Semi-Analytical Solutions for Dynamic Portfolio Choice in Jump-Diffusion Models and the Optimal Bond-Stock Mix

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Abstract

This paper studies the optimal portfolio selection problem in jump-diffusion models where an investor has a HARA utility function, and there are potentially a large number of assets and state variables. More specifically, we incorporate jumps into both stock returns and state variables, and then derive semi-analytical solutions for the optimal portfolio policy up to solving a set of ordinary differential equations to greatly facilitate economic insights and empirical applications of jump-diffusion models. To examine the effect of jump risk on investors' behavior, we apply our results to the bond-stock mix problem and particularly revisit the bond/stock ratio puzzle in jump-diffusion models. Our results cast new light on this puzzle that unlike pure-diffusion models, it cannot be rationalized by the hedging demand assumption due to the presence of jumps in stock returns.

JEL Classification: G11

Keywords: Finance, optimal portfolio selection, jump-diffusion models, HARA utility functions, bond-stock mix

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1 Introduction

As prompted by the seminal work of Merton (1969), there is a large literature on the dynamic portfolio choice problem that has typically been studied in continuous-time models primarily due to their analytical tractability. There are two popular methods that are widely employed to solve this problem. The first one is the HJB-based approach proposed by Merton (1969), and the other is the martingale approach advanced by Karatzas, Lehoczky and Shreve (1987) and Cox and Huang (1989). In both approaches, the investor's utility function plays a fundamental role in seeking the optimal portfolio policy.[1] Unfortunately, it is well known that semi-analytical solutions to the dynamic portfolio choice problem are generally unavailable, although they are vitally important to facilitate economic insights and empirical applications. In this paper, we solve the optimal asset allocation problem in closed form for multi-asset jump-diffusion models in the way that the solutions provide a new instrument to analyze the behavior of investors with general HARA preferences towards distinct risk factors.

In a growing literature, numerous efforts have been made to solve the portfolio choice problem in closed form. Specifically, Bajeux-Besnainou and Portait (1998) extend the static setup in Markowitz (1952) to a much more challenging dynamic version and explicitly solve the dynamic mean-variance problem in a complete pure-diffusion model. Recently, by using the martingale approach, Lioui and Poncet (2016) provide closed-form solutions to the dynamic mean-variance problem in a complete affine diffusion model. As remarked by the authors, the dynamic mean-variance model in Section 2.3 of Lioui and Poncet (2016) may result in time-inconsistent portfolio strategies, showing that the investor may find it optimal to deviate from her initial policy. In contrast, Basak and Chabakauri (2010)[2] explicitly solve the time-consistent dynamic mean-variance policy based on a recursive representation. In a continuous-time mean variance model with constraints on portfolio policy, Wang and Forsyth (2011) develop a numerical scheme to determine the optimal time-consistent asset allocation strategy[3]. For a von Neumann-

[1] The widely used utility functions belong to the so-called hyperbolic absolute risk aversion (HARA) family, including quadratic (with restrictions on parameters), exponential, logarithmic, and power forms.

[2] We thank an anonymous referee for pointing this out to us.

[3] For a good discussion on time-inconsistent portfolio strategies, see Dang and Forsyth (2016).

Morgenstern utility, Detemple, Garcia and Rindisbacher (2003) also use the martingale approach to solve the portfolio choice problem in a complete pure-diffusion model which may include a large number of assets and state variables with non-affine structures. They obtain the optimal portfolio strategy using the Monte Carlo simulation, yet which may be time-consuming in the presence of a large number of assets and state variables.

As discussed in Bardhanand and Chao (1996), a jump-diffusion model with random jump sizes is inherently incomplete. One of the key assumptions in the aforementioned papers is the completeness of the market. In general, it is a daunting task to explicitly solve the optimal portfolio choice problem in an incomplete market. One usually resorts to either the HJB equation or the martingale method. As is well known, it is difficult to apply the HJB equation to a high-dimensional problem in both complete and incomplete markets. Furthermore, it is very challenging to use the martingale method in an incomplete market since there are infinitely many martingale measures. To solve the optimal portfolio problem in incomplete pure-diffusion models, approximation methods are proposed in Bick, Kraft and Munk (2013) and Haugh, Kogan and Wang (2006), respectively. Yet, their solutions are numerically approximated and thus may suffer inaccuracy.

In contrast, by assuming quadratic conditions in pure-diffusion models, Liu (2007) explicitly solves the optimal dynamic portfolio choice problem in both complete and incomplete markets, up to the solutions to a set of ordinary differential equations (ODEs). Specifically, he solves a set of ODEs by guessing the exponential linear form of the indirect value function without simulation. This method is widely used in the asset allocation literature of pure-diffusion models nowadays. However, much less is known about the conditions that can lead to the ODE-based analytic solution to the optimal portfolio choice problem in jump-diffusion models especially when both stock prices and state variables are allowed to jump.[4] The objective of the present paper is then to generalize the afore-

[4]Mounting empirical evidence suggests that the jump risk needs to be captured in asset price processes and other risk factors, such as volatility processes, in addition to the diffusion risk. For example, Eraker, Johannes and Polson (2003) and Eraker (2004) among many others find strong evidence for co-jumps in volatility and stock returns, i.e., that a big jump in stock prices is likely to be associated with a big jump in volatility. Besides, Das (2002) shows that a class of Poisson-Gaussian models offer a good statistical description of short rate behavior and capture empirical features of the data which would not be captured by Gaussian models (We thank an anonymous referee for bringing this issue to our attention). In the meantime, it is well understood that jump risk in stock prices has a substantial impact on portfolio

mentioned ODE-based approach in pure-diffusion models to jump-diffusion models which nest the former (e.g., Liu (2007)) as special cases.

More specifically, we first consider constant relative risk aversion (CRRA) utility functions and provide the conditions under which the indirect value function in jump-diffusion models has an exponential linear form. The indirect value function and the optimal portfolio strategy can then be obtained by solving a set of ODEs. By providing an efficient two-step approach, we further extend our ODE-based method to more general HARA utility functions given their popularity in financial economics.[5] Our results show that the indirect utility function for a HARA utility takes a form significantly different from the exponential linear one for a CRRA utility. To the best of our knowledge[6], we are not aware of any semi-analytical solution to the dynamic asset allocation problem in jump-diffusion models where risk-averse investors face jumps in multiple risky assets and state variables. More importantly, the semi-analytical solutions may greatly facilitate economic insights and enhance our understanding of investors' behavior towards jump risks.

Our paper is closely related to the work of Jin and Zhang (2012) in that they use a decomposition approach based on an HJB equation to solve a portfolio selection problem that includes a large number of risky assets and state variables. But their state variables are pure-diffusion processes and the indirect value function is evaluated by the Monte Carlo simulation. Our paper also relates to the work of Das and Uppal (2004) and Aït-Sahalia, Cacho-Diaz and Hurd (2009). These studies solve the portfolio selection problem in jump-diffusion models, but without state variables. In contrast, we obtain semi-analytical solutions to the optimal portfolio strategy under jump-diffusion models that include a large number of assets and state variables. These solutions therefore allow

selection, see, for example, Liu, Longstaff and Pan (2003) and Das and Uppal (2004).

[5]More importantly, Perets and Yashiv (2016) show that the HARA utility is more fundamental to economic analysis. This functional form is the unique one which satisfies basic economic principles in an optimization context. Therefore, the use of HARA utility functions is not just a matter of convenience or tractability, but rather emerges from economic reasoning, i.e., it is inherent in the economic optimization problem.

[6]It should be noted that for the logarithmic utility maximization under jump diffusions, semi-analytical solutions are generally available primarily due to its myopic nature of the optimal portfolio strategy. For example, in a general semimartingale market model, Goll and Kallsen (200) explicitly solve the problem of maximizing the expected logarithmic utility from consumption or terminal wealth. We thank an anonymous referee for suggesting this discussion.

us to solve in a computationally efficient way the dynamic portfolio selection problem in jump-diffusion models where both stock returns and state variables can jump.

