Heterogeneous Managers, Distribution Picking and Competition

A thesis presented for the degree of Doctor of Philosophy In Economics

By: Tony Xiao Liu

Department of Economics, University of Warwick

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Abstract

The first chapter of this thesis develops a model where a number of new hedge funds with unknown and varying ability compete to enhance their reputations by registering high performance relative to their peers. The funds' choice variable is their return distribution, which financial engineering gives them complete control over subject to a constraint on their means that proxies for ability. This approach has the advantage of not requiring knowledge of fund moneymaking strategies. In all equilibria, funds play tail risk in expectation, and increasing the number of competitors causes tail risk and fund failure rates to rise. This is because a higher number of competitors makes it more difficult to stand out with high relative performance. In the second chapter, a variant of the model where the fund with the greatest Bayesian probability of being a high ability type wins the reputational boost is analysed as a robustness check. Funds still play tail risk, but the results from chapter 1 are weakened by the existence a class of equilibria where tail risk does not increase with the number of funds. Some equilibria of this new model correspond to the setting of Foster and Young (2010), with low ability funds mimicking high ability funds. This is because the more rational version is less like a Blotto Game and closer to a pure signalling model. In the last chapter, an incentive bonus scheme (2 and 20) commonly used in the hedge fund industry is added to the model. When funds play probability mass above the bonus threshold, such a scheme raises failure risk compared to the basic model from part 1 under some mild conditions. When financial engineering that enables return manipulation is available and managers are constrained by innate ability, such a bonus scheme gives funds incentives to play probability mass at high return levels at the cost of tail risk. With the bonus scheme, funds play less probability mass at higher variance above the bonus threshold. The model also returns a restriction on the minimum amount of tail risk.

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Declaration

I hereby declare that all the work in this thesis is my own and has not been published or submitted as part of a dissertation anywhere else.

CHAPTER 1

A Basic Model of Hedge Fund Competition

1.1. Introduction

In the last three decades, there has been an explosion in the level of complexity in financial markets around the world. This has been bought about by a combination of deregulation and (consequently) a very large amount of financial innovation. The most well known example is the enormous growth in the number and sophistication of derivatives available since the 80's. Other types of financial product have also massively increased in variety and complexity, with an example being the proliferation of mortgage backed securities with complex tranching and CDOs of various types in the 2000's. The value of this increasing complexity has been debated, with a variety of arguments for and against it. Generally, arguments for this development have focused on how the increased variety of financial products completes markets and gives financial institutions better tools to manage and spread risk, leading to more efficient allocations, with Alan Greenspan being the most famous proponent of that line. Arguments against have focused on how in a world of asymmetric information complexity can be used to hide leverage and risk taking and how there can be severe mispricing of securities investors do not fully understand (see the mortgage backed security bubble of the mid-late 2000s).

One particular setting in which complex financial instruments can be misused is in the world of hedge funds. Due to their light regulation, they are free to execute intricate trading and moneymaking strategies and are thus able to take maximum advantage of the opportunities afforded to them by these products. In addition, hedge funds have few, if any disclosure requirements. Consequently, the return characteristics of their portfolios are extremely difficult to unravel, and fund managers probably want it this way: this prevents their highly proprietary trading strategies being copied by competitors.

Hedge funds have high attrition rates. This results in a fund population where many funds are young, with only a few years of history (Fung and Hsieh, (2006)). Also, there is evidence that strong performance relative to their competitors matters heavily (Brown, Goetzmann and Park (2000)). If a new hedge fund outperforms its peers, it can boost its reputation by ranking highly in performance standings and convince investors that it has skilled managers. Even though there are no requirements to disclose performance, a fund that performs well can choose to publish its performance when it does well, which can be interpreted by investors as a signal that the fund has skilled managers. As Foster and Young (2010) point out, when there is such a low degree of transparency and limited sampling of return histories due to funds being young, it is possible for low ability managers to use complex financial engineering to manipulate return distributions to make themselves appear more skilled than they actually are, and enhance their reputation by appearing to outperform their peers. An example they give is when the fund manager writes and sells puts on some unlikely market event, and invests the money gained from these puts in a risk free investment, giving the appearance of being able to generate riskless returns above the risk free rate for a significant period of time. In the case of the unlikely market event coming to pass, all their money is lost. Although it may be easy to detect this specific strategy, they point out there are much more complex and opaque trades that can give the same result. Other authors have also pointed out this possibility too, an example being Weisman (2002), who calls these trades "informationless" strategies and notes that they can be used to game performance measures even though the expected return of such strategies may be low. This creates an interesting set of incentives which are important to understand, since hedge funds play a significant role in financial markets due to their significant investments in illiquid assets and their high leverage. When there are so many young funds of unknown ability, a large proportion of them may be incentivised to disguise their ability using financial engineering, which may have negative implications on their stability and survival rates. In addition, with the aforementioned large number of young hedge funds seeking to register strong relative performances, competition is clearly an important consideration when analysing what may incentivise them to use financial instruments to distort return distributions. This is not something considered in Foster and Young's analysis, and is the main subject of investigation in this paper: the effect of competition on hedge funds failure rates in the environment of relative performance competition and high financial complexity.

This paper formalises the problem of return manipulation using opaque financial instruments raised by the aforementioned authors in a simple theoretical model, using an adaption of the models used in Spiegler (2006) and Myerson (1993) to consider problems in behavioural economics and political economy respectively. It is also related to the all-pay auction, the models used in the sales literature started by Varian (1980), and the literature on Blotto Games. This approach replicates some crucial features of the problem at hand, while still being tractable. The key feature is that there is limited return sampling in determining performance ranking, (only one sample for simplicity) and that the choice variable for hedge funds is their return distribution, subject to a few constraints. This proxies for the ability of hedge funds to manipulate their return distribution using complex financial products. Innate fund ability generates these constraints: it is assumed that high ability funds can always achieve a higher expected return than low ability funds, and the use of financial products cannot affect this. Such an approach has the advantage of not requiring explicit modelling of hedge fund trading strategies, which are incredibly diverse and complex. However, the framework includes a skill differential between funds, while still allowing for an analysis of inter-firm competition.

The main result that comes from this model is that increasing the number of competing firms raises the expected fund failure rate, which is directly related to the amount of tail risk (risk of extreme losses) in the return distributions of each fund. This is despite the infinite number of pure strategy equilibria. Tail risk is played in every symmetric pure strategy equilibrium, which is a prediction somewhat consistent with some empirical work on hedge funds (Agarwal (2004)) and their observed short lifespans. This is due to it being increasingly less likely for a single fund to outperform all its peers as the number of competing funds increases. In the presence of ability constraints, incentives are thus generated to distort return distributions to offer the chance of drawing very high returns with a good chance to beating its competitors at the cost of there being significant tail risk and therefore significant chances of firm failure. The ingredients critical to this result are that the reputational benefit is only gained for the top few performers in the population of funds, funds being ability constrained, and that the funds have some power via complex financial engineering to alter their return distributions to respond to these incentives. This effect persists in more complex distribution picking settings as long these two features are present.

Trivially, it also follows that a policy implication of this result is that failure risk and tail risk can be lowered by introducing a high cost to altering return distributions. Clearly, imposing a high cost on shifting probability mass will make it unprofitable for a fund to alter its return distribution in response to incentives provided by increased competition. Introducing costs can be interpreted as adding regulation that makes it costly or difficult to freely alter return distributions or imposing some sort of limitation on the complexity of the securities available to the funds.

The two papers that are most closely related in modelling terms are Spiegler (2006) and Myerson (1993). Spiegler considers the problem of firms selling a complex, multifaceted product that has different prices and values in different world states to a consumer who is boundedly rational and can only consider a single world state. Myerson uses a model of this type to analyse electoral candidate behaviour when voters will prefer whoever promises them more transfers out of a budget whose size is fixed by taxation constraints. Both of these two papers have a continuous probability distribution as a choice variable, albeit with different interpretations in each paper. In Spiegler, the cumulative probability distribution represents the probability that the world will be in a state where the product has a price less than that number, while in Myerson, the cumulative probability distribution represents the probability that a randomly selected group of voters will be offered transfers less than some value. This paper is more similar to Myerson in that there are restrictions on the mean of the probability distributions, which are not present in Spiegler. Where this paper differs is that the distributions clearly have different interpretations (they are distributions of returns) and the introduction of different types of player with different mean restrictions on their played return distributions, which is a new feature. Clearly, the two different firm types end up playing different strategies in equilibrium, while all players play the same strategy in symmetric equilibria in the two aforementioned papers. This results in an infinity of symmetric equilibria, but with significant restrictions on the played distributions. There are similarities between the predictions made by this model and those made in Spiegler and Myerson: the variance of the distributions played must increase with the number of players, and the linearity of the probability that a firm that draws a return r will be picked are the two most important ones. In addition, this paper is linked to a far older literature on Colonel Blotto games such as Wagner and Gross (1950), since the problem of picking a continuous distribution subject to constraints is somewhat analogous to a Colonel Blotto game over infinitely many battlefields.

The model is also related to the wider literature on price dispersion, namely the famous model by Varian (1980), where shops face a population of customers where some search and compare prices actively, while others do not exert effort in searching and pick a shop at random. Their choice variable is a price distribution. The key methodological similarity is that the choice variable of the economic agents is a probability distribution while being subject to some constraints (reservation price and marginal cost in the case of Varian). The distribution happens to be over prices in that case, and is over returns on a portfolio in this model. In particular, some of the equilibria that are eventually found bear some resemblance to the continuum of asymmetric equilibria found in Varian's model by Baye and Kovenock et al. (1992), with mass atoms at extreme values of price (return in this case). This is not entirely surprising, given the presence of two different types in this model. The model is also related to the literature on all-pay auctions (Baye and Kovenock et al. (1996)). In a first price sealed all-pay auction, the winner of the auction is clearly determined by the highest bid, and every player will have to pay the same cost. This is exactly analogous to the situation here, where the highest performing fund receives a reputational gain and all funds have symmetric cost functions. It comes as no surprise that some of equilibria of this model resemble some of those found in the all-pay auction. However, the ability constraint in this model has no exact analogue in the all-pay auction.

Some other papers are related thematically, despite not being related methodologically. There has been some theoretical work on actively managed funds, mainly in the area of mutual funds. An example is Taylor (2001), who constructs a simple model of the incentives mutual fund managers face when competing for cash inflows. That model constructs a simple game in a tournament setting, where two managers who receive their mid-year performance results compete for incoming investor funds at the end of the year, and finds that managers in a stronger position in the middle of the year are more likely to gamble due to the pressure exerted from their competition being likely to gamble too. A feature that drives the results of that paper and this one is the tournament-like setting - the money flows into the fund that shows the best returns performance by the end of the year. This is a strong incentive to use riskier return distributions when faced with competition. Taylor (2001) considers mutual funds however, which do not use many complex financial instruments. Thus, it does not feature the much greater freedom to manipulate returns that are allowed in this paper. Berk and Green (2002) is a well known paper that is related because it features a market where mutual funds of differing skills compete with each other for funding, and it features investors who try and put their money into funds that are perceived to be higher skill. They use this framework to consider the problem of why the past performance of mutual funds is not a predictor of their future performance and why actively managed funds do not outperform passive benchmarks, and they explain this phenomenon by assuming that the funds have diminishing returns to scale. Since they are focusing on a different problem, they do not consider any return manipulation and simply assume that the funds generate a constant return per period plus a mean zero error process. For the same reason, the decision variable for the fund managers is also different - they decide the scale of their operation and what proportion of funds are managed actively. In all, their model is markedly different from the one presented here, since they are not concerned about return manipulation. Finally, this paper is related to Murdock and Stiglitz (2000) which also makes the point that increased competition and financial liberalisation increase bankruptcy amongst banks. The mechanism in their model is different from the one presented here: competition and liberalisation lower future profits, eroding the franchise value of said banks, making it more likely that they will make very risky investments and fail. The mechanism in this paper does not revolve around falling profits or franchise values, and so is significantly different. However, their model is in some senses isomorphic to the one presented here: banks choose between a risky investment which runs a significant risk of causing firm failure and a less risky investment.

This paper has three main sections. The settings and assumptions of the model will be laid out, and the equilibria will be solved for. Then the results and possible refinements are discussed, and finally there will be a conclusion.

1.2. The Model

1.2.1. General Setting. There are N "new" risk-neutral hedge funds of unknown ability. The funds can be of either high ability or low ability with prior probabilities β and $1-\beta$ respectively. The funds know their own type, but not the types of other competitors in the market. Each of the funds has access to complex financial instruments that allow them to manipulate their return distributions as they see fit, subject to a mean return constraint that is dependent on their type. This setup allows us to sidestep the issue of modelling their money making strategies explicitly, while still allowing for a distinction in hedge fund abilities. Once they have decided upon a return distribution, the funds invest some of their own wealth, I_f , into their decided return distribution, generating a draw from their return distribution that is observed. Since they are new hedge funds, they are competing for publicity, and will derive some benefit by performing well relative to their peers. To simplify things, it is assumed they get a payoff α from drawing the highest return out of the group of competing funds. In reality, funds are often are aiming to be at a high rank in a performance list. The assumption that they only derive benefit from being first amongst their competitors is an extreme assumption, but simplifies the analysis greatly while preserving the importance of strong relative performance. Firm failure is modelled by there being an exogenous return threshold, $r_f > 0$, below which the firm fails. If a hedge fund draws a return below this threshold, the firm fails, and is shut down by the manager and therefore cannot take a place on the performance list even if it draws the highest return of all the observed funds. This can also be interpreted as the existence of a safe outside option - if a fund draws less than some threshold level, they can appear worse than a safe and known investment option, and registering strong relative performance will not bring a benefit.

1.2.2. Game Structure. There are two time periods. To simplify analysis, there is no discounting. The time structure of the game is as follows:

- $t = 0$. Fund types are drawn randomly and independently. They are either of the H type with probability β or the L type with probability $1 - \beta$. They then simultaneously pick their return distributions without knowledge of the types of the other funds, and invest I_f of their own money into their picked distributions.

- $t = 1$. The investment made by each fund matures, and generates a return drawn from the distribution chosen in the previous phase. This return is drawn independently for each fund. The return for fund i is denoted by $r_i I_f$. The fund that draws the highest return will receive some benefit (derived from reputational considerations) α, related to appearing to perform well. The fund keeps the return generated from its own money ($r_i I_f$) regardless of whether it was successful in gaining the benefit from drawing the highest return. If a fund draws $r_i < r_f$, the fund fails and cannot gain any benefit from appearing to outperform its competitors.

1.2.3. Constraints on Return Distributions. What we are interested in is how the funds use their access to complex financial products to affect their return distributions. Therefore, we assume that funds have a lot of power to shape their return distributions. As stated, there are some restrictions on the distributions they can choose, and these depend on the type of the fund in question. The pool of possible funds contains both ones of high ability, the H type, and low ability funds, the L type. Explicitly, if fund i is of the H type, then

$$E(r_i) \leq r_H \tag{1.2.1}$$

And if fund i is of the L type,

$$E(r_i) \leq r_L \tag{1.2.2}$$

These are the $type\ constraints$. And

$$r_H > r_L > 0 \tag{1.2.3}$$

These constraints can be effectively interpreted as ability constraints - funds of lower ability have a lower limit on the maximum expected return of their distributions than high ability funds. This to some extent proxies for the situation in reality, where the difference between the return levels of high and low ability funds arises from skill differences in finding and executing trading strategies. Using these constraints, we can somewhat sidestep the issue of explicitly modelling the moneymaking strategies used by different funds, which are tricky to model due to their variety and complexity. The use of complex financial products is assumed to change the shape of the return distributions.

A further restriction on the return distribution is required due to the limited liability of the funds. Essentially, they cannot lose more money than they invest. Thus, if I units of money in is invested in fund i, the investor will get $r_i I$ in total when the investment matures. Limited liability is represented by the following restriction:

$$r_i \geq 0 \tag{1.2.4}$$

This applies to funds of both types. There is no limit on the upper bound of the supports of played return distributions.

1.2.4. Objective Functions. As stated, the fund that generates the highest return at $t = 1$ without failing (drawing $r_i < r_f$) gets to the top of something like a performance list and gains α. If there are W funds tied at the highest return value, the winner is drawn at random out of these W funds. In that case, the expected benefit to a firm that ties for top performance is $\frac{\alpha}{W}$. Due to risk neutrality, this is equivalent to the benefit α being split evenly amongst these funds. Before proceeding to write down the objective function, a key variable must be introduced: the probability that fund i gets to the top of the performance ranking given the return that the firm draws at $t = 1$. Let this be $P_i(r_i)$. As noted before, to simplify the analysis, there is no discounting. The ex-ante objective function for the fund i is:

$$\pi_i = E(r_i)I_f + \alpha E(P_i(r_i))$$

They chose $f_i(r_i)$ subject to the constraints (1.2.1 for H type, 1.2.2 for L type, and 1.2.4 for both) listed in a previous section to maximise the above function.

It is immediately obvious that the type constraints must bind. If they did not, it is possible to shift probability mass in $f_i(r_i)$ so that the average return $E(r_i)$ increases. Given that the investor always picks the fund that generates the highest return, P_i is a an increasing function of r_i. One can always increase $E(r_i)$ by shifting some probability mass to a higher r_i. Given that. P_i is an increasing function of r_i, this increases $E(r_i)$ while leaves $E(P_i(r_i))$ either unchanged or strictly higher, resulting in a strict increase in π_i. It follows that any $f_i(r_i)$ which does not have binding type constraints will not be a profit maximising return distribution.

It then follows that if the type constraints must bind, then maximising the objective function is achieved as long as $E(P_i(r_i))$ is maximised, since $E(r_i)$ is maximised as long the type constraints bind. Thus the problem simplifies to:

$$\max E(P_i(r_i))$$

(1.2.5) $$s.t. \quad E(r_i) = r_H$$

for the H type, and

$$\max E(P_i(r_i))$$

$$s.t. \quad E(r_i) = r_L \tag{1.2.6}$$

for the L type.

1.2.5. Model Solution. We want to solve for symmetric pure equilibria where all firms of the same type play the same distribution. This is natural, since all firms of the same type face a symmetric problem. Although there appears to be a dauntingly large range of possible probability distributions that can be played, a series of propositions will be shown that narrow down the distributions that can result in an equilibrium. All detailed proofs of the presented propositions are in the appendix. Note that due to the requirement that ability constraints are binding, only deviations from equilibrium that involve mean preserving probability mass movements need to be considered. This is the main method used to deduce the following results: if the ability constraints are binding for both types and no type can find a profitable mean preserving probability mass movement, then that combination of return distributions is an equilibrium. The results give conditions on the played return distributions for no profitable mean preserving mass movements to be possible for any type.

PROPOSITION 1.1. *For either fund type, there can be no probability mass in the interval* $(0,\, r_f)$.

PROOF. See appendix 1. □

This is a fairly intuitive result. Since any return drawn between zero and r_f can never win the reputational reward, all funds avoid playing any probability mass there. Any mass played there can be shifted down to $r_i = 0$ with no effect on $E(P_i(r_i))$, and mean preservation allows some mass elsewhere to be moved up, resulting in a strict increase in $E(P_i(r_i))$.

PROPOSITION 1.2. *In a symmetric equilibrium, no fund can play distributions with any atoms of probability mass in the open interval* $[r_f, \infty)$. *All cdfs played must be continuous within this range.*

PROOF. See appendix 1. □

This is a crucial step in deriving further results. A simple example will help clarify the intuition for why this must be the case. Consider a scenario where there are only two funds, and both fund types are playing an atom of probability mass at the same return level, say r'. Due to the mass atom, both funds will draw r' with some non-zero probability. Recall that in the event of a tie on returns, the winner of the reward is chosen at random out of the two funds. This gives an incentive for both funds to deviate, since they can both make a mean preserving deviation (thus keeping the type constraints binding) where all the mass played at r' is shifted up by an infinitesimal amount. Due to the infinitesimally small shift, only an infinitesimally small amount of mass must be shifted to a lower return level elsewhere to preserve the mean. The latter only results in a negligible decrease in $P_i(r_i)$, while the shift of all the mass at r' upwards means that the random tiebreaker between the two funds is avoided in the states of the world where both firms draw the mass atoms (as mentioned before, these happen with non-zero probability), giving a finite increase in $P_i(r_i)$. Effectively, if any fund plays a distribution with mass atoms, it allows all other funds to make a deviation where an infinitesimally small shift in mass locations gives a finite increase in winning probabilities.

To solve the model, we must calculate $P_i(r_i)$ in symmetric equilibrium. Given that failure prevents the firm from winning the reputational boost, it is clear that $P_i(r_i) = 0$ for $r_i \in [0,\, r_f)$. Calculating $P_i(r_i)$ in all other regions is simplified considerably by the knowledge that there are no atoms in the interval $[r_f, \infty)$, since the probability that more than one firm ties for the highest return is now effectively zero. The firm that draws the highest return at $t = 1$ will gain benefit α. The probability of being the top performer depends on how many H and L types there are in the population of hedge funds.

Consider a return draw for firm i in the interval $[r_f, \infty)$. If out of the other $N-1$ funds, there are m high type funds (obviously with $N-1 \geq m$), then $N-1-m$ funds must be low type funds. In a symmetric equilibrium, all funds of identical type must be playing the same return distribution. If F_L is the return cdf of L type funds and F_H is the return cdf of all H type funds, then the probability of registering the strongest performance out of all the young hedge funds is the probability that all other funds ($N-1$ of them) draw lower returns than r_i. Due to the independent return draws, the expression for this is simple:

$$F_H^m(r_i)F_L^{N-1-m}(r_i)$$

This is apparent from the definition of a cdf as the probability that a draw from the distribution is below some value. The next question to ask is what $P_i(r_i)$, the ex-ante probability of drawing the highest return is. We need to take into account how likely it is that there are m high ability funds. Given that the probability of a fund being H type is β and the probability of a fund being L type is $1-\beta$, it follows that, the probability that there are m high ability funds out of $N-1$ total funds is:

$$\frac{(N-1)!}{m!(N-1-m)!}\beta^m(1-\beta)^{N-1-m}$$

and thus, assuming that the draws for fund types and returns are made independently, the probability of there being m high type funds in the other $N-1$ firms followed by a draw r_i being the winning one is

$$\frac{(N-1)!}{m!(N-1-m)!}\beta^m(1-\beta)^{N-1-m}F_H^m(r_i)F_L^{N-1-m}(r_i)$$

Finally, to find $P_i(r_i)$, we need to sum these probabilities over the different possible compositions of fund types:

$$E(P_i(r_{i1})) = \sum_{m=0}^{N-1} \frac{(N-1)!}{m!(N-1-m)!}\beta^m(1-\beta)^{N-1-m}F_H^m(r_i)F_L^{N-1-m}(r_i)$$

This can be simplified using the binomial expansion, giving the key result that

$$P_i(r_i) = (\beta F_H(r_i) + (1-\beta)F_L(r_i))^{N-1} \quad for\ r_i \in [r_f,\ \infty) \tag{1.2.7}$$

Overall, this gives the following form for $P_i(r_i)$ in symmetric equilibrium:

$$P_i(r_i) \begin{cases} (\beta F_H(r_i) + (1-\beta)F_L(r_i))^{N-1} & r_i \in [r_f,\ \infty) \\ 0 & r_i \in [0,\ r_f) \end{cases} \tag{1.2.8}$$

An important conclusion can be drawn from this:

PROPOSITION 1.3. *$P_i(r_i)$ is an increasing function of r_i.*

This is immediately apparent since it is an increasing function of two cdfs, which are increasing functions themselves. It immediately follows that if $f_i(r_i)$ is the return pdf played by firm i, then

$$E(P_i(r_i)) = \int_{r_f}^{\infty} (\beta F_H(r_i) + (1-\beta)F_L(r_i))^{N-1} f_i(r_i)dr_i$$

PROPOSITION 1.4. *Assume that the H type plays a return distribution with a support upper bound of k_H and that the L type plays a return distribution with a support upper bound of k_L. Let the greatest upper bound of the two different types be $k_u = \max[k_H,\ k_L]$. For every $r_i \in [r_f,\ k_u]$, at least one of the two types must be playing probability mass.*

PROOF. See appendix 1. □

If there is a region within $[r_f,\ k_u]$ where no firm type plays mass, it results in that entire region sharing the same $P_i(r_i)$ - all return values drawn within that region share the same probability of being the highest draw, since the next highest return value that any fund can draw is not within the region where there is no mass. Consequentially, it is possible to make a profitable deviation by moving mass from the upper edge of this region down into the massless region while moving some mass elsewhere up, since the downwards mass movement has no effect on $E(P_i(r_i))$. Crucially, this logic relies on proposition 1.2, the continuity of the played cdfs in $[r_f,\ k_u]$. If there are mass atoms, then it is not necessarily true that a mass movement from the next highest drawn return downwards into the massless region results in no effect on $E(P_i(r_i))$.

A direct consequence of this is the following:

COROLLARY. $P_i(r_i)$ *is strictly increasing in* $r_i \in [r_f,\ k_u]$.

PROPOSITION 1.5. *At least one of the firms must have an atom of probability mass at* $r_i = 0$.

PROOF. See appendix 1. □

This is an important result that generates some of the key predictions of the model. It is derived from similar reasoning to that used in proposition 1.3. If neither fund is playing probability mass at the greatest loss level, then profitable mean preserving mass movements are possible: if mass is moved from the lowest return level where there is mass down to zero, there is no negative effect on $E(P_i(r_i))$, since by definition $P_i(r_i) = 0$ at the lowest return level where there is mass. The decrease in the mean from such a movement means that there must be a movement of probability mass upwards somewhere else to preserve the mean, which if picked appropriately will have a positive effect on $E(P_i(r_i))$. The net result is a strict increase in $E(P_i(r_i))$. These kinds of deviation can be made non-profitable if at least one of the two types plays probability mass below return level r_f, since the next lowest return where there is mass will have $P_i(r_i) > 0$. The only place where it is possible to play mass below r_f without violating proposition 1.2 is at zero. There, the reasoning used in proposition 1.2 does not work since moving a finite amount of mass upwards by an infinitesimal amount ϵ from $r_i = 0$ does not generate a finite gain in $E(P_i(r_i))$, since $P_i(\epsilon) = 0$ still due to firm failure.

Note that this immediately gives one of our key findings: there will always be hedge funds that play probability mass at the highest possible loss level, $r_i = 0$, which corresponds to all investments being lost. The next proposition places strong constraints on the forms of the distributions played, and is probably the most important in determining the key results of the paper.

PROPOSITION 1.6. *$P_i(r_i) = Zr_i$ for $r_i \in [r_f,\ k_u]$, with $Z > 0$.*

PROOF. See appendix 1. □

The key point to realise here is that $P_i(r_i)$ being linear immediately makes all firms indifferent to any mean preserving shift in probability mass within $[r_f,\ k_u]$. When this proposition holds, any mass shift within the aforementioned region results in a linear change in $P_i(r_i)$. Crucially, an innate property of the mean is that any shift in mass also has a linear effect on it. The result is that when the mass shift is mean preserving (to keep the type constraints binding), the the effect on $P_i(r_i)$ is zero since the linear changes in $P_i(r_i)$ due to the mass movements in opposite directions cancel each other out. The fact that $P_i(r_i)$ is directly proportional to r_i (rather than just being linearly related) is because funds must be indifferent to deviations that involve moving mass between the mass atom at $r_i = 0$ and the rest of the return distribution. This imposes an additional condition which pins down $P_i(r_i) = Zr_i$. Propositions 1.6, 1.5 and 1.1 determine the form of the equilibria.

With these key propositions stated, the form of the symmetric pure Nash Equilibria can now be considered.

1.2.6. Form of Equilibria. It can be immediately seen that the support of both return distributions must be finite. This stems from the fact that $P_i(r_i)$ is a probability, and must be capped at 1. If $P_i(r_i) = (\beta F_H(r_i) + (1-\beta)F_L(r_i))^{N-1} = Zr_i$, then unless Z is infinitesimally small, then the support of both F_H and F_L must be finite.

Assume that the H type plays a return distribution with a support upper bound of k_H and that the L type plays a return distribution with a support upper bound of k_L. As before, let the greatest upper bound of the two different types be $k_u = \max[k_H,\ k_L]$.

From proposition 1.6 and 1.2.7, it must be that for $r_i \in [r_f,\ k_u]$,

$$(\beta F_H(r_i) + (1-\beta)F_L(r_i))^{N-1} = Zr_i \tag{1.2.9}$$

And that $P_i(r_i)$ must be of the following form:

$$P_i(r_i)\begin{cases} 1 & r_i \in [k_u, \infty) \\ Zr_i & r_i \in [r_f, k_u] \\ 0 & r_i \in [0, r_f) \end{cases}$$

From this we can deduce that if the H type is playing a return cdf of form $G(r_i)$, it must be that the L type plays a cdf of the form of $\frac{(Zr)^{\frac{1}{N-1}}}{1-\beta} - \frac{\beta}{1-\beta}G(r_i)$. Similarly, if the L type is playing a cdf of form $G(r_i)$, it must be that the H type plays a return cdf of the form $\frac{(Zr)^{\frac{1}{N-1}}}{\beta} - \frac{1-\beta}{\beta}G(r_i)$. This result leads to four different cases for pure strategy equilibria, which are given below. They vary depending on which of the two types plays the more freely picked return distribution $G(r_i)$, and whether k_u is equal to k_H or k_L. Both types could be playing mass atoms at $r_i = 0$. Most of these forms are obtained by applying proposition 1.6 together with 1.2.8.

Case 1. Here, $k_u = k_H$, i.e. $k_H \geq k_L$. It is assumed that the L type plays a mass atom of weight a_L at $r_i = 0$ and picks a cumulative density function $G(r_i)$ $g(r_i)$ for $r_i \in [r_f,\, k_L]$ with $G(r_f) = 0$. Thus, F_L is given by

$$F_L(r_i) \begin{cases} 1 & r_i \in [k_L,\, \infty) \\ G(r_i) + a_L & r_i \in [r_f,\, k_L] \\ a_L & r_i \in (0,\, r_f) \\ a_L & r_i = 0 \end{cases}$$

To ensure that proposition 1.6 is satisfied, F_H must be $\frac{(Zr_{i1})^{\frac{1}{N-1}}}{\beta} - \frac{(1-\beta)}{\beta} F_L(r_i)$. Assume that a_H is the mass atom played by the H type at $r_i = 0$.

$$F_H(r_i) \begin{cases} 1 & r_i \in [k_H,\, \infty) \\ \frac{(Zr_i)^{\frac{1}{N-1}}}{\beta} - \frac{(1-\beta)}{\beta} & r_i \in [k_L,\, k_H] \\ \frac{(Zr_i)^{\frac{1}{N-1}}}{\beta} - \frac{(1-\beta)}{\beta}(G(r_i) + a_L) & r_i \in [r_f,\, k_L] \\ a_H = \frac{(Zr_f)^{\frac{1}{N-1}}}{\beta} - \frac{(1-\beta)}{\beta} a_L & r_i \in (0,\, r_f) \\ a_H = \frac{(Zr_f)^{\frac{1}{N-1}}}{\beta} - \frac{(1-\beta)}{\beta} a_L & r_i = 0 \end{cases} \tag{1.2.10}$$

Both of these cdfs are consistent with propositions laid out above. The propositions laid out previously ensure both types are indifferent to all mean preserving mass movements.

Case 2. Like in the first case, the L type picks a return cdf G and the H type plays a cdf of the form $\frac{(Zr_{i1})^{\frac{1}{N-1}}}{\beta} - \frac{1-\beta}{\beta} G(r_i)$. However, we now assume that $k_H \leq k_L$. The forms of the played return cdfs are similar to the previous case, albeit with an alteration for the L type. Recall that $P_i(r_i)$ linearity 1.2.9 is required for all values of r_i, which in this case is guaranteed for $r_i \in [r_f,\ k_H]$ due to the functional forms stated above. However, to ensure this for $r_i \in [k_H,\ k_L]$ where the H type does not play any probability mass, the L type must play $\frac{(Zr_{i1})^{\frac{1}{N-1}}}{1-\beta} - \frac{\beta}{1-\beta}$ here as a return cdf (since $F_H = 1$ for $r_i \in [k_H,\ k_L]$). This time $k_u = k_L$, i.e. $k_L \geq k_H$. For the L type:

$$F_L(r_i)\begin{cases} 1 & r_i \in [k_L,\ \infty) \\ \frac{(Zr_i)^{\frac{1}{N-1}}}{1-\beta} - \frac{\beta}{1-\beta} & r_i \in [k_H,\ k_L] \\ G(r_i) + a_L & r_i \in [r_f,\ k_H] \\ a_L & r_i \in (0,\ r_f) \\ a_L & r_i = 0 \end{cases}$$

For the H type:

$$F_H(r_i)\begin{cases} 1 & r_i \in [k_H,\ \infty) \\ \frac{(Zr_i)^{\frac{1}{N-1}}}{\beta} - \frac{(1-\beta)}{\beta}(G(r_i) + a_L) & r_i \in [r_f,\ k_H] \\ a_H = \frac{(Zr_f)^{\frac{1}{N-1}}}{\beta} - \frac{(1-\beta)}{\beta}a_L & r_i \in (0,\ k_H) \\ a_H = \frac{(Zr_f)^{\frac{1}{N-1}}}{\beta} - \frac{(1-\beta)}{\beta}a_L & r_i = 0 \end{cases}$$

Case 3. Here, the H type plays a pdf with a mass atom at zero a_H and a continuous density function $g(r_i)$ for $r_i \in [r_f,\ r_H]$. Thus, the L type must play a cdf of the form $\frac{(Zr_{i1})^{\frac{1}{N-1}}}{1-\beta} - \frac{\beta}{1-\beta}G(r_i)$ to satisfy proposition 1.6. This time, we assume $k_L \geq k_H$, so that $k_u = k_L$. $F_L$is of the form $\frac{(Zr_{i1})^{\frac{1}{N-1}}}{1-\beta} - \frac{\beta}{1-\beta}$ between k_u and k_H to satisfy proposition 1.6 because the H type does not play mass there. Let the mass atom at zero played by the the L type be a_L, and finally note that in this case $k_u = k_L$, i.e. $k_L \geq k_H$. This gives the following cdfs:

$$F_L(r_i)\begin{cases} 1 & r_i \in [k_L,\ \infty) \\ \frac{(Zr_{i1})^{\frac{1}{N-1}}}{1-\beta} - \frac{\beta}{1-\beta} & r_i \in [k_H,\ k_L] \\ \frac{(Zr_i)^{\frac{1}{N-1}}}{1-\beta} - \frac{\beta}{1-\beta}(G(r_i)+a_H) & r_i \in [r_f,\ k_H] \\ a_L = \frac{(Zr_f)^{\frac{1}{N-1}}}{(1-\beta)} - \frac{\beta}{1-\beta}a_H & r_i \in (0,\ r_f) \\ a_L = \frac{(Zr_f)^{\frac{1}{N-1}}}{(1-\beta)} - \frac{\beta}{1-\beta}a_H & r_i = 0 \end{cases}$$

$$F_H(r_i)\begin{cases} 1 & r_i \in [k_H,\ \infty) \\ G(r_i)+a_H & r_i \in [r_f,\ k_H] \\ a_H & r_i \in (0,\ r_f) \\ a_H & r_i = 0 \end{cases}$$

Case 4. This is essentially the same as case 3, but with $k_u = k_H$, i.e. $k_H > k_L$ instead. The notation is the same as in the previous case. The played cdfs are:

$$F_L(r_i)\begin{cases} 1 & r_i \in [k_L,\ \infty) \\ \frac{(Zr_i)^{\frac{1}{N-1}}}{1-\beta} - \frac{\beta}{1-\beta}(G(r_{i1})+a_H) & r_i \in [r_f,\ k_L] \\ a_L = \frac{(Zr_f)^{\frac{1}{N-1}}}{(1-\beta)} - \frac{\beta}{1-\beta}a_H & r_i \in (0,\ r_f) \\ a_L = \frac{(Zr_f)^{\frac{1}{N-1}}}{(1-\beta)} - \frac{\beta}{1-\beta}a_H & r_i = 0 \end{cases}$$

$$F_H(r_i)\begin{cases} 1 & r_i \in [k_L,\ \infty) \\ \frac{(Zr_i)^{\frac{1}{N-1}}}{\beta} - \frac{1-\beta}{\beta} & r_i \in [k_H,\ k_L] \\ G(r_i)+a_H & r_i \in [r_f,\ k_L] \\ a_H & r_i \in (0,\ r_f) \\ a_H & r_i = 0 \end{cases}$$

Though there are some restrictions on the played distributions (see appendix), this clearly results in an infinity of equilibria. However, it is still possible to extract some useful information about the characteristics of the equilibria. Each case gives a set of four simultaneous equations that must be satisfied. Two of these are obtained by imposing the ability/type constraints 1.2.5 for the H type and 1.2.6 for the L type, while the remaining two come from the fact that cumulative probability mass must integrate to 1 for a density function to be a valid pdf. Applying this to the H and L types individually gives two more equations. There are also a number of continuity conditions between different return regions to ensure that proposition 1.1 (cdf continuity except at zero) hold. Despite the four separate cases, some critical features and variables are the same for all of them:

PROPOSITION 1.7. *In all pure symmetric Nash Equilibria, the coefficient Z is given by*

$$Z = \frac{1}{k_u} \tag{1.2.11}$$

where k_u is independent of the distributions played, and is given by the solution of the equation

$$\bar{r}N + \left(\frac{r_f^N}{k_u}\right)^{\frac{1}{N-1}} = k_u \tag{1.2.12}$$

where $\bar{r} = \beta r_H + (1-\beta) r_L$. It can be shown this equation (1.2.12) has a unique solution for $k_u \geq 0$.

