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Well-Quasi-Ordering versus Clique-Width: New Results on Bigenic Classes

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Abstract Daligault, Rao and Thomassé asked whether a hereditary class of graphs well-quasi-ordered by the induced subgraph relation has bounded clique-width. Lozin, Razgon and Zamaraev recently showed that this is not true for classes defined by infinitely many forbidden induced subgraphs. However, in the case of finitely many forbidden induced subgraphs the question remains open and we conjecture that in this case the answer is positive. The conjecture is known to hold for classes of graphs defined by a single forbidden induced subgraph H, as such graphs are well-quasi-ordered and are of bounded clique-width if and only if H is an induced subgraph of P_4. For bigenic classes of graphs, i.e. ones defined by two forbidden induced subgraphs, there are several open cases in both classifications. In the present paper we obtain a number of new results on well-quasi-orderability of bigenic classes, each of which supports the conjecture.

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1 Introduction

Well-quasi-ordering is a highly desirable property and frequently discovered concept in mathematics and theoretical computer science [18, 22]. One of the most remarkable results in this area is Robertson and Seymour's proof of Wagner's conjecture, which states that the set of all finite graphs is well-quasi-ordered by the minor relation [27]. One of the first steps towards this result was the proof of the fact that graph classes of bounded treewidth are well-quasi-ordered by the minor relation [26] (a graph parameter π is said to be bounded for some graph class $\mathcal{G}$ if there exists a constant c such that $\pi(G) \leq c$ for all $G \in \mathcal{G}$).

The notion of clique-width generalizes that of treewidth in the sense that graph classes of bounded treewidth have bounded clique-width, but not necessarily vice versa. The importance of both notions is due to the fact that many algorithmic problems that are NP-hard on general graphs become polynomial-time solvable when restricted to graph classes of bounded treewidth or clique-width. For treewidth this follows from the meta-theorem of Courcelle [6], combined with a result of Bodlaender [2]. For clique-width this follows from combining results from several papers [8, 17, 20, 25] with a result of Oum and Seymour [24].

In the study of graph classes of bounded treewidth, we can restrict ourselves to minor-closed graph classes, because from the definition of treewidth it immediately follows that the treewidth of a graph is never smaller than the treewidth of its minor. This restriction, however, is not justified when we study graph classes of bounded clique-width, as the clique-width of a graph can be much smaller than the clique-width of its minor. In particular, Courcelle [7] showed that if $\mathcal{G}$ is the class of graphs of clique-width 3 and $\mathcal{G}'$ is the class of graphs obtainable from graphs in $\mathcal{G}$ by applying one or more edge contraction operations, then $\mathcal{G}'$ has unbounded clique-width. On the other hand, the clique-width of a graph is never smaller than the clique-width of any of its induced subgraphs (see, for example, [9]). This allows us to restrict ourselves to classes of graphs closed under taking induced subgraphs. Such graph classes are also known as *hereditary* classes.

It is well-known (and not difficult to see) that a class of graphs is hereditary if and only if it can be characterized by a set of minimal forbidden induced subgraphs. Due to the minimality, the set $\mathcal{F}$ of forbidden induced subgraphs is always an antichain, that is, no graph in $\mathcal{F}$ is an induced subgraph of another graph in $\mathcal{F}$. For some hereditary classes this set is finite, in which case we say that the class is *finitely defined*, whereas for other hereditary classes (such as, for instance, bipartite graphs) the set of minimal forbidden induced subgraphs forms an infinite antichain. The presence of these infinite antichains immediately shows that the induced subgraph relation is not a well-quasi-order. In fact there even exist graph classes of bounded clique-width that are not well-quasi-ordered by the induced subgraph relation: take, for example, the class of cycles, which all have clique-width at most 4. What about the inverse implication: does well-quasi-ordering imply bounded clique-width? This was stated as an open problem by Daligault, Rao and Thomassé [15] and a negative answer to this question was recently given by Lozin, Razgon and Zamaraev [23]. However, the latter authors disproved the conjecture by giving a hereditary class of graphs whose set of minimal forbidden induced subgraphs is infinite. Hence, for finitely defined classes the question remains open and we conjecture that in this case the answer is positive.

Conjecture 1 *If a finitely defined class of graphs $\mathcal{G}$ is well-quasi-ordered by the induced subgraph relation, then $\mathcal{G}$ has bounded clique-width.*

We emphasize that our motivation for verifying Conjecture 1 is not only mathematical but also algorithmic. Should Conjecture 1 be true, then for finitely defined classes of graphs the aforementioned algorithmic consequences of having bounded clique-width also hold for the property of being well-quasi-ordered by the induced subgraph relation.

A class of graphs is *monogenic* or *H-free* if it is characterized by a single forbidden induced subgraph H. For monogenic classes, the conjecture is true. In this case, the two notions even coincide: a class of graphs defined by a single forbidden induced subgraph H is well-quasi-ordered if and only if it has bounded clique-width if and only if H is an induced subgraph of P_4 (see, for instance, [14, 16, 21]).

A class of graphs is *bigenic* or *(H_1, H_2)-free* if it is characterized by two incomparable forbidden induced subgraphs H_1 and H_2. The family of bigenic classes is more diverse than the family of monogenic classes. The questions of well-quasi-orderability and having bounded clique-width still need to be resolved. Recently, considerable progress has been made towards answering the latter question for bigenic classes; there are currently seven (non-equivalent) open cases [13]. With respect to well-quasi-orderability of bigenic classes, Korpelainen and Lozin [21] left all but 14 cases open. Since then, Atminas and Lozin [1] proved that the class of (K_3, P_6)-free graphs is well-quasi-ordered by the induced subgraph relation and that the class of ($\overline{2P_1 + P_2}$, P_6)-free graphs is not. Very recently, we used the techniques developed in the present paper to prove that the classes of ($\overline{P_1 + P_3}$, $P_2 + P_4$)-free graphs and ($\overline{P_1 + P_3}$, $P_1 + P_5$)-free graphs are also well-quasi-ordered by the induced subgraph relation [13]. The class of ($\overline{P_1 + P_3}$, $P_1 + 2P_2$)-free graphs, which is contained in both of these classes, was also an open case. This reduced the number of remaining open cases to nine. All available results for bigenic classes verify Conjecture 1. Moreover, six of the nine open cases have bounded clique-width (and thus immediately verify Conjecture 1), leaving three remaining open cases of bigenic classes for which we still need to verify Conjecture 1.

1.1 Our Results

Our first goal is to obtain more (bigenic) classes that are well-quasi-ordered by the induced subgraph relation and to support Conjecture 1 with further evidence. Our second and more general goal is to increase our general knowledge on well-quasi-ordered graph classes and the relation to the possible boundedness of their clique-width.

Towards our first goal we prove in Section 4 that the class of ($\overline{2P_1 + P_2}$, $P_2 + P_3$)-free graphs (which has bounded clique-width [11]) is well-quasi-ordered by the induced subgraph relation. In Section 5 we also determine, by giving infinite antichains, two bigenic classes that are not, namely the class of ($\overline{2P_1 + P_2}$, $P_2 + P_4$)-free graphs, which has unbounded clique-width [11], and the class of ($\overline{P_1 + P_4}$, $P_1 + 2P_2$)-free graphs, for which boundedness of the clique-width is unknown (see Fig. 1 for drawings of the five forbidden induced subgraphs). Consequently, there are six classes of (H_1, H_2)-free graphs for which we do not know whether they are well-quasi-ordered by the induced subgraph relation, and there is one open case left for the verification of Conjecture 1 for bigenic classes. We refer to Open Problems 1 and 3, respectively, in Section 6 where we also give an exact description of the state-of-the-art for results on well-quasi-orderability and boundedness of clique-width for bigenic classes of graphs.

Towards our second goal, we aim to develop general techniques as opposed to tackling specific cases in an ad hoc fashion. Our starting point is a very fruitful technique used for determining (un)boundedness of the clique-width of a graph class $\mathcal{G}$. We transform a given

$$\overline{2P_1 + P_2} \qquad \overline{P_1 + P_4} \qquad P_1 + 2P_2 \qquad P_2 + P_3 \qquad P_2 + P_4$$
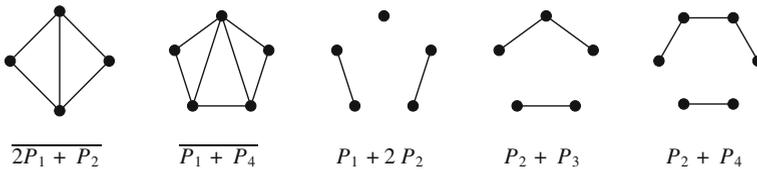
Fig. 1 The forbidden induced subgraphs considered in this paper

graph from $\mathcal{G}$ via a number of elementary graph operations that do not modify the clique-width by "too much" into a graph from a class for which we do know whether or not its clique-width is bounded.

It is a natural question to research how the above modification technique can be used for well-quasi-orders. We do this in Section 3. The permitted elementary graph operations that preserve (un)boundedness of the clique-width are vertex deletion, subgraph complementation and bipartite complementation. As we will explain in Section 3, these three graph operations do not preserve well-quasi-orderability. We circumvent this by investigating whether these three operations preserve boundedness of a graph parameter called uniformicity. This parameter was introduced by Korpelainen and Lozin [21], who proved that every graph class $\mathcal{G}$ of bounded uniformicity is well-quasi-ordered by the so-called labelled induced subgraph relation, which in turn implies that $\mathcal{G}$ is well-quasi-ordered by the induced subgraph relation. Korpelainen and Lozin [21] proved that boundedness of uniformicity is preserved by vertex deletion. We prove that this also holds for the other two graph operations.

