UNIQUENESS OF g-MEASURES AND THE INVARIANCE OF THE BETA-FUNCTION UNDER FINITARY ISOMORPHISMS, WITH FINITE EXPECTED CODE LENGTHS, BETWEEN g-SPACES.

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DECLARATION

All work within this thesis is original except where explicitly stated.

ABSTRACT

The following is split into two chapters. The first chapter gives a brief history concerning g-measures, their state of investigation and under what conditions, on g, unique g-measures exist. It concludes by giving equivalent conditions for a g-function to have a unique g-measure. This will, possibly, lead to a solution to Keane's original problem about the uniqueness of a g-measure for an arbitrary g-function.

The second chapter generalises the result of Prof. K. Schmidt that the Beta-function is invariant under finitarily isomorphic (with finite expected code length) Markov spaces, to g-spaces with certain conditions on the g-function. The approach adopted is essentially that of Schmidt with slight modifications due to the more restrictive nature of the problem. The condition on the g-function, that of finite first moment variational sum, fits nicely between the two more commonly used conditions, finite variation sum and exponentially decreasing variation.

CHAPTER 1

EQUIVALENT CONDITIONS FOR THE UNIQUENES OF g-MEASURES

1. INTRODUCTION

The study of g-measures was derived from trying to understand equilibrium states and phase transitions, which have direct applications in the field of statistical mechanics.

The problem as to whether a g-function has a unique g-measure was originally posed by Keane [6] in 1974, where he studied so-called "covering transformations". The problem, derived from his work, is an example of a covering transformation the one-sided subshift of finite type. This uniqueness problem, I'm afraid, I was unable to solve. However, in the process of trying to provide a solution, I was able to produce equivalent conditions for the uniqueness of g-measure. In Walters [12] a sufficient condition for uniqueness was given. However this, unfortunately, was not a necessary condition as exhibited by Hofbauers example. I produce in this paper a new class of examples, generalizing Hofbauers example, which again have unique g-measures but do not satisfy the Walters condition.

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2. ONE SIDED SUBSHIFTS OF FINITE TYPE AND g-MEASURES

Let X_0 be a set of symbols (states) of finite cardinality $|X_0|$. Denote by $X+$ the one sided full shift

$$X+ := \prod_{\mathbb{N}} X_0 = \{(x_n)_0^\infty | x_i \in X_0\}.$$

The shift transformation, denoted by σ, operates on $X+$ as follows:

$$\sigma : X+ \to X+, \quad \text{where } (\sigma(x))_n = X_{n+1}$$

(i.e. σ -moves the coordinates of x 1-place nearer the zeroth coordinate, the zeroth coorindate dropping off the end.)

A closed subset $X \subset X+$ is said to be a subshift of finite type if $\sigma X = X$ and the points of X are completely determined by a finite collection, G, of sets, if each $C \in G$ is a member of X_0^N, some $N > 0$. (i.e. when we look at N coordinates, of a point in X, it defines a set in G.)

X is said to be one-sided topologically transitive if for each non-empty sets U,V, in X, $\exists\, n \geq 1$ st $\sigma^{-n}U \cap V \neq \emptyset$. X is said to be topologically mixing if $\exists\, N > 1$ with $\sigma^{-n}U \cap V \neq \emptyset \quad \forall n \geq N$.

It is well known, when X is a one-sided topologically transitive subshift of finite type, that X can be represented as a disjoint union of closed subsets, $\{X_i\}_{i=1}^d$, with $\sigma X_i = X_{i+1 \bmod d}$, $\sigma^d X_i = X_i$, and σ^d is topologically mixing when restricted to X_i. (see Chung [1]). The number $d \geq 1$ is called the period of X.

Definition 2.1

With X+as above we shall denote by $M(X)$ $M(X,\sigma)$ the set of all probability and σ-invariant Borel-probabilities on X. (i.e. $\mu \in M(X,\sigma)$ if $\mu(\sigma^{-1}B) = \mu(B)$ for all Borel subsets of X.)

Keane [6] originally defined a g-measure in terms of, what has become known as, the Ruelle operator. However, since that time, many equivalent conditions have been found so we therefore shall define a g-measure by the results of the next theorem. First, however, we shall need to know what is meant by the Ruelle operator.

For $\phi \in C(x)$ define the Ruelle operator $L_\phi : C(X) \to C(X)$ by $(L_\phi f(x) = \sum_{y\in\sigma^{-1}x} e^{\phi(y)} f(y)$. We look at special functions of the form $\phi = \log g$ where $g \in G = \{g \in C(x) | g > 0$ and $\sum_{y\in\sigma^{-1}x} g(y) = 1$ for all $x \in X+\}$. Thus $(L_{\log g} 1)(x) = \sum_{y\in\sigma^{-1}x} g(y)$, $L_{\log g} 1 = 1$, and $L_{\log g} U_\sigma f = f$ where $U_\sigma f = f \circ \sigma$. Such g are called g-functions. The following Theorem is used as a definition of a g-measure, giving several equivalent conditions for a g-measure. In this Theorem $L^* : C(X)^* \to C(X)^*$ denotes the adjoint of $L_{\log g} : C(X) \to C(X)$ and $E_\mu(f|\sigma^{-1}\mathcal{B})$ denotes the conditional expectation of $f \in L'(\mu)$ relative to the σ-algebra $\sigma^{-1}\mathcal{B}$, where $\mathcal{B}$ denotes the Borel σ-algebra of X.

Theorem 2.1 (Ledrappier [8])

Let $g \in G$ and $\mu \in M(X) = \{$the probability measures on X$\}$. If L denotes $L_{\log g}$ the following are equivalent

(i) $\mathcal{L}^*\mu = \mu$.

(ii) $\mu \in M(X,\sigma)$ and $E_\mu(f|\sigma^{-1}\mathcal{B})(x) = \sum_{z\in\sigma^{-1}\sigma x} g(z)f(z)$ a.e. μ

for $f \in L'(\mu)$

(iii) $\mu \in M(X,\sigma)$ and μ is an equilibrium state for $\log g$. In fact $h_\mu(\sigma) + \mu(\log g) = 0$.

(iv) $\frac{d\mu\sigma}{d\mu} = \frac{1}{g}$. □

A μ satisfying any of (i), (iv) and hence all is called a g-measure.

<u>Lemma 2.1</u>

(a) If $g \in G$ then μ has full support, i.e. each g-measure μ gives positive measure to each non-empty open set.

(b) If $g_1, g_2 \in G$ and some g_1-measure coincides with some g_2-measure then $g_1 = g_2$.

<u>Proof</u>

See Walters [12].

By the Schauder-Tychanoff fixed point theorem, (Dunford and Schwartz [5], page 456) $\mathcal{L}^*$ always has a fixed point in $M(X)$ so a g-measure always exists. The question immediately posed is as to whether there is a unique g-measure given an arbitrary $g \in G$. The following partial result is due to P. Walters although I have extended it slightly from the topologically mixing case to the transitive.

Theorem 2.2

Let $\sigma: X \to X$ be a topoligically transitive one sided subshift of finite type and $g \in G$. Denote $L_{\log g}$ by L. Then, if

$\sum_{m=1}^{\infty} \operatorname{var}_m(\log g) < \infty$ (p = period of the transformation),

$\sum_{n=1}^{N-1} \frac{L^n f}{N}$ converges uniformly to a constant $\mu(f)$ $\forall f \in C(X)$.

The μ is the unique g-measure.

Proof

Walters, in [12], proves the result for the topologically mixing case so essentially all we have to do, in the transitive case, is to reduce this to the topologically mixing one. By earlier comments we can represent a transitive $X = \bigcup_{i=1}^{p} X_i$ such that $\sigma^p|X_i$ is a topologically mixing map. The function $p(x) = g(x)g(\sigma x)\ldots g(\sigma^{p-1}(x)$ is a g-function with respect to σ^p (i.e. $\sum_{y \in \sigma^{-p}x} g(x)\cdots g(\sigma^{p-1}x) = 1$). Thus, if we can verify $\sum \operatorname{var}_m p < \infty$, we can apply Walters Theorem on X_i w.r.t. σ^p, assume $m > p$

$$\operatorname{var}_m \log p = \operatorname{var}_m \left(\sum_{k=0}^{p} \log g(\sigma^k x)\right)$$

$$\leq \sum_{k=m-p}^{m} \operatorname{var}_k \log g$$

$$\text{so } \sum_{m>p} \operatorname{var}_m \log p \leq \sum_{m=p+1}^{\infty} \sum_{k=m-p}^{m} \operatorname{var}_k \log g < \infty$$

$$\leq \sum_{m=1}^{\infty} (p+1) \operatorname{var}_m \log g < \infty.$$

Then by Walters for each $1 \le i \le p$ we can find a σ^p-invariant measure $\mu_i \in M(X_i, \sigma^p)$ such that if $f \in C(X_i)$, $\lim_{m\to\infty} L^m_{(\log P, \sigma^p)} f \to \mu_i(f)$.

$$\left(\lim_{m\to\infty} \sum_{y \in \sigma^{-mp}x} P(y)P(\sigma^p y)\ldots P(\sigma^{p(m-1)}y)f(y) = \mu_i(f)\right).$$

Let $f \in C(X)$. We can express f as $f = f_1 + \ldots + f_p$ where $f_i = f|X_i \in C(X_i)$, p the period of the subshift.

Then if $x \in X_p$

$$\frac{1}{mp} \sum_{k=0}^{mp-1} L^k_{\log g} f(x)$$

$$= \frac{1}{mp} \sum_{k=0}^{m-1} \Big(\sum_{i=1}^{p} \sum_{y \in \sigma^{-(p-i)}(x)} \big(\sum_{z \in \sigma^{-kp}y} g(z)\ldots g(\sigma^{kp-1}z) f_i(z)\big) g(y) g(\sigma y) \ldots g(\sigma^{(p-i-1)}y)$$

$$= \frac{1}{mp} \sum_{i=1}^{p} \sum_{y \in \sigma^{-(p-i)}(x)} \sum_{k=0}^{m-1} L^{kp} f_i(y) g(y) g(\ y) \ldots g(\sigma^{(p-i-1)}y)$$

letting $m \to \infty$ this converges to

$$\frac{1}{p} \sum_{i=1}^{p} \Big(\sum_{y \in \sigma^{-(p-i)}(x)} g(y) \ldots g(\sigma^{(p-i-1)}y) \mu_i(f)$$

$$= \int f \, d\frac{(\mu_1 + \cdots + \mu_p)}{p} \quad .$$

We can show by a similar method if $x \in X_i$

$$\frac{1}{mp} \sum_{k=0}^{mp-1} L^k_{\log g} f(x) \to \int f \quad d\frac{(\mu_1 + \cdots + \mu_p)}{p}$$

This convergence is uniform on each X_i and thus uniform on X. We thus have proved $\frac{1}{mp}\sum_{k=0}^{mp-1} L_{\log g}^k f$ converges uniformly to

$$\int f\,d\mu \text{ if } \mu = \frac{\mu_1 + \dots + \mu_p}{p},$$

as $n \to \infty$. I claim this implies $h_N = \frac{1}{N}\sum_{n=0}^{N-1} L_{\log g}^n f$ converges uniformly to $\int f\,d\mu$ as $N \to \infty$.

Proof of Claim

If $h \in C(x)$. Let $\alpha(h) = \min h$ and $\beta(h) = \max h$. Then we have

$$\alpha(h_0) \le \alpha(h_1) \le \int f\,d\mu \le \quad \le \beta(h_0) \le \beta(h_0)$$

by the above. Also since $\alpha(h_{mp}) \to \int f\,d\mu$ and $\beta(h_{mp}) \to \int f\,d\mu$, as $m \to \infty$, it follows that $\lim_{N\to\infty} h_N = \int f\,d\mu$ and the convergence is uniform.

By the above convergence property it follows that $L_{\log g}^* \mu = \mu$ and μ is a g-measure.

If ν is another g-measure

$$\int f\,d\nu = \int \frac{1}{N}\sum_{n=0}^{N-1} L_{\log g}^n f\,d\nu \text{ for each } N$$

But as $N \to \infty$ the integrand converges to $\int f\,d\mu$. Thus $\nu = \mu$ and μ is the unique g-measure

Remark

Note that if $p > 1$ then $L^n_{\log g} f \not\to \int f\,d\mu$. For if

$f = f_1 = \chi_{X_1}$ then $L^{np} f_1 \to \int f_1 d\mu = \dfrac{\mu_1(X_1) = \mu(X_1)}{P}$ on X_i

but $L^{np} f_1 \to 0$ on $X \backslash X_1$. □

It follows however, from the above, in the topologically mixing case that $L^n_{\log g} f$ converges to $\int f\,d\mu$. The condition that $\sum_{n=1}^{\infty} \mathrm{var}_n \log g < \infty$ is not however a necessary condition. There are a class of functions where this sometimes fails, these functions are called Grid functions (see Markley-Paul[2]).

Let $\underline{0} = (0,\dots 0 \dots) \subset \{0,1\}^N$ be the point with all coordinates zero. Let $\{M_n\}$ be a partition of $[0] \backslash \underline{0}$ with the following properties.

(i) $\lim M_n = \underline{0}$

i.e. $d(M_n, \underline{0}) \to 0$ as $m \to \infty$ (i.e. $\max\limits_{x \in M_n} d(x, \underline{0}) \to 0$ as $n \to \infty$).

(ii) Each M_n is closed and open (i.e. a finite union of cylinders).

(iii) k_0 st. if $B \subset [0]_0 \backslash \underline{0}$ is a cylinder, of length greater than k_0, there is a $M_j(B)$ with $B \subset M_{j(B)}$.

Let $1 > a_n > 0$ be a decreasing sequence of reals with a_n converging to a.

Define $g = \sum_{n=1}^{\infty} M_n a_n + a\,\chi_{\underline{0}}$ on $[0]_0$ and $g(1x) = 1 - g(0x)$

(i.e. g is defined to be a g-function).

I claim g has a unique g-measure. I will show that if B is a cylinder, contained in $[0]\backslash\underline{0}$, of length greater than k_0, μ a g-measure, then $\mu(B)$ is uniquely determined by M_n and g. Since μ is non-atomic by Lemma 2.1, this shows μ is uniquely determined on [0].

<u>Lemma 2.2</u> (Markley-Paul)

Let $A \subset [0] \backslash \underline{0}$ be a cylinder of length ℓ and $B \subset [0]\backslash\underline{0}$ a cylinder of length $\geq K_0$ Then $\mu(AB) = K(A,B)\ \mu(B)$.

(K (A,B) is a constant depending also on g and the partition).

<u>Proof</u>

$$\mu(AB) = \int_{\substack{X=\Pi\{0,1\} \\ N}} L^{\ell}\chi_{AB}\, d\mu.$$

$$= \int \sum_{y\in\sigma^{-\ell}x} g(y)\cdots g(\sigma^{\ell-1}y)\chi_A(y)\chi_B(x)\, d\mu\ .$$

Let $A = [a_0,\ldots,a_{\ell-1}]$ and $a = (a_0,\ldots,a_{\ell-1})$. Then

$$\int \sum_{y\in\sigma^{-\ell}x} g(y)\ldots g(T^{\ell-1}y)\chi_A(y)\ \chi_B(x)\ d\mu$$

$$= \int g(ax)\ldots g(\sigma^{\ell-1}ax)\ \chi_B(x)\ d\mu$$

$= K(A,B)\ \mu(B)$ by property (iii) on M_n and definition of g. □

Thus by Lemma 2.2

$$\mu(BAB) = K(BA,B)\mu(B) \qquad 2.(i)$$

Now by Kac's Theorem (see [13]), if $r(z)$ denotes the return time of a point in B to B, we have

$$1 = \int_B r(z)d\mu = \Sigma \binom{\text{numbers depending}}{\text{only on } g \text{ and } M_n} \mu(B)$$

(by expression 2.(i)).