PROOF. See appendix 1. $\square$

1.2.7. Comparative Statics of Fund Failure. Recall that from proposition 1.5 that in expectation (before type determination) funds play a mass atom at $r_i = 0$, and that the firm i fails when they draw a return $r_i \in [0, r_f)$. In addition, from proposition 1.1, no fund is playing mass in the interval $(0, r_f)$. So, the probability of fund failure is just the probability that a fund will draw $r_i = 0$, i.e. the size of the mass atom per firm in expectation before type determination. It is then clear that failure risk is synonymous with tail risk in the context of this model. Let T denote this probability, then $T = \beta a_H + (1-\beta)a_L$. The following can be proved about this probability T, and is the key result of this paper.

PROPOSITION 1.8. *The average amount of mass at $r_i = 0$ played by each fund before type determination in equilibrium $T = \beta a_H + (1-\beta)a_L$ is invariant of the exact played distributions, and is given by $(Zr_f)^{\frac{1}{N-1}}$.* Also, as long as $r_H > r_L > r_f$, the probability of fund failure T is increasing in N for all valid numbers of competing firms, i.e. $\frac{dT}{dN} > 0$ for $N > 2$.

PROOF. See appendix 1. □

This result implies that the risk of firm failure increases endogenously with the degree of competition amongst these new hedge funds.

Although these results on comparative statics of N apply to all symmetric equilibria, it needs to be verified that these symmetric equilibria actually exist. It is impossible to find these equilibria for certain parameter combinations. This is easily illustrated by a case where there are two funds, and r_f is small compared to $\bar{r}$. In addition, the H type is very rare, but has a very high maximum return r_H. The formula that determines k_u is mainly dependent on the term $2\bar{r}$, so it could give a value that is below r_H, making it impossible to choose a return distribution for the H type that keeps the ability constraint binding while having a support that has an upper bound less than k_u. It is difficult to find necessary and sufficient conditions on the parameters that ensure the existence of these symmetric equilibria for all $N \geq 2$. However, the critical comparative static result on how N affects failure risk is independent of the exact functional forms of F_H and F_L, and applies to all symmetric equilibria that might exist. Therefore, it suffices to find parameter conditions that ensure the existence of equilibria for some functional forms for all $N \geq 2$ and for some $r_f > 0$ (since fund failure is a key issue studied by the model). This can indeed be shown, and leads to proposition 1.9:

PROPOSITION 1.9. *A sufficient parameter condition to guarantee the existence of symmetric equilibria of the form analysed for $N \geq 2$ is*

$$r_f \geq 2M\bar{r} + M^2$$

where

$$M = \frac{(1-\beta)(r_H - r_L)}{r_H}$$

This condition is not actually as restrictive as one might think. There are four variables in the inequality: r_f, r_H, r_L and β. Although there are a few restrictions on these parameters, there is still a large amount of variation possible. We need $0 < r_f < r_L < r_H$, and $\beta \in [0,1]$. Note that $M < 1$ since the denominator is always greater than the numerator. Thus it is possible to find parameter values such that $r_f \geq 2M\bar{r} + M^2$.The main way of finding parameters that guarantee equilibria for $N \geq 2$ is to find combinations of r_H, r_L and β that give small values of M (they definitely need to be less than $\frac{1}{2}$). M is just a product of the proportional difference between r_H and r_L, and the fraction of funds that are of the$L$ type. It is consistent with the example of a set of parameter conditions with no symmetric equilibria given earlier: in that scenario, both $(1-\beta)$ and $\frac{r_H - r_L}{r_H}$ were close to one. When the condition is not satisfied, either $k_u < r_H$, forcing negative probability mass for the H type at zero and thus $\int_{r_f}^{k_u} f_H(r)dr > 1$ greater than to make $\int_0^{k_u} r f_H(r)dr = r_H$, or r_L is so low compared to r_H that the L type must play a mass atom $C_L > \frac{(Zr_f)^{\frac{1}{N-1}}}{(1-\beta)}$ to allow the L type to play probability mass up to k_u. This also forces negative mass from the H type to keep the total expected probability mass at zero equal to $(Zr_f)^{\frac{1}{N-1}}$.

.Fortunately, in reality, $\frac{r_H - r_L}{r_H}$ is not very large. A very optimistic estimate for the returns of the best managers is 1.4 (an incredible 40% interest rate), while low skill funds might make slightly more than the risk free rate, say 1.05 or so. This tends to make M small, even when H type funds are rare. In addition, r_f tends to be surprisingly high in reality: not only does this represent a failure threshold, but it can also be interpreted as being related to an outside option for both the fund managers and investors. Thus, r_f might be close to the risk free rate, and is almost certainly close to 1. In these kinds of conditions, there are actually no restrictions on the value of β, as long as it is a valid probability - $2M\bar{r} + M^2$ tends to a maximum of value around 0.5875 (when β tends to zero) with $r_L = 1.05$, $r_H = 1.4$, which is considerably below a sensible value of r_f (around one). Thus, the conditions that ensure existence of equilibria for $N \geq 2$ and thus the conditions on the validity of comparative statics are not very restrictive. There are plenty of examples of parameter values where proposition 1.9 holds. An example is $\beta = 0.25$, $r_L = 1.05$, $r_H = 1.2$. With these values, $r_f \geq 0.213$ is sufficient to satisfy the condition in proposition 1.9.

In addition, it is possible to show the following:

Proposition 1.10. *For every value of the parameters that satisfy $0 < r_f < r_L < r_H$, there exists a sufficiently large value of N where equilibria exist.*

Proof. See appendix 1. □

This means that even for extreme values of parameters where say, $\bar{r}$ is much larger than r_f and r_L, and β is also small, the comparative statics are still valid to some extent since symmetric equilibria above a certain threshold for N will always exist, making the increase of tail risk with the level of competition a meaningful result still. This is because

k_u increases almost linearly with N, and a large enough k_u ensures it is possible to find a valid pair f_H and $f_L$of the form used to prove proposition 1.9.

1.3. Discussion

1.3.1. Discussion of Results and a Trivial Policy Implication. As one might expect, the high degree of freedom in picking return distributions for both types leads to a vast number of possible equilibria. Despite this, the model produces the consistent prediction that ex-ante firm failure risk increases with the number of competing firms under some fairly mild assumptions: it is not unreasonable to assume that the returns of low and high ability managers are both above the failure threshold. The intuition for this result is that as the number of competitors increases, it becomes less and less likely for a fund drawing a given return r_i to be the highest performing fund in its peer group. As N rises, there are more and more other funds that must be outdrawn by r_i to win the reputational reward. Essentially, increasing N increases the curvature of P, making it increasingly convex in r_i for a given pair of H and L type return distributions. This generates incentives for funds to shift more probability mass to higher r_i values, and the only way this is possible while staying within the ability constraint is to move some probability mass down to lower r_i values, with the most efficient way of doing so being to move mass from low values of r_i to zero. Consequently, as competition increases, firms are incentivised to play distributions with high failure risk and more mass at very high values of r_i. The introduction of more competing funds makes it increasingly difficult to stand out as a strong performer, and this creates incentives to use financial engineering to give a chance of generating exceptional returns at the cost of a significant chance of firm failure.

Clearly, part of what is driving this is the extreme assumption that a new fund only gains a benefit when it outperforms all its competitors, i.e. the probability of winning the award α is tied to the Nth order statistic. This ensures that the addition of more firms makes P more convex, i.e. it increases more rapidly at higher values of r. However, the result is more robust than it appears. Even if funds benefit from being in the top performing M out of N funds rather than only from being the fund with the highest return draw, the addition of competitors will still make P more convex for any given played return distributions. This is because P in that case will depend on the sum of a few order statistics, which are all become more convex as N rises. Thus, extending the reward for high performance to the top few funds will generate the same incentives to move probability mass upwards to high values of r at the cost of playing tail risk.

In the light of the mechanism described above, it is important to think about how financial complexity affects the result. Recall that financial complexity is proxied in the model by the immense freedom the funds have to change their return distribution. The reasoning presented in the previous two paragraphs can be applied without complete freedom to alter return distributions: consider the case when funds have control over the mean and variance of the return distributions, but not their exact shape, which is constrained by being in a certain family of distributions. N increasing will still raise the convexity of the P function, and thus will still generate incentives to raise the variance of returns. So, this result probably does not require the complete flexibility to engineer return distributions they have here, but the funds certainly need some power to change their return distribution shapes, since they must be able to shift probability mass to respond to these incentives. What the complete flexibility to engineer return distributions does do is ensure that all funds can increase variance while preserving return mean without limit, which is impossible with some functional forms. It is also responsible for firms playing a point mass at the $r = 0$, the maximum loss level allowed by limited liability, and gives strong results about firm failure/tail risk. It allows funds to make the most efficient mean preserving mass movements that raise $E(P_i(r))$, since moving mass to $r = 0$ gives the smallest amount of mass that has to be moved downwards to compensate a given upwards mass movement elsewhere. In that sense, the effect of having almost total power to manipulate return distributions is somewhat ambiguous. It may actually decrease failure risk for some functional forms, since in some special cases the loss of the power to move probability mass to exactly $r = 0$ may cause funds to end up moving more mass downwards into the failure region for every upwards mass movement.

A fairly trivial policy implication comes from the discussion on the effect of the power to shape return distributions. It is clear that in the extreme case of all power to manipulate return distributions being removed, it becomes impossible for funds of either type to respond to the incentive to increase the spread and variance as competition increases. This indicates that restrictions on the ability to alter return distributions can remove the increase in failure risk with competition. One way of introducing these restrictions is to introduce a cost for shifting probability mass from its initial location. If sufficiently high, this will cause there to be no failure risk in equilibrium, since it is expensive to create return distributions that with large amounts of mass at low r unless the initial return distribution already has large amounts of mass in those regions already. The policy interpretation of this is that restricting the use of complex instruments lowers failure risk, making it difficult or costly to respond to the incentive to play more risky distributions as competition increases. In a way however, the assumption of costly distribution alteration somewhat forces the above conclusions.

This model is missing some features such as risk aversion or an explicit penalty for firm failure. Adding these certainly makes the results weaker, with failure risk no longer monotonically increasing in number of competing firms when there is a continuation payoff if a fund does not fail. The results are weakened by the addition of these features, since in both cases there will be a trade off to moving probability mass to below the failure threshold in order to move some other mass upwards in pursuit of high relative performance. The mechanisms that generate the increase in failure rate still exist if risk aversion or continuation payoffs are considered, however.

1.3.2. Limitations and Extensions. The assumptions used in this paper are quite extreme: return manipulation using complex financial products is costless, and these instruments serve no other purpose. In reality, skilful use of derivatives and other products can be the source of the higher ability funds' higher returns. As such, this paper only looks at incentives to manipulate return distributions when faced with competition, and does not capture the dependence of returns on the use of these complex securities in the first place. On a related note, another criticism is that in reality leverage is a vital part of the operation of hedge funds. Increasing leverage generally increases expected return at the cost of increased downside risk. Due to the ability constraint limiting the mean, it is difficult to see a return manipulation that is equivalent to increasing leverage. This is the trade off for using this particular setting to bypass the complexity and variety of hedge funds. An extension would therefore be to create a similar model but without mean constraints. The issue then is that it is unclear how to distinguish between high ability and low ability funds, although a differential cost of return distribution changing may be one way of doing this.

Assuming that benefits go only to the firm with the highest return is in some sense dubious. If a new hedge fund attains the highest return out of all its competitors, the benefit that it obtains should be some sort of reputational boost. However, hedge funds are assumed to have sophisticated investors rather than people who will always be naively attracted to investing in the apparently highest performing new hedge fund. In the context of the model, the main benefit to an investor of knowing fund performance is as a signal of fund type. They would like to maximise the probability of selecting a fund with a high ability manager to invest in. It follows that a reasonable extension of the model is to evaluate the Bayesian probability of each fund being a H type, and to give a reputational boost to the fund with the highest probability of being a H type. This is done in the next chapter, and alleviates some of the concerns about the implied naivety of investor behaviour in this model, as well as making the framing and interpretation more like a signalling game where firms are trying to deceive a rational investor of their skill. This is more like the scenario described in Foster and Young (2010). The main results of this paper mostly carry over in a slightly weakened form - there are some additional restrictions on played distributions, and there are a class of equilibria with probability mass atoms above the failure threshold the results in this paper do not necessarily apply to.

Finally, a very interesting feature to investigate would be if there are any general results in this setting on the effect of more financial complexity. This particular paper makes the assumption of complete freedom to manipulate return distributions within ability constraints, which is a scenario corresponding to an unrealistically high level of financial engineering power. It would clearly be interesting to consider how risk taking and failure rate behaves at a lower level of financial engineering power, and it would be even more interesting to see if there are any general results on how varying the power of financial engineering and complexity affects risk taking and failure rates. The issue is that in this distribution picking environment, it is difficult to quantify the level of financial freedom and complexity without introducing a cost to shifting probability mass, which produces the fairly obvious result discussed in the policy implication. It may be better to start by finding a result on risk taking and failure rate in this setting (new hedge funds) that is unique to the scenario where funds are assumed to have maximum financial engineering power, but does not apply when financial engineering power is restricted.

1.4. Conclusions

To sum up, this paper formalises the problem of new hedge funds of differing but unknown ability levels competing to achieve strong relative performance in an attempt to improve their reputation. They key features of the model are limited return sampling and a high degree of power to manipulate return distributions, subject to ability constraints. The problem is formulated in a way that bypasses the need to explicitly model the money-making strategies of the funds themselves, since doing so is problematic due to the enormous variety of money-making strategies hedge funds employ.

Despite the wide variety of return distributions that can be played in equilibrium, this paper predicts that each firm in expectation (before type is decided) will play some probability mass at the worst possible return level (all initial investment lost due to limited liability), which can be interpreted as tail risk and fund failure failure rate. This is somewhat consistent with empirical evidence (Agarwal (2004), Fung and Hsieh, (2006)). The model also predicts that fund failure risk or tail risk will increase with the number of competitors under some mild assumptions, and is thus detrimental to financial stability given the proportion of young hedge funds in the industry. This is because an increase in the number of competitors makes it more difficult for a fund to outperform all of them without using a return distribution that has the possibility of generating a very high return. Given that ability constrains the expected return for the funds, the only way to do that is to use financial engineering to balance out the high returns with tail risk. The ability to use complex financial products to manipulate return distributions is required for the result to some extent, since funds need the ability to change their return distribution in response to the incentive to increase the probability of drawing a high return at the cost of increased tail risk.

These incentives to increase failure risk/tail risk identified by this paper will exist as long as new hedge funds are strongly rewarded for getting the top few highest returns amongst their peers, and have the power to alter their return distributions in response to a change in the number of competing firms. These identified incentives still persist even if features like risk aversion and a continuation payoff for non-failure are added. Trivially, a policy implication of this model is that a strong enough cost on return manipulation will remove failure risk and tail risk.

Appendix 1

Proof of Proposition 1.1. Assume that at least one of the fund types has some probability mass δ in this interval in equilibrium, and that the fund i is of this type. The contribution to $E(P_i(r_i))$ from this probability mass is zero, since if the return draw r_i lands in $(0,\, r_f)$, then $P_i(r_i) = 0$ due to the return being below the failure threshold r_f. Any equilibrium distribution must have 1.2.5 and 1.2.6 binding. Consider an unilateral deviation by fund i. It is always possible to move some of this mass in $(0,\, r_f)$, say ϵ, up to a $r' > r_f$ for which $P_i(r') > 0$, while also moving $\delta - \epsilon$ down to $r_i = 0$. This shift preserves total probability mass, and can always be made mean preserving by picking ϵ appropriately. Thus, this still satisfies 1.2.5 for the H type and 1.2.6 for the L type. Given that at least one of the types is playing some probability mass in $(0,\, r_f)$, moving this mass ϵ up to just r_f makes the contribution to $E(P_i(r_i))$ from this mass strictly positive, since drawing r_f can result in an investor pick if all the other funds are drawn as the same type as i, and all draw a return in $(0,\, r_f)$. So, a suitable r' is any $r \geq r_f$. The movement of the rest of the mass $\delta - \epsilon$ down to $r_i = 0$ has no negative effect on $E(P_i(r_i))$, since $P_i(0) = 0$, the same as for any r_i in the interval $[0,\, r_f)$. Therefore, the total contribution to $E(P_i(r_i))$ from this mass δ is now strictly greater than zero, and this deviation is strictly profitable for firm i, since $E(P_i(r_i))$ has increased, while $E(r_i)$ has remained constant. Hence any distribution with mass in $(0,\, r_f)$ cannot be played in equilibrium, since it is not profit maximising.

Proof of Proposition 1.2. Assume one of the types is playing a return cdf that is discontinuous, with an atom of probability mass, say u, at r^*, where $r^* \geq r_f$. Assume that firm i is of this type. There will be a finite probability that some of the other $N-1$ firms will be of the same type. In the event that a draw of $r^*$is made, it follows that there is a finite probability that some other firms will also draw r^*. In such a case, if r^* is the highest drawn return, the winning fund is picked at random from the funds that draw r^*, meaning that fund i will be only be picked in a fraction, say $\frac{1}{q}$, of these scenarios, which is a finite probability still. Consider a deviation in which firm i moves $u-\delta$ of the mass at r^* up to $r^*+\epsilon$, where ϵ is arbitrarily small, while moving δ of it down to $r_i = 0$ so that $E(r_i)$ is preserved. Now, when fund i draws $r^*+\epsilon$, in the case when the next highest draw is r^* (occurs with some finite probability), firm i wins all the time rather than merely $\frac{1}{q}$ of the time, a finite increase in $E(P_i(r_i))$. Given that ϵ is arbitrarily small, the amount of mass moved down, δ is also arbitrarily small. Thus, the decrease in $E(P_i(r_i))$ from moving δ down $r_i = 0$ is infinitesimally small, while the gain from moving $u-\delta$ up to $r^*+\epsilon$ is finite. There is therefore a net increase in $E(P_i(r_i))$, and the deviation is profitable. Thus, there can be no atoms of probability mass for $r_i \geq r_f$. Thus, in equilibrium, neither firm type can have atoms of probability mass in $r_i \in [r_f,\, \infty)$

Proof of Proposition 1.4. Assume that there is a region, say $r_i \in (u_1,\, u_2)$ within $[r_f,\, k_u]$ where neither type is playing any probability mass. By propositions 1.2 and 1.3, it follows that $P_i(u_1) = P_i(u_2)$, since if there is no probability mass played from either of the types and there are no discontinuities in F_L or F_H, $F_L(u_1) = F_L(u_2)$ and likewise $F_H(u_1) = F_H(u_2)$. Consider a deviation of the following form: moving mass δ from a small region around u_2 down to a small region around u_1, while moving mass ϵ from a small region around a location where there is mass x to a small region around a location $y > x$. Recall again that we require the type constraints 1.2.1 and 1.2.2 to be binding for any equilibrium to be optimal. This implies that the above variables must be picked to preserve the mean return, and thus to first order:

$$\epsilon(y-x) = \delta(u_2 - u_1)$$

This becomes exact as the regions we move mass from become very small, and implies that if ϵ is positive, so must δ. To deduce whether this deviation is profitable or not, consider its impact on $E(P_i(r_i))$, which is $-\delta P_i(u_2) + \delta P_i(u_1) - \epsilon P_i(x) + \epsilon P_i(y)$ for mass movements between very small regions. Since $P_i(u_1) = P_i(u_2)_i$, the effect actually becomes

$$\epsilon(P_i(y) - P_i(x))$$

Given that $P_i(r_i)$ is an increasing function from proposition 1.3, $P_i(y) \geq P_i(x)$. As long as $x < k_u$, it is always possible to find a y for which $y > x$ and $P_i(y) > P_i(x)$. Thus, this deviation is always profitable for either type, giving individual incentives to deviate for all firms. Thus, choices of return distribution with no mass from either type in regions $r_i \in [r_f,\ k_u]$ cannot be an equilibrium.

Proof of Proposition 1.5. This is an extension of the reasoning used to prove proposition 1.4. To satisfy proposition 1.4, at least one of the two types must be playing mass at $r_i = r_f$. Consider the case when neither of the two types plays mass at $r_i = 0$, and a deviation by firms that are playing mass at $r_i = r_f$. Mass δ from the region around $r_i = r_f$ is moved down to $r_i = 0$, and to preserve the mean so that the type constraints bind, mass ϵ is moved from a region $x \geq r_f$ to a region $y > x$. Given that $P_i(r_i)$ is continuous and strictly increasing in $r_i \in [r_f,\ k_u]$ from proposition 1.2 and the corollary of proposition 1.4, it is always possible to find a x and y such that $P_i(x) < P_i(y)$. Mean preservation requires:

$$\epsilon(y - x) = \delta(r_f)$$

The change in $E(P_i(r_i))$ is $\epsilon(P_i(y) - P_i(x)) - \delta(P_i(r_f) + P_i(0))$. Crucially, no firm is playing any mass at $P_i(r_i) = 0$, and by proposition 1.1, no firm is playing any mass in $(0,\ r_f)$ either. In addition, there can be no mass atom at $r_i = r_f$. All of this implies that $P_i(r_f) = 0$, so that the effect on $E(P_i(r_i))$ becomes

$$\epsilon(P_i(y) - P_i(x)) > 0$$

Due to $P_i(x) < P_i(y)$. Thus it follows that this type of deviation is always profitable if no firms are playing probability mass at r_i, and so these cannot be Nash Equilibria. In fact, by the reasoning above, any distributions where $P_i(r_f) = 0$ cannot be Nash Equilibria. Thus, at least one of the fund types must be playing mass below r_f, so that $P_i(r_f) > 0$. The only way this is possible without contradicting proposition 1.1 is for at least one firm type to play an atom at $r_i = 0$, where the reasoning used to prove proposition 1.2 does not hold. In that proof, we considered a deviation where a mass movement upwards from a mass atom from $r_i = 0$ (in this case) to $r_i = \epsilon$ where ϵ is infinitesimally small generates a finite increase in $E(P_i(r_i))$. This kind of deviation can never increase $E(P_i(r_i))$ since it would violate proposition 1.1, and firm failure means that $P_i(0) = P_i(\epsilon) = 0$.

Proof of Proposition 1.6. For the distributions over r_i played by the two fund types to be an equilibrium, there must be no mean preserving shift in probability mass that will result in an increase in $E(P_i(r_i))$. Consider shifting the probability mass of either of the two types. As stated, such a shift must be mean preserving to keep constraints 1.2.5 and 1.2.6 binding. There are two types of mean preserving deviation to consider: one that involves moving mass around in the interval $[r_f,\ k_u]$, and another type where mass can be moved to and from $r_i = 0$.

Let us examine the case of a general unilateral deviation in return distribution by firm i where mass is moved around only in the interval $[r_f,\ k_u]$. If we move probability mass ϵ from a very small region around $r_i = x \geq r_f$ where there is probability mass, to a small region around $r_i = x + \triangle x \leq k_u$, with $\triangle x > 0$. The resulting shift in $E(r_i)$ will be equal to $\epsilon\triangle x$, a strict increase in $E(r_i)$. Although this is first order, the assumption of moving mass from a very small region around x to a very small region around $x + \triangle x$ means that this is correct. Likewise, consider the effect of this shift on $E(P_i(r_i))$, which will be $\epsilon P_i(x+\triangle x)-\epsilon P_i(x)$ to first order. To ensure mean preservation, mass δ must be shifted from a point $r_i=y+\triangle y\geq r_f$, where there is probability mass, down to $r_i=y$, with $\triangle y>0$. This shift has an impact $-\delta\triangle y$ on $E(r_i)$, and an effect of $\delta P_i(y)-\delta P_i(y+\triangle y)$ on $E(P_i(r_i))$. The mean preservation condition thus requires the net effect of the two shifts in probability mass on $E(r_i)$ to be zero, which gives

$$\epsilon\triangle x=\delta\triangle y \tag{1.4.1}$$

For the deviation to be profitable, we require that the net impact on $E(P_i(r_i))$ be positive. This condition gives

$$\epsilon(P_i(x+\triangle x)-P_i(x))>\delta(P_i(y+\triangle y)-P_i(y))$$

If we substitute in 1.4.1, we get

$$\frac{P_i(x+\triangle x)-P_i(x)}{\triangle x}>\frac{P_i(y+\triangle y)-P_i(y)}{\triangle y} \tag{1.4.2}$$

This is an important condition. Consider the case when $\triangle x$ and $\triangle y$ tend to 0. From proposition 1.1, $P_i(r_i)$ must be continuous in $[r_f,\ k_u]$. No atoms of probability mass means that there are no discontinuities in F_H and F_L. 1.4.2 then becomes approximately

$$\frac{dP_i}{dr_i}\mid_{r_i=x}>\frac{dP_i}{dr_i}\mid_{r_i=y} \tag{1.4.3}$$

Now assume that $P_i(r_i)$ is nonlinear. It must be continuous from proposition 1.2. Given proposition 1.4 and continuity, both fund types must play probability mass in some intervals above 0 so that 1.2.5 and 1.2.6 are met (type restrictions binding). Thus, in regions where at least one of the two fund types is playing probability mass, the fund that is playing mass there can always find a x and y (recall that there must be probability mass in these two locations) such that 1.4.3 is true, due to non-linearity implying that $\frac{dP_i}{dr_i}$ is not constant. Thus this contradicts the equilibrium condition that no unilateral profitable deviations must be possible. It follows that $\frac{dP_i}{dr_i}$ must be constant in regions where at least one of the types is playing probability mass, and $\frac{dP_i}{dr_{i1}}$ constant implies that $P_i(r_i)$ is linear there, i.e. of the form $Zr_i + c$. This is a sufficient condition for all mean preserving mass movements to be non profitable, since any mass movement can be broken down into these infinitesimal mass movements. If there is indifference to any infinitesimal mean preserving mass movement within $[r_f,\ k_u]$, then there will be indifference to any mean preserving mass movement within that range. This gives a partial result:

LEMMA 1.1. *In any Equilibrium,* $P_i(r_i) = Zr_i + c$ *for* $r_i \in [r_f,\ k_u]$

We can eliminate the need to consider deviations where firms move mass to $r_i > k_u$. This is helpful in the next step. The linear form of $P_i(r_i)$ implies that the only case that needs (due to the indifference to any mean preserving shift within $0 \leq r_i \leq k_{upper}$ and the fact that other cases can be broken down into a shift of mass up to k_u and a shift of that mass beyond k_u) to be checked is the case where mass is moved by a type from k_u to a $r' > k_u$above it. Since $P_i(k_u) = P_i(r') = 1$ by proposition 1.3 and the definition of k_u, the movement of mass upwards has no positive effect on $E(P_i(r_i))$, while a movement of mass downwards somewhere else is required to preserve the mean. The downwards shift always results in a negative effect on $E(P_i(r_i))$, given the form of $P_i(r_i)$. This means such deviations always have a net negative effect on $E(P_i(r_i))$, which means that can never be profitable, and thus deviations where mass is moved to $r_i > k_u$ can be ignored. Thus we can state

LEMMA 1.2. *Deviations where mass is moved to* $r_i > k_u$ *can never be profitable, and therefore can be ignored.*

Finally, we need to consider deviations where mass can be moved to and from $r_i = 0$, due to proposition 1.5. If at least one of the two firm types is playing mass at $r_i = 0$, then all types must be indifferent to a mean preserving mass shift to or from the atom at $r_i = 0$. We only need to consider a deviation where mass ϵ is moved up from $r_i = 0$ to $r^* \in [r_f, k_u]$ by lemma 1.2. To preserve the mean, mass δ must be moved down from $x \geq r_f$ to $y < x$. There are two cases here. Firstly, if $y \geq r_f$, mean preservation gives

$$\epsilon r^* = \delta(x - y)$$

and the requirement for the fund to be indifferent to the deviation (net effect on $E(P_i(r_i)) = 0$) gives

$$\epsilon(P_i(r^*) - P_i(0)) = \delta(P_i(x) - P_i(y))$$

Substituting in the mean preservation condition, $P_i(r_i) = Zr_{i1} + c$ from lemma 1 and $P_i(0) = 0$ gives

$$c = 0 \tag{1.4.4}$$

This is the the only value of c that will make funds indifferent to this deviation type for general values of r^*, x and y. Secondly, consider the case when $y < r_f$. By proposition 1.1, the only value y can take is 0. In that case, the mean preservation condition gives

$$\epsilon r^* = \delta x$$

while the requirement for fund indifference gives

$$\epsilon(P_i(r^*) - P_i(0)) = \delta(P_i(x) - P_i(0))$$

Once again, subbing in the mean preservation condition, $P_i(r_i) = Zr_i + c$ and $P_i(0) = 0$ gives

$$cx = cr^*$$

$c = 0$ makes this true for general x and r^*, and is thus a consistent condition on $P_i(r_1)$. The conditions for indifference to deviations that involve moving mass down to $r_i = 0$ from elsewhere are identical, since all one has to is to change the sign of both the mass movements ϵ and δ. These give exactly the same equations as above, and so $c = 0$ generates indifferences to all mass movements to and from $r_i = 0$. So, this together with lemma 1.1 gives the result we sought.

Proof of Proposition 1.7. The strategy to prove this is simply to go through the cases one by one and solve the equations that determine some of the characteristics of equilibrium.

Case 1. The cdfs played are:

$$F_L(r_i) \begin{cases} 1 & r_i \in [k_L,\ \infty) \\ G(r_i) + a_L & r_i \in [r_f,\ k_L] \\ a_L & r_i \in (0,\ r_f) \\ a_L & r_i = 0 \end{cases}$$

$$F_H(r_i) + \begin{cases} 1 & r_i \in [k_H,\ \infty) \\ \frac{(Zr_i)^{\frac{1}{N-1}}}{\beta} - \frac{(1-\beta)}{\beta} & r_i \in [k_L,\ k_H] \\ \frac{(Zr_i)^{\frac{1}{N-1}}}{\beta} - \frac{(1-\beta)}{\beta}(G(r_i) + a_L) & r_i \in [r_f,\ k_L] \\ \frac{(Zr_f)^{\frac{1}{N-1}}}{\beta} - \frac{(1-\beta)}{\beta}a_L & r_i \in (0,\ r_f) \\ \frac{(Zr_f)^{\frac{1}{N-1}}}{\beta} - \frac{(1-\beta)}{\beta}a_L & r_i = 0 \end{cases}$$

To close the solution, we need to ensure that total probability mass in both these two distributions sum to 1, and that the type constraints are binding. This gives the following equations:

For the L type:

$$a_L + G(k_L) = 1 \tag{1.4.5}$$

$$\int_{r_f}^{k_L} rg(r)dr = r_L \tag{1.4.6}$$

Where $g = \frac{dG}{dr_i}$. For the H type:

$$\frac{(Zk_H)^{\frac{1}{N-1}}}{\beta} - \frac{1-\beta}{\beta} = 1 \tag{1.4.7}$$

$$\int_{r_f}^{k_H} r\frac{Z(Zr)^{\frac{1}{N-1}-1}}{\beta(N-1)}dr - \frac{(1-\beta)}{\beta}\int_{r_f}^{k_L} rg(r)dr = r_H \tag{1.4.8}$$

Using 1.4.7 gives

$$Z = \frac{1}{k_H} = \frac{1}{k_u}$$

This previous result together with 1.4.6 and 1.4.8, allows us to deduce that

$$\bar{r}N = k_H - (\frac{r_f^N}{k_H})^{\frac{1}{N-1}}$$

This equation, although not solvable analytically in the general case, will give upper bound of the H type support and by extension the coefficient Z. Some conditions on G must be imposed to ensure that F_L and F_H are valid cdfs - the pdf must be greater or equal to zero everywhere for both the L and H types. For the L type:

$$g(r_i) \geq 0 \; for \; r_i \in [r_f,\, k_L]$$

$$a_L \geq 0$$

Likewise, F_H must be a valid cdf:

$$Z\frac{(Zr_i)^{\frac{1}{N-1}-1}}{(N-1)} - (1-\beta)g(r_i) \geq 0 \; for \; r_i \in [r_f,\, k_L]$$
$$\frac{(Zr_f)^{\frac{1}{N-1}}}{\beta} - \frac{(1-\beta)}{\beta}a_L \geq 0$$

$f_H = \frac{dF_H}{dr_i} \geq 0$ elsewhere already since $Z\frac{(Zr_i)^{\frac{1}{N-1}-1}}{(N-1)} \geq 0$.

