Lines on Intersections of Three Quadrics

by

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Declarations

In Chapter 1, the section with Motivation is my own work developed through discussions with my supervisor, unless otherwise stated. The main focus of this thesis is Conjecture 1.2.1 which (as far as the I know) has not been published before and is the result of numerous discussions with myself and my supervisor.

Chapter 2 largely consists of a summary of necessary background material which can be found in many different common sources which have been thoroughly cited throughout the section.

Chapter 3 consists of a summary of a relevant recent paper, and is hence cited throughout. It is here to emphasise that the important points from the case of quartic hypersurfaces also apply in the case of three quadrics.

Everything in Chapter 4 (unless explicity stated otherwise) is my own work, developed with the guidance and influence of my supervisor. As far as I am aware, none of the material here is published elsewhere, and thus forms the bulk of the original content of this thesis.

Furthermore, I confirm that this thesis has not been submitted for a degree at any other university.

Abstract

In this document we formulate and discuss conjecture 1.2.1, giving an upper bound for the number of lines on K3 surfaces occurring as complete intersections of three quadrics in $\mathbb{P}^5$. In the case that these quadrics contain in their span a quadric of rank 4, we construct a pair of elliptic fibrations, each of which realises the lines on the surface as either sections or line components within the singular fibres, and the general fibre is realised as an intersection of two quadrics in $\mathbb{P}^3$.

The possibilities for singular fibres are limited by the Euler number of the surface, while the rank of the group of sections is bounded by the rank of its Picard group. In the cases where this rank is low, these bounds are enough to prove the stated conjecture in the torsion-free case by utilising the height-pairing.

In the remaining cases, if a surface has more lines than the stated conjecture, we discuss how these techniques can be used to construct necessary conditions on the configurations of the lines on the surface, along with an example of how this could work in practice.

Chapter 1

Introduction

1.1 Motivation

The famous result of Cayley and Salmon of 1849 goes as follows:

Theorem 1.1.1. [Cay69] Let $X \subset \mathbb{P}^3$ be a smooth cubic surface over $\mathbb{C}$. Then X contains exactly 27 lines.

I first heard this result as an undergraduate, and found it most curious. What does the '27' represent exactly? Why 'exactly'? were obvious and immediate questions that arose in my head. Of course, these surfaces have now been extensively studied and the answers related to this surface have been known for a long time. The most immediate follow-up question is therefore: "What about lines on other surfaces? Does a similar result follow for quartics surfaces?". In 1943, Segre answers this question:

Theorem 1.1.2. [Seg43] Let $X \subset \mathbb{P}^3$ be a smooth quartic surface over $\mathbb{C}$. Then X contains at most 64 lines.

This is also quite an interesting result! Note that for the quartic we can only say "at most 64 lines", whereas the cubic was "exactly 27". In fact, there do exist such examples of quartic hypersurfaces over $\mathbb{C}$ with 64 lines, as well as examples with fewer:

- Schur's quartic $\{x^4 - xy^3 = z^4 - zw^3\} \subset \mathbb{P}^3$ contains precisely 64 lines.
- The Fermat quartic $\{x^4 + y^4 + z^4 + w^4 = 0\} \subset \mathbb{P}^3$ contains precisely 48 lines.
- On the other hand, if X is a "random" choice of smooth quartic hypersurface in $\mathbb{P}^3$, it will contain no lines at all. That is, the set of surfaces with no lines

is an open dense subset of the moduli space parametrising smooth quartic hypersurfaces in $\mathbb{P}^3$.

The number of lines can be checked using methods found in [BS07].
This change of behaviour in the minimum number of lines represents the fact that the quartic hypersurface is a "K3" surface, while the cubic is a del Pezzo surface. Indeed, a much more recent result shows that Segre's result can be extended to more general fields:

Theorem 1.1.3. [RS15b] Let k be any field of characteristic not 2 or 3. Then any smooth quartic surface $S \subset \mathbb{P}^3_k$ contains no more than 64 lines.

There are also corresponding results for the missing characteristics. In characteristic 3, the Fermat quartic contains 112 lines which is maximal [RS15a], while in characteristic 2 the maximal number on a smooth surface is 60 lines [Deg16], though whether this is sharp is unknown.
The methods used in these papers are much more modern than the geometric arguments used by Segre and Cayley/Salmon, allowing the results over more general fields.
It is therefore natural to ask "If you know the result for quartics, what about other types of K3 surface?". In this document, we concentrate on the case that X is a smooth complete intersection of three quadrics in $\mathbb{P}^5$.
Consider the following example: Let $x_0, x_1, \cdots, x_5$ be homogeneous coordinates for $\mathbb{P}^5$ over the field $\mathbb{C}$. Let $X \subset \mathbb{P}^5$ be the surface defined by the simultaneous vanishing of:

$$\begin{aligned}
Q_1 &: x_0^2 - 2x_1^2 + x_2^2 - 2x_5^2 \\
Q_2 &: x_1^2 - 2x_2^2 + x_3^2 - 2x_5^2 \\
Q_3 &: x_2^2 - 2x_3^2 + x_4^2 - 2x_5^2
\end{aligned}$$

note that X is smooth and a complete intersection. Notice also that the line L parametrised by

$$(t : u) \mapsto (t : t + u : t + 2u : t + 3u : t + 4u : u)$$

lies on X; it vanishes on Q_1, Q_2 and Q_3.
Observe that Q_1, Q_2 and Q_3 are all diagonal quadrics, so replacing x_i with $-x_i$, all three polynomials remain unchanged. Hence there is a $(\mathbb{Z}/2\mathbb{Z})^5$-group action on X, and the orbit of L under this group action yields $2^5 = 32$ lines, whose parametrisations can easily be obtained from that of L above. (Indeed, if one is interested in

questions regarding fields of definition for lines, note that all of the lines above in the orbit are defined over $\mathbb{Q}$.)

This document was written to discuss the following question: "Does this example for X have the maximal possible number of lines for an intersection of three quadrics?". In fact, one quite powerful result related to lines on intersections of quadrics whose proof requires no complicated tools is the following:

Lemma 1.1.4. Any smooth surface $X \subset \mathbb{P}^5$ which is a complete intersection of three quadrics in $\mathbb{P}^5$ cannot contain a "triangle" (that is, a 3-cycle of lines).

Proof. Let $I(X)$ be the ideal associated to the variety X, generated by three degree two homogeneous polynomials f_1, f_2 and f_3. Let $L \subset \mathbb{P}^5$ be a choice of a line. If L is suitably general, then $f_i|_L$ vanishes either at two distinct points or a single point with multiplicity two. Otherwise, it is possible is that $f_i|_L$ is identically zero, and hence f_i vanishes along the entire line L. In particular, if $f_i|_L$ vanishes at three distinct points, then $f_i|_L$ is identically zero. In terms of varieties, this means that if $L \cap Q_i$ contains at least three distinct points, then $L \subset Q_i$, and hence if $L \cap X$ contains at least three distinct points, then $L \subset X$.

Suppose now that X contains a triangle T of lines. The lines themselves are called "edges" and denoted by $\{L_1, L_2, L_3\}$, the "vertices" consist of the points $p_1 := L_2 \cap L_3$, $p_2 := L_1 \cap L_3$ and $p_3 := L_1 \cap L_2$. Note that there exists a unique two dimensional plane P with $T \subset P \subset \mathbb{P}^5$. Now $p \in P$ be any point and let L be a line in P passing through p, which does not meet p_1, p_2 and p_3 (this is the general case). Then $L \cap T$ consists of three distinct points x_1, x_2, x_3. These points lie on $L \cap X$, so by the argument above, $L \subset X$, and in particular, $p \in X$.

This argument works for any point $p \in P \setminus \{p_1, p_2, p_3\}$. As $T \subset X$ by definition, it follows that the entire plane $P \subset X$; that is, the two dimensional smooth complete intersection X contains a two plane P. Since X is smooth, we conclude that $P = X$, but this contradicts the fact that X is a K3 surface. We conclude that no such triangle $T \subset X$ exists. $\square$

Indeed, the configuration of lines on the example with 32 lines contains no "triangles", instead having many "squares"; i.e. four-cycles. It is this restrictive property combined with some of the ideas from Rams-Schütt which inspired the results and methods that I present in this document.

1.2 Overview of Results

The main question we address in this thesis is the following conjecture:

Conjecture 1.2.1. Let k be a field of characteristic not 2 or 3 and let $X \subset \mathbb{P}^5$ be a smooth complete intersection of three quadric hypersurfaces over k. Then X cannot contain any more than 32 lines.

Remark 1.2.2. Since we are looking for an upper bound for the number of lines on the surface, it is reasonable to assume that k is algebraically closed: we can only increase the number of lines if we include those that are not only defined on k but also on all extensions.

As far as the author knows, this result is not known. This conjecture is in the same spirit as the result of [RS15b, Theorem 1.2], which I state here:

Theorem 1.2.3 (Rams, Schütt)**.** Let k be a field of characteristic not 2 or 3. Then any geometrically smooth quartic surface over k contains at most 64 lines.

While I was not able to prove this conjecture in the required time to submit this in my PhD Thesis, this document contains a useful construction of an elliptic fibration on complete intersections of three quadric hypersurfaces, which allows for a study of their lines. From this we are able to deduce the following theorems, all of which (to the best of my knowledge) are new results.

Theorem 1.2.4. Let X be a smooth complete intersection of quadrics over k. If X contains a square of lines, then it can contain no more than 48 lines.

This is Lemma 4.1.10 together with Proposition 4.5.2.
In order to get closer to the ultimate goal of Conjecture 1.2.1, we later separate into a number of cases based upon the rank of the associated Mordell Weil Group of sections Φ which is a finitely generated abelian group (see Definition 4.6.1). Under the computational requirement of being torsion free, we use the height-pairing to improve the above estimate into the following theorem:

Theorem 1.2.5. Let X be a smooth complete intersection of quadrics over k as before. Assume that X contains a square of lines, it contains more than 32 lines in total, and its Mordell-Weil Group Φ is torsion free. It then follows that $\mathrm{rank}(\Phi) > 1$.

This is proved as Corollary 4.7.14.

Chapter 2

Background

This chapter contains a summary of necessary common background assumed in this document.

2.1 K3 Surfaces

Throughout this document k shall be assumed to be algebraically closed field (see Remark 1.2.2). We shall be dealing only with *varieties*; that is, geometrically integral separated schemes of finite type over k. Basic theory of varieties can be found in (for instance) [Rei88, Sha74, Mum95], while more advanced theory of schemes can be found in (for instance) [Sha95, Mum99]. What follows will be a short introduction into K3 surfaces.

2.1.1 Definitions and Examples

Definition 2.1.1. For a smooth variety $X \subset \mathbb{P}^n$ of dimension 2, consider the sheaf of differentials Ω_X. By definition this is a locally free sheaf of rank 2. The *canonical sheaf* ω_X is the invertible sheaf defined to be the determinant of Ω_X. Its associated divisor will be denote by K_X and called the *canonical divisor*.
In terms of differential forms, $K_X = \operatorname{div} s$, where s is any rational 2-form on X. In local coordinates $\{x, y\}$, this is $s = g dx \wedge dy$ for a non-zero rational function $g \in k(X)$.

Definition 2.1.2. A *K3 surface* is a smooth projective 2 dimensional variety over a field k such that $\omega_X \cong \mathscr{O}_X$ and $H^1(X, \mathscr{O}_X) = 0$.

Example 2.1.3. 1. A smooth quartic $X \subset \mathbb{P}^3$ is a K3 surface. Indeed from the short exact sequence

$$0 \to \mathscr{O}(-4) \to \mathscr{O} \to \mathscr{O}_X \to 0$$

on $\mathbb{P}^3$, consider the long exact sequence of cohomology, and use the fact that

$$H^1(\mathbb{P}^3, \mathscr{O}) = H^2(\mathbb{P}^3, \mathscr{O}(-4)) = 0$$

to deduce that $H^1(X, \mathscr{O}_X) = 0$.
Taking determinants of the cotangent bundle sequence ([Har77] Proposition 8.12)

$$0 \to \mathscr{O}(-4)|_X \to \Omega_{\mathbb{P}^3}|_X \to \Omega_X \to 0$$

yields the adjunction formula $\omega_X \cong \omega_{\mathbb{P}^3} \otimes \mathscr{O}(4)|_X \cong \mathscr{O}_X$. In local coordinates with X given by the vanishing of a quartic polynomial f, a trivialising section of ω_X is

$$\frac{\sum (-1)^i x_i dx_0 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_3}{f}$$

where $\widehat{dx_i}$ means "omit this term from the sum".

2. Similarly, any smooth complete intersection of type $(d_1, \cdots d_n)$ in $\mathbb{P}^{n+2}$ is a K3 surface if and only if $\sum d_i = n+3$. Note that this implies that there are only three cases: (4) in $\mathbb{P}^3$ (as above), $(2,3)$ in $\mathbb{P}^4$ and $(2,2,2)$ in $\mathbb{P}^5$. For the majority of Chapter 4 we shall be concentrating exclusively on the final of these three cases.

Let $h^i(X, \mathscr{F})$ be the rank of the sheaf cohomology group $H^i(X, \mathscr{F})$.

Remark 2.1.4. Note that if X is a K3 surface, then by definition $h^0(X, \mathscr{O}_X) = 1$, and $h^1(X, \mathscr{O}_X) = 0$. By Serre Duality, we have $h^2(X, \mathscr{O}_X) = h^0(X, \mathscr{O}_X) = 1$. This implies that $\chi(X, \mathscr{O}_X) = 2$.

2.1.2 Linear, Algebraic and Numerical equivalence

Let X be a smooth surface over k and write $\mathrm{Div}(X)$ for its group of Weil divisors. Let $- \cdot - : \mathrm{Div}(X) \times \mathrm{Div}(X) \to \mathbb{Z}$ denote the intersection pairing on X. Recall the following equivalence relations on $\mathrm{Div}(X)$:

Definition 2.1.5. 1. $C, D \in \mathrm{Div}(X)$ are *linearly equivalent* if $C = D + \mathrm{div}(f)$ for some rational function $f \in k(X)$.

2. $C, D \in \mathrm{Div}(X)$ are *algebraically equivalent* if there is a connected curve T, two closed points 0 and $1 \in T$, and a divisor E in $X \times T$, flat over T, such that $E_0 - E_1 = C - D$.

3. C, D are *numerically equivalent* if $C \cdot E = D \cdot E$ for all $E \in \mathrm{Div}(X)$.

For general surfaces, these relations are related by the following implications:

$$\text{Linear Equivalence} \Rightarrow \text{Algebraic Equivalence} \Rightarrow \text{Numerical Equivalence}$$

Here is a brief explanation:
For the first implication, if C and D are linearly equivalent, then $C = D + \operatorname{div}(f)$ so we can take $T = \mathbb{P}^1$ with local coordinates t, u and then $E = \operatorname{div}(tf - u)$ in $X \times \mathbb{P}^1$, showing that C and D are algebraically equivalent.
For the second, suppose that C, D are algebraically equivalent, given by a divisor $E \subset X \times T$. Choose an embedding $X \hookrightarrow \mathbb{P}^n$ given by a very ample divisor H; this allows us to embed $X \times T$ and hence E in $\mathbb{P}^n_T$. As $C \cdot H$ is the degree of C in the embedding induced by H, by flatness of the fibres E over T and as T is connected, we see that $C \cdot H = D \cdot H$. From [Har77, page 359] (as we assume k is algebraically closed, Remark 1.2.2), we can write any divisor on X as a difference of ample divisors, completing the argument.

Definition 2.1.6. The *Picard Group* is the abelian group $\operatorname{Pic}(X)$ for the equivalence classes of $\operatorname{Div}(X)$ by linear equivalence.
Let $\operatorname{Pic}^{\tau}(X) \subseteq \operatorname{Pic}(X)$ be the set of classes which are numerically trivial; that is

$$\operatorname{Pic}^{\tau}(X) = \{L \in \operatorname{Pic}(X) | L \cdot L' = 0 \text{ for all } L' \in \operatorname{Pic}(X)\}$$

Finally, let $\operatorname{Pic}^0(X)$ be the set of classes algebraically equivalent to zero.
The *Neron-Severi Group* is the abelian group $\operatorname{NS}(X) := \operatorname{Pic}(X)/\operatorname{Pic}^0(X)$, and denote $\operatorname{Num}(X) := \operatorname{Pic}(X)/\operatorname{Pic}^{\tau}(X)$.

In the special case of X being a K3 surface, things become much simpler. First, by Riemann-Roch for surfaces [Har77, Theorem V.1.6] we have:

Lemma 2.1.7. Let X be a K3 surface and let $L \in \operatorname{Pic}(X)$. Then

$$\chi(X, L) = \frac{L^2}{2} + 2$$

Proof. For any divisor on a surface X we have, by Riemann-Roch

$$\chi(X, L) = \frac{1}{2} L \cdot (L - K_X) + \chi(X, \mathcal{O}_X).$$

X being a K3 surface yields $K_X = 0$ and since we showed above in Remark 2.1.4 that $\chi(X, \mathcal{O}_X) = 2$ this immediately yields the result. □

Proposition 2.1.8. Let X be a K3 surface over a field. Then the natural surjections

$$\mathrm{Pic}(X) \to \mathrm{NS}(X) \to \mathrm{Num}(X)$$

are isomorphisms.

Proof. Since X is projective, there is an ample sheaf L' on X. If $L \in \ker(\mathrm{Pic}(X) \to \mathrm{Num}(X))$ then $L.L' = 0$ and thus if $L \neq \mathcal{O}_X$ then $H^0(X, L) = 0$. Serre duality implies that $H^2(X, L) \cong H^0(X, L^{\otimes -1})^* = 0$. Hence $\chi(X, L) \leq 0$. On the other hand, by Lemma 2.1.7, we have $\chi(X, L) = \frac{1}{2}L^2 + 2$ and hence $L^2 < 0$, meaning that L cannot be numerically trivial. □

The above proposition allows us to freely switch between different viewpoints with no confusion, and we will often be doing so implicitly.
To help us compute the self-intersection numbers of curves on X, we have the adjunction formula, see [Har77, V, Proposition 1.5]:

Proposition 2.1.9. If C is a nonsingular curve of genus g on the surface X, then

$$2g - 2 = C \cdot (C + K_X).$$

In the case of X being a K3 surface, we simply have

$$2g - 2 = C^2.$$

and in particular, if $L \cong \mathbb{P}^1$ is a line, then $g(L) = 0$ so $L^2 = -2$.

We shall now turn our attention to the Singular Cohomology of K3 surfaces, allowing us to obtain important data on $\mathrm{Pic}(X)$.

2.1.3 Singular Cohomology of Complex K3 surfaces

In the special case where $k = \mathbb{C}$, every K3 surface as defined above is also given the structure of a complex manifold. This process is known as Serre's GAGA Principle [Ser56] This allows us to study K3 surfaces via their singular cohomology.
Let $e(X)$ denote the topological Euler characteristic of a space X, and $c_i(X)$ denote the i-th chern class of the tangent bundle of X for $i = 1$ and 2. Again, from Remark 2.1.4, $\chi(X, \mathcal{O}_X) = 2$. On the other hand Noether's formula states that

$$\chi(X, \mathcal{O}_X) = \frac{1}{12}(c_1(X)^2 + c_2(X)) = \frac{1}{12}(K_X \cdot K_X + e(X))$$

(see [BHPVdV04, Theorem I.5.5]). As $K_X = 0$ (or equivalently $\omega_X \cong \mathscr{O}_X$), we have $c_1(X)^2 = 0$ and hence $e(X) = c_2(X) = 24$.
For the singular cohomology groups of X we have $H^0(X,\mathbb{Z}) \cong \mathbb{Z}$ since X is connected and $H^4(X,\mathbb{Z}) \cong \mathbb{Z}$ because X is oriented (see, for example [Hat01, Section 3.3]).
In order to relate terms of singular cohomology with sheaf cohomology, we use the exponential sequence:

$$0 \to \mathbb{Z} \to \mathscr{O}_X \to \mathscr{O}_X^\times \to 0$$

giving rise to the long exact sequence:

$$\begin{aligned} 0 &\to H^0(X,\mathbb{Z}) \to H^0(X,\mathscr{O}_X) \to H^0(X,\mathscr{O}_X^\times) \to H^1(X,\mathbb{Z}) \to H^1(X,\mathscr{O}_X)\cdots \\ &\to H^1(X,\mathscr{O}_X^\times) \xrightarrow{c_1} H^2(X,\mathbb{Z}) \to H^2(X,\mathscr{O}_X) \to H^2(X,\mathscr{O}_X^\times) \to H^3(X,\mathbb{Z}) \end{aligned} \tag{2.1}$$

By using Cellular or Simplicial homology (and the fact that X here is a complex manifold, and hence is constructed only from cells of even dimension), the equivalence of cohomology theories and the Universal Coefficient theorem (for example, see [Hat01, Section 2.2]) we can see that $H^1(X,\mathbb{Z}) = H^3(X,\mathbb{Z}) = 0$. From the Universal Coefficient Theorem we have that

$$0 \to \mathrm{Ext}(H_1(X,\mathbb{Z}),\mathbb{Z}) \to H^2(X,\mathbb{Z}) \to \mathrm{Hom}(H_2(X,\mathbb{Z}),\mathbb{Z}) \to 0$$

the Ext term is 0 and as $H_2(X,\mathbb{Z})$ is finitely generated, we see that $H^2(X,\mathbb{Z})$ is torsion free and of the same rank as $H_2(X,\mathbb{Z})$. As $e(X) = 24$, we deduce that $\operatorname{rank} H^2(X,\mathbb{Z}) = 24 - 1 - 1 = 22$. Summarising these results about the singular cohomology of a complex K3 surface, we deduce that:

Proposition 2.1.10. Let X be a K3 surface over the $\mathbb{C}$. Then its singular cohomology groups are:

$$H^i(X,\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if } i = 0,4 \\ \mathbb{Z}^{22} & \text{if } i = 2 \\ 0 & \text{otherwise} \end{cases}$$

As $H^4(X,\mathbb{Z}) \cong \mathbb{Z}$, Poincaré Duality tells us that the cup product induces a perfect bilinear pairing:

$$\smile : H^2(X,\mathbb{Z}) \times H^2(X,\mathbb{Z}) \to \mathbb{Z}$$

for this case, this simply means that for any primitive x there is a y such that $x \smile y = 1$.
We shall investigate this paring f on $H^2(X,\mathbb{Z})$, but first we recall some lattice

theory.

2.2 Lattices

In this section we shall contain all the necessary background on lattice theory. For more detailed information, see the paper [Nik80] covering the classification of unimodular lattices, or one of the books [CS99, Ebe13, MH73].
For the time being we treat lattices in their pure and most abstract form, and begin with a series of algebraic definitions.

Definition 2.2.1. Let M be a free $\mathbb{Z}$-module and $f : M \times M \to \mathbb{Z}$ be a bilinear form on M.

1. f is said to be *degenerate* if there exists $x \in M$ with $x \neq 0$ such that for any $y \in M$, $f(x,y) = 0$. Otherwise, f is said to be *non-degenerate*.

2. f is said to be symmetric if for any $x, y \in M$ we have $f(x,y) = f(y,x)$.

Definition 2.2.2. A *lattice* S is a free $\mathbb{Z}$-module of finite rank, together with a non-degenerate symmetric bilinear form with values in $\mathbb{Z}$. In this document, we shall often use the notation $x \cdot y := f(x,y)$ for the bilinear form on S (the same notation as for the intersection form on a surface). When $x = y$ we shall often use $x^2 := f(x,x)$.

Let S be a lattice of finite rank r. Choose a basis $\{e_1, \cdots, e_r\}$ to generate the free $\mathbb{Z}$-module. Note that by convention any element of S can be written as a sum of integer multiples of e_i.

Definition 2.2.3. The *Gram matrix of S with respect to this basis* is the $r \times r$ matrix A, whose (i,j)-th entry is the integer $e_i \cdot e_j$.

Remark 2.2.4.

1. As the bilinear form on S is assumed to be symmetric, any Gram matrix will be a symmetric matrix.

2. As the bilinear form on S is assumed to be non-degenerate, any Gram matrix will always have full rank. This is because after an orthogonal change of basis we can assume that the Gram matrix is diagonal with eigenvectors $x_1, \cdots, x_n$. But then $x_i \cdot x_j = 0$ if $i \neq j$, so the Gram matrix has full rank if and only if its eigenvectors have $x_i . x_i \neq 0$ for each i. It follows that if the Gram matrix is not full rank, then the bilinear form is degenerate. (See Lemma 2.2.10.)

