the operators $U_{\mathfrak{p}}$, as required.

The following two lemmas are practically identical in spirit to Lemma 7.4.6, but they are included here for completeness.

Lemma 9.1.6. *Let $a_1, a_2 \in \mathcal{O}_L$.*

(i) Suppose $\mu \in [\mathbb{D}_k \otimes V_k^*](\mathcal{O}_L)$ with $\pi_1^0(\mu) \in V_{k,k,\mathfrak{p}}^{\lambda}(\mathcal{O}_L)$. Then

$$\mu\Big|_k \left[\begin{pmatrix} 1 & a_1 \\ 0 & \beta \end{pmatrix}, \begin{pmatrix} 1 & a_2 \\ 0 & \overline{\beta} \end{pmatrix}\right] \in \lambda[\mathbb{D}_k \otimes V_k^*](\mathcal{O}_L).$$

(ii) Suppose $v(\lambda) < k+1$. Then for $\mu \in F^N[\mathbb{D}_k \otimes V_k^*](\mathcal{O}_L)$, we have

$$\mu\Big|_k \left[\begin{pmatrix} 1 & a_1 \\ 0 & \beta \end{pmatrix}, \begin{pmatrix} 1 & a_2 \\ 0 & \overline{\beta} \end{pmatrix}\right] \in \lambda F^{N+1}[\mathbb{D}_k \otimes V_k^*](\mathcal{O}_L).$$

Proof. Take some $\mu \in [\mathbb{D}_k \otimes V_k^*](\mathcal{O}_L)$. Then

$$\mu\Big|_k \left[\begin{pmatrix} 1 & a_1 \\ 0 & \beta \end{pmatrix}, \begin{pmatrix} 1 & a_2 \\ 0 & \overline{\beta} \end{pmatrix}\right](x^m y^n) = \mu((a_1 + \beta x)^m (a_2 + \overline{\beta} y)^n)$$

$$= \sum_{i=0}^{m} \sum_{j=0}^{n} \binom{m}{i} \binom{n}{j} a_1^{m-i} a_2^{n-j} \beta^i \overline{\beta}^j \mu(x^i y^j).$$

(i) Suppose that $\pi^0(\mu)$ lies in $V_{k,k,\mathfrak{p}}^{\lambda}(\mathcal{O}_L)$, so that $\mu(x^i y^j) \in \lambda p^{-i}\mathcal{O}_L$ for any $i \leq \lfloor v(\lambda) \rfloor$. As $\beta^i \in p^i \mathcal{O}_L$, it follows that each term of the sum above lies in $\lambda\mathcal{O}_L$, and hence we have the result. If instead i is greater than $\lfloor v(\lambda) \rfloor$, it follows that $i > v(\lambda)$, so that $\beta^i \in \lambda\mathcal{O}_L$, and hence the result follows as $\mu(x^i y^j) \in \mathcal{O}_L$.

(ii) Now suppose $\mu \in F^N[\mathbb{D}_k \otimes V_k^*](\mathcal{O}_L)$. Again considering the sum above, the terms where $i \leq k$ vanish. If $i > k$, then

$$i \geq k + 1 > v(\lambda),$$

since λ has p-adic valuation $< k+1$. As β^i and λ are divisible by integral powers of π_L, it follows that $\beta^i \in \pi_L \lambda \mathcal{O}_L$. Hence, as $\mu(x^i y^j) \in \pi_L^{N-i}\mathcal{O}_L$, it follows that

$$\beta^i \mu(x^i y^j) \in \lambda \pi_L^{(N+1)-i}\mathcal{O}_L,$$

which completes the proof. $\square$

Now let $\overline{\mathfrak{p}}^n = (\delta)$, with image $(\overline{\delta}, \delta)$ in $\mathcal{O}_L^2$. Note that $v(\delta) = n$, whilst $\overline{\delta}$ is a unit in $\mathcal{O}_L$.

Lemma 9.1.7. *Let $a_1, a_2 \in \mathcal{O}_L$.*

(i) Suppose $\mu \in \mathbb{D}_{k,k}(\mathcal{O}_L)$ with $\pi_2^0(\mu) \in [\mathbb{D}_k \otimes V_k^*]^{\lambda}(\mathcal{O}_L)$. Then

$$\mu\Big|_k \left[\begin{pmatrix} 1 & a_1 \\ 0 & \overline{\delta} \end{pmatrix}, \begin{pmatrix} 1 & a_2 \\ 0 & \delta \end{pmatrix}\right] \in \lambda\mathbb{D}_{k,k}(\mathcal{O}_L).$$

(ii) Suppose $v(\lambda) < k+1$. Then for $\mu \in F_{\mathfrak{p}}^N \mathbb{D}_{k,k}(\mathcal{O}_L)$, we have

$$\mu\Big|_k \left[\begin{pmatrix} 1 & a_1 \\ 0 & \overline{\delta} \end{pmatrix}, \begin{pmatrix} 1 & a_2 \\ 0 & \delta \end{pmatrix}\right] \in \lambda F_{\mathfrak{p}}^{N+1}\mathbb{D}_{k,k}(\mathcal{O}_L).$$

Proof. Identical to that of Lemma 9.1.6 – up to notation – but with j's replacing i's where appropriate. □

We are hence in exactly the situation of Theorem 8.1.1, and applying it twice gives us:

Lemma 9.1.8. *Let $K/\mathbb{Q}$ be an imaginary quadratic field, p a rational prime that splits as $\mathfrak{p}\overline{\mathfrak{p}}$ in K, n an integer such that $\mathfrak{p}^n$ is principal, and $L/\mathbb{Q}_p$ the finite extension defined in Notation 7.2.1. Let $\Gamma = \Gamma_1^i(\mathfrak{n})$ be a twist of $\Gamma_1(\mathfrak{n}) \leq \mathrm{SL}_2(\mathcal{O}_K)$ with $(p)|\mathfrak{n}$. Let $\lambda \in L^\times$. Then, when $v(\lambda) < n(k+1)$, we have:*

(i) The restriction of the specialisation map

$$\rho_1^0 : \mathrm{Symb}_\Gamma([\mathbb{D}_k \otimes V_k^*](L))^{U_{\mathfrak{p}}^n=\lambda} \longrightarrow \mathrm{Symb}_\Gamma(V_{k,k}(L)^*)^{U_{\mathfrak{p}}^n=\lambda}$$

(where the superscript $(U_{\mathfrak{p}} = \lambda)$ denotes the λ-eigenspace for $U_{\mathfrak{p}}$) is an isomorphism.

(ii) The restriction of the specialisation map

$$\rho_2^0 : \mathrm{Symb}_\Gamma(\mathbb{D}_{k,k}(L))^{U_{\mathfrak{p}}^n=\lambda} \longrightarrow \mathrm{Symb}_\Gamma([\mathbb{D}_k \otimes V_k^*](L))^{U_{\mathfrak{p}}^n=\lambda}$$

is an isomorphism.

Proof. The only remaining details to fill in are formalities regarding tensor products, which are analogous to before and are omitted. □

We require results about the $U_{\mathfrak{p}}$ operator, whilst the results above are for the $U_{\mathfrak{p}}^n$ operator. We work around this using:

Lemma 9.1.9. *Suppose we have two L-vector spaces D and V, each equipped with an operator U acting on the right, and a U-equivariant surjection $\rho : D \to V$ such that, for some positive integer n and $\lambda \in L^\times$, the restriction of ρ to the λ^n-eigenspaces of the U^n operator is an isomorphism. Then the restriction of ρ to the λ-eigenspaces of the U operator is an isomorphism.*

Proof. Suppose $\phi \in V$ is a U-eigensymbol with eigenvalue λ. Then ϕ is also a U^n-eigensymbol with eigenvalue λ^n, and accordingly there is a unique lift Ψ of ϕ to a U^n-eigensymbol with eigenvalue λ^n. We claim that this is in fact a U-eigensymbol with eigenvalue λ. Indeed, by U-equivariance we have

$$\rho(\Psi|U) = \rho(\Psi)|U = \phi|U = \lambda\phi,$$

so that $\Psi|U$ is a lift of $\lambda\phi$. But clearly $\lambda\Psi$ is also a lift of $\lambda\phi$. Now as $\lambda\Psi$ and $\lambda\phi$ are both U^n-eigensymbols with eigenvalue λ^n, it follows that $\lambda\Psi$ is the unique eigenlift of $\lambda\phi$ under ρ; but then it follows that $\Psi|U = \lambda\Psi$, as required. The lemma follows easily. $\square$

This then gives:

Theorem 9.1.10. *(i) Let K, p, $\mathfrak{p}$, n, Γ and L be as in Lemma 9.1.8. Take $\lambda_1, \lambda_2 \in L^*$ with $v(\lambda_1), v(\lambda_2) < k+1$. Then the restriction of the specialisation map*

$$\rho^0 : \mathrm{Symb}_\Gamma(\mathbb{D}_{k,k}(L))^{U_{\mathfrak{p}}^n=\lambda_1^n, U_{\overline{\mathfrak{p}}}^n=\lambda_2^n} \longrightarrow \mathrm{Symb}_\Gamma(V_{k,k}(L)^*)^{U_{\mathfrak{p}}^n=\lambda_1^n, U_{\overline{\mathfrak{p}}}^n=\lambda_2^n}$$

(where the superscript denotes the simultaneous λ_1^n-eigenspace of $U_{\mathfrak{p}}^n$ and λ_2^n-eigenspace of $U_{\overline{\mathfrak{p}}}^n$) is an isomorphism.

(ii) In the set up of part (i), the restriction of the specialisation map

$$\rho^0 : \mathrm{Symb}_{\Omega_1(\mathfrak{n})}(\mathbb{D}_{k,k}(L))^{U_{\mathfrak{p}}=\lambda_1, U_{\overline{\mathfrak{p}}}=\lambda_2} \longrightarrow \mathrm{Symb}_{\Omega_1(\mathfrak{n})}(V_{k,k}(L)^*)^{U_{\mathfrak{p}}=\lambda_1, U_{\overline{\mathfrak{p}}}=\lambda_2}$$

is an isomorphism.

Proof. (i) Take a simultaneous $U_{\mathfrak{p}}^n$- and $U_{\overline{\mathfrak{p}}}^n$-eigensymbol ϕ^0, with eigenvalues λ_1^n, λ_2^n respectively. Then Lemma 9.1.8(i) says that we can lift ϕ^0 uniquely to some

$$\varphi^0 \in \mathrm{Symb}_\Gamma([\mathbb{D}_k \otimes V_k^*](L))^{U_{\mathfrak{p}}^n=\lambda_1^n}.$$

We claim that φ^0 is a $U_{\overline{\mathfrak{p}}}^n$-eigensymbol with eigenvalue λ_2^n. Indeed, consider the action of the operator $\lambda_2^{-n}U_{\overline{\mathfrak{p}}}^n$. When applied to φ^0, the result is a $U_{\mathfrak{p}}^n$-eigensymbol with eigenvalue λ_1^n, since

$$\begin{aligned}\left(\varphi^0\middle|\lambda_2^{-n}U_{\overline{\mathfrak{p}}}^n\right)\middle|U_{\mathfrak{p}}^n &= \left(\varphi^0\middle|U_{\mathfrak{p}}^n\right)\middle|\lambda_2^{-n}U_{\overline{\mathfrak{p}}}^n\\ &= \lambda_1^n\varphi^0\middle|\lambda_2^{-n}U_{\overline{\mathfrak{p}}}^n,\end{aligned}$$

as $U_{\mathfrak{p}}^n$ and $U_{\overline{\mathfrak{p}}}^n$ commute. Then by the $\Sigma_0(p)^2$-equivariance of ρ_1^0 we have

$$\begin{aligned}\rho_1^0\left(\varphi^0\middle|\lambda_2^{-n}U_{\overline{\mathfrak{p}}}^n\right) &= \rho_1^0(\varphi^0)\middle|\lambda_2^{-n}U_{\overline{\mathfrak{p}}}^n\\ &= \phi^0\middle|\lambda_2^{-n}U_{\overline{\mathfrak{p}}}^n = \phi^0,\end{aligned}$$

since ϕ^0 is a $U_{\overline{\mathfrak{p}}}^n$-eigensymbol with eigenvalue λ_2^n. But then by uniqueness, we must have

$$\varphi^0 \Big| \lambda_2^{-n} U_{\overline{\mathfrak{p}}}^n = \varphi^0,$$

that is, φ^0 is a $U_{\overline{\mathfrak{p}}}^n$-eigensymbol with eigenvalue λ_2^n, as required. Now we can use Lemma 9.1.8(ii) to lift φ^0 to some

$$\phi \in \mathrm{Symb}_\Gamma(\mathbb{D}_{k,k}(L))^{U_{\overline{\mathfrak{p}}}^n = \lambda_2^n}.$$

By an identical argument to that above, ϕ is a $U_{\mathfrak{p}}^n$-eigensymbol with eigenvalue λ_1^n, and since by construction $\rho^0(\phi) = \phi^0$, this is the result.

(ii) From part (i), it is easy to see that we have an isomorphism

$$\rho^0 : \mathrm{Symb}_{\Omega_1(\mathfrak{n})}(\mathbb{D}_{k,k}(L))^{U_{\mathfrak{p}}^n = \lambda_1^n, U_{\overline{\mathfrak{p}}}^n = \lambda_2^n} \longrightarrow \mathrm{Symb}_{\Omega_1(\mathfrak{n})}(V_{k,k}(L)^*)^{U_{\mathfrak{p}}^n = \lambda_1^n, U_{\overline{\mathfrak{p}}}^n = \lambda_2^n}.$$

The result then follows directly from Lemma 9.1.9, since there are well-defined $U_{\mathfrak{p}}$ and $U_{\overline{\mathfrak{p}}}$ operators on each of the spaces, and ρ^0 is equivariant with respect to these operators.

□

9.2. The action of $\Sigma_0(p)^2$ and locally analytic distributions

The following results are proved in an almost identical manner to those of Section 8.3.2.

Lemma 9.2.1. *Let p split as $\mathfrak{p}\overline{\mathfrak{p}}$ in K, with $\mathfrak{p}^n = (\beta)$ principal, and recall the definition of the embedding σ in equation (7.2).*

(i) Let $g \in \mathbb{A}_{k,\ell}[L,r,s]$, and $a_1, a_2 \in \mathcal{O}_L$. Then

$$[\sigma(\gamma)] \cdot_{(k,\ell)} g, \quad \gamma = \begin{pmatrix} 1 & a_i \\ 0 & \beta^m \end{pmatrix},$$

naturally extends to $B(\mathcal{O}_{\mathfrak{p}}, rp^m, s)$ and thus gives an element of $\mathbb{A}_{k,\ell}[L, rp^m, s]$.

(ii) Let $\mu \in \mathbb{D}_{k,\ell}[L,r,s]$, and γ_i as above for $i = 1,2$. Then $\mu|_{(k,\ell)}[\sigma(\gamma)]$ naturally gives an element of the smaller space $\mathbb{D}_{k,\ell}[L, rp^{-m}, s]$.

We also have an entirely analogous result for the $U_{\overline{\mathfrak{p}}}$ operator. Combining the two then gives the following:

Proposition 9.2.2. *Suppose that $\Psi \in \mathrm{Symb}_\Gamma(\mathbb{D}_{k,k}(L))$ is simultaneously a $U_{\mathfrak{p}}^n$- and $U_{\overline{\mathfrak{p}}}^n$-eigensymbol with non-zero eigenvalues. Then Ψ is an element of $\mathrm{Symb}_\Gamma(\mathcal{D}_{k,k}(L))$.*

Corollary 9.2.3. *Let $K/\mathbb{Q}$ be an imaginary quadratic field, p a rational prime that splits as $\mathfrak{p}\overline{\mathfrak{p}}$ in K, and $L/\mathbb{Q}_p$ the finite extension defined in Notation 7.2.1. Let $\Omega_1(\mathfrak{n})$ be as defined in Definition 2.3.*

Take $\lambda_1, \lambda_2 \in L^*$ with $v(\lambda_1), v(\lambda_2) < k+1$. Then the restriction of the specialisation map

$$\rho^0 : \mathrm{Symb}_{\Omega_1(\mathfrak{n})}(\mathcal{D}_{k,k}(L))^{U_{\mathfrak{p}}=\lambda_1, U_{\overline{\mathfrak{p}}}=\lambda_2} \longrightarrow \mathrm{Symb}_{\Omega_1(\mathfrak{n})}(V_{k,k}(L)^*)^{U_{\mathfrak{p}}=\lambda_1, U_{\overline{\mathfrak{p}}}=\lambda_2}$$

is an isomorphism.

9.3. Admissibility for p split

In this new setting, we need a new definition of admissibility - namely one that encodes the slope at both $\mathfrak{p}$ and $\overline{\mathfrak{p}}$.

Definition 9.3.1. Let $\mu \in \mathcal{D}_{k,k}(L)$ be a locally analytic distribution. We say μ is (h_1, h_2)-*admissible* if

$$||\mu||_{r,s} = O(r^{-h_1})$$

uniformly in s as $r \to 0^+$, and

$$||\mu||_{r,s} = O(s^{-h_2})$$

uniformly in r as $s \to 0^+$.

Proposition 9.3.2. *Suppose p splits in K as $\mathfrak{p}\overline{\mathfrak{p}}$, with $\mathfrak{p}^n = (\beta)$ principal, and suppose that $\Psi \in \mathrm{Symb}_\Gamma(\mathcal{D}_{k,k}(L))$ is a $U_{\mathfrak{p}}^n$-eigensymbol with eigenvalue λ_1^n and a $U_{\overline{\mathfrak{p}}}^n$-eigensymbol with eigenvalue λ_2^n, with slopes $h_i = v(\lambda_i)$. Then, for every $D \in \Delta_0$, the distribution $\Psi(D)$ is (h_1, h_2)-admissible.*

Proof. For any r and s and a positive integer n, we have

$$\begin{aligned}
||\Psi(D)||_{\frac{r}{p^{mn}},s} &= |\lambda_1|^{-mn} \left|\left| \left(\Psi\Big|_{(k,\ell)} U_{\mathfrak{p}}^{mn}\right)(D) \right|\right|_{\frac{r}{p^{mn}},s} \\
&\leq |\lambda_1|^{-mn} \max_{[a]\in\mathcal{O}_K/\mathfrak{p}^{mn}} \left|\left| \Psi\left(\begin{pmatrix}1 & a\\ 0 & \beta^{mn}\end{pmatrix} D\right)\Big|_{k,\ell} \sigma \left[\begin{pmatrix}1 & a\\ 0 & \beta^{mn}\end{pmatrix}\right] \right|\right|_{\frac{r}{p^{mn}},s},
\end{aligned}$$

where σ is the embedding $\mathcal{O}_K \hookrightarrow \mathcal{O}_L \times \mathcal{O}_L$,

$$\begin{aligned}
&\leq |\lambda_1|^{-mn} \max_{[a]\in\mathcal{O}_K/\mathfrak{p}^{mn}} \left|\left| \Phi\left(\begin{pmatrix}1 & a\\ 0 & \beta^{mn}\end{pmatrix} D\right) \right|\right|_{r,s} \\
&\leq |\lambda_1|^{-mn} ||\Phi||_{r,s},
\end{aligned}$$

the norm defined in Definition-Proposition 8.3.12. Again, here the second to last inequality holds since we have, for $\gamma = \left(\begin{smallmatrix} 1 & a \\ 0 & \beta^{mn} \end{smallmatrix}\right)$ and for any $\mu \in \mathbb{D}_{k,\ell}[L, r, s]$,

$$\left|\left| \mu \right|_{k,\ell} [\sigma(\gamma)] \right|\right|_{\frac{r}{p^{mn}}, s} \leq ||\mu||_{r,s}.$$

An identical argument using $U_{\overline{\mathfrak{p}}}$ shows that

$$||\Phi(D)||_{r, \frac{s}{p^{mn}}} \leq |\lambda_2|^{mn} ||\Phi||_{r,s},$$

and combining the two inequalities gives the result. □

Remark: We again summarise the results of this section, noting the similarity to Remark 8.3.14. If we take an eigensymbol ϕ associated to a Bianchi cuspidal eigenform of small slopes h_1 and h_2 at $\mathfrak{p}$ and $\overline{\mathfrak{p}}$ respectively, then we can lift it to some overconvergent eigensymbol Ψ using the control theorem, and the components of the distribution $\Psi(\{0\} - \{\infty\})$ are (h_1, h_2)-admissible locally analytic distributions.

Chapter 10

The p-adic L-function

Let Φ be a small slope Bianchi modular form. In previous chapters, we have associated to Φ a canonical overconvergent modular symbol $\Psi = (\Psi_1, ..., \Psi_h)$ using the control theorem. In this chapter, the last of Part II, we show that the distributions $\Psi_i(\{0\} - \{\infty\})$ can be combined into a ray class distribution μ_Φ that interpolates critical L-values of Φ, and define it to be the p-adic L-function *of Φ.*

10.1. Evaluating at $\{0\} - \{\infty\}$

Consider the rational case. In particular, let f be a small slope cuspidal eigenform of p-divisible level, with associated classical modular symbol ϕ_f and with overconvergent lift Ψ_f. Then $\Psi_f(\{0\} - \{\infty\})$ is a distribution on $\mathbb{Z}_p$, and the restriction of this distribution to $\mathbb{Z}_p^\times$ is the p-adic L-function of f. We want to emulate this result in the Bianchi case. The analogue of $\mathbb{Z}_p$ will be $\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{Z}_p$, and the analogue of $\mathbb{Z}_p^\times$ will be the ray class group $\mathrm{Cl}_K(p^\infty)$.

Notation: To ease notation, we write $\mathcal{O}_{K,p} := \mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{Z}_p$.

We start by defining some fundamental locally analytic functions on $\mathcal{O}_{K,p}$. Recall that we fixed representatives $I_1, ..., I_h$ for the class group that are coprime to our level $\mathfrak{n}$ (and hence to (p)). Let $\mathfrak{f}$ be an ideal of $\mathcal{O}_K$ with $\mathfrak{f}|(p^\infty)$, and for each $b \pmod{\mathfrak{f}}$, take an element $d_b \in \mathcal{O}_K$ such that $d_b \in I_1, ..., I_h$ and $d_b \equiv b \pmod{\mathfrak{f}}$ using the Chinese Remainder Theorem. Then for integers $q, r \geq 0$ define

$$P_{b,\mathfrak{f}}^{q,r}(z) := z^q \overline{z}^r \mathbb{1}_{b \pmod{\mathfrak{f}}},$$

a locally polynomial function on $\mathcal{O}_{K,p}$, where $\mathbb{1}_{b \pmod{\mathfrak{f}}}$ is the indicator function for the minimal open subset of $\mathcal{O}_{K,p}^\times$ containing the image of $b + \mathfrak{f} \subset \mathcal{O}_K^\times$ under the canonical embedding of $\mathcal{O}_K^\times$ into $\mathcal{O}_{K,p}^\times$. One should see such an open set as the analogue of the set $b + p^j\mathbb{Z}_p \subset \mathbb{Z}_p$ in the rational case.

We define an operator $U_{\mathfrak{f}}$ as follows:

$$U_{\mathfrak{f}} := \prod_{\mathfrak{p}^n || \mathfrak{f}} U_{\mathfrak{p}}^n.$$

In this chapter, we will also need to work with the $U_{\mathfrak{f}}$ operator. Since $\mathfrak{f}$ is not necessarily principal, to facilitate an explicit description of this operator, we need some extra notation. To this end let $j_i \in \{1, ..., h\}$ be the unique integer such that $[\mathfrak{f}I_i] = [I_{j_i}]$ in the class group, and choose $\alpha_i \in K^\times$ such that we have an equality

$$\mathfrak{f}I_i = (\alpha_i)I_{j_i}$$

of ideals of K. Then from Chapter 7.3.2, we have:

Lemma 10.1.1. *Let $(\Psi_1, ..., \Psi_h) \in \mathrm{Symb}_{\Omega_1(\mathfrak{n})}(\mathcal{D}_{k,k}(L))$ be an overconvergent modular symbol. We can explicitly describe the $U_{\mathfrak{f}}$ operator on this symbol as*

$$(\Psi_1, ..., \Psi_h)\Big| U_{\mathfrak{f}} = \left(\sum_{b \pmod{\mathfrak{f}}} \Psi_{j_1} \Bigg| \begin{pmatrix} 1 & d_b \\ 0 & \alpha_1 \end{pmatrix}, ..., \sum_{b \pmod{\mathfrak{f}}} \Psi_{j_h} \Bigg| \begin{pmatrix} 1 & d_b \\ 0 & \alpha_h \end{pmatrix} \right).$$

Definition 10.1.2. Let $(\phi_1, ..., \phi_h)$ be a classical Bianchi eigensymbol (resp. Φ a classical Bianchi eigenform), with $U_{\mathfrak{p}}$ eigenvalue(s) $a_{\mathfrak{p}}$ for $\mathfrak{p}|p$. We can canonically see $a_{\mathfrak{p}}$ as living in $\overline{\mathbb{Q}}_p$ under our fixed embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$, and thus it is meaningful to take its p-adic valuation. We say that $(\phi_1, ..., \phi_h)$ (resp. Φ) has *small slope* if $v(a_{\mathfrak{p}}) < (k+1)/e_{\mathfrak{p}}$ for all $\mathfrak{p}|p$, where $e_{\mathfrak{p}}$ is the ramification index of $\mathfrak{p}$ in K. Note that this is precisely the condition that allows us to lift ϕ using one of the control theorems above. We say that the *slope* is $(v(a_{\mathfrak{p}}))_{\mathfrak{p}|p}$.

Take some small slope classical Bianchi eigensymbol $(\phi_1, ..., \phi_j) \in \mathrm{Symb}_{\Omega_1(\mathfrak{n})}(V_{k,k}^*(L))$ with $U_{\mathfrak{f}}$-eigenvalue $\lambda_{\mathfrak{f}}$, and lift it to an overconvergent eigensymbol $(\Psi_1, ..., \Psi_h)$ using the control theorem. We now describe the value of $(\Psi_1, ..., \Psi_h)(\{0\} - \{\infty\})$ at the fundamental polynomials defined above. Indeed, we have

$$\begin{aligned}(\Psi_1, .., \Psi_h)(\{0\} - \{\infty\})(P_{b,\mathfrak{f}}^{q,r}) &= \lambda_{\mathfrak{f}}^{-1}[(\Psi_1, ..., \Psi_h)|U_{\mathfrak{f}}](\{0\} - \{\infty\})(z^q \overline{z}^r \mathbb{1}_{b \pmod{\mathfrak{f}}}) \\ &= \lambda_{\mathfrak{f}}^{-1} \left(\sum_{b \pmod{\mathfrak{f}}} \Psi_{j_1} \Bigg| \begin{pmatrix} 1 & d_b \\ 0 & \alpha_1 \end{pmatrix}, ..., \sum_{b \pmod{\mathfrak{f}}} \Psi_{j_h} \Bigg| \begin{pmatrix} 1 & d_b \\ 0 & \alpha_h \end{pmatrix} \right) \{0\} - \{\infty\} \\ &= \lambda_{\mathfrak{f}}^{-1} \Big(\Psi_{j_i} \left(\{d_b/\alpha_i\} - \{\infty\} \right) [(\alpha_i z + d_b)^q (\overline{\alpha_i z} + \overline{d_b})^r] \Big)_{i=1}^{h}. \end{aligned} \tag{10.1}$$

Note here that the sum for $U_{\mathfrak{f}}$ is 'absorbed' by the indicator function; indeed, we have

$$\begin{aligned}\left[\begin{pmatrix}1 & d_b\\ 0 & \sigma\end{pmatrix},\begin{pmatrix}1 & \overline{\partial}\\ 0 & \overline{\sigma}\end{pmatrix}\right]\cdot\left(z^q\overline{z}^r\mathbb{1}_{b\,(\mathrm{mod}\ \mathfrak{f})}\right)(x,y) &= z^q\overline{z}^r\mathbb{1}_{b\,(\mathrm{mod}\ \mathfrak{f})}(\partial+\sigma x,\overline{\partial}+\sigma y)\\ &= 0 \text{ unless } \partial\equiv b\,(\mathrm{mod}\,\mathfrak{f}).\end{aligned}$$

Suppose now that for $d\in K$ and $\alpha\in K^\times$ we set

$$\phi(\{d/\alpha\}-\{\infty\}) = \sum_{i,j=0}^{k} c_{i,j}\left(\frac{d}{\alpha}\right)\left(\mathcal{Y}-\frac{d}{\alpha}\mathcal{X}\right)^{k-i}\mathcal{X}^i\left(\overline{\mathcal{Y}}-\frac{\overline{d}}{\overline{\alpha}}\overline{\mathcal{X}}\right)^{k-j}\overline{\mathcal{X}}^j, \tag{10.2}$$

where $\mathcal{X}^i\mathcal{Y}^{k-i}\overline{\mathcal{X}}^j\overline{\mathcal{Y}}^{k-j}$ is the basis element of $V_{k,k}^*(L)$ such that

$$\mathcal{X}^i\mathcal{Y}^{k-i}\overline{\mathcal{X}}^j\overline{\mathcal{Y}}^{k-j}(X^IY^{k-I}\overline{X}^J\overline{Y}^{k-J}) = \delta_{iI}\delta_{jJ}.$$

Note that this is chosen so that under the change of basis for $V_{k,k}(L)$ defined by

$$X^iY^{k-i}\overline{X}^j\overline{Y}^{k-j}\longmapsto(\alpha X+dY)^iY^{k-i}(\overline{\alpha}\overline{X}+\overline{dY})^j\overline{Y}^{k-j},$$

the corresponding change of dual basis is given by

$$\mathcal{X}^i\mathcal{Y}^{k-i}\overline{\mathcal{X}}^j\overline{\mathcal{Y}}^{k-j}\longmapsto\mathcal{X}^i(\alpha\mathcal{Y}-d\mathcal{X})^{k-i}\overline{\mathcal{X}}^j(\overline{\alpha}\overline{\mathcal{Y}}-\overline{d\mathcal{X}})^{k-j}.$$

Since Ψ is a lift of ϕ, for $0\leq q,r\leq k$, we can substitute equation (10.2) into (10.1), using the obvious dictionary between the two spaces $\mathcal{D}_{k,k}(L)$ and $V_{k,k}^*(L)$. We find that:

Proposition 10.1.3. *We can explicitly describe the value of the distribution Ψ_i at a fundamental polynomial as*

$$\begin{aligned}\Psi_i(\{0\}-\{\infty\})(P_{b,\mathfrak{f}}^{q,r}) &= \lambda_{\mathfrak{f}}^{-1}\alpha_i^q\overline{\alpha_i}^r c_{q,r}^{j_i}\left(\frac{d_b}{\alpha_i}\right)\\ &= \lambda_{\mathfrak{f}}^{-1}\psi(t_i)^{-1}\psi(t_{j_i})\psi_{\mathfrak{f}}(d_b)^{-1}\psi_{\mathfrak{f}}\left(\frac{d_b}{\alpha_i}\right)c_{q,r}^{j_i}\left(\frac{d_b}{\alpha_i}\right).\end{aligned}$$

Here we've used that

$$\alpha_i^q\overline{\alpha_i}^r = \psi_\infty(\alpha_i) = \psi(t_i)^{-1}\psi(t_{j_i})\psi_{\mathfrak{f}}(\alpha_i)^{-1}.$$

Note that

$$\alpha_i^{-1}\in\mathfrak{f}^{-1}I_i^{-1}I_{j_i}\subset\mathfrak{f}^{-1}I_i^{-1}.$$

Accordingly, since $d_b\in I_i$ for each i, we have $d_b/\alpha_i\in\mathfrak{f}^{-1}$. In fact, we have:

Proposition 10.1.4. *For each i, the set*

$$\{d_b/\alpha_i : b \in (\mathcal{O}_K/\mathfrak{f})^\times\}$$

forms a full set of coset representatives for the set $\{[a] \in \mathfrak{f}^{-1}/\mathcal{O}_K : (a)\mathfrak{f}$ *coprime to* $\mathfrak{f}\}$.

Proof. We have $(d_b/\alpha_i)\mathfrak{f} = (d_b)I_i^{-1}I_{j_i}$, which is coprime to $\mathfrak{f}$ since d_b is a unit $\pmod{\mathfrak{f}}$. Now suppose $b \neq b' \pmod{\mathfrak{f}}$. If d_b/α_i and $d_{b'}/\alpha_i$ gave the same element of $\mathfrak{f}^{-1}/\mathcal{O}_K$, then $d_b - d_{b'} \in (\alpha_i) = \mathfrak{f}I_i^{-1}I_{j_i}$. Since d_b and $d_{b'}$ are integral and $\mathfrak{f}, I_i$ and I_{j_i} are coprime, we must have $d_b - d_{b'} \in \mathfrak{f}I_{j_i} \subset \mathfrak{f}$, so that $b \cong b' \pmod{\mathfrak{f}}$, a contradiction. Since both sets have the same size, we are done. □

Thus as b varies over $(\mathcal{O}_K/\mathfrak{f})^\times$, the values $c_{q,r}^{j_i}(d_b/\alpha_i)$ are precisely what we need to access the L-values, since they occur in an integral formula for the critical values of the L-function.

10.2. Ray class groups

The p-adic L-function of a modular form should be a function on characters in a suitable sense. To make this more precise, we recall the theory of ray class groups.