Case 2. The cdfs played are:

$$F_L(r_i)\begin{cases} 1 & r_i \in [k_L,\, \infty) \\ \frac{(Zr_i)^{\frac{1}{N-1}}}{1-\beta} - \frac{\beta}{1-\beta} & r_i \in [k_H,\, k_L] \\ G(r_i) + a_L & r_i \in [r_f,\, k_H] \\ a_L & r_i \in (0,\, r_f) \\ a_L & r_i = 0 \end{cases}$$

$$F_H(r_i)\begin{cases} 1 & r_i \in [k_H,\ \infty) \\ \frac{(Zr_i)^{\frac{1}{N-1}}}{\beta} - \frac{(1-\beta)}{\beta}(G(r_i)+a_L) & r_i \in [r_f,\ k_H] \\ \frac{(Zr_f)^{\frac{1}{N-1}}}{\beta} - \frac{(1-\beta)}{\beta}a_L & r_i \in (0,\ k_H) \\ \frac{(Zr_f)^{\frac{1}{N-1}}}{\beta} - \frac{(1-\beta)}{\beta}a_L & r_i = 0 \end{cases}$$

The same logic applies as before, total probability mass must sum to 1, cdfs must be continuous, and the type (mean) constraints must bind, giving the following equations that must hold. For the L type:

$$\frac{(Zk_L)^{\frac{1}{N-1}}}{1-\beta} - \frac{\beta}{1-\beta} = 1 \tag{1.4.9}$$

$$\int_{r_f}^{k_L} rg(r)dr + \int_{k_L}^{k_H} r\frac{Z(Zr)^{\frac{1}{N-1}-1}}{(1-\beta)(N-1)} = r_L \tag{1.4.10}$$

$$G(k_H) + a_L = \frac{(Zk_H)^{\frac{1}{N-1}}}{1-\beta} - \frac{\beta}{1-\beta} \tag{1.4.11}$$

The first two are the probability mass and type constraints, while the the last equation is cdf continuity constraint.

$$\frac{(Zk_H)^{\frac{1}{N-1}}}{\beta} - \frac{1-\beta}{\beta}(G(k_H)+a_L) = 1 \tag{1.4.12}$$

$$\int_{r_f}^{k_H} r\frac{Z(Zr)^{\frac{1}{N-1}-1}}{\beta(N-1)}dr - \frac{(1-\beta)}{\beta}\int_{r_f}^{k_H} rg(r)dr = r_H \tag{1.4.13}$$

1.4.9 immediately gives

$$Z = \frac{1}{k_L} = \frac{1}{k_u}$$

Using this together with 1.4.10 and 1.4.13, we can deduce that

$$\bar{r}N = k_L - (\frac{r_f^N}{k_L})^{\frac{1}{N-1}}$$

There are a few conditions that must be applied to ensure that F_H and F_L are valid cdfs. For the L type:

$$g(r_i) \geq 0 \; for \; r_i \in [r_f,\ k_L]$$

$$a_L \geq 0$$

For the H type:

$$Z\frac{(Zr_i)^{\frac{1}{N-1}-1}}{(N-1)} - (1-\beta)g(r_i) \geq 0 \; for \; r_i \in [r_f,\ k_L]$$

$$\frac{(Zr_f)^{\frac{1}{N-1}}}{\beta} - \frac{(1-\beta)}{\beta}a_L \geq 0$$

Case 3. The cdfs played are:

$$F_L(r_i)\begin{cases} 1 & r_{i1}\in[k_L,\ \infty) \\ \frac{(Zr_i)^{\frac{1}{N-1}}}{1-\beta}-\frac{\beta}{1-\beta} & r_{i1}\in[k_H,\ k_L] \\ \frac{(Zr_i)^{\frac{1}{N-1}}}{1-\beta}-\frac{\beta}{1-\beta}(G(r_i)+a_H) & r_{i1}\in[r_f,\ k_H] \\ \frac{(Zr_f)^{\frac{1}{N-1}}}{(1-\beta)}-\frac{\beta}{1-\beta}a_H & r_{i1}\in(0,\ r_f) \\ \frac{(Zr_f)^{\frac{1}{N-1}}}{(1-\beta)}-\frac{\beta}{1-\beta}a_H & r_{i1}=0 \end{cases}$$

$$F_H(r_i)\begin{cases} 1 & r_{i1}\in[k_H,\ \infty) \\ G(r_i)+a_H & r_{i1}\in[r_f,\ k_H] \\ a_H & r_{i1}\in(0,\ r_f) \\ a_H & r_{i1}=0 \end{cases}$$

Again, we require cdf continuity, total probability mass summing to 1, and the type constraints to be binding. This gives the following equations. For the L type:

$$\frac{(Zk_L)^{\frac{1}{N-1}}}{1-\beta}-\frac{\beta}{1-\beta}=1 \tag{1.4.14}$$

$$\int_{r_f}^{k_L} r\frac{Z(Zr)^{\frac{1}{N-1}-1}}{(1-\beta)(N-1)}dr-\frac{\beta}{1-\beta}\int_{r_f}^{k_H} rg(r)dr=r_L \tag{1.4.15}$$

For the H type:

$$G(k_H)+a_H=1 \tag{1.4.16}$$

$$\int_{r_f}^{k_H} rg(r)dr = r_H \tag{1.4.17}$$

Solving for critical variables is done using identical methods to the ones used previously. 1.4.14 immediately gives

$$Z = \frac{1}{k_L} = k_u$$

And this result together with 1.4.17 and 1.4.15 gives

$$\bar{r}N \quad = \quad k_L - (\frac{r_f^N}{k_L})^{\frac{1}{N-1}}$$

As before, there are restrictions that must placed on G to ensure that F_H and F_L are valid probability distributions:

$$g(r_i) \geq 0 \; for \; r_i \in [r_f, \; k_H]$$

$$a_H \geq 0$$

and

$$Z\frac{(Zr_i)^{\frac{1}{N-1}-1}}{(N-1)} - \beta g(r_i) \geq 0 \; for \; r_i \in [r_f, \; k_H]$$

$$\frac{(Zr_f)^{\frac{1}{N-1}}}{(1-\beta)} - \frac{\beta}{1-\beta}a_H \geq 0$$

Case 4. The cdfs played are:

$$F_L(r_i)\begin{cases} 1 & r_i \in [k_L,\ \infty) \\ \frac{(Zr_i)^{\frac{1}{N-1}}}{1-\beta} - \frac{\beta}{1-\beta}(G(r_i)+a_H) & r_i \in [r_f,\ k_L] \\ \frac{(Zr_f)^{\frac{1}{N-1}}}{(1-\beta)} - \frac{\beta}{1-\beta}a_H & r_i \in (0,\ r_f) \\ \frac{(Zr_f)^{\frac{1}{N-1}}}{(1-\beta)} - \frac{\beta}{1-\beta}a_H & r_i = 0 \end{cases}$$

$$F_H(r_i)\begin{cases} 1 & r_i \in [k_L,\ \infty) \\ \frac{(Zr_i)^{\frac{1}{N-1}}}{\beta} - \frac{1-\beta}{\beta} & r_i \in [k_H,\ k_L] \\ G(r_i)+a_H & r_i \in [r_f,\ k_L] \\ a_H & r_i \in (0,\ r_f) \\ a_H & r_i = 0 \end{cases}$$

As before, cdf continuity, probability mass summing to 1, and binding type constraints give for the L type:

$$\frac{(Zk_L)^{\frac{1}{N-1}}}{1-\beta} - \frac{\beta}{1-\beta}(G(k_L)+a_H) = 1 \tag{1.4.18}$$

$$\int_{r_f}^{k_L} r\frac{Z(Zr)^{\frac{1}{N-1}-1}}{(1-\beta)(N-1)}dr - \frac{\beta}{1-\beta}\int_{r_f}^{k_L} rg(r)dr = r_L \tag{1.4.19}$$

And for the H type:

$$\frac{(Zk_H)^{\frac{1}{N-1}}}{\beta} - \frac{1-\beta}{\beta} = 1 \tag{1.4.20}$$

$$\int_{r_f}^{k_L} rg(r)dr + \int_{k_L}^{k_H} r\frac{Z(Zr)^{\frac{1}{N-1}-1}}{\beta(N-1)}dr = r_H \tag{1.4.21}$$

$$G(k_L) + a_H = \frac{(Zk_L)^{\frac{1}{N-1}}}{\beta} - \frac{1-\beta}{\beta} \tag{1.4.22}$$

As before, 1.4.20 implies that

$$Z = \frac{1}{k_H} = \frac{1}{k_u}$$

And this with 1.4.21 and 1.4.19 once more give

$$\bar{r}N \quad = \quad k_H - (\frac{r_f^N}{k_H})^{\frac{1}{N-1}}$$

As before, there are restrictions that must placed on G to ensure that F_H and F_L are valid probability distributions:

$$g(r_i) \geq 0 \; for \; r_i \in [r_f, \; k_L]$$

$$a_H \geq 0$$

and

$$Z\frac{(Zr_i)^{\frac{1}{N-1}-1}}{(N-1)} - \beta g(r_i) \geq 0 \; for \; r_i \in [r_f, \; k_L]$$

$$\frac{(Zr_f)^{\frac{1}{N-1}}}{(1-\beta)} - \frac{\beta}{1-\beta}a_H \geq 0$$

In every one of these four cases, it is apparent that $Z = \frac{1}{k_u}$, and also $\bar{r}N = k_u - (\frac{r_f^N}{k_u})^{\frac{1}{N-1}}$, which is what the proposition states.

We also need to prove that the equation $\bar{r}N + (\frac{r_f^N}{k_u})^{\frac{1}{N-1}} = k_u$ has a unique solution in the region $k_u \geq 0$. Rearrange it into the form

$$\bar{r}N = k_u - (\frac{r_f^N}{k_u})^{\frac{1}{N-1}}$$

The solution to the equation is thus when the function $k_u - (\frac{r_f^N}{k_u})^{\frac{1}{N-1}}$ crosses the horizontal line $\bar{r}N$. It is clear that when $k_u \geq 0$, $k_u - (\frac{r_f^N}{k_u})^{\frac{1}{N-1}}$ is strictly increasing in k_u. Thus, it will cross $\bar{r}N$ once only. For there to be a unique solution in the region $k_u \geq 0$, it only needs to be verified that $k_u - (\frac{r_f^N}{k_u})^{\frac{1}{N-1}}$ begins below $\bar{r}N$ at $k_u = 0$. It is clear that for any finite N, $k_u - (\frac{r_f^N}{k_u})^{\frac{1}{N-1}} \to -\infty$ as $k_u \to 0$. This is clearly below $\bar{r}N$, which has a lower bound of $2\bar{r} > 0$. Thus, there will always be a unique, positive solution to this equation for $N \geq 2$.

Proof of Proposition 1.8. Denote the failure probability as T. From proposition 1.1, no fund will play any probability mass in the interval $(0,\, r_f)$, and before type is drawn, each firm will play a mass atom at $r_i = 0$ in expectation. Thus, the only way a firm will fail is if it draws $r_i = 0$. In expectation (before type is drawn), this probability, and thus the failure probability is equal to

$$T = \beta a_H + (1 - \beta)a_L$$

Where a_H and a_L are the mass atoms played at zero by the H and L types respectively. From the results obtained during the proof of proposition 1.7, we can calculate this on a case by case basis, which gives $\beta a_H + (1-\beta)a_L = (Zr_f)^{\frac{1}{N-1}}$ in every one of the four cases examined there. Proposition 1.7 also finds that Z is a function of k_u, and that k_u in turn depends only on the base parameters of the model. This immediately implies that

$\beta a_H + (1-\beta)a_L$ is independent of the shape of the played distributions in equilibrium. Thus,

$$T = (Zr_f)^{\frac{1}{N-1}}$$

Recall that

$$Z = \frac{1}{k_u}$$

(1.4.23) $$\bar{r}N + (\frac{r_f^N}{k_u})^{\frac{1}{N-1}} = k_u$$

We can use these to write

$$T = \frac{k_u - \bar{r}N}{r_f}$$

To analyse how firm failure probability responds to N, the number of competing firms, we can evaluate $\frac{dT}{dN}$. By using implicit differentiation to find $\frac{dk_u}{dN}$, it is possible to arrive at the following expression for $\frac{dT}{dN}$:

$$\frac{dT}{dN} = (\frac{r_f}{k_u})^{\frac{1}{N-1}} \left(\frac{\frac{(\ln k_u - \ln r_f)}{(N-1)^2} - \bar{r}\frac{1}{k_u(N-1)}}{1 + \frac{r_f^{1+\frac{1}{N-1}}}{(N-1)k_u^{1+\frac{1}{N-1}}}} \right)$$

From this, it is straightforward to show that the conditions for $\frac{dT}{dN} > 0$ are:

(1.4.24) $$\frac{k_u}{\bar{r}} \ln\left(\frac{k_u}{r_f}\right) > N-1$$

Given equation 1.4.23 and that $r_f > 0$ and $k_u > 0$, it is clear that $k_u > \bar{r}N$. It follows that

$$\frac{k_u}{\bar{r}} \ln\left(\frac{k_u}{r_f}\right) > N \ln\left(\frac{\bar{r}N}{r_f}\right)$$

since $\frac{k_u}{\bar{r}} \ln\left(\frac{k_u}{r_f}\right)$ is increasing in k_u. Thus if we can show that

$$N \ln\left(\frac{\bar{r}N}{r_f}\right) > N - 1$$

Then 1.4.24 holds too. Consider the conditions required for $N \ln\left(\frac{\bar{r}N}{r_f}\right) > N - 1$. It can be shown that the both sides of this inequality increase with N, and also that the LHS of this inequality increases more quickly with N than the RHS. Thus, we only need to consider $N = 2$, the smallest possible value for N. This gives

$$2 \ln\left(\frac{2\bar{r}}{r_f}\right) > 1$$

We can immediately see that if both $r_H > r_f$ and $r_L > r_f$, then $\bar{r} > r_f$, and this inequality holds. Consequently, $r_H > r_f$ and $r_L > r_f$ implies that
$\frac{k_u}{\bar{r}} \ln\left(\frac{k_u}{r_f}\right) > N \ln\left(\frac{\bar{r}N}{r_f}\right) > N - 1$ for all $N \geq 2$ and thus that $\frac{dT}{dN} > 0$ for all valid numbers of competing firms.

Proof of Proposition 1.9. Consider the equations that specify equilibrium pairs of distributions f_L and f_H in proposition 1.7. If f_L is a valid pdf, has no mass in the interval $(0,\, r_f)$ and causes the L ability constraint to be binding, then the choice of Z and $k_u$will ensure that the mass and ability constraint conditions are satisfied for the H type. We just need to check that F_H is a valid cdf, i.e. $f_H \geq 0$ for all r_i. Since the comparative static results over N are independent of the functional forms of the return distributions, it suffices to show that there exists a functional form for which equilibria exist for all $N \geq 2$ and for some $r_f > 0$.

A convenient functional form for F_L is:

$$F_L(r)\begin{cases} 1 & r\in[k_u,\,\infty) \\ A(Zr)^{\frac{1}{N-1}} - A(Zr_f)^{\frac{1}{N-1}} + C_L & r\in[r_f,\,k_u) \\ C_L & r\in[0,\,r_f) \end{cases}$$

Where A and C_L are constants to be determined such that the ability constraint is binding and total probability mass integrates to 1:

$$F_L(k_u) = 1$$

$$\int_0^{k_u} r f_L(r)dr = r_L$$

Substituting the functional form in gives the two following equations:

(1.4.25) $$1 - C_L = AZ^{\frac{1}{N-1}}(k_u^{\frac{1}{N-1}} - r_f^{\frac{1}{N-1}})$$

(1.4.26) $$AZ^{\frac{1}{N-1}}(k_u^{\frac{1}{N-1}+1} - r_f^{\frac{1}{N-1}+1}) = Nr_L$$

Dividing the two gives an expression for $1 - C_L$:

$$1 - C_L = \frac{r_L N(k_u^{\frac{1}{N-1}} - r_f^{\frac{1}{N-1}})}{(k_u^{\frac{1}{N-1}+1} - r_f^{\frac{1}{N-1}+1})}$$

And

$$A = \frac{1 - C_L}{Z^{\frac{1}{N-1}}(k_u^{\frac{1}{N-1}} - r_f^{\frac{1}{N-1}})}$$

Substituting the expression for $1 - C_L$ gives:

$$A = \frac{r_L N}{Z^{\frac{1}{N-1}}(k_u^{\frac{1}{N-1}+1} - r_f^{\frac{1}{N-1}+1})}$$

This can be greatly simplified by recalling that $Z = \frac{1}{k_u} \Rightarrow (Zk_u)^{\frac{1}{N-1}} = 1$, and using $k_u = \bar{r}N + r_f(\frac{r_f}{k_u})^{\frac{1}{N-1}} = \bar{r}N + r_f(Zr_f)^{\frac{1}{N-1}}$. This gives

(1.4.27) $$A = \frac{r_L}{\bar{r}}$$

We can double check that F_L is a valid cdf. Clearly the density for $r > 0$ is always positive. It just needs to verified that $1 \leq C_L \leq 0$. Subbing in equation1.4.27 into equation $1.4.25$ and using equation 3.2.11 gives

$$1 - C_L = \frac{r_L}{\bar{r}k_u^{\frac{1}{N-1}}}(k_u^{\frac{1}{N-1}} - r_f^{\frac{1}{N-1}})$$

In chapter 1, I assumed that $r_H > r_L > r_f > 0$. This clearly implies $0 < \frac{(k_u^{\frac{1}{N-1}} - r_f^{\frac{1}{N-1}})}{k_u^{\frac{1}{N-1}}} < 1$, and clearly $0 < \frac{r_L}{\bar{r}} < 1$ as well. So it follows that $0 < 1 - C_L < 1$, and therefore that C_L is within the acceptable bounds for F_L to be a valid cdf.

Now we need to find conditions such that $f_H \geq 0$ for all r. From equation 1.2.10, F_H must be of the following form to satisfy proposition 1.7 and be an equilibrium pair of distributions with F_L:

(1.4.28) $$F_H(r)\begin{cases} 1 & r \in (k_u,\, \infty) \\ \frac{(Zr_i)^{\frac{1}{N-1}}}{\beta} - \frac{(1-\beta)}{\beta}[AZ^{\frac{1}{N-1}}(r^{\frac{1}{N-1}} - r_f{}^{\frac{1}{N-1}}) + C_L] & r \in [r_f,\, k_u] \\ \frac{(Zr_f)^{\frac{1}{N-1}}}{\beta} - \frac{(1-\beta)}{\beta}C_L & r \in [0,\, r_f) \end{cases}$$

LEMMA. $f_H(r) \geq 0$ *for all* $r \in (0,\, \infty)$.

PROOF. Consider the density in $r \in (0,\, \infty)$. For now, ignore the density at zero. We can just differentiate the expression for $F_H(r)$ for $r > 0$.

$$f_H(r)\begin{cases} 0 & r \in (k_u,\, \infty) \\ \frac{Z^{\frac{1}{N-1}} r^{\frac{1}{N-1}-1}}{\beta(N-1)} - \frac{(1-\beta)}{\beta(N-1)} A Z^{\frac{1}{N-1}} r^{\frac{1}{N-1}-1} & r \in [r_f,\, k_u] \\ 0 & r \in (0,\, r_f) \end{cases}$$

So, clearly we only require $\frac{Z^{\frac{1}{N-1}} r^{\frac{1}{N-1}-1}}{\beta(N-1)} - \frac{(1-\beta)}{\beta(N-1)} A Z^{\frac{1}{N-1}} r^{\frac{1}{N-1}-1} \geq 0$ for $r \in [r_f,\, k_u]$ to complete the proof. Simplifying this gives:

$$A \leq \frac{1}{1-\beta}$$

Given $A = \frac{r_L}{\bar{r}}$ and $0 \leq \beta \leq 1$, it's clear that this is always satisfied. $\square$

Now we just need to find the parameter conditions that ensure the mass atom played by the H type at $r = 0$ is between zero and one. From 1.4.28, this is:

$$0 \leq \frac{(Zr_f)^{\frac{1}{N-1}}}{\beta} - \frac{(1-\beta)}{\beta} C_L < 1$$

The following lemma establishes the conditions required to fulfil this set of inequalities.

LEMMA. *$r_f \geq 2M\bar{r} + M^2$, where $M = \frac{(1-\beta)(r_H - r_L)}{r_H}$ is a necessary and sufficient condition for $0 \leq \frac{(Zr_f)^{\frac{1}{N-1}}}{\beta} - \frac{(1-\beta)}{\beta} C_L < 1$ for $N \geq 2$.*

PROOF. First, consider the conditions required for $\frac{(Zr_f)^{\frac{1}{N-1}}}{\beta} - \frac{(1-\beta)}{\beta} C_L < 1$. Sub in the expression for C_L from 1.4.25 and use $(Zk_u)^{\frac{1}{N-1}} = 1$ from 1.2.10to simplify, this gives:

$$1 > (1-\beta)\frac{r_L}{\bar{r}} + (Zr_f)^{\frac{1}{N-1}}\left(1 - (1-\beta)\frac{r_L}{\bar{r}}\right)$$

Note $(Zr_f)^{\frac{1}{N-1}} = (\frac{r_f}{k_u})^{\frac{1}{N-1}}$. Given equation 1.2.12, it is clear that $(Zr_f)^{\frac{1}{N-1}} < 1$ since $k_u > r_f$. So, we can write:

$$1 = (1-\beta)\frac{r_L}{\bar{r}} + (1-(1-\beta)\frac{r_L}{\bar{r}}) > (1-\beta)\frac{r_L}{\bar{r}} + (Zr_f)^{\frac{1}{N-1}}(1-(1-\beta)\frac{r_L}{\bar{r}})$$

Since $0 < (1-\beta)\frac{r_L}{\bar{r}} < 1$. Thus, $\frac{(Zr_f)^{\frac{1}{N-1}}}{\beta} - \frac{(1-\beta)}{\beta}C_L < 1$ is always satisfied. This is consistent with $f_H \geq 0$ for all $r \neq 0$.

Consider the conditions required for $\frac{(Zr_f)^{\frac{1}{N-1}}}{\beta} - \frac{(1-\beta)}{\beta}C_L \geq 0$. Sub in the expression for C_L and use $(Zk_u)^{\frac{1}{N-1}} = 1$ to simplify:

$$(Zr_f)^{\frac{1}{N-1}} - (1-\beta)(1 - \frac{r_L}{\bar{r}}(1-(Zr_f)^{\frac{1}{N-1}}) \geq 0$$

Let $T = (Zr_f)^{\frac{1}{N-1}}$, simplifying further:

$$T(1-(1-\beta)\frac{r_L}{\bar{r}}) \geq (1-\beta)(1-\frac{r_L}{\bar{r}})$$

Since $0 < (1-\beta)\frac{r_L}{\bar{r}} < 1$ and $T > 0$, we can say that if this holds for the minimum value of T, it must hold for all other values of T. Note that T is just the ex-ante fund failure probability. In deriving proposition 1.8, it was shown that $\frac{dT}{dN} \geq 0$ for $N \geq 2$ when $r_L > r_f$. Thus, the minimum relevant of T is reached when $N = 2$. Using the expression for k_u equation 1.2.12, it can be shown that for $N = 2$

$$T_{|N=2} = \frac{r_f}{\bar{r} + \sqrt{\bar{r}^2 + r_f^2}}$$

Thus we require

$$\frac{r_f}{\bar{r} + \sqrt{\bar{r}^2 + r_f^2}} \geq \frac{(1-\beta)(1-\frac{r_L}{\bar{r}})}{1-(1-\beta)\frac{r_L}{\bar{r}}}$$

This simplifies to

$$\frac{r_f}{\bar{r}+\sqrt{\bar{r}^2+r_f^2}} \geq \frac{(1-\beta)(r_H-r_L)}{r_H}$$

To make things less cluttered, write the RHS as M, a constant. Rearranging:

$$r_f - M\bar{r} \geq M\sqrt{\bar{r}^2+r_f^2}$$

If $r_f < M\bar{r}$, then this inequality can never be satisfied. If $r_f \geq M\bar{r}$, we can proceed by squaring both sides. Simplifying, we obtain:

$$r_f \geq 2M\bar{r} + M^2 \tag{1.4.29}$$

Where

$$M = \frac{(1-\beta)(r_H-r_L)}{r_H}$$

Since $\frac{(Zr_f)^{\frac{1}{N-1}}}{\beta} - \frac{(1-\beta)}{\beta}C_L < 1$ is always satisfied, the condition for $\frac{(Zr_f)^{\frac{1}{N-1}}}{\beta} - \frac{(1-\beta)}{\beta}C_L \geq 0$ is the required condition for $0 \leq \frac{(Zr_f)^{\frac{1}{N-1}}}{\beta} - \frac{(1-\beta)}{\beta}C_L < 1$ for $N \geq 2$. This is the above condition. □

$0 \leq M \leq 1$, so it is possible to set values of r_f, r_L, r_H and β such that this inequality equation 1.4.29 is satisfied. Together, these two lemmas ensure that $f_H(r) \geq 0$ everywhere. That will guarantee the existence of equilibria using these functional forms for $N \geq 2$.

Proof of Proposition 1.10. In the proof of proposition 1.9, the only condition which required parameter restrictions to be fulfilled was

(1.4.30) $$T(1-(1-\beta)\frac{r_L}{\bar{r}}) \geq (1-\beta)(1-\frac{r_L}{\bar{r}})$$

Where $T = (Zr_f)^{\frac{1}{N-1}}$. In proving proposition 8, it was shown that $\frac{dT}{dN} > 0$ for $N \geq 2$ if $0 < r_f < r_L < r_H$. So, in principle, it is possible for there to exist a large enough N that results in a large enough T to always satisfy this inequality. The only thing that may prevent this from being true is if T reaches an upper bound where this inequality is not always satisfied. Clearly, the given $\frac{dT}{dN} > 0$ for $N \geq 2$, the maximum value of T must be $\lim_{N\to\infty} T$. Let us evaluate this limit:

$$\lim_{N\to\infty} T = \lim_{N\to\infty} (\frac{r_f}{k_u})^{\frac{1}{N-1}}$$

Given $N \geq 2$, it is clear that $\lim_{N\to\infty}(r_f)^{\frac{1}{N-1}} \to 1$. To complete the calculation, we need to find $\lim_{N\to\infty}(k_u)^{\frac{1}{N-1}}$. Recall that $k_u = \bar{r}N + r_f(\frac{r_f}{k_u})^{\frac{1}{N-1}}$. Because we can always pick the unique positive solution to k_u given by this equation, it is clear that $0 < (\frac{r_f}{k_u})^{\frac{1}{N-1}} < 1$ since $k_u > \bar{r}N > r_f$. This is because $N \geq 2$ and $r_f < r_L \leq \bar{r}$. Using the fact that $0 < (\frac{r_f}{k_u})^{\frac{1}{N-1}} < 1$, we can write

$$\bar{r}N + r_f > k_u > \bar{r}N$$

And since $N \geq 2$,

$$(\bar{r}N + r_f)^{\frac{1}{N-1}} > k_u^{\frac{1}{N-1}} > (\bar{r}N)^{\frac{1}{N-1}}$$

We can evaluate the limits of the logarithms of these expressions more easily. $\lim_{N\to\infty} \ln(\bar{r}N)^{\frac{1}{N-1}} = \lim_{N\to\infty} \frac{\ln(\bar{r}N)}{N-1}$. We can use L'Hôpital's rule to evaluate this limit:

$$\lim_{N\to\infty} \ln(\bar{r}N)^{\frac{1}{N-1}} = \lim_{N\to\infty} \frac{1}{N} = 0$$

Which implies that

$$\lim_{N\to\infty} (\bar{r}N)^{\frac{1}{N-1}} = e^0 = 1$$

A similar method can be used to show that

$$\lim_{N\to\infty} (\bar{r}N + r_f)^{\frac{1}{N-1}} = 1$$

Since $\bar{r}N + r_f > k_u > \bar{r}N$ and limits of the expressions greater than and less than $k_u$converge at $N \to \infty$, by the squeeze theorem it must be the case that

$$\lim_{N\to\infty} (k_u)^{\frac{1}{N-1}} = 1$$

Thus it is clear that

$$\lim_{N\to\infty} T = \lim_{N\to\infty} (\frac{r_f}{k_u})^{\frac{1}{N-1}} = 1$$

Note that when $T = 1$, inequality 1.4.30 is always satisfied for any valid values of r_H, r_L and β. Thus given that $\frac{dT}{dN} > 0$ for $N \geq 2$ and $\lim_{N\to\infty} T = 1$, there will always be a value of N where T becomes large enough to assure the existence of a least one symmetric equilibrium with the functional forms used to prove proposition 1.9.

CHAPTER 2

Robustness Check: Introducing Bayesian Type Evaluation

2.1. Introduction

Hedge funds are highly opaque entities with relatively high turnover and short average lifespans. They are lightly regulated and have access to powerful financial instruments that can manipulate their return distributions. New hedge funds strive for strong relative performance (Brown, Goetzmann, Park (2001)), which is important to get them off the ground. In such an environment, when funds can be of varying ability, they have incentives to use their financial instruments to manipulate their return distributions in order to compete for reputational benefits by attaining high returns relative to their peers. This comes at the cost of tail risk, and is a point made by Foster and Young (2010). An interesting question to ask in this setting is how differing levels of competition affects these incentives and fund behaviour.

The previous chapter investigated the problem of new hedge funds of unknown ability with access to complex financial instruments competing to improve their reputations by achieving high returns relative to their peers. The framework used is related to the sales literature on price dispersion (Varian (1980), Spiegler (2006), all pay auctions (Baye and Kovenock et al. (1996)), Blotto Games (Gross and Wagner (1950) and formation of favoured minority groups in electoral situations (Myerson (1993)). In the previous chapter, hedge funds are randomly drawn as high ability or low ability, and their ability level affects their maximum expected return, obviously with high ability types being able to deliver higher returns than low ability types. There is limited return sampling, and the fund that draws the highest return will gain a benefit. The benefit can be interpreted as the reputational gain from being able to appear high on performance rankings, or the benefit from having a strong enough performance relative to its peers that the fund can improve its reputation by choosing to disclose return numbers. Funds can use complex financial instruments to manipulate their return distributions to improve their chances of winning this benefit, which is proxied by the high degree of freedom the funds have to manipulate their return distributions. Effectively, they choose any return distribution as long as their expected returns stay within their ability constraints. The main finding of the model is that increasing the number of competing firms in this scenario increases the amount of tail risk played by the average fund and raises the firm failure rate. This is because increasing the number of funds makes it more difficult for them to stand out and win the reputational reward, since there are more funds they have to beat to win it. Combining this with the ability for funds to shape their return distributions means that incentives are generated to move probability mass up to high return levels, raising the chance of winning the reputational reward. Given the ability constraint that limits the mean return, the funds must counterbalance this upwards mass movement with a movement of mass down to the largest loss level allowed by limited liability, raising tail risk. Mathematically, this result is due to an increase in competition making the probability of winning function, which is related to the highest order statistic, more convex. This gives funds an incentive to move mass upwards to high levels of return, which in turn must be compensated by mass movements downwards to zero to satisfy the ability constraint.

A major criticism of this model is the assumption that a reputational boost is gained by the fund that has the best performance in the peer group of new funds of unknown ability. Hedge funds are supposed to cater to smart, sophisticated investors - that is one of the reasons they are so lightly regulated and so opaque. Their investors are assumed to be smart and aware enough to be less susceptible to foul play, and are assumed to be able to take responsibility for their own money. Related to this is the fact that most hedge funds often have a minimum investment level that is set to be very high, so only very wealthy people who can be assumed to have some experience in investing their money in more complex financial institutions can participate. This is at odds with the way reputational gain is awarded in the model in chapter 1, which is somewhat naive and simplistic. This calls into question whether that assumption is suitable given the context and setting of the model. More sophisticated investors will not blindly consider the new fund that draws a high return to automatically be a good investment prospect.

Obviously, an adjustment to the model that can be made in light of this criticism is to make the awarding of the reputational gain more sophisticated and rational, and this is what this chapter attempts to do. A natural way of doing this is to assume that more sophisticated observers will be interested in knowing what new funds have the highest chance of being run by high ability managers, preferring those with the highest chance. However, we still want to keep the feature of rewarding strong performance relative to the peer group. To implement reputationally rewarding the funds judged to have the highest ranked probabilities of being a high ability type, a rational observer who records the performance of each fund and uses this to calculate the Bayesian probability of each fund being a high ability type is introduced. The observer ranks the funds in order of these probabilities, and the fund that is the most likely to be a high ability type gains a reputational boost. Only if this procedure does not produce a clear winner does the reputational boost go to fund that draws the highest return. The other assumptions of the model are mostly unchanged from the previous chapter.

The question to be answered is if the main results from chapter 1 hold when rational evaluation of type probabilities, a more sophisticated method of deciding which fund gains a boost to their reputation is used as the primary method of choosing the best fund, rather than an algorithm that picks the fund that draws the highest return. This is a pertinent question to ask, since this alteration to the model radically changes the way the reward mechanism is evaluated, and the solution concept changes. After this change, the model bears more resemblance to a signalling scenario than before, and the beliefs of the rational observer must be taken into account. Due to the latter fact, we need to solve for a modified symmetric perfect Bayesian equilibria, which is a significant departure from the original model.

The main findings of this analysis is that although the resulting model superficially resembles the model in chapter 1, the results from it are significantly weaker. The new rational model resembles the non-rational one due to the derived requirement that the played return distributions f_H and f_L, of the high and low ability types respectively, must have a monotone likelihood ratio property, which causes the probability of being judged most likely to be a high ability type to be increasing in r_i. Thus the probability of winning the reputational boost is related to the highest order statistic, like in the model from chapter 1. In addition, like in the previous chapter, at least one fund plays a mass atom at the zero return level, leading to every equilibrium having tail risk and fund failure. The two models thus unsurprisingly share some of the same equilibria where the played return distributions are continuous above r_f, and the results on the positive relation between the number of competing funds and the amount of tail risk/probability of fund failure apply exactly to these equilibria. Also, because of the way the equilibrium distribution shapes still depend on the order statistic, the same mechanism that causes this result in the previous chapter still applies to many other equilibria types as well.

The big difference is that in this model, equilibria where both types play a mass atom anywhere above the fund failure threshold are possible. This is important since this it means that equilibria exist where the link between competition and fund failure probability does not apply. In particular, the result does not apply when both fund types play probability mass atoms at two points only. In these equilibria, the number of competing funds has no effect on tail risk because the ability constraint must be binding in equilibrium, making the mean return constant for both fund types. There is only one way of reaching this required mean when funds play mass at only two points. In addition, in equilibria where there are a lot of mass atoms, the upper bounds of their supports are exogenously determined, unlike in the continuous equilibria. This is another reason why raising competition does not raise the fund failure rate in some of these mass atom equilibria. Some of these mass atom equilibria also resemble the return mimicking scenario described by Foster and Young, and were not present in the original non-rational version of the model. This is not surprising, since this version of the model bears more resemblance to a Bayesian signalling model, although the monotone likelihood ratio property of the equilibrium distributions means that the highest order statistic in still important in determining equilibria, giving them some characteristics like that of a Blotto Game with spread out probability mass and different types playing contrasting equilibria. The two point equilibria are the most like Bayesian signalling equilibria out of all of them.

Section 2 details the model and its setup in detail, and proceeds to derive a number key restrictions on the form of the equilibria and some other key results needed to analyse the equilibria of the model, section 3 discusses whether the results from the previous chapter hold in detail, and section 4 concludes.

2.2. The Model

2.2.1. Model Description. N new risk neutral hedge funds of unknown ability, each of which is run by a high ability manager (H type) with probability β or a low ability manager (L type) with probability $1 - \beta$, compete to try and enhance their reputations. Each of the funds has access to complex financial instruments that they can use the alter their return distributions as they see fit, subject to constraints on their means that are imposed by manager ability. A H type fund will be able to deliver a higher expected return than a L type fund. A rational, sophisticated observer records the funds' performance and evaluates the Bayesian probability that each fund is a H type, albeit with the restriction of limited sampling of return histories. It is assumed for simplicity that the funds all invest their own funds to generate one return from their picked distributions, and that the observer can see each fund's draw. Reputational benefit is then gained by the fund that is evaluated as being the most likely to be a high ability H type fund. In the event of several funds tying for the highest probability of being a H type, the fund that draws the highest return out of the tying funds gains the reputational benefit. Firm failure is incorporated into this model using an exogenous threshold r_f. If a fund draws a return $r < r_f$, then the firm fails and cannot claim the reputational reward for being judged the most likely to be a high ability type. The solution concept to be used is a refinement of a symmetric Bayesian Equilibrium.