Thus $\mu(B)$ is uniquely determined by g and M_n). Similarly we can construct the partition of $[1]_0$
$n \geq 1$, $M_n' = \{(1x) : 0x \in M_n\}$, $(1\ \underline{0})$ and obtain that μ is uniquely determined on cylinders B of length greater than K_0. It thus follows that μ is uniquely determined.

An example of a grid function is an adaptation of Hofbauer's (see [3]) example where we take $M_n = [\underbrace{0 \ldots 0}_{n} 1]$, $n > 1$. An easy computation shows that $\mathrm{var}_n \log g = \log a_n/a$. Therefore, if we choose $a_n = \exp(\frac{1}{n} + \log a)$ we have that $\Sigma \log a_n$ does not exist, the condition in Theorem 2.2 is, therefore, not a necessary one (see P. Hulse, Ph.D. Thesis) [4].

Theorem 2.3 - THE MAIN THEOREM AND ITS PROOF.

Let $g \in G$ then the following are equivalent

(i) g has a unique g-measure .

(ii) $\frac{1}{N} \sum_{n=0}^{N-1} L^n_{\log g} f(x) \to \mu(f)$ for all x and each $f \in C(X)$.

(iii) As (ii) but the convergence is uniform.

(iv) $C(X) = \mathbb{C} \oplus \bar{B}$ where $B = \{L_{\log g} f - f \mid f \in C(X)\}$.

Proof

The proof is essentially the same as when we are looking for a uniquely ergodic shift invariant measure. (See Parry [10]). First note that if $f \in \mathbb{C}$ (i.e. a constant function)

$\frac{1}{N} \sum_{n=0}^{N-1} L^n_{\log g} f = f = \mu(f)$ and convergence is trivial.

Similarly, if $f = L_{\log} h - h \in B$; we have

$$\left\| \frac{1}{N} \sum_{n=0}^{N-1} L^n_{\log g} f \right\|_\infty = \left\| \frac{L^N_{\log g} h - h}{N} \right\|_\infty \leq \left[\frac{\| L_{\log g} \|^N + 1}{N} \right] \|h\|_\infty$$

which clearly tends to zero as $N \to \infty$. Therefore $\frac{1}{N} \sum_{n=0}^{N-1} L^n_{\log g} f$ tends uniformly to 0 when $f \in B$ or, by approximation, when $f \in \bar{B}$. Clearly $\bar{B} \cap \mathbb{C} = \{0\}$, and $\frac{1}{N} \sum_{n=0}^{N-1} L^n_{\log g}$ leaves functions in $\mathbb{C}$ unaltered and converges to zero for functions in $\bar{B}$. These remarks show that (iv) $\Rightarrow$ (iii), (iii) $\Rightarrow$ (ii). (ii) $\Rightarrow$ (i) follows from the fact that if μ_1 is another g-measure, $f \in C(X)$, then

$$\mu_1(f) = \lim_{N\to\infty} \int \frac{1}{N} \sum_{n=0}^{N-1} L_{\log g}^n f \, d\mu_1 = \mu(f)$$

and $\mu_1 = \mu$ (see Walters [11] Theorem 6.2 page 147). It thus remains to show (i) $\Rightarrow$ (iv).

Let $x \in X \subseteq \prod_{\mathbb{N}} X_0$, since X is a compact metric space we can choose a dense set $\{f_n\}_{n=1}^{\infty}$ of functions in $(C(X), \| \; \|_\infty)$ (see Kelley [7]).

Note that, since $\|\frac{1}{N} \sum_{n=0}^{N-1} L_{\log g}^n f_1(x)\| \le \|f_1\|$; we can choose a subsequence $\mathbb{N}_1 \subset \mathbb{N}$ such that $\lim_{N\in\mathbb{N}_1} \frac{1}{N} \sum_{n=0}^{N-1} L_{\log g}^n f_1(x)$ converges. Again, since $\frac{1}{N} \sum_{n=0}^{N-1} L_{\log g}^n f_2(x)$, $N \in \mathbb{N}_1$, is a bounded sequence, we can choose a subsequence $\mathbb{N}_2 \subset \mathbb{N}_1$ such that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} L_{\log g}^n f_2(x)$$

exists. Repeating this argument for each f_i we obtain sequence of integers $\mathbb{N}_1 \supset \mathbb{N}_2 \supset \mathbb{N}_3$, where $\mathbb{N}_i = M_1^i, M_2^i \ldots$ such that $\lim_{N\in\mathbb{N}_i} \frac{1}{N} \sum_{n=0}^{N-1} L_{\log g}^n f_j(x)$ exists for $j \le i$. Taking the diagonal sequence $N = M_1^1, m_2^2, \ldots$ we have that $\lim_{N\in\mathcal{N}} (1/N) \sum_{n=0}^{N-1} L_{\log g}^n f_i(x)$ exists for all i. Since $\{f_i\}$ is dense in $C(X)$.

$$\lim_{N\in\mathcal{N}} (1/N) \sum_{n=0}^{N-1} L^n_{\log g} f(x) = J(f) \text{ exists for all } f \in C(X).$$

This defines a continuous linear functional which is clearly positive. Moreover, since $g \in G$, $J(1) = 1$ and $J(L_{\log g} f) = J(f)$ for $f \in C(X)$. Thus using the Riesz Representation Theorem and Theorem 2.1 J defines the unique g-measure μ. Then for any other point $y \in X$ we can construct a subsequence $N' \subset \mathcal{N}$ such that

$$\lim_{\substack{N\to\infty \\ N\in\mathcal{N}'}} 1/N \sum_{n=0}^{N-1} L^n_{\log g} f(y) = \mu(f).$$

Therefore, if we do this for a dense set of points in X, the diagonal sequence produces a sequence $\mathcal{N}''$ such that

$$\lim_{\substack{N\to\infty \\ N\in\mathcal{N}'}} \frac{1}{N} \sum_{n=0}^{N-1} L^n_{\log g} f(x) = \mu(f) \quad \forall f \in C(X), \quad \forall x \in X. \qquad 2.(ii)$$

Let $f \in C(X)$ then we can write

$$f = f - \mu(f) + \mu(f) \text{ and } f-\mu(f) \in \ker J \quad \{h \in C(X) | J(h) = 0\}$$

so in order to prove (i) $\Rightarrow$ (iv) it will be sufficient to show $\ker J = \bar{B}$. By the above Ker $J \supset \bar{B}$ is closed, J being continuous. Hence, by the extension theorem for continuous functionals on closed subsets (Dunford, Schwartz, [5])) we need only show that any continuous linear functional, on C(X), annihilates Ker J when it annihilates $\bar{B}$ (or equivalently B). So suppose $P \in C(X)^*$ is such that $P(L_{\log g} f - f) = 0)$, $\forall f \in C(X)$. Let $f \in \ker \mathcal{J}$ then, by using Lebesgues dominated convergence Theorem and 2.(ii) we have

$$\frac{1}{N}\sum_{n=0}^{N-1} P(\mathcal{L}^n_{\log g} f) \to 0 \text{ as } N \to \infty \quad N \in \mathcal{N}'',$$

and since $P(\mathcal{L}_{\log g} f - f) = 0$

$$\frac{1}{N}\sum_{n=0}^{N-1} P(f) \to 0, \ N \in \mathcal{N}''.$$

In other words $P(f) = 0$ $\forall f \in \ker J$ and the Theorem is proven.

Corollary 2.4

The set of $\{g \in G \mid$ there is a unique g-measure$\}$ is a dense G_δ in G.

Proof

The proof is essentially Palmers [9] but I include it for completeness.

Let $\{f_n\}_{n=1}^{\infty}$ be dense in $C(X)$. For natural numbers n, m, N and $c \in \mathbb{R}$. Let $\mathcal{U}_{n,m,c,N}$

$$= \{g \in G \mid \|\frac{1}{N}\sum_{k=0}^{N-1} \mathcal{L}^k_{\log g} f_n - c\| < \frac{1}{m}\} \ .$$

This is an open subset of G and therefore

$$\tilde{G} = \bigcap_n \bigcap_m \bigcup_c \bigcup_N \mathcal{U}_{n,m,c,N} \text{ is a } G_\delta.$$

I claim in fact $\tilde{G} = \{g \in G \mid$ for all $f \in C(X)$ there exists $c(f) \in \mathbb{R}$ with $\|\frac{1}{N}\sum_{k=0}^{N-1} \mathcal{L}^k_{\log g} f - c(f)\| \to 0\}$.

If we assume the claim then, by Theorem 2·3 $\tilde{G}$ is the set of g with unique g-measures. It therefore remains to prove the claim. If g belongs to this set then $g \in \tilde{G}$. Conversely if $g \in \tilde{G}$ then for all n, m there exists $C_m(n)$, N, such that

$$\left\| \frac{1}{N} \sum_{k=0}^{N-1} L^k_{\log g} f_n - c_m(n) \right\|_\infty < \frac{1}{m} . \qquad 2.(iii)$$

If μ is any g-measure

$$\int \frac{1}{N} \sum_{k=0}^{N-1} L^k_{\log g} f \, d\mu = \mu(f).$$

Thus using 2.(iii)

$$|\mu(f_n) - c_m(n)| < \frac{1}{m} \text{ and thus}$$

$$\left\| \frac{1}{N} \sum_{k=0}^{N-1} L^k_{\log g} f_n - \mu(f_n) \right\|_\infty < \frac{2}{m} \text{ for } M > N.$$

Therefore

$$\left\| \frac{1}{M} \sum_{k=0}^{M-1} L^k_{\log g} f_n - \mu(f_n) \right\|_\infty$$

$$\leq \frac{N}{M} \left\| \frac{1}{N} \sum_{k=0}^{N-1} L^k_{\log} f_n - \mu(f_n) \right\|_\infty + \left\| \frac{1}{M} \sum_{k=N}^{M-1} L^k_{\log g} f_n - \left(\frac{M-N}{N}\right) \mu(f_n) \right\|_\infty .$$

Using the fact that $\| L_{\log g} \| \leq 1$ we obtain

$$\leq \frac{N}{M} \left\| \frac{1}{N} \sum_{k=0}^{N-1} L^k_{\log g} f_n - \mu(f_n) \right\|_\infty + \left(\frac{M-N}{M}\right) \left\| \frac{1}{M-N} \sum_{k=0}^{M-N-1} L^k_{\log g} f_n - \mu(f_n) \right\|_\infty$$

Thus if $M = 2N$ we obtain using 2(iii)

$$\leq \frac{1}{2}\ \frac{2}{m} + \frac{1}{2}\frac{2}{m} = \frac{2}{m} \quad .$$

By induction of ℓ if $M = \ell N$ we have

$$\left\| \frac{1}{\ell N} \sum_{k=0}^{N-1} L^k_{\log g}(f_n) - \mu(f_n) \right\|_\infty$$

$$\leq \frac{1}{\ell}\ \frac{2}{m} + \left(\frac{\ell-1}{\ell}\right) \frac{2}{m} = \frac{2}{m} \ .$$

Therefore $\lim\limits_{m\to\infty} \left\| \frac{1}{mN} \sum\limits_{k=0}^{mN-1} L^k_{\log g}\, f_n - \mu(f_n) \right\| = 0$

and it follows that $\lim\limits_{N\to\infty} \left\| \frac{1}{N} \sum\limits_{k=0}^{N-1} L^k_{\log g}\, f_n - \mu(f_n) \right\| = 0.$

Therefore $\left\| \frac{1}{N} \sum\limits_{k=0}^{N-1} L^k_{\log g} f - \mu(f) \right\| \to 0$ for all $f \in C(X)$.

Since there are a dense set of g with unique g-measures the result follows.

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CHAPTER 2

INVARIANCE OF THE BETA-FUNCTION UNDER FINITARY ISOMORPHISMS WITH FINITE EXPECTED CODE LENGTH

INTRODUCTION

The following sections are a generalization of the work of K. Schmidt and W. Parry concerning the invariance, under finitary isomorphism with finite expected code length, of the β-function for Markov shifts, see Parry-Schmidt [1], Schmidt [1].

The result has interesting applications to Axiom A flows concerning the invariance of the β-function when looking at the associated suspension of the flow. For further details, about axiom A flows, see Pollicott [1]. For details about the β-function and its properties look at Tuncel [1] and Parry-Tuncel [1].

The result is as follows:

If $\phi: X_1 \to X_2$, is a finitary isomorphism with finite expected code length, between g-spaces, where the g-functions have finite first moment variational sum (i.e. $\sum_{r=1} r \operatorname{var}_r \phi < \infty$) then the β-function is an invariant, if X_1, X_2 are topologically mixing.

The first interest, in finitary isomorphisms, came about because of the paper of M. Keane and M. Smorodinsky concerning the fact that two Markov shifts, which have the same entropy and period, are finitarily isomorphic.

This Theorem led to people investigating as to whether the period and entropy where complete invariants under finitary isomorphisms with finite expected code length. This was found to be false see (Parry [3]), and people sought after further invariants to solve this completeness problem. One such invariant that arose was the β-function, $\beta: \mathbf{R} \to \mathbf{R}$ defined by $\beta(t) = \exp P(-tIp)$ (where $P(-tIp)$ denotes the pressure of $-t$ times the information function with respect to a Markov measure. (See S. Tuncel [1]). A discussion of the β-function, as an invariant, can be found in Tuncel, S.[1]as well as further information.

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3. ISOMORPHISMS WITH FINITE EXPECTED CODE LENGTH

Let (X,σ) be the two-sided shift space on k-symbols. Let $g_i \in C(X)$, $i = 1,2$, be such that it depends only on coordinates to the right of zero and $g_i \in G$, when restricted to $X+$. We also assume that the g_i's have finite first moment variational sum. Then by Theorem 2.1 we can choose $\mu_i (i = 1,2)$. Since the μ_i are members of $M(X+,\sigma)$ they can be extended uniquely to σ-invariant measures on X. For brevity these measures will also be called μ_1,μ_2. The subshifts (X_1,σ_1,μ_1), (X_2,σ_2,μ_2) are said to be isomorphic if there exists a measure preserving isomorphism $\phi:(X_1,\sigma_1\cdot\mu_1) \to (X_2,\sigma_2,\mu_2)$ with $\phi\sigma_1 = \sigma_2\phi$, this isomorphism ϕ is called finitary if there exists null sets $E_1 \subset X_1$, $E_2 \subset X_2$ such that the restrictions of ϕ and ϕ^{-1} to $X_1 \backslash E_1$ and $X_2 \backslash E_2$ are continuous. If ϕ is a finitary isomorphism we can find measurable, non-negative, integer valued functions a_ϕ and m_ϕ on X with

$$(\phi(x))_0 = (\phi(x'))_0$$

whenever $x, x' \in X_1 \backslash E_1$ satisfy $x_i = x'_i$ for all $i \in \mathbb{Z}$ with $-m_\phi(x) \leq i \leq a_\phi(x)$. We can similarly define analogous objects $a_{\phi^{-1}}, m_{\phi^{-1}}$ for ϕ^{-1}.