Definition 10.2.1. Let K be a number field with ring of integers $\mathcal{O}_K$, and take an ideal $\mathfrak{f} \subset \mathcal{O}_K$. The *ray class group of K modulo* $\mathfrak{f}$, denoted $\mathrm{Cl}_K(\mathfrak{f})$, is the group $I_\mathfrak{f}$ of fractional ideals of K that are coprime to $\mathfrak{f}$ modulo the group $K_\mathfrak{f}^1$ of principal ideals that have a generator congruent to 1 mod $\mathfrak{f}$. We can also define the ray class group adelically; if we let $U(\mathfrak{f}) = 1 + \mathfrak{f}\widehat{\mathcal{O}_F}$, then

$$\mathrm{Cl}_K(\mathfrak{f}) \cong K^\times \backslash \mathbb{A}_K^\times / U(\mathfrak{f})\mathbb{C}^\times.$$

The ray class group fits into a useful exact sequence; we have

$$0 \longrightarrow \mathcal{O}_K^\times \pmod{\mathfrak{f}} \longrightarrow (\mathcal{O}_K/\mathfrak{f})^\times \xrightarrow{\quad\beta\quad} \mathrm{Cl}_K(\mathfrak{f}) \longrightarrow \mathrm{Cl}_K \longrightarrow 0,$$

where the map β takes an element $\alpha + \mathfrak{f}$ to the class $(\alpha) + K_\mathfrak{f}^1$ and the surjection is the natural quotient map. Now let $\mathfrak{f} = (p^n)$. Piecing this together as we let n vary, taking the inverse limit of this family of exact sequences, we obtain an exact sequence

$$0 \longrightarrow \mathcal{O}_K^\times \longrightarrow (\mathcal{O}_K \otimes_\mathbb{Z} \mathbb{Z}_p)^\times \longrightarrow \mathrm{Cl}_K(p^\infty) \xrightarrow{\quad\delta\quad} \mathrm{Cl}_K \longrightarrow 0,$$

where here

$$\mathrm{Cl}_K(p^\infty) := \varprojlim_n \mathrm{Cl}(K, p^n)$$

is defined to be the inverse limit. (Note here that although taking inverse limits is not in general a right-exact functor, here we have right-exactness of the limit since the final term is constant).

10.3. Explicit Hecke characters on $\mathrm{Cl}_K(p^\infty)$

Let φ be a Hecke character for K of conductor $\mathfrak{f}|(p^\infty)$ and infinity type (q,r), and recall that in Chapter 1.3.2 we associated to φ a locally analytic function $\varphi_{p-\mathrm{fin}}$ on $\mathrm{Cl}_K(p^\infty)$. In this situation, we can give a simple description of $\varphi_{p-\mathrm{fin}}$ in terms of the fundamental polynomials defined above.

First, we describe what $\mathrm{Cl}_K(p^\infty)$ actually looks like. By choosing a set of representatives for the class group, we are choosing a section of the map $\mathrm{Cl}_K(p^\infty) \to \mathrm{Cl}_K$, and thus, going back to the exact sequence above, we can identify $\mathrm{Cl}_K(p^\infty)$ with a disjoint union of h copies of $\mathcal{O}_{K,p}^\times/\mathcal{O}_K^\times$, indexed by our class group representatives. On each of the h components, the character $\varphi_{p-\mathrm{fin}}$ gives a locally polynomial function on $\mathcal{O}_{K,p}^\times$; from the definition, on the ith component this is given by

$$P_i(z) = \varphi(t_i) \sum_{b \in (\mathcal{O}_K/\mathfrak{f})^\times} \varphi_{\mathfrak{f}}(b) P_{b,\mathfrak{f}}^{q,r}(z),$$

where t_i is an idele representing I_i and $P_{b,\mathfrak{f}}^{q,r}$ is the fundamental polynomial considered above. Accordingly, we have an identity

$$\varphi_{p-\mathrm{fin}} = \sum_{i=1}^h P_i \mathbb{1}_i,$$

where $\mathbb{1}_i$ is the indicator function for the ith component of $\mathrm{Cl}_K(p^\infty)$.

10.4. Constructing the p-adic L-function

10.4.1. Construction

Let Φ be a small slope cuspidal Bianchi eigenform with associated (canonical) overconvergent modular symbol $\Psi = (\Psi_1, ..., \Psi_h)$. Define, for each i, a distribution

$$\mu_i := \Psi_i(\{0\} - \{\infty\})\Big|_{\mathcal{O}_{K,p}^\times}.$$

Then we know that, for $\mathfrak{f}|(p^\infty)$ and when b is a unit $\pmod{\mathfrak{f}}$, we have

$$\mu_i\left(P_{b,\mathfrak{f}}^{q,r}(z)\right) = \lambda_{\mathfrak{f}}\varphi(t_i)\varphi(t_{j_i})\varphi_{\mathfrak{f}}(d_b)^{-1}\varphi_{\mathfrak{f}}(d_b/\alpha_i)c_{q,r}^{j_i}(d_b/\alpha_i),$$

where $\lambda_{\mathfrak{f}}$ is the $U_{\mathfrak{f}}$-eigenvalue of Φ. The appearance of terms of form $c_{q,r}^{j_i}(d_b/\alpha_i)$ leads us, in the spirit of equation (6.4), to define a distribution

$$\mu_p := \sum_{i=1}^h \mu_i \mathbb{1}_i$$

on $\mathrm{Cl}_K(p^\infty)$, for $\mathbb{1}_i$ as above.

10.4.2. Interpolation of L-values

Now take a Hecke character φ of infinity type (q,r) – where $0 \leq q,r \leq k$ – and conductor $\mathfrak{f}|(p^\infty)$. We make the additional stipulation that $\mathfrak{f}$ is divisible by all the primes above p. Then we have

$$\begin{aligned}
\mu_p(\varphi_{p-\mathrm{fin}}) = \mu_p\left(\sum_{i=1}^{h} P_i(z)\mathbb{1}_i\right) &= \sum_{i=1}^{h}\sum_{b\in(\mathcal{O}_K/\mathfrak{f})^\times} \varphi(t_i)\varphi_{\mathfrak{f}}(d_b)\mu_i\left(P_{b,\mathfrak{f}}^{q,r}(z)\right) \\
&= \lambda_{\mathfrak{f}}^{-1}\sum_{i=1}^{h}\varphi(t_{j_i})\sum_{b\in(\mathcal{O}_K/\mathfrak{f})^\times}\varphi_{\mathfrak{f}}\left(\frac{d_b}{\alpha_i}\right)c_{q,r}^{j_i}\left(\frac{d_b}{\alpha_i}\right) \\
&= (-1)^{k+q+r}\left[\frac{\varphi(x_{\mathfrak{f}})Dw\tau(\varphi^{-1})}{2\varphi_{\mathfrak{f}}(x_{\mathfrak{f}})\lambda_{\mathfrak{f}}}\right]\Lambda(\Phi,\varphi),
\end{aligned}$$

using equation (6.4). In this equation, recall that $\varphi_{\mathfrak{f}} = \prod_{\mathfrak{p}|\mathfrak{f}}\varphi_{\mathfrak{p}}$, the idele $x_{\mathfrak{f}}$ is as defined in Section 1.2.3, $\lambda_{\mathfrak{f}}$ is the $U_{\mathfrak{f}}$-eigenvalue of Φ, $-D$ is the discriminant of K, w is the size of the unit group of K and $\tau(\varphi^{-1})$ is a Gauss sum as defined in Section 1.3.1. This is the interpolation property that a p-adic L-function should satisfy.

10.4.3. Admissibility

We earlier gave a definition of admissibility for locally analytic distributions on $\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{Z}_p$. We need an appropriate definition for locally analytic distributions on $\mathrm{Cl}_K(p^\infty)$. This is covered in detail in [Loe14], where the condition is defined very similarly. Indeed, one can exhibit a family of norms on the space of locally analytic distributions on $\mathrm{Cl}_K(p^\infty)$, and then say that a distribution is admissible if it satisfies a suitable growth condition with respect to these norms. One such way to do so is as follows: let $f : \mathrm{Cl}_K(p^\infty) \to \mathbb{C}_p$ be any continuous function. To f one can associate a collection of h functions $f_1, ..., f_h : \mathcal{O}_{K,p} \to \mathbb{C}_p$, all supported on $\mathcal{O}_{K,p}^\times$ and invariant under multiplication by $\mathcal{O}_K^\times$ (see Chapter 14.1.2). Then one says f is *locally analytic of radius* (r,s) if each f_i is on $\mathcal{O}_{K,p}$. As in the case of functions on $\mathcal{O}_{K,p}$, there is a natural norm on the space of locally analytic functions of radius (r,s), and the space of locally analytic functions on $\mathrm{Cl}_K(p^\infty)$ is the union of these spaces as $r,s \to 0$. For each pair (r,s), one obtains an operator norm on the space of locally analytic distributions by

$$||\mu||_{r,s} = \sup_f \frac{|\mu(f)|_p}{|f|_{r,s}},$$

where the supremum is taken over all locally analytic functions of radius (r,s). The admissibility conditions are then the same as before.

We see from this definition that if each of the distributions $\Psi_i(\{0\}-\{\infty\})$ are $(h_{\mathfrak{p}})_{\mathfrak{p}|p}$-admissible on $\mathcal{O}_{K,p}$, then the distribution μ_p is also $(h_{\mathfrak{p}})_{\mathfrak{p}|p}$-admissible on $\mathrm{Cl}_K(p^\infty)$. This admissibility condition means that the distribution is uniquely determined by its values on locally polynomial functions of sufficiently small degree, and in particular, that when $h_{\mathfrak{p}} < k+1$, that μ_p is uniquely determined by the interpolation condition above. This uniqueness property is proved in [Loe14] in the case where each $h_{\mathfrak{p}} < 1$, which he assumes merely for simplicity. For a more detailed example of the general situation in the one variable case, see [Col10].

10.4.4. Summary of results

The interpolation result above, combined with the admissibility conditions of previous sections, mean we have now proved:

Theorem 10.4.1. *Let $K/\mathbb{Q}$ be an imaginary quadratic field of class number h and discriminant $-D$, and let p be a rational prime. Let Φ be a cuspidal Bianchi eigenform of weight (k,k) and level $\Omega_1(\mathfrak{n})$, where $(p)|\mathfrak{n}$, with $U_{\mathfrak{p}}$-eigenvalues $a_{\mathfrak{p}}$, where $v(a_{\mathfrak{p}}) < (k+1)/e_{\mathfrak{p}}$ for all $\mathfrak{p}|p$. Then there exists a locally analytic distribution μ_p on $\mathrm{Cl}(K,p^\infty)$ such that for any Hecke character φ of K of conductor $\mathfrak{f}|(p^\infty)$ and infinity type $0 \leq (q,r) \leq (k,k)$, with $\mathfrak{f}$ divisible by each prime above p, we have*

$$\mu_p(\varphi_{p-\mathrm{fin}}) = \left[\frac{(-1)^{k+q+r}2\varphi_{\mathfrak{f}}(x_{\mathfrak{f}})\lambda_{\mathfrak{f}}}{\varphi(x_{\mathfrak{f}})Dw\tau(\varphi^{-1})}\right]^{-1}\Lambda(\Phi,\varphi),$$

for $\varphi_{p-\mathrm{fin}}$ as defined in Chapter 1.3.2. The distribution μ_p is $(h_{\mathfrak{p}})_{\mathfrak{p}|p}$-admissible in the sense of Definitions 8.3.10 and 9.3.1, where $h_{\mathfrak{p}} = v_p(a_{\mathfrak{p}})$, and hence is unique.

We call μ_p the p-adic L-function *of Φ.*

Proof. The eigenform Φ corresponds to a collection of h cusp forms $\mathcal{F}^1, ..., \mathcal{F}^h$ on $\mathcal{H}_3$; associate to each $\mathcal{F}^i$ a classical Bianchi eigensymbol $\phi_{\mathcal{F}^i}$ with coefficients in a p-adic field L, and lift each to its corresponding unique overconvergent Bianchi eigensymbol Ψ_i. Define

$$\mu_i := \Psi_i(\{0\}-\{\infty\})\Big|_{\mathcal{O}_{K,p}^\times},$$

and define a locally analytic distribution μ_p on $\mathrm{Cl}(K,p^\infty)$ by $\mu_p := \sum_{i=1}^h \mu_i \mathbb{1}_i$. Then by the work above, μ_p satisfies the interpolation and admissibility properties. These determine the distribution uniquely; see, for example, [Loe14] for this result in the weight $(0,0)$ case. □

Remark: As an example of where this theorem applies, suppose p splits in K and let E/K be a modular elliptic curve with supersingular reduction at both primes above p. Then to E we can associate a modular symbol which will have slope $1/2$ at each of the primes above p. Accordingly, our construction will give the p-adic L-function of E.

Part III

General Number Fields

(Joint with Daniel Barrera Salazar)

In this section, which contains joint work with Daniel Barrera Salazar (Université de Montréal), we construct p-adic L-functions for small slope automorphic forms over a completely general number field, generalising the results of Part II of this thesis.

Whilst the overall strategy – using overconvergent modular symbols – is morally very similar to that used in Part II, the theory of modular symbols in the general setting does not lend itself to the same level of explicit study. In particular, the methods and techniques used in this section of the text are very different from those used in Part II. In particular, modular symbols over general number fields are elements of higher degree compactly supported cohomology spaces, and accordingly their study is considerably more abstract than over imaginary quadratic fields. As a consequence, the proofs in this section are not constructive.

We start by working classically; in particular, via the Eichler–Shimura isomorphism, we can associate a modular symbol to an automorphic form. Through Dimitrov's theory of automorphic cycles, we link this modular symbol to critical values of the L-function of the automorphic form. After this, we prove a control theorem in the general setting by using BGG resolutions. In the final section, we show how to use automorphic cycles to canonically attach a distribution to an overconvergent modular symbol, and then show that this distribution interpolates the critical L-values of the automorphic form. We define the p-adic L-function of the automorphic form to be this distribution.

The results of this section appear in the paper "P-adic L-functions for GL_2" *(see [BSW16]).*

Chapter 11

Classical Modular Symbols

In this chapter, we discuss how to generalise the theory of classical modular symbols to the setting of completely general number fields. In particular, we introduce a cohomological approach. We give the general form of the Eichler-Shimura isomorphism, which allows us to attach a modular symbol to an automorphic form. We end by describing a method for attaching algebraic and p-adic modular symbols to an automorphic form using the theory of periods.

11.1. Generalising Symb_Γ to number fields

In the work of Pollack and Stevens over $\mathbb{Q}$ in [PS11], and over imaginary quadratic fields in Part II of this thesis, we were able to describe modular symbols as functions on paths between cusps in a natural and explicit manner. When we pass to more general number fields, this is no longer an approach that works. In this section, we discuss an equivalent formulation of the theory in the rational and imaginary quadratic setting that generalises easily to other number fields.

The key is to think of modular symbols as *cohomology classes* rather than as functions on cusps. We have the following result of Ash–Stevens:

Theorem 11.1.1. *Let $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ be a congruence subgroup. There is a Hecke-equivariant isomorphism*

$$\mathrm{Symb}_\Gamma(V_k(\mathbb{C})^*) \cong \mathrm{H}^1_{\mathrm{c}}\left(\Gamma\backslash\mathcal{H}, \mathcal{L}(V_k(\mathbb{C})^*)\right),$$

where $\mathrm{H}^\bullet_{\mathrm{c}}$ denotes the compactly supported cohomology and $\mathcal{L}(V_k(\mathbb{C})^*)$ is the local system on $\Gamma\backslash\mathcal{H}$ corresponding to the Γ-module $V_k(\mathbb{C})^*$.

Proof. See [AS86]. $\square$

Similarly, for an imaginary quadratic field K and a discrete subgroup $\Gamma \subset \mathrm{SL}_2(K)$, we have a Hecke-equivariant isomorphism

$$\mathrm{Symb}_\Gamma(V_{k,k}(\mathbb{C})^*) \cong \mathrm{H}^1_{\mathrm{c}}(\Gamma\backslash\mathcal{H}_3, \mathcal{L}(V_{k,k}(\mathbb{C})^*)).$$

Recall that in the imaginary quadratic setting, we considered modular symbols to live most naturally in the space

$$\mathrm{Symb}_{\Omega_1(\mathfrak{n})}(V_{k,k}(\mathbb{C})^*) := \bigoplus_{i=1}^{h} \mathrm{Symb}_{\Gamma_1^i(\mathfrak{n})}(V_{k,k}(\mathbb{C})^*).$$

There is a cohomological analogue here, too; namely

$$\mathrm{H}^1_{\mathrm{c}}\left(Y_1(\mathfrak{n}), \mathcal{L}(V_{k,k}(\mathbb{C})^*)\right) \cong \bigoplus_{i=1}^{h} \mathrm{H}^1_{\mathrm{c}}\left(\Gamma_1^i(\mathfrak{n})\backslash\mathcal{H}_3, \mathcal{L}(V_{k,k}(\mathbb{C})^*)\right),$$

where $Y_1(\mathfrak{n})$ is the locally symmetric space of level $\Omega_1(\mathfrak{n})$ (see Definition 5.4.6).

Now let F be a number field of degree $d = r_1 + 2r_2$, and recall that we defined $q := r_2 + r_2$ to be the number of infinite places of F (noting that $q = 1$ if and only if F is $\mathbb{Q}$ or imaginary quadratic). Recall Theorem 3.2.2, in which we attached a harmonic differential q-form to an automorphic form over F. This suggests that the 'correct' spaces to consider in the general setting are the cohomology groups $\mathrm{H}^q_{\mathrm{c}}(Y_1(\mathfrak{n}), \mathcal{L}(V))$, where $Y_1(\mathfrak{n})$ is the analogous locally symmetric space of level $\Omega_1(\mathfrak{n})$ and for an appropriate choice of V.

In the rest of this chapter, we will define classical modular symbols over general number fields by making this more precise.

11.2. Set-up and notation

For convenience, we recap some of the major notation required in this section of the thesis. Let F, r_1, r_2 and q be as above. Let $\mathfrak{n} \subset \mathcal{O}_F$ be an ideal, and recall that we defined

$$\Omega_1(\mathfrak{n}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2\left(\widehat{\mathcal{O}}_F\right) : c \in \mathfrak{n}\widehat{\mathcal{O}}_F, d \equiv 1 \pmod{\mathfrak{n}} \right\}.$$

Recall also that we defined $K_\infty^+ := \mathrm{SO}_2(\mathbb{R})^{r_1} \times \mathrm{SU}_2(\mathbb{C})^{r_2}$ and $Z_\infty := Z(\mathrm{GL}_2(\mathbb{C}))$. We define the *locally symmetric space of level* $\Omega_1(\mathfrak{n})$ to be

$$Y_1(\mathfrak{n}) := \mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbb{A}_F)/\Omega_1(\mathfrak{n})K_\infty^+ Z_\infty.$$

This plays the role of the modular curve in the general setting.

For a weight $\lambda = (\mathbf{k}, \mathbf{v}) \in \mathbb{Z}[\Sigma]^2$ and a ring R, define

$$V_{\mathbf{k}}(R) := \bigotimes_{v\in\Sigma} V_{k_v}(R),$$

equipped with a left action of $\mathrm{GL}_2(R)^d$ given on each component by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot P(X,Y) = P(dX+bY, cX+aY).$$

We write $V_\lambda(R)$ for the space $V_{\mathbf{k}}(R)$ equipped with this action twisted by $\det^{\mathbf{v}}$.

11.3. Local systems

We'll need to study the interplay between complex and p-adic coefficients. We give two ways of defining local systems on $Y_1(\mathfrak{n})$.

Definition 11.3.1. For all modules M below, we suppose that the centre of $\mathrm{GL}_2(F)\cap\Omega_1(\mathfrak{n}) \cong \{\varepsilon \in \mathcal{O}_F^\times : \varepsilon \equiv 1 \pmod{\mathfrak{n}}\}$ acts trivially on M. If this were not the case, the following local systems would not be well-defined.

(i) Suppose M is a right $\mathrm{GL}_2(F)$-module. Then define $\mathcal{L}_1(M)$ to be the locally constant sheaf on $Y_1(\mathfrak{n})$ given by the fibres of the projection

$$\mathrm{GL}_2(F)\backslash(\mathrm{GL}_2(\mathbb{A}_F)\times M)/\Omega_1(\mathfrak{n})K_\infty^+Z_\infty \longrightarrow Y_1(\mathfrak{n}),$$

where the action is given by

$$\gamma(g,m)ukz = (\gamma gukz, m|\gamma^{-1}).$$

(ii) Suppose M is a right $\Omega_1(\mathfrak{n})$-module. Then define $\mathcal{L}_2(M)$ to be the locally constant sheaf on $Y_1(\mathfrak{n})$ given by the fibres of the projection

$$\mathrm{GL}_2(F)\backslash(\mathrm{GL}_2(\mathbb{A}_F)\times M)/\Omega_1(\mathfrak{n})K_\infty^+Z_\infty \longrightarrow Y_1(\mathfrak{n}),$$

where the action is given by

$$\gamma(g,m)ukz = (\gamma gukz, m|u).$$

Remarks 11.3.2: (i) Note that if M is a right $\mathrm{GL}_2(F\otimes_{\mathbb{Q}}\mathbb{R})$-module or a right $\mathrm{GL}_2(F\otimes_{\mathbb{Q}}\mathbb{Q}_p)$-module, then M can be given a $\mathrm{GL}_2(F)$-module structure by restriction in the natural way, giving a sheaf $\mathcal{L}_1(M)$ as in (i) above.

(ii) Similarly, for any right $\mathrm{GL}_2(F\otimes_{\mathbb{Q}}\mathbb{Q}_p)$-module, we have an action of $\Omega_1(\mathfrak{n})$ on M via the projection $\mathrm{Pr} : \mathrm{GL}_2(\mathbb{A}_F) \to \mathrm{GL}_2(F\otimes_{\mathbb{Q}}\mathbb{Q}_p)$, and we get a sheaf $\mathcal{L}_2(M)$ as above. In this case, the sheaves $\mathcal{L}_1(M)$ and $\mathcal{L}_2(M)$ are naturally isomorphic via the map

$$(g,m) \longmapsto (g, m|g_p)$$

of local systems, where g_p is the image of g under the map Pr above.

(iii) Note that, for a number field K containing the normal closure of F, the space $V_\lambda(K)^*$ is naturally a $\mathrm{GL}_2(F)$-module via the embedding of $\mathrm{GL}_2(F)$ in $\mathrm{GL}_2(F\otimes_{\mathbb{Q}}\mathbb{R})$, whilst if $L/\mathbb{Q}_p$ is a finite extension containing $\mathrm{inc}_p(K)$, then $V_\lambda(L)^*$ is naturally a $\mathrm{GL}_2(F\otimes_{\mathbb{Q}}\mathbb{Q}_p)$-module. So our above comments apply and we get sheaves attached to $V_\lambda(A)^*$ for suitable A.

It will usually be clear which sheaf we must take. However, when the coefficient system is $V_\lambda(L)^*$ (for a sufficiently large finite extension $L/\mathbb{Q}_p$) we can associate two *different* (though isomorphic) local systems. As we'll later (in Lemma 14.2.1) need to keep track of precisely what this isomorphism does to cohomology elements, throughout the text we'll retain the subscript for clarity.

11.4. Operators on cohomology groups

11.4.1. Hecke operators

Recall $q := r_1 + r_2$. We can define actions of the Hecke operators on the cohomology groups $\mathrm{H}^q_{\mathrm{cusp}}(Y_1(\mathfrak{n}), \mathcal{L}_i(V_\lambda(A)^*))$. This is described fully in [Hid88], Chapter 7, pages 346–347, and [Dim05], Section 1.14, page 518. We give a very brief description of the definition, following Dimitrov.

For each prime ideal $\mathfrak{p}$ of $\mathcal{O}_F$, we have a Hecke operator $T_\mathfrak{p}$ induced by the double coset $[\Omega_1(\mathfrak{n})a_\mathfrak{p}\Omega_1(\mathfrak{n})]$, where $a_\mathfrak{p} \in \mathrm{GL}_2(\mathbb{A}_F)$ is defined by

$$(a_\mathfrak{p})_v = \begin{cases} \left(\begin{smallmatrix} 1 & 0 \\ 0 & \pi_\mathfrak{p} \end{smallmatrix}\right) & : v = \mathfrak{p} \\ \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) & : \text{otherwise.} \end{cases}$$

When $\mathfrak{p}|\mathfrak{n}$ we write $U_\mathfrak{p}$ in place of $T_\mathfrak{p}$ in the usual manner.

11.4.2. Action of the Weyl group

We also have an action of the Weyl group $\{\pm 1\}^{\Sigma(\mathbb{R})}$ on the cohomology, again described by Dimitrov. Note that via strong approximation, there is a decomposition

$$Y_1(\mathfrak{n}) = \bigsqcup_{i=1}^{h} Y_1^i(\mathfrak{n}), \tag{11.1}$$

where

$$\begin{aligned} Y_1^i(\mathfrak{n}) &= \mathrm{GL}_2(F)\backslash \mathrm{GL}_2(F) g_i \Omega_1(\mathfrak{n}) \mathrm{GL}_2^+(F_\infty)/\Omega_1(\mathfrak{n}) K_\infty^+ Z_\infty^+ \\ &= \Gamma_1^i(\mathfrak{n})\backslash \mathcal{H}_F, \end{aligned}$$

where $\Gamma_1^i(\mathfrak{n})$ is as defined in equations (2.11) and $\mathcal{H}_F = \mathcal{H}^{\Sigma(\mathbb{R})} \times \mathcal{H}_3^{\Sigma(\mathbb{C})}$. Now, let $\iota = (\iota_v)_{v\in\Sigma(\mathbb{R})} \in \{\pm 1\}^{\Sigma(\mathbb{R})}$. Then ι acts on $\mathcal{H}_F$ by $\iota \cdot \mathbf{z} = [(\iota_v \cdot z_v)_{v\in\Sigma(\mathbb{R})}, (z_v)_{v\in\Sigma(\mathbb{C})}]$, where for $v \in \Sigma(\mathbb{R})$ we define

$$\iota_v \cdot z_v := \begin{cases} z_v & : \iota_v = 1 \\ -\overline{z_v} & : \iota_v = -1. \end{cases}$$

This action induces an action of $\{\pm 1\}^{\Sigma(\mathbb{R})}$ on $Y_1^i(\mathfrak{n})$ for each i and hence on $Y_1(\mathfrak{n})$. The action of $\{\pm 1\}^{\Sigma(\mathbb{R})}$ on $\mathrm{H}^q_{\mathrm{cusp}}(Y_1(\mathfrak{n}), \mathcal{L}_1(V_\lambda(\mathbb{C})^*))$ is then induced by the map of local systems

$$\iota \cdot (g, P) \longmapsto (\iota \cdot g, P).$$

We write this action on the right by $\phi \mapsto \phi|\iota$. The actions of the Hecke operators and the Weyl group commute.

11.5. The Eichler–Shimura isomorphism

The major step in the construction of a modular symbol attached to an automorphic form is the *Eichler–Shimura isomorphism.* This is a Hecke-equivariant isomorphism between spaces of automorphic forms and the cuspidal cohomology of the associated locally symmetric space.

Definition 11.5.1. Let $X_1(\mathfrak{n})$ denote the Borel-Serre compactification of $Y_1(\mathfrak{n})$, and let $\partial X_1(\mathfrak{n})$ denote its boundary. Then for a sheaf $\mathcal{M}$ on $Y_1(\mathfrak{n})$, the *cuspidal cohomology group* $\mathrm{H}^i_{\mathrm{cusp}}(Y_1(\mathfrak{n}), \mathcal{M})$ is defined to be the kernel of the natural restriction map

$$\mathrm{res}^i : \mathrm{H}^i(Y_1(\mathfrak{n}), \mathcal{M}) \cong \mathrm{H}^i\left(X_1(\mathfrak{n}), \overline{\mathcal{M}}\right) \longrightarrow \mathrm{H}^i\left(\partial X_1(\mathfrak{n}), \overline{\mathcal{M}}\right),$$

where $\overline{\mathcal{M}}$ is a suitable extension of $\mathcal{M}$ and the first isomorphism is induced by the inclusion $Y_1(\mathfrak{n}) \hookrightarrow X_1(\mathfrak{n})$ (which is a homotopy equivalence). See [Sen14] for more details.

The theorem is then:

Theorem 11.5.2 (Eichler–Shimura)**.** *There is a Hecke-equivariant injection*

$$S_\lambda(\Omega_1(\mathfrak{n})) \hookrightarrow \mathrm{H}^q_{\mathrm{cusp}}\left(Y_1(\mathfrak{n}), \mathcal{L}_1(V_\lambda(\mathbb{C})^*)\right),$$

where $S_\lambda(\Omega_1(\mathfrak{n}))$ is the space of cuspidal automorphic forms defined in Definition 3.5.2.

Proof. An explicit recipe is given in [Hid94]. □

Remarks: (i) In fact, one can define *automorphic forms of type* J for a subset $J \subset \Sigma(\mathbb{R})$, and then if we replace the left-hand side with $\oplus_{J\subset\Sigma(\mathbb{R})} S_{\lambda,J}(\Omega_1(\mathfrak{n}))$, this becomes an isomorphism. In general, an automorphic form of type J satisfies a holomorphicity condition at the places in J and an anti-holomorphicity condition at the remaining real places. In the case where $F = \mathbb{Q}$, the case where $J = \Sigma(\mathbb{R})$ defines the usual theory of modular forms, whilst if $J = \varnothing$, we get the theory of anti-holomorphic modular forms. We stay exclusively with the case $J = \Sigma(\mathbb{R})$ for simplicity, but the results should carry over to more general J with only minor modification.

(ii) In this more general case, there is also a natural action of the Weyl group on the direct sum $\oplus_{J\subset\Sigma(\mathbb{R})} S_{\lambda,J}(\Omega_1(\mathfrak{n}))$, and it permutes the factors in a natural way. The isomorphism is also equivariant with respect to this action.

(iii) The cuspidal cohomology injects into the compactly supported cohomology. We will use this in the sequel to define modular symbols attached to automorphic forms.

Under the decomposition of equation (11.1), we see that for sufficiently large extensions A of $\mathbb{Q}$ or $\mathbb{Q}_p$, there is a (non-canonical) decomposition

$$\mathrm{H}^q_{\mathrm{cusp}}\left(Y_1(\mathfrak{n}), \mathcal{L}_1(V_\lambda(A)^*)\right) \cong \bigoplus_{i=1}^{h} \mathrm{H}^q_{\mathrm{cusp}}\left(Y_1^i(\mathfrak{n}), \mathcal{L}_1(V_\lambda(A)^*)\right). \tag{11.2}$$

11.6. Modular symbols

Let $L/\mathbb{Q}_p$ be a finite extension.

Definition 11.6.1. The space of *modular symbols of weight λ and level $\Omega_1(\mathfrak{n})$ with values in L* is the compactly supported cohomology space $\mathrm{H}^q_{\mathrm{c}}(Y_1(\mathfrak{n}), \mathcal{L}_2(V_\lambda(L)^*))$.

Let $\Phi \in S_\lambda(\Omega_1(\mathfrak{n}))$ be a Hecke eigenform. Then via Theorem 11.5.2 we can attach to Φ an element

$$\phi_{\mathbb{C}} \in \mathrm{H}^q_{\mathrm{cusp}}\left(Y_1(\mathfrak{n}), \mathcal{L}_1(V_\lambda(\mathbb{C})^*)\right).$$

We want to pass from a cohomology class with complex coefficients to one with p-adic coefficients. To do this, we use the theory of periods described earlier in Section 4.4.

Definition 11.6.2. Let ε be a character of the Weyl group $\{\pm 1\}^{\Sigma(\mathbb{R})}$. Then define

$$\mathrm{H}^q_{\mathrm{cusp}}\left(Y_1(\mathfrak{n}), \mathcal{L}_1(V_\lambda(\mathbb{C})^*)\right)[\varepsilon] \subset \mathrm{H}^q_{\mathrm{cusp}}\left(Y_1(\mathfrak{n}), \mathcal{L}_1(V_\lambda(\mathbb{C})^*)\right)$$

to be the subspace on which $\{\pm 1\}^{\Sigma(\mathbb{R})}$ acts by ε.

Proposition 11.6.3. *Let K be a number field containing the normal closure of F and the Hecke eigenvalues of Φ, and let ε be as above. Let Ω_Φ^ε be the period appearing in Theorem*

4.4.1. Define

$$\phi_{\mathbb{C}}^{\varepsilon} := 2^{-r_1} \sum_{\iota \in \{\pm 1\}^{\Sigma(\mathbb{R})}} \varepsilon(\iota)\phi_{\mathbb{C}}|\iota.$$

Then $\phi_{\mathbb{C}}^{\varepsilon} \in \mathrm{H}^q_{\mathrm{cusp}}(Y_1(\mathfrak{n}), \mathcal{L}_1(V_\lambda(\mathbb{C})^*))[\varepsilon]$*, and*

$$\phi_K^{\varepsilon} := \phi_{\mathbb{C}}^{\varepsilon}/\Omega_{\Phi}^{\varepsilon} \in \mathrm{H}^q_{\mathrm{cusp}}\left(Y_1(\mathfrak{n}), \mathcal{L}_1(V_\lambda(K)^*)\right)[\varepsilon].$$

Proof. See [Hid94], Chapter 8. □

Definition 11.6.4. Define

$$\theta_K := \sum_{\varepsilon} \phi_K^{\varepsilon} \in \mathrm{H}^q_{\mathrm{cusp}}(Y_1(\mathfrak{n}), \mathcal{L}_1(V_\lambda(K)^*)),$$

where the sum is over all possible characters of the Weyl group $\{\pm 1\}^{\Sigma(\mathbb{R})}$.