The above enables us to focus on boundedness of uniformicity. However, we cannot always do this: there exist graph classes of unbounded uniformicity that are well-quasi-ordered by the labelled induced subgraph relation. As such, we sometimes need to rely only on the labelled induced subgraph relation directly. Hence, in Section 3 we also show that the three permitted graph operations, vertex deletion, subgraph complementation and bipartite complementation, preserve well-quasi-orderability by the labelled induced subgraph relation.

As explained in Section 6, we believe that this graph modification technique will also be useful for proving well-quasi-orderability of other graph classes. As such, we view the results in Section 3 as the second main contribution of our paper.

2 Preliminaries

The *disjoint union* $(V(G) \cup V(H), E(G) \cup E(H))$ of two vertex-disjoint graphs G and H is denoted by $G + H$ and the disjoint union of r copies of a graph G is denoted by rG. The *complement* of a graph G, denoted by $\overline{G}$, has vertex set $V(\overline{G}) = V(G)$ and an edge between two distinct vertices if and only if these vertices are not adjacent in G. For a subset $S \subseteq V(G)$, we let $G[S]$ denote the subgraph of G *induced* by S, which has vertex set S and edge set $\{uv \mid u, v \in S, uv \in E(G)\}$. If $S = \{s_1, \dots, s_r\}$ then, to simplify notation, we may also write $G[s_1, \dots, s_r]$ instead of $G[\{s_1, \dots, s_r\}]$. We use $G \setminus S$ to denote the graph obtained from G by deleting every vertex in S, i.e. $G \setminus S = G[V(G) \setminus S]$. We write $H \subseteq_i G$ to indicate that H is isomorphic to an induced subgraph of G.

The graphs C_r, K_r, $K_{1,r-1}$ and P_r denote the cycle, complete graph, star and path on r vertices, respectively. The graph $K_{1,3}$ is also called the *claw*. The graph $S_{h,i,j}$, for $1 \le h \le$

$i \leq j$, denotes the *subdivided claw*, that is, the tree that has only one vertex x of degree 3 and exactly three leaves, which are of distance h, i and j from x, respectively. Observe that $S_{1,1,1} = K_{1,3}$. We let $\mathcal{S}$ denote the class of graphs, each connected component of which is either a subdivided claw or a path. For a set of graphs $\{H_1, \ldots, H_p\}$, a graph G is $(H_1, \ldots, H_p)$-*free* if it has no induced subgraph isomorphic to a graph in $\{H_1, \ldots, H_p\}$; if $p = 1$, we may write H_1-free instead of (H_1)-free.

For a graph $G = (V, E)$ and a vertex $u \in V$, the set $N_G(u) = \{v \in V \mid uv \in E\}$ denotes the (open) *neighbourhood* of u in G and $N_G[u] = N_G(u) \cup \{u\}$ denotes the *closed neighbourhood* of u. We may write $N(u)$ and $N[u]$ instead of $N_G(u)$ and $N_G[u]$ if this is unambiguous. A graph is *bipartite* if its vertex set can be partitioned into (at most) two independent sets. The *biclique* $K_{r,s}$ is the bipartite graph with sets in the partition of size r and s respectively, such that every vertex in one set is adjacent to every vertex in the other set.

Let X be a set of vertices of a graph $G = (V, E)$. A vertex $y \in V \setminus X$ is *complete* to X if it is adjacent to every vertex of X and *anti-complete* to X if it is non-adjacent to every vertex of X. Similarly, a set of vertices $Y \subseteq V \setminus X$ is *complete* (resp. *anti-complete*) to X if every vertex in Y is complete (resp. anti-complete) to X. A vertex $y \in V \setminus X$ *distinguishes* X if y has both a neighbour and a non-neighbour in X. The set X is a *module* of G if no vertex in $V \setminus X$ distinguishes X. A module X is *non-trivial* if $1 < |X| < |V|$, otherwise it is *trivial*. A graph is *prime* if it has only trivial modules.

A *quasi order* $\leq$ on a set X is a reflexive, transitive binary relation. Two elements $x, y \in X$ in this quasi-order are *comparable* if $x \leq y$ or $y \leq x$, otherwise they are *incomparable*. A set of elements in a quasi-order is a *chain* if every pair of elements is comparable and it is an *antichain* if every pair of elements is incomparable. The quasi-order $\leq$ is a *well-quasi-order* if any infinite sequence of elements $x_1, x_2, x_3, \ldots$ in X contains a pair (x_i, x_j) with $x_i \leq x_j$ and $i < j$. Equivalently, a quasi-order is a well-quasi-order if and only if it has no infinite strictly decreasing sequence $x_1 \gneq x_2 \gneq x_3 \gneq \cdots$ and no infinite antichain.

For an arbitrary set M, let M^* denote the set of finite sequences of elements of M. Any quasi-order $\leq$ on M defines a quasi-order $\leq^*$ on M^* as follows: $(a_1, \ldots, a_m) \leq^* (b_1, \ldots, b_n)$ if and only if there is a sequence of integers $i_1, \ldots, i_m$ with $1 \leq i_1 < \cdots < i_m \leq n$ such that $a_j \leq b_{i_j}$ for $j \in \{1, \ldots, m\}$. We call $\leq^*$ the *subsequence relation*.

Lemma 1 (Higman's Lemma [19]) *If $(M, \leq)$ is a well-quasi-order then $(M^*, \leq^*)$ is a well-quasi-order.*

2.1 Labelled Induced Subgraphs and Uniformicity

To define the notion of labelled induced subgraphs, let us consider an arbitrary quasi-order $(W, \leq)$. We say that G is a *labelled* graph if each vertex v of G is equipped with an element $l_G(v) \in W$ (the *label* of v). Given two labelled graphs G and H, we say that G is a *labelled induced subgraph* of H if G is isomorphic to an induced subgraph of H and there is an isomorphism that maps each vertex v of G to a vertex w of H with $l_G(v) \leq l_H(w)$. Clearly, if $(W, \leq)$ is a well-quasi-order, then a class of graphs X cannot contain an infinite sequence of labelled graphs that is strictly-decreasing with respect to the labelled induced subgraph relation. We therefore say that a class of graphs X is well-quasi-ordered by the *labelled* induced subgraph relation if it contains no infinite antichains of labelled graphs whenever $(W, \leq)$ is a *well*-quasi-order. Such a class is readily seen to also be well-quasi-ordered by the induced subgraph relation.

We will use the following three results.

Lemma 2 ([1]) *The class of P_6-free bipartite graphs is well-quasi-ordered by the labelled induced subgraph relation.*

Lemma 3 ([1]) *Let k, ℓ, m be positive integers. Then the class of $(P_k, K_\ell, K_{m,m})$-free graphs is well-quasi-ordered by the labelled induced subgraph relation.*

Lemma 4 ([1]) *Let X be a hereditary class of graphs. Then X is well-quasi-ordered by the labelled induced subgraph relation if and only if the set of prime graphs in X is. In particular, X is well-quasi-ordered by the labelled induced subgraph relation if and only if the set of connected graphs in X is.*

Let k be a natural number, let K be a symmetric square $(0, 1)$-matrix of order k, and let F_k be a graph on the vertex set $\{1, 2, \ldots, k\}$. Let H be the disjoint union of infinitely many copies of F_k, and for $i = 1, \ldots, k$, let V_i be the subset of $V(H)$ containing vertex i from each copy of F_k. Now we construct from H an infinite graph $H(K)$ on the same vertex set by applying a subgraph complementation to V_i if and only if $K(i, i) = 1$ and by applying bipartite complementation to a pair V_i, V_j if and only if $K(i, j) = 1$. In other words, two vertices $u \in V_i$ and $v \in V_j$ are adjacent in $H(K)$ if and only if $uv \in E(H)$ and $K(i, j) = 0$ or $uv \notin E(H)$ and $K(i, j) = 1$. Finally, let $\mathcal{P}(K, F_k)$ be the hereditary class consisting of all the finite induced subgraphs of $H(K)$.

Let k be a natural number. A graph G is *k-uniform* if there is a matrix K and a graph F_k such that $G \in \mathcal{P}(K, F_k)$. The minimum k such that G is k-uniform is the *uniformicity* of G.

The following result was proved by Korpelainen and Lozin. The class of disjoint unions of cliques is a counterexample for the reverse implication.

Theorem 1 ([21]) *Any class of graphs of bounded uniformicity is well-quasi-ordered by the labelled induced subgraph relation.*

3 Permitted Graph Operations

It is not difficult to see that if G is an induced subgraph of H, then $\overline{G}$ is an induced subgraph of $\overline{H}$. Therefore, a graph class X is well-quasi-ordered by the induced subgraph relation if and only if the set of complements of graphs in X is. In this section, we strengthen this observation in several ways.

First, we define the operation of subgraph complementation as follows.

Definition 1 *Subgraph complementation* in a graph G is the operation of complementing a subgraph of G induced by a subset of its vertices.

