Dynamic Properties of Condensing Particle Systems

by

Thomas Keith Rafferty

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Declarations

This work has been composed by myself and has not been submitted for any other degree of professional qualification.

- The work done in Chapters 3 and 4 has been submitted to Annales de l'Institut Henri Poincaré and is under the third round of reviews.
- Work done in Chapters 5 and 6 will be submitted for publication.
- The construction of the growth process given in Appendix A.2 was first given in my Master's thesis, which can be found on `www.warwick.ac.uk/trafferty`.

Abstract

Condensation transitions are observed in many physical and social systems, ranging from Bose-Einstein condensation to traffic jams on the motorway. The understanding of the critical phenomena prevalent in these systems presents many interesting mathematical challenges. We are interested in understanding the various definitions of condensation which are suitable in the field of stochastic particle systems and how they are related. Furthermore, we are also interested in dynamic properties of processes that undergo the condensation transition, such as typical convergence time scales and monotonicity properties.

Condensation can be defined in many different ways; considering the thermodynamic limit, a weak law of large numbers for the maximum occupation number, and an infinite particle limit on fixed finite lattices. For the latter definition, and processes that exhibit a family of stationary product measures, we prove an equivalent characterisation in terms of sub-exponential distributions generalising previous known results.

All known examples of condensing processes that exhibit homogeneous stationary product measures are non-monotone, i.e. the dynamics do not preserve a partial ordering of the state space. This non-monotonicity is typically characterised by an overshoot of the canonical current, which on a heuristic level is related to metastability. We prove that these processes with a finite critical density are necessarily non-monotone confirming a previous conjecture. If the critical density is infinite, condensation can still occur on finite lattices. We present partial evidence that there also exist monotone condensing processes.

We also study the typical convergence time scales of condensing inhomogeneous zero-range processes. Our results represent a first rigours calculation of the relaxation time of a condensing zero-range process, where we prove a dynamic transition in the order of the relaxation time as the density crosses a critical value. We also derive bounds for homogeneous condensing models and obtain results consistent with known metastable time scales.

CHAPTER 1

Introduction

Condensation is ubiquitous in nature. In addition to the classical definition of vapour condensation, it has also been observed in the context of quantum mechanics. For example, first predicted by Einstein in 1924 [1] and since experimentally observed in 1995 [2] Bose-Einstein condensation occurs in a dilute gas of bosons cooled to absolute zero. Condensation can also occur in population models and most notably in Kingman's model of the distribution of the fitness of a population undergoing selection and mutation [3]. When the rate of selection dominates the rate of mutation, a condensation transition occurs since a positive proportion of the population in later generations takes an optimal fitness [4]. The growth of complex networks such as the World Wide Web may also exhibit a condensation transition due to a preferential effect as new nodes (or links) aim to connect to nodes which are already popular to increase visibility, resulting in the "rich get richer" or "winner takes all" phenomena [5]. Condensation and the "rich get richer" phenomena have also been observed in simple economic models of the distribution of wealth in a population [6]. Simple particle models of traffic dynamics with applications to the transport of mass in cells and on the road network, known as stochastic particle systems, have been shown to condense/jam due to local interactions [7] or system defects [8, 9]. In this thesis, we study condensation in stochastic particle systems through system defects or particle interactions.

Stochastic particle systems, also known as lattice gases, are probabilistic models describing transport of a conserved quantity on discrete geometries or lattices where the time evolution is normally specified by giving the infinitesimal rates at which transitions occur. Many well known examples are introduced in [10], including zero-range processes and exclusion processes. These are both special cases of the more general family of misanthrope processes introduced in [11]. Originally motivated by statistical mechanics to gain a better understanding of critical phenomena such as symmetry breaking and phase transitions, stochastic particle systems also represent a natural extension to the theory of Markov processes. Typically, the global evolution of stochastic particle systems is Markovian, however, the trajectories of single particles are not. This local non-Markovian behaviour is due to local interactions and leads to difficulties in calculating typical convergence time scales of the process and hydrodynamic limits, which describe the large scale dynamics of the processes [12].

Recent research has focused on understanding the dynamic properties of condensing stochastic particle systems. An example of such a system is the zero-range process, which is a stochastic particle system without restriction on the local occupation numbers and the jump rate only depends on the number of particles on the departure site. It is known that condensation occurs if the jump rate decreases sufficiently slowly with the number of particles, see for example [13, 7]. First results on the nature of the condensate in zero-range processes are discussed in [14, 15]. In the condensed phase the canonical current typically exhibits a large overshoot above its value in the thermodynamic limit [16], which leads to a metastable switching (hysteresis) between a fluid and condensed phase. This switching phenomenon is related to a separation of time scales and therefore metastability, which has been rigorously established for a condensing zero-range process in [17] on finite lattices and in [18] for the thermodynamic limit. Before reaching stationarity the dynamics of the process in the condensed phase correspond to a coarsening process as increasingly large clusters appear on a decreasing number of sites [19, 20, 21]. Coarsening, hysteresis, and metastability are not only features of the zero-range process but are also found in other condensing stochastic particle systems. The inclusion process is similar to the zero-range process with the addition of an attractive component where now the jump rates depend on departure and arrival sites. Coarsening and metastability results for the condensing inclusion process have been established rigorously in [22, 23] for finite lattices and heuristically in [24] in the thermodynamic limit.

A classic and important problem in the theory of Markov processes and stochastic particle systems is characterising typical convergence times to stationarity. For condensing systems this is heuristically dominated by the motion of the condensate. In general, two important measures of convergence are the mixing time and relaxation times, where the mixing time is the time it takes for the processes to reach the invariant measure and the relaxation time controls the exponential rate of decay of correlations. In statistical physics, Markov chains arise for example in Monte Carlo simulations of complex processes, where the mixing time not only controls the number of steps needed to sample from the stationary measure but also has deep connections to the spatial properties spin systems [25, 26, 27, 28]. For stochastic particle systems such as the exclusion and zero-range processes sharp bounds on the relaxation time are only known in certain cases and results typically rely on monotonicity and coupling tools. By a simple mapping, the symmetric exclusion process and the symmetric constant rate zero-range process on the one dimensional torus can be shown to be equivalent and therefore have the same relaxation time [29]. Furthermore, the relaxation time for the constant rate zero-range process on any geometry can be calculated from comparisons of the process on the complete graph [30], provided the process is reversible. In [31] a recursive method of bounding the relaxation time was found by a decomposition of the state space, a method which was first introduced for the Kawasaki Ising model [32] and has since been used to give crude bounds for general Markov chains [33].

The mixing time of a Markov chain characterises how fast a process approaches stationarity and can exhibit highly non-trivial behaviour. An example of this is the cutoff phenomenon, where the distance stays close to its maximal value, then drops suddenly to a small value and then tends to zero exponentially fast, which is characterised by the relaxation time. The cutoff phenomenon was first discovered in card shuffling problems [34], which heuristically implies that it takes roughly 7 riffle shuffles to adequately mix a deck of 52 cards. Cutoff then allows us to restate the question of convergence from asking "how close to stationarity are we after say 10^6 steps?" to asking "will 7 steps suffice?" For an early review of the cutoff phenomenon in card shuffling and urn models see [35]. For stochastic particle systems establishing a cutoff is extremely difficult and results are only known for the exclusion process on various underlying geometries. For the complete graph exclusion process, a result was first obtained in the case of n particles and $2n$ sites by a comparison of the process to the Bernoulli-Laplace diffusion model [36], and has since been generalised in [37] by restating the problem as a birth-death process. Cutoff for the symmetric simple exclusion process on the one dimensional torus was established in the sequence of papers [38, 39, 40] by comparing the process to the discrete heat equation and coupling the process with dynamics of an interface first developed in [41].

In this thesis, we calculate the relaxation time of a condensing inhomogeneous zero-range process by a decomposition of the state space and give a heuristic description of the mixing time and cutoff properties. In addition, we study the monotonicity properties of homogeneous condensing processes and how an overshoot of the canonical current can lead to a transition of the relaxation and mixing times of a projection of the underlying process.

This thesis is organised as follows. In Chapter 2 we construct continuous time Markov processes and the interacting particle systems we study in this thesis. We also give a brief overview of the coupling techniques and define the relaxation, mixing, and hitting times. We characterise condensation in the thermodynamic limit and infinite particle limit in Chapter 3 and discuss the links between the two definitions. The main result of this chapter is an equivalence between the stationary measures of condensing processes and measures with sub-exponential tails. In Chapter 4 we prove that all condensing particle systems with stationary product measures with a finite critical density are necessarily non-monotone by showing non-monotone behaviour of the expected value of a test function related to the canonical current. We also discuss the monotonicity properties of misanthrope, long-range misanthrope, generalised zero-range, and chipping processes. We compute the relaxation time for a special class of condensing zero-range processes in Chapter 5 and give heuristic results on the mixing time and cutoff properties. In Chapter 6 we provide sharp bounds on the mixing and relaxation times for the projection chain of a condensing homogeneous zero-range process. Finally, in Chapter 7 we give a brief review of the work done in this thesis followed by a short discussion on possible directions of future research.

CHAPTER 2

Interacting Particle Systems

In this chapter, we construct the interacting particle systems which are studied in this thesis and summarise the results which are most relevant. We also introduce the key concepts and results treated in this work, such as couplings and monotonicity, and relaxation times.

The construction of interacting particle systems presented here closely follows [42, 43]. To construct couplings of interacting particle systems we follow [42, 44] and review [45]. Discussions on relaxation, mixing and hitting times largely follow [44, 46].

2.1 Markov processes, semi-groups, generators and the master equation

Interacting particle systems are continuous time Markov processes denoted by $(\eta(t))_{t\geq 0}$ with state space $\Omega = E^\Lambda$ where Λ is a finite or countable lattice and E is a countable set. The dynamics are typically specified by giving the (infinitesimal) rates for transitions to occur between two states in the state space.

The state space $\Omega = E^\Lambda$ of the process is the set of all possible configurations, for example the set E is given by $\mathbb{N}$ for zero-range dynamics or $\{0, 1\}$ for exclusion processes, which are discussed in Section 2.3. Configurations are denoted by Greek letters $\eta = (\eta_x)_{x\in\Lambda} \in \Omega$, where $\eta_x \in E$ denotes the occupation of the site x for each $x \in \Lambda$. In addition, $\eta_x(t)$ denotes the occupation of site $x \in \Lambda$ at time t. The state space Ω is endowed with the product topology which is metrizable, with measurable structure given by the Borel σ-algebra $\mathcal{B}$.

The time evolution of the processes is given by sample paths from the canonical path space

$$D[0,\infty) = \{\eta(\cdot) : [0,\infty) \to \Omega \mid \eta(\cdot) \text{ is right continuous and has left limits}\} \ .$$

Let $\mathcal{F}$ be the smallest σ-algebra on $D[0,\infty)$ relative to which all functions $\eta(\cdot) \mapsto \eta(s)$ for $s \geq 0$ are measurable. For $t \in [0,\infty)$, let $\mathcal{F}_t$ be the smallest σ-algebra relative to which all functions $\eta(\cdot) \mapsto \eta(s)$ for $0 \leq s \leq t$ are measurable. The filtered space $(D[0,\infty), \mathcal{F}, \mathcal{F}_t)$ serves as a generic choice for the probability space of the process.

Definition 2.1.1. *A Markov process on Ω is a collection of $\{\mathbb{P}_\eta : \eta \in \Omega\}$ of probability measures on $D[0,\infty)$ indexed by initial configurations $\eta \in \Omega$ with the following properties;*

(i) $\mathbb{P}_\eta[\xi(\cdot) \in D[0,\infty) : \xi(0) = \eta] = 1$ *for all* $\eta \in \Omega$.

(ii) $\mathbb{E}_\eta[\xi(s+\cdot) \in A \,|\, \mathcal{F}_s] = \mathbb{P}_{\xi(s)}[A]$ *for every* $\eta \in A$ *and* $A \in \mathcal{F}$.

(iii) The mapping $\eta \mapsto \mathbb{P}_\eta[A]$ *is measurable for every* $A \in \mathcal{F}$.

For a Markov process the expectation corresponding to $\mathbb{P}_\eta$ will be denoted by $\mathbb{E}_\eta$ which is given by

$$\mathbb{E}_\eta[A] = \int_{D[0,\infty)} A \, d\mathbb{P}_\eta \ ,$$

for any measurable function A on $D[0,\infty)$ which is integrable with respect to $\mathbb{P}_\eta$.

Property (i) states that $\mathbb{P}_\eta$ is normalised on paths with initial configuration $\eta \in \Omega$. (ii) is the Markov property which ensures that the probability of some future event, conditioned on the history of the process up to some time s only depends on the configuration at time s. Property (iii) allows us to consider the process with arbitrary initial distribution ν on Ω, defined by

$$\mathbb{P}_\nu = \int_\Omega \mathbb{P}_\eta \nu(d\eta) \ .$$

The dynamics of the process are specified by the rates at which transitions from η to $\eta' \in \Omega$ denoted $c(\eta,\eta') \geq 0$, called transition rates. Intuitively the transition rates have the following meaning

$$\mathbb{P}_\eta[\eta(dt) = \eta'] = c(\eta,\eta')dt + o(dt) \quad \text{as} \quad dt \searrow 0 \quad \text{for} \quad \eta \neq \eta' \ , \tag{2.1}$$

i.e. in a small time window dt the process transitions from η to η' with probability approximately given by $c(\eta,\eta')dt$. The transition rate $c(\cdot,\cdot)$ is assumed to be a non-negative, uniformly bounded and continuous function of η and η' in the product topology on Ω.

In this thesis we study processes which conserve the particle number, on finite lattices ($|\Lambda| < \infty$), called driven diffusive systems or lattice gases. On finite lattices the state space is not necessarily compact but countable, and therefore the processes we consider are Markov chains and their constructions can be found in [42, 47]. The construction of theses processes on infinite lattices must be done on a case-by-case basis, for example [48, 49] and [50] for a zero-range process. Throughout this work, we focus on processes defined on finite state spaces with a fixed number of particles and consider their properties in the thermodynamic limit or as the particle number diverges.

We now define Markov semigroups and state the main results which show one-to-one correspondence between Markov semigroups and processes. Let $C(\Omega)$ denote the set of continuous bounded functions $f : \Omega \to \mathbb{R}$, which is regarded as a Banach space with $||f|| := \sup_{\eta\in\Omega} |f(\eta)|$.

Definition 2.1.2. *For a given process $\{\mathbb{P}_\eta : \eta \in \Omega\}$, for each $t \geq 0$ we define the operator $S(t) : C(\Omega) \to C(\Omega)$ by*

$$(S(t)f)(\eta) := \mathbb{E}_\eta\left[f(\eta(t))\right] \ . \tag{2.2}$$

A Markov process is said to be a Feller process if $S(t)f \in C(\Omega)$ for every $t \geq 0$ and $f \in C(\Omega)$.

The properties of the linear operators $\{S(t), t \geq 0\}$ arising from Feller processes $\{P_\eta : \eta \in \Omega\}$ are given in the following proposition.

Proposition 2.1.3. *Suppose $\{P_\eta : \eta \in \Omega\}$ is a Feller process on Ω. Then the collection of linear operators $\{S(t) : t \geq 0\}$ on $C(\Omega)$ has the following properties;*

(i) $S(0) = I$, *the identity operator on* $C(\Omega)$.

(ii) The mapping $t \mapsto S(t)f$ *from* $[0, \infty) \to C(\Omega)$ *is right continuous for every* $f \in C(\Omega)$.

(iii) $S(s + t)f = S(s)S(t)f$ *for every* $s, t \geq 0$ *and* $f \in C(\Omega)$.

(iv) $S(t)\mathbb{1} = \mathbb{1}$ *for all* $t \geq 0$.

(v) $S(t)f \geq 0$ *for all non-negative* $f \in C(\Omega)$.

Proof. See, for example, [42, Proposition 1.3]. □

Definition 2.1.4. *A family $\{S(t) : t \geq 0\}$ of linear operators on $C(\Omega)$ that satisfies conditions $(i) - (v)$ of Proposition 2.1.3 is called a Markov semigroup.*

The operator $S(t)$ determines the time evolution of functions $f \in C(\Omega)$, which are interpreted as observables. Markov semigroups are in one-to-one correspondence with Markov processes outlined in Proposition 2.1.3 and the following theorem.

Theorem 2.1.5. *Suppose $\{S(t) : t \geq 0\}$ is a Markov semigroup on $C(\Omega)$. Then there exists a unique Feller Markov process $\{\mathbb{P}_\eta : \eta \in \Omega\}$ such that* (2.2) *holds for all $t \geq 0$.*

Proof. See for example [42, 47] $\square$

Therefore, the semigroup provides a full representation of the Markov process, dual to the path measures $\{\mathbb{P}_\eta : \eta \in \Omega\}$ since $C(\Omega)$ is dual to the set $\mathcal{P}(\Omega)$ of probability measures on Ω. The expectation of observables at $t \geq 0$ with respect to the initial distribution $\nu \in \mathcal{P}(\Omega)$ is given by

$$\mathbb{E}_\nu[f(\eta(t))] = \int_\Omega (S(t)f)(\xi)\nu[d\xi] = \int_\Omega S(t)f \, d\nu \quad \text{for all} \quad f \in C(\Omega) \ .$$

From property (iii) of Proposition 2.1.3 we expect $\{S(t) : t \geq 0\}$ has an exponential form characterised by $S'(0)$, the time derivative of $S(t)$ at zero, in the sense

$$S(t) = \text{``}e^{S'(0)t}\text{''} = I + S'(0)t + o(t) \quad \text{with } S(0) = I \ . \tag{2.3}$$

This is made precise as follows.

Definition 2.1.6. *The (infinitesimal) generator $\mathcal{L} : D_{\mathcal{L}} \to C(\Omega)$ for the process $\{S(t) : t \geq 0\}$ is given by*

$$\mathcal{L}f = \lim_{t \searrow 0} \frac{S(t)f - f}{t} \quad \text{for all } f \in D_{\mathcal{L}} \ , \tag{2.4}$$

where the domain $D_{\mathcal{L}} \subseteq C(\Omega)$ is the set of all functions for which the limit exists.

For finite state spaces, Ω, $D_{\mathcal{L}} = C(\Omega)$, otherwise one often has to restrict to bounded cylinder test functions [42].

Proposition 2.1.7. *$\mathcal{L}$ as defined by* (2.4) *is a Markov generator, i.e.*

(i) $\mathbb{1} \in D_{\mathcal{L}}$ and $\mathcal{L}\mathbb{1} = \mathbb{1}$ (Conservation of probability).

(ii) For all $f \in D_{\mathcal{L}}$ and $\lambda > 0$, $\min_{\xi\in\Omega} f(\xi) \geq \min_{\xi\in\Omega}(f - \lambda\mathcal{L}f)(\xi)$ (Positivity).

(iii) $D_{\mathcal{L}}$ is dense in $C(\Omega)$ and the range $\mathcal{R}(I - \lambda\mathcal{L}) = C(\Omega)$ for λ sufficiently small.

Proof. See for example [42, Proposition 2.2] □

Theorem 2.1.8. *(Hille-Yosida) There is a one-to-one correspondence between Markov generators and semigroups on* $C(\Omega)$*, given by* (2.4) *and*

$$S(t)f = e^{t\mathcal{L}}f := \lim_{n\to\infty}\left(I - \frac{t}{n}\mathcal{L}\right)^{-n} f \text{ for } f \in C(\Omega) \text{ and } t \geq 0 \ . \tag{2.5}$$

Furthermore, if $f \in D_{\mathcal{L}}$ *then* $S(t)f \in D_{\mathcal{L}}$ *for all* $t \geq 0$*, and*

$$\frac{d}{dt}S(t)f = S(t)\mathcal{L}f = \mathcal{L}S(t)f \ , \tag{2.6}$$

called the forward and backward equations respectively.

Proof. See for example [42, Theorem 2.9] □

Finally, for finite systems with a finite state space, the generator can be computed directly from (2.2) and the heuristic (2.1) as follows

$$\begin{aligned} S(dt)f = \mathbb{E}_\eta[f(\eta(t)] &= \sum_{\eta'\in\Omega} f(\eta')\mathbb{P}_\eta[\eta(dt) = \eta'] \\ &= \sum_{\eta'\in\Omega} c(\eta,\eta')f(\eta')dt + f(\eta)\left(1 - \sum_{\eta'\neq\eta} c(\eta,\eta')dt\right) + o(dt) \ . \end{aligned}$$

Then (2.4) implies

$$\mathcal{L}f(\eta) = \sum_{\eta'\neq\eta} c(\eta,\eta')\left(f(\eta') - f(\eta)\right) \ . \tag{2.7}$$

Markov processes and semigroups are therefore characterised by the transition rates $c(\eta,\eta')$ between states η and η'. We use the convention that $c(\eta,\eta') = 0$ for all $\eta = \eta' \in \Omega$.

From the generator and semigroup definitions of finite state Markov processes, we can immediately construct the master equation as follows: Consider the indicator functions $\mathbb{1}_\eta : \Omega \to \{0,1\}$, which are bounded and form a basis of $C(\Omega)$ for finite Ω, defined by

$$\mathbb{1}_\eta(\xi) = \begin{cases} 1 & \text{if } \xi = \eta \\ 0 & \text{otherwise} \end{cases} \ .$$

Let $p_t = \mu S(t)$ denote the distribution on Ω at time t characterised by

$$p_t[f] = \int_\Omega S(t) f d\nu \ . \tag{2.8}$$

Substituting $f(\eta) = \mathbb{1}(\eta)$ into the forward equation (2.6) we have

$$\begin{aligned} \frac{d}{dt} p_t[\eta] &= \int_\Omega S(t)\mathcal{L}\mathbb{1}_\eta d\nu = \sum_{\xi\in\Omega} p_t[\xi] \sum_{\xi'\in\Omega} c(\xi,\xi')\left(\mathbb{1}_\eta(\xi') - \mathbb{1}_\eta(\xi)\right) \\ &= \sum_{\xi\in\Omega} p_t[\xi] c(\xi,\eta) - p_t[\eta] \sum_{\xi'\in\Omega} c(\eta,\xi') \ , \end{aligned} \tag{2.9}$$

which is called the master equation.

2.2 Stationary measures, reversibility and ergodicity

Definition 2.2.1. *A probability measure $\mu \in \mathcal{P}(\Omega)$ is said to be stationary or invariant if it satisfies $\mu\left(S(t)f\right) = \mu(f)$ for all $t \geq 0$ and $f \in C(\Omega)$. The measure is called reversible if $\mu\left(fS(t)g\right) = \mu\left(gS(t)f\right)$ for all $f, g \in C(\Omega)$.*

Here and throughout this thesis we use the notation $\mu(f) = \int_\Omega f\, d\mu$ to denote the expectation of $f \in C(\Omega)$ with respect to the measure μ on Ω.

It is clear that every reversible measure μ is stationary (taking $g \equiv 1$). The probabilistic interpretation of a stationary measure μ is given by a process $\eta(t)$ with initial distribution μ has the same distribution as $\eta(t+s)$ for every $s \in [0,\infty)$, or formally

$$\mathbb{P}_\mu[\eta(\cdot) \in A] = \mathbb{P}_\mu[\eta(t+\cdot) \in A] \text{ for all } t \geq 0,\ A \in \mathcal{F} \ .$$

Equivalently, using (2.8) if $p_0 = \mu$ then $p_t = \mu$ for all $t \geq 0$.

Proposition 2.2.2. *A measure $\mu \in \mathcal{P}(\Omega)$ is stationary if and only if*

$$\mu(\mathcal{L}f) = 0 \ \textit{for all } f \in D_\mathcal{L} \ .$$

Furthermore, the measure μ is reversible if and only if

$$\mu(f\mathcal{L}g) = \mu(g\mathcal{L}f) = 0 \ \textit{for all } f, g \in D_\mathcal{L} \ .$$

Proof. See for example [42, Propisition 2.13]. □

By a similar approach to the construction of the master equation, reversible measures can be characterised via the following proposition.

Proposition 2.2.3. *A measure μ on a countable state space Ω is reversible for the process with transition rates $c(\cdot,\cdot)$ if and only if it fulfils the detailed balance conditions*

$$\mu[\eta]c(\eta,\xi) = \mu[\xi]c(\xi,\eta) \quad \textit{for all} \quad \eta,\xi \in \Omega \ .$$

Definition 2.2.4. *A Markov process with semigroup $\{S(t) : t \geq 0\}$ is ergodic if there exists a unique stationary measure $\pi \in \mathcal{P}(\Omega)$ and*

$$\lim_{t\to\infty} p_t = \pi \ \textit{ for all initial distributions } p_0 \ ,$$

where p_t is the distribution of the process at time t given by (2.8)*.*

Definition 2.2.5. *A Markov process $\{\mathbb{P}_\eta : \eta \in \Omega\}$ is called irreducible if for all $\eta, \eta' \in \Omega$*

$$\mathbb{P}_\eta[\eta(t) = \eta'] > 0 \ \textit{ for all } t \geq 0 \ .$$

The interpretation of an irreducible Markov process is as follows; an irreducible process can sample the entire state space from any initial condition. Irreducibility implies the uniqueness of the stationary measure and if the state space Ω is finite then the process is ergodic as outlined in the following theorem.

Theorem 2.2.6. *An irreducible Markov process with finite state space Ω is ergodic.*

Proof. See for example [47] □

2.3 Example processes

2.3.1 The zero-range process

The zero-range process (ZRP), introduced in [10], is a stochastic particle system on the state space $\Omega_L = \mathbb{N}^\Lambda$ where $\Lambda = \{1,\ldots,L\}$. A single particle at site x leaves at rate $g_x(\eta_x)$ and jumps to site y with probability $p(x,y)$ where the dynamics are defined by the generator

$$\mathcal{L}f(\eta) = \sum_{x,y\in\Lambda} g_x(\eta_x)p(x,y)\left(f(\eta^{x,y}) - f(\eta)\right) \tag{2.10}$$

for $f \in C(\Omega_L)$. Here $\eta^{x,y}$ denotes the configuration after a single particle has jumped from site x to y and is given by

$$\eta^{x,y}_z = \begin{cases} \eta_x - 1 & \text{if } z = x \\ \eta_y + 1 & \text{if } z = y \\ \eta_z & \text{otherwise} \end{cases} .$$

To ensure the process is non-degenerate and irreducible, the jump rates $g_x(n)$ satisfy $g_x(n) = 0$ for all $x \in \Lambda$ if and only if $n = 0$. The process is called homogeneous if $g_x(n) = g_y(n)$ for all $x, y \in \Lambda$ and $n \in \mathbb{N}$ and inhomogeneous otherwise. Throughout this thesis we study both homogeneous and inhomogeneous zero-range processes. Zero-range processes and similar models are often studied in a translation invariant setting ($p(x, y) = q(|y - x|)$) on a regular lattice with periodic boundary conditions. Typical choices in one dimension are symmetric and totally asymmetric transition probabilities with $p(x, y) = \frac{1}{2}\delta_{y,x+1} + \frac{1}{2}\delta_{y,x-1}$, $p(x, y) = \delta_{y,x+1}$, or fully connected transition probabilities $p(x, y) = (1 - \delta_{y,x})\frac{1}{L-1}$, respectively.

Well studied zero-range processes include the constant rate zero-range process where $g_x(n) = 1$ for all $x \in \Lambda$ and $n > 0$, which is a system of L server queues with mean one exponential random service times. The constant rate zero-range process can be extended to include site defects where $g_x(n) = r < 1$ for some $x \in \Delta \subseteq \Lambda$, which are studied in Chapter 5. If $g(k) = k$ then the zero-range process is a system of independent random walkers on Λ.

Under translation invariant $p(x, y)$, zero-range processes defined by the generator (2.10) exhibit a family of stationary product measures $\{\nu^L_\phi : \phi \in D_\phi\}$ on Ω_L, where ϕ is called the fugacity and $D_\phi = [0, \phi_c)$ or $[0, \phi_c]$ is the domain of the family of measures [48]. $\nu^L_\phi[\cdot]$ is given by

$$\nu^L_\phi[\eta] = \prod_{x\in\Lambda} \nu^x_\phi[\eta_x] \quad \text{where} \quad \nu^x_\phi[n] = \frac{w_x(n)\phi^n}{z_x(\phi)} . \tag{2.11}$$

The measures exist for all $\phi \in D_\phi$ where ϕ_c is the determined by the radii of convergence of the single site partition function

$$z_x(\phi) = \sum_{k=0}^{\infty} w_x(k)\phi^k . \tag{2.12}$$

Technically, $\phi_c = \min_{x\in\Lambda} \phi^x_c$ where ϕ^x_c is the radius of converge of $z_x(\phi)$ and is given by

$$\phi_c^x = \left(\limsup_{n\to\infty} \sqrt[n]{w_x(n)} \right)^{-1} . \tag{2.13}$$

The stationary weights $w_x(n)$ are given by

$$w_x(n) = \prod_{k=1}^{n} \frac{1}{g_x(k)} \ \text{ for } n > 0 \ \text{ and } w_x(0) = 1 \ \text{ for all } x \in \Lambda . \tag{2.14}$$

The family $\{\nu_\phi[\cdot] : \phi \in D_\phi\}$ is called the grand-canonical ensemble and $z_x(\phi)$ are called the grand-canonical partition functions. The (singe site) grand-canonical densities are functions of the fugacity $\phi \in D_\phi$ and are given by

$$\rho_x(\phi) := \nu_\phi^x(\eta_x) = \frac{1}{z_x(\phi)} \sum_{n=1}^{\infty} n w_x(n) \phi^n . \tag{2.15}$$

Since the zero-range process conserves the particle number the process is irreducible on the finite state space

$$\Omega_{L,N} := \left\{ \eta \in \Omega_L \ : \ \sum_{x\in\Lambda} \eta_x = N \right\} . \tag{2.16}$$

Therefore, the process restricted to $\Omega_{L,N}$ is ergodic with a unique stationary measure on $\Omega_{L,N}$ given by

$$\pi_{L,N}[\cdot] := \nu_\phi^L \left[\cdot \middle| \eta \in \Omega_{L,N} \right] . \tag{2.17}$$

The family $\{\pi_{L,N} : N \in \mathbb{N}\}$ is called the canonical ensemble and these measures are independent of the fugacity ϕ. The measure $\pi_{L,N}[\cdot]$ can be easily shown to have the mass function

$$\pi_{L,N}[\eta] = \frac{1}{Z_{L,N}} \prod_{x\in\Lambda} w_x(\eta_x) \, \mathbb{1}(\eta \in \Omega_{L,N}) , \tag{2.18}$$

where canonical partition function is given by

$$Z_{L,N} = \sum_{\eta\in\Omega_{L,N}} \prod_{x\in\Lambda} w_x(\eta_x) . \tag{2.19}$$

2.3.2 The misanthrope process

Misanthrope processes are generalisations of zero-range processes, where the jump rate now depends on the exit and entry sites, and are defined by the generator

$$\mathcal{L}f(\eta) = \sum_{x,y\in\Lambda} r(\eta_x,\eta_y)p(x,y)\left(f(\eta^{x,y}) - f(\eta)\right)\ , \tag{2.20}$$

for $f \in C(\Omega_L)$. Again, the configuration $\eta^{x,y} \in \Omega_L$ denotes the configuration after a single particle jumps from site x to $y \in \Omega_L$. To ensure the process is non-degenerate the jump rate $r(n,m) = 0$ for all $m \geq 0$ if and only if $n = 0$, and $r(n,m) > 0$ for all $n > 0$ and $m \geq 0$.

Misanthrope processes include many well-known examples of interacting particle systems, such as zero-range processes [10], the inclusion process [51, 52], and the explosive condensation model [53]. It is known [11, 54] that misanthrope processes with translation invariant dynamics $p(x,y) = q(x-y)$ exhibit stationary product measures of the form (2.11) if and only if the rates fulfil

$$\frac{r(n,m)}{r(m+1,n-1)} = \frac{r(n,0)r(1,m)}{r(m+1,0)r(1,n-1)} \quad \text{for all} \quad n \geq 1, m \geq 0\ , \tag{2.21}$$

and, in addition, either

$$\begin{cases} q(z) = q(-z) \text{ for all } z \in \Lambda \text{ or,} \\ r(n,m) - r(m,n) = r(n,0) - r(m,0) \text{ for all } n,m \geq 0\ . \end{cases} \tag{2.22}$$

The corresponding stationary weights satisfy

$$\frac{w(k+1)}{w(k)} = \frac{w(1)}{w(0)}\frac{r(1,k)}{r(k+1,0)} \quad \text{and} \quad w(n) = \prod_{k=1}^{n} \frac{r(1,k-1)}{r(k,0)}\ . \tag{2.23}$$

In [45] generalised misanthrope processes have been introduced where more than one particle is allowed to jump simultaneously. They are defined via transitions $\eta \to \eta + n(\delta_y - \delta_x)$ for $n \in \{0,\ldots,\eta_x\}$ at rate $\Gamma^n_{\eta_x,\eta_y}(y-x)$ and the authors give necessary and sufficient conditions on the jump rates for the processes to be monotone.

2.3.3 Generalised zero-range processes

Zero-range processes can also be generalised to allow more than one particle to exit from site x in a single transition. The generalised zero-range process (gZRP) [45] is a stochastic particle system on the state space $\Omega_L = \mathbb{N}^\Lambda$ defined by the generator

$$\mathcal{L}^{gZRP} f(\eta) = \sum_{x,y\in\Lambda} \sum_{k=1}^{\eta_x} \alpha_k(\eta_x) p(x,y) \big(f(\eta^{x\to(k)y}) - f(\eta) \big) \ . \tag{2.24}$$

Here $\eta^{x\to(k)y} \in \Omega_L$ is the configuration after k particles have jumped from x to $y \in \Lambda$. The jump rates $\alpha_k(n)$ satisfy $\alpha_k(n) = 0$ if $k > n$, and we use the convention that empty summations are zero. We consider translation invariant $p(x,y)$ on a finite lattice $\Lambda = \{1, \ldots, L\}$ with periodic boundary conditions and we also note that the process preserves particle number $\sum_x \eta_x = N$.