2.2.2. Bayesian Type Probabilities and the Awarding of Reputational Reward α. When the observer see that fund i draws return r_i, then the probability that fund i is a H type can be calculated. The assumption that the observer is sophisticated and rational means that it knows the structure of the game and is therefore able to predict what distributions H and L types play, and in equilibrium calculates the Bayesian probability of fund i being a H type upon seeing a return r_i assuming that H types play f_H and L types play f_L.

In a symmetric equilibrium, all H type funds play the same return distribution as each other, and all the L type funds play the same distribution as each other too. Let f_L be the return pdf selected by the L type and f_H be the return pdf selected by the H type. In equilibrium, the observer beliefs match the equilibrium distributions. Let ϵ be a very small interval around r_i. If a fund i draws a return within this interval, then the probability of that fund being a H type, $\phi_i(r_i)$ is approximately

(2.2.1)
$$prob(H|f_H, f_L, \beta, r_i) = \phi_i(r_i) = \frac{\beta f_H(r_i)\epsilon}{\beta f_H(r_i)\epsilon + (1-\beta) f_L(r_i)\epsilon} = \frac{\beta f_H(r_i)}{\beta f_H(r_i) + (1-\beta) f_L(r_i)}$$

Where ϵ is the width of the interval. Consider the case when $\epsilon \to 0$. This gives the exact probability of fund i being an H type when it draws r_i. We also need to consider what happens to ϕ_i when there are probability mass atoms, since they are not excluded from existing. At a return r^* where there are mass atoms, if the size of the atom played by the H type is a_H and the size of the atom played by the L type is a_L, then if firm i draws r^*,

$$\phi_i(r*) = \frac{\beta a_H}{\beta a_H + (1-\beta) a_L} \tag{2.2.2}$$

If one of the types is not playing a mass atom at r^* while the other is, clearly $\phi(r^*) = 1$ if the type playing a mass atom is the H type, and $\phi(r^*) = 0$ if the L type is the one playing a mass atom there while the H type is not.

Upon observing the return draws by all the firms, the following procedure is used to pick which fund the award α is given to:

- Funds that draw returns which give $\phi_i = 0$ cannot gain reputational benefits, since they are judged to be low ability types. So, if all funds draw returns that give $\phi_i = 0$, then no fund gains a reputational boost.
- If there are no funds tied for the highest value of $\phi_i(r_i)$ and $\max \phi_i(r_i) \neq 0$, then the reward α is given to the unique fund with the highest $\phi_i(r_i)$.
- If there are several funds tied for the highest value of $\phi_i(r_i)$ and $\max \phi_i(r_i) \neq 0$, then α is awarded to the fund that has the highest return r_i out of the funds that are tied for the highest $\phi_i(r_i)$.
- If there are M funds tied for highest $\phi_i(r_i)$, all with the same return $r*$ and $\phi_i(r*) \neq 0$, each of the M funds with the return $r*$ has a $\frac{1}{M}$ change of getting the reputational benefit, i.e. the winner is picked at random out of the tied funds.

The refinement to use returns as a tie-breaker if ϕ_i does not produce a unique winner is to simplify the analysis. There is also a lot of empirical evidence that investors do pick funds heavily based on past performance. Effectively, the funds are sorted lexicographically - funds that have the highest probability of being a H type $\phi_i(r_i)$ are always preferred. If there is a tie in that, the next tie breaker is to pick the highest return fund out of all the ones that are tied with the highest $\phi_i(r_i)$. If that criteria does not produce a unique winner, the observer picks a fund with tied highest ϕ_i and r_i at random.

2.2.3. Model Timing and Objective Functions.

- $t = 0$: The type of each fund is drawn randomly and independently (H with probability β or L with probability $1 - \beta$), and all funds simultaneously pick a return distribution subject to the appropriate ability constraints. All funds invest I_f of their own money into this.
- $t = 1$: Each fund independently realises a return from the distribution they picked in $t = 0$. It is kept by the fund, and noted by the observer. If the return $r_i < r_f$, the failure threshold, then the fund closes and cannot claim the reputational reward for being the most likely to be a high ability type. A fund that is judged to have $\phi_i = 0$ can also never claim the reputational reward either. The observer ranks the funds in order of ϕ_i for each, excluding the ones that draw returns that indicate $\phi_i = 0$. The one that draws $r_i \geq r_f$ (and thus does not fail) and also gets the highest ϕ_i in this ranking obtains a reputational gain α. If several funds tie for highest ϕ_i, then the fund with the highest return out of them gets the award α. If there is a further tie, winner is picked at random between the funds tied with the highest return and $\phi_i(r_i)$ (in this case it is the same as the reward being split evenly between them).

As in the last chapter, there are constraints on return means to proxy for differences in ability between the H and L types:

$$\int_{s_H} r f_H(r) dr \leq r_H \tag{2.2.3}$$

$$\int_{S_L} r f_L(r) dr \leq r_L \tag{2.2.4}$$

Where the integrals are over S_H, the support of f_H, and S_L, the support of f_L. In addition, $r_i \geq 0$, which is an assumption of limited liability. Also, assume that both fund types can in expectation make more than the failure threshold, i.e.

$$r_H > r_L > r_f \tag{2.2.5}$$

There is no discounting to simplify the analysis. To write down an objective function for each fund, a function $P_i(r_i)$ needs to be introduced. This is the probability that firm i gets the highest $\phi_i(r_i)$ and wins the reputational benefit when it draws a return r_i. Then, the objective function for a fund is:

$$\pi_i = E(r_i)I_f + \alpha E(P_i(r_i))$$

All funds maximise this subject to either 2.2.3 if the fund is H type, or 2.2.4 if the fund is L type.

2.2.4. Equilibrium Properties.

2.2.4.1. *Off Equilibrium Path Assumptions.* When reputational rewards were based on the highest return level, there is no need to consider what happens off equilibrium path since the fund that gains reputational benefit is simply decided by an algorithm. However, when a rational observer gives the reputational award to the firm most likely to be a H type, what happens off equilibrium path must be considered. A particular issue is that it becomes impossible to exclude the possibility of funds playing mass atoms in equilibrium in $r_i \in [r_f, \infty)$ unless very particular and strong assumptions are made about how the observer evaluates ϕ_i when a firm draws a r_i value off the equilibrium path.

Unfortunately, there is not an intuitive or natural way of evaluating ϕ_i off equilibrium path. For example, it is not possible to apply the intuitive criterion, the benefit of moving mass out of the supports of f_H and f_L is entirely dependent on how ϕ_i is evaluated there. This is because it is costless move to probability mass, and so it is impossible to rule out mass movements for one of the types as unprofitable. In fact, the marginal benefit/loss of moving some probability mass out of the region of the equilibrium supports of f_H and f_L is symmetric for both types, making it difficult to deduce what type a deviating firm is likely to be.

In such a circumstance, it seems to reasonable to assume that the observer uses prior probabilities when it sees a return value not in the equilibrium supports of f_H and f_L. This implies that:

$$\phi_i(r_i) = \beta \qquad for\ r_i \notin S_H \bigcup S_L \tag{2.2.6}$$

Where S_H is the support of f_H and $S_L$is the support of f_L. This is a somewhat extreme assumption, but is not a entirely contrived one. From this, a number of results that narrow down the range of equilibria can be derived.

2.2.4.2. *Equilibrium Restrictions.* In this section, a number of propositions important to determining the form of the equilibria and the key results will be derived using the above assumption on how ϕ_i is determined off-equilibrium path. Each one places restrictions on the form of the equilibria, and will be needed to prove some of they key results of this paper and to describe the equilibria of the model. Following this, we will use some of these results to calculate P_i, which is needed to determine some features of the equilibria.

PROPOSITION 2.1. *In equilibrium, no funds play any mass in the interval $r_i \in (0,\, r_f)$.*

PROOF. The proof is almost identical to that of a similar result in chapter 1. If a fund type is playing mass at a point $r' \in (0,\, r_f)$ in equilibrium, it can make a mean preserving deviation by moving some mass from r' upwards to a $r^* \geq r_f$, while simultaneously moving some mass from r' downwards to $r_i = 0$. $\phi_i(r^*) \geq \beta$ can be obtained due to 2.2.6 by picking r^* to be either off equilibrium path or at point where the H type is playing more density than the L type. The net effect of this deviation on $E(P_i(r_i))$ is unambiguously positive, since $P_i(r') = P_i(0) = 0$ due to firm failure, and $P_i(r^*) > 0$ since drawing r^* will always win the reputational benefit if all other funds draw r'. The mean preserving feature of the mass movement means that it can be done even if the type constraints 2.2.3 and 2.2.4 are binding, and does not affect $E(r_i)$, which implies it will definitely increase utility. Thus, at least one of the fund types will find it profitable to make this unilateral deviation and any f_H and f_L that has probability density in $r_i \in (0,\, r_f)$ cannot be played in equilibrium. $\square$

Crucially, note the logic used to prove proposition 2.1 does not apply to probability mass at $r_i = 0$, since moving probability mass from that point only in a mean preserving way is not possible because mass cannot be moved downwards from $r_i = 0$ due to limited liability. This will turn out to be important later on. The second proposition eliminates equilibria where f_H is a single point mass at a return level r^* and f_L is a single point point mass also at r^*, with $r^* \leq r_L$ to meet the type constraints 2.2.3 and 2.2.4. If such a equilibrium existed, it would make some of the later proofs much more difficult or impossible to execute, since the existence of equilibria where the union of the supports of f_H and f_L consists of a single value r_i invalidates arguments involving mass shifting while staying on equilibrium path.

Proposition 2.2. *$f_H = \delta(r_i - r^*)$ and $f_L = \delta_i(r - r^*)$ cannot be an equilibrium, where δ is the delta function. Therefore the union of the supports of f_H and f_L must contain at least two different points.*

Proof. There are two cases to consider, $r^* \geq r_f$ and $r^* = 0$. All cases with $r^* \in (0,\, r_f)$ are eliminated by proposition 2.1.

Case 1. Consider the case when $r^* \geq r_f$. If $f_H = \delta(r_i - r^*)$ and $f_L = \delta_i(r - r^*)$, then there is only one mass atom with weight 1 at r^* for both f_H and f_L. Thus we need to use equation 2.2.2 to calculate $\phi_i(r^*)$, which gives $\phi_i(r^*) = \beta$. Note this is equivalent to $\phi_i(r_i)$ when $r_i \geq r_f$ from assumption 2.2.6 on how ϕ_i is evaluated off equilibrium path. Also, recall that given $r_H > r_L$, it must the case that $r^* \leq r_L$, otherwise 2.2.4 is violated. Therefore, the type/ability constraint is not binding for at least the H type. It follows that given $\phi_i(r_i) = \beta$ away from r^*, at least the H type can move mass from r^* upwards, which does not affect $E(P_i(r_i))$ but increases $E(r_i)$, which unambiguously increases utility. Thus at least the H type has an incentive to unilaterally deviate, which means that $f_H = \delta(r_i - r^*)$ and $f_L = \delta_i(r - r^*)$ cannot be an equilibrium.

Case 2. Consider the case where $r^* = 0$. We do not need to calculate ϕ_i here, since $r^* = 0$ implies that both types will always fail, i.e. $P_i(r^*) = 0$. The type constraints are not binding for either type, and so both types can move mass upwards to any $r' > 0$ and the effect on $E(P_i(r_i))$ will be zero or positive, while $E(r_i)$ rises. The same logic that prevents this from being an equilibrium from the previous case carries through.

□

This eliminates equilibria where r_i is completely uninformative about the type of the fund. Such combinations of return distributions must mean that the type constraints are not binding, and so at least one of the fund types can just shift mass upwards to increase their expected return without harming their expected chances of winning the reputational reward, since r_i is uninformative about the type. The next proposition eliminates equilibria where the L type plays mass in areas where the H type does not.

PROPOSITION 2.3. *Any return distributions where there is a $r' \geq r_f$ with $f_L(r') > 0$ and $f_H(r') = 0$ cannot be an equilibrium.*

PROOF. If there exists a $r' \geq r_f$ in equilibrium where $f_L(r') > 0$ and $f_H(r') = 0$ then $\phi_i(r') = 0$. If $r^* \geq r_f$ with $\phi_i(r^*) > 0$ does not exist, $f_H = \delta(r_i)$ to satisfy proposition 2.1, i.e. the H type plays all its probability mass at $r_i = 0$. In that case, apply the same logic as in case 2 of proposition 2.2 to show that the H type can always move probability mass upwards from $r_i = 0$ and be strictly better off. Therefore, there must be a $r^* \geq r_f$ where $f_H \neq 0$ and thus $\phi_i(r^*) > 0$. It follows that $P_i(r^*) > P_i(r') = 0$, since a fund drawing r^* will always be ranked as more likely to be a high ability fund than a fund that draws r', even if $\phi_i(r^*)$ is lower than ϕ_i for all other r_i but r'. It is then always possible for the L type to make a mean preserving mass movement where some mass is moved to r^*, i.e. where the H type is actually playing mass. The movement of mass to r^* will increase $E(P_i(r_i))$ since $P_i(r^*) > P_i(r')$, and any counterbalancing mass movement will at worst be made to a r_- where $P_i(r_-) = 0$, which has no effect on $E(P_i(r_i))$ since $P_i(r') = P_i(r_-) = 0$. Thus such a deviation will unambiguously raise $E(P_i(r_i))$, and its mean preserving nature implies that it can always be made even if the type/ability constraint 2.2.4 is binding, and will also not affect the $E(r_i)$ component of utility. Consequently, such a deviation is always welfare increasing, and implies that the L type will always have a unilateral incentive to deviate if there exists a r' in equilibrium where $f_L(r') > f_H(r') = 0$, meaning that such a case cannot be an equilibrium. $\square$

This is related to the fact that moving mass is costless. When this is true, the H and L types cannot be fully separated, since the L type can costlessly move some probability mass (using a mean preserving mass movement) to regions where the H type is playing mass and increase their chance of gaining the reputational reward. Importantly, proposition 2.3 implies that in $[r_f, \infty)$, either no type plays mass, both types play mass, or just the H type plays mass. There can never be a region where only the L type is playing mass.

Finding solutions for the model is still somewhat problematic. Unlike in the previous chapter, the form of $P_i(r_i)$ cannot be immediately deduced from the assumptions of the model. For any given r_i drawn by a fund, the probability of winning the reputational reward is the chance that all other $N-1$ funds draw returns that with lower values of ϕ_i. Therefore the form of P_i is dependent on how ϕ_i varies over r_i, which in turn depends on the ratio of f_H to f_L at any given point. It is necessary to place some restrictions on the form of P_i to get an idea of what the equilibria of this model look like. This is what the next proposition does, and it is vital to the key results of the chapter.

PROPOSITION 2.4. $P_i(r_i)$ *must be a strictly increasing function of* r_i *over* $S_H \bigcup S_L$ *the union of the supports of* f_H *and* f_L.

PROOF. See appendix 2. □

This is a key result for simplifying equilibrium analysis. It immediately follows that if the probability of winning the reputational reward is strictly increasing, it must be based on the Nth order statistic (i.e. highest r_i draw). This is the reason why many of the results from the previous chapter carry over to the fully rational version of the model. The reason this proposition must hold is because if P_i is non-monotonic, then there will be areas where P_i will either be at a local maximum or minimum. Consider the case of a local maximum, which must be surrounded by regions where P_i is lower. If that is the case, it will always be possible to make a profitable mean preserving deviation that moves probability mass from above and below the local maximum towards from the local maximum of P_i. In the case of local minima, just reverse the direction of these mass movements.

Recall that the funds are ranked by the observer in order of the ϕ_i values that result from their return draws. If P_i is strictly increasing over $S_H \bigcup S_L$, the union of the supports of f_H and f_L, then ϕ_i must also be increasing over $S_H \bigcup S_L$. This imposes a strong restriction over the form of f_H and f_L:

COROLLARY. *f_H and f_L must have a monotone likelihood ratio in r_i over $S_H \bigcup S_L$ at all points other than $r_i = 0$, i.e.*

$$\frac{f_H(r')}{f_L(r')} \geq \frac{f_H(r)}{f_L(r)}$$

where $r' > r$, $r' \in S_H \bigcup S_L$ and $r \in S_H \bigcup S_L$, and $r > 0$.

This naturally follows from the requirement that $\phi_i = \frac{\beta f_H(r_i)}{\beta f_H(r_i)+(1-\beta)f_L(r_i)}$ be increasing for $r_i \in S_H \bigcup S_L$. The likelihood ratio does not need to be strictly increasing over $S_H \bigcup S_L$ since the tie-break procedure will automatically select higher draws of r_i when there are ties for the highest ϕ_i, ensuring that $P_i(r_i)$ is still strictly increasing over $S_H \bigcup S_L$ even if the likelihood ratio is only weakly increasing. There is an exception for $r_i = 0$, since fund failure implies that $P_i(0) = 0$ regardless of what $\phi_i(0)$ is there . We can use this to calculate P_i, which is needed to obtain the equilibrium conditions. This will be done later.

PROPOSITION 2.5. *In all equilibria, there must be at least one type playing a mass atom at $r_i = 0$.*

PROOF. See appendix 2. □

This proposition is also vital for the results of the paper. It ensures that every fund has a chance to fail in expectation and that there will always be tail risk in equilibrium. It is due to similar reasons for the analogous result in the previous chapter. Essentially, there must be a mass atom at zero to ensure that the next highest point in $S_H \bigcup S_L$ has $P_i > 0$, otherwise a mean preserving deviation that moves mass down to zero from the next highest point in $S_H \bigcup S_L$ while moving some other mass upwards will always be profitable. This requires that the lowest point in $S_H \bigcup S_L$ be a mass atom, and the possibility that this is above r_f is excluded using the assumption of $\phi_i = \beta$ off equilibrium path and propositions 2.3 and 2.4. Proposition 2.4 requires $\frac{f_H}{f_L}$ to be increasing, which causes a contradiction with proposition 2.3 unless $\phi_i < \beta$ at the lowest point in $S_H \bigcup S_L$. This is however itself not an equilibrium due to incentives to move mass from the lowest point in $S_H \bigcup S_L$ off equilibrium path.

PROPOSITION 2.6. *The type/ability constraints 2.2.3 and 2.2.4 must be binding for both types in equilibrium.*

PROOF. See appendix 2. □

This is somewhat intuitive, since P_i is strictly increasing, so if the type constraints are not binding, then a fund can always move mass upwards to increase the mean return as well as $E(P_i)$. However, we need the assumption that ϕ_i is constant off equilibrium path to prove this in the special case where a fund type plays mass at only a single point.

PROPOSITION 2.7. *When $S_H \bigcap S_L$ includes more than one point,$P_i(r_i) = Zr_i$, with $Z > 0$ for all $r_i \in S_H \bigcup S_L$.*

PROOF. See appendix 2. □

This is analogous to the result that allows the form of the equilibria to be calculated in chapter 1, and is simply an extension of the result to allow for the fact that there can be a finite number of points in $S_H \bigcup S_L$. The logic is very similar, and enshrines the fact that all equilibria must be robust to mean preserving mass movements, since proposition 2.6 must hold. It will determine the shape of the return distributions in equilibrium.

PROPOSITION 2.8. *If there is a mass atom played by either fund at a point r^*, then the points $r^* - \epsilon$ and $r^* + \epsilon$ where $\epsilon \to 0$ must be off equilibrium path, i.e. $f_H = f_L = 0$ there.*

PROOF. See appendix 2. □

So, any mass atoms must be separated from the rest of $S_H \bigcup S_L$ by regions that are off equilibrium path where $f_H = 0$ and $f_L = 0$. Now, we can use some of these propositions to calculate P_i.

2.2.4.3. *Calculating* $P_i(r_i)$. Proposition 2.4 and its corollary force P_i to be strictly increasing in r_i when $r_i \in S_H \bigcup S_L$. If we ignore the case when several funds tie for the highest return draw, given a return draw r_i, a fund i will win the reputational reward if all its $N-1$ competitors draw returns lower than r_i, since those will be either be less likely to be H types due to increasing ϕ_i, or will be selected over other return draws with the same ϕ_i by the tie-breaking mechanism of choosing the highest return fund. Therefore, when there is no chance of tie for top return, we can write for $r_i \in S_H \bigcup S_L$ in symmetric equilibrium:

$$P_i(r_i) = (\beta F_H(r_i) + (1-\beta)F_L(r_i))^{N-1} \tag{2.2.7}$$

Where F_H and F_L are the cdfs of the H and L types respectively. The procedure for obtaining this is the same as used in the previous chapter, and the expression obtained is identical. It can be rationalised by observing that the probability of one particular fund drawing less than r_i is $\beta F_H(r_i) + (1-\beta)F_L(r_i)$, and that type determination and return draw is independent for every fund. Thus, the probability that the $N-1$ competitors all draw worse than r_i is simply the product of $N-1$ funds all individually drawing lower than r_i. However, the expression for P_i is not complete: it is not possible to exclude return distributions with probability mass atoms in the model with the rational observer. When there are mass atoms, there is the possibility that several funds tie for the highest return. In this case, the reputational reward is randomly given to one of the top funds (risk neutrality implies that this is equivalent to the reputational gain being split evenly between them).

Assume that there is a mass atom at return level $r_a \geq r_f$ so that $P_i(r_a)$ is not trivially zero due to firm failure, and that no fund has deviated from the equilibrium path. Let the probability of the H type drawing r_a be a_H and the probability of the L type drawing r_a be a_L. Consider the probability of fund i winning the reputational reward by obtaining the highest ϕ_i when it draws return r_a. Clearly, we don't need to consider a scenario where at least one other fund draws more than r_a due to proposition 2.4. So, there are two scenarios to consider: one where every other fund draws less than r_a, and a second where all the other funds draw r_a or less, with at least one other fund drawing r_a. The latter scenario forces a random tiebreaker between the firms that drew r_a.

Consider the first scenario. All other $N-1$ funds must draw less than r_a We need to obtain the probability of a fund drawing less than r_a before type determination. Denote this as $u(r_a)$. This is:

$$u(r_a) = \beta(F_H(r_a) - a_H) + (1-\beta)(F_L(r_a) - a_L) \tag{2.2.8}$$

$F_H(r_a)$ is the probability that the H type draws less than or equal to r_a, so we need to subtract a_H, the probability of drawing r_a, from this to get the probability that they draw strictly less than r_a. Similar logic applies to the L type's chances of drawing less than r_a. The type and return draw of each fund is determined independently, so we can write the probability of the first scenario happening as

$$(u(r_a))^{N-1}$$

Now, consider the second case. Assume that W other funds draw r_a, while the rest of the other funds ($N-1-W$ of them) draw less than r_a. Using the fact that the types and draws are independently determined again, the probability of this happening must be

$$\frac{(N-1)!}{(N-1-W)!(W!)}(\beta a_H + (1-\beta)a_L)^W (u(r_a))^{N-1-W}$$

To further simplify the expression, we can define $\overline{a} = \beta a_H + (1-\beta)a_L$, which is the expected probability of drawing r_a before type determination, so that the above expression can be written as

$$\frac{(N-1)!}{(N-1-W)!(W!)}\overline{a}^{W}(u(r_a))^{N-1-W}$$

$(\beta a_H + (1-\beta)a_L)^W (u(r_a))^{N-1-W}$ only gives the probability that the first W funds draw r_a. Since the order of the funds that draw r_a does not matter, it is functionally identical to any other combination of fund drawing r_a. Thus, we need to adjust the probability by the binomial coefficient to account for the number of combinations of W out of $N-1$ funds drawing r_a.

In the first scenario, the probability of fund i winning with a draw of r_a is one. In the second scenario, it is $\frac{1}{W+1}$ due to the random tiebreaker between those that draw r_a - fund i draws it, and W of the other funds also do it. To calculate $P_i(r_a)$, we need to sum over W, the number of funds that draw r_a. This gives

$$P_i(r_a) = \sum_{W=0}^{W=N-1}(\frac{1}{W+1})\frac{(N-1)!}{(N-1-W)!(W)!}\overline{a}^{W}(u(r_a))^{N-1-W} \tag{2.2.9}$$

This can be written in a more condensed form by factorising:

$$P_i(r_a) = \frac{1}{\overline{a}N}\sum_{W=0}^{W=N-1}\frac{N((N-1)!)}{(N-1-W)!(W+1)!}\overline{a}^{W+1}(u(r_a))^{N-1-W}$$

Noting that $N((N-1)!) = N!$ and relabelling using $W+1 = K$ gives

$$P_i(r_a) = \frac{1}{\overline{a}N}\sum_{K=1}^{K=N}\frac{N!}{(N-K)!(K)!}\overline{a}^{K}(u(r_a))^{N-K}$$

Now, the binomial theorem states that $\sum_{K=0}^{K=N}\frac{N!}{(N-K)!(K)!}\overline{a}^{K}(u(r_a))^{N-K} = (\overline{a}+u(r_a))^N$, and so we can write

$$P_i(r_a) = \frac{1}{\overline{a}N}\left[(\sum_{K=0}^{K=N}\frac{N!}{(N-K)!(K)!}\overline{a}^{K}(u(r_a))^{N-K}) - (u(r_a))^N\right]$$

which gives as a final result for mass atoms on equilibrium path in symmetric equilibrium:

$$P_i(r_a) = \frac{1}{\overline{a}N}\left((\overline{a} + u(r_a))^N - (u(r_a))^N\right) \tag{2.2.10}$$

for return values r_a where there are mass atoms. This is a general expression that holds when both types play mass atoms at r_a, or just when one type plays mass atoms there. If one type is not playing mass atoms at r_a, simply set the appropriate a_H or a_L to zero. Note that as expected, this is finitely greater than the probability that all funds draw less than r_a, since that is $W = 0$ term in the sum $\sum_{W=0}^{W=N-1}(\frac{1}{W+1})\frac{(N-1)!}{(N-1-W)!(W)!}\overline{a}^W(u(r_a))^{N-1-W}$, but finitely less than the probability that every other fund draws r_a or less, which is given by the sum $\sum_{W=0}^{W=N-1}\frac{(N-1)!}{(N-1-W)!(W)!}\overline{a}^W(u(r_a))^{N-1-W}$.

Finally, we need to consider what P_i is off equilibrium path, assuming other funds stay on equilibrium path. From the assumption on how ϕ_i is calculated off equilibrium path, $\phi_i = \beta$ when $r_i \notin S_H \bigcup S_L$. Let r_o be the highest value of $r_i \in S_H \bigcup S_L$ for which $\phi_i(r_o) \leq \beta$. If $\phi_i > \beta$ for all $S_H \bigcup S_L$, then by proposition 2.5, $r_o = 0$ since there must be at least one fund type playing mass at zero, and fund failure means that drawing zero leaves firms unable to get the reputational reward anyway. Firstly, consider when $r_o = 0$. If any fund draws $r' \neq 0$ and $r' \in S_H \bigcup S_L$, then due to proposition 2.1, a fund drawing $r_i \notin S_H \bigcup S_L$ will not win since if $r_o = 0$, then $\phi_i(r_i) = \beta < \phi_i(r')$. Then, $r_i \notin S_H \bigcup S_L$ will only win the reputational reward if all other funds draw zero. So, for $r_i \notin S_H \bigcup S_L$, $P_i(r_i) = (\beta F_H(0) + (1-\beta)F_L(0))^{N-1}$.

When $r_o \neq 0$, then by proposition 2.1, $r_o \in [r_f,\ \infty)$. In this case, if r_o is at a point where there are no probability mass atoms, then the probability of a random tiebreaker if $r_i \notin S_H \bigcup S_L$ is zero. The draw r_i when off equilibrium path will win if all other funds draw less than it. So, in that case $P_i(r_i) = (\beta F_H(r_o) + (1-\beta)F_L(r_o))^{N-1}$ when $r_i \notin S_H \bigcup S_L$.

If r_o is at a point where there is a mass atom, then drawing $r_i \notin S_H \bigcup S_L$ gives an identical ϕ_i to actually drawing from the mass atom at r_o, and whether it wins or not depends on $r_i > r_o$ or not. It will win if against firms that draw r_o if $r_i > r_o$, and lose if $r_i < r_o$ ($r_i = r_o$ is impossible since r_i is off equilibrium path). Effectively, when $r_o > r_i$, the off equilibrium path draw will only win if all other funds draw returns in $S_H \bigcup S_L$ less than r_o, and if $r_i > r_o$, it will win if all other funds draw returns from $S_H \bigcup S_L$ that are equal to or less than r_o.

To summarise, P_i is a complex function that takes different forms in different regions. In $S_H \bigcup S_L$, the union of the supports of the H and L types, it takes the following form:

$$P_i(r_i) \begin{cases} (\beta F_H(r_i) + (1-\beta)F_L(r_i))^{N-1} & no\ atom\ at\ r_i \\ \frac{1}{\overline{a}N}\left((\overline{a} + u(r_i))^N - (u(r_i))^N\right) & atom\ at\ r_i \end{cases}$$

With the size of the mass atom at r_i being a_H for the H type and $a_L$for the L type, $\overline{a} = \beta a_H + (1-\beta)a_L$ and $u(r_a)$ defined by 2.2.8. Off equilibrium path (not in the support of the return distribution of either type), i.e. $r_i \notin S_H \bigcup S_L$, it takes the following form:

$$P_i(r_i) \begin{cases} (\beta F_H(r_o) + (1-\beta)F_L(r_o))^{N-1} & no\ atom\ at\ r_o \\ (\beta F_H(r_o) + (1-\beta)F_L(r_o))^{N-1} & atom\ at\ r_o\ and\ r_i > r_o \\ (\beta[F_H(r_o) - a_H] + (1-\beta)[F_L(r_o) - a_L])^{N-1} & atom\ at\ r_o\ and\ r_i < r_o \end{cases}$$

With r_o being the maximum value of r_i in $S_H \bigcup S_L$ for which $\phi_i(r_o) \le \beta$ or $r_o = 0$ if such a value does not exist. The size of the mass atom at r_o being a_H for the H type and $a_L$for the L type and other symbols are defined as before. Of course,

$$P_i(r_i) = 0 \quad for\ r_i \in [0,\ r_f)$$

due to firm failure when $r_i < r_f$.

There are an infinity of equilibria. Unfortunately, unlike when the reputational reward is automatically obtained by the fund with the highest return, the common properties of these equilibria cannot be described succinctly. Instead, some equilibria types important to the results shall be characterised and discussed.

2.3. Results and Discussion

2.3.1. Equilibria Shared with the Model from Chapter 1. There are many equilibria in this variant of the model that are not equilibria in the non-rational version and vice-versa, mainly due to the lack of a proposition that excludes the existence of equilibria with mass atoms at or above the failure threshold r_f. This allows there to be a whole range of equilibria that involve mass atoms in the interval $[r_f, \infty)$ that would not have been equilibria when the winner of the reputational benefit was decided solely by the highest return draw. However, some of the equilibria from the non-rational model still hold, and therefore the results from that model apply to these equilibria exactly. To see this, we need to recall some of the properties of the equilibria in the non-rational version of the model. In the non-rational model, all equilibria have return distributions with a mass atom in expectation at $r_i = 0$, and have continuous cumulative distribution functions in the range $[r_f, \infty)$. In that model, P_i takes the same form as it does here on equilibrium path (when $r_i \in S_H \bigcup S_L$). The similarity is forced by the fact that P_i must be increasing in r_i on equilibrium path (proposition 2.4). Proposition 2.5 means that like in the non-rational case of the model, every fund plays mass at $r_i = 0$ in expectation before type determination. Proposition 2.6 also applies in the non-rational model, making the type constraints binding, and proposition 2.7 is a version of the result that forces P_i to be linear in the non-rational version of the model generalised to a scenario where there may be mass atoms. Analogous or identical results (though requiring different proofs) that restrict the form of the equilibria apply in both models, and so it is apparent that finding equilibria that have mass atoms in expectation at $r_i = 0$ and have continuous cdfs elsewhere will give at least some of the same equilibria. There are some restrictive conditions from the mechanics of the non-rational model that remove some of the equilibria from the non-rational model, however. The next step is to describe the equilibria common to both the rational and non-rational model.

As stated, the equilibria both models share have continuous cdfs everywhere but at $r_i = 0$. By proposition 2.6, both type/ability constraints must be binding, and by proposition 2.5, at least one of the two fund types must be playing a mass atom at $r_i = 0$. Since cdfs of both types are continuous everywhere but at $r_i = 0$, there are no mass atoms for $r_i \geq r_f$, and proposition 2.6 excludes equilibria where a fund type only plays mass at $r_i = 0$. Given both these facts and the assumption that both types can in expectation make more than the failure threshold 2.2.5, both types must have non-zero pdfs in regions where $r_i \geq r_f$, and not just single points there. Let k_H and k_L be the upper bounds of the supports of f_H and f_L respectively, and the let the upper bound of the two supports be $k_u = \max(k_H,\ k_L)$. Assume that $F_H(r_i)$ and $F_L(r_i)$ are the cdfs of the H and L types respectively. Now, note that $k_u = k_H$, since if this is not the case, there will be a region (or point) in $r_i \in [r_f, \infty)$ where $f_L > 0$ and $f_H = 0$, which violates proposition 2.4 - P_i will not be strictly increasing in $r_i$over $S_H \bigcup S_L$. If there are no mass atoms above or at r_f, proposition 2.3 and proposition 2.4 force the H type to be playing mass at every point in the interval $[r_f, k_u]$. Since the L type must be playing mass in some interval in $r_i \in [r_f, \infty)$, this will force $S_H \bigcap S_L$ to contain more than one point. Thus, proposition 2.7 must apply to all of $S_H \bigcup S_L$, forcing P_i to be the same linear function over the supports of both f_H and f_L. We can then write some equations that most hold in symmetric equilibria down in a similar manner to chapter one. Assume that F_H is given by the following:

$$F_H(r_i) \begin{cases} 1 & r_i \in [k_u,\ \infty) \\ \frac{(Zr_i)^{\frac{1}{N-1}}}{\beta} - \frac{1-\beta}{\beta} & r_i \in [k_L,\ k_u] \\ G(r_i) + a_H & r_i \in [r_f,\ k_L] \\ a_H & r_i \in [0,\ r_f) \end{cases}$$

Where $G(r_i)$ is a strictly increasing, continuous function with $G(r_f) = 0$ and $G(k_H) = 1 - a_H$. It is playing a mass atom at $r_i = 0$. Thus, $F_L$must be

$$F_L(r_i)\begin{cases} 1 & r_i \in [k_L,\ \infty) \\ \frac{(Zr_i)^{\frac{1}{N-1}}}{1-\beta} - \frac{\beta}{1-\beta}(G(r_i) + a_H) & r_i \in [r_f,\ k_L] \\ a_L = \frac{(Zr_f)^{\frac{1}{N-1}}}{(1-\beta)} - \frac{\beta}{1-\beta}a_H & r_i \in (0,\ r_f) \\ a_L = \frac{(Zr_f)^{\frac{1}{N-1}}}{(1-\beta)} - \frac{\beta}{1-\beta}a_H & r_i = 0 \end{cases}$$

So that proposition 2.7 holds. We can also formulate an equilibrium where the L type has a more free choice over its equilibrium distribution:

$$F_L(r_i)\begin{cases} 1 & r_i \in [k_L,\ \infty) \\ G(r_i) + a_L & r_i \in [r_f,\ k_L] \\ a_L & r_i \in (0,\ r_f) \\ a_L & r_i - 0 \end{cases}$$

Locking in the following form for F_H to ensure that proposition 2.7 holds:

$$F_H(r_i)\begin{cases} 1 & r_i \in [k_u,\ \infty) \\ \frac{(Zr_i)^{\frac{1}{N-1}}}{\beta} - \frac{1-\beta}{\beta} & r_i \in [k_L,\ k_u] \\ a_H = \frac{(Zr_i)^{\frac{1}{N-1}}}{-\beta} - \frac{1-\beta}{\beta}(G(r_i) + a_L) & r_i \in [r_f,\ k_H] \\ a_H = \frac{(Zr_f)^{\frac{1}{N-1}}}{\beta} - \frac{(1-\beta)}{\beta}a_L & r_i \in [0,\ r_f) \end{cases}$$

These are clearly a special case of the equilibria in the non-rational version of the model, and like in that model, the same equations will determine Z, a_H and a_L. To recap, using these forms for F_L and F_H with the fact that the type/ability constraints 2.2.3 and 2.2.4 are binding gives the following equations regardless of which type is playing the G

function:

$$Z = \frac{1}{k_u}$$

$$\bar{r}N + (\frac{r_f^N}{k_u})^{\frac{1}{N-1}} = k_u$$

$$\beta a_H + (1-\beta)a_L = (Zr_f)^{\frac{1}{N-1}}$$

where $\bar{r} = \beta r_H + (1-\beta)r_L$. The procedure to obtain these is identical to that used in the non-rational version of the model. The result from chapter 1 on fund failure rate before type determination, $T = \beta a_H + (1-\beta)a_L$, applies identically - it can be shown that $\frac{dT}{dN} > 0$ since we assumed that $r_H > r_L > r_f$ 2.2.5. So the results from the previous chapter hold identically in this class of equilibria. There are identical restrictions on the choice of G, a_H and a_L due to the requirement that $f_H \geq 0$ and $f_L \geq 0$ at all points - there cannot be negative mass in a pdf. These state that the function g needs be picked so that $Z\frac{(Zr_i)^{\frac{1}{N-1}-1}}{(N-1)} - (1-\beta)g(r_i) \geq 0$ or $Z\frac{(Zr_i)^{\frac{1}{N-1}-1}}{(N-1)} - \beta g(r_i) \geq 0$ in the second case for all $r_i \in [r_f,\ k_u]$, and that $a_H \in [0,1)$ and $a_L \in [0,1)$. This is related to the fact that Z and k_u are independent of the choice of g, and choosing a g function too steep makes it impossible for comply with proposition 2.7 without needing negative mass.