Applied to the entire vertex set of G, this operation coincides with the usual complementation of G. However, applied to a pair of vertices, it changes the adjacency of these vertices only. Clearly, repeated applications of this operation can transform G into any other graph on the same vertex set. Therefore, unrestricted applications of subgraph complementation may transform a well-quasi-ordered class X into a class containing infinite antichains. However, if we bound the number of applications of this operation by a constant, we preserve many nice properties of X, including well-quasi-orderability with respect to the labelled induced subgraph relation.

Next, we introduce the following operation:

Definition 2 *Bipartite complementation* in a graph G is the operation of complementing the edges between two disjoint subsets $X, Y \subseteq V(G)$.

Note that applying a bipartite complementation between X and Y has the same effect as applying a sequence of three subgraph complementations: with respect to X, Y and $X \cup Y$.

Finally, we define the following operation:

Definition 3 *Vertex deletion* in a graph G is the operation of removing a single vertex v from a graph, together with any edges incident to v.

3.1 Operations on Labelled Graphs

Let $k \geq 0$ be a constant and let γ be a graph operation. A graph class $\mathcal{G}'$ is (k, γ)-*obtained* from a graph class $\mathcal{G}$ if the following two conditions hold:

(i)   every graph in $\mathcal{G}'$ is obtained from a graph in $\mathcal{G}$ by performing γ at most k times, and
(ii) for every $G \in \mathcal{G}$ there exists at least one graph in $\mathcal{G}'$ obtained from G by performing γ at most k times.

We say that γ *preserves* well-quasi-orderability by the labelled induced subgraph relation if for any finite constant k and any graph class $\mathcal{G}$, any graph class $\mathcal{G}'$ that is (k, γ)-obtained from $\mathcal{G}$ is well-quasi-ordered by this relation if and only if $\mathcal{G}$ is.

Lemma 5 *The following operations preserve well-quasi-orderability by the labelled induced subgraph relation:*

(i)   *Subgraph complementation,*
(ii) *Bipartite complementation and*
(iii) *Vertex deletion.*

Proof We start by proving the lemma for subgraph complementations.

Let X be a class of graphs and let Y be a set of graphs obtained from X by applying a subgraph complementation to each graph in X. More precisely, for each graph $G \in X$ we choose a set Z_G of vertices in G; we let G' be the graph obtained from G by applying a complementation with respect to the subgraph induced by Z_G and we let Y be the set of graphs G' obtained in this way. Clearly it is sufficient to show that X is well-quasi-ordered by the labelled induced subgraph relation if and only if Y is.

Suppose that X is not well-quasi-ordered under the labelled induced subgraph relation. Then there must be a well-quasi-order $(L, \leq)$ and an infinite sequence $G_1, G_2, \ldots$ of graphs in $\mathcal{X}$ with vertices labelled with elements of L, such that these graphs form an infinite antichain under the labelled induced subgraph relation. Let $(L', \leq')$ be the quasi-order with $L' = \{(k, l) : k \in \{0, 1\}, l \in L\}$ and $(k, l) \leq' (k', l')$ if and only if $k = k'$ and $l \leq l'$ (so L' is the disjoint union of two copies of L, where elements of one copy are incomparable with elements of the other copy). Note that $(L', \leq')$ is a well-quasi-order since $(L, \leq)$ is a well-quasi-order.

For each graph G_i in this sequence, with labelling l_i, we construct the graph G_i' (recall that G_i' is obtained from G_i by applying a complementation on the vertex set Z_{G_i}). We label the vertices of $V(G_i')$ with a labelling l_i' as follows:

–   set $l_i'(v) = (1, l_i(v))$ if $v \in Z_{G_i}$ and
–   set $l_i'(v) = (0, l_i(v))$ otherwise.

We claim that when $G'_1, G'_2, \ldots$ are labelled in this way they form an infinite antichain with respect to the labelled induced subgraph relation. Indeed, suppose for contradiction that G'_i is a labelled induced subgraph of G'_j for some $i \neq j$. This means that there is a injective map $f : V(G'_i) \to V(G'_j)$ such that $l'_i(v) \leq' l'_j(f(v))$ for all $v \in V(G'_i)$ and $v, w \in V(G'_i)$ are adjacent in G'_i if and only if $f(v)$ and $f(w)$ are adjacent in G'_j. Now since $l'_i(v) \leq' l'_j(f(v))$ for all $v \in V(G'_i)$, by the definition of $\leq'$ we conclude the following:

- $l_i(v) \leq l_j(f(v))$ for all $v \in V(G'_i)$ and
- $v \in Z_{G_i}$ if and only if $f(v) \in Z_{G_j}$.

Suppose $v, w \in V(G_i)$ with $w \notin Z_{G_i}$ (v may or may not belong to Z_{G_i}) and note that this implies $f(w) \notin Z_{G_j}$. Then v and w are adjacent in G_i if and only if v and w are adjacent in G'_i if and only if $f(v)$ and $f(w)$ are adjacent in G'_j if and only if $f(v)$ and $f(w)$ are adjacent in G_j.

Next suppose $v, w \in Z_{G_i}$, in which case $f(v), f(w) \in Z_{G_j}$. Then v and w are adjacent in G_i if and only if v and w are non-adjacent in G'_i if and only if $f(v)$ and $f(w)$ are non-adjacent in G'_j if and only if $f(v)$ and $f(w)$ are adjacent in G_j.

It follows that f is an injective map $f : V(G_i) \to V(G_j)$ such that $l_i(v) \leq l_j(f(v))$ for all $v \in V(G_i)$ and $v, w \in V(G_i)$ are adjacent in G_i if and only if $f(v)$ and $f(w)$ are adjacent in G_j. In other words G_i is a labelled induced subgraph of G_j. This contradiction means that if $G_1, G_2, \ldots$ is an infinite antichain then $G'_1, G'_2, \ldots$ must also be an infinite antichain.

Therefore, if the class X is not well-quasi-ordered by the labelled induced subgraph relation then neither is Y. Repeating the argument with the roles of $G_1, G_2, \ldots$ and $G'_1, G'_2, \ldots$ reversed shows that if Y is not well-quasi-ordered under the labelled induced subgraph relation then neither is X. This completes the proof for subgraph complementations.

Since a bipartite complementation is equivalent to doing three subgraph complementations one after another, the result for bipartite complementations follows.

Finally, we prove the result for vertex deletions.

Let X be a class of graphs and let Y be a set of graphs obtained from X by deleting exactly one vertex z_G from each graph G in X. We denote the obtained graph by $G - z_G$. Clearly it is sufficient to show that X is well-quasi-ordered by the labelled induced subgraph relation if and only if Y is.

Suppose that Y is well-quasi-ordered by the labelled induced subgraph relation. We will show that X is also well-quasi-ordered by this relation. For each graph $G \in X$, let G' be the graph obtained from G by applying a bipartite complementation between $\{z_G\}$ and $N(z_G)$, so z_G is an isolated vertex in G'. Let Z be the set of graphs obtained in this way. By Lemma 5.(ii), Z is well-quasi-ordered by the labelled induced subgraph relation if and only if X is. Suppose G_1, G_2 are graphs in Z with vertices labelled from some well-quasi-order $(L, \leq)$. Then for $i \in \{1, 2\}$ the vertex z_{G_i} has a label from L and the graph $G_i - z_{G_i}$ belongs to Y. Furthermore if $G_1 - z_{G_1}$ is a labelled induced subgraph of $G_2 - z_{G_2}$ and $l_{G_1}(z_{G_1}) \leq l_{G_2}(z_{G_2})$ then G_1 is a labelled induced subgraph of G_2. Now by Lemma 1 it follows that Z is well-quasi-ordered by the labelled induced subgraph relation. Therefore X is also well-quasi-ordered by this relation.

Now suppose that Y is not well-quasi-ordered by the labelled induced subgraph relation. Then Y contains an infinite antichain $G_1, G_2, \ldots$ with the vertices of G_i labelled by functions l_i which takes values in some well-quasi-order $(L, \leq)$. For each G_i, let G'_i be a corresponding graph in X, so $G_i = G'_i - z_{G'_i}$. Then in G'_i we label $z_{G'_i}$ with a new label $*$

and label all other vertices $v \in V(G'_i)$ with the same label as that used in G_i. We make this new label $*$ incomparable to all the other labels in L and note that the obtained quasi order $(L \cup \{*\}, \leq)$ is also a well-quasi-order. It follows that $G'_1, G'_2, \ldots$ is an antichain in X when labelled in this way. Therefore, if Y is not well-quasi-ordered by the labelled induced subgraph relation then neither is X. This completes the proof. $\qquad\square$

Note that the above lemmas only apply to well-quasi-ordering with respect to the *labelled* induced subgraph relation. Indeed, if we take a cycle and delete a vertex, complement the subgraph induced by an edge or apply a bipartite complementation to two adjacent vertices, we obtain a path. However, while the set of cycles is an infinite antichain with respect to the induced subgraph relation, the set of paths is not.

3.2 Operations on *k*-Uniform Graphs

We now show that our graph operations do not change uniformicity by "too much" either. The result for vertex deletion was proved by Korpelainen and Lozin.