Here we use the usual rules of matrix multiplication as a short-hand while working in the lattice S. In this way, by definition A can be expressed as

$$A = \begin{pmatrix} e_1 \\ \vdots \\ e_r \end{pmatrix} \begin{pmatrix} e_1 & \cdots & e_r \end{pmatrix}$$

then if $\{f_1, \cdots, f_r\}$ is some other basis, with a change of basis matrix P, we have that

$$\begin{pmatrix} f_1 & \cdots & f_r \end{pmatrix} = \begin{pmatrix} e_1 & \cdots & e_r \end{pmatrix} P$$

and hence

$$P^T A P = P^T \begin{pmatrix} e_1 \\ \vdots \\ e_r \end{pmatrix} \begin{pmatrix} e_1 & \cdots & e_r \end{pmatrix} P = \begin{pmatrix} f_1 \\ \vdots \\ f_r \end{pmatrix} \begin{pmatrix} f_1 & \cdots & f_r \end{pmatrix}$$

so the associated Gram matrix for S with respect to $\{f_1, \cdots, f_r\}$ is the conjugate $P^T AP$.

We can now define a number of invariants of the lattice, with corresponding properties of their Gram matrices (of course these properties of the matrices will be invariant under conjugation).

Definition 2.2.5. The lattice S is said to be *even* if $x^2 := x \cdot x \in 2\mathbb{Z}$ is even for each $x \in S$. Otherwise, S is said to be *odd*.

Lemma 2.2.6. Let A be the Gram matrix of S with respect a choice of basis. S is even if and only if the diagonal entries $A_{ii} \in 2\mathbb{Z}$ for all $1 \leq i \leq r$.

Proof. Let $\{e_1, \cdots e_r\}$ be the basis associated to A. Note that $x^2 \in 2\mathbb{Z}$ for all $x \in S$ if and only if $e_i^2 \in 2\mathbb{Z}$ for all $1 \leq i \leq r$. This happens if and only if $A_{ii} = e_i^2 \in 2\mathbb{Z}$ for each i. □

Definition 2.2.7. For two lattices S_1, S_2 we denote by $S := S_1 \oplus S_2$ to be their *orthogonal sum.*

That is, if B_1 is a basis for S_1 with quadratic form f_1 and B_2 is a basis for S_2 with quadratic form f_2 then a basis for S is the union $B_1 \cup B_2$. The quadratic form f for S is then defined on basis elements by:

$$f(x,y) := \begin{cases} f_1(x,y) & \text{if } x \text{ and } y \in B_1 \\ f_2(x,y) & \text{if } x \text{ and } y \in B_2 \\ 0 & \text{otherwise} \end{cases}$$

If A_1 and A_2 are the Gram matrices of S_1 and S_2 with respect to the bases B_1 and B_2 respectively, then the associated Gram matrix of S with respect to B is simply the diagonal sum of matrices $A := A_1 \oplus A_2$.

Definition 2.2.8. Let S be a lattice with basis B and A be the Gram matrix of S with respect to B.

1. The *discriminant* of S is the value $\mathrm{discr}(S) := \det(A)$.

2. S is said to be *unimodular* if $\mathrm{discr}(S) = \pm 1$.

Note that the discriminant of a lattice is independent of the choice of basis; any change of basis matrix and its inverse are both integral and their determinants are hence ± 1.

Definition 2.2.9. Let S be a lattice with quadratic form f. For $a \in \mathbb{Q}$, write $S(a)$ for the lattice with quadratic form af, under the assumption that af remains integral. We say that $S(a)$ is obtained from S by *twisting* by a.

If S has Gram matrix A with respect to some basis, we can compute its eigenvalues. Being symmetric, we have the following well-known result from Linear Algebra (which we do not prove here):

Lemma 2.2.10. A square matrix A is symmetric if and only if there exists an orthogonal matrix Q and a diagonal matrix D such that $Q^T AQ = D$. In particular, the diagonal elements of D are eigenvalues. If 0 is an eigenvalue the form is degenerate.

Let λ_+ be the number of positive eigenvalues of A, while λ_- is the number of negative eigenvalues of A.

Definition 2.2.11. The *rank* of a lattice S, denoted $\mathrm{rank}(S)$ is defined to be its rank as a $\mathbb{Z}$-module.
If S has Gram matrix A with respect to some basis, then the *signature* of S, is defined to be the pair $\mathrm{sign}(S) := (\lambda_+, \lambda_-)$.

Lemma 2.2.12. The signature of a lattice S is independent of the choice of basis.

Proof. Extend the quadratic form of the lattice to the vector space $V := S \otimes_{\mathbb{Z}} \mathbb{R}$. Here we use Sylvester's Law on Inertia (for example, [Nor86]) implies that the induced quadratic form on V can be diagonalised into the form

$$f = y_1^2 + y_2^2 + \cdots + y_p^2 - y^p \overline{S_1} = q\overline{S_2} 2_{p+1} - \cdots - y_r^2$$

and the number p of positive squares appearing in this reduction does not depend on the choice of basis. This number p is hence the value of λ_+, while $\lambda_- = r - p$. $\square$

Definition 2.2.13. A lattice S is called:

1. *positive definite* if for any $0 \neq x \in S, x^2 > 0$.
2. *negative definite* if for any $0 \neq x \in S, x^2 < 0$.
3. *positive semi-definite* if for any $x \in S, x^2 \geq 0$.
4. *negative semi-definite* if for any $x \in S, x^2 \leq 0$.

Equivalently, one sees that a non-degenerate lattice S of signature (λ_+, λ_-) is either positive or negative definite if and only if $\lambda_- = 0$ or $\lambda_+ = 0$ respectively. The following classification of even unimodular lattices is due to Milnor:

Theorem 2.2.14. [Ser70, V.2.2][Classification of Indefinite Even Unimodular Lattices] We have an existence and uniqueness criteria for even indefinite unimodular lattices as follows:

1. An even unimodular lattice of signature (λ_+, λ_-) exists if and only if $l_+ - l_- \equiv 0 \mod 8$.
2. If $\lambda_+ > 0$ and $\lambda_- > 0$ then an even unimodular lattice with these invariants is isomorphic to the lattice

$$U^{\oplus \lambda_+} \oplus E_8(-1)^{\oplus(\lambda_+ - \lambda_) - /8}$$

where U and E_8 are the lattices that are constructed by the Gram matrices

$$U := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad E_8 := \begin{pmatrix} 2 & -1 & & & & & & \\ -1 & 2 & -1 & & & & & \\ & -1 & 2 & -1 & & & & \\ & & -1 & 2 & -1 & & & \\ & & & -1 & 2 & -1 & & -1 \\ & & & & -1 & 2 & -1 & \\ & & & & & -1 & 2 & \\ & & & & -1 & & & 2 \end{pmatrix} \tag{2.2}$$

2.2.1 Intersection Form on a K3 Surface

We recall Proposition 2.1.10, which implies that any K3 surface X over $\mathbb{C}$ has $H^2(X, \mathbb{Z}) \cong \mathbb{Z}^{22}$ and that Poincaré Duality induces a perfect bilinear pairing $\smile$. Indeed, as it is a perfect pairing, the quadratic form on $H^2(X, \mathbb{Z})$ is non-degenerate and in particular it is unimodular. Moreover, the cup product $\smile$: $H^2(X,\mathbb{Z})\times H^2(X,\mathbb{Z})\to H^4(X,\mathbb{Z})$ is symmetric on H^2, so we get the following:

Lemma 2.2.15. Let X be a K3 surface over $\mathbb{C}$. The cup product induces a lattice structure on $H^2(X,\mathbb{Z})$.

We shall now compute various invariants of this lattice structure.

Proposition 2.2.16. [BHPVdV04, VIII.3.1] The pairing $\smile$ is even; that is $x\smile x\in 2\mathbb{Z}$ for all $x\in H^2(X,\mathbb{Z})$.

The cup product thus gives rise to an even integral quadratic form

$$q : H^2(X,\mathbb{Z})\to\mathbb{Z} \qquad z\mapsto x\smile x$$

Extend q by $\mathbb{R}$-linearity to a form $q_{\mathbb{R}} : H^2(X,\mathbb{Z})\otimes\mathbb{R}\to\mathbb{R}$. Let λ_+ and λ_- denote respectively the number of positive and negative eigenvalues of $q_{\mathbb{R}}$. The Thom-Hirzebruch index theorem [Hir66, Page 86] says that

$$b_+ - b_- = \frac{1}{3}(c_1(X)^2 - 2c_2(X)) = -16.$$

On the other hand, $\lambda_+ + \lambda_- = \mathrm{rank}(H^2(X,\mathbb{Z})) = 22$. Combining this discussion with Theorem 2.2.14, we have proved the following:

Proposition 2.2.17. Let X be a K3 surface over $\mathbb{C}$. The cup product induces the structure of a lattice Λ_{K3} on $H^2(X,\mathbb{Z})$. Moreover:

$$\Lambda_{K3}\cong U^{\oplus 3}\oplus E_8(-1)^{\oplus 2}$$

2.2.2 Néron-Severi Lattice of a Complex K3 surface

For a complex K3 surface X, the long exact sequence (2.1) associated to the exponential sequence and the vanishing of $H^1(X,\mathscr{O}_X)=0$ give an injection

$$c_1 : H^1(X,\mathscr{O}_X^\times)\hookrightarrow H^2(X,\mathbb{Z})$$

which is also called the first Chern Class.
Recall the standard isomorphism $\mathrm{Pic}(X)\cong H^1(X,\mathscr{O}_X^\times)$ (see [Har77, III Ex4.5]).

Definition 2.2.18. Let X be a complex surface. We define by $H^{p,q}(X)$ the *Dolbeault cohomology group of complex differential forms of type* (p,q). This is isomorphic by Dolbeault's theorem to

$$H^{p,q}(X)\cong H^q(X,\Omega_X^p)$$

and satisfy the relations

$$H^{p,q}(X) = \overline{H^{q,p}(X)} \quad \text{and} \quad \bigoplus_{p+q=k} H^{p,q}(X) = H^k(X, \mathbb{C}).$$

For more information on the Hodge decomposition of complex surfaces, see [Voi07, Chapter 6].

When X is a K3 surface, from $\omega_X \cong \mathscr{O}_X$ we get $\mathrm{rank}(H^{2,0}(X)) = 1$, generated by ω_X. Serre duality and $\omega_X \cong \mathscr{O}_X$ again give that $\mathrm{rank}(H^{0,2}(X)) = 1$. As $\mathrm{rank}(H^2(X, \mathbb{C})) = 22$, it follows that $\mathrm{rank}(H^{1,1}(X)) = 20$.
In particular $\mathbb{C}\omega_X = H^{2,0}(X)$ determines the decomposition of $H^2(X, \mathbb{C})$ into its Dolbeault cohomology groups. The cup product on $H^2(X, \mathbb{Z})$ extends to a symmetric bilinear pairing on $H^2(X, \mathbb{C})$ equal to the bilinear form $(\alpha, \beta) \mapsto \int_X \alpha \wedge \beta$.
Let $i_* : H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{C})$ be the canonical map. The Lefschetz (1,1)-theorem says that the image of $i_* \circ c_1$ is $H^{1,1}(X) \cap i_* H^2(X, \mathbb{Z})$. By Proposition 2.1.8 and the GAGA Principle [Ser56], the intersection $H^{1,1}(X) \cap i_* H^2(X, \mathbb{Z})$ coincides with $\mathrm{Pic}(X)$.
In particular, we have the following:

Proposition 2.2.19. Let X be a complex K3 surface. Then $0 \leq \mathrm{rank}(\mathrm{Pic}(X)) \leq 20$.

Proof. From the discussion above, the Néron-Severi lattice consists of integral classes in $H^2(X, \mathbb{Z})$ that are closed $(1,1)$ forms. In particular, the Picard number is at most the dimension of $H^{1,1}(X)$, which is 20 for a K3 surface. □

2.3 Elliptic Fibrations

In this section we introduce elliptic surfaces, and explain the relationship between elliptic curves over one-dimensional function fields using the generic fibre and the Kodaira-Néron model. General sources for this section are [Mir89, SS10, CD89].
As always, we shall work over a field of characteristic not 2 or 3.
Let C be a smooth projective curve over k.

Definition 2.3.1. An *elliptic surface* S over C is a smooth projective surface S together with an elliptic fibration over C; that is a surjective morphism $f : S \to C$ such that:

1. almost every fibre is a smooth connected curve of genus 1.

2. there are no -1 curves in the fibres of f.

Remark 2.3.2. The second condition comes from the classification of algebraic surfaces. -1-curves occur as exceptional divisors of blow-ups to surfaces at smooth points (this is Castelnuovo's Theorem, see [Har77, V 5.7]). In fact, it is always possible to blow-down any -1 curves on a surface X to reduce to a smooth minimal model Y. In other words, for any surface X there exists a surface Y and a birational map $f : X \to Y$ such that Y has no -1 curves, and that f can be factored as a finite number of blow-downs. For more details on this, and a proof, see [Har77, V 5.8].

Remark 2.3.3. Note in the definition, (which is consistent with the majority of literature on elliptic surfaces) we have not said that "the general fibre of f is an elliptic curve". This would imply that in each fibre there is a chosen special point acting as the trivial element for the fibre's group law, and hence that a section for the map f would be given. For the application we have in mind, we will not only assume the existence of a section, but will also assume be this section occurs as a line on the surface. Indeed, the cases where there are no sections become rather trivial!

Definition 2.3.4. We say that an elliptic surface $f : X \to C$ has *section* S if there is a map $s : C \to X$ with $f \circ s = \mathrm{id}_C$ where S is the image of s.

Assumption 2.3.5. In all that follows in this document, any elliptic fibration will take the following two assumptions:

1. there exists at least one section, which shall usually be denoted by S_0, and called the *zero section*; on each smooth genus 1 fibre F of X, $S_0 \cap F$ is the chosen point which forms the trivial element in the group law on F.
2. there exists at least one singular fibre; this rules out uninteresting cases, such as $X \cong E \times C$ where E is an elliptic curve.

The following remark is an important technical point:

Remark 2.3.6. Note that for a given surface X, there is no a priori reason why this choice of section would be unique; whenever a section is chosen the reader should be wary that the choice is not necessary "natural". For our purposes for finding bounds on the numbers of lines, this ambiguity is not so much of a problem, rather the flexibility that it offers is possibly an advantage.

Definition 2.3.7. Let $p : X \to \mathbb{P}^1$ be any elliptic fibration on X and C an irreducible curve on X. If C is contained in some fibre F of p, we say that C is *vertical with respect to p*. Otherwise, we say that C is *horizontal with respect to p*. When there is no confusion as to which elliptic fibration we refer to, we simply say C is *vertical* or *horizontal* as needed.

Assuming the section is very convenient for one more reason: we can work with a Weierstrass form

$$y^2 = x^3 + ax + b, \quad a, b \in k(C)$$

where we regard the generic fibre E as an elliptic curve over the function field $k(C)$. In particular, this implies that the sections form an abelian group, with the chosen section S_0 the trivial element of this group.
In terms of the Weierstrass form, the assumption that X contains at least 1 singular fibre guarantees that the coefficients are not contained in the ground field k, but are non-constant elements of the function field $k(C)$.

2.3.1 Interplay with Elliptic Curves over function fields

Constructing a curve over a function field: An elliptic surface X over C gives rise to an elliptic curve E over the function field $k(C)$ by way of using the generic fibre. Explicitly, a section $s : C \to X$ produces a $k(C)$ rational point P on E as follows: Let S denote the image of C under s in X. Then $P = E \cap S$.
Conversely, let P be a $k(C)$-rational point on the generic fibre E. A priori, P is only defined on the smooth fibres, but we can consider the closure S of P in X (so that $S \cap E = P$). Restricting the fibration to S, we obtain a birational morphism of S onto the non-singular curve C.
By Zariski's main theorem ([Har77, V 5.2]), (which summarises as "Along birational morphisms, there is only one branch at normal points",) $f|_S$ is an isomorphism. The inverse gives the unique section associated to the $k(C)$-rational point P.

Constructing the surface: Suppose we are given an elliptic curve E over the function field $k(C)$ of a curve C. The Kodaira-Néron model describes how to associate an elliptic surface $S \to C$ over k to E whose generic fibre returns exactly E.

At first, we omit the singular fibres. Remove all points from C where the discriminant of E over $k(C)$ vanishes. Denote the resulted open subset of C by C^0. Above every point of C^0, the fibre is therefore a smooth elliptic curve over k from E. This gives a quasi-projective surface X^0 with smooth elliptic fibration

$$f^0 : X^0 \to C^0.$$

It is best to simply think of the Weierstrass equation of E being restricted to C^0 (after adding the usual point at ∞ to every smooth fibre). All that remains is to fill

in suitable singular fibres at the points omitted from C.
For instance, if the Weierstrass form defines a smooth surface everywhere, then all fibres are irreducible. The singular fibres are either nodal or cuspidal rational curves. If the Weierstrass form is not smooth somewhere, then we resolve these singularities minimally, and the resulting surface X is called the *Kodaira-Néron* model for E. The desingularisation process is called *Tate's algorithm*, and this can be found in detail in [SS10, Section 4.2]. Uniqueness of the Kodaira-Néron model should be clear: Assume there are two desingularisations X and $\widetilde{X}$ which are both smooth elliptic surfaces over C. In particular, neither of them contain (-1)-curves in the fibres.

$$\begin{tikzcd} X \arrow[r, squiggly, "\text{bir}"'] \arrow[dr] & X^0 \arrow[r, squiggly, "\text{bir}"'] & \widetilde{X} \arrow[dl] \\ & C & \end{tikzcd}$$

and hence there is a birational morphism $X \to \widetilde{X}$. The surface classification [Har77, Section V 5.5] relies on the fact that every birational morphism is a succession of smooth blow-ups and blow-downs. By construction these two desingularisations X and $\widetilde{X}$ are isomorphic outside the singular fibres, and neither do the fibres contain any $(-1))$ curves. Hence $X \cong \widetilde{X}$ and the Kodaira-Néron model is unique.
From this construction we can identify the usual Mordell-Weil group Φ of points on the elliptic curve E over $k(C)$ with the group of sections. This is looked at later in sections 4.6 and 4.7.
All we need is the following result which is a special case of the Mordell-Weil theorem for abelian varieties over suitable global fields (see [Ser89, Section 4]):

Theorem 2.3.8. [SS10, Theorem 6.1] Let $X \to C$ be an elliptic surface with a section. The associated elliptic curve E over $k(C)$ is a finitely generated group.

2.3.2 Classification of Singular Fibres

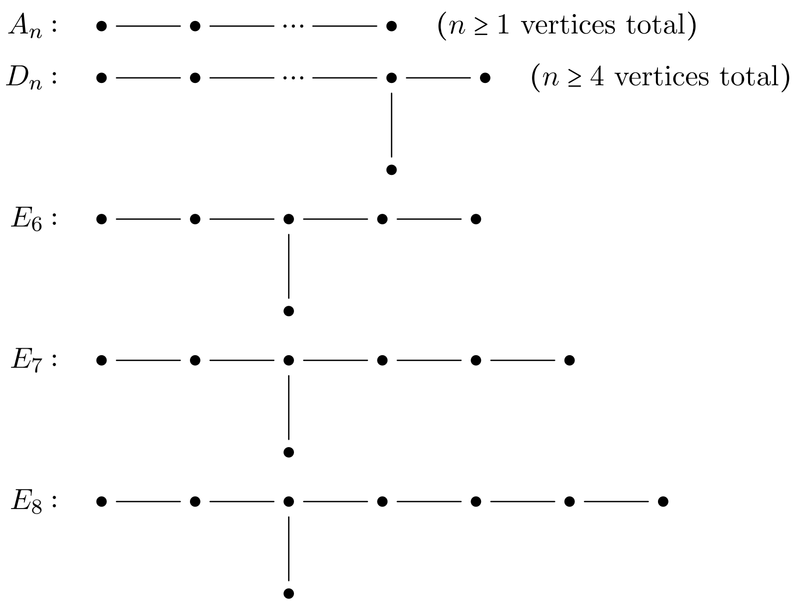
The possible singular fibres of a smooth elliptic surface were classified by Kodaira and we shall briefly summarise the ideas here, following [Mir89]. The intersection form on X (as discussed in previous sections above) induces a symmetric bilinear form on the $\mathbb{Q}$-vector space V whose basis is the set of irreducible components of singular fibres.
It turns out [Mir89, Lemma I.6.1] that by using the Hodge Index Theorem this form is negative semi-definite on V and its kernel is dimension 1, spanned by the class of a smooth fibre, X_0.
Now, for any graph G, (possibly containing loops or multiple edges) one can form

the $\mathbb{Q}$-vector space V_G with basis the vertices of G, and define a symmetric bilinear form on V_G by declaring that on the vertices:

$$v \cdot w := \begin{cases} -2 + 2(\#\{\text{loops at } v\}) & \text{if } v = w \\ \text{the number of edges joining } v \text{ and } w & \text{if } v \neq w \end{cases}$$

The space V_G, together with this form is called the *associated lattice* to the graph G.

Consider the following list of graphs, called the *Dynkin diagrams*:

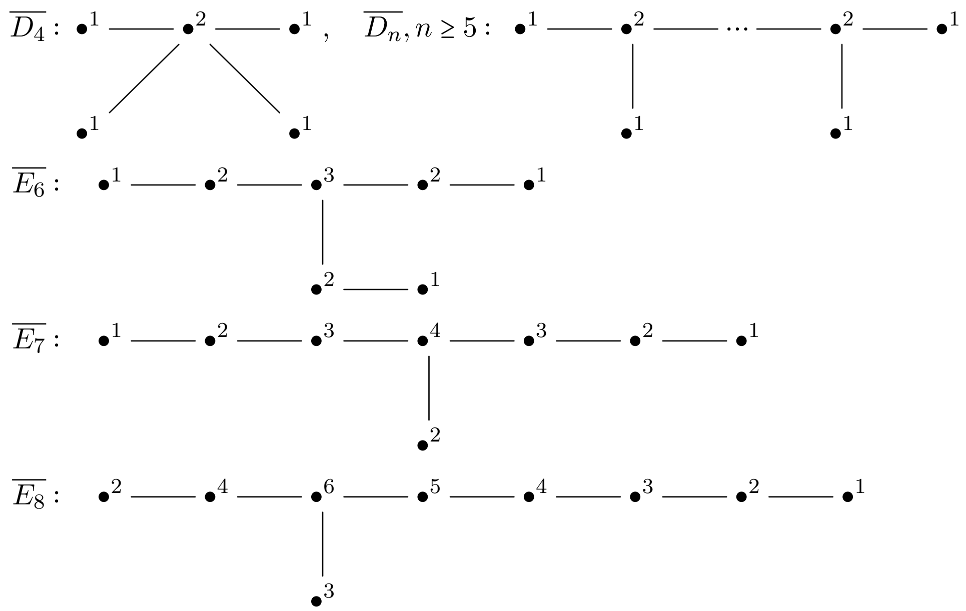


Note in particular that the associated lattice V_{E_8} to the Dynkin Diagram E_8 above is precisely the lattice "$E_8(-1)$" in the notation of Theorem 2.2.14 in the previous section.

Indeed, all of the Dynkin diagrams above have negative definite associated lattices.

We also should consider the *Extended Dynkin Diagrams with multiplicities*:

$\overline{A_0}: \bullet^1$, $\overline{A_1}: \bullet^1 = \bullet^1$, $\overline{A_n}, n \geq 2:$ a cycle of $n+1$ vertices, all labelled 1.



For the extended Dynkin Diagrams G, note that the element of V_G consisting of multiplying each vertex by its label, has square 0 in the lattice.
Thus each extended Dynkin diagram contains an ordinary Dynkin diagram and an additional vertex. We see therefore that the associated lattice to each extended Dynkin diagram is negative semi-definite, with a dimension 1 kernel, spanned by this vector labelled above.
The following lemma is a curious fact:

Lemma 2.3.9. [Mir89, Lemma I.6.3]

1. Every connected graph either is contained in or contains an extended Dynkin diagram.

2. Every connected graph without loops or multiple edges either is contained in or contains an extended Dynkin diagram without loops or multiple edges.