By using the theoretical framework developed in this paper, we study the problem of how jumps in stock returns affect the optimal cash-bond-stock portfolio in a dynamic asset allocation model where an investor can trade one stock, two bonds, and cash. Especially, we revisit the asset allocation puzzle raised in Canner, Markiw and Weil (1997). They document the empirical evidence that strategic asset allocation advices tend to recommend a higher bond/stock ratio for a more risk-averse investor. Several studies have attempted to explain the rationality of this puzzle. For instance, Brennan and Xia (2000) and Bajeux-Besnainou and Portait (2001) relate the puzzle to a hedging component in the stochastic interest rate and provide elegant solutions to the asset allocation puzzle. All of these studies assume that both the short-term interest rate and stock returns follow pure diffusion processes. Our framework generalizes these studies by incorporating jumps into stock returns and examining the role of risk aversion in determining the optimal cash-bond-stock portfolio. In particular, we show both theoretically and numerically that unlike the pure-diffusion models in Brennan and Xia (2000), Bajeux-Besnainou and Portait (2001) and Lioui (2007), there is no clear-cut answer to the bond/stock ratio puzzle in jump-diffusion models even despite the aforementioned hedging assumption. In other words, the puzzle itself cannot be rationalized by the hedging assumption in the presence of jumps in stock returns. The underlying reason for this is that an investor responds distinctly to diffusion risk premium and jump risk premium when there is an increase in the investors's relative risk aversion coefficient.

In summary, our paper makes three contributions to the literature on portfolio choice. First, our work generalizes the popular ODE-based approach used in pure-diffusion models to jump-diffusion models for CRRA utility functions, which may greatly alleviate computational efforts in seeking the optimal portfolio strategy. Second, we provide an efficient two-step method for solving HARA preference-based ODEs. This then extends the applicability of our approach within a family of general utility functions. Finally, we illustrate that the hedging assumption in pure-diffusion models fails to resolve the asset allocation puzzle in jump-diffusion models, which further provides a new channel for us to understand the nature of this well-known puzzle.

The rest of the paper is organized as follows. In Section 2, we present the framework for Merton's dynamic portfolio selection problem in jump-diffusion models and then present affine conditions in the jump-diffusion models. In Section 3, we use the affine conditions to explicitly solve the indirect value function and the optimal portfolio strategy in terms of the solutions to a set of ODEs for general HARA preferences. In Section 4, we derive semi-analytical solutions to the optimal bond-stock mix and especially investigate how jump risk in stock returns affects bond/stock ratios. Section 5 is devoted to a calibration exercise in order to illustrate numerically the theoretical results in Section 4. We conclude in Section 6. All proofs are collected in Appendices.

2 The Economy

In this section, we formulate a model of incomplete financial markets in a continuous time economy where asset prices and state variables follow a multidimensional jump-diffusion process on the fixed time horizon $[0, T]$ $(0 < T < \infty)$. We consider a complete probability space (Ω, F, P), where Ω is the set of states of nature with generic elements ω s, and F is the -algebra of observable events, while P is a probability measure on (Ω, F).

We use an l-dimensional vector $X_t = (X_{1t}, \ldots, X_{lt})$ to denote the state variables of the economy where the convention stands for the transpose of a vector or a matrix. The state variables X_t may include stochastic volatility and stochastic interest rate as its components. We assume that state variables X_t follow a jump-diffusion process

$$dX_t = b^x(X_t)dt + {}^x(X_t)dB^X(t) + {}^x_J(X_t)(Y^x \bullet dN(t)) \tag{1}$$

where $b^x(X_t)$ is an l-dimensional vector function, ${}^x(X_t)$ is an $l \times l$ matrix function of X_t, and ${}^x_J(X_t)$ is an $l \times m$ matrix function of X_t, respectively. $B^X(t) = (B^X_1(t), \ldots, B^X_l(t))$ is an l-dimensional standard Brownian motion; $N(t) = (N_1(t), \ldots, N_m(t))$ is an m-dimensional multivariate Poisson process with $N_k(t)$ denoting the number of type k jumps up to time t, while $Y^x = (Y^x_1, \ldots, Y^x_m)$ represents an m-dimensional jump size process with Y_k^x denoting the amplitude of the type k jump conditional on the occurrence of the k-th jump. For any two n-dimensional vectors $x = (x_1; \ldots; x_n)'$ and $y = (y_1; \ldots; y_n)'$, we denote the component-wise multiplication as $x \bullet y = (x_1 y_1; \ldots; x_n y_n)'$. Note that unlike Liu (2007) and Jin and Zhang (2012), the above specification of X_t includes jumps in state variables. For instance, we can incorporate jumps into a volatility process. By letting $Y^x = 0$, our jump-diffusion model reduces to its pure-diffusion counterpart for the state variables X_t.

The uncertainty of the economy is also generated by a d-dimensional standard Brownian motion $B^S(t) = (B_1^S(t); \ldots; B_d^S(t))'$ which drives stock prices defined below. Assume $B^S(t)$ and $B^X(t)$ are correlated and $E[dB^X(t) d(B^S(t))'] = \rho_t dt$, for some $l \times d$ matrix ρ_t. The information flow in the economy is given by the natural filtration, i.e., the right-continuous and augmented filtration $\{\mathcal{F}_t\}_{t \in [0;T]} = \{\mathcal{F}_t^S \vee \mathcal{F}_t^X \vee \mathcal{F}_t^N; t \in [0;T]\}$, where $\mathcal{F}_t^S = \sigma(B^S(s); 0 \leq s \leq t)$, $\mathcal{F}_t^X = \sigma(B^X(s); 0 \leq s \leq t)$ and $\mathcal{F}_t^N = \sigma(N(s); 0 \leq s \leq t)$. We suppose that observable events are eventually known, i.e., $\mathcal{F} = \mathcal{F}_T$. For illustrative purposes, we assume that N_k admits stochastic intensity $\lambda_k(X_t)$ that represents the rate of the jump process at time t.

The market includes $n+1$ assets traded continuously on the time horizon $[0;T]$. One of these assets, risk-free, has a price $S_0(t)$ which evolves according to the differential equation

$$dS_0(t) = S_0(t) r(X_t) dt; \qquad S_0(0) = 1.$$

The remaining n assets, called stocks, are risky, and their prices are modeled by the linear stochastic differential equation

$$\frac{dS_i(t)}{S_i(t-)} = b_i(X_t)dt + \sigma_i^p(X_t) dB^S(t) + \sigma_i^q(X_t)(Y^s \bullet dN^S(t))$$

where $i = 1; \ldots; n$, $N^S(t) = (N_1(t); \ldots; N_{n-d}(t))'$, and $Y^s = (Y_1^s; \ldots; Y_{n-d}^s)'$, with Y_k^s denoting the amplitude of the type k jump conditional on the occurrence of the k-th jump. Here $\sigma_i^p(X_t)$ is the d-dimensional diffusion coefficient row vector and $\sigma_i^q(X_t)$ is the $(n-d)$-dimensional jump coefficient row vector. In particular, the Brownian motions represent frequent small movements in stock prices, while the jump processes represent infrequent large shocks to the market. Assuming $n - d \leq m$, the jumps $N^S(t)$ can be regarded as common jumps in stock returns and state variables.