It is known [55, 54] that these processes exhibit stationary product measures if and only if the jump rates have the explicit form

$$\alpha_k(n) = g(k) \frac{h(n-k)}{h(n)} \ , \tag{2.25}$$

where $g, h : \mathbb{N} \to [0, \infty)$ are arbitrary non-negative functions with h strictly positive. The stationary weights are then given by $w(n) = h(n)$.

2.4 Monotonicity and couplings

In this section, we will review the relevant results on monotone (attractive) interacting particle systems and give details on how to construct coupling which preserve a partial order of the state space.

We use the natural partial order on the state space $\Omega_L = \mathbb{N}^\Lambda$ given by $\eta \leq \zeta$ if $\eta_x \leq \zeta_x$ for all $x \in \Lambda$. A function $f : \Omega_L \to \mathbb{R}$ is said to be increasing if and only if $\eta \leq \zeta$ implies $f(\eta) \leq f(\zeta)$. Two measures μ_1, μ_2 on Ω are stochastically ordered (monotone) with $\mu_1 \leq \mu_2$, if for all increasing functions $f : \Omega_L \to \mathbb{R}$ we have for expectations $\mu_1(f) \leq \mu_2(f)$.

A stochastic particle system on Ω_L with generator $\mathcal{L}$ and semi-group $(S(t) = e^{t\mathcal{L}} : t \geq 0)$ is called monotone (attractive) if it preserves stochastic order in time, i.e.

$$\mu_1 \leq \mu_2 \quad \Longrightarrow \quad \mu_1 S(t) \leq \mu_2 S(t) \quad \text{for all } t \geq 0 \ ,$$

which is equivalent to

$$S(t)f(\eta) \leq S(t)f(\xi) \ \text{ for all } \eta \leq \xi \ \text{ and all } f \in C(\Omega) \ \text{ increasing and } t \geq 0 \ . \tag{2.26}$$

Utilising the Hille-Yosida Theorem 2.1.8 and the definition of the generator (2.4) we see that the process is monotone if and only if

$$\mathcal{L}f(\eta) \leq \mathcal{L}f(\xi) \ , \tag{2.27}$$

for all $\eta \leq \xi$ and all $f \in C(\Omega_L)$ increasing such that $f(\eta) = f(\xi)$. The condition $f(\eta) = f(\xi)$ is necessary since for the following to hold

$$\mathcal{L}f(\eta) = \lim_{t \searrow 0} \frac{S(t)f(\eta) - f(\eta)}{t} \leq \lim_{t \searrow 0} \frac{S(t)f(\xi) - f(\xi)}{t} = \mathcal{L}f(\xi) \ , \tag{2.28}$$

we need $f(\eta) \geq f(\xi)$, however, f is increasing and $\eta \leq \xi$ implies the equality.

Coupling techniques for monotone processes are important tools to derive rigorous results on the large scale dynamics of such systems such as hydrodynamic limits [45]. Let $(\eta(t))_{t\geq 0}$ be an interacting particle system on Ω_L. A Markov coupling of $(\eta(t))_{t\geq 0}$ with itself is a processes $(\xi(t), \zeta(t))_{t\geq 0}$ on $\Omega_L \times \Omega_L$ such that each marginal $\xi(t)$ and $\zeta(t)$ is distributed as the original process $(\eta(t))_{t\geq 0}$, *i.e.* if we observe one of the processes without observing the other, the process behaves as it is originally constructed.

The link between stochastic monotonicity and couplings is given by Strassen's theorem [56]:

Theorem 2.4.1. *(Strassen) For probability measures μ_1, μ_2 on a common state space Ω_L, $\mu_1 \leq \mu_2$ if and only if there exists a coupling μ on the product state space $\Omega_L \times \Omega_L$ such that*

$$\mu\left(\{\eta = (\eta_1, \eta_2) : \eta_1 \leq \eta_2\}\right) = 1 \ ,$$

i.e. the probability of observing the partial order is 1.

Strassen's theorem has the natural extension to couplings of stochastic processes and monotone processes.

2.4.1 Constructing a coupling for the zero-range process

In Chapter 5, we rely on known results of monotone zero-range process, therefore, here we include here a detailed construction of a coupling for the ZRP. Furthermore, understanding the construction of a coupled process is also necessary for results obtained in Chapter 4 on generalised zero-range and misanthrope processes.

Let $(\xi(t))_{t\geq 0}$ and $(\zeta(t))_{t\geq 0}$ be two zero-range processes defined via the same generator (2.10) such that their initial conditions satisfy $\xi \in \Omega_{L,N}$ and $\zeta \in \Omega_{L,N+1}$. It is sufficient, for our applications, to construct a coupling on the joint state space

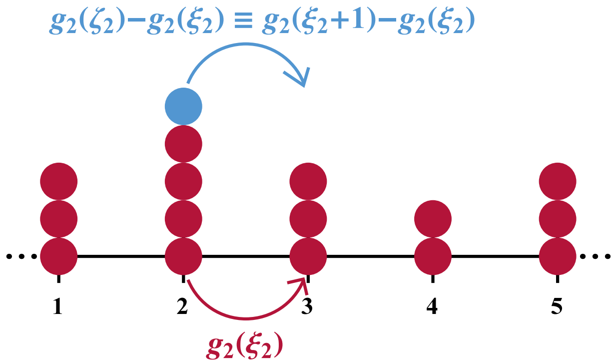

Figure 2.1: Example configuration of the coupled dynamics. The ξ process is shown in red and the second class particle is shown in blue. The jump rates are defined according to equation (2.29) and it is clear that the coupling can only be constructed for increasing jump rates $g_x(\cdot)$ as we need $g_x(m)-g_x(n) \geq 0$ for all $x \in \Lambda$ and $m \geq n$.

$(\Omega_{L,N}, \Omega_{L,N+1})$ between the processes $\xi(t)$ and $\zeta(t)$ such that, $\xi(t) = \eta(t) + \delta_y$ for some $y \in \Lambda$, *i.e.* there is an extra particle in the $\zeta(t)$ process at site y, often called a second class particle. The coupling is constructed via the following rules called a basic coupling

1. The marginals of the coupled process are two zero-range processes with N and $N+1$ particles respectively and each are defined by the generator (2.10). As a consequence, when the process has converged to its stationary measure for the joint process this is a coupling of the measures $\pi_{L,N}$ and $\pi_{L,N+1}$ [48].

2. Particles move together as much as possible.

If $g_x(n)$ is non-decreasing for each $x \in \Lambda$ the coupled process behaves via the following transition rates, which is illustrated in Figure 2.1; for the site with the second class particle

$$\left.\begin{array}{l} \xi_y = n+1 \\ \eta_y = n \end{array}\right\} \xrightarrow{g_y(\xi_y)-g_y(\eta_y)} \left\{\begin{array}{l} \xi_y = n \\ \eta_y = n \end{array}\right.$$
$$\left.\begin{array}{l} \xi_y = n+1 \\ \eta_y = n \end{array}\right\} \xrightarrow{\quad g_y(\eta_y) \quad} \left\{\begin{array}{l} \xi_y = n \\ \eta_y = n-1 \end{array}\right. , \tag{2.29}$$

and for the remaining sites, both processes jump at rate $g_x(\eta_x) = g_y(\xi_x)$.

In general, the generator for the basic coupling of the zero-range process on

$\Omega_L \times \Omega_L$ is given by

$$\begin{aligned}\mathcal{L}f(\xi,\zeta) = &\sum_{\substack{x,y\\ \xi_x\le\zeta_x}} g_x(\xi_x)p(x,y)\left(f(\xi^{x,y},\zeta^{x,y}) - f(\xi,\zeta)\right)\\ &+ \sum_{\substack{x,y\\ \xi_x\le\zeta_x}} \left(g_x(\zeta_x) - g_x(\xi_x)\right)p(x,y)\left(f(\xi,\zeta^{x,y}) - f(\xi,\zeta)\right)\\ &+ \sum_{\substack{x,y\\ \zeta_x\le\xi_x}} g_x(\zeta_x)p(x,y)\left(f(\xi^{x,y},\zeta^{x,y}) - f(\xi,\zeta)\right)\\ &+ \sum_{\substack{x,y\\ \zeta_x\le\xi_x}} \left(g_x(\xi_x) - g_x(\zeta_x)\right)p(x,y)\left(f(\xi^{x,y},\zeta) - f(\xi,\zeta)\right)\ . \end{aligned} \tag{2.30}$$

For the coupling constructed in (2.29) and (2.30) to exist, the jump rate $g_x(n)$ has to be non-decreasing.

Theorem 2.4.2. *The zero-range process on $\Omega_L = \mathbb{N}^\Lambda$ defined by the generator* (2.10) *is monotone if and only if the jump rates satisfy $g_x(m) \ge g_x(n)$ for all $m \ge n \in \mathbb{N}$ and $x \in \Lambda$.*

Proof. ($\Longleftarrow$) The condition $g_x(m) \ge g_x(n)$ for all $m \ge n \in \mathbb{N}$ and $x \in \Lambda$ implies (2.30) is a generator of a Markov process with non-negative rates. By substituting the functions $f(\xi,\zeta) = f_1(\xi)$ and $f(\xi,\zeta) = f_2(\zeta)$ into (2.30) it is clear that the marginals are zero-range processes with the generator (2.10). Therefore, by Strassen's theorem the ZRP with non-decreasing jump rates is monotone.

($\Longrightarrow$) Consider the increasing test function $f(\eta) = \eta_y$ and two configurations $\eta = n\delta_x$ and $\xi = m\delta_x$ such that $x \ne y$, $m \ge n$, and $p(x,y) > 0$. Clearly $\eta \le \xi$ and assuming the process is monotone the inequality (2.27) implies $g_x(n)p(x,y) \le g_x(m)p(x,y)$. Since the choice of x, y were arbitrary and $p(x,y) > 0$, we have $g_x(n) \le g_x(m)$ for all $m \ge n$ and $x \in \Lambda$, which completes the proof of Theorem 2.4.2. □

2.5 Mixing, hitting, and relaxation times

In this section, we define the relaxation, mixing, and hitting times used to measure how a stochastic process converges to its stationary measure. We also review the main results and techniques for calculating such convergence times for interacting particle systems. Throughout this discussion we only consider finite state spaces Ω.

The following definitions are necessary throughout this section. For a measure π on Ω and $f, g \in C(\Omega)$ the inner product is given by

$$\langle f, g\rangle_\pi = \sum_{\eta\in\Omega} \pi[\eta] f(\eta) g(\eta) \ ,$$

and

$$||f||_{2,\pi} = \left(\sum_{\eta\in\Omega} \pi[\eta] f(\eta)^2\right)^{1/2} \ .$$

2.5.1 The relaxation time and the spectral gap

The relaxation time of an ergodic Markov process characterises the exponential rate of convergence to the stationary measure. For reversible processes on a countable state space, the relaxation time is given by the reciprocal of the smallest non-zero eigenvalue of $-\mathcal{L}$, called the spectral gap, where $\mathcal{L}$ is the generator of the process.

For a Markov process $(\eta(t))_{t\geq 0}$ with generator $\mathcal{L}$ on Ω with stationary measure π, the Dirichlet form is given by

$$\mathcal{D}_\mathcal{L}(f) = \langle f, -\mathcal{L} f\rangle_\pi = -\sum_{\eta\in\Omega} \pi[\eta] f(\eta) \mathcal{L} f(\eta) \ ,$$

for $f \in C(\Omega)$. For a reversible process with generator (2.7) the Dirichlet form can be rewritten as

$$\mathcal{D}_\mathcal{L}(f) = \frac{1}{2} \sum_{\eta,\xi} \pi[\eta] c(\eta, \xi) \left(f(\xi) - f(\eta)\right)^2 \ . \tag{2.31}$$

Furthermore, it is easy to show [46, Lemma 2.1.2]

$$\frac{d}{dt} ||S(t)f||^2_{2,\pi} = -\mathcal{D}_\mathcal{L}(S(t)f) \ .$$

Let $\mathrm{Var}_\pi(f)$ denote the variance of a function $f : \Omega \to \mathbb{R}$ with respect to the measure π then the spectral gap and relaxation time are defined by the Rayleigh-Ritz principle as follows.

Definition 2.5.1. *The spectral gap $\lambda_\mathcal{L}$ of the generator $\mathcal{L}$ on Ω is given by the variational principle*

$$\lambda_\mathcal{L} = \inf_f \left\{ \frac{\mathcal{D}_\mathcal{L}(f)}{\mathrm{Var}_\pi(f)} \ : \ \mathrm{Var}_\pi(f) \neq 0 \right\} \ , \tag{2.32}$$

and the relaxation time is given by the inverse spectral gap $T^{rel}_\mathcal{L} := \frac{1}{\lambda_\mathcal{L}}$.

For a reversible process the value $\lambda_{\mathcal{L}}$ is then the difference between the smallest eigenvalues of the generator $-\mathcal{L}$.

Proposition 2.5.2. *Let $(\eta(t))_{t\geq 0}$ be an irreducible Markov process with stationary measure π on the state space Ω then $\lambda_{\mathcal{L}}$ is the optimal constant appearing in the inequality*

$$Var_\pi(S(t)f) \leq e^{-2\lambda_{\mathcal{L}} t} Var_\pi(f) \tag{2.33}$$

Proof. See for example [46, Lemma 2.1.4]. □

From Proposition 2.5.2, we see that the inverse spectral gap gives the characteristic time scale of the contraction of the variance of the kernel $S(t)$ towards stationarity, where $S(t)f(\eta) \to \pi(f)$ for all $\eta \in \Omega$.

2.5.2 Mixing times

The mixing time of a Markov process is another measure of how far the process is from the stationary distribution, which are measured by the total variation distance. The total variation distance between two measures μ and ν on Ω is given by

$$||\mu - \nu||_{TV} = \max_{A \in \Omega} |\mu[A] - \nu[A]| \ . \tag{2.34}$$

By Proposition 4.2 of [44] the total variation distance can be rewritten in the form

$$||\mu - \nu||_{TV} = \frac{1}{2} \sum_{\eta \in \Omega} |\mu[\eta] - \nu[\eta]| \ .$$

For a Markov process with semigroup $\{S(t) : t \geq 0\}$ on Ω let the distribution at time t and initial condition η be given by $P_t(\eta, \cdot) = \delta_\eta S(t)$. Let π be the stationary measure then the distance from stationarity is given by

$$d(t) = \max_{\eta \in \Omega} ||P_t(\eta, \cdot) - \pi||_{TV} \ , \tag{2.35}$$

and the ε-mixing time is defined as follows.

Definition 2.5.3. *The total variation ε-mixing time of a process generated by $\mathcal{L}$ on Ω with stationary measure π is given by*

$$T_{mix}(\epsilon) = \inf\{t \geq 0 \, : \, d(t) \leq \varepsilon\} \ .$$

In practice, mixing times of interacting particle systems are extremely difficult to calculate however there exists many methods of bounding mixing times by quantities which are easier to compute. For example, the total variation distance $||\mu - \nu||_{TV}$ can be given in terms of a coupling between the measures μ and ν (see Proposition 4.7 of [44])

$$||\mu - \nu||_{TV} = \inf\{\mathbb{P}(X \neq Y) \,:\, (X,Y) \text{ is a coupling of } \mu \text{ and } \nu\} \ .$$

Therefore, the total variation and mixing times of a process can be well approximated by the coupling time T_{couple} of a coupled process $(\xi(t), \zeta(t))_{t\geq 0}$, which is given by

$$T_{couple} = \inf\{t \geq 0 \,:\, \xi(t) = \zeta(t)\} \ .$$

In addition, the relaxation time gives upper and lower bounds for the mixing time of the form

$$\log\left(\frac{1}{2\varepsilon}\right) T_{\mathcal{L}}^{rel} \leq T_{mix}(\varepsilon) \leq \log\left(\frac{1}{\varepsilon\pi_\star}\right) T_{\mathcal{L}}^{rel} \ ,$$

where $\pi_\star = \min_{\eta\in\Omega} \pi[\eta]$ (see for example [44]). However, due to the inclusion of $\pi_\star$ in the upper bound this method typically does not give sharp bounds. Sharp upper and lower bounds can be found via hitting times of large sets [57], which are introduced in the next section.

2.5.3 Hitting times

For a Markov process $(\eta(t))_{t\geq 0}$ on the state space Ω the hitting time τ_A of a subset $A \subseteq \Omega$ is given by

$$\tau_A := \inf_{t\geq 0}\{t \geq 0 \,:\, \eta(t) \in A\} \ ,$$

and for simplicity we write $\tau_\eta = \tau_{\{\eta\}}$. The expected hitting time $H_A(\eta)$ of a set $A \subseteq \Omega$ and initial condition $\eta \in \Omega$ is given by

$$H_A(\eta) = \mathbb{E}_\eta[\tau_A] \ . \tag{2.36}$$

Theorem 2.5.4. *For an irreducible Markov process on a finite state space Ω the vector of expected hitting times $H_A = (H_A(\eta) : \eta \in \Omega)$ is the minimal non-negative solution to the system of linear equations*

$$\begin{cases} H_A(\eta) = 0 & \textit{for } \eta \in A \ , \\ -\sum_{\xi\in\Omega} c(\eta,\xi) H_A(\xi) = 1 & \textit{for } \eta \notin A \ . \end{cases}$$

Proof. This proof is a simply application of the Markov property, see for example Theorem 3.3.3. of [47] □

The following theorem [57, 58] relates the hitting times of large sets with the 1/4-mixing time of the Markov chain.

Theorem 2.5.5. *For every irreducible and reversible Markov process on a finite state space* Ω*, and for each* $\alpha < 1/2$ *there exists constants* c_α, c'_α *so that*

$$c_\alpha \max_{\eta \in \Omega : \pi[A] \geq \alpha} H_A(\eta) \leq T_{mix}\left(\frac{1}{4}\right) \leq c'_\alpha \max_{\eta \in \Omega : \pi[A] \geq \alpha} H_A(\eta) \ . \tag{2.37}$$

Whilst this theorem is useful in itself, calculating the upper bound appearing (6.16) is highly non-trivial. This result, however, allows for a study of the mixing time via a method of decomposing the state space Ω into disjoint unions of sets Ω_i for $i \in [n] = \{1, \ldots, n\}$, *i.e.* $\Omega = \bigcup_{i \in [n]} \Omega_i$ and $\Omega_i \cap \Omega_j = \emptyset$ for each $i \neq j$. Understanding how the process behaves in each set Ω_i and how it transitions from $\Omega_i \to \Omega_j$ give rise to good bounds of the mixing time via (6.16) [59]. In this thesis, we use a similar method of decomposing the state space to calculate relaxation times given in Chapter 5.

CHAPTER 3

Characterisation of Condensation

3.1 Introduction

A condensation transition occurs when the particle density exceeds a critical value and the system phase separates into a fluid phase and a condensate. The fluid phase is distributed according to the maximal invariant measure at the critical density, and the excess mass concentrates on a single lattice site, called the condensate. Most results on condensation so far focus on zero-range or more general misanthrope processes in thermodynamic limits where the lattice size and the number of particles diverge simultaneously. Initial results are contained in [13, 60, 7], and for summaries of recent progress in the probability and theoretical physics literature see e.g. [61, 62, 63]. Condensation has also been shown to occur for processes on finite lattices in the limit of infinite density, where the tails of the single site marginals of the stationary product measures behave like a power law [64]. In general, condensation results from a sub-exponential tail of the maximal invariant measure [65], and so far most studies focus on power law and stretched exponential tails [65, 66, 67]. As a first result, we generalize the work in [64] and provide a characterization of condensation on finite lattices in terms of a class of sub-exponential tails that has been well studied in the probabilistic literature [68, 69, 70, 71].

In this chapter we discuss various definitions of condensation for finite systems and in the thermodynamic limit for processes that exhibit homogeneous stationary product measures. We state our main result linking condensation on finite lattices and sub-exponential distributions in Section 3.3. In Section 3.5, we review key results for processes which exhibit condensation in the thermodynamic limit and provide an example where condensation occurs in the thermodynamic limit but not on finite lattices. We give a short discussion on the links between various definitions of condensation on finite lattices in Section 3.6 and provide a proof of our main result in Section 3.7. In Section 3.8 we review a process that does not exhibit stationary product measures and discuss further the differences between condensation on a finite lattice and in the thermodynamic limit.

3.2 Definitions

Condensation appears in the mathematical and physical literature in many different forms, and therefore, one global definition which encompasses all relevant results is difficult to come by. In this section, we discuss various definitions of condensation in the thermodynamic limit and on finite lattices, which are appropriate for the interacting particle systems discussed in this thesis.

Formally, we consider interacting particle systems on the countable state space $\mathbb{N}^\Lambda$ where $|\Lambda| = L < \infty$. We assume the interacting particle system conserves particle density, is translation invariant, and is irreducible on the state space

$$\Omega_{L,N} = \left\{ \eta \in \mathbb{N}^\Lambda : \sum_{x \in \Lambda} \eta_x = N \right\} , \tag{3.1}$$

which implies that the process exhibits a unique invariant measure $\pi_{L,N}$ on $\Omega_{L,N}$. Furthermore, from translation invariance we have $\pi_{L,N}[\{\eta_x \in \cdot\}] = \pi_{L,N}[\{\eta_y \in \cdot\}]$ for all $x, y \in \Lambda$.

In the thermodynamic limit, where

$$N,\, L \to \infty \text{ such that } \frac{N}{L} \to \rho , \tag{3.2}$$

we define condensation via a local weak limit of the sequence of probability measures $\pi_{L,N}$ to a measure μ_ρ (if it exists) on $\mathbb{N}^{\mathbb{N}}$. For a sequence of probability measures $\pi_{L,N}$, local weak convergence means that

$$\pi_{L,N}(f) \to \mu_\rho(f) \text{ for all } f \in C_b^0\left(\mathbb{N}^{\mathbb{N}}\right) , \tag{3.3}$$

where $C_b^0\left(\mathbb{N}^{\mathbb{N}}\right)$ is the set of bounded cylinder functions on $\mathbb{N}^{\mathbb{N}}$. Note that local weak convergence is equivalent to convergence in distribution of all finite dimensional marginals. For a more complete discussion of weak convergence in the context of interacting particle systems see [12]. Condensation in the thermodynamic limit is then defined as follows:

Definition 3.2.1. *A stochastic particle system with canonical measures $\pi_{L,N}$ ex-*

hibits ***condensation in the thermodynamic limit*** (3.2) *for some* ρ *if there exists a measure* μ_ρ *on* $\mathbb{N}^{\mathbb{N}}$ *such that the sequence of measures* $\pi_{L,N}$ *converges to* μ_ρ *in the sense of* (3.3) *with*

$$\mu_\rho\,(\eta_0) < \rho\ .$$

Heuristically, this definition indicates that mass has been lost in the thermodynamic limit. Large finite systems phase separate into a condensate, where a finite fraction of particles concentrates in a vanishing volume fraction, and a fluid or bulk phase where the remaining particles are homogeneously distributed.

This has been established rigorously for interacting particle systems that exhibit stationary product measures (2.11) with stationary weights $w(n) > 0$ which decay sub-exponentially, *i.e.*

$$\frac{1}{n}\log\left(w(n)\right) \to 0 \text{ as } n \to \infty\ .$$

For such models, $D_\phi = [0, \phi_c]$ and there is an invariant product measure ν_{ϕ_c} with density $\rho_c := \rho(\phi_c) < \infty$. Then for $\rho > \rho_c$ the limit measure ν_ρ is the grand canonical measure at a critical density ρ_c, which satisfies $\mu_\rho(\eta_0) = \nu_{\phi_c}(\eta_0) = \rho_c < \rho$. Then condensation according to Definition 3.2.1 occurs as a continuous phase transition at $\rho = \rho_c$. In [72, 73], heuristic computations and numerical simulations show a condensation transition for processes that exhibit stationary measures that are finite range Gibbs measures on $\Omega_{L,N}$.

Condensation has also been established as a discontinuous phase transition for processes that exhibit stationary product measures with size-dependent stationary weights $w_L(n)$ [74]. In this case, there exists a transition density, ρ_{trans}, and critical density, ρ_c, such that for all $\rho > \rho_{\text{trans}}$ the system separates into a condensate and fluid region distributed according a critical measure with density ρ_c.

In the infinite particle limit $N \to \infty$ on fixed lattices Λ, *i.e.* $|\Lambda| = L < \infty$, condensation could be defined by the same approach as in the thermodynamic limit by excluding condensed sites using order statistics or cut-off. This approach, however, fails to capture examples of condensing systems as we will discuss in Section 3.6. Therefore, for finite systems we outline two definitions of condensation by first defining the maximum occupation numbers

$$M_L(\eta) := \max_{x\in\Lambda} \eta_x\ . \tag{3.4}$$

First, consider a definition of condensation by using the weak law of large number for the maximum occupation number.

Definition 3.2.2. *A stochastic particle system with canonical measure $\pi_{L,N}$ on $\Omega_{L,N}$ with $L \geq 2$ exhibits **weak condensation** (on finite lattices) if*

$$\frac{M_L}{N} \xrightarrow{\pi_{L,N}} 1 \text{ as } N \to \infty \ ,$$

where $\xrightarrow{\pi_{L,N}}$ denotes convergence in probability, i.e.

$$\lim_{N\to\infty} \pi_{L,N} \left[\left| \frac{M_L}{N} - 1 \right| > \varepsilon \right] = 0 \tag{3.5}$$

for all $\varepsilon > 0$.

A second definition, which was first used in [64], is given as follows.

Definition 3.2.3. *A stochastic particle system with canonical measure $\pi_{L,N}$ on $\Omega_{L,N}$ with $L \geq 2$ exhibits **condensation** on fixed finite lattices if*

$$\lim_{K\to\infty} \lim_{N\to\infty} \pi_{L,N} \left[M_L \geq N - K \right] = 1 \ . \tag{3.6}$$

In [64] condensation according to Definition 3.2.3 was proved for processes that exhibit stationary (conditional) product measures on $\Omega_{L,N}$ with stationary weights of the form $w(n) \sim n^{-b}$ for $b > 1$. Furthermore, it was proved that the distribution $\pi_{L,N}$ with the maximum occupation $M_L(\eta)$ removed converges weakly (or equivalently in total variation) to the critical grand-canonical measure on $L-1$ sites. In Section 3.3 we generalise this result for processes that exhibit stationary (conditional) product measure with stationary weights that have a general sub-exponential tail. It is immediate that Definition 3.2.3 implies Definition 3.2.2 (condensation implies weak condensation) since (3.6) implies a weak law of large numbers for the rescaled maximum occupation M_L/N. However, the two definitions are not equivalent as we will discuss in Section 3.6.

A law of large numbers analogous to (3.5) has also been proved in the thermodynamic limit for particular models with stationary product measures in [65, 66], which implies that the condensed phase actually concentrates on a single lattice site.

3.3 Results

In this section, we outline our main result on characterising condensation on finite lattices for processes that exhibit stationary product measures as stated in Definition 3.2.3 (condensation). We also give a brief discussion on this result and state some common distributions that exhibit condensation.

Recall the (homogeneous) conditional product measures $\pi_{L,N}$ on the finite state space $\Omega_{L,N}$ introduced in Section 2.3.1 with mass function

$$\pi_{L,N}[\eta] = \frac{1}{Z_{L,N}} \prod_{x\in\Lambda} w(\eta_x) \mathbb{1}\left(\eta \in \Omega_{L,N}\right) \ . \tag{3.7}$$

Our results hold for systems with general stationary weights, $w(n) > 0$ for each $n \in \mathbb{N}$, subject to the regularity assumption that

$$\lim_{n\to\infty} w(n-1)/w(n) \in (0, \infty] \tag{3.8}$$

exists. Under this regularity condition, this limit is given by the radius of convergence ϕ_c of the grand canonical partition function $z(\phi)$ (2.12). If $\phi_c < \infty$ then weights that satisfy (3.8) are sometimes called long-tailed [75], which is discussed in more detail in Section 3.5.

Proposition 3.3.1. *Consider a stochastic particle system as defined in Chapter 2 with (conditional) stationary product measures as defined by* (3.7) *satisfying regularity assumption* (3.8)*. Then the process exhibits condensation according to Definition 3.2.3 (condensation) if and only if* $\phi_c < \infty$, $D_\phi = [0, \phi_c]$, *and*

$$\lim_{N\to\infty} \frac{\nu_{\phi_c}^2[\eta_1 + \eta_2 = N]}{\nu_{\phi_c}[\eta_1 = N]} = \lim_{N\to\infty} \frac{Z_{2,N}}{w(N) z(\phi_c)} \in (0, \infty) \quad \textit{exists} \ . \tag{3.9}$$

In that case, the distribution of particles outside of the maximum converges weakly (equivalently in total variation) to the critical product measure $\nu_{\phi_c}^{L-1}$*, i.e. for fixed* $n_1, \dots, n_{L-1} \geq 0$ *we have*

$$\pi_{L,N}[\eta_1 = n_1, \dots, \eta_{L-1} = n_{L-1} | M_L = \eta_L] \to \prod_{i=1}^{L-1} \nu_{\phi_c}[\eta_i = n_i] \quad \textit{as} \quad N \to \infty \ . \tag{3.10}$$

Proof. See Section 3.7. □

Note that for $\phi_c \in (0, \infty)$ we may rescale the exponential part of the weights to get $\phi_c = 1$ and we can further multiply with a constant, so that in the following we can assume without loss of generality that

$$w(0) = 1 \quad \text{and} \quad \phi_c = \lim_{n\to\infty} w(n-1)/w(n) = 1 \ . \tag{3.11}$$

The condition (3.9) can also be written as

$$\lim_{N\to\infty} \frac{Z_{2,N}}{w(N)} = \lim_{N\to\infty} \frac{(w*w)(N)}{w(N)} \in (0,\infty) \quad \text{exists} \ , \tag{3.12}$$

where $(w*w)(N) = \sum_{k=0}^{N} w(k)w(N-k)$ is the convolution product. This is a standard characterization to define a class of distributions with sub-exponential tail (see e.g. [76, 77]). Implications and simpler necessary conditions on $w(n)$ which imply (3.12) have been studied in detail, and we provide a short discussion in Section 3.4.

Proposition 3.3.1 provides a generalization of previous results on condensation on finite lattices [64]. The class of sub-exponential distributions that fulfil (3.9) and therefore exhibit condensation on finite lattices is large (see e.g. [68, Table 3.7]), and includes in particular

- power law tails $w(n) \sim n^{-b}$ where $b > 1$,
- log-normal distribution

  $$w(n) = \frac{1}{n} \exp\{-(\log(n) - \mu)^2/(2\sigma^2)\} \ , \tag{3.13}$$

  where $\mu \in \mathbb{R}$ and $\sigma > 0$, which always has finite mean,
- stretched exponential tails $w(n) \sim \exp\{-Cn^\gamma\}$ for $0 < \gamma < 1$, $C > 0$,
- almost exponential tails $w(n) \sim \exp\big\{-\frac{n}{log(n)^\beta}\big\}$ for $\beta > 0$.

For the last two examples, all polynomial moments are finite. This covers all previously studied models of condensation on fixed finite lattices according to Definition 3.2.3 and in the thermodynamic limit for zero-range processes [7, 64, 67]. It can also be shown that the limit in (3.12) is necessarily equal to $2z(\phi_c)$ and that in fact $\frac{Z_{L,N}}{w(N)} \to Lz(\phi_c)^{L-1}$ for any fixed $L \geq 2$ [71].

Since we consider a fixed lattice Λ, $\rho_c = \rho(\phi_c) < \infty$ is not a necessary condition for condensation as opposed to systems in the thermodynamic limit. Even if the distribution of particles outside the maximum has infinite mean, condensation in the sense of Definition 3.2.3 (condensation) can occur. However, if $z(\phi_c) = \infty$ (e.g. for power law tails with $b \leq 1$), the distribution of particles outside the maximum cannot be normalized, condition (3.9) fails, and there is no condensation in the sense of Definition 3.2.3 (condensation).

3.4 Sub-exponential distributions

In the previous section, we saw an equivalence between condensation on finite lattices for processes that exhibit stationary product measures and sub-exponential distributions. In this section, we give an overview of sub-exponential distributions and their properties.

Sub-exponential distributions are a special class of heavy-tailed distributions, the following characterization was introduced in [78] with applications to branching random walks, and has been studied systematically in later work (see e.g. [71, 69, 70, 76]). For a review see for example [68] or [77].