Lemma 6 *Let G be a graph of uniformicity k. Let G', G'' and G''' be graphs obtained from G by applying one vertex deletion, subgraph complementation or bipartite complementation, respectively. Let ℓ', ℓ'' and ℓ''' be the uniformicities of G', G'' and G''', respectively. Then the following three statements hold:*

  (i) $\ell' < k < 2\ell' + 1$ [21]*;*
  (ii) $\frac{k}{2} \leq \ell'' \leq 2k$*;*
  (iii) $\frac{k}{3} \leq \ell''' \leq 3k$*.*

Proof The first inequality of Part (i) is trivial. The second inequality of Part (i) was proved in [21]. We prove Parts (ii) and (iii) below.

Let G be a graph of uniformicity k, let X be a set of vertices in G and let G'' be the graph obtained from G by applying a complementation with respect to the subgraph induced by X. Let ℓ'' be the uniformicity of G''. By symmetry, to prove Part (ii), it is sufficient to prove that $\ell'' \leq 2k$.

Since G is a k-uniform graph, it must belong to $P(F_k, K)$ for some F_k and some K, so it is an induced subgraph of $H(K)$.

Consider the graph obtained from F_k by replacing each vertex v of F_k by two non-adjacent vertices v and v'. Apply a complementation with respect to $\{1', 2', \ldots, k'\}$ and let F'_k be the obtained graph. In other words, if $v \in V(F_k)$ then

$$N_{F'_k}(v) = N_{F_k}(v) \cup \{w' \mid w \in N_{F_k}(v)\} \text{ and}$$
$$N_{F'_k}(v') = N_{F_k}(v) \cup \{w' \mid w \in V(F_k) \setminus N_{F_k}[v]\}.$$

Let K' be a $2k \times 2k$ matrix indexed by $\{1, 2, \ldots, k, 1', 2', \ldots, k'\}$ with $K'_{i,j} = K'_{i,j'} = K'_{i',j} = 1 - K'_{i',j'} = K_{i,j}$.

This means that $H(K')$ is formed from $H(K)$ by adding a copy of each vertex that has the same neighbourhood and then applying a complementation with respect to the set of newly-created vertices.

Similarly, G'' can be obtained from G by replacing each vertex in X by a copy with the same neighbourhood and then applying a complementation with respect to the set of

newly-created vertices. Therefore G'' is an induced subgraph of $H(K')$. Therefore G'' is a $2k$-uniform graph.

Part (iii) follows from similar arguments to those for Part (ii). (Also note that if we weaken the bounds in Part (iii) to $\frac{k}{8} \leq \ell''' \leq 8k$ then the result follows immediately from combining Part (ii) with the fact that a bipartite complementation is equivalent to a sequence of three subgraph complementations). □

4 A New Well-Quasi-Ordered Class

In this section we show that $(\overline{2P_1 + P_2}, P_2 + P_3)$-free graphs are well-quasi-ordered by the labelled induced subgraph relation. We divide the proof into several sections, depending on whether or not the graphs under consideration contain certain induced subgraphs or not. We follow the general scheme that Dabrowski, Huang and Paulusma [11] used to prove that this class has bounded clique-width, but we will also need a number of new arguments.

4.1 Graphs containing a K_5

We first consider graphs containing a K_5 and prove the following lemma.

Lemma 7 *The class of $(\overline{2P_1 + P_2}, P_2 + P_3)$-free graphs that contain a K_5 is well-quasi-ordered by the labelled induced subgraph relation.*

Proof Let G be a $(\overline{2P_1 + P_2}, P_2 + P_3)$-free graph. Let X be a maximal (by set inclusion) clique in G containing at least five vertices.

Claim 1. Every vertex not in X has at most one neighbour in X.
This follows from the fact that X is maximal and G is $\overline{2P_1 + P_2}$-free.

Suppose there is a P_3 in $G \setminus X$, say on vertices x_1, x_2, x_3 in that order. Since $|X| \geq 5$, we can find $y_1, y_2 \in X$ none of which are adjacent to any of x_1, x_2, x_3. Then $G[y_1, y_2, x_1, x_2, x_3]$ is a $P_2 + P_3$. This contradiction shows that $G \setminus X$ is P_3-free and must therefore be a union of disjoint cliques $X_1, \ldots, X_k$. We say that a clique X_i is *large* if it contains at least two vertices and that it is *small* if it contains exactly one vertex.

Claim 2. If $x \in X$ is adjacent to $y \in X_i$ and X_j (with $i \neq j$) is large, then x can have at most one non-neighbour in X_j.
For contradiction, assume that x is non-adjacent to $z_1, z_2 \in X_j$. Since $|X| \geq 5$ and each vertex that is not in X has at most one neighbour in X, there must be a vertex $x' \in X$ that is non-adjacent to y, z_1 and z_2. Then $G[z_1, z_2, x', x, y]$ is a $P_2 + P_3$, a contradiction.

We consider several cases:

Case 1: $G \setminus X$ contains at most one clique.
Then the complement of G is a bipartite graph. Moreover, since the complement of $\overline{2P_1 + P_2}$ is an induced subgraph of P_6, we conclude that the complements of graphs in our class form a subclass of P_6-free bipartite graphs, which are well-quasi-ordered by the labelled induced subgraph relation by Lemmas 2 and 5.(i).

Case 2: $G \setminus X$ does not contain large cliques.
In this case, the structure of graphs can be described as follows: take a collection of stars and create a clique on their central vertices and then add a number (possibly zero) of isolated vertices. In other words, applying subgraph complementation once (to the clique X), we obtain a graph which is a disjoint union of stars and isolated vertices. Clearly, a graph every connected component of which is a star or an isolated vertex is P_6-free bipartite and hence by Lemmas 2 and 5.(i) we conclude that graphs in our class are well-quasi-ordered by the labelled induced subgraph relation.

Case 3: $G \setminus X$ contains exactly one large clique.
Without loss of generality, assume that X_1 is large and the remaining cliques $X_2, \ldots, X_k$ are small. Suppose there are ℓ distinct vertices $x_1, \ldots, x_\ell \in X$, each of which has a neighbour in $X_2 \cup \cdots \cup X_k$. By Claim 2, each of these vertices has at most one non-neighbour in X_1. But then $\ell \leq 2$, since otherwise a vertex of X_1 has more than one neighbour in X.

Therefore, by deleting at most two vertices from G we transform it to a graph from Case 1 plus a number of isolated vertices. Lemma 4 allows us to ignore isolated vertices, while Lemma 5.(iii) allows the deletion of finitely many vertices. Therefore, in Case 3 we deal with a set of graphs which is well-quasi-ordered by the labelled induced subgraph relation.

Case 4: $G \setminus X$ contains at least two cliques that are large.
Suppose there is a vertex $x \in X$ that has a neighbour outside of X. By Claim 2, x has at most one non-neighbour in each large clique. Therefore, at most two vertices of X have neighbours outside of X, since otherwise each large clique would have a vertex with more than one neighbour in X. But then by deleting at most two vertices we transform G into a P_3-free graph (i.e. a graph every connected component of which is a clique). It is well-known (and also follows from Lemma 4) that the set of P_3-free graphs is well-quasi-ordered by the labelled induced subgraph relation. Therefore, by Lemma 5.(iii), the same is true for graphs in Case 4. □

4.2 Graphs containing a C_5

By Lemma 7, we may restrict ourselves to looking at K_5-free graphs in our class. We now consider the case where these graphs have an induced C_5.

Lemma 8 *The class of $(\overline{2P_1 + P_2}, P_2 + P_3, K_5)$-free graphs that contain an induced C_5 has bounded uniformicity.*

Proof To prove this, we modify the proof from [11], which shows that this class of graphs has bounded clique-width. Let G be a $(\overline{2P_1 + P_2}, P_2 + P_3, K_5)$-free graph containing a C_5, say on vertices v_1, v_2, v_3, v_4, v_5 in order. Our goal is to show that the graph G has bounded uniformicity and hence, by Lemma 6.(i), in the proof we can neglect any set of vertices that is bounded in size by a constant.

Let Y be the set of vertices adjacent to v_1 and v_2 (and possibly to other vertices on the cycle). If $y_1, y_2 \in Y$ are non-adjacent, then $G[v_1, v_2, y_1, y_2]$ would be a $\overline{2P_1 + P_2}$. Therefore, Y is a clique. This clique has at most two vertices, since otherwise three vertices of Y together with v_1 and v_2 would create a K_5. Therefore, the set of all vertices with two consecutive neighbours on the cycle is finite and hence can be neglected (removed from the graph). We may therefore assume that each vertex not on the cycle has at most two neighbours on the cycle and if it has two such neighbours, they must be non-consecutive vertices of the cycle.

Now let W be the set of vertices whose unique neighbour on the cycle is v_1. If $y_1, y_2 \in W$ are non-adjacent, then $G[v_3, v_4, y_1, v_1, y_2]$ would be a $P_2 + P_3$. If $y_1, y_2 \in W$ are adjacent, then $G[y_1, y_2, v_2, v_3, v_4]$ would be a $P_2 + P_3$. Therefore, W contains at most one vertex, and hence the set of all vertices with exactly one neighbour on the cycle can be removed from the graph.

Let X be the set of vertices with no neighbours on the cycle. X must be an independent set, since if two vertices in $x_1, x_2 \in X$ are adjacent, then $G[x_1, x_2, v_1, v_2, v_3]$ would induce a $P_2 + P_3$ in G.