To obtain the semi-analytical solutions to the optimal portfolio choice problem, we now turn to the assumption for affine models. In this paper, we focus on Merton's problem of maximizing the expected utility from the terminal wealth.[7] In this section, for illustrative purposes, we follow the literature to consider the CRRA utility function given by

$$U(x) = \begin{cases} \frac{x^{1-\gamma}}{1-\gamma}, & \forall x > 0, \\ -\infty, & \forall x \leq 0, \end{cases} \tag{2}$$

where $\gamma(\neq 1)$ is the relative risk aversion (RRA) coefficient. We will solve the optimal portfolio choice problem for more general HARA utility functions in the next section. Specifically, we consider an investor with the utility function $U(x)$, endowed with some initial wealth w_0 that is invested in the above-mentioned $n+1$ assets. Let $\pi(t) = (\pi_1(t), \ldots, \pi_n(t))^\top$ denote a trading strategy, where the $\mathcal{F}_t$-predictable $\pi_i(t)$ is the proportion of the total wealth invested in the i-th risky asset held at time t. Furthermore, $\pi(t)$ satisfies the standard square-integrability condition discussed in Bremaud (1981). Moreover, the portfolio policy $\pi(t)$ has an associated wealth process W_t that evolves as

$$\begin{aligned} W_t &= W_0 + \int_0^t r(s) W_s ds + \int_0^t W_s \pi^\top(s)(b(s) - r(s)\mathbf{1}_n) ds \\ &\quad + \int_0^t W_s \pi^\top(s) \Sigma_b(X_s) dB^S(s) + \int_0^t W_{s-} \pi^\top(s-) \Sigma_q(X_s)(Y^s \bullet dN^S(s)) \end{aligned} \tag{3}$$

where $b(t) \equiv (b_1(X_t), \ldots, b_n(X_t))^\top$, $\Sigma_b(X_t)$ is an $n \times d$ matrix with σ_i^b being its i-th row, $\Sigma_q(X_t)$ is the $n \times (n-d)$ matrix with σ_i^q being its i-th row. Here we use $\mathbf{1}_n$ to denote the n-dimensional column vector of ones. The portfolio policy $\pi(t)$ is said to be admissible if the corresponding wealth process satisfies $W_t \geq 0$ almost surely. We use $\mathcal{A}(w_0)$ to denote the set of all admissible trading strategies. Then, Merton's portfolio choice problem states

[7] A semi-analytical solution can be obtained for the optimal portfolio choice problem with the utility function defined by (2) in Liu (2007) when the Brownian motions in prices and state variables are the same, namely, $B^X(t) = B^S(t)$. This condition is satisfied in the applications in Section 4.

that the investor attempts to maximize the following quantity

$$u(w_0;X_0) = \max_{\in A(w_0)} J(w_0;X_0) = E\,[U(W_T)]:$$

We consider the general case: $n - d < m$ because, by letting $Y_k^x = 0; k = m_0 + 1; ::: ; m$, we can get the model where there are only m_0 ($\leq n - d$) types of jumps in state variables. Using the standard approach to stochastic control and an appropriate Ito's lemma for jump-diffusion processes, the optimal portfolio policy and the corresponding indirect value function J of the investor's problem then follow the HJB equation:

$$
\begin{aligned}
0 \;=\; \max\Big\{ & J_t + \frac{1}{2} W_t^2 \quad \Sigma_b \Sigma_b \quad J_{WW} + W_t [\quad (b(t) - r\mathbf{1}_n) + r] J_W \\
& + b^x(t) J_X + W_t \quad \Sigma_b \;_t \quad {}^x \; (t) J_{WX} + \frac{1}{2} Tr(\;{}^x(t) \;{}^x\; (t) J_{XX} \;) \\
& + \sum_{k=1}^{n-d} {}_k E\,[J(W_t + W_t \quad {}_{qk} Y_k^s ; X_t + Y_k^x ; t) - J(W_t;X_t;t)] \Big\} \\
& + \sum_{k=n-d+1}^{m} {}_k E\,[J(W_t;X_t + Y_k^x;t) - J(W_t;X_t;t)]
\end{aligned} \tag{4}
$$

where ${}_{qk}$ denotes the k-th column of ${}_q$. The above HJB equation nests the HJB equation (3) for the pure-diffusion model in Liu (2007) as a special case by letting $n - d = 0$. In other words, we generalize the models in Liu (2007) by incorporating jumps into stock returns and state variables. It is well-known that in the pure-diffusion model in Liu (2007), the indirect value function $J(W_t;X_t;t)$ is conjectured to have the form: $J(W_t;X_t;t) = \frac{W_t^{1-}}{1-} \left[e^{A(t)+B(t) \;\; X_t} \right]$, where $A(t)$ is a scalar and $B(t)$ is an $l \times 1$ vector. Then, under the quadratic conditions, a set of ODEs for the functions $A(t)$ and $B(t)$ are obtained by substituting the function J and the optimal portfolio strategy into the HJB equation (4). As shown below, the argument in Liu (2007) does not trivially apply to jump-diffusion models because the portfolio policy may depend on the state variables X_t.

We now illustrate the difficulty caused by jumps. More specifically, compared with the HJB equation (3) in Liu (2007) for pure-diffusion models, the jump terms in the above HJB equation create new difficulties for semi-analytical solutions to the optimal portfolio choice problem in jump-di□usion models. We now consider a simple case where there are no jumps in the state variables X_t by letting $Y_k^x = 0; k = 1; :::; n-d$. As in Liu (2007), we substitute the indirect value function $J(W_t; X_t; t) = \frac{W_t^{1-}}{1-} \quad (f(t; X_t))$ into (4) and obtain the following form for the last term:

$$
\begin{aligned}
&\sum_{k=1}^{-d} \quad E[J(W_t + W_t \quad {}_{qk} Y_k^s; X_t; t) - J(W_t; X_t; t)] \\
&= \frac{W_t^{1-}}{1-} \quad (f(t; X_t)) \sum_{k=1}^{-d} \quad {}_k(X_t) E[(1 + \quad {}_{qk} Y_k^s)^{1-} \quad - 1]:
\end{aligned}
$$

As is well-understood from, for instance, Liu (2007), in order to gain an explicit solution for the indirect value function $J(W_t; X_t; t)$ of the form $J(W_t; X_t; t) = \frac{W^{1-}}{1-} \left[e^{A(t)+B(t) \quad X_t} \right]$, the term $E[(1+ \quad {}_{qk} Y_k^s)^{1-} \quad]$ should be an a□ne function of the state variables X_t. This term, however, is hard to be an a□ne function of the state variables X_t unless the optimal jump exposure $\quad {}_{qk}$ is a deterministic function of time t, because the function x^{1-} is generally not an a□ne function. Based on this observation and inspired by the results in Liu (2007) and the result of decomposition of optimal portfolio weights in Jin and Zhang (2012), we are able to specify an a□ne model[8] which leads to ODEs for $A(t)$ and $B(t)$ given in Proposition 1 in Section 3.

More specifically, by setting $a_k = E(Y_k^s); k = 1; :::; n-d$, we assume that the matrix $\Sigma = [\Sigma_b; \Sigma_q]$ is invertible. The market price of risk is then represented by

$$
\begin{matrix} b \\ q \end{matrix} \quad = \Sigma^{-1}(b(t) - r\mathbf{1}_n + \Sigma_q(\quad \bullet a)); \tag{5}
$$

where $\bullet a = (\;{}_1 a_1; :::; \;{}_{n-d} a_{n-d}) \;;\; {}^b = ({}^b_1; :::; {}^b_d)$ and ${}^q = ({}^q_1; :::; {}^q_{n-d})$. As shown in Section 4, ${}^b_i$ denotes the risk premium for the Brownian motion $B_i^S; i = 1; :::; d$, while ${}^q_k$ represents the risk premium for the jump $N_k^S; k = 1; :::; n-d$, in the stock returns. We further make the following assumptions:

[8] Here, for expositional purposes, we consider a□ne models only as it is straightforward to generalize our results to quadratic processes defined in Liu (2007).

Assumption 1

$$
\begin{aligned}
b^x(X) &= k - KX; \quad {}^{x}\ {}^{x} = h_0 + h_1 \cdot X; \\
r &= {}_0 + {}_1 X; \quad {}^{b}\ {}^{b} = H_0 + H_1 X; \\
{}^{x}\ {}_{t}\ {}^{b} &= g_0 + g_1 X; \quad {}^{x}\ {}_{t}\ {}_{t}\ {}^{x} - {}^{x}\ {}^{x} = l_0 + l_1 \cdot X; \\
&= {}_0 + {}_1 X; \quad {}^{q}_{k} = {}^{0}_{k}\ {}_{k}; k = 1; :::; n - d;
\end{aligned}
\tag{6}
$$

where $k;\ {}_1;H_1$ and g_0 are $l \times 1$ constant vectors; $K;h_0;g_1$ and l_0 are $l \times l$ constant matrices; ${}_0;H_0$ and ${}^{0}_{k}$ are constants; ${}_0$ is an $(n-d)\times 1$ constant vector; ${}_1$ is an $(n-d)\times l$ constant matrix; $h_1 = h^i_{1jk}; i;j;k = 1;:::;l$ and $l_1 = l^i_{1jk}; i;j;k = 1;:::;l$ are constant tensors with three indices (one upper index and two lower indices). In particular, $h_1 \cdot X$ is an $l \times l$ matrix whose $(j;k)$ element is given as follows:

$$
(h_1 \cdot X)_{jk} = \sum_{i=1}^{l} h^i_{1jk} X_{it}:
$$

The $l \times l$ matrix $l_1 \cdot X$ is defined exactly in the same manner. The above assumptions except the last two are similar to those made in Liu (2007), while the last two assumptions on jump intensity and jump risk premium are also standard in literature, and the last assumption states that the risk premium for the k-th jump is proportional to its intensity.