Definition 3.4.1. *A non-negative random variable X with distribution function $F(x) = \mathbb{P}[X \leq x]$ is called heavy tailed if $F(0+) = 0$, $F(x) < 1$ for all $x > 0$, and*

$$e^{\lambda x}(1 - F(x)) \to \infty \;\; \textit{as } x \to \infty \;\; \textit{for all } \lambda > 0 \; . \tag{3.14}$$

It is called sub-exponential if $F(0+) = 0$, $F(x) < 1$ for all $x > 0$, and

$$\frac{1 - F^{\star 2}(x)}{1 - F(x)} \to 2 \quad \textit{as } x \to \infty \; . \tag{3.15}$$

Here $F^{\star 2}(x) = \mathbb{P}[X_1 + X_2 \leq x]$ denotes the convolution product, the distribution function of the sum of two independent copies X_1 and X_2. It has been shown [78, 79] that (3.15) is equivalent to either of the following conditions,

$$\lim_{x\to\infty} \frac{1 - F^{\star L}(x)}{1 - F(x)} = L \quad \text{for all} \quad L \geq 2 \; , \quad \text{or} \tag{3.16}$$

$$\lim_{x\to\infty} \frac{\mathbb{P}\big[\sum_{i=1}^{L} X_i > x\big]}{\mathbb{P}\big[\max\{X_i : i \in \{1, \dots, L\}\} > x\big]} = 1 \quad \text{for all} \quad L \geq 2 \; . \tag{3.17}$$

The second characterization shows that a large sum of independent sub-exponential random variables X_i is typically realized by one of them taking a large value, which is of course reminiscent of the condensation phenomenon. It was further shown in [78, 68] that sub-exponential distributions also have the following properties,

$$\lim_{x\to\infty} \frac{1 - F(x - y)}{1 - F(x)} = 1 \quad \forall y \in \mathbb{R}, \tag{3.18}$$

$$(1 - F(x))e^{\epsilon x} \to \infty \quad \forall \epsilon > 0 \quad \text{(heavy tailed in the sense of (3.14))} \; . \tag{3.19}$$

Most results in the literature are formulated in terms of distribution functions and tails and apply to discrete as well as continuous random variables. [71] provides a valuable connection to discrete random variables in terms of their mass functions $w(n)$, $n \in \mathbb{N}$. Assume the following properties for a sequence $\{w(n) > 0 : n \in \mathbb{N}\}$,

(a) $\frac{w(n-1)}{w(n)} \to 1$ as $n \to \infty$,

(b) $z(1) := \sum_{n=0}^{\infty} w(n) < \infty$ (normalizability),

(c) $\lim_{N\to\infty} \frac{(w*w)(N)}{w(N)} = C \in (0, \infty)$ exists.

Then [71, Theorem 1] asserts that $C = 2z(1)$ and $w(n)/z(1)$ is the mass function of a discrete, sub-exponential distribution. The implication

$$\frac{(w^{\star L})(N)}{w(N)} \to Lz(1)^{L-1} \quad \text{as } N \to \infty \text{ for } L > 2$$

is given in [71, Lemma 5]. Sufficient (but not necessary) conditions for assumption (c) to hold are given in [71, Remark 1]. Provided $z(1) < \infty$, then (c) holds if either of the following conditions are met:

(i) $\sup_{1\leq k\leq n/2} \frac{w(n-k)}{w(n)} \leq K$

for some constant $K > 0$, or

(ii) $w(n) = e^{-n\psi(n)}$

where $\psi(x)$ is a smooth function on $\mathbb{R}$ with $\psi(x) \searrow 0$ and $x^2|\psi'(x)| \nearrow \infty$ as $x \to \infty$, and $\int_0^{\infty} dx\, e^{-\frac{1}{2}x^2|\psi'(x)|} < \infty$.

Case (i) includes distributions with power law tails, $w(n) \sim n^{-b}$ with $b > 1$. The stretched exponential with $\psi(x) = x^{\gamma-1}$, $\gamma \in (0, 1)$, and the almost exponential with $\psi(x) = (\log(x))^{-\beta}$, $\beta > 0$, are covered by case (ii). The class of sub-exponential distributions includes many more known examples than the list given in Section 3.3 (see e.g. [68, Table 3.7]). Analogous to the characterisation of sub-exponential distributions, given by (3.17), for discrete distributions the existence of the limit $(w * w)(N)/w(N)$ is equivalent to the following condition

$$\frac{\mathbb{P}[X_1 + X_2 = N]}{\mathbb{P}\big[\max\{X_1, X_2\} = N\big]} \to 1 \text{ as } N \to \infty\ . \tag{3.20}$$

This holds, since we have the following equality of ratios

$$\frac{\mathbb{P}[X_1 + X_2 = N]}{\mathbb{P}\big[\max\{X_1, X_2\} = N\big]} = \frac{Z_{2,N}}{2w(N)\sum_{n=0}^{N} w(n)} = \frac{(w * w)(N)}{2w(N)\sum_{n=0}^{N} w(n)}\ .$$

Specific properties of power law tails $w(n)$ are used in [64] to show condensation for finite systems in the sense of Definition 3.2.3. In Proposition 3.3.1, proved in the Section 3.7, we extend this result to stationary product measures with general sub-exponential tails. In this context, condensation is basically characterized by the property (3.17) which assures emergence of a large maximum when the sum of independent variables is conditioned on a large sum. As summarized in the introduction, condensation in stochastic particle systems has mostly been studied in the thermodynamic limit with particle density $\rho \geq 0$, where $L, N \to \infty$ such that $N/L \to \rho$. In that case conditions on the sum of L independent random variables become large deviation events, which have been studied in detail in [80, 81].

3.5 Connection with the thermodynamic limit

In the thermodynamic limit, we have defined condensation by a weak limit of measures as given by Definition 3.2.1 (condensation in the thermodynamic limit). Equivalently, the approach presented in [65, 62] follows the classical paradigm for phase transitions in statistical mechanics via the equivalence of ensembles (see e.g. [82] for more details). A system with stationary product measures (2.11) exhibits condensation if the critical density (2.15) is finite, *i.e.* $\rho_c < \infty$ and the canonical measures $\pi_{L,N}$ are equivalent to the critical product measure ν_{ϕ_c} in the limit $L, N \to \infty$ such that $N/L \to \rho$ for all super-critical densities $\rho \geq \rho_c$. The interpretation is again that the bulk of the system (any finite set of sites) is distributed as the critical product measure in the limit. It has been shown in [65] (see also [62] for a more complete presentation) that the regularity condition (3.8) and $\rho_c < \infty$ imply the equivalence of ensembles, which has therefore been used as a definition of condensation in [62, Definition 2.1]. Condensation on fixed finite lattices in the sense of Definition 3.2.3 implies the regularity condition (3.8) and therefore, if in addition $\rho_c < \infty$, this implies condensation in the thermodynamic limit. This includes all previously studied examples [7, 67], however there exist distributions that satisfy (3.8) with $\rho_c < \infty$ but do not satisfy the conditions of Proposition 3.3.1 and do not condense for fixed Λ. This is illustrated by an example given below. It is also discussed in [62, Section 3.2] that assumption (3.8) is not necessary to show equivalence of ensembles, but weaker conditions are of a special, less general nature and are not discussed here. Note also that equivalence of ensembles does not imply that the condensate concentrates on a single lattice site, the latter has been shown so far only for stretched exponential and power-law tails with $\rho_c < \infty$ in [66, 67].

The condensation phenomena can also be studied for continuous random variables on the local state space $[0, \infty)$, see for example [81]. The following continuous example, taken from [70] is shown to satisfy (3.8) but is not sub-exponential. We show that the distribution has a finite mean and exhibits condensation in the thermodynamic limit, as shown in [65], but not on a fixed finite lattice in the sense of Definition 3.2.3. For a real-valued random variable X with distribution function $F(x) = \mathbb{P}[X \le x]$, assume $F'(x) = g'(x)e^{-g(x)}$. Let $(x_n)_{n\in\mathbb{N}}$ be an increasing sequence with $x_0 = 0$ and $g(x)$ be a continuous and piecewise linear function such that $g(0) = 0$ and $g'(x) = 1/n$ for $x \in (x_{n-1}, x_n)$. The sequence $(x_n)_{n\in\mathbb{N}}$ is defined iteratively as follows

$$\begin{aligned} x_n - x_{n-1} &= 2ne^{g(x_{n-1})} \ , \\ g(x_n) - g(x_{n-1}) &= 2e^{g(x_{n-1})} \ , \end{aligned} \tag{3.21}$$

and $g(x) - g(x_{n-1}) = \frac{x - x_{n-1}}{n}$ for $x \in [x_{n-1}, x_n)$. The mean can be computed as follows

$$\int_0^\infty xF'(x)dx = \sum_{n=1}^\infty \frac{1}{n} \int_{x_{n-1}}^{x_n} xe^{-g(x)}dx = \sum_{n=1}^\infty \frac{e^{-g(x_{n-1})}}{n} \int_{x_{n-1}}^{x_n} xe^{-\frac{x-x_{n-1}}{n}} dx \ .$$

Evaluating the integral we find

$$\begin{aligned} \int_0^\infty xF'(x)dx &= \sum_{n=1}^\infty e^{-g(x_{n-1})} \left(n + x_{n-1} - (n + x_n)e^{-\frac{x_n - x_{n-1}}{n}} \right) \\ &= \sum_{n=1}^\infty ne^{-g(x_{n-1})} + \sum_{n=1}^\infty x_{n-1}e^{-g(x_{n-1})} - \sum_{n=1}^\infty ne^{-g(x_n)} - \sum_{n=1}^\infty x_n e^{-g(x_n)} \ . \end{aligned}$$

Using (3.21) we can simplify the final term to show

$$\int_0^\infty xF'(x)dx = \sum_{n=0}^\infty e^{-g(x_n)} < \infty \ ,$$

where the final step uses the relation $g(x_n) - g(x_{n-1}) = 2e^{g(x_{n-1})} \ge 2(1 + g(x_{n-1}))$ and $g(x_0) = 0$, which implies $g(x_n) \ge 2(2^n - 1)$, to bound the series from above.

For all long-tailed but not sub-exponential measures $Z_{L,N}/w(N)$ does not have a limit in $(0, \infty)$ as $N \to \infty$ and with Proposition 3.3.1 there is no condensation on finite lattices according to Definition 3.2.3 (condensation). In the following, we adapt the example above for discrete random variables on $\mathbb{N}$. First redefine the sequences $(x_n)_{n\in\mathbb{N}}$ and $(g(x_n))_{n\in\mathbb{N}}$ as follows

$$\begin{aligned} x_n - x_{n-1} &= n 2^{g(x_{n-1})} \\ g(x_n) - g(x_{n-1}) &= 2^{g(x_{n-1})} \ , \end{aligned}$$

with $x_0 = 0$ and $g(x_0) = 0$, which ensures $x_n \in \mathbb{N}$ and $g(x_n) \in \mathbb{N}$ for all $n \in \mathbb{N}$. For $k \in [x_{n-1}, x_n)$ let the weights be given by

$$w(k) = 2^{-g(k)} = 2^{-g(x_{n-1}) - \left(\frac{k - x_{n-1}}{n}\right)} \ . \tag{3.22}$$

Following the approach given in [70] we show $Z_{2,N}/w(N) \to \infty$ for $N = x_n$ as $n \to \infty$. By dropping the terms $k \in \{0, \dots, x_{n-1} - 1\}$ we have

$$\frac{Z_{2,x_n}}{w(x_n)} \geq \sum_{k=x_{n-1}}^{x_n} 2^{-g(k) - g(x_n - k) + g(x_n)} \ .$$

Since $g(\cdot)$ is linearly increasing and $g'(k) = \frac{1}{n}$ for $k \in [x_{n-1}, x_n)$, which is decreasing, we have

$$g(x_n) - g(x_n - k) \geq k g'(x_n) = \frac{k}{n} \ .$$

Therefore,

$$-g(k) + g(x_n) - g(x_n - k) \geq -g(x_{n-1}) - \frac{k - x_{n-1}}{n} + \frac{k}{n} \geq -g(x_{n-1}) \ ,$$

which implies

$$\begin{aligned} \frac{Z_{2,x_n}}{w(x_n)} &\geq \sum_{k=x_{n-1}}^{x_n} 2^{-g(k) - g(x_n - k) + g(x_n)} \\ &\geq \sum_{k=x_{n-1}}^{x_n} 2^{-g(x_{n-1})} \\ &\geq 2^{-g(x_{n-1})} (x_n - x_{n-1}) = n \ , \end{aligned}$$

which diverges as $n \to \infty$. For this example, following the proof of Proposition 3.3.1, this implies that $\pi_{2,N}[\eta_1 \wedge \eta_2 \leq K] \to 0$ along the subsequence $N = x_n$ as $N \to \infty$ (and $n \to \infty$) for all $K \geq 0$. Therefore, the $L = 2$ bulk occupation number $\eta_1 \wedge \eta_2$ diverges in distribution as $N \to \infty$ by receiving a diverging excess mass from the condensate due to the light tail. Since the distribution is long-tailed the model does exhibit condensation in the thermodynamic limit according to Definition 3.2.1

(condensation in the thermodynamic limit) where the excess mass can be distributed on a diverging number of sites.

3.6 A law of large numbers for the rescaled maximum occupation number $M_L(\eta)/N$

In this section, we discuss the links between Definition 3.2.3 (condensation) and condensation defined by the weak law of large numbers for the rescaled maximum occupation M_L/N in Definition 3.2.2 (weak condensation). As previously discussed, it is clear that assuming condensation holds according to Definition 3.2.3 (condensation) then a weak law of large numbers holds for the rescaled maximum, *i.e.* $M_L/N \xrightarrow{\pi_{L,N}} 1$ as $N \to \infty$, and condensation holds according to Definition 3.2.2 (weak condensation). However, the converse is not true as we will show with the following example.

Consider a conditional product measure with weights of the form

$$w(n) = \frac{1}{n+1} \ ,$$

then the critical partition function $z(\phi_c) = z(1) = \sum_{n=0}^{\infty} \frac{1}{n+1} = \infty$ and the critical measure does not exist. Also the ratio $\frac{Z_{2,N}}{w(N)} \to \infty$ as $N \to \infty$ and, therefore, by Proposition 3.3.1 condensation does not occur according to Definition 3.2.3 (condensation) on two sites. We now show that the weak law of large numbers is satisfied for $L = 2$ and, therefore, condensation occurs according to Definition 3.2.2 (weak condensation). We have $M_2(\eta) \in \{\lceil N/2 \rceil, \dots, N\}$, and then (3.5) holds if

$$\lim_{N\to\infty} \pi_{2,N} \left[\lceil N/2 \rceil \le M_2 < N - \lfloor \varepsilon N \rfloor \right] = 0 \ ,$$

for all $\varepsilon > 0$ small enough. For simplicity consider the case when N is odd, then we have

$$\pi_{2,N} \left[\lceil N/2 \rceil \le M_2 < N - \lfloor \varepsilon N \rfloor \right] \le \frac{1}{Z_{2,N}} \sum_{n=\frac{N+1}{2}}^{N-\lfloor \varepsilon N \rfloor} w(n) w(N-n) \ . \tag{3.23}$$

To find a lower bound for the partition function we note $w(n)w(N-n)$ is symmetric under $n \leftrightarrow N-n$, and $w(n)w(N-n)$ is decreasing for $n \in \{0, \dots \lfloor \frac{N}{2} \rfloor\}$, so

$$Z_{2,N} = 2 \sum_{n=0}^{\frac{N-1}{2}} w(n)w(N-n) \geq 2 \int_0^{\frac{N+1}{2}} \frac{1}{x+1} \frac{1}{N+1-x} dx$$
$$= 2 \frac{\log(N+3)}{N+2} \ .$$

The numerator in (3.23) can be bounded above by

$$\frac{1}{\lfloor \varepsilon N \rfloor + 1} \frac{1}{N+1-\lfloor \varepsilon N \rfloor} + \frac{\log\left(\frac{(N+1)(N-\lfloor \epsilon N \rfloor+1)}{(N+3)(\lfloor \epsilon N \rfloor+1)}\right)}{N+2} \ ,$$

which implies

$$\pi_{2,N}\left[\lceil N/2 \rceil \leq M_2 < N - \lfloor \varepsilon N \rfloor\right] \to 0 \ \text{ as } N \to \infty$$

and the weak law of large numbers holds for the sequence M_2/N. Therefore, Definition 3.2.2 (weak condensation) does not imply Definition 3.2.3 (condensation) and the two statements are not equivalent. For this example, the minimum occupation number holds an $o(N)$ number of particles which diverges as $N \to \infty$, in contrast to processes that condense according to Definition 3.2.3, where a finite number of particles occupy the minimum in the limit as $N \to \infty$.

3.7 Proof of Proposition 3.3.1

Let us first assume that the process exhibits condensation according to Definition 3.2.3 (condensation) and has canonical distributions of the form (2.18) where the weights fulfil (3.8), i.e. $w(n-1)/w(n) \to \phi_c \in (0, \infty]$ as $n \to \infty$. In this part of the proof we establish that;

1. $\phi_c < \infty$,
2. $\frac{Z_{L,N}}{w(N)}$ has a limit as $N \to \infty$,
3. $z(\phi_c) < \infty$, which also implies $\frac{Z_{L,N}}{w(N)} \to Lz(\phi_c)^{L-1}$ as $N \to \infty$, and
4. convergence of $\frac{Z_{L,N}}{w(N)} \to Lz(\phi_c)^{L-1}$ for some $L \geq 2$ implies convergence for $L = 2$ and therefore (3.9) holds.

Step (1), show $\phi_c < \infty$. Assume first that $w(n-1)/w(n) \to \infty$ as $n \to \infty$.

For all $K \in \mathbb{N}$ and $N > K$ we have

$$\begin{aligned}\pi_{L,N}[M_L \geq N-K] &= \frac{L}{Z_{L,N}} \sum_{n=0}^{K} Z_{L-1,n} w(N-n) \\ &\leq L \frac{K+1}{Z_{L,N}} \max_{0 \leq n \leq K} (Z_{L-1,n}) \max_{0 \leq n \leq K} (w(N-n)) \ .\end{aligned}$$

Let $n^\star = \mathrm{argmax}_{0 \leq n \leq K} (w(N-n))$. The partition function $Z_{L,N}$ is trivially bounded below by the event that site 1 takes $N - n^\star - 1$ particles and the second site takes the remaining $n^\star + 1$ particles, *i.e.*

$$Z_{L,N} \geq w(0)^{L-2} w(n^\star + 1) w(N - n^\star - 1) \ .$$

Therefore

$$\pi_{L,N}[M_L \geq N-K] \leq \frac{L}{w(0)^{L-2}} \frac{K+1}{w(n^\star+1)} \frac{w(N-n^\star)}{w(N-n^\star-1)} \max_{0 \leq n \leq K} (Z_{L-1,n}) \to 0$$

as $N \to \infty$, which implies condensation cannot occur in the sense of Definition 3.2.3 (condensation) contradicting the initial assumption, therefore $\phi_c < \infty$.

Step (2), prove $Z_{L,N}/w(N)$ converges as $N \to \infty$. By Definition 3.2.3 the limit

$$a_K := \lim_{N \to \infty} \pi_{L,N}[M_L \geq N-K] \ , \tag{3.24}$$

exists and $a_K > 0$ for K sufficiently large. For $N > K$ we have

$$\pi_{L,N}[M_L \geq N-K] = L \frac{w(N)}{Z_{L,N}} \sum_{n=0}^{K} Z_{L-1,n} \frac{w(N-n)}{w(N)} \ . \tag{3.25}$$

Since $w(N-n)/w(N) \to \phi_c^n$, K is fixed, and $a_K > 0$, (3.25) implies the convergence of $Z_{L,N}/w(N)$ as $N \to \infty$.

Step (3), prove $z(\phi_c) < \infty$. By (3.6) we have $a_K \to 1$ as $K \to \infty$, with the limit as $N \to \infty$ of (3.25) this implies

$$\lim_{K \to \infty} \sum_{n=0}^{K} Z_{L-1,n} \phi_c^n = \sum_{n=0}^{\infty} Z_{L-1,n} \phi_c^n < \infty \ . \tag{3.26}$$

Since we also have $\sum_{n=0}^\infty Z_{L-1,n}\phi_c^n = z(\phi_c)^{L-1}$, this implies $z(\phi_c) < \infty$. Using $a_K \to 1$, (3.25) then also implies $Z_{L,N}/w(N) \to L z(\phi_c)^{L-1}$ as $N \to \infty$.

Step (4). We have seen above that condensation implies $\phi_c < \infty$, $z(\phi_c) < \infty$, and $Z_{L,N}/w(N) \to Lz(\phi_c)^{L-1}$ as $N \to \infty$, then [83, Theorem 2.10] implies

$$\lim_{N\to\infty} \frac{Z_{2,N}}{w(N)} = 2z(\phi_c) \ ,$$

completing this part of the proof.

To prove (3.10), let us consider a stochastic particle system with canonical distributions of the form (2.18) which fulfil (3.11) and (3.12) with $\phi_c = 1$ and $z(1) < \infty$. We keep the notation for $\phi_c = 1$ general in the following to clarify the argument. From [71, Theorem 1 and Lemma 5], we have

$$\frac{Z_{L,N}}{w(N)} \to Lz(\phi_c)^{L-1} \ \text{as } N \to \infty \ , \tag{3.27}$$

therefore it is immediate that

$$\pi_{L,N}[M_L = N] = Lw(N)/Z_{L,N} \to z(\phi_c)^{-(L-1)} > 0 \ .$$

Then we have for all fixed K and $N > K$

$$\begin{aligned}\pi_{L,N}[M_L \geq N-K] = L\sum_{n=0}^{K} \frac{w(N-n)Z_{L-1,n}}{Z_{L,N}} &= \sum_{n=0}^{K} Z_{L-1,n}\frac{w(N-n)}{w(N)}\frac{Lw(N)}{Z_{L,N}} \\ \to \sum_{n=0}^{K} \frac{Z_{L-1,n}\phi_c^n}{z(\phi_c)^{L-1}} &= \nu_{\phi_c}(\eta_1+\ldots+\eta_{L-1} \leq K)\end{aligned}$$

as $N \to \infty$. Since ν_{ϕ_c} is a non-degenerate probability distribution, this implies that $\nu_{\phi_c}(\eta_1+\ldots+\eta_{L-1} \leq K) \to 1$ as $K \to \infty$, which is (3.6).

To compute the distribution outside the maximum we get for fixed $n_1, \ldots, n_{L-1}$ and large enough N

$$\begin{aligned}&\pi_{L,N}[\eta_1{=}n_1,\ldots,\eta_{L-1}{=}n_{L-1}|M_L{=}\eta_L] = \frac{w(n_1)\cdots w(n_{L-1})w(N-n_1-\ldots-n_{L-1})}{\pi_{L,N}[M_L{=}\eta_L]\ Z_{L,N}} \\ &\quad = \frac{1}{L\pi_{L,N}[M_L=\eta_L]} w(n_1)\cdots w(n_{L-1})\frac{w(N-n_1-\ldots-n_{L-1})}{w(N)}\frac{Lw(N)}{Z_{L,N}} \\ &\quad \to w(n_1)\cdots w(n_{L-1})\phi_c^{n_1+\ldots+n_{L-1}}/z(\phi_c)^{L-1} \ ,\end{aligned} \tag{3.28}$$

as $N \to \infty$. Here we have used that spatial homogeneity of the measure and asymptotic uniqueness of the maximum according to (3.6) imply $\pi_{L,N}[M_L = \eta_L] \to 1/L$. This completes the proof of Proposition 3.3.1.

3.8 Condensation beyond stationary product measures

We have defined the critical density ρ_c only for systems with product stationary measures (see (2.15)). In general, the critical density on a fixed system of size $L \geq 2$, with unique invariant measure $\pi_{L,N}$, can be defined as

$$\rho_c(L) := \limsup_{N\to\infty} R_L^{bg}(N) \ , \tag{3.29}$$

where the background density is defined as

$$R_L^{bg}(N) := \frac{1}{L-1}\pi_{L,N}(N - M_L) \ . \tag{3.30}$$

Notice if $\pi_{L,N}$ are conditional product measures (2.18) then $\rho_c(L)$ is consistent with (2.15) and in-particular independent of L, which follows from Proposition 3.3.1 (or more explicitly (3.10)).

In this section, we introduce the chipping model, which is a process that does not exhibit stationary product measures. This process has been shown to exhibit condensation via heuristic computations in the thermodynamic limit [84, 85, 86]. By computing the stationary measure for the process on two sites we show that the process exhibits condensation according to Definition 3.2.3 (condensation). We also compute the critical density (3.29), $\rho_c(2)$, which leads to the suggestion that the critical density can be dependent on the system size for processes without stationary product measures. We further demonstrate this claim with numerics from simulations of the process shown in Figure 3.1.

The chipping model is a stochastic particle system on the state space $\Omega_{L,N}$ introduced in [84, 85]. The dynamics of the chipping model are constructed such that either all particles at site $x \in \Lambda$ jump collectively at rate 1 or a single particle jumps at rate $w > 0$. Once a transition occurs at site x the collection of particles jump to site $y \in \Lambda$ according to an irreducible random walk $p(x,y)$. The dynamics are defined by the generator

$$\begin{aligned}\mathcal{L}_{L,N}^{\text{chip}} f(\eta) = &\sum_{x,y\in\Lambda} w\mathbb{1}(\eta_x > 0)p(x,y)\big(f(\eta^{x,y}) - f(\eta)\big) \\ &+ \sum_{x,y\in\Lambda} \mathbb{1}(\eta_x > 0)p(x,y)\big(f(\eta + \eta_x(\delta_y - \delta_x)) - f(\eta)\big) \ ,\end{aligned} \tag{3.31}$$

where $\eta + \eta_x(\delta_y - \delta_x)$ denotes the configuration after all the particles at site x have jumped to site y.

For the chipping model in the case $L = 2$ with $p(1,2) = p(2,1) = 1$, the process reduces to a 1-dimensional process on $\{0,\dots,N\}$ defined by the generator

$$\begin{aligned}\mathcal{L}^{\text{chip}}_{2,N} f(n) =& \mathbb{1}(n>0)\left(f(0)-f(n)\right)+w\mathbb{1}(n>0)\left(f(n-1)-f(n)\right)\\ &+\mathbb{1}(n<N)\left(f(N)-f(n)\right)+w\mathbb{1}(n<N)\left(f(n+1)-f(n)\right)\ . \end{aligned}\tag{3.32}$$

The stationary measure of the two site chipping model can be computed exactly and is given in the following proposition.

Proposition 3.8.1. *Let $\pi_{2,N}[\cdot]$ be the stationary measure of the two site chipping model defined by the generator* (3.32) *on $\Omega_{2,N}$. Define $\mu_N[n] := \pi_{2,N}[(n, N-n)]$ then*

$$\mu_N[n] = \frac{A_+^n\left(A_+ - 1 + A_-^N - A_-^{N+1}\right) - A_-^n\left(A_- - 1 + A_+^N - A_+^{N+1}\right)}{2\left(A_+^{N+1} - A_-^{N+1}\right)}\ , \tag{3.33}$$

where

$$A_\pm = \frac{1+w\pm\sqrt{1+2w}}{w}\ .$$

Proof. Directly from the master equation (2.9) and since the process is homogeneous we have that the measure $\mu_N[n]$ is stationary if it satisfies the following conditions;

(i) $2(1+w)\mu_N[n] = w\mu_N[n-1] + w\mu_N[n+1]$ for $n\in\{1,\dots,N-1\}$,
(ii) $(1+w)\mu_N[0] = \sum_{k=1}^{N}\mu_N[k] + w\mu_N[1]$, and
(iii) $(1+w)\mu_N[N] = \sum_{k=0}^{N-1}\mu_N[k] + w\mu_N[N-1]$.

To simplify the later parts of this proof we first show that the measure $\mu_N[n]$ is symmetric. To see this, first note that $A_+A_- = 1$, which implies

$$\begin{aligned}\mu_N[N-n] &= \frac{A_+^{N-n}\left(A_+ - 1 + A_-^N - A_-^{N+1}\right) - A_-^{N-n}\left(A_- - 1 + A_+^N - A_+^{N+1}\right)}{2\left(A_+^{N+1} - A_-^{N+1}\right)}\\ &= \frac{A_+^{-n}\left(A_+^{N+1} - A_+^N + 1 - A_-\right) - A_-^{-n}\left(A_-^{N+1} - A_-^N + 1 - A_+\right)}{2\left(A_+^{N+1} - A_-^{N+1}\right)}\ .\end{aligned}$$

Multiplying by $A_+^nA_-^n = 1$ we see that (i) holds for (3.33).

Condition (i) holds since

$$w\mu_N[n-1] + w\mu_N[n+1] =$$
$$\frac{A_+^n\left(A_+ - 1 + A_-^N - A_-^{N+1}\right)\left(\frac{w}{A_+} + wA_+\right) - A_-^n\left(A_- - 1 + A_+^N - A_+^{N+1}\right)\left(\frac{w}{A_-} + wA_-\right)}{2\left(A_+^{N+1} - A_-^{N+1}\right)}\ ,$$

and

$$\left(\frac{w}{A_\pm} + wA_\pm\right) = 2(1+w) \ .$$

By the geometric series formula it is easy to see $\sum_{n=0}^{N} \mu_N[n] = 1$ and condition (ii) holds. Finally, condition (iii) holds by the symmetry condition given above. □

By equation (3.29) and Proposition 3.8.1 the two site critical density $\rho_c(2)$ can be computed explicitly to find

$$\rho_c(2) = \lim_{N\to\infty} 2\sum_{n=1}^{N/2} n\mu_N[n] = \frac{1}{2}\left(\sqrt{1+2w} - 1\right) \ . \tag{3.34}$$

Condensation, according to Definition 3.2.3 (condensation), occurs in the $L = 2$ chipping model since

$$\begin{aligned}\lim_{N\to\infty} \mu_{2,N}[M_2 \geq N-K] &= \lim_{N\to\infty} 2\mu_{2,N}\left[\eta_2 \geq N-K \,\middle|\, \eta_1 \leq \eta_2\right] \\ &= \lim_{N\to\infty} 2\sum_{n=0}^{K} \mu_N[n] \\ &= \frac{A_+ - (A_-)^K}{A_+} \ .\end{aligned}$$

Where the final term tends to 1 as $K \to \infty$ since $0 < A_- < 1$ for all $w > 0$.

In [84, 85, 86] the critical density in the thermodynamic limit is defined as

$$\rho_c := \sup\left\{\rho : \frac{\mu_{L,N}(\eta_x^2)}{L} \to 0 \text{ as } N, L \to \infty \text{ such that } \frac{N}{L} \to \rho\right\} \ ,$$

and heuristic computations show that

$$\rho_c = \sqrt{w+1} - 1 \ . \tag{3.35}$$

This suggests that the critical density can depend on the system size L for distributions with non-product stationary measures, which is highlighted in Figure 3.1 for the chipping model on a complete graph.

The $\sqrt{w}$ scaling of the critical density can be intuitively understood in the two site chipping model with N particles. This process can be interpreted as a symmetric random walk on the state space $\{0, \dots, N\}$ with jumps $i \to i \pm 1$ at rate w and random jumps to either boundary (resetting, $i \to 0$ or N) at rate 1. After

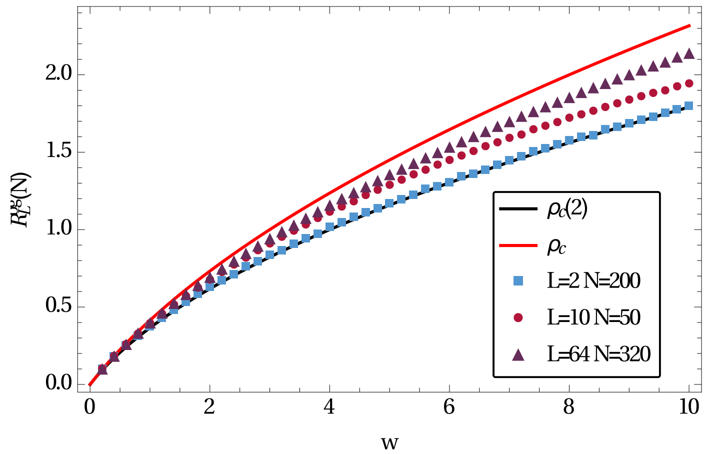

Figure 3.1: System size dependence of the background density $R_L^{bg}(N)$ (3.30) for the chipping model on a complete graph with $L = 2$ $N = 200$, $L = 10$ $N = 50$, and $L = 64$ $N = 320$ as a function of w. The two site critical density $\rho_c(2)$ (3.34) and thermodynamic limit critical density ρ_c (3.35) and are given by black and red lines. Results show a dependence on the system size L for the critical density $\rho_c(L)$ (3.29), which appears to approach ρ_c as L increases. All simulations of the chipping model were performed using the algorithm described in Appendix A.3.1.

a reset, the particle diffuses at rate w and reaches a typical distance $\sqrt{w}$ from the boundary until the next reset. This model is a spatially homogeneous process that heuristically exhibits a condensation transition with finite (size dependent) critical density, but it does not exhibit stationary product measures. Condensation is also observed in models where chipping is absent ($w = 0$) and the dynamics result in a single block of particles jumping on the lattice $\{1, \ldots, L\}$ corresponding to the critical density $\rho_c = 0$.

CHAPTER 4

Monotonicity and Condensation in Stochastic Particle Systems

4.1 Introduction

In this chapter, we focus on spatially homogeneous models with stationary product measures as discussed in Chapter 2, which can exhibit a condensation transition that has recently been studied intensively.

Monotone or attractive particle systems preserve the partial order on the state space in time, which enables the use of powerful coupling techniques to derive rigorous results on large scale dynamic properties such as hydrodynamic limits (see [45] and references therein). These techniques have also been used to study the dynamics of condensation in attractive zero-range processes with spatially inhomogeneous rates [9, 87, 88, 89, 90], and more recently [91, 92]. As we discuss in Section 4.5, non-monotonicity in homogeneous systems with finite critical density can be related on a heuristic level to convexity properties of the canonical entropy. For condensing systems with zero-range dynamics, it has been shown that this is related to the presence of metastable states resulting in the non-monotone behaviour of the canonical stationary current/diffusivity [16]. This corresponds to a first order correction of a hydrodynamic limit leading to an ill-posed equation with negative diffusivity in the case of reversible dynamics. Heuristically, this is of course consistent with the concentration of mass in a small, vanishing volume fraction, but poses great technical difficulties to any rigorous proof of hydrodynamic limits for such particle systems. First results in this direction only hold for sub-critical systems under restrictive conditions [93], and due to lack of monotonicity there are no results for non-reversible dynamics.