This implies that the extended Dynkin diagrams are the only graphs whose associated lattice is negative semi-definite with dimension 1 kernel.
From this, the classification of singular fibres is straightforward; each singular fibre is a collection of rational curves whose intersection graph is an extended Dynkin diagram. The complete list is below, for the proof of completeness see [Mir89, Theorem I.6.6]

Name	Description
I_0	smooth elliptic curve
I_1	nodal rational curve
I_2	two smooth rational curves meeting transversally at two points
$I_n, n \geq 3$	n smooth rational curves meeting in a cycle; with dual graph $\overline{A_n}$
$I_n^*, n \geq 0$	$n+5$ smooth rational curves meeting with dual graph $\overline{D_{n+4}}$
II	a cuspidal rational curve
III	two smooth rational curves meeting at one point to order 2
IV	three smooth rational curves all meeting at one point
IV^*	7 smooth rational curves meeting with dual graph $\overline{E_6}$
III^*	8 smooth rational curves meeting with dual graph $\overline{E_7}$
II^*	9 smooth rational curves meeting with dual graph $\overline{E_8}$
${}_mI_n, n \geq 0$	topologically an I_n fibre, but each curve has multiplicity m.

Table 2.1: Kodaira's Classification of Singular Fibres

Remark 2.3.10. In fact, for the applications we have in mind, things are a lot simpler than this. We will primarily be looking at surfaces with only *semi-stable* fibres; these are simply of the form I_n for $n \geq 0$. The others will not be needed so much, I only state them here for the sake of completeness.

Chapter 3

Lines on a Quartic Surface

In this chapter, we summarise the methods of the paper [RS15b].
Consider a smooth degree 4 hypersurface $S \subset \mathbb{P}^3$. The aim is to prove

Theorem. Let k be any field of characteristic not 2 or 3. Then any smooth quartic surface $S \subset \mathbb{P}^3_k$ contains no more than 64 lines.

3.1 Elliptic Fibrations

For this section, we will assume that the field k has characteristic not 2 or 3.
Let S be a smooth quartic in $\mathbb{P}^3 = \mathbb{P}^3_k$ and suppose that there exists a line l on S. Take a pencil of planes in $\mathbb{P}^3$ that contain this line. After a change of coordinates, we can assume that the line l is given by $\{x_3 = x_2 = 0\}$ and so any plane passing through l is described uniquely by an equation of the form $\{\tau x_0 = \upsilon x_1\}$ for $(\tau : \upsilon) \in \mathbb{P}^1$. We call this element in $\mathbb{P}^1$ the *gradient* of the plane. Note that the gradient of the plane depends on the choice of coordinates.
For $t \in \mathbb{P}^1$ denote by F_t the intersection of $S \smallsetminus l$ with the plane of gradient t.
From the pencil of planes containing l we get a fibration:

$$\pi : S \to \mathbb{P}^1$$

mapping each point $x \in S \smallsetminus l$ to the element $t \in \mathbb{P}^1$ such that $x \in F_t$. In particular, F_t is the fibre of t and for almost all choices of $t \in \mathbb{P}^1$, F_t consists of a smooth cubic plane curve.
By construction, any other line l_1 in S meeting l will be contained in a plane meeting l and so l_1 will be a reduced component of F_t for some t. In this case, F_t will be a singular cubic plane curve.

The possible types of singularities that can occur have been classified by Kodaira. Using his notation we list them in the table below:

Fibre	**Description**	e(**Fibre**)
I_1	Cubic curve with a node	1
I_2	A conic and a line meeting transversally in two points	2
I_3	Three lines meeting pairwise in three different points	3
II	Cubic curve with a cusp	2
III	A conic and line meeting tangentially at one point	3
IV	Three lines all meeting transversally at one point	4

Table 3.1: Classification of singularities on plane cubics and their Euler number

Note that for a general smooth plane curve, there are exactly nine points of inflection (intersection of determinant of 3×3 Hessian and degree 3 original equation). We now make the following definitions:

Definition 3.1.1. A line l on S is said to be of the *second kind* if and only if it intersects every smooth fibre of the fibration $\pi : S \to \mathbb{P}^1$ in a point of inflection. Otherwise, we say l is a line of the *first kind.*

Definition 3.1.2. Let $\mathscr{F} \subset \mathbb{P}^3$ be the closure of the set of inflection points on all the smooth cubics. We call $\mathscr{F}$ the *flex locus* of the fibration $\pi : S \to \mathbb{P}^1$.

If the original line l was a line of the second kind, then l meets each singular fibre at points lying in $\mathscr{F}$.
Using the group structure present on each smooth fibre, Néron's minimal model for Abelian Varieties [Nér64] tells us where $\mathscr{F}$ can intersect each singular fibre. Using the fact that each inflection point on a fibre is precisely a point of order 3, the table below tells us the intersection of $\mathscr{F}$ with each singular fibre in Table 3.1:

Fibre type	**Fibre** $\cap \mathscr{F}$
I_1	Three Smooth Points and the node
I_2	Three smooth Points (all on one component) and both nodes
I_3	Three Smooth Points on each Component
II	One Smooth Point and the cusp
III	One Smooth Point and the node
IV	1 Smooth Point on each Component and the node.

Table 3.2: Intersection of Singular Fibres with the Flex Locus

For example, in the case of smooth curves degenerating into an I_1 fibre, we have 6 inflection points meeting at the node.

In the I_2 case, we have 3 inflection points meeting at each one of the nodes.
In this way we can see that we can attach an elliptic fibration to the quartic hypersurface in the natural way, and that the most special type of line, a line of the second kind, is forced to intersect each singular fibre in a very limited way. The lines of the second kind are the most difficult to deal with, as the number of lines meeting a line of the first kind is very easy to compute algebraically.

3.2 Lines of the First Kind

In this section, we will bound the number of lines that can meet a line of the first kind. First we have the following algebraic lemma:

Lemma 3.2.1. [Seg43] Let l be a line of the first kind in a non-singular quartic surface $S \subset \mathbb{P}^3$. Then at most 18 points of l can be points of inflection for some fibre F_t of $\pi : S \to \mathbb{P}^1$.

Proof. We suppose l is the line $x_2 = x_3 = 0$. Then S is given by an equation of the form

$$\begin{aligned} f: \quad & x_2\alpha_0(x_0,x_1) + x_3\alpha_1(x_0,x_1) \\ & + x_2^2\beta_0(x_0,x_1) + x_2x_3\beta_1(x_0,x_1) + x_3^2\beta_2(x_0,x_1) \\ & + x_2^3\gamma_0 + x_2^2x_3\gamma_1 + x_2x_3^2\gamma_2 + x_3^3\gamma_3 \end{aligned}$$

where $\deg \alpha_i = 3$, $\deg \beta_i = 2$ and $\deg \gamma_i = 1$ in x_0, x_1. The fibre F_λ is therefore given by the intersection of the plane $x_3 = \lambda x_2$ and f. This decomposes into the line l itself and the cubic

$$C_\lambda : \alpha_0(x_0,x_1) + \lambda\alpha_1(x_0,x_1) + x_2(\beta_0 + \lambda\beta_1 + \lambda^2\beta_2) + x_2^2(\widetilde{\gamma}_0 + \widetilde{\gamma}_1 + \widetilde{\gamma}_2 + \widetilde{\gamma}_3) \tag{3.1}$$

and so $C_\lambda \cap l$ is given by $x_2 = x_3 = 0$ and $\alpha_0(x_0,x_1) + \lambda\alpha_1(x_0,x_1) = 0$.
A point in this set is an inflection point of C_λ if and only if the determinant of the Hessian of 3.1 vanishes. I.e. taking second derivatives with respect to x_0, x_1, x_2 then setting $x_2 = 0$ amounts to:

$$\det \begin{pmatrix} a_1 & b_1 & d_2 \\ b_1 & c_1 & e_2 \\ d_2 & e_2 & f_3 \end{pmatrix} = 0 \tag{3.2}$$

where $a_1, b_1, c_1, d_2, e_2, f_3$ are polynomials in λ of degree represented by their subscript. The coefficients of these polynomials are all degree 1 in x_0, x_1. The determinant is therefore a degree 5 polynomial in λ whose coefficients are all homogeneous

cubics in x_0, x_1.
Multiplying by α_1^5 and then substituting in $\alpha_0(x_0,x_1) + \lambda\alpha_1(x_0,x_1) = 0$ gives a homogeneous polynomial of degree 18 in x_0, x_1, whose solutions are the points of l for which meet C_λ in a point of inflection. Note that if equation 3.2 identically vanishes, then this implies that every point of l meets every fibre at a point of inflection and so l was not a line of the first kind. $\square$

Now, note that any reducible plane cubic containing a line is (after a change of coordinates) of the form $xq_2(x,y,z) = 0$ for some quadratic form q_2. Then the Hessian is

$$\begin{pmatrix} 2\frac{\partial q_2}{\partial x} + \frac{\partial^2 q_2}{\partial x^2}x & \frac{\partial q_2}{\partial y} + \frac{\partial^2 q_2}{\partial x \partial y}x & \frac{\partial q_2}{\partial z} + \frac{\partial^2 q_2}{\partial x \partial z}x \\ \frac{\partial q_2}{\partial y} + \frac{\partial^2 q_2}{\partial x \partial y}x & \frac{\partial^2 q_2}{\partial y^2}x & \frac{\partial^2 q_2}{\partial y \partial z}x \\ \frac{\partial q_2}{\partial z} + \frac{\partial^2 q_2}{\partial x \partial z}x & \frac{\partial^2 q_2}{\partial y \partial z}x & \frac{\partial^2 q_2}{\partial z^2}x \end{pmatrix}$$

and so the determinant vanishes at $x = 0$ (to order 1); in particular the entire line is contained in the vanishing of the Hessian determinant. Combining this with the previous lemma gives us:

Corollary 3.2.2. [Seg43] Any line l of the first kind on a non-singular quartic surface $S \subset \mathbb{P}^3$ can be met by no more than 18 lines of S.

3.3 Base Change and Ramification

We have seen that any line l on the quartic surface S yields an elliptic fibration. However, such a line also gives a natural morphism

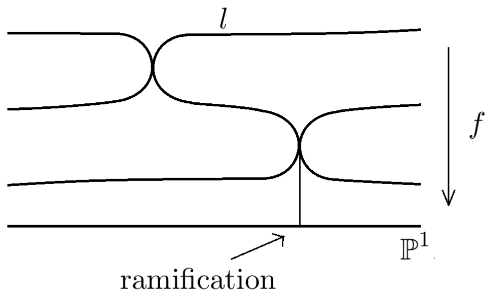
$$f : l \to \mathbb{P}^1.$$

Take $x \in l \subset S$. Since S is a smooth surface, x has a well-defined tangent plane in $\mathbb{P}^3$. This tangent plane certainly contains l, and so is one of the planes in the elliptic fibration $\pi : S \to \mathbb{P}^1$. In this plane, the planar cubic curve F_t intersects l at x and so we set $f(x) := t$ (the "slope" of the plane). Note that since F_t will (in general) intersect l in three points, the map $f : l \to \mathbb{P}^1$ is a degree 3 morphism (F_t intersects l in three points; all these points therefore have the same tangent plane).
Note that by varying the slope t of the plane smoothly, the points $F_t \cap l$ vary, describing curves which may ramify at points when the curve F_t happens to become singular. Hence any attempt at a section might get permuted at ramification points, so these do not trace out well-defined sections $\mathbb{P}^1 \to S$. We seek to remedy this situation:

Given $f : l \to \mathbb{P}^1$, the corresponding map of function fields therefore corresponds to a degree 3 field extension $k(l)/k(t)$. Therefore we may write the Galois closure of this field extension in the form $K = k(t, \alpha_1, \alpha_2, \alpha_3)$ where $\alpha_1, \alpha_2, \alpha_3$ are the roots of the defining cubic polynomial $\phi \in k(t)[s]$. Hence $K/k(t)$ is a field extension of order 3 or 6 depending whether $\sqrt{\operatorname{disc}(\phi)} \in k(l)$. Let B denote the curve with function field K.
We now perform a base change of the fibration $\pi : S \to \mathbb{P}^1$ as follows:

$$\begin{tikzcd} S_2 \arrow[r] \arrow[d] & S_1 \arrow[r] \arrow[d, "\pi_1"] & S \arrow[d, "\pi"] \\ B \arrow[r] & l \arrow[r, "f"] & \mathbb{P}^1 \end{tikzcd}$$

where $S_1 = l \times_{\mathbb{P}^1} S$ and $S_2 = S_1 \times_l B$ are formed by taking the fibred product.
Above each point of $x \in l$ is a fibre of $\pi : S \to \mathbb{P}^1$: $\pi_1^{-1}(x) = \{(x, y) \in l \times S | \pi(y) = f(x)\}$.



By construction the fibration $S_1 \to l$ is now a proper elliptic fibration; each fibre is a cubic curve with a marked point (i.e. an elliptic curve), and this choice of marked point remains consistent as we smoothly range over the fibres, tracing out a well-defined section O.
For the fibration $S_2 \to B$, we now have three well-defined sections, $O, P_1, P_2 : \mathbb{P}^1$.
Suppose now that l is a line of the second kind. Therefore l meets each smooth fibre F_t at a point of inflection. In particular, the sections P_1 and P_2 consist of 3-torsion points which are inverse to each other (O, P_1, P_2 all lie on the l for each t). Looking at the fibration $S_2 \to B$, these sections limit the number of cases for the singular fibres of $S_2 \to B$. In particular, the 3-torsion sections meet any fibre of $S_2 \to B$ (smooth or non-smooth) in three distinct smooth points by construction. From Table 3.2 the only possibilities for singular fibres are I_1, I_2, I_3 or IV.
We can now distinguish between these fibre types according to how the morphism $f : l \to \mathbb{P}^1$ ramifies.

Lemma 3.3.1. Let l be a line of the second kind, and F a singular fibre of $\pi : S \to l$ such that the map $f : l \to \mathbb{P}^1$ is unramified at F. Then F has type I_1, I_3 or IV.

Proof. Since the fibre is unramified, l meets F is 3 distinct points, all necessarily smooth. On S_2 the fibre F is replaced by $n = 3$ or 6 fibres of the same type. These fibres all accommodate non-trivial 3-torsion sections. From the table, I_1, I_2, I_3 or IV are possibilities.
However, I_2 (consisting of a line and a conic) is not possible, since the 3 torsion sections would all meet the same fibre component, while the line l inducing them meets both fibre components (the curves l, the residual line and the conic all lie in the same plane). In particular, l does not intersect a single component in three smooth points, contradicting the Table 3.2. □

We now deal with the ramified fibres (which are automatically singular). At such a fibre, the morphism $f : l \to \mathbb{P}^1$ has either 2 preimages (case 1) or it has a single preimage (case 2).

Lemma 3.3.2. Let l be a line of the second kind, and F a singular fibre of $\pi : S \to \mathbb{P}^1$ such that the map $f : l \to \mathbb{P}^1$ ramifies at F. Then F has type I_1, I_2, II or IV according to the ramification type as follows:

Fibre type	II	I_1, I_2, IV
Ramification type	case 1	case 2

Table 3.3: Fibre type according to Ramification type

Proof. First of all, F cannot be of type I_3 since the every intersection with $\mathscr{F}$ is a smooth point, so the fibre admits 3-torsion, so that $l \to \mathbb{P}^1$ cannot ramify.
By a similar argument, if $f : l \to \mathbb{P}^1$ ramifies at a fibre of type IV then l must meet the node of the fibre (since l is a line of the second kind) and hence the ramification type is case 2.
For other fibre types, we argue with the base change $S_2 \to B$. Here, the fibre F is replaced by fibre which admit 3-torsion (see Table 3.2). Tate's Algorithm [Tat75] is used to describe the behaviour of singular fibres under a cyclic base change of degree d. This normally consists of repeated blow-ups, but outside of characteristic 2 and 3, there is a much quicker procedure:
We may assume that the singular fibre in the elliptic fibration is given locally by a short Weierstrass equation

$$y^2 = x^3 + a_4 x + a_6$$

Where a_4 and a_6 are polynomials depending on a local parameter t (where the singular fibre occurs at $t = 0$). Let $\Delta = -27a_6^3 - 4a_4^2$ be the discriminant of this equation. Then denote by $n := v(\Delta)$ the order of vanishing of Δ at $t = 0$. Similarly,

$v(a_4)$ and $v(a_6)$ are the order of vanishing of a_4 and a_6 (at $t = 0$). Then the fibre type is completely determined by table 3.4 below.

If the form is not in one of the fibre types indicated in the table, then $v(a_4) \geq 4$

Fibre type	$v(a_4)$	$v(a_6)$
I_0	$\begin{cases} 0 \\ \geq 0 \end{cases}$	$\begin{cases} \geq 0 \\ 0 \end{cases}$
$I_n (n > 0)$	0	0
II	≥ 1	1
III	1	≥ 2
IV	≥ 2	2
I_0^*	$\begin{cases} 2 \\ \geq 2 \end{cases}$	$\begin{cases} \geq 3 \\ 3 \end{cases}$
$I_{n-6}^* (n > 6)$	2	3
IV^*	≥ 3	4
III^*	3	≥ 5
II^*	≥ 4	5

Table 3.4: Fibre Type According to Orders of Vanishing in local Weierstrass form: [SS10]

and $v(a_6) \geq 6$ and so we can apply the change of variables

$$(x', y') := (t^2 x, t^3 y)$$

to obtain an isomorphic equation. This transformation drops $v(\Delta)$ by 12 and decreases $v(a_4)$ and $v(a_6)$. Since $v(\Delta)$ must remain positive, we only need to apply the transformation a finite number of times.

We now perform a cyclic base change of degree d (that is, in the local Weierstrass form of the singular fibres, replace $a_4(t)$ and $a_6(t)$ with $a_4(t^d)$ and $a_6(t^d)$ respectively); the results of Tate's Algorithm on this process are contained in the following table 3.5:

Fibre type	$d = 2$	$d = 3$
I_n	I_{2n}	I_{3n}
II	IV	I_0^*
III	I_0^*	III^*
IV	IV^*	I_0

Table 3.5: Singular fibres after cyclic base Change: [RS15b]

In particular, the behaviour illustrated in the table rules out the fibre type

$F = III$ since it cannot be base changed to fibres I_0^* or III^* as they are not possible fibres in Table 3.1.
When F is of type II, we see that the only possibility is to replace it by three fibres of type IV in S_2. On $l \to \mathbb{P}^1$ this corresponds to the non-Galois case of ramification type 1 (l meets a smooth point of a fibre transversally and a node). (See Table 3.2).
We also see that when F is of type IV, on S_2 F can only be replaced by a smooth fibre. Then l meets it at at one point with multiplicity 3 (since l is a line of the second kind). Hence the fibre is of ramification type 2.
We now analyse fibres of type I_1 and I_2. Assume that the ramification type is 1; that is, l meets a smooth point of the fibre transversally and a node. In S_2 this fibre would be replaced by fibres of type I_2 or I_4. By assumption the section O would meet one fibre component on S_2 (the original smooth intersection point) while the other sections P_1 and P_2 meet a different fibre component (corresponding to the node). However, the structure of the fibre components is $\mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Z}/4\mathbb{Z}$ which do not admit proper 3-torsion; therefore all 3-torsion sections must have the same component, which is a contradiction. □

3.4 Consequences for lines of the second kind

Assume $l \subset S$ is a line of the second kind. Recall the degree 3 map $f : l \to \mathbb{P}^1$. By the Hurwitz Theorem, we have

$$\sum_{p \in l} (e_p - 1) = 4$$

where e_p is the ramification index of p. The ramification type of f has three possibilities:

1. 4 points in $\mathbb{P}^1$ each have 2 preimages, and so correspond to ramification type 1 (one of the preimages has $e_p = 2$, the other $e_q = 1$).
2. 1 point in $\mathbb{P}^1$ has 1 preimage, corresponding to ramification type 2 (the preimage has $e_p = 3$), 2 others have ramification type 1
3. 2 points in $\mathbb{P}^1$ each have 1 preimage, and so both correspond to ramification type 2.

We refer to these cases as $1^4, 2 \cdot 1^2$ and 2^2 respectively. We now have the following proposition:

Proposition 3.4.1. Let $l \subset S$ be a line of the second kind on a smooth quartic S with ramification type R. Then l meets exactly $12, 15, 16, 18, 19$ or 20 lines, depending on the ramification type as follows:

R	**Number of lines meeting l**
1^4	12
$2 \cdot 1^2$	15 or 16
2^2	18,19 or 20

Table 3.6: Number of lines meeting a line of the second kind: [RS15b]

Proof. To prove the proposition, we use the 3-torsion sections on S_2. These induce an isogeny on the generic fibres and the resulting surface is denoted by S_2'. S_2 and S_2' have the same topological invariants (Euler-Poincaré) characteristic, Betti numbers, geometric genus, Picard number). This puts severe restrictions on the singular fibres of S_2 and S_2'. Recalling the possible singular fibres of S from Lemmas 3.3.1 and 3.3.2. The corresponding fibres on S_2 and S_2' are given in the following table. Here, $n = 3$ or 6 is the degree of the splitting field of $k(l)/k(t)$. In particular, since S_2 and

	Fibre on S	Fibre on S_2	Fibre on S_2'
unramified	I_1	$n \times I_1$	$n \times I_3$
	I_3	$n \times I_3$	$n \times I_1$
	IV	$n \times IV$	$n \times IV$
ramified	II	$3 \times IV$	$3 \times IV$
	I_1	I_3	I_9
	I_2	I_6	I_{18}
	IV	I_0	I_0

Table 3.7: Singular Fibres after Base Change: [RS15b]

S_2' have the same Euler characteristic, and it is the sum of the local contributions of singular fibres, we deduce that the unramified fibres of type I_3 must balance out the other fibres of type I_1 since by Table 3.7 we see that on unramified fibres they are interchanged after the base change from S_2 to S_2'. That is, in the unramified case, each I_1 fibre is balanced out by a I_3 fibre. Moreover, for every I_2 fibre, there are two I_3 fibres.

Knowing this, we will use Theorem 6.10 of [SS10] which says that if $\mathrm{char}(k) \neq 2$ or 3 then

$$24 = e(S) = \sum_{t \in \mathbb{P}^1} e(F_t)$$

Case 1: $R = 1^4$ By Lemma 3.3.2 S has 4 ramified fibres of type II these contribute

8 to the Euler characteristic $e(S) = 24$. Therefore there are 16 local contributions among the unramified singular fibres. This consists of a total of 4 pairs (I_1, I_3) and IV. Any possibility contributes 3 lines each, so l meets a total of 12 lines.
Case 2: $R = 2 \cdot 1^2$ In this case there are two ramified fibres of type II. These contribute 4 to the Euler characteristic, so there are 20 remaining contributions over a total number of five pairs (I_1, I_3) and IV, or possible a triple (I_2, I_3, I_3) in which case there are three leftover pairs and type IV fibres. In any possibility, there are five fibres consisting of 3 lines, and if there is a triple (I_2, I_3, I_3) we get one more line from the I_2 fibre. This results in either 15 or 16 lines meeting l.
Case 3: $R = 2^2$ In this case there can be no fibres of type II. In order to reach $e(S) = 24$, the local contributions from the singular fibres are distributed over a total number of 6 pairs of (I_1, I_3) and fibres of type IV, with the possibility of one or two triples (I_2, I_3, I_3) (each counting for two). Note that the presence of each triple accounts for 8 towards the Euler characteristic.
In all cases, the fibration has 6 fibres consisting of 6 lines each, and each triple (I_2, I_3, I_3) gives an extra line from the I_2 fibre; therefore there are $18, 19$ or 20 lines meeting l. $\square$

3.5 64 lines

Lemma 3.5.1. If l on S is met by more than 12 lines, then it is met by 3 coplanar lines.

Proof. If l is met by 3 coplanar lines, then the corresponding elliptic fibration must have singular fibres of type I_3 or type IV. Therefore assume instead that no such fibres appear.
If the number of singular fibres of type $I_1, I_2, I_3, II, III, IV$ is given by a_1,a_2,a_3,b,c,d respectively, then from Table 3.3, and [SS10] Theorem 6.10 we have:

$$e(S) = 24 = a_1 + 2a_2 + 2b + 3c$$

and the number of lines meeting l is therefore $a_2 + c$. Maximising $a_2 + c$ we therefore have that $a_1 = 0, b = 0$ so $24 = 2(a_2 + c) + c$. Since $a_2 + c$ is maximal, we may assume c is minimal (i.e. $c = 0$) and so $a_2 = 12$; that is l meets 12 lines, all coming from type I_2 singularities. $\square$

We need one more definition:

Definition 3.5.2. Let $S \subset \mathbb{P}^3$ be a nonsingular quartic surface. The *flecnodal divisor*

$\mathscr{F}$ is the subset of points $p \in S$ such that the order of vanishing of the tangent plane T_pS is order 4.