3 The Portfolio Choice Problem

Given the a□ne models in the proceeding section, we now explicitly solve the optimal portfolio choice problem for hyperbolic absolute risk aversion (HARA) utility functions up to solving a set of ODEs. The most popular utility functions used in almost all applied theories and empirical studies in finance belong to the class of linear risk tolerance (LRT) or HARA utility functions, including the quadratic function (with restrictions on parameters), the CRRA utility, the exponential utility and the logarithmic utility as special cases. Therefore, the explicit solutions to the portfolio choice problem for HARA preferences may cast new light on investors' behavior towards distinct risk factors in a stochastic investment environment. More specifically, a HARA utility function is given by

$$U(x) = \begin{cases} \frac{0}{1-}(x - \;)^{1-}\,; & \forall x > \\ -\infty\,; & \forall x \leq \end{cases} : \tag{7}$$

For $\;= 0;\ U(x)$ reduces to a CRRA utility function (2). Here we consider a realistic case with $\;> 0$,[9] that is, the relative risk aversion is decreasing with wealth. In Bajeux-Besnainou and Portait (2001), they interpret the constant as a "subsistence level".

Canakoglu and Ozekici (2012) consider the optimal portfolio selection problem in a continuous-time pure-diffusion setting where the market states follow Markov pro- cesses. They utilize the HJB-based approach to obtain semi-analytical solutions for the CRRA utility, the exponential utility and the logarithmic utility, respectively. In Bajeux-Besnainou and Portait (2001), they obtain closed-form solutions to the optimal dynamic portfolios for the HARA utility in pure-diffusion models. Specifically, they employ the duality results developed by Karatzas, Lehoczky and Shreve (1987), substantially rooted in the key assumption of the existence of a unique equivalent martingale measure in a complete market. In contrast, the markets in this paper are incomplete due to random jump sizes and thus there exist infinitely many equivalent martingale measures. As in Jin, Luo and Zeng (2016), to solve an optimal dynamic portfolio problem for the HARA un-tility, we resort to the duality results for incomplete markets developed by Kramkov and Schachermayer (1999) in combination with the results developed for the CRRA utility. But our results differ from Jin, Luo and Zeng (2016) in that we incorporate jumps into state variables and solve the optimal portfolio problem based on a set of ODEs instead of a simulation-based approach used in their paper. Our main results are summarized in the following two propositions.

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[9] For the case $\;< 0$, similar to the results in Section 6.3 of Merton (1990), the unconstrained policies derived by the method in the present paper may violate the nonnegativity condition on wealth. Thus, we need to solve the constrained problem with a positive wealth process. This is beyond the scope of the present paper and we leave it as a future research.

Proposition 1 *Under Assumption 1, the indirect value function is represented as*

$$J(W_t;X_t;t) = \frac{\left(W_t - \ e^{\ (t)- \ A(t)+(\ (t)- \ B(t)) \ X_t} \right)^{1-}}{1-} \left[e^{A(t)+B(t) \ X_t} \right] \tag{8}$$

where $A(t)$, $B(t)$, (t) and (t) are obtained by ODEs in Appendix A.

Proof. See Appendix A. ■

The result in (8) suggests that unlike the indirect utility function for a CRRA utility by setting $\ = 0$, the one for a HARA preference cannot be separated into a product of two functions, one depending on the wealth W and the other on the state variables X_t and time t. This result extends the literature on the optimal portfolio choice with a HARA utility. For detailed discussions, for example, Merton (1990) and Perets and Yashiv (2016) suggest that the above decomposition holds true due to constant investment opportunities.

Proposition 2 *Under Assumption 1, the optimal portfolio weight $\ = (\ _1;:::;\ _n)$ is given by*

$$= \left(\mathrm{e}_{b1};:::;\mathrm{e}_{bd};\mathrm{e}_{q1};:::;\mathrm{e}_{q(n-d)} \right) \Sigma^{-1} \tag{9}$$

where the optimal e_b is given by

$$\mathrm{e}_b = \frac{W - \ g(t;X_t)}{W} \left[\frac{\mathrm{e}b}{} + \ _t \ ^x \ B(t) \right] + \frac{\Sigma_b \ _t \ ^x \ (\ (t) - \ B(t))g(t;X_t)}{W} \tag{10}$$

and e_{qk} solves the following optimization problem:

$$\max_{\mathrm{e}_{qk} \in F_k} \Bigg(\mathrm{e}_{qk} W(W - \ g(t;X_t))^{-} \ (\mathrm{e}^q_k - \ _k a_k) + \frac{k}{1-} E \left[\left(W(1+ \ _{qk} Y^s_k) - \ g(t;X_t) e^{\ (t) \ ^x_{Jk} Y^x_k} \right)^{1-} e^{\ B(t) \ ^x_{Jk} Y^x_k} \right] \Bigg) \tag{11}$$

for $k = 1;:::;n-d$, where F_k is the set of feasible k-th jump exposures satisfying the jump induced no-bankruptcy condition, namely, $F_k = \{x | x \cdot y > -1; \forall \ \in A_k\}$, with A_k denoting the support of the k-th jump size Y^s_k, and $g(t;X_t) = e^{\ (t)- \ A(t)+(\ (t)- \ B(t)) \ X_t}$.

Proof. See Appendix B. ■

The second term in (10) indicates that as opposed to a CRRA utility ($\eta = 0$), a HARA utility ($\eta \neq 0$) has a separate hedging demand for the interest rate related risk. This term will disappear if the interest rate is a constant since in this case, $\beta(t) = \gamma B(t)$ as can seen in the proof of Appendix A. Furthermore, letting $\eta = 0$ in (11) and using Assumption 1 gives the optimal jump exposure problem for a CRRA utility:

$$\max_{\widetilde{\sigma}_{qk} \in \mathcal{F}_k} \left(\widetilde{\sigma}_{qk} \left(\theta_k^0 - a_k \right) + \frac{1}{1 - } E \left[(W(1 + \quad {}_{qk} Y_k^s))^{1-} \; e^{B(t) \quad {}^{x}_{Jk} Y_k^x} \right] \right) \tag{12}$$

The objective function in the optimization problem in (12) does not include the state variables X_t and thus, for each k, the optimal jump exposure $\widetilde{\ }^{q}_{k}$ is deterministic.[10] This justifies the conjectured exponential linear form of the indirect value function for a CRRA utility. It is worth mentioning that despite the deterministic jump exposure $\widetilde{\ }^{q}_{k}$, the optimal portfolio policy is still dependent on the state variables X_t through the optimal diffusion exposures ($\widetilde{\ }^{b}_{1}; \ldots; \widetilde{\ }^{b}_{d}$) and the matrix Σ. This state-dependent portfolio strategy reflects the investor's market timing behavior.

As we discuss in Appendix B, the conjecture-based approach used in Liu (2007) is very likely inapplicable to a HARA utility in jump-diffusion models as it is hard to substitute the optimal jump exposure in (11) into the HJB equation. Two reasons account for this difficulty. On the one hand, as shown in the first-order condition for $\widetilde{\ }^{q}_{k}$ in Appendix A, it is generally impossible to solve the optimal $\widetilde{\ }^{q}_{k}$ in closed form unless all jumps are constants. On the other hand, the optimization problem in (11) shows that the jump exposure $\widetilde{\ }^{q}_{k}$ depends on both the wealth W and the state variables X_t and thus is not deterministic, making it hard to use the conjecture-based method. As a result, we propose a two-step approach to solving the optimal asset allocation problem for the HARA utility function specified in (7) summarized as follows:

(i) In the first step, the functions $(t);\ (t); A(t)$ and $B(t)$ are determined by solving the optimal asset allocation problem for a CRRA utility function in (2);

[10] It will be shown in Appendix A that the result of the deterministic jump exposure $\widetilde{\ }^{q}_{k}$ of the CRRA utility function is particularly useful when we solve the optimal portfolio choice problem in closed form with a more general HARA utility function.