Condensing monotone particle systems would, therefore, provide interesting examples of homogeneous systems for which coupling techniques could be used to derive stronger results on hydrodynamic limits. However, our result implies that this is not possible for condensing models with stationary product measures and a finite critical density on finite lattices. In the thermodynamic limit condensation occurs if the critical density is finite, which implies the tail of the stationary measure is long-tailed. In the previous chapter, we proved that condensation occurs on fixed finite lattices according to Definition 3.2.3 if and only if the stationary measure is sub-exponential. However, as discussed in Section 3.5, since there exist long-tailed measures which are not sub-exponential, condensation can occur in the thermodynamic limit but not of finite lattices. Therefore, possible monotonicity for such models remains an open problem, see Sections 3.4 and 3.5, for a more detailed discussion.

For systems with infinite critical density condensation can still occur on finite lattices, and since the point at which monotonicity is broken typically occurs above the critical density such processes can also be monotone. For power law tails of the stationary measure that decay faster than $n^{-3/2}$ with the occupation number n, we prove that such processes are still non-monotone. In Section 4.4 we present preliminary results for tails that decay slower than $n^{-3/2}$, for which we present partial and numerical evidence that a monotone and condensing particle system exists. The existence of a monotone processes with stationary weights of the form $n^{-3/2}$ has recently been proved in [54].

The chapter is organised as follows. In Section 4.2 we state our main result. In Section 4.3 we prove our main theorem by induction over the size of the lattice, showing that the family of canonical stationary measures is necessarily not monotonically ordered in the number of particles. In Section 4.4 we review examples of homogeneous processes that have been shown to exhibit condensation, and present some explicit computations for misanthrope processes and processes with power law tails.

4.2 Notation and results

4.2.1 Condensing stochastic particle systems

We consider stochastic particle systems on fixed finite lattices $\Lambda = \{1, \ldots, L\}$, which are continuous-time Markov chains on the countable state space $\Omega_L = \mathbb{N}^\Lambda$ defined by the generator (2.7), *i.e.*

$$\mathcal{L}f(\eta) = \sum_{\{\xi\in\Omega_L:\xi\neq\eta\}} c(\eta,\xi)\big(f(\xi)-f(\eta)\big) \ , \tag{4.1}$$

for all continuous functions $f:\Omega_L\to\mathbb{R}$. We assume the process conserves the total number of particles $N=\sum_{x\in\Lambda}\eta_x$ and is irreducible on the finite state space $\Omega_{L,N}=\{\eta\in\Omega_L:\sum_{x\in\Lambda}\eta_x=N\}$. On $\Omega_{L,N}$ the process has a unique stationary distribution $\pi_{L,N}$, and the family $\{\pi_{L,N}:N\geq 0\}$ is called the canonical ensemble as discussed in Section 2.3.1. We focus on systems for which the stationary distributions are spatially homogeneous, *i.e.* the marginal distributions $\pi_{L,N}[\eta_x\in\cdot]$ are identical for all $x\in\Lambda$. This typically results from translation invariant dynamics on translation invariant lattices with periodic boundary conditions, but the actual details of the dynamics are not needed for our results.

In this chapter, we focus on systems that condense on fixed finite lattices according to Definition 3.2.3 (condensation), *i.e.*

$$\lim_{K\to\infty}\lim_{N\to\infty}\pi_{L,N}[M_L\geq N-K]=1\ ,$$

where $M_L(\cdot)$ is the maximum occupation number (3.4).

4.2.2 Monotonicity and product measures

Recall, from Section 2.4, the natural partial order of configurations and the stochastic order of measures on the state space Ω_L.

A stochastic particle system on Ω_L with generator $\mathcal{L}$ and semi-group $(S(t)=e^{t\mathcal{L}}:t\geq 0)$ is called monotone (attractive) if it preserves stochastic order in time, i.e.

$$\mu\leq\mu' \quad\Longrightarrow\quad \mu S(t)\leq\mu' S(t) \quad \text{for all } t\geq 0.$$

Coupling techniques for monotone processes are an important tool to derive rigorous results on the large scale dynamics of such systems such as hydrodynamic limits. There are sufficient conditions on the jump rates (4.1) to ensure monotonicity for a large class of processes (see e.g. [45] for more details), however for our results we only need a simple consequence for the stationary measures of the process.

Lemma 4.2.1. *If the stochastic particle system as defined in Section 4.2.1 is mono-*

tone, then the canonical distributions $\pi_{L,N}$ *are ordered in* N*, i.e.*

$$\pi_{L,N} \le \pi_{L,N+1} \quad \textit{for all } N \ge 0 \ . \tag{4.2}$$

The proof is completely standard but short, so we include it for completeness.

Proof. Consider two initial distributions μ and μ', concentrating on $\Omega_{L,N}$ and $\Omega_{L,N+1}$ respectively, given by

$$\mu[\eta] = \mathbb{1}(\eta_1 = N) \quad \text{and} \quad \mu'[\xi] = \mathbb{1}(\xi_1 = N+1) \ ,$$

for $\eta \in \Omega_{L,N}$ and $\xi \in \Omega_{L,N+1}$. Clearly $\mu \le \mu'$, and so by monotonicity of the process this implies $\mu S(t) \le \mu' S(t)$ for all $t \ge 0$. Furthermore, by ergodicity we have

$$\pi_{L,N} = \lim_{t\to\infty} \mu S(t) \le \lim_{t\to\infty} \mu' S(t) = \pi_{L,N+1} \ .$$

□

All rigorous results on condensing particle systems so far have been achieved for processes for which the measures $\pi_{L,N}$ take a simple factorized form. In this case, the process (4.1) has a stationary product measure with un-normalized single-site weights $w(n) > 0$, $n \in \mathbb{N}$, which we first discussed in Section 2.3.1 for general zero-range processes. For clarity, we redefine these (homogeneous) measures here.

Due to the conservation of particle number the process exhibits a whole family of homogeneous product measures

$$\nu_\phi^L[\eta] = \prod_{x\in\Lambda} \nu_\phi[\eta_x] \quad \text{with marginals} \quad \nu_\phi[\eta_x] = \frac{w(\eta_x)}{z(\phi)}\, \phi^{\eta_x} \ , \tag{4.3}$$

which are defined whenever the normalization $z(\phi) = \sum_{n=0}^{\infty} \phi^n w(n)$ is finite. The family $\{\nu_\phi : \phi \in D_\phi\}$ is also called the grand-canonical ensemble. Since the process is irreducible on $\Omega_{L,N}$ for all $N \in \mathbb{N}$ we have $w(n) > 0$ for all $n \ge 0$. The canonical distribution can be written as

$$\pi_{L,N}[\eta] = \nu_\phi^L[\eta | \sum_{x\in\Lambda} \eta_x = N] \quad \text{for all } \phi \in D_\phi \ , \tag{4.4}$$

which is independent of the choice of ϕ. Equivalently

$$\pi_{L,N}[\eta] = \frac{1}{Z_{L,N}} \prod_{x\in\Lambda} w(\eta_x) \quad \text{where} \quad Z_{L,N} = \sum_{\eta\in\Omega_{L,N}} \prod_{x\in\Lambda} w(\eta_x) \tag{4.5}$$

is the (canonical) partition function. Note that throughout this chapter we characterize all measures by their mass functions since we work only on a countable state space Ω_L and the measures $\pi_{L,N}$ concentrate on finite state spaces $\Omega_{L,N}$.

4.2.3 Results

In this chapter, our results hold for systems with general stationary weights, $w(n) > 0$ for each $n \in \mathbb{N}$, subject to the regularity assumption that

$$\lim_{n\to\infty} w(n-1)/w(n) \in (0,\infty] \tag{4.6}$$

exists, which is then necessarily equal to ϕ_c by (2.13).

Theorem 4.2.2. *Consider a spatially homogeneous stochastic particle system as defined in Section 4.2.1 which exhibits condensation on fixed finite lattices in the sense of Definition 3.2.3, has stationary product measures that satisfy* (4.6)*, and has finite critical density*

$$\rho_c := \nu_{\phi_c}(\eta_1) = \frac{1}{z(\phi_c)} \sum_{n=0}^{\infty} n w(n) \phi_c^n < \infty \ . \tag{4.7}$$

Then the canonical measures are not stochastically ordered so the process is necessarily not monotone (see Lemma 4.2.1).
The same is true if (4.7) is replaced by the assumption that[1] $w(n) \sim n^{-b}$ *as* $n \to \infty$ *with* $b \in (3/2, 2]$*, i.e. stationary weights have a power law tail with infinite first moment (*$\rho_c = \infty$*).*

4.2.4 Discussion

We will prove non-monotonicity in the next section by showing that expectations for a particular monotone decreasing observable $f : \Omega_L \to \mathbb{R}$ under $\pi_{L,N}$ are not decreasing in N. The chosen function is related to (but not equal to) the number of particles outside the maximum (condensate), which has been shown previously to exhibit non-monotone behaviour for a class of condensing zero-range processes in the thermodynamic limit (3.2) [16, 67]. When the number of particles $N > \rho_c L$ just exceeds the critical value, typical configurations still appear homogeneous. Only when the number of particles is increased further the system switches to a condensed state that contains almost all of the particles in the system. We present numerical

[1] For functions $g, h : \mathbb{N} \to \mathbb{R}$ we use the notation $g(n) \sim h(n)$ if $\frac{g(n)}{h(n)} \to c \in (0, \infty)$ as $n \to \infty$.

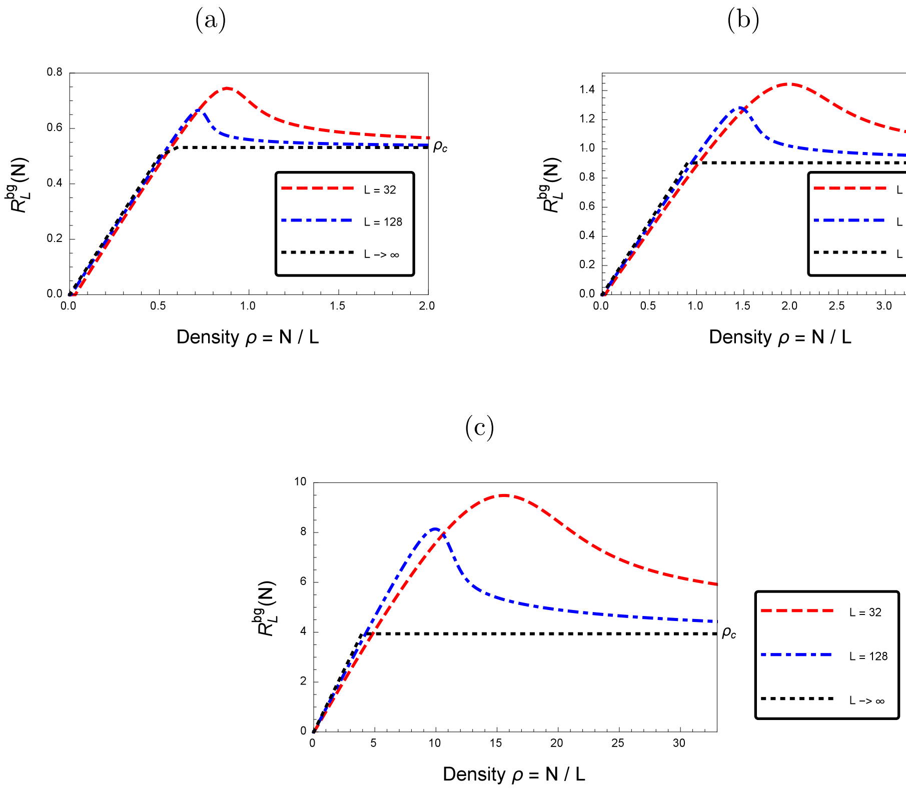

Figure 4.1: Non-monotone behaviour of the expected background density $R_L^{bg}(N)$ (4.8) for lattice sizes $L = 32$ and $L = 128$; (a) (finite mean) power law tails with $b = 5$, (b) log-normal tails with $\mu = 0$ and $\sigma = 1/\sqrt{2}$, and (c) almost exponential tails with $\beta = 1$ (see Section 3.3 for details). The dotted black line shows the limit as $L, N \to \infty$ and $N/L \to \rho$, which is monotone and non-decreasing.

evidence of this non-monotone switching behaviour for the background density

$$R_L^{bg}(N) := \frac{1}{L-1}\pi_{L,N}(N - M_L) \tag{4.8}$$

in Figure 4.1. This is a finite size effect which disappears with increasing L, and for specific models it has been shown to be related to the existence of super-critical homogeneous metastable states [16, 74]. For large L, the switching to condensed states occurs abruptly over a relatively small range of values for N/L. Our result implies that this behaviour is generic for all condensing systems with finite critical density. We also discuss a connection to convexity properties of the entropy of the system in Section 4.5.

There are several examples of homogeneous, monotone particle systems with finite critical density that condense in the thermodynamic limit, which have been studied on a heuristic level as summarized in Section 4.4. Their stationary measures are not of product form and no explicit formulas are known, so these systems are therefore hard to analyse rigorously. In models with product measures different parts of the lattice are uncorrelated and can therefore independently accommodate fluctuations of occupation numbers beyond the critical value, which intuitively explains the presence of metastable fluid states with densities higher than ρ_c around the critical point. For systems with non-product stationary measures, upward fluctuations in the density which are homogeneously distributed may be suppressed strongly enough, so that those metastable states do not exist. Such models may then be also monotone.

We excluded the case $\phi_c = 0$ in the presentation in Section 4.2.3 for notational convenience, but it is easy to see that our results also hold in this case. With the convention $0^0 = 1$ we have $z(0) = w(0) = 1$ and $\rho_c = 0$, and then existence of the limit $Z_{2,N}/w(N)$ is equivalent to

$$\pi_{2,N}[M_2 = N] = 2\frac{w(N)w(0)}{Z_{2,N}} \to \frac{2w(0)}{2z(0)} = 1 \ ,$$

i.e. condensation of all N particles on a single site. This can easily be extended to all $L \geq 2$ with Proposition 4.3.2. Considering only events with all N particles on one site, or $N-1$ particles on one site and 1 particle elsewhere, we have convergence from above

$$\frac{Z_{L,N}}{w(N)} - Lw(0)^{L-1} > L(L-1)w(0)^{L-2}w(1)\frac{w(N-1)}{w(N)} > 0 \ .$$

This includes non-monotonicity of $\pi_{L,N}$ as we will see in Section 4.3. Examples of this kind have been studied in [94] for zero-range dynamics with rates which asymptotically decay to 0 as the occupation number diverges. This leads to super-exponential stationary weights $w(n)$ with $\phi_c = 0$. A further example is given by the condensing inclusion process studied in [22] and [23].

4.3 Proof of Theorem 4.2.2

We assume that the process exhibits condensation in the sense of Definition 3.2.3 and has stationary product measures, so the canonical measures $\pi_{L,N}$ are of the form (4.5). Furthermore, we assume the weights satisfy the regularity assumption (3.11), where without loss of generality $\phi_c = 1$. We show that the family of canonical measures is not stochastically ordered in N, which implies non-monotonicity of the process by Lemma 4.2.1. To achieve this, we use the test function

$$f(\eta) := \mathbb{1}(\eta_1 = \eta_2 = \ldots = \eta_{L-1} = 0)\ , \tag{4.9}$$

which indicates the event where all particles concentrate in the maximum at site L.

Lemma 4.3.1. *The function $f : \Omega_L \to \mathbb{R}$ defined in (4.9) is monotonically decreasing, which implies that*

$$\frac{Z_{L,N}}{w(N)} \leq \frac{Z_{L,N+1}}{w(N+1)} \quad \textit{for all } N \geq 0\ , \tag{4.10}$$

whenever the canonical measures $\pi_{L,N}$ are stochastically ordered in N.

Proof. Fix configurations $\eta, \zeta \in \Omega_L$ such that $\eta \leq \zeta$. If $f(\eta) = 0$ then η has at least one particle outside of site L, therefore so does ζ which implies $f(\zeta) = 0$. If $f(\eta) = 1$ then necessarily $f(\eta) \geq f(\zeta)$ since $f(\zeta) \in \{0, 1\}$. Therefore f is a decreasing function. Using (4.5) and the convention (3.11), we find that the canonical expectation of the function (4.9) is given by

$$\pi_{L,N}(f) = \frac{w(0)^{L-1} w(N)}{Z_{L,N}} = \frac{w(N)}{Z_{L,N}}\ .$$

So if the canonical measures are monotone in N, monotonicity of f implies (4.10). □

By Proposition 3.3.1 we know that for condensing systems the ratio $Z_{2,N}/w(N)$ converges and then by [71, Theorem 1 and Lemma 5], which is summarised below in Proposition 4.3.2, the sequence $Z_{L,N}/w(N)$ in Lemma 4.3.1 converges for all $L \geq 2$.

Proposition 4.3.2. *Consider conditional product measures (4.5) with weights $w(n) > 0$ for all $n \in \mathbb{N}$, which satisfy*

- $\frac{w(n-1)}{w(n)} \to \phi_c$ *as $n \to \infty$, the regularity assumption (3.11),*
- $z(\phi_c) < \infty$,
- $\frac{Z_{2,N}}{w(N)} \to C$ *as $N \to \infty$.*

Then $C = 2z(\phi_c)$ and furthermore,

$$\frac{Z_{L,N}}{w(N)} \to L z(\phi_c)^{L-1} \quad \textit{as } N \to \infty \textit{ for all } L \geq 2\ . \tag{4.11}$$

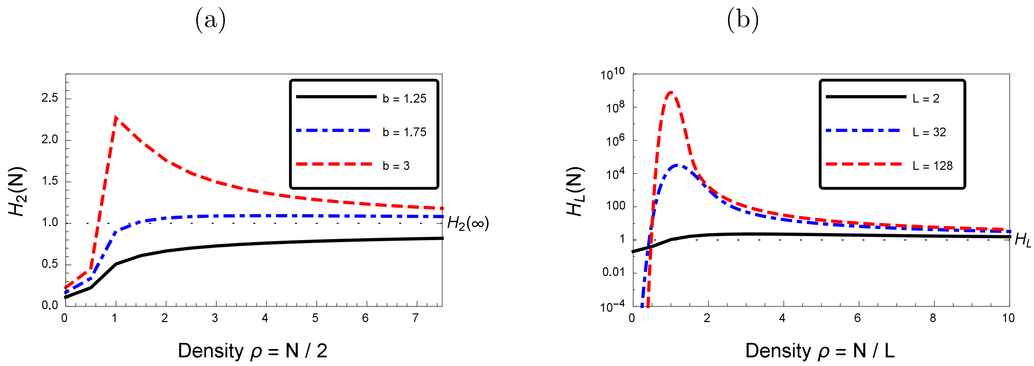

Figure 4.2: Non-monotone behaviour of $H_L(N)$ (4.12), which is the expected value of the observable (4.9) rescaled by its limit; (A) power law tails $w(n) \sim n^{-b}$ for $L = 2$ with $b = 3$, 1.75 and 1.35, where the latter is conjectured to be monotone (see Section 4.4); (B) log-normal tails (3.13) with $\mu = 0$ and $\sigma = 1/\sqrt{2}$ for $L = 2$, $L = 32$ and $L = 128$.

Note that the limit in (4.11) states that the probability of observing a large value of N under the critical product measure is asymptotically equivalent to the probability of observing a large value of N on any one of the L sites, precisely

$$\lim_{N\to\infty} \frac{\nu_{\phi_c}^L[S_L(\eta) = N]}{L\,\nu_{\phi_c}[\eta_1 = N]} = 1\,.$$

This is further equivalent to the canonical probability of the maximum containing the total mass converges to the critical probability that $L-1$ sites are empty, i.e. $\pi_{L,N}[M_L = N] \to \nu_{\phi_c}^{L-1}[\eta \equiv 0]$.

To complete the proof we show a sub-sequence of $Z_{L,N}/w(N)$ converges from above, which contradicts the assumption of monotonicity. We present a numerical illustration for the monotonicity properties of the function

$$H_L(N) := \frac{1}{Lz(1)^{L-1}} \frac{Z_{L,N}}{w(N)}\,. \tag{4.12}$$

in Figure 4.2, which is normalized such that $H_L(N) \to 1$ as $N \to \infty$. The proof of the following Lemma represents the most significant part of the proof of Theorem 4.2.2 and is given in Section 4.3.1 for the case of finite mean and in Section 4.3.2 for the power law case.

Lemma 4.3.3. *Under the conditions of Theorem 4.2.2, and assuming (3.11) without loss of generality, for each $L \geq 2$ there exists a sequence $N_m \in \mathbb{N}$ with $N_m \to \infty$ as*

$m \to \infty$ *such that*

$$\min_{n \le N_m} \left(\frac{Z_{L,n}}{w(N_m)} - Lz(1)^{L-1} \right) \ge C/N_m \ .$$

Therefore, we know that there exists some $N^* \in \mathbb{N}$ such that $Z_{L,N^*}/w(N^*) > Lz(1)^{L-1}$, contradicting the monotonicity assumption. By Lemma 4.3.1 this implies that the canonical measures are not stochastically ordered in N, and thus the process can not be monotone by Lemma 4.2.1. This completes the proof of Theorem 4.2.2.

4.3.1 Proof of Lemma 4.3.3: The finite mean case

In order to prove Lemma 4.3.3 we first specify a non-decreasing subsequence $\{N_m : m \in \mathbb{N}\}$ on which we can bound the ratio $\frac{w(N_m-n)}{w(N_m)}$ below.

Claim 4.3.4. *For weights $\{w(n) : n \in \mathbb{N}\}$ with finite and non-zero first moment, i.e. $0 < \rho_c < \infty$, there exists a sequence $N_m \in \mathbb{N}$ with $N_m \to \infty$ as $m \to \infty$ such that for all $k \in \{0, \dots, N_m - 1\}$*

$$\frac{w(N_m - k)}{w(N_m)} \ge 1 + \frac{k}{N_m}. \tag{4.13}$$

Proof. For each $m \in \mathbb{N}$, define N_m as follows

$$N_m = \max\{n \le m : n\, w(n) = \min_{j \le m} j\, w(j)\} \ .$$

By definition N_m is a non-decreasing sequence. Assume for contradiction that N_m is bounded above, then for all $j \in \mathbb{N}$ we would have $jw(j) \ge j^\star w(j^\star) > 0$ for some $j^\star \in \mathbb{N}$, and therefore $\sum_n nw(n) \to \infty$ contradicting the assumption of finite first moment. For $k \in \{0, \dots N_m - 1\}$ we have

$$\begin{aligned} &(N_m - k)w(N_m - k) \ge N_m w(N_m) \\ \text{and thus} \quad &\frac{w(N_m - k)}{w(N_m)} \ge \frac{N_m}{(N_m - k)} \ge 1 + \frac{k}{N_m}. \end{aligned}$$

□

Claim 4.3.5. *For weights $\{w(n) : n \in \mathbb{N}\}$ with finite first moment, there exists a subsequence $\{N_\ell : \ell \in \mathbb{N}\}$ of the sequence defined in Claim 4.3.4 such that $\frac{Z_{2,N_\ell - n}}{w(N_\ell)} > 2z(1)$ for all $n \in \{0, \dots, N_\ell\}$ and ℓ sufficiently large.*

Proof. By neglecting at most a single term in the sum defining $Z_{L,N}$, the ratio $\frac{Z_{2,N}}{w(N)}$

can be bounded below as follows,

$$\frac{Z_{2,N}}{w(N)} = \sum_{n=0}^{N} w(n)\frac{w(N-n)}{w(N)} \geq 2 \sum_{n=0}^{\lfloor N/2 \rfloor -1} w(n)\frac{w(N-n)}{w(N)}. \tag{4.14}$$

We define

$$K_m := \max\Big\{k^* \leq N_m : Z_{2,k^*} = \min_{0\leq k\leq N_m} Z_{2,k}\Big\}$$

to be the largest index where the ratio $\frac{Z_{2,k}}{w(N_m)}$ is minimized. In particular

$$\frac{Z_{2,N_m-n}}{w(N_m)} \geq \frac{Z_{2,K_m}}{w(N_m)} \quad \text{for all} \quad m \geq 0 \ \text{ and } \ n \in \{0,\dots,N_m\}. \tag{4.15}$$

By definition $K_m \leq N_m$, and so $r := \limsup_{m\to\infty} K_m/N_m \leq 1$. There exists a subsequence $(m_\ell : \ell \geq 0)$ such that $K_{m_\ell}/N_{m_\ell} \to r$, with a slight abuse of notation we denote the subsequences N_{m_ℓ} and K_{m_ℓ} simply by N_ℓ and K_ℓ. Suppose $r < 1$, by Claim 4.3.4 we have

$$\frac{Z_{2,K_\ell}}{w(N_\ell)} \geq \frac{Z_{2,K_\ell}}{w(K_\ell)}\left(2 - \frac{K_\ell}{N_\ell}\right) \to 2z(1)(2-r) > 2z(1)\,,$$

which together with (4.15) contradicts Proposition 4.3.2, therefore $K_\ell/N_\ell \nearrow 1$.

By (4.14) and applying Claim 4.3.4 we then have

$$\frac{Z_{2,K_\ell}}{w(N_\ell)} \geq 2\sum_{k=0}^{\lfloor K_\ell/2\rfloor-1} w(k)\frac{w(K_\ell-k)}{w(N_\ell)} \geq \frac{2}{N_\ell}\sum_{k=0}^{\lfloor K_\ell/2\rfloor-1} k\,w(k) + 2\Big(2-\frac{K_\ell}{N_\ell}\Big)\sum_{k=0}^{\lfloor K_\ell/2\rfloor-1} w(k)\ .$$

Subtracting $2z(1)$ we get

$$\begin{aligned}\frac{Z_{2,K_\ell}}{w(N_\ell)} &- 2z(1)\\ &\geq \frac{2}{N_\ell}\sum_{k=0}^{\lfloor K_\ell/2\rfloor-1} k\,w(k) - 2\sum_{k=\lfloor K_\ell/2\rfloor}^{\infty} w(k) + 2\Big(1-\frac{K_\ell}{N_\ell}\Big)\sum_{k=0}^{\lfloor K_\ell/2\rfloor-1} w(k)\ .\end{aligned} \tag{4.16}$$

Neglecting the final term in (4.16) we have

$$\begin{aligned}N_\ell\Big(\frac{Z_{2,K_\ell}}{w(N_\ell)} - 2z(1)\Big) &> 2\sum_{k=0}^{\lfloor K_\ell/2\rfloor-1} k\,w(k) - 2N_\ell\sum_{k=\lfloor K_\ell/2\rfloor}^{\infty} w(k)\\ &> 2\sum_{k=0}^{\lfloor K_\ell/2\rfloor-1} k\,w(k) - \frac{4N_\ell}{K_\ell}\sum_{k=\lfloor K_\ell/2\rfloor}^{\infty} kw(k) \to 2\rho_c z(1) > 0\,,\end{aligned}$$

using $K_\ell/N_\ell \to 1$ as $\ell \to \infty$, where ρ_c is the critical density defined in (4.7). Together with (4.15) this completes the proof of Claim 4.3.5. $\square$

To complete the proof of Lemma 4.3.3 we proceed by induction on the system size, L. We make the following inductive hypothesis;

(H) there exists a sequence $\{N_m : m \in \mathbb{N}\}$ such that $\frac{Z_{L,N_m-n}}{w(N_m)} > Lz(1)^{L-1}$ for all $n \in \{0, \dots, N_m\}$ and m sufficiently large.

The case $L = 2$ is given by Claim 4.3.5. Analogously to the proof of Claim 4.3.5 we define

$$K_m := \max\Big\{k^* \le N_m : Z_{L,k^*} = \min_{0\le k\le N_m} Z_{L,k}\Big\}. \tag{4.17}$$

By the same argument as in the proof of Claim 4.3.5 there exists a subsequence $(m_\ell : \ell \ge 0)$ such that $K_{m_\ell}/N_{m_\ell} \nearrow 1$, again we denote the respective subsequences by K_ℓ and N_ℓ. For ℓ sufficiently large, we have

$$\begin{aligned}
\frac{Z_{L+1,K_\ell}}{w(N_\ell)} &= \sum_{k=0}^{K_\ell} w(k)\frac{Z_{L,K_\ell-k}}{w(N_\ell)} = \sum_{k=0}^{\lfloor K_\ell/2\rfloor} w(k)\frac{Z_{L,K_\ell-k}}{w(N_\ell)} + \sum_{k=0}^{\lfloor K_\ell/2\rfloor-1} Z_{L,k}\frac{w(K_\ell-k)}{w(N_\ell)} \\
&> L\,z(1)^{L-1}\sum_{k=0}^{\lfloor K_\ell/2\rfloor} w(k) + \sum_{k=0}^{\lfloor K_\ell/2\rfloor-1} Z_{L,k}\Big(1+\frac{N_\ell-K_\ell+k}{N_\ell}\Big),
\end{aligned}$$

where the final inequality follows from the inductive hypothesis (H) and Claim 4.3.4. Subtracting $(L+1)z(1)^L$ we get

$$\begin{aligned}
\frac{Z_{L+1,K_\ell}}{w(N_\ell)} - (L+1)z(1)^L > &- L\,z(1)^{L-1}\sum_{k=\lfloor K_\ell/2\rfloor+1}^{\infty} w(k) \\
&+ \frac{1}{N_\ell}\sum_{k=0}^{\lfloor K_\ell/2\rfloor-1} k\,Z_{L,k} + \Big(1-\frac{K_\ell}{N_\ell}\Big)\sum_{k=0}^{\lfloor K_\ell/2\rfloor-1} Z_{L,k} \\
&+ \sum_{k=0}^{\lfloor K_\ell/2\rfloor-1} Z_{L,k} - z(1)^L\ .
\end{aligned} \tag{4.18}$$

Now, following the proof of Claim 4.3.5, multiply (4.18) by N_ℓ and neglect the second term on the second line. Then the first term vanishes, since

$$0 \le N_\ell \sum_{k=\lfloor K_\ell/2\rfloor+1}^{\infty} w(k) \le \frac{2N_\ell}{K_\ell}\sum_{k=\lfloor K_\ell/2\rfloor+1}^{\infty} k\,w(k) \to 0 \quad \text{as } \ell\to\infty\ .$$

In terms of the normalized grand-canonical measure $Z_{L,k} = z(1)^L \nu_1^L[S_L = k]$, so we have

$$\sum_{k=0}^{\infty} k\, Z_{L,k} = z(1)^L \nu_1^L(S_L) = \rho_c L z(1)^L \in (0,\infty) \ , \tag{4.19}$$

where ρ_c is the critical density as defined in (4.7). This implies that the first term in the second line of (4.18), after multiplication with N_ℓ, converges to a strictly positive constant. Finally, the third line in (4.18) converges to zero after multiplying by N_ℓ since we have $\sum_{n=0}^{\infty} Z_{L,n} = z(1)^L$, which implies

$$0 \geq N_\ell \bigg(\sum_{k=0}^{\lfloor K_\ell/2 \rfloor -1} Z_{L,k} - z(1)^L \bigg) = -N_\ell \sum_{k=\lfloor K_\ell/2 \rfloor}^{\infty} Z_{L,k} \geq -\frac{N_\ell}{\lfloor K_\ell/2 \rfloor} \sum_{k=\lfloor K_\ell/2 \rfloor}^{\infty} k Z_{L,k} \to 0$$

as $\ell \to \infty$, by (4.19) and using that K_ℓ/N_ℓ converges to 1. Using the definition of K_ℓ in (4.17), this implies that there exists a constant $C > 0$ such that for all ℓ large enough

$$\min_{n \leq N_\ell} \left(\frac{Z_{L+1,n}}{w(N_\ell)} - (L+1) z(1)^L \right) \geq C/N_\ell \ ,$$

so (H) holds for $L+1$, completing the induction, which concludes the proof of Lemma 4.3.3 for the case where the critical measure has finite mean.

4.3.2 Proof of Lemma 4.3.3: The infinite mean power law case

We consider stationary weights of the form $w(n) = n^{-b} h(n)$ with $w(0) = 1$, $h(n) \to c \in (0,\infty)$, and $b \in (1,2)$. We prove non-monotonicity of $Z_{L,N}/w(N)$ for $b \in (3/2, 2)$ and $h(n) = 1$ for all $n \in \mathbb{N}$ via an exact computation. The case $b = 2$ can be done completely analogously but involves different expressions with logarithms in the resulting limits, and is presented in Section 4.3.4. The proof remains valid for general converging $h(n)$ with only minor differences, which we explain in a remark at the end of this section. Convergence of $Z_{2,N}/w(N) \to 2z(1)$ from above or below for the exact power law depends on the parameter $b \in (1,2)$, as summarized in the next result.

Lemma 4.3.6. *For stationary weights of the form $w(n) = n^{-b}$ and $w(0) = 1$ with $b \in (1,2)$*

$$N^{b-1} \bigg(\frac{Z_{2,N}}{w(N)} - 2z(1) \bigg) \to F_2(b) \ \text{as } N \to \infty \ , \tag{4.20}$$

where

$$F_2(b) = 2\sum_{i=1}^{\infty} \frac{1}{i!} \prod_{j=0}^{i-1} \big(j+b\big) \frac{2^{b-1-i}}{1-b+i} - 2\frac{2^{b-1}}{b-1} \begin{cases} > 0 & \text{if } b \in (\frac{3}{2}, 2) \\ < 0 & \text{if } b \in (1, \frac{3}{2}) \end{cases} .$$

For $L > 2$ we have

$$\lim_{N\to\infty} N^{b-1}\bigg(\frac{Z_{L,N}}{w(N)} - Lz(1)^{L-1}\bigg) = F_L(b) := z(1)F_{L-1}(b) + (L-1)z(1)^{L-2}F_2(b) \ , \tag{4.21}$$

which has the same sign as $F_2(b)$ and is given by $F_L(b) = \frac{1}{2}L(L-1)z(1)^{L-2}F_2(b)$.