We collect some facts about the flecnodal divisor:

Proposition 3.5.3. [Seg43]

1. $\mathscr{F}$ is obtained as a complete intersection of S with a surface of order 20.
2. Any line l on S is a component of $\mathscr{F}$.
3. If l is a line on S, contained in $\mathscr{F}$ with multiplicity d then $l \cdot (K - dl) = l \cdot K - dl^2 = 20 + 2d$.

We are now ready to prove

Theorem 3.5.4. If there are no lines of the second kind, then there are no more than 64 lines on a nonsingular quartic surface $S \subset \mathbb{P}^3$.

Proof. We prove the theorem in a number of cases.
Case 1 First, assume that S contains 4 lines A, B, C, D in plane π. Note that these four lines all meet each other in π. In addition, any other line $l \subset S \subset \mathbb{P}^3$ meets π and thus meets (at least) one of the lines A, B, C, D. By Corollary 3.2.2, A, B, C and D can only meet 18 lines in total; 3 of these are already accounted for so each one of A, B, C and D can (at most) meet 15 other lines. This gives a total of $4 + 4 \times 15 = 64$ lines.
Otherwise, no 4 lines are in a plane. Hence any plane π spanned by two lines $C \subset S$ and $D \subset S$ with $C \cap D \neq \varnothing$ meets S again in a residual nonsingular conic Ω.
Case 2 We assume that $\pi \cap S = C + D + \Omega$ and Ω is not a component of the flecnodal divisor $\mathscr{F}$. In that case, Ω meets $\mathscr{F}$ in $2 \times 20 = 40$ points. Four of these points are where Ω meets the lines C and D and so $\mathscr{F}$ meets Ω in 36 other points. Any other line of S would meet π and so meets Ω in one of these points. (If multiple lines go through a point, this point is counted with multiplicity in the $36 = \Omega \cdot (K - C - D)$). Moreover, by Lemma 3.5.1 we have assumed that C and D meet at most 11 lines on S each, so the total number of lines on S cannot exceed $2 + 11 + 11 + 36 = 60$ lines.
Otherwise, any plane spanned by any pair of intersecting lines meets a conic Ω which is also a component of the flecnodal divisor $\mathscr{F}$.
Case 3 We assume that S contains $n \geq 8$ pairs of intersecting lines. By assumption, $\mathscr{F}$ therefore contains at least $n \geq 8$ irreducible conics, in addition to the lines of S. Since $\mathscr{F}$ has degree 80, the number of lines cannot exceed $80 - 2 \times 8 = 64$ lines.
Case 4 The final case is when S contains only $n < 8$ pairs of incident lines. Then

any other line meets no lines at all. In terms of the intersection form, classes of pairs of incident lines span a sublattice of $H^2(S,\mathbb{Z})$ of rank at least $n+1$. $\mathrm{Pic}(S)$ cannot exceed 22, so the m classes of the disjoint lines span a sublattice of rank $m \leq 22-(n+1)$, and the number of lines on S cannot exceed $2n+(22-(n+1)) = 21+n \leq 29$ lines. □

We now turn to the case where S contains a line l of the second kind. Then we have the following lemma:

Lemma 3.5.5. [RS15b] If $l \subset S$ is a line of the second kind, then a line l_0 in a fibre of $\pi : S \to \mathbb{P}^1$ is of the second kind if and only if S is the Schur quartic and the fibre is ramified (of Kodaira type IV at 0 or ∞).

That is, unless S is the Schur quartic, every line of the second kind meeting more than 16 lines is disjoint to all other lines of the second kind.
With this, we can easily prove the following corollary:

Corollary 3.5.6. Any quartic surface S contains at most 66 lines.

Proof. Since the Schur quartic contains exactly 64 lines, we deduce from Lemma 3.5.5 that any line contained in a fibre of π is of the first kind. Take a fibre of type I_3 or IV contained in a plane H. (Proposition 3.4.1 gurantees that there are either 4,5 or 6 such fibres to choose from depending whether the degree 3 map $f : l \to \mathbb{P}^1$ ramifies as 1^4, $2\cdot 1^2$ or 2^2 respectively.) Each fibre component meets at most 18 other lines on S by Corollary 3.2.2. Thus (arguing similar to Case 1 of the proof of Theorem 3.5.4) we see that any other line in S meets π and so meets on of the components of the fibre, or l. Since all the 4 lines in the plane H meet each other, there can be at most

$$4 + 3 \times \underbrace{(18-3)}_{\text{first kind}} + \underbrace{(20-3)}_{\text{second kind}} = 66$$

lines on S. □

In fact, having more than 64 lines implies the existence of I_2 fibres and by means of a second elliptic fibration along the lines in those fibres, we can improve the bound of Corollary 3.5.6. We shall not discuss this in detail here, since it is not necessary for the results we hope to attain for the case of three quadrics.

Chapter 4

Lines on a Complete Intersection of three Quadrics

In this chapter, we will prove the main result of this document, using the method explained in Chapter 3 as inspiration. Throughout this chapter, we will work in a field k which is not of characteristic 2 or 3.

Remark 4.0.7. We shall be looking at upper bounds for number of lines on the surface X. For this reason, it is reasonable to assume that k is algebraically closed (see Remark 1.2.2): we can only increase the number of lines if we include those that are not only defined on k but also on all extensions.

4.1 Existence of Quadrics of Rank 4

For our field k, denote by $x_0, x_1, \cdots, x_5$ a choice of homogeneous coordinates on $\mathbb{P}^5$.

Definition 4.1.1. Let $f = \sum_{i,j=0}^{5} a_{ij} x_i x_j$ be a homogeneous degree two polynomial in the x_i. Let $A \in M_{6\times 6}(k)$ be the matrix:

$$A := \begin{pmatrix} a_{00} & a_{01} & a_{02} & a_{03} & a_{04} & a_{05} \\ a_{10} & a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{20} & a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{30} & a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{40} & a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{50} & a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{pmatrix}$$

We then define the *rank* of f to be the rank of the symmetric matrix $\frac{1}{2}(A + A^T)$. For a quadric hypersurface $Q \subset \mathbb{P}^n$ we define $\mathrm{rank}(Q)$ to be the rank of its defining polynomial.

We note that there are many such matrices A that a single f could yield, but $\frac{1}{2}(A + A^T)$ is the unique symmetric matrix that satisfies the matrix equation

$$f = x^T \frac{1}{2}(A + A^T)x, \qquad \text{where } x = \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}$$

Since allowing changes of coordinates greatly allows us to simplify calculations, the following lemma is necessary:

Lemma 4.1.2. The rank of a quadric hypersurface Q is unchanged under linear changes of the coordinates $x_0, \cdots, x_5$.

Proof. Let f be the defining equation of Q in $x_0, \cdots, x_5$. If $y_0, \cdots, y_5$ is a linear change of coordinates (so that the degree of f is unchanged) then there exists an invertible matrix B such that $By = x$. Then for some symmetric matrix A, we have $f = x^T Ax = (By)^T A(By) = y^T B^T ABy$ and so the rank of f in the y coordinates is the rank of the matrix

$$\frac{1}{2}\left(\left(B^T AB\right) + \left(B^T AB\right)^T\right) = B^T \frac{1}{2}\left(A + A^T\right) B = B^T AB$$

By Lemma 2.2.10 applied to the symmetric matrices A and $B^T AB$, there exist orthogonal matrices P_1, P_2 and diagonal matrices D_1, D_2 such that $P_1^T AP_1 = D_1$ and $(BP_2)^T ABP_2 = P_2^T B^T ABP_2 = D_2$. In particular, $\mathrm{rank}(A) = \mathrm{rank}(D_1)$ and $\mathrm{rank}(B^T AB) = \mathrm{rank}(D_2)$.
It follows that $(P_1^T BP_2)^T D_1 (P_1^T BP_2) = D_2$ and hence D_1 and D_2 are two diagonal matrices with the same eigenvalues. In particular, their ranks are equal, and so

$$\mathrm{rank}(f(x_i)) = \mathrm{rank}(A) = \mathrm{rank}(D_1) = \mathrm{rank}(D_2) = \mathrm{rank}(B^T AB) = \mathrm{rank}(f(y_i))$$

hence $\mathrm{rank}(f)$ (and hence $\mathrm{rank}(Q)$) is independent of the choice of coordinates. $\square$

Now let Q_1, Q_2 and Q_3 be three quadrics in $\mathbb{P}^5$, defined by polynomials f_1, f_2 and f_3 respectively. In some coordinate system, let A_1, A_2, A_3 be their corresponding symmetric matrices. We consider the ideal I generated by f_1, f_2, f_3. For the construction of the elliptic fibration in the next section, we will be subject to the following assumption:

Assumption 4.1.3. If $X = Q_1 \cap Q_2 \cap Q_3$ is a smooth complete intersection of three quadrics in $\mathbb{P}^5$ and I is the ideal generated by the defining equations of the quadrics Q_1, Q_2 and Q_3, then there exists a degree two polynomial $f \in I$ such that its vanishing set Q is a quadric with $\mathrm{rank}(Q) \leq 4$.

For the remainder of this section, we will turn our attention to justify why this assumption is reasonable, given the context of trying to prove Conjecture 1.2.1.

Remark 4.1.4. For general choice of Q_1, Q_2 and Q_3 we see that the general Q in the ideal I is of rank 6, but there certainly exist quadrics of lower rank; consider the pencil $xQ_1 + yQ_2$ and the degree 6 polynomial $f = \det(xA_1 + yA_2)$. Roots of this polynomial correspond precisely to elements of the pencil (and hence the ideal I) whose rank at most 5. It follows (see Lemma 4.1.6) that a stronger condition on the pencil of quadrics will be sufficient to guarantee a matrix of rank at most 4. Unfortunately, this condition remains unknown to the author at this present time.

This assumption is crucial to the construction of an explicit elliptic fibration that we will make in the next section, which is then used throughout the rest of the document.
Indeed, we can obtain an abstract elliptic fibration with the help of the following theorem:

Proposition 4.1.5. [Huy16][Chapter 11, Proposition 1.3 2] Let X be a K3 surface over a field k with $\mathrm{char}(k) \neq 2, 3$. If $\mathrm{rank}(\mathrm{Pic}(X) > 5$ then X admits an elliptic fibration.

This implies that for any smooth surface $X = Q_1 \cap Q_2 \cap Q_3$ which has many lines, hence large Picard rank, X admits an elliptic fibration. While this elliptic fibration may not have anything to do with the one constructed in Section 4.2, its singular fibres are still restricted by the topology of X. It would be nice to prove that the general smooth fibre must be an intersection of two quadrics, since then we can use the ideas in this document.
The result below relates existence of a quadric of low rank if the determinant planar curve contains a singularity.

Lemma 4.1.6. Let $M_1 = (a_{ij})$, $M_2 = (b_{ij})$ and M_3 be $n \times n$ matrices, and let C be the degree n curve defined by $f := \det(xM_1 + yM_2 + zM_3) = 0$ on $\mathbb{P}^2_{\langle x,y,z\rangle}$. Let Q_1, Q_2, Q_3 be the quadric hypersurfaces defined by M_1, M_2, M_3 respectively and assume that $X = Q_1 \cap Q_2 \cap Q_3$ is smooth. Assume also that $p = (0:0:1)$ lies on C (that is, $\operatorname{rank}(M_3) \le n-1$). If this point p is a singular point of the curve then $\operatorname{rank}(M_3) \le n-2$.

Proof. The result is essentially a direct computation. Let $r = \operatorname{rank}(M_3)$. First, we will assume that M_3 is in Smith Normal Form. If it is not, find matrices P and Q so that PM_3Q is in smith normal form. P and Q are invertible since they can be obtained by row and column operations (which are invertible). Then

$$\phi := \det(PM_1Qx + PM_2Qy + PM_3Qz) = \det(P)\det(xM_1 + yM_2 + zM_3)\det(Q)$$

Note that $\operatorname{rank}(PM_2Q) = \operatorname{rank}(M_2)$ and $\operatorname{rank}(PM_1Q) = \operatorname{rank}(M_1)$, and the curve defined by ϕ is equal to the original curve.
Since M_3 is in smith normal form,

$$M_3 = \begin{pmatrix} I_r & 0_{r,n-r} \\ 0_{n-r,r} & 0_{n-r,n-r} \end{pmatrix}$$

smoothness of X implies that either $a_{nn} \neq 0$ or $b_{nn} \neq 0$: Assume temporarily that $a_{nn} = b_{nn} = 0$. Hence $p := (0:0:0:0:\cdots:1)$ lies on X. Note that

$$\frac{\partial Q_1}{\partial x_k} = 2\sum_{j=0}^{n} a_{jk}x_j, \qquad \frac{\partial Q_2}{\partial x_k} = 2\sum_{j=0}^{n} b_{jk}x_j, \qquad \frac{\partial Q_3}{\partial x_k} = \begin{cases} 2x_k & k \le r (\le n-1) \\ 0 & \text{otherwise} \end{cases}$$

We see that at the point p all these partial derivatives vanish (since we assume $a_{nn} = b_{nn} = 0$ and thus p is a singular point of X. Since X is smooth, this is a contradiction.
We now have all the ingredients for the proof: work locally on the affine chart containing $(0:0:1)$: let s and t be local parameters for x, y at $(0,0)$. Since M_3 is a singular point, $\frac{\partial f}{\partial s}$ and $\frac{\partial f}{\partial t}$ vanish at $(0,0)$; by assumption the polynomial f therefore has no terms of degree one or zero in s and t.
f is given by the determinant of the matrix

$$\begin{pmatrix} (sa_{ij} + tb_{ij} + 1)_{1\le i,j\le r} & (sa_{ij} + tb_{ij})_{1\le i\le r<j\le n} \\ (sa_{ij} + tb_{ij})_{1\le j\le r<i\le n} & (sa_{ij} + tb_{ij})_{r<i,j\le n} \end{pmatrix}$$

Hence f is a sum of products of n entries of this matrix, and so each term of f is exactly of degree n in s, t except those using entries from the top-left $r \times r$ submatrix, where it is possible to have terms of lower degree. We see that the lowest possible degree is when r terms are taken from this $r \times r$ submatrix, and so the lowest degree

terms of f yield the determinant of the bottom-right $n - r \times n - r$ submatrix in the expression above.
If M_3 has rank $n - 1$ then f can be written as

$$f = sa_{nn} + tb_{nn} + \text{ higher order terms}$$

since $a_{nn} \neq 0$ or $b_{nn} \neq 0$ or we obtain non-zero linear terms for f, contradicting that f is singular at $(0,0)$. To avoid a contradiction, Hence $\operatorname{rank}(M_3) \leq n - 2$. $\square$

Remark 4.1.7. 1. In fact, the converse to Lemma 4.1.6 also holds; existence of a quadric of rank less than $n-2$ in the pencil implies existence of a singularity on the curve C. In other words, singularities on C correspond precisely to quadrics of low rank in the pencil. It is therefore relatively easy to check whether a pencil of quadrics on $\mathbb{P}^5$ contains a quadric of rank 4 in their span; simply search the planar curve for a singular point.

2. Of course, a random choice of a smooth surface X is unlikely to result in any singularities on the curve C. However with regard to the next Lemma 4.1.10, it is the idea that a surface with many lines on it will cause such a situation. We will justify this idea in a moment.

Why should we think this assumption is reasonable? We will see this after Lemma 4.1.10. First, we define a *square* of lines and prove some easy statements about them:

Definition 4.1.8. A *square* on X is defined to be a subvariety consisting of a union of four lines $\{l_0, l_1, l_2, l_3\}$ on X whose non-empty intersections are precisely $p_{01} := l_0 \cap l_1, p_{12} := l_1 \cap l_2, p_{23} := l_2 \cap l_3$ and $p_{30} := l_3 \cap l_0$.

This implies the following easy result:

Lemma 4.1.9. Let S be a square in $\mathbb{P}^5$ whose edges are defined over k. There is a coordinate system $x_0, \cdots, x_5$ on $\mathbb{P}^5$ such that the ideal defining S is generated by x_0x_2, x_1x_3, x_4, x_5.

Proof. Let $p_{01}, p_{12}, p_{23}, p_{30}$ be the four points of S corresponding respectively to the non-empty intersections of the lines as in Definition 4.1.8. Note that $p_{01}, p_{12}, p_{23}, p_{30}$ cannot all be contained in the same plane; if they were then all four lines of S would be contained in this same plane and hence any two edges of the square would meet, which is impossible; l_0 does not meet l_2.

Moreover, no three of these intersection points are collinear, and hence these intersection points are linearly independent and therefore we can define homogeneous coordinates $x_0, \cdots, x_5$ on $\mathbb{P}^5$ such that

$$\begin{aligned} p_{01} &= (1:0:0:0:0:0) \\ p_{12} &= (0:1:0:0:0:0) \\ p_{23} &= (0:0:1:0:0:0) \\ p_{30} &= (0:0:0:1:0:0) \end{aligned}$$

Hence the ideal defining S is thus generated in these coordinates by x_0x_2, x_1x_3, x_4, and x_5. $\square$

In particular, we see that a square defines a 3 dimensional linear subspace. This implies the following result about quadrics of rank 4:

Lemma 4.1.10. Let $X \subset \mathbb{P}^5$ be a smooth complete intersection of three linearly independent quadrics Q_1, Q_2, Q_3. If X contains a square, then there exists a quadric of rank at most 4 containing X.

Proof. Let S be a square in X and choose coordinates $x_0, \cdots, x_5$ for $\mathbb{P}^5$ as in Lemma 4.1.9. In this coordinate system, any quadric containing S has an equation of the form

$$\alpha x_0 x_2 + \beta x_1 x_3 + \lambda x_4 + \mu x_5 = 0$$

where $\alpha, \beta \in k$ and λ, μ are linear forms. In particular, any quadric in the ideal (Q_1, Q_2, Q_3) defining X has to be of this form, since these all vanish on X, hence S. In particular, we can take linear combinations of Q_1, Q_2 and Q_3 (by way of Gaussian Elimination) to obtain at least one quadric of the form $Q = \lambda' x_4 + \mu' x_5$ in the ideal (Q_1, Q_2, Q_3).
We see that the symmetric matrix corresponding to Q has a 4×4 zero submatrix, and hence the rank$(Q) \leq 4$. $\square$

Indeed, if there is a quadric of rank at most 4, then the smoothness of X implies that this quadric has rank exactly 4:

Lemma 4.1.11. If X is the smooth intersection of three quadrics in $\mathbb{P}^5$ then the rank of every quadric vanishing on X is at least four.

Proof. Let q_0 be a quadratic form vanishing on X and let $L \subset \mathbb{P}^5$ be the singular subvariety of the quadric $q_0 = 0$. By Lemma 2.2.10 after an orthogonal change of coordinates we may assume that q_0 is of the form $\sum_{i=1}^{r} x_i^2$ where r is the rank of q_0.

Observe that L is a linear subspace and to prove the statement it suffices to show that the dimension of L is at most 1, since $\dim(L) = \dim(q_0) - r = 5 - r$.
Choose quadratic forms q_1, q_2 such that X is the vanishing set of q_0, q_1 and q_2. For each $i \in \{0, 1, 2\}$ denote by Q_i the quadric corresponding to $q_i = 0$. On one hand, the intersection $L \cap Q_1 \cap Q_2 \subseteq X$, on the other hand, this intersection consists of singular points of X since Q_0 is singular along L and X is a complete intersection. Since X is smooth, we conclude that the varieties L and $Q_1 \cap Q_2$ are disjoint.
As the codimension of $Q_1 \cap Q_2$ is 2, we conclude that the dimension of L is at most 1, as required. □

In particular, if Assumption 4.1.3 is not satisfied, then X contains no squares. Indeed, if a surface X would have a maximal number of lines, then increasing its Picard rank only allows more room for more lines. Consider Proposition 4.1.5, hence we may assume the existence of an abstract elliptic fibration on X. Any lines on X are contained as either fibres or sections of the elliptic fibration, and by considering how the singular fibres of the elliptic fibration are limited by the global topology of X, if the configuration of lines is square-free then certainly no more than 16 lines can appear as fibres (8 fibres of type I_3). Now lines appearing as sections are described uniquely by which irreducible components of singular fibres they intersect. If two different section lines meet the same pair of fibre lines then this results in a square. We see that to avoid squares, two section lines can only share at most one fibre line, but on the other hand, each section line must meet every fibre somewhere.

4.2 Construction of the Elliptic Fibration

As before, assume $X = Q_1 \cap Q_2 \cap Q_3$ is a smooth complete intersection of three quadrics, embedded in $\mathbb{P}^5$.
For this section, we will be assuming that Assumption 4.1.3 holds. Let $Q_0 \in I(X)$ denote a quadric of rank 4 in the ideal spanned by Q_1, Q_2, Q_3. It is clear that the ideal of X is therefore generated by Q_0 and two of the remaining three quadrics, therefore we may assume during the following construction that $\operatorname{rank}(Q_3) = 4$.

Construction 4.2.1. $Q_3 \subset \mathbb{P}^5$ is a quadric of rank 4 in $\mathbb{P}^5$. After a suitable coordinate change, it is immediate that Q_3 is a cone over a quadric $Q' \subset \mathbb{P}^3_{\langle s,t,u,v\rangle}$, where s, t, u, v are homogeneous coordinates for $\mathbb{P}^3$ and can be identified with linear forms in $\mathbb{P}^5$. Since $Q' \cong \mathbb{P}^1 \times \mathbb{P}^1$, we can make a further coordinate change and we may assume that Q_3 is given by the equation $su - tv = 0$ in $\mathbb{P}^5$.
In particular, we see that the singular subscheme of Q_3 consists of the line

$l_0 := \{s = t = u = v = 0\}$ in $\mathbb{P}^5$. Note that this line l_0 does not lie on X, since X is a smooth complete intersection.

Let $\pi_0 : \mathbb{P}^5 \to \mathbb{P}^3$ map defined by projecting away from l_0. If $\mathbb{P}^5$ is chosen to have coordinates $\{s, t, u, v, x_4, x_5\}$, then π_0 amounts to "forgetting" coordinates x_4 and x_5. We see that $\pi_0(Q_3) = Q'$.

Since $Q' \cong \mathbb{P}^1 \times \mathbb{P}^1$, let $\pi_1, \pi_2 : Q' \to \mathbb{P}^1$ be the projections onto the first and second factor respectively.

Now consider the following diagram:

$$\begin{tikzcd}
 & X = Q_1 \cap Q_2 \cap Q_3 \arrow[ddl, "p_1"'] \arrow[d, hook] \arrow[ddr, "p_2"] & \\
 & Q_3 \arrow[d, "\pi_0"] & \\
\mathbb{P}^1 & Q' \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & \mathbb{P}^1
\end{tikzcd}$$

By composing, we therefore obtain two maps $p_1, p_2 : X \to \mathbb{P}^1$. For a point $m \in \mathbb{P}^1$, let $F_i(m)$ denote $p_i^{-1}(m)$. Then for each i and m, $(\pi_i \circ \pi_0)^{-1}(m)$ consists of a 3-dimensional linear subspace $\mathbb{P}$ of $Q_3 \subset \mathbb{P}^5$ and so $F_i(m) = \mathbb{P} \cap Q_1 \cap Q_2$ is an intersection of two quadrics in $\mathbb{P}^3$.

In particular, each fibre is a curve of arithmetic genus 1.

We conclude that a quadric of rank 4 containing X induces two different morphisms $X \to \mathbb{P}^1$ induced by linear projections from the ambient projective space. Each fibre of the morphisms is isomorphic to an intersection of two quadrics in $\mathbb{P}^3$. Lemma 4.2.3 below ensures that the fibration satisfies the first condition in Definition 2.3.1. This is a non-degeneracy condition that, for example, ensures that the surface X is not the product of a singular fibre and $\mathbb{P}^1$.

Remark 4.2.2. Note that for a line $l \subset X$, the projections π_0, π_1, π_2 either map the line to a point or map the line isomorphically to another line. As a result, under the induced elliptic fibration $\pi : X \to \mathbb{P}^1$, each line is mapped either isomorphically to the base curve (l is horizontal) or the line collapses onto a single point (l is vertical). In particular, there is no need to perform base changes as in the case of lines on the quartic surface in Section 3.3; there is no ramification to consider when constructing sections.

Lemma 4.2.3. Let X be a smooth complete intersection of three quadrics Q_1, Q_2, Q_3 in $\mathbb{P}^5$ and assume that the rank of Q_3 is 4. Let $\mathbb{P} \subset \mathbb{P}^5$ be a linear subspace of dimension 3 contained in Q_3 and let Π be the pencil in $\mathbb{P}$ generated by the quadrics $Q_1 \cap \mathbb{P}$ and $Q_2 \cap \mathbb{P}$. A general quadric in this pencil Π is smooth.