In addition, there are a number of extra restrictions that apply to this class of equilibria here, so clearly the results are not as strong as they are in the non-rational case - not all the equilibria which the failure rate result applies to in the non-rational model exist in the rational case. The additional restrictions are:

- As stated above, $k_u = k_H$.
- $\phi_i(k_u) \geq \beta$. This is to ensure that there are no incentives to move probability mass off equilibrium path. If this does not hold, then funds will find it profitable to make mean preserving deviations that include a mass movement to $r_i \geq k_u$, i.e. a movement off equilibrium path.

- $\frac{d(\frac{f_H}{f_L})}{dr_i} \geq 0$ for all $r_i \in [r_f,\ k_u]$. This is imposed by proposition 2.4 and its corollary, which says that P_i must be strictly increasing over $S_H \bigcup S_L$, which in turn implies that ϕ_i must be (weakly) increasing over $S_H \bigcup S_L$. Consequently, we get the condition that f_H and f_L must have the monotone likelihood ratio property in r_i, and this can be written in derivative form since f_H and f_L are continuous in $[r_f,\ k_u]$ for these types of equilibria. Writing out the conditions, we get

$$(\frac{Z^{\frac{1}{N-1}} r_i^{\frac{2-N}{N-1}}}{(1-\beta)(N-1)} - \frac{\beta g}{(1-\beta)})g' - (\frac{(2-N)Z^{\frac{1}{N-1}} r_i^{\frac{1}{N-1}-2}}{(1-\beta)(N-1)^2} - \frac{\beta g'}{(1-\beta)})g \geq 0$$

  in the case where the H type picks g. Something similar is obtained when the L type picks g. This imposes a further condition on the shape of g due to the requirement of a monotone likelihood ratio.

The last extra restriction is particular noteworthy, since it imposes strong restrictions in on the shape of the possible return distributions, invalidating a number of equilibria from the non-rational model where the ratio of f_H and f_L did not necessarily increase in the interval $[r_f,\ k_u]$. This is due to proposition 2.4, the requirement that P_i be monotonically increasing in $S_H \bigcup S_L$, which exists due to the ability constraints. In a setting where there is distribution shifting subject to mean constraints and risk neutrality, non-monotonic P_i cannot exist since any equilibria must be robust to a mean preserving mass movement, which requires all points in $S_H \bigcup S_L$ to be on a straight line.

So to summarise, a restricted subset of the equilibria from the non-rational version exist, to which the results on the effect of competition from chapter 1 apply exactly. Again, we need to consider whether these equilibria exist or not. Fortunately, the functional forms for F_H and F_L used to prove propositions 1.9 and 1.10 conform with the three extra conditions stated above: $k_u = k_H$, and the ratio $\frac{f_H(r_i)}{f_L(r_i)} = \frac{r_H}{r_L}$ for $r_i \in [r_f,\ k_u]$, a constant. This satisfies the monotone likelihood ratio property required by proposition

2.4, and also ensures that $\frac{f_H(k_u)}{f_L(k_u)} > 1$ so that $\phi_i(k_u) \geq \beta$ as well. Thus for these continuous equilibria, we can just perform the same proofs to show that propositions 1.9 and 1.10 still hold. Therefore, the same sufficient conditions on parameters for equilibrium existence for $N \geq 2$ (and thus the validity of any discussion on comparative statics) apply for these equilibria too. For a discussion on these parameter restrictions, refer to chapter 1.

2.3.2. Other Equilibria and the Results from Chapter 1. After determining that the results from the non-rational model apply to a subset of the equilibria of the rational model, a relevant question to ask is whether they also apply to the other equilibria in the rational model. A fairly obvious observation is that proposition 2.5 implies that there must always be a mass atom at $r_i = 0$, which implies that there is always be a possibility that a particular fund will fail in expectation, i.e there will always be tail risk played in equilibrium. This is a result that carries over from the previous chapter. However, we need to consider if the result that failure rates increase with competition applies to these other equilibria.

Section 2.3.1 described the equilibria that correspond to the ones present in the non-rational version of the model, and the key feature of these is that return cdfs are continuous in the interval $[r_f,\ k_u]$. The other equilibria in the model all have mass atoms in this interval, which mean that k_u is not just a function of exogenous parameters, but also depends on where the mass atoms in $[r_f,\ k_u]$ are located. Therefore, these other equilibria do not have common values of k_u. Consequently, common values of $T = \beta a_H + (1-\beta)a_L$, the size of the mass atom at $r_i = 0$ that is also the ex-ante fund failure probability, will also not exist. It will still be useful to consider the exact form of these other equilibria, and it is still possible to make some generalisations about their features.

Collecting the conditions that must be satisfied in equilibrium, we can verbally describe these equilibria that have mass atoms above zero:

- To comply with proposition 2.5, at least one of the two fund types must play a mass atom at $r_i = 0$.
- $P_i(r_i) = Zr_i$ with $Z > 0$ for all points in $S_H \bigcup S_L$ if there are at least two points in $S_H \bigcap S_L$. This also applies trivially to when there are two points in $S_H \bigcup S_L$. This is due to proposition 2.7.
- Any $r^* \geq r_f$ where there is a mass atom must be separated from the rest of $S_H \bigcup S_L$ by regions above and below r^* that have $f_H = f_L = 0$, i.e. are not in $S_H \bigcup S_L$.

There are some additional requirements so that there are no incentives for funds to move mass off equilibrium path, often involving conditions that make sure that both types are indifferent to mean-preserving mass movements from $r_i = 0$ up to r_f while moving some other mass from k_u downwards to r_f, which puts some limits on how high k_u can actually be. These equilibria can have both regions where F_H and F_L are continuous, punctuated by mass atoms where both f_H and f_L are delta functions, as long as these are surrounded above and below by regions that are not in $S_H \bigcup S_L$. There is in principle no limit on the number of mass atoms above r_f. This is what makes it very difficult to derive a general result on if fund failure rates increase with competition.

2.3.2.1. *Equilibria where the results from Chapter 1 do not apply.* A place to begin is to find some equilibria in the fully rational model where the results from chapter 1 do not hold. In particular, there is an entire class of equilibria for which fund failure and tail risk are unaffected by the number of competing funds.

Because mass atoms can be played anywhere in this model, it is possible to have equilibria with a only a finite number of points in $S_H \bigcup S_L$. This is done by having both types play only mass atoms at certain values of r_i, while still complying with propositions 2.5, 2.6

and 2.7. So, most importantly, at least one type must play a mass atom at $r_i = 0$, and both type/ability constraints must be binding. A trivial way of doing this is to consider a scenario where $S_H \bigcup S_L$ consists only of two points, $r_i = 0$ and $r_i = r^* \geq r_H$ so it is possible to satisfy proposition 2.6. Proposition 2.5 is satisfied by this, and proposition 2.4 will be too. Given that there are only two points in $S_H \bigcup S_L$, proposition 2.7 is trivially satisfied - if there are only two values of P_i to consider, there will always be a straight line that goes through both. Thus all types are indifferent to mean preserving mass movements where mass is around moved only within $S_H \bigcup S_L$. To satisfy proposition 2.6, there is only one possible f_H and f_L that can be played: if we let the probability of the H and L type drawing r^* be t_H and t_L respectively, $t_H = \frac{r_H}{r^*}$ and $t_L = \frac{r_L}{r^*}$, and the consequently the probability of the H and L types drawing $r_i = 0$ must be $1 - \frac{r_H}{r^*}$ and $1 - \frac{r_L}{r^*}$ respectively. This is the only combination of mass atom probabilities that make the ability/type constraints binding. Since proposition 2.7 is satisfied trivially, the only thing to check is if there is an incentive to move mass to a point off equilibrium path. This will be the case as long as $\frac{P_i(r_f)}{r_f} = \frac{(\beta(1-\frac{r_H}{r^*})+(1-\beta)(1-\frac{r_L}{r^*}))^{N-1}}{r_f} \leq \frac{P_i(r^*)-P_i(r_f)}{r^*-r_f}$ so that no fund type finds it profitable to move mass from r^* down to r_f while moving mass up from zero to r_f.

The key thing to note is that increasing the number of competitors does not increase the probability of fund failure, since the probabilities of the H and L types drawing r^* or zero are locked in by the requirement that the type/ability constraints 2.2.3 and 2.2.4 are binding, and cannot vary as the number of funds increases. The results from chapter 1 do not apply to this class of equilibria.

In addition, there is one other case where the results from the non-rational model do not apply fully. Note that P_i need not be a single linear function over $S_H \bigcup S_L$ when there are only three points in it. This special case is when the three points are $r_i = 0$, $r_i = r_m$ and $r_i = r_t$. The L type plays mass only at zero and r_m, and the H type only plays mass at r_m and r_t. Thus, this requires $r_t > r_m > r_L$, and $r_t > r_H$ for proposition 2.6 to be satisfied. There are some additional requirements for there to be no incentive for there mass to be moved off the equilibrium path, which are that $\phi_i(r_m) > \beta$ and $\frac{P_i(r_f)}{r_f} \leq \frac{P_i(r_t)-P_i(r_f)}{r_t-r_f}$. In a way, this is related to the special case of $S_H \bigcup S_L$ only containing two points, since S_H and S_L both only contain two points each. What makes this equilibrium special is that if there is no incentive to move mass off equilibrium path, then there are only two possible mean-preserving mass movements by either type that change $E(P_i)$. One of these is a movement of some mass from r_m up to r_t, while moving some mass from r_m down to zero for either type. This is never profitable as long as $\frac{P_i(r_t)-P_i(r_m)}{r_t-r_m} \leq \frac{P_i(r_m)}{r_m}$. The other one is the H type moving mass from r_t down to zero, while moving mass from r_m up to r_t, which is never profitable as long as $\frac{P_i(r_t)-P_i(r_m)}{r_t-r_m} \leq \frac{P_i(r_t)}{r_t}$. $\frac{P_i(r_t)-P_i(r_m)}{r_t-r_m} \leq \frac{P_i(r_m)}{r_m}$ is actually sufficient for both mass movements to be unprofitable, since if it holds, $\frac{P_i(r_m)}{r_m} \geq \frac{P_i(r_t)}{r_t}$. Note that proposition 2.7 does not need to hold here, since $S_H \bigcap S_L$ contains only one point. Due to the lack of a binding constraint on P_i over $S_H \bigcup S_L$, although increasing N does change P_i by raising $P_i(r_t) - P_i(r_m)$ relative to $P_i(r_m)$, it may not cause the violation of the inequality required to hold for it to be an equilibrium until N is sufficiently large, and the played distributions will not change. Thus, for some $N < N'$ (N' depending on the parameters), there will be no change in the amount of mass played at $r_i = 0$ and thus no change in the fund failure rate. Once this N' is exceeded, this equilibrium can no longer hold with the parameters r_t and r_m. The fact that S_H and S_L are individually two points each means that both funds are trivially indifferent to mass movements within S_H and S_L when N changes - the two point support will always be on the same straight line in $P_i(r_i)$ and r_i space even when N changes.

2.3.2.2. *Other Equilibria where the results from Chapter 1 apply.* However, the result of increasing fund failure probability with the number of competitors still holds in a lot of equilibria with mass atoms, because the same incentive to move probability mass upwards when competition increases from chapter 1 still exists in them. Consider the case when are at least three separate points, $r_b < r_m < r_t$ in one of S_H or S_L. Assume that the model is in equilibrium. To create indifference to mean preserving mass movements, f_H and f_L will be picked so that $P_i = Zr_i$ over the support of the type that has at least three points in its support. Now, consider increasing the number of competing funds from N to $N+1$. It is clear from 2.2.10 and 2.2.7 that the expression for P_i is decreasing in N for a given f_H and f_L, regardless of whether there is a mass atom at that point or not. This is because there will be more fund returns that the draw must be higher than to win the reputational boost, with lower r_i seeing larger falls in P_i due to the increasing the power index N making P_i more convex - low draws will experience bigger falls in winning probabilities, since increasing N will make it comparatively much more likely that at least one out of the other funds will beat a low draw, compared to when the fund draws high. P_i will no longer be linear, and the model will not be in equilibrium. Due to the convex nature of P_i after the increase in competition, all fund types have incentives to make mean preserving deviations that move mass from r_m the upwards to r_t while moving some mass from r_m downwards to r_b, since the convexity of P_i means that the downwards movement of mass from the middle of the support to the lower end of the support decreases $E(P_i)$ less than the increase from moving mass from the middle to the top.

This mechanism will increase the fund failure rate when competition increases as long as $r_i = 0$ is in the support of a fund type that has at least three points in its support, and therefore applies to a very wide range of return distributions. It automatically applies to any equilibrium where f_H and f_L are continuous in some regions since that implies that both $f_H > 0$ and $f_L > 0$ there due to propositions 2.3 and 2.4, and so both S_H and S_L will have more than three points, and at least one of them must include $r_i = 0$. Consider a scenario when there are at least four points in $S_H \bigcup S_L$. Propositions 2.3 and 2.4 require S_H to contain every point in $S_H \bigcup S_L$ apart from $r_i = 0$. If S_H contained $r_i = 0$, the above mechanism will cause the fund failure rate to rise. In the case when

$r_i = 0$ is not in S_H, as long as S_L has at least three points in it, (this must include $r_i = 0$), increasing competition will cause the L type to move mass down to $r_i = 0$, raising the fund failure rate. Given the incentive to move mass to extreme values as competition rises, no type will move mass away from $r_i = 0$ as competition rises. Thus this mechanism will cause the size of the mass atom at $r_i = 0$ to rise, and thus the fund failure to increase with competition when there are at least four points in $S_H \bigcup S_L$.

There is a caveat to this though. When probability mass is played entirely in the form of atoms at exogenously picked points, then this mechanism only increases fund failure rate up to a point. As the number of competing funds rises, more and more probability mass is moved towards extreme values, and there will be a level of competition where there is no more mass left to move to extreme values. In such a case, both types will be playing distributions where they are playing mass at only two points, at zero and at one other point above r_H, like in the equilibria described in 2.3.2.1. Once this level of competition is reached, increasing competition no longer increases the fund failure rate, since if the points at which they can play mass are exogenously picked, then both fund types are already playing the maximum mass that can be played at zero without violating proposition 2.6. The logic from 2.3.2.1 then applies to explain why increasing competition even further has no effect on failure rates.

2.3.3. Discussion. At some level, the results from chapter 1 still apply. This is due to propositions 2.4 and 2.5. Proposition 2.5 in particular is a direct analogue of a result from the non-rational model, and critically, it implies that in expectation (before type determination), all funds play probability mass at zero. This guarantees that $r_i = 0$ is always in the support of at least one of the fund types, and therefore some of the implications for fund failure and financial stability still hold. Proposition 2.4 is also vital, since it forces P_i to be monotonically increasing in r_i, which together with the reputational reward only being given to the fund that has the strongest return relative to its competitors, causes P_i to be related to the Nth order statistic when there are N competing funds. This means that the mechanism driving the results in the previous chapter, which is detailed in section 2.3.2.2., is also in effect here. Like in the non-rational model, the order statistic nature of P_i means that increasing N increases its convexity, giving an incentive to move probability mass to high return levels while moving mass downwards to ensure that the played distribution still satisfies the ability/type constraint. This can be interpreted as it being increasingly difficult to stand out with the highest return and appear to be the most likely to be a high ability type as there are more funds in the market. The fact that P_i is increasing in r_i forces a monotone likelihood ratio property on f_H and f_L, and makes the model superficially resemble the non-rational version.

Proposition 2.5 is true for the same reasons in the non-rational case - the assumption of fund failure together with a degree of freedom to move probability mass. Proposition 2.4 is due to the ability constraints together with the ability to move probability mass around. Effectively, non-monotonic P_i cannot exist in the environment of return distribution picking subject to mean constraints, since mean preserving mass movements where mass is moved from above and below local peaks in P_i towards said peaks would always exist if P_i were non-monotonic, with the opposite applying for local minima in P_i. This will still be the case as long as there is some restriction on $\int g(r)f_H(r)dr$ and $\int g(r)f_L(r)dr$, where $g(r)$ is some increasing function of r, so this is somewhat robust to functional form assumptions.

However, the importance of how ϕ_i is calculated off equilibrium path, as stated in section 2.4.1, is somewhat troubling. It is required to prove proposition 2.5 in particular, and is require to show that P_i must be monotonically increasing rather than decreasing in r_i, although an assumption that ϕ_i is merely constant off equilibrium path suffices to some extent for proposition 2.4. Without this assumption, proposition 2.5 may not hold and a mass atom at $r_i = 0$ need not be in every equilibrium. It is however difficult to propose an alternative, more rational equilibrium refinement in this setting, and may be something that can be improved on in future work.

The results from the previous chapter are undoubtedly weakened here, since they only apply to a restricted subset of the equilibria. Changing the model to be something closer to a signalling model where the observer rationally deduces which fund is most likely to be of a high ability type means that equilibria with mass atoms above r_f are no longer excluded. Allowing equilibria with mass atoms above r_f means that the range (in particular the maximum return value) of the supports of equilibrium return distributions can be arbitrarily determined. Contrast this to how they were always endogenously determined by the parameters in the non-rational model. This is a major reason why increased competition does not necessarily increase fund failure and tail risk in this model. In the extreme case where the entirety of $S_H \bigcup S_L$ consists of a finite number of points, (incidentally the strongest exceptions to the results from chapter 1 all fall into this category), the supports of the return distributions are determined entirely exogenously. Raising competition has less of an effect in cases like this, since the maximum value in $S_H \bigcup S_L$, k_u, is exogenously fixed. In the non-rational version of the model and in the continuous equilibria of this model, the rise in k_u with competition means that more mass is played at higher return levels, and so to preserve the mean, more mass must also be played at zero. This mechanism does not operate in some scenarios with mass atoms, where return supports are exogenously picked. This is most clearly seen in the scenario where there are just two points in $S_H \bigcup S_L$, and the return distributions are decided by needing the type/ability constraints to be binding. The exogenous support there forces the played distributions to take a certain form that is invariant of the number of competitors.

An interesting feature to note is how some of the equilibria in this model (in particular the two point equilibrium of section 2.3.2.1) strongly resemble the story laid out by Foster and Young (2010), much more so than the non-rational version of the model. In their setting, low ability managers essentially write disaster insurance using a put option on an unlikely market event and obtain a return that is high and stable most of the time, but occasionally causes the total loss of the investment. This corresponds exactly to a two-point equilibrium where the L type plays mass at zero and r_H, while the H type plays mass only at r_H. There is no possibility of fund types playing distributions with mass in different regions, which is possible in the non-rational version. Why this happens is an interesting question to ask, and almost certainly has to do with the fact that this incarnation of the model is more like a Bayesian signalling game, which Foster and Young's story has more in common with. The non-rational version of the model is more like a Blotto Game, where different fund types try to spread out their probability mass and can to some extent play return distributions in which there are regions where one type is playing mass, but the other is not. In a signalling game with costless return distribution manipulation, equilibria where the L type is playing mass but the H type does not cannot exist, since the L type can always move some of that mass to where $\phi_i > 0$. The equilibria of the rational model have some characteristics of the non-rational version due to P_i being identical on equilibrium path and the presence of the same mean/ability constraints, while still having some of the constraints you would expect from a Bayesian signalling equilibrium imposed. The continuous equilibria detailed in section 2.3.1 are most like the Blotto Game ones, and exist in both models. Equilibria that are entirely mass atoms are the most unlike the Blotto Game type equilibria, and are most like Bayesian signalling equilibria, and are unsurprisingly the ones most similar to the scenario detailed in Foster and Young.

2.4. Conclusions

In this paper, a possible criticism of chapter 1 was addressed. Hedge funds in reality service sophisticated investors. This makes the assumption that new funds gain a boost to their reputation by posting high returns seem somewhat naive. To address this, a version of the model from chapter 1 with the reputational reward falling to the fund that is evaluated as having the highest Bayesian probability of being a high ability type is analysed, which addresses criticism of naive behaviour to some extent.

Although the model is different in essence from the case when all funds are racing for the highest return, a number of results can be derived that make the rational version of the model behave in a superficially similar way. The probability of winning of the reputational boost is decided in an identical way to chapter 1, by a function related to the highest order statistic. There are however substantial differences in the restrictions on the equilibria allowed. Crucially, in the fully rational model, equilibria with mass atoms at arbitrary return levels are allowed. This means that there are a much larger variety of equilibria with different features, making it much more difficult to derive general results, unlike in the non-rational model.

The main finding of this analysis is that some of the key results from the previous chapter carry through in a weakened form, with some reliance on how fund type probabilities are calculated off equilibrium path. In expectation, all funds play distributions where they can draw a return of zero and fail, a result that is shared in common with the non-rational model. The result that increasing the number of competing funds raises the expected amount of probability mass played at zero and thus the fund failure rate holds in a substantially weaker form. This is in particular due to a class of equilibria where there are no mass atoms above the failure threshold existing in both the rational and non-rational versions of this model. The mechanism of competition raising the convexity of the probability of winning function operates in the rational model too, making an increase in competition still raise the probability of fund failure in a considerable number of cases up to a limit. This mechanism gives funds an incentive to move mass to extreme values as competition increases, but there will be a level of competition where mass is only being played at extreme values, meaning that there will be no mass left to move from intermediate values. Then, the return distributions become two point, at which point raising competition has no effect any more. This is the main exception to the results from chapter 1. When funds play mass at two points only, at zero and some other exogenously defined point, there will only one be a single way of making the ability constraint binding. Thus, such equilibria will be invariant to the number of competing funds. This invariance is also related to the fact that the supports of such equilibria are exogenously determined: the upper bound of the supports of the return distributions does not rise with competition, which would have raised the amount of mass played at zero.

These equilibria bear some resemblance to the return mimicking scenario described in Foster and Young (2010), and this is not really surprising because the model appears more like a signalling game than the non-rational version does, and these two point equilibria are the most signalling-like equilibria. The equilibria where the results from chapter 1 still apply have signalling elements in some of the restrictions on distribution shapes (monotone likelihood ratios are required), but retain a lot of the Blotto-game like character of the equilibria in the non-rational model, where funds try to spread out their probability mass and can play distributions that are differentiated from each other.

Appendix 2

Proof of Proposition 2.4. The strategy to proving this has several steps. Firstly, prove that P_i must be monotonic over the supports of both types. Then, show that ϕ_i cannot be a decreasing function of r_i. These two steps will show that P_i must be an increasing function of r_i over $S_H \bigcup S_L$. Then, it can be shown that can be no interval $[r', r'']$ in $S_H \bigcup S_L$ where P_i is constant, which will force P_i to be strictly increasing over $S_H \bigcup S_L$.

LEMMA. *P_i must be monotonic over the support of the H type and the L type.*

PROOF. Consider a case where P_i is non-monotonic over the supports of f_H and f_L. For P_i to be non-monotonic over the supports of both f_H and f_L, there must be at least 3 points in the supports of both types, which holds true even if both f_H and f_L consist entirely of probability mass atoms. Let these 3 points be labelled d_l, d_m and d_u, with $d_l < d_m < d_u$. Given that P_i is non-monotonic, it is always possible to pick d_l, d_m and d_h such that either

(1) $P_i(d_l) < P_i(d_m),\ P_i(d_m) > P_i(d_u)$
(2) $P_i(d_l) > P_i(d_m),\ P_i(d_m) < P_i(d_l)$

It can be shown that in both cases, it is possible for at least one of the fund types to make a profitable mean preserving mass movement. A mean preserving mass movement will always be possible to execute regardless of whether the ability/type constraints (2.2.3 and 2.2.4) are slack or binding. In addition, they will not affect the expected return component of the funds' objective function. So, if a mean preserving deviation unambiguously increases $E(P_i(r_i))$, then it will also unambiguously increase utility. Thus, if a profitable mean preserving mass movement by one of the types is possible under some conditions, then a combination of return distributions where these conditions can never be an equilibrium.

Case 1. In case 1, the L type must be playing probability mass at d_l and d_u, otherwise $\phi_i = 1$ at both aforementioned points, and $P_i(d_l) < P_i(d_m)$ is impossible. Consider moving mass ϵ upwards from d_l to d_m. To preserve the mean, mass δ can be moved down from d_u to d_m. The overall effect on $E(P_i(r_i))$, ΔP is

$$\Delta P = \epsilon(P_i(d_m) - P_i(d_l)) + \delta(P_i(d_m) - P_i(d_u))$$

From the conditions on this case $P_i(d_l) < P_i(d_m),\ P_i(d_m) > P_i(d_u)$, it is clear that $\Delta P > 0$. To preserve the mean, we can adjust the relative size of ϵ and

δ, imposing the following the condition:

$$\delta = \epsilon \frac{(d_m - d_l)}{(d_u - d_m)}$$

Given that the L type must be playing mass at d_l and d_u, at least the L type can always make this deviation, and so case 1 cannot be an equilibrium.

Case 2. In case 2, the L type must be playing mass at d_m, otherwise $\phi_i(d_m) = 1$ and $P_i(d_m) < P_i(d_l)$ is impossible. Consider moving mass ϵ downwards from d_m to d_l. To preserve the mean, simultaneously move mass δ upwards from d_m to d_u. The effect on $E(P_i(r_i))$, ΔP is:

$$\Delta P = \epsilon(P_i(d_l) - P_i(d_m)) + \delta(P_i(d_u) - P_i(d_m))$$

From the conditions on this case $P_i(d_l) > P_i(d_m),\ P_i(d_m) < P_i(d_l)$, it must be the case that $\Delta P > 0$. To preserve the mean, again adjust the relative sizes of δ and ϵ according to the following condition:

$$\delta = \epsilon \frac{(d_m - d_l)}{(d_u - d_m)}$$

Given that the L type must be playing mass at d_m, at least the L type can always make this deviation, and so case 2 cannot be an equilibrium either.

It follows that any case where P_i is non-monotonic in r_i cannot be an equilibrium, at least one of the firm types will have an unilateral incentive to deviate. □

LEMMA. *Assuming ϕ_i is constant off equilibrium path, P_i must be increasing over the support of the H type and the L type.*

PROOF. Again, the proof is by contradiction. Given that P_i must be monotonic over regions where at least one type is playing mass, it must be either a decreasing or an increasing function of r_i over the supports of both types. Let k_H be the upper bound of the support of f_H and k_L be the upper bound of the support of f_L. Then, designate

$k_u = \max(k_H, k_L)$ as the greatest support upper bound of either fund type. Assume that P_i is a decreasing function of r_i over the supports of both the types. From proposition 2.2, there must be at least two points in the supports of both types. Let these two points be k_u and d_l. Given that $P_i(r_i)$ is decreasing in r_i, it should always be possible to find two points that satisfy $d_l < k_u$ and $P_i(d_l) > P_i(k_u)$. There are two cases to consider:

(1) $\phi_i(r') = \beta > \phi_i(k_u)$ for $r' \in (k_u, \infty)$
(2) $\phi_i(r') = \beta \leq \phi_i(k_u)$ for $r' \in (k_u, \infty)$

This is due to the assumption on how ϕ_i is calculated off equilibrium path, although this proof does not require this assumption to work. Again, we are looking for the existence of profitable mean preserving derivations to rule out each case as an equilibrium.

Case 1. At least one of the fund types can make a profitable mean preserving deviation. The funds (be they H or L type) that are playing probability mass at k_u can move mass from a small region around there ϵ upwards to a small region around $r' \in (k_u, \infty)$, while simultaneously moving mass δ from a small area around k_u to a small area d_l. The overall change in $E(P_i(r_i))$ that results from this, ΔP, is:

$$\Delta P = \epsilon(P_i(r') - P_i(k_u)) + \delta(P_i(d_l) - P_i(k_u))$$

It is clear that $\Delta P > 0$. Given that the observer ranks funds in order of ϕ_i for each fund, $\phi_i(r') > \phi_i(k_u) \Rightarrow P_i(r') > P_i(k_u)$. $P_i(d_l) > P_i(k_u)$ then ensures that $\Delta P > 0$. The deviation can be made mean preserving using

$$\delta = \epsilon \frac{(r' - k_u)}{(k_u - d_l)}$$

Thus case 1 cannot be an equilibrium.

Case 2. Consider a similar deviation to the previous case, with the fund types that are playing probability mass at k_u moving mass from k_u up to some $r' > k_u$, and some mass downwards from k_u down to d_l. Mean preservation again gives

$$\delta = \epsilon \frac{(r' - k_u)}{(k_u - d_l)}$$

Substituting this into the expression for the overall change in $E(P_i(r_i))$, ΔP, gives

$$\Delta P = \epsilon(P_i(r') - P_i(k_u)) + \epsilon \frac{(r' - k_u)}{(k_u - d_l)}(P_i(d_l) - P_i(k_u))$$

Although $\phi_i(r') \leq \phi_i(k_u) \Rightarrow P_i(r') \leq P_i(k_u)$, making the first part of the expression negative or zero, the second part is positive due to P_i being decreasing in r_i. Note that ΔP can always be made positive by picking a sufficiently large r' to make the second term dominate the first term. Thus case 2 cannot be an equilibrium either.

There are always unilateral incentives to make a mean preserving derivation when P_i is decreasing over the supports of f_H and f_L. □

If P_i must be monotonic and not decreasing over the supports of f_H and f_L, it must be that P_i is increasing over the supports of both types. To complete the proof, we need to show that P_i is strictly increasing over $S_H \bigcup S_L$ by excluding the possibility of intervals in $S_H \bigcup S_L$ existing where P_i is constant.

LEMMA. *There can be no interval $[r',\, r'']$ in $S_H \bigcup S_L$ the union of the supports of f_H and f_L, where P_i is constant.*

PROOF. The tie-breaking procedure that selects the top fund as the one that draws the highest return ensures that this is impossible. For $P_i(r_i)$ to be constant in the interval $[r', r'']$, then $\phi_i(r_i)$ must be constant within this interval, since if it is increasing then $P_i(r'') > P_i(r')$. Even if it is constant, $P_i(r_b) > P_i(r_a)$ where $r_a \in [r', r'']$, $r_b \in [r', r'']$ and $r_b > r_a$. This is because the tie-breaking procedure will always select r_b over r_a

(higher return) if they are drawn by two different funds. Hence $P_i(r_b) > P_i(r_a)$ for any r_b and r_a, making it so that P_i is always strictly increasing within $S_H \bigcup S_L$. $\square$

Together, these three lemmas force P_i to be strictly increasing over $S_H \bigcup S_L$, the union of the supports of f_H and f_L.

Proof of Proposition 2.5. Firstly, define $\underline{b}$ as the lower bound of the union of the supports of f_H and f_L. In other words, $\underline{b} = \min(b_H,\, b_L)$, where b_L is the lower bound of the support of f_L and b_H is the lower bound of the support of f_H. Proposition 2.1 implies that that $\underline{b}$ cannot be in the interval $(0,\, r_f)$. Therefore, $\underline{b}$ must either be at $r_i = 0$ or in $[r_f,\, \infty)$. Obviously, we need to eliminate the second case as a possibility. Firstly, consider the case when there is no mass atom at $\underline{b}$, and $\underline{b} \in [r_f,\, \infty)$. It can be shown that this is never an equilibrium.

LEMMA. *If $\underline{b} \in [r_f,\, \infty)$, there must a mass atom played by both types at $\underline{b}$.*

PROOF. Assume that there is no mass atom at $\underline{b}$. There must therefore be a $r' > \underline{b}$ where both types are playing probability mass. By proposition 2.4, $P_i(r_i)$ is an increasing function of r_i over the union of the supports of f_H and f_L. Thus $P_i(\underline{b}) = 0$, since if P_i is an increasing function of r_i and $\underline{b}$ is the lower bound of the union of the supports of f_H and f_L. $P_i(\underline{b}) = 0$ may not be the case if there are mass atoms at $\underline{b}$, but this is excluded by our initial assumption. By proposition 2.3, both fund types must be playing mass at $\underline{b}$. Both types can always make the following mean preserving deviation that will increase welfare: take mass from $\underline{b}$ and move it downwards to $r_i = 0$, while moving mass from $\underline{b}$ upwards to $r^* > \underline{b}$ but still within $S_H \bigcup S_L$, which is possible given that there must be at least one other point in $S_H \bigcup S_L$ above $\underline{b}$ by proposition 2.2. Given that $P_i(\underline{b}) = 0$ and $P_i(0) = 0$, the downwards mass movement will not affect $E(P_i(r_i))$, but given the requirement that $P_i(r_i)$ must be strictly increasing over $S_H \bigcup S_L$, the upwards mass movement will raise $E(P_i(r_i))$. Thus, this deviation will be profitable for both types by raising $E(P_i(r_i))$ while leaving $E(r_i)$ unaffected, and can be executed even if type/ability constraints are binding due to mean preservation. Thus it is not an equilibrium if $\underline{b} \in [r_f, \infty)$ and there is no mass atom there. $\square$

LEMMA. *If $\underline{b} \in [r_f, \infty)$ and both types play a mass atom there, it is also not an equilibrium.*

PROOF. By proposition 2.3, it must be the case that both types are playing mass atoms at $\underline{b}$. If only the H type was playing a mass atom, P_i cannot be strictly increasing over $S_H \bigcup S_L$ since $\phi_i(\underline{b}) \approx 1$, and must have the highest P_i in all of $S_H \bigcup S_L$. This violates proposition 2.4. A scenario where only the L type is playing mass is ruled out by proposition 2.3, since $\phi_i(\underline{b}) = 0$ if that is the case.