This result implies that whenever $w(n) = n^{-b}$ for $n \geq 1$ and $w(0) = 1$ with $b \in (3/2, 2)$ Lemma 4.3.3 holds with $C = F_2(b)$. This completes the proof of Lemma 4.3.3 in the case $h(n) = 1$.

Proof of Lemma 4.3.6. To prove this result we make use of the full Taylor series of $(1-x)^{-b}$ at $x = 0$ and integral approximations to compute the asymptotic behaviour of summations. To simplify notation we assume that N is even. For odd N there is no term with multiplicity one and there exists an obvious modification. First note that $w(n)$ fulfils the regularity assumption (3.11) and $Z_{L,N}/w(N) \to Lz(1)^{L-1}$ as $N \to \infty$ for all $L \geq 2$ [64], so by Proposition 3.3.1 a process with stationary measures $\pi_{L,N}$ will exhibit condensation. For $L = 2$ we subtract $2z(1)$ from $Z_{2,N}/w(N)$ to get

$$\begin{aligned} \frac{Z_{2,N}}{w(N)} - 2z(1) &= 2\sum_{n=0}^{N/2} w(n)\frac{w(N-n)}{w(N)} - 2\sum_{n=0}^{\infty} w(n) - \frac{w(N/2)w(N/2)}{w(N)} \\ &= 2\sum_{n=1}^{N/2} n^{-b}\Big(1 - \frac{n}{N}\Big)^{-b} - 2\sum_{n=1}^{\infty} n^{-b} - 2^{2b}N^{-b} \ . \end{aligned} \tag{4.22}$$

Substituting the Taylor expansion of $(1-x)^{-b}$ we find

$$\begin{aligned} \frac{Z_{2,N}}{w(N)} - 2z(1) &= 2\sum_{n=1}^{N/2} n^{-b} \sum_{i=0}^{\infty} \frac{1}{i!}\Big(\frac{n}{N}\Big)^{i} \prod_{j=0}^{i-1}(j+b) - 2\sum_{n=1}^{\infty} n^{-b} - 2^{2b}N^{-b} \\ &= 2\sum_{i=1}^{\infty} \frac{1}{i!} \prod_{j=0}^{i-1} \big(j+b\big) \frac{1}{N^i} \sum_{n=1}^{N/2} n^{-b+i} - 2\sum_{n=N/2+1}^{\infty} n^{-b} - 2^{2b}N^{-b} \ . \end{aligned} \tag{4.23}$$

In the last line the $i = 0$ term was combined with the second term, and we adopt the usual convention that empty products are equal to one. Both summations in n are over continuous and monotone functions $g : \mathbb{R} \to (0, \infty)$, therefore we can use the usual integral approximation for decreasing (increasing) functions

$$\int_c^{d+1} g(x)\,dx \leq (\geq) \sum_{n=c}^{d} g(n) \leq (\geq) g(c) + \int_c^d g(x)\,dx \tag{4.24}$$

for all $c \in \mathbb{N}$ and $d \in \mathbb{N} \cup \{\infty\}$. Multiplying by N^{b-1} we find the limit as $N \to \infty$ of (4.23) to be

$$F_2(b) = 2\sum_{i=1}^{\infty} \frac{1}{i!} \prod_{j=0}^{i-1} (j+b) \frac{2^{b-1-i}}{1-b+i} - 2\frac{2^{b-1}}{b-1} \ . \tag{4.25}$$

It is shown in Section 4.3.3 that this is positive (and finite) in the region $b \in (3/2, 2)$ and negative (and finite) in the region $b \in (1, 3/2)$, completing the proof of Lemma 4.3.6 for $L = 2$. The result holds for general system size, $L \geq 2$, and is proved by induction. The inductive hypothesis states

$$\lim_{N\to\infty} N^{b-1}\left(\frac{Z_{L,N}}{w(N)} - Lz(1)^{L-1}\right) = F_L(b) = z(1)F_{L-1}(b) + (L-1)z(1)^{L-2}F_2(b) \ . \tag{4.26}$$

Similar to the case $L = 2$ we write

$$N^{b-1}\left(\frac{Z_{L+1,N}}{w(N)} - (L+1)z(1)^L\right)$$
$$= \underbrace{N^{b-1}\left(\sum_{n=0}^{N/2} Z_{L,n}\frac{w(N-n)}{w(N)} - z(1)^L\right)}_{\Xi_{L,N}} + \underbrace{N^{b-1}\left(\sum_{n=0}^{N/2-1} w(n)\frac{Z_{L,N-n}}{w(N)} - Lz(1)^L\right)}_{\Theta_{L,N}} \ . \tag{4.27}$$

We first establish the limit of the function $\Theta_{L,N}$ in equation (4.27). The inductive hypothesis (4.26) can be written as

$$\frac{Z_{L,n}}{w(n)} = \frac{F_L(b) + o_n(1)}{n^{b-1}} + Lz(1)^{L-1} \ , \tag{4.28}$$

which implies $\Theta_{L,N}$ can be written as

$$\begin{aligned}\Theta_{L,N} &= N^{b-1}\left(\sum_{n=0}^{N/2-1} w(n)\frac{w(N-n)}{w(N)}\frac{Z_{L,N-n}}{w(N-n)} - Lz(1)^L\right)\\ &= N^{b-1}\left(\sum_{n=0}^{N/2-1} w(n)\frac{w(N-n)}{w(N)}\left[\frac{F_L(b)+o_N(1)}{(N-n)^{b-1}} + Lz(1)^{L-1}\right] - Lz(1)^L\right) \ .\end{aligned}$$

Rearranging terms and noting that $\frac{w(N-n)}{w(N)}\frac{N^{b-1}}{(N-n)^{b-1}} = \left(\frac{N-n}{N}\right)^{1-2b}$ we then have

$$\begin{aligned}\Theta_{L,N} =&(F_L(b)+o_N(1))\sum_{n=0}^{N/2-1} w(n)\left(\frac{N-n}{N}\right)^{1-2b}\\ &+ Lz(1)^{L-1}N^{b-1}\left(\sum_{n=0}^{N/2-1} w(n)\frac{w(N-n)}{w(N)} - z(1)\right) \ .\end{aligned}$$

After Taylor expanding $(1-x)^{1-2b}$ appearing in the first line above, and using (4.24) we see that the limit of the first line is given by $F_L(b)z(1)$ as $N\to\infty$. Using the $L=2$ result to calculate the limit of the second line we find

$$\Theta_{L,N} \to F_L(b)z(1) + \frac{Lz(1)^{L-1}F_2(b)}{2} \text{ as } N\to\infty \ . \tag{4.29}$$

To identify the limit of $\Xi_{L,N}$ in (4.27), we again make use of the Taylor expansion of $(1-x)^{-b}$ similarly to the two site case and we write

$$\Xi_{L,N} = N^{b-1}\left(Z_{L,0} + \sum_{n=1}^{N/2} Z_{L,n}\sum_{i=0}^{\infty}\frac{1}{i!}\prod_{j=0}^{i-1}(j+b)\left(\frac{n}{N}\right)^i - z(1)^L\right) \ .$$

Changing the order of summations, separating the $i=0$ term and using $\sum_{n=0}^{\infty} Z_{L,n} = z(1)^L$ we have

$$\Xi_{L,N} = N^{b-1}\left(\sum_{i=1}^{\infty}\frac{1}{i!}\prod_{j=0}^{i-1}(j+b)\frac{1}{N^i}\sum_{n=1}^{N/2} n^i Z_{L,n} - \sum_{n=N/2+1}^{\infty} Z_{L,n}\right) \ . \tag{4.30}$$

For all $i\geq 1$ and $b\in(1,2)$ we have $N^{b-1-i}\to 0$ as $N\to\infty$, which implies that for any fixed $N_1\in\mathbb{N}$ we have $N^{b-1-i}\sum_{n=1}^{N_1-1} n^i Z_{L,n}\to 0$. Therefore, the following limits are equal

$$\lim_{N\to\infty} \Xi_{L,N} = \lim_{N\to\infty} N^{b-1}\Bigg(\sum_{i=1}^{\infty} \frac{1}{i!} \prod_{j=0}^{i-1} (j+b) \frac{1}{N^i} \sum_{n=N_1}^{N/2} n^i Z_{L,n} - \sum_{n=N/2+1}^{\infty} Z_{L,n} \Bigg) .$$

Using the inductive hypothesis (4.28) we have $\lim_{N\to\infty} \Xi_{L,N}$ is given by

$$\begin{aligned} &\lim_{N\to\infty} N^{b-1}(F_L(b)+o_N(1))\Bigg(\sum_{i=1}^{\infty} \frac{1}{i!} \prod_{j=0}^{i-1} (j+b) \frac{1}{N^i} \sum_{n=N_1}^{N/2} n^i \frac{w(n)}{n^{b-1}} - \sum_{n=N/2+1}^{\infty} \frac{w(n)}{n^{b-1}} \Bigg) \\ &\quad + \lim_{N\to\infty} N^{b-1} L z(1)^{L-1}\Bigg(\sum_{i=1}^{\infty} \frac{1}{i!} \prod_{j=0}^{i-1} (j+b) \frac{1}{N^i} \sum_{n=N_1}^{N/2} n^i w(n) - \sum_{n=N/2+1}^{\infty} w(n) \Bigg) . \end{aligned} \tag{4.31}$$

Now applying the $L=2$ result, by identifying (4.25) in the second line and proving that the first line converges to 0, we have

$$\Xi_{L,N} \to \frac{Lz(1)^{L-1}F_2(b)}{2} , \tag{4.32}$$

where the limit of the first line of (4.31) was 0 by the additional factor $1/n^{b-1}$ appearing in the summations. Combining (4.29) and (4.32) we have

$$N^{b-1}\Bigg(\frac{Z_{L+1,N}}{w(N)} - (L+1)z(1)^L\Bigg) \to z(1)F_L(b) + Lz(1)^{L-1}F_2(b) \text{ as } N\to\infty ,$$

concluding the induction so the result holds for all $L \geq 2$. From the recursion (4.21) it is obvious that

$$F_L(b) = \frac{1}{2}L(L+1)z(1)^{L-2}F_2(b) ,$$

which will have the same sign as $F_2(b)$, completing the proof of Lemma 4.3.6. □

A slightly modified version of Lemma 4.3.6 also holds if the stationary weights are of the form $w(n) = n^{-b}h(n)$ where $\lim_{n\to\infty} h(n) = c \in (0,\infty)$. The limit in (4.20) only depends on the tail behaviour of the weights and is now given by $cF_2(b)$. Briefly, this can be seen as follows, (4.22) becomes

$$2\sum_{n=0}^{N/2} n^{-b}h(n)\frac{h(N-n)}{h(N)}\left(1-\frac{n}{N}\right)^{-b} - 2\sum_{n=0}^{\infty} n^{-b}h(n) + 2^{2b}N^{-b}\frac{h(N/2)h(N/2)}{h(N)} .$$

Taylor expanding $(1-x)^{-b}$ and rearranging to find terms of the form

$N^{1-b-i}\sum_{n=1}^{N/2} h(n)n^{-b+i}$ and using the same argument to calculate the limit of $\Xi_{L,N}$ we have

$$\lim_{N\to\infty} N^{b-1-i}\sum_{n=1}^{N/2} h(n)n^{-b+i} = \lim_{N\to\infty} N^{b-1-i}\sum_{n=N_1}^{N/2} cn^{-b+i} < \infty$$

for all $i \geq 1$ and any $N_1 \in \mathbb{N}$, and the result follows. Similar modifications are required in the inductive step and the new limit in (4.21) is given by $c^{L-1}F_L(b)$ for all $L \geq 2$. This does not change the sign of the limit in (4.21) and therefore Lemma 4.3.3 still holds.

4.3.3 On the sign of $F_2(b)$

In this section, we compute the sign of $F_2(b)$ for $b \in (1, 2)$, where

$$F_2(b) = 2\sum_{i=1}^{\infty}\frac{1}{i!}\prod_{j=0}^{i-1}(j+b)\frac{2^{b-1-i}}{1-b+i} - 2\frac{2^{b-1}}{(b-1)} \ .$$

To compute the sign we first show

$$F_2(b) = -\frac{\sqrt{\pi}2^{2b-1}\Gamma(2-b)}{(b-1)\Gamma\left(\frac{3}{2}-b\right)} \ .$$

Recall the definition of the Pochhammer symbol

$$(q)_n = \begin{cases} 1 & \text{if } n = 0 \\ (q)(q+1)\dots(q+i-2)(q+i-1) & \text{for } n \geq 1 \end{cases} ,$$

and the hypergeometric function

$${}_2F_1(c,d,e,z) = \sum_{i=0}^{\infty}\frac{z^i}{i!}\frac{(c)_i(d)_i}{(e)_i} \ .$$

We now show

$$F_2(b) = -\frac{2^b}{b-1}{}_2F_1\left(1-b, b, 2-b, \frac{1}{2}\right) \ . \tag{4.33}$$

Factorising the term $2^b/(b-1)$ from $F_2(b)$ and rearranging terms inside the summation we have

$$\begin{aligned} F_2(b) &= \frac{2^b}{b-1}\left(\sum_{i=1}^{\infty}\frac{1}{i!}\left(\frac{1}{2}\right)^i\prod_{j=0}^{i-1}(j+b)\frac{b-1}{1-b+i}-1\right) \\ &= \frac{2^b}{b-1}\sum_{i=0}^{\infty}\frac{1}{i!}\left(\frac{1}{2}\right)^i\prod_{j=0}^{i-1}(j+b)\frac{b-1}{1-b+i} \ . \end{aligned}$$

Now use the following identities to simplify the terms inside the summation

$$\prod_{j=0}^{i-1}(j+b)=(b)_i \quad \text{and} \quad (1-b+i)=(1-b)\frac{(2-b)_i}{(1-b)_i} \ ,$$

which gives the required result (4.33).

To complete the proof, we use the following two relations for hypergeometric functions, Euler's transform [95, 15.3.3]

$${}_2F_1(c,d,e,z)=(1-z)^{e-d-c}{}_2F_1(e-c,e-d,e,z) \ ,$$

and Gauss's second summation theorem [95, 15.1.24]

$${}_2F_1\left(c,d,\frac{1}{2}(1+c+d),\frac{1}{2}\right)=\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}(1+c+d)\right)}{\Gamma\left(\frac{1}{2}(1+c)\right)\Gamma\left(\frac{1}{2}(1+d)\right)} \ .$$

Therefore,

$$\begin{aligned} F_2(b) &= -\frac{2^b}{b-1}{}_2F_1\left(1-b,b,2-b,\frac{1}{2}\right) \\ &= -\frac{2^{2b-1}}{b-1}{}_2F_1\left(1,2-2b,2-b,\frac{1}{2}\right) \\ &= -\frac{\sqrt{\pi}2^{2b-1}\Gamma(2-b)}{(b-1)\Gamma\left(\frac{3}{2}-b\right)} \ . \end{aligned}$$

To calculate the sign of $F_2(b)$ we first note that the gamma function is given by [95, 6.1.3]

$$\Gamma(x)=\frac{e^{-\gamma x}}{x}\prod_{n=1}^{\infty}\left[\left(1+\frac{x}{n}\right)^{-1}e^{\frac{x}{n}}\right] \ \text{for some } \gamma>0 \ .$$

This implies $\Gamma(x) < 0$ for $x \in (-1, 0)$ and $\Gamma(x) > 0$ for $x > 0$ and we have

$$F_2(b) \begin{cases} < 0 \text{ for } b \in (1, 3/2) \\ > 0 \text{ for } b \in (3/2, 2) \end{cases} .$$

4.3.4 Proof of Lemma 4.3.3: The infinite mean power law case with $b = 2$

Consider stationary weights of the form $w(n) = n^{-2}$ with $w(0) = 1$, we prove the non-monotonicity of $Z_{L,N}/w(N)$ in a similar fashion to the proof of Lemma 4.3.6 summarised in the following lemma.

Lemma 4.3.7. *For stationary weights of the form $w(n) = n^{-2}$ with $w(0) = 1$ we have*

$$\frac{N}{\log(N)} \left(\frac{Z_{2,N}}{w(N)} - 2z(1) \right) \to \hat{F}_2 = 4 \text{ as } N \to \infty .$$

For $L > 2$ we have

$$\frac{N}{\log(N)} \left(\frac{Z_{L,N}}{w(N)} - Lz(1)^{L-1} \right) \to \hat{F}_L := z(1)\hat{F}_{L-1} + (L-1)z(1)^{L-2}\hat{F}_2 \text{ as } N \to \infty , \tag{4.34}$$

which is positive for all $L \geq 2$ since $\hat{F}_2 > 0$.

Proof. First consider the case $L = 2$. As in the proof of Lemma 4.3.6 we will utilise the full Taylor expansion of $(1 - x)^{-2}$, integral bounds on monotone series, and assume N is even, for N odd there exist obvious modifications to the proof. We have from (4.22)

$$\frac{Z_{2,N}}{w(N)} - 2z(1) = 2\sum_{n=1}^{N/2} n^{-2} \left(1 - \frac{n}{N}\right)^{-2} - 2\sum_{n=1}^{\infty} n^{-2} - 2^4 N^{-2} .$$

Where the terms $n = 0$ in the above summations cancel. Substituting the Taylor expansion of $(1 - x)^{-2}$ we find from (4.23)

$$\frac{Z_{2,N}}{w(N)} - 2z(1) = 2\sum_{i=1}^{\infty} (i + 1)N^{-i} \sum_{n=1}^{N/2} n^{-2+i} - 2 \sum_{n=N/2+1}^{\infty} n^{-2} - 2^4 N^{-2} . \tag{4.35}$$

Now we are in a position to apply the integral bounds (4.24), first noting that n^{-2+i} is decreasing for $i = 1$, constant and equal to 1 for $i = 2$, and increasing for $i \geq 3$. Multiplying both sides of (4.35) and applying the integral bounds it is easy to show

$$\frac{N}{\log(N)}\left(\frac{Z_{2,N}}{w(N)} - 2z(1)\right) \to 4 \text{ as } N \to \infty\ . \tag{4.36}$$

Now consider the case $L > 2$ and make the following inductive hypothesis

$$\lim_{N\to\infty} \frac{N}{\log(N)}\left(\frac{Z_{L,N}}{w(N)} - Lz(1)^{L-1}\right) = \hat{F}_L = z(1)\hat{F}_{L-1} + (L-1)z(1)^{L-2}\hat{F}_2\ .$$

As in the proof of Lemma 4.3.6 write

$$\begin{aligned}&\frac{N}{\log(N)}\left(\frac{Z_{L+1,N}}{w(N)} - (L+1)z(1)^L\right)\\ &= \underbrace{\frac{N}{\log(N)}\left(\sum_{n=0}^{N/2} Z_{L,n}\frac{w(N-n)}{w(N)} - z(1)^L\right)}_{\hat{\Xi}_{L,N}} + \underbrace{\frac{N}{\log(N)}\left(\sum_{n=0}^{N/2-1} w(n)\frac{Z_{L,N-n}}{w(N)} - Lz(1)^L\right)}_{\hat{\Theta}_{L,N}}\ .\end{aligned} \tag{4.37}$$

We first establish the limit of $\hat{\Theta}_{L,N}$ in (4.37). The inductive hypothesis can be rewritten as

$$\frac{Z_{L,N}}{w(N)} = \left(\hat{F}_L + o_N(1)\right)\frac{\log(N)}{N} + Lz(1)^{L-1} \tag{4.38}$$

Similar to the proof of Lemma 4.3.6 $\hat{\Theta}_{L,N}$ can be written in the form

$$\begin{aligned}\hat{\Theta}_{L,N} =& \frac{N}{\log(N)}\left(\hat{F}_L + o_N(1)\right)\left(\sum_{n=0}^{N/2-1} w(n)\frac{w(N-n)}{w(N)}\frac{\log(N-n)}{N-n}\right)\\ &+ Lz(1)^{L-1}\frac{N}{\log(N)}\left(\sum_{n=0}^{N/2-1} w(n)\frac{w(N-n)}{w(N)} - z(1)\right)\ .\end{aligned} \tag{4.39}$$

Since $\log(N-n)$ is decreasing for $n \in \{0, \ldots N/2-1\}$ we can find upper and lower bounds of the first term, by pulling out the logarithm, of the form

$$\begin{aligned}&\frac{\log(N/2-1)}{\log(N)}\left(\hat{F}_L + o_N(1)\right)\left(\sum_{n=0}^{N/2-1} w(n)\frac{w(N-n)}{w(N)}\frac{N}{N-n}\right)\\ &\le \frac{N}{\log(N)}\left(\hat{F}_L + o_N(1)\right)\left(\sum_{n=0}^{N/2-1} w(n)\frac{w(N-n)}{w(N)}\frac{\log(N-n)}{N-n}\right)\\ &\le \left(\hat{F}_L + o_N(1)\right)\left(\sum_{n=0}^{N/2-1} w(n)\frac{w(N-n)}{w(N)}\frac{N}{N-n}\right)\ .\end{aligned}$$

Applying (4.24) to the upper and lower bounds above, and (4.36) to the second term in (4.39) we have

$$\lim_{N\to\infty} \hat{\Theta}_{L,N} = z(1)\hat{F}_L + \frac{1}{2}Lz(1)^{L-1}\hat{F}_2 \ . \tag{4.40}$$

To identify the limit of $\hat{\Xi}_{L,N}$ in (4.37) we again follow the steps given in the proof of Lemma 4.3.6, which implies

$$\lim_{N\to\infty} \hat{\Xi}_{L,N} = \frac{1}{2}Lz(1)^{L-1}\hat{F}_2 \ . \tag{4.41}$$

Combining this with (4.40), we have

$$\frac{N}{\log(N)}\left(\frac{Z_{L+1,N}}{w(N)} - (L+1)z(1)^L\right) \to \hat{F}_{L+1} = z(1)\hat{F}_L + Lz(1)^{L-1}\hat{F}_2 \ \text{as} \ N\to\infty \ .$$

From the recursion (4.34), it is obvious that $\hat{F}_L$ will have the same sign as $\hat{F}_2$, completing the proof of Lemma 4.3.7.

□

4.4 Examples of homogeneous condensing processes

In this section, we review several stochastic particle systems that exhibit condensation. By Theorem 4.2.2, if these processes are homogeneous and monotone with a finite critical density they do not have stationary product measures. To prove monotonicity for the examples mentioned below, it is sufficient to construct a basic coupling of the stochastic process on the state space $(\Omega_{L,N}, \Omega_{L,N+1})$ where particles jump together with maximal rate. For a definition of a coupling see [44] and for the statement of Strassen's theorem linking stochastic monotonicity and the coupling technique see [56]. The steps to construct a basic coupling are outlined in [45].

4.4.1 Misanthrope processes and generalizations

Condensation in homogeneous particle systems has mostly been studied in the framework of misanthrope processes [11, 45]. At most one particle is allowed to jump at a time and the rate that this occurs depends on the number of particles in the exit and entry sites. The misanthrope process is a stochastic particle system on the state space $\Omega_L = \mathbb{N}^\Lambda$ defined by the generator

$$\mathcal{L}^{mis} f(\eta) = \sum_{x,y\in\Lambda} r(\eta_x, \eta_y) p(x,y) \big(f(\eta^{x,y}) - f(\eta) \big) \ . \tag{4.42}$$

Here $\eta^{x,y} = \eta - \delta_x + \delta_y$ denotes the configuration after a single particle has jumped from site x to site y, which occurs with rate $r(\eta_x, \eta_y)$. The purely spatial part of the jump rates, $p(x,y) \geq 0$, are translation invariant transition probabilities of a random walk on Λ.

As discussed in Sections 2.3.1 and 2.3.2 misanthrope processes exhibit stationary product measures (2.11) and we recall that the stationary weights satisfy

$$\frac{w(k+1)}{w(k)} = \frac{w(1)}{w(0)} \frac{r(1,k)}{r(k+1,0)} \quad \text{and} \quad w(n) = \prod_{k=1}^{n} \frac{r(1,k-1)}{r(k,0)} \ . \tag{4.43}$$

Misanthrope processes are monotone (attractive) [11] if and only if the jump rates satisfy

$$\begin{aligned} &r(n,m) \leq r(n+1,m) \ \textit{i.e.} \text{ non-decreasing in } n, \\ &r(n,m) \geq r(n,m+1) \ \textit{i.e.} \text{ non-increasing in m} \ . \end{aligned} \tag{4.44}$$

In Theorem 4.2.2, we have proved that processes that exhibit stationary product measures and condensation with finite mean or power law tails, $w(n) \sim n^{-b}$, with $b \in (3/2, 2]$ are necessarily not monotone. For power law tails with $b \in (1, 3/2]$ convergence of $Z_{L,N}/w(N)$ is from below and our method does not disprove monotonicity of the measures $\pi_{L,N}$ or monotonicity of the underlying process. Using the specific form of the stationary measures (4.43), it is clear that possible examples of monotone processes with stationary product measures of this form cannot be of misanthrope type.

Lemma 4.4.1. *A misanthrope process defined by the generator* (4.42)*, that has stationary product measures and exhibits condensation on fixed finite lattices according to Definition 3.2.3 is not monotone.*

Proof. Proposition 3.3.1 implies the existence of the critical measure ν_{ϕ_c} with $\phi_c < \infty$ as a necessary condition for condensation. (4.44) gives necessary conditions for the monotonicity of the misanthrope process and implies with (4.43) that

$$\frac{w(n-1)}{w(n)} = \frac{r(n,0)}{r(1,n-1)} \tag{4.45}$$

is non-decreasing. This implies that the ratio converges to ϕ_c and we have

$$\frac{w(n-1)}{w(n)} \leq \phi_c \qquad \Longrightarrow \qquad w(n) \geq w(n-1)\phi_c^{-1} \tag{4.46}$$

for all $n \in \mathbb{N}$. Therefore, $w(n) \geq w(0)\phi_c^{-n}$ which implies

$$\sum_{n=0}^{N} w(n)\phi_c^n \geq w(0)\sum_{n=0}^{N} \phi_c^n \phi_c^{-n} \to \infty \text{ as } N \to \infty \ . \tag{4.47}$$

We conclude that the critical partition function diverges, hence Proposition 3.3.1 fails and therefore, condensation does not occur in misanthrope processes with stationary product measures. □

In [45] generalised misanthrope processes have been introduced where more than one particle is allowed to jump simultaneously. They are defined via transitions $\eta \to \eta + n(\delta_y - \delta_x)$ for $n \in \{0, \dots, \eta_x\}$ at rate $\Gamma^n_{\eta_x,\eta_y}(y-x)$ and conditions on the jump rates for monotonicity are characterized. This class provides candidates for possible monotone, condensing processes with product measures as we discuss in the next subsection.

4.4.2 Generalised zero-range processes

Recall the definition of the generalised zero-range process (gZRP) from Section 2.3.3 on the state space $\Omega_L = \mathbb{N}^\Lambda$ with the generator

$$\mathcal{L}^{gZRP} f(\eta) = \sum_{x,y\in\Lambda} \sum_{k=1}^{\eta_x} \alpha_k(\eta_x) p(x,y) \big(f(\eta^{x\to(k)y}) - f(\eta)\big) \ . \tag{4.48}$$

Further recall that these processes exhibit stationary product measures if and only if the jump rates have the explicit form

$$\alpha_k(n) = g(k)\frac{h(n-k)}{h(n)} \ , \tag{4.49}$$

where $g, h : \mathbb{N} \to [0,\infty)$ are arbitrary non-negative functions with h strictly positive and the stationary weights are then given by $w(n) = h(n)$. Monotonicity of the gZRP can be characterized in terms of

$$R_k(n) := \sum_{m=0}^{n-k} \left(\alpha_{n-m}(n) - \alpha_{n+1-m}(n+1)\right) \ . \tag{4.50}$$

The gZRP is monotone if and only if

$$\begin{aligned} R_k(n) \geq 0 \text{ for all } n \geq 1 \text{ and } k \in \{1, \dots, n\} \\ \alpha_k(n+1) \geq R_k(n) \text{ for all } n \geq 1 \text{ and } k \in \{1, \dots, n\} \ . \end{aligned} \tag{4.51}$$

These conditions on the transition rates arise from a basic coupling of the gZRP, which is given in Section 4.4.2.1. We note these conditions arise from a special case of the results in [45, Theorem 2.11] on generalised misanthrope models, since $\alpha_k(n)$ depends only on the occupation of the exit site and not the entry site.

In this class, which is discussed in detail in [54], there exist processes which condense on fixed finite lattices according to Definition 3.2.3 which are monotone, homogeneous, and have stationary product measures with a power tail $w(n) \sim n^{-b}$ with $b \in (1, 3/2]$. As an example, consider the gZRP with rates given by

$$\alpha_k(n) = \begin{cases} 0 & \text{if } k = 0 \text{ or } n = 0 \\ k^{-b}(1 - \frac{k}{n})^{-b} & \text{if } k \in \{1, \dots, n-1\} \\ 1 & \text{otherwise .} \end{cases} \tag{4.52}$$

Since $\alpha_k(n)$ is of the form (4.49), the process exhibits stationary product measures with weights of the form

$$w(n) = \begin{cases} 1 & \text{if } n = 0 \\ n^{-b} & \text{otherwise} \end{cases} \ .$$

For all $b > 1$ and $L \geq 2$ the ratio $\frac{Z_{L,N}}{w(N)}$ converges to $Lz(1)^{L-1}$ as $N \to \infty$ [64] so by Proposition 3.3.1 the process exhibits condensation. To prove the process is monotone, we must show the rates satisfy the conditions given in equation (4.51). We first prove $R_k(n) \geq 0$ for all $k \in \{1, \dots, n-1\}$ and $n > 1$. Since $\alpha_n(n) - \alpha_{n+1}(n) = 0$ for all $n \geq 1$ we can drop the $m = 0$ term from the definition of $R_k(n)$. We have

$$R_k(n) = \sum_{m=1}^{n-k} m^{-b} \left[\left(1 - \frac{m}{n}\right)^{-b} - \left(1 - \frac{m}{n+1}\right)^{-b} \right] \ . \tag{4.53}$$

Since $(1-x)^{-b}$ is increasing in $x \in (0, 1)$ and $b > 0$ we have

$$R_k(n) > 0 \text{ for all } k \in \{1, \dots, n\} \text{ and } n \geq 1 \ .$$

We also need to show $\alpha_k(n+1) \geq R_k(n)$ for all $k \in \{1, \dots, n-1\}$ and $n > 1$. Taking

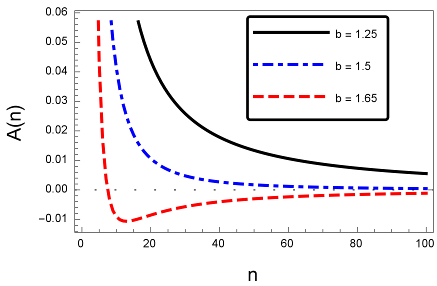

Figure 4.3: Monotonicity condition (4.54) for $b = 1.25$, $b = 1.5$ and $b = 1.65$. For $b = 1.65$ the function $A(n)$ falls below zero, implying the process is non-monotone. For $b = 1.25$ and $b = 1.5$ the function $A(n)$ is positive, indicating the process is monotone.

discrete derivatives in k and using (4.53)

$$\begin{aligned}\alpha_{k+1}(n+1) - R_{k+1}(n) - (\alpha_k(n+1) - R_k(n)) &= \alpha_k(n) - \alpha_k(n+1) \\ &= k^{-b}\left(1 - \frac{k}{n}\right)^{-b} - k^{-b}\left(1 - \frac{k}{n+1}\right)^{-b} > 0 \ ,\end{aligned}$$

so $\alpha_k(n+1) - R_k(n)$ is an increasing function in k. Therefore,

$$\alpha_k(n+1) - R_k(n) \geq \alpha_1(n+1) - R_1(n) \ ,$$

and it suffices to show

$$A(n) := \alpha_1(n+1) - R_1(n) \geq 0 \text{ for all } n \geq 1 \ . \tag{4.54}$$

We present numerical evidence in Figure 4.3 which corroborates our claim that the process with rates (4.52) is indeed monotone for $b \in (1, 3/2]$ and is not for $b > 3/2$.

Instead, consider the alternative formulation of condensation given by Definition 3.2.2 (weak condensation) given in Section 3.2, and the gZRP with rates (4.52) and $b = 1$. In Section 3.6, we showed that the sequence $M_2/N \to 1$ as $N \to \infty$ in probability and, therefore, the process exhibits weak condensation according to Definition 3.2.2. We claim that this process is monotone. For this example, the function $R_k(n)$ remains non-negative for all $k \in \{1, \ldots, n\}$ and $n \geq 0$. Therefore, to prove the process is monotone we have to show (4.54) holds, which is equivalent to the sequence $\sum_{k=1}^{n-1} k^{-1}(n-k)^{-1}$ being non-increasing in n. Since

$$\sum_{k=1}^{n-1} k^{-1}(n-k)^{-1} = \frac{2}{n}\sum_{k=1}^{n-1}\frac{1}{k} \ ,$$

taking a discrete derivative in n and a simple application of the integral test shows the sequence is non-increasing. Therefore, $A(n) \geq 0$ for all n, which implies the process is monotone.