Proof. Choose homogeneous coordinates $x_0, \cdots, x_5$ in $\mathbb{P}^5$ so that $\mathbb{P}$ is the linear subspace defined by $x_0 = x_1 = 0$. In this coordinate system, an equation for the quadric Q_3 takes the form $x_0 l_0 + x_1 l_1 = 0$ where l_0 and l_1 are linear forms; denote by q_1 a quadratic form defining Q_1 and by q_2 a quadratic form defining Q_2. In what follows, for a polynomial f in $x_0, \cdots, x_5$ let $\widetilde{f}$ be the polynomial obtained by setting $x_0 = x_1 = 0$. we compute the matrix J obtained from the Jacobian matrix of the polynomials $x_0 l_0 + x_1 l_1, q_1, q_2$ setting to zero the variables x_0, x_1:

$$J = \begin{pmatrix} \widetilde{l_0} & \widetilde{l_1} & 0 & 0 & 0 & 0 \\ \widetilde{\dfrac{\partial q_1}{\partial x_0}} & \widetilde{\dfrac{\partial q_1}{\partial x_1}} & \widetilde{\dfrac{\partial q_1}{\partial x_2}} & \widetilde{\dfrac{\partial q_1}{\partial x_3}} & \widetilde{\dfrac{\partial q_1}{\partial x_4}} & \widetilde{\dfrac{\partial q_1}{\partial x_5}} \\ \widetilde{\dfrac{\partial q_2}{\partial x_0}} & \widetilde{\dfrac{\partial q_2}{\partial x_1}} & \widetilde{\dfrac{\partial q_2}{\partial x_2}} & \widetilde{\dfrac{\partial q_2}{\partial x_3}} & \widetilde{\dfrac{\partial q_2}{\partial x_4}} & \widetilde{\dfrac{\partial q_2}{\partial x_5}} \end{pmatrix}$$

Suppose that p is a point in $Q_1 \cap Q_2 \cap \mathbb{P}$ where $Q_1 \cap \mathbb{P}$ and $Q_2 \cap \mathbb{P}$ are both singular. At the point p, the rank of the matrix J is at most two, since its last four columns vanish and hence p is a singular point of the surface X, contradicting the smoothness assumption. Hence each point of the base locus of the pencil Π is contained in the smooth locus of some quadric in the pencil. By Bertini's Theorem over an algebraically closed (hence infinite) field we conclude that the quadrics in the pencil Π which are smooth forms an open dense subset, as required. $\square$

We shall use the following terminology while working with the elliptic fibrations constructed in this section:

Definition 4.2.4. Consider the two elliptic fibrations $p_1, p_2 : X \to \mathbb{P}^1$ in Construction 4.2.1. Due to the symmetry present in their construction, we will not be specific as to whether we choose the first or second projection. In this situation we say that p_2 is the *dual* fibration to p_1.
Furthermore, for $m \in \mathbb{P}^1$ we say that the curve $F_1(m)$ is a fibre and $F_2(m)$ is the *dual fibre* of m. On the other hand given a fibre or dual fibre $F_i(m)$, we say that m is the *slope* of the fibre.

The following remark is an easy consequence of the construction, nevertheless it will be crucial in proving Theorem 1.2.4:

Remark 4.2.5. Note that the projections π_0 and π_1, π_2 are linear morphisms. As a result if l is a line on X, then $\pi_0(l)$ is a line on $Q' = \mathbb{P}^1 \times \mathbb{P}^1$ and so lies in one of

the two families. In particular, if it lies in the first family then $\pi_1|_l : l \to \mathbb{P}^1$ is an isomorphism, and so l is a section of the map $p_1 : X \to \mathbb{P}^1$. On the other hand, l is contained in the fibre of p_2 with slope $p_2(l)$.
By symmetry, if l lies in the second family of Q' then l is contained in a fibre of p_2 while being a section of p_1.
Thus with respect to a particular elliptic fibration, since each line l on X occurs either in a fibre or as a section, each line on X occurs as an irreducible component in either a fibre or a dual fibre.

4.3 j-Invariant of the Fibres

We consider the elliptic fibration for $X \to \mathbb{P}^1$ in Construction 4.2.1 with the assumption that Q_3 is a quadric of rank 4. In this section we will be investigating the j-function $j : \mathbb{P}^1 \to \mathbb{P}^1$ that sends m to the j-invariant of the fibre of slope m in this given elliptic fibration.
We will study it directly using a particularly nice set of coordinates that come from the following algebraic lemma:

Lemma 4.3.1. [Fuk90, 2.3, p. 31] Suppose that M_1 and M_2 are symmetric matrices over an algebraically closed field k and that M_1 has full rank. Then we can diagonalise both of them as follows:

$$A^T M_1 A = I \quad \text{and} \quad A^T M_2 A = D$$

where D is the matrix of eigenvalues and A is the corresponding matrix of eigenvectors of $M_1^{-1}M_2$, that is $M_1^{-1}M_2 A = AD$.

Remark 4.3.2.
1. The matrix of eigenvectors A is not necessarily an orthogonal matrix.
2. As M_1 is invertible, $\det(M_1)\det(zI_n - M_1^{-1}M_2) = \det(zM_1 - M_2)$, so the eigenvalues of $M_1^{-1}M_2$ are precisely the roots of the equation $\det(zM_1 - M_2)$.
3. When k is algebraically closed we are free to assume that M_1 is an identity matrix.

In the case of $X = Q_1 \cap Q_2 \cap Q_3$ being a smooth complete intersection of three quadrics with $\mathrm{rank}(Q_3) = 4$, the fact that the theorem requires Q_1 to be invertible is no big deal, there is at least one quadric Q in the span of Q_1 and Q_2 with full rank; smoothness of X guarantees that the polynomial $\det(zQ_1 - Q_2) \in k[z]$ is not constantly zero (see the proof of Lemma 4.1.6). Then $X = Q \cap Q_2 \cap Q_3$ with Q full rank.

Then, for any intersection of two quadrics Q_1 and Q_2 in $\mathbb{P}^3$, from this lemma, we choose coordinates $x_0, \cdots, x_3$ for $\mathbb{P}^3$ so that Q_1 and Q_2 are written as

$$Q_1 := \{x_0^2 + x_1^2 + x_2^2 + x_3^2 = 0\} \quad \text{and} \quad Q_2 := \{\lambda_0 x_0^2 + \lambda_1 x_1^2 + \lambda_2 x_2^2 + \lambda_3 x_3^2 = 0\}$$

where the $\lambda_i \in k$.

We know that $Q_1 \cap Q_2$ is a genus one curve; in order for it to be a true elliptic curve we need to pick a point. The j-invariant can then be obtained by putting the curve into Weierstrass form with this point being the point at infinity.

For an explicit formula in terms of the λ_i, we may let

$$P = (p_0 : p_1 : p_2 : p_3) := (\sqrt{\lambda_2 - \lambda_1} : \sqrt{\lambda_0 - \lambda_2} : \sqrt{\lambda_1 - \lambda_0} : 0) \in Q_1 \cap Q_2$$

(this point is well-defined unless all the λ_i coincide, in which case $Q_1 = Q_2$ and so $Q_1 \cap Q_2$ is not a curve.) From here, (by Magma, which puts $Q_1 \cap Q_2$ into Weierstrass form), one may check that $(Q_1 \cap Q_2, P)$ is isomorphic to the elliptic curve E with j-invariant

$$\begin{aligned} j(E) = 2^8 \Bigg[\sum_{i=0}^{3} \frac{\lambda_i^2}{2} \Bigg(\sum_{j<k,\, j \neq i \neq k} (\lambda_j - \lambda_k)^2 \Bigg) - (\lambda_0\lambda_1 - \lambda_2\lambda_3)^2 \\ - (\lambda_0\lambda_2 - \lambda_1\lambda_3)^2 - (\lambda_0\lambda_3 - \lambda_1\lambda_2)^2 \Bigg]^3 \Bigg(\prod_{i<j} (\lambda_i - \lambda_j)^2 \Bigg)^{-1} \end{aligned} \tag{4.1}$$

In particular, note that this expression for the j-invariant is independent of the specific choice of coordinates in Lemma 4.3.1, and of the choice of the point $P \in Q_1 \cap Q_2$. Since the denominator of the j-invariant for an elliptic curve is equal to the discriminant of the curve, from the above expression we conclude that the curve $Q_1 \cap Q_2$ is smooth if and only if the eigenvalues $\lambda_0, \cdots, \lambda_3$ of $M_1^{-1} M_2$ are distinct.

Note that equation (4.1) is a symmetric rational function of degree 12 in terms of the roots $\lambda_0, \lambda_1, \lambda_2, \lambda_3$. We wish to obtain the same expression without having to compute the roots of $\det(xM_1 - M_2)$ or utilise the specific coordinate system that simultaneously diagonalises both quadrics.

Let σ_i be the elementary symmetric polynomial of degree i in $\{\lambda_0, \cdots, \lambda_3\}$. That is:

$$\begin{aligned}
&\sigma_0 := 1, \qquad\qquad \sigma_1 := \lambda_0 + \lambda_1 + \lambda_2 + \lambda_3,\\
&\sigma_2 := \lambda_0\lambda_1 + \lambda_0\lambda_2 + \lambda_0\lambda_3 + \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3,\\
&\sigma_3 := \lambda_0\lambda_1\lambda_2 + \lambda_0\lambda_1\lambda_3 + \lambda_0\lambda_2\lambda_3 + \lambda_1\lambda_2\lambda_3,\\
&\sigma_4 := \lambda_0\lambda_1\lambda_2\lambda_3
\end{aligned}$$

By definition of λ_i, and elementary symmetric polynomials we have

$$\det(xM_1 - yM_2) = \prod_{i=0}^{3}(x - \lambda_i) = \sum_{i=0}^{4}(-1)^{4-i}\sigma_{4-i}x^i y^{4-i}$$

and so the j-invariant (4.1) can be written as:

$$\begin{aligned}
j(E) = 2^8 \left(\sigma_2^2 - 3\sigma_3\sigma_1 + 12\sigma_4\sigma_0\right)^3 \big[&\sigma_3^2(\sigma_2^2\sigma_1^2 - 4\sigma_2^3\sigma_0 - 4\sigma_3\sigma_1^3 + 18\sigma_3\sigma_2\sigma_1\sigma_0 - 27\sigma_3^2\sigma_0^2)\\
&+ \sigma_4(-4\sigma_2^3\sigma_1^2 + 16\sigma_2^4\sigma_0 + 18\sigma_3\sigma_2\sigma_1^3 - 80\sigma_3\sigma_2^2\sigma_1\sigma_0 - 6\sigma_3^2\sigma_1^2\sigma_0 + 144\sigma_3^2\sigma_2\sigma_0^2)\\
&+ \sigma_4^2(-27\sigma_1^4 + 144\sigma_2\sigma_1^2\sigma_0 - 128\sigma_2^2\sigma_0^2 - 192\sigma_3\sigma_1\sigma_0^2) + 256\sigma_4^3\sigma_0^3\big]^{-1}
\end{aligned} \tag{4.2}$$

which has the advantage of being directly computable from the coefficients of the polynomial $\det(xM_1 - yM_2)$, with the disadvantage of being more unwieldy and less symmetric for computations by hand. Note that both the numerator and the denominator for this expression are homogeneous polynomials in the σ_i's of degree 6. We will use this fact in the proof of the next proposition, which is the main result of this section.

Proposition 4.3.3. Let X be a smooth complete intersection of three quadrics Q_1, Q_2, Q_3 in $\mathbb{P}^5$, with $\operatorname{rank}(Q_3) = 4$. For a fibre F_t of slope t in the associated elliptic fibration 4.2.1, the j-invariant of $j(F_t)$ is expressible as a rational function whose numerator and denominator are both polynomials of degree (at most) 24 in the slope of the fibre t.

Proof. Recall that we may perform a linear change of coordinates to assume Q_3 is of the standard form $x_0x_3 - x_1x_2$. Then the fibration π is the composition of the projection $\pi_0 : \mathbb{P}^5 \to \mathbb{P}^3$ obtained by eliminating the coordinates x_4, x_5, together with the projection onto, say, the first factor $\pi_1 : \pi_0(Q_3) \to \mathbb{P}^1$. The fibre F_t is then given by $\pi^{-1}(t) = \pi_0^{-1}(\pi_1^{-1}(t)) = Q_1 \cap Q_2 \cap \pi_1^{-1}(t)$. Now, on $\mathbb{P}^3$, $\pi_0(Q_3)$ is isomorphic to

$\mathbb{P}^1 \times \mathbb{P}^1$ via the Segre map, and so $\pi_1^{-1}(t)$ is given by the line

$$L_t := \begin{cases} x_0 = tx_2 \\ x_1 = tx_3 \end{cases} \subset \mathbb{P}^3_{\langle x_0, x_1, x_2, x_3 \rangle}$$

Therefore, F_t is given by intersecting Q_1 and Q_2 with $\pi_0^{-1}(L_t)$, this amounts to restricting Q_1 and Q_2 to the linear subspace $\mathbb{P}$ isomorphic to $\mathbb{P}^3$ defined by $x_0 = tx_2$ and $x_1 = tx_3$. Therefore, the corresponding symmetric matrices M_1 and M_2 for the fibre F_t are 4×4 matrices, whose entries are polynomials of degree at most

$$\begin{pmatrix} 2 & 2 & 1 & 1 \\ 2 & 2 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

Therefore $\det(xM_1 - yM_2) \in (k[t])[x, y]$ is a homogeneous polynomial of degree y in x, y, whose coefficients are polynomial of (at most) degree 4 in the slope t. In particular, in the notation of equation (4.2),

$$\det(xM_1 - yM_2) = \sum_{i=0}^{4} (-1)^{4-i} \sigma_{4-i} x^i y^{4-i}$$

we see that each σ_i is a degree 4 polynomial in t, and hence, by equation (4.2), $j(F_t)$ is a rational function, whose numerator and denominator are both degree 24 polynomials in t.
Since $j(F_t)$ is unchanged by isomorphisms of the fibre, in particular this is independent of the choice of coordinates using in the simplifying expression for Q_3. □

For any elliptic curve E, the denominator of $j(E)$ is precisely the discriminant $\Delta(E)$ of the curve. Using the following results will prove to be very useful:

Proposition 4.3.4. [MP86, Corollary 1.2] The Euler number of a singular fibre $e(F_t)$ is equal to the order of vanishing of the discriminant Δ at t.

Since the construction 4.2.1 and the formula obtained by Proposition 4.3.3 for the j-invariant are explicit in terms of the defining polynomials Q_1, Q_2, Q_3 for X, this proposition immediately allows us to find and then identify the singular fibres by analysing the discriminant. In order to emphasise this, in the next section we will shortly see an example with all the calculations performed transparently.
The following result relates the Euler number of a fibre, the rank of the associated

sublattice of the Néron-Severi group and the type of the singular fibre. For a singular fibre F_t, let $e(F_t)$ denote its Euler number. Let S_0 be a distinguished section for the elliptic fibration $\pi : X \to \mathbb{P}^1$. (S_0 is the zero-section of π). Let $\alpha : X \to \widehat{X}$ contract all components of fibres not meeting S_0. Then (with respect to S_0,) $\widehat{X}$ can locally be presented in Weierstrass form, and the sections defined by the three roots of the cubic make up a cubic curve $\overline{T}$ consisting of three sections. They can be considered as sublattices of the Néron-Severi group of X and as such have rank $r(F_t)$.

Proposition 4.3.5. [MP86, Corollary 1.3] For all cases of singular fibres F_t, $0 \leq e(F_t) - r(F_t) < 2$. Moreover,

1. When $e(F_t) - r(F_t) = 0$, the fibre F_t is smooth (type I_0).
2. When $e(F_t) - r(F_t) = 1$, the fibre F_t is semistable (type I_n), for some $n \geq 1$.

We will soon see how the fixed structure of the Néron-Severi lattice and the j-invariant determines which combinations of singular fibres are permitted. We shall use this information in the next section to start to achieve bounds on the number of lines present in the fibres of Construction 4.2.1.

4.4 Singular Fibres

As usual, X is a non-singular complete intersection of three quadrics in $\mathbb{P}^5$. In the fibration $\pi : X \to \mathbb{P}^1$ from Construction 4.2.1, any vertical lines present in X will occur in singular fibres, which can only occur when the expressions for the j-invariant in Equations (4.1) and (4.2) are not defined; that is when the denominator vanishes.
For an abstract elliptic fibration, the singular fibres are completely classified in [Kod63], and this process is summarised in section 2.3.2. Since all fibres in the Construction 4.2.1 are intersections of two quadrics in $\mathbb{P}^3$, we can narrow down the possibilities as follows:

Lemma 4.4.1. [Bak23, Chapter 3, Page 122] The following table lists all the possible cases for a singular fibre π of the elliptic fibration in Construction 4.2.1, together with the associated Euler number of that fibre. The notation used is the Kodaira notation with added decoration.

Proof. We give a short justification to the completeness of the table, and examples for the cases that do exist.
Let Q_1 and Q_2 be two quadrics in $\mathbb{P}^3$. $Q_1 \cap Q_2$ is a degree 4 curve in $\mathbb{P}^3$ (with

Type	**Description**	**Euler Number**	**Lines**
I_1	Nodal Quartic Curve	1	0
I_2a	Line meeting Twisted Cubic in two points	2	1
I_2b	Two nonsingular conics meeting in two points	2	0
I_3	A nonsingular conic meeting a singular conic at two smooth points, one point on each component	3	2
I_4	Four lines meeting in a square	4	4
II	Cuspidal Quartic Curve	2	0
$IIIa$	Line meeting Twisted Cubic in one point at a tangent	3	1
$IIIb$	Two nonsingular conics meeting at one point tangentially	3	0
IV	A nonsingular conic meeting a singular conic at its singular point	4	2
R_3	A double line meeting two skew lines	4	3
R_2	Two double lines meeting at one point	3	2
R_1	A double conic	2	0

Table 4.1: Singular Fibres in the Elliptic Fibration

homogeneous coordinates x, y, z, t), and if irreducible, it meets a general plane in four points.

Take any two smooth points on the curve and consider the pencil of planes passing through the line joining these two points. Each plane therefore intersects the curve (generally) in two other points. When the curve has exactly one singular point, any plane passing through it and another smooth base point meets the curve in one more point and so the points of the curve are in one-to-one correspondence with the points of a line; that is, the curve is rational. In this case we get a node I_1, (for example $2xy + z^2 - t^2, 2xy + y^2 + (1 - m^2)z^2 - (1 - n^2)t^2$, or a cusp II e.g. $2xy + z^2 + t^2, 2xy + 2yz + z^2 + (1 - n^2)t^2$.

If the curve has more than one singular point, then the curve is reducible. Indeed, if it was irreducible, any general plane passing through two of its singular points would miss the rest of the curve; but the union of such planes is an open dense subset of $\mathbb{P}^3$ and so the entire curve is contained in a union of finitely many planes. In particular, the curve restricted to one of these planes is a degree 4 planar curve, and thus meets any general line in four points. Being an intersection of two quadrics, any quadric that vanishes on three collinear points vanishes on the line containing them, and so the curve contains a plane, which is a contradiction.

We can now assume that the curve decomposes into at least two components. If

there is a component of degree 3, then the curve consists of a line and a twisted cubic (otherwise $Q_1 \cap Q_2$ is planar), and they intersect at either two points transversally in case I_2a (e.g. $xy+zt, 2xy+2czt+y^2-y^2$) or at one point tangentially (this is case $IIIa$ (e.g. $xy+zt$, $2xy+2zt+2yt+z^2$).
All other cases consist of a union of two conics: the conics will not intersect in four points, since then the union of them would be planar. If they are both nonsingular, then this gives cases I_2b (for example $x^2+y^2+z^2+t^2, x^2+y^2+cz^2+dt^2$) and $IIIb$ (for example $2xy+y^2+z^2-t^2, 2xy+z^2+t^2$). If one is singular and the other is not then we get cases I_3 (for example $2xy+z^2-t^2, 2xy+y^2+c(z^2-t^2)$) and IV (e.g. $2xy+z^2-t^2, 2xy+2yz+z^2-t^2$) from the table.
In all other cases, both conics are singular. We can have a square of four lines I_4 ($x^2+y^2+z^2+t^2, x^2+y^2+c(z^2+t^2)$), or when one of the lines from one conic coincides with one the lines from the other conic, giving case R_3 ($xy+zt, xy+yt$). R_2 and R_1 are the most degenerate cases when the two conics coincide, which can only happen if one of the quadrics is in fact a double plane (either tangent or general). □

Now that we have established the notation for the singular fibres, we present a detailed example clarifying the construction of an elliptic fibration and computation of the j-map.

Example 4.4.2. Let $X \subset \mathbb{P}^5_{\langle x_0,x_1,\cdots,x_5\rangle}$ be the smooth surface in Section 1.1 formed by the intersection of:

$$Q_1 := x_0^2 - 2x_1^2 + x_2^2 - 2x_5^2$$
$$Q_2 := x_1^2 - 2x_2^2 + x_3^2 - 2x_5^2$$
$$Q_3 := x_2^2 - 2x_3^2 + x_4^2 - 2x_5^2$$

Note that in this choice of coordinates, the symmetric matrix associated to Q_1 is diagonal $\operatorname{diag}(1,-2,1,0,0,-2)$ and so is a quadric of rank 4 with singular subscheme consisting of the line $l_0 := (x_0 = x_1 = x_2 = x_5 = 0)$.
Let $\pi_0 := \mathbb{P}^5 \to \mathbb{P}^3_{\langle s,t,u,v\rangle}$ be the projection away from l_0. Q_1 is then a cone over a quadric $\pi_0(Q_1)$, which is expressed in these coordinates as $sv = tu$. This $\mathbb{P}^3$ can be embedded explicitly in $\mathbb{P}^5$ by setting

$$s = x_0 - \sqrt{2}x_1, \quad t = -x_2 - \sqrt{2}x_5, \quad u = x_2 - \sqrt{2}x_5, \quad v = x_0 + \sqrt{2}x_1$$

and thus:

$$sv - tu = (x_0 - \sqrt{2}x_1)(x_0 + \sqrt{2}x_1) - (-x_2 - \sqrt{2}x_5)(x_2 - \sqrt{2}x_5) = x_0^2 - 2x_1^2 + x_2^2 - 2x_5^2$$

as required.