Now let $L/\mathbb{Q}_p$ be a finite extension containing $\mathrm{inc}_p(K)$ (for our fixed embedding $\mathrm{inc}_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$). Then inc_p induces an inclusion

$$\mathrm{H}^q_{\mathrm{cusp}}\left(Y_1(\mathfrak{n}), \mathcal{L}_1(V_\lambda(K)^*)\right) \longhookrightarrow \mathrm{H}^q_{\mathrm{cusp}}\left(Y_1(\mathfrak{n}), \mathcal{L}_1(V_\lambda(L)^*)\right) \cong \mathrm{H}^q_{\mathrm{cusp}}(Y_1(\mathfrak{n}), \mathcal{L}_2(V_\lambda(L)^*)). \quad (11.3)$$

Finally, there is a canonical inclusion

$$\mathrm{H}^q_{\mathrm{cusp}}\left(Y_1(\mathfrak{n}), \mathcal{L}_2(V_\lambda(L)^*)\right) \longhookrightarrow \mathrm{H}^q_{\mathrm{c}}\left(Y_1(\mathfrak{n}), \mathcal{L}_2(V_\lambda(L)^*)\right). \quad (11.4)$$

Definition 11.6.5. Let Φ be an eigenform of weight λ and level $\Omega_1(\mathfrak{n})$, and let L be as above. The *modular symbol attached to* Φ *with values in* L is the image

$$\theta_L \in \mathrm{H}^q_{\mathrm{c}}\left(Y_1(\mathfrak{n}), \mathcal{L}_2(V_\lambda(L)^*)\right)$$

of the symbol θ_K under the inclusion of equations (11.3) and (11.4).

Remark: To give some brief motivation for this definition, we'll later define an evaluation map $\mathrm{H}^q_{\mathrm{c}}(Y_1(\mathfrak{n}), \mathcal{L}_2(V_\lambda(L)^*)) \to L$ corresponding to a critical character φ such that if φ corresponds to the character ε_1 of $\{\pm 1\}^{\Sigma(\mathbb{R})}$, the image of $\phi_L^{\varepsilon_2}$ gives the algebraic part of the critical L-value at φ if $\varepsilon_1 = \varepsilon_2$ and vanishes otherwise. So by taking the sum, we allow ourselves to see *all* critical values.

Chapter 12

Automorphic Cycles and L-values

In this chapter, we use Dimitrov's theory of automorphic cycles to give a link between the modular symbol attached to an automorphic form Φ and critical values of its L-function. In the process, we give an integral formula for such L-values. We start by working exclusively over $\mathbb{C}$, before showing that we can actually work over a sufficiently large finite extension of $\mathbb{Q}$ or $\mathbb{Q}_p$. In particular, we link the p-adic modular symbol attached to Φ with the algebraic parts of its critical L-values.

12.1. Automorphic cycles, evaluation maps and L-values

Let Φ be a cuspidal automorphic form over F. In this section, we give a connection between the cohomology class $\phi_{\mathbb{C}}$ associated to Φ via the Eichler-Shimura isomorphism and critical values of its L-function. We do so via *automorphic cycles*. The cycles we define here are a generalisation of the objects Dimitrov uses in [Dim13] in the totally real case. As a consequence of this section, we also get an integral formula for the L-function of Φ, generalising the results of [Hid94], Section 7, where such a formula is obtained for Hecke characters with trivial conductor.

12.1.1. Automorphic cycles

Let $\mathfrak{f}$ be an integral ideal of F. We begin with some essential definitions:

Definition 12.1.1. Define $F_\infty^+ \subset (F \otimes_{\mathbb{Q}} \mathbb{R})^\times$ to be the connected component of the identity in the subgroup of infinite ideles, and let F_∞^1 be the subset defined by

$$F_\infty^1 := \{x \in F_\infty^+ : |x_v|_v = 1 \text{ for all } v|\infty.\}.$$

Definition 12.1.2. (i) Define an open compact subgroup of $\mathbb{A}_{F,f}^\times$ by

$$U(\mathfrak{f}) := \{x \in \widehat{\mathcal{O}_F}^\times : x \equiv 1 \pmod{\mathfrak{f}}\},$$

and the global equivalent

$$E(\mathfrak{f}) := \{x \in \mathcal{O}_{F,+}^{\times} : x \equiv 1 \pmod{\mathfrak{f}}\} = U(\mathfrak{f}) \cap F^{\times}.$$

(ii) We define the *automorphic cycle of level* $\mathfrak{f}$ to be

$$X_{\mathfrak{f}} := F^{\times}\backslash \mathbb{A}_F^{\times}/U(\mathfrak{f})F_{\infty}^{1}.$$

Proposition 12.1.3. *There is a natural decomposition*

$$X_{\mathfrak{f}} = \bigsqcup_{\mathbf{y}\in \mathrm{Cl}_F^{+}(\mathfrak{f})} X_{\mathbf{y}}.$$

There is a natural embedding

$$\eta_{\mathfrak{f}} : X_{\mathfrak{f}} \hookrightarrow Y_1(\mathfrak{n})$$

induced by

$$\begin{aligned} \eta : \mathbb{A}_F^{\times} &\hookrightarrow \mathrm{GL}_2(\mathbb{A}_F) \\ x &\longmapsto \begin{pmatrix} x & (x\mathfrak{f}^{-1})_{v|\mathfrak{f}} \\ 0 & 1 \end{pmatrix}, \end{aligned}$$

where $(\mathfrak{f}^{-1})_{v|\mathfrak{f}}$ is the idele defined in Definition 1.3.2. This map is shown to be well-defined in Proposition 12.1.4 below.

Recall that we have a decomposition $Y_1(\mathfrak{n}) = \bigsqcup_{i=1}^{h} Y_1^i(\mathfrak{n})$, where $Y_1^i(\mathfrak{n})$ is as defined in equation (11.1). In particular, $Y_1^i(\mathfrak{n})$ can be described as $\{[g] \in Y_1(\mathfrak{n}) : \det(g) \text{ represents } i \text{ in } \mathrm{Cl}_F^{+}\}$.

Proposition 12.1.4. *The map* η *induces a well-defined map*

$$\eta_{\mathfrak{f}} : X_{\mathfrak{f}} \longrightarrow Y_1(\mathfrak{n}).$$

Moreover, the restriction of $\eta_{\mathfrak{f}}$ *to* $X_{\mathbf{y}}$ *has image in* $Y_1^{i_{\mathbf{y}}}(\mathfrak{n})$*, where* $i_{\mathbf{y}}$ *denotes the element of the narrow class group given by the image of* $\mathbf{y}$ *under the natural projection* $\mathrm{Cl}_F^{+}(\mathfrak{f}) \to \mathrm{Cl}_F^{+}$*. Finally,* $\eta_{\mathfrak{f}}$ *is independent of the choice of uniformisers* π_v *for* $v|\mathfrak{f}$*.*

Proof. Suppose γxur is a different representative of $[x] \in X_{\mathfrak{f}}$. Then

$$\begin{aligned}
[\eta_{\mathfrak{f}}(\gamma xur)] &= \left[\begin{pmatrix} \gamma xur & (\gamma xur\mathfrak{f}^{-1})_{v|\mathfrak{f}} \\ 0 & 1 \end{pmatrix}\right] \\
&= \left[\begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x & (x\mathfrak{f}^{-1})_{v|\mathfrak{f}} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} u & ((u-1)\mathfrak{f}^{-1})_{v|\mathfrak{f}} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} r & 0 \\ 0 & 1 \end{pmatrix}\right] \qquad (12.1) \\
&= [\eta_{\mathfrak{f}}(x)] \in Y_1(\mathfrak{n}),
\end{aligned}$$

showing that the induced map is well-defined. To see that the restriction to $X_{\mathbf{y}}$ lands in $Y_1^{i_{\mathbf{y}}}(\mathfrak{n})$, note that $\det(\eta_{\mathfrak{f}}(x)) = x$, so that if x represents $\mathbf{y} \in \mathrm{Cl}_F^+(\mathfrak{f})$, we see that $\eta_{\mathfrak{f}}(x)$ represents $i_{\mathbf{y}} \in \mathrm{Cl}_F^+$, and in particular, $\eta_{\mathfrak{f}}$ induces a map

$$\{x \in \mathbb{A}_F^\times : [x] = \mathbf{y} \in \mathrm{Cl}_F^+(\mathfrak{f})\} \longrightarrow Y_1^{i_{\mathbf{y}}}(\mathfrak{n}),$$

which then descends as claimed.

To see that $\eta_{\mathfrak{f}}$ is independent of the choice of uniformisers, suppose that $\{\varpi_v : v|\mathfrak{f}\}$ is a different collection of uniformisers at the places dividing $\mathfrak{f}$, and write $\eta'_{\mathfrak{f}}$ for the corresponding map. Then

$$\eta'_{\mathfrak{f}}(x)\eta_{\mathfrak{f}}(x)^{-1} = \begin{pmatrix} 1 & (x(\varpi_v^{-v_v(\mathfrak{f})} - \pi_v^{-v_v(\mathfrak{f})}))_{v|\mathfrak{f}} \\ 0 & 1 \end{pmatrix} \in \Omega_1(\mathfrak{n}),$$

so that $\eta_{\mathfrak{f}}(x)$ and $\eta'_{\mathfrak{f}}(x)$ determine the same element of $Y_1(\mathfrak{n})$. □

Remark 12.1.5: Later, we will choose $a_{\mathbf{y}}$ as follows. Choose $\{a_i\}$ to be representatives of Cl_F^+ as above, with $(a_i)_{\mathfrak{f}} = (a_i)_\infty = 1$. Now for each $[j] \in (\mathcal{O}_F/\mathfrak{f})^\times$, choose $\alpha_j \in \mathcal{O}_{F,+}^\times$ such that $\alpha_j \equiv j \pmod{\mathfrak{f}}$, and define an idele b_j by

$$(b_j)_v = \begin{cases} \alpha_j^{-1} & : v \nmid \mathfrak{f}\infty, \\ 1 & : v|\mathfrak{f}\infty. \end{cases}$$

Then the set

$$\{a_{ij} := a_i b_j : i \in \mathrm{Cl}_F^+, j \in (\mathfrak{f}^{-1}/\mathcal{O}_F)^\times\}$$

is a multiset of representatives of $\mathrm{Cl}_F^+(\mathfrak{f})$, where each class is represented $\#\mathrm{Im}(\mathcal{O}_{F,+}^\times \to (\mathcal{O}/\mathfrak{f})^\times)$ times, via the exact sequence

$$\mathcal{O}_{F,+}^\times \longrightarrow (\mathcal{O}_F/\mathfrak{f})^\times \longrightarrow \mathrm{Cl}_F^+(\mathfrak{f}) \longrightarrow \mathrm{Cl}_F^+ \longrightarrow 0.$$

If a_{ij} represents the same element of $\mathrm{Cl}_F^+(\mathfrak{f})$ as $a_{\mathbf{y}}$, then we denote

$$X_{ij} := F^\times \backslash F^\times a_{ij} U(\mathfrak{f}) F_\infty^+ / F_\infty^1 = X_{\mathbf{y}}$$

and write Δ_{ij} for the corresponding map. The benefit of this approach is that whilst we do count representatives multiple times, in our later calculations we will be able to write down Gauss sums more effectively, as we can isolate the components coming from $(\mathcal{O}_F/\mathfrak{f})^\times$, and means we can use general theory of Gauss sums as developed by Deligne. The authors apologise for the slightly cumbersome work of carrying around both sets of notation; however, the interplay between them should be apparent, and from now on we will use whichever of the two approaches suits best in particular situations. This will typically be a_{ij} in situations where we develop general theory for individual components (so as to do this in the greatest generality), then using $a_{\mathbf{y}}$ when we want to talk about these objects as a whole (indexed by $\mathrm{Cl}_F^+(\mathfrak{f})$).

12.1.2. Evaluation maps

We now use these automorphic cycles to define *evaluation maps*

$$\mathrm{Ev} : \mathrm{H}_c^q\left(Y_1(\mathfrak{n}), \mathcal{L}_1(V_\lambda(\mathbb{C})^*)\right) \longrightarrow \mathbb{C}.$$

This will be done in several stages.

Pulling back to $X_\mathfrak{f}$

First, we pullback under the inclusion $\eta_\mathfrak{f} : X_\mathfrak{f} \hookrightarrow Y_1(\mathfrak{n})$. The corresponding sheaf $\mathcal{L}_{\mathfrak{f},1}(V_\lambda(\mathbb{C})^*) := \eta_\mathfrak{f}^* \mathcal{L}_1(V_\lambda(\mathbb{C})^*)$ can be seen, via equation (12.1), to be given by the sections of the natural map

$$F^\times \backslash (\mathbb{A}_F^\times \times V_\lambda(\mathbb{C})^*)/U(\mathfrak{f}) F_\infty^1 \longrightarrow X_\mathfrak{f},$$

where the action is given by

$$f(x,P)ur = \left(fxur, P \middle| \begin{pmatrix} f^{-1} & 0 \\ 0 & 1 \end{pmatrix} \right).$$

Passing to individual components

We can explicitly write

$$X_{\mathbf{y}} := F^\times \backslash F^\times a_{\mathbf{y}} U(\mathfrak{f}) F_\infty^+ / F_\infty^1,$$

for $\{a_{\mathbf{y}} : \mathbf{y} \in \mathrm{Cl}_F^+(\mathfrak{f})\}$ a (henceforth fixed) set of class group representatives. Note here that there is an isomorphism

$$\begin{aligned} E(\mathfrak{f})F_\infty^1\backslash F_\infty^+ &\xrightarrow{\sim} X_{\mathbf{y}}, \\ r &\longmapsto a_{\mathbf{y}}r. \end{aligned} \tag{12.2}$$

Pulling back under this isomorphism composed with the inclusion $X_{\mathbf{y}} \subset X_{\mathfrak{f}}$, we see that the corresponding sheaf $\mathcal{L}_{\mathfrak{f},\mathbf{y},1} := \tau_{a_{\mathbf{y}}}^*\mathcal{L}_{\mathfrak{f},1}(V_\lambda(\mathbb{C})^*)$ is given by the sections of

$$E(\mathfrak{f})F_\infty^1\backslash(F_\infty^+ \times V_\lambda(\mathbb{C})^*) \longrightarrow E(\mathfrak{f})F_\infty^1\backslash F_\infty^+,$$

where now the action is by

$$es(r,P) = \left(esr, P\middle|\begin{pmatrix} e^{-1} & 0 \\ 0 & 1\end{pmatrix}\right).$$

Evaluating

Let $\mathbf{j} \in \mathbb{Z}[\Sigma]$ be such that there is a Hecke character φ of conductor $\mathfrak{f}$ and infinity type $\mathbf{j}+\mathbf{v}$. Note that in this case, for all $e \in E(\mathfrak{f})$, we have $e^{\mathbf{j}+\mathbf{v}} = 1$; indeed, $e^{\mathbf{j}+\mathbf{v}} = \varphi_\infty(e) = \varphi_f(e)^{-1} = 1$, since $e \equiv 1 \pmod{\mathfrak{f}}$. Now let $\rho_{\mathbf{j}}$ denote the map

$$\rho_{\mathbf{j}} : V_\lambda(\mathbb{C})^* \longrightarrow \mathbb{C}$$

given by evaluating at the polynomial $\mathbf{X}^{\mathbf{k}-\mathbf{j}}\mathbf{Y}^{\mathbf{j}}$. Then $\rho_{\mathbf{j}}$ induces a map $(\rho_{\mathbf{j}})_*$ of local systems on $E(\mathfrak{f})F_\infty^1\backslash F_\infty^+$, as

$$P\middle|\begin{pmatrix} e^{-1} & 0 \\ 0 & 1\end{pmatrix}(\mathbf{X}^{\mathbf{k}-\mathbf{j}}\mathbf{Y}^{\mathbf{j}}) = (e^{\mathbf{j}+\mathbf{v}})^{-1}P(\mathbf{X}^{\mathbf{k}-\mathbf{j}}\mathbf{Y}^{\mathbf{j}}) = P(\mathbf{X}^{\mathbf{k}-\mathbf{j}}\mathbf{Y}^{\mathbf{j}}).$$

We see that the sheaf $(\rho_{\mathbf{j}})_*\mathcal{L}_{\mathfrak{f},\mathbf{y},1}(V_\lambda(\mathbb{C})^*)$ is the constant sheaf attached to $\mathbb{C}$ over $E(\mathfrak{f})F_\infty^1\backslash F_\infty^+$. But note that this space is a connected orientable real manifold of dimension q, and hence that there is an isomorphism

$$\mathrm{H}_{\mathrm{c}}^q\left(E(\mathfrak{f})F_\infty^1\backslash F_\infty^+, \mathbb{C}\right) \cong \mathbb{C},$$

given by integration over $E(\mathfrak{f})F_\infty^1\backslash F_\infty^+$.

Definition 12.1.6. Define

$$\mathrm{Ev}_{\mathfrak{f},\mathbf{j},1}^{a_{\mathbf{y}}} : \mathrm{H}_{\mathrm{c}}^q(Y_1(\mathfrak{n}), \mathcal{L}_1(V_\lambda(\mathbb{C})^*)) \longrightarrow \mathbb{C}$$

to be the composition of the maps

$$\mathrm{H}^q_c(Y_1(\mathfrak{n}), \mathcal{L}_1(V_\lambda(\mathbb{C})^*)) \xrightarrow{\eta_\mathfrak{f}^*} \mathrm{H}^q_c(X_\mathfrak{f}, \mathcal{L}_{\mathfrak{f},1}(V_\lambda(\mathbb{C})^*)) \xrightarrow{\tau^*_{a_\mathbf{y}}} \cdots$$
$$\mathrm{H}^q_c(E(\mathfrak{f})F^1_\infty \backslash F^+_\infty, \mathcal{L}_{\mathfrak{f},\mathbf{y},1}(V_\lambda(\mathbb{C})^*)) \xrightarrow{(\rho_\mathbf{j})_*} \mathrm{H}^q_c(E(\mathfrak{f})F^1_\infty \backslash F^+_\infty, \mathbb{C}) \cong \mathbb{C}.$$

Remarks: (i) Note that this definition is not restricted to polynomials with coefficients in $\mathbb{C}$. Indeed, the evaluation maps are well-defined for cohomology with coefficients in a number field or an extension of $\mathbb{Q}_p$. We will distinguish between the various cases by using a subscript on the cohomology class (for example, $\phi_\mathbb{C}$ is a complex modular symbol).

(ii) The subscript 1 in $\mathrm{Ev}^{a_\mathbf{y}}_{\mathfrak{f},\mathbf{j},1}$ dictates that this is an evaluation map from the cohomology with coefficients in $\mathcal{L}_1(V_\lambda(\mathbb{C})^*)$. Later, we'll define an evaluation map $\mathrm{Ev}^{a_\mathbf{y}}_{\mathfrak{f},\mathbf{j},2}$.

12.1.3. An explicit description of $\phi_\mathbb{C}$

We now give an explicit description of the cohomology class $\phi_\mathbb{C}$ attached to an automorphic form Φ. We do this by utilising the isomorphism between Betti and de Rham cohomology at the level of complex coefficients (see [Del79], Section 0.4), which allows us to describe this class as a differential, as in [Hid94].

Let $\delta_{ij} := \tau^*_{a_{ij}}\eta^*_\mathfrak{f}\phi_\mathbb{C}$. Then we can write

$$\delta_{ij}(z) = \sum_{0\leq\mathbf{j}\leq\mathbf{k}} \delta^\mathbf{j}_{ij}(z)\mathcal{X}^{\mathbf{k}-\mathbf{j}}\mathcal{Y}^\mathbf{j}$$

(as elements of the de Rham cohomology), where $\delta^\mathbf{j}_{ij} \in \mathrm{H}^q_c\left(E(\mathfrak{f})F^1_\infty\backslash F^+_\infty, \mathbb{C}\right)$. Moreover, we see that

$$\int_{E(\mathfrak{f})F^1_\infty\backslash F^+_\infty} \delta^\mathbf{j}_{ij} = \mathrm{Ev}^{a_{ij}}_{\mathfrak{f},\mathbf{j},1}(\phi_\mathbb{C}).$$

Finally, before giving $\delta^\mathbf{j}_{ij}$ explicitly, we comment on the structure of $E(\mathfrak{f})F^1_\infty\backslash F^+_\infty$. We can parametrise this space as the quotient of $\mathbb{R}^q_{>0}$ by units, with one copy of $\mathbb{R}_{>0}$ coming from each real embedding and one from each pair of complex embeddings. This is then isomorphic to $\mathbb{R}_{>0} \times (S^1)^{q-1}$. The reader should think of this as being an analogue of the path $\{iy \in \mathcal{H} : y \in \mathbb{R}_{>0}\}$ (as seen in the rational case when evaluating modular symbols at $\{0\} - \{\infty\}$) in the general setting.

Definition 12.1.7. (i) We parametrise $E(\mathfrak{f})F^1_\infty\backslash F^+_\infty$ as $E(\mathfrak{f})\backslash \left\{\mathbf{y} = (y_v)_{v\in\Sigma(\mathbb{R})\sqcup\Sigma(\mathbb{C})} : y_v \in \mathbb{R}_{>0}\right\}$. The use of $\mathbf{y}$ to mean this rather than a class group representative will be clear from context.

(ii) If $\mathbf{j} \in \mathbb{Z}[\Sigma]$, then we define $\mathbf{j}(\Sigma(\mathbb{R})) := \sum_{v\in\Sigma(\mathbb{R})} j_v \in \mathbb{Z}$ (and similarly for $\mathbf{j}(\Sigma(\mathbb{C}))$).

Remark: We will use $\mathbf{y}$ interchangeably to mean an element of F_∞^+ or $F_\infty^1\backslash F_\infty^+$, in the style of Hida. This is for convenience purposes, since the Fourier expansion takes input from the former, whilst the differential has values on the latter. There is, of course, a canonical quotient map between the two, which corresponds to taking norms at complex places. We also use $\mathbf{y}$ to mean a representative of $E(\mathfrak{f})F_\infty^1\backslash F_\infty^+$, that is, an element of $F_\infty^1\backslash F_\infty^+$ representing a class of this modulo $E(\mathfrak{f})$. The reader is urged not to get hooked up on the details of this notation!

Proposition 12.1.8. *We can explicitly describe $\delta_{ij}^{\mathbf{j}}$ as follows:*

$$\delta_{ij}^{\mathbf{j}}(\mathbf{y}) = c_1 \mathbf{y}^{\mathbf{j}+\mathbf{v}} \Phi_{\mathbf{n}} \begin{pmatrix} a_{ij}\mathbf{y} & (\mathfrak{f}^{-1})_{v|\mathfrak{f}} \\ 0 & 1 \end{pmatrix} \bigwedge_{v\in\Sigma(\mathbb{C})} |y_v|_{\mathbb{C}}^{-1} d|y_v|_{\mathbb{C}} \bigwedge_{v\in\Sigma(\mathbb{R})} y_v^{-1} dy_v,$$

where:

- $c_1 := 2^{r_2} \prod_{v\in\Sigma(\mathbb{C})} (-1)^{k_v+j_{vc}+1} \begin{pmatrix} k_v^* \\ n_v \end{pmatrix} \prod_{v\in\Sigma(\mathbb{R})} (-1)^{k_v+j_v} i^{j_v+1}$, *for* $[\cdot]$ *as in Definition 1.2.5,*
- $\mathbf{y} \in F_\infty^+$ *is considered as an idele by setting* $y_v = 1$ *for all finite places* v*,*
- $\mathbf{n} := \sum_{v\in\mathbb{C}} (k_v + j_v - j_{vc} + 1)v \in \mathbb{Z}[\Sigma(\mathbb{C})]$*, and* $\Phi^u(g) := \sum_{0\leq \mathbf{r}\leq \mathbf{k}^*} \Phi_{\mathbf{r}}^u(g)\mathbf{X}^{\mathbf{k}^*-\mathbf{r}}\mathbf{Y}^{\mathbf{r}}$.

Proof. This is proved in a similar fashion to the analogous result in the Bianchi case (see Chapter 6.2.2). Most of this more general formulation is proved in [Hid94], Section 2.5, though with some notable differences, which we will detail here. One change is purely cosmetic; we have rescaled by $\mathbf{y}^{\mathbf{k}/2+\mathbf{v}}$ using the automorphy condition (since Hida evaluates at the matrix $y_\infty^{-1/2}\left(\begin{smallmatrix} a_{ij}\mathbf{y} & 0 \\ 0 & 1\end{smallmatrix}\right)$).

A more important change is that Hida's results have no dependence on j. To see where this comes in, note that we are restricting Φ to elements of form $\left(\begin{smallmatrix} \mathbf{y} & \mathbf{x} \\ 0 & 1\end{smallmatrix}\right)$ where $\mathbf{x} = (a_{ij}\mathfrak{f}^{-1})_{v|\mathfrak{f}}$. But by definition $(a_{ij})_v = 1$ for all $v|\mathfrak{f}$. Thus $(a_{ij}\mathfrak{f}^{-1})_{v|\mathfrak{f}} = (\mathfrak{f}^{-1})_{v|\mathfrak{f}}$.

The rest of the proof follows exactly as in [Hid94]. Note that he never explicitly uses the dual module $V_\lambda(\mathbb{C})^*$, instead working implicitly with a particular basis of $V_\lambda(\mathbb{C})$ that corresponds identically to the basis of $V_\lambda(\mathbb{C})^*$ we use under the canonical isomorphism between the two as $\mathrm{SU}_2(\mathbb{C})$-modules (see Lemma 5.4.9). □

12.1.4. An integral formula for the L-function

Let φ be a Hecke character of conductor $\mathfrak{f}$ and infinity type $\mathbf{j}+\mathbf{v}$ for some $0 \leq \mathbf{j} \leq \mathbf{k}$. We now look at the image of $\phi_{\mathbb{C}}$ under the evaluation maps, obtaining an integral formula for the L-function at φ. As this calculation is long and messy, for clarity of writing, we have split the work into subsections.

Notation

Let $s \in \mathbb{C}$ be an auxiliary variable, and consider the integral

$$C_{ij}^{\mathbf{j}}(s) := \int_{E(\mathfrak{f})F_\infty^1 \backslash F_\infty^+} \delta_{ij}^{\mathbf{j}}(\mathbf{y})|\mathbf{y}|_{\mathbb{A}_F}^s,$$

where $\mathbf{y}$ denotes an element of F_∞^+ (and can be considered as an idele by setting all finite components equal to 1). Note here that

$$C_{ij}^{\mathbf{j}}(0) = \mathrm{Ev}_{\mathfrak{f},\mathbf{j},1}^{a_{ij}}(\phi_{\mathbb{C}}). \tag{12.3}$$

Substituting known expressions

Now we substitute the explicit value of $\delta_{ij}^{\mathbf{j}}(\mathbf{y})$ into $C_{ij}^{\mathbf{j}}(s)$. This gives

$$C_{ij}^{\mathbf{j}}(s) = c_1 \int_{E(\mathfrak{f})F_\infty^1 \backslash F_\infty^+} \mathbf{y}^{\mathbf{j}+\mathbf{v}} \Phi_{\mathbf{n}} \begin{pmatrix} a_{ij}\mathbf{y} & (\mathfrak{f}^{-1})_{v|\mathfrak{f}} \\ 0 & 1 \end{pmatrix} |\mathbf{y}|_{\mathbb{A}_F}^s d^\times|\mathbf{y}|,$$

where here we've written

$$d^\times|\mathbf{y}| = \bigwedge_{v\in\Sigma(\mathbb{C})} |y_v|_{\mathbb{C}}^{-1} d|y_v|_{\mathbb{C}} \bigwedge_{v\in\Sigma(\mathbb{R})} y_v^{-1} dy_v.$$

We can use the Fourier expansion described in Theorem 4.2.2; since the ideal corresponding to $\zeta a_{ij}\mathbf{y}\mathfrak{D}$ is just $\zeta a_{ij}\mathfrak{D}$, this yields

$$\begin{aligned} C_{ij}^{\mathbf{j}}(s) = c_1 \int_{E(\mathfrak{f})F_\infty^1 \backslash F_\infty^+} \mathbf{y}^{\mathbf{j}+\mathbf{v}} |a_{ij}\mathbf{y}|_{\mathbb{A}_F} \sum_{\zeta \in F_+^\times} c(\zeta a_{ij}\mathfrak{D}, \Phi) e_F(\zeta(\mathfrak{f}^{-1})_{v|\mathfrak{f}}) \\ \times \left[\prod_{v\in\Sigma(\mathbb{C})} i^{j_v - j_{vc}} \begin{pmatrix} k_v^* \\ n_v \end{pmatrix} (\zeta y_v)^{j_{vc}-j_v} |\zeta y_v|_{\mathbb{C}}^{j_v - j_{vc} - v_c - v_{cv}} K_{j_v - j_{vc}}(4\pi|\zeta y_v|_{\mathbb{C}}) \right] \\ \times \left[\prod_{v\in\Sigma(\mathbb{R})} (|\zeta|_{\mathbb{R}} y_v)^{-v_v} e^{-2\pi|\zeta|_{\mathbb{R}} y_v} \right] |\mathbf{y}|_{\mathbb{A}_F}^s d^\times|\mathbf{y}|. \end{aligned}$$

For simplicity, let $\mathbf{j}^\# := (\mathbf{j} - c\mathbf{j})/2$. Grouping together similar terms and rearranging, the above expression simplifies to

$$\begin{aligned} C_{ij}^{\mathbf{j}}(s) = c_1 |a_{ij}|_{\mathbb{A}_F} \prod_{v\in\Sigma(\mathbb{C})} i^{j_v - j_{vc}} \begin{pmatrix} k_v^* \\ n_v \end{pmatrix} \\ \times \int_{E(\mathfrak{f})F_\infty^1 \backslash F_\infty^+} \sum_{\zeta \in F_+^\times} c(\zeta a_{ij}\mathfrak{D}, \Phi) e_F(\zeta(\mathfrak{f}^{-1})_{v|\mathfrak{f}}) \left[\prod_{v\in\Sigma(\mathbb{C})} |y_v|_{\mathbb{C}}^{2+j_v+j_{vc}+2s} K_{j_v - j_{vc}}(4\pi|\zeta y_v|_{\mathbb{C}}) \right] \\ \times \left[\prod_{v\in\Sigma(\mathbb{R})} (y_v)^{1+j_v+s} e^{-2\pi|\zeta|_{\mathbb{R}} y_v} \right] \zeta^{\mathbf{j}^\#} |\zeta|_\infty^{-\mathbf{v}} d^\times|\mathbf{y}|, \end{aligned}$$

where we write

$$|\zeta|_\infty^{-\mathbf{v}} = \prod_{v\in\Sigma} |\zeta|_{F_v}^{-v_v}.$$

Rearranging sums and integrals

For Re(s) >> 0, we have absolute convergence of the sum; hence we can exchange the order of the sum and integral. Note that in this case, we have

$$\int_{E(\mathfrak{f})F_\infty^1\backslash F_\infty^+} \sum_{\zeta\in F_+^\times} = \sum_{\zeta\in F_+^\times} \int_{E(\mathfrak{f})F_\infty^1\backslash F_\infty^+} = \sum_{\text{ideals } \zeta a_{ij}\mathfrak{D}} \sum_{\epsilon\in\mathcal{O}_{F,+}^\times} \int_{E(\mathfrak{f})F_\infty^1\backslash F_\infty^+}.$$

Then we have:

Lemma 12.1.9. *Let $\epsilon \in E(\mathfrak{f}) \subset \mathcal{O}_{F,+}^\times$. Then replacing ζ with $\epsilon\zeta$ in the expression above leaves the integrand unchanged.*

Proof. Firstly, it's clear that for any unit, we have $|\epsilon\zeta| = |\zeta|$. There are two other terms involving ζ. One sees from the definition of e_F that if $\epsilon \equiv 1 \pmod{\mathfrak{f}}$, then $e_F(\epsilon\zeta(\mathfrak{f}^{-1})_{v|\mathfrak{f}}) = e_F(\zeta(\mathfrak{f}^{-1})_{v|\mathfrak{f}})$.