For $i \in \{1, 2, 3, 4, 5\}$, let V_i be the set of vertices not on the cycle that are adjacent to v_{i-1} and v_{i+1}, but non-adjacent to all other vertices of the cycle (subscripts on vertices and vertex sets are interpreted modulo 5 throughout this proof). For each i, the set V_i must be independent, since if $x, y \in V_i$ are adjacent then $G[x, y, v_{i-1}, v_{i+1}]$ is a $\overline{2P_1 + P_2}$.

We say that two sets V_i and V_j are *consecutive* (respectively *opposite*) if v_i and v_j are distinct adjacent (respectively non-adjacent) vertices of the cycle. We say that a set X or V_i is *large* if it contains at least three vertices, otherwise it is *small*. We say that a bipartite graph with bipartition classes A and B is a *matching* (*co-matching*) if every vertex in A has at most one neighbour (non-neighbour) in B, and vice versa.

Dabrowski, Huang and Paulusma proved the following claims about the edges between these sets (see Appendix A for a proof).

Claim 1. ([11]) $G[V_i \cup X]$ is a matching.
Claim 2. ([11]) If V_i and V_j are opposite, then $G[V_i \cup V_j]$ is a matching.
Claim 3. ([11]) If V_i and V_j are consecutive, then $G[V_i \cup V_j]$ is a co-matching.
Claim 4. ([11]) If V_i is large, then X is anti-complete to $V_{i-2} \cup V_{i+2}$.
Claim 5. ([11]) If V_i is large, then V_{i-1} is anti-complete to V_{i+1}.
Claim 6. ([11]) If V_{i-1}, V_i, V_{i+1} are large, then V_i is complete to $V_{i-1} \cup V_{i+1}$.

We also prove the following claim:

Claim 7. Suppose two consecutive sets V_i and V_{i+1} are large and a vertex $y \in V_i$ is not adjacent to a vertex $z \in V_{i+1}$. Then every vertex $x \in X \cup V_{i+3}$ is either complete or anti-complete to $\{y, z\}$.
To prove this, suppose for contradiction that this is not the case. Without loss of generality, we may assume that that x is adjacent to y but not to z. Since V_i is large it contains at least two other vertices, say a and b. Then z is adjacent to both a and b (since $G[V_i \cup V_{i+1}]$ is a co-matching), while x is adjacent neither to a nor to b (since $G[V_i \cup X]$ and $G[V_i \cup V_{i+3}]$ are matchings). But then $G[x, y, a, z, b]$ is a $P_2 + P_3$, a contradiction.

We are now ready to prove the lemma. We may delete from G the vertices $v_1, \ldots, v_5$ and all vertices in every small set X or V_i. Let G' be the resulting graph. In order to show that G' has bounded uniformicity, we split the analysis into the following cases.

Case 1: All sets $V_1, \ldots, V_5$ are large.
From the above claims we conclude that any two consecutive sets are complete to each other and any two opposite sets are anti-complete to each other. Also, X is anti-complete to each of them. Therefore G' is 6-uniform.

Case 2: Four sets V_i are large, say $V_1, \ldots, V_4$.
Then V_1 and V_4 form a matching, while any other pair of these sets are either complete or anti-complete to each other. Also, X is anti-complete to each of them. Therefore G' is 5-uniform.

Case 3: Three consecutive sets V_i are large, say V_1, V_2, V_3.
Then V_2 is complete to V_1 and V_3, while V_1 and V_3 are anti-complete to each other. Also, X is anti-complete to V_1 and V_3 forms a matching with V_2. Therefore G' is 4-uniform.

Case 4: Three non-consecutive sets V_i are large, say V_1, V_3, V_4.
Then X is anti-complete to each of them. From the above claims we know that V_1 forms a matching with both V_3 and V_4, while V_3 and V_4 form a co-matching. Also, from Claim 7 we conclude that whenever two vertices $y \in V_3$ and $z \in V_4$ are non-adjacent, then either none of them has a neighbour in V_1 or they both are adjacent to the same vertex of V_1. Therefore, if we complement the edges between V_3 and V_4, then G' transforms into a graph in which every connected component is one of the following graphs: K_3, P_3, P_2, P_1. Each of these graphs is an induced subgraph of $\overline{P_1 + P_3}$ (also known as the *paw*), so the obtained graph is 4-uniform. By Lemma 6.(iii), it follows that $G[V_1 \cup V_3 \cup V_4]$ is 12-uniform and so G' is 13-uniform.

Case 5: Two consecutive sets V_i are large, say V_3, V_4.
This case is similar to the previous one, where the role of V_1 is played by X. Thus G' is 12-uniform.

Case 6: Two non-consecutive sets V_i are large.
Then X is anti-complete to each of them and hence the graph is obviously 3-uniform.

Case 7: At most one set V_i is large.
Then G' is obviously 2-uniform.
   Since the above cases cover all possibilities, this completes the proof. □

4.3 Graphs Containing a C_4

By Lemmas 7 and 8, we may restrict ourselves to looking at (K_5, C_5)-free graphs in our class. We prove the following structural result.

Lemma 9 *Let G be a $(\overline{2P_1 + P_2}, P_2 + P_3, K_5, C_5)$-free graph containing an induced C_4. Then by deleting at most 17 vertices and applying at most two bipartite complementations, we can modify G into the disjoint union of a $(P_2 + P_3)$-free bipartite graph and a 3-uniform graph.*

Proof In order to prove the lemma, we again modify the proof from [11], which shows that this class of graphs has bounded clique-width. Let G be a $(\overline{2P_1 + P_2}, P_2 + P_3, K_5, C_5)$-free graph containing a C_4 induced by the vertices v_1, v_2, v_3, v_4 in order. We interpret subscripts on vertices modulo 4 in this proof.
   Let Y be the set of vertices adjacent to v_1 and v_2 (and possibly to other vertices on the cycle). If y_1, $y_2 \in Y$ are non-adjacent, then $G[v_1, v_2, y_1, y_2]$ would be a $\overline{2P_1 + P_2}$. Therefore, Y is a clique. This clique has at most two vertices, since otherwise three vertices of Y together with v_1 and v_2 would create a K_5. Therefore, after deleting at most $2 \times 4 = 8$ vertices, we may assume that no vertex of the graph contains two consecutive neighbours on the cycle.
   Let V_1 denote the set of vertices adjacent to v_2 and v_4, and let V_2 denote the set of vertices adjacent to v_1 and v_3. For $i \in \{1, 2, 3, 4\}$, let W_i denote the set of vertices whose only neighbour on the cycle is v_i. If a set W_i contains at most one vertex then we may delete this vertex. Thus, deleting at most four vertices, we may assume that every set W_i is either

empty of contains at least two vertices. Finally, let X be the set of vertices with no neighbour on the cycle.

Dabrowski, Huang and Paulusma proved the following claims about the edges between these sets (see Appendix B for a proof).

Claim 1. ([11]) V_i is independent for $i \in \{1, 2\}$.
Claim 2. ([11]) W_i is independent for $i \in \{1, 2, 3, 4\}$.
Claim 3. ([11]) X is independent.
Claim 4. ([11]) W_i is anti-complete to X for $i \in \{1, 2, 3, 4\}$.
Claim 5. ([11]) For $i \in \{1, 2\}$ either W_i or W_{i+2} is empty. Therefore, we may assume by symmetry that $W_3 = \emptyset$ and $W_4 = \emptyset$.

Note that in our arguments so far we have deleted at most 12 vertices. We now argue as follows:

Any vertices of X that do not have neighbours in $V_1 \cup V_2$ must be isolated vertices of the graph. Since adding isolated vertices to a $P_2 + P_3$-free bipartite graph maintains the property of it being $P_2 + P_3$-free and bipartite, we may therefore assume that every vertex in X has a neighbour in $V_1 \cup V_2$. Let X_0 denote the subset of X whose vertices have neighbours both in V_1 and V_2, let X_1 denote the subset of X whose vertices have no neighbours in V_1 and let X_2 denote the subset of X whose vertices have no neighbours in V_2.

Let V_0 denote the set of vertices in $V_1 \cup V_2$ adjacent to at least one vertex of X_0 and let $V_{10} = V_1 \cap V_0$ and $V_{20} = V_2 \cap V_0$. If V_{10} or V_{20} contains at most one vertex then we may delete this vertex. This would cause X_0 to become empty. Therefore, by deleting at most one vertex, we may assume that either both V_{10} and V_{20} each contain at least two vertices or else V_{10}, V_{20} and X_0 are all empty. We will show that $G[X_0 \cup V_0]$ is 3-uniform and can be separated from the rest of the graph using at most two bipartite complementations. To do this, we first prove the following additional claims.

Claim 6. Every vertex of X_0 has exactly one neighbour in V_{10} and exactly one neighbour in V_{20} and these neighbours are adjacent.
First, we observe that if a vertex $x \in X_0$ is adjacent to $y \in V_1$ and to $z \in V_2$, then y is adjacent to z, since otherwise $G[x, y, v_2, v_1, z]$ is an induced C_5. This implies that if x has the third neighbour $y' \in V_1 \cup V_2$, then $G[x, z, y, y']$ is a $\overline{2P_1 + P_2}$. This contradiction proves the claim.