We see that $\pi_0(Q_1)$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, so for $(\lambda : \mu) \in \mathbb{P}^1$, let $L_{(\lambda:\mu)}$ be the line on $\pi_0(Q_1)$ defined by $(\mu s = \lambda u, \mu t = \lambda v)$. As $(\lambda : \mu) \in \mathbb{P}^1$ varies, $L_{(\lambda:\mu)}$ traces out one of the family of lines in $\pi_0(Q_1)$. Hence:

$$\pi_0^{-1}(L_{(\lambda:\mu)}) := \begin{cases} \mu(x_0 - \sqrt{2}x_1) = \lambda(x_2 - \sqrt{2}x_5) \\ \mu(-x_2 - \sqrt{2}x_5) = \lambda(x_0 + \sqrt{2}x_1) \end{cases} \subset Q_1$$

which is simply a linear subspace in $\mathbb{P}^5$ isomorphic to $\mathbb{P}^3$. Hence the fibre with slope $(\lambda : \mu)$ is the curve given by

$$F(\lambda : \mu) := \pi_0^{-1}(L_{(\lambda:\mu)}) \cap Q_2 \cap Q_3 \subset X$$

We note that when both $\lambda \neq 0$ and $\mu \neq 0$, $F(\lambda : \mu)$ is 3 dimensional linear subspace determined by setting

$$\left(x_0 = \frac{\lambda^2 - \mu^2}{2\lambda\mu}x_2 - \sqrt{2}\frac{\lambda^2 + \mu^2}{2\lambda\mu}x_5\right) \cap \left(x_1 = \frac{-\lambda^2 - \mu^2}{2\sqrt{2}\lambda\mu}x_2 + \frac{\lambda^2 - \mu^2}{2\lambda\mu}x_5\right)$$

and so

$$F(\lambda : \mu) = \begin{cases} \frac{\lambda^4 - 14\lambda^2\mu^2 + \mu^4}{8\lambda^2\mu^2}x_2^2 + x_3^2 + \frac{-\lambda^4 + \mu^4}{4\lambda^2\mu^2}\sqrt{2}x_2x_5 + \frac{\lambda^4 - 10\lambda^2\mu^2 + \mu^4}{4\lambda^2\mu^2}x_5^2 \\ x_2^2 - 2x_3^2 + x_4^2 - 3x_5^2 \end{cases}$$

i.e.

$$\pi_0(F(\lambda : \mu)) = \begin{cases} \frac{1}{8}(v^2 - 2sv + s^2) + \frac{1}{4}(-3u^2 + 2tu - 3t^2) \\ -tu \end{cases}$$

which gives us symmetric matrices:

$$M_2 := \begin{pmatrix} \frac{\lambda^4 - 14\lambda^2\mu^2 + \mu^4}{8\lambda^2\mu^2} & 0 & 0 & \sqrt{2}\frac{-\lambda^4 + \mu^4}{8\lambda^2\mu^2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \sqrt{2}\frac{-\lambda^4 + \mu^4}{8\lambda^2\mu^2} & 0 & 0 & \frac{\lambda^4 - 10\lambda^2\mu^2 + \mu^4}{4\lambda^2\mu^2} \end{pmatrix}, \qquad M_3 := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}$$

then

$$\begin{aligned} \det(xM_2 - yM_3) &= -y(x + 2y)\left(\frac{-3\lambda^4 + 18\lambda^2\mu^2 - 3\mu^4}{4\lambda^2\mu^2}x^2 - xy + 2y^2\right) \\ &= \frac{3\lambda^4 - 18\lambda^2\mu^2 + 3\mu^4}{4\lambda^2\mu^2}x^3y + \frac{3\lambda^4 - 16\lambda^2\mu^2 + 3\mu^4}{2\lambda^2\mu^2}x^2y^2 + 4xy^3 + 4y^4 \end{aligned}$$

Using expression (4.2), we have

$$\sigma_0 := 0$$
$$\sigma_1 := \frac{-3\lambda^4 + 18\lambda^2\mu^2 - 3\mu^4}{4\lambda^2\mu^2}$$
$$\sigma_2 := \frac{3\lambda^4 - 16\lambda^2\mu^2 + 3\mu^4}{2\lambda^2\mu^2}$$
$$\sigma_3 := -4$$
$$\sigma_4 := 4$$

and so the j-invariant of the general fibre is:

$$j(F) := \frac{2^6(9\lambda^8 - 132\lambda^6\mu^2 + 490\lambda^4\mu^4 - 132\lambda^2\mu^6 + 9\mu^8)^3}{-3^4\lambda^2\mu^2(\lambda^2 - 2\lambda\mu - \mu^2)^4(\lambda^2 + 2\lambda\mu - \mu^2)^4(\lambda^2 - 6\mu^2)(6\lambda^2 - \mu^2)}$$

in particular, the only fibres $F(\lambda:\mu)$ which are singular are when the denominator of $j(F)$ vanishes:

Slope λ/μ of singular fibre	0	∞	$1 \pm \sqrt{2}$	$-1 \pm \sqrt{2}$	$\pm\sqrt{6}$	$\frac{\pm 1}{\sqrt{6}}$
Multiplicity	2	2	4, 4	4, 4	1, 1	1, 1

Table 4.2: Slopes of Singular Fibres and their multiplicity

We now analyse the singular fibres one by one, and decide what type of fibre they are:

1. When $\lambda = 0$,

$$F = \begin{cases} x_0 = \sqrt{2}x_1 \\ x_2 = -\sqrt{2}x_5 \\ x_1^2 + x_3^2 - 6x_5^2 \\ -2x_3^2 + x_4^2 \end{cases}$$

   which consists of two smooth conics intersecting at the two points $(\pm\sqrt{12} : \pm\sqrt{6} : -\sqrt{2} : 0 : 0 : 1)$.

2. When $\mu = 0$,

$$F = \begin{cases} x_0 = -\sqrt{2}x_1 \\ x_2 = \sqrt{2}x_5 \\ x_1^2 + x_3^2 - 6x_5^2 \\ -2x_3^2 + x_4^2 \end{cases}$$

which consists of two smooth conics intersecting at the two points $(\mp\sqrt{12} : \pm\sqrt{6} : \sqrt{2} : 0 : 0 : 1)$.

3. When $\lambda/\mu = 1 + \sqrt{2}$,

$$F = \begin{cases} x_0 = x_2 + 2x_5 \\ x_1 = -x_2 + x_5 \\ -x_2^2 + x_3^2 - 2x_2x_5 - x_5^2 \\ x_2^2 - 2x_3^2 + x_4^2 - 2x_5^2 \end{cases}$$

which consists of a square of 4 lines

$$(x_3 + (x_2 + x_5))(x_3 - (x_2 + x_5)) \cap (x_4 + x_2 + 2x_5)(x_4 - x_2 - 2x_5)$$

meeting at the points: $(1 : 2 : -1 : 0 : \pm 1 : 1), (0 : 3 : -2 : \pm 1 : 0 : 1)$.

4. When $\lambda/\mu = 1 - \sqrt{2}$,

$$F = \begin{cases} x_0 = x_2 - 2x_5 \\ x_1 = x_2 + x_5 \\ -x_2^2 + x_3^2 + 2x_2x_5 - x_5^2 \\ x_2^2 - 2x_3^2 + x_4^2 - 2x_5^2 \end{cases}$$

which consists of a square of 4 lines

$$(x_3 + (x_2 - x_5))(x_3 - (x_2 - x_5)) \cap (x_4 + x_2 - 2x_5)(x_4 - x_2 + 2x_5)$$

meeting at the points: $(0 : 3 : 2 : \pm 1 : 0 : 1), (-1 : 2 : 1 : 0 : \pm 1 : 1)$.

5. When $\lambda/\mu = -1 + \sqrt{2}$,

$$F(\lambda : \mu) = \begin{cases} x_0 = -x_2 + 2x_5 \\ x_1 = -x_2 - x_5 \\ -x_2^2 + x_3^2 + 2x_2x_5 - x_5^2 \\ x_2^2 - 2x_3^2 + x_4^2 - 2x_5^2 \end{cases}$$

which consists of a square of four lines

$$(x_3 + (x_2 - x_5))(x_3 - (x_2 - x_5)) \cap (x_4 + x_2 - 2x_5)(x_4 - x_2 + 2x_5)$$

meeting at the four points: $(0:-3:2:\pm1:0:1),(1:-2:1:0:\pm1:1)$.

6. When $\lambda/\mu = -1-\sqrt{2}$,

$$F(\lambda:\mu) = \begin{cases} x_0 = -x_2 - 2x_5 \\ x_1 = -x_2 + x_5 \\ -x_2^2 + x_3^2 - 2x_2x_5 - x_5^2 \\ x_2^2 - 2x_3^2 + x_4^2 - 2x_5^2 \end{cases}$$

which consists of a square of four lines

$$(x_3 + x_2 + x_5)(x_3 - (x_2 + x_5)) \cap (x_4 + x_2 + 2x_5)(x_4 - x_2 - 2x_5)$$

meeting at the four points: $(-1:2:-1:0:\pm1:1),(0:3:-2:\pm1:0:1)$.

7. When $\lambda/\mu = \sqrt{6}$.

$$F(\lambda:\mu) = \begin{cases} x_0 = \frac{5\sqrt{6}}{12}x_2 + \frac{7\sqrt{12}}{12}x_5 \\ x_1 = -\frac{7\sqrt{12}}{24}x_2 + \frac{5\sqrt{6}}{12}x_5 \\ \frac{-47}{48}x_2^2 + x_3^2 - \frac{35}{24}\sqrt{2}x_2x_5 - \frac{23}{24}x_5^2 \\ x_2^2 - 2x_3^2 + x_4^2 - 2x_5^2 \end{cases}$$

is singular at the one point $(-\sqrt{6}\sqrt{2}:\sqrt{6}:-\sqrt{2}:0:0:1)$, so the fibre consists of an irreducible quartic curve with a node.

8. When $\lambda/\mu = -\sqrt{6}$.

$$F(\lambda:\mu) = \begin{cases} x_0 = -\frac{5\sqrt{6}}{12}x_2 - \frac{7\sqrt{12}}{12}x_5 \\ x_1 = \frac{7\sqrt{12}}{24}x_2 - \frac{5\sqrt{6}}{12}x_5 \\ \frac{-47}{48}x_2^2 + x_3^2 - \frac{35}{24}\sqrt{2}x_2x_5 - \frac{23}{24}x_5^2 \\ x_2^2 - 2x_3^2 + x_4^2 - 2x_5^2 \end{cases}$$

is singular at the one point $(-\sqrt{6}\sqrt{2}:-\sqrt{6}:-\sqrt{2}:0:0:1)$, so the fibre consists of an irreducible quartic curve with a node.

9. When $\lambda/\mu = \sqrt{6}^{-1}$.

$$F(\lambda:\mu) = \begin{cases} x_0 = -\frac{5\sqrt{6}}{12}x_2 + \frac{7\sqrt{12}}{12}x_5 \\ x_1 = -\frac{7\sqrt{12}}{24}x_2 - \frac{5\sqrt{6}}{12}x_5 \\ -\frac{47}{48}x_2^2 + x_3^2 + \frac{35\sqrt{2}}{24}x_2x_5 - \frac{23}{24}x_5^2 \\ x_2^2 - 2x_3^2 + x_4^2 - 2x_5^2 \end{cases}$$

is singular at the one point $(\frac{\sqrt{12}}{6} : -\sqrt{6} : \sqrt{2} : 0 : 0 : 1)$, so the fibre consists of an irreducible quartic curve with a node.

10. When $\lambda/\mu = -\sqrt{6}^{-1}$

$$F(\lambda:\mu) = \begin{cases} x_0 = \frac{5\sqrt{6}}{12}x_2 - \frac{7\sqrt{12}}{12}x_5 \\ x_1 = \frac{7\sqrt{12}}{24}x_2 + \frac{5\sqrt{6}}{12}x_5 \\ -\frac{47}{48}x_2^2 + x_3^2 + \frac{35\sqrt{2}}{24}x_2x_5 - \frac{23}{24}x_5^2 \\ x_2^2 - 2x_3^2 + x_4^2 - 2x_5^2 \end{cases}$$

is singular at the one point $(\frac{-\sqrt{12}}{6} : \sqrt{6} : \sqrt{2} : 0 : 0 : 1)$, so the fibre consists of an irreducible quartic curve with a node.

In summary, we see that there are 10 fibres contributing 4 squares, 4 nodes and 2 pairs of conics, and so we see here 16 of the 32 lines of X. The other 16 appear as sections; for example the line $x_1 - x_0 - x_5, x_2 - x_0 - 2x_5, x_3 - x_0 - 3x_5, x_4 - x_0 - 4x_5$ is mapped to the section corresponding to the line $s = (\sqrt{2}-1)t, (\sqrt{2}-1)v = u$ (from the other pairing of $\pi_0(Q_1)$).

4.5 Semi-Stable Fibres

Now that we have the full table, we will begin to make some simplifying assumptions. From the Proposition4.3.4 in the last section, we saw that singular fibres occur precisely when the discriminant of the j-map vanishes and so we see that singular fibres are isolated and we can proceed locally.
With regard to investigating Conjecture 1.2.1, we will be looking for as many lines as possible with limitation that the sum of all the Euler numbers of the singular fibres is at most 24.
Let $\pi : X \to \mathbb{P}^1$ be an elliptic fibration. We say this fibration is *semistable* if each singular fibre is of type I_n in the Kodaira notation.

From checking the table we see that we get lines with the greatest "efficiency" on the Euler number in the semistable fibres. From the duality between the fibres and sections of Construction 4.2.1, when assuming maximality, we first choose to maximise the number of lines that appear in fibres, and study the possibilities for the lines in sections/dual-fibres. That is, if the discriminant vanishes with order 4, we might as well assume that the corresponding singular fibre contains as many lines as possible, which in this case is I_4.
This leads us to the simplifying assumption:

Assumption 4.5.1. The elliptic fibration on X will be *semistable*; it will only contain singular fibres of type $I_1, I_2a, I_2b, I_3, I_4$. (In the notation of Table 4.1).

Note that we allow the presence of I_1 and I_2b fibres, despite containing no lines as fibres.
Note that from [MP89], an elliptic fibration is stable if and only if the j-map has degree exactly 24. From Lemma 4.3.3, this is the general case.
It follows from Lemma 4.3.4 that we can classify all possible semistable elliptic K3 surfaces by the 1575 partitions of 24.
Returning to the case where X is an intersection of three quadrics in $\mathbb{P}^5$, we have seen from Lemma 4.4.1 that the greatest local contribution to the Euler characteristic comes from an I_4 fibre, which has an Euler number itself of 4. Thus, we are only interested in the 169 partitions of 24 where each element of the partition is an integer $1 \leq n \leq 4$.
In fact, we are only interested in these partitions that yield many lines; using Lemma 4.2.1 we see that in the semistable case:

$$e(F) = \begin{cases} 4 \Rightarrow \text{ 4 lines on } F. \\ 3 \Rightarrow \text{ 2 lines on } F. \\ 2 \Rightarrow \text{ 1 or 0 lines on } F\text{, corresponding to } I_2a \text{ or } I_2b \text{ accordingly.} \\ 1 \Rightarrow \text{ no lines in F.} \end{cases}$$

In order to address Conjecture 1.2.1, we will focus only on those surfaces which can have more than 32 lines total. The duality between the vertical and horizontal lines means we can always assume (by changing which elliptic fibration we take from Construction 4.2.1 if necessary) that there are more lines appearing in the fibres than the sections. As a result we only need to look at those cases where there are 17 or more lines appearing within singular fibres. This leaves us with the following 46 semistable cases, which we list below. This list can be created using the magma code written in section A

The five cases with strictly more than 20 lines occurring as fibres are:

$$\begin{array}{rl} [4,4,4,4,4,4] & \text{24 lines} \\ [4,4,4,4,4,2a,2a] & \text{22 lines} \\ [4,4,4,4,4,3,1] & \text{22 lines} \\ [4,4,4,4,4,2a,1,1] & \text{21 lines} \\ [4,4,4,4,4,2a,2b] & \text{21 lines} \end{array}$$

The six cases with exactly 20 lines occurring as fibres are:

$$\begin{array}{r} [4,4,4,4,4,2b,2b] \\ [4,4,4,4,4,2b,1,1] \\ [4,4,4,4,3,3,1,1] \\ [4,4,4,4,3,2a,2a,1] \\ [4,4,4,4,2a,2a,2a,2a] \\ [4,4,4,4,4,1,1,1,1] \end{array}$$

The six cases with exactly 19 lines occurring as fibres are:

$$\begin{array}{r} [4,4,4,4,3,2a,2b,1] \\ [4,4,4,4,2a,2a,2a,2b] \\ [4,4,4,3,3,3,2a,1] \\ [4,4,4,3,3,2a,2a,2a] \\ [4,4,4,4,3,2a,1,1,1] \\ [4,4,4,4,2a,2a,2a,1,1] \end{array}$$

The thirteen cases with exactly 18 lines occurring as fibres are:

$$[4,4,4,4,3,2b,2b,1] \qquad [4,4,4,4,2a,2a,2b,2b]$$
$$[4,4,4,3,3,3,2b,1] \qquad [4,4,4,3,3,2a,2a,2b]$$
$$[4,4,4,4,3,2b,1,1,1] \qquad [4,4,4,4,2a,2a,2b,1,1]$$
$$[4,4,4,4,2a,2a,1,1,1,1] \qquad [4,4,4,4,3,1,1,1,1,1]$$
$$[4,4,4,3,3,3,1,1,1] \qquad [4,4,4,3,3,2a,2a,1,1]$$
$$[4,4,4,3,2a,2a,2a,2a,1] \qquad [4,4,4,2a,2a,2a,2a,2a,2a]$$
$$[4,4,3,3,3,3,2a,2a]$$

The remaining sixteen cases with exactly 17 lines occurring as fibres are:

$$[4,4,4,4,2a,2b,2b,2b] \qquad [4,4,4,3,3,2a,2b,2b]$$
$$[4,4,4,4,2a,2b,2b,1,1] \qquad [4,4,4,4,2a,2b,1,1,1,1]$$
$$[4,4,4,3,3,2a,2b,1,1] \qquad [4,4,4,3,2a,2a,2a,2b,1]$$
$$[4,4,4,2a,2a,2a,2a,2a,2b] \qquad [4,4,3,3,3,3,2a,2b]$$
$$[4,4,4,4,2a,1,1,1,1,1,1] \qquad [4,4,4,3,3,2a,1,1,1,1]$$
$$[4,4,4,3,2a,2a,2a,1,1,1] \qquad [4,4,4,2a,2a,2a,2a,2a,1,1]$$
$$[4,4,3,3,3,3,2a,1,1] \qquad [4,4,3,3,3,2a,2a,2a,1]$$
$$[4,4,3,3,2a,2a,2a,2a,2a] \qquad [4,3,3,3,3,3,3,2a]$$

In particular, there can be no more than 24 lines appearing as fibres. Using the duality between the fibres and sections in Construction 4.2.1, this yields:

Proposition 4.5.2. Let X be a smooth K3 surface defined as an intersection of three quadrics in $\mathbb{P}^5$, whose ideal contains a quadric of rank 4. Then X contains no more than 48 lines.

4.6 Shioda-Tate Formula

Note that if $\pi : X \to \mathbb{P}^1$ is an elliptic fibration, this is equivalent to considering X as an elliptic curve E over the base function field $k(\mathbb{P}^1) = k(t)$. $E(k(t))$ has an abelian group structure of $k(t)$-rational points, which is finitely generated by Theorem 2.3.8. From the correspondence discussed in Section 2.3.1 the sections of $\pi : X \to \mathbb{P}^1$ form a finitely generated abelian group of irreducible curves on X.

Definition 4.6.1. Denote by Φ the abelian group of sections, described above. We call it the *Mordell-Weil* group of sections.

Remark 4.6.2. Note that it is possible that a surface X can contain only vertical $\mathbb{P}^1$s. In these cases we conclude that X can contain at most 24 lines, and X is not a counterexample to Conjecture 1.2.1. We therefore continue with the assumption that the group Φ contains at least one section which is a line.

Any singular fibres in X correspond to sublattices within the Néron-Severi group $\mathrm{NS}(X)$. In particular, recall that an I_n fibre (with $n \geq 1$) corresponds to a copy of the sublattice A_{n-1} which is a rank $n-1$ sublattice with Gram Matrix of the form:

$$\begin{pmatrix} -2 & 1 & 0 & 0 & 1 \\ 1 & -2 & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & -2 & 1 \\ 1 & 0 & 0 & 1 & -2 \end{pmatrix}$$

Also note that for any section S, it is birational to $\mathbb{P}^1$, and hence from Proposition 2.1.9, we have $S^2 = -2$.

Definition 4.6.3. Let $\Sigma := \{\sigma \in \mathrm{NS}(X) | \sigma \cdot F = 1, \sigma^2 = -2\}$ be the set of *numerical sections*. Viewing Φ as a subset of $\mathrm{NS}(X)$, the set of numerical sections Σ contains Φ.

Let U be the sublattice of $\mathrm{NS}(X)$ generated by the zero section S_0 and the class of a smooth fibre F. Then $F^2 = 0$ (any two smooth fibres are disjoint), and $S_0 \cdot F = 1$, meaning that U is a unimodular lattice. It follows that we have a decomposition $\mathrm{NS}(X) = U \oplus U^\perp$ and we therefore have a projection map $p : \Sigma \to U^\perp$.

Lemma 4.6.4. The map p restricts to a bijection $p : \Sigma \to U^\perp$.

Proof. Take $\sigma \in \Sigma$. Then

$$\begin{aligned} p(\sigma) &= \det \begin{pmatrix} S_0 & \sigma & F \\ S_0^2 & \sigma \cdot S_0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \\ &= \sigma + (S_0^2 - \sigma \cdot S_0)F - S_0 \end{aligned}$$

Then for $\tau \in U^\perp$ define

$$p'(\tau) = \tau - \frac{1}{2}(\tau^2)F + S_0$$

and we see that

$$\begin{aligned}
p'(p(\sigma)) &= \sigma + [(S_0)^2 - (\sigma \cdot S_0)]F - S_0 - \frac{1}{2}[\sigma + [(S_0)^2 - (\sigma \cdot S_0)]F - S_0]^2 F + S_0 \\
&= \sigma + [(S_0)^2 - (\sigma \cdot S_0)]F - S_0 - \frac{1}{2}[S_0^2 - 2\sigma \cdot S_0 - 2]F + S_0 \\
&= \sigma + \left[\frac{1}{2}S_0^2 + 1\right]F = \sigma
\end{aligned}$$

and

$$\begin{aligned}
p(p'(\tau)) &= p(\tau - \frac{1}{2}(\tau^2)F + S_0) \\
&= \tau - \frac{1}{2}(\tau^2)F + S_0 + \left[(S_0)^2 - \left(\tau - \frac{1}{2}(\tau^2)F + S_0\right) \cdot S_0\right]F - S_0 \\
&= \tau - \frac{1}{2}(\tau^2)F + \left[(S_0)^2 - \left(-\frac{1}{2}(\tau^2) + S_0^2\right)\right]F = \tau - \frac{1}{2}(\tau^2)F + \left[\frac{1}{2}(\tau^2)\right]F \\
&= \tau
\end{aligned}$$

so p and p' are inverse bijections. $\square$

We also have a projection $q : \Sigma \to \Phi$: any numerical section σ may be decomposed as $\sigma = \sigma_0 + r$ where σ_0 is an irreducible section and $rF = 0$ (that is, r consists of components of fibres). Then define $q(\sigma) := \sigma_0$.
Let R be the sublattice of N generated by components of fibres not meeting S_0. R is therefore a negative definite sublattice with a natural decomposition as a direct sum of the reducible fibres. Note $R \subseteq U^\perp$, so by Lemma 4.6.4 we have a well defined map $p' : R \to \Sigma$. Then

Lemma 4.6.5. The sequence

$$0 \to p'(R) \hookrightarrow \Sigma \xrightarrow{q} \Phi \to 0$$

is exact.

Proof. Certainly $p'(R) \subseteq \ker(q)$. If $q(\sigma) = S_0$ then $\sigma = S_0 + \sum_i n_i E_i + lF$ where $E_i \in U^\perp$. As $\sigma^2 = S_0^2$ we must have $l = -\frac{1}{2}\left(\sum_i n_i E_i\right)^2$. Thus $\sigma = p'\left(\sum_i n_i E_i\right)$ by definition of p'. $\square$

Taking ranks of the lattices in Lemma 4.6.5 gives us:

Theorem 4.6.6 (Shioda-Tate Formula)**.** [Shi72][Tat66] Let X be an elliptic K3 surface and let Φ be the set of irreducible sections. Then Φ is an abelian group with rank(Φ) finite and

$$\mathrm{rank}(\mathrm{Pic}(X)) = 2 + \sum_F r(F) + \mathrm{rank}(\Phi)$$

where the sum is taken over fibres, and $r(F)$ is the rank of the sublattice of $\mathrm{NS}(X)$ corresponding to F. In particular, when X is semistable, corresponding to the partition $[n_1, \ldots, n_s]$ of 24, this formula reduces to:

$$\mathrm{rank}(\mathrm{Pic}(X)) = 26 - s + \mathrm{rank}(\Phi).$$

From the fact that the Picard rank $\mathrm{rank}(\mathrm{Pic}(X)) \leq 20$ for a K3 surface, for the fibre configuration corresponding to a partition $[n_1, \cdots, n_s]$ we get that $\mathrm{rank}(\Phi) \leq s - 6$. Table 4.3 shows what the maximal rank of Φ can be for each of the 46 cases. In particular, we note in all cases that the maximum possible rank of Φ is 5, and that the case $[4,4,4,4,4,4]$ corresponding to fibres of type I_4 has Φ as a finite group.
In order to yield bounds on the number of lines, we first discuss the relationship between the group law on Φ with the structure of the Néron-Severi Group.
In what follows, use $\overline{S}$ do describe the section in the Mordell-Weil Group Φ associated to a divisor S in the Néron-Severi Group. Moreover, for $n \in \mathbb{N}$, denote by $n\overline{S} := \underbrace{\overline{S} + \cdots + \overline{S}}_{n \text{ times}}$.
The following lemma is the key to understand how the group laws interact with each other:

Lemma 4.6.7. Let $X \to \mathbb{P}^1$ be an elliptic fibration. Let $S_0 \in \mathrm{NS}(X)$ denote the section corresponding to the zero element $O \in \Phi$. Then for any $P, Q \in \Phi$, we have

$$P + Q = R \text{ in } \Phi \quad \Leftrightarrow \quad \overline{P} + \overline{Q} - S_0 = \overline{R} + V \text{ in } \mathrm{NS}(X)$$

where V is a vertical divisor.