This just leaves the term $\zeta^{\mathbf{j}^\#}$. To deal with this term, recall that we took φ to be a Hecke character of conductor $\mathfrak{f}$ and infinity type $\mathbf{j}+\mathbf{v}$. Now define

$$\psi := \varphi|\cdot|_{\mathbb{A}_F}^{-[\mathbf{j}+\mathbf{v}]}.$$

Then we see that ψ has conductor $\mathfrak{f}$ and infinity type $-\mathbf{j}^\#$. In particular, we see that

$$(\epsilon\zeta)^{\mathbf{j}^\#} = \psi_\infty(\epsilon\zeta)^{-1} = \psi_f(\epsilon\zeta).$$

But $\psi_f(\epsilon) = 1$. Thus it follows that $\zeta^{\mathbf{j}^\#}$ is invariant under multiplication by ϵ, and we are done. □

In particular, this invariance now allows us to rewrite the integral as

$$[\mathcal{O}_{F,+}^\times : E(\mathfrak{f})] \sum_{\text{ideals } \zeta a_{ij}\mathfrak{D}} \int_{F_\infty^1\backslash F_\infty^+}.$$

Computing standard integrals

Using the above, and still assuming that Re(s) >> 0, we rearrange further. We can identify F_∞^+/F_∞^1 with $(\mathbb{R}_{>0})^q$, and hence the integral breaks down into the product of integrals from 0 to ∞ at each infinite place. We get:

$$
\begin{aligned}
C_{ij}^{\mathbf{j}}(s) = c_1 |a_{ij}|_{\mathbb{A}_F} [\mathcal{O}_{F,+}^{\times} : E(\mathfrak{f})] \prod_{v \in \Sigma(\mathbb{C})} i^{j_v - j_{vc}} \begin{pmatrix} k_v^* \\ n_v \end{pmatrix} \\
\times \sum_{\text{ideals } \zeta a_{ij} \mathfrak{D}} c(\zeta a_{ij} \mathfrak{D}, \Phi) e_F(\zeta(\mathfrak{f}^{-1})_{v|\mathfrak{f}}) \zeta^{\mathbf{j}^\#} |\zeta|_{\infty}^{-\mathbf{v}} \left[\prod_{v \in \Sigma(\mathbb{R})} \int_0^\infty (y_v)^{j_v + s} e^{-2\pi |\zeta|_{\mathbb{R}} y_v} dy_v \right] \\
\times \left[\prod_{v \in \Sigma(\mathbb{C})} \int_0^\infty K_{j_v - j_{vc}} (4\pi |\zeta y_v|_{\mathbb{C}}) |y_v|_{\mathbb{C}}^{1 + j_v + j_{vc} + 2s} d|y_v|_{\mathbb{C}} \right].
\end{aligned}
$$

These are standard integrals; indeed, we have

$$
\int_0^\infty K_{j_{cv} - j_v} (4\pi |\zeta y_v|_{\mathbb{C}}) |y_v|_{\mathbb{C}}^{j_v + j_{vc} + 2s + 1} d|y_v|_{\mathbb{C}} = (2\pi |\zeta|_{\mathbb{C}})^{-j_v - j_{cv} - 2s - 2} 2^{-2} \Gamma(j_v + s + 1) \Gamma(j_{cv} + s + 1),
$$

whilst

$$
\int_0^\infty e^{-2\pi |\zeta y_v|_{\mathbb{R}}} |y_v|_{\mathbb{R}}^{j_v + s} dy_v = (2\pi |\zeta|_{\mathbb{R}}^{-j_v - s - 1} \int_0^\infty e^{-x} x^{j_v + s} dx = (2\pi |\zeta|_{\mathbb{R}})^{-j_v - s - 1} \Gamma(j_v + s + 1).
$$

By substituting these integrals in, we get

$$
\begin{aligned}
C_{ij}^{\mathbf{j}}(s) = c_2 \sum_{\text{ideals } \zeta a_{ij} \mathfrak{D}} c(\zeta a_{ij} \mathfrak{D}) e_F(\zeta(\mathfrak{f}^{-1})_{v|\mathfrak{f}}) \zeta^{\mathbf{j}^\#} |\zeta|_{\infty}^{-\mathbf{v}} \\
\times \left[\prod_{v \in \Sigma(\mathbb{C})} (2\pi |\zeta|_{\mathbb{C}})^{-j_v - j_{cv} - 2s - 2} 2^{-2} \Gamma(j_v + s + 1) \Gamma(j_{vc} + s + 1) \right] \\
\times \left[\prod_{v \in \Sigma(\mathbb{R})} (2\pi |\zeta|_{\mathbb{R}}^{-j_v - s - 1} \Gamma(j_v + s + 1) \right],
\end{aligned}
$$

which simplifies to

$$
\begin{aligned}
C_{ij}^{\mathbf{j}}(s) = c_2 (2\pi)^{-\mathbf{j} - (s+1)\mathbf{t}} \Gamma(\mathbf{j} + (s+1)\mathbf{t}) \\
\times \sum_{\text{ideals } \zeta a_{ij} \mathfrak{D}} c(\zeta a_{ij} \mathfrak{D}) e_F(\zeta(\mathfrak{f}^{-1})_{v|\mathfrak{f}}) \zeta^{\mathbf{j}^\#} |\zeta|_{\infty}^{-\mathbf{v} - \mathbf{j} - (s+1)\mathbf{t}}.
\end{aligned}
$$

Here we've written

$$
c_2 = c_1 [\mathcal{O}_{F,+}^{\times} : E(\mathfrak{f})] |a_{ij}|_{\mathbb{A}_F} \prod_{v \in \Sigma(\mathbb{C})} i^{j_v - j_{vc}} \begin{pmatrix} k_v^* \\ n_v \end{pmatrix} \tag{12.4}
$$

and defined

$$
\Gamma(\mathbf{j} + (s+1)\mathbf{t}) := \prod_{v \in \Sigma} \Gamma(j_v + s + 1),
$$

$$
(2\pi)^{\mathbf{j} + (s+1)\mathbf{t}} := \prod_{v \in \Sigma} (2\pi)^{j_v + s + 1}
$$

for simplicity.

Simplifying the constant

We now focus on the term c_2. Recall we defined c_1 in Proposition 12.1.8. Substituting this into equation (12.4) above, we see that the binomial coefficients cancel, and the signs reduce to give

$$\begin{aligned} c_2 &= |a_{ij}|_{\mathbb{A}_F}[\mathcal{O}_{F,+}^\times : E(\mathfrak{f})]2^{r_2}(i)^{\mathbf{j}+\mathbf{t}} \prod_{v\in\Sigma(\mathbb{C})}(-1)^{k_v}\prod_{v\in\Sigma(\mathbb{R})}(-1)^{k_v+j_v} \\ &= N(a_{ij}\mathfrak{D})^{-1}|D|[\mathcal{O}_{F,+}^\times : E(\mathfrak{f})]2^{r_2}(-i)^{\mathbf{j}+\mathbf{t}} \prod_{v\in\Sigma(\mathbb{C})}(-1)^{k_v}\prod_{v\in\Sigma(\mathbb{R})}(-1)^{k_v+j_v}, \end{aligned}$$

using the fact that $|D| = N(\mathfrak{D})$ and $|a_{ij}|_{\mathbb{A}_F} = N(I(a_{ij}))^{-1}$.

Rearranging further

We can massage our formula a bit further; note that

$$\begin{aligned} |\zeta|_\infty^{-\mathbf{v}-\mathbf{j}-(s+1)\mathbf{t}} &= N((\zeta))^{-[\mathbf{j}+\mathbf{v}]-s-1} \\ &= N(a_{ij}\mathfrak{D})^{[\mathbf{j}+\mathbf{v}]+s+1}N(\zeta a_{ij}\mathfrak{D})^{-[\mathbf{j}+\mathbf{v}]-s-1}, \end{aligned}$$

where the first equality follows from the definition of $[\cdot]$ (see Definition 1.2.5). When we multiply this by $|D|N(a_{ij}\mathfrak{D})^{-1}$, we obtain $|D|N(a_{ij}\mathfrak{D})^{[\mathbf{j}+\mathbf{v}]+s}N(\zeta a_{ij}\mathfrak{D})^{-[\mathbf{j}+\mathbf{v}]-s-1}$. Incorporating all of this, we end up with the formula

$$C_{ij}^{\mathbf{j}}(s) = c_3 N(a_{ij}\mathfrak{D})^{[\mathbf{j}+\mathbf{v}]+s}\sum_{\text{ideals } \zeta a_{ij}\mathfrak{D}} c(\zeta a_{ij}\mathfrak{D},\Phi)\zeta^{\mathbf{j}^\#}e_F(\zeta(\mathfrak{f}^{-1})_{v|\mathfrak{f}})N(\zeta a_{ij}\mathfrak{D})^{-[\mathbf{j}+\mathbf{v}]-s-1},$$

where

$$c_3 := (-2\pi i)^{-\mathbf{j}-\mathbf{t}}2^{-r_2}(2\pi)^{-s\mathbf{t}}|D|\Gamma(\mathbf{j}+(s+1)\mathbf{t})[\mathcal{O}_{F,+}^\times : E(\mathfrak{f})]\prod_{v\in\mathbb{C}}(-1)^{k_v}\prod_{v\in\Sigma(\mathbb{R})}(-1)^{k_v+j_v}.$$

Gauss sums

We now sum over class group representatives and use Gauss sums to obtain the correct twisted L-function. Recall that we defined φ to be a Hecke character of F of conductor $\mathfrak{f}$ and infinity type $\mathbf{j}+\mathbf{v}$, and defined $\psi := \varphi|\cdot|^{-[\mathbf{j}+\mathbf{v}]}$, which has infinity type $-\mathbf{j}^\#$. In particular, note that $\zeta^{\mathbf{j}^\#} = \psi_\infty(\zeta)^{-1}$ (since ζ is totally positive) and we have

$$\varphi(a_{ij})N(a_{ij}\mathfrak{D})^{[\mathbf{j}+\mathbf{v}]} = N(\mathfrak{D})^{[\mathbf{j}+\mathbf{v}]}\psi(a_{ij}).$$

Now define

$$
\begin{aligned}
C_\varphi(s) &:= \sum_{i,j} \varphi(a_{ij}) N(a_{ij}\mathfrak{D})^{-s} C_{ij}^{\mathbf{j}}(s) \\
&= c_3 \sum_{i,j} \psi(a_{ij}) N(\mathfrak{D})^{[\mathbf{j}+\mathbf{v}]} \sum_{\text{ideals } \zeta a_{ij}\mathfrak{D}} c(\zeta a_{ij}\mathfrak{D}, \Phi)\psi_\infty(\zeta)^{-1} \\
&\qquad\qquad e_F(\zeta(\mathfrak{f}^{-1})_{v|\mathfrak{f}}) N(\zeta a_{ij}\mathfrak{D})^{-[\mathbf{j}+\mathbf{v}]-s-1}.
\end{aligned}
$$

Here the sum is over $i \in \mathrm{Cl}_F^+$ and $j \in (\mathcal{O}_F/\mathfrak{f})^\times$. Ideally, we want the Fourier coefficient to be independent of j, so that we can break up the sum and leave a Gauss sum. To achieve this, we scale by b_j. Recall b_j is an idele defined to be 1 at places dividing $\mathfrak{f}\infty$ and α_j^{-1} everywhere else, where $\alpha_j \in \mathcal{O}_{F,+}^\times$ is congruent to $j \pmod{\mathfrak{f}}$. Replace ζ with $\zeta'\alpha_j$ (noting that we still have absolute convergence). Then $\zeta a_{ij}\mathfrak{D} = \zeta' a_i \mathfrak{D}$ (as ideals). This gives

$$
\begin{aligned}
C_\varphi(s) = c_3 &\sum_{i,j} \psi(a_{ij}) N(\mathfrak{D})^{[\mathbf{j}+\mathbf{v}]} \\
&\sum_{\text{ideals } \zeta' a_i\mathfrak{D}} c(\zeta' a_i\mathfrak{D}, \Phi)\psi_\infty(\zeta'\alpha_j)^{-1} e_F(\zeta'\alpha_j(\mathfrak{f}^{-1})_{v|\mathfrak{f}}) N(\zeta' a_i\mathfrak{D})^{-[\mathbf{j}-\mathbf{v}]-s-1}.
\end{aligned}
$$

We need to fix a further piece of notation.

Notation: Recall: we took d to be a (finite) idele representing the different $\mathfrak{D}$. We choose a specific d. Write $\mathfrak{D} = I_{i_\mathfrak{D}}(\delta)$, where $\delta \in F_\infty^+$. Then we can taken $d = a_{i_\mathfrak{D}}\delta_f$, where δ_f is the finite idele with every component equal to δ. It follows that

$$
(\zeta'\delta)\alpha_j d^{-1}(\mathfrak{f})_{v|\mathfrak{f}} = \zeta'\alpha_j(\mathfrak{f})_{v|\mathfrak{f}}. \tag{12.5}
$$

Incorporating equation (12.5), breaking up $\psi(a_{ij}) = \psi(a_i)\psi(b_j)$, and introducing the term $\psi(d)\psi(d)^{-1}$ (for the Gauss sum), this becomes

$$
\begin{aligned}
C_\varphi(s) = c_3\psi(d) \sum_i \psi(a_i) N(\mathfrak{D})^{[\mathbf{j}+\mathbf{v}]} &\sum_{\text{ideals } \zeta' a_i\mathfrak{D}} c(\zeta' a_i\mathfrak{D}, \Phi)\psi_\infty(\zeta')^{-1} N(\zeta' a_i\mathfrak{D})^{-[\mathbf{j}+\mathbf{v}]-s-1} \\
&\times \psi(d)^{-1} \sum_j \psi(b_j)\psi_\infty(\alpha_j)^{-1} e_F((\zeta'\delta)\alpha_j d^{-1}(\mathfrak{f}^{-1})_{v|\mathfrak{f}}).
\end{aligned}
$$

Now, we see that

$$
\begin{aligned}
\psi(b_j)\psi_\infty(\alpha_j)^{-1} &= \left[\prod_{v\nmid\infty} \psi(\alpha_j^{-1}) \prod_{v|\mathfrak{f}} \psi(\alpha_j)\right] \left[\prod_{v\nmid\infty} \psi(\alpha_j)\right] \\
&= \prod_{v|\mathfrak{f}} \psi(\alpha_j) = \psi_\mathfrak{f}(\alpha_j),
\end{aligned}
$$

so that the second sum becomes

$$\psi(d)^{-1}\sum_{j\in(\mathcal{O}_F/\mathfrak{f})^\times}\psi_{\mathfrak{f}}(\alpha_j)e_F((\zeta'\delta)\alpha_j(\mathfrak{f}^{-1})_{v|\mathfrak{f}}) = \begin{cases}\psi_{\mathfrak{f}}(\zeta'\delta)^{-1}\tau(\psi) & : ((\zeta'\delta),\mathfrak{f})=1\\ 0 & : \text{otherwise,}\end{cases}$$

using the theory of Gauss sums in Section 1.3.1. Now note that when $((\zeta'\delta),\mathfrak{f})=1$, we have

$$\psi((\zeta'\delta))\psi_{\mathfrak{f}}(\zeta'\delta)\psi_\infty(\zeta'\delta)=1,$$

by the definition of $\psi_{\mathfrak{f}}$. The sum now becomes

$$\begin{aligned}C_\varphi(s) &= c_3\psi(d)\tau(\psi)\sum_i \psi(a_i)N(\mathfrak{D})^{[\mathbf{j}+\mathbf{v}]}\\ &\qquad\times\sum_{\substack{\text{ideals } \zeta' a_i\mathfrak{D}\\ \text{coprime to } \mathfrak{f}}} c(\zeta' a_i\mathfrak{D},\Phi)\psi((\zeta'\delta))\psi_\infty(\delta)N(\zeta' a_i\mathfrak{D})^{-[\mathbf{j}+\mathbf{v}]-s-1}.\end{aligned}$$

Note that

$$\psi(d)\psi_\infty(\delta)=\psi(a_{i_{\mathfrak{D}}})\psi_f(\delta)\psi_\infty(\delta)=\psi(a_{i_{\mathfrak{D}}}).$$

Rearranging again, and consolidating the terms involving the different and noting that $\psi(\zeta' a_i\mathfrak{D}) = 0$ when $((\zeta'\delta),\mathfrak{f})\neq 1$, this becomes

$$\begin{aligned}C_\varphi(s) &= c_3\tau(\psi)N(\mathfrak{D})^{[\mathbf{j}+\mathbf{v}]}\\ &\qquad\times\sum_i\sum_{\text{ideals } \zeta' a_i\mathfrak{D}} c(\zeta' a_i\mathfrak{D},\Phi)\psi(\zeta' a_i\mathfrak{D})N(\zeta' a_i\mathfrak{D})^{-[\mathbf{j}+\mathbf{v}]-s-1}.\end{aligned}$$

The sum now collapses to one over all ideals of F.

Obtaining L-values

We have $\psi(d)^{-1}N(\mathfrak{D})^{[\mathbf{j}+\mathbf{v}]}=\varphi(d)^{-1}$; hence, it's easy to see that $N(\mathfrak{D})^{[\mathbf{j}+\mathbf{v}]}\tau(\psi)=\tau(\varphi)$. Thus we have

$$\begin{aligned}C_\varphi(s) &= c_3\tau(\varphi)L(\Phi,\psi,[\mathbf{j}+\mathbf{v}]+s+1)\\ &= c_3\tau(\varphi)L(\Phi,\varphi,s+1).\end{aligned}$$

With a little extra work, we see that this formula gives an analytic continuation of $L(\Phi,\varphi,s)$ to the complex plane. In particular, setting $s=0$, and recalling that

$$C^{\mathbf{j}}_{ij}(0)=\mathrm{Ev}^{a_{ij}}_{\mathfrak{f},\mathbf{j},1}(\phi_{\mathbb{C}}),$$

we see that

$$L(\Phi,\varphi):=L(\Phi,\varphi,1)=\frac{1}{c_3\tau(\varphi)}\sum_{i,j}\varphi(a_{ij})\mathrm{Ev}^{a_{ij}}_{\mathfrak{f},\mathbf{j},1}(\phi_{\mathbb{C}}).$$

We see that we've proved the following theorem:

Theorem 12.1.10. *Let $F/\mathbb{Q}$ be a number field, and let Φ be a cuspidal eigenform over F of weight $\lambda = (\mathbf{k}, \mathbf{v}) \in \mathbb{Z}[\Sigma]^2$, where $\mathbf{k}+2\mathbf{v}$ is parallel, and let φ be a Hecke character of conductor $\mathfrak{f}$ and infinity type $\mathbf{j}+\mathbf{v}$, where $0 \leq \mathbf{j} \leq \mathbf{k}$. Let $\Lambda(\Phi, \cdot)$ be the normalised L-function attached to Φ defined in Definition 4.3.6. Then there is an integral formula*

$$\sum_{i,j} \varphi(a_{ij})\mathrm{Ev}_{\mathfrak{f},\mathbf{j},1}^{a_{ij}}(\phi_{\mathbb{C}}) = (-1)^{R(\mathbf{j},\mathbf{k})} \left[\frac{[\mathcal{O}_{F,+}^{\times} : E(\mathfrak{f})]|D|\tau(\varphi)}{2^{r_2}}\right] \cdot \Lambda(\Phi, \varphi),$$

where:

- *The sum is over $i \in \mathrm{Cl}_F^+$ and $j \in (\mathcal{O}_F/\mathfrak{f})^\times$,*
- $R(\mathbf{j}, \mathbf{k}) := \sum_{v\in\Sigma(\mathbb{C})} k_v + \sum_{v\in\Sigma(\mathbb{R})} k_v + j_v$,
- *$\tau(\varphi)$ is the Gauss sum attached to φ defined in Definition 1.3.2,*
- *D is the discriminant of the number field F,*
- *$\mathrm{Ev}_{\mathfrak{f},\mathbf{j},1}^{a_{ij}}$ is the classical evaluation map from Definition 12.1.6,*
- *and $\phi_{\mathbb{C}}$ is the modular symbol attached to Φ under the Eichler-Shimura isomorphism.*

12.1.5. Evaluating at ideals other than the conductor

In the sequel, we will need to look at evaluation maps at ideals other than the conductor of the relevant Hecke character. For example, let φ be a Hecke character of conductor $\mathfrak{f}$ and infinity type $\mathbf{j} + \mathbf{v}$, and let $\mathfrak{p}$ be a prime not dividing $\mathfrak{f}$; then we will need to consider the expression

$$\sum_{\mathbf{y}\in\mathrm{Cl}_F^+(\mathfrak{f}\mathfrak{p})} \varphi(a_{\mathbf{y}})\mathrm{Ev}_{\mathfrak{f}\mathfrak{p},\mathbf{j},1}^{a_{\mathbf{y}}}(\phi_{\mathbb{C}}).$$

In particular, we need to know how this relates to the evaluation maps at $\mathfrak{f}$ considered above in the case that $\mathfrak{p}$ divides the level $\mathfrak{n}$. In this section, we provide a formula for this case.

We start by making the following simple, but crucial, observation about Gauss sums.

Lemma 12.1.11. *Let φ be a Hecke character of conductor $\mathfrak{f}$, and let $\mathfrak{p}$ be a prime not dividing $\mathfrak{f}$. Let B be a complete set of representatives in $\mathcal{O}_F$ for the set*

$$\{b \,(\mathrm{mod}\, \mathfrak{f}\mathfrak{p}) : b \,(\mathrm{mod}\, \mathfrak{f}) \in (\mathcal{O}_F/\mathfrak{f})^\times\}.$$

Then we have

$$\varphi(d^{-1})\sum_{b\in B} \varphi_{\mathfrak{f}}(b)e_F(\zeta b d^{-1}(\mathfrak{f}\mathfrak{p})_{v|\mathfrak{f}\mathfrak{p}}^{-1}) = \begin{cases} N(\mathfrak{p})\varphi_{\mathfrak{f}}(\zeta)^{-1}\tau(\varphi) & : ((\zeta), \mathfrak{f}) = 1 \text{ and } \mathfrak{p}|(\zeta), \\ 0 & : \text{otherwise.} \end{cases}$$

Proof. The sum splits as a product

$$\left[\sum_{\alpha\in\mathcal{O}_F/\mathfrak{p}} e_F(\zeta\alpha d^{-1}(\mathfrak{p}^{-1})_{v|\mathfrak{p}})\right]\cdot\left[\varphi(d^{-1})\sum_{\beta\in(\mathcal{O}_F/\mathfrak{f})^\times}\varphi_{\mathfrak{f}}(\beta)e_F(\zeta\beta d^{-1}(\mathfrak{f}^{-1})_{v|\mathfrak{f}})\right].$$

The first term is non-zero if and only if $\mathfrak{p}|(\zeta)$, in which case $e_F(\zeta\alpha d^{-1}(\mathfrak{p}^{-1})_{v|\mathfrak{p}})=1$ and the sum is $N(\mathfrak{p})$. The second term is just the usual Gauss sum. The result follows. □

Let $a_{ij}=a_ib_j$, as before, form the usual multiset of representatives for $\mathrm{Cl}_F^+(\mathfrak{f})$. We extend this slightly. To this end, let

$$\{c_k\in\mathcal{O}_{F,+}:k\in\mathcal{O}_F/\mathfrak{p}\}$$

form a complete set of representatives for $\mathcal{O}_F/\mathfrak{p}$ (noting in particular that we cannot have $c_0=0$). Define

$$a_{ijk}=a_{ij}c_k,$$

and note that

$$\{a_{ijk}:i\in\mathrm{Cl}_F^+,j\in(\mathcal{O}_F/\mathfrak{f})^\times,k\in(\mathcal{O}_F/\mathfrak{p})^\times\}$$

forms a multiset of representatives for $\mathrm{Cl}_F^+(\mathfrak{fp})$, with each representative counted $\#\mathrm{Im}(\mathcal{O}_{F,+}^\times\longrightarrow(\mathcal{O}_F/\mathfrak{fp})^\times)$ times. In particular, we see that

$$\sum_{i,j}\sum_{k\in(\mathcal{O}_K/\mathfrak{p})^\times}\varphi(a_{ijk})\mathrm{Ev}_{\mathfrak{fp},\mathbf{j},1}^{a_{ijk}}(\phi_{\mathbb{C}})=$$
$$\sum_{i,j}\sum_{k\in\mathcal{O}_K/\mathfrak{p}}\varphi(a_{ijk})\mathrm{Ev}_{\mathfrak{fp},\mathbf{j},1}^{a_{ijk}}(\phi_{\mathbb{C}})-\sum_{i,j}\varphi(a_{ij}c_0)\mathrm{Ev}_{\mathfrak{fp},\mathbf{j},1}^{a_{ij}c_0}(\phi_{\mathbb{C}}),$$

where in all three expressions we sum over $i\in\mathrm{Cl}_F^+$ and $j\in(\mathcal{O}_F/\mathfrak{f})^\times$. We study each of these terms separately.

Lemma 12.1.12. *We have*

$$\sum_{i,j}\varphi(a_{ij}c_0)\mathrm{Ev}_{\mathfrak{fp},\mathbf{j},1}^{a_{ij}c_0}(\phi_{\mathbb{C}})=\frac{[\mathcal{O}_{F,+}^\times:E(\mathfrak{fp})]}{[\mathcal{O}_{F,+}^\times:E(\mathfrak{f})]}\sum_{i,j}\varphi(a_{ij})\mathrm{Ev}_{\mathfrak{f},\mathbf{j},1}^{a_{ij}}(\phi_{\mathbb{C}}),$$

where the sum is over $i\in\mathrm{Cl}_F^+$ and $j\in(\mathcal{O}_F/\mathfrak{f})^\times$ in both expressions.

Proof. We compute both sides in an almost identical manner to the proof of the integral formula. Firstly, we obtain the ratio in the unit indices since we're now integrating over $E(\mathfrak{fp})F_\infty^1\backslash F_\infty^+$ rather than $E(\mathfrak{f})F_\infty^1\backslash F_\infty^+$. We see that the only other step that changes is the one involving the Gauss sum. To see that this really doesn't affect the final result, note that we have

$$e_F((\zeta'\delta)\alpha_jc_0d^{-1}(\mathfrak{fp})^{-1}_{v|\mathfrak{fp}})=e_F((\zeta'\delta)\alpha_jc_0d^{-1}(\mathfrak{f}^{-1})_{v|\mathfrak{f}})e_F((\zeta'\delta)\alpha_jc_0d^{-1}(\mathfrak{p}^{-1})_{v|\mathfrak{p}}),$$

and that the second term of this product is equal to 1 as $c_0 \cong 0 \pmod{\mathfrak{p}}$. The result then follows, since $\{\alpha_j c_0\}$ is again a full set of representatives for $(\mathcal{O}_F/\mathfrak{f})^\times$. $\square$

The study of the first term is a little more involved. We give a sketch; the calculations are almost identical to those in the integral formula proved previously, and for the reader's sanity, we do not wish to repeat them.

Lemma 12.1.13. *Suppose that $\phi_{\mathbb{C}}$ is an eigenform for all the Hecke operators. We have*

$$\sum_{i,j}\sum_{k\in\mathcal{O}_K/\mathfrak{p}} \varphi(a_{ijk})\mathrm{Ev}_{\mathfrak{fp},\mathbf{j},1}^{a_{ijk}}(\phi_{\mathbb{C}}) = \lambda_{\mathfrak{p}}\varphi(\mathfrak{p})\frac{[\mathcal{O}_{F,+}^\times : E(\mathfrak{fp})]}{[\mathcal{O}_{F,+}^\times : E(\mathfrak{f})]}\sum_{i,j}\varphi(a_{ij})\mathrm{Ev}_{\mathfrak{f},\mathbf{j},1}^{a_{ij}}(\phi_{\mathbb{C}}),$$

where $\lambda_{\mathfrak{p}} = c(\mathfrak{p},\Phi)$ is the eigenvalue of $\phi_{\mathbb{C}}$ at the Hecke operator $U_{\mathfrak{p}}$ (recalling that $\mathfrak{p}|\mathfrak{n}$).

Proof. Again, we examine the proof of the integral formula; the term involving ratios of unit indices is introduced exactly as in the previous lemma. By following the remaining steps in deriving the integral formula, we see again that the only major change is in the Gauss sum, and indeed that we end up with the 'modified Gauss sum' of Lemma 12.1.11. In particular, in the calculation of the integral formula, we are left with a sum over ideals that are divisible by $\mathfrak{p}$, in addition to introducing a factor of $N(\mathfrak{p})$. Since $\phi_{\mathbb{C}}$ is an eigenform, and as $\mathfrak{p}|\mathfrak{n}$, the Fourier coefficients satisfy

$$c(I\mathfrak{p},\Phi) = c(I,\Phi)c(\mathfrak{p},\Phi),$$

so that the summands are multiplicative and we can recover a sum over all ideals by factoring out the expression $c(\mathfrak{p},\Phi)\varphi(\mathfrak{p})N(\mathfrak{p})^{-(s+1)}$. After setting $s = 0$ and incorporating the extra factor of $N(\mathfrak{p})$ coming from the Gauss sum, we recover the result. $\square$

We now conclude this section by stating the compatibility results we need. We find we've proved the following:

Theorem 12.1.14. *Let φ be a Hecke character of conductor $\mathfrak{f}$ and infinity type $\mathbf{j}+\mathbf{v}$, and let $\mathfrak{p}$ be a prime dividing $\mathfrak{n}$ but* not *dividing $\mathfrak{f}$. Then we have*

$$\sum_{i,j,k}\varphi(a_{ijk})\mathrm{Ev}_{\mathfrak{fp},\mathbf{j},1}^{a_{ijk}}(\phi_{\mathbb{C}}) = (\varphi(\mathfrak{p})\lambda_{\mathfrak{p}} - 1)\frac{[\mathcal{O}_{F,+}^\times : E(\mathfrak{fp})]}{[\mathcal{O}_{F,+}^\times : E(\mathfrak{f})]}\sum_{i,j}\varphi(a_{ij})\mathrm{Ev}_{\mathfrak{f},\mathbf{j},1}^{a_{ij}}(\phi_{\mathbb{C}}),$$

where the sums are over $i \in \mathrm{Cl}_F^+$, $j \in (\mathcal{O}_F/\mathfrak{f})^\times$ and $k \in (\mathcal{O}_F/\mathfrak{p})^\times$.

In the next section, we will remove the terms of form $[\mathcal{O}_{F,+}^\times : E(\mathfrak{fp})]$ by summing only over the respective narrow ray class groups (and, in the process, eliminating the double counting of class group representatives; see Theorem 12.2.7). With this in mind, we record the following corollary, which is proved by a simple induction:

Corollary 12.1.15. *Suppose $(p)|\mathfrak{n}$, and let φ be a Hecke character of conductor $\mathfrak{f}|(p^\infty)$ and infinity type $\mathbf{j}+\mathbf{v}$. Let B be the set of primes above p for which φ is not ramified, and define $\mathfrak{f}' := \mathfrak{f}\prod_{\mathfrak{p}\in B}\mathfrak{p}$. Then $\mathfrak{f}'$ is divisible by every prime above p and we have*

$$\sum_{\mathbf{y}\in \mathrm{Cl}_F^+(\mathfrak{f}')} \varphi(a_\mathbf{y})\mathrm{Ev}_{\mathfrak{f}',\mathbf{j},1}^{a_\mathbf{y}}(\phi_\mathbb{C}) = \left(\prod_{\mathfrak{p}\in B}(\varphi(\mathfrak{p})\lambda_\mathfrak{p}-1)\right)\sum_{\mathbf{y}\in \mathrm{Cl}_F^+(\mathfrak{f})} \varphi(a_\mathbf{y})\mathrm{Ev}_{\mathfrak{f},\mathbf{j},1}^{a_\mathbf{y}}(\phi_\mathbb{C}).$$

12.2. Algebraicity results

So far, all of our work has been done over $\mathbb{C}$. We will now refine these results to show that the *algebraic* modular symbol define in Chapter 11.6 also sees all of the critical L-values above.

Recall Theorem 4.4.1, which said that the normalised L-value $\Lambda(\Phi,\varphi)$ is an algebraic multiple of a period $\Omega_\Phi^{\varepsilon_\varphi}$, where ε_φ is the character of $\{\pm1\}^{\Sigma(\mathbb{R})}$ attached to φ (see Chapter 1.2.2). In Chapter 11.6, for a character ε of $\{\pm1\}^{\Sigma(\mathbb{R})}$ we defined a modular symbol $\phi_\mathbb{C}^\varepsilon$ and stated a result that $\phi_K^\varepsilon := \phi_\mathbb{C}^\varepsilon/\Omega_\Phi^\varepsilon$ lived in an algebraic subspace. We now relate $\phi_\mathbb{C}^\varepsilon$ (and, by scaling, ϕ_K^ε) to the L-function using our above formula.

Definition 12.2.1. Let $A_\mathfrak{f} = \{a_\mathbf{y} : \mathbf{y}\in \mathrm{Cl}_F^+(\mathfrak{f})\}$ denote a fixed set of representatives for $\mathrm{Cl}_F^+(\mathfrak{f})$, with components at infinity that are not necessarily trivial. For a Hecke character φ of conductor $\mathfrak{f}$ and infinity type $\mathbf{j}+\mathbf{v}$, where $0\leq \mathbf{j}\leq \mathbf{k}$, define a function

$$\mathrm{Ev}_\varphi^{A_\mathfrak{f}} : \mathrm{H}_\mathrm{c}^q\left(Y_1(\mathfrak{n}),\mathcal{L}_1(V_\lambda(\mathbb{C})^*)\right) \longrightarrow \mathbb{C}$$

by

$$\mathrm{Ev}_\varphi^{A_\mathfrak{f}}(\phi) = \sum_{\mathbf{y}\in \mathrm{Cl}_F^+(\mathfrak{f})} \varepsilon_\varphi\varphi_f(a_\mathbf{y})\mathrm{Ev}_{\mathfrak{f},\mathbf{j}}^{a_\mathbf{y}}(\phi),$$

where as previously we write ε_φ as a function on the ideles by composing it with the natural sign map $\mathbb{A}_F^\times \to \{\pm1\}^{\Sigma(\mathbb{R})}$.

This definition is intimately related to the locally analytic function $\varphi_{p-\mathrm{fin}}$ we defined in Section 1.3.2. In particular, note that $\varepsilon_\varphi\varphi_f = \varphi/\varphi_\infty^{\mathrm{alg}}$, where $\varphi_\infty^{\mathrm{alg}}(x) = x_\infty^{\mathbf{j}+\mathbf{v}}$ is the unique algebraic function on F_∞ that agrees with φ_∞ on F_∞^+.