4.4.2.1 Coupling the gZRP

In this section, we construct a coupling of the gZRP to establish conditions on the jump rates for monotonicity. Consider $\eta \in \Omega_{L,N}$ and $\xi \in \Omega_{L,N+1}$ such that $\xi = \eta + \delta_i$ for some $i \in \Lambda$. In order for the coupled process to preserve the order it is necessary that when k particles leave site i in the η process we have either k or $k+1$ particles leave in the ξ process. At sites where $\eta_j = \xi_j$ for $j \in \Lambda$ the same number of particles leave the η and ξ processes. The coupled jump rates are defined hierarchically and are given below. Let $R_k(n)$ be as (4.50), then

$$\begin{cases}\eta_i = n \\ \xi_i = n+1\end{cases} \longrightarrow \begin{cases}\eta_i = n \\ \xi_i = n\end{cases} \quad \text{at rate } \alpha_1(n+1) - R_1(n)$$

$$\begin{cases}\eta_i = n \\ \xi_i = n+1\end{cases} \longrightarrow \begin{cases}\eta_i = n-1 \\ \xi_i = n\end{cases} \quad \text{at rate } R_1(n)$$

$$\begin{cases}\eta_i = n \\ \xi_i = n+1\end{cases} \longrightarrow \begin{cases}\eta_i = n-1 \\ \xi_i = n-1\end{cases} \quad \text{at rate } \alpha_2(n+1) - R_2(n)$$

$$\begin{cases}\eta_i = n \\ \xi_i = n+1\end{cases} \longrightarrow \begin{cases}\eta_i = k-1 \\ \xi_i = k\end{cases} \quad \text{at rate } R_{n+1-k}(n)$$

$$\begin{cases}\eta_i = n \\ \xi_i = n+1\end{cases} \longrightarrow \begin{cases}\eta_i = k \\ \xi_i = k\end{cases} \quad \text{at rate } \alpha_{n+1-k}(n+1) - R_{n+1-k}(n)$$

$$\begin{cases}\eta_i = n \\ \xi_i = n+1\end{cases} \longrightarrow \begin{cases}\eta_i = k \\ \xi_i = k+1\end{cases} \quad \text{at rate } R_{n-k}(n)$$

$$\begin{cases}\eta_i = n \\ \xi_i = n+1\end{cases} \longrightarrow \begin{cases}\eta_i = 1 \\ \xi_i = 1\end{cases} \quad \text{at rate } \alpha_n(n+1) - R_n(n)$$
$$\begin{cases}\eta_i = n \\ \xi_i = n+1\end{cases} \longrightarrow \begin{cases}\eta_i = 0 \\ \xi_i = 1\end{cases} \quad \text{at rate } R_n(n)$$
$$\begin{cases}\eta_i = n \\ \xi_i = n+1\end{cases} \longrightarrow \begin{cases}\eta_i = 0 \\ \xi_i = 0\end{cases} \quad \text{at rate } \alpha_{n+1}(n+1) - R_{n+1}(n) = \alpha_{n+1}(n+1) \; .$$

The transition rates are constructed in this hierarchical way to ensure the marginals of the coupled process correspond to the individual processes η on $\Omega_{L,N}$ and ξ on $\Omega_{L,N+1}$, which are defined by the generator (4.48). This can be seen by calculating the total rate $n-k$ particles leave site i in the η process, which is given by

$$\alpha_{n+1-k}(n+1) - R_{n+1-k}(n) + R_{n-k}(n) = \alpha_{n-k}(n) \; ,$$

and the total rate $n+1-k$ particles leave site i in the ξ process, which is given by

$$R_{n+1-k}(n) + \alpha_{n+1-k}(n+1) - R_{n+1-k}(n) = \alpha_{n+1-k}(n+1) \; .$$

Since all transitions in the coupled process must be non-negative we see that this coupling construction implies the conditions (4.51).

4.4.3 Homogeneous monotone processes without product measures

The chipping model is a stochastic particle system on the state space $\Omega_L = \mathbb{N}^\Lambda$, introduced in [84, 85], see Chapter 3 for more details.

It is easy to see that a basic coupling will preserve the partial order on the state space Ω_L as defined in Section 2.4. Therefore, by Strassen's theorem [56], the chipping model is a monotone process and Lemma 4.2.1 implies that conditional stationary measures of the process are ordered in N, with monotonic convergence of the background density $R_2^{bg}(N)$ shown explicitly in Figure 4.4. The condensation transition in the chipping model was established on a heuristic level in [84, 85, 86] in the thermodynamic limit. In Section 3.8 we showed that the process exhibits condensation according to Definition 3.2.3 for the process on two sites. Therefore, this model is a monotone and spatially homogeneous process that heuristically exhibits a condensation transition with finite (size dependent) critical density, but it does not exhibit stationary product measures.

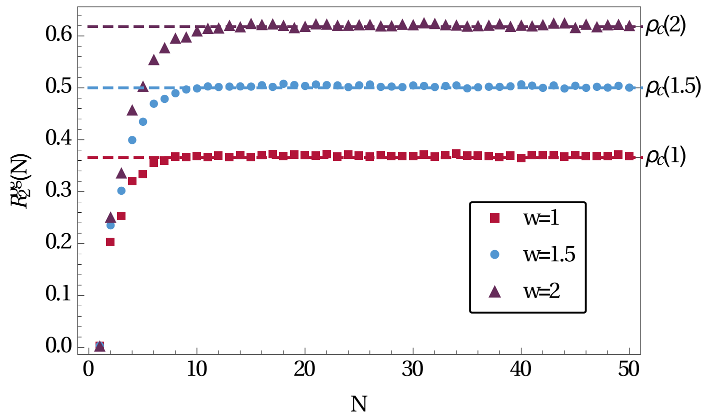

Figure 4.4: Simulation results calculating the background density $R_2^{bg}(N)$ (4.8) for the two site chipping model with $w = 1$, $w = 1.5$, and $w = 2$. $R_2^{bg}(N)$ converges monotonically to the critical density $\rho_c(w) = \frac{1}{2}\left(\sqrt{1+2w}-1\right)$ given in Section 3.8.

4.4.4 Non-monotonicity of processes beyond misanthrope dynamics

We can further generalise zero-range and misanthrope processes, here called long range misanthrope processes (LRMP), to stochastic particle system on the state space $\Omega_L = \mathbb{N}^\Lambda$ defined by the generator

$$\mathcal{L}^{LRMP} f(\eta) = \sum_{x,y\in\Lambda} r_{x,y}(\eta) p(x,y) \left(f(\eta^{x,y}) - f(\eta) \right) \ , \tag{4.55}$$

where the jump rate $r_{x,y} : \Omega_L \to \mathbb{R}$ is dependent on the full configuration $\eta \in \Omega_L$.

For processes with totally asymmetric dynamics, $p(x, x+1) = 1$, then it is known that they can exhibit pair-factorised or cluster-factorised stationary measures of the form

$$\pi_{L,N}[\eta] = \frac{\prod_{x\in\Lambda} w(\eta_{A_x})}{Z_{L,N}} \text{ where } \eta_{A_x} = \{\eta_y : y \in A_x\} \text{ and } A_x \subseteq \Lambda \ ,$$

provided the jump rate $r_{x,y}$ satisfies certain relations [96]. The sets A_x for each $x \in \Lambda$ are typically of the form $\{x, x+1\}$ for processes with pair factorised stationary measures or $\{x-d, \dots, x+d\}$, where $d \in \mathbb{N}$, for processes with cluster factorised stationary measures. Condensation has been observed in totally asymmetric processes that exhibit pair-factorised stationary measures [72, 97, 98] where due to the enhanced correlations the condensate no longer sits on a single site (for numerical evidence see [99]).

In Lemma 4.4.1, we proved that no monotone misanthrope process with sta-

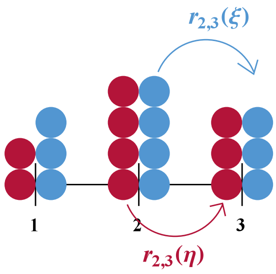

Figure 4.5: Three examples of ordered configurations for the three site, $L = 3$, totally asymmetric LRMP (4.55). The configuration $\eta = (2, 4, 3)$ (red particles) is fixed for each example. The configurations ξ (blue particles) each satisfy $\eta \leq \xi$. A coupling of the totally asymmetric LRMP must both preserve the partial order of the state space and each marginal must behave as originally constructed. Looking at the second and third examples, $\xi = (2, 5, 3)$ and $\xi = (2, 4, 4)$ respectively, the construction of a basic coupling would imply that the rate $r_{2,3}(\cdot)$ must be non-decreasing in the exit occupation and non-increasing in the entry occupation. However, if the rate $r_{2,3}(\cdot)$ is not independent of the occupation of the first site the the basic coupling cannot preserve the partial order. Since either a blue particle exits site 2 breaking the order at site 2, or a red particle leaves site 2 breaking the order at site 3.

tionary product measures can exhibit condensation. We now prove that, under totally asymmetric dynamics, monotone LRMPs are in fact monotone misanthrope processes, and therefore, the conditions derived in Lemma 4.4.1 hold for these processes. In Figure 4.5, we give the intuition for this result, which is based on the construction of the basic coupling as discussed in Section 2.4.

Lemma 4.4.2. *If the totally asymmetric LRMP defined by the generator* (4.55) *is monotone then the jump rate $r_{x,y}(\eta)$ only depends on the occupation numbers of the exit, x, and entry sites, y, i.e. $r_{x,y}(\eta) = \tilde{r}(\eta_x, \eta_y)$.*

Proof. Since monotone processes preserve the natural order of the state space, the evolution of increasing observables must remain ordered under the dynamics of the process (2.26). Therefore, to prove that monotone totally asymmetric LRMPs are in fact monotone misanthrope processes it is enough to consider how multiple observables behave under the dynamics of the process, starting from the same ordered initial conditions.

Monotonicity implies that for all $\eta \leq \xi$ and $f : \Omega_L \to \mathbb{R}$ increasing we have

$$S(t)f(\eta) \leq S(t)f(\xi) \ ,$$

where $(S(t) = e^{t\mathcal{L}^{LRMP}} : t \geq 0)$ is the semi-group of the LRMP on $C(\Omega_L)$.

Consider $f : \Omega_L \to \mathbb{R}$ increasing such that

$$f(\eta) = \mathbb{1}(\eta_a \geq K) \text{ for some } a \in \Lambda \text{ and } K \in \mathbb{N} .$$

Fix η and ξ such that $\eta_a = \xi_a$, which implies $f(\eta) = f(\xi)$. Furthermore, since the process is monotone we can write the following inequality (see Section 2.4)

$$\mathcal{L}^{LRMP} f(\eta) = \lim_{t \downarrow 0} \frac{S(t)f(\eta) - f(\eta)}{t} \leq \lim_{t \downarrow 0} \frac{S(t)f(\xi) - f(\xi)}{t} = \mathcal{L}^{LRMP} f(\xi) . \quad (4.56)$$

Consider the configurations $\eta \leq \xi$ which satisfy

$$\eta_x = \begin{cases} K & \text{if } x = a \\ Q & \text{if } x = a+1 \\ \eta_x & \text{otherwise} \end{cases} \quad \text{and} \quad \xi_x = \begin{cases} K & \text{if } x = a \\ Q & \text{if } x = a+1 \\ \xi_x & \text{otherwise} \end{cases} .$$

Since $f(\eta) = \mathbb{1}(\eta_a \geq K)$ we have

$$\mathcal{L}^{LRMP} f(\eta) = -r_{a,a+1}(\eta) \quad \text{and} \quad \mathcal{L}^{LRMP} f(\xi) = -r_{a,a+1}(\xi) ,$$

which by (4.56) implies

$$\begin{aligned} r_{a,a+1}(\eta) &= r_{a,a+1}(\eta_1, \ldots, K, Q, \eta_{a+1}, \ldots \eta_L) \\ &\geq r_{a,a+1}(\xi_1, \ldots, K, Q, \xi_{a+1}, \ldots \xi_L) = r_{a,a+1}(\xi) . \end{aligned} \quad (4.57)$$

Now consider the function $h : \Omega_L \to \mathbb{R}$ increasing such that

$$h(\eta) = \mathbb{1}(\eta_{a+1} \geq Q+1) \text{ for some } a \in \Lambda \text{ and } Q \in \mathbb{N} .$$

Fix η and ξ as before, which implies $h(\eta) = h(\xi)$. Furthermore, since the process is monotone we can write the following inequality $\mathcal{L}^{LRMP} h(\eta) \leq \mathcal{L}^{LRMP} h(\xi)$, see (4.56). Since

$$\mathcal{L}^{LRMP} h(\eta) = r_{a,a+1}(\eta) \quad \text{and} \quad \mathcal{L}^{LRMP} h(\xi) = r_{a,a+1}(\xi) ,$$

we have

$$\begin{aligned} r_{a,a+1}(\eta) &= r_{a,a+1}(\eta_1, \ldots, K, Q, \eta_{a+1}, \ldots \eta_L) \\ &\leq r_{a,a+1}(\xi_1, \ldots, K, Q, \xi_{a+1}, \ldots \xi_L) = r_{a,a+1}(\xi) . \end{aligned} \quad (4.58)$$

Combining (4.57) and (4.58), we see that the jump rate $r_{a,a+1}(\eta)$ is both non-increasing and non-decreasing in each coordinate $x \in \Lambda$ not equal to a and $a+1$, which implies

$$r_{a,a+1}(\eta) = \tilde{r}(\eta_a, \eta_{a+1}) \text{ for all } a \in \Lambda \text{ and } \eta \in \Omega_L \ .$$

Therefore, the jump rate depends only on the target and exit sites $a+1$ and a respectively.

□

Whilst Lemma 4.4.2 holds for totally asymmetric misanthrope type processes it is not immediately clear how this statement can be generalised. For example, the Kawasaki Ising model can be monotone (attractive) (see for example [42]) even though transition rates depend on a weighted average of the neighbouring configurations.

4.5 Connection to statistical mechanics

Condensation and non-monotonicity are also related to convexity properties of the entropy, which we briefly describe in the following in a non-rigorous discussion. In the thermodynamic limit, the canonical entropy is defined as

$$s(\rho) := \lim_{\substack{L\to\infty \\ N/L\to\rho}} \frac{1}{L} \log Z_{L,N} \ . \tag{4.59}$$

For the processes we consider, equivalence of canonical and grand-canonical ensembles has been established in [65] for condensing or non-condensing systems, so $s(\rho)$ is given by the (logarithmic) Legendre transform of the pressure

$$p(\phi) := \log z(\phi) \ . \tag{4.60}$$

This takes a particularly simple form since the grand-canonical measures are factorisable, and is a strictly convex function for $\phi < \phi_c$. General results then imply that $s(\rho)$ also has to be strictly convex below the critical density ρ_c (see e.g. [100]), which holds for non-condensing systems and condensing systems with $\rho_c = \infty$. For condensing systems with finite critical density, $s(\rho)$ is linear for $\rho > \rho_c$, consistent with phase separation phenomena, where in this case the condensed phase formally exhibits density ∞ (see e.g. [74] for a general discussion).

It is not possible to derive general results for finite L and N, but if we assume that the ratio of weights $w(n-1)/w(n)$ is monotone increasing in n, we can show that a monotone order of $\pi_{L,N}$ implies that $N \mapsto \frac{1}{L}\log Z_{L,N}$ is necessarily convex. Note that with (4.6), our assumption implies that $w(n)$ has exponential tails with $\phi_c \in (0,\infty)$ or decays super-exponentially with $\phi_c = \infty$, and in both cases the system does not exhibit condensation. We can define $w(-1) = 0$ so that $w(\eta_x - 1)/w(\eta_x)$ is a monotone increasing test function on Ω_L. It is easy to see that for its canonical expectation we have for all $L \geq 2$ and $N \geq 2$

$$\pi_{L,N}\Big(\frac{w(\eta_x - 1)}{w(\eta_x)}\Big) = \frac{Z_{L,N-1}}{Z_{L,N}} \ . \tag{4.61}$$

Therefore, monotonicity of the canonical measures implies that the ratio of partition functions (4.61) is increasing and the discrete derivative of $\log Z_{L,N}$ in N is decreasing. We expect that in the limit $L \to \infty$, the monotonicity assumption on $w(n-1)/w(n)$ is not necessary, and $\frac{1}{L}\log Z_{L,N}$ is convex in N for all non-condensing systems, consistent with strict convexity of $s(\rho)$.

For condensing systems, the weights w decay sub-exponentially, and if $w(n-1)/w(n)$ is monotone then it has to be decreasing in n. Therefore the choice $w(-1) = 0$ which implies $f(\eta) = w(\eta_x - 1)/w(\eta_x)$ is not a monotone test function, and the above general arguments cannot be used to relate non-convexity of $\frac{1}{L}\log Z_{L,N}$ to the absence of a monotone order in $\pi_{L,N}$. For particular condensing systems, however, it has been shown that $\frac{1}{L}\log Z_{L,N}$ is typically convex for small $N < \rho_c L$ and concave for larger $N > \rho_c L$ [16, 67]. These results focus on power law and stretched exponential tails for $w(n)$, and have been derived for zero-range processes where the ratio $Z_{L,N-1}/Z_{L,N}$ is equal to the canonical current. Non-monotone behaviour around the critical density has therefore implications for finite-size corrections and derivations of hydrodynamic limits as mentioned in the introduction.

CHAPTER 5

The Relaxation Time of a Condensing Zero-Range Process with Site Defects

5.1 Introduction

The relaxation time for an ergodic Markov process characterises the exponential rate of convergence to the stationary measure. For reversible processes on a countable state space, the relaxation time is given by the reciprocal of the smallest non-zero eigenvalue of $-\mathcal{L}$ where $\mathcal{L}$ is the generator of the process, see Section 2.5 for more details. Typically, sharp bounds for the relaxation time are derived using powerful coupling tools that require the process to be monotone (attractive) (see [31, 101, 30] for zero-range processes, [38, 39] for exclusion processes and [44] for a general discussion). Coupling methods typically rely on the coupled process preserving a partial order of the state space, which does not occur for non-monotone processes. In Chapter 4, we proved that homogeneous particle systems that exhibit a condensation transition are necessarily non-monotone, which implies that these coupling tools cannot be applied to bound the relaxation and mixing times.

For spatially homogeneous (non-condensing) zero-range processes on the complete graph with jump rate $g : \mathbb{N} \to (0, \infty)$, bounds on the relaxation time T^{rel} are known only in the following cases[1];

- $g(k) = \mathbb{1}(k \geq 1)$ then $T^{\text{rel}} \asymp (1 + \rho)^2$ [30],
- $g(k) = k^\gamma$ for $\gamma \in (0, 1)$ then $T^{\text{rel}} \asymp (1 + \rho)^{1-\gamma}$ [101][2],

[1] For functions $f, g : \mathbb{R} \to \mathbb{R}$ we say $f \asymp g$ if there exists a positive constant C such that $\frac{1}{C} f(x) \leq g(x) \leq C f(x)$ for all $x \in \mathbb{R}$.

[2] The current version of this proof contains a mistake, which has not been corrected at the time

- $g(k)$ is asymptotically linear in the sense that $\sup_k |g(k+1) - g(k)| < \infty$ and there exists $k_0 \in \mathbb{N}$ and $a_2 > 0$ such that $g(k) - g(j) \geq a_2$ for $k \geq j + k_0$ then T^{rel} is a bounded function of the density $\rho > 0$ [31],

- there exists some $M > 0$ such that $g(k) = \mathbb{1}(1 \leq k \leq M)$ and $\eta_x \leq M$ for all $x \in \Lambda$, then T^{rel} is a bounded function of the density $\rho \in (0, M/L)$ [102].

For each example above with symmetric dynamics a factor L^2 appears in the scaling of T^{rel} due to single particle diffusions associated with the transport of mass.

In contrast to the homogeneous case, inhomogeneous processes that exhibit stationary product measures can be both monotone and condensing, which allows the use of coupling arguments to bound important quantities of interest. A simple example of condensation arising due to system inhomogeneities is the constant rate ZRP with a single site defect [8, 9, 87, 103]. This process can be interpreted as a system of server queues each processing jobs at rate 1 with a single defect server which processes a job at a slower rate. As the density increases jobs (particles) will accumulate (condense) on the defect server whilst the density of the remaining servers will remain fixed. Coupling constructions and necessary conditions for such processes to be monotone are given in [11, 104, 45] and are reviewed in Section 2.4 with an explicit discussion for zero-range processes.

In this chapter, we calculate the relaxation time of two spatially inhomogeneous condensing zero-range processes on the complete graph. In both cases, the relaxation time exhibits a dynamic transition as the density varies on 'large systems'. To calculate sharp bounds for the relaxation time of these process, we decompose the state space into a finite number of disjoint subspaces. Understanding the dynamics of the process restricted to each subspace and a projection of the process, which transitions between the disjoint subspaces under some 'average dynamics', allows for a recursive method of calculating the relaxation time. This is the same approach used for a non-condensing zero-range process [31] and the Kawasaki Ising model [32], and for general Markov chains in [33]. In our case, the projection is a birth-death chain which exhibits different behaviour depending on the density $\rho < \rho_c$ or $\rho \geq \rho_c$. The restriction chain, the process restricted to a subspace, is a constant rate zero-range process previously studied in [30].

This chapter is organised as follows. In Section 5.2, we introduce the processes, define the relaxation time in terms of the Dirichlet form and variance, and state the theorems necessary for our proof. We state our main results in Section 5.4 and in Section 5.5 we construct the projection and restriction chains. The proofs of

of writing this thesis.

our main results are given in Sections 5.6, 5.7 and 5.8. In Section 5.9, we present preliminary results on the mixing times of the zero-range processes with one and two defects.

5.2 Background and notation

Consider a spatially inhomogeneous zero-range process (ZRP) on the lattice $\Lambda = \{1, \ldots, L\}$ with N particles, which is irreducible on the finite state space

$$\Omega_{L,N} = \left\{\eta \in \mathbb{N}^\Lambda : \sum_{x=1}^{L} \eta_x = N\right\} . \tag{5.1}$$

Particles exit a site $x \in \Lambda$ containing η_x particles at rate $g_x(\eta_x)$ and move to a neighbouring site y according to an irreducible random walk $p(x, y)$. The dynamics are defined by the generator (2.10) which we recall here

$$\mathcal{L}f(\eta) = \sum_{x,y\in\Lambda} g_x(\eta_x) p(x, y) \left(f(\eta^{x\to y}) - f(\eta)\right) . \tag{5.2}$$

We consider the case when $p(x, y) = \frac{1}{L-1}$ for all $x \neq y$, here called complete graph dynamics. In this chapter, we consider the constant rate ZRP with a set of defect sites $\Delta \subset \Lambda$, with jump rates given by

$$g_x(n) = \begin{cases} \mathbb{1}(n \geq 1) & \text{if } x \in \Lambda \setminus \Delta \\ r\,\mathbb{1}(n \geq 1) & \text{if } x \in \Delta \end{cases} \quad \text{where } 0 < r < 1 . \tag{5.3}$$

The process exhibits a family of stationary product measures $\{\nu_\phi : \phi \in [0, r)\}$ on the state space $\Omega_L = \mathbb{N}^\Lambda$ satisfying

$$\nu_\phi[\{\eta = \mathbf{n}\}] = \prod_{x\in\Lambda} \nu^x_\phi[\{\eta_x = n_x\}] .$$

The single site marginals are given by

$$\nu^x_\phi[\{\eta_x = n\}] = \frac{w_x(n)\phi^n}{z_x(\phi)} \text{ where } w_x(n) = \begin{cases} 1 & \text{if } \in \Lambda \setminus \Delta \\ r^{-n} & \text{if } x \in \Delta \end{cases} . \tag{5.4}$$

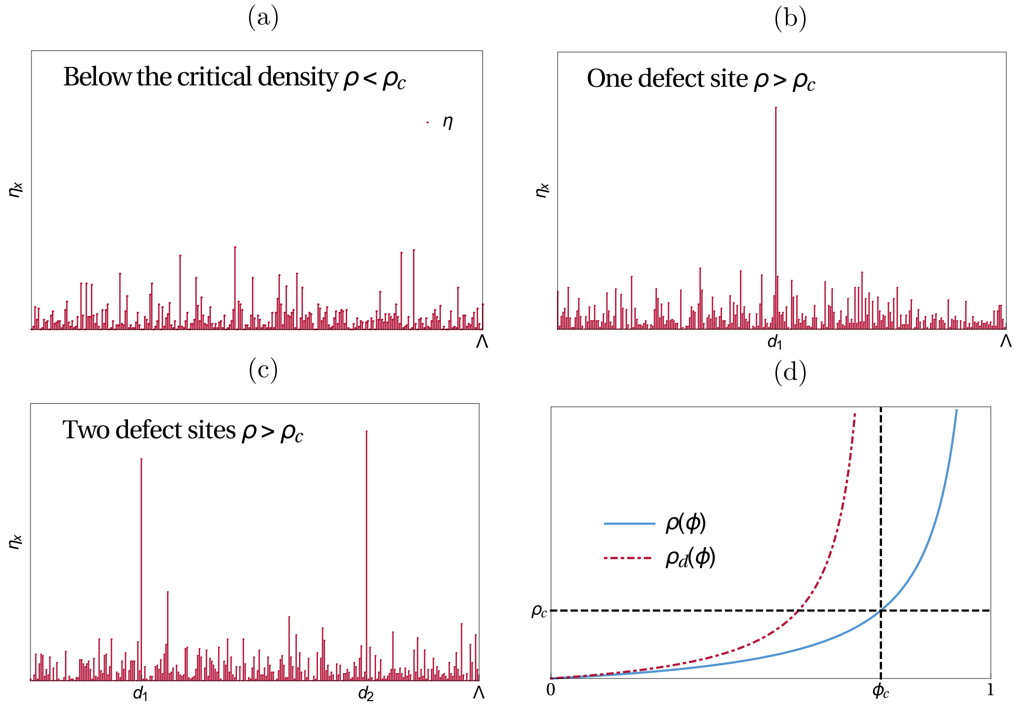

Figure 5.1: Typical configurations of a condensing zero range processes with multiple defect sites and the grand canonical density functions with $r = 0.75$. Figure 5.1a is a typical configuration below the critical density for systems with defect sites. Figure 5.1b and 5.1c are typical configurations above the critical density for processes with 1 or 2 defects respectively. Figure 5.1d shows the grand canonical densities (5.5) $\rho(\phi)$, the density for a non defect site, and $\rho_d(\phi)$, the density for a defect site.

The grand-canonical partition functions and densities are given by

$$z_x(\phi) = \begin{cases} \frac{1}{1-\phi} & \text{if } x \in \Lambda \setminus \Delta \\ \frac{r}{r-\phi} & \text{if } x \in \Delta \end{cases} \quad \rho_x(\phi) = \begin{cases} \frac{\phi}{1-\phi} & \text{if } x \in \Lambda \setminus \Delta \\ \frac{\phi}{r-\phi} & \text{if } x \in \Delta \end{cases} . \tag{5.5}$$

As $\phi \nearrow \phi_c = r$ and for each $d \in \Delta$ the density $\rho_d(\phi)$ diverges whilst the background densities $\rho_x(\phi)$ for each $x \in \Lambda\setminus\Delta$ tend to $\frac{r}{1-r} =: \rho_c$. If $N > \rho_c L$ the system separates into a condensed phase on the defect sites and a fluid phase in the background, see for example [62] and references therein. Typical configurations and grand canonical densities of a condensing zero range process with multiple defect sites are shown in Figure 5.1.

The process is irreducible on $\Omega_{L,N}$ with a unique stationary measure $\pi^{\Delta}_{\Lambda,N}[\cdot]$, which depends on the set of defect sites Δ and is given by

$$\pi^{\Delta}_{\Lambda,N}[\{\eta = \mathbf{n}\}] = \frac{1}{Z^{\Delta}_{\Lambda,N}} r^{-\sum_{d\in\Delta} n_d} \mathbb{1}(\mathbf{n} \in \Omega_{L,N}) \ , \tag{5.6}$$

and the partition function (normalisation) is given by

$$Z^{\Delta}_{\Lambda,N} = \sum_{\mathbf{n}\in\Omega_{L,N}} r^{-\sum_{d\in\Delta} n_d} \ . \tag{5.7}$$

Since we only consider the process on the complete graph the location of the defects is not important, therefore with slight abuse of notation we let

$$\begin{aligned} \pi^{0}_{L,N}[\{\eta = \mathbf{n}\}] &= \pi^{\emptyset}_{\Lambda,N}[\{\eta = \mathbf{n}\}] && \text{for no defect sites,} \\ \pi^{K}_{L,N}[\{\eta = \mathbf{n}\}] &= \pi^{\{1,\dots,K\}}_{\Lambda,N}[\{\eta = \mathbf{n}\}] && \text{for } K \text{ defect sites,} \end{aligned}$$

and similarly $Z^K_{L,N}$ denotes the respective partition function. The variance of a function $f : \Omega_{L,N} \to \mathbb{R}$ with respect to the measure $\pi^K_{L,N}$ is denoted $\mathrm{Var}^K_{L,N}(f)$ and the Dirichlet form is given by

$$\mathcal{D}^K_{L,N}(f) = -\pi^K_{L,N}(f\mathcal{L}f) = \frac{1}{2(L-1)} \sum_{\substack{x,y \\ x\neq y}} \pi^K_{L,N}\left(g_x(\eta_x)\left(f(\eta^{x,y}) - f(\eta)\right)^2\right) \ . \tag{5.8}$$

The spectral gap of the generator (5.2) on $\Omega_{L,N}$ with K defects is denoted $\lambda^K_{L,N}$ and is given by the variational principle given in Definition 2.5.1, *i.e.*

$$\lambda^K_{L,N} = \inf_f \left\{ \frac{\mathcal{D}^K_{L,N}(f)}{\mathrm{Var}^K_{L,N}(f)} : \mathrm{Var}^K_{L,N}(f) \neq 0 \right\} \ . \tag{5.9}$$

The relaxation time is defined by $T^K_{L,\rho} := 1/\lambda^K_{L,N}$ where $\rho := N/L$.

5.3 Preliminary results

Calculating an upper bound on the relaxation time of the zero-range process with defect sites relies on the following theorems for the relaxation time of a constant rate zero-range process [30, Theorem 1], and birth-death chains [105, Theorem 1] and [106, Theorem 1.1].

Theorem 5.3.1 (Morris). *There exist (universal) constants $c_1, c_2 > 0$ such that the relaxation time for a constant rate zero-range process ($\Delta = \emptyset$) on the complete graph* *satisfies*

$$c_1\,(1+\rho)^2 \le T^0_{L,\rho} \le c_2\,(1+\rho)^2\ , \tag{5.10}$$

for all $L \geq 2$.

Theorem 5.3.2 (Mufa)**.** *Consider a birth-death process on* $\mathbb{N}$ *with death rate* a_i *and birth rate* b_i *with spectral gap* λ*. Let* $v \in \mathcal{V} = \{w = (w_i)_{i\in\mathbb{N}} : w_i > 0 \text{ for all } i \in \mathbb{N}\}$ *and define*

$$R_i(v) = a_{i+1} + b_i - \frac{a_i}{v_{i-1}} - b_{i+1}v_i\ , \tag{5.11}$$

where $a_0 = 0$*. Then*

$$\lambda = \sup_{v\in\mathcal{V}} \inf_{i\geq 0} R_i(v)\ . \tag{5.12}$$

Theorem 5.3.3 (Chen and Saloff-Coste)**.** *Consider a reversible birth-death process on* $\mathbb{N}$ *with death rates* a_i *and birth rate* b_i *with spectral gap* λ *and unique stationary measure* μ*. Define*

$$B_+(i) = \sup_{x>i}\left(\sum_{y=i+1}^{x}\frac{1}{\mu_y a_y}\right)\sum_{y\geq x}\mu_y\ ,$$
$$B_-(i) = \sup_{x<i}\left(\sum_{y=x}^{i-1}\frac{1}{\mu_y b_y}\right)\sum_{y\leq x}\mu_y\ .$$

Let m *be the median of* μ *and*

$$B = \sup\{B_+(m), B_-(m)\}\ ,$$

then

$$\frac{1}{4B} \leq \lambda \leq \frac{2}{B}\ .$$

5.4 Results

Our main results are stated in the following two theorems.

Theorem 5.4.1. *There exist universal constants* $c_1, c_1', c_2, c_2' > 0$ *such that the relaxation time of the ZRP with a single defect, on the complete graph, satisfies*

$$c_1\left(\sqrt{r} - \sqrt{\frac{\rho}{\rho+1}}\right)^{-2} \leq T^1_{L,\rho} \leq c_1'\left(\sqrt{r} - \sqrt{\frac{\rho}{\rho+1}}\right)^{-2}\left(1 + \frac{(1+\rho_c)^2}{L}\right)\ ,$$

for all $\rho < \rho_c$*. For all* $\rho \geq \rho_c$ *we have*

$$c_2\,(1+\rho_c)^2\,L \leq T^1_{L,\rho} \leq c_2'\,(1+\rho_c)^2\,L\left(1+\frac{C_\rho}{L}\right)\ , \tag{5.13}$$

where $C_\rho = (1+\rho)^2\left(\frac{1}{(1+\rho_c)^2}+1\right)$.

In the thermodynamic limit (3.2) we have

$$T^1_{L,\rho} \asymp \begin{cases} \left(\sqrt{r}-\sqrt{\frac{\rho}{\rho+1}}\right)^{-2} & \text{for all } \rho < \rho_c\ , \\ (1+\rho_c)^2 L & \text{for all } \rho \geq \rho_c\ , \end{cases} \tag{5.14}$$

i.e. a transition in the behaviour of the relaxation time of the constant rate zero-range process with one defect at density ρ_c.