<u>Definition</u>

ϕ is said to have finite expected code length if

$$\int (a_\phi + m_\phi)d\mu_1 < \infty \quad \text{and} \int (a_{\phi^{-1}} + m_{\phi^{-1}})d\mu_2 < \infty \quad .$$

For the remainder of this paper we shall assume $\phi:(X_1,\sigma_1,\mu_1) \to (X_2,\sigma_2,\mu_2)$ is a finitary isomorphism with finite expected code length.

Following Krieger [1] we observe that

$$a_\phi^*(x) = \sup_{n\geq 0} \ (a_\phi(\sigma^{-n}(x))-n) < \infty$$

$$m_\phi^*(x) = \sup_{n\geq 0} \ (m_\phi(\sigma^{n}(x))-n) < \infty \qquad \text{a.e.}$$

From this Krieger draws the following conclusions.

<u>Proposition 3.1</u>

(i) There exists a null set E_1^* such that if $x,x' \in X_1\backslash E_1^*$ satisfy $x = x_i'$ for all $i \in \mathbb{Z}$ with $-\infty < i \leq a_\phi^*(x)$ $(-m_\phi^*(x) \leq i < \infty)$ then we have $\phi(x)_i = \phi(x')_i$ for all $i \leq 0 (i \geq 0)$. An analogous statement can be made about ϕ^{-1}.

(ii) If $x,x' \in X_1\backslash E_1^*$ and $x_i = x_i'$ for $i \geq N$ for some $N \in \mathbb{Z}$ then there exists an integer M, depending on x, N with $\phi(x)_i = \phi(x')_i$ for $i \geq M$;

(iii) Similarly, if $x,x' \in X_1\backslash E_1^*$ satisfy $x_i = x_i'$ for $i \leq N$ there exists $M = M(N,x)$ with $\phi(x)_i = \phi(x')_i$ for $i \leq M$.

If $x,x' \in X\backslash E_1^*$ satisfy $x_i = x_i'$ for $|i| \geq N \geq 0$ there exists $M' = M'(N,x)$ with $\phi(x)_i = \phi(x')_i$ for $|i| \geq M'$. Similar results for ϕ^{-1}.

Since the functions a_ϕ^* and m_ϕ^* are finite a.e. we can find an integer $M \geq 0$ and a cylinder set $C = [i_{-m}, \quad i_m]_{-m}^{m} \subset X$ such that

$$D = C \cap \{x : a_\phi^*(x) \leq M \text{ and } M_\phi^*(x) \leq M\}$$

has positive measure.

4. LOCALLY FINITE DIMENSIONAL AUTOMORPHISM.

An automorphism $V: X_1 \to X_1$ is said to be locally finite dimensional if V is non-singular and fixes all but a finite number of coordinates for a.e. (μ) $x \in X$. Krieger essentially expresses this in the following manner; Define an equivalence relation on X by $x \sim x'$ if $x_n = x'_n$ for all but finitely many $n \in \mathbb{Z}$. A non-singular automorphism V of X is then locally finite dimensional if for μ-a.e. x, $Vx \sim x$. Denote by F_1, F_2 the group of all non-singular locally finite dimensional automorphisms of (X_1, μ_1), (X_2, μ_2), then (iv) of Proposition 3.1 tells us that $\phi F_1 \phi^{-1} = F_2$. For further reference we shall denote by E_i the group generated by F_i $(i = 1,2)$ and σ.

For calculation purposes, concerning the invariance of the β-function, we are really interested in the elements of F_i which leave the set C (as described in Section 3) invariant. Thus we define a subgroup of F_i, with this property, as follows:

$$H_i^+ = \{v \in F_i \mid V(x)_j = x_j \text{ for } j \le M\}$$

$$H_i^- = \{v \in F_i \mid V(x)_j = x_j \text{ for } j \ge -M\}$$

$H_i = H_i^+ H_i^-$ is then a subgroup of F_i with the above property concerning C. We can thus discuss the way in which H acts on $\{C, \mu_C^i\}$ (μ_C^i is the measure induced on C by μ_i) with the following result.

Proposition 4.1

The action of H on the space (C,μ_C^i) is ergodic , $i = (1,2)$

Proof

We have to prove if B is a non-trivial Borel subset of C $\exists V \in H$ st $\mu_C^i(VB \cap C\backslash B) > 0$.

We shall in fact prove if B_1, B_2 are non-trivial Borel subsets of C (i.e. $B_1, B_2 \in \mathcal{B}_{\mathcal{B}_C} \cap C$) then $\exists V \in H$ with $\mu_c^i(VB_1) \cap B_2) > 0$.

I claim if $C_1, C_2 \in \mathcal{B}_c$ are clopen $\exists V \in H \cap P$ with $(VC_1) \cap C_2) = C_2$. Assuming the claim, for the moment, given $B_1, B_2 \in \mathcal{B}_c$ we can choose clopen sets $C_1, C_2 \in \mathcal{B}_c$ with $\mu_c^i(C_1 \Delta B_1) < \varepsilon$, $\mu_c^i(C_2 \Delta B_2) < \varepsilon$, where $\varepsilon > 0$ is arbitrary. We can thus find $V \in H$ with $\mu_i(V(C_1) \cap (C_2)) > 0$. I claim in fact if ε is chosen small enough this implies $\mu_i(V(B_1) \cap B_2) > 0$. For if $\mu_i(V(B_1) \cap B_2) = 0$ for every $\varepsilon > 0$

$$\begin{aligned}
\mu_c^i(V(C_1) \cap C_2) &\leq \mu_c^i(V(B_1 \cup (B_1 \Delta C_1)) \cap (B_2 \cup B_2 \Delta C_2) \\
&\leq \mu_c^i(V(B_1) \cap B_2) + \mu_c^i(V(B_1) \cap (B_2 \Delta C_2)) \\
&\quad + \mu_c^i(V(B_1 \Delta C_1) \cap B_2) \\
&\quad + \mu_c^i(V(B_1 \Delta C_1) \cap B_2 \Delta C_2)
\end{aligned}$$

By the above assumptions this is

$$\leq 2\varepsilon + \varepsilon E \qquad \text{(E as in Lemma below)}$$

which implies $\mu_i(V(C_1) \cap C_2) = \mu_i(C_2)$ can be chosen to be arbitrarily small a contradiction. It thus suffices to prove the claim.

Let C_1', C_2' be two arbitrary cylinders. And assume $\ell(C_1') \geq \ell(C_2')$ and choose a subcylinder, of C_2', of the same length as C_1'. We thus can, in effect for the proof, assume that C_1', C_2' have the same length. Now define $V': X_i \to X_i$ which fixes C by mapping the co-ordinates of C_1' to C_2' and vice-versa. (By similarly looking at subclinders we can assume the images of V do in fact lie in X_i), leaving all other co-ordinates which are not part of the determining co-ordinates of the C_i''s, fixed. We can assume C_1 is a union of more cylinders than C_2. We then construct V by using V' on a subset of cylinders of C_1 until $V(C_1) = C_2$. V is clearly uniformly locally finite dimensional and we need just to verify it is non-singular.

Lemma

If $V: X_i \to X_i$ is uniformly locally finite dimensional automorphism then $\frac{d\mu_i V}{d\mu_i}$, $\frac{d\mu_i}{d\mu_i V}$ both exist, $(i = 1,2)$. (i.e. μV and μ are equivalent).

Proof

I claim it is sufficient to prove $\exists$ D,E > 0 such that

$$(*) \quad D\mu_i[x_{-n},\ldots,x_0,\ldots,x_n] \leq \mu_i V[x_{-n},\ldots,x_0,\ldots,x_n] \leq E\,\mu_i[x_{-n},\ldots,x_0,\ldots,x_n]$$

exists for all $n \geq 0$ and $x = (x_n)_{n=1}^{\infty} \in X_i$. This is so since (*) shows clearly that $\mu_i V$ and μ_i have the same sets of zero measure. It remains to prove (*). Assume V fixes coordinates uniformly for $|n| \geq N$. Then since $L^*\mu_i = \mu_i$ we obtain that

$$\mu_i[(Vx)_{-n},\ldots,(Vx)_0,\ldots,(Vx)_n]$$

$$= \int g_i(\sigma^{-n} y) \ldots g_i(y) \ldots g_i(\sigma^n y)\ d\mu_i(y) \text{ where } B=[(Vx)_{-n},\ldots,(Vx)_0,\ldots (Vx)_n]$$

$$\leq \exp\left[\sum_{k=0}^{n-N} \mathrm{var}_k \log g_i + 2N \|\log g_i\|_\infty\right]\mu_i[x_{-n},\ldots,x_0,\ldots,x_n]$$

Thus if $C = \exp\left[\sum_{k=0}^{\infty} \mathrm{var}_k \log g_i + 2N \|\log g_i\|_\infty\right]$ then one side of the inequality follows. Similarly if

$$D = \exp\left[-\left(\sum_{k=0}^{\infty} \mathrm{var}_k \log g_i + 2N \|\log g_i\|_\infty\right)\right]$$

the other side of the inequality follows.

From now on X is an arbitrary topologically mixing subshift of finite type, X+ its obvious restriction.

The following lemma is essentially Sinai's result (Sinai [1] page 28). The finite first moment variational sum being the condition essentially used in Sinai's proof.

Lemma 4.2

If $\phi \in C(X)$ st. $\sum_{k=1}^{\infty} k \operatorname{var}_k \phi < \infty$ then ϕ is cohomologous to a function $\phi_+ \in C(X+)$ (where $\sum_{k=1}^{\infty} k \operatorname{var}_k \phi_+ < \infty$ and $C(X+) \subset C(X)$ is identified with functions of $C(X)$, which depend only on coordinates to the right of zero.)

Proof

Sinai's proof shall be included for completeness.

Define $g_n(z) = \sup_{x \in [z_{-n}, \ldots, z_n]_{-n}^{n}} \phi(x)$ and

$$\phi_n = g_n - g_{n-1} \text{ for } n \geq 1.$$

Then $\|\phi_n\|_\infty \leq \operatorname{var}_n \phi$, ϕ_n depends only on $(z_{-n}, \ldots, z_0, \ldots, z_n)$ and $\lim \phi_n = 0$, $n \geq 1$.

Therefore if we let, $\phi_0 = g_0$ then

$$\phi = \phi_0 + \sum_{n\geq 1}^{\infty} \phi_n$$

If $\psi = \sum_{n=0}^{\infty} \phi_n \sigma^n$, then $\psi \in C(X+)$. Let

$$u = \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} \phi_n \sigma^k. \text{ Then } \|u\|_\infty \leq \sum_{k=1}^{\infty} K \operatorname{var}_k \phi < \infty .$$

Since, given $x, y \in X$,

$$|u(x)-u(y)| \leq \sum_{n=1}^{\infty} \left| \sum_{k=0}^{n-1} \phi_n \sigma^k(x) - \sum_{k=0}^{n-1} \phi_n \sigma^k(x) \right|$$

$$\leq \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} | \phi_n \sigma^k(x) - \phi_n \sigma^k(x)|$$

$$\leq \sum_{n=1}^{\infty} n \operatorname{var}_n \phi < \infty \quad .$$

Moreover,

$$u\sigma - u = \sum_{n=0}^{\infty} \phi_n \sigma^n - \sum_{n=0}^{\infty} \phi_n = \psi - \phi,$$

the lemma is thus complete.

5. THE INFORMATION COCYCLE

Let α be the time zero partition of X (i.e. $\alpha = \{[i]_0 \mid i \in \{1,\ldots,k\}\} \cap X$ and let A be the σ-algebra generated by $\bigcup_{n=0}^{\infty} \sigma^{-n}\alpha$ (i.e. $A = \bigvee_{i=1}^{\infty} \sigma^{-i}\alpha$).

As, in Butler-Schmidt [1], Schmidt-Parry [1] we shall define the information cocycle, for $V \in E$, as follows:

$$J_\mu(A,V) = I_\mu(A|V^{-1}A) - I_\mu(V^{-1}A|A)$$

$$- \log E_\mu\left(\frac{d\mu V^{-1}}{d\mu} \,\middle|\, A\right) \circ V.$$

Where $E_\mu(\cdot|A)$ denotes the conditional expectation with respect to A and $I_\mu(A|V^{-1}A)$ is the information about A given $V^{-1}A$ (see Parry [2] for full information). The information cocycle has the following properties and values, for the case under consideration, as given by the following proposition.

Proposition 5.1

(i) $J_\mu(A,\cdot)$ is indeed an information cocycle on E namely:

$$J_\mu(A,\cdot) : E \to \mathbb{R} \quad \text{and}$$

$$J_\mu(A,ph) = J_\mu(A,p) \circ h + J_\mu(A,h) \quad \text{a.e. } \mu \qquad p,h \in E.$$

(where μ is, and always shall be, from here on, a g-measure for a g with finite first moment variational sum which by Lemma 4.2 can be assumed to depend only on coordinates to the right of zero and as such $g \in G$).

(ii) For $V \in E$, μ and g as in (i)

$$J_\mu(A,V) = \log \prod_{n=0}^{\infty} \frac{g(\sigma^n V)}{g(\sigma^n)} \quad \text{a.e. } \mu \quad .$$

Proof

(i) See Butler-Schmidt [1] Theorem 4.13.

(ii) For this we shall need the following Lemmas.

Lemma 5.2

$$J_\mu(A,\sigma)(x) = I_\mu(A|\sigma^{-1}A)(x) = \log \frac{1}{g(x)} \quad \text{a.e. } \mu \quad .$$

Lemma 5.3 (Butler-Schmidt [1] Theorem 4.18.)

Let $P = \{v \in E$ such that V fixes all coordinates for $|i| \geq N$

where N is independent of $x\}$

let $[P] = \{v \in E | Vx \in Px$ for μ. a.e. $x \in X\}$.

Then for every $V \in [P]$ and $p \in P$ we have

$$J_\mu(A,V)(x) = J_\mu(A,p)(x)$$

μ. a.e. on $B_p = \{x \in X | Vx = px\}$.

Remark 5.4

It is easy to see that in fact $[P] = E$ and thus in order to compute $J_\mu(A,V)$, for $V \in E$, we can assume V fixes all coordinates for $|i| \geq N$ (some $N \in \mathbb{N}$) independent of x.