Lemma 12.2.2. *The function $\mathrm{Ev}_\varphi^{A_\mathfrak{f}}$ is independent of class group representatives.*

Proof. Let $a_\mathbf{y}'$ be an alternative representative corresponding to $\mathbf{y}\in \mathrm{Cl}_F^+(\mathfrak{f})$. Then $a_\mathbf{y} = fa_\mathbf{y}ur$, where $f\in F^\times, u\in U(\mathfrak{f})$ and $r\in F_\infty^+$. Looking at the description of the evaluation maps, we see that

$$\mathrm{Ev}_{\mathfrak{f},\mathbf{j}}^{a_\mathbf{y}'}(\phi) = f^{\mathbf{j}+\mathbf{v}}\mathrm{Ev}_{\mathfrak{f},\mathbf{j}}^{a_\mathbf{y}}(\phi).$$

But

$$\begin{aligned}\varepsilon_\varphi \varphi_f(a'_{\mathbf{y}}) = \varepsilon_\varphi \varphi_f(f a_{\mathbf{y}} u r) &= \varepsilon_\varphi \varphi_f(f) \varepsilon_\varphi \varphi_f(a_{\mathbf{y}}) \\ &= f^{-\mathbf{j}-\mathbf{v}} \varepsilon_\varphi \varphi_f(a_{\mathbf{y}}),\end{aligned}$$

since $\varepsilon_\varphi \varphi_f$ is trivial on $U(\mathfrak{f})F_\infty^+$ and by our earlier comment, we have

$$\varepsilon_\varphi \varphi_f(f) = \varphi(f)/\varphi_\infty^{\mathrm{alg}}(f) = f^{-\mathbf{j}-\mathbf{v}}.$$

Putting this together, we find that

$$\varepsilon_\varphi \varphi_f(a'_{\mathbf{y}}) \mathrm{Ev}_{\mathfrak{f},\mathbf{j}}^{a'_{\mathbf{y}}}(\phi) = \varepsilon_\varphi \varphi_f(a_{\mathbf{y}}) \mathrm{Ev}_{\mathfrak{f},\mathbf{j}}^{a_{\mathbf{y}}}(\phi),$$

which is the required result. $\square$

Definition 12.2.3. Define Ev_φ to be the map $\mathrm{Ev}_\varphi^{A_\mathfrak{f}}$ for any choice of class group representatives $A_\mathfrak{f}$. This is well-defined by the above lemma.

We'll combine this with the following to deduce the result we desire.

Proposition 12.2.4. *Let $\iota \in \{\pm 1\}^{\Sigma(\mathbb{R})}$. Then for any idele a, we have*

$$\mathrm{Ev}_{\mathfrak{f},\mathbf{j}}^{\iota a}\left(\phi \middle| \iota\right) = \mathrm{Ev}_{\mathfrak{f},\mathbf{j}}^{a}(\phi).$$

Proof. Recall that the definition of the action of $\iota \in \{\pm 1\}^{\Sigma(\mathbb{R})}$ on the cohomology of $Y_1(\mathfrak{n})$ was described in Section 11.4.2. There is a well-defined action of $\{\pm 1\}^{\Sigma(\mathbb{R})}$ on the local system corresponding to $\mathcal{L}_{\mathfrak{f},1}(V_\lambda(\mathbb{C})^*)$ given by

$$\iota \cdot (x, P) = (\iota x, P),$$

where here we've considered ι to be an idele by setting $\iota_v = 1$ for all complex and finite places v. A simple check shows that if $\phi \in \mathrm{H}_c^q(Y_1(\mathfrak{n}), \mathcal{L}_1(V_\lambda(\mathbb{C})^*))$ then we have

$$\eta_\mathfrak{f}^*(\phi|\iota) = \eta_\mathfrak{f}^*(\phi)|\iota$$

coming from the commutative diagram

$$\begin{tikzcd} (g, P) \arrow[r, "\eta_\mathfrak{f}^*"] \arrow[d, "|\iota"] & (x, P) \arrow[d, "|\iota"] \\ (\iota \cdot g, P) \arrow[r, "\eta_\mathfrak{f}^*"] & (\iota x, P) \end{tikzcd}$$

of local systems. Continuing to work at the level of local systems, suppose x is an idele that, under the natural quotient map, lies in the component of $X_{\mathfrak{f}}$ corresponding to $a_{\mathbf{y}}$. Then the image of ιx lies in the component corresponding to $\iota a_{\mathbf{y}}$ (where here we note that if $\{a_{\mathbf{y}} : \mathbf{y} \in \mathrm{Cl}_F^+(\mathfrak{f})\}$ is a complete set of representatives for $\mathrm{Cl}_F^+(\mathfrak{f})$, then so is the set $\{\iota a_{\mathbf{y}} : \mathbf{y} \in \mathrm{Cl}_F^+(\mathfrak{f})\}$). Thus we see that there is a commutative diagram of maps of local systems

$$\begin{tikzcd}
(x,P) \arrow[r, "\tau_{a_{\mathbf{y}}}^*"] \arrow[d, "|\iota"] & (r,P) \arrow[r, "\text{ev. at } \mathbf{X}^{\mathbf{k}-\mathbf{j}}\mathbf{Y}^{\mathbf{j}}"] \arrow[d] & (r,c) \arrow[d] \\
(\iota x, P) \arrow[r, "\tau_{\iota a_{\mathbf{y}}}^*"] & (r,P) \arrow[r, "\text{ev. at } \mathbf{X}^{\mathbf{k}-\mathbf{j}}\mathbf{Y}^{\mathbf{j}}"] & (r,c),
\end{tikzcd}$$

where the local system on the far right hand side defines the constant sheaf given by sections of $(E(\mathfrak{f})F_\infty^1\backslash F_\infty^+) \times \mathbb{C} \to E(\mathfrak{f})F_\infty^1\backslash F_\infty^+$. The result follows. □

Corollary 12.2.5. *We have the relation*

$$\mathrm{Ev}_\varphi(\phi|\iota) = \varepsilon_\varphi(\iota)\mathrm{Ev}_\varphi(\phi).$$

Proof. Considering ι as an idele in the usual way, we have

$$\begin{aligned}
\mathrm{Ev}_\varphi(\phi|\iota) &= \sum_{\mathbf{y}\in\mathrm{Cl}_F^+(\mathfrak{f})} \varepsilon_\varphi \varphi_f(\iota a_{\mathbf{y}})\mathrm{Ev}_{\mathbf{j},\mathfrak{f},1}^{\iota a_{\mathbf{y}}}(\phi|\iota) \\
&= \varepsilon_\varphi(\iota) \sum_{\mathbf{y}\in\mathrm{Cl}_F^+(\mathfrak{f})} \varepsilon_\varphi \varphi_f(a_{\mathbf{y}})\mathrm{Ev}_{\mathfrak{f},\mathbf{j},1}^{a_{\mathbf{y}}}(\phi) \\
&= \varepsilon_\varphi(\iota)\mathrm{Ev}_\varphi(\phi),
\end{aligned}$$

as required. □

Corollary 12.2.6. *We have*

$$\mathrm{Ev}_\varphi(\phi_{\mathbb{C}}^{\varepsilon}) = \begin{cases} \mathrm{Ev}_\varphi(\phi_{\mathbb{C}}) & : \varepsilon = \varepsilon_\varphi \\ 0 & : \textit{otherwise}. \end{cases}$$

Proof. By definition,

$$\begin{aligned}
\mathrm{Ev}_\varphi(\phi_{\mathbb{C}}^{\varepsilon}) &= \mathrm{Ev}_\varphi\left(2^{-r_1} \sum_{\iota\in\{\pm1\}^{\Sigma(\mathbb{R})}} \varepsilon(\iota)\phi_{\mathbb{C}}|\iota\right) \\
&= \left[2^{-r_1} \sum_{\iota\in\{\pm1\}^{\Sigma(\mathbb{R})}} \varepsilon(\iota)\varepsilon_\varphi(\iota)\right] \mathrm{Ev}_\varphi(\phi_{\mathbb{C}}),
\end{aligned}$$

using linearity of the evaluation maps and Corollary 12.2. The result is then clear from basic representation theory, as $\varepsilon_\varphi^{-1} = \varepsilon_\varphi$. □

Recall that in Definition 11.6.4, we set $\theta_K := \sum_\varepsilon \phi_K^\varepsilon$. Note that here θ_K is an element of the cohomology with *algebraic* coefficients in the number field K.

Theorem 12.2.7. *Let φ be a Hecke character of conductor $\mathfrak{f}$ and infinity type $\mathbf{j}+\mathbf{v}$, where $0 \leq \mathbf{j} \leq \mathbf{k}$, and write ε_φ for the associated character of $\{\pm 1\}^{\Sigma(\mathbb{R})}$ defined in Chapter 1.2.2. Let* Ev_φ *be as in Definition 12.2.3. We have*

$$\mathrm{Ev}_\varphi(\theta_K) = (-1)^{R(\mathbf{j},\mathbf{k})} \left[\frac{|D|\tau(\varphi)}{2^{r_2}\Omega_\Phi^{\varepsilon_\varphi}}\right] \cdot \Lambda(\Phi,\varphi),$$

where $R(\mathbf{j},\mathbf{k}) = \sum_{v\in\Sigma(\mathbb{R})} j_v + k_v + \sum_{v\in\Sigma(\mathbb{C})} k_v$.

Proof. We use Theorem 12.1.10. In particular, note that in the statement of the theorem, we use the multiset $\{a_{ij}\}$ of class group representatives in which each element of the class group is represented $[\mathcal{O}_{F,+}^\times : E(\mathfrak{f})] = \#\mathrm{Im}(\mathcal{O}_{F,+}^\times \to (\mathcal{O}_F/\mathfrak{f})^\times)$ times, so we can cancel this term from the result. Then we see that for these representatives,

$$\varepsilon_\varphi \varphi_f(a_{ij}) = \varphi(a_{ij}),$$

since we chose $(a_{ij})_\infty = 1$, so that the sum we obtained in the statement of Theorem 12.1.10 is nothing but $[\mathcal{O}_{F,+}^\times : E(\mathfrak{f})]\mathrm{Ev}_\varphi(\phi_\mathbb{C})$. The result follows. □

To summarise: we've now defined an *algebraic* cohomology class that sees the algebraic parts of all of the critical L-values that we hope to interpolate. In particular, by embedding K into a sufficiently large finite extension $L/\mathbb{Q}_p$, we get a p-adic modular symbol θ_L that sees all of these critical values.

Chapter 13

Overconvergent Cohomology

In this chapter, we explore p-adic deformations of the spaces of classical modular symbols. As in the Bianchi case, this will be done by replacing the module of coefficients with a space of p-adic distributions. We begin by defining these distribution modules, closely following the analogous section of [BS13]. This allows us to define the space of overconvergent modular symbols *as the compactly supported cohomology of $Y_1(\mathfrak{n})$ with coefficients in distributions. We then use compactness of the U_p operator to show that slope decompositions exist in wide generality for overconvergent modular symbols. This is crucially important to the main result of this chapter, where we prove a control theorem in the general setting. In particular, we use an argument of Urban to prove that the natural specialisation map from overconvergent to classical cohomology becomes an isomorphism upon restriction to appropriate 'small slope' subspaces.*

13.1. Distributions and overconvergent cohomology

Throughout this section, L is a finite extension of $\mathbb{Q}_p$ containing the image of $\mathrm{inc}_p \circ \sigma : F \hookrightarrow \overline{\mathbb{Q}}_p$ for each embedding σ of F into $\overline{\mathbb{Q}}$. First, we give some motivation by reformulating the definition of the space $V_\lambda(L)$. We previously defined this to be the d-fold tensor product of the polynomial spaces $V_{k_v}(L)$, with an action of $\mathrm{GL}_2(L)$ depending on λ. Note that $\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p$ embeds naturally in $\overline{\mathbb{Q}}_p^d$, and in particular, we can see an element of $V_\lambda(L)$ as a function on $\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p$ in a natural way. We see that the following definition agrees with the definition we gave in Section 11.2.

Definition 13.1.1. Let $L/\mathbb{Q}_p$ be a finite extension and let $\lambda = (\mathbf{k}, \mathbf{v}) \in \mathbb{Z}[\Sigma]$ be admissible in the sense of Definition 2.3.2 (so that, in particular, $\mathbf{k} \geq 0$). Define $V_\lambda(L)$ to be the space of functions on $\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p$ that are polynomial of degree $\mathbf{k}$ with coefficients in L, with a left action of $\mathrm{GL}_2(\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p)$ given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot P(x) = (ad - bc)^{\mathbf{v}} (a + cx)^{\mathbf{k}} P\left(\frac{b + dx}{a + cx}\right).$$

We've passed to a non-homogeneous version here. This definition is more easily seen to be compatible with the rest of this section. In particular, it is compatible with the following:

Definition 13.1.2. Let $\mathcal{A}(L)$ be the space of locally analytic functions on $\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p$ that are defined over L.

We'd like to define an action of $\mathrm{GL}_2(\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p)$ on this space, analogously to above. Unfortunately, the action above doesn't extend to the full space $\mathcal{A}(L)$. We can, however, define an action of a different semigroup.

Definition 13.1.3. (i) Let $\Sigma_0(p)$ be the semigroup

$$\Sigma_0(p) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p) : c \in p\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p, a \in (\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p)^\times, ad - bc \neq 0 \right\}.$$

(ii) Define $\mathcal{A}_\lambda(L)$ to be the space $\mathcal{A}(L)$ equipped with a left 'weight λ action' of $\Sigma_0(p)$ given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot f(z) = (ad - bc)^{\mathbf{v}}(a + cz)^{\mathbf{k}} P\left(\frac{b + dz}{a + cz}\right).$$

Note in particular that this semigroup contains the image of $\Gamma_1(\mathfrak{n})$ under the natural embedding $M_2(\mathcal{O}_F) \subset M_2(\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p)$ as well as the matrices that we'll need to define a Hecke action at primes above p. It is *not* a subset of $\mathrm{GL}_2(\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p)$, but the action of this different semigroup also extends naturally to $V_\lambda(L)$, since both live inside $\mathrm{GL}_2(F \otimes_{\mathbb{Q}} \mathbb{Q}_p)$.

We're now in a position to define the distribution spaces.

Definition 13.1.4. Define $\mathcal{D}_\lambda(L) := \mathrm{Hom}_{\mathrm{cts}}(\mathcal{A}_\lambda(L), L)$ to be the topological dual of $\mathcal{A}_\lambda$, with a right action of $\Sigma_0(p)$ defined by

$$(\mu|\gamma)(f) := \mu(\gamma \cdot f).$$

Note that $\Omega_1(\mathfrak{n})$ acts on $\mathcal{D}_\lambda(L)$ via its projection to $\mathrm{GL}_2(\mathbb{Q}_p)$, giving rise to a local system $\mathcal{L}_2(\mathcal{D}_\lambda(L))$ on $Y_1(\mathfrak{n})$, as in Chapter 11.3.

Definition 13.1.5. The space of *overconvergent modular symbols* is the compactly supported cohomology group $\mathrm{H}^q_c(Y_1(\mathfrak{n}), \mathcal{L}_2(\mathcal{D}_\lambda(L)))$.

By dualising the inclusion $V_\lambda(L) \subset \mathcal{A}_\lambda(L)$, we get a $\Sigma_0(p)$-equivariant surjection

$$\mathcal{D}_\lambda(L) \longrightarrow V_\lambda(L)^*.$$

This gives rise to a $\Sigma_0(p)$-equivariant *specialisation map*, a map

$$\rho : \mathrm{H}^q_c(Y_1(\mathfrak{n}), \mathcal{L}_2(\mathcal{D}_\lambda(L))) \longrightarrow \mathrm{H}^q_c(Y_1(\mathfrak{n}), \mathcal{L}_2(V_\lambda(L)^*)).$$

The space of overconvergent modular symbols is, in a sense, a p-adic deformation of the space of classical modular symbols. They were introduced by Glenn Stevens in [Ste94].

We conclude this section with a result that will be crucial in the following section, where we prove that the space of overconvergent modular symbols admits a slope decomposition with respect to the Hecke operators. For the relevant definitions, see [Urb11], Section 2.3.12. The space $\mathcal{D}_\lambda(L)$ is naturally a nuclear Fréchet space[1]; indeed, let $\mathcal{A}_{n,\lambda}(L)$ be the space of functions that are *locally analytic of radius* n, that is, functions that are analytic on each open set of the form $a+p^n\mathcal{O}_F\otimes_{\mathbb{Z}}\mathbb{Z}_p$. Each $\mathcal{A}_{n,\lambda}(L)$ is a Banach space, and the inclusions $\mathcal{A}_{n,\lambda}(L) \hookrightarrow \mathcal{A}_{n+1,\lambda}(L)$ are compact ([Urb11], Lemma 3.2.2). We write $\mathcal{D}_{n,\lambda}(L)$ for the topological dual of $\mathcal{A}_{n,\lambda}(L)$. Then $\mathcal{D}_\lambda(L) \cong \varprojlim \mathcal{D}_{n,\lambda}(L)$ is equipped with a family of norms coming from the Banach spaces $\mathcal{D}_{n,\lambda}(L)$.

Definition 13.1.6. Let $M \cong \varprojlim M_n$ be a nuclear Fréchet space. We say that an endomorphism U of M is *compact* if it is continuous and there are continuous maps U'_n making the diagram

$$\begin{tikzcd} M \arrow[r] \arrow[d, "U"] & M_{n-1} \arrow[d, "U'_n"] \\ M \arrow[r] & M_n \end{tikzcd}$$

commute, where the horizontal maps are the natural projections.

In this situation, we obtain compact[2] endomorphisms U_n on M_n by composing U'_n with the natural map $M_n \to M_{n-1}$. In [Ser62], it is proved that if M_n is a Banach space equipped with a compact endomorphism U_n, then M_n admits a slope decomposition with respect to U_n, and in [Urb11], Section 2.3.10, Urban uses this – and compactness of U – to deduce the existence of a slope decomposition for M with respect to U. In particular, the following lemma will be crucial in the next section.

Lemma 13.1.7. *Let $\eta \in \mathrm{GL}_2(F) \cap \Sigma_0(p)$, which acts naturally on $\mathcal{D}_\lambda(L)$. This action is compact. In particular, the action of $\left(\begin{smallmatrix}1 & 0\\ 0 & p\end{smallmatrix}\right)$ is compact on $\mathcal{D}_\lambda(L)$.*

Proof. See [Urb11], Lemma 3.2.8. ☐

[1] That is, an inverse limit of Banach spaces in which the projection maps are compact. In [Urb11], Urban calls this a *compact Fréchet space*. We instead follow the terminology utilised in [Sch02].

[2] Urban uses 'compact' and 'completely continuous' interchangeably to describe endomorphisms of Banach spaces that map bounded subsets into relatively compact subsets.

13.2. Slope decompositions

We start by recalling the relevant definitions about slope decompositions. This makes the notion of the 'subspace on which an operator acts with small slope' rigorous.

Definition 13.2.1. Let L be a finite extension of $\mathbb{Q}_p$, and let $h \in \mathbb{Q}$. We say a polynomial $Q(X) \in L[X]$ has *slope* $\leq h$ if $Q(0) \in \mathcal{O}_L^\times$ and if $\alpha \in \overline{L}$ is a root of $Q^*(X) := X^{\deg(Q)}Q(1/X)$, then $v_p(\alpha) \leq h$.

Definition 13.2.2. Let M be an L-vector space equipped with the action of an L-linear endomorphism U. We say that M has a *slope* $\leq h$ *decomposition with respect to* U if there is a decomposition $M \cong M_1 \oplus M_2$ such that:

(i) M_1 is finite-dimensional,

(ii) The polynomial $\det(1 - UX)|_{M_1}$ has slope $\leq h$, and

(iii) For all polynomials $P \in L[X]$ with slope $\leq h$, the polynomial $P^*(U)$ acts invertibly on M_2.

We write $M^{\leq h, U} := M_1$ for the elements of *slope* $\leq h$ in M. Where the operator U is clear, we drop it from the notation and just write $M^{\leq h}$.

In this section, we will prove the following theorem:

Theorem 13.2.3. *Let $\lambda = (\mathbf{k}, \mathbf{v})$ be an admissible weight. Then for each $i \in \mathbb{N}$ and any $h \in \mathbb{Q}$, the L-vector space $\mathrm{H}^i_{\mathrm{c}}(Y_1(\mathfrak{n}), \mathcal{L}_2(\mathcal{D}_\lambda(L)))$ admits a slope $\leq h$ decomposition with respect to the Hecke operator U_p.*

To prove this theorem we follow the arguments given in [Urb11] and [BS15], where the same statement is proved in the cases of the cohomology without compact support and GL_2 over a totally real field respectively. Both of these rely on general results from earlier in [Urb11], where Urban proves that any nuclear Fréchet space M equipped with a compact endomorphism U admits a slope decomposition with respect to U. Given this, the key step is to construct a complex whose cohomology is $\mathrm{H}^\bullet_{\mathrm{c}}(Y_1(\mathfrak{n}), \mathcal{L}_2(\mathcal{D}_\lambda(L)))$ and such that each term of the complex is isomorphic to finitely many copies of $\mathcal{D}_\lambda(L)$. We can find a lift of the Hecke operators on the cohomology to this complex, and then we use the fact that the action of $\left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right)$ on $\mathcal{D}_\lambda(L)$ is compact to deduce that this lift acts compactly on the complex. Using Urban's results, we deduce the theorem.

13.2.1. Compactly supported cohomology

Complexes

Let Γ be a torsion-free arithmetic subgroup of $\mathrm{SL}_2(F)$ and consider the manifold $\Gamma \setminus \mathcal{H}_F$. We denote by $\Gamma \setminus \overline{\mathcal{H}}_F$ the Borel-Serre compactification of $\Gamma \setminus \mathcal{H}_F$ (See [BS74]). Let $\mathfrak{B}$ be the set of proper parabolic $\mathbb{Q}$-subgroups of $\mathrm{SL}_2(F \otimes \mathbb{R})$. To construct the Borel-Serre compactification, recall that we first enlarge $\mathcal{H}_F$ to a space $\overline{\mathcal{H}}_F$ by adding a euclidean space $e(P)$ of dimension d to each $P \in \mathfrak{B}$. The boundary of $\overline{\mathcal{H}}_F$ is given by

$$\partial\overline{\mathcal{H}}_F = \bigsqcup_{P \in \mathfrak{B}} e(P). \tag{13.1}$$

The group Γ acts on $\overline{\mathcal{H}}_F$. The quotient $\Gamma \setminus \overline{\mathcal{H}}_F$ is a compact, smooth and C^∞-variety with boundary, so there exists a finite triangulation (see [Mun67]) that induces a triangulation on the boundary $\partial(\Gamma \setminus \overline{\mathcal{H}}_F)$. From this, we obtain a triangulation of $\overline{\mathcal{H}}_F$ which contains a triangulation of $\partial(\overline{\mathcal{H}}_F)$. We consider the complexes of simplicial chains attached to those triangulations, denoted by

$$C_\bullet(\Gamma) \quad \text{and} \quad C_\bullet^\partial(\Gamma).$$

These complexes satisfy the following properties:

- If $i \in \mathbb{N}$, then $C_i(\Gamma)$ and $C_i^\partial(\Gamma)$ are $\mathbb{Z}[\Gamma]$-free modules of finite rank. Since the group Γ acts properly on $\overline{\mathcal{H}}_F$ and its boundary, this is a consequence of the fact that Γ is torsion-free and the (fixed) triangulation of $\Gamma \setminus \overline{\mathcal{H}}_F$ is finite.
- The complex $C_\bullet(\Gamma)$ is a resolution of the trivial $\mathbb{Z}[\Gamma]$-module $\mathbb{Z}$, since this complex gives the homology of $\overline{\mathcal{H}}_F$, which is contractible.
- The complex $C_\bullet^\partial(\Gamma)$ is a resolution of the $\mathbb{Z}[\Gamma]$-module $\mathbb{Z}^{\mathfrak{B}}$, where the action of Γ on $\mathbb{Z}^{\mathfrak{B}}$ is given by the action on $\mathfrak{B}$. In fact, $C_\bullet^\partial(\Gamma)$ gives the homology of $\partial\overline{\mathcal{H}}_F$, and in decomposition (13.1), each $e(P)$ is contractible.

Suppose now that the image of Γ in $\mathrm{GL}_2(F \otimes_{\mathbb{Q}} \mathbb{Q}_p)$ is contained in the Iwahori group (that is, the matrices that are upper-triangular modulo p), as is the case for the groups $\Gamma_1^i(\mathfrak{n})$. Then any right $\Omega_1(\mathfrak{n})$-module M, as in Definition 11.3.1 (ii), has an action of Γ (the reader should keep the case $M = \mathcal{D}_\lambda(L)$ in mind). We define the complexes $C^\bullet(\Gamma, M)$ and $C_\partial^\bullet(\Gamma, M)$ by:

$$C^i(\Gamma, M) := \mathrm{Hom}_{\mathbb{Z}[\Gamma]}(C_i(\Gamma), M),$$

$$C_\partial^i(\Gamma, M) := \mathrm{Hom}_{\mathbb{Z}[\Gamma]}(C_i^\partial(\Gamma), M).$$

The properties given above for $C_\bullet(\Gamma)$ and $C_\bullet^\partial(\Gamma)$ have the following consequences:

- $C^i(\Gamma, M)$ and $C_\partial^i(\Gamma, M)$ are isomorphic to finitely many copies of M. In particular they are nuclear Fréchet L-vector spaces.

- The cohomology of $C^\bullet(\Gamma, M)$ is isomorphic to $\mathrm{H}^\bullet(\Gamma \setminus \mathcal{H}_F, \mathcal{L}_2(M))$.
- The cohomology of $C^\bullet_\partial(\Gamma, M)$ is isomorphic to $\mathrm{H}^\bullet(\partial(\Gamma \setminus \mathcal{H}_F), \mathcal{L}_2(M))$.

Via the natural map $C^\partial_\bullet(\Gamma) \hookrightarrow C_\bullet(\Gamma)$, we obtain a map of complexes $C^\bullet(\Gamma, M) \to C^\bullet_\partial(\Gamma, M)$. We define:

$$C^\bullet_c(\Gamma, M) \coloneqq \mathrm{Cone}\left[C^\bullet(\Gamma, M) \to C^\bullet_\partial(\Gamma, M)\right].$$

Proposition 13.2.4. *For each $i \in \mathbb{N}$, the L-vector space $C^i_c(\Gamma, M)$ is a Fréchet space. The cohomology of the complex $C^\bullet_c(\Gamma, M)$ is $\mathrm{H}^\bullet_c(\Gamma \setminus \mathcal{H}_F, \mathcal{L}(M))$.*

Proof. By construction we have $C^i_c(\Gamma, M) = C^i(\Gamma, M) \oplus C^{i-1}_\partial(\Gamma, M)$. Thus this is a Fréchet space as $C^i(\Gamma, M)$ and $C^{i-1}_\partial(\Gamma, M)$ both are.

For the remainder of the proposition, note that there are isomorphisms

$$\mathrm{H}^i(C^\bullet(\Gamma, M)) \cong \mathrm{H}^i(\Gamma \setminus \mathcal{H}_F, \mathcal{L}_2(M)) \quad \text{and}$$

$$\mathrm{H}^i(C^\bullet_\partial(\Gamma, M)) \cong \mathrm{H}^i(\partial(\Gamma \setminus \mathcal{H}_F), \mathcal{L}_2(M)).$$

Moreover we have two long exact sequences

$$... \to \mathrm{H}^i(C^\bullet_c(\Gamma, M)) \to \mathrm{H}^i(C^\bullet(\Gamma, M)) \to \mathrm{H}^i(C^\bullet_\partial(\Gamma, M)) \to ... \quad \text{and}$$

$$... \to \mathrm{H}^i_c(\Gamma \setminus \mathcal{H}_F, \mathcal{L}(M)) \to \mathrm{H}^i(\Gamma \setminus \mathcal{H}_F, \mathcal{L}(M)) \to \mathrm{H}^i(\partial(\Gamma \setminus \mathcal{H}_F), \mathcal{L}(M)) \to ... \quad .$$

Applying the five-lemma to this gives the result. □

Hecke operators

Let Γ and Γ' be torsion-free arithmetic subgroups of $\mathrm{SL}_2(F)$, let $h : \Gamma \to \Gamma'$ be a group homomorphism and let $f : M \to M$ be a linear transformation such that $f(h(\gamma)\mu) = \gamma f(\mu)$. Using h we can consider the complex $C_\bullet(\Gamma')$ as a resolution of $\mathbb{Z}$ by $\mathbb{Z}[\Gamma]$-modules. Since $C_\bullet(\Gamma)$ is a projective resolution of $\mathbb{Z}$ by $\mathbb{Z}[\Gamma]$-modules, we obtain a map $h_\bullet : C_\bullet(\Gamma) \to C_\bullet(\Gamma')$ compatible with h. Using this last morphism and f we obtain a map:

$$\begin{aligned} C^\bullet(\Gamma', M) &\longrightarrow C^\bullet(\Gamma, M), \\ \varphi &\longmapsto f \circ \varphi \circ h_\bullet. \end{aligned}$$

In the same way as before, we consider the complex $C^\partial_\bullet(\Gamma')$ as a resolution of $\mathbb{Z}^{\mathfrak{B}}$ by $\mathbb{Z}[\Gamma]$-modules, giving a map of complexes $C^\partial_\bullet(\Gamma) \to C^\bullet_i(\Gamma')$. Then we obtain maps

$$C^\bullet_\partial(\Gamma', M) \longrightarrow C^\bullet_\partial(\Gamma, M).$$

From these maps we obtain and the natural compatibility with the maps $C^\bullet(\Gamma, M) \to C^\bullet_\partial(\Gamma, M)$ and $C^\bullet(\Gamma', M) \to C^\bullet_\partial(\Gamma', M)$, we obtain maps

$$C^\bullet_c(\Gamma', M) \longrightarrow C^\bullet_c(\Gamma, M').$$

Let $\eta \in \mathrm{GL}_2(F) \cap \Sigma_0(p)$ (again, the reader should keep in mind $\eta = \left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right)$), and define maps

$$\begin{aligned} h_1 : \Gamma \cap \eta\Gamma\eta^{-1} &\longrightarrow \Gamma \cap \eta^{-1}\Gamma\eta, \\ \gamma &\longmapsto \eta^{-1}\gamma\eta \end{aligned}$$

and $f_1 : M \to M, f_1(\mu) = \mu|\eta$. Let $h_2 : \Gamma \cap \eta^{-1}\Gamma\eta \hookrightarrow \Gamma$ be the inclusion and f_2 the identity map. By considering the pairs (h_1, f_1) and (h_2, f_2) in the situation above, we obtain maps

$$[\eta] : C^\bullet_c(\Gamma \cap \eta^{-1}\Gamma\eta, M) \to C^\bullet_c(\Gamma \cap \eta\Gamma\eta^{-1}, M),$$

$$\mathrm{res}^\Gamma_{\Gamma\cap\eta^{-1}\Gamma\eta} : C^\bullet_c(\Gamma, M) \to C^\bullet_c(\Gamma \cap \eta^{-1}\Gamma\eta, M).$$

We define corestriction maps $\mathrm{cor}^\Gamma_{\Gamma\cap\eta\Gamma\eta^{-1}}$ for the complexes $C^\bullet$ and $C^\bullet_\partial$ in the same way as in [Urb11, §4.2.5] and [BS15, §2.2.2]. Then we obtain maps

$$\mathrm{cor}^\Gamma_{\Gamma\cap\eta\Gamma\eta^{-1}} : C^i_c(\Gamma \cap \eta\Gamma\eta^{-1}, M) \to C^i_c(\Gamma, M).$$

Denote by $[\Gamma\eta\Gamma] = \mathrm{cor}^\Gamma_{\Gamma\cap\eta\Gamma\eta^{-1}} \circ [\eta] \circ \mathrm{res}^\Gamma_{\Gamma\cap\eta^{-1}\Gamma\eta}$ the composition

$$[\Gamma\eta\Gamma] : C^i_c(\Gamma, M) \to C^i_c(\Gamma, M).$$

Proposition 13.2.5. *The operator $[\Gamma\eta\Gamma]$ is compact. Moreover, if $\eta = \left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right)$, it is a lift of the U_p operator on the cohomology to the level of complexes.*

Proof. The action of η on $\mathcal{D}_\lambda(L)$ is compact. Using this property, Proposition 13.2.4, and the fact that composition of a compact map with a continuous map is again compact, we deduce that $[\Gamma\eta\Gamma]$ is compact on $C^i_c(\Gamma, M)$ (see [Urb11], Section 4.2.9).