Claim 7. Every vertex of V_0 is adjacent to exactly one vertex of X_0.
Let v be a vertex in V_0. Without loss of generality, assume that v belongs to V_{10}. Suppose that v has at least two neighbours in X_0, say x and x', and at least one non-neighbour, say x''. Let v'' be the neighbour of x'' in V_{10}. Then $G[v'', x'', x, v, x']$ is a $P_2 + P_3$. This contradiction shows that if v has at least two neighbours in X_0, then it must be adjacent to all the vertices of X_0. Since every vertex of X_0 has exactly one neighbour in V_{10} it follows that v is the only vertex of V_{10}, a contradiction. We conclude that v (and hence every other vertex of V_0) has exactly one neighbour in X_0.

Claim 8. V_{10} is complete to V_2 and V_{20} is complete to V_1.
Suppose $v \in V_{10}$ is non-adjacent to $y \in V_2$ and let x be the unique neighbour of v in X_0. Since y is non-adjacent to v, it cannot be the unique neighbour of x in V_2. Therefore y must be non-adjacent to x. It follows that $G[x, v, v_1, y, v_3]$ is a $P_2 + P_3$, a contradiction. The second part of the claim follows by symmetry.

Claim 9. Every vertex in $W_1 \cup W_2 \cup X_1 \cup X_2$ is either complete or anti-complete to V_{i0} for $i = 1, 2$.

Suppose a vertex $w \in W_1 \cup W_2 \cup X_1 \cup X_2$ has both a neighbour v and a non-neighbour v' in V_{10}. Let x and x' be the neighbours of v and v', respectively, in X_0. Recall that x and x' must be non-adjacent to w. Then $G[v', x', w, v, x]$ is a $P_2 + P_3$, a contradiction.

By Claims 8 and 9, every vertex outside $V_{10} \cup V_{20} \cup X_0$ is either complete or anti-complete to V_{10} and either complete or anti-complete to V_{20}. Applying at most two bipartite complementations, we may therefore disconnect $G[V_{10} \cup V_{20} \cup X_0]$ from the rest of the graph i.e. remove all edges between vertices in $V_{10} \cup V_{20} \cup X_0$ and vertices outside $V_{10} \cup V_{20} \cup X_0$. By Claims 6, 7 and 8 it follows that $G[V_{10} \cup V_{20} \cup X_0]$ is a 3-uniform graph.

We may now assume that X_0 is empty. Let H be the graph obtained from G by deleting the vertices of the original cycle. Note that $V(H) = X_1 \cup V_1 \cup W_1 \cup X_2 \cup V_2 \cup W_2$. It remains to show that H is a $(P_2 + P_3)$-free bipartite graph.

We claim that H is bipartite with independent sets $X_1 \cup V_1 \cup W_1$ and $X_2 \cup V_2 \cup W_2$. To show this, it suffices to prove that H has no triangles, because all other odd cycles are forbidden in G (and hence in H). We know that $X_1 \cup V_1$ is an independent set and $X_2 \cup V_2$ is an independent set. Also, W_1 and W_2 are independent and the vertices of X have no neighbours in $W_1 \cup W_2$. It follows that in H vertices in X_1 an can only have neighbours in V_2 and vertices of X_2 can only have neighbours in V_1, so no triangle in H contains a vertex of X. By symmetry, if $H[x, y, z]$ is a triangle then we may therefore assume that $x \in W_1$, $y \in V_2$ and $z \in V_1 \cup W_2$. Now $G[x, y, z, v_1]$ is a $\overline{2P_1 + P_2}$, a contradiction. It follows that H is bipartite. Moreover, it is $(P_2 + P_3)$-free. This completes the proof. $\qquad\square$

Since $P_2 + P_3$ is an induced subgraph of P_6, it follows that every $(P_2 + P_3)$-free graph is P_6-free. Combining Lemma 9 with Theorem 1 and Lemmas 2, 5.(ii) and 5.(iii) we therefore obtain the following corollary.

Corollary 1 *The class of $(\overline{2P_1 + P_2}, P_2 + P_3, K_5, C_5)$-free graphs containing an induced C_4 is well-quasi-ordered by the labelled induced subgraph relation.*
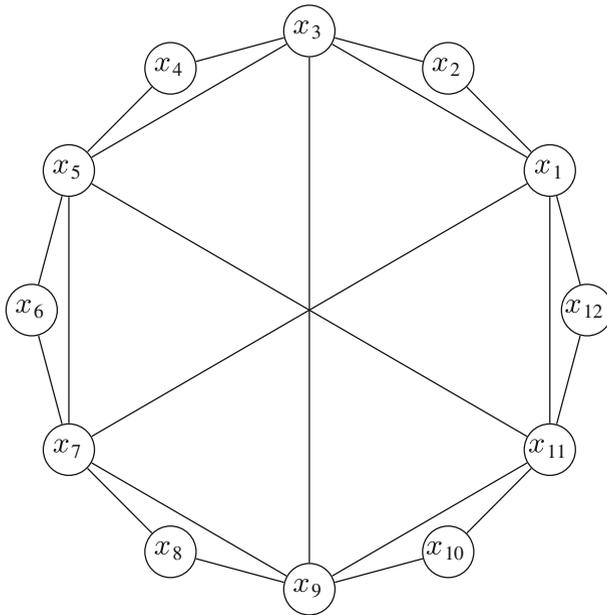
4.4 $(\overline{2P_1 + P_2}, P_2 + P_3)$-free graphs

Theorem 2 *The class of $(\overline{2P_1 + P_2}, P_2 + P_3)$-free graphs is well-quasi-ordered by the labelled induced subgraph relation.*

Proof Graphs in the class under consideration containing an induced subgraph isomorphic to K_5, C_5 or C_4 are well-quasi-ordered by the labelled induced subgraph relation by Lemmas 7 and 8 and Corollary 1, respectively. The remaining graphs form a subclass of $(P_6, K_5, K_{2,2})$-free graphs, since $C_4 = K_{2,2}$ and $P_2 + P_3$ is an induced subgraph of P_6. By Lemma 3, this class of graphs is well-quasi-ordered by the labelled induced subgraph relation. Therefore, the class of $(\overline{2P_1 + P_2}, P_2 + P_3)$-free graphs is well-quasi-ordered by the labelled induced subgraph relation. $\qquad\square$

5 Two New Non-Well-Quasi-Ordered Classes

In this section we show that the classes of $(\overline{2P_1 + P_2}, P_2 + P_4)$-free graphs and $(\overline{P_1 + P_4}, P_1 + 2P_2)$-free graphs are not well-quasi-ordered by the induced subgraph relation. The antichain used to prove the first of these cases was previously used by Atminas

Fig. 2 The Graph G_{4n} from Theorem 3 when $n = 3$

and Lozin to show that the class of $(\overline{2P_1 + P_2}, P_6)$-free graphs is not well-quasi-ordered with respect to the induced subgraph relation. Because of this, we can show show a stronger result for the first case.

Theorem 3 *The class of $(\overline{2P_1 + P_2}, P_2 + P_4, P_6)$-free graphs is not well-quasi-ordered by the induced subgraph relation.*

Proof Let $n \geq 2$ be an integer and consider a cycle C_{4n}, say $x_1 - x_2 - \cdots - x_{4n} - x_1$. We partition the vertices of this cycle into three sets as follows:

$$X = \{x_i \mid i \equiv 0 \text{ or } 2 \mod 4\},$$
$$Y = \{x_i \mid i \equiv 1 \mod 4\},$$
$$Z = \{x_i \mid i \equiv 3 \mod 4\}.$$

Let G_{4n} be the graph obtained from C_{4n} by connecting every vertex of Y to every vertex of Z (see also Fig. 2). Atminas and Lozin showed that the resulting graphs are $(\overline{2P_1 + P_2}, P_6)$-free and form an infinite antichain with respect to the induced subgraph relation [1].

It remains to prove that G_{4n} is $(P_2 + P_4)$-free. We argue as in the proof of [11, Theorem 1 (iv)]. For contradiction, suppose that G_{4n} contains an induced subgraph I isomorphic to $P_2 + P_4$. Let I_1 and I_2 be the connected components of I isomorphic to P_2 and P_4, respectively. Since $G_{4n}[Y \cup Z]$ is complete bipartite, I_2 must contain at least one vertex of X. Since the two neighbours of any vertex of X are adjacent, any vertex of X in I_2 must be an end-vertex of I_2. Then, as Y and Z are independent sets, I_2 contains a vertex of both Y and Z. As I_1 can contain at most one vertex of X (because X is an independent set), I_1 contains a vertex $u \in Y \cup Z$. However, $G_{4n}[Y \cup Z]$ is complete bipartite and I_2 contains a vertex of both Y and Z. Hence, u has a neighbour in I_2, which is not possible. This completes the proof. □
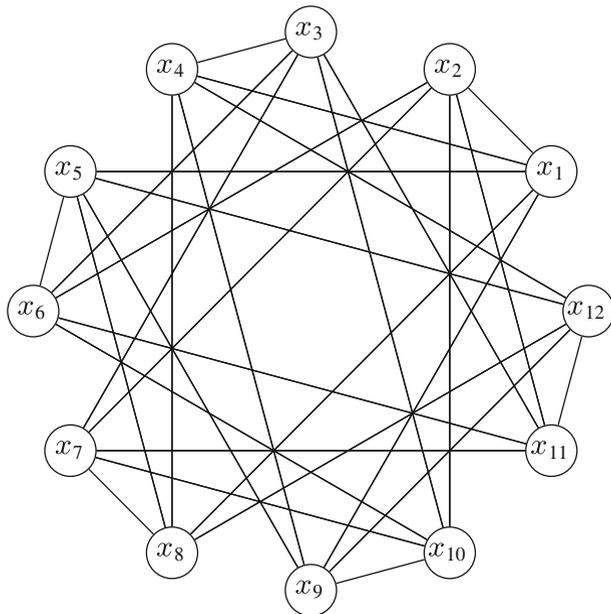
Fig. 3 The Graph G_{4n} from Theorem 4 when $n = 3$

The graphs G_{4n} in the above proof are obtained from cycles in the same way that walls were transformed in [11] to show unboundedness of clique-width for $(\overline{2P_1 + P_2}, P_2 + P_4)$-free graphs (The sets A, B and C in [11] correspond to the sets Y, X and Z, respectively, in the proof above). In fact the construction in [11] is also P_6-free by the same arguments as in [1], so we obtain the following:

Remark 1 The class of $(\overline{2P_1 + P_2}, P_2 + P_4, P_6)$-free graphs has unbounded clique-width.