Assuming these Lemma's, for the moment, we shall continue with the proof of Proposition 5.1(ii). By Remark 6.4 we can choose $N > 0$ such that V fixes coordinates a.e.μ for $|n| \geqslant N$, thus $\sigma^N V \sigma^{-N}$ fixes all coordinates to the right of zero and $\sigma^N V \sigma^{-N} A = A$. Thus

$$J_\mu(A\varsigma^N V\sigma^{-N}) = I_\mu(A|\sigma^N V^{-1}\sigma^{-N}A) - I_\mu(\sigma^N V^{-1}\sigma^{-N}A|A)$$

$$- \log E_\mu\left(\frac{d\mu\sigma^N V^{-1}\sigma^{-N}}{d\mu}|A\right) \circ \sigma^N V\sigma^{-N}.$$

$$= -\log E_\mu\left(\frac{d\mu\sigma^N V^{-1}\sigma^{-N}}{d\mu}|A\right) \circ \sigma^N V\sigma^{-N}$$

$$= -\log E_\mu\left(\frac{d\mu\sigma^N V^{-1}\sigma^{-N}}{d\mu} \circ \sigma^N V\sigma^{-N}|A\right) \quad \text{(By Expectation property).}$$

$$= -\log E_\mu\left(\frac{d\mu}{d\mu\sigma^N V\sigma^{-N}}|A\right)$$

$$= -\log E_\mu\left(\frac{d\mu}{d\mu V\sigma^{-N}}|A\right) \quad \text{since } \mu \text{ is } \sigma\text{-invariant.}$$

$$= -\log E_\mu\left(\frac{d\mu}{d\mu V} \circ \sigma^{-N}|A\right).$$

I claim this is in fact equal to zero in other words:

$$E_\mu\left(\frac{d\mu}{d\mu V} \circ \sigma^{-N}|A\right) = 1.$$

It will suffice to prove if $[x_0, \ \ ,x_m]_0$ is a cylinder starting at 0 then

$$\int_{[x_o, \quad ,x_m]_o} \frac{d\mu V}{d\mu} \circ \sigma^{-N} d\mu = \mu([x_o, \quad ,x_m]_o). \qquad (*)$$

this implies

For, then,

$$E_\mu(\frac{d\mu V}{d\mu} \circ \sigma^{-N} | A) = 1$$

and since $J_\mu(A, 1_{id}) = 0$, using the cocycle property:

$$0 = J_\mu(A, 1_{id}) = J_\mu(A, \sigma^{-N} V^{-1} \sigma^{-N}) \circ \sigma^N V \sigma^{-N} + J_\mu(A, \sigma^N V \sigma^{-N}).$$

Since $\sigma^N V^{-1} \sigma^{-N}$ also fixes A we have that

$$J_\mu(A, \sigma^N V^{-1} \sigma^{-N}) = -\log E_\mu(\frac{d\mu}{d\mu V^{-1}} \circ \sigma^{-N} | A).$$

Thus $\log E_\mu(\frac{d\mu}{d\mu V^{-1}} \circ \sigma^{-N} | A) \circ \sigma^N V \sigma^{-N}$

$$= - \log E_\mu(\frac{d\mu}{d\mu V} \circ \sigma^{-N} | A)$$

i.e. $(E_\mu(\frac{d\mu V}{d\mu} \circ \sigma^{-N} | A) = [E_\mu(\frac{d\mu}{d\mu V} \circ \sigma^{-N} | A)]^{-1}$

$= 1)$

It remains just to prove (*).

$$\int_{[x_o, \; ,x_m]_o} \frac{d\mu V}{d\mu} \circ \sigma^{-N} d\mu = \int_{[x_o, \, ,x_m]_N} \frac{d\mu V}{d\mu} \; d\mu$$

$$= \mu V[x_o, \; ,x_m]_N .$$

Since V fixes coordinates for $|n| > N$ then

$$V[x_o, \; ,x_m]_N = [x_o, \quad ,x_m]_N$$

and $$\mu V[x_o, \; ,x_m]_N = \mu[x_o, \quad ,x_m]_N = \mu[x_o, \quad ,x_m]_o .$$

Thus $$J_\mu(A, \sigma^N V \sigma^{-N}) = 0 \qquad 5.2(iii)$$

Using the cocycle property we obtain that

$$J_\mu(A, \sigma^N V \sigma^{-N}) = J_\mu(A, \sigma^N) \circ V\sigma^{-N} + J_\mu(A, V) \circ \sigma^{-N} + J_\mu(A, \sigma^{-N})$$

Note that since $J_\mu(A, 1_{id}) = 0$ we obtain again using the cocycle property, that

$$J_\mu(A, \sigma^{-N}) = -J_\mu(A, \sigma^N) \circ \sigma^{-N}$$

Thus, using these two identities we obtain from 5.2(iii)

$$0 = J_\mu(A, \sigma^N) \circ V\sigma^{-N} + J_\mu(A, V) \circ \sigma^{-N} - J_\mu(A, \sigma^N) \circ \sigma^{-N}$$

This implies

$$J_\mu(A,V) = J_\mu(A,\sigma^N) - J_\mu(A,\sigma^N) \circ V.$$

By Lemma 5.2 and the cocycle property we can obtain that

$$J_\mu(A,\sigma^N) = \log \prod_{m=0}^{N-1} \frac{1}{g(\sigma^m)} .$$

Thus

$$J_\mu(A,V) = \log \prod_{m=0}^{N-1} \frac{g(\sigma^m V)}{g(\sigma^m)} .$$

Thus since V fixes coordinates to the right of N Proposition 5.1 is proven. To complete the proof we need to verify Lemma's 5.2 and 5.3.

<u>Proof of Lemma 5.2</u>

If α_1,α_2 are two partitions of X, denote by $\alpha_1 \vee \alpha_2$ their refinement and $\hat{\alpha}_1$ the σ-algebra generated by α_1, $\hat{\alpha}_1 \vee \hat{\alpha}_2$ the σ-algebra generated by $\alpha_1 \vee \alpha_2$.

By definition (see Parry [1])

$$I(A|\sigma^{-1}A) = \lim_{m\to\infty} I(\bigvee_{i=0}^{m} \sigma^{-i}\alpha|\sigma^{-1}A).$$

$$I_\mu(\bigvee_{i=0}^{m} \sigma^{-i}\alpha|\sigma^{-1}A)$$

$$= I_\mu(\bigvee_{i=0}^{m} \sigma^{-i}\alpha \mid \bigvee_{i=1}^{\infty} \hat{\sigma}^{-i}\alpha)$$

$$= I_\mu(\alpha| \bigvee_{i=1}^{\infty} \hat{\sigma}^{-i}\alpha) + I_\mu(\bigvee_{i=1}^{m} \sigma^{-i}\alpha \mid \bigvee_{i=1}^{\infty} \hat{\sigma}^{-i}\alpha)$$

$$= \lim_{k\to\infty} I_\mu(\alpha \mid \bigvee_{i=1}^{k} \hat{\sigma}^{-i}\alpha) \quad \text{(By Theorem 6 Parry [1]).}$$

Let μ_+ be the restriction of μ of X+, then μ+ is a g-measure for the restriction of g to X+ by the way in which μ was defined.

$I_\mu(\alpha| \bigvee_{i=1}^{k} \hat{\sigma}^{-i}\alpha)$ depends only on coordinates to the right

of zero and thus is equal to

$$I_{\mu_+}(\alpha| \bigvee_{i=1}^{k} \hat{\sigma}^{-i}\alpha).$$

By definition

$$I_{\mu_+}(\alpha| \bigvee_{i=1}^{m} \sigma^{-i}\hat{\alpha})$$

$$= - \sum_{i\in\{1,\ldots,k\}} \chi_{[i]} \log \mu([i]| \bigvee_{i=1}^{m} \sigma^{-i}\hat{\alpha}).$$

where $\mu([i] | \bigvee_{i=1}^{m} \sigma^{-i}\hat{\alpha}) = E(\chi_{[i]} | \bigvee_{i=1}^{m} \sigma^{-i}\hat{\alpha})$

$$= \sum_{x_1, x_m \in \{1,\dots,k\}} \chi_{[x_1,\dots,x_m]_1} \left(\int_{[x_1,\dots,x_m]_1} \chi_{[i]}\, d\mu\right) \frac{1}{\mu[x_1,\dots,x_m]}$$

$$= \sum_{x_1\, x_m} \chi_{[x_1\, x_m]} \frac{\mu[ix_1,\dots,x_m]}{\mu[x_1,\dots,x_m]} .$$

So

$$I_{\mu+}(\alpha | \bigvee_{i=1}^{m} \sigma^{-i}\hat{\alpha})$$

$$= - \sum_{x_0, \, ,x_m} \chi_{[x_0,\dots,x_m]} \log \frac{\mu[x_0,x_1,\dots,x_m]}{\mu[x_1,\dots,x_m]}$$

$$= \log E\left(\frac{d\mu_+\sigma}{d\mu_+} \Big| \bigvee_{i=0}^{m} \sigma^{-i}\hat{\alpha}\right).$$

Thus by taking limits and using the increasing martingale Theorem (Parry [1] Theorem 2 Page 30).

$$I_\mu(\alpha | \bigvee_{i=1}^{\infty} \sigma^{-i}\hat{\alpha})$$

$$= \log \frac{d\mu_+\sigma}{d\mu_+}$$

$= \log \frac{1}{g}$ by Theorem 2.1.

Thus $I_\mu(A|\sigma^{-1}A) = \log \frac{1}{g}$.

Proof of Lemma 5.3

As already indicated the proof is to be found in Schmidt-Butler Theorem 4.18.

Proposition 5.5

Using the notation of earlier we have

$$J_{\mu_1}(A,V) - J_{\mu_1}(\phi^{-1}A,V) = fV-f \quad \forall\, V \in E_1$$

where $f = I_{\mu_1}(A|\phi^{-1}A) - I_{\mu_1}(\phi^{-1}A|A)$ is measurable.

Proof

The proof is exactly the same as for the Markov measure case given in Parry [3] (finitary isomorphisms with finite expected code length using Butler-Schmidt Theorem 4.14.

Lemma 5.6

For $V \in E$, μ, g as in previous notation, then

$$\frac{d\mu V}{d\mu} = \prod_{n=-\infty}^{\infty} g\,\frac{(\sigma^n V)}{g(\sigma^n)} .$$

Proof

For $m \geq N$ $\sigma^{-m}V\sigma^m(A) = A$, therefore

$$J_\mu(A,\sigma^{-m}V\sigma^m) = -\log E_\mu \frac{(d\mu\sigma^{-m}V^{-1}\sigma^m|A)}{d\mu} \circ \sigma^{-m}V\sigma^m$$

$$= -\log E_\mu \frac{(d\mu \quad |A)}{d\mu V\sigma^m} .$$

By Proposition 5.1 we know that

$$J_\mu(A,\sigma^{-n}V\sigma^n) = \log \prod_{n=0}^{\infty} g\,\frac{(\sigma^n\sigma^{-m}V\sigma^m)}{g(\sigma^n)}$$

$$= \log \prod_{n=-m}^{\infty} \frac{g(\sigma^n V)}{g(\sigma^n)} \circ \sigma^m .$$

Also

$$-\log E_\mu\left(\frac{d\mu}{d\mu V\sigma^m}\middle| A\right)$$

$$= -\log E_\mu\left(\frac{d\mu}{d\mu V}\middle|\sigma^m A\right) \circ \sigma^m \quad \text{(since } \mu\ \sigma\text{-invariant)}$$

$$= \log \left(E_\mu\left(\frac{d\mu}{d\mu V}\middle|\sigma^m A\right)\right)^{-1} \circ \sigma^m$$

$\Longrightarrow$

$$\prod_{n=-m}^{\infty} \frac{g(\sigma^n V)}{g(\sigma^n)} = \left(E_\mu\left(\frac{d\mu}{d\mu V}\middle|\sigma^m A\right)\right)^{-1}.$$

Taking the limit as m tends to infinity and using the increasing Martingale Theorem (see Parry [1]).

This implies

$$\left(\frac{d\mu}{d\mu V}\right)^{-1} = \prod_{n=-\infty}^{\infty} \frac{g(\sigma^n V)}{g(\sigma^n)}$$

and $$\frac{d\mu V}{d\mu} = \prod_{n=-\infty}^{\infty} \frac{g(\sigma^n V)}{g(\sigma^n)} .$$

Remark 5.6(i)

Note that $\prod_{n=-\infty} \frac{g(\sigma^n V)}{g(\sigma^n)}$ exists since V is locally finite dimensional and g has finite first moment variational sum.

Lemma 5.7

If $V \in H$ then $|f\circ V - f| < k = \sum_{n=1}^{\infty} \mathrm{var}_n \log g_1 + \sum_{n=1}^{\infty} \mathrm{var}_n \log g_2 < \infty$ on D.

(f as in Proposition 5.5).

Proof

By Proposition 5.5 we have that

$$f \circ V - f = J_{\mu_1}(A,V) - J_{\mu_1}(\phi^{-1}A,V)$$

Thus estimates, involving the expression $f\circ V - f$, can be computed by studying the information cocycles which we have expressions for. The computation of $J_\mu(A,V)$, for $V \in H$, can be reduced even further as follows:

Each $V \in H$ is of the form $V = V^- V^+$ and, by the cocycle equation for $J_{\mu_1}(A,\cdot)$,

$$J_{\mu_1}(A,V) = J_{\mu_1}(A,V^- V^+)$$

$$= J_\mu(A,V^-) \circ V^+ + J_\mu(A,V^+)$$

Thus in order to prove fV-f is bounded we need only prove that

$$J_{\mu_1}(A,V^+) - J_{\mu}(\phi^{-1}A,V^+),\ J_{\mu_1}(A,V^-) - J_{\mu_1}(\phi^{-1}A,V^-)$$

are bounded.

$$J_{\mu_1}(A,V^-) = \log \prod_{n=0}^{\infty} g\ \frac{(\sigma^n V^-)}{g(\sigma^n)} \quad \text{(by Proposition 5.1(ii))}$$

and, since V^- fixes coordinates to the right of -M and g depends only on coordinates to the right of zero this equals zero.

Analogously, since $J_{\mu_1}(\phi^{-1}A,V^-)$

$$= J_{\mu_2}(A,\phi V^-\phi^{-1}) \circ \phi ,$$

it follows that $J_{\mu_1}(\phi^{-1}A,V^-) = 0$.

By Lemma 5.6 and Proposition 5.1(ii)

$$\left|\log \frac{d\mu_1 V^-}{d\mu_1} - J_{\mu_1}(A,V^+)\right|$$

$$= \left|\sum_{n=-1}^{\infty} \log \frac{g_1(\sigma^n V^+)}{g_1(\sigma^n)}\right|$$

$$\leq \sum_{n=1}^{\infty} \operatorname{var}_n \log g_1 \quad \text{(since } V^+ \text{ fixes coordinates to the left of M).}$$

Since $J_{\mu_1}(\phi^{-1}A,V^+) = J_{\mu_2}(A,\phi V^+\phi^{-1}) \circ \phi$ and

$$\frac{d\mu_2\phi V^+\phi^{-1}}{d\mu_2} \circ \phi = \frac{d\mu_1 V^+}{d\mu_1} \quad \text{(recall that } \mu_2\phi = \mu_1\text{)}$$

we have that

$$\left|\log \frac{d\mu_1 V^+}{d\mu_1} - J_{\mu_1}(\phi^{-1}A, V^+)\right|$$

$$= \left|\left(J_{\mu_2}(A, \phi V^+\phi^{-1}) - \log \frac{d\mu_2\phi V^+\phi^{-1}}{d\mu_2}\right) \circ \phi\right|$$

$$= \left|\sum_{n=-1}^{\infty} \log g_2 \frac{(\sigma^n\phi V^+\phi^{-1})}{g_2(\sigma^n)} \circ \phi\right| .$$

On D, since $a_\phi^* \leq M$ and V^+ fixes coordinates to the left of M, $\{(\phi(V^+x))_{-n} = (\phi(x))_{-n} \mid n = 0,1,\dots\}$.

Thus

$$\left|\sum_{n=1}^{\infty} \log \frac{g_2(\sigma^n\phi V^+)}{g_2(\sigma^n\phi)}\right|$$

$$\leq \sum_{n=1}^{\infty} \operatorname{var}_n \log g_2.$$

Thus

$$\left|J_{\mu_1}(A, V^+) - J_{\mu_1}(\phi^{-1}A, V^+)\right|$$

$$\leq \left|J_{\mu_1}(A, V^+) - \log \frac{d\mu_1 V^+}{d\mu_1}\right| + \left|\log \frac{d\mu_1 V^+}{d\mu_1} - J_{\mu_1}(\phi^{-1}A, V^+)\right|$$

$$\leq \sum_{n=1}^{\infty} \operatorname{var}_n \log g_1 + \operatorname{var}_n \log g_2.$$

Therefore we obtain Lemma 5.7. We are now in a position to prove the main proposition of this section.