(ii) In the second step, the indirect utility function $J(W_t; X_t; t)$ of the HARA utility function is evaluated by (8) and then the optimal portfolio weights are determined through (9), (10) and (11).

Our two-step approach therefore contributes to the literature in solving the optimal portfolio choice problem for HARA preferences efficiently in jump-diffusion models.

4 Dynamic Asset Allocation for Stocks, Bonds and Cash

We now apply the results in Section 3 to examine the impact of jumps in stock returns on the optimal cash-bond-stock mix in a dynamic model where an investor can trade one stock, two bonds, and cash (or the called money market account). A closely related problem is the asset allocation puzzle raised in Canner, Markiw and Weil (1997). They empirically document that the strategic asset allocation advice tends to recommend a higher bond/stock ratio for an investor with more risk aversion. This finding, however, is inconsistent with Tobin (1958)'s Separation Theorem that the ratio of bonds to stocks in the optimal portfolio is the same for all investors regardless of their risk aversion.

Brennan and Xia (2000) and Bajeux-Besnainou and Portait (2001) relate this puzzle to a hedging component in the stochastic interest rate and provide elegant solutions to the asset allocation puzzle. More specifically, as pointed out by Lioui (2007), the puzzle can be resolved under the assumption that one or several bonds can perfectly hedge the risk from the interest rate and the market price of risk. Yet, Lioui (2007) argues that there is no clear-cut answer to the puzzle if the hedging assumption is invalid. All of these studies assume that the short-term interest rate and stock returns follow pure-diffusion processes. This section attempts to generalize these studies by incorporating jumps into stock returns[11] and examining the role of risk aversion in determining the optimal cash-bond-stock mix in the presence of jump risk. Interestingly, we will show that unlike the pure-diffusion model in Lioui (2007), there is no clear-cut answer to the bond/stock ratio

[11] For simplicity, we do not include jumps in the short-term interest rate which is a state variable in this section. In Hong and Jin (2016), by using Propositions 1 and 2 developed in the present paper, they show that jumps in volatility process play a significant role in variance swap investments in a model where volatility is a state variable.

puzzle in a jump-diffusion model even despite the aforementioned hedging assumption. This finding demonstrates that the puzzle cannot be rationalized by the hedging assumption in the presence of jumps and thus strengthens the claim made by Lioui (2007) that the asset allocation puzzle is still a puzzle.

Like Lioui (2007), we adopt a two-factor term structure model that is a simplified version of the multi-factor models in Sangvinatsos and Wachter (2005). We extend it by adding a jump component in the stock price. The model assumes the following dynamics under the physical measure P:

$$
\begin{aligned}
r(X(t);t) &= {}_0 + \quad X(t);\\
dX(t) &= K(\quad - X(t))dt + {}_X\, dZ(t);
\end{aligned}
\tag{13}
$$

where $r(t)$ is the short-term interest rate; $X(t)$ is a 2×1 vector of state variables; $Z(t) = (Z_1(t);Z_2(t))$ is a standard 2-dimensional Brownian motion; ${}_0 \in R$; $\in R^{2\times1}$; $K \in R^{2\times2}$; $\in R^{2\times1}$; ${}_X = ({}_{X\,ij})_{1\le i;j\le 2}$ is a 2×2 non-singular matrix, and all of these parameters are assumed to be constants.

For simplicity, we incorporate only one type of jump into the stock returns. We specify the Radon-Nikodym derivative as $\frac{dQ}{dP} = {}_t = {}^Z_t\ {}^N_t$ as follows:

$$
\begin{aligned}
{}^Z_t &= {}^Z_0 \exp\left(-\bar{\Lambda}(t)\ dZ(t) - \frac{1}{2}\int_0^t \bar{\Lambda}(t)\ \bar{\Lambda}(t)\ dt \right)\\
{}^N_t &= {}^N_0 \prod_{i=1}^{N(t)} \#(t_i)\ (t_i;z_i)\ \exp\left(\int_0^t \int_A (1 - \#(s)\ (s;z))\ (X_s)\Phi(s;dz)ds \right)
\end{aligned}
$$

where $\bar{\Lambda}(t) = \bar{}_1 + \bar{}_2 X(t)$, $\bar{}_1 \in R^{2\times1}$ is a constant vector; $\bar{}_2 \in R^{2\times2}$ is a constant matrix; t_i is the $i-$th jump time up to t; z_i is the corresponding jump size; $\#(s)$ and $(s;z)$ are positive stochastic processes, and $(s;z)$ satisfies the relationship of $\int_A (t;z)\Phi(t;dz) = 1$,where A and $\Phi(t;dz)$ are the support and distribution of the jump size, respectively. By Theorem T10 of Bremaud (1981), under the probability measure Q, the intensity ^{Q} is $\#$ and the density function $\Phi^Q(t;dz)$ is $(z)\Phi(t;dz)$.

Due to no jumps in the interest rate, a zero-coupon can be priced by using Radon-

Nikodym derivative ξ^Z. As shown in Sangvinatsos and Wachter (2005), the nominal bond price evolves as follows:

$$\frac{dP_i(t)}{P_i(t)} = (A_2(\tau_i)\sigma_X\bar{\Lambda}(t) + r(t))dt + A_2(\tau_i)\sigma_X dZ(t); i = 1; 2; \tag{14}$$

where $\tau_i = T_i - t$ and T_i denotes the maturity date of bond i with $\tau_1 \neq \tau_2$, while $A_2(\tau_i) = (A_{21}(\tau_i), A_{22}(\tau_i))$ is a 1×2 row vector for $i = 1, 2$. And moreover, from Appendix A in Sangvinatsos and Wachter (2005), $A_2(\tau)$ solves the following ODE

$$\frac{dA_2(\tau)}{d\tau} = -A_2(\tau)(K + \sigma_X\bar{\Lambda}_2) - \delta_X; \tag{15}$$

with the boundary condition $A_2(0) = 0_{1\times 2}$.

To explain the asset allocation puzzle, Lioui (2007) assumes that only the short rate is stochastic while the market prices are deterministic. For comparison, we follow Lioui (2007) to assume that the price of risk $\bar{\Lambda}(t)$ is a constant vector by setting $\bar{\Lambda}_2 = 0_{2\times 2}$, and then solve the equation in(15) to obtain the following

$$A_2(\tau) = \delta_X(e^{-K\tau} - 1)K^{-1}: \tag{16}$$

Denote the vectors of volatility and risk premia of the two bonds by

$$\sigma_P = \begin{pmatrix} A_2(\tau_1)\sigma_X \\ A_2(\tau_2)\sigma_X \end{pmatrix} = \begin{pmatrix} A_2(\tau_1) \\ A_2(\tau_2) \end{pmatrix}\sigma_X = \mathbf{A}_2\sigma_X;$$

and $\mu_P = \sigma_P\bar{\Lambda}(t)$, respectively.

To compare with the results of the bond/stock ratio in a pure-diffusion model in Brennan and Xia (2000), we assume that the investor who has a CRRA utility function is allowed to invest in two bonds, one stock, and cash. In addition to the above two bonds, we assume there exist both an instantaneously riskfless money market account with the price $B(t)$ and one stock index with the price $S(t)$ where $B(t)$ and $S(t)$ satisfy

$$\frac{dB(t)}{B(t)} = r(t)dt; \tag{17}$$

$$\frac{dS(t)}{S(t)} = (\mu_S + r(t))dt + \sigma_S dZ(t) + J\,dN(t) - g^P \lambda^P dt; \tag{18}$$

where $\mu_S = \sigma_S \overline{\Lambda}(t) + g^P \lambda^P - g^Q \lambda^Q$; $\sigma_S = (\sigma_{S1}; \sigma_{S2})$; g^P and λ^P are the expected jump size and jump intensity under the physical measure P, respectively; g^Q and λ^Q are the expected jump size and jump intensity under the risk neutral measure Q, respectively. Specifically, μ_S is the total risk premium for the stock with the term $\sigma_S \overline{\Lambda}(t)$ compensating for the diffusion risk, while the term $g^P \lambda^P - g^Q \lambda^Q$ compensates for the jump risk.