Let the size of the mass atoms played by the H and L types at $\underline{b}$ be a_H and a_L respectively. There are a number of scenarios to consider: firstly, the case where $a_H < a_L$, and secondly the case where $a_H \geq a_L$. It can be shown that both cases cannot be an equilibrium.

Case 1. When $a_H \leq a_L$, $\phi_i \leq \beta$ from using 2.2.2. Consider $P_i(\underline{b}+\epsilon)$, where $\epsilon \to 0$. If $\underline{b}+\epsilon$ is on equilibrium path, then given that P_i must be increasing on equilibrium path (i.e. within $S_H \bigcup S_L$), then $\phi_i(\underline{b}+\epsilon) \geq \phi(\underline{b})$ (the tie-breaking rule selects $\phi_i(\underline{b}+\epsilon)$ in case of equality). If $\underline{b}+\epsilon$ is not on equilibrium path, our assumption that $\phi_i = \beta$ off equilibrium path 2.2.6 also ensures that $\phi_i(\underline{b}+\epsilon) \geq \phi(\underline{b})$, and a fund that draws $\underline{b}+\epsilon$ will always win over a fund that draws $\underline{b}$ (via the tiebreaker if $a_H = a_L$). Given that there are mass atoms at $\underline{b}$ and that it is the lowest point in $S_H \bigcup S_L$, $P_i(\underline{b})$ must be equal to $\frac{1}{N}(\beta a_H + (1-\beta)a_L)^{N-1}$ since winning the reputational reward is only possible if all other firms draw $\underline{b}$ and the winner is drawn at random. Note that $P_i(\underline{b}+\epsilon) \geq (\beta a_H + (1-\beta)a_L)^{N-1}$ since $\phi_i(\underline{b}+\epsilon) \geq \phi(\underline{b})$ and drawing $\underline{b}+\epsilon$ will always beat every other fund if they all draw $\underline{b}$. Crucially, $P_i(\underline{b}+\epsilon)$ is finitely larger than $P_i(\underline{b})$. This allows both firms to make a profitable mean preserving deviation by moving mass

δ from $\underline{b}$ upwards to $\underline{b}+\epsilon$ while moving some mass w from $\underline{b}$ downwards to $r_i = 0$. Mean preservation implies

$$w = \delta \frac{\epsilon}{\underline{b}}$$

Thus the overall effect on $E(P_i(r_i))$ of this deviation, ΔP, has to be

$$\Delta P = \delta(P_i(\underline{b}+\epsilon) - P_i(\underline{b})) - \delta\frac{\epsilon}{\underline{b}}(P_i(\underline{b}) - P_i(0))$$

Given our above statements, $P_i(\underline{b}+\epsilon) - P_i(\underline{b}) \geq \frac{N-1}{N}(\beta a_H + (1-\beta)a_L)^{N-1}$, and using $P_i(0) = 0$ and $\epsilon \to 0$ gives

$$\Delta P \geq \delta(\frac{N-1}{N}(\beta a_H + (1-\beta)a_L)^{N-1})$$

This is clearly always finite and positive for finite N, and so gives an unambiguous increase in welfare. Thus the case where $a_H \leq a_L$ cannot be an equilibrium.

Case 2. When $a_H > a_L$, $\phi_i > \beta$. This means the logic from the previous case does not work. However, given that P_i must be strictly increasing in $S_H \bigcup S_L$, ϕ_i must be (at least weakly) increasing in r_i on equilibrium path. This implies that $f_H > f_L$ for all other points in $S_H \bigcup S_L$. However, $\int_{\underline{b}}^{k_H} f_H(r)dr = 1$ and $\int_{\underline{b}}^{k_L} f_L(r)dr = 1$, where k_H and k_L are the upper bounds of the support of the H type and L type respectively. It follows that $k_H < k_L$ due to the requirement that $f_H > f_L$ for all other points in $S_H \bigcup S_L$, which violates propositions 2.3 and 2.4. The only way of avoiding $k_H < k_L$ is to have some regions in the interval $[\underline{b},\ \max(k_H,\ k_L)]$ where $f_L < f_H$, but this violates propositions 2.3 and 2.4 as well. Thus this means that this case cannot be an equilibrium either.

□

So, we have shown that if $\underline{b} \in [r_f, \infty)$, it cannot be an equilibrium regardless if f_H and f_L are continuous there or not. Therefore, only $\underline{b} = 0$ can be an equilibrium. This implies that at least one of the fund types plays mass there at equilibrium.

To show that there must be a mass atom at $r_i = 0$, define b', the next highest point in $S_H \bigcup S_L$ after $\underline{b} = 0$, i.e. the second lowest point in the union of the return distribution supports. Clearly from the two lemmas above, $b' \geq r_f$. If there is no mass atom at $\underline{b}$ and there is no mass atom at b' either, then $P_i(b') = 0 = P_i(\underline{b})$ in equilibrium if it exists, which means that the logic used to prove the first lemma in this section can be applied to show that this cannot be an equilibrium. If there is a mass atom at b' but no mass atom at $\underline{b}$, then the logic from the second lemma in this section can be applied, since the integrated probability at $\underline{b}$ is effectively zero if there are no mass atoms there. If b_H and b_L are the size of the mass atom played at b' by the H type and the L type respectively, then if $b_H > b_L$, then propositions 2.3 and 2.4 cannot be satisfied, and $b_L < b_H$ means that there will always a profitable deviation by any type that plays mass at b' by shifting mass from there infinitesimally upwards. This completes the proof by showing that there must be a mass atom at $r_i = 0$.

Proof of Proposition 2.6. The strategy is to split the proof into two lemmas. First, show that the ability constraints must be binding for any type when their support has at least two different points in it. Then, show that the ability constraints must be binding for a fund type when there is only one point in their return support.

LEMMA. *The ability constraint must be binding for the H type fund as long as there are at least two points in the support of its return distribution. The same applies to the L type fund.*

PROOF. Assume that the type/ability constraint is slack for the H type, and that the support of f_H, S_H, has at least two points in it. Given that the type/ability constraint is slack, the H type can move some mass from the lower of the two points to the higher of the two points. Because P_i strictly increases with r_i over $S_H \bigcup S_L$, this deviation strictly increases not only $E(r_i)$ but $E(P_i(r_i))$, and is therefore always profitable. The same deviation can be done if there more than two points in S_H. Therefore the if the H type ability/type constraint is slack and there are at least two points in S_H, then it cannot be an equilibrium. Similarly, it is also not an equilibrium if the type/ability constraint is slack for the L type and there are at least two points in S_L. □

LEMMA. *The ability constraint must be binding for the H type when there is only one point in its return distribution. The same applies to the L type.*

PROOF. First, assume that the H ability constraint is slack, and that there is only point in S_H, the support of f_H. If S_H only has one point in it, then it must be at the highest point in $S_H \bigcup S_L$, otherwise P_i cannot be strictly increasing over $S_H \bigcup S_L$ as required by proposition 2.4. Let this point be at $r^* < r_H$. Given that the H type constraint is not binding, consider moving mass ϵ upwards from r^* to a point $r' > r^*$. The effect on the utility , $\Delta\pi$, of this movement is:

$$\Delta\pi = \epsilon(r' - r^*) + \epsilon\alpha(P_i(r') - P_i(r^*))$$

Crucially r' is not on equilibrium path (since r^* is the highest point of $S_H \bigcup S_L$), by our assumption that $\phi_i(r_i) = \beta$ off equilibrium path, it follows that $P_i(r')$ is constant for all $r_i > r^*$. Therefore, although $P_i(r') - P_i(r^*) < 0$, note that it is constant for $r' > r^*$. It follows that we can always pick a r' large enough so that $(r' - r^*) > \alpha(P_i(r^*) - P_i(r'))$, making this upwards mass movement profitable. Although we need to satisfy the ability constraint, ϵ can always be picked so that it is satisfied. Using the assumption that there is only point in S_H and rearranging the ability constraint 2.2.3 gives:

$$\frac{r_H - r^*}{r' - r^*} \geq \epsilon$$

Thus this deviation is always feasible and profitable, and a scenario where S_H is a single point and the H ability constraint is slack cannot be an equilibrium. Consequently, from this conclusion and the previous lemma, the H ability constraint must be binding. Now, consider the case when the L ability constraint is slack and the support of f_L, S_L, consists of a single point. Let this be r^* again. Clearly $r^* < r_L$. However, given that the H ability constraint is binding, the H type must be playing mass at $r' \geq r_H$. Given proposition 2.4, $P_i(r') > P_i(r^*)$ since $r' > r^*$. Thus, given the L type's slack ability constraint, it is always profitable for it to move mass up to r' since it raises both $E(r_i)$ and $E(P_i(r_i))$. Therefore, it is not an equilibrium when the L type has a slack ability constraint and S_L consists of one point. □

The two lemmas together imply that the ability constraint must be binding for both types at all times in equilibrium.

Proof of Proposition 2.7.

LEMMA. *If there are at least three points in S_H, the support of f_H, then P_i must be linear over all of S_H in equilibrium. Likewise, if there are at least three points in S_L, the support of f_L, then P_i must be linear over all of S_L in equilibrium.*

PROOF. Given proposition 2.6, we only need to consider mean preserving deviations in equilibrium. Consider the case when P_i is not linear over all of S_H. If P_i is not linear over all of S_H and there at least three points in S_H, we can always find at least three points in S_H over which P_i is not linear. Label these three points r_b, r_m and r_t, with $r_b > r_m > r_t$. If P_i is not linear over all of S_H, we must be able to find a set of points in S_H such that either

$$\frac{P_i(r_m) - P_i(r_b)}{r_m - r_b} > \frac{P_i(r_t) - P_i(r_m)}{r_t - r_m} \tag{2.4.1}$$

or

$$\frac{P_i(r_m) - P_i(r_b)}{r_m - r_b} < \frac{P_i(r_t) - P_i(r_m)}{r_t - r_m} \tag{2.4.2}$$

These conditions imply that the chord joining the point $(r_b,\ P_i(r_b))$ and $(r_m,\ P_i(r_m))$ does not have the same gradient as the chord joining $(r_m, P_i(r_m))$ and $(r_t, P_i(r_t))$, which must be the case if r_b, r_m and r_t have been picked so that P_i is not linear over them. Consider making a mean preserving mass movement, moving mass ϵ from r_m upwards to r_t while moving mass δ from r_m downwards to r_b. Mean preservation means that

$$\delta = \epsilon \frac{(r_t - r_m)}{(r_m - r_b)}$$

using the above result, the effect on on $E_i(P_i(r_i))$ of this deviation, ΔP, is

$$\Delta P = \epsilon(r_t - r_m)\left(\frac{P_i(r_t) - P_i(r_m)}{r_t - r_m} - \frac{P_i(r_m) - P_i(r_b)}{r_m - r_b}\right)$$

Given 2.4.1 and 2.4.2, it follows that $\Delta P \neq 0$. If $\Delta P > 0$ (when 2.4.2 is true), then the deviation is profitable, and P_i being nonlinear with S_H containing at least 3 points cannot be an equilibrium. If $\Delta P < 0$ (when 2.4.1 is true), then the reverse movement is profitable (moving mass ϵ from r_t down to r_m and mass δ from r_b up to r_m), and this scenario is also not an equilibrium. Thus, P_i must be linear over S_H since if $P_i(r_i) = Zr_i + c$, then $\frac{P_i(r_m)-P_i(r_b)}{r_m-r_b} = \frac{P_i(r_t)-P_i(r_m)}{r_t-r_m} = Z$ for all possible combinations of r_b, r_m and r_t, and the H type is indifferent to such a deviation. Identical logic can be applied to the case for the L type to give the result that P_i must be linear over S_L when there at least three points in S_L. □

If there are at least two points in $S_H \bigcap S_L$, there must be at least two points in $S_H \bigcup S_L$. When there are exactly two points in $S_H \bigcup S_L$ that are both in $S_H \bigcap S_L$, then the result trivially applies since a straight line can be drawn through any two points. Note however that the straight line must pass through $P_i(0) = 0$, since one of the two points in

$S_H \bigcup S_L$ must be $r_i = 0$ by proposition 2.5, resulting in the form $P_i(r_i) = Zr_i$ over $S_H \bigcup S_L$.

When there are three points in $S_H \bigcup S_L$, there are two points above or at r_f and one point at zero (by propositions 2.1 and 2.5). The two points above r_f must be in S_H by propositions 2.3 and 2.4. If $r_i = 0$ is also in S_H, we can use the previous lemma to show that P_i must be linear over all three points in $S_H \bigcup S_L$. Given that it must pass through $P_i(0) = 0$, its form is clearly $P_i(r_i) = Zr_i$ over $r_i \in S_H \bigcup S_L$. If $r_i = 0$ is in S_L only, all three points in $S_H \bigcup S_L$ must be in S_L, since there are at least two points in $S_H \bigcap S_L$. We can therefore use the lemma in the same way on the L type to deduce that $P_i(r_i) = Zr_i$ over $r_i \in S_H \bigcup S_L$ (no constant as it has to pass through zero again).

When there are at least four points in $S_H \bigcup S_L$, note that all but one of the points must be in $r_i \in [r_f, \infty)$ by proposition 2.1 (the other one must be $r_i = 0$ by proposition 2.5). Consequently, all but one of the points ($r_i = 0$) must be in S_H due to propositions 2.4 and 2.3. We can apply the previous lemma to deduce that P_i must be of the form $P_i = Z_H r_i + c_H$ over S_H. If the final point $r_i = 0$ is also in S_H, then the proposition will clearly hold. If $r_i = 0$ is on S_L only, then for $S_H \bigcap S_L$ to have at least two points, S_L must include at least two other points out of the ones in $r_i \in [r_f, \infty)$. Then we can apply the lemma to show that $P_i = Z_L r_i + c_L$ over S_L. However, if there are at least two points in $S_H \bigcap S_L$, then to make the previous lemma apply to both the H and L types simultaneously over $S_H \bigcap S_L$, P_i must be the same linear function over S_H and S_L, i.e. $Z_L = Z_H$ and $c_L = c_H$.

Furthermore, $c_H = c_L = 0$ since the point $r_i = 0$ must in $S_H \bigcup S_L$, and the linear function that is P_i must pass through $P_i(0) = 0$.

Finally, in all cases, clearly $Z > 0$ to satisfy proposition 2.4, and clearly $Z \neq 0$ in equilibrium since probability mass is being played in $r_i \geq r_f$. This completes the proof.

Proof of Proposition 2.8.

PROOF. Assume that at least one fund type is playing mass at $r^* - \epsilon$. Consider a mean preserving deviation by that type from point $r^* - \epsilon$ that consists of moving mass δ from there down to $r_i = 0$ and mass m from there up to r^*. Mean preservation implies $\delta = \frac{m\epsilon}{(r^* - \epsilon)}$. Using this and the fact that $P_i(0) = 0$ gives the total effect on $E(P_i)$, ΔP, which is

$$\Delta P = m(P_i(r^*) - P_i(r^* - \epsilon)) - \frac{m\epsilon}{(r^* - \epsilon)}(P_i(r^* - \epsilon))$$

Crucially, from the result 2.2.9, $P_i(r^*) = \sum_{W=0}^{W=N-1}(\frac{1}{W+1})\frac{(N-1)!}{(N-1-W)!(W)!}\overline{a}^W(u(r^*))^{N-1-W}$, where $\overline{a}$ is the size of the mass atom in expectation at r^*. One can see that $P_i(r^* - \epsilon) \leq (u(r^*))^{N-1}$, i.e. the $W = 0$ term of the sum (when all funds draw less than r^*). Thus, $P_i(r^*)$ is $finitely$ greater than $P_i(r^* - \epsilon)$ as long as N is also finite, even if $\epsilon \to 0$. Given the mean preservation requirement, $\delta = \frac{m\epsilon}{(r* - \epsilon)} \to 0$ also and so there is effectively no loss in P_i from this mass movement, while the part of the expression containing the gain in P_i does not tend to zero due to $P_i(r^*)$ being finitely greater than $P_i(r^* - \epsilon)$. Therefore this deviation is always profitable, and no fund type can play mass at $r^* - \epsilon$.

Assume that at least one type plays mass at $r^* + \epsilon$. Consider a mean preserving deviation by that type that involves moving mass δ from r^* down to zero, and an upward movement of mass m from r^* to $r^* + \epsilon$. Crucially, note that

$$P_i(r^* + \epsilon) \geq \sum_{W=0}^{W=N-1} \frac{(N-1)!}{(N-1-W)!(W)!}\overline{a}^W(u(r^*))^{N-1-W}$$

by definition, since a draw of $r^* + \epsilon$ will always win when all other funds draw r^* or less. Every term of this sum is greater than or equal to

$$P_i(r^*) = \sum_{W=0}^{W=N-1}(\frac{1}{W+1})\frac{(N-1)!}{(N-1-W)!(W)!}\overline{a}^W(u(r^*))^{N-1-W}$$

Therefore, $P_i(r^* + \epsilon)$ is $finitely$ greater than $P(r^*)$ even if $\epsilon \to 0$. Then, we can apply similar logic to the previous case to show that the net effect on $E(P_i)$ is always positive, since there is a finite gain in $E(P_i)$ by moving mass upwards by an infinitesimal amount while the amount of mass that needs to be moved downwards to compensate, δ, tends to zero as $\epsilon \to 0$. So this deviation is always profitable, and no fund type can play mass at $r^* + \epsilon$ either.

The logic used in these arguments do not apply to $r^* < r_f$, but no fund type plays mass there except at $r_i = 0$ anyway by propositions 2.1 and 2.5. Although proposition 2.5 states there must be a mass atom at $r_i = 0$, this proposition applies to it given that no mass is played in $(0,\ r_f)$. $\square$

CHAPTER 3

Model with Incentive Bonuses

3.1. Introduction

Real life hedge funds are heavy users of high powered incentive schemes. A particular one that is very common is known as the 2 and 20 scheme. Abstractly, schemes combine a flat fee for funds under management with a bonus scheme that allows the fund to keep a significant cut of any profits above a certain threshold. In practice, the 2 and 20 scheme involves a flat 2% of funds under management fee and a 20% bonus fee for any returns above the risk free threshold.

There are clear benefits to using a high powered incentive scheme like this. The most obvious one is that it if the fund manager is able to exert variable effort and returns depend on this, the bonus scheme helps to align the interests of the manager and the investors by incentivising effort. This is especially important given the existence of the flat management fee, which might otherwise cause issues by allowing managers to collect a large payoff while exerting little effort. Often the bonus scheme is combined with other measures such as forcing the manager to invest a significant amount of their own money into the fund, which is another method of incentivising effort and decreasing moral hazard. Some other less obvious benefits to these bonus schemes have also been suggested, for example by Arya and Mittendorf (2005) who point out that high powered incentive schemes can be used as a signalling device by high ability managers, who are more inclined to take these bonuses because they are more likely to be able to trigger them. Das and Sundram (2002) show theoretically that funds with stronger incentive fees should exhibit better performance in general.

However, the empirical evidence on this is mixed, and this casts some doubt on claims that these bonus schemes actually improve performance. Some studies find that these bonuses are indeed linked with better performance (Ackermann McEnally and Ravenscraft (1999)), while some other studies (Brown, Goetzmann and Ibbotson (1999)) find that there is no link between performance and incentive schemes. At the very least, the link between the two is somewhat tenuous. One of the reasons this may be the case is that there incentive bonus schemes may have also have detrimental effects that mitigate their positive effects. Some theoretical literature that points out unexpected negative effects of these bonus schemes. An example is Hodder and Jackwerth (2005), who examine the dynamic risk taking decisions of a utility maximising manager. They use numerical simulations to show that the fine details of the incentive schemes can have drastic effects on managerial risk taking, and under such incentive schemes fund managers do not behave in way desired by investors at all. Incentive schemes like the 2 and 20 one used by a lot of hedge funds are effectively like out of the money options, and this can distort fund manager behaviour as it may cause them to take extreme risks to push the incentive bonus into the money.

This chapter contributes to the latter strand of literature by highlighting a possible negative effect of these incentive bonuses. It considers the consequences of using bonuses like the "20" portion of the commonly used 2 and 20 scheme, i.e. when a fund manager keeps a cut of all profits above a certain threshold in the setting of the model in part 1, where new hedge funds of unknown ability with access to complex financial engineering compete to enhance their reputation by performing well relative to their peers. The model is kept simple to isolate potential negative effects, and therefore some features like variable effort are not included, since that will not change the potential negative effects of incentive bonuses while adding modelling complexity.

The main finding is that as one might expect, incentive bonuses encourage risk taking in such a setting. Comparing the tail risk and fund failure rates in equilibria where mass is played above the incentive bonus return threshold in this model with the equivalent equilibria from the model in chapter 1 shows that adding bonuses increases failure probabilities and tail risk under mild conditions. When funds have the ability to manipulate return distributions via financial engineering and are incentivised to do so by a bonus scheme, they will use the tools at their disposal to move probability mass into return regions where they gain a bonus. If their innate ability constrains them from generating high return investment strategies, then moving mass to bonus regions will result in higher failure risk and tail risk.

In addition, with this setup, funds actually play less total probability mass above the bonus threshold than without it. The probability mass they do play above the bonus threshold is higher variance, however. Intuitively, this is because the bonus scheme rewards high return values that are delivered by having high variance above r'. Again, this is also related to the assumption that funds are constrained by their innate ability: to run expected return and variance above r', they must offset this with more probability mass below r' in total to keep the mean return within their ability constraint. A second interesting finding is that this setting imposes a minimum amount of tail risk per fund. This is due to the combination of the freedom to pick return distributions with the bonus and the chasing of relative performance. Funds must be indifferent to moving mass above r', and therefore the loss from moving mass downwards in the region below r' must be as high as the gain in bonuses from moving mass upwards in the region above r', and the only way of achieving this is to have funds play steep cdfs below r', so the reduction in the probability of winning the reputational boost offsets this.

This part is organised in the following way: section 2 introduces the model and analyses some of the features of the equilibria, and derives the key equations necessary for the results. Section 3 discusses and explains these results, and section 4 is a brief summary and conclusion.

3.2. The Model

3.2.1. Model Setup. The model is very similar to the one from chapter 1. N hedge funds of unknown ability level and financial engineering powerful enough to allow them to pick their return distributions with complete freedom subject to an ability constraint compete to win a reputational boost, with the winning fund decided by strong performance relative to its peers. Unlike in the model in part 1, they have some of their own funds I_f but also have funds I from another source, e.g. from investors. The key addition to this model is the effect of a compensation scheme like the 2 and 20 scheme commonly used by hedge funds. This states that funds keep a cut s of all returns on the money they manage (i.e. money that is not their own) above a certain threshold r'. In practice this threshold is the risk free return rate, effectively stating that a fund keeps 20% of any returns it can make in excess of it.

Otherwise, the funds have the same options and payoff structure as they do in part 1. They are risk neutral, and consider the returns on their own money, the odds of winning the reputational boost, and the cut of the return they make above r' due to the incentive scheme. The timing of the game is as follows:

- $t = 0$: Nature draws the types of the N funds independently, with each fund being a H ability type with probability β and a L ability type with probability $1-\beta$. Each fund knows its own type, but not the types of its competitors. Each fund picks a return distribution subject to an ability constraint that allows the H type to play a distribution with a greater mean than the L type, and invests I_f of its own funds and I funds from other investors into it.
- $t = 1$: The investment matures, paying out $r_i(I + I_f)$ in total for fund i, following the distribution picked at $t = 0$. The fund keeps its own investment $r_i I_f$, and as a consequence of its incentive scheme, it also gets a share $s(r_i - r')I$ of the profits on the investors' funds if r_i is greater than some threshold r'.

The winner of the reputational boost is the fund that shows the best relative performance in its peer group, and therefore the fund that draws the highest return out of the N competing entities gets this award. Like in part 1, if a fund draws a return below an exogenous threshold r_f, it fails. If it fails, it cannot win the reputational boost.

Assume for simplicity that there is no discounting. Given this setting, we can write the objective function for each fund:

$$\pi_i = E(r_i I_f) + \alpha E(P_i(r_i)) + L(r_i)I$$

With the special function L defined to replicate the incentive scheme described previously:

$$L(r_i) \begin{cases} L = s(r_i - r') & for\ r_i \geq r' \\ L = 0 & for\ r_i < r' \end{cases}$$

Where s, I and α are all positive numbers. As before, P_i is the probability of winning the reputational boost given a return draw r_i, and α is a parameter that determines the size of the reputational boost. The type constraints, or ability constraints the funds are subject to are the same as in part 1:

(3.2.1) $$E(r_i) \leq r_H \quad for\ H\ type$$

(3.2.2) $$E(r_i) \leq r_L \quad for\ L\ type$$

And to give the H type higher "ability" than the L type,

$$r_H > r_L$$

A number of other assumptions are also made. Firstly, $r_f < r_L < r_H$, i.e. both fund types can produce mean returns above the failure threshold. Secondly, $r' > r_f$. Intuitively, the bonus threshold should be above the failure threshold. Limited liability, i.e. $r_i \geq 0$ is assumed. Also, $\frac{sIr'}{\alpha} < 1$. The last assumption is so that there are equilibria where mass is played above r' exist.

P_i is unchanged from part 1 since the way the winner of the reputational reward is determined is unchanged, and so is given by the probability that a draw of r_i is the highest return draw out of the N funds. Thus, it is given by

$$P_i(r_i) = (\beta F_H(r_i) + (1-\beta)F_L(r_i))^{N-1} \tag{3.2.3}$$

This is obtained using the same reasoning as for the model in part 1.

3.2.2. Limits on the Form of the Equilibria. This section contains a number of propositions that restrict the form of the equilibria.

PROPOSITION 3.1. *The ability constraints 3.2.1 and 3.2.2 must be binding for both types.*

PROOF. This proof is very similar that of the analogous result in the first section. If the type constraint is not binding for one type, then it will always be possible to move probability mass upwards, which is a deviation that raises the expected return. Since P_i must be an increasing function of r_i, this deviation must either raise $E(P_i)$ or keep it the same. Thus, this deviation must be strictly welfare increasing since it increases the expected mean return while at least keeping $E(P_i)$ constant, and so it cannot be an equilibrium if any fund has a slack ability/type constraint. Thus the type constraint must be binding for all funds in equilibrium. □

The implication of this is that if the type/ability constraints bind, then funds in equilibrium only need to be indifferent to mean preserving mass movements. There can be no mean preserving mass movements that raise $E(P_i)$, which gives us a useful method for finding the equilibria.

PROPOSITION 3.2. *No fund types play mass in the interval* $(0,\ r_f)$.

PROOF. The proof of this is extremely similar to that of the analogous results from the previous sections. Given proposition 3.1, we only need to consider mean preserving deviations. Assume that in equilibrium, a fund plays probability mass in this interval. It can always move some mass upwards to any $r_i \geq r_f$ while simultaneously moving mass downwards to $r_i = 0$ to preserve the mean return. Such a deviation will always strictly increase $E(P_i)$ since $P_i = 0$ for $r_i \in [0,\ r_f)$ and $P_i > 0$ for $r_i \geq r_f$ since we have assumed that funds play probability mass in $(0,\ r_f)$. Therefore, there will be incentives for unilateral deviation if funds play probability mass in $(0,\ r_f)$, and this will not happen in equilibrium. $\square$

This must be true because moving mass from that interval downwards has no effect on $E(P_i)$ since the contribution to $E(P_i)$ from mass in $[0,\ r_f)$ is always zero. If this can be done, mean preservation allows some other mass to be moved upwards to where $P_i > 0$.

PROPOSITION 3.3. *No fund type can play a mass atom in the open interval* $[r_f,\ \infty)$.

PROOF. This is identical to the proof for the analogous result for the model in chapter 1. Assume that one of the fund types is playing a mass atom at a point r'. This cannot be an equilibrium since a profitable deviation is always possible: a fund that is playing mass at point r' can move mass upwards from there to $r' + \epsilon$, where $\epsilon \to 0$, while preserving the mean by moving mass downwards elsewhere. The gain in $E(P_i)$ from moving mass upwards infinitesimally is finite, which can be deduced using the formula for P_i at a mass atom from chapter 2 (2.2.10), or by noting that drawing $r' + \epsilon$ will beat any draw of r' or less compared to before the mass movement, when other funds drawing r' can only be beaten by winning a random tiebreaker. The effect of the downwards mass movement on $E(P_i)$ tends to zero due to ϵ tending to zero, making this deviation always profitable. This applies both for $r_i < r'$, and $r_i \geq r_f$, since the extra payoff gained by moving mass upwards is continuous in both $r_i \geq r'$ and $r_f \leq r_i \leq r'$. $\square$

The winner of the reputational boost is determined by the highest order statistic. When there is a mass atom, moving a finite quantity of probability mass infinitesimally upwards from the mass atom gives a finite gain in $E(P_i)$, and because of the infinitesimally small size of the upwards movement, only a correspondingly small amount of mass needs to be moved downwards to preserve the mean. The effect on $E(P_i)$ of the downwards movement thus tends to zero, and this deviation is overall always profitable.

PROPOSITION 3.4. *At least one of the two fund types must play a mass atom at* $r_i = 0$.

PROOF. This proof is identical to the proof for proposition 1.5 in chapter 1. Let b be the lower bound of the union of the supports of f_H and f_L. If there are no mass atoms at $r_i = 0$, then $P_i(b) = 0$, since by propositions 3.2 and 3.3 P_i must be continuous above $r_i = 0$. It is then always possible to make a profitable mean preserving deviation: move some mass from $r_i = b$ down to $r_i = 0$, while moving some probability mass from b upwards. The downwards mass movement does not decrease $E(P_i)$ due to $P_i(b) = P_i(0) = 0$, and the upwards mass movement will increase $E(P_i)$. Thus there must be a mass atom at $r_i = 0$. $\square$

Proposition 3.4 follows a similar logic to proposition 3.2. If there is no mass atom at $r_i = 0$, then the lowest bound of the union of the supports of the two different fund types, b, has $P_i(b) = 0$ due to the winner of the reputational boost being decided by the Nth order statistic. It is then possible to move mass downwards from b without incurring any losses to $E(P_i)$, while keeping mean preservation with a strictly profitable upwards mass movement. This result ensures that there will be failure/tail risk.

PROPOSITION 3.5. *Mass must be played for all $r_i \in [r_f,\ k_u]$, where k_u is the greatest upper bound of the union of the supports of f_H and f_L.*

PROOF. If there is an interval $(b_1,\ b_2)$ in which no fund type is playing mass within $[r_f,\ k_u]$, then P_i must be constant within $(b_1,\ b_2)$. Then it is always possible to make a profitable mean preserving mass movement: move some mass from b_2 downwards to b_1 and some mass upwards elsewhere from r' to r'', where $r' < r''$ and $P_i(r') < P_i(r'')$. Such a r' and r'' will always exist due to propositions 3.1, 3.2 and 3.3. The first movement has no effect on $E(P_i)$ due to P_i being constant here, while the second movement strictly improves $E(P_i)$. Hence this cannot be an equilibrium. □

This follows similar logic to the previous proposition. If no fund plays mass in some regions within $[r_f,\ k_u]$, then P_i is constant there, allowing the same type of mean preserving mass distribution to be always profitable.

3.2.3. Calculating Equilibria. Like in the basic model presented in chapter 1, we can use the previous propositions to narrow down some of the properties of the equilibria. Although there are an infinity of equilibria, they share a number of common properties relating to failure risk/tail risk that can be calculated. We are looking for symmetric equilibria where at least one of the funds has the possibility of being affected by the incentive scheme, i.e. at least one of the funds is playing mass above r'.

Given proposition 3.1, all fund types must be indifferent to a mean preserving shift of probability mass in equilibrium, while obeying the propositions detailed in the previous subsection. So, in equilibrium, to obey proposition 3.1, we must have $\int r f_H dr = r_H$ and $\int r f_L dr = r_L$. There must also always be a mass atom in expectation at $r_i = 0$, while $f_H = f_L = 0$ in $r_i \in (0,\ r_f)$, and both F_H and F_L must be continuous in $r_i \in [r_f,\ \infty)$. We need to set out the conditions for all fund types to be indifferent to mean preserving mass shifts in equilibrium. The bonus for drawing $r_i > r'$ means that there are two distinct regions of r_i draws with differing payoffs, giving rise to a number of different combinations of mass movements between regions that must be considered. In equilibrium, both fund types must be indifferent to all these different combinations of movements. Let k_u be the greatest upper bound of the supports of f_H and f_L. Since we only want to consider equilibria where $k_u > r'$, this can be assumed.

Firstly, consider making a mean preserving deviation in which mass ϵ is moved upwards from x to $x + \Delta x$ and mass δ down from $y + \Delta y$ to y. For now, assume that this mass movement only involves moving mass around in the interval $[r_f,\ r']$, i.e. $r_f \leq x < x + \Delta x \leq r'$ and $r_f \leq y < y + \Delta y \leq r'$. There are no conditions on the levels of y and x relative to each other. The effect of such a movement on the utility of a fund is

$$\Delta\pi_i = \epsilon((x+\Delta x)I_f - xI_f + \alpha P_i(x+\Delta x) - \alpha P_i(x)) + \delta(yI_f - (y+\Delta y)I_f + \alpha P_i(y) - \alpha P_i(y+\Delta y))$$

Mean preservation gives:

$$\delta[(y + \Delta y) - y] = \epsilon[(x + \Delta x) - x]$$

$$\delta = \epsilon\frac{\Delta x}{\Delta y}$$

Setting $\Delta\pi_i = 0$ and substituting the mean preservation condition into $\Delta\pi_i$ gives

$$\frac{P_i(x + \Delta x)}{\Delta x} = \frac{P_i(y + \Delta y)}{\Delta y}$$

The analysis can be simplified by only considering this condition when $\Delta x \to 0$ and $\Delta y \to 0$. This can be done without loss of generality because P_i must be continuous in this region due to proposition 3.3. The effect of any mean preserving mass movement can be broken down into the effects of a series of smaller ones, and this takes that logic to the extreme to simplify matters. Like in the previous sections, the result is a condition on the derivative of P_i:

$$\frac{dP_i}{dr_i}|_{r_i=y} = \frac{dP_i}{dr_i}|_{r_i=x}$$

This must hold for every $y \in [r_f,\ r']$ and $x \in [r_f,\ r']$. If $\Delta\pi_i < 0$ anywhere then it is possible to possible to reverse the directions of the mass movements while still preserving the mean to get a profitable deviation. Recall that P_i must be an increasing function. The above condition together with the requirement that P_i be continuous from proposition 3.3 implies that P_i must be linear in $[r_f,\ r']$, since that is the only functional form for P_i that has a constant derivative over the interval. If $\frac{dP_i}{dr_I}$ is not constant over this interval, then it is always possible to find a mean preserving mass movement such that $\Delta\pi_i > 0$. Thus $P_i(r_i) = Zr_i + c$ for $r_i \in [r_f,\ r']$.