Proposition 5.8

f (as in Proposition 5.5) is bounded a.e. μ_1 on D.

Proof

Choose $\alpha \in R$ such that $A_\varepsilon = \{x:|f(x) - \alpha|<\varepsilon\} \cap D$ has positive measure for every $\varepsilon > 0$ and such that $|\alpha|$ is minimal. If f is not bounded by $|\alpha|$ we can choose $\beta \in R$ ($|\beta| > |\alpha|$) with the property that $B_\varepsilon\{x| \ |f(x) - \beta|<\varepsilon\} \cap D$ has positive measure for all $\varepsilon > 0$.
By Proposition 4.1 we can choose $V \in H$ st $VA_\varepsilon \cap B_\varepsilon \neq \phi$.
Lemma 5.7 tells us that $|f(Vx) - f(x)| < k$ a.e. $x \in D$. This implies, since $VA_\varepsilon \cap B_\varepsilon \neq \phi$, that there exists x st $|fV(x) - \beta| < \varepsilon$ and $|f(x) - \alpha| < \varepsilon$.
Therefore:

$$|\beta - \alpha| \leq |\beta - fV(x)| + |f(V(x)) - f(x)| + |f(x) - \alpha| .$$

$$\leq 2\varepsilon + \sum_{n=1}^{\infty} \text{var}_n \log g_1 + \text{var}_n \log g_2 .$$

But $\varepsilon > 0$ was arbitrary therefore

$$|\beta - \alpha| \leq k.$$

Therefore f is bounded a.e. μ_1 on D and

$$|f| \leq 2 \max \{K, |\alpha|\} \quad .$$

Let C' be an upper bound for f on D.

Definition

Let $A' = \{x \in X_1 : |f(x))| \leq 2C'\}$.

By Proposition 5.8 $D \subset A'$ a.e. μ_1 and thus A has positive μ_1 measure.

Proposition 5.9

We have $\mu_1(VD \backslash A') = 0$ for every $V \in H_1^+ \cup H_1^-$.

Proof

By Lemma 5.7 we have the relation:

$$|fV-f| \leq k \quad \text{a.e. } x \in D \text{ and } \forall\, V \in H_1.$$

By the proof of Lemma 5.7 we can deduce $K \leq C'$. By definition we have

$$|f(x)| \leq C' \quad \text{a.e. } x \in D.$$

Thus

$$|fV(x)| \leq |fV(x) - f(x)| + |f(x)| \leq 2C'.$$

Corollary 5.9(i)

There exists a null set $\Delta \subset D$ such that for every $x \in D\backslash\Delta$, $n \geq 0$, $x' \in X_1$ with $x_i = x_i'$ for $i \geq -M$ and $i \leq -M-n$ (or for $i \leq M$ and $i \geq M+n$) we have $x' \in A$.

Proof

The proof is exactly the same as in Proposition 3.4 of Schmidt [1] using Proposition 5.9.

6. PRESSURE

The concept of pressure was considered, as a quantity for subshifts, by Ruelle [1]. Walters [1] generalised the concept to arbitrary dynamical systems (X,T) and verified the so-called "variational formula" for pressure namely:

If $f \in C(X)$ $P(f) =$

$$= \sup_{\mu \in M(X,T)} \int f d\mu + h_\mu(T).$$

There are now many equivalent definitions of pressure but, for convenience, the above variational formula shall be taken as the definition of pressure.

Definition

Let X be an arbitrary topologically mixing subshift of finite type; Pressure is a function:

$P : C(X,\mathbb{R}) \to \mathbb{R}$ described as follows.

$\phi \in C(X,\mathbb{R})$

$$P(\phi) = \sup_{\mu \in M(X,\sigma)} \int \phi \, d\mu + h_\mu(\sigma)$$

$$= \sup_{\mu \in M(X,\sigma)} \int \phi + I_\mu \, d\mu .$$

We shall be needing the following Lemma to be found in Walters [1]).

Lemma 6.1

If $\phi \in C((X+,\mathbb{R}) \subset C(X,\mathbb{R})$ is such that $\sum_{n=1}^{\infty} \mathrm{var}_n\phi < \infty$.

Then $P(\phi) = \log\lambda$ where λ is the spectral radius of $L_\phi : C(X+) \to C(X+)$. (i.e. the maximum eigenvalue.) □

Pressure has the following properties for subshifts of finite type.

Theorem 6.2

If $f,g \in C(X,\mathbb{R})$ then

(i) $|P(f) - P(g)| \leq \|f-g\|_\infty$.

(ii) $P(\cdot)$ is a convex function.

Proof

See Walters [2] page 214.

Condition (i) implies that P is a Lipschitz continuous function with respect to the sup metric on $C(X,\mathbb{R})$. We would like to define some sort of "differentiability" of the pressure function. Thus if, $f,h \in C(X)$ consider the map

$$\lambda \longrightarrow \frac{P(f+\lambda h) - P(f)}{\lambda} .$$

Since P is convex, (ii) above, this is monotonely increasing and by (i) it has upper and lower bounds $\pm \|h\|_\infty$. We may thus define a derivative from the right of f in the direction h by

$D_f^+ P(h) = \lim_{\lambda \downarrow 0} \frac{P(f+\lambda h)-P(f)}{\lambda} \geq - \|h\|_\infty$. Similarly we may define a left derivative. In general these are not known to be equal. However Ruelle [1] showed if we restrict to $f_{\mathbb{R}}^\theta = \{f \in C(X+,\mathbb{R}) | \mathrm{var}_n f \leq C\theta^n , c \in R, 0 < \theta < 1\}$ (the set of exponentially decreasing variation functions) the left and right derivatives are equal and P is said to be "real analytic".

There are several equivalent definitions of real analyticity of a function $f:B \to \mathbb{C}$, where B is a Banach space. An alternative to the above is to define analyticity in terms of normal complex analysis. Namely defining $\ell:\Omega \to B$ (Ω open domain in $\mathbb{C}$) to be analytic if ℓh is analytic for all $h \in B^*$; f is then said to be analytic if ℓf is analytic for all analytic ℓ. ℓf being analytic is, of course, equivalent to ℓf having a power series expansion at every point of Ω. (See M.J. Field [1]).

7. β-FUNCTION

The β-function was originally introduced by S. Tuncel [1] in the form $\beta_\mu(t) = \exp P(-tI_\mu)$ (where P denotes the pressure and I_μ is the information function w.r.t. the measure μ), however the basic ideas, concerning the pressure and the β-function, are due to D. Ruelle. Parry-Tuncel [1] proved, in the Markov case, that the β-function can be represented in the following form.

$$\beta(t) = \lim_{n\to\infty} \left(\int \exp(1-t)J_{\mu_P}(A,\sigma^n)d\mu_P \right)^{1/n}$$

(μ_P being the Markov measure associated with the stochastic matrix P). This characterisation of the β-function can in fact be generalised to g-measures as the following result shows.

Lemma 7.1

Let $g \in G$ be such that $\sum_{n=1}^{\infty} n \operatorname{var}_n \log g < \infty$ then if μ is the unique g-measure,

$$\lim_{n\to\infty} \frac{1}{n} \log \mu\,(\exp t\, J_\mu(A,\sigma^n)) = \log B_\mu(1-t) \ \forall\, t \in R$$

pointwise convergence.

Proof

$$J_\mu(A,\sigma^n) = -\sum_{m=1}^{n-1} \log g(\sigma^m) =: S_n(-\log g).$$

Therefore,

$$\exp tJ_\mu(A,\sigma^n) = \exp S_n(-t \log g),$$

and

$$\lim_{n\to\infty} \frac{1}{n} \log \mu(\exp t\ J_\mu(A,\sigma^n))$$

$$= \lim_{n\to\infty} \frac{1}{n} \log \mu\ (\exp S_n(-t \log g)).$$

By definition $\beta_\mu(1-t) = \exp P(-(1-t)I_\mu(A|\sigma^{-1}A))$

$$= \exp P((1-t)\log g).$$

The result thus follows from the following Lemma.

<u>Lemma 7.2</u>

If $g \in G$, $\sum_{n=1}^{\infty} n\ \mathrm{var}_n \log g < \infty$, μ the unique g-measure, $f \in C(X+), f \geq 0$.

such that $\sum_{n=1}^{\infty} \mathrm{var}_n f < \infty$ then $\lim \frac{1}{n} \log \mu\ (\exp S_n f) = P(\log g + f)$.

<u>Proof</u>

By the definition of pressure applying the spectral radius formula we obtain

$$P(\log g + f) = \lim_{n\to\infty} \log\ (\|L_\phi^n\|)^{1/n} = \lim_{n\to\infty} \log L_\phi^n 1. \ (\phi = \log g + f)$$

$$= \lim_{n\to\infty} \frac{1}{n} \log L_{\log g}^n\ (e^{S_n f})\ .$$

I claim the last limit is the same as $\lim \frac{1}{n} \log \mu\ (\exp S_n f)$. Assuming this the proof is complete.

Proof of claim

We have to show $\lim\limits_{n\to\infty} [\mathcal{L}^n_{\log g}(e^{S_n f})]^{1/n} = \lim\limits_{n\to\infty} [\mu(e^{S_n f})]^{1/n}$.

Because μ is a g-measure

$$\mu(e^{S_n f}) = \mu(\mathcal{L}^n_{\log g}(e^{S_n f})).$$

And thus $\lim\limits_{n\to\infty} (\mu(e^{S_n f}))^{1/n} = \lim\limits_{n\to\infty} [\mu(\mathcal{L}^n_{\log}(e^{S_n f})]^{1/n}$.

By Holders inequality

$$\geq \lim_{n\to\infty} \mu\{[\mathcal{L}^n_{\log g}(e^{S_n f})]^{1/n}\} = \lim_{n\to\infty} [\mathcal{L}^n_{\log g}(e^{S_n f}))^{1/n}$$

By Ruelle's Operator Theorem (See Walters [1] Theorem 3.1)

$$\lim_{n\to\infty} [\mu(e^{S_n f})]^{1/n} = \lim_{n\to\infty} \lim_{m\to\infty} [\mathcal{L}^m_{\log g}(e^{S_n f})]^{1/n}$$

$$\leq \lim_{n\to\infty} [\mathcal{L}^n_{\log g}(e^{S_n f})]^{1/n} \qquad 2(i).$$

Remark

Lemma 7.2 is in fact true under the conditions $f, \phi \in C(X+)$, $L_\phi^p \mu = \lambda\mu$ for some $\lambda > 0$. See Tuncel, S. [1]. The result then takes the form:

$$\lim \frac{1}{n} \log \mu \, (e^{S_n f}) = P(f+\phi) - P(\phi).$$

Thus Lemma 1 actually holds under the conditions $g \in G$, μ a g-measure.

In Ruelle [1] a proof that the β-function, in the case of $\log g$ having exponentially decreasing variation, is analytic can be found. The method , essentially adopted by Ruelle, is to show that the Pressure function is analytic on $f_{\mathbb{R}}^\theta = \{f \in C(X+,\mathbb{R}) | \mathrm{var}_n f \le C\theta^n$, $c \in R$ $0 < \theta < 1\}$. This is deduced from the fact that, for $\phi \in f_{\mathbb{R}}^\theta$, L_ϕ has a maximal isolated eigenvalue. It therefore seemed quite reasonable, in order to prove the analyticity of the β-function for ϕ with finite first moment variation, to adopt an analogous line of reasoning. In order to achieve this goal we shall need the following definitions.

Definition 7.3

Let $f_{\mathrm{var}}^+ = \{f \in C(X+) | \; \|f\|_{\mathrm{var}} = \sum_{k=0}^{\infty} \mathrm{var}_k f < \infty\}$.

We shall show that, if $\phi \in C(X+)$ has finite first moment variational sum then $L_\phi(f_{\mathrm{var}}^+) \subset f_{\mathrm{var}}^+$. By defining the norm

$$|||f|||_{\mathrm{var}} = \max \; \{ \|f\|_\infty \quad , \quad \|f\|_{\mathrm{var}} \}$$

we can make f^+_{var} into a Banach space. For if $\lim_{m,n} |||f_n - f_m|||_{var} = 0$ by completeness of $(C(X+), \| \ \|_\infty)$ there exist $f \in C(X+)$ with $\|f_n - f\|_\infty$ tending to zero. I now claim that $f \in f^+_{var}$ and $|||f_n - f|||_{var}$ tends to zero.

<u>Proof of Claim</u>

$$\|f_n - f_m\|_{var} \geq \left| \, \|f_n\|_{var} - \|f_m\|_{var} \right| .$$

Thus $\{\|f_n\|_{var}\}_{n=1}^{\infty}$ is bounded above by some constant $C'' \in R$. Thus for each $M \geq 0$

$$\sum_{n=0}^{M} var_n f_m \leq C'' .$$

Letting m tend to infinity this implies

$$\sum_{n=0}^{M} var_n f \leq C''.$$

Letting M tend to infinity we obtain $\|f\|_{var} \leq C''$ and $f \in f^+_{var}$. Thus $\|f_n - f\|_{var} \to 0$ as $n \to \infty$. Then $|||f_n - f|||_{var} \to 0$ and the claim is proven.

<u>Lemma 7.4</u>

The closed unit $||| \ \ |||_{var}$ -ball is $\| \ \ \|_\infty$- compact.

Proof

The map $f \to |||f|||_{var}$ is lower semicontinuous on f^+_{var} with respect to $\| \ \|_\infty$ (i.e. given $\varepsilon > 0, f$, we can find $\delta > 0$ such if $h \in f^+_{var}$ $|||h - f|||_{var} < \delta$ implies $|||h|||_{var} \geq |||f|||_{var} - \varepsilon$.)

For given $\varepsilon > 0$ we can choose M such that

$$\sum_{n=0}^{\infty} var_n f - \frac{\varepsilon}{2} \leq \sum_{n=0}^{M} var_n f \leq 2M|||f-h|||_\infty + \sum_{n=0}^{M} var_n h.$$

Then $\delta = \frac{\varepsilon}{2M}$ and the lower semi-continuity follows.

The lower semi-continuity of this map implies the compactness of the $||| \ |||_{var}$ one ball, B^1_{var}, as follows:

Lower semi-continuity implies that B^1_{var} is closed in the $\| \ \|_\infty$-norm. For, if $f_n \in B^1_{var}$ and $|||f_n - f|||_{var} \to 0$ as n tends to infinity, by lower semi-continuity given $\varepsilon > 0$ we can choose $\delta > 0$ such that $\|f_n - f\|_\infty < \delta$ implies:

$$|||f|||_{var} \leq \varepsilon + |||f_n|||_{var} \leq 1 + \varepsilon.$$

But, $\varepsilon > 0$, is arbitrary and B^1_{var} is closed in the $\| \ \|_\infty$-norm. B^1_{var} is clearly bounded in the $\| \ \|_\infty$-norm and therefore, by the Arzela-Ascoli Theorem, is $\| \ \|_\infty$-compact.