For our second class, we need a new construction.

Theorem 4 *The class of $(\overline{P_1 + P_4}, P_1 + 2P_2)$-free graphs is not well-quasi-ordered by the induced subgraph relation.*

Proof Let $n \geq 3$ be an integer. Consider a cycle C_{4n}, say $x_1 - x_2 - \cdots - x_{4n} - x_1$. We partition the vertices of C_{4n} as follows:

$$X = \{x_i \mid i \equiv 0 \text{ or } 1 \mod 4\},$$
$$Y = \{x_i \mid i \equiv 2 \text{ or } 3 \mod 4\}.$$

Next we apply a complementation to each of X and Y, so that in the resulting graph X and Y each induce a clique on $2n$ vertices with a perfect matching removed. Let G_{4n} be the resulting graph (see also Fig. 3).

Suppose, for contradiction that G_{4n} contains an induced $P_1 + 2P_2$. Without loss of generality, the set X must contain three of the vertices v_1, v_2, v_3 of the $P_1 + 2P_2$. Since every component of $P_1 + 2P_2$ contains at most two vertices, without loss of generality we may assume that v_1 is non-adjacent to both v_2 and v_3. However, every vertex

of $G_{4n}[X]$ has exactly one non-neighbour in X. This contradiction shows that G_{4n} is indeed $(P_1 + 2P_2)$-free.

Every vertex in X has exactly one neighbour in Y and vice versa. This means that any K_3 in G_{4n} must lie entirely in $G_{4n}[X]$ or $G_{4n}[Y]$. Since $G_{4n}[X]$ or $G_{4n}[Y]$ are both complements of perfect matchings and every vertex of $\overline{P_1 + P_4}$ lies in one of three induced K_3's, which are pairwise non-disjoint, it follows that G_{4n} is $\overline{P_1 + P_4}$-free.

It remains to show that the graphs G_{4n} form an infinite antichain with respect to the induced subgraph relation. Since $n \geq 3$, every vertex in X (resp. Y) has at least two neighbours in X (resp. Y) that are pairwise adjacent. Therefore, given x_1, we can determine which vertices lie in X and which lie in Y. Every vertex in X (resp. Y) has a unique neighbour in Y (resp. X) and a unique non-neighbour in X (resp. Y). Therefore, by specifying which vertex in G_{4n} is x_1, we uniquely determine $x_2, \ldots, x_{4n}$. Suppose G_{4n} an induced subgraph of G_{4m} for some $m \geq 3$. Then $n \leq m$ due to the number of vertices. By symmetry, we may assume that the induced copy of G_{4n} in G_{4m} has vertex x_1 of G_{4n} in the position of vertex x_1 in G_{4m}. Then the induced copy of G_{4n} must have vertices $x_2, \ldots, x_{4n}$ in the same position as $x_2, \ldots, x_{4n}$ in G_{4m}, respectively. Now x_1 and x_{4n} are non-adjacent in G_{4n}. If $n < m$ then x_1 and x_{4n} are adjacent in G_{4m}, a contradiction. We conclude that if G_{4n} is an induced subgraph of G_{4m} then $n = m$. In other words $\{G_{4n} \mid n \geq 3\}$ is an infinite antichain with respect to the induced subgraph relation. $\qquad\square$

6 State of the Art and Future Work

In this section we summarise what is currently known about well-quasi-orderability and boundedness of clique-width, taking into account the results proved in this paper. We also give a number of directions for future work.

Given four graphs H_1, H_2, H_3, H_4, the class of (H_1, H_2)-free graphs and the class of (H_3, H_4)-free graphs are *equivalent* if the unordered pair H_3, H_4 can be obtained from the unordered pair H_1, H_2 by some combination of the operations (i) complementing both graphs in the pair and (ii) if one of the graphs in the pair is K_3, replacing it with $\overline{P_1 + P_3}$ or vice versa. If two classes are equivalent, then one of them is well-quasi-ordered with respect to the induced subgraph relation if and only if the other one is [21]. Similarly, if two classes are equivalent, then one of them has bounded clique-width if and only if the other one does [14]. We use this terminology in the remainder of this section.

6.1 Well-Quasi-Ordering

Atminas and Lozin [1] proved that the class of (K_3, P_6)-free graphs is well-quasi-ordered by the induced subgraph relation, while the class of $(\overline{2P_1 + P_2}, P_6)$-free graphs is not. In a recent paper [13] we proved that the classes of $(\overline{P_1 + P_3}, P_2 + P_4)$-free graphs and $(\overline{P_1 + P_3}, P_1 + P_5)$-free graphs are well-quasi-ordered by the labelled induced subgraph relation. Updating the classification in [21] with these four results and the three results proved in this paper (Theorems 2–4) leads to the following classification:

Theorem 5 *Let $\mathcal{G}$ be a class of graphs defined by two forbidden induced subgraphs. Then:*

1. *$\mathcal{G}$ is well-quasi-ordered by the labelled induced subgraph relation if it is equivalent to a class of (H_1, H_2)-free graphs such that one of the following holds:*

   (i) *H_1 or $H_2 \subseteq_i P_4$;*

   (ii) $H_1 = s P_1$ and $H_2 = K_t$ for some s, t;
   (iii) $H_1 \subseteq_i \overline{P_1 + P_3}$ and $H_2 \subseteq_i P_2 + P_4$, $P_1 + P_5$ or P_6.
   (iv) $H_1 \subseteq_i \overline{2P_1 + P_2}$ and $H_2 \subseteq_i P_2 + P_3$ or P_5.

2. $\mathcal{G}$ is not well-quasi-ordered by the induced subgraph relation if it is equivalent to a class of (H_1, H_2)-free graphs such that one of the following holds:

   (i)   neither H_1 nor H_2 is a linear forest (disjoint union of paths);
   (ii) $H_1 \supseteq_i K_3$ and $H_2 \supseteq_i 3P_1 + P_2$, $3P_2$ or $2P_3$;
   (iii) $H_1 \supseteq_i \overline{2P_2}$ and $H_2 \supseteq_i 4P_1$ or $2P_2$;
   (iv) $H_1 \supseteq_i \overline{2P_1 + P_2}$ and $H_2 \supseteq_i 4P_1$, $P_2 + P_4$ or P_6;
   (v) $H_1 \supseteq_i \overline{P_1 + P_4}$ and $H_2 \supseteq_i P_1 + 2P_2$.

Note that in Theorem 5 every class that is well-quasi-ordered with respect to the induced subgraph relation is also well-quasi-ordered with respect to the *labelled* induced subgraph relation (see [1, 21] and Theorem 2). This agrees with a conjecture of Atminas and Lozin [1] stating that these concepts coincide for hereditary classes X precisely when X is defined by a finite collection of forbidden induced subgraphs. Theorem 5 leaves us with six open cases.

Open Problem 1 Is the class of (H_1, H_2)-free graphs well-quasi-ordered by the induced subgraph relation when:

(i) $H_1 = \overline{2P_1 + P_2}$ and $H_2 \in \{P_1 + 2P_2, P_1 + P_4\}$;
(ii) $H_1 = \overline{P_1 + P_4}$ and $H_2 \in \{P_1 + P_4, 2P_2, P_2 + P_3, P_5\}$.

In relation to Open Problem 1, we mention that the infinite antichain for $(\overline{P_1 + P_4}, P_1 + 2P_2)$-free graphs was initially found by a computer search. This computer search also showed that similar antichains do not exist for any of the remaining six open cases. As such, constructing antichains for these cases is likely to be a challenging problem and this suggests that many of these classes may in fact be well-quasi-ordered. Some of these remaining classes have been shown to have bounded clique-width [3–5, 10]. We believe that some of the structural characterizations for proving these results may be useful for showing well-quasi-orderability.

6.2 Clique-Width

In the aforementioned paper [13] we also proved that the class of $(\overline{P_1 + P_3}, P_2 + P_4)$-free graphs has bounded clique-width. The following theorem, obtained after updating the theorem from [10] with the above result, describes exactly for which pairs (H_1, H_2) the (un)boundedness of the clique-width of (H_1, H_2)-free graphs has been determined.