The value of c can also be found by considering a mean preserving deviation that moves mass from r_f down to zero while moving some mass up. Due to proposition 3.2, there is no incentive to move mass downwards to $(0,\ r_f)$, and there is no need to consider mass movements from some other $r_i > r_f$ downwards to zero since such a movement can be decomposed into two separate mean preserving mass movements: firstly a movement where mass is moved to r_f, and then a further movement where this mass is moved down to zero. The linear form of P_i in $[r_f,\ r']$ means funds will be indifferent to the first of the two mass movements, and so only the second movement needs to be analysed. Consider a mass movement where mass ϵ is moved from $r^* \in [r_f,\ r']$ to a return level r^{**}, where $r^{**} \in [r_f,\ r']$ and $r^{**} > r^*$, while mass δ is moved downwards from r_f down to zero. The effect on the objective function $\Delta\pi_i$ is

$$\Delta\pi_i = \epsilon[I_f(r^{**} - r^*) + \alpha(Zr^{**} + c - (Zr^* + c))] - \delta[I_f r_f + \alpha(Zr_f + c)]$$

Mean preservation gives

$$\delta = \epsilon \frac{(r^{**} - r^{*})}{r_f}$$

Substituting this, using $P_i(0) = 0$ in and requiring that $\Delta\pi_i = 0$ gives

$$\frac{c}{r^{**} - r^{*}} = \frac{c}{r_f}$$

This can only satisfied for all r^{**} and r^{*} only if $c = 0$. Intuitively, $P_i(0)$ and $P_i(r_f)$ must be on the same straight line for funds to be indifferent to this deviation, giving $c = 0$. Thus

$$P_i(r_i) = Zr_i \quad for\ r_i \in [r_f,\ r'] \tag{3.2.4}$$

This must be the case for funds to be indifferent to mean preserving deviations that move mass around within $[0,\ r']$.

However, given that $k_u > r'$, funds must also be indifferent to mean preserving mass movements that shift mass only in the interval $[r',\ k_u]$. Consider the effect on the objective function of such a movement.

$$\Delta\pi_i = \epsilon[(x+\Delta x)I_f - xI_f + \alpha P_i(x+\Delta x) - \alpha P_i(x) + sI((x+\Delta x) - r') - sI(x - r')]$$
$$+ \delta[yI_f - (y+\Delta y)I_f + \alpha P_i(y) - \alpha P_i(y+\Delta y) + sI(y - r') - sI((y+\Delta y) - r')]$$

Using mean preservation and setting $\Delta\pi_i = 0$ as before gives:

$$\frac{P_i(x+\Delta x)}{\Delta x} = \frac{P_i(y+\Delta y)}{\Delta y}$$

Using similar reasoning to before, it follows that P_i is linear for this region too. Let

$$P_i(r_i) = Z'r_i + c' \tag{3.2.5}$$

for $r_i \in [r',\ k_u]$, with Z' and c' denoting the possibility that these coefficients are different from P_i in $r_i \in [r_f,\ r']$. So, P_i must be linear in both $[r_f,\ r']$ and $[r',\ k_u]$, although the same linear relation does not need to hold in both regions.

We also need to make funds indifferent to mean preserving mass movements that move mass in both regions simultaneously. Consider a mass movement where some mass is moved from x upwards to $x + \Delta x$ within $[r',\ k_u]$, i.e. $r' \leq x < x + \Delta x$ and mass δ is simultaneously moved downwards from $y + \Delta y$ to y within $[r_f,\ r']$, i.e. $r_f \leq y < y + \Delta y \leq r'$. The effect of this on the objective function of a fund is

$$\Delta\pi_i = \epsilon[(x+\Delta x)I_f - xI_f + \alpha P_i(x+\Delta x) - \alpha P_i(x) + sI((x+\Delta x) - r') - sI(x - r')] \\ + \delta[yI_f - (y+\Delta y)I_f + \alpha P_i(y) - \alpha P_i(y+\Delta y)]$$

Setting this to zero for indifference and using mean preservation gives

$$\alpha \frac{P_i(x+\Delta x)}{\Delta x} + sI = \alpha \frac{P_i(y+\Delta y)}{\Delta y}$$

Using the result on P_i in $[r_f,\ r']$, 3.2.4 and the general linearity of P_i in $[r',\ k_u]$ 3.2.5, a relation between Z, the gradient of P_i in $[r_f,\ r']$ and Z', the gradient of P_i in $[r',\ k_u]$ can be obtained:

$$Z' = Z - \frac{sI}{\alpha} \tag{3.2.6}$$

It is immediately possible to deduce the value of c' and thus the form of P_i in $[r',\ k_u]$. Recall that due to proposition 3.2, P_i must be continuous in this interval. A continuity argument then implies that the expression for $P_i(r')$ in the lower region must be equal to the expression for $P_i(r')$ from the upper region, i.e. that $Zr' = Z'r' + c'$. Using 3.2.6, this gives

$$c' = \frac{sIr'}{\alpha} \tag{3.2.7}$$

From this , we can obtain a complete expression for $P_i(r_i)$:

$$P_i(r_i) \begin{cases} P_i = 0 & r_i \in [0,\ r_f) \\ P_i = Zr_i & r_i \in [r_f,\ r'] \\ P_i = (Z - \frac{sI}{\alpha})r_i + \frac{sIr'}{\alpha} & r_i \in [r',\ k_u] \\ P_i = 1 & r_i \in [k_u,\ \infty) \end{cases} \tag{3.2.8}$$

Clearly the probability of gaining a reputational boost is zero below r_f due to the fund failure, and will be 1 above k_u, the upper bound of the union of the supports of both the types. To complete the calculation of P_i, it needs to be verified that fund types are indifferent to mass movements where mass is moved between the two regions as well. An example of this is a mass movement where mass is moved upwards within $[r',\ k_u]$ but some mass is moved downwards from the upper region into the lower region to maintain mean preservation. Fortunately, this step can be skipped by noting that any pair of mass movements that involves shifting mass between the two regions can be broken down into a series of smaller movements that do not cross the boundary between the two regions. For example, a movement of mass upwards in the upper region only coupled with a downwards movement from the upper region into the lower region can be broken down into a movement of the same mass upwards by a smaller distance in the upper region and a downwards movement of mass in the upper region to the boundary r', followed by a downwards movement in the lower region only from r' downwards, and a mass movement in the upper region only. Indifference to movements between regions can also be simply verified using the same method used to consider movements that do not cross boundaries by calculating the effect on objective functions of all the different possible mass movements across the boundary r', and finding that any mean preserving movement gives $\Delta\pi_i = 0$.

Note that $Z' = Z - \frac{sI}{\alpha}$ means that $Z > \frac{sI}{\alpha}$, since $Z' > 0$ by definition. If this is not true, then it is always possible to find profitable mean preserving deviations where mass is simply moved to above r' to take advantage of the bonus. This shows why the assumption $\frac{sIr'}{\alpha} < 1$ is necessary to ensure equilibria where mass is played above r' exist. If the lowest Z can be is $\frac{sI}{\alpha}$, then the assumption is necessary to ensure that $P_i(r') < 1$ so that mass is still played above r'.

3.2.4. Tail Risk and Failure Risk. Like in the model in part 1, there are an infinity of equilibria. However, the form of P_i can still be used to deduce a number of important equilibrium properties, namely failure probabilities, which are equivalent to tail risk due to proposition 3.4. These end up being independent of the exact return distributions played in equilibrium. The results calculated in the previous section apply to symmetric equilibria where $k_u > r'$, which are the ones of interest.

3.2.4.1. *Calculating Z and its Relation to Failure Rate.* If it is assumed that one of the fund types chooses a return distribution with a mass atom of size a_H or a_L at zero and a probability density function g elsewhere, then the form of P_i, given by 3.2.8, together with the expression for P_i in terms of F_H and F_L, 3.2.3, dictates what the return distribution of the other type must be in equilibrium. Like in the model in part 1, the exact form of F_H and F_L varies according to if k_u is k_H or k_L, which of the two types has a (relatively) free choice of pdf by playing the g function, and if the support upper bound of the type that is not determining k_u is less than r'. This creates a multitude of different cases each with a slightly different form for F_H and F_L. The details of what these are and how they are derived are contained in appendix 3.

From the cdf forms mentioned before, pdfs and thus expressions for $E(r_i)$ can be found, and because proposition 3.1 states that the type constraints must bind in equilibrium, these expressions for $E(r_i)$ can be substituted into the ability/type constraint equations (3.2.1 and 3.2.2) to generate two equations that give relationships between the key endogenous variables. These can be used to eliminate the free pdf component g, resulting in a single equation derived from every different form of F_H and F_L. Like in the analysis in chapter 1, every different form of F_H and F_L ends up generating the same key relation between the parameters of the model, Z, Z' and k_u:

$$\overline{r} = \int_{r_f}^{r'} r\frac{Z(Zr)^{\frac{1}{N-1}-1}}{(N-1)}dr + \int_{r'}^{k_u} r\frac{Z'(Z'r+c')^{\frac{1}{N-1}-1}}{(N-1)}dr \tag{3.2.9}$$

Where $\overline{r} = \beta r_H + (1-\beta)r_L$, and Z' and c' are given by the results derived previously: 3.2.6 and 3.2.7 respectively. The details showing this calculation are contained within appendix 3.

Given that the relation between Z' and Z is known, only k_u needs to be eliminated to give an equation in one variable. To do so, consider the nature of P_i - it is a probability, the probability of winning the reputational boost. Given that there are no mass atoms, P_i must be equal to one at the upper bound of the union of the supports of f_H and f_L, which is k_u. This is because if there is no mass atom at k_u, then drawing k_u means that the reputational boost is always won - by definition a draw higher than k_u in equilibrium is impossible, and another draw equal to k_u happens with zero probability due to the lack of mass atoms. Thus we can write:

$$Z'k_u + c' = 1$$

Subbing in $Z' = Z - \frac{sI}{\alpha}$ from 3.2.6 and $c' = \frac{sIr'}{\alpha}$ from 3.2.7 gives an expression for k_u in terms of Z and the parameters:

$$k_u = \frac{1 - \frac{sIr'}{\alpha}}{Z - \frac{sI}{\alpha}} \tag{3.2.10}$$

This can be used to eliminate k_u from 3.2.9 to give an expression for Z, which is the most important variable for determining failure rate and tail risk. Simplifying 3.2.9 by performing the integrals gives:

$$\overline{r} = [\frac{Z^{\frac{1}{N-1}} r^{\frac{1}{N-1}+1}}{N}]_{r_f}^{r'} + [r(Z'r + c')^{\frac{1}{N-1}}]_{r'}^{k_u} - [\frac{(N-1)(Z'r + c')^{\frac{1}{N-1}+1}}{NZ'}]_{r'}^{k_u}$$

Noting that $Z'k_u + c'$, $Z' = Z - \frac{sI}{\alpha}$, $c' = \frac{sIr'}{\alpha}$, using 3.2.10, and simplifying gives the final result of:

$$Z = \frac{sI}{\alpha} + \frac{1}{\overline{r}N + (Zr_f^N)^{\frac{1}{N-1}}}[1 + (N-1)\frac{sI}{\alpha} Z^{\frac{1}{N-1}} r'^{\frac{1}{N-1}+1} - \frac{NsIr'}{\alpha}] \tag{3.2.11}$$

Given equation 3.2.10, it is clear that $Z > \frac{sI}{\alpha}$ for k_u to be a valid upper bound for the return distributions of the H and L types. Although it is not clear that the equation for Z (3.2.11) produces $Z > \frac{sI}{\alpha}$, the following proposition can be proved:

PROPOSITION 3.6. $2\overline{r} \geq r' > r_f$ *and* $\frac{sIr'}{\alpha} \in (0, 1)$ *are sufficient to ensure that equation 3.2.11 gives a unique* $Z > \frac{sI}{\alpha}$ *for* $N \geq 2$.

PROOF. See appendix 3. □

These are necessary assumptions to ensure that the equations governing symmetric equilibria are well behaved. What remains to be done is finding the failure rate and tail risk in terms of Z and the parameters. This is done in an identical fashion to chapter 1. Assume that the H type fund plays an atom of size a_H. Due to 3.2.8 and cdf continuity for $r_i \geq r_f$, $a_L = \frac{(Zr_f)^{\frac{1}{N-1}}}{1-\beta} - \frac{\beta}{1-\beta} a_H$. Likewise, if the L type is playing a_L, then by identical reasoning $a_H = \frac{(Zr_f)^{\frac{1}{N-1}}}{\beta} - \frac{1-\beta}{\beta} a_L$. Because funds do not play any mass below the failure threshold except for a mass atom at zero, the failure probability and tail risk per fund is given by the expected size of the mass atom at zero before type determination, $\overline{a} = \beta a_H + (1-\beta) a_L$. Both the expressions previously listed give the same expression for this:

$$\overline{a} = (Zr_f)^{\frac{1}{N-1}} \tag{3.2.12}$$

This is precisely same expression as the corresponding one in part 1. However, the value of Z is clearly different here.

3.2.4.2. *Comparative Statics and a Parameter Condition on Higher Failure Rates.* To isolate the effect of introducing an incentive scheme like the 2 and 20 used by a lot of hedge funds, the fund failure probability here needs to be compared to the failure probability for the same model without the incentive scheme. Fortunately, this has already been analysed in chapter 1. As stated before, the expression for failure rate in the model without the incentive scheme is identical to the one obtained in this chapter, and is $\overline{a} = (Zr_f)^{\frac{1}{N-1}}$. This only depends on one endogenous variable, Z. Therefore, the only factor that determines which failure risk is higher is Z. The higher Z is, the higher the failure/tail risk.

Let Z^* be the expression for the equivalent endogenous variable in the model in part 1. Although chapter 1 does not derive an expression for Z, it is straightforward to derive it from the relation $\frac{1}{k_u} = Z$ and the equation relating k_u to the exogenous parameters. Z^* is given by:

$$Z^* = \frac{1}{\overline{r}N + (Z^* r_f^N)^{\frac{1}{N-1}}} \tag{3.2.13}$$

To recap, failure/tail risk is greater when there is an incentive scheme if $Z > Z^*$, with Z given by 3.2.11 and Z^* given by the above equation. Although it is difficult to analyse which is greater exactly due to lack of closed forms for Z and Z^*, it is possible to get concrete conditions on when $Z > Z^*$.

PROPOSITION 3.7. *If $\frac{sIr'}{\alpha}r' + \frac{sIr_f}{\alpha}r_f \geq r'$, $2\bar{r} > r'$ is a sufficient condition for $Z > Z^*$. If $\frac{sIr'}{\alpha}r' + \frac{sIr_f}{\alpha}r_f < r'$, $2\bar{r} + \frac{sIr'}{\alpha}r' + \frac{sIr_f}{\alpha}r_f > 2r'$ is a sufficient condition for $Z > Z*$. Under these conditions, failure/tail risk is always greater in symmetric equilibria where at least one type is playing mass above r' when there is an incentive scheme that gives funds a cut s of profits above r'.*

PROOF. See appendix 3. □

However, like in chapter 1, it must be shown that symmetric equilibria actually exist. A similar strategy to that employed in chapter 1 is used: a computationally convenient functional form for F_L is selected, and the equilibrium is computed. From this, sufficient conditions on the parameters are obtained for the equilibria to be valid. The following is proved in the appendix:

PROPOSITION 3.8. *Under the conditions required for proposition 3.6 to hold, the condition*

$$\frac{sIr_f}{\alpha} \geq \frac{(1-\beta)(r_H - r_L)}{r_H}$$

on the exogenous parameters is sufficient to ensure existence of equilibria for all N of interest, i.e $N \geq 2$.

PROOF. See appendix 3. □

This condition is not inconsistent with the assumptions needed for proposition 3.6. The expression $\frac{(1-\beta)(r_H - r_L)}{r_H} \in (0,1)$, and $\frac{sIr_f}{\alpha} \in (0,1)$ since $\frac{sIr'}{\alpha} \in (0,1)$ and $r_f < r'$. Thus it is possible to pick parameter values such that this holds, since $\frac{sI}{\alpha}$ is a free variable.

In reality, one might expect the potential future gains from outperforming all competitors, α, to be larger than a fraction s, which is around 0.2, of investor money now for a new fund of unknown reputation (I will not large). Thus, $\frac{sI}{\alpha} \in (0,1)$ is not a particularly extreme assumption. Given that r' is going to be around the risk free rate and is this only slightly greater than one, $\frac{sIr'}{\alpha} < 1$, a required assumption, does not seem unreasonable. Also, $\frac{(1-\beta)(r_H-r_L)}{r_H}$, the right hand side of the inequality that ensures the existence of symmetric equilibria for $N \geq 2$ in proposition 3.8 is low in reality. This is because r_L and r_H are not likely to be extremely far apart in absolute terms, r_H might be something like 1.4, while r_L might be something like 1.07 or so. This is still a huge 33% difference in interest rates (probably smaller in reality), but also ensures that $\frac{(1-\beta)(r_H-r_L)}{r_H}$ is small even for relatively low values of β. For example, with $r_L = 1.07$, $r_H = 1.40$ and a small value of β such as 0.01, the expression is still small (0.233). This eases the restrictiveness of the inequality in proposition 3.8 by allowing for a larger range of values for $\frac{sIr_f}{\alpha}$. Given that r_f is not much lower than r' due to its interpretation as an outside option, $\frac{sIr_f}{\alpha}$ will only be slightly smaller than $\frac{sIr_f}{\alpha}$. However, we are still restricted in what values of $\frac{I}{\alpha}$ we can take if we want to ensure existence of equilibria for $N \geq 2$. This cannot be too small and cannot be too large (we require $\frac{sIr'}{\alpha} \in (0,1)$). Note that these conditions are only sufficient, and $\frac{sIr_f}{\alpha}$ being only slightly less than $\frac{(1-\beta)(r_H-r_L)}{r_H}$ might still be enough for equilibria for $N \geq 2$.

3.3. Discussion

The conditions that ensure higher tail risks in symmetric equilibria from proposition 3.7 do not require that $r' \leq \overline{r}$. However, $r' \leq \overline{r}$ is a condition that is not too far fetched. One might expect r_H, the return level of high ability funds to be reasonably larger than r', which could be interpreted as the risk free rate, and r_L to be at worst around the risk free rate. If one assumes that r' is also going around the risk free rate, then $r' \leq \overline{r}$ is likely to be satisfied or almost satisfied, which will ensure proposition 3.7 holds. Given that $\frac{sIr'}{\alpha}$ and $\frac{sIr_f}{\alpha}$ cannot be too small (due to proposition 3.8), the conditions from proposition 3.7 are likely to be satisfied even if $r' > \bar{r}$, since r' is unlikely to be much more than the risk free rate (which is itself close to r_L) . Also, since proposition 3.7 only provides sufficient conditions for $Z > Z*$, there are many more parameter combinations for which this is true too.

So, for many reasonable parameter values, failure risk is greater when bonus schemes are used. This is somewhat unsurprising given the setup of the model. The agents have the power to manipulate their return distributions subject to an ability constraint. This gives funds the power to move probability mass into the region above r' where they receive their incentive bonus at the cost of moving some mass downwards to keep the ability constraint binding, and since they are risk neutral, they are happy with getting the possibility of large bonuses at the risk of of failing. In a sense, this is what the incentive scheme is designed to do: encourage fund managers to deliver high returns. In other models, the bonus scheme encourages greater effort, leading to higher returns. When financial engineering allows return distributions to be manipulated costlessly, funds will be incentivised to move mass to above r'. What this model makes clear is that if the expected return of funds are constrained by ability, then this mass movement will be accompanied by an increase in tail risk. This particular assumption is an interesting one - to what extent are funds in reality constrained by ability rather than effort cost? It is conceivable that above a certain level of bonus managers will be constrained by their innate ability, and so bonuses above that level merely induce heavy risk taking.

In addition, the mixture of risk neutrality and the linear incentive bonus means that there is less mass played above r' than when there is no incentive bonus. This can be seen by noting that $P_i(r_i)$ is strictly increasing in the amount of probability mass that a fund plays below r_i in expectation (before type determination), and that P_i for $r_i < r'$ is Zr_i when there is a bonus scheme and Z^*r_i where there is not. As stated before, $Z > Z^*$ is definitely true under mild conditions. Then, there will be less mass played above r' when there is a bonus, since there is more mass below r'. If the conditions in proposition 3.7 hold, to ensure that type constraints are binding, the mass above r' must contribute more to the mean than without the bonus since there is more mass below r'. It follows that mass above r' must be higher variance with the bonus. The reason for this is the incentive bonus linearly rewarding extremely high return draws, meaning funds prefer higher return values. The bonus rewards distributions with high expected value from probability mass above r', which is exactly what running high variance above r' gives. However, since the funds are constrained by their ability, they must offset the higher expected value of the mass played above r' with more mass below it. Technically, the gradient of P_i above r' is lower than below r' to compensate for the additional return due to incentive bonus in this region, which makes the total probability mass above r' lower but its variance higher, since k_u will be higher.

In this simple model, competing for strong relative performance and return distribution manipulation actually puts a limit to the amount of risk taken. In principle, one might expect the funds to play mass at extremely high values of r_i to maximise $sI(r_i - r')$ if they are bound by their ability constraints. This is actually limited by the setting of the model requiring return cdf continuity (proposition 3.1) and proposition 3.5 implying that mass must be played from r_f all the way up to k_u. This prevents a distribution being played that has a lot of mass at zero, with a small amount of mass at a very high r, since mass must be played all the way up to k_u. The is caused by some of the Blotto game like characteristics of the setting - the types of deviations that make these propositions true are analogous to moving resources out from battlefields that are losing badly and moving enough to battlefields that are not winning to swing them. The expected return is like the resource constraint in this context, and being able to move mass downwards while still keeping the same P_i for that mass is like being able to move mass away from battlefields that are losing heavily. Another characteristic that the setting gives is a minimum degree of tail risk. Like in the model from chapter 1, the rise of tail risk with competition is driven by the $\frac{1}{N-1}$ power term, which pushes $\overline{a}$ to one. Z falls with competition, but not fast enough to offset the rise due to the power term. However, now Z takes a minimum value of $\frac{sI}{\alpha}$ from proposition 3.6 (to ensure indifference to moving mass above r' anyway), even if say $\overline{r}$ is very large (this causes Z^* to be very small, lowering failure risk when there are no incentive bonuses). Thus, the minimum tail risk/failure risk in such an equilibrium must be

$$\frac{sI}{\alpha} r_f$$

Since failure risk is lowest when N is at a minimum. This is because the ability to move probability mass around subject to mean constraints implies that P_i, which is related to how much mass funds play below that r_i value, must be high enough to cause funds to be indifferent to moving mass to above r', where P_i cannot be less than zero. This is a detrimental effect due to introducing these incentive bonuses into the environment of the model in part 1.

There are of course plenty of problems with this approach. It is missing key features like variable effort, but as mentioned earlier, that would not have eliminated the negative effects that this analysis finds. Adding variable effort makes the analysis considerably more complex, weakens the magnitude of the negative effects, but does not remove them. Likewise, risk aversion would have mitigated the risk taking but not eliminated it. More fundamental problems with the approach are really the same as the model in part 1: the setting itself may not be the best description of how hedge funds operate. There is no denying that financial engineering allows return distributions to be manipulated, but it in reality it does so in a way that trades off risk against mean return. The assumption that managers' innate ability restricts mean returns is therefore a strong assumption that is critical in generating a lot of the results, and as mentioned before is an interesting assumption that can be debated.

3.4. Conclusion

This part analysed the effects of adding an incentive bonus scheme similar to the "20" portion of the common 2 and 20 scheme (where a fund keeps a cut of any profits above a certain return threshold) to a model like the one introduced in chapter 1. As one might expect, amongst new hedge funds competing for strong relative performance to enhance their reputations, under some mild assumptions, failure risk is higher in equilibria where mass is being played above the bonus threshold when there are bonuses. In an environment where funds are risk neutral, they are incentivised by the bonuses to use financial engineering to move probability mass to above the bonus threshold. Critically, in this setting, funds are constrained by the innate ability of their managers. So, to play return distributions with probability mass in bonus regions, they must also play tail risk to keep the return distributions within their ability constraints. This also leads to less probability mass, but with higher variance being played above the bonus threshold compared to the case without the incentive scheme due to a combination of the linear incentive scheme and the ability constrained nature of the funds. A unique feature of this model is the minimum tail risk that must exist in equilibria where mass is played above the bonus threshold, which arises due to the need to be indifferent to moving mass to above the bonus threshold, which at the very least gives some extra return due to the bonus. The return pdfs must have a minimum steepness below the bonus threshold for this indifference to hold.

This is a somewhat crude attempt to analyse the effects of bonus schemes in an environment of new, competing hedge funds. To extend this work further, an alternative, less extreme framework of return manipulation by financial engineering will need to be developed, since that is the most controversial part of this model. The assumption that funds have their mean returns restricted by their ability is a very strong one that is critical in driving the results of the model, and this framework does not allow return manipulation equivalent to changing leverage to happen, which would be a desirable feature to have. Overall, this is merely a first step to analysing the effects of incentive bonuses on a world of competing hedge funds.

Appendix 3

Deriving Forms of F_H and F_L and Equilibrium Equations. The form of P_i, 3.2.8, can be used together with $3.2.3$ to derive the form of F_H and F_L when one of the types is assumed to have a free choice of return distribution. As stated in section 2.4, a number of different cases that give different forms for F_H and F_L need to be analysed, and they are done here case by case.

Case 1. $k_u = k_H$. Assume the H type is playing a cumulative distribution function with a mass atom a_H at zero, and a function G above that. $G(r_f)$ must be zero at r_f to comply with proposition 3.2, i.e. F_H and F_L must be continuous apart from at zero. Assume that for now $k_L \geq r'$. Because P_i must be linear up to k_u (equation 3.2.8) and $k_L < k_u$, F_H must take on a different form between k_L and k_u since F_L is set at one and can no longer be picked to give the correct form for P_i there. So, F_H must have the following form:

$$F_H(r_i)\begin{cases} a_H & r_i \in [0,\ r_f) \\ G(r_i) + a_H & r_i \in [r_f,\ k_L] \\ \frac{(Z'r_i+c')^{\frac{1}{N-1}}}{\beta} - \frac{1-\beta}{\beta} & r_i \in [k_L,\ k_u] \\ 1 & r_i \geq k_u \end{cases}$$

Where $Z' = Z - \frac{sI}{\alpha}$ from 3.2.6. It follows from 3.2.8 that F_L must be:

$$F_L(r_i)\begin{cases} \frac{(Zr_f)^{\frac{1}{N-1}}}{1-\beta} - \frac{\beta}{1-\beta}a_H & r_i \in [0,\ r_f) \\ \frac{(Zr_i)^{\frac{1}{N-1}}}{1-\beta} - \frac{\beta}{1-\beta}(G(r_i) + a_H) & r_i \in [r_f,\ r'] \\ \frac{(Z'r_i+c')^{\frac{1}{N-1}}}{1-\beta} - \frac{\beta}{1-\beta}(G(r_i) + a_H) & r_i \in [r',\ k_L] \\ 1 & r_i \geq k_L \end{cases}$$

From proposition 3.1, the type/ability constraints 3.2.1 and 3.2.2 must be binding. We can use the derived forms for F_H and F_L above to find expressions for f_H and f_L, the pdfs, and substitute these into the ability constraints. This gives the following two equations:

$$r_H = \int_{r_f}^{k_L} rg(r)dr + \int_{k_L}^{k_u} r\frac{Z'(Z'r_i + c')^{\frac{1}{N-1}-1}}{\beta(N-1)}$$

$$r_L = \int_{r_f}^{r'} r[\frac{Z(Zr)^{\frac{1}{N-1}-1}}{(1-\beta)(N-1)} - \frac{\beta}{1-\beta}g(r)]dr + \int_{r'}^{k_L} r[\frac{Z'(Z'r + c')^{\frac{1}{N-1}-1}}{(N-1)(1-\beta)} - \frac{\beta g(r)}{(1-\beta)}]dr$$

Where $g = \frac{dG}{dr}$. Combining these two equations allows g to be eliminated, giving a simplified condition derived from the type constraints. $\overline{r} = \beta r_H + (1-\beta)r_L$.

$$\overline{r} = \int_{r_f}^{r'} r\frac{Z(Zr)^{\frac{1}{N-1}-1}}{(N-1)}dr + \int_{r'}^{k_u} r\frac{Z'(Z'r + c')^{\frac{1}{N-1}-1}}{(N-1)}dr$$

If $k_L < r'$, using 3.2.8 and the same reasoning, F_H must be

$$F_H(r_i) \begin{cases} a_H & r_i \in [0,\ r_f) \\ G(r_i) + a_H & r_i \in [r_f,\ k_L] \\ \frac{(Zr_i)^{\frac{1}{N-1}}}{\beta} - \frac{1-\beta}{\beta} & r_i \in [k_L,\ r'] \\ \frac{(Z'r_i+c')^{\frac{1}{N-1}}}{\beta} - \frac{1-\beta}{\beta} & r_i \in [r',\ k_u] \\ 1 & r_i \geq k_u \end{cases}$$

And $F_L$must be:

$$F_L(r_i)\begin{cases} \frac{(Zr_f)^{\frac{1}{N-1}}}{1-\beta} - \frac{\beta}{1-\beta}a_H & r_i \in [0,\ r_f) \\ \frac{(Zr_i)^{\frac{1}{N-1}}}{1-\beta} - \frac{\beta}{1-\beta}(G(r_i)+a_H) & r_i \in [r_f,\ k_L] \\ 1 & r_i \geq k_L \end{cases}$$

Using F_L and F_H to derive f_H and f_L and substituting these into the binding ability constraints 3.2.1 and 3.2.2 gives:

$$r_H = \int\limits_{r_f}^{k_L} rg(r)dr + \int_{k_L}^{r'} \frac{Z(Zr)^{\frac{1}{N-1}-1}}{\beta(N-1)}dr + \int\limits_{r'}^{k_u} r\frac{Z'(Z'r_i+c')^{\frac{1}{N-1}-1}}{\beta(N-1)}dr$$

$$r_L = \int_{r_f}^{k_L} r[\frac{Z(Zr)^{\frac{1}{N-1}-1}}{(1-\beta)(N-1)} - \frac{\beta}{1-\beta}g(r)]dr$$

The condensed condition with g eliminated is the same as when $k_L \geq r'$:

$$\overline{r} = \int_{r_f}^{r'} r\frac{Z(Zr)^{\frac{1}{N-1}-1}}{(N-1)}dr + \int_{r'}^{k_u} r\frac{Z'(Z'r+c')^{\frac{1}{N-1}-1}}{(N-1)}dr$$

Case 2. $k_u = k_H$, but the L type has the freedom to pick a G function now. Assume that $k_L \geq r'$. Applying the same logic as case 1 gives:

$$F_L(r_i)\begin{cases} a_L & r_i \in [0,\ r_f) \\ G(r_i)+a_L & r_i \in [r_f,\ k_L] \\ 1 & r_i \geq k_L \end{cases}$$

And therefore F_H must be:

$$F_H(r_i)\begin{cases} \frac{(Zr_f)^{\frac{1}{N-1}}}{\beta}-\frac{1-\beta}{\beta}a_L & r_i\in[0,\ r_f) \\ \frac{(Zr_i)^{\frac{1}{N-1}}}{\beta}-\frac{1-\beta}{\beta}(G(r_i)+a_L) & r_i\in[r_f,\ r'] \\ \frac{(Z'r_i+c')^{\frac{1}{N-1}}}{\beta}-\frac{1-\beta}{\beta}(G(r_i)+a_L) & r_i\in[r',\ k_L] \\ \frac{(Z'r_i+c')^{\frac{1}{N-1}}}{\beta}-\frac{1-\beta}{\beta} & r_i\in[k_L,\ k_u] \\ 1 & r_i\geq k_u \end{cases}$$

Using F_L and F_H to derive f_H and f_L and substituting these into the binding ability constraints 3.2.1 and 3.2.2 gives:

$$r_L=\int_{r_f}^{k_L} rg(r)dr$$

$$r_H=\int_{r_f}^{r'} r[\frac{Z(Zr)^{\frac{1}{N-1}-1}}{\beta(N-1)}-\frac{1-\beta}{\beta}g(r)]dr+\int_{r'}^{k_L} r[\frac{Z'(Z'r+c')^{\frac{1}{N-1}-1}}{(N-1)\beta}-\frac{1-\beta}{\beta}g(r)]dr$$
$$+\int_{k_L}^{k_u} r\frac{Z'(Z'r+c')^{\frac{1}{N-1}-1}}{(N-1)\beta}dr$$

Eliminating g gives the same condition as in case 1:

$$\overline{r}=\int_{r_f}^{r'} r\frac{Z(Zr)^{\frac{1}{N-1}-1}}{(N-1)}dr+\int_{r'}^{k_u} r\frac{Z'(Z'r+c')^{\frac{1}{N-1}-1}}{(N-1)}dr$$

If $k_L<r'$, then F_L must be:

$$F_L(r_i)\begin{cases} a_L & r_i\in[0,\ r_f) \\ G(r_i)+a_L & r_i\in[r_f,\ k_L] \\ 1 & r_i\geq k_L \end{cases}$$

And from 3.2.8, $F_H$can be obtained:

$$F_H(r_i)\begin{cases} \frac{(Zr_f)^{\frac{1}{N-1}}}{\beta} - \frac{1-\beta}{\beta}a_L & r_i \in [0,\ r_f) \\ \frac{(Zr_i)^{\frac{1}{N-1}}}{\beta} - \frac{1-\beta}{\beta}(G(r_i)+a_L) & r_i \in [r_f,\ k_L] \\ \frac{(Zr_i)^{\frac{1}{N-1}}}{\beta} - \frac{1-\beta}{\beta} & r_i \in [k_L,\ r'] \\ \frac{(Z'r_i+c')^{\frac{1}{N-1}}}{\beta} - \frac{1-\beta}{\beta} & r_i \in [k_L,\ k_u] \\ 1 & r_i \geq k_u \end{cases}$$

Using F_L and F_H to derive f_H and f_L and substituting these into the binding ability constraints 3.2.1 and 3.2.2 gives:

$$r_L = \int_{r_f}^{k_L} rg(r)dr$$

$$r_H = \int_{r_f}^{k_L} r[\frac{Z(Zr)^{\frac{1}{N-1}-1}}{\beta(N-1)} - \frac{1-\beta}{\beta}g(r)]dr + \int_{k_L}^{r'} r\frac{Z(Zr)^{\frac{1}{N-1}-1}}{(N-1)\beta}dr + \int_{r'}^{k_u} r\frac{Z'(Z'r+c')^{\frac{1}{N-1}-1}}{(N-1)\beta}dr$$

Eliminating g once again gives the same condition as case 1.

$$\overline{r} = \int_{r_f}^{r'} r\frac{Z(Zr)^{\frac{1}{N-1}-1}}{(N-1)}dr + \int_{r'}^{k_u} r\frac{Z'(Z'r+c')^{\frac{1}{N-1}-1}}{(N-1)}dr$$

Cases 3 and 4. These are very similar to cases 1 and 2 respectively, but with $k_u = k_L > k_H$. The H type has the freedom to choose the G function in case 3, while the L type has the ability to choose the G function in case 4. The calculations will not be gone over in detail, since case 3 is highly similar to case 2, but with the H type playing the role of the L type. In both scenarios, the fund type that has a support upper bound less than k_u is playing G. To get the forms of the cdfs and equations, take the cdfs and equations from case 2, switch F_H with F_L and r_L with r_H, replace k_L with k_H, switch

β with $1-\beta$, and replace a_L with a_H. The resulting equations give the same equation involving Z, Z' and k_u 3.2.9 as cases 1 and 2, as required.