Theorem 7.5

Let $\phi \in \{\phi \in C(X+) | \sum_{r=1}^{\infty} r\mathrm{var}_r\phi < \infty\}$, then there exists $\lambda > 0$, $\nu \in M(X+)$ and $h \in f_{var}^{+}$ such that

(i) $L_\phi h = \lambda h$.

(ii) All other eigenvalues of the operator $L_\phi : f_{var}^{+} \to f_{var}^{+}$ have strictly smaller modulus.

(iii) $L_\phi^{*}\nu = \lambda\nu$.

(iv) If $f \in C(X+)$ then $\frac{L_\phi^n f}{\lambda^n} \to \nu(f)h$ uniformly.

(v) λ is a simple eigenvalue.

(vi) $P(\phi) = \log \lambda$.

Proof

(i) Parts (i), (iii), (iv), (v) are already known under finite variation sum (Due to P. Walters [1]). A different proof, based on M. Pollicott's thesis, is given from which we can deduce (ii), not known under finite first moment variational sum.

(ii) Let $s = \{f \in C(X+) \,|\, f(x) \le \exp\left(\sum_{r=k+1}^{\infty} \mathrm{var}_r\phi\right)f(y)$,

if $d(x,y) \le \frac{1}{k+1}$, $f \ge 0$ and $\|f\|_\infty \le 1\}$.

The first inequality means that for $d(x,y) \le \frac{1}{k+1}$

$$f(x) - f(y) \le f(y)(\exp(\sum_{r=k+1}^{\infty} \mathrm{var}_r\phi) - 1) \le \|f\|_\infty (\exp(\sum_{r=k+1}^{\infty} \mathrm{var}_r\phi) - 1).$$

Therefore,

$$\mathrm{var}_k f \le \|f\|_\infty (\exp(\sum_{r=k+1}^{\infty} \mathrm{var}_r\phi) - 1)$$

$$\le \|f\|_\infty (\exp(\sum_{r=1}^{\infty} \mathrm{var}_r\phi) \sum_{r=k+1}^{\infty} \mathrm{var}_r\phi).$$

Thus

$$|\|f\||_{var} \le \|f\|_\infty \exp(\sum_{r=1}^{\infty} \mathrm{var}_r\phi) \sum_{r=1}^{\infty} r\mathrm{var}_r\phi$$

and so

$$|\|f\||_{var} \le C \|f\|_\infty, \ C = \exp(\sum_{r=1}^{\infty} \mathrm{var}_r\phi) \sum_{r=1}^{\infty} r\mathrm{var}_r\phi.$$

Thus S is contained in the $|\| \ \||_{var}$-ball of radius Max(C,1). As S is $\| \ \|_{var}$-closed it is thus $\| \ \|_\infty$-compact by Lemma 7.4. Define

$$L_n : S \to S \text{ by } L_n f = \frac{L_\phi(f+1/n)}{\|L_\phi(f+1/n)\|_\infty}$$

which is well defined since:

$$L_\phi(f + 1/n) \ge \frac{1}{n} e^{-\|\phi\|_\infty}.$$

To show S is L_n-invariant notice that if $d(x,y) \leq \frac{1}{k+1}$ then

$$L_\phi(f + 1/n)(x) = \sum_{ix \in X+} e^{\phi(ix)}(f + 1/n)(ix)$$

$$\leq \sum_{ix \in X+} \exp(\phi(iy) + \mathrm{var}_{k+1}\phi)(f + 1/n)(iy) \exp\left(\sum_{r=k+2}^{\infty} \mathrm{var}_r\phi\right)$$

$$= \exp\left(\sum_{r=k+1}^{\infty} \mathrm{var}_r\phi\right) L_\phi(f + 1/n)(y).$$

Since S is convex and L_n is $\| \ \|_\infty$ continuous there exists a fixed point $f_n \in S$ (see the Schauder-Tychanoff fixed point theorem). If we let $\lambda_n = \| L_\phi(f_n + 1/n) \|_\infty$ then

$$f_n \geq (\inf f_n + 1/n) \frac{e^{-\|\phi\|_\infty}}{\lambda_n}.$$

In particular

$$\inf f_n \geq \left(\frac{\inf f_n}{\lambda_n} + \frac{1}{n\lambda_n}\right) e^{-\|\phi\|_\infty} > 0.$$

By rearranging this we see that

$$\lambda_n \geq \left(1 + \frac{1}{n \inf f_n}\right) e^{-\|\phi\|_\infty} > 0.$$

Choose subsequence $f_{n_i} \to h$ then by continuity $\lim_{i \to \infty} \lambda_{n_i} = \lambda > 0$ and $L_\phi h = \lambda h$.

By compactness $h \in S \subset f^{+}_{var}$. If $h(x) = 0$ for some $x \in X+$ then $L^{n}_{\phi}h(x) = 0$ for all $n \geq 0$ (i.e. $h(y) = 0$ for all $y \in \bigcup_{n=0}^{\infty} \sigma^{-n}x$.

Since X+ is topologically mixing this set is dense and so h must be strictly greater than zero since it has positive norm.

We may define from this result a g-function, $g \in G$, by $\log g = \phi + \log h - \log \lambda - \log h\sigma$

$$\mathrm{var}_n \log g \leq 2 \sum_{r=n}^{\infty} \mathrm{var}_r \phi \ .$$

Therefore $\sum_{n=0}^{\infty} \mathrm{var}_n \log g \leq 2 \sum_{1}^{\infty} n\mathrm{var}_n \phi < \infty$.

(ii) Let $f \in f^{+}_{var}$, and define g as in the first part. In order to prove (ii) it suffices to show all other eigenvalues of $L_{\log g}$ are strictly less than one. (The calculations from here on are essentially due to Walters [1]). Let $d(x,y) \leq \frac{1}{k+1}$

$$| L^{n}_{\log g} f(x) - L^{n}_{\log g} f(y)|$$

$$= | \sum_{p \in S^n = \{p \in \{0,1,\dots,k\}^n | px \in X+\}} g(px)g(\sigma px)g(\sigma^2 px) \ \dots \ g(\sigma^{n-1}px)f(px) - g(py)\dots g(\sigma^{n-1}py)f(py)|$$

$$\leq | \sum_{p \in S^n} g(px)g(\sigma px)\dots g(\sigma^{-n}px) \ [f(px) - f(py)]|$$

$$+ | \sum_{p \in S^n} f(py) \ [g(px)\dots g(\sigma^{n-1}px) - g(py)\dots g(\sigma^{n-1}py)]|$$

$$\leq \sup_{p \in S^n} |f(px) - f(py)| + \|f\|_{\infty} \sum_{p \in S^n} |g(px)\dots g(\sigma^{n-1}px) - g(py)\dots g(\sigma^{n-1}py)| .$$

However

$$|g(px)\cdots g(\sigma^{n-1}px) - g(py)\ \ldots\ g(\sigma^{n-1}py)|$$

$$= \quad g(py)\cdots g(\sigma^{n-1}py)\ \left|\frac{g(px)}{g(py}\ \frac{g(\sigma^{n-1}px)}{g(\sigma^{n-1}py)} - 1\right|$$

$$\le \quad g(py)\cdots g(\sigma^{n-1}py)\max(e^{\sum_{r=k}^{n+k} \mathrm{var}_r \log} - 1,\ 1 - e^{-\sum_{r=k}^{n+k} \mathrm{var}_r \log g})$$

$$\le \quad g(py)\cdots g(\sigma^{n-1}py)e^{\sum_{r=k}^{\infty} \mathrm{var}_r \log g} \sum_{r=k}^{\infty} \mathrm{var}_r \log g.$$

Therefore

7.5(a) $$\mathrm{var}_k L^n_{\log g} f \le \mathrm{var}_{n+k} f + \|f\|_\infty\ e^{\sum_{r=k}^{\infty} \mathrm{var}_r \log g} \sum_{r=k}^{\infty} \mathrm{var}_r \log g$$

$$\sum_{k=0}^{\infty} \mathrm{var}_k L^n_{\log g} f \le C(n)\ |||f|||_{\mathrm{var}} + \|f\|_\infty\ C'''$$

where $$C''' = e^{\sum_{r=0}^{\infty} \mathrm{var}_r \log g} \sum_{r=1}^{\infty} r\,\mathrm{var}_r \log g$$

and $$C(n) = \frac{\sum_{k=0}^{\infty} \mathrm{var}_{n+k} f}{\sum_{k=0}^{\infty} \mathrm{var}_k f}\ ,\ \text{if } f \ne \text{const}.$$

This gives

7.5(b) $\| L^n_{\log g} f \|_{var} \le C(n) \ |\!|\!| f |\!|\!|_{var} + \| f \|_\infty \ C''$

if $f \neq$ const. and $C''' = e^{\sum_{r=0}^{\infty} \overline{var_r \log g}} \sum_{r=1}^{\infty} r var_r \log g$.

Since $\| L_{\log g} f \|_\infty \le \| f \|_\infty$. This implies

7.5(c) $|\!|\!| L^n_{\log g} f |\!|\!|_{var} \le C(n) \ |\!|\!| f |\!|\!|_{var} + \| f \|_\infty C'''' (C'''' = \max\{1, C'''\})$

Equation 7.5(a) implies $\{L^n_{\log g} f\}_{n=0}^{\infty}$ is an equicontinuous family. $\| L^n_{\log g} f \|_\infty$ is also bounded above by $\| f \|_\infty$ so, by the Arzela-Ascoli Theorem, the closure of $\{L^n f | n \ge 0\}$ is compact. Hence there exists $f_* \in C(X)$ and a sequence n_i of positive integers such that;

$$\lim_{n_i \to \infty} \| L^{n_i}_{\log g} f - f_* \|_\infty = 0.$$

If $\alpha(h)$, $\beta(h)$ denote the maximum and minimum values of $h \in C(X+)$ we have:

$$\alpha(f) \le \alpha(L_{\log} f) \quad \le \alpha(f_*) \le \beta(f_*) \le \beta(Lf) \le \beta(f).$$

Note that, since $\lim_{n_i \to \infty} \alpha(L^{n_i} f) = \alpha(f_*)$, $\lim_{n \to \infty} \alpha(L^n f) = \alpha(f_*)$

$\alpha(f_*) = \alpha(Lf_*)$: This implies

$$f_*(y) = \alpha(f_*) \text{ for } y \in \sigma^{-1} z$$

if

$$\alpha(f_*) = \alpha(Lf_*) = Lf_*(z).$$

Similarly if $\alpha(f_*) = \alpha(L^k f_*) = Lf_*(\omega)$ then

$$f_*(y) = \alpha(f_*) \text{ for } y \in \sigma^{-k}\omega \ .$$

Thus, using topological mixing of X+, we see that f_* attains its minimum on every cylinder set as follows:

Let ω^n minimise $L^n f_*$. The set of sequences $\{(\omega^n)\}$ has a symbol i such that

$$(\omega^n)_0 = i \text{ for infinitely many } n.$$

Thus if C is a cylinder there exist N such that for $n \geq N$

$$C \cap \sigma^{-n}[i] \neq 0 \quad \forall\, n \geq N.$$

Thus we can choose $n_0 \geq N$ such that

$$C \cap \sigma^{-n_0}\omega^{n_0} \neq \emptyset.$$

And thus f attains its minimum on C.

Therefore f_* is a constant and clearly $L^n f$ converges to f_* uniformly.

Since $L_{\log g}(1) = 1$, $L_{\log g}$ defines an operator P on the quotient space $f^+_{var}/\mathbb{R}$ and inequality 7.5(c) becomes:

$$|||P^n f|||_{var} \leq C(n)\ |||f|||_{var} + C'\ \mathrm{var}_0 f.$$

By the above $\mathrm{var}_0 P^n f$ converges to zero so we have, for large n, since $C(n) \to 0$ as n tends to infinity:

$$|||P^{2n}f|||_{\mathrm{var}} \leq C' \,\mathrm{var}_0\, P^n f + C(n)\,(C'\mathrm{var}_0 f + C(n)\, |||f|||_{\mathrm{var}})$$

$$< 1 \qquad 7.5(d).$$

By Lemma 7.4 $B'_{\mathrm{var}}=\{f \mid |||f|||_{\mathrm{var}} \leq 1\}$ is $\| \;\|_\infty$-compact, so we may choose n so that 7.5(d) holds for all f in this ball. Equation 7.5(d) with the above thus implies that the operator $P: f^+_{\mathrm{var}}/\mathbb{R} \to f^+_{\mathrm{var}}/\mathbb{R}$ has norm strictly less than 1. This, with the observation that all eigenvalues have modulus less than or equal to the norm, gives the result.

(iii), (iv), (v)

These results are standard and can be found in Walters [1], and M. Pollicott's Thesis.

(vi)

Two functions $f,g \in C(X+)$ are said to be cohomologous if there exists $u \in C(X+)$ such that $f = g + u\sigma - u$. We then write $f \sim g$. By earlier parts we know that $\phi \sim \log g + \log \lambda$. It can be shown, see Walters [2], that two cohomologous functions have the same pressure, $P(\log g) = 0$ and that $P(\log g + \log \lambda) = P(\log g) + P(\log \lambda) = \log \lambda$. $\square$

Theorem 7.6

The function $\phi \to P(\phi)$ is a real analytic function on the space $\{t\psi | t \in \mathbb{R}\}$ if $\sum_{n=1}^{\infty} n \, \mathrm{var}_n \psi < \infty$, $\psi \in C(X+)$.

Proof

The theorem above, for the exponentially decreasing variation case, was proven by Ruelle ([1], p.92). The proof adopted here is based on the elaborated version, of Ruelle's proof, given by M. Pollicott. I shall denote by $\{z\Psi | z \in \mathbb{C}\}$ and $f_{var}^{\mathbb{C}}$ the complex version of f_{var}^{+} and $\{t\psi | t \in \mathbb{R}\}$.

Let $\Phi: \{\psi z : z \in \mathbb{C}\} \to f_{var}^{\mathbb{C}}$ be given by:

$$\Phi(\phi) = \{x \to \exp_{ix \in X+} \phi(ix)\}\}$$

I claim Φ is $\||\ \||_{var}$-analytic.

Proof of claim

By earlier comments it is sufficient to show, if $\ell : \Omega \to (\Psi, \||\ \||_{var})$ $\{\Psi = \{z\psi | z \in \mathbb{C}\}\}$ and $u \in (f_{var}^{\mathbb{C}})^{*}$,

that $u \Phi \ell$ has a power series expansion about zero of $\Sigma(u\Phi\ell)1/n!$.