Theorem 5.4.2. *There exist universal constants* $c_1, c_1', c_2, c_2', c_3, c_3' > 0$ *such that the relaxation time of the ZRP with two defect sites, on the complete graph, satisfies*

$$c_1\left(\sqrt{r}-\sqrt{\frac{\rho}{\rho+1}}\right)^{-2} \leq T^2_{L,\rho} \leq c_1'\left(\sqrt{r}-\sqrt{\frac{\rho}{\rho+1}}\right)^{-2}\left(1+\frac{T^1_{L-1,\rho}}{L}\right)\ ,$$

for all $\rho < \rho_c$*. For* $\rho = \rho_c$ *we have*

$$c_2(1+\rho_c)^2 L \leq T^2_{L,\rho} \leq c_2'(1+\rho_c)^2 L\left(1+\frac{T^1_{L-1,\rho}}{L}\right)\ . \tag{5.15}$$

For all $\rho > \rho_c$

$$c_3\frac{1}{r}(\rho-\rho_c)^2 L^2 \leq T^2_{L,\rho} \leq c_3'\frac{1}{r}(\rho-\rho_c)^2 L^2\left(1+\left(\frac{1}{L}+\frac{1}{(\rho-\rho_c)^2 L^2}\right)T^1_{L-1,\rho}\right)\ . \tag{5.16}$$

In the thermodynamic limit (3.2) we have

$$T^2_{L,\rho} \asymp \begin{cases} \left(\sqrt{r}-\sqrt{\frac{\rho}{\rho+1}}\right)^{-2} & \text{for all } \rho < \rho_c\ , \\ (1+\rho_c)^2 L & \text{for all } \rho = \rho_c\ , \\ (\rho-\rho_c)^2 L^2 & \text{for all } \rho > \rho_c\ , \end{cases} \tag{5.17}$$

i.e. two transitions in the behaviour of the relaxation time of the constant rate zero-range process with two defect sites at density ρ_c.

To prove these theorems, we consider a decomposition of the state space $\Omega_{L,N}$ via conditioning the ZRP to have n particles at a defect site. This method allows us to compare the relaxation time of the full ZRP with a copy of the ZRP on $\Omega_{L-1,N-n}$, called the restriction chain, and a projection chain which controls how the process transitions from one set in the decomposition to another, *i.e.* $n \to n \pm 1$. We see a transition in the relaxation time of the projection chain when ρ crosses the critical density ρ_c, which leads to a transition in the relaxation time of the full ZRP. If the process has one defect site then the projection chain above ρ_c behaves like a random walk with drift towards a point $n^\star \asymp (\rho - \rho_c)L$. For the ZRP with two defect sites, the projection chain above ρ_c then behaves like a symmetric random walk, which leads to the relaxation time scaling like L^2.

Theorems 5.4.1 and 5.4.2 can be easily extended to zero-range processes with more defect sites. If $K = |\Delta|$ is the number of defect sites, then we conjecture that if K is independent of L the scaling of the relaxation time above the critical density ρ_c is $\frac{1}{K-1}\frac{1}{r}(\rho - \rho_c)^2 L^2$. We expect this to be the case since the region that the projection chain behaves like a symmetric random walk decreases with the number of defects since more mass can be accommodated in defect sites. Alternatively, the typical number of particles in the background goes from $(L-2)\rho_c$ to $(L-K)\rho_c$. If instead we have order L defects, $K \asymp L$, then we do not expect the relaxation time to be of order L^2. In this case, it is natural to expect that the relaxation time behaves like the that of a constant rate ZRP since a projection processes that moves mass between defect and non-defect sites will exhibit bias towards the defect sites. Heuristically, this projection chain will have a gap independent of the system size L. Furthermore, the processes restricted to the defect or non-defect sites are constant rate ZRPs with jump rates r or 1 respectively.

5.5 Decomposition: Projection and restriction chain

In this section, we decompose the state space $\Omega_{L,N}$ into a disjoint union of subspaces and construct the projection and restriction chains used to prove our main results. This method has previously been used to calculate sharp bounds for the relaxation time of the zero-range process with asymptotically linear jump rates [31].

Lemma 5.5.1. *Consider the complete graph zero-range process* (5.2) *with K defect sites on $\Omega_{L,N}$. For all $f : \Omega_{L,N} \to \mathbb{R}$ we have*

$$Var^K_{L,N}(f) \le \sup_{0 \le n \le N} \left\{ T^{K-1}_{L-1,N-n} \right\} \mathcal{D}^K_{L,N}(f) + \widetilde{T}^K_{L,N} \widetilde{\mathcal{D}}^K_{L,N}\left(H_f(\cdot)\right) \ , \tag{5.18}$$

where $H_f(n) := \pi^K_{L,N}(f|\eta_1 = n)$ and $\widetilde{T}^K_{L,N}$ is the relaxation time of a birth death *process on* $\{0,\ldots,N\}$ *with rates*

$$\hat{q}(i,j) = \begin{cases} \pi^{K-1}_{L-1,N-i}\left(g_x(\cdot)\right) & \text{if } j = i+1 \ , \\ r\mathbb{1}(i \geq 1) & \text{if } j = i-1 \ , \\ 0 & \text{otherwise} \ . \end{cases} \tag{5.19}$$

Proof. By the law of total variance, we have

$$\mathrm{Var}^K_{L,N}(f) = \pi^K_{L,N}\left(\mathrm{Var}^K_{L,N}\left(f \,|\, \eta_1\right)\right) + \mathrm{Var}^K_{L,N}\left(\pi^K_{L,N}\left(f \,|\, \eta_1\right)\right) \ . \tag{5.20}$$

Since $\pi^K_{L,N}$ is a product measure conditioned on the total number of particles, if $K \geq 1$, the conditional variance appearing in the first term can be rewritten as

$$\mathrm{Var}^K_{L,N}\left(f \,|\, \eta_1\right) = \mathrm{Var}^{K-1}_{L-1,N-\eta_1}\left(f(\eta_1,\cdot)\right) \ . \tag{5.21}$$

We can now apply a Poincaré inequality for the zero-range process on $\Omega_{L-1,N-\eta_1}$ as follows

$$\mathrm{Var}^{K-1}_{L-1,N-\eta_1}\left(f(\eta_1,\cdot)\right) \leq T^{K-1}_{L-1,N-\eta_1}\mathcal{D}^{K-1}_{L-1,N-\eta_1}\left(f(\eta_1,\cdot)\right) \ . \tag{5.22}$$

Taking the supremum over $\eta_1 \in \{0,\ldots N\}$ and the expectation with respect to $\pi^K_{L,N}$ we have

$$\pi^K_{L,N}\left(\mathrm{Var}^K_{L,N}\left(f|\eta_1\right)\right) \leq \sup_{0\leq n\leq N}\left\{T^{K-1}_{L-1,N-n}\right\}\pi^K_{L,N}\left(\mathcal{D}^{K-1}_{L-1,N-\eta_1}\left(f(\eta_1,\cdot)\right)\right) \ . \tag{5.23}$$

Finally, it is easy to see that (see for example [31])

$$\pi^K_{L,N}\left(\mathcal{D}^{K-1}_{L-1,N-\eta_1}\left(f(\eta_1,\cdot)\right)\right) \leq \mathcal{D}^K_{L,N}(f) \ . \tag{5.24}$$

We now turn to the second term of (5.20). Defining

$$H_f(n) = \pi^K_{L,N}\left(f \,|\, \eta_1 = n\right) \ , \tag{5.25}$$

then the second term is the variance of a function of one variable. Applying the Dirichlet form (5.8) to a function $h : \{0,\ldots,N\} \to \mathbb{R}$ which depends only on η_1 we have

$$
\begin{aligned}
\mathcal{D}^K_{L,N}(h) = & \frac{1}{2(L-1)} \sum_{x \neq 1} \pi^K_{L,N} \left(g_x(\eta_x) \left[h(\eta_1+1) - h(\eta_1) \right]^2 \right) \\
& + \frac{1}{2(L-1)} \sum_{y \neq 1} \pi^K_{L,N} \left(g_1(\eta_1) \left[h(\eta_1-1) - h(\eta_1) \right]^2 \right) \\
= & \frac{1}{2} \pi^K_{L,N} \left(\pi^{K-1}_{L-1,N-\eta_1}(g_y(\eta_y)) \left[h(\eta_1+1) - h(\eta_1) \right]^2 \right) \\
& + \frac{1}{2} \pi^K_{L,N} \left(g_1(\eta_1) \left[h(\eta_1-1) - h(\eta_1) \right]^2 \right) \\
& =: \widetilde{\mathcal{D}}^K_{L,N}(h) \ ,
\end{aligned} \tag{5.26}
$$

where we have used $\pi^{K-1}_{L-1,M}(g_y(\eta_y)) = \pi^{K-1}_{L-1,M}(g_z(\eta_z))$ for all $y, z \in \Lambda$. $\widetilde{\mathcal{D}}^K_{L,N}(h)$ is the Dirchlet form of a birth-death chain on $\{0, \ldots N\}$ with stationary measure

$$
\pi^K_{L,N}[\{\eta_1 = n\}] =: \mu^K_{L,N}[n] \ , \tag{5.27}
$$

the marginal of the measure $\pi^K_{L,N}$ on site 1. The jump rates of the birth-death chain (projection chain) are given by

$$
\hat{q}(i,j) = \begin{cases} \pi^{K-1}_{L-1,N-i}(g_x(\cdot)) & \text{if } j = i+1 \ , \\ r\mathbb{1}(i \geq 1) & \text{if } j = i-1 \ , \\ 0 & \text{otherwise} \ . \end{cases} \tag{5.28}
$$

By applying a Poincaré inequality of the birth-death process, we have

$$
\mathrm{Var}^K_{L,N}(h) \leq \widetilde{T}^K_{L,N} \widetilde{\mathcal{D}}^K_{L,N}(h) \ , \tag{5.29}
$$

where $\widetilde{T}^K_{L,N}$ is the relaxation time of the birth-death chain. Applying this estimate to the test function (5.25) and combining the estimates (5.23) and (5.24) we have

$$
\mathrm{Var}^K_{L,N}(f) \leq \sup_{0 \leq n \leq N} \left\{ T^{K-1}_{L-1,N-n} \right\} \mathcal{D}^K_{L,N}(f) + \widetilde{T}^K_{L,N} \widetilde{\mathcal{D}}^K_{L,N}\left(H_f(\cdot) \right) \ . \tag{5.30}
$$

□

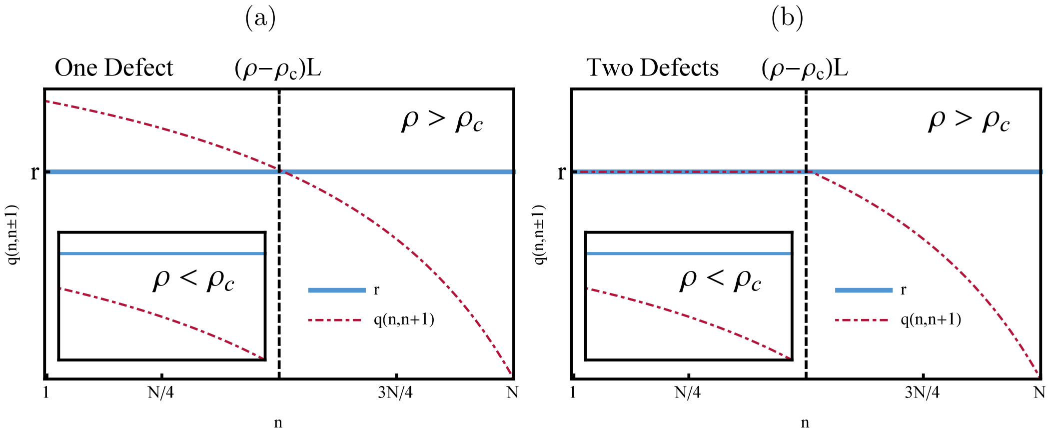

Figure 5.2: Jump rates for the birth-death chain (5.19) for $\rho > \rho_c$ and $\rho < \rho_c$. Figure 5.2a: rates for the projection chain with one defect site $K = 1$, the process exhibits drift towards a point $(\rho - \rho_c)L$ above the critical density. Figure 5.2b: rates for the projection chain with two defect sites $K = 2$, above the critical density the process behaves like a symmetric random walk in the region $\{0, \ldots, (\rho - \rho_c)L\}$.

5.6 Single site defect: Proof of Theorem 5.4.1

In this section, we calculate upper bounds for the relaxation times for the cases $\rho < \rho_c$ and $\rho \geq \rho_c$ separately. Lower bounds of the same order are given in Section 5.8. For $\rho < \rho_c$, the projection chain exhibits a strong drift to the origin giving rise to a L independent relaxation time. For $\rho \geq \rho_c$, the associated birth-death chain instead exhibits a drift towards a value $n^\star \asymp (\rho - \rho_c)L$ which grows linearly with the system size and the relaxation time has a linear dependence on L. We show numerical solutions and the upper bound of $\widetilde{T}^1_{L,N}$ for the relaxation time of the birth-death chain in Figure 5.3.

In light of Lemma 5.5.1 and Theorem 5.3.1 we only have to bound the relaxation time of the projection chain and compare the Dirichlet forms of the projection chain and the full process. This is contained in the following lemmas, which will be proved in the following two subsections.

Lemma 5.6.1. *There exists (universal) constants $c_1, c_2 > 0$ such that for all $\rho < \rho_c$*

$$\widetilde{T}^1_{L,N} \leq c_1 \left(\sqrt{r} - \sqrt{\frac{\rho}{1+\rho}} \right)^{-2} , \tag{5.31}$$

and for $\rho \geq \rho_c$ we have

$$\widetilde{T}^1_{L,N} \leq c_2 (1+\rho_c)^2 \, L \ . \tag{5.32}$$

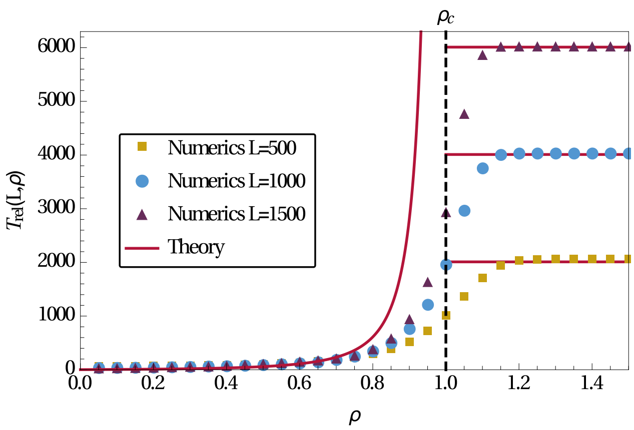

Figure 5.3: The relaxation time of the projection chain, a birth-death chain, for the zero-range process with one defect site. For $L = 500, 1000$, and 1500 with $r = 0.5$ we plot the numerical solution for the reciprocal of the smallest non-zero eigenvalue of the generator (gold, blue, and purple markers) with the scaling form of the upper bound given in Lemma 5.6.1 (red line). Above the critical density the relaxation grows linearly with the system size L

Lemma 5.6.2. *For all* $f : \Omega_{L,N} \to \mathbb{R}$

$$\widetilde{\mathcal{D}}^1_{L,N}(H_f(\eta_1)) \leq \left(2 + \frac{c_2(1+\rho)^2}{2L}\right) \mathcal{D}^1_{L,N}(f) \ ,$$

where c_2 *is the constant appearing in Theorem 5.3.1.*

Therefore, the upper bound in Theorem 5.4.1 follows by combining (5.18), Lemmas 5.6.1 and 5.6.2, and using the inequality

$$(1+\rho)^2 \leq \left(\sqrt{r} - \sqrt{\frac{\rho}{1+\rho}}\right)^{-2}$$

for all $r \in (0,1)$ and $\rho < \rho_c$, which is used to compare the scaling forms appearing in Theorem 5.3.1 and Lemma 5.6.1.

5.6.1 Proof of Lemma 5.6.1: The relaxation time of the projection chain with one defect

Recall that the projection chain (5.19) is a birth death process $(X_t)_{t\geq 0}$ on $\{0, ..., N\}$ with death rate $\hat{q}(n, n-1) = r\mathbb{1}(n > 0)$ and birth rate $\hat{q}(n, n+1) = \pi^0_{L-1,N-n}(g_x(\cdot))$. Since $\pi^0_{L-1,N-n}$ is the uniform measure on $\Omega_{L-1,N-n}$ and the partition function is given by the size of $\Omega_{L,N}$, *i.e.*

$$Z^0_{L,N} = |\Omega_{L,N}| = \binom{N+L-1}{N} ,$$

then the birth rate $\hat{q}(n, n+1)$ is given by

$$\hat{q}(n, n+1) = \pi^0_{L-1,N-n}(g_x(\cdot)) = \frac{Z^0_{L-1,N-n-1}}{Z^0_{L-1,N-n}} = \frac{N-n}{N+L-n-2} .$$

We first calculate the spectral gap of the birth death process $(X_t)_{t\geq 0}$ for the case $\rho < \rho_c$ using Theorem 5.3.2. Define

$$v_i = v = \sqrt{\frac{r(1+\rho)}{\rho}} \tag{5.33}$$

for all $i \in \{0, \ldots, N\}$. We note that this is the same function used to calculate a sharp upper bound for relaxation time of a birth-death process with death rate r and birth rate $\frac{\rho}{1+\rho}$ in [105]. Then

$$R_i(v) = \frac{N-i}{N+L-i-2} + r - r\sqrt{\frac{\rho}{r(1+\rho)}} - \frac{N-i-1}{N+L-i-3}\sqrt{\frac{r(1+\rho)}{\rho}} ,$$

which is increasing on $i \in \{0, \ldots N\}$ and positive for all $\rho < \rho_c$, therefore the minimum is attained at $R_0(v)$. Writing $\rho = N/L$ we have

$$\begin{aligned} R_0(v) &= \frac{\rho}{1+\rho-2/L} + r - r\sqrt{\frac{\rho}{r(1+\rho)}} - \frac{\rho - 1/L}{1+\rho-3/L} \\ &> r + \frac{\rho}{1+\rho} - 2r\sqrt{\frac{\rho}{1+\rho}} = \left(\sqrt{r} - \sqrt{\frac{\rho}{1+\rho}}\right)^2 . \end{aligned}$$

Therefore, the inverse of the relaxation time for the birth-death chain has the lower bound

$$\left(\widetilde{T}^1_{L,N}\right)^{-1} \geq \inf_{i\geq 0} R_i(v) = R_0(v) > \left(\sqrt{r} - \sqrt{\frac{\rho}{\rho+1}}\right)^2 ,$$

which completes the first part of the proof of Lemma 5.6.1 for $\rho < \rho_c$.

For $\rho = \rho_c$ the function $v_i = 1$ for all i gives the required lower bound since $R_i(v)$ is increasing on $\{0, \ldots, N\}$ and

$$R_0(v) = \frac{N}{N+L-2} - \frac{N-1}{N+L-3} . \tag{5.34}$$

Setting $N/L = \rho_c$ we have

$$R_0(v) = \frac{L-2}{(L(1+\rho_c)-3)(L(1+\rho_c)-2)} > \frac{L-2}{L^2(1+\rho_c)^2} \ , \tag{5.35}$$

which implies

$$\widetilde{T}^1_{L,N} < c(1+\rho_c)^2 L \tag{5.36}$$

for some constant $c > 0$.

For $\rho > \rho_c$ define $n^\star_{L,N}$ to be the smallest integer such that the difference between the birth and death rates is minimised, which leads to

$$n^\star_{L,N} = \left\lfloor \frac{N - r\,(N+L-2)}{1-r} \right\rfloor \ .$$

Define

$$v_i = \begin{cases} 1 - \frac{1}{r\,L} & \text{if } i \in \{0, \dots, n^\star_{L,N} - 1\} \ , \\ \quad 1 & \text{if } i \in \{n^\star_{L,N}, \dots, N\} \ . \end{cases} \tag{5.37}$$

then

$$R_i(v) = \begin{cases} \frac{N-i}{N+L-i-2} + r - \frac{r}{1-\frac{1}{r\,L}} - \frac{N-i-1}{N+L-i-3}\left(1 - \frac{1}{r\,L}\right) & \text{if } i < n^\star_{L.N} \\ \frac{N-i}{N+L-i-2} - \frac{N-i-1}{N+L-i-3} & \text{if } i \geq n^\star_{L,N} \end{cases} .$$

$R_i(v)$ is decreasing in i on the interval $\{0, \dots, n^\star_{L,N}-1\}$ and increasing on $\{n^\star_{L,N}, \dots, N\}$, which implies the the minimum is given by

$$\inf_{i \geq 0} R_i(v) = \min\left\{ R_{n^\star_{L,N}-1}\,(v)\ ,\ R_{n^\star_{L,N}}(v) \right\} \ .$$

Therefore, taking the limit $L, N \to \infty$ such that $N/L \to \rho > \rho_c$ we find

$$L\left(R_{n^\star_{L,N}-1}\,(v)\right) \to (1-r)^2 \quad \text{and} \quad L\left(R_{n^\star_{L,N}}\,(v)\right) \to (1-r)^2$$

The critical density is given by $\rho_c = \frac{r}{1-r}$, so we have $1 + \rho_c = \frac{1}{1-r}$, which implies

$$\widetilde{T}^1_{L,N} < c(1+\rho_c)^2\, L \ ,$$

for some constant $c > 0$. This completes the proof of Lemma 5.6.1.

5.6.2 Proof of Lemma 5.6.2: Comparison of the Dirichlet forms

To compare the Dirichlet form of the projection chain and zero-range process we follow the approach used in [31]. The major differences in this chapter are variance estimates of the jump rate $g_x(\eta_x)$ with respect to the measure $\pi^{|\Delta|}_{L,N}$ and comparisons with the zero-range process on $\Omega_{L-1,N-n}$.

Recall, the Dirichlet forms for the projection chain and the complete graph zero-range process, see (5.26) and (5.8), with a single site defect are given by

$$\widetilde{\mathcal{D}}^1_{L,N}(h) = \frac{1}{2}\sum_{n=0}^{N-1} \mu^1_{L,N}[n+1]r\left(h(n+1)-h(n)\right)^2 \text{ for } h:\{0,\ldots,N\}\to\mathbb{R} \tag{5.38}$$

$$\mathcal{D}^1_{L,N}(f) = \frac{1}{2(L-1)}\sum_{x\neq y}\pi^1_{L,N}\left(g_x(\eta_x)\left(f(\eta^{x,y})-f(\eta)\right)^2\right) \text{ for } f:\Omega_{L,N}\to\mathbb{R}\ . \tag{5.39}$$

Using the same approach as [31], define $M:\Omega_{L,N}\to\mathbb{R}$ as

$$M(\eta) = \frac{1}{r}\frac{\pi^1_{L,N}[\eta_1]}{\pi^1_{L,N}[\eta_1+1]}\frac{1}{L-1}\sum_{x=2}^{L} g_x(\eta_x)\ . \tag{5.40}$$

Since

$$\frac{1}{r}\frac{\pi^1_{L,N}[\eta_1]}{\pi^1_{L,N}[\eta_1+1]} = \frac{1}{\hat{q}(\eta_1,\eta_1+1)}\ ,$$

the function $M(\eta)$ can be interpreted as the ratio of the actual rate into site 1 under the zero-range dynamics and the rate into site 1 under the projection defined in Section 5.5. By [31, Lemma 4.4] we can write

$$\begin{aligned}
&\pi^1_{L,N}(f|\eta_1=n+1)-\pi^1_{L,N}(f|\eta_1=n)\\
&\quad=\frac{1}{r}\frac{1}{\pi^1_{L,N}[\{\eta_1=n+1\}]}\frac{1}{L-1}\sum_{x=2}^{L}\pi^1_{L,N}\left(g_x(\eta_x)\left(f(\eta^{x,1})-f(\eta)\right)\mathbb{1}(\eta_1=n)\right)\\
&\qquad\qquad+\pi^1_{L,N}\left(M;f|\eta_1=n\right)\ ,
\end{aligned}$$

where $\pi^1_{L,N}(M;f|\eta_1=n)$ is the covariance of M and f with respect to $\pi^1_{L,N}[\cdot|\eta_1=n]$.

Therefore, using the Cauchy-Schwarz inequality we have $\widetilde{D}^1_{L,N}(H_f(\cdot))$ is bounded above by

$$\sum_{n=0}^{N-1} \frac{1}{r\pi^1_{L,N}[\{\eta_1 = n+1\}]} \left[\pi^1_{L,N} \left(\frac{1}{L-1} \sum_{x=2}^{L} g_x(\eta_x) \left(f(\eta^{x,1}) - f(\eta) \right) \mathbb{1}(\eta_1 = n) \right) \right]^2 \tag{5.41}$$

$$+ \sum_{n=0}^{N-1} r\pi^1_{L,N}[\{\eta_1 = n+1\}] \left[\pi^1_{L,N} \left(M; f | \eta_1 = n \right) \right]^2 .$$

Define the first line as $A_1^f(L,N)$ and the second as $A_2^f(L,N)$. Using the Cauchy-Schwarz inequality we can show

$$\begin{aligned} A_1^f(L,N) &\leq \pi^1_{L,N} \left(\frac{1}{L-1} \sum_{x=2}^{L} g_x(\eta_x) \left(f(\eta^{x,1}) - f(\eta) \right)^2 \right) \\ &\leq \frac{1}{L-1} \sum_{x,y} \pi^1_{L,N} \left(g_x(\eta_x) \left(f(\eta^{x,y}) - f(\eta) \right)^2 \right) = 2\mathcal{D}^1_{L,N}(f) \ . \end{aligned} \tag{5.42}$$

We now turn to calculating an upper-bound for $A_2^f(L,N)$. We first rewrite $A_2^f(L,N)$ as

$$A_2^f(L,N) = \pi^1_{L,N} \left(\frac{1}{\pi^0_{L-1,N-\eta_1}(g_x(\eta_x))} \left[\pi^0_{L-1,N-\eta_1} \left(f; \frac{1}{L-1} \sum_{x=2}^{L} g_x(\eta_x) \right) \right]^2 \right) ,$$

and using the Cauchy-Schwarz inequality we have an upper bound given by

$$A_2^f(L,N) \leq \frac{1}{(L-1)^2} \pi^1_{L,N} \left(\frac{\mathrm{Var}^0_{L-1,N-\eta_1} \left(\sum_{x=2}^{L} g_x(\eta_x) \right)}{\pi^0_{L-1,N-\eta_1}(g_x(\eta_x))} \mathrm{Var}^0_{L-1,N-\eta_1}(f) \right) .$$

By Theorem 5.3.1 the first variance can be bounded above by the relaxation time of the constant rate ZRP on the complete graph and its associated Dirichlet form. Since $g_x(n) = \mathbb{1}(n \geq 1)$ for $x \in \{2, \ldots L\}$ and $\pi^0_{L-1,N-\eta_1}$ is the uniform measure on $\Omega_{L-1,N-\eta_1}$, we can compute the following upper bound of the second variance scaled by expected jump rate of the form

$$\frac{1}{(L-1)^2}\frac{\mathrm{Var}^0_{L-1,N-\eta_1}\left(\sum_{x=2}^{L} g_x(\eta_x)\right)}{\pi^0_{L-1,N-\eta_1}(g_x(\eta_x))} = \frac{(L-2)(N-\eta_1-2)}{(L-1)(N+L-\eta_1-3)(N+L-\eta_1-2)} \tag{5.43}$$

$$\leq \begin{cases} c\frac{\rho}{(1+\rho)^2}\frac{1}{L} & \text{if } \rho \leq 1 \ , \\ c\frac{1}{4L} & \text{if } \rho > 1 \ . \end{cases} . \tag{5.44}$$

Therefore, combining (5.44) and Theorem 5.3.1 $A_2^f(L,N)$ is bounded above by

$$\begin{cases} c_2\frac{\rho}{L}\mathcal{D}^1_{L,N}(f) & \text{if } \rho \leq 1 \ , \\ c_2\frac{(1+\rho)^2}{4L}\mathcal{D}^1_{L,N}(f) & \text{if } \rho > 1 \ , \end{cases} \tag{5.45}$$

Combining (5.42) and (5.45) we have

$$\widetilde{\mathcal{D}}^1_{L,N}(H_f(\eta_1)) \leq \left(2 + \frac{c_2(1+\rho)^2}{4L}\right)\mathcal{D}^1_{L,N}(f)$$

for all $f : \Omega_{L,N} \to \mathbb{R}$, completing the proof of Lemma 5.6.2.

5.7 The complete graph zero-range process with two defect sites: Proof of Theorem 5.4.2

For the constant rate ZRP on the complete graph with two defect sites the relaxation time exhibits different behaviour for $\rho < \rho_c$, $\rho = \rho_c$ and $\rho > \rho_c$. By the same method as Section 5.6, we calculate the relaxation time by conditioning the process to have n particles on one of the defect sites. This allows us to use results on the relaxation time of the background process, a ZRP with one defect site. For $\rho < \rho_c$ the birth-death chain exhibits a strong drift to the origin giving rise to a L independent relaxation time. Unlike the birth-death process defined in Section 5.6, for $\rho > \rho_c$ the birth-death process behaves like a symmetric random walk, which gives rise to a L^2 dependence on the relaxation time.

In light of Lemma 5.5.1 and Theorem 5.4.1 we only have to bound the relaxation time of the projection chain and compare the Dirichlet forms of the projection chain and the full process. This is contained in the following lemmas, which will be proved in the following two subsections.

Lemma 5.7.1. *There exists (universal) constants $c_1, c_2, c_3 > 0$ such that for all*

$\rho < \rho_c$

$$\widetilde{T}^2_{L,N} \leq c_1 \left(\sqrt{r} - \sqrt{\frac{\rho}{1+\rho}}\right)^{-2} \ , \tag{5.46}$$

for $\rho = \rho_c$

$$\widetilde{T}^2_{L,N} \leq c_2 \left(1+\rho_c\right)^2 L \ , \tag{5.47}$$

and for $\rho \geq \rho_c$ we have

$$\widetilde{T}^2_{L,N} \leq c_3 \frac{1}{r} (\rho - \rho_c)^2 L^2 \ . \tag{5.48}$$

Lemma 5.7.2. *For all $L > 3$, $N \in \mathbb{N}$, and $f : \Omega_{L,N} \to \mathbb{R}$*

$$\widetilde{\mathcal{D}}^2_{L,N}(H_f(\eta)) \leq \left(2 + \frac{2}{L} T^1_{L-1,\rho}\right) \mathcal{D}^2_{L,N}(f) \ .$$

Since the background process is a constant rate ZRP with a single site defect on $\Omega_{L-1,N-\eta_{d_1}}$, the upper bound in Theorem 5.4.2 follows from (5.18), Theorem 5.4.1, as well as Lemmas 5.7.1 and 5.7.2.

5.7.1 Proof of Lemma 5.7.1: The relaxation time of the projection chain with two defects

Recall that the projection chain (5.19) is a birth death process $(X_t)_{t \geq 0}$ on $\{0, ..., N\}$ with death rate $\hat{q}(n, n-1) = r\mathbb{1}(n > 0)$ and birth rate $\hat{q}(n, n+1) = \pi^1_{L-1,N-n}(g_x(\cdot))$. The birth rate is of the form

$$\hat{q}(n, n+1) = \pi^1_{L-1,N-n}(g_x(\cdot)) = \frac{Z^1_{L-1,N-n-1}}{Z^1_{L-1,N-n}} \ ,$$

where

$$Z^1_{L,N} = \sum_{n=0}^{N} r^{-n} \binom{N-n+L-2}{N-n} \ . \tag{5.49}$$

By an equivalence of ensembles argument (see for example [62]) we have

$$\lim_{\substack{L \to \infty \\ N/L \to \rho}} \pi^1_{L-1,N}(g_y(\eta_y)) = \begin{cases} \phi(\rho) = \frac{\rho}{\rho+1} & \text{if } \rho < \rho_c \\ \phi(\rho_c) = r & \text{if } \rho \geq \rho_c \end{cases} \ . \tag{5.50}$$

Furthermore, for $\rho(1-x) < \rho_c$

$$\lim_{\substack{L \to \infty \\ N/L \to \rho \text{ and } n/N \to x}} \pi^1_{L-1,N-n}(g_y(\eta_y)) = \phi\left(\rho(1-x)\right) = \frac{\rho(1-x)}{\rho(1-x)+1} \ . \tag{5.51}$$

As in the proof of Lemma 5.6.1 let $v_i = v = \sqrt{\frac{r(\rho+1)}{\rho}}$ for all $i \in \{0,\ldots,N\}$. By (5.51) and assuming $\rho < \rho_c$

$$\lim_{\substack{L\to\infty \\ N/L\to\rho \text{ and } i/N\to x}} R_i(v) = \frac{r(1+\rho(1-x))\left(\sqrt{\frac{\rho}{r(1+\rho)}}-1\right)+(1-x)\left(1-\sqrt{\frac{r(1+\rho)}{\rho}}\right)}{1+\rho(1-x)}$$

is increasing in $x \in [0,1]$. Therefore, by Theorem 5.3.2 for all $\rho < \rho_c$ the relaxation time of the birth-death chain is bounded above by

$$\widetilde{T}^2_{L,N} \le c_1\left(\sqrt{r}-\sqrt{\frac{\rho}{\rho+1}}\right)^{-2} .$$

The case $N/L \to \rho = \rho_c$ follows as before, setting $v_i = 1$ for all $i \in \{0,\ldots,N\}$ we have

$$\widetilde{T}^2_{L,N} \le c_2(1+\rho_c)^2 L \tag{5.52}$$

Now consider the case $\rho > \rho_c$. In this case, for large L and $n \le (\rho-\rho_c)L$ the birth-death chain behaves like a symmetric random walk, and for $n > (\rho-\rho_c)L$ the chain behaves like the random walk with the rates at the critical density $\rho = \rho_c$. Therefore, we partition $\{0,1,\ldots,N\}$ into two disjoint subsets

$$\{0,1,\ldots,N\} = \underbrace{\{0,1,\ldots,\lfloor(\rho-\rho_c)L\rfloor\}}_{:=\Omega_1} \cup \underbrace{\{\lfloor(\rho-\rho_c)L\rfloor+1,\ldots,N\}}_{:=\Omega_2} . \tag{5.53}$$

A simple change of coordinates shows the birth-death chain restricted to Ω_2 is the same as the chain when $\rho = \rho_c$ and therefore, the relaxation time scales like $(1+\rho_c)^2 L$. Since $\pi_{L-1,N-\eta_1}(g(\cdot)) \to \phi_c = r$ as $N/L \to \rho > \rho_c$ the birth-death chain restricted to Ω_1 behaves like a simple-random walk on the lattice $\{0,1,\ldots,(\rho-\rho_c)L\}$ and the relaxation time scales like $\frac{1}{r}(\rho-\rho_c)^2L^2$. Therefore, we expect the relaxation time for the birth-death chain to scale like $\frac{1}{r}(\rho-\rho_c)^2L^2$. This can be shown explicitly by first noting that in the limit $N, L \to \infty$ such that $N/L \to \rho$ and $n/N \to x$ we have

$$\frac{1}{L}\log Z^0_{L,N-n} \to (1+\rho(1-x))\log\left(1+\rho(1-x)\right) - \rho(1-x)\log\left(\rho(1-x)\right) .$$

Via integral approximations of (5.49) we have

$$Z^1_{L,N} \asymp \begin{cases} \frac{(1+\rho)^{L(1+\rho)}}{\rho^{L\rho}} \frac{\left(1-\left(\frac{\rho}{r(1+\rho)}\right)^{L\rho}\right)}{\log\left(\frac{r(1+\rho)}{\rho}\right)} & \text{for } \rho \le \rho_c \\ \frac{\sqrt{2\pi L}}{\sqrt{r}} \rho_c^{L-1} r^{-L(\rho+1)} & \text{for } \rho > \rho_c \end{cases} . \tag{5.54}$$

Since the stationary measure for the birth-death chain is given by

$$\mu^2_{L,N}[n] = \pi^2_{L,N}[\{\eta_1 = n\}] = \frac{1}{Z^2_{L,N}} r^{-n} Z^1_{L-1,N-n} ,$$

for large L, N such that $N/L \to \rho > \rho_c$ and $n/N \to x < 1 - \frac{\rho_c}{\rho}$

$$r^{-n} Z^1_{L-1,N-n} \asymp \frac{\sqrt{2\pi L}}{\sqrt{r}} \rho_c^{L-1} r^{-L(\rho+1)} ,$$

which is independent of n. For $x > 1 - \frac{\rho_c}{\rho}$, using the integral approximation (5.54) $r^{-n} Z^1_{L-1,N-n}$ can be shown to decay with x, and the functions appearing in Theorem 5.3.3 can be easily approximated to give a bound which scales like $\frac{1}{r}(\rho-\rho_c)^2 L^2$.