Now fix $\eta = \left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right)$. The operators $[\Gamma\eta\Gamma]$ on $C^\bullet(\Gamma, M)$ and on $C^i_\bullet(\Gamma, M)$ lift the corresponding U_p operators on $\mathrm{H}^\bullet(\Gamma \setminus \mathcal{H}_F, \mathcal{L}_2(M))$ and $\mathrm{H}^\bullet_\partial(\Gamma \setminus \mathcal{H}_F, \mathcal{L}_2(M))$. Hence we deduce the same result for the compact support situation. ☐

13.2.2. Proof of Theorem 13.2.3

Proof. (Theorem 13.2.3). Recall that we have the decomposition

$$\mathrm{H}^i_{\mathrm{c}}(Y_1(\mathfrak{n}), \mathcal{L}_2(\mathcal{D}_\lambda(L))) = \bigoplus_{j=1}^{h} \mathrm{H}^i_{\mathrm{c}}(Y_1^j(\mathfrak{n}), \mathcal{L}_2(\mathcal{D}_\lambda(L))). \tag{13.2}$$

Moreover, we can describe the action of the U_p operator on $\mathrm{H}^i_{\mathrm{c}}(Y_1(\mathfrak{n}), \mathcal{L}_2(\mathcal{D}_\lambda(L)))$ with respect to this decomposition; indeed, we have

$$U_p = \bigoplus_{j=1}^{h} \left[\Gamma_1^j(\mathfrak{n}) \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \Gamma_1^j(\mathfrak{n}) \right], \tag{13.3}$$

where $\Gamma_1^j(\mathfrak{n})$ is as defined in equation (2.11). For each $i \in \mathbb{N}$, let

$$C^i_c(\mathfrak{n}, \mathcal{D}_\lambda(L)) := \bigoplus_{j=1}^{h} C^i_c(\Gamma_1^j(\mathfrak{n}), \mathcal{D}_\lambda(L)).$$

Using Proposition 13.2.4 and equation (13.2), we deduce that each term $C^i_c(\mathfrak{n}, \mathcal{D}_\lambda(L))$ is a Fréchet L-vector space and that the cohomology of the complex $C^\bullet_c(\mathfrak{n}, \mathcal{D}_\lambda(L))$ is $\mathrm{H}^\bullet_{\mathrm{c}}(Y_1(\mathfrak{n}), \mathcal{L}_2(\mathcal{D}_\lambda(L)))$.

For each j we have an operator

$$\left[\Gamma_1^j(\mathfrak{n}) \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \Gamma_1^j(\mathfrak{n}) \right] : C^i_c\left(\Gamma_1^j(\mathfrak{n}), \mathcal{D}_\lambda(L)\right) \to C^i_c\left(\Gamma_1^j(\mathfrak{n}), \mathcal{D}_\lambda(L)\right),$$

lifting the corresponding operator on the cohomology. We define the operator

$$U_p : C^i_c(\mathfrak{n}, \mathcal{D}_\lambda(L)) \to C^i_c(\mathfrak{n}, \mathcal{D}_\lambda(L))$$

by $U_p = \bigoplus_{j=1}^{h} [\Gamma_1^j(\mathfrak{n}) \left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right) \Gamma_1^j(\mathfrak{n})]$. The decomposition of equation (13.3) implies that U_p is a lift of the corresponding Hecke operator on the cohomology to the level of complexes. Moreover, from Proposition 13.2.5 we deduce that U_p is a compact operator on $C^i_c(\mathfrak{n}, \mathcal{D}_\lambda(L))$.

Finally, we complete the proof of Theorem 13.2.3 by applying [Urb11, Lemma 2.3.13] to U_p. □

13.3. A control theorem

In this section, we prove a *control theorem*, showing that the restriction of the specialisation map from overconvergent to classical modular symbols to the ‘small slope’ subspaces is an isomorphism. We actually need a slightly finer definition of slope decomposition; namely, we define the slope decomposition with respect to a finite set of operators rather than just one.

To this end, let I be a finite set, and suppose that for each $i \in I$, we have an endomorphism U_i on the L-vector space M. Write $A := L[U_i, i \in I]$ for the algebra of polynomials in the variables U_i. Then A acts on M, and for $\mathbf{h} = (h_i) \in \mathbb{Q}^I$ we write

$$M^{\leq \mathbf{h}} := \bigcap_{i \in I} M^{\leq h_i, U_i}.$$

We call $M^{\leq \mathbf{h}, A}$ the *slope* $\leq \mathbf{h}$ *subspace with respect to* A. Where the choice of operators is clear, we will drop the A from the notation and just write $M^{\leq \mathbf{h}}$.

13.3.1. Preliminary results

We start by stating some properties of slope decompositions that will be required in the proof.

Lemma 13.3.1. *(i) Let M, N and P be L-vector spaces equipped with an action of A, and suppose that M, N and P each admit a slope $\leq \mathbf{h}$ decomposition with respect to A. If $0 \to M \to N \to P \to 0$ is an exact sequence of A-modules, then we have an exact sequence*

$$0 \to M^{\leq \mathbf{h}} \to N^{\leq \mathbf{h}} \to P^{\leq \mathbf{h}} \to 0.$$

(ii) Let $M \cong \varprojlim M_n$ be a nuclear Fréchet space equipped with a compact endomorphism U that induces compact operators U_n on M_n for each n. Then for each n there is an isomorphism

$$M^{\leq \mathbf{h}, U} \cong M_n^{\leq \mathbf{h}, U_n}.$$

(iii) Let $(M, ||\cdot||)$ be an L-Banach space equipped with an action of A, where $||\cdot||$ denotes the norm on M, and suppose that there is a $\mathcal{O}_L$-submodule

$$\mathcal{M} \subset \{m \in M : ||m|| \geq 0\}$$

that is stable under the action of A. Let $\mathbf{h} = (h_i)_{i \in I}$ with $h_{i_0} < 0$ for some $i_0 \in I$. Then $M^{\leq \mathbf{h}} = 0$.

Proof. Part (i) is simple (see Corollary 2.3.5 of [Urb11]). Part (ii) is proved in [Urb11], Lemma 2.3.13. For part (iii), suppose that $M^{\leq \mathbf{h}} \neq 0$. Then, after possibly replacing L with a finite extension, we can find $\alpha \in L$ and $x \in \mathcal{M}$ such that $v_p(\alpha) < 0$ and $U_{i_0} x = \alpha x$. Then there exists $n \in \mathbb{Z}$ such that $\alpha^n x \notin \mathcal{M}$. This is a contradiction because $\alpha^n x = U_{i_0}^n x \in \mathcal{M}$ by A-stability of $\mathcal{M}$. $\square$

In particular, we have the following corollary.

Definition 13.3.2. For each $\sigma \in \Sigma$, denote by $\mathfrak{p}(\sigma)$ the unique prime $\mathfrak{p}|p$ such that the embedding $\sigma : F \hookrightarrow \overline{\mathbb{Q}} \subset \mathbb{C}$ extends to an embedding $F_{\mathfrak{p}} \hookrightarrow \overline{\mathbb{Q}_p} \subset \mathbb{C}_p$ that is compatible with the fixed embedding $\mathrm{inc}_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}_p}$. If σ corresponds to $\mathfrak{p}$ under this identification, we write $\sigma \sim \mathfrak{p}$.

Definition 13.3.3. Let $\nu = (\mathbf{k}, \mathbf{v}) \in \mathbb{Z}[\Sigma]^2$ be an admissible weight. Define

$$\omega_{\mathfrak{p}}(\nu) := \sum_{\sigma \sim \mathfrak{p}} v_\sigma.$$

Corollary 13.3.4. *(i) Let $\nu = (\mathbf{k}, \mathbf{v}) \in \mathbb{Z}[\Sigma]^2$ be a weight with $\mathbf{k} + 2\mathbf{v}$ parallel (but allowing for negative values of k_σ). Let $\mathbf{h} \in \mathbb{Q}^{\{\mathfrak{p}|p\}}$ be such that*

$$h_{\mathfrak{p}} < \frac{\omega_{\mathfrak{p}}(\nu)}{e_{\mathfrak{p}}}$$

for some prime $\mathfrak{p}$ above p. Then for all r, we have $\mathrm{H}^r_{\mathrm{c}}(Y_1(\mathfrak{n}), \mathcal{L}_2(\mathcal{D}_\nu(L)))^{\leq \mathbf{h}} = \{0\}$.

(ii) Under the same hypotheses, the same result holds if we replace $\mathcal{D}_\nu(L)$ with any $\Sigma_0(p)$-stable submodule or by quotients by such submodules.

Proof. From Chapter 13.1, we know that

$$\mathcal{D}_\lambda(L) \cong \varprojlim \mathcal{D}_{\lambda,n}(L),$$

where $\mathcal{D}_{\lambda,n}(L)$ is the (L-Banach space) of distributions that are locally analytic of radius n. We also know (from results in the previous section) that the cohomology group $\mathrm{H}^r_{\mathrm{c}}(Y_1(\mathfrak{n}), \mathcal{L}_2(\mathcal{D}_{\nu,0}(L)))$ is an L-Banach space, and we see that $\mathrm{H}^r_{\mathrm{c}}(Y_1, \mathcal{L}_2(\mathcal{D}_{\nu,0}(\mathcal{O}_L)))$ is a $\mathcal{O}_L$-submodule of the elements of non-negative norm. This space is *not* necessarily preserved by the Hecke operators at p, but it *is* preserved by the modified operators

$$U'_{\mathfrak{p}} := \pi_{\mathfrak{p}}^{-\omega_{\mathfrak{p}}(\nu)} U_{\mathfrak{p}},$$

where we scale by $\pi_{\mathfrak{p}}^{-\omega_{\mathfrak{p}}(\nu)}$ to ensure integrality in the case $\omega_{\mathfrak{p}}(\nu)$ is large and negative. Write $A' := L[U'_{\mathfrak{p}}]$ for the algebra generated by these modified operators. Applying part (iii) of the above lemma, we see that if $\mathbf{h}' \in \mathbb{Q}^{\{\mathfrak{p}|p\}}$ is chosen such that $h'_{\mathfrak{p}} < 0$ for some prime $\mathfrak{p}$ above p, we have

$$\mathrm{H}^r_{\mathrm{c}}(Y_1(\mathfrak{n}), \mathcal{L}_2(\mathcal{D}_\nu(L)))^{\leq \mathbf{h}', A'} = \{0\}.$$

Now note that for any operator U on a nuclear Fréchet space M, we have a relation

$$M^{h, p^k U} \cong M^{h-k, U}.$$

In particular, define $\mathbf{h} \in \mathbb{Q}^{\{\mathfrak{p}|p\}}$ by

$$h_{\mathfrak{p}} := h'_{\mathfrak{p}} + \frac{\omega_{\mathfrak{p}}(\nu)}{e_{\mathfrak{p}}}.$$

Note that $h'_{\mathfrak{p}} < 0$ for some $\mathfrak{p}$ above p if and only if $h_{\mathfrak{p}} < \frac{\omega_{\mathfrak{p}}(\nu)}{e_{\mathfrak{p}}}$ for some $\mathfrak{p}$ above p, and that the space on which the Hecke operators at p act with slope $\leq \mathbf{h}$ is isomorphic to the space on which the operators $U'_{\mathfrak{p}}$ act with slope $\leq \mathbf{h}'$. Part (i) follows.

The proof for submodules is identical. The case of quotients then follows by taking a long exact sequence, applying Lemma 13.3.1(i), and using the result for submodules. □

13.3.2. Theta maps and partially overconvergent coefficients

We now introduce modules of partially overconvergent coefficients that will play a key role in the proof.

For any $\sigma \in \Sigma$, let $\lambda_\sigma = (\mathbf{k}', \mathbf{v}')$ be the weight defined by

$$k'_\tau = \begin{cases} k_\tau & : \tau \neq \sigma, \\ -2 - k_\sigma & : \tau = \sigma. \end{cases}, \qquad v'_\sigma = \begin{cases} v_\tau & : \tau \neq \sigma, \\ v_\sigma + k_\sigma + 1 & : \tau = \sigma. \end{cases} \tag{13.4}$$

Let f be a locally analytic function on $\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p$, and let $\{V\}$ be an open cover of $\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p$ such that $f|_V$ is analytic for each V. Then we can consider $f|_V$ as a power series in the d variables $\{z_\sigma : \sigma \in \Sigma\}$. We can consider the operator $(d/dz_\sigma)^{k_\sigma+1}$ on such power series in the natural way, and note that this induces a map

$$\Theta_\sigma : \mathcal{A}_\lambda(L) \longrightarrow \mathcal{A}_{\lambda_\sigma}(L).$$

For more details about this map, see [Urb11, Prop. 3.2.11]. Taking the continuous dual of this map, we obtain a map

$$\Theta_\sigma^* : \mathcal{D}_{\lambda_\sigma}(L) \longrightarrow \mathcal{D}_\lambda(L).$$

Remark: This map is equivariant with respect to the action of $\Sigma_0(p)$. Note, however, that the action of the $U_{\mathfrak{p}}$ operator is *different* on $\mathcal{D}_{\lambda_\sigma}(L)$ and $\mathcal{D}_\lambda(L)$, due to the scaling of $\mathbf{v}$ at σ. Indeed, we introduce a factor of the determinant of the component at σ to the power of $k_\sigma + 1$.

Now label the elements of Σ as $\sigma_1, \sigma_2, ..., \sigma_d$, where we can choose any ordering of the elements. We write $\Theta_0^* : \{0\} \to \mathcal{D}_\lambda$, and for each $s = 1, ..., d$, we denote by Θ_s^* the map

$$\Theta_s^* := \sum_{i=1}^{s} \Theta_{\sigma_i}^* : \bigoplus_{i=1}^{s} \mathcal{D}_{\lambda_{\sigma_i}}(L) \longrightarrow \mathcal{D}_\lambda(L).$$

The cokernels of the maps Θ_s^* play a crucial role in the sequel. In particular, from the definition it is clear that $\mathrm{coker}(\Theta_0^*) = \mathcal{D}_\lambda(L)$. Consider now the map Θ_1^*. If $\mu \in \mathcal{D}_{\lambda_{\sigma_1}}(L)$, then $\Theta_1^*(\mu)$ is 0 on elements of $\mathbb{A}_\lambda(L)$ that are locally polynomial in z_{σ_1} of degree at most k_{σ_1}. Hence, for $\mu \in \mathcal{D}_\lambda(L)$, we have $\mu \notin \mathrm{Im}(\Theta_1^*)$ if and only if there exists a monomial $\mathbf{z}^{\mathbf{r}} := \prod_{\sigma\in\Sigma} z_\sigma^{r_\sigma}$ with $r_{\sigma_1} \leq k_{\sigma_1} + 1$ such that $\mu(\mathbf{z}^{\mathbf{r}}) \neq 0$. From this one can see that $\mathrm{coker}(\Theta_1^*)$ can be seen as the module of coefficients that are classical at σ_1 and overconvergent at $\sigma_2, ..., \sigma_d$. This motivates the following:

Definition 13.3.5. Let $J \subset \Sigma$. For $\nu = (\mathbf{k}, \mathbf{v}) \in \mathbb{Z}[\Sigma]^2$, define $\mathcal{A}_\nu^J(L)$ to be the space of functions on $\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p$ defined over L that are locally analytic in the variables z_σ for $\sigma \notin J$ and locally algebraic of degree at most $\max(k_\sigma, 0)$ in the variables z_σ for $\sigma \in J$. Define $\mathcal{D}_\nu^J(L)$ be the topological dual of $\mathcal{A}_\nu^J(L)$.

Thus we see that $\mathrm{coker}(\Theta_1^*) = \mathcal{D}_\lambda^{\{\sigma_1\}}(L)$. Continuing in the same vein, we see that $\mathrm{coker}(\Theta_s^*) = \mathcal{D}_\lambda^{J_s}(L)$, where $J_s := \{\sigma_1, ..., \sigma_s\}$. In particular, if we write $V_{\lambda,\mathrm{loc}}(L)$ for the space of locally algebraic polynomials on $\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p$ of degree at most $\mathbf{k}$, with the natural action of $\Sigma_0(p)$ depending on λ, then we get:

Proposition 13.3.6. *There is an exact sequence*

$$\bigoplus_{\sigma\in\Sigma} \mathcal{D}_{\lambda_\sigma}(L) \xrightarrow{\Theta_d^*} \mathcal{D}_\lambda(L) \longrightarrow V_{\lambda,\mathrm{loc}}(L) \longrightarrow 0.$$

In particular, we have

$$\mathrm{coker}(\Theta_d^*) = \mathcal{D}_\lambda^\Sigma(L) \cong V_{\lambda,\mathrm{loc}}(L)^*.$$

These are the last terms of the locally analytic BGG resolution introduced in [Urb11], Section 3.3. See Proposition 3.2.12 of Urban's paper for further details of this exact sequence.

Remark: This bears comparison to the results in Chapter 9, where we use this notion of 'half-overconvergent' modular symbols.

13.3.3. The control theorem

The following theorem is the main result of this part of the chapter, and allows us to canonically lift small-slope classical modular symbols to overconvergent modular symbols.

Theorem 13.3.7. *Let $\lambda = (\mathbf{k}, \mathbf{v})$ be an admissible weight, and let $\mathbf{h} = (h_\mathfrak{p})_{\mathfrak{p}|p} \in \mathbb{Q}^{\{\mathfrak{p}|p\}}$. Let $k_\mathfrak{p}^0 := \min\{k_\sigma : \sigma \sim \mathfrak{p}\}$ and recall the definition of $\omega_\mathfrak{p}(\lambda)$ from Definition 13.3.3. If for each prime $\mathfrak{p}$ above p we have*

$$h_\mathfrak{p} < \frac{k_\mathfrak{p}^0 + \omega_\mathfrak{p}(\lambda) + 1}{e_\mathfrak{p}}, \tag{13.5}$$

then, for each r, the restriction

$$\rho : \mathrm{H}^r_{\mathrm{c}}(Y_1(\mathfrak{n}), \mathcal{L}_2(\mathcal{D}_\lambda(L)))^{\leq \mathbf{h}} \xrightarrow{\sim} \mathrm{H}^r_{\mathrm{c}}(Y_1(\mathfrak{n}), \mathcal{L}_2(V_\lambda(L)^*))^{\leq \mathbf{h}}$$

of the specialisation map to the slope $\leq$ $\mathbf{h}$ subspaces with respect to the $U_{\mathfrak{p}}$-operators is an isomorphism.

To prove this, we make use of:

Lemma 13.3.8. *In the set-up of Theorem 13.3.7, if $\mathbf{h}$ satisfies equation (13.5), then for any s there is an isomorphism*

$$\mathrm{H}^r_{\mathrm{c}}(Y_1(\mathfrak{n}), \mathcal{L}_2(\mathcal{D}_\lambda^{J_{s-1}}(L)))^{\leq \mathbf{h}} \xrightarrow{\sim} \mathrm{H}^r_{\mathrm{c}}(Y_1(\mathfrak{n}), \mathcal{L}_2(\mathcal{D}_\lambda^{J_s}(L)))^{\leq \mathbf{h}}$$

induced from the natural specialisation maps.

Proof. We follow [Urb11]. For any $\sigma \in \Sigma$, let $\lambda_\sigma = (\mathbf{k}', \mathbf{v}')$ be the weight defined in equation (13.4), and recall the theta maps

$$\Theta_s^* : \bigoplus_{i=1}^{s} \mathcal{D}_{\lambda_{\sigma_i}}(L) \to \mathcal{D}_\lambda(L).$$

Recall that $\mathrm{coker}(\Theta_s^*) = \mathcal{D}_\lambda^{J_s}(L)$ can be viewed as a module of distributions that are classical at $\sigma_1, ..., \sigma_s$ and overconvergent at $\sigma_{s+1}, ..., \sigma_d$. In particular, there are natural projection maps $\mathcal{D}_\lambda^{J_{s-1}}(L) \to \mathcal{D}_\lambda^{J_s}(L)$ given by specialising from overconvergent to classical coefficients at σ_s. Moreover, from the definition of $\Theta_{\sigma_s}^*$ there is an exact sequence

$$\mathcal{D}_{\lambda_s}(L) \xrightarrow{\Theta_{\sigma_s}^*} \mathcal{D}_\lambda^{J_{s-1}}(L) \longrightarrow \mathcal{D}_\lambda^{J_s}(L) \longrightarrow 0,$$

and a closer inspection shows that the sequence

$$0 \longrightarrow \mathcal{D}_{\lambda_{\sigma_s}}^{J_{s-1}}(L) \longrightarrow \mathcal{D}_\lambda^{J_{s-1}}(L) \longrightarrow \mathcal{D}_\lambda^{J_s}(L) \longrightarrow 0 \tag{13.6}$$

is exact for the quotient $\mathcal{D}_{\lambda_{\sigma_s}}^{J_{s-1}}(L)$ of $\mathcal{D}_{\lambda_{\sigma_s}}(L)$.

Using Lemma 13.3.1 on the exact sequence of equation (13.6), we obtain the exact sequence

$$\begin{aligned} \cdots \longrightarrow \mathrm{H}^i_{\mathrm{c}}(Y_1(\mathfrak{n}),&\mathcal{L}_2(\mathcal{D}_{\lambda_{\sigma_s}}^{J_{s-1}}(L)))^{\leq \mathbf{h}} \longrightarrow \mathrm{H}^i_{\mathrm{c}}(Y_1(\mathfrak{n}), \mathcal{L}_2(\mathcal{D}_\lambda^{J_{s-1}}(L)))^{\leq \mathbf{h}} \\ &\longrightarrow \mathrm{H}^i_{\mathrm{c}}(Y_1(\mathfrak{n}), \mathcal{L}_2(\mathcal{D}_\lambda^{J_s}(L)))^{\leq \mathbf{h}} \longrightarrow \mathrm{H}^{i+1}_{\mathrm{c}}(Y_1(\mathfrak{n}), \mathcal{L}_2(\mathcal{D}_{\lambda_{\sigma_s}}^{J_{s-1}}(L)))^{\leq \mathbf{h}} \longrightarrow \cdots, \end{aligned}$$

where here we're taking slope decompositions with respect to the Hecke operators at p.

If $h_\mathfrak{p} < (k_\mathfrak{p}^0 + \omega_\mathfrak{p}(\lambda) + 1)/e_\mathfrak{p}$ for all primes above p, it follows that

$$h_{\mathfrak{p}(\sigma_s)} < \frac{k_{\sigma_s} + \omega_{\mathfrak{p}(\sigma_s)}(\lambda) + 1}{e_{\mathfrak{p}(\sigma_s)}} = \frac{\omega_{\mathfrak{p}(\sigma_s)}(\lambda_{\sigma_s})}{e_{\mathfrak{p}(\sigma_s)}}.$$

Now, by Corollary 13.3.4 (ii), as $\mathcal{D}_{\lambda_{\sigma_s}}^{J_{s-1}}$ is a quotient of $\mathcal{D}_{\lambda_{\sigma_s}}$, we must have

$$\mathrm{H}_\mathrm{c}^r(Y_1(\mathfrak{n}), \mathcal{L}_2(\mathcal{D}_{\lambda_{\sigma_s}}^{J_{s-1}}(L)))^{\leq \mathbf{h}} = \{0\}$$

for all r. Then, using the long exact sequence, for all r we have

$$\mathrm{H}_\mathrm{c}^r(Y_1(\mathfrak{n}), \mathcal{L}_2(\mathcal{D}_\lambda^{J_{s-1}}(L)))^{\leq \mathbf{h}} \cong \mathrm{H}_\mathrm{c}^r(Y_1(\mathfrak{n}), \mathcal{L}_2(\mathcal{D}_\lambda^{J_s}(L)))^{\leq \mathbf{h}},$$

as required. □

Proof. (Theorem 13.3.7). Recall that we defined $V_{\lambda,\mathrm{loc}}(L) \subset \mathcal{A}(L)$ to be the subspace of functions which are locally polynomial of degree at most $\mathbf{k}$. We see that $V_{\lambda,\mathrm{loc}}(L) \cong \varprojlim V_{\lambda,n}(L)$, where $V_{\lambda,n}(L) \coloneqq \mathcal{A}_{\lambda,n}(L) \cap V_{\lambda,\mathrm{loc}}(L)$. Note that $V_\lambda(L) = V_{\lambda,0}(L)$. In particular, using Lemma 13.3.1, we have

$$\mathrm{H}_\mathrm{c}^q(Y_1(\mathfrak{n}), \mathcal{L}_2(V_{\lambda,\mathrm{loc}}(L)^*))^{\leq \mathbf{h}} \cong \mathrm{H}_\mathrm{c}^q(Y_1(\mathfrak{n}), \mathcal{L}_2(V_\lambda(L)^*))^{\leq \mathbf{h}}.$$

Hence it suffices to prove the theorem by considering the coefficients of the target space to be in $V_{\lambda,\mathrm{loc}}(L)^*$ instead of $V_\lambda(L)^*$.

We use the lemma. For this, note that $\mathcal{D}_\lambda^\Sigma(L) = V_{\lambda,\mathrm{loc}}(L)^*$ and $\mathcal{D}_\lambda^\varnothing(L) = \mathcal{D}_\lambda(L)$. A simple induction on s then shows that we have the required isomorphism. □

Chapter 14

Constructing the p-adic L-function

In this chapter, we conclude this part of the thesis by showing how to construct the p-adic L-function of a small slope automorphic form for GL_2 *over a number field. In particular, in the first section, we use overconvergent analogues of the classical evaluation maps of Chapter 12.1.2 to attach a canonical ray class distribution to an overconvergent modular eigensymbol* Ψ*. In the case where* Ψ *is attached to a small slope automorphic form* Φ *via the control theorem, we then show that this distribution interpolates critical values of the L-function of* Φ*. To do this, we relate the overconvergent evaluations with their classical analogues. Unlike in the imaginary quadratic case, this interpolation property does not necessarily determine the distribution uniquely; we conclude by remarking on this lack of uniqueness. To get around this, we note that our construction is canonical at every step, and that it thus makes sense to define the p-adic L-function of* Φ *to be the distribution we construct.*

14.1. Construction of the distribution

Let Φ be a cuspidal eigenform over F that has small slope (in the sense of the previous section). Then via Eichler–Shimura, we can attach to Φ a small slope p-adic classical modular eigensymbol, and using the results of previous sections, we can lift this to a unique small slope overconvergent eigensymbol. In the work of Pollack and Stevens in [PS11] and [PS12], and the work in Part II of this thesis, once one has such a symbol, one can evaluate it at the cycle $\{0\} - \{\infty\}$ to obtain the p-adic L-function we desire. To generalise this to an arbitrary number field, we use automorphic cycles to define overconvergent analogues of the evaluation maps of Chapter 12.1.2, following the work of Barrera in [BS13] in the totally real case. The notation we use here was fixed in Chapter 12.1.1.

14.1.1. Evaluating overconvergent classes

Recall that in Section 12.1.2, we used automorphic cycles to define evaluation maps on the space of classical modular symbols with complex coefficients. Here, we adapt these evaluation maps to the case of *overconvergent* modular symbols with *p-adic* coefficients. As we are considering a different local system on $Y_1(\mathfrak{n})$, this will be slightly different. In the sequel, we will link all of the various evaluation maps together by explicitly examining the interplay between the various local systems.

Suppose $\Psi \in \mathrm{H}^q_c(Y_1(\mathfrak{n}), \mathcal{L}_2(\mathcal{D}_\lambda(L)))$. Here recall that we consider the local system given by fibres of

$$\mathrm{GL}_2(F)\backslash(\mathrm{GL}_2(\mathbb{A}_F) \times \mathcal{D}_\lambda(L))/\Omega_1(\mathfrak{n})K_\infty^+ Z_\infty \longrightarrow Y_1(\mathfrak{n}),$$

where the action is by

$$\gamma(x,\mu)uk = (\gamma xuk, \mu * u).$$

In this setting, the evaluation maps will allow us to associate a distribution to such a class.

Step 1: Pulling back to $X_\mathfrak{f}$

First we pullback along the map $\eta_\mathfrak{f} : X_\mathfrak{f} \to Y_1(\mathfrak{n})$. We have

$$\eta_\mathfrak{f}^*\Psi \in \mathrm{H}^q_c(X_\mathfrak{f}, \eta_\mathfrak{f}^*\mathcal{L}_2(\mathcal{D}_\lambda(L))).$$

We can see (by examining equation (12.1)) that here the local system corresponding to $\mathcal{L}'_{\mathfrak{f},2}(\mathcal{D}_\lambda(L)) \coloneqq \eta_\mathfrak{f}^*\mathcal{L}_2(\mathcal{D}_\lambda(L))$ is given by the fibres of

$$F^\times\backslash(\mathbb{A}_F^\times \times \mathcal{D}_\lambda(L))/U(\mathfrak{f})F_\infty^1 \longrightarrow X_\mathfrak{f},$$

with action

$$\gamma(x,\mu)ur = \left(\gamma xur, \mu * \begin{pmatrix} u & ((u-1)\mathfrak{f}^{-1})_{v|p} \\ 0 & 1 \end{pmatrix}\right).$$

Step 2: Twisting the action

Unlike in the complex case described earlier, the action describing the local system above is *not* a nice action, so we twist to get a nicer action of units. To this end, the matrix

$$\begin{aligned} \begin{pmatrix} 1 & -1 \\ 0 & (\mathfrak{f})_{v|p} \end{pmatrix} &\in \mathrm{GL}_2\Bigg(\prod_{\mathfrak{p}|p} F_\mathfrak{p}\Bigg) \\ &= \mathrm{GL}_2(\mathcal{O}_F \otimes \mathbb{Z}_p) \end{aligned}$$

lies in $\Sigma(F \otimes_\mathbb{Q} \mathbb{Q}_p)$. So we twist our local system by this; denote this twist on distributions by

$$\begin{aligned} \zeta : \mathcal{D}_\lambda(L) &\longrightarrow \mathcal{D}_\lambda(L), \\ \mu &\longmapsto \mu * \begin{pmatrix} 1 & -1 \\ 0 & (\mathfrak{f})_{v|p} \end{pmatrix}, \end{aligned}$$

and consider

$$\zeta_*\eta_{\mathfrak{f}}^*\Psi \in \mathrm{H}_c^q(X_{\mathfrak{f}}, \mathcal{L}_{\mathfrak{f},2}(\mathcal{D}_\lambda(L))),$$

where now the local system $\mathcal{L}_{\mathfrak{f},2}(\mathcal{D}_\lambda(L))$ is given by

$$F^\times\backslash(\mathbb{A}_F^\times \times \mathcal{D}_\lambda(L))/U(\mathfrak{f})F_\infty^1 \longrightarrow X_{\mathfrak{f}},$$

$$\gamma(x,\mu)ur = \left(\gamma xur, \mu * \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}\right).$$

Step 3: Passing to individual components

In identical fashion to Section 12.1.2, we pull back under the isomorphism $\tau_{a_{\mathbf{y}}} : E(\mathfrak{f})F_\infty^1\backslash F_\infty^+ \xrightarrow{\sim} X_{\mathbf{y}} \hookrightarrow X_{\mathfrak{f}}$ given by multiplication by $a_{\mathbf{y}}$. Then we have

$$\tau_{a_{\mathbf{y}}}^*\zeta_*\eta_{\mathfrak{f}}^*\Psi \in \mathrm{H}_c^q(E(\mathfrak{f})F_\infty^1\backslash F_\infty^+, \mathcal{L}_{\mathfrak{f},\mathbf{y},2}(\mathcal{D}_\lambda(L))),$$

where the local system $\mathcal{L}_{\mathfrak{f},\mathbf{y},2}(\mathcal{D}_\lambda(L))$ is given by

$$E(\mathfrak{f})F_\infty^1\backslash(F_\infty^+ \times \mathcal{D}_\lambda(L)) \longrightarrow E(\mathfrak{f})F_\infty^1\backslash F_\infty^+,$$

$$er(z,\mu) = \left(erz, \mu * \begin{pmatrix} e^{-1} & 0 \\ 0 & 1 \end{pmatrix}\right).$$

(Note here that whilst $u \in U(\mathfrak{f})$ acts as $\left(\begin{smallmatrix} u & 0 \\ 0 & 1 \end{smallmatrix}\right)$, in this step we now have an inverse. This because u is considered as an element of the finite ideles whilst we instead see e as a diagonal infinite idele, which is equivalent under multiplication by $F^\times$ to e^{-1} as a diagonal finite idele and thus an element of $U(\mathfrak{f})$).

Step 4: Restricting the coefficient system

We would like a constant local system. This would allow us to evaluate the cohomology class easily. We see that if we restrict to a quotient of $\mathcal{D}_\lambda(L)$ such that, for all $e \in E(\mathfrak{f})$, the matrix $\left(\begin{smallmatrix} e & 0 \\ 0 & 1 \end{smallmatrix}\right)$ acts trivially, then we have precisely this. With this in mind, we make the following definitions:

Definition 14.1.1. (i) Define $\mathcal{A}_\lambda^+(L)$ to be the subspace of $\mathcal{A}_\lambda(L)$ given by

$$\mathcal{A}_\lambda^+(L) := \left\{ f \in \mathcal{A}_\lambda(L) : \begin{pmatrix} e & 0 \\ 0 & 1 \end{pmatrix} * f = f \ \forall e \in E(1) \right\}.$$

Note that equivalently this is the set of all $f \in \mathcal{A}_\lambda(L)$ such that $f(ez) = e^{\mathbf{k}+\mathbf{v}} f(z)$.

(ii) Define $\mathcal{D}_\lambda^+(L)$ to be the topological dual of $\mathcal{A}_\lambda^+(L)$. Note that $\mathcal{D}_\lambda^+(L)$ is a quotient of

$\mathcal{D}_\lambda(L)$.

Now, if we pushforward via the map

$$\begin{aligned}\nu : \mathcal{D}_\lambda(L) &\longrightarrow \mathcal{D}_\lambda^+(L),\\ \mu &\longmapsto \mu|_{\mathcal{A}_\lambda^+(L)},\end{aligned}$$

then the resulting local system is constant. We see that

$$\begin{aligned}\nu_*\tau_{a_{\mathbf{y}}}^*\zeta_*\eta_{\mathfrak{f}}^*\Psi \in &\mathrm{H}_{\mathrm{c}}^q(E(\mathfrak{f})F_\infty^1\backslash F_\infty^+, \mathcal{D}_\lambda^+(L))\\ &\cong \mathcal{D}_\lambda^+(L),\end{aligned}$$

where the isomorphism is given by integrating over $E(\mathfrak{f})F_\infty^1\backslash F_\infty^+$.