It is in fact sufficient to check $\sum_{n=0}^{\infty} \frac{\psi^n}{n!}$ is convergent in $(f_{var}^{\mathbb{C}}, \||\ \||_{var})$. We shall show in fact that it is in fact absolutely convergent namely

$$\sum_{n=0}^{\infty} \frac{|||\psi|||^n \ \mathrm{var}}{n!} < \infty$$

$$\mathrm{var}_k\, \psi^2 = \sup\{|\psi^2(x) - \psi^2(y)| \ |d(x,y) \le \frac{1}{k+1}\}$$

$$= \sup\{\psi(x)(\psi(x) - \psi(y)) + \psi(y)(\psi(x) - \psi(y)| \,|d(x,y) \le \frac{1}{k+1}\}$$

$$\le \ \|\psi\|_\infty 2 \ \mathrm{var}_k \psi .$$

By induction this implies

$$\mathrm{var}_k\, \psi^n \le \|\psi\|_\infty^{n-1}\, 2^{n-1}\, \mathrm{var}_k \psi \ .$$

Therefore

$$\sum_{k=0}^{\infty} \mathrm{var}_k\, \psi^n = \|\psi^n\|_{\mathrm{var}} \le |||\psi|||_\infty^{n-1}\, 2^{n-1} \quad \|\psi\|_{\mathrm{var}} .$$

And it follows

$$|||\psi^n|||_{\mathrm{var}} \le 2^{n-1} \quad |||\psi|||^n_{\mathrm{var}} \ .$$

Thus

$$\sum_{n=0}^{\infty} \frac{|||\psi^n|||_{\mathrm{var}}}{n!} \le \sum_{n=0}^{\infty} \frac{2^{n-1} \quad |||\psi|||^n_{\mathrm{var}}}{n!}$$

and $\lim\limits_{n\to\infty} \dfrac{2^n \quad |||\psi|||^{n+1}_{\mathrm{var}}}{(n+1)!} \quad \dfrac{n!}{2^{n-1} \quad |||\psi|||^n_{\mathrm{var}}} = \lim \dfrac{2}{n+1} \quad |||\psi|||_{\mathrm{var}} = 0.$

Therefore,

$$\sum_{n=0}^{\infty} 2^{n-1} \frac{|||\psi|||_{var}^{n}}{n!}, \text{ and thus } \sum_{n=0}^{\infty} \frac{|||\psi^n|||_{var}}{n!},$$

exists by the ratio test. □

Thus, using the above and the linearity of $h \to \{f \to \sum_{ix \in X+} \exp (h((x))f(ix)\}$, we see that

$$J: \Psi = \{z\psi | z \in \mathbb{C}\} \to \mathcal{B}(f_{var}^{\mathbb{C}})$$

given by

$$\phi \to L_\phi \text{ is } |||\quad|||_{var} \text{ analytic.}$$

If $\phi \in \{t\psi | t \in \mathbb{R}, \psi \in C(X+) \Sigma \, nvar_n \psi < \infty\}$ then, by Theorem 7.5, we have that the spectrum of L , $sp(L\ ) = \Sigma \cup \{\lambda_\phi\}$, where $\{\lambda_\phi\}$ and Σ are disjoint. Choose a path Γ in $\mathbb{C}$, about λ_ϕ , separating these two parts of the spectrum. Define an operator P on $f_{var}^{\mathbb{C}}$ by

$$P = -\frac{1}{2\pi i} \int_\Gamma \frac{1}{L_\phi - z} \, dz.$$

This operator has the following properties.

<u>Theorem 7.7</u>

$P^2 = P$ (i.e. P is a projection). We can decompose $f_{var}^{\mathbb{C}}$ as a direct sum of two L_ϕ invariant subspaces, $M = P(f_{var}^{\mathbb{C}})$, $N = (1-P)(f_{var}^{\mathbb{C}})$, such that the spectrum of L_ϕ, when restricted to M and N, is Σ_1 and Σ_2 respectively where Σ_1 is contained within Γ.

Proof

See Kato [1], Theorem 6.17, p. 178.

By the above Σ_1 is in fact just λ, furthermore since λ is a simple eigenvalue dim M = 1.

Let $\omega \in \Phi = \{z\psi | z \in \mathbb{C} \ \ \Sigma n \ \mathrm{var}_n \phi < \infty\}$ and let $S = L_\omega - L_\phi$ and $f \in f_{var}^{\mathbb{C}}$.

Let us first consider an heuristic argument about when, and under what conditions, the operator $(L_\omega - \lambda)^{-1}$ exists, $\lambda \in \mathbb{C}$.

$$\frac{1}{L_\omega - \lambda} = \frac{1}{L_\omega - L_\phi + L_\phi - \lambda} = \frac{1}{(L_\phi - \lambda)} \left(\frac{1}{1 - \frac{(-L_\omega + L_\phi)}{L_\phi - \lambda}} \right)$$

$$= \frac{1}{L_\omega - \lambda} \sum_{n=0}^{\infty} [(L_\phi - \lambda)^{-1}(L_\phi - L_\omega)]^n .$$

This series is absolutely convergent provided $\lambda \in \mathbb{C} \setminus Sp(L_\phi)$ and

$$\sup_{|||f|||_{var} \le 1} |||L_\phi f - L_\omega f|||_{var} < \sup_{\{|||f|||_{var} \le 1\}} |||(L_\phi - \lambda) f|||_{var} .$$

Thus if $\sup_{\{f: |||f|||_{var} \le 1\}} |||(L_\phi - L_\omega) f|||_{var} < \min_{\lambda \in \Gamma} \sup_{\{f: |||f|||_{var} \le 1\}} |||(L_\phi - \lambda) f|||_{var}$

then $\Gamma \subseteq \mathbb{C} \setminus Sp(L_\omega)$ (Note that $\min_{\lambda \in \Gamma} \sup_{\{f | \, |||f|||_{var} \le 1\}} |||(L_\phi - \lambda) f|||_{var} > 0$ for Γ is compact and $\lambda \to \sup_{\{f | \, |||f|||_{var} \le 1\}} |||L_\phi f - \lambda f|||_{var}$ is a continuous map).

Thus if $|||L_\omega - L_\phi|||_{var} =: \sup_{f \mid |||f|||_{var} \le 1} |||L_\phi f - \lambda f|||_{var}$ is small enough we may define

$$P_s = -\frac{1}{2\pi i} \int_\Gamma \frac{1}{L_\omega - \lambda} \, d\lambda.$$

By Theorem 7.7, $Sp(L_\omega) = \Sigma_1(L_\omega) \cup \Sigma_2(L_\omega)$ where Γ separates Σ_1 from Σ_2. I claim that if $L_\omega - L_\phi = S$ is small enough this implies

$$|||P_s - P|||_{var} < 1.$$

Assuming this for the moment then $\dim P_s f^{\mathbb{C}}_{var} = \dim M = 1$. This can be seen as follows:
If $\dim P_s f^{\mathbb{C}}_{var} > 1$ we may choose $f_1, f_2 \in P_s f^{\mathbb{C}}_{var}$ such that $Pf_1 = f_1$ and $Pf_2 = 0$. Then

$$(P_s - P)f_2 = P_s f_2 = f_2.$$

This contradicts $|||P_s - P|||_{var} < 1$.
It remains to show that if $L_\omega - L_\phi$ is small enough $|||P_s - P|||_{var} < 1$.
By an earlier computation

$$|||P_s - P||| = ||| - \frac{1}{2\pi i}\int_\Gamma (L_\phi-\lambda)^{-1} \sum_{n=0}^{\infty} [(L_\phi-\lambda)^{-1}(L_\phi-L_\omega)]^n - (L_\phi-\lambda)^{-1} d\lambda|||_{var}$$

$$= ||| - \frac{1}{2 i}\int_\Gamma (L_\phi-\lambda)^{-1} \sum_{n=1}^{\infty} [(L_\phi-\lambda)^{-1}(L_\phi-L_\omega)]^n \, d\lambda|||_{var}$$

$$\le \frac{1}{2\pi}\int_\Gamma \sum_{n=1}^{\infty} |||L_\phi-\lambda|||_{var}^{-(n+1)} |||L_\phi-L_\omega|||_{var}^n \, d\lambda .$$

Let $D' = \max_{\lambda\in\Gamma} |||L_\phi-\lambda|||_{var}^{-1}$

Then

$$\frac{1}{2\pi}\int_\Gamma \sum_{n=1}^{\infty} |||L_\phi-\lambda|||_{var}^{-(n+1)} |||L_\phi-L_\omega|||_{var}^n \, d\lambda.$$

$$\le \frac{1}{2\pi}\left[\frac{D'}{(1-|||L_\phi-L_\omega|||_{var}D')} - D'\right]$$

so that if

$$\frac{D'}{2\pi(1-|||L_\phi-L_\omega|||_{var}D')} - \frac{D'}{2\pi} < 1$$

it implies

$$|||P_s - P|||_{var} < 1.$$

which is equivalent to

$$|||L_\phi-L_\omega|||_{var} < \frac{2\pi}{(2\pi+D')D'}$$

Thus if $|||L_\phi - L_\omega|||_{var}$ is small enough $|||P_s - P||| < 1$ and the proof is complete.

Furthermore since

$$L_\omega(L_\omega-\lambda)^{-1} = L_\omega(L_\omega)^{-1} \sum_{n=0}^{\infty} \lambda^n (L_\omega)^{-n}$$

$$= 1 + \lambda \sum_{n=1}^{\infty} \lambda^{n-1}(L_\omega)^{-n}$$

$$= 1 + \lambda\ (L_\omega-\lambda)^{-1}.$$

Thus $\quad L_\omega(L_\omega-\lambda)^{-1} = 1 + \lambda(L_\omega-\lambda)^{-1} \qquad\qquad$ 7.7(a).

Also since L_ω is a closed operator and the fact that we can approximate an integral by a finite sum we obtain using 7.7(a):

$$L_\omega P_s = -\frac{1}{2\pi i}\int_\Gamma L_\omega(L_\omega-\lambda)^{-1}d\lambda$$

$$= -\frac{1}{2\pi i}\int_\Gamma 1 + \lambda(L_\omega-\lambda)^{-1}d\lambda$$

$$= -\frac{1}{2\pi i}\int_\Gamma \lambda(L_\omega-\lambda)^{-1}d\lambda$$

$$= -\frac{1}{2\pi i}\int_\Gamma \frac{\lambda}{L_\phi+S-\lambda}\ d\lambda$$

$$(s = L_\omega - L_\phi).$$

Expanding the integrand as a power series and passing the integral through the summation we obtain

$$L_\omega P_s = \sum_{n=0}^{\infty} \frac{(-1)^n}{2\pi i} \int_\Gamma \frac{\lambda}{L_\phi - \lambda} \left(\frac{S}{L_\phi - \lambda}\right)^n d\lambda \ .$$

Therefore, since $\omega \to L_\omega$ is an analytic map in a neighbourhood of $\{t\psi | t \in R. \ \Sigma n \ \mathrm{var}_n \psi < \infty\}$ within its complexification, $\omega \to L_\omega P_s$ is an analytic map.

By the use of perturbation theory (see Kato [1]), it can be shown that

$$\phi \longmapsto \mathrm{tr}(L_\phi P_s) = \lambda_\phi$$

is a real analytic map on

$$\{t\psi \mid t \in \mathbb{R}, \psi \in C(X+), \ \sum n \, \mathrm{var}_n \psi < \infty\}.$$

Therefore, by restricting to $\{t\psi | t \in \mathbf{R}, \ \psi \in C(X+). \ \Sigma n \ \mathrm{var}_n \psi < \infty\}$ and using the analyticity of log and trace (tr), we see that

$$\phi \to \log \mathrm{tr}(L_\phi P_s) = \log \lambda_\phi = P(\phi)$$

is a real analytic map.

Corollary 7.8

Let μ be the unique g-measure, for a $g \in G$ with $\sum_{n=0}^{\infty} n \, \mathrm{var}_n \log g < \infty$, then $b_\mu : R \to R$

$$b_\mu(t) = \exp P(-t \ I_\mu(A|\sigma^{-1}A)$$

is a real analytic function.

Proof

By an earlier computation $I_\mu(A|\sigma^{-1}A) = -\log g$. Therefore, since $t \to -t\log g$ is analytic and P is analytic on $\{t \log g | \ \Sigma n \ \mathrm{var}_n \log g < \infty\}$ we have that

$$B_\mu(t) = \exp P(-tI_\mu(A|\sigma^{-1}A)$$

is real analytic.

8. THE MAIN THEOREM AND ITS PROOF

Theorem 8.1

Let $\phi:(X_1,\mu_1) \to (X_2,\mu_2)$ be a finitary isomorphism with finite code length and inverse code length between two topological mixing subshifts of finite type X_1 and X_2, (μ_1 and μ_2 are the unique g-measures of g_1, g_2 which have finite first moment variational sum)then the β-functions $\beta_1(t)$ $\beta_2(t)$ are identical.

Proof

This proof uses the Techniques of Schmidt [1]. Let C, D the sets as described in Section 3. By Proposition 5.5 we can find a measurable function $f:X_1 \to R$ satisfying Proposition 5.5 and 5.8. Define $A' \supset D$ as in Section 5 then $\mu_1(A') > 0$ and, by Proposition 5.8 and definition of A'

$$|J_{\mu_1}(A,\sigma^n) - J_{\mu_2}(A,\sigma^n)o\phi|$$

8.(i) $= |fo\sigma^n - f| \le 4C'$ on $A' \cap \sigma^{-n}A'$ a.e. μ_1 and for every $n \ge 1$.

I shall prove

8.(ii) $\limsup\limits_{n\to\infty} \left(\int_{A'\cap\sigma^{-n}A'} \operatorname{expt} J_{\mu_1}(A,\sigma^n)d\mu_1\right)^{1/n}$

$$= \lim_{n\to\infty} \left(\int \operatorname{expt} J_{\mu_1}(A,\sigma^n)d\mu_1\right)^{1/n} = \beta_1(1-t) \quad \forall\, t > 0.$$

Assuming 8.(ii), for the moment, we can complete the proof as follows:

$$\left(\int_{A' \cap \sigma^{-n}A'} \exp t\, J_{\mu_1}(A,\sigma^n)\, d\mu_1\right)^{1/n}$$

$$= \left(\int_{\phi(A' \cap \sigma^{-n}A')} \exp t\, J_{\mu_1}(A,\sigma^n) \circ \phi^{-1}\, d\mu_2\right)^{1/n}$$

(by change of variables Parthasarathy p. 135)

$$\leq \left(\int \exp\,[t(J_{\mu_2}(A,\sigma^n) + 4C')]\, d\mu_2\right)^{1/n} \quad \text{by 8.(i).}$$

Thus taking limits as n tends to infinity we obtain

$$\beta_1(1-t) \leq \lim_{n\to\infty} \left(\int \exp[t(J_{\mu_2}(A,\sigma^n) + 4C')]\, d\mu_2\right)^{1/n}$$

$$= \lim_{n\to\infty} (\exp t4C)^{1/n} \left(\int \exp t\, J_{\mu_2}(A,\sigma^n)\, d\mu_2\right)^{1/n}$$

$$= \beta_2(1-t) \quad \forall\, t \in \mathbb{R},\ t > 0.$$

The inequality is symmetric in 1 and 2 and we have that

$$\beta_1(1-t) = \beta_2(1-t) \quad \forall\, t \in \mathbb{R},\ t > 0.$$

The analyticity of the β-function (Corollary 7.8) reveals that $\beta_1(t) = \beta_2(t)$ for all $t \in \mathbb{R}$.

It just remains for us to verify relation 8.(ii). Consider the partition C generated by $\bigcup_{i=n-M}^{\infty} \sigma^{-i}\alpha \cup \bigcup_{i=-M} \sigma^{i}\alpha$, $n > 2M$.

Let $[x]_C$ be an atom of this partition containing the element $x \in X$. Denote by $\mu^C_{1,x}$ the conditional measure of x, with respect to the measure μ_1, given by this partition. The conditional measure on $[x]_C$ with respect to the partition C is defined to be the measure $\mu^C_{1,x}$ which satisfies the following two conditions.

(i) $\mu^C_{1,x}([x]_C) = 1$

(ii) $\mu_1(B) = \int_{X_1} \mu^C_{1,x}(B)d\mu$ for all $B \in \mathcal{B}_X$.