Proof. See [Mir89, VII.2]. □

From this relationship, we get the following result:

Proposition 4.6.8. If $nP = Q$ for $n \in \mathbb{Z}$ then $n\overline{P} - (n-1)S_0 = \overline{Q} + V$, where V is vertical.

Proof. We proceed by induction on n. For $n = 1$ the result is clear (in this case $V = 0$). Assume true for $n = k$. Then for $n = k+1$, $Q = (k+1)P = kP + P$ so $k\overline{P} - (k-1)S_0 + \overline{P} - S_0 = V' + \overline{Q}$. Hence $(k+1)\overline{P} - kS_0 = V' + \overline{Q}$ as required.

Case	Number of lines as fibres	**Maximal** rank(Φ)
$[4,4,4,4,4,4]$	24	0
$[4,4,4,4,4,2a,2a]$	22	1
$[4,4,4,4,4,3,1]$	22	1
$[4,4,4,4,4,2a,1,1]$	21	2
$[4,4,4,4,4,2a,2b]$	21	1
$[4,4,4,4,4,2b,2b]$	20	1
$[4,4,4,4,4,2b,1,1]$	20	2
$[4,4,4,4,3,3,1,1]$	20	2
$[4,4,4,4,3,2a,2a,1]$	20	2
$[4,4,4,4,2a,2a,2a,2a]$	20	2
$[4,4,4,4,4,1,1,1,1]$	20	3
$[4,4,4,4,3,2a,2b,1]$	19	2
$[4,4,4,4,2a,2a,2a,2b]$	19	2
$[4,4,4,3,3,3,2a,1]$	19	2
$[4,4,4,3,3,2a,2a,2a]$	19	2
$[4,4,4,4,3,2a,1,1,1]$	19	3
$[4,4,4,4,2a,2a,2a,1,1]$	19	3
$[4,4,4,4,3,2b,2b,1]$	18	2
$[4,4,4,4,2a,2a,2b,2b]$	18	2
$[4,4,4,3,3,3,2b,1]$	18	2
$[4,4,4,3,3,2a,2a,2b]$	18	2
$[4,4,4,4,3,2b,1,1,1]$	18	3
$[4,4,4,4,2a,2a,2b,1,1]$	18	3
$[4,4,4,4,2a,2a,1,1,1,1]$	18	4
$[4,4,4,4,3,1,1,1,1,1]$	18	4
$[4,4,4,3,3,3,1,1,1]$	18	3
$[4,4,4,3,3,2a,2a,1,1]$	18	3
$[4,4,4,3,2a,2a,2a,2a,1]$	18	3
$[4,4,4,2a,2a,2a,2a,2a,2a]$	18	3
$[4,4,3,33,3,2a,2a]$	18	2
$[4,4,4,4,2a,2b,2b,2b]$	17	2
$[4,4,4,3,3,2a,2b,2b]$	17	2
$[4,4,4,4,2a,2b,2b,1,1]$	17	3
$[4,4,4,4,2a,2b,1,1,1,1]$	17	4
$[4,4,4,3,3,2a,2b,1,1]$	17	3
$[4,4,4,3,2a,2a,2a,2b,1]$	17	3
$[4,4,4,2a,2a,2a,2a,2a,2b]$	17	3
$[4,4,3,3,3,3,2a,2b]$	17	2
$[4,4,4,4,2a,1,1,1,1,1,1]$	17	5
$[4,4,4,3,3,2a,1,1,1,1]$	17	4
$[4,4,4,3,2a,2a,2a,1,1,1]$	17	4
$[4,4,4,2a,2a,2a,2a,2a,1,1]$	17	4
$[4,4,3,3,3,3,2a,1,1]$	17	3
$[4,4,3,3,3,2a,2a,2a,1]$	17	3
$[4,4,3,3,2a,2a,2a,2a,2a]$	17	3
$[4,3,3,3,3,3,3,2a]$	17	2

Table 4.3: Maximal Ranks of Φ for each combination of singular fibres

For the negative case, if $-P = Q$ then by definition $P+Q = O$ in Φ. Hence $\overline{P}+\overline{Q}-S_0 = V + S_0$ and so $-\overline{P} - (-1-1)S_0 = \overline{Q} + (-V)$ as required. Combining this with the result in the positive case proves the proposition. □

This above result also works when working over $\mathbb{Q}$ rather than $\mathbb{Z}$. For this we will write the shorthand $\frac{p}{q}P = Q$ to mean that $pP = qQ$ in the Mordell-Weil Group Φ.

Proposition 4.6.9. If $aP = Q$ for $a \in \mathbb{Q}$ then $a\overline{P} - (a-1)S_0 = \overline{Q} + V$, where V is vertical.

Proof. We write $a = \frac{m}{n}$. By definition we have

$$mP = nQ$$

and from the previous proposition (applied twice) we have

$$m\overline{P} - (m-1)S_0 = n\overline{Q} - (n-1)S_0 + V \in \mathrm{Pic}(X) \otimes \mathbb{Q}$$

dividing both sides of this equality by n and rearranging gives

$$a\overline{P} - (a-1)S_0 = \overline{Q} + V$$

for some vertical divisor V. □

Unfortunately, the vertical divisors that appear in all three of the above lemmas make computations of intersection numbers between different sections quite difficult, since the lattice structure of the Néron-Severi Group $\mathrm{NS}(X)$ is quite mysterious. One way of avoiding having to deal with the vertical sections is by utilising the height pairing on Φ, which we shall introduce in the next section.

4.7 Height Pairing on the Mordell-Weil Group of Sections

The main reference for this section is the survey article [SS10], Chapters 6 and 11. If X is an elliptic surface of the type contained in Table 4.3, while we know very little about the structure of the lattice $\mathrm{NS}(X)$ other than the bound for the rank $\mathrm{rank}(\mathrm{Pic}(X)) \leq 20$, it does contain a large sublattice of a very specific type thanks to Theorem 4.6.6.

Let $f : X \to \mathbb{P}^1$ be an elliptic surface with zero section S_0. We will utilise the following notation in the remainder of this section:

F a general fibre

F_v the fibre $f^{-1}(v)$ above $v \in \mathbb{P}^1$

m_v the number of components in the fibre F_v

R the points of $\mathbb{P}^1$ whose fibre is reducible.

$\Theta_{v,0}$ the component of F_v met by the zero section. This component will also be called "the trivial component".

$\Theta_{v,i}$ the other components of $F_v (i = 1, \dots, m_v - 1)$

T_v the lattice generated by the fibre components in F_v not meeting the zero section: $T_v = \langle \Theta_{v,i} : 1 \leq i < m_v - 1 \rangle$.

A_v the Gram matrix of T_v with respect to the generators $\Theta_{v,i}$.

Definition 4.7.1. In the above notation we define the *trivial lattice* $T \subset \mathrm{NS}(X)$ to be the orthogonal sum

$$T = \langle S_0, F \rangle \oplus \bigoplus_{v \in R} T_v$$

We observe that as X is a K3 surface, $\langle S_0, F \rangle$ has Gram Matrix $\begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix}$. In all of the semistable cases in Table 4.3 all of the summands T_v are A_n lattices for $n \in \{1, 2, 3\}$, and so for these, T has signature $(1, 1 + \sum_{v \in R}(m_v - 1))$.

Theorem 4.7.2. The map $\Phi \to \mathrm{NS}(X)/T$ sending a point P to its section $\overline{P} \mod T$ gives an isomorphism

$$\Phi \cong \mathrm{NS}(X)/T$$

Proof. We shall exhibit the inverse of the map

$$P \to \overline{P} \mod T$$

Let E denote the generic curve and for convenience view E as an elliptic curve on the elliptic surface X. Define a homomorphism

$$\mathrm{Div}(X) \to \mathrm{Div}(E)$$

as follows: take any divisor D on X; this decomposes into horizontal and vertical parts. The horizontal part D' intersects E properly, giving a divisor $D|_E$ of degree $D'.E$ on E called the *restriction* of D to E. Note that $D|_E$ is linearly equivalent to

$0 \in \mathrm{Div}(E)$ if and only if D is linearly equivalent to a vertical divisor V on X.
By Abel's Theorem on E the divisor D thus determines a unique point $P \in E$ by the following linear equivalence of degree zero divisors

$$D|_E - (D'.E)S_0 = P - S_0$$

Write $\psi(D) = P$. We therefore have obtained a homomorphism $\psi : \mathrm{Div}(X) \to \Phi$, whose kernel is generated by vertical divisors and the zero section S_0.
In particular, ψ is therefore defined on the quotient $\mathrm{NS}(X)/T \to \Phi$, and gives the required isomorphism $\mathrm{NS}(X)/T \cong \Phi$. □

We now aim to endow the Mordell-Weil group Φ (up to torsion) with the structure of a positive-definite lattice. The intuition we have in mind from Theorem 4.7.2 is that the sections are 'complementary' to the trivial lattice T. This motivates the following definition:

Definition 4.7.3. The *essential lattice* L is the orthogonal complement of the trivial lattice T inside $\mathrm{NS}(X)$.

Since $\mathrm{NS}(X)$ is even and negative definite of rank $\mathrm{rank}(\mathrm{Pic}(X))$, and T is a sublattice of rank $2 + \sum_v (m_v - 1)$, the following lemma is immediate:

Lemma 4.7.4. The essential lattice L is even and negative definite. Furthermore,

$$\mathrm{rank}(L) = \mathrm{rank}(\mathrm{Pic}(X)) - 2 - \sum_v (m_v - 1).$$

Note that each section S does not precisely lie in the essential lattice L since S meets each fibre and some fibre components non-trivially. To deal with this, we shall use an orthogonal projection to map points of the Mordell-Weil group Φ to elements of the essential lattice.
For simplicity, we shall tensor these lattices by $\mathbb{Q}$ and work in the vectorspaces $\mathrm{NS}(X)_{\mathbb{Q}} := \mathrm{NS}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ and the corresponding vectorspaces $T_{\mathbb{Q}}, L_{\mathbb{Q}}$. As $\mathrm{NS}(X)$ is torsion free this will not affect calculations in any way, only to avoid dealing with subtleties with finite-index lattice embeddings. The intersection pairing on the lattice extends to a negative definite bilinear form on these vectorspaces which takes values in $\mathbb{Q}$.
Define the vector space homomorphism

$$\phi : \mathrm{NS}(X)_{\mathbb{Q}} \to L_{\mathbb{Q}}$$

as the orthogonal projection with respect to the subspace $T_{\mathbb{Q}}$. For an element $D \in \mathrm{NS}(X)$, the image $\phi(D)$ is uniquely determined by the properties $\phi(D) \perp T_{\mathbb{Q}}$ and $\phi(D) - D \in T_{\mathbb{Q}}$. To show the uniqueness, suppose A and B are two divisors, both orthogonal to $T_{\mathbb{Q}}$ and $A - D \in T_{\mathbb{Q}}$, $B - D \in T_{\mathbb{Q}}$. Then $0 = A.(B - D) = A.(A - D) = B.(A - D)$ so $A^2 = A.D = A.B$. If $A \neq B$ then $A.B \geq 0$ but $A^2 < 0$ (by Lemma 4.7.4), giving a contradiction. Moreover, this is given on points $P \in E$ by the formula

$$\phi(P) = \overline{P} - S_0 - (\overline{P}.S_0 + 2)F - \sum_v \begin{pmatrix} \Theta_{v,1} & \cdots & \Theta_{v,m_v-1} \end{pmatrix} A_v^{-1} \begin{pmatrix} \overline{P}.\Theta_{v,1} \\ \vdots \\ \overline{P}.\Theta_{v,m_v-1} \end{pmatrix} \tag{4.3}$$

As we see formula (4.3), the integral matrices A_v do not necessarily have determinant ± 1 and so their inverses may well not be integral. In particular, for an elliptic surface of type $[n_1, ..., n_s]$ in Table 4.3 above, $\phi(P)$ will at most need coefficients in $\frac{1}{m}\mathbb{Z}$, where $m = \mathrm{lcm}(n_1, \cdots, n_s)$.

Proposition 4.7.5. The orthogonal projection $\phi : \Phi \to L_{\mathbb{Q}}$ given by formula (4.3) is a group homomorphism with kernel $\mathrm{Tors}(\Phi)$.

Proof. Note that by Lemma 4.6.7, $\overline{P} + \overline{Q} - S_0 = \overline{P + Q} + V$ for a vertical divisor V. Now compute $\phi(P + Q) - \phi(P) - \phi(Q)$ using formula (4.3) above. After cancellation we are left with the expression

$$-V + (V.S_0)F + \sum_v \begin{pmatrix} \Theta_{v,1} & \cdots & \Theta_{v,m_v-1} \end{pmatrix} A_v^{-1} \begin{pmatrix} V.\Theta_{v,1} \\ \vdots \\ V.\Theta_{v,m_v-1} \end{pmatrix}$$

Write the vertical divisor $V = aF + \sum_v b_{v,0}\Theta_{v,0} + \cdots + b_{v,m_v-1}\Theta_{v,m_v-1}$, and we can compute the intersection numbers in the expression above.
If $F_v = I_2$, then the element in the sum corresponding to this fibre equates to $(b_{v,1} - b_{v,0})\Theta_{v,1}$. For $F_v = I_3$ the corresponding element is

$$\begin{pmatrix} \Theta_{v,1} & \Theta_{v,2} \end{pmatrix} \begin{pmatrix} \frac{-2}{3} & \frac{-1}{3} \\ \frac{-1}{3} & \frac{-2}{3} \end{pmatrix} \begin{pmatrix} b_{v,0} - 2b_{v,1} + b_{v,2} \\ b_{v,0} + b_{v,1} - 2b_{v,2} \end{pmatrix} = b_{v,1}\Theta_{v,1} + b_{v,2}\Theta_{v,2} - b_{v,0}(\Theta_{v,1} + \Theta_{v,2})$$

While for the case $F_v = I_4$ the corresponding term in the sum is

$$\begin{pmatrix}\Theta_{v,1} & \Theta_{v,2} & \Theta_{v,3}\end{pmatrix}\begin{pmatrix}\frac{-3}{4} & \frac{-1}{2} & \frac{-1}{4}\\ \frac{-1}{2} & -1 & \frac{-1}{2}\\ \frac{-3}{4} & \frac{-1}{2} & \frac{-1}{4}\end{pmatrix}\begin{pmatrix}b_{v,0}-2b_{v,1}+b_{v,2}\\ b_{v,1}-2b_{v,2}+b_{v,3}\\ b_{v,0}+b_{v,2}-2b_{v,3}\end{pmatrix}$$
$$= b_{v,1}\Theta_{v,1} + b_{v,2}\Theta_{v,2} + b_{v,3}\Theta_{v,3} - b_{v,0}(\Theta_{v,1} + \Theta_{v,2} + \Theta_{v,3})$$

and hence for any singular fibre F_v, the related term in the sum can be written in a shorthand as $V|_{F_v} - b_{v,0}F$.
Moreover, $(V.S_0)F = aF + \sum_v b_{v,0}F$; we see that all the terms in the expression cancel giving $\phi(P+Q) - \phi(P) - \phi(Q) = 0$ as required, so ϕ is a homomorphism.
Certainly we have that $\mathrm{Tors}(\Phi) \subset \ker(\phi)$, since any torsion element of the image is trivial after being tensored with $\mathbb{Q}$.
Now take a point P such that $\phi(P) = 0$. Then if P is not torsion then from the formula, $\overline{P} \in T_{\mathbb{Q}}$ is an element of the trivial lattice. From the isomorphism in Theorem 4.7.2, it follows that $P = S_0$. □

We are now in a position to endow Φ with the structure of a lattice by pulling back the lattice structure on $L_{\mathbb{Q}}$:

Definition 4.7.6. Let $\phi : \Phi \to L_{\mathbb{Q}}$ be the orthogonal projection. Define a bilinear map on Φ, called the *height-pairing*:

$$P, Q \in \Phi \Rightarrow \langle P, Q\rangle := -\phi(P)\cdot\phi(Q) \in \frac{1}{m}\mathbb{Z}$$

where m is defined as above. In the special case of $P = Q$, we denote $h(P) := \langle P, P\rangle$ to be called the *height* of the point P.

Theorem 4.7.7. The height pairing is a symmetric bilinear pairing on Φ (equivalently the generic elliptic curve E). It induces the structure of an even positive-definite lattice on the free abelian group $\frac{\Phi}{\mathrm{Tors}(\Phi)}$.

Proof. Symmetry and bilinearity follow immediately from the properties of the intersection pairing on $\mathrm{NS}(X)$ and that ϕ is a homomorphism. As L is negative-definite, we have

$$\langle P, P\rangle = 0 \Leftrightarrow \phi(P) = 0 \Leftrightarrow P \in \mathrm{Tors}(\Phi)$$

showing that the height-pairing is positive-definite. □

This lattice structure is only defined on the quotient $\frac{\Phi}{\mathrm{Tors}(\Phi)}$. For this reason, any information on sections in only valid under the following assumption:

Assumption 4.7.8. The Mordell-Weil Group of sections Φ for the surface X is torsion-free.

Since we are concerned primarily here with surfaces whose singular fibres are only I_1, I_2, I_3 and I_4, this gives the following easy result.

Corollary 4.7.9. Let $P \in \Phi$ with $\overline{P} \neq S_0$ any section. Then $h(P) \geq \frac{1}{12}$. If furthermore the elliptic surface X contains no fibres of type I_3 then $h(P) \geq \frac{1}{4}$.

Proof. This follows immediately from the fact that the height pairing is positive definite, and for an elliptic surface of type $[n_1, \cdots, n_s]$, it takes values in $\frac{1}{m}\mathbb{Z}$, where $m := \operatorname{lcm}\{n_1, \cdots, n_s\}$. Since all the cases in Table 4.3 contain at least one fibre of type I_4 this means $m \in \{4, 12\}$, depending whether the surface also contains at least one fibre of type I_3. □

The height pairing is remarkably easy to compute, and can be given as an exact formula as follows in terms of intersection numbers:

Proposition 4.7.10. Let $P, Q \in \Phi$, and write $\overline{P}, \overline{Q}$ for the corresponding sections in $\mathrm{NS}(X)$. The height-pairing is then given by the formula:

$$\langle P, Q\rangle = 2 + \overline{P}.S_0 + \overline{Q}.S_0 - \overline{P}.\overline{Q} - \sum_v \mathrm{contr}_v(P, Q).$$

In the special case when $P = Q$ this formula reduces to

$$h(P) = 4 + 2\overline{P}.S_0 - \sum_v \mathrm{contr}_v(P).$$

Here the correction terms $\mathrm{contr}_v(P, Q)$ depend only on the fibre components of F_v met by the sections. Specifically, let P meet $\Theta_{v,i}$ and Q meet $\Theta_{v,j}$ (with S_0 meeting $\Theta_{v,0}$ as usual). Then the local correction term at v is

$$\mathrm{contr}_v(P, Q) := \begin{cases} 0, & \text{if } ij = 0 \\ -(A_v^{-1})_{i,j}, & \text{if } ij \neq 0 \end{cases}$$

The correction term for a single section P is defined by $\mathrm{contr}_v(P, P)$. As the only singular fibres are of type I_1, I_2, I_3, I_4, it follows that in every case $\mathrm{contr}_v(P, Q) \geq 0$.

Proof. The formula follows from direct computation. Let us concentrate first on the terms involving the sum. Since $\overline{P}$ meets $\Theta_{v,i}$ and no other $\Theta_{v,k}$, while $\overline{Q}$ meets $\Theta_{v,j}$

the sum in the expression for $\phi(P)$ is

$$\sum_v \sum_{k=1}^{m_v-1} \Theta_{v,k}(A_v^{-1})_{k,i}$$

if $ij \neq 0$.
Since $F^2 = F.\Theta_{v,k} = 0$ for any choice of v, k, F, $S_0.F = \overline{P}.F = \overline{Q}.F = 1$ and the conditions of $\overline{P}$ and $\overline{Q}$ meeting irreducible fibre components with A_v^{-1} symmetric, we have, for $ij \neq 0$:

$$-\langle P, Q\rangle = \overline{P}.\overline{Q} - \overline{P}.S_0 - \overline{Q}.S_0 + S_0^2 - 2\sum_v (A_v^{-1})_{i,j} + \sum_v \sum_{k,l=1}^{m_v-1} \Theta_{v,k}.\Theta_{v,l}(A_v^{-1})_{k,i}(A_v^{-1})_{l,j} \tag{4.4}$$

By definition, $\Theta_{v,k}\cdot\Theta_{v,l} = (A_v)_{i,j}$, while usual matrix multiplication gives the formula $(AB)_{i,j} = \sum_k A_{i,k}B_{k,j}$, hence

$$\sum_{k,l=1}^{m_v-1} \Theta_{v,k}.\Theta_{v,l}(A_v^{-1})_{k,i}(A_v^{-1})_{l,j} = \sum_{l=1}^{m_v-1}(A_v A_v^{-1})_{i,l}(A_v^{-1})_{l,j} = (A_v^{-1})_{i,j}$$

and so

$$-\langle P, Q\rangle = \overline{P}.\overline{Q} - \overline{P}.S_0 - \overline{Q}.S_0 - 2 - \sum_v (A_v^{-1})_{i,j}$$

as required.
In the case that $ij = 0$ we take the values of all the sums to be 0 in formula 4.4 instead, and the required result follows immediately. $\square$

We now turn our attention to lines occurring as sections. Let X be as usual; a K3 surface formed as a smooth complete intersection of three quadrics in $\mathbb{P}^5$, one of whom is a quadric of rank 4 giving rise to the pair of dual elliptic fibrations as usual. If there are no lines on X that appear as sections, then the work here is done. Otherwise, we will take one of them and assume that this line is the zero section S_0. Now, the explicit formula easily gives results when the section $\overline{P}$ is a line on X:

Corollary 4.7.11. If L is any line on X appearing as a section, then $h(L) \leq 6$. Moreover, $h(L) \leq 4$ for all but at most two of these lines.

Proof. Since we assumed that both L and S_0 are lines, $L.S_0$ is at most 1. As each local contribution $\text{contr}_v(P) \geq 0$, it follows from the explicit formula that $h(P) \leq 4 + 2\overline{P}.S_0 \leq 6$.
On the other hand, each time we have a choice of two elliptic fibrations on X,

and X is assumed to be semistable. In particular L and S_0 can be considered as sections or alternatively as irreducible components of fibres of the dual fibration. Therefore S_0 can at most meet 2 other irreducible components of fibres in the dual fibration, at most two of these can be lines. We conclude that $L.S_0 = 0$ except for at most two choices of section lines L and in these cases we obtain an upper bound of $h(P) \leq 4$. $\square$

We can use Corollaries 4.7.9 and 4.7.11 by performing a count of points satisfying these bounds. These give us crude upper bounds for the numbers of lines that can appear as sections.

In fact, this point count for lines in only needs to take into account the effective cone of points in Φ:

Lemma 4.7.12. The set $\{L \in \Phi | \overline{L} \text{ is a line}\}$ is contained in the effective cone generated by a choice of effective generators for Φ.

Proof. Take a finite generating set for Φ. For each generator P, if $\overline{P}$ is effective then $\overline{-P}$ cannot be effective. Since Φ is torsion-free, by Proposition 4.7.5 we get an embedding of Φ into $\frac{1}{m}L$. Then $\overline{-P}.H = -\overline{P}.H < 0$, where H is the hyperplane class of X. As the effective sections form a cone within the essential lattice, it follows that any point $Q \in \Phi$ can correspond to an effective curve if it is a positive sum of effective generators. $\square$

This works nicely for the case when $\text{rank}(\Phi) = 1$:

Proposition 4.7.13. When Φ has rank 1 and is torsion-free, the elliptic surface X can contain at most 9 lines as sections.

Proof. Let P be a generator of Φ. Then $h(P) \geq \frac{1}{12}$ and so by bilinearity of the height pairing $h(nP) \geq \frac{n^2}{12}$. For nP to be a line we need $h(nP) \leq 4$, so $n^2 \leq 48$ and hence $|n| \leq 6$ since n is an integer.

From Lemma 4.7.12, we may assume that the generator P gives an effective section (otherwise replace P by $-P$) and hence we assume that $n \geq 0$. Therefore there are at most six non-trivial points $P \in \Phi$ that can be lines. These together with the zero section and the other possible two lines (see Corollary 4.7.11) gives an upper bound of $6 + 2 + 1$ lines. $\square$

Corollary 4.7.14. If X has more than 32 lines and Φ is torsion free then $\text{rank}(\Phi) > 1$.