This specification implies that the two bonds and cash are relatively safer than stock during a turbulent period when jump occurs. As is well understood, jumps in stock returns have significant impacts on the optimal portfolio choice. For instance, Liu, Longstaff and Pan (2003) demonstrate that in the presence of jumps in stock returns investors are less willing to take levered or short positions than in a standard diffusion model. Furthermore, even when the chance of a large jump is remote, an investor has strong incentives to significantly reduce her exposure to the stock market. The reason is that, if a jump occurs, invested wealth can change significantly from its current value, and such changes cannot be hedged through continuous rebalancing, resulting in potentially large losses for investors with levered or short positions. In stark contrast, the changes in bond prices can be hedged through continuous rebalancing as they follow pure-diffusion processes. A natural question is: how does a risk-averse investor choose her bond-stock mix when facing uncertain abrupt changes in stock returns? More concretely, does a more risk-averse investor hold more bonds and/or cash than a less risk-averse investor does? To answer these questions, we let $\omega_{B1}; \omega_{B2}$ and ω_S denote the fractions of the wealth invested in the two bonds and the stock, respectively. And hence, the remainder $\omega_C = 1 - \omega_{B1} - \omega_{B2} - \omega_S$ is invested in cash. The following proposition presents a semi-analytical solution to the optimal strategy.

Proposition 3 *The optimal portfolio weight* $= (\ _{B1};\ _{B2};\ _S)$ *is given by*

$$(\ _{B1};\ _{B2}) \;=\; \left[\frac{\overline{\Lambda(t)}}{} + \frac{f_X}{f}\ _X\right]\ _p^{-1} - \tilde{\ }_q\ _S\ _p^{-1}; \tag{19}$$

$$_S \;=\; \tilde{\ }_q; \tag{20}$$

where the function $f(t;X_t)$ *is given in Appendix A, and* e_q *solves the following optimization problem:*

$$\sup_{e_q \in F}\ e_q(-g^Q\ ^Q) + \frac{P}{1-}\int_A (1+e_q z)^{1-}\ \Phi(dz); \tag{21}$$

where F speci es the set of feasible jump exposures satisfying the jump induced no-bankruptcy condition, and A and $\Phi(dz)$ are the support and distribution of the jump size.

Proof. See Appendix C. ■

Interestingly, Equation (20) shows that the demand for the stock index has a speculative component to gain the risk premium only from jumps as suggested by the static optimization problem for e_q, while the burden of hedging the interest rate risk and the market price of risk is borne by the two bonds. This result holds true regardless of whether or not $_1 = T$, namely, the maturity of a bond equal to the investment horizon. The reason underlying the results in Proposition 3 is that the two bonds span the risk of the interest rate and the market price of risk while only stock spans the jump risk. In contrast, the bond portfolio weights have three components. The first is the myopic demand for the risk premia of two diffusion risks; the second is the hedging demand against the risk stemming from the two diffusion risks; the third one is another myopic demand for the jump risk premium. More specifically, as shown in Appendix C, the first two components are identical to the optimal weights in the market where the stock is not available for trading. And thus, the third component determines more or fewer bonds the investor holds when she can trade the stock. Although the two bonds are independent of jumps, the investor can gain the jump risk premium by investing more in the two bonds, as the two bonds and the stock are correlated via diffusion, suggested by the term $_S\ _p^{-1}$.

To make the intuition behind the results as clear as possible, we concentrate on a simple case by further assuming that the jump sizes $J = g^P$ and $J = g^Q$ are negative constants under both the physical measure P and the risk-neutral measure Q. We follow Sangvinatsos and Wachter (2005) to assume that the state variables X_1 and X_2 follow the equations below.

$$dX_1(t) = K_1(\ _1 - X_1(t))dt + \ _{X_{11}} dZ_1(t);$$
$$dX_2(t) = K_2(\ _2 - X_2(t))dt + \ _{X_{22}} dZ_2(t); \tag{22}$$

where K_1 and K_2 are positive constants. In this case, by (16), we have

$$A_{2i}(\ ) = \frac{e^{-K_i\ } - 1}{K_i}\ _i; i = 1; 2:$$

We further assume that X_1 is a permanent state variable with a low value of K_1 while X_2 is a transitory state variable with a high one of K_2. Like Table II in Sangvinatsos and Wachter (2005), we let $\ _{X_{11}} > 0;\ _{X_{22}} > 0;\ _{S1} < 0;\ _{S2} > 0$ and $\ _{S1}\ _{X_{11}} + \ _{S2}\ _{X_{22}} < 0$ so that the stock returns are negatively correlated with both the state variable $X_1(t)$ and the interest rate $r(t)$. The negative correlation between stock returns and interest rates has been documented in the literature (see, for example, Fama (1981) and Sangvinatsos and Wachter (2005)). From (14), it is easy to check that the bond return and the interest rate are negatively correlated as $A_{21}(\ ) < 0$ and $A_{22}(\ ) < 0$. Furthermore, in order to investigate whether or not the explanation of Lioui (2007) for the bond/stock ratio puzzle is still valid in our jump-diffusion model, we assume that the maturity $\ _1$ of the first bond is equal to the investment horizon T. Then, the optimal portfolio weights in Proposition 3 are given explicitly in the following result.

Proposition 4 *The optimal portfolio weight $\ = (\ _{B_1};\ _{B_2};\ _S)$ is given by*

$$\ _{B1} = \frac{1}{|A_2|}\left(\frac{\bar{\Lambda}_1(t)}{\ _{X_{11}}} A_{22}(\ _2) - \frac{\bar{\Lambda}_2(t)}{\ _{X_{22}}} A_{21}(\ _2)\right) + \left(1 - \frac{1}{\ }\right) - \frac{\tilde{\ }_q}{|A_2|}\left(\frac{\ _{S1}}{\ _{X_{11}}} A_{22}(\ _2) - \frac{\ _{S2}}{\ _{X_{22}}} A_{21}(\ _2)\right);$$
$$\ _{B2} = \frac{1}{|A_2|}\left(-\frac{\bar{\Lambda}_1(t)}{\ _{X_{11}}} A_{22}(\ _1) + \frac{\bar{\Lambda}_2(t)}{\ _{X_{22}}} A_{21}(\ _1)\right) - \frac{\tilde{\ }_q}{|A_2|}\left(-\frac{\ _{S1}}{\ _{X_{11}}} A_{22}(\ _1) + \frac{\ _{S2}}{\ _{X_{22}}} A_{21}(\ _1)\right);$$
$$\ _S = \tilde{\ }_q = \frac{1}{g^P}\left[\left(\frac{g^Q\ ^Q}{g^P\ ^P}\right)^{-\frac{1}{\ }} - 1\right]; \tag{23}$$

where $|A_2| = A_{21}(\ _1)A_{22}(\ _2) - A_{21}(\ _2)A_{22}(\ _1) < 0.$

Proof. See Appendix C. ■

The above results suggest that Bond 1 perfectly hedges the interest rate risk, which is the same as a pure-di□usion model in Lioui (2007). Using the facts that $A_{21}(\ ) < 0; A_{22}(\ ) < 0; |A_2| < 0$ and $_{S1} < 0$, we can verify that the coe□cient of $\tilde{}_q$ in the first equation in (23) is positive while the one in the second equation in (23) is negative. In other words, to gain jump risk premia, the investor holds more short-term bonds (Bond 1) and less long-term bonds (Bond 2) to o□set the position in Bond 1. Meanwhile, the total demand for the two bonds due to jump risk is positive, which can be rewritten as

$$-\frac{\tilde{}_q}{|A_2|}\left[\frac{_{S1}}{X_{11}}(A_{22}(\ _2) - A_{22}(\ _1)) + \frac{}{X_{22}}(A_{21}(\ _1) - A_{21}(\ _2))\right] \tag{24}$$

and the coe□cient of $\tilde{}_q$ is positive.