Definition of the evaluation map

Definition 14.1.2. We write $\mathrm{Ev}_{\mathfrak{f},\dagger}^{a_{\mathbf{y}}}$ for the composition

$$\mathrm{Ev}_{\mathfrak{f},\dagger}^{a_{\mathbf{y}}} : \mathrm{H}_{\mathrm{c}}^q(Y_1(\mathfrak{n}), \mathcal{L}_2(\mathcal{D}_\lambda(L))) \longrightarrow \mathcal{D}_\lambda^+(L)$$

of the maps

$$\begin{aligned}\mathrm{H}_{\mathrm{c}}^q(Y_1(\mathfrak{n}),&\mathcal{L}_2(\mathcal{D}_\lambda(L))) \xrightarrow{\zeta_*\eta_{\mathfrak{f}}^*} \mathrm{H}_{\mathrm{c}}^q(X_{\mathfrak{f}}, \mathcal{L}_{\mathfrak{f},2}(\mathcal{D}_\lambda(L))) \xrightarrow{\tau_{a_{\mathbf{y}}}^*} \cdots\\ &\mathrm{H}_{\mathrm{c}}^q(E(\mathfrak{f})F_\infty^1\backslash F_\infty^+, \mathcal{L}_{\mathfrak{f},\mathbf{y},2}(\mathcal{D}_\lambda(L))) \xrightarrow{\nu_*} \mathrm{H}_{\mathrm{c}}^q(E(\mathfrak{f})F_\infty^1\backslash F_\infty^+, \mathcal{D}_\lambda^+(L)) \cong \mathcal{D}_\lambda^+(L).\end{aligned}$$

In particular, we have maps $\mathrm{Ev}_{\mathfrak{f},\dagger}^{a_{\mathbf{y}}}$ for each $\mathbf{y} \in \mathrm{Cl}_F^+(\mathfrak{f})$. Note that these maps are dependent on the choice of representatives. Despite this, we have then proved:

Proposition 14.1.3. *There is, for a fixed choice of representatives $\{a_{\mathbf{y}} \in \mathbb{A}_F^\times : \mathbf{y} \in \mathrm{Cl}_F^+(\mathfrak{f})\}$, a map*

$$\mathrm{H}_{\mathrm{c}}^q(Y_1(\mathfrak{n}), \mathcal{L}_2(\mathcal{D}_\lambda(L))) \xrightarrow{\oplus_{\mathbf{y}}\mathrm{Ev}_{\mathfrak{f},\dagger}^{a_{\mathbf{y}}}} \bigoplus_{\mathrm{Cl}_F^+(\mathfrak{f})} \mathcal{D}_\lambda^+(L).$$

To summarise the above construction: to an overconvergent modular symbol, for a fixed ideal $\mathfrak{f}$, we attach a collection of distributions (indexed by $\mathrm{Cl}_F^+(\mathfrak{f})$) with a compatible action of the totally positive units of F. This construction depends on a choice of idelic representatives for $\mathrm{Cl}_F^+(\mathfrak{f})$.

14.1.2. Locally analytic functions on $\mathrm{Cl}_F^+(p^\infty)$

The p-adic L-function should be a locally analytic distribution on the ray class group $\mathrm{Cl}_F^+(p^\infty)$. Before constructing such a distribution, we must take a digression to describe what locally analytic functions on this space actually look like.

The geometry of $\mathrm{Cl}_F^+(p^\infty)$

We first recall the geometry of $\mathrm{Cl}_F^+(p^\infty)$. It is defined as follows:

$$\mathrm{Cl}_F^+(p^\infty) := F^\times \backslash \mathbb{A}_F^\times / U(p^\infty) F_\infty^+.$$

Letting $\mathfrak{f}$ range over all ideals dividing $(p)^\infty$ and taking the inverse limit of the series of exact sequences

$$\mathcal{O}_{F,+}^\times \longrightarrow (\mathcal{O}_F/\mathfrak{f})^\times \longrightarrow \mathrm{Cl}_F^+(\mathfrak{f}) \longrightarrow \mathrm{Cl}_F^+ \longrightarrow 0,$$

we see that we have an exact sequence

$$\overline{\mathcal{O}_{F,+}^\times} \longrightarrow (\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p)^\times \longrightarrow \mathrm{Cl}_F^+(p^\infty) \longrightarrow \mathrm{Cl}_F^+ \longrightarrow 0,$$

so that – after picking a choice of representatives for Cl_F^+ – we have

$$\mathrm{Cl}_F^+(p^\infty) \cong \bigsqcup_{\mathrm{Cl}_F^+} (\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p)^\times / \overline{E(1)}.$$

(Here note that $E(1) = \mathcal{O}_{F,+}^\times$, and we've taken $\overline{E(1)}$ to be the closure of the image of $E(1)$ in $(\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p)^\times$). Indeed, for any $\mathfrak{f}$, we can go further, and write

$$\mathrm{Cl}_F^+(p^\infty) \cong \bigsqcup_{\mathbf{y} \in \mathrm{Cl}_F^+(\mathfrak{f})} G_{\mathbf{y}},$$

where

$$G_{\mathbf{y}} := \{z \in \mathrm{Cl}_F^+(p^\infty) : z \mapsto \mathbf{y} \text{ under the map } \mathrm{Cl}_F^+(p^\infty) \to \mathrm{Cl}_F^+(\mathfrak{f})\}. \tag{14.1}$$

Note that multiplication by $a_{\mathbf{y}}^{-1}$ gives an isomorphism

$$G_{\mathbf{y}} \cong G := \{z \in (\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p)^\times : z \equiv 1 \ (\mathrm{mod}\ \mathfrak{f})\}/\overline{E(\mathfrak{f})}.$$

Analytic functions

Let L be a (not necessarily finite) extension of $\mathbb{Q}_p$ contained in $\mathbb{C}_p$, the completion of the algebraic closure of $\mathbb{Q}_p$. Suppose the L is large enough to contain the image of all completions of F at primes above p under their embeddings into $\mathbb{C}_p$.

Definition 14.1.4. A locally analytic function on $\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p$ defined over L is a function $f : \mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p \to L$, such that for each $z_0 \in \mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p$, there is a neighbourhood $U \subset \mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p$ of z_0 such that

$$f\Big|_U (z) = \sum_{\mathbf{r} \geq 0} a_{\mathbf{r}} (z - z_0)^{\mathbf{r}}, \quad a_{\mathbf{r}} \in L,$$

where $\mathbf{r} \in \mathbb{Z}[\Sigma]$.

For a choice of idelic representatives $\{a_i\} \subset \mathbb{A}_F^\times$ of Cl_F^+, we can consider any function

$$\varphi : \mathrm{Cl}_F^+(p^\infty) \longrightarrow L$$

as a collection

$$\varphi_{a_i} : (\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p)^\times / \overline{E(1)} \longrightarrow L,$$

where

$$\varphi_{a_i}(z) := \varphi(a_i^{-1} z).$$

Then, in a slight abuse of notation, φ_{a_i} can be thought of as a function

$$\varphi_{a_i} : \mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p \longrightarrow L$$

with support on $(\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p)^\times$ and with $\varphi_{a_i}(ez) = \varphi_{a_i}(z)$ for all $e \in \overline{E(1)}$.

Definition 14.1.5. We say that φ is *locally analytic on* $\mathrm{Cl}_F^+(p^\infty)$ if each φ_{a_i} is locally analytic on $\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p$.

This is independent of the choice of class group representatives. Before we prove this, we rephrase the condition slightly. If $U \subset \mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p$ is an open set, then we say a function $f : U \to L$ is *analytic* if it can be written as a single convergent power series. By definition, as $\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p$ is compact, φ_{a_i} is locally analytic as a function on $\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p$ if and only if there exists an ideal $\mathfrak{f} | p^\infty$ such that φ_{a_i} is analytic on each of the sets $\{a + \mathfrak{f}(\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p) : a \in \mathcal{O}_F \otimes \mathbb{Z}_p\}$. To make this more precise: given a function $\varphi : \mathrm{Cl}_F^+(p^\infty) \to L$, and an ideal $\mathfrak{f} | p^\infty$, pick some choice of idelic representatives $a_{\mathbf{y}}$ for $\mathrm{Cl}_F^+(\mathfrak{f})$, and define

$$\begin{aligned} \varphi_{a_{\mathbf{y}}} : G &\longrightarrow L, \\ z &\longmapsto \varphi(a_{\mathbf{y}}^{-1} z), \end{aligned}$$

where $G \cong G_{\mathbf{y}}$ is as above. Also as above, $\varphi_{a_{\mathbf{y}}}$ can be viewed as a function on $\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p$. We say that φ is locally analytic if there exists some $\mathfrak{f}$ such that $\varphi_{a_{\mathbf{y}}}$ is analytic for each $\mathbf{y}$.

Proposition 14.1.6. *Let $\varphi : \mathrm{Cl}_F^+(p^\infty) \to L$ be a function. Then in the above construction, if φ is locally analytic for some choice of idelic class group representatives for $\mathrm{Cl}_F^+(\mathfrak{f})$, then it is locally analytic for* any *choice of representatives.*

Proof. Suppose there exists an $\mathfrak{f}$ and a set $\{a_{\mathbf{y}}\}$ of representatives such that each $\varphi_{a_{\mathbf{y}}}$ is analytic. Consider a different choice of representatives $\{a'_{\mathbf{y}}\}$. Then, for a fixed $\mathbf{y}$, we have $a'_{\mathbf{y}} = a_{\mathbf{y}} \gamma u r$, where $\gamma \in F^\times, u \in U(\mathfrak{f})$, and $r \in F_\infty^+$. Then

$$\begin{aligned}\varphi_{a'_{\mathbf{y}}}(z) = \varphi((a'_{\mathbf{y}})^{-1}z) &= \varphi(a_{\mathbf{y}}^{-1}u^{-1}\gamma^{-1}r^{-1}z)\\ &= \varphi(a_{\mathbf{y}}^{-1}u^{-1}z)\\ &= \varphi_{a_{\mathbf{y}}}(\widetilde{u}^{-1}z),\end{aligned}$$

where $\widetilde{u}$ is the image in $(\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p)^\times$ of $u \in U(\mathfrak{f})$ in $U(\mathfrak{f})/U(p^\infty)$ (which is naturally isomophic to a subset of $(\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p)^\times$). Then, if $\varphi_{a_{\mathbf{y}}}(z) = \sum a_{\mathbf{r}} z^{\mathbf{r}}$, we have

$$\varphi_{a'_{\mathbf{y}}}(z) = \sum (a_{\mathbf{r}}\widetilde{u}^{-r})z^{\mathbf{r}}.$$

As we picked L large enough to contain the image of $\widetilde{u}$ under any embedding into $\mathbb{C}_p$, and $\widetilde{u}$ is a unit, this is a convergent power series over L.

Thus $\varphi_{a_{\mathbf{y}}}$ is analytic if and only if $\varphi_{a'_{\mathbf{y}}}$ is analytic, and as $\mathbf{y}$ was arbitrary, this proves the proposition. $\square$

Remark 14.1.7: Note that we have the dictionary

$$\varphi_{a_{\mathbf{y}}}(z) = \varphi_{a'_{\mathbf{y}}}(\widetilde{u}z)$$

as functions $\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p \to L$, where $a'_{\mathbf{y}} = ua_{\mathbf{y}}\gamma r$. We will use this later to prove that the distribution we obtain is canonical.

Definition 14.1.8. We write $\mathcal{A}(\mathrm{Cl}_F^+(p^\infty), L)$ for the space of locally analytic functions on $\mathrm{Cl}_F^+(p^\infty)$ defined over L. We also define the space of *locally analytic distributions* on the ray class group to be

$$\mathcal{D}(\mathrm{Cl}_F^+(p^\infty), L) := \mathrm{Hom}_{\mathrm{cts}}(\mathcal{A}(\mathrm{Cl}_F^+(p^\infty), L), L).$$

14.1.3. Constructing μ_Ψ in $\mathcal{D}(\mathrm{Cl}_F^+(p^\infty), L)$

With our family of maps $\mathrm{Ev}_{\mathfrak{f},\dagger}^{a_{\mathbf{y}}}$ from $\mathrm{H}_c^q(Y_1(\mathfrak{n}), \mathcal{L}_2(\mathcal{D}_\lambda(L)))$ to $\mathcal{D}_\lambda^+(L)$, we can construct a candidate distribution for the p-adic L-function. Fix an ideal $\mathfrak{f}|p^\infty$.

Notation: We write $A_{\mathfrak{f}} = \{a_{\mathbf{y}}\}$ to denote our system of class group representatives for $\mathrm{Cl}_F^+(\mathfrak{f})$.

We now construct a distribution $\mu_\Psi^{A_{\mathfrak{f}}}$ associated to *this* choice of representatives. Let φ be a locally analytic function on $\mathrm{Cl}_F^+(p^\infty)$. Via the above construction, we obtain functions $\varphi_{a_{\mathbf{y}}} : G_{\mathbf{y}} \to L$, which we can view as a function

$$\varphi_{a_{\mathbf{y}}} : \mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p \longrightarrow L$$

with support on the open subset $\{z \in (\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p)^\times : z \equiv 1 \,(\mathrm{mod}\, \mathfrak{f})\}$ and satisfying

$$\varphi_{a_{\mathbf{y}}}(ez) = \varphi_{a_{\mathbf{y}}}(z) \quad \forall e \in E(\mathfrak{f}).$$

Now, $\mathrm{Ev}_{\mathfrak{f},\dagger}^{a_{\mathbf{y}}}(\Psi) \in \mathcal{D}_\lambda^+(L)$. This is a distribution that takes as input functions $\psi : (\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p)^\times \to L$ with $\psi(ez) = e^{\mathbf{k}+\mathbf{v}}\psi(z)$. So to force $\varphi_{a_{\mathbf{y}}}$ to satisfy this condition, we twist it.

Definition 14.1.9. If $\psi : \mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p \to L$ is a function with support on elements congruent to $1 \,(\mathrm{mod}\, \mathfrak{f})$ and that satisfies $\psi(ez) = \psi(z)$ for all $e \in E(\mathfrak{f})$, then we define $\psi^* \in \mathcal{A}_\lambda^+(L)$ by

$$\psi^*(z) = z^{\mathbf{k}+\mathbf{v}}\psi(z)^{-1}.$$

It is simple to see that this now satisfies the condition required. We use ψ^{-1} rather than ψ for reasons of compatibility in later calculations.

Now we can evaluate $\mathrm{Ev}_{\mathfrak{f},\dagger}^{a_{\mathbf{y}}}(\Psi)$ at $\varphi_{a_{\mathbf{y}}}^*$. This motivates:

Definition 14.1.10. Define $\mu_\Psi^{A_\mathfrak{f}} \in \mathcal{D}(\mathrm{Cl}_F^+(p^\infty), L)$ by

$$\mu_\Psi^{A_\mathfrak{f}}(\varphi) = \sum_{\mathbf{y} \in \mathrm{Cl}_F^+(\mathfrak{f})} \mathrm{Ev}_{\mathfrak{f},\dagger}^{a_{\mathbf{y}}}(\Psi)(\varphi_{a_{\mathbf{y}}}^*) \in L.$$

Proposition 14.1.11. *For fixed $\mathfrak{f}$, this is independent of the choice of class group representatives.*

Proof. There are two layers to this. Choosing representatives fixes:

(a) The collection of maps $\{\mathrm{Ev}_{\mathfrak{f},\dagger}^{a_{\mathbf{y}}}(\Psi) : a_{\mathbf{y}} \in A_\mathfrak{f}\}$, and

(b) The identification of φ with $(\varphi_{a_{\mathbf{y}}})_{\mathbf{y} \in \mathrm{Cl}_F^+(\mathfrak{f})}$.

We prove that these choices cancel each other out. To do so, we examine the local systems; see Section 12.1.1 for descriptions of each local system.

Recall that we have $\zeta_*\eta_\mathfrak{f}^*\Psi \in \mathrm{H}_c^q(X_\mathfrak{f}, \mathcal{L}_{\mathfrak{f},2}(\mathcal{D}_\lambda(L)))$ (canonically), and then that we can pull back to $X_{\mathbf{y}}$ under the canonical inclusion. At the first stage where our representatives come into play, the map of local systems induced by

$$\tau_{a_{\mathbf{y}}} : E(\mathfrak{f})F_\infty^1 \backslash F_\infty^+ \xrightarrow{\sim} X_{\mathbf{y}}$$

can be described by the map

$$F^\times\backslash(F^\times a_{\mathbf{y}}U(\mathfrak{f})F_\infty^+ \times \mathcal{D}_\lambda(L))/U(\mathfrak{f})F_\infty^1 \longrightarrow E(\mathfrak{f})F_\infty^1\backslash(F_\infty^+ \times \mathcal{D}_\lambda(L)) \tag{14.2}$$
$$(\gamma a_{\mathbf{y}}ur, \mu) \longmapsto \left(r, \mu * \begin{pmatrix} \widetilde{u}^{-1} & 0 \\ 0 & 1 \end{pmatrix}\right),$$

recalling that $\tau_{a_{\mathbf{y}}}$ is given by $z \mapsto a_{\mathbf{y}}z$ and that $\widetilde{u}$ is the image of u in $(\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p)^\times$. This map is well-defined; indeed, consider

$$\begin{aligned}\gamma'[(\gamma a_{\mathbf{y}}ur, \mu)]vs &= \left[\left(\gamma'\gamma a_{\mathbf{y}}uvrs, \mu * \left(\begin{smallmatrix} \widetilde{v} & 0 \\ 0 & 1 \end{smallmatrix}\right)\right)\right] \\ &\longmapsto \left[\left(rs, \left(\mu * \left(\begin{smallmatrix} \widetilde{v} & 0 \\ 0 & 1 \end{smallmatrix}\right)\right) * \left(\begin{smallmatrix} (\widetilde{uv})^{-1} & 0 \\ 0 & 1 \end{smallmatrix}\right)\right)\right] \\ &= \left[\left(r, \mu * \left(\begin{smallmatrix} \widetilde{u}^{-1} & 0 \\ 0 & 1 \end{smallmatrix}\right)\right)\right] = \mathrm{Im}([\gamma a_{\mathbf{y}}ur, \mu]).\end{aligned}$$

Now suppose we choose a different set of representatives $\{a'_{\mathbf{y}}\}$, with, as before,

$$a'_{\mathbf{y}} = a_{\mathbf{y}}\gamma ur, \quad \gamma \in F^\times, u \in U(\mathfrak{f}), r \in F_\infty^1.$$

Then under the map of equation (14.2), we have

$$[(a'_{\mathbf{y}}, \mu)] = [(a_{\mathbf{y}}\gamma ur, \mu)] \longmapsto \left[\left(r, \mu * \left(\begin{smallmatrix} \widetilde{u}^{-1} & 0 \\ 0 & 1 \end{smallmatrix}\right)\right)\right].$$

Thus, when we restrict, we find that

$$\mathrm{Ev}_{\mathfrak{f},\dagger}^{a'_{\mathbf{y}}}(\Psi) = \mathrm{Ev}_{\mathfrak{f},\dagger}^{a_{\mathbf{y}}}(\Psi) * \begin{pmatrix} \widetilde{u}^{-1} & 0 \\ 0 & 1 \end{pmatrix}.$$

We've already shown that, for $\varphi \in \mathcal{A}(\mathrm{Cl}_F^+(p^\infty), L)$, we have

$$\varphi_{a'_{\mathbf{y}}}(\widetilde{u}z) = \varphi_{a_{\mathbf{y}}}(z).$$

Then

$$\begin{aligned}\varphi_{a_{\mathbf{y}}}^*(z) &= z^{\mathbf{k}+\mathbf{v}}\varphi_{a_y}(z)^{-1} \\ &= z^{\mathbf{k}+\mathbf{v}}\varphi_{a'_{\mathbf{y}}}(\widetilde{u}z)^{-1} \\ &= \widetilde{u}^{-\mathbf{k}-\mathbf{v}}(\widetilde{u}z)^{\mathbf{k}+\mathbf{v}}\varphi_{a'_{\mathbf{y}}}(\widetilde{u}z)^{-1} \\ &= \widetilde{u}^{-\mathbf{k}-\mathbf{v}}\varphi_{a'_{\mathbf{y}}}^*(\widetilde{u}z) \\ &= \begin{pmatrix} \widetilde{u}^{-1} & 0 \\ 0 & 1 \end{pmatrix} * \varphi_{a'_{\mathbf{y}}}^*(z).\end{aligned}$$

Thus

$$\varphi^*_{a'_{\mathbf{y}}}(z) = \begin{pmatrix} \widetilde{u} & 0 \\ 0 & 1 \end{pmatrix} * \varphi^*_{a_{\mathbf{y}}}(z).$$

Accordingly,

$$\begin{aligned}\mathrm{Ev}^{a'_{\mathbf{y}}}_{\mathfrak{f},\dagger}(\Psi)(\varphi^*_{a'_{\mathbf{y}}}) &= \mathrm{Ev}^{a_{\mathbf{y}}}_{\mathfrak{f},\dagger}(\Psi) * \begin{pmatrix} \widetilde{u}^{-1} & 0 \\ 0 & 1 \end{pmatrix} \left(\begin{pmatrix} \widetilde{u} & 0 \\ 0 & 1 \end{pmatrix} * \varphi^*_{a_{\mathbf{y}}} \right) \\ &= \mathrm{Ev}^{a_{\mathbf{y}}}_{\mathfrak{f},\dagger}(\Psi)(\varphi^*_{a_{\mathbf{y}}}).\end{aligned}$$

Thus this is independent of the choice of representatives, as desired. □

Definition 14.1.12. For some choice of representatives $A_{\mathfrak{f}} = \{a_{\mathbf{y}}\}$ of $\mathrm{Cl}^+_F(\mathfrak{f})$, define

$$\mu^{\mathfrak{f}}_{\Psi} := \mu^{A_{\mathfrak{f}}}_{\Psi}.$$

(Note that, by the proposition, this is well-defined for each $\mathfrak{f}$).

14.1.4. Compatibility over choice of $\mathfrak{f}$

We have defined, for each $\mathfrak{f}|p^\infty$ with $(p)|\mathfrak{f}$, a distribution $\mu^{\mathfrak{f}}_{\Psi} \in \mathcal{D}(\mathrm{Cl}^+_F(p^\infty), L)$. We now investigate how this distribution varies with the choice of $\mathfrak{f}$. Since we have proved that the distribution we obtain for each $\mathfrak{f}$ is independent of class group representatives, we now choose class group representatives that are compatible in the following sense:

Notation: Throughout this section, take $\mathfrak{f}|p^\infty$ with $(p)|\mathfrak{f}$ and let $\mathfrak{p}|p$ be a prime. Let $A_{\mathfrak{f}} = \{a_{\mathbf{y}}\}$ be a full set of representatives for $\mathrm{Cl}^+_F(\mathfrak{f})$, and let $\{u_r \in U(\mathfrak{f}) : r \in R\}$, for $R = U(\mathfrak{f})/E(\mathfrak{f})U(\mathfrak{f}\mathfrak{p})$, be elements of $U(\mathfrak{f})$ such that the set

$$A_{\mathfrak{f}\mathfrak{p}} := \{a_{\mathbf{y}} u_r : \mathbf{y} \in \mathrm{Cl}^+_F(\mathfrak{f}), r \in R\}$$

is a full set of representatives for $\mathrm{Cl}^+_F(\mathfrak{f}\mathfrak{p})$.

Lemma 14.1.13. *(i) There is a commutative diagram*

$$\begin{tikzcd} \mathrm{H}^q_c(Y_1(\mathfrak{n}), \mathcal{L}_2(\mathcal{D}_\lambda(L))) \arrow[r, "U_{\mathfrak{p}}"] \arrow[d, "\zeta_*\eta^*_{\mathfrak{f}\mathfrak{p}}"] & \mathrm{H}^q_c(Y_1(\mathfrak{n}), \mathcal{L}_2(\mathcal{D}_\lambda(L))) \arrow[d, "\zeta_*\eta^*_{\mathfrak{f}}"] \\ \mathrm{H}^q_c(X_{\mathfrak{f}\mathfrak{p}}, \mathcal{L}_{\mathfrak{f}\mathfrak{p},2}(\mathcal{D}_\lambda(L))) \arrow[r, "\mathrm{Tr}"] & \mathrm{H}^q_c(X_{\mathfrak{f}}, \mathcal{L}_{\mathfrak{f},2}(\mathcal{D}_\lambda(L))) \end{tikzcd},$$

where the bottom map is the natural trace map on cohomology (see, for example, [Hid93], Section 7).

(ii) Write $\mathrm{Ev}^{A_{\mathfrak{f}}}_{\mathfrak{f},\dagger} = \oplus_{\mathbf{y}\in\mathrm{Cl}^+_F(\mathfrak{f})}\mathrm{Ev}^{a_{\mathbf{y}}}_{\mathfrak{f},\dagger}$, *and similarly for* $\mathfrak{f}\mathfrak{p}$ *with respect to* $A_{\mathfrak{f}\mathfrak{p}}$. *Then we have the*

commutative diagram

$$\begin{tikzcd}
\mathrm{H}^q_c(Y_1(\mathfrak{n}), \mathcal{L}_2(\mathcal{D}_\lambda(L))) \arrow[r, "\mathrm{Ev}^{A_{\mathfrak{f}\mathfrak{p}}}_{\mathfrak{f}\mathfrak{p},\dagger}"] \arrow[d, "U_{\mathfrak{p}}"] & \bigoplus_{\mathrm{Cl}_F^+(\mathfrak{f}\mathfrak{p})} \mathcal{D}_\lambda^+(L) \arrow[d, "\mathrm{tr}_{\mathfrak{f}}"] \\
\mathrm{H}^q_c(Y_1(\mathfrak{n}), \mathcal{L}_2(\mathcal{D}_\lambda(L))) \arrow[r, "\mathrm{Ev}^{A_{\mathfrak{f}}}_{\mathfrak{f},\dagger}"] & \bigoplus_{\mathrm{Cl}_F^+(\mathfrak{f})} \mathcal{D}_\lambda^+(L)
\end{tikzcd},$$

where

$$\mathrm{tr}_{\mathfrak{f}}\Big((\mu_{a_{\mathbf{y}}u_r})_{\mathbf{y},r}\Big) = \left(\sum_{r\in R} \mu_{a_{\mathbf{y}}u_r}\Big| \begin{pmatrix} \widetilde{u_r}^{-1} & 0 \\ 0 & 1 \end{pmatrix}\right)_{\mathbf{y}\in \mathrm{Cl}_F^+(\mathfrak{f})}.$$

Proof. We construct commutative diagrams at each step in the definition of the evaluation maps. For convenience, we drop $\mathcal{D}_\lambda(L)$ from the notation and instead write $\mathcal{L} = \mathcal{L}(\mathcal{D}_\lambda(L))$, for appropriate subscripts on $\mathcal{L}$.

Note that there is a natural projection map $\mathrm{pr} : X_{\mathfrak{f}\mathfrak{p}} \to X_{\mathfrak{f}}$ that induces a trace map on the cohomology. We wish to construct a map $\mathrm{H}^q_c(X_{\mathfrak{f}\mathfrak{p}}, \mathcal{L}'_{\mathfrak{f}\mathfrak{p},2}) \to \mathrm{H}^q_c(X_{\mathfrak{f}}, \mathcal{L}'_{\mathfrak{f},2})$, and for this it hence suffices to construct a map of sheaves

$$\mathrm{pr}_*\mathcal{L}'_{\mathfrak{f}\mathfrak{p},2} \longrightarrow \mathcal{L}'_{\mathfrak{f},2}.$$

We do this as follows. Note that there is a natural map $\alpha : \mathcal{L}'_{\mathfrak{f}\mathfrak{p},2} \to \mathcal{L}'_{\mathfrak{f},2}$ given by

$$(x,\mu) \longmapsto \left(x, \mu * \begin{pmatrix} 1 & 0 \\ 0 & \pi_{\mathfrak{p}} \end{pmatrix}\right).$$

Let $G := \mathrm{Gal}(X_{\mathfrak{f}\mathfrak{p}}/X_{\mathfrak{f}}) \cong \mathcal{O}_F/\mathfrak{p}$. Now let $\mathcal{U} \subset X_{\mathfrak{f}}$ be open and sufficiently small that

$$\mathrm{pr}^{-1}(\mathcal{U}) = \bigsqcup_{g\in G} \mathcal{U}_g \subset X_{\mathfrak{f}\mathfrak{p}},$$

where pr induces a homeomorphism $i_g : \mathcal{U} \to \mathcal{U}_g$. Then

$$\mathrm{pr}_*\mathcal{L}'_{\mathfrak{f},2}(\mathcal{U}) := \mathcal{L}'_{\mathfrak{f}\mathfrak{p},2}(\mathrm{pr}^{-1}(\mathcal{U})) = \bigoplus_{g\in G} \mathcal{L}'_{\mathfrak{f}\mathfrak{p},2}(\mathcal{U}_g).$$

Then define a map

$$\begin{aligned} \mathrm{pr}_*\mathcal{L}'_{\mathfrak{f}\mathfrak{p},2}(\mathcal{U}) &\longrightarrow \mathcal{L}'_{\mathfrak{f},2}(\mathcal{U}), \\ s = (s_g)_{g\in G} &\longmapsto \sum \alpha \circ s_g \circ i_g. \end{aligned}$$

We see that we have defined a map

$$\mathrm{H}^q_c(X_{\mathfrak{f}\mathfrak{p}}, \mathcal{L}'_{\mathfrak{f}\mathfrak{p},2}) \xrightarrow{*\left(\begin{smallmatrix}1 & 0\\ 0 & \pi_{\mathfrak{p}}\end{smallmatrix}\right)} \mathrm{H}^q_c(X_{\mathfrak{f}}, \mathcal{L}'_{\mathfrak{f},2}),$$

and moreover we see that this makes the diagram

$$\begin{tikzcd}
\mathrm{H}^q_c(Y_1(\mathfrak{n}), \mathcal{L}_2(\mathcal{D}_\lambda(L))) \arrow[r, "U_{\mathfrak{p}}"] \arrow[d, "\eta^*_{\mathfrak{f}\mathfrak{p}}"] & \mathrm{H}^q_c(Y_1(\mathfrak{n}), \mathcal{L}_2(\mathcal{D}_\lambda(L))) \arrow[d, "\eta^*_{\mathfrak{f}}"] \\
\mathrm{H}^q_c(X_{\mathfrak{f}\mathfrak{p}}, \mathcal{L}'_{\mathfrak{f}\mathfrak{p},2}) \arrow[r, "*\left(\begin{smallmatrix}1 & 0\\ 0 & \pi_{\mathfrak{p}}\end{smallmatrix}\right)"] & \mathrm{H}^q_c(X_{\mathfrak{f}}, \mathcal{L}'_{\mathfrak{f},2})
\end{tikzcd}$$

commute.

Via a similar construction, and by replacing the map α with the map $\alpha' : \mathcal{L}_{\mathfrak{f}\mathfrak{p},2} \to \mathcal{L}_{\mathfrak{f},2}$ defined by $(x,\mu) \mapsto (x,\mu)$, we see that we construct a map $\mathrm{H}^q_c(X_{\mathfrak{f}\mathfrak{p}}, \mathcal{L}_{\mathfrak{f}\mathfrak{p},2}) \to \mathrm{H}^q_c(X_{\mathfrak{f}}, \mathcal{L}_{\mathfrak{f},2})$ that is nothing but the trace map on cohomology, and in particular that we have the following commutative diagram:

$$\begin{tikzcd}
\mathrm{H}^q_c(X_{\mathfrak{f}\mathfrak{p}}, \mathcal{L}'_{\mathfrak{f}\mathfrak{p},2}) \arrow[r, "*\left(\begin{smallmatrix}1 & 0\\ 0 & \pi_{\mathfrak{p}}\end{smallmatrix}\right)"] \arrow[d, "\zeta_*"] & \mathrm{H}^q_c(X_{\mathfrak{f}}, \mathcal{L}'_{\mathfrak{f},2}) \arrow[d, "\zeta_*"] \\
\mathrm{H}^q_c(X_{\mathfrak{f}\mathfrak{p}}, \mathcal{L}_{\mathfrak{f}\mathfrak{p},2}) \arrow[r, "\mathrm{tr}"] & \mathrm{H}^q_c(X_{\mathfrak{f}}, \mathcal{L}_{\mathfrak{f},2})
\end{tikzcd}.$$

Finally, we bring in the dependence on our choice of class group representatives. We see that there is commutative diagram

$$\begin{tikzcd}
\mathrm{H}^q_c(X_{\mathfrak{f}\mathfrak{p}}, \mathcal{L}_{\mathfrak{f}\mathfrak{p},2}) \arrow[r, "\mathrm{tr}"] \arrow[d, "\tau^*_{a_{\mathbf{y}}u_r}"] & \mathrm{H}^q_c(X_{\mathfrak{f}}, \mathcal{L}_{\mathfrak{f},2}) \arrow[d, "\tau^*_{a_{\mathbf{y}}}"] \\
\mathrm{H}^q_c(E(\mathfrak{f}\mathfrak{p})F^1_\infty\backslash F^+_\infty, \mathcal{L}_{\mathfrak{f}\mathfrak{p},\mathbf{x},2}) \arrow[r, "\left[*\left(\begin{smallmatrix}\widetilde{u_r}^{-1} & 0\\ 0 & 1\end{smallmatrix}\right)\right]_* \circ \mathrm{pr}_*"] & \mathrm{H}^q_c(E(\mathfrak{f})F^1_\infty\backslash F^+_\infty, \mathcal{L}_{\mathfrak{f},\mathbf{y},2})
\end{tikzcd},$$

where $\mathbf{x}$ denotes the class in $\mathrm{Cl}^+_F(\mathfrak{f}\mathfrak{p})$ represented by $a_{\mathbf{y}}u_r$, and pr is the natural projection map. Here we have used the results of the previous section. This shows that for $\Psi \in \mathrm{H}^q_c(Y_1(\mathfrak{n}), \mathcal{L}_2(\mathcal{D}_\lambda(L)))$, we have

$$\mathrm{Ev}^{a_{\mathbf{y}}u_r}_{\mathfrak{f}\mathfrak{p},\dagger}(\Psi) * \begin{pmatrix}\widetilde{u_r}^{-1} & 0\\ 0 & 1\end{pmatrix} = \mathrm{Ev}^{a_{\mathbf{y}}}_{\mathfrak{f},\dagger}(\Psi|U_{\mathfrak{p}})\Big|_{G_{\mathbf{x}}},$$

where $G_r = \{z \in \mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p : z \cong u_r e \,(\mathrm{mod}\, \mathfrak{f}\mathfrak{p}) \text{ for some } e \in E(\mathfrak{f})\}$. Summing over the relevant narrow ray class groups gives the diagram as stated. □

Proposition 14.1.14. *Let $\mathfrak{f}|p^\infty$ with $(p)|\mathfrak{f}$, and let $\mathfrak{p}$ be a prime above p. Let*

$$\Psi \in \mathrm{H}^q_c(Y_1(\mathfrak{n}), \mathcal{L}_2(\mathcal{D}_\lambda(L)))$$

be an eigensymbol for all the Hecke operators at p, with $U_{\mathfrak{p}}$-eigenvalue $\lambda_{\mathfrak{p}}$. Then

$$\mu_\Psi^{\mathfrak{f}\mathfrak{p}} = \lambda_{\mathfrak{p}}\mu_\Psi^{\mathfrak{f}}.$$

Proof. Let $\varphi \in \mathcal{A}(\mathrm{Cl}_F^+(p^\infty))$. We evaluate $\mu_\Psi^{\mathfrak{f}\mathfrak{p}}$ at φ by using the class group representatives $A_{\mathfrak{f}\mathfrak{p}}$, and then evaluate $\mu^{\mathfrak{f}}_{\Psi|U_{\mathfrak{p}}}$ at φ using the representatives $A_{\mathfrak{f}}$, and use the previous lemma to show that they are equal.