Likewise, case 4 is highly similar to case 1, but with the L type playing the role of the H type in case 1. The type that has the freedom to choose the G function is the type that has the greatest support upper bound in both cases. The cdfs and equations are obtained in an identical manner to the other cases, and the forms of the cdfs and equations can be obtained from the case 1 cdfs and equations by switching F_L with F_H and r_H with r_L, replacing k_H with k_L, switching $1-\beta$ with β, and replace a_H with a_L. Once again, the final equation that results is 3.2.9, the same equation as in cases 1 and 2.

The analysis of the above four cases still applies exactly when $k_u = k_H = k_L$. The intervals such as $[k_L,\ k_u]$ and $[k_H,\ k_u]$ vanish and when considering the ability constraints, the integrals over these intervals are zero. Simply replace k_L and k_H with k_u. The resulting equation from the ability constraints is still 3.2.9, as required. All these different cases generate the same equation with Z, Z' and k_u.

Proof of Proposition 3.6. We want to prove that $\frac{sIr'}{\alpha} \in (0,1)$ and $2\bar{r} \geq r' > r_f$ are sufficient conditions for equation 3.2.11 to give $Z > \frac{sI}{\alpha}$. Take equation 3.2.11:

$$Z = \frac{sI}{\alpha} + \frac{1}{\bar{r}N + (Zr_f^N)^{\frac{1}{N-1}}}[1 + (N-1)\frac{sI}{\alpha}Z^{\frac{1}{N-1}}r'^{\frac{1}{N-1}+1} - \frac{NsIr'}{\alpha}]$$

To simplify things, denote $Z - \frac{sI}{\alpha} = \hat{Z}$. Then rearrange the above equation into the following form:

(3.4.1) $$\hat{Z}\bar{r}N + \hat{Z}(\hat{Z} + \frac{sI}{\alpha})^{\frac{1}{N-1}}r_f^{\frac{N}{N-1}} - (N-1)\frac{sI}{\alpha}(\hat{Z} + \frac{sI}{\alpha})^{\frac{1}{N-1}}r'^{\frac{N}{N-1}} = 1 - \frac{NsIr'}{\alpha}$$

Denote the LHS of the above expression as

$$f(\hat{Z}) = \hat{Z}\bar{r}N + \hat{Z}(\hat{Z} + \frac{sI}{\alpha})^{\frac{1}{N-1}}r_f^{\frac{N}{N-1}} - (N-1)\frac{sI}{\alpha}(\hat{Z} + \frac{sI}{\alpha})^{\frac{1}{N-1}}r'^{\frac{N}{N-1}}$$

so that equation 3.4.1 becomes

$$f(\hat{Z}) = 1 - \frac{NsIr'}{\alpha}$$

LEMMA. *If , $2\bar{r} \geq r' > r_f$ and $\frac{sIr'}{\alpha} \in (0,1)$, then satisfying the inequality*

$$0 < (1 - \frac{sIr'}{\alpha}) - (N-1)[\frac{sIr'}{\alpha} - (\frac{sIr'}{\alpha})^{\frac{N}{N-1}}]$$

ensures equation 3.4.1 gives a unique $\hat{Z} > 0$.

PROOF. For the lemma to be true, we need f to cross the horizontal line $1 - \frac{NsIr'}{\alpha}$ once in the region $\hat{Z} > 0$. Firstly prove that $\frac{df}{d\hat{Z}} \geq 0$ for all $\hat{Z} \geq 0$.

Take the first and second derivative of f wrt to $\hat{Z}$:

$$\frac{df}{d\hat{Z}} = \bar{r}N + (\hat{Z} + \frac{sI}{\alpha})^{\frac{1}{N-1}} r_f^{\frac{N}{N-1}} + \frac{\hat{Z}(\hat{Z} + \frac{sI}{\alpha})^{\frac{1}{N-1}-1}}{N-1} r_f^{\frac{N}{N-1}} - \frac{sIr'^{\frac{N}{N-1}}}{\alpha}(\hat{Z} + \frac{sI}{\alpha})^{\frac{1}{N-1}-1}$$

$$\frac{d^2 f}{d\hat{Z}^2} = \frac{2(\hat{Z} + \frac{sI}{\alpha})^{\frac{1}{N-1}-1} r_f^{\frac{N}{N-1}}}{N-1} - \frac{(N-2)}{(N-1)^2}\hat{Z}(\hat{Z} + \frac{sI}{\alpha})^{\frac{1}{N-1}-2} r_f^{\frac{N}{N-1}} + \frac{sIr'^{\frac{N}{N-1}}}{\alpha}\frac{(N-2)}{(N-1)}(\hat{Z} + \frac{sI}{\alpha})^{\frac{1}{N-1}-2}$$

Consider when $\frac{d^2 f}{d\hat{Z}^2} = 0$. There is a unique value of $\hat{Z}$ where this is true is given by

$$\hat{Z} = \frac{[-\frac{sIr'^{\frac{N}{N-1}}}{\alpha}\frac{(N-2)}{(N-1)} - \frac{2sIr_f^{\frac{N}{N-1}}}{\alpha(N-1)}](N-1)^2}{Nr_f^{\frac{N}{N-1}}} < 0$$

This is clearly (all parameters are positive numbers) at a value of $\hat{Z} < 0$. Also, note that $\hat{Z} = 0$, $\frac{d^2 f}{d\hat{Z}^2} > 0$ since the only negative term is zero. It follows from continuity of $\frac{d^2 f}{d\hat{Z}^2}$ that $\frac{d^2 f}{d\hat{Z}^2} \geq 0$ for all $\hat{Z} \geq 0$. This is because there are no discontinuities in $\frac{d^2 f}{d\hat{Z}^2}$ for $\hat{Z} \geq 0$, and the only $\hat{Z}$ where $\frac{d^2 f}{d\hat{Z}^2}$ crosses the zero line is at a value of $\hat{Z} < 0$.

Now, consider $\frac{df}{d\hat{Z}}$ evaluated at $\hat{Z} = 0$:

$$\frac{df}{d\hat{Z}}|_{\hat{Z}=0} = \bar{r}N + (\frac{sIr_f}{\alpha})^{\frac{1}{N-1}} r_f - (\frac{sIr'}{\alpha})^{\frac{1}{N-1}} r'$$

Given that it was assumed that $\frac{sIr'}{\alpha} \in (0,1)$, $r_f < r'$ and $N \geq 2$, $0 < (\frac{sIr_f}{\alpha}) < (\frac{sIr'}{\alpha}) < 1$ and thus $(\frac{sIr_f}{\alpha})^{\frac{1}{N-1}} < 1$ and $(\frac{sIr'}{\alpha})^{\frac{1}{N-1}} < 1$. Also, since $N \geq 2$ and, we can write

$$\frac{df}{d\hat{Z}}|_{\hat{Z}=0} = \bar{r}N + (\frac{sIr_f}{\alpha})^{\frac{1}{N-1}} r_f - (\frac{sIr'}{\alpha})^{\frac{1}{N-1}} r' > 2\bar{r} + (\frac{sIr_f}{\alpha})^{\frac{1}{N-1}} r_f - r'$$

It is clear that $2\bar{r} \geq r'$ is sufficient to ensure $\frac{df}{d\hat{Z}}|_{\hat{Z}=0} > 0$. Given that $\frac{d^2f}{d\hat{Z}^2} \geq 0$ for all $\hat{Z} \geq 0$ as well, it follows that $\frac{df}{d\hat{Z}} > 0$ for all $\hat{Z} \geq 0$ given our assumptions.

This assures that if there is a solution in the region $\hat{Z} > 0$, it will be unique since $f(\hat{Z})$ is strictly increasing in $\hat{Z} \in [0, \infty)$. Furthermore, $\frac{df}{d\hat{Z}} \geq 0$ for all $\hat{Z} \geq 0$ and $\frac{df}{d\hat{Z}}$ being infinite at as $\hat{Z} \to \infty$ implies that equation 3.4.1

$$f = \hat{Z}\bar{r}N + \hat{Z}(\hat{Z} + \frac{sI}{\alpha})^{\frac{1}{N-1}} r_f^{\frac{N}{N-1}} - (N-1)\frac{sI}{\alpha}(\hat{Z} + \frac{sI}{\alpha})^{\frac{1}{N-1}} r'^{\frac{N}{N-1}} = 1 - \frac{NsIr'}{\alpha}$$

will have a unique solution in $\hat{Z} \in (0, \infty)$ providing that $f(0) < 1 - \frac{NsIr'}{\alpha}$. This condition is:

$$0 < (1 - \frac{sIr'}{\alpha}) - (N-1)[\frac{sIr'}{\alpha} - (\frac{sIr'}{\alpha})^{\frac{N}{N-1}}] \tag{3.4.2}$$

This completes the proof of the lemma. □

Now, the next lemma will complete the proof of the proposition:

LEMMA. $\frac{sIr'}{\alpha} \in (0,1)$ *is a sufficient condition for*

$$0 < (1 - \frac{sIr'}{\alpha}) - (N-1)[\frac{sIr'}{\alpha} - (\frac{sIr'}{\alpha})^{\frac{N}{N-1}}]$$

to be true for $N \geq 2$.

PROOF. Let the RHS of inequality 3.4.2 be denoted by

$$g(N) = (1 - \frac{sIr'}{\alpha}) - (N-1)[\frac{sIr'}{\alpha} - (\frac{sIr'}{\alpha})^{\frac{N}{N-1}}]$$

. The first step is to prove that $\frac{dg}{dN} < 0$ for $N \geq 2$.

Calculate the first and second derivatives of the function g wrt to N:

$$\frac{dg}{dN} = (\frac{sIr'}{\alpha})^{\frac{N}{N-1}}[1 - \frac{1}{N-1}\ln(\frac{sIr'}{\alpha})] - \frac{sIr'}{\alpha}$$

$$\frac{d^2g}{dN^2} = \frac{1}{(N-1)^3}(\frac{sIr'}{\alpha})^{\frac{N}{N-1}}(\ln(\frac{sIr'}{\alpha}))^2$$

The second derivative of g wrt to N is clearly always positive for $N \geq 2$.

Consider the limit of $\frac{dg}{dN}$ as N tends to infinity. Write $\frac{dg}{dN}$ as:

$$\frac{dg}{dN} = (\frac{sIr'}{\alpha})[(1 - \frac{\ln(\frac{sIr'}{\alpha})}{N-1})(\frac{sIr'}{\alpha})^{\frac{1}{N-1}} - 1]$$

We can see that $\lim_{N\to\infty} \frac{dg}{dN} \to 0$ from the negative side since $(\frac{sIr'}{\alpha})^{\frac{1}{N-1}} \to 1$. Given that $\frac{d^2g}{dN^2} > 0$ for $N \geq 2$, it must the case that $\frac{dg}{dN} < 0$ for $N \geq 2$.

Given $\frac{dg}{dN} < 0$ for $N \geq 2$, the minimum value of g is attained when $N \to \infty$. It follows that given $\frac{dg}{dN} < 0$ for $N \geq 2$, we only require $g > 0$ for $N \to \infty$ to ensure $g > 0$ for $N \geq 2$. Rearranging condition 3.4.2 and noting that $\frac{sIr'}{\alpha} \in (0,1) \Rightarrow \frac{sIr'}{\alpha} - (\frac{sIr'}{\alpha})^{\frac{N}{N-1}} > 0$, we get

$$1 < \frac{\frac{(1-\frac{sIr'}{\alpha})}{N-1}}{\frac{sIr'}{\alpha} - \left(\frac{sIr'}{\alpha}\right)^{\frac{N}{N-1}}} \tag{3.4.3}$$

We can use L'Hôpital's rule to evaluate the limit of the RHS of this expression as $N \to \infty$:

$$\lim_{N\to\infty} \frac{\frac{(1-\frac{sIr'}{\alpha})}{N-1}}{\frac{sIr'}{\alpha} - \left(\frac{sIr'}{\alpha}\right)^{\frac{N}{N-1}}} = \frac{(1-\frac{sIr'}{\alpha})}{-\frac{sIr'}{\alpha}\ln(\frac{sIr'}{\alpha})}$$

Note that this is indeed positive as expected, since $\frac{sIr'}{\alpha} \in (0,1) \Rightarrow \ln(\frac{sIr'}{\alpha}) < 0$. So the as $N \to \infty$, inequality 3.4.3 becomes:

$$\frac{\frac{sIr'}{\alpha} - 1}{\frac{sIr'}{\alpha}} < \ln(\frac{sIr'}{\alpha})$$

To simplify things, write $K = \frac{sIr'}{\alpha}$. The above inequality is then

$$\frac{K-1}{K} < \ln(K)$$

This can be shown to be true for $\frac{sIr'}{\alpha} \in (0,1)$. Note that at $K = 1$, $\frac{K-1}{K} = \ln(K)$. The derivative of the LHS of this expression is $\frac{1}{K^2}$, while the derivative of the RHS is $\frac{1}{K}$. So for $\frac{sIr'}{\alpha} = K \in (0,1)$, the LHS must increase faster than the RHS. Given that they cross at $K = 1$, it must be the case that $\frac{K-1}{K} < \ln(K)$ for $\frac{sIr'}{\alpha} = K \in (0,1)$. Therefore, given $\frac{sIr'}{\alpha} \in (0,1)$, condition 3.4.2 must hold for $N \to \infty$ and $\frac{dg}{dN} < 0$ for $N \geq 2$ ensures that it will also hold for $N \in [2,\infty)$. □

The first lemma states that if $\frac{sIr'}{\alpha} \in (0,1)$ and $2\bar{r} \geq r' > r_f$, satisfying 3.4.2 ensures that 3.4.1 gives a unique $\hat{Z} > 0$ if it has a solution there. The second lemma states that $\frac{sIr'}{\alpha} \in (0,1)$ and $2\bar{r} \geq r' > r_f$ are sufficient to ensure 3.4.2 is satisfied for $N \geq 2$, making equation 3.4.1 give a unique $\hat{Z} > 0$. This is exactly the same as 3.2.11 giving a $Z > \frac{sI}{\alpha}$, which completes the proof of the proposition.

Proof of Proposition 3.7.

Proof. Rearrange 3.2.11 into:

$$Z(\bar{r}N + (Zr_f^N)^{\frac{1}{N-1}}) = \frac{sI}{\alpha}(\bar{r}N + (Zr_f^N)^{\frac{1}{N-1}}) + 1 + (N-1)\frac{sI}{\alpha}Z^{\frac{1}{N-1}}r'^{\frac{1}{N-1}+1} - \frac{NsIr'}{\alpha}$$

and rearrange 3.2.13 into:

$$Z^*(\overline{r}N + (Z^* r_f^N)^{\frac{1}{N-1}}) = 1$$

For any r_H, r_L, β and N, $Z(\overline{r}N + (Zr_f^N)^{\frac{1}{N-1}}) > Z^*(\overline{r}N + (Z^* r_f^N)^{\frac{1}{N-1}})$ is a necessary and sufficient condition for $Z > Z^*$ given that both functions are increasing in Z and Z^* when $Z > 0$ and $Z^* > 0$. The similarity of the expressions and the fact that Z and Z^* are the only endogenous variables ensures this. Applying this condition gives:

(3.4.4) $$N(\bar{r} - r') + (Zr_f^N)^{\frac{1}{N-1}} + (N-1)Z^{\frac{1}{N-1}} r'^{\frac{1}{N-1}+1} > 0$$

From proposition 3.6, we can obtain a lower bound on Z, that $Z > \frac{sI}{\alpha}$. Thus $Z^{\frac{1}{N-1}} r'^{\frac{1}{N-1}} > (\frac{sIr'}{\alpha})^{\frac{1}{N-1}}$. Then, given that $\frac{sIr'}{\alpha} \in (0,1)$ and $N \geq 2$, then $(\frac{sIr'}{\alpha})^{\frac{1}{N-1}} > \frac{sIr'}{\alpha}$. Noting that $\frac{sIr_f}{\alpha} < \frac{sIr'}{\alpha}$, we can write

$$N(\bar{r} - r') + (Zr_f^N)^{\frac{1}{N-1}} + (N-1)Z^{\frac{1}{N-1}} r'^{\frac{1}{N-1}+1} > N(\bar{r} - r') + (N-1)\frac{sIr'}{\alpha} r' + \frac{sIr_f}{\alpha} r_f$$

Consider the conditions needed for $N(\bar{r} - r') + (N-1)\frac{sIr'}{\alpha} r' + \frac{sIr_f}{\alpha} r_f > 0$ for $N \geq 2$. A necessary condition is that the LHS of the expression increases with N, which gives $2\bar{r} + 2\frac{sIr'}{\alpha} r' > 2r'$. We also need this expression to be true for $N = 2$, which gives $2\bar{r} + \frac{sIr'}{\alpha} r' + \frac{sIr_f}{\alpha} r_f > 2r'$. Given that $r' > r_f$, $2\bar{r} + \frac{sIr'}{\alpha} r' + \frac{sIr_f}{\alpha} r_f < 2\bar{r} + 2\frac{sIr'}{\alpha} r'$. Thus only the second condition needs to be met to satisfy 3.4.4:

(3.4.5) $$2\bar{r} + \frac{sIr'}{\alpha} r' + \frac{sIr_f}{\alpha} r_f > 2r'$$

Now consider the initial condition for proposition 3.6 to hold: $2\bar{r} \geq r' > r_f$. Taking the condition $2\bar{r} \geq r \Rightarrow 2\bar{r} + r' \geq 2r'$ and comparing this with the above condition 3.4.5 leads to two scenarios:

Case 1. $\frac{sIr'}{\alpha}r' + \frac{sIr_f}{\alpha}r_f \geq r'$, then $2\bar{r} \geq r' \Rightarrow 2\bar{r} + \frac{sIr'}{\alpha}r' + \frac{sIr_f}{\alpha}r_f \geq 2r'$. So the required condition is:

$$2\bar{r} > r$$

Case 2. $\frac{sIr'}{\alpha}r' + \frac{sIr_f}{\alpha}r_f < r'$, then $2\bar{r} + \frac{sIr'}{\alpha}r' + \frac{sIr_f}{\alpha}r_f > 2r' \Rightarrow 2\bar{r} > r'$. So the required condition is:

$$2\bar{r} + \frac{sIr'}{\alpha}r' + \frac{sIr_f}{\alpha}r_f > 2r'$$

This completes the proof. □

Proof of Proposition 3.8. Use a similar strategy to that employed in chapter 1. Assume the cdf $F_L(r)$ played by the L type is:

$$F_L(r) \begin{cases} 1 & r \in [k_u, \infty) \\ A[(Z - \frac{sI}{\alpha})r + \frac{sIr'}{\alpha}]^{\frac{1}{N-1}} - A(Zr_f)^{\frac{1}{N-1}} + C_L & r \in [r', k_u] \\ A(Zr)^{\frac{1}{N-1}} - A(Zr_f)^{\frac{1}{N-1}} + C_L & r \in [r_f, r'] \\ C_L & r \in [0, r_f] \end{cases} \tag{3.4.6}$$

We pick A_L and C_L so that the ability constraint for the L type is binding and the probability mass integrates to 1:

$$A[(k_u - \frac{sI}{\alpha})r + \frac{sIr'}{\alpha}]^{\frac{1}{N-1}} - A(Zr_f)^{\frac{1}{N-1}} + C_L = 1$$

$$\int_{r'}^{k_u} \frac{rA(Z-\frac{sI}{\alpha})[(Z-\frac{sI}{\alpha})r+\frac{sIr'}{\alpha}]^{\frac{1}{N-1}-1}}{(N-1)}dr + \int_{r_f}^{r'} \frac{rAZ^{\frac{1}{N-1}}r^{\frac{1}{N-1}-1}}{(N-1)}dr = r_L$$

By noting that $(Z-\frac{sI}{\alpha})k_u + \frac{sIr'}{\alpha} = 1$ from the definition of $P_i(r_i)$ (equation3.2.8), these two equations can be simplified to:

(3.4.7) $$1 = A - A(Zr_f)^{\frac{1}{N-1}} + C_L$$

(3.4.8)
$$r_L = A[(k_u - (Zr')^{\frac{1}{N-1}}r') - (\frac{N-1}{N})\frac{(1-(Zr')^{\frac{1}{N-1}+1})}{(Z-\frac{sI}{\alpha})}] + \frac{AZ^{\frac{1}{N-1}}(r'^{\frac{1}{N-1}+1} - r_f^{\frac{1}{N-1}+1})}{N}$$

We can divide the two equations to get

$$1 - C_L = \frac{r_L(1-(Zr_f)^{\frac{1}{N-1}})}{[(k_u - (Zr')^{\frac{1}{N-1}}r') - (\frac{N-1}{N})\frac{(1-(Zr')^{\frac{1}{N-1}+1})}{(Z-\frac{sI}{\alpha})}] + \frac{Z^{\frac{1}{N-1}}(r'^{\frac{1}{N-1}+1} - r_f^{\frac{1}{N-1}+1})}{N}}$$

And then rearrange 3.4.7 and sub in the expression for $1 - C_L$ to get

$$A = \frac{r_L}{[(k_u - (Zr')^{\frac{1}{N-1}}r') - (\frac{N-1}{N})\frac{(1-(Zr')^{\frac{1}{N-1}+1})}{(Z-\frac{sI}{\alpha})}] + \frac{Z^{\frac{1}{N-1}}(r'^{\frac{1}{N-1}+1} - r_f^{\frac{1}{N-1}+1})}{N}}$$

Fortunately, this can be greatly simplified. Noting that $k_u = \frac{1-\frac{sIr'}{\alpha}}{Z-\frac{sI}{\alpha}}$ from 3.2.10 and simplifying, we get

$$A = \frac{r_L}{[\frac{1-\frac{NsIr'}{\alpha}-(Zr')^{\frac{1}{N-1}+1}+\frac{NsI}{\alpha}Z^{\frac{1}{N-1}}r'^{\frac{1}{N-1}+1}}{N(Z-\frac{sI}{\alpha})}] + \frac{Z^{\frac{1}{N-1}}(r'^{\frac{1}{N-1}+1} - r_f^{\frac{1}{N-1}+1})}{N}}$$

We can write this as

(3.4.9)
$$A = \frac{r_L}{[\frac{1-\frac{NsIr'}{\alpha}+(N-1)(Zr')^{\frac{1}{N-1}}r'+(\frac{sI}{\alpha}-Z)(Zr')^{\frac{1}{N-1}}r'}{N(Z-\frac{sI}{\alpha})}] + \frac{Z^{\frac{1}{N-1}}(r'^{\frac{1}{N-1}+1}-r_f^{\frac{1}{N-1}+1})}{N}}$$

Recalling the equation for Z from 3.2.11:

$$Z - \frac{sI}{\alpha} = \frac{1}{\bar{r}N + (Zr_f^N)^{\frac{1}{N-1}}}[1 + (N-1)\frac{sI}{\alpha}Z^{\frac{1}{N-1}}r'^{\frac{1}{N-1}+1} - \frac{NsIr'}{\alpha}]$$

and substituting this into 3.4.9 gives

(3.4.10)
$$A = \frac{r_L}{\bar{r}}$$

To see that $F_L$must be a valid cdf, note that proposition 3.6 ensures that $Z > \frac{sI}{\alpha} > 0$ given $\frac{sIr'}{\alpha} \in (0,1)$ and $\bar{r} > r' > r_f$. Then, it is clear that the pdf $f_L = \frac{dF_L}{dr}$ is positive or zero for $r > 0$. From the expression for P (equation 3.2.8), it is clear that $Zr_f \in (0,1)$, since $Z > \frac{sI}{\alpha}$. From 3.4.7 and the expression for A, we can obtain $1 - C_L = \frac{r_L}{\bar{r}}(1 - (Zr_f)^{\frac{1}{N-1}})$. $Zr_f \in (0,1) \Rightarrow (Zr_f)^{\frac{1}{N-1}} \in (0,1)$, and $\frac{r_L}{\bar{r}} \in (0,1)$ by definition. This implies that $C_L \in (0,1)$ as well. Therefore F_L is a valid cdf.

Given the form of F_L and the requirement that $P_i(r_i)$ must be of the form given in 3.2.8, F_H must be given by the following expression in symmetric equilibrium:

(3.4.11)
$$F_H(r)\begin{cases} 1 & r \in [k_u, \infty) \\ \frac{([Z-\frac{sI}{\alpha}]r+\frac{sIr'}{\alpha})^{\frac{1}{N-1}}}{\beta} - \frac{(1-\beta)}{\beta}(A[(Z-\frac{sI}{\alpha})r + \frac{sIr'}{\alpha}]^{\frac{1}{N-1}} - A(Zr_f)^{\frac{1}{N-1}} + C_L) & r \in [r', k_u] \\ \frac{(Zr)^{\frac{1}{N-1}}}{\beta} - \frac{(1-\beta)}{\beta}[A(Zr)^{\frac{1}{N-1}} - A(Zr_f)^{\frac{1}{N-1}} + C_L] & r \in [r_f, r'] \\ \frac{(Zr_f)^{\frac{1}{N-1}}}{\beta} - \frac{(1-\beta)}{\beta}C_L & r \in [0, r_f] \end{cases}$$

Now, what needs to be done is to show that this is a valid cdf:

LEMMA. *Given that proposition 3.6 holds and the form for F_H 3.4.11, the following condition on the exogenous parameters is sufficient for $f_H(r) \geq 0$ for $N \geq 2$:*

$$\frac{sIr_f}{\alpha} \geq \frac{(1-\beta)(r_H - r_L)}{r_H}$$

PROOF. There are two parts to the pdf f_H implied by F_H, a delta function corresponding to the mass atom at $r = 0$, and a finite density elsewhere. We need to find conditions on the parameters so that $f_H \geq 0$ for $r > 0$ and the mass atom at $r = 0$ is between zero (inclusive) and one, i.e.

$$0 \leq \frac{(Zr_f)^{\frac{1}{N-1}}}{\beta} - \frac{(1-\beta)}{\beta} C_L < 1$$

Begin by considering $r > 0$. Differentiate the expression for F_H, 3.4.11, over $r > 0$ to get:

$$f_H(r) \begin{cases} 0 & r \in [k_u, \infty) \\ \frac{(Z-\frac{sI}{\alpha})([Z-\frac{sI}{\alpha}]r+\frac{sIr'}{\alpha})^{\frac{1}{N-1}-1}}{\beta(N-1)} - \frac{(1-\beta)}{\beta} A \frac{(Z-\frac{sI}{\alpha})([Z-\frac{sI}{\alpha}]r+\frac{sIr'}{\alpha})^{\frac{1}{N-1}-1}}{(N-1)} & r \in [r', k_u] \\ \frac{Z(Zr)^{\frac{1}{N-1}-1}}{(N-1)\beta} - \frac{(1-\beta)}{\beta} A \frac{Z(Zr)^{\frac{1}{N-1}-1}}{(N-1)} & r \in [r_f, r'] \\ 0 & r \in (0, r_f] \end{cases}$$

From this, the conditions for $f_H(r) \geq 0$ are:

$$\frac{(Z-\frac{sI}{\alpha})([Z-\frac{sI}{\alpha}]r+\frac{sIr'}{\alpha})^{\frac{1}{N-1}-1}}{\beta(N-1)} - \frac{(1-\beta)}{\beta} A \frac{(Z-\frac{sI}{\alpha})([Z-\frac{sI}{\alpha}]r+\frac{sIr'}{\alpha})^{\frac{1}{N-1}-1}}{(N-1)} \geq 0$$

and

$$\frac{Z(Zr)^{\frac{1}{N-1}-1}}{(N-1)\beta} - \frac{(1-\beta)}{\beta} A \frac{Z(Zr)^{\frac{1}{N-1}-1}}{(N-1)} \geq 0$$

Given that $Z > \frac{sI}{\alpha}$ from proposition 3.6 and $N \geq 2$, it is straightforward to show that these both simplify to

$$A \leq \frac{1}{1-\beta}$$

Given $A = \frac{r_L}{\bar{r}}$ from 3.4.10, this is clearly true. What remains to be done is to show that the mass atom at zero played by the H type is between zero and one:

$$0 \leq \frac{(Zr_f)^{\frac{1}{N-1}}}{\beta} - \frac{(1-\beta)}{\beta} C_L < 1$$

First, consider $\frac{(Zr_f)^{\frac{1}{N-1}}}{\beta} - \frac{(1-\beta)}{\beta} C_L < 1$. Substituting in $C_L = 1 - \frac{r_L}{\bar{r}}[1 - (Zr_f)^{\frac{1}{N-1}}]$ and simplifying gives:

$$0 < (1 - (Zr_f)^{\frac{1}{N-1}})[1 - \frac{r_L}{\bar{r}}(1-\beta)]$$

Given that $Zr_f \in (0,1) \Rightarrow (Zr_f)^{\frac{1}{N-1}} \in (0,1)$, then this always holds. It is also possible to see that $\frac{(Zr_f)^{\frac{1}{N-1}}}{\beta} - \frac{(1-\beta)}{\beta} C_L < 1$ because if this was not the case, then $f_H < 0$ somewhere for $r > 0$, since that is the only way for the probability mass to integrate to 1. The fact that $f_H(r) \geq 0$ for $r > 0$ means that $\frac{(Zr_f)^{\frac{1}{N-1}}}{\beta} - \frac{(1-\beta)}{\beta} C_L > 1$ is impossible. The possibility of $\frac{(Zr_f)^{\frac{1}{N-1}}}{\beta} - \frac{(1-\beta)}{\beta} C_L = 1$ is ruled out by the fact that $A < \frac{1}{1-\beta}$ always, which means that $f_H > 0$ from the expression derived previously.

Finally, consider $\frac{(Zr_f)^{\frac{1}{N-1}}}{\beta} - \frac{(1-\beta)}{\beta} C_L > 0$. Subbing in $C_L = 1 - \frac{r_L}{\bar{r}}[1 - (Zr_f)^{\frac{1}{N-1}}]$ and simplifying gives:

$$(Zr_f)^{\frac{1}{N-1}}[1 - \frac{r_L}{\bar{r}}(1-\beta)] \geq (1-\beta)(1 - \frac{r_L}{\bar{r}})$$

Note that the RHS of this inequality is strictly increasing in $(Zr_f)^{\frac{1}{N-1}}$. Thus, to find conditions on the parameters such that this is true for $N \geq 2$, we just need to find a version of this expression with the lowest possible value of $(Zr_f)^{\frac{1}{N-1}}$ for $N \geq 2$. Note that $(Zr_f)^{\frac{1}{N-1}}$ is increasing in Z. Since it is assumed that the conditions required for proposition 3.6 hold, we can use it to form a lower bound for Z, which is $Z > \frac{sI}{\alpha}$. Thus $(Zr_f)^{\frac{1}{N-1}} > (\frac{sI}{\alpha}r_f)^{\frac{1}{N-1}}$. Given that $0 < r_f < r'$ and that $\frac{sIr'}{\alpha} \in (0,1)$, $\frac{sIr_f}{\alpha} \in (0,1)$ too. Then, note that $(\frac{sI}{\alpha}r_f)^{\frac{1}{N-1}}$ is increasing in N. Thus, the lowest bound of $(\frac{sI}{\alpha}r_f)^{\frac{1}{N-1}}$ must be when N is its minimum value, i.e.$N = 2$. Thus we can write $(Zr_f)^{\frac{1}{N-1}} > \frac{sIr_f}{\alpha}$, and that a sufficient condition for equilibria to exist for $N \geq 2$ is:

$$\frac{sIr_f}{\alpha} \geq \frac{(1-\beta)(r_H - r_L)}{r_H} \tag{3.4.12}$$

Thus the mass atom at $r = 0$ will be between zero and one in size, and therefore the density $f_H(r) \geq 0$ everywhere. This completes the proof of the lemma. □

The proof of this lemma immediately implies that F_H is a valid cdf since Z and k_u are solved such that $\int f_H(r)dr = 1$. In addition, the choice of Z and k_u ensure that $\int rf_H(r)dr = r_H$. Therefore, as long as the previous lemma holds, it will be possible to construct a pair of cdfs F_L and F_H of the form given in this proof that constitute an equilibrium for any N of interest ($N \geq 2$): they are valid cdfs that make the ability constraints binding and also ensure that $P_i(r_i)$ is of the form required by 3.2.8.

Bibliography

[1] Ackermann, Carl, Richard McEnally, and David Ravenscraft. "The performance of hedge funds: Risk, return, and incentives." Journal of Finance (1999): 833-874.

[2] Agarwal, Vikas, and Narayan Y. Naik. "Risks and portfolio decisions involving hedge funds." Review of Financial studies 17.1 (2004): 63-98.

[3] Arya, Anil, and Brian Mittendorf. "Offering stock options to gauge managerial talent." Journal of Accounting and Economics 40.1 (2005): 189-210.

[4] Baye, Michael R., Dan Kovenock, and Casper G. De Vries. "The all-pay auction with complete information." Economic Theory 8.2 (1996): 291-305.

[5] Baye, Michael R., Dan Kovenock, and Casper G. de Vries. "It takes two to tango: equilibria in a model of sales." Games and Economic Behavior 4.4 (1992): 493-510.

[6] Berk, Jonathan B., and Richard C. Green. Mutual fund flows and performance in rational markets. No. w9275. National Bureau of Economic Research, 2002

[7] Brown, Stephen J., William N. Goetzmann, and Roger G. Ibbotson. "Offshore hedge funds: Survival and performance, 1989-95." Journal of Business 72.1 (1999).

[8] Brown, Stephen J., William N. Goetzmann, and James Park. "Careers and survival: Competition and risk in the hedge fund and CTA industry." The Journal of Finance 56.5 (2001): 1869-1886.

[9] Das, Sanjiv Ranjan, and Rangarajan K. Sundaram. "Fee Speech: Signaling, Risk Sharing, and the Impact of Fee Structures on Investor Welfare." Review of Financial Studies 15.5 (2002): 1465-1497.

[10] Foster, Dean P., and H. Peyton Young. "Gaming performance fees by portfolio managers." The Quarterly Journal of Economics 125.4 (2010): 1435-1458.

[11] Fung, William KH, and David A. Hsieh. "Hedge funds: An industry in its adolescence." Economic Review-Federal Reserve Bank Of Atlanta 91.4 (2006): 1.

[12] Gross, Oliver, and Robert Wagner. A continuous Colonel Blotto game. No. RAND-RM-408. RAND PROJECT AIR FORCE SANTA MONICA CA, 1950.

[13] Hellmann, Thomas F., Kevin C. Murdock, and Joseph E. Stiglitz. "Liberalization, moral hazard in banking, and prudential regulation: Are capital requirements enough?." American economic review (2000): 147-165.

[14] Hodder, James E., and Jens Carsten Jackwerth. "Incentive contracts and hedge fund management." Journal of Financial and Quantitative Analysis 42.04 (2007): 811-826.

[15] Myerson, Roger B. "Incentives to cultivate favored minorities under alternative electoral systems." American Political Science Review 87.04 (1993): 856-869.

[16] Spiegler, Ran. "Competition over agents with boundedly rational expectations." (2006).

[17] Taylor, Jonathan. "Risk-taking behavior in mutual fund tournaments." Journal of Economic Behavior & Organization 50.3 (2003): 373-383.

[18] Varian, Hal R. "A model of sales." The American Economic Review (1980): 651-659.

[19] Weisman, Andrew B. "Informationless investing and hedge fund performance measurement bias." The Journal of Portfolio Management 28.4 (2002): 80-91.