We now turn to the impact of the risk aversion coe□cient on the bond/stock ratio. From Proposition 4, the bond/stock ratio is separated into three terms that correspond to three parts in the portfolio on the bonds: mean-variance allocation, hedging demand for interest risk, and myopic demand for jump risk. The second term is actually exploited to explain the asset allocation puzzle in the literature (e.g., Brennan and Xia (2000), Bajeux-Besnainou and Portait (2001) and Lioui (2007)). It is interesting to investigate whether the ratio increases with the relative risk aversion coe□cient in our model here. For this purpose, we follow Brennan and Xia (2000) to rewrite the total demand for the two bonds in Proposition 4 as:

$$_B = \frac{a}{} + 1 - \frac{1}{} - b\tilde{}_q;$$

with

$$a = \frac{}{|A_2|}\left[\frac{_1(t)}{X_{11}}(A_{22}(\ _2) - A_{22}(\ _1)) + \frac{_2(t)}{X_{22}}(A_{21}(\ _1) - A_{21}(\ _2))\right];$$

$$b = \frac{1}{|A_2|}\left[\frac{_{S1}}{X_{11}}(A_{22}(\ _2) - A_{22}(\ _1)) + \frac{_{S2}}{X_{22}}(A_{21}(\ _1) - A_{21}(\ _2))\right]:$$

And hence, the bond/stock ratio is obtained as:

$$f(\ ) = \frac{B}{\tilde{S}} = \left(\frac{a-1}{\ } + 1\right)\frac{1}{\tilde{\ }_q} - b; \tag{25}$$

implying that by using the third equation in (23),

$$f'(\ ) = \frac{df(\ )}{d} = \frac{1}{{}^{2}\tilde{\ }_q}\left[1 - a - \frac{1}{g^P\ \tilde{\ }_q}\left(\frac{a-1}{\ } + 1\right)\left(\frac{g^Q\ \ ^Q}{g^P\ \ ^P}\right)^{-\frac{\ }{\ }+\ } ln\left(\frac{g^Q\ \ ^Q}{g^P\ \ ^P}\right)\right]:$$

As shown below, the function $f'(\ )$ can be either positive or negative depending on the model parameters. For instance, we show that it can be negative under certain conditions. For this, we consider the case of $a > 1$ in which the investor takes highly levered positions in bonds as documented in Table VI of Sangvinatsos and Wachter (2005) and in the numerical analysis in the following section (Section 5).

We now rewrite $f'(\ )$ as

$$f'(\ ) = \frac{1}{{}^{2}\tilde{\ }_q}\left[1 - a - \left(\frac{a-1}{\ } + 1\right)\frac{ln\left(\frac{g^Q\ \ ^Q}{g^P\ \ ^P}\right)}{1 - \left(\frac{g^Q\ \ ^Q}{g^P\ \ ^P}\right)^{\frac{1}{\ }}}\right]:$$

Assuming $1 \leq \ \leq 3$, we can show that $f'(\ ) < 0$ when $\frac{g^Q\ \ ^Q}{g^P\ \ ^P} > g(a) = \left(\frac{a+2}{a-1}\right)^3$, that is, the ratio related to the jump risk premium is higher than $g(a)$ which is a function of the diffusion risk premia. Therefore, in this case, the ratio $\frac{B}{\tilde{\ }_q}$ is a decreasing function of in the range of $[1;3]$. The reason for this is that unlike a pure-diffusion model, the demand $\tilde{\ }_q$ for the stock is not proportional to $1=$ as indicated by the third equation in (23). In fact, $\tilde{\ }_q$ decreases slower than $1=$ when increases in that $\frac{@\tilde{\ }_q}{@} = -\frac{1}{{}^{2}g^P} ln\left(\frac{g^Q\ \ ^Q}{g^P\ \ ^P}\right)\left(\frac{g^Q\ \ ^Q}{g^P\ \ ^P}\right)^{-1=}$ and $\left(\frac{g^Q\ \ ^Q}{g^P\ \ ^P}\right)^{-1=}$ increases with for $\frac{g^Q\ \ ^Q}{g^P\ \ ^P} > 1$. In other words, the investor with more risk aversion holds relatively more stocks than bonds to exploit the jump risk premium when the premia compensated for both the jump risk and diffusion risks satisfy the aforementioned condition. This is in contrast with the observations in a pure-diffusion model. Specifically, our jump-diffusion model reduces to a pure-diffusion model by replacing the jump component in stock returns with a diffusion one $Z_3(t)$. Then, the results in Proposition 4 except $_S$ remain unchanged. Specifically, $_S = \Lambda_3 = $, where $\Lambda_3 > 0$ is the risk premium for the diffusion term $Z_3(t)$. As a result, $f(\ ) = (a - 1 + \ )\frac{1}{3} - b$, which is an increasing function of . And thus, as argued in Lioui (2007), this leads to the resolution of the asset allocation puzzle in pure-diffusion models. In short, the rationality of the bond/stock ratio puzzle cannot be explained by the intertemporal hedging demand in the presence of jumps in stock returns, and thus our jump-diffusion model provides another channel to strengthen the issue addressed by Lioui (2007) that the asset allocation puzzle is still a puzzle.

Finally, we conduct a comparative static analysis to investigate the effect of the jump parameters on the cash-bond-stock mix. For simplicity, we just vary the jump intensity ^{P} while keeping the other parameters fixed. The third equation in (23) suggests that:

$$\frac{@q}{@^P} = \frac{1}{g^P}(\ ^P)_1^{-1}\left(\overline{g^{QPQ}}\right)^{-\frac{1}{g}} < 0;$$

implying that the total demand ^{B} for the two bonds decreases with ^{P} from (24) while the cash holding increases with ^{P}. In contrast, the bond/stock ratio increases with ^{P} by (25) if $a > 1$. The investor hence holds less in stocks when facing more frequent jumps. Namely, the investor reduces her position in stocks during a turbulent time of the stock market, and also reduces her bond holding ^{B} based on the above discussion. Interestingly, the investor holds more bonds relative to stocks as indicated by the increasing bond/stock ratio. As a result, the investor holds more cash and relatively more bonds, reflecting the phenomenon of flight-to-safety, when facing a high possibility of jump risk.

5 Numerical Results

In this section, we use a numerical example to illustrate the theoretical findings in the preceding section. Especially, we investigate the effect of the extreme negative jump risk on the bond/stock ratio. The recent financial crises have fuelled a renewed interest in modeling, estimating, and deriving the implications of extreme tail events. It has been documented in the literature that the distribution for extreme events can be well approximated by a power law that captures the slow tail decay in financial returns. More specifically, we adopt the single power law distribution of Barro and Jin (2011). Namely, let Y denote the jump size in stock returns, and the density function of a random variable $Z = \frac{1}{1+Y}$ is given by

$$v(z) = \quad z_0 z^{-(\;+1)}; z \geq z_0 > 1; \quad > 0. \tag{26}$$

This implies that Y is a negative jump with domain of $(-1; 1/z_0 - 1]$ and the density function of Y can be obtained as follows:

$$f_Y(y) = \quad z_0 (1+y) \;\; ^{-1}; y \in (-1; 1/z_0 - 1]. \tag{27}$$

Furthermore, it can be shown that for $y \in (-1; 1/z_0 - 1)$,

$$P(Y \leq y) = z_0 (1+y) \;\; .$$

Thus, the parameter measures the fatness of the left tail of stock returns. In particular, the smaller the value of is, the fatter the tail is, provided that the probability $P(Y \leq y)$ decreases with since $z_0(1+y) \;\; < 1$ for $y \in (-1; 1/z_0 - 1)$. The left panel of Figure 1 depicts the left tail for three cases: $= 5; 10$ and 15, showing that the jump tail for $= 5$ is much fatter than the one in the other two cases.