Since $[X]_C$ is a finite set the measure which satisfies (i) and (ii) is the atomic measure which assigns to the point $(y_n)^{\infty}_{-\infty} \in [x]_C$

$$\mu^C_{1,x}(\{(y)\}) = \frac{\mu_1([Y_M, \ldots, Y_{n-M}]^{n-M}_M)}{\mu_1([x_M]_M \cap [x_{n-M}]_{n-M})}$$

(Note that $X_m = Y_m$ and $X_{n-M} = Y_{n-M}$ and that $\mu_1([x_m]_M \cup [X_{n-M}]_{n-M}) > 0$ since μ_1 has full support).

The measure which satisfies (i) and (ii) is unique (see V.A. Rohlin [1]).

Thus by property (ii)

$$\int_{A'\cap\sigma^{-n}A'} \exp[tJ_{\mu_1}(A,\sigma^n)]d\mu_1$$

$$= \int \int_{A'\cap\sigma^{-n}(A')} \exp[t\ J_{\mu_1}(A,\sigma^n)]d\mu^C_{1,x}\ d\mu_1$$

$$\geq \int_{D\cap\sigma^{-n}D} \int \exp[t\ J_{\mu_1}(A,\sigma^n)]d\mu^C_{1,x}\ \ d\mu_1$$

(since $\mu^C_{1,x}\ (A'\cap\sigma^{-n}A') = 1$ a.e. $x \in D \cap \sigma^{-n}D$ by Corollary 5.9(i))

$$= \int_{D\cap\sigma^{-n}D} \sum_{y\in[x]_C} \exp[t\ J_{\mu_1}(A,\sigma^n)(y)]\ \frac{\mu_1[X_M,\ldots,X_{n-M}]}{\mu_1([X_M]_M\cap[X_{n-M}]_M)}d\mu_1$$

Since $x \in D \cap \sigma^{-n}D$, $X_m = i_M$, $X_{n-M} = i_{-M}$. Thus since also

$$\text{ex}\ \ [t\ J_{\mu_1}(A,\sigma^n)(y)] = \Big(\prod_{k=0}^{n-1} \frac{1}{g_1(\sigma^k y)}\Big)^t$$

$$\int_{D\cap\sigma^{-n}D} \sum_{y\in[X]_C} \exp\ t[J_{\mu_1}(A,\sigma^n)(y)]\ \frac{\mu_1[x_m,\ldots,X_{n-M}]^{n-M}_M}{\mu_1([X_M]_M\cap[X_{n-M}]_M}\ d\mu_1$$

$$\geq \frac{1}{\mu([i_M]_0\cap[i-M])}\ Q^{(2M-2)t} \int_{D\cap\sigma^{-n}D} \sum_{y\in[x]_C} \Big(\prod_{k=M}^{n-M} \frac{1}{g_1(\sigma^k y)}\Big)^t\ [X_M,\ldots,X_{n-M}]d\mu_1$$

where $Q > 0$ is a real number chosen such that $\frac{1}{g} > Q$.

$$\text{Define } P_S(x) = \inf_{z\in[x_0,\ \ x_S]} \prod_{k=0}^{S} \frac{1}{g_1(\sigma^k z)}\ .$$

Then if $x_0 = y_0, \ldots, x_s = y_s$, $y \in X$,

$$P_S(y) = \inf_{z\in[y_0,\ldots,y_s]} \prod_{k=0}^{S} \frac{1}{g_1(\sigma^k z)} = P_s(x), \text{ i.e.}$$

P_s is constant on cylinders of length s+1. Also

$$P_{n-2M}(x) \le \prod_{k=0}^{n-2M} \frac{1}{g_1(\sigma^k x)} ,$$

so that

$$P_{n-2m}(\sigma^M x) \le \prod_{k=M}^{n-M} \frac{1}{g_1(\sigma^k x)} .$$

Therefore

$$\frac{1}{\mu_1[i_M]_o \cap [i_{-M}]_{n-2M})} Q^{(2M-2)t} \int_{D\cap\sigma^{-n}D} \sum_{y\in[x]_c} \Big(\prod_{k=M}^{n-M} \frac{1}{g_1(\sigma^k y)}\Big)^t \mu_1[x_M,\ldots,x_{n-M}])d\mu_1$$

$$\ge \frac{Q^{(2M-2)t}}{\mu_1([i_M]_o \cap [i_{-M}]_{n-2M}} \int_{D\cap\sigma^{-n}D} \sum_{y\in[x]_c} [P_{n-2M}(\sigma^M y)]^t \mu_1([x_M,\ldots,x_{n-M}])d\mu_1 .$$

Now;

$$\int_{[i_M]_M \cap [i-M]_{n-M}} P_{n-2M}(\sigma^M)\, d\mu_1 = \sum_{y\in[x]_c} [P_{n-2M}(\sigma^M y)]\, \mu_1([i_M, y_{m+1},\ldots,i_{-M}])$$

The above equality being true since on cylinders of length n-2M+1, between M'th and n-M'th co-ordinates, $P_{n-2M}(\sigma^M)$ is a constant. Therefore;

$$\frac{Q^{(2M-2)t}}{\mu([i_M]_o \cap [i\text{-}M]_{n-2M})} \int_{D\cap\sigma^{-n}D} \sum_{y\in[x]_c} [P_{n-2M}(\sigma^M y)]^t \, \mu[X_M, [X_{n-M}] d\mu_1$$

$$= \frac{Q^{(2M-2)t}}{\mu_1([i_M]_o \cap [i_{-M}]_{n-2M})} \int_{D\cap\sigma^{-n}D} \int_{[i_M]_M \cap [i_{-M}]_{n-M}} [P_{n-2M}(\sigma^M)]^t \, d\mu_1 \, d\mu_1$$

$$= \frac{\mu_1(D\cap\sigma^{-n}D) Q^{(2M-2)t}}{\mu_1([i_M]_o \cap [i\text{-}M]_{n-M})} \int_{[i_M]_M \cap [i_{-M}]_{n-M}} [P_{n-2M}(\sigma^M)]^t \, d\mu_1$$

$$= \frac{\mu_1(D\cap\sigma^{-n}D) Q^{(2M-2)t}}{\mu_1(\ i_M\ {}_o \cap\ i_{-M\ n-M})} \int_{[i_M]_o \cap [i_{-M}]_{n-2M}} (P_{n-2M})^t \, d\mu_1$$

(The last equality is by the σ-invariance of μ_1 and the change of variables formula Parthasarathy [1] p. 135).

$$P_{n-2M}(x) = \inf_{z\in[x_o,\dots,x_{n-2M}]} \left(\prod_{k=0}^{n-2M} \frac{1}{g_1(\sigma^k z)} \right)$$

$$\log P_{n-2M}(x) = \inf_{z\in[x_o,\dots,x_{n-2M}]} \sum_{k=0}^{n-2M} \log \frac{1}{g_1(\sigma^k z)}$$

for $z \in [x_o, \ ,x_{n-2M}]$ we have the following:

$$\left| \sum_{k=0}^{n-2M} \log \frac{1}{g_1(\sigma^k z)} - \sum_{k=0}^{n-2M} \log \frac{1}{g_1(\sigma^k x)} \right| \le \sum_{k=0}^{n-2M} \mathrm{var}_n \log g_1 \ .$$

Therefore

$$\sum_{k=0}^{n-2M} \log \frac{1}{g_1(\sigma^k z)} \ge \sum_{k=0}^{n-2M} \log \frac{1}{g_1(\sigma^k x)} - \sum_{k=0}^{n-2M} \mathrm{var}_n \log g_1 .$$

Thus

$$\log P_{n-2M}(x) \ge J_{\mu_1}(A,\sigma^{n-2M})(x) - \sum_{k=0}^{n-2M} \mathrm{var}_n \log g_1 .$$

Therefore:

$$(P_{n-2M}(x))^t \ge \exp[t\, J_{\mu_1}(A,\sigma^{n-2M})] \exp[-t\, (\sum_{k=0}^{n-2M} \mathrm{var}_n \log g_1)]$$

Thus putting all these inequalities together

$$\int_{A\cap\sigma^{-n}A} \exp t\, J_{\mu_1}(A,\sigma^n)d\mu_1$$

$$\ge \frac{\mu_1(D\cap\sigma^{-n}D)Q^{(2M-2)t}}{\mu_1([i_M]_o\cap[i_{-M}]_{n-2M}} \exp(-t(\sum_{k=0}^{n-2M} \mathrm{var}_n \log g_1)) \int_{[i_M]_o\cap[i_{-M}]_{n-2M}} \exp[t\, J_{\mu_1}(A,\sigma^{n-2M})]d\mu$$

Thus taking lim sup's we obtain

$$\limsup_n \left(\int_{A\cap\sigma^{-n}A} \exp[t\, J_{\mu_1}(A,\sigma^n)]d\mu_1\right)^{1/n}$$

$$\ge \limsup_n \left(\int_{[i_M]_o\cap[i_{-M}]_{n-2M}} \exp(t\, J_\mu(A,\sigma^{n-2M}))d\mu_1\right)^{1/n}$$

Lemma 8.1

With μ_1, g_1 as above

$$\limsup_{n\to\infty} \left(\int_{[i_M]_0 \cap [i_{-M}]_{n-2M}} \exp[t\, J_{\mu_1}(A,\sigma^{n-2M})]d\mu_1\right)^{1/n}$$

$$= \lim_{n\to\infty} \left(\int \exp[t\, J_{\mu_1}(A,\sigma^{n-})^{2M}]\, d\mu_1\right)^{1/n}.$$

Assuming Lemma 8.1, for the moment, 8(ii) is then proven and Theorem 8.1 is complete.

Proof of Lemma 8.1

Clearly, since $\exp[t\, J_{\mu_1}(A,\sigma^{n-2M})] > 0$

$$\limsup_{n\to\infty} \left(\int_{[i_M]_0 \cap [i_{-M}]_{n-2M}}) \exp[t\, J_{\mu_1}(A,\sigma^{n-2M})]d\mu_1\right)^{1/n}$$

$$\leq \lim_{n\to\infty} \left(\int_{X_1} \exp[t\, J_{\mu_1}(A,\sigma^{n-2M})]d\mu_1\right)^{1/n}$$

It thus suffices to prove the converse inequality.

$$\int \exp[t\, J_{\mu_1}(A,\sigma^{n-2M})]d\mu_1$$

$$= \sum_{i,j} \int_{[i]\cap[j]_{n-2M}} \exp[t\, J_{\mu_1}(A,\sigma^{n-2M})]d\mu_1 \qquad 8.(iii).$$

Consider an estimate for

$$\int_{[i]\cap[j]_{n-2M}} \exp[t\, J_{\mu_1}(A,\sigma^{n-2M})]d\mu_1 - \int_{[i_M]\cap[i_{-M}]_{n-2M}} \exp[t\, J_{\mu_1}(A,\sigma^{n-2M})]d\mu_1$$

8.(iv).

$$= \sum_{x_1,\ldots,x_{n-2M-1}} \int_{[i,x_1,\ldots,x_{n-2M-1},j]} \exp[t\, J_{\mu_1}(A,\sigma^{n-2M})]d\mu_1$$

$$- \sum_{x_1,\ldots,x_{n-2M-1}} \int_{[i_M,x_1,\ldots,x_{n-2M-1},i_{-M}]} \exp[tJ_{\mu_1}(A,\sigma^{n-2M})]d\mu_1$$

let $a(x) = (x,x_1 \quad x_{n-2M-1},j,x_{2M} \ldots)$

$b(x) = (i_M x_1, \quad x_{n-2M-1},i_{-M},x_{n-2M}\ldots),\ x = (x_i)_{i=1}^{\infty}.$

8.(v) $= \sum_{x_1,\ldots,x_{n-2M-1}} \int \exp[t\, J_{\mu_1}(A,\sigma^{n-2M})](a(x)) - \exp[t\, J_{\mu}(A,\sigma^{n-2M})](b(x))d\mu_1$.

$$\exp \quad t[J_{\mu_1}(A,\sigma^{n-2M}](a(x))$$

$$= \Big(\prod_{\ell=0}^{n-2M} \frac{1}{g(\sigma^{\ell}(a(x))}\Big)^t$$

Therefore since:

$$\Big|\log \prod_{\ell=0}^{n-2M} \frac{1}{g_1(\sigma^{\ell}a(x))} - \log \prod_{\ell=0}^{n-2M} \frac{1}{g_1(\sigma^{\ell}b(x))}\Big|$$

$$\leq \sum_{k=0}^{n-2M-1} \operatorname{var}_k \log g_1 + \operatorname{var}_o \log g_1$$

$$\left(\prod_{\ell=0}^{n-2M} \frac{1}{g_1(\sigma^\ell a(x))}\right)^t \le \exp\left[t\left(\prod_{k=0}^{n-2M-1} \mathrm{var}_k \log g_1 + \mathrm{var}_o \log g_1\right)\right]\left(\prod_{\ell=0}^{n-2M} \frac{1}{g_1(\sigma^\ell(b(x)))}\right)^t$$

Putting this inequality into expression 8.(v) we obtain:

$$\sum_{x_1, \quad x_{n-1-2M}} \int \exp[t\, J_{\mu_1}(A,\sigma^{n-2M})](a(x)) - \exp[t\, J_{\mu_1}(A,\sigma^{n-2M})(b(\dot{x}))]d\mu_1$$

$$\le \sum_{x_1, \quad x_{n-1-2M}} \int [\exp[t(\sum_{k=0}^{n-2M-1} \mathrm{var}_k \log g_1 + \mathrm{var}_o \log g_1)]-1)]\exp t\, J_{\mu_1}(A,\sigma^{n-2M})(b(x))d\mu_1$$

$$= [\exp[t(\sum_{k=0}^{n-2M-1} \mathrm{var}_k \log g_1 + \mathrm{var}_o \log g_1)]-1]\int_{[i_M]\cap[i_{-M}]_{n-2M}} \exp[t\, J_{\mu_1}(A,\sigma^{n-2M})]d\mu_1$$

Therefore putting these inequalities together it follows that:

$$\int_{[i]_o\cap[j]_{n-2M}} \exp[t\, J_{\mu_1}(A,\sigma^{n-2M})]d\mu_1$$

$$\le \quad [\exp[t(\sum_{k=0}^{n-2M-1} \mathrm{var}_k \log g_1 + \mathrm{var}_o \log g_1)] - 1] \int_{[i_m]\cap[i_{-M}]_{n-2M}} \exp[t - J_{\mu_1}(A,\sigma^{n-2M})]d\mu_1$$

$$+ \int_{[i_M]\cap[i_{-M}]_{n-2M}} \exp[t - J_{\mu_1}(A,\sigma^{n-2M})]d\mu_1$$

Subing this expression into 8.(iii) and taking n'th roots we obtain the inequality.

$$[k^2(\exp[t(\sum_{k=0}^{n-2M-1} \mathrm{var}_k \log g_1 + \mathrm{var}_o \log g_1)]) - 1] + 1]^{1/n}$$

$$(\int_{[i_M]\cap[i_{-M}]_{n-2M}} \exp[t\, J_{\mu_1}(A,\sigma^{n-2M})]d\mu_1)^{1/n}$$

$$\geq (\int \exp[t\, J_{\mu_1}(A,\sigma^{n-2M})]d\mu_1)^{1/n}$$

Taking lim sup as n tends to infinity we obtain the converse inequality and Lemma 8.1 follows. Thus Theorem 8.1 is complete.

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