Theorem 6 *Let $\mathcal{G}$ be a class of graphs defined by two forbidden induced subgraphs. Then:*

1. $\mathcal{G}$ has bounded clique-width if it is equivalent to a class of (H_1, H_2)-free graphs such that one of the following holds:

   (i) H_1 or $H_2 \subseteq_i P_4$;
   (ii) $H_1 = s P_1$ and $H_2 = K_t$ for some s, t;
   (iii) $H_1 \subseteq_i P_1 + P_3$ and $\overline{H_2} \subseteq_i K_{1,3} + 3P_1$, $K_{1,3} + P_2$, $P_1 + P_2 + P_3$, $P_1 + P_5$,
          $P_1 + S_{1,1,2}$, $P_2 + P_4$, P_6, $S_{1,1,3}$ or $S_{1,2,2}$;
   (iv) $H_1 \subseteq_i 2P_1 + P_2$ and $\overline{H_2} \subseteq_i P_1 + 2P_2$, $2P_1 + P_3$, $3P_1 + P_2$ or $P_2 + P_3$;

(v) $H_1 \subseteq_i P_1 + P_4$ *and* $\overline{H_2} \subseteq_i P_1 + P_4$ *or* P_5;
(vi) $H_1 \subseteq_i 4P_1$ *and* $\overline{H_2} \subseteq_i 2P_1 + P_3$;
(vii) $H_1, \overline{H_2} \subseteq_i K_{1,3}$.

2. *$\mathcal{G}$ has unbounded clique-width if it is equivalent to a class of (H_1, H_2)-free graphs such that one of the following holds:*

   (i) $H_1 \notin \mathcal{S}$ *and* $H_2 \notin \mathcal{S}$;
   (ii) $\overline{H_1} \notin \mathcal{S}$ *and* $\overline{H_2} \notin \mathcal{S}$;
   (iii) $H_1 \supseteq_i K_{1,3}$ *or* $2P_2$ *and* $\overline{H_2} \supseteq_i 4P_1$ *or* $2P_2$;
   (iv) $H_1 \supseteq_i 2P_1 + P_2$ *and* $\overline{H_2} \supseteq_i K_{1,3},\ 5P_1,\ P_2 + P_4$ *or* P_6;
   (v) $H_1 \supseteq_i 3P_1$ *and* $\overline{H_2} \supseteq_i 2P_1 + 2P_2,\ 2P_1 + P_4,\ 4P_1 + P_2,\ 3P_2$ *or* $2P_3$;
   (vi) $H_1 \supseteq_i 4P_1$ *and* $\overline{H_2} \supseteq_i P_1 + P_4$ *or* $3P_1 + P_2$.

This leaves us with the following seven non-equivalent open cases.

Open Problem 2 Does the class of (H_1, H_2)-free graphs have bounded or unbounded clique-width when:

(i) $H_1 = 3P_1$ and $\overline{H_2} \in \{P_1 + S_{1,1,3},\ S_{1,2,3}\}$;
(ii) $H_1 = 2P_1 + P_2$ and $\overline{H_2} \in \{P_1 + P_2 + P_3,\ P_1 + P_5\}$;
(iii) $H_1 = P_1 + P_4$ and $\overline{H_2} \in \{P_1 + 2P_2,\ P_2 + P_3\}$ or
(iv) $H_1 = \overline{H_2} = 2P_1 + P_3$.

A potential direction for future research related to determining boundedness of clique-width is investigating linear clique-width for classes defined by two forbidden induced subgraphs. Indeed, it is not hard to show that k-uniform graphs have bounded linear clique-width. Again, we can use complementations and vertex deletions when dealing with this parameter.

6.3 Well-Quasi-Ordering versus Clique-Width

Recall that all bigenic classes known to be well-quasi-ordered by the induced subgraph relation are also known to have bounded clique-width. Hence, to verify Conjecture 1 for bigenic classes, we only need to check the six open cases mentioned in Open Problem 1. Note that Conjecture 1 is verified directly if a graph class has bounded clique-width. Brandstädt, Le and Mosca [4] proved that the class of $(\overline{P_1 + P_4}, P_1 + P_4)$-free graphs (and thus the class of $(\overline{2P_1 + P_2}, P_1 + P_4)$-free graphs) has bounded clique-width. Using results from [3], the same authors proved in [5] that the class of $(\overline{P_1 + P_4}, P_5)$-free graphs (and thus the class of $(\overline{P_1 + P_4}, 2P_2)$-free graphs) has bounded clique-width. Dabrowski, Dross and Paulusma [10] showed that the class of $(\overline{2P_1 + P_2}, P_1 + 2P_2)$-free graphs has bounded clique-width. Hence, there is only one bigenic class of graphs left for which Conjecture 1 needs to be verified.

Open Problem 3 Is Conjecture 1 true for the class of (H_1, H_2)-free graphs when $H_1 = \overline{P_1 + P_4}$ and $H_2 = P_2 + P_3$?

As can be seen from Open Problems 1 and 2, we known neither whether the class of $(\overline{P_1 + P_4}, P_2 + P_3)$-free graphs is well-quasi-ordered by the induced subgraph relation nor whether its clique-width is bounded.

Appendix A: Proofs of the Claims in Lemma 8 from [11, Lemma 10]

Claim 1. $G[V_i \cup X]$ is a matching.
Indeed, if some vertex x in V_i (respectively X) is adjacent to two vertices y_1, y_2 in X (respectively V_i), then $G[v_{i+2}, v_{i+3}, y_1, x, y_2]$ is a $P_2 + P_3$.
Claim 2. If V_i and V_j are opposite, then $G[V_i \cup V_j]$ is a matching.
Suppose for contradiction that $x \in V_1$ is adjacent to two vertices $y, y' \in V_3$. Then $G[v_2, x, y, y']$ would be a $\overline{2P_1 + P_2}$, a contradiction.
Claim 3. If V_i and V_j are consecutive, then $G[V_i \cup V_j]$ is a co-matching.
Suppose for contradiction that $x \in V_1$ is non-adjacent to two vertices $y, y' \in V_2$. Then $G[x, v_5, y, v_3, y']$ is a $P_2 + P_3$, a contradiction.
Claim 4. If V_i is large, then X is anti-complete to $V_{i-2} \cup V_{i+2} \int$.
Suppose for contradiction that V_3 is large and $x \in X$ has a neighbour $y \in V_1$. Then since V_3 is large and both $G[X \cup V_3]$ and $G[V_1 \cup V_3]$ are matchings, there must be a vertex $z \in V_3$ that is non-adjacent to both x and y. Then $G[x, y, v_3, v_4, z]$ is a $P_2 + P_3$, a contradiction.
Claim 5. If V_i is large, then V_{i-1} is anti-complete to V_{i+1}.
Suppose for contradiction that V_2 is large and $x \in V_1$ has a neighbour $y \in V_3$. Since V_2 is large and each vertex in $V_1 \cup V_3$ has at most one non-neighbour in V_2, there must be a vertex $z \in V_2$ that is adjacent to both x and y. Now $G[x, y, v_2, z]$ is a $\overline{2P_1 + P_2}$, a contradiction.
Claim 6. If V_{i-1}, V_i, V_{i+1} are large, then V_i is complete to $V_{i-1} \cup V_{i+1}$.
Suppose for contradiction that V_1, V_2, V_3 are large and some vertex $x \in V_1$ is non-adjacent to a vertex $y \in V_2$. Since V_3 is large and $G[V_2 \cup V_3]$ is a co-matching, there must be two vertices $z, z' \in V_3$, adjacent to y. By the previous claim, since V_2 is large, z, z' must be non-adjacent to x. Therefore $G[x, v_5, z, y, z']$ is a $P_2 + P_3$, which is a contradiction.

Appendix B: Proofs of the Claims in Lemma 9 from [11, Lemma 11]

Claim 1. V_i is independent for $i \in \{1, 2\}$.
If $x, y \in V_i$ were adjacent then $G[x, y, v_{i+1}, v_{i+3}]$ would be a $\overline{2P_1 + P_2}$.
Claim 2. W_i is independent for $i \in \{1, 2, 3, 4\}$.
If $x, y \in W_i$ were adjacent then $G[x, y, v_{i+1}, v_{i+2}, v_{i+3}]$ would be a $P_2 + P_3$.
Claim 3. X is independent.
If $x, y \in X$ were adjacent then $G[x, y, v_1, v_2, v_3]$ would be a $P_2 + P_3$.
Claim 4. W_i is anti-complete to X for $i \in \{1, 2, 3, 4\}$.
If $x \in X$ were adjacent to $y \in W_i$ then $G[x, y, v_{i+1}, v_{i+2}, v_{i+3}]$ would be a $P_2 + P_3$.
Claim 5. For $i \in \{1, 2\}$ either W_i or W_{i+2} is empty. Therefore, we may assume by symmetry that $W_3 = \emptyset$ and $W_4 = \emptyset$.
To show this, first suppose that vertices $x \in W_1$ and $y \in W_3$ are adjacent. Then $G[v_1, v_2, v_3, y, x]$ is a C_5, which is a contradiction. Therefore, W_1 is anti-complete to W_3. If both W_1 and W_3 are non-empty then by our earlier assumption they must each contain at least two vertices. Suppose that $x \in W_1$ and $y, z \in W_3$. In this case $G[x, v_1, y, v_3, z]$ is a $P_2 + P_3$, a contradiction. We conclude that at least one of W_1 and W_3 must be empty. Without loss of generality, we assume that W_3 is empty. Similarly, we assume that W_4 is empty.

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