To estimate the parameters in this model, the calibration exercise below is based on the estimates reported in Table I and II of Sangvinatsos and Wachter (2005). Specifically, we first initialize the parameters of the two-factor model in (22):

$$_1 = \;\; _2 = 0; \;\; _{X11} = \;\; _{X22} = 1; K_1 = 0.576; K_2 = 3.343.$$

Next, for the interest rate, we let $_0 = 0.056$ as in Table I of Sangvinatsos and Wachter (2005) and set $= (_1; _2)'$ by matching the volatilities of interest rates both in our model and in their model. According to Table I in Sangvinatsos and Wachter (2005), the volatility of the interest rate they used is equal to 0.0217 (= $\sqrt{0.018^2 + 0.007^2 + 0.01^2}$),

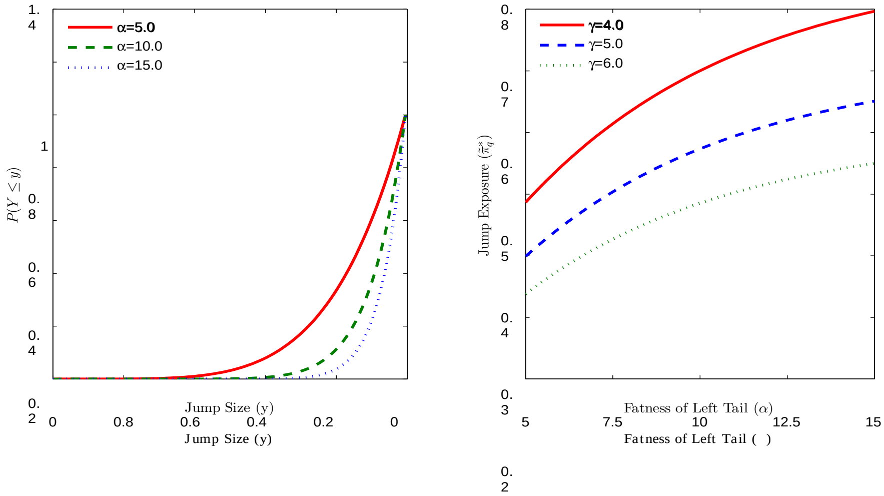

Figure 1: **Tail Fatness and Jump Exposure.** The first panel plots the left tail of stock returns with the various values of $= 5.0; 10.0$ and 15.0, respectively. In the second panel illustrates the jump exposure of ˜ corresponding to $= 4.0; 5.0$ ~~and 6.0 within a range of~~ s.

while the corresponding volatility in our model is $\sqrt{\;^2_1 \,{}^2_{11} + \;^2_2 \,{}^2_{22}} = \sqrt{\;^2_1 + \;^2_2}$. Then $= (\;_1; \;_2)'$ satisfies $\sqrt{\;^2 + \;^2} = 0.0217$.[12]

For the stock return process, we choose its parameters by equating the risk premium in stock returns in our model to the one of the model in Sangvinatsos and Wachter (2005). For simplicity, we assume the jump size distributions are the same under both the physical probability P and the risk-neutral probability Q while the jump frequency ^{Q} under Q is larger than the jump frequency ^{P} under P reflecting a positive jump risk premium for the investor to hold jump risk. We set $_{S1} = -0.10$, $_{S2} = 0.10$. In Sangvinatsos and Wachter (2005), they report the stock return's risk premium in Tables II and III as $[-1.255 \times (-0.563) + 0.572 \times (-0.245) + (-2.946) \times (-0.219) + 14.277 \times 0.44]/100 = 7.49\%$. Thus, ^{P} and g^P satisfy the following equation

$$_{S1} \times (-0.563) + \;_{S2}(-0.245) + \;^P g^P - \;^Q g^Q = 7.49\%;$$

where $g^Q = g^P = \frac{1}{1+ \;z_0} - 1$. With the parameters calibrated above, Table 1 reports the optimal bond/stock ratios. To investigate how the bond/stock ratio changes with the

[12] The solutions for $= (\;_1; \;_2)'$ are clearly not unique. The results reported in Tables 1 and 2 remain qualitatively similar when we vary the parameters $_1$ and $_2$. This is also the case for the parameters in the stock return process detailed below.

	No jumps (= 0)	Jumps (> 0) with various s				
		5	10	15	20	25
2	43.9201	67.1450	61.4360	61.4360	61.4360	61.4360
3	44.1629	64.3945	53.0696	50.4196	50.4196	50.4196
4	44.4057	63.3998	52.4153	49.3810	48.2053	47.6686
5	44.6485	63.0248	52.1821	49.2377	48.1044	47.5888
6	44.8913	62.9416	52.1441	49.2433	48.1316	47.6270
7	45.1341	63.0192	52.2130	49.3314	48.2302	47.7313

Table 1: **Bond/Stock Ratio**. This table reports the optimal bond/stock ratios. The relative risk aversion coefficient varies in the set $\{2,3,4,5,6,7\}$ and the tail parameter ranges in the set $\{5,10,15,20,25\}$ with the other fixed parameters as follows: $T = 5;\ {}_1 = {}_2 = 0;\ {}_{X_{11}} = {}_{X_{22}} = 1; K_1 = 0.5760; K_2 = 3.3430;\ {}_0 = 0.0560;\ {}_1 = 0.0180;\ {}_2 = 0.0122;\ {}_P = 0.2500;\ {}_Q = 0.5000;\ {}_{S1} = -0.1000;\ {}_{S2} = 0.1000; \Lambda_1 = -0.5630; \Lambda_2 = -0.2450$. The maturities of two bonds are T and $2T$, respectively. In addition, ${}_{S3} = 0.1023$ and $\Lambda_3 = 0.4215$ in the pure-diffusion model.

relative risk aversion coefficient and the tail parameter , we vary and to test their effects on the optimal bond/stock ratio in Table 1. In our setting, as shown in each column where we vary only from two to seven, the optimal bond/stock ratio first decreases with and then increases with . This confirms the prediction of the theoretical results, that is, the hedging demand assumption loses its explanatory power for the asset allocation puzzle in the presence of jumps in stock returns. Next, we vary the parameter . As shown in each row of Table 1, the optimal bond/stock ratio decreases with across all s. The underlying reason is that the left tail of the stock returns becomes fatter when decreases and thus the investor reduces her jump exposure $\tilde{}_q$ in stocks reflecting her fear of jump risks. As a result, by (25), the bond/stock ratio is bigger for smaller . This is also confirmed by the right panel of Figure 1, illustrating how the jump exposure $\tilde{}_q$ responds to and . It is clearly shown that that for a given , $\tilde{}_q$ increases with due to the less fear of tail risk and that for a given , $\tilde{}_q$ decreases with due to more risk aversion.

To compare with the pure-diffusion model discussed in the second paragraph from the end of the previous section, we estimate the model by matching the first two moments in the pure-diffusion model and jump-diffusion model with $= 5.0$. The second column under "No jumps" in Table 1 reports the bond/stock ratios, clearly indicating that the asset allocation is resolved. Interestingly, given a value of , the bond/stock ratios in the pure-diffusion model are much smaller than the ones in the jump-diffusion model reported in the rest columns of Table 1, as the stock holding $_S = \Lambda_3 =$ in the pure-diffusion model is much larger than the one $\tilde{}_q$ in the jump-diffusion model, again by (25), leading to smaller bond/stock ratios in the pure-diffusion model.

6 Conclusion

In this paper, we obtain the semi-analytical solutions to the optimal dynamic portfolio choice problem in multi-asset affine jump-diffusion models where both stock returns and state variables may exhibit time-varying jumps. More specifically, our semi-analytical formulas for the indirect value function and the optimal portfolio weights are obtained in terms of the solutions to a set of ODEs for HARA preferences. Our results extend the pure-diffusion models in Liu (2007) by incorporating jumps into both stock returns and state variables.

We further apply the theoretical results to investigate the bond-stock mix puzzle. In particular, our analysis shows that unlike in pure-diffusion models, there is no clear- cut answer to the bond/stock ratio puzzle in jump-diffusion models despite the hedging assumption. This result then provides a new channel to understand the nature of this well-known problem, and accordingly, the result further strengthens the claim made by Lioui (2007) that the asset allocation puzzle is still a puzzle.

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