Fix $\mathbf{y} \in \mathrm{Cl}_F^+(\mathfrak{f})$ and $r \in R$. Then we see that

$$\varphi_{a_{\mathbf{y}}u_r}(z) = \varphi_{a_{\mathbf{y}}}(u_r^{-1}z)$$

for $z \in G_r$. In particular, we have

$$\varphi^*_{a_{\mathbf{y}}}\Big|_{G_r} = \begin{pmatrix} \widetilde{u_r}^{-1} & 0 \\ 0 & 1 \end{pmatrix} * \varphi^*_{a_{\mathbf{y}}u_r}(z)$$

Observe now that by the previous lemma, we have

$$\begin{aligned} \mathrm{Ev}_{\mathfrak{f},\dagger}^{a_{\mathbf{y}}}(\Psi|U_{\mathfrak{p}})(\varphi^*_{a_{\mathbf{y}}}) &= \sum_{r\in R} \mathrm{Ev}_{\mathfrak{f}\mathfrak{p}}^{a_{\mathbf{y}}u_r}(\Psi)\Big| \begin{pmatrix} \widetilde{u_r} & 0 \\ 0 & 1 \end{pmatrix} \left(\varphi^*_{a_{\mathbf{y}}}\Big|_{G_r}\right) \\ &= \sum_{r\in R} \mathrm{Ev}_{\mathfrak{f}\mathfrak{p}}^{a_{\mathbf{y}}u_r}(\Psi)\left(\varphi^*_{a_{\mathbf{y}}u_r}\right). \end{aligned}$$

Summing over $\mathbf{y} \in \mathrm{Cl}_F^+(\mathfrak{f})$ on both sides, and replacing $\Psi|U_{\mathfrak{p}}$ with $\lambda_{\mathfrak{p}}\Psi$ on the left hand side, now shows the result. □

We've now proved the following:

Theorem 14.1.15. *Let $\Psi \in \mathrm{H}^q_{\mathrm{c}}(Y_1(\mathfrak{n}), \mathcal{L}_2(\mathcal{D}_\lambda(L)))$ be an eigenclass for the $U_{\mathfrak{p}}$ operators for all $\mathfrak{p}|p$, and let $\mathfrak{f}|(p^\infty)$ be some choice of ideal with $(p)|\mathfrak{f}$. Write $\lambda_{\mathfrak{f}}$ for the eigenvalue of $U_{\mathfrak{f}}$, and define*

$$\mu_\Psi := \lambda_{\mathfrak{f}}^{-1}\mu_\Psi^{\mathfrak{f}}.$$

This is well-defined and independent of choices up to a fixed choice of uniformisers at primes above p.

Thus for such Ψ there is a canonical way of attaching an element μ_Ψ of $\mathcal{D}(\mathrm{Cl}_F^+(p^\infty), L)$ to Ψ.

14.1.5. Evaluating at Hecke characters

Let φ be a Hecke character of infinity type $\mathbf{r} \in \mathbb{Z}[\Sigma]$ and conductor $\mathfrak{f}|(p^\infty)$, where $(p)|\mathfrak{f}$, and recall that in Section 1.3.2, we associated to φ a function $\varphi_{p-\mathrm{fin}}$ on $\mathrm{Cl}_F^+(p^\infty)$. It is simple to see that this function is in fact locally analytic. In this section we describe the evaluation of the distribution μ_Ψ at $\varphi_{p-\mathrm{fin}}$.

Recall: $\varphi_{p-\mathrm{fin}}$ was defined to be the function on the ideles defined by

$$\varphi_{p-\mathrm{fin}}(x) = \varepsilon_\varphi \varphi_f(x) w_p^{\mathbf{r}}(x),$$

where $w_p^{\mathbf{r}}$ is an algebraic function (for example, when $F = \mathbb{Q}$, we have $w_p^r(x) = x_p^r$). In particular, choosing representatives $\{a_{\mathbf{y}}\}$ for $\mathrm{Cl}_F^+(\mathfrak{f})$, we see that

$$(\varphi_{p-\mathrm{fin}})_{a_{\mathbf{y}}} = \mathbb{1}_{G_{\mathbf{y}}} \varepsilon_\varphi \varphi_f(a_{\mathbf{y}}) \mathbf{z}^{\mathbf{r}},$$

where $\mathbb{1}_{G_{\mathbf{y}}}$ is the indicator function of the open subset of $\mathrm{Cl}_F^+(p^\infty)$ corresponding to $\mathbf{y} \in \mathrm{Cl}_F^+(\mathfrak{f})$ (see equation (14.1)), and $\mathbf{z}$ is a variable on $\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p$.

We see that, for Ψ as above,

$$\mu_\Psi(\varphi_{p-\mathrm{fin}}) = \lambda_{\mathfrak{f}}^{-1} \sum_{\mathbf{y}} \varepsilon_\varphi \varphi_f(a_{\mathbf{y}}) \mathrm{Ev}_{\mathfrak{f},\dagger}^{a_{\mathbf{y}}}(\Psi)(\mathbf{z}^{\mathbf{k}+\mathbf{v}-\mathbf{r}}). \tag{14.3}$$

14.2. Interpolation of L-values

In previous sections, we have defined the maps denoted by solid arrows in the following diagram:

$$\begin{tikzcd}
\mathrm{H}_{\mathrm{c}}^q(Y_1(\mathfrak{n}), \mathcal{L}_1(V_\lambda(L)^*)) \arrow[rr, "\mathrm{Ev}_{\mathfrak{f},\mathbf{j},1}^{a_{\mathbf{y}}}"] \arrow[d, equal, "\wr"] & & L \arrow[d, dotted, "\beta"] \\
\mathrm{H}_{\mathrm{c}}^q(Y_1(\mathfrak{n}), \mathcal{L}_2(V_\lambda(L)^*)) \arrow[rr, dotted, "\mathrm{Ev}_{\mathfrak{f},\mathbf{j},2}^{a_{\mathbf{y}}}"] & & L \\
\mathrm{H}_{\mathrm{c}}^q(Y_1(\mathfrak{n}), \mathcal{L}_2(\mathcal{D}_\lambda(L))) \arrow[u, "\rho"] \arrow[r, "\mathrm{Ev}_{\mathfrak{f},\dagger}^{a_{\mathbf{y}}}"] & \mathcal{D}_\lambda^+(L) \arrow[r, "\text{ev. at } z^{\mathbf{k}-\mathbf{j}}"] & L \arrow[u, dotted, "\delta"]
\end{tikzcd} \tag{14.4}$$

In particular, the isomorphism is induced by the isomorphism of local systems given in Remark 11.3.2, the top (classical) evaluation map was defined in Section 12.1.2, the map ρ is induced from the specialisation $\mathcal{D}_\lambda(L) \to V_\lambda(L)^*$, and the bottom (overconvergent) evaluation map was defined in Section 14.1.1. In this section, we define the maps above denoted by dotted arrows in a manner such that the diagram commutes. By doing so, we'll be able to use our previous results to relate the evaluation of the distribution μ_Φ at Hecke characters with critical L-values of Φ.

14.2.1. Classical evaluations, II

We start by defining the "missing" evaluation map. We've already touched on all of the key points of this construction; it is essentially a blend of our previous two evaluation maps. Taking notation from Section 12.1, we pullback along $\eta_{\mathfrak{f}}$, giving a local system $\eta_{\mathfrak{f}}^*\mathcal{L}_2(V_\lambda(L)^*)$ on $X_{\mathfrak{f}}$ that can be described by sections of the projection

$$F^\times\backslash(\mathbb{A}_F^\times \times V_\lambda(L)^*)/U(\mathfrak{f})F_\infty^1,$$

with action

$$f(x,P)ur = \left(fxur, P * \begin{pmatrix} u & ((u-1)\mathfrak{f}^{-1})_{v|\mathfrak{f}} \\ 0 & 1 \end{pmatrix}\right).$$

This bears relation with the overconvergent case, in that we have an action of units that isn't particularly nice. As in that case, we 'untwist' this action using the map $(\zeta_{\mathfrak{f}})_*$ from Section 14.1.1, so that units act via the matrix $\left(\begin{smallmatrix} u & 0 \\ 0 & 1 \end{smallmatrix}\right)$. We can then pull-back under the injection

$$\tau_{a_{\mathbf{y}}} : E(\mathfrak{f})F_\infty^1\backslash F_\infty^+ \hookrightarrow X_{\mathfrak{f}}$$

of previous sections. Finally, as in the classical case, we pushforward under evaluation at the polynomial $\mathbf{X^{k-j}Y^j}$, which lands us in a cohomology group with coefficients in a constant sheaf (see Section 12.1.2). Combining all of these maps, we get a map

$$\mathrm{Ev}_{\mathfrak{f},\mathbf{j},2}^{a_{\mathbf{y}}} : \mathrm{H}_{\mathrm{c}}^q(Y_1(\mathfrak{n}), \mathcal{L}_2(V_\lambda(L)^*)) \longrightarrow L,$$

which gives the definition of the dotted horizontal arrow in the diagram.

The following lemma determines the definition of the map β in the diagram. For ease of notation, write Ev_k for the map $\mathrm{Ev}_{\mathfrak{f},\mathbf{j},k}^{a_{\mathbf{y}}}$.

Lemma 14.2.1. *Let α denote the isomorphism*

$$\alpha : \mathrm{H}_{\mathrm{c}}^q(Y_1(\mathfrak{n}), \mathcal{L}_1(V_\lambda(L)^*)) \xrightarrow{\sim} \mathrm{H}_{\mathrm{c}}^q(Y_1(\mathfrak{n}), \mathcal{L}_2(V_\lambda(L)^*))$$

induced by the isomorphism $\mathcal{L}_1(V_\lambda(L)^*) \xrightarrow{\sim} \mathcal{L}_2(V_\lambda(L)^*)$ of local systems given by

$$(g,P) \longmapsto (g, P|g_p)$$

(see Remark 11.3.2). Then

$$\mathrm{Ev}_2(\alpha(\phi)) = \mathfrak{f}^{\mathbf{j}+\mathbf{v}}\mathrm{Ev}_1(\phi).$$

Remark: Here, in an abuse of notation, we write $\mathfrak{f}$ for the natural element of L corresponding to $(\mathfrak{f})_{v|p} \in \mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p$ under our fixed choice of uniformisers at primes above p. In particular, 'multiplication by $\mathfrak{f}^{\mathbf{j}+\mathbf{v}}$' is a well-defined concept.

Proof. We look at the local systems in each case. A simple check shows that there is a commutative diagram

$$\begin{tikzcd} \mathrm{H}^q_c(Y_1(\mathfrak{n}), \mathcal{L}_1(V_\lambda(L)^*)) \arrow[r, "\eta_{\mathfrak{f}}^*"] \arrow[d, "\alpha"] & \mathrm{H}^q_c(X_{\mathfrak{f}}, \mathcal{L}_{\mathfrak{f},1}(V_\lambda(L)^*)) \arrow[d, "\alpha'"] \\ \mathrm{H}^q_c(Y_1(\mathfrak{n}), \mathcal{L}_2(V_\lambda(L)^*)) \arrow[r, "(\zeta_{\mathfrak{f}})_*\eta_{\mathfrak{f}}^*"] & \mathrm{H}^q_c(X_{\mathfrak{f}}, \mathcal{L}_{\mathfrak{f},2}(V_\lambda(L)^*)) \end{tikzcd},$$

where α' is the map induced by the map

$$(x, P) \longmapsto \left(x, P \middle| \begin{pmatrix} x_p & 0 \\ 0 & (\mathfrak{f})_{v|\mathfrak{f}} \end{pmatrix} \right)$$

of local systems. Then continuing, we see that there is a commutative diagram

$$\begin{tikzcd} \mathrm{H}^q_c(X_{\mathfrak{f}}, \mathcal{L}_{\mathfrak{f},1}(V_\lambda(L)^*)) \arrow[r, "\tau_{a_{\mathbf{y}}}^*"] \arrow[d, "\alpha'"] & \mathrm{H}^q_c(E(\mathfrak{f})F^1_\infty \backslash F^+_\infty, \mathcal{L}_{\mathfrak{f},\mathbf{y},1}(V_\lambda(L)^*)) \arrow[d, "\alpha''"] \\ \mathrm{H}^q_c(X_{\mathfrak{f}}, \mathcal{L}_{\mathfrak{f},2}(V_\lambda(L)^*)) \arrow[r, "\tau_{a_{\mathbf{y}}}^*"] & \mathrm{H}^q_c(E(\mathfrak{f})F^1_\infty \backslash F^+_\infty, \mathcal{L}_{\mathfrak{f},\mathbf{y},2}(V_\lambda(L)^*)) \end{tikzcd}$$

where α'' is the map induced by the map

$$(r, P) \longmapsto \left(r, \middle| \begin{pmatrix} 1 & 0 \\ 0 & (\mathfrak{f})_{v|\mathfrak{f}} \end{pmatrix} \right)$$

of local systems. Finally, there is a commutative diagram

$$\begin{tikzcd} \mathrm{H}^q_c(E(\mathfrak{f})F^1_\infty \backslash F^+_\infty, \mathcal{L}_{\mathfrak{f},\mathbf{y},1}(V_\lambda(L)^*)) \arrow[r, "(\text{ev. at } \mathbf{X}^{\mathbf{k}-\mathbf{j}}\mathbf{Y}^{\mathbf{j}})_*"] \arrow[d, "\alpha''"] & L \arrow[d, "\times \mathfrak{f}^{\mathbf{j}+\mathbf{v}}"] \\ \mathrm{H}^q_c(E(\mathfrak{f})F^1_\infty \backslash F^+_\infty, \mathcal{L}_{\mathfrak{f},\mathbf{y},2}(V_\lambda(L)^*)) \arrow[r, "(\text{ev. at } \mathbf{X}^{\mathbf{k}-\mathbf{j}}\mathbf{Y}^{\mathbf{j}})_*"] & L \end{tikzcd}.$$

Putting these diagrams together gives the required result. □

Recall the definition of Ev_φ in Definition 12.2.3, and relabel $\mathrm{Ev}_{\varphi,1} := \mathrm{Ev}_\varphi$. Similarly define

$$\mathrm{Ev}_{\varphi,2} := \sum_{\mathbf{y} \in \mathrm{Cl}^+_F(\mathfrak{f})} \varepsilon_\varphi \varphi_f(a_{\mathbf{y}}) \mathrm{Ev}^{a_{\mathbf{y}}}_{\mathfrak{f},\mathbf{j},2},$$

where this makes sense, and note that by an identical argument to previously this is independent of class group representatives. Using the results above with the results in Section 12.2, we obtain:

Corollary 14.2.2. *Recall the definition of* $\theta_K \in \mathrm{H}^q_{\mathrm{cusp}}(Y_1(\mathfrak{n}), \mathcal{L}_1(V_\lambda(K)^*))$ *from Definition 11.6.4, and recall that we set* θ_L *to be its image in* $\mathrm{H}^q_c(Y_1(\mathfrak{n}), \mathcal{L}_2(V_\lambda(L)^*))$ *under the inclusions of equation (11.3) and (11.4). Then*

$$\mathrm{Ev}_{\varphi,2}(\theta_L) = \mathfrak{f}^{\mathbf{j}+\mathbf{v}} \mathrm{Ev}_{\varphi,1}(\theta_K) = (-1)^{R(\mathbf{j},\mathbf{k})} \left[\frac{|D|\tau(\varphi)\mathfrak{f}^{\mathbf{j}+\mathbf{v}}}{2^{r_2}\Omega_\Phi^{\varepsilon_\varphi}} \right] \cdot \Lambda(\Phi, \varphi),$$

where $R(\mathbf{j},\mathbf{k}) = \sum_{v\in\Sigma(\mathbb{R})} j_v + k_v + \sum_{v\in\Sigma(\mathbb{C})} k_v$.

14.2.2. Relating classical and overconvergent evaluations

Returning to the commutative diagram in equation (14.4), we now show that the map δ is actually nothing but the identity map. For a suitable automorphic form Φ, this will then allow us to prove the required interpolation property for the distribution μ_Φ.

Proposition 14.2.3. *There is a commutative diagram*

$$\begin{tikzcd}
\mathrm{H}^q_c(Y_1(\mathfrak{n}),\mathcal{L}_2(\mathcal{D}_\lambda(L))) \arrow[r, "\mathrm{Ev}^{a_\mathbf{y}}_{\mathfrak{f},\dagger}"] \arrow[d, "\rho"] & \mathcal{D}^+_\lambda(L) \arrow[d, "ev.\ at\ z^{\mathbf{k}-\mathbf{j}}"] \\
\mathrm{H}^q_c(Y_1(\mathfrak{n}),\mathcal{L}_2(V_\lambda(L)^*)) \arrow[r, "\mathrm{Ev}^{a_\mathbf{y}}_{\mathfrak{f},\mathbf{j},2}"] & L
\end{tikzcd},$$

where the left vertical arrow is the specialisation map and the right vertical arrow is evaluation at the polynomial $z^{\mathbf{k}-\mathbf{j}}$.

Proof. This is easily shown by looking at each step of the construction of the maps $\mathrm{Ev}^{a_\mathbf{y}}_{\mathfrak{f},\dagger}$ and $\mathrm{Ev}^{a_\mathbf{y}}_{\mathfrak{f},\mathbf{j},2}$ in the previous sections. At each of steps 1, 2 and 3 we can write down a specialisation map by restricting the coefficients, and by looking at the level of local systems, we can clearly see that these specialisations commute with the maps $\eta_\mathfrak{f}$, $\zeta_\mathfrak{f}$ and $\tau_{a_\mathbf{y}}$. It remains to show compatibility over step 4, where the construction is slightly different. This amounts to showing that the diagram

$$\begin{tikzcd}
\mathrm{H}^q_c(E(\mathfrak{f})F^1_\infty\backslash F^+_\infty,\mathcal{L}_{\mathfrak{f},\mathbf{y},2}(\mathcal{D}_\lambda(L))) \arrow[r, "res"] \arrow[d] & \mathcal{D}^+_\lambda(L) \arrow[d, "ev.\ at\ z^{\mathbf{k}-\mathbf{j}}"] \\
\mathrm{H}^q_c(E(\mathfrak{f})F^1_\infty\backslash F^+_\infty,\mathcal{L}_{\mathfrak{f},\mathbf{y},2}(V_\lambda(L)^*)) \arrow[r, "\mathrm{ev.\ at\ }\mathbf{X}^{\mathbf{k}-\mathbf{j}}\mathbf{Y}^\mathbf{j}"] & L
\end{tikzcd}$$

commutes, where the lefthand map is restriction of the coefficients, the map *res* is the restriction of coefficients to $\mathcal{D}^+_\lambda(L)$ followed by integration over a fixed de Rham cohomology class, and the bottom map is the composition of $(\rho_\mathbf{j})_*$ with integration over the same de Rham cohomology class. Since $V_\lambda(L)^* \hookrightarrow \mathcal{A}_\lambda(L)$ via $P(\mathbf{X},\mathbf{Y}) \mapsto P(z,1)$, we see that when we look at the corresponding local systems, we are evaluating at the same element in each case; thus the diagram commutes. $\square$

By combining this with the formula (14.3) for $\mu_\Psi(\varphi_{p-\mathrm{fin}})$, we get the following corollary:

Corollary 14.2.4. *Let* $\phi \in \mathrm{H}^q_c(Y_1(\mathfrak{n}),\mathcal{L}_2(V_\lambda(L)^*))$ *be a small slope Hecke eigensymbol with* $U_\mathfrak{f}$*-eigenvalue* $\lambda_\mathfrak{f}$ *and with (unique) overconvergent eigenlift* Ψ*, and let* μ_Ψ *be the corresponding ray class distribution. Then for a Hecke character* φ *of infinity type* $\mathbf{j}+\mathbf{v}$ *and conductor* $\mathfrak{f}|(p^\infty)$, *where* $0 \leq \mathbf{j} \leq \mathbf{k}$ *and* $(p)|\mathfrak{f}$*, we have*

$$\mu_\Psi(\varphi_{p-\mathrm{fin}}) = \lambda_{\mathfrak{f}}^{-1}\mathrm{Ev}_{\varphi,2}(\phi).$$

14.2.3. Interpolating at unramified characters

We now consider interpolation of L-values at Hecke characters that are not necessarily ramified at all primes above p. For this, we use Corollary 12.1.15. Whilst the results of this section up until now have been for arbitrary modular symbols, to use this corollary we need to restrict to the case where the cohomology classes we consider are those attached to automorphic forms via the Eichler-Shimura isomorphism. Let Φ be such an automorphic form of weight λ and level $\Omega_1(\mathfrak{n})$, and suppose that Φ is a Hecke eigenform that has small slope at the primes above p. Let ϕ_L be the (p-adic) modular symbol attached to Φ, and let Ψ be the associated (unique) overconvergent modular symbol corresponding to ϕ_L under the control theorem. Then we have the following lemma:

Lemma 14.2.5. *Let* φ *be a Hecke character of conductor* $\mathfrak{f}|(p^\infty)$ *(with no additional conditions on* $\mathfrak{f}$*) and infinity type* $\mathbf{j}+\mathbf{v}$*, where* $0 \leq \mathbf{j} \leq \mathbf{k}$*. Let* B *be the set of primes above* p *that do not divide* $\mathfrak{f}$*, and define* $\mathfrak{f}' := \mathfrak{f}\prod_{\mathfrak{p}\in B}\mathfrak{p}$*, so that* $\mathfrak{f}'$ *is divisible by all the primes above* p*. Then we have*

$$\begin{aligned}\mu_\Psi(\varphi_{p-\mathrm{fin}}) &= \lambda_{\mathfrak{f}'}^{-1}(\mathfrak{f}')^{\mathbf{j}+\mathbf{v}}\left[\prod_{\mathfrak{p}\in B}(\varphi(\mathfrak{p})\lambda_{\mathfrak{p}}-1)\right]\mathrm{Ev}_{\varphi,1}(\phi_L)\\ &= \lambda_{\mathfrak{f}}^{-1}(\mathfrak{f})^{\mathbf{j}+\mathbf{v}}\left[\prod_{\mathfrak{p}\in B}\varphi_{p-\mathrm{fin}}(\pi_{\mathfrak{p}})(1-\lambda_{\mathfrak{p}}^{-1}\varphi(\mathfrak{p})^{-1})\right]\mathrm{Ev}_{\varphi,1}(\phi_L). \qquad (14.5)\end{aligned}$$

Proof. By definition, $\mu_\Psi := \lambda_{\mathfrak{f}'}^{-1}\mu_\Psi^{\mathfrak{f}'}$ (which is canonical since $\mathfrak{f}'$ is divisible by every prime above p). Hence we see that

$$\mu_\Psi(\varphi_{p-\mathrm{fin}}) = \lambda_{\mathfrak{f}'}^{-1}\sum_{\mathbf{y}\in\mathrm{Cl}_F^+(\mathfrak{f}')}\varepsilon_\varphi\varphi_f(a_\mathbf{y})\mathrm{Ev}_{\mathfrak{f}',\dagger}^{a_\mathbf{y}}(\Psi)(\mathbf{z}^{\mathbf{k}-\mathbf{j}}).$$

Using the results of Section 14.2.2, we can replace the overconvergent evaluations with classical ones, and then using the results of Section 14.2.1, we get

$$\mu_\Psi(\varphi_{p-\mathrm{fin}}) = \lambda_{\mathfrak{f}'}^{-1}(\mathfrak{f}')^{\mathbf{j}+\mathbf{v}}\sum_{\mathbf{y}\in\mathrm{Cl}_F^+(\mathfrak{f}')}\varepsilon_\varphi\varphi_f(a_\mathbf{y})\mathrm{Ev}_{\mathfrak{f}',\mathbf{j},1}^{a_\mathbf{y}}(\phi_L).$$

We now use Corollary 12.1.15, which directly gives the first equality. The second equality follows since for $\mathfrak{p}$ not dividing $\mathfrak{f}$, we have $\mathfrak{p}^{\mathbf{j}+\mathbf{v}} = \varphi_{p-\mathrm{fin}}(\pi_{\mathfrak{p}})\varphi(\mathfrak{p})^{-1}$, an identity which follows from the definition of $\varphi_{p-\mathrm{fin}}$. $\square$

14.3. Summary of results

The results of the previous section, and in particular Corollaries 14.2.2 and 14.2.4 and Lemma 14.2.5, give the desired interpolation property for our distribution. The following is a summary of the main results of this part of this thesis.

Recall the set-up. Let $F/\mathbb{Q}$ be a number field and p a rational prime. Let Φ be a small slope cuspidal eigenform over F of weight $\lambda = (\mathbf{k}, \mathbf{v}) \in \mathbb{Z}[\Sigma]^2$, where $\mathbf{k} + 2\mathbf{v}$ is parallel, and level $\Omega_1(\mathfrak{n})$, where $(p)|\mathfrak{n}$. Let $\Lambda(\Phi, \cdot)$ be the normalised L-function attached to Φ in Definition 4.3.6. Write θ_L for the p-adic modular symbol associated to Φ in Definition 11.6.5, where L is a sufficiently large extension of $\mathbb{Q}_p$. Using the control theorem, we may lift θ_L to a unique small slope overconvergent eigensymbol Ψ, and using Theorem 14.1.15 we may construct a canonical distribution

$$\mu_\Psi \in \mathcal{D}(\mathrm{Cl}_F^+(p^\infty), L)$$

attached to Ψ.

Theorem 14.3.1. *Let φ be a Hecke character of conductor $\mathfrak{f}|(p^\infty)$ and infinity type $\mathbf{j}+\mathbf{v}$, where $0 \leq \mathbf{j} \leq \mathbf{k}$, and let ε_φ be the character of $\{\pm 1\}^{\Sigma(\mathbb{R})}$ attached to φ in Chapter 1.2.2. As described in Section 1.3.2, φ gives rise to a canonical locally algebraic function $\varphi_{p-\mathrm{fin}} \in \mathcal{A}(\mathrm{Cl}_F^+(p^\infty), L)$. Let B be the set of primes above p that do not divide $\mathfrak{f}$. Then*

$$\mu_\Psi(\varphi_{p-\mathrm{fin}}) = (-1)^{R(\mathbf{j},\mathbf{k})} \left[\frac{|D|\tau(\varphi)\mathfrak{f}^{\mathbf{j}+\mathbf{v}}}{2^{r_2}\lambda_{\mathfrak{f}}\Omega_\Phi^{\varepsilon_\varphi}}\right] \left(\prod_{\mathfrak{p}\in B} Z_{\mathfrak{p}}\right) \Lambda(\Phi, \varphi),$$

where

$$Z_{\mathfrak{p}} := \varphi_{p-\mathrm{fin}}(\pi_{\mathfrak{p}})(1 - \lambda_{\mathfrak{p}}^{-1}\varphi(\mathfrak{p})^{-1})$$

(noting here that $\varphi(\mathfrak{p})$ is well-defined since φ is unramified at $\mathfrak{p}$).

Here $R(\mathbf{j}, \mathbf{k}) = \sum_{v\in\Sigma(\mathbb{R})} j_v + k_v + \sum_{v\in\Sigma(\mathbb{C})} k_v$, D is the discriminant of F, $\tau(\varphi)$ is the Gauss sum of Definition 1.3.2, r_2 is the number of pairs of complex embeddings of F, $\lambda_{\mathfrak{f}}$ is the $U_{\mathfrak{f}}$-eigenvalue of Φ, $\Omega_\Phi^{\varepsilon_\varphi}$ is the fixed period attached to Φ and ε_φ in Theorem 4.4.1, and $\Lambda(\Phi, \cdot)$ is the normalised L-function of Φ as defined in Definition 4.3.6.

Definition 14.3.2. In the set-up of above, we call μ_Ψ the *p-adic L-function of* Φ.

14.4. Remarks on uniqueness

When F is a totally real or imaginary quadratic field, we can prove a *uniqueness* property of this distribution. In particular, we prove that the distribution constructed above is *admissible* in a certain sense, and any admissible distribution is uniquely determined by its values at functions coming from critical Hecke characters (see [Col10] and [Loe14]). For further details of admissibility conditions in these cases, see [BS13] and Chapter 8.3.3 for the totally real and imaginary quadratic situations respectively. In the general case, things are more subtle. There is a good notion of admissibility for distributions on $\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p$, but it is not at all clear how this descends to a 'useful' admissibility condition on $\mathrm{Cl}_F^+(p^\infty)$.

In particular, recall that

$$\mathrm{Cl}_F^+(p^\infty) = \bigsqcup_{\mathrm{Cl}_F^+} (\mathcal{O}_F \otimes \mathbb{Z}_p)^\times / \overline{E(1)}.$$

When F is imaginary quadratic, the unit group is finite, and in particular in passing to the quotient we do not change the rank. In this case, growth properties pass down almost unchanged. When F is totally real, the unit group is in a sense 'maximal' if we assume Leopoldt's conjecture. In particular, provided this, the quotient is just one dimensional, and we have a canonical 'direction' with which to check growth properties.

Let us illustrate the difficulties of the general case with a conceptual example, for which the authors would like to thank David Loeffler. Let $F = \mathbb{Q}(\sqrt[3]{2})$, and note that F is a cubic field of mixed signature. We see that $(\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p)^\times$ is a p-adic Lie group of rank 3, and that the quotient by $\overline{E(1)}$ has rank 2 (since the unit group has rank 1 by Dirichlet's unit theorem). In particular, a distribution on $\mathrm{Cl}_F^+(p^\infty)$ can 'grow' in two independent directions.

As the maximal CM subfield of F is nothing but $\mathbb{Q}$, it follows that the only possible infinity types of Hecke characters of F are parallel. In particular, there is only *one* 'dimension' of Hecke characters. In this sense, even though we have constructed a distribution that interpolates all critical Hecke characters, there are simply not enough Hecke characters to hope that we can uniquely determine a ray class distribution by this interpolation property.

One might be able to obtain nice growth properties using the extra structure that we obtain from our overconvergent modular symbol; in particular, one might expect the overconvergent cohomology classes we construct to take values in the smaller space of admissible distributions on $\mathcal{O}_F \otimes \mathbb{Z}_p$, which makes sense *before* we quotient to obtain distributions on $\mathrm{Cl}_F^+(p^\infty)$. Without the theory of admissibility at hand in the latter situation, however, we cannot show that the distribution constructed in this paper is (in general) unique. We have tried to rectify this by proving that the distribution we obtain is canonical. As seen in the previous sections, we were able to do this up to a (fixed) choice of uniformisers at the primes above p. Hence, in the spirit of Pollack and Stevens in [PS12], we simply *define* the p-adic L-function to be this distribution.

It remains to comment on the dependence on choices of uniformisers. This dependence seems to be intrinsic to this more explicit approach; indeed, the evaluation maps at the level of p-adic coefficients depend on the choice of uniformiser, and accordingly the distribution we've defined the be the p-adic L-function does as well. However, the interpolation property *also* has an explicit dependence on the uniformisers (coming from the Gauss sum and the term $\mathfrak{f}^{\mathbf{j}+\mathbf{v}}$), so by changing the uniformisers we are changing both the distribution and its interpolating property, so don't 'break' any potential uniqueness property. Despite this, it would be interesting to remove these dependences if possible.

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