Proof. Consider Table 4.3. In order to maximise lines on X under the restriction that Φ is torsion free and $\text{rank}(\Phi) \le 1$, the elliptic surface X is of type $[4,4,4,4,4,2a,2a]$ or $[4,4,4,4,4,3,1]$ and by Proposition 4.7.13, X contains at most $9 + 22 = 31$ lines. $\square$

We can then do the same for the other possible ranks. Let $\{P_1, \cdots, P_r\}$ be a generating set for Φ.
Note that by bilinearity $h(a_1P_1 + a_2P_2) = a_1^2 + 2a_1a_2 \langle P_1, P_2\rangle + a_2^2 h(P_2)$. In order to maximise the results of the point count, we assume that $h(P_1) = h(P_2) = \frac{1}{m}$ is hence minimal. Since the height-pairing is positive definite and $\langle P_1, P_2\rangle \in \frac{1}{m}\mathbb{Z}$, it follows that $\langle P_1, P_2\rangle \ge 0$. Indeed, the point-count is therefore bounded above by the case that the height-pairing of $\langle P_1, P_2\rangle = 0$.
The same argument applies to all other ranks; we can therefore attain maximal possible bounds when the Gram matrix of Φ with respect to the generating set $\{P_1, \cdots, P_r\}$ is $\frac{1}{m}I_r$.
Subject to these conditions, the point count can be performed.

Proposition 4.7.15. The number of points height less than 4 for various cases is given in the table below. In the same table, we use these for crude estimates on the number of lines.

Case	$\#\{P, h(P) \le 4\}$	**Number of lines**
$r = 2$, no I_3 fibres	17	$17 + 2 + 21 = 40$
$r = 2$, with an I_3 fibre	43	$43 + 2 + 20 = 65$
$r = 3$, no I_3 fibres	54	$54 + 2 + 20 = 76$
$r = 3$, with an I_3 fibre	230	$230 + 2 + 19 = 251$
$r = 4$, no I_3 fibres	165	$165 + 2 + 18 = 185$
$r = 4$, with an I_3 fibre	1110	$1110 + 2 + 18 = 1130$
$r = 5$, (only case has no I_3 fibres)	482	$482 + 2 + 17 = 501$

Table 4.4: Upper bounds on the number of points of bounded height in the remaining cases.

We notice that all the results for lines are bigger than 32, and all results for lines other than the first are even bigger than 48 which was attained by Proposition 4.5.2 and so further analysis is needed.

4.8 Further Ideas

Is it possible to find the generating set for the lattice $\frac{1}{m}I_n$ above? If so, we can try and establish the intersection data for these generators. Here, we present a computational method that in principle could work for all cases, and presents necessary configurations of lines to achieve these bounds in the previous table.

Example 4.8.1. Let us look in detail at the case for the surface $[4,4,4,4,4,2a,1,1]$ (21 lines as fibres) and assume that $\Phi \cong \mathbb{Z}^2$ as an abstract group, generated by $\{P, Q\}$. We may assume that $\overline{P}$ and $\overline{Q}$ are lines. Note that, as is always the case when using the height-pairing, we have to assume that Φ is torsion free, and we get no information about torsion sections.
From Proposition 4.7.10, the explicit formula tells us that the height of P is computed from knowing $\overline{P}.S_0$ and the local contributions of the section at each singular fibre. In order to maximise the values of these point counts, we wish to minimise therefore the value of $h(P)$, and so we try to maximise the sum $\sum_v \mathrm{contr}_v(P)$ for each case of when $\overline{P}.S_0 \in \{0,1\}$. The value of $\sum_v \mathrm{contr}_v(P)$ can be calculated for each case depending on which type of irreducible fibre components the section $\overline{P}$ meets.
In this example there are 42 options (up to permutations) for how a given section intersects the singular fibres for each case of $\overline{P}.S_0 = 0$ and the case $\overline{P}.S_0 = 1$, giving 84 cases in total (up to permutations of the I_4 fibres). The candidates for minimal height are in the table below, along with how which components of each singular fibre is met. For instance, the first case meets 3 of the I_4 fibres at their trivial component $\Theta_{4,0}$, meets one at $\Theta_{4,1}$ and the remaining one at $\Theta_{4,2}$ which is disjoint from the trivial component in that fibre. This case also meets the I_2 fibre at the trivial component $\Theta_{2,0}$.
In the table below each row corresponds to a section and contains a sequence of numbers. The number in column indexed by $\#\Theta_{n,i}$ indicated how many irreducible components of type $\Theta_{v,i}$ the section meets, when F_v is of type I_n. Next denotes the height of the section when the value of $L_i.S_0$ is as stated in the final column.

This data allows us to compute the value of $\mathrm{contr}_v(P,Q)$; we are looking for the cases where this vanishes. The idea is to go through each case and see what configurations are forced in order for the height lattice to take the required shape of $\frac{1}{4}I_2$.

For example, assume L_1 and L_2 are the generators. In order for $\langle L_1, L_2\rangle$ to be the required value of 0, we need $\overline{L_1 L_2} = 0$, since $\overline{L_1}.S_0 = \overline{L_2}.S_0 = 1$ and having a triangle $\{L_1, L_2, S_0\}$ is forbidden by Lemma 1.1.4. Hence we require $\sum_v \mathrm{contr}_v(L_1, L_2) = 4$. It is clear that for the two I_1 fibres and also the I_2 fibre that the local contributions here are all zero since both L_1 and L_2 meet the trivial components of each fibre. We only need to consider the five I_4 fibres. From the table, L_1 intersects one of these in component $\Theta_{4,1}$, and another in $\Theta_{4,2}$ and the

Name	$\#\Theta_{4,0}$	$\#\Theta_{4,1}$	$\#\Theta_{4,2}$	$\#\Theta_{4,3}$	$\#\Theta_{2,0}$	$\#\Theta_{2,1}$	$\#\Theta_{1,0}$	$h(P)$	$\overline{L_i}.S_0$
L_1	3	1	1	0	1	0	2	$\frac{1}{4}$	1
L_2	3	0	1	1	1	0	2	$\frac{1}{4}$	1
L_3	1	3	1	0	0	1	2	$\frac{1}{4}$	0
L_4	1	2	1	1	0	1	2	$\frac{1}{4}$	0
L_5	1	1	1	2	0	1	2	$\frac{1}{4}$	0
L_6	1	0	1	3	0	1	2	$\frac{1}{4}$	0
L_7	1	1	3	0	1	0	2	$\frac{1}{4}$	0
L_8	0	5	0	0	1	0	2	$\frac{1}{4}$	0
L_9	0	4	0	1	1	0	2	$\frac{1}{4}$	0
L_{10}	0	3	0	2	1	0	2	$\frac{1}{4}$	0
L_{11}	0	2	0	3	1	0	2	$\frac{1}{4}$	0
L_{12}	0	1	0	4	1	0	2	$\frac{1}{4}$	0
L_{13}	0	0	0	5	1	0	2	$\frac{1}{4}$	0

Table 4.5: The thirteen candidates for lines of minimal height with their intersections of singular fibres on the surface $[4,4,4,4,4,2,1,1]$.

remaining three in the trivial component each. L_2 is similar, except it meets an $\Theta_{4,3}$ component instead of $\Theta_{4,1}$. Note that we have not yet specified exactly which I_4 fibres are met in what way, only the number of types of component met. We need to enumerate and go through all the possibilities for which I_4 fibres are met my each section and see if the sum of local contributions can ever reach 4.

For the case of $\{L_1, L_2\}$, we quickly see that the target of $\sum_v \mathrm{contr}_v(L_1, L_2) = 4$ is impossible to attain; there are only two singular fibres for which the local contribution at those fibres is non-zero and this can only occur when neither section L_1 or L_2 meets the trivial component; there are two cases, either there is an I_4 singular fibre where the two sections both meet at component $\Theta_{4,1}$ (giving $\sum_v \mathrm{contr}_v(L_1, L_2) = \frac{3}{4} + \frac{1}{2}$) or instead there is a singular fibre where the two sections L_1, L_2 meet at components $\Theta_{4,2}, \Theta_{4,1}$ (giving $\sum_v \mathrm{contr}_v(L_1, L_2) = \frac{1}{4} + \frac{1}{2}$) respectively. Either way, the height-pairing $\langle L_1, L_2 \rangle > 0$.

We can repeat this argument for each choice of generators $\{L_i, L_j\}$ with $1 \le i < j \le 13$. In each argument, we see that L_1 cannot be a generator since the maximum possible $\sum_v \mathrm{contr}_v(L_1, L_j)$ can be $1 + \frac{3}{4}$, attained only if L_j meets in at least one $\Theta_{4,2}$ and a $\Theta_{4,1}$. But then in whenever $j > 2$, we have $\langle L_1, L_j \rangle = 2 + 1 + 0 - \overline{L_1} \cdot \overline{L_j} - (1 + \frac{3}{4}) = 1 + \frac{3}{4} - \overline{L_1} \cdot \overline{L_j} > 0$, since L_1 and L_j are assumed to be lines.

The same argument shows that $\{L_2, L_j\}$, $j > 2$ cannot be a generating set for the lattice $\frac{1}{4}I_2$.

Let us look at the case $\{L_3, L_4\}$. Here we need $\sum_v \mathrm{contr}_v(L_3, L_4) + \overline{L_3}.\overline{L_4} = 2$. As both L_3 and L_4 meet the I_2 fibre non-trivially, the contribution from the I_4 fibres must total $\frac{3}{2}$ or $\frac{1}{2}$. We can go through and check each choice of pairing between L_3 and L_4.

It turns out that $\overline{L_3}.\overline{L_4} = 0$. If L_3 meets the five I_4 fibres at $\Theta_0, \Theta_1, \Theta_1, \Theta_1, \Theta_2$ respectively then we can assume L_4 meets these fibres in one of the four configurations (and 10 other permutations obtained by permuting the middle three terms).

$$[\Theta_2, \Theta_0, \Theta_1, \Theta_3, \Theta_1], [\Theta_1, \Theta_1, \Theta_2, \Theta_3, \Theta_0]$$

Now we use Lemma 7.4 on page 33 of the survey article:

Lemma 4.8.2. [SS10](Lemma 7.4) Consider the map $\psi : E(K) \to \prod_{v \in R} G(F_v)$, taking a section to the respective fibre components that it meets. Then ψ is a group homomorphism.

In particular, if L_4 is forced to have one of these two configurations above, then this determines the configuration for $L_3 + L_4$ and hence we can compute the height. Since we are assuming L_3 and L_4 are generators for the height lattice $\frac{1}{4}I_2$, we require that $h(L_3 + L_4) = \frac{1}{2}$. On the other hand, from the lemma for each of the two cases, $L_3 + L_4$ is forced to intersect the I_4 fibres in the following configurations respectively:

$$[\Theta_2, \Theta_1, \Theta_2, \Theta_0, \Theta_3], [\Theta_1, \Theta_2, \Theta_3, \Theta_0, \Theta_2]$$

while from Table 4.5, $L_3 + L_4$ meets the I_2 fibre trivially.
It follows that the sums of local contributions $\sum_v \mathrm{contr}_v(L_3 + L_4)$ are $1 + 1 + \frac{3}{4} + \frac{3}{4} = \frac{14}{4}$ in both cases.
Then the height $h(L_3 + L_4) = 4 + 2\overline{L_3 + L_4}.S_0 - \frac{14}{4} = \frac{1}{2}$ only when $\overline{L_3 + L_4}.S_0 = 0$.
The same argument works for $\{L_3, L_5\}$: this time there are four cases; the only cases when $h(L_3 + L_5) = \frac{1}{2}$ is when L_5 intersects the I_4 fibres in the configuration

$$[\Theta_2, \Theta_0, \Theta_1, \Theta_3, \Theta_3], [\Theta_3, \Theta_1, \Theta_2, \Theta_3, \Theta_0]$$

and $\overline{L_3 + L_5}.S_0 = 0$.
For $\{L_3, L_6\}$ the height-pairing of $\langle L_3, L_6 \rangle$ can only be zero if L_6 meets the fibres one of the configurations

$$[\Theta_3, \Theta_0, \Theta_3, \Theta_3, \Theta_2], [\Theta_0, \Theta_2, \Theta_3, \Theta_3, \Theta_3]$$

the corresponding heights $h(L_3 + L_6)$ are both $4 + 2\overline{L_3 + L_6}.S_0 - \frac{13}{4} \neq \frac{1}{2}$.
For $\{L_3, L_7\}$ there are two possibilities for L_7, but each gives $\sum_v (L_3 + L_7) = 3$ hence $h(L_3 + L_7) \neq \frac{1}{2}$.
For the $\{L_3, L_j\}$ with $j \geq 8$ the height-pairing always satisfies $\langle L_3, L_j \rangle \neq 0$.
By repeating the same argument for the remaining $\{L_i, L_j\}$ with $4 \leq i < j$ we see

that this can only be a generating set for $\frac{1}{4}I_2$ if $i, j \in \{4, 5, 6, 7\}$; and those cases are when $i = 4$ and L_j is:

$$L_5 \in \{[\Theta_1, \Theta_2, \Theta_3, \Theta_0, \Theta_3], [\Theta_2, \Theta_1, \Theta_3, \Theta_3, \Theta_0], \\ [\Theta_2, \Theta_0, \Theta_3, \Theta_1, \Theta_3,], [\Theta_3, \Theta_1, \Theta_3, \Theta_0, \Theta_2]\}$$

or $i = 4$ and L_j is:

$$L_6 \in \{[\Theta_3, \Theta_2, \Theta_3, \Theta_0, \Theta_3], [\Theta_2, \Theta_0, \Theta_3, \Theta_3, \Theta_3]\}$$

or $i = 4$ and L_j is:

$$L_7 \in \{[\Theta_0, \Theta_2, \Theta_2, \Theta_1, \Theta_2], [\Theta_1, \Theta_0, \Theta_2, \Theta_2, \Theta_2], [\Theta_1, \Theta_2, \Theta_2, \Theta_2, \Theta_0]\}.$$

Alternatively, $i = 5$ and j is either 6 or 7. In the case $j = 6$ we have:

$$L_6 \in \{\Theta_2, \Theta_3, \Theta_3, \Theta_0, \Theta_3], [\Theta_3, \Theta_3, \Theta_0, \Theta_2, \Theta_3]\}$$

while in the case $j = 7$ we have instead:

$$L_7 \in \{[\Theta_1, \Theta_0, \Theta_2, \Theta_2, \Theta_2], [\Theta_1, \Theta_2, \Theta_2, \Theta_0, \Theta_2], [\Theta_0, \Theta_2, \Theta_1, \Theta_2, \Theta_2]\}.$$

The final cases are when $i = 6$ and L_7 is:

$$L_7 \in \{[\Theta_1, \Theta_2, \Theta_0, \Theta_2, \Theta_2], [\Theta_0, \Theta_1, \Theta_2, \Theta_2, \Theta_2]\}$$

More similar calculations perhaps could be done to eliminate these possibilities for being generating sets.

The conclusions from these calculations can be summarised in the following theorem:

Theorem 4.8.3. Let X be an elliptic surface $[4, 4, 4, 4, 4, 2, 1, 1]$, whose Mordell-Weil Group Φ is torsion-free, and has rank 2, generated by P and Q with $\overline{P}, \overline{Q}$ lines. Then the height pairing on X gives the lattice structure $\frac{1}{4}I_2$ only if P and Q intersect the singular fibres according to the data listed in Table 4.6.

In this table, each row is a section, and contains a sequence of numbers. A number i in the column denote by the fibre fibre F_v means "this section meets $\Theta_{v,i}$, where $\Theta_{0,v}$ denotes the trivial component, i.e. the component meeting the zero section.

Fibre type:	I_4	I_4	I_4	I_4	I_4	I_2
P:	0	1	1	1	2	1
Q:	2	1	2	0	3	1
	1	2	3	0	2	1
	2	0	1	3	3	1
	3	1	2	3	0	1
P:	0	1	1	2	3	1
Q:	1	2	3	0	3	1
	2	1	3	3	0	1
	2	0	3	1	3	1
	3	1	3	0	2	1
	3	2	3	0	3	1
	2	0	3	3	3	1
	0	2	2	1	2	0
	1	0	2	2	2	0
	1	2	2	2	0	0
P:	0	1	2	3	3	1
Q:	2	3	3	0	3	1
	3	3	0	2	3	1
	1	0	2	2	2	0
	1	2	2	0	2	0
	0	2	1	2	2	0
P:	0	2	3	3	3	1
Q:	1	2	0	2	2	0
	0	1	2	2	2	0

Table 4.6: The required configurations for generators of the Mordell-Weil group when surface $[4,4,4,4,4,2,1,1]$ has 40 lines.

Corollary 4.8.4. If X is an elliptic surface of type $[4,4,4,4,4,2,1,1]$ with Mordell-Weil Group Φ torsion free then X has at most 40 lines, attaining this bound only if $\Phi \cong \mathbb{Z}^2$ generated by P, Q with $\overline{P}, \overline{Q}$ lines in configurations listed in Table 4.6.

Of course, similar tables can equally be created for the other surfaces when $\text{rank}(\Phi) > 1$. The reason I have not done this here is due to time constraints on the submission deadline for this document.

Appendix A

Code for Table 4.3

This section contains the functions I used in Magma to quickly calculate the data present in Table 4.3. Note that the full table includes separate cases to distinguish between I_2a and I_2b fibres which are described in Table 4.1, but the code below assumes any I_2 fibres are of type I_2a. In order to compute the missing cases, note that replacing an I_2a fibre with an I_2b removes a vertical line from the fibration, while leaving the combinatorial data in the partition of 24 unchanged.

```
// First we create a sequence of all partitions of 24 that contain
// only the numbers 1..4
// As the numbers in each partition from the "Partitions()" function
// are sorted from highest to lowest, we only need to check the
// first element.
S := Partitions(24);
S1 := [];
for s in S do;
if s[1] le 4 then;
Append(~S1,s);
end if;
end for;

// The function MaxLines takes s, a partition of 24 and returns the
// maximum possible number of vertical lines in the corresponding
// singular fibres.
function MaxLines(s)
Output := 0;
for i in [1..#s] do;
```

```
if s[i] eq 4 then Output := Output +4;
end if;
if (s[i] eq 3) or (s[i] eq 2) or (s[i] eq 1) then;
 Output := Output + s[i] - 1;
end if;
end for;
return Output;
end function;

// S2 consists only of the partitions in S1 that generate 17 or more
// vertical lines. This leaves us with 30 possibilities.
S2 := [];
for s in S1 do;
if MaxLines(s) gt 16 then;
Append(~S2,s);
end if;
end for;

// The function MaxRank takes s, a partition of 24 and returns the
// maximum rank of the group of sections, as given in the Shioda-
// Tate formula.
function MaxRank(s)
return #s - 6;
end function;


// The following command prints the results of Table 3.4:

for i in [1..#S2] do;
print S2[i], MaxLines(S2[i]), MaxRank(S2[i]);
end for;
```

Appendix B

Code for Example 4.8.1

This section contains the functions I used in Magma to quickly calculate all the cases in Example 4.8.1, where I determine necessary conditions for an elliptic surface $[4,4,4,4,4,2,1,1]$ to have more than 32 lines. Nothing in this code contains any advanced theory, this code could be reworked to deal with the other surfaces. This code can be used to generate the data in Table 4.6 as follows: For each $1 \leq i < j \leq 13$ we aim to search for the possibilities for the height-pairing $\langle L_i, L_j \rangle = 0$. We note that in order for this to happen, $\sum_v \mathrm{contr}_v(L_i, L_j) = 2 + \overline{L_i}.S_0 + \overline{L_j}.S_0 - \overline{L_i}.\overline{L_j}$. We can use the value of $L_i.S_0$ and $L_j.S_0$ and the assumption of $\overline{L_i}.\overline{L_j} \in \{0,1\}$ to get two target values $t1, t2$ for $\sum_v \mathrm{contr}_v(L_i, L_j)$.
After pasting in the code below, simply enter in the commands:

```
SumLocalContributionsCases(PotentialLines[i],PotentialLines[j],t1);
SumLocalContributionsCases(PotentialLines[i],PotentialLines[j],t2);
```

Magma will return for you a sequence of triples. The first value in each triple corresponds to the value of P in Table 4.6, while the second value in each triple corresponds to a choice of Q. The third value in the triple simply serves as a reminder for what value you assigned for the target value of $\sum_v \mathrm{contr}_v(L_i, L_j)$.

```
//PotentialLines contains the thirteen cases from Table 12,
// consisting of combinatorial data of how many components of each
// type are met from among the I_4 fibres.
PotentialLines := [[0,0,0,1,2],[0,0,0,2,3],[0,1,1,1,2],[0,1,1,2,3],
[0,1,2,3,3],[0,2,3,3,3],[0,1,2,2,2],[1,1,1,1,1],
[1,1,1,1,3],[1,1,1,3,3],[1,1,3,3,3],[1,3,3,3,3],
[3,3,3,3,3]];
```

```
//The function "AddSequences" simply adds two sequences term-by-term.
AddSequences := function(SeqA,SeqB);
assert(#SeqA eq #SeqB);
SeqC := [];
for i in [1..#SeqA] do;
Append(~SeqC,SeqA[i]+SeqB[i]);
end for;
return SeqC;
end function;

//The function ListPermutations takes a sequence of integers (which
// may contain repeated values) and lists all permutations of that
// given sequence. This list is filtered, with repeated permutations
// deleted.
ListPermutations := function(Input);
//The sequence of integers is copied, then we modify it by adding on
// rational numbers between 0 and 1 to ensure that in what follows,
// the sequence has no repeated terms. Note that this action is
// temporary and will be undone by flooring all these terms,
// returning to the original values.
//SeqB contains the distinct list of rational numbers between 0 and
// 1.
SeqB := [Rationals()|];
for i in [1..#Input] do;
Append(~SeqB,i/(2*(#Input)));
end for;
//Make Unique adds this list of rationals to the integer-valued
// input to guarantee all terms are distinct.
MakeUnique := AddSequences(Input ,SeqB);
//Once all terms in the sequence are unique, we typecase the
// sequence to a set. We then call the "Permutations" command which
// lists all permutations of that set, typecasting all these outputs
// back to sequences. As all entries are distinct by construction,
// this typecasting preserves all the data.
Perms := Setseq(Permutations(Seqset(MakeUnique)));
//Now we have a list of permutations we floor all of these rational
// valued sequences, returning them to the integer values we started
```

```
// with.
for i in [1..#Perms] do;
for j in [1..#Input] do;
Perms[i][j] := Floor(Perms[i][j]);
end for;
end for;
//Perms now contains many repeated permutations. We delete these by
// typecasting the sequence of sequences into a set of sequences and
// back again, automatically deleting repeated values in "Perms".
return Setseq(Seqset(Perms));
end function;

//The function "MatchVectorsI4" deal with local contributions. You
// input two vectors each corresponding to the combinatorial data of
// which irreducible component of the singular I4 fibre is met. The
// function then returns the sum of the local contributions
// \sum_v contr_v(. , .) for the computation of the height pairing.
MatchVectorsI4 := function(Vect1,Vect2);
//I4 is simply the matrix of local correction terms for the I4
// singular fibre. See the definition of the local contribution at a
// fibre contr_v(P,Q) in the proof of a Proposition.
I4 := Matrix(Rationals(),4,4,[
[0,0,0,0],
[0,3/4,1/2,1/4],
[0,1/2,1,1/2],
[0,1/4,1/2,3/4]]);
assert(#Vect1 eq #Vect2);
sum := 0;
for i in [1..#Vect1] do;
sum := sum + I4[Integers()!(Vect1[i]+1)][Integers()!(Vect2[i]+1)];
end for;
return sum;
end function;

//The function "SumLocalContributionsCases" takes in two sets of
// combinatorial data for the intersections of two sections with the
// I4 fibres. We wish to calculate the sum of local contributions
```

```
// between these two sections in order to get a value of the height-
// pairing. In order to do this, we need to range over all
// permutations for this combinatorial data (we may assume one is
// fixed and simply vary the other one). Then, given a target value
// for the sum of local contributions, we search for a configuration
// that returns it. Any successful configurations are stored.
SumLocalContributionsCases := function(Vect1,Vect2,Target);
Output := [];
//We assume Vect1 is fixed and go through all permutations of Vect2.
Perms := ListPermutations(Vect2);
for V in Perms do;
//We calculate the sum of local contributions between each
// permutation V of Vect2 and the fixed value Vect1.
Contr := MatchVectorsI4(Vect1,V);
if Contr eq (Target) then;
//Any pair which yields the desired value is stored...
Append(~Output,[Vect1,V,[Contr]]);
end if;
end for;
//...and returned.
return Output;
end function;
```

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