5.7.2 Proof of Lemma 5.7.2: Comparison of the Dirichlet forms

We now turn to the proof of Lemma 5.7.2. We follow the same approach outlined in Section 5.6 and here we only note the major differences in the proof.

Again, define

$$M(\eta) = \frac{1}{r} \frac{\pi^2_{L,N}[\eta_1]}{\pi^2_{L,N}[\eta_1 + 1]} \frac{1}{L-1} \sum_{x=2}^{L} g_x(\eta_x) , \tag{5.55}$$

which gives the same upper bound for $\widetilde{\mathcal{D}}^2_{L,N}(H_f(\eta_1))$ as equation (5.41). Splitting the terms into $A^f_1(L, N)$ and $A^f_2(L, N)$ as before, we have the same bound

$$A^f_1(L, N) \le 2\mathcal{D}^2_{L,N}(f) . \tag{5.56}$$

We again use the Cauchy-Schwarz inequality to bound $A^f_2(L, N)$ above by

$$\frac{1}{(L-1)^2} \pi^2_{L,N} \left(\frac{1}{\pi^1_{L-1,N-\eta_1}(g_x(\eta_x))} \mathrm{Var}^1_{L-1,N-\eta_1}(f) \mathrm{Var}^1_{L-1,N-\eta_1} \left(\sum_{x=2}^{L} g_x(\eta_x) \right) \right) . \tag{5.57}$$

Let $J = L - 1$ and $M = N - \eta_1$. For some $y \notin \Delta$, a simple computation on the canonical measure of the constant rate ZRP with single site defect shows

$$\begin{aligned}
&\frac{1}{\pi^1_{J,M}(g_y(\eta_y))}\mathrm{Var}^1_{J,M}\left(\sum_{x=2}^{L} g_x(\eta_x)\right) \\
&\qquad = \sum_{x=1}^{J} \pi^1_{J,M-1}(g_x(\eta_x+1)) + J(J-1)\pi^1_{L,M-1}(g_y(\eta_y)) - J^2\pi^1_{J,M}(g_y(\eta_y)) \\
&\qquad = J-1+r+J(J-1)\pi^1_{J,M-1}(g_y(\eta_y)) - J^2\pi^1_{J,M}(g_y(\eta_y)) \ ,
\end{aligned}$$

where in the final step we have used $g_x(\eta_x+1)$ is equal to 1 for all $x \in \Lambda \setminus \Delta$ and r for $x \in \Delta$. Since $g_y(n) = \mathbb{1}(n \geq 1)$ for $y \in \Lambda \setminus \Delta$ we can rewrite the expected jump rate as

$$\pi^1_{J,M}\left(g_y(\eta_y)\right) = 1 - \pi^1_{J,M}[\{\eta_y = 0\}] \ ,$$

which implies

$$\begin{aligned}
&\frac{1}{\pi^1_{J,M}(g_y(\eta_y))}\mathrm{Var}^1_{J,M}\left(\sum_{x=2}^{L} g_x(\eta_x)\right) \\
&\qquad = \sum_{x=2}^{L} \pi^1_{J,M}\left(g_x(\eta_x+1)\right) + J(J-1)\pi^1_{J,M-1}\left(g_y(\eta_y)\right) - J^2\pi^1_{J,M}\left(g_y(\eta_y)\right) \\
&\qquad = J-1+r+J(J-1)\left(1-\pi^1_{J,M-1}[\{\eta_y=0\}]\right) - J^2\left(1-\pi^1_{J,M}[\{\eta_y=0\}]\right) \\
&\qquad = J^2\left(\pi^1_{J,M}[\{\eta_y=0\}] - \pi^1_{J,M-1}[\{\eta_y=0\}]\right) + J\pi^1_{J,M-1}[\{\eta_y=0\}] + r - 1 \ .
\end{aligned}$$

The zero-range process with one defect site is monotone (attractive), which implies

$$\pi^1_{J,M}(f) \leq \pi^1_{J,M+1}(f) \text{ for all increasing functions } f : \mathbb{N}^\Lambda \to \mathbb{R} \ .$$

Fix $y \in \Lambda$, the test function $k(\eta) = \mathbb{1}(\eta_y = 0)$ is decreasing and we have

$$\pi^1_{J,M}[\{\eta_y=0\}] - \pi^1_{J,M-1}[\{\eta_y=0\}] \leq 0 \text{ for all } J, M \in \mathbb{N} \ , \tag{5.58}$$

therefore

$$\begin{aligned}
J^2\left(\pi^1_{J,M}[\{\eta_y=0\}] - \pi^1_{J,M-1}[\{\eta_y=0\}]\right) + J\pi^1_{J,M-1}[\{\eta_y=0\}] + r - 1& \\
\leq J\pi^1_{J,M-1}[\{\eta_y=0\}] \ .&
\end{aligned}$$

Now substituting $J = L - 1$ and $M = N - \eta_1$ we have

$$\frac{1}{\pi^1_{L-1,N-\eta_1}(g_y(\eta_y))}\mathrm{Var}^1_{L-1,N-\eta_1}\left(\sum_{x=2}^{L} g_x(\eta_x)\right) \leq (L-1)\pi^1_{L-1,N-\eta_1}[\{\eta_y = 0\}] \ , \tag{5.59}$$

and since $\pi^1_{L,N}$ is a probability measure we have

$$\pi^1_{L-1,N-\eta_1}[\{\eta_y = 0\}] \leq 1 \ .$$

Therefore, we have

$$\frac{1}{(L-1)^2}\frac{1}{\pi^1_{L-1,M-\eta_1}(g_y(\eta_y))}\mathrm{Var}^1_{L-1,N-\eta_1}\left(\sum_{x=2}^{L} g_x(\eta_x)\right) \leq \frac{1}{L-1} \ , \tag{5.60}$$

which leads to

$$A_2^f(L,N) \leq \frac{2}{L}T^1_{L-1,\rho}\mathcal{D}^2_{L,N}(f) \ . \tag{5.61}$$

where $T^1_{L-1,\rho}$ is the relaxation time of the complete graph zero-range process with one defect site. Combining (5.56) and (5.61) we have

$$\widetilde{\mathcal{D}}^2_{L,N}(H_f(\eta)) \leq \left(2 + \frac{2}{L}T^1_{L-1,\rho}\right)\mathcal{D}^2_{L,N}(f)$$

completing the proof of Lemma 5.7.2.

5.8 Lower bounds of the relaxation times

To complete the proofs of Theorems 5.4.1 and 5.4.2, we must derive lower bounds of the relaxation times, which can be achieved by analysing the Dirichlet forms and variance of appropriate test functions. We consider the sub and super critical cases separately.

5.8.1 $\rho < \rho_c$

Consider the models with one and two defects when $\rho < \rho_c$ and define $F : \Omega_{L,N} \to \mathbb{R}$ as follows

$$F(\eta) = \left(\frac{\phi(\rho)}{r}\right)^{\alpha\eta_1} \tag{5.62}$$

where $\alpha > -1/2$, $\rho = N/L$, $\phi(\rho) = \frac{\rho}{1+\rho}$ and let $K = |\Delta| \in \{1, 2\}$. Since $F(\eta)$ is a function of the first site only, the Dirichlet form reduces to

$$\begin{aligned}
\mathcal{D}^K_{L,N}(F) =& \frac{1}{2}\pi^K_{L,N}\left(r\mathbb{1}(\eta_1 > 0)\left(F(\eta^{1,y}) - F(\eta)\right)^2\right) \\
&+ \frac{1}{2}\pi^K_{L,N}\left(\pi^{K-1}_{L-1,N-\eta_1}\left(g_y(\eta_y)\right)\left(F(\eta^{y,1}) - F(\eta)\right)^2\right) \\
\asymp& \frac{1}{2}\nu^1_\phi\left(r\mathbb{1}(\eta_1 > 0)\left(F(\eta^{1,y}) - F(\eta)\right)^2\right) \\
&+ \frac{1}{2}\nu^1_\phi\left(\nu_\phi\left(g_y(\eta_y)\right)\left(F(\eta^{y,1}) - F(\eta)\right)^2\right) \\
=& \frac{\phi\left(1 - \frac{\phi}{r}\right)\left(1 - \left(\frac{\phi}{r}\right)^\alpha\right)^2}{1 - \left(\frac{\phi}{r}\right)^{1+2\alpha}} .
\end{aligned}$$

Where we have used [62, Theorem 4.2] to replace the expectation with respect to the canonical measure with that of the grand canonical measure for sub-critical systems.

The variance of $F(\eta) = \left(\frac{\phi}{r}\right)^{\alpha\eta_1}$ is given by

$$\begin{aligned}
\mathrm{Var}^K_{L,N}(F) =& \pi^K_{L,N}\left(\left(\frac{\phi}{r}\right)^{2\alpha\eta_1}\right) - \left(\pi^K_{L,N}\left(\left(\frac{\phi}{r}\right)^{\alpha\eta_1}\right)\right)^2 \\
\asymp& \nu^1_\phi\left(\left(\frac{\phi}{r}\right)^{2\alpha\eta_1}\right) - \left(\nu^1_\phi\left(\left(\frac{\phi}{r}\right)^{\alpha\eta_1}\right)\right)^2 = \mathrm{Var}^1_\phi(F) .
\end{aligned}$$

Again, we have replaced the expectation with respect to the canonical measure with that of the grand canonical measure for sub-critical systems using [62, Theorem 4.2]. Since the measure $\nu^1_\phi[\cdot]$ is geometric with parameter ϕ/r we can compute the variance of F exactly to find

$$\mathrm{Var}^1_\phi(F) = \frac{\phi\left(1 - \frac{\phi}{r}\right)\left(1 - \left(\frac{\phi}{r}\right)^\alpha\right)^2}{r\left(1 - \left(\frac{\phi}{r}\right)^{1+\alpha}\right)^2\left(1 - \left(\frac{\phi}{r}\right)^{1+2\alpha}\right)} .$$

Therefore, there exists a constant $c > 0$ such that the spectral gap is bounded above by

$$\begin{aligned}
\lambda^K_{L,N} \le \frac{\mathcal{D}^K_{L,N}(F)}{\mathrm{Var}^K_{L,N}(F)} \le& c \inf_{\alpha > -1/2} r\left(1 - \left(\frac{\phi}{r}\right)^{1+\alpha}\right)^2 = c\left(\sqrt{r} - \sqrt{\phi}\right)^2 \\
=& c\left(\sqrt{r} - \sqrt{\frac{\rho}{1+\rho}}\right)^2 .
\end{aligned}$$

5.8.2 $\rho \geq \rho_c$

Consider the case $\rho \geq \rho_c$ and the zero-range process with one defect site. Define $F : \Omega_{L,N} \to \mathbb{R}$ as

$$F(\eta) = \eta_1 \ .$$

It is easy to see that the Dirichlet form is given by

$$\mathcal{D}^1_{L,N}(F) = \pi^1_{L,N}\left(g_1(\eta_1)\right) \asymp \phi_c = r \ .$$

The variance is given by

$$\begin{aligned}
\mathrm{Var}^1_{L,N}(F) = \mathrm{Var}^1_{L,N}(\eta_1) =& \mathrm{Var}^1_{L,N}\left(N - \sum_{x=2}^{L} \eta_x\right) \\
=& \mathrm{Var}^1_{L,N}\left(\sum_{x=2}^{L} \eta_x\right) \\
\asymp& L\mathrm{Var}^1_{\phi_c}(\eta_2) \\
=& r(1+\rho_c)^2 L \ .
\end{aligned}$$

Where we have replaced the expectation with respect to the canonical measure with that of the critical grand canonical measure for super critical systems [62, Theorem 4.3]. Therefore, the spectral gap is bounded above by

$$\lambda^1_{L,N} \leq \frac{\mathcal{D}^1_{L,N}(\eta_1)}{\mathrm{Var}^1_{L,N}(\eta_1)} < \frac{c}{(1+\rho_c)^2 L}$$

for some constant $c > 0$.

For the zero-range process with two defect sites, an upper bound for the spectral gap follows from the variational principle (5.9), Theorem 5.3.3 and the scaling arguments at the end of Section 5.7.1. By (5.9) the spectral gap of the full process is bounded above by the spectral gap of the projection chain (5.19) as constructed in Section 5.5, *i.e.*

$$\begin{aligned}
\lambda^K_{L,N} &= \inf_f \left\{ \frac{\mathcal{D}^K_{L,N}(f)}{\mathrm{Var}^K_{L,N}(f)} : \mathrm{Var}^K_{L,N}(f) \neq 0 \right\} \\
&\leq \inf_h \left\{ \frac{\mathcal{D}^K_{L,N}(h)}{\mathrm{Var}^K_{L,N}(h)} : \mathrm{Var}^K_{L,N}(h) \neq 0 \ \text{ and } \ h(\eta) = h(\eta_1) \right\} = \widetilde{\lambda}^K_{L,N} \ ,
\end{aligned}$$

where $\widetilde{\lambda}^K_{L,N}$ is the spectral gap of the projection chain (5.19). By Theorem 5.3.3 and computations in Section 5.7.1 we have

$$\widetilde{\lambda}^2_{L,N} \leq c\frac{r}{(\rho-\rho_c)^2 L^2} \tag{5.63}$$

for some constant $c > 0$.

5.9 Coupling times and cut-off

In this section, we discuss convergence to the stationary measure in total-variation. For two measures μ and ν on a countable state space Ω, the total-variation distance is given by

$$||\nu - \mu||_{\mathrm{TV}} = \sum_{\eta\in\Omega} |\nu[\eta] - \mu[\eta]| \ .$$

For the defect site ZRP with stationary measure $\pi^{\Delta}_{L,N}$ (5.6), the total-variation mixing time is given by

$$d(t) := \sup_{\eta\in\Omega_{L,N}} ||P_t(\eta,\cdot) - \pi^{\Delta}_{L,N}||_{\mathrm{TV}} \ , \tag{5.64}$$

where $P_t(\eta, A) = \mathbb{P}\left(\eta(t) \in A \,|\, \eta(0) = \eta\right)$ for some $A \in \Omega_{L,N}$.

Consider a coupling of the defect site ZRP $(\eta(t), \zeta(t))_{t\geq 0}$ on the state space $\Omega_{L,N} \times \Omega_{L,N}$ such that $\zeta(0) \sim \pi^{\Delta}_{L,N}$, *i.e.* $\zeta(t)$ is stationary for all $t \geq 0$. Define the coupling time as

$$T^{\mathrm{coup}}_{L,N} := \inf\left\{t \geq 0 \,:\, \eta(t) = \zeta(t)\right\} \ .$$

Since $\zeta(t)$ is stationary the total-variation mixing time is bounded above by the coupling time in the following sense

$$d(t) \leq \sup_{\eta\in\Omega_{L,N}} \mathbb{P}_\eta(T^{\mathrm{coup}}_{L,N} > t) \ , \tag{5.65}$$

where $\mathbb{P}_{\eta,\zeta}\left(T^{\mathrm{coup}}_{L,N} > t\right) = \mathbb{P}\left(T^{\mathrm{coup}}_{L,N} > t \,|\, \eta(0) = \eta \text{ and } \zeta(0) \sim \pi^{\Delta}_{L,N}\right)$.

In Figures 5.4 and 5.6, we present numerical evidence for the coupling time of a ZRP with one and two defect sites respectively. We see vastly different types of behaviour above and below the critical density, and with one and two defect sites. In Theorems 5.6 and 5.7, we saw that the relaxation time of the full process is given by the relaxation of the projection chain. In both cases, below the critical density the projection chain behaves like a biased random walk. Above the critical density, the projection chains behave differently; with one defect the chain exhibits drift towards a value which scales like $(\rho - \rho_c)L$, and for two defect the chain behaves like a symmetric random walk. In the latter case, it is known that the symmetric random walk does not exhibit total-variation cut-off, and therefore, we expect to not see total-variation cut-off in the ZRP with two defects.

In Figures 5.5 and 5.7, we present numerical evidence for the coupling time of the projection chains with one and two defect sites respectively. In both cases, above the critical density the coupling time of the projection chain behaves remarkably similar to that of the ZRP. Heuristically, above the critical density the background has enough time to reach a quasi stationary regime before the maximum occupation becomes macroscopic. Therefore, the slow site occupation (which is typically the maximum) really behaves like the projection chain and the mixing time of the projection chain is the mixing time of the ZRP.

5.10 Conclusion

In this chapter, we have studied a zero-range process that exhibits condensation due to single site defects under complete graph dynamics. To calculate sharp upper bounds of the relaxation time, we partition the state space into a disjoint union of subspaces and analyse the restriction and projection chains. For the zero-range process with one and two defect sites, the relaxation time exhibits a transition as the density crosses a critical value as summarised in (5.14) and (5.17). This scaling of the relaxation time arises from a transition in the relaxation time of the projection chain. For one defect site, the projection chain below the critical density behaves like a random walk driven to the left boundary and above the critical density, the projection chain behaves like a mean reverting process. For two defects above the critical density the projection chain behaves like a simple symmetric random walk.

We also give initial heuristic results on the mixing time of this process. By Monte Carlo simulation, we numerically calculate the coupling time, which is known to give good bounds for the mixing time of the process. We also numerically calculate the coupling time of the projection chain and show that it exhibits remarkably similar behaviour to the full process above the critical density. For the one defect case, the coupling time appears to show a cutoff at $L\log(L)$, whilst for the two defect case the coupling time appears to scale like L^2 with no cutoff, which is expected since the projection chain behaves like a symmetric random walk.

(a) (b)

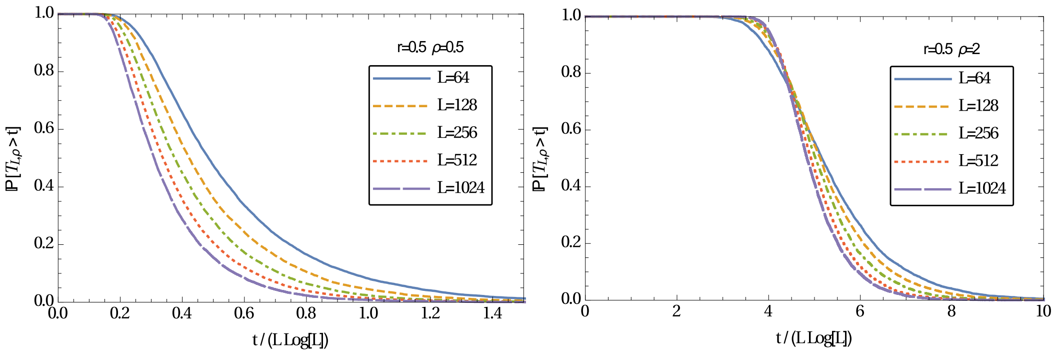

Figure 5.4: Coupling time for the complete graph zero-range process with one defect site. Figure 5.4a: coupling time rescaled by $L\log(L)$ suggesting the process exhibits total-variation cutoff. Figure 5.4b: coupling time rescaled by $L\log(L)$ suggesting the process exhibits total-variation cutoff.

(a) (b)

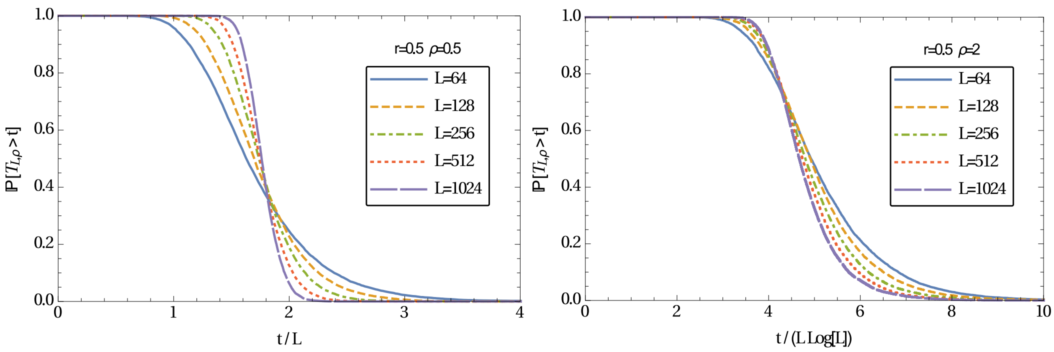

Figure 5.5: Coupling time for the projection chain (5.19) generated from the complete graph ZRP with one defect site. Figure 5.5a: coupling time rescaled by L. Comparisons with the biased birth-death chain shows explicitly that the process exhibits cutoff at $c_{\rho,r}L$ with a window of order $\sqrt{L}$, see [44] for more details. Figure 5.5b: coupling time rescaled by $L\log(L)$ suggesting the process exhibits total-variation cutoff.

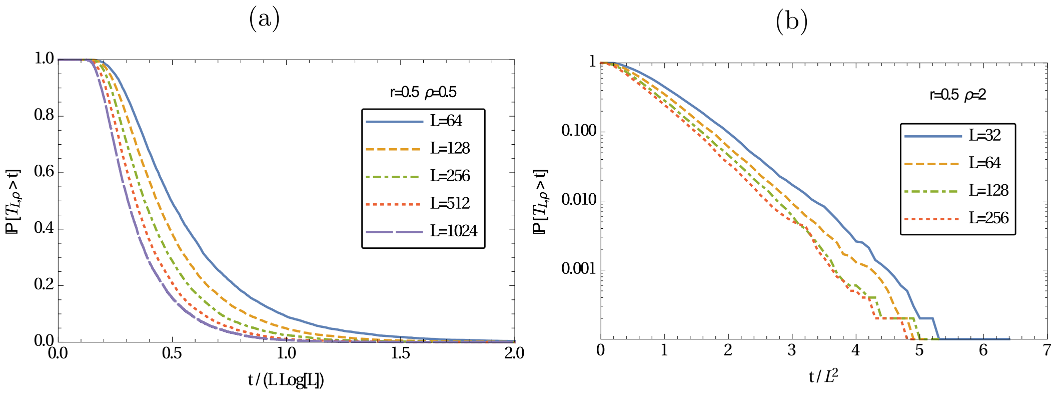

Figure 5.6: Coupling time for the complete graph zero-range process with two defect sites. Figure 5.6a: coupling time rescaled by $L\log(L)$ for $\rho < \rho_c$ suggesting the process exhibits total-variation cutoff. Figure 5.6b: coupling time rescaled by L^2 for $\rho > \rho_c$ suggesting that the process does not exhibit total-variation cutoff, as predicted by the dynamics of the projection chain.

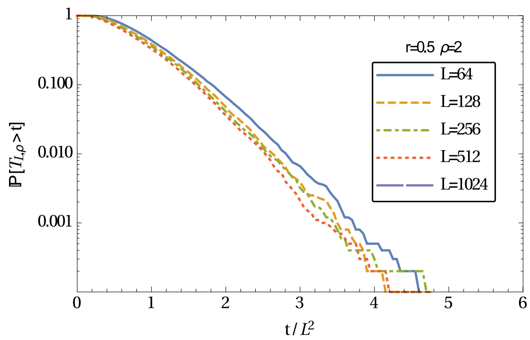

Figure 5.7: Coupling time rescaled by L^2 for the projection chain (5.19) generated from the complete graph ZRP with two defect sites above the critical density. As in the case for one defect site the coupling time of the projection chain is remarkably similar to the coupling time of the ZRP with the same density ρ and rate r. For this example, above the critical density the projection chain behaves like a symmetric random walk leading to a mixing time of $O(L^2)$ and exhibits no cutoff.

CHAPTER 6

Birth-Death Chains: Relaxation, Hitting and Mixing Times

6.1 Introduction

In Chapter 5, we have seen that by understanding the dynamics of a zero-range process under certain projections, we can find sharp bounds for the relaxation time and heuristically give good estimates on the mixing time. For the defect site zero-range process, it was natural to project the process onto the coordinate of the slow site, the typical maximum, since it is natural to expect that the dynamics of the defect site is the slowest mode in the system, and therefore dominates the relaxation time. For homogeneous zero-range processes which condense via local interactions, the location of the condensate is typically uniformly distributed on the lattice Λ and, heuristically, the slowest time scale is the motion of the condensate and not the single site dynamics. Intuitively, under homogeneous dynamics, the stationary measure of a condensing process will contain $|\Lambda|$ disjoint wells, which are each associated with the condensate being located at a site $x \in \Lambda$. The dynamics within these wells can then be understood by a projection on a single birth-death chain, which is an important step in estimating metastable time scales of these processes [107, 18]. In this chapter we calculate the relaxation, hitting, and mixing times for the birth-death chains which are associated with condensing homogeneous zero-range processes.

This chapter is organised as follows. In Section 6.2, we introduce general birth-death processes and state our main results. We also give a brief outline of the main methods used to prove our results. We calculate relaxation times in Section

6.3.1 and hitting and mixing times in Section 6.3.2.

6.2 Notation and results

A birth-death chain $(X_t)_{t\geq 0}$ is a continuous time Markov processes with at most two possible transitions called births and deaths. We focus on finite state space $\mathbb{Z}_N = \{0, \ldots, N\}$. The process is defined by the generator

$$\mathcal{L}_N f(n) = \alpha_n \left(f(n-1) - f(n) \right) + \beta_n \left(f(n+1) - f(n) \right) \ . \tag{6.1}$$

The death rate $\alpha_n \geq 0$ satisfies $\alpha_n = 0$ if and only if $n = 0$ and the birth rate $\beta_n \geq 0$ satisfies $\beta_n = 0$ if and only if $n = N$. This implies the process is irreducible on $\mathbb{Z}_N$ and therefore ergodic with unique stationary (reversible) measure μ_N on $\mathbb{Z}_N$ which is given by the mass function

$$\mu_N[n] = \frac{1}{Z_N} \prod_{k=1}^{n} \frac{\beta_{k-1}}{\alpha_k} \quad \text{where} \quad Z_N = \sum_{n=0}^{N} \prod_{k=1}^{n} \frac{\beta_{k-1}}{\alpha_k} \ . \tag{6.2}$$

Before stating our main results, we first define the notation for the spectral gap, relaxation time, and mixing time of birth-death processes. We also give a short review of the methods and theorems we use in this chapter.

We denote the variance of a function $f : \mathbb{Z}_N \to \mathbb{R}$ with respect to the stationary measure μ_N as $\mathrm{Var}_N(f)$ and the Dirichlet form is given by

$$\mathcal{D}_N(f) = -\mu_N(f \mathcal{L}_N f) = \frac{1}{2} \sum_{n=1}^{N} \alpha_n \mu_N[n] \left(f(n-1) - f(n) \right)^2 \ . \tag{6.3}$$

The spectral gap of the generator (6.1) on $\mathbb{Z}_N$ denoted by λ_N is given by the variational principle given in Definition 2.5.1, *i.e.*

$$\lambda_N := \inf_f \left\{ \frac{\mathcal{D}_N(f)}{\mathrm{Var}_N(f)} \ : \ \mathrm{Var}_N(f) \neq 0 \right\} \ . \tag{6.4}$$

The relaxation time is defined as the reciprocal of the spectral gap $T_N^{\mathrm{rel}} := 1/\lambda_N$. The spectral gap λ_N is then also the optimal constant satisfying

$$\mathrm{Var}_N \left(P_t f \right) \leq e^{-2\lambda_N t} \mathrm{Var}_N(f) \ , \tag{6.5}$$

where $P_t(x, \cdot) = \mathbb{P}\left(X_t \in \cdot \,|\, X_0 = x \right)$, as stated in Proposition 2.5.2. To find a lower bound for the spectral gap, we construct a coupling of the birth death process and utilise the following theorem [105, 101], which reduces the variational problem appearing in (6.4) to a problem of finding a 'good' test function on the coupled state space.

Theorem 6.2.1 (Chen). *Let $\mathcal{L}$ be the generator of a Markov process on the (finite) state space Ω with spectral gap λ_{gap}, and $\widehat{\mathcal{L}}$ be a Markovian coupling of $\mathcal{L}$ on the state space $\Omega \times \Omega$. Furthermore, let $F : \Omega \times \Omega \to \mathbb{R}$ be any positive function such that $F(\eta, \xi) = 0$ if and only if $\eta = \xi$ then*

$$\lambda_{gap} \geq \min_{\substack{(\eta,\xi)\in\Omega\times\Omega \\ \eta\neq\xi}} \frac{-\widehat{\mathcal{L}}F(\eta,\xi)}{F(\eta,\xi)} \tag{6.6}$$

In this chapter, we are also interested in the the mixing time, which is defined as the maximal distance $d_N(t)$ (over initial state $x \in \Omega$) between $P_t(x, \cdot)$ and the stationary measure μ_N in total variation as defined in Section 2.5.2, *i.e.*

$$d_N(t) := \sup_{x\in\mathbb{Z}_N} ||P_t(x,\cdot) - \mu_N||_{\mathrm{TV}} \ . \tag{6.7}$$

For each $\varepsilon > 0$ the ε-mixing time is given by

$$T_N^{\mathrm{mix}}(\varepsilon) := \inf\{t \geq 0 : d_N(t) < \varepsilon\} \ . \tag{6.8}$$

Coupling methods can also be used to give good bounds for the mixing time of a finite Markov process, and for birth death processes this can be further extended to a calculation of expected hitting times by the following method. Let $Z_t = (X_t, Y_t)$ be a coupling of two copies of the same process, such that $X_t \leq Y_t$ for all $t \geq 0$, and define the coupling time as

$$T_N^{\mathrm{coup}} := \inf\{t \geq 0 \ : \ X_t = Y_t\} \ . \tag{6.9}$$

The distance from the stationary measure μ_N at time t (6.7) is bounded above by the probability that the two processes have not coupled in time t, *i.e.*

$$d_N(t) \leq \sup_{x,y\in\mathbb{Z}_N} \mathbb{P}_{x,y}\left(T_N^{\mathrm{coup}} > t\right) \ , \tag{6.10}$$

where $\mathbb{P}_{x,y}\left(T_N^{\mathrm{coup}} > t\right) = \mathbb{P}\left(T_N^{\mathrm{coup}} > t \ : \ X_0 = x \text{ and } Y_0 = y\right)$. Since the birth death process is one dimensional the time, T_N^{coup} is bounded above by the time it takes for the left most particle to hit N starting at 0 or the right most particle to hit 0 starting at N, *i.e.*

$$\sup_{x,y\in\mathbb{Z}_N} \mathbb{P}_{x,y}\left(T_N^{\mathrm{coup}} > t\right) \leq \min\left\{\mathbb{P}_0\left(\tau_N > t\right), \mathbb{P}_N\left(\tau_0 > t\right)\right\} \ , \tag{6.11}$$

where $\tau_A = \inf\{t \geq 0 : X_t \in A\}$ is the hitting time of a set $A \subseteq \mathbb{Z}_N$ and we write $\tau_{\{x\}} = \tau_x$ for point sets. Finally, a simple application of Markov's inequality shows

$$d_N(t) \leq \min\left\{\frac{\mathbb{E}_0\left(\tau_N\right)}{t}, \frac{\mathbb{E}_N\left(\tau_0\right)}{t}\right\} \ , \tag{6.12}$$

where $\mathbb{E}_x(\tau_A) = \mathbb{E}\left(\tau_A \,|\, X_0 = x\right)$ is the expected hitting time of the set A with initial condition $X_0 = x$. Define $H_A : \mathbb{Z}_N \to \mathbb{R}$ as

$$H_A(x) = \mathbb{E}_x\left(\tau_A\right) \ ,$$

the expected hitting time of the set A with initial condition $X_0 = x$. For a Markov processes on a finite state space, the expected hitting time is given by the minimal solution to the system

$$\begin{cases} H_A(n) = 0 & \text{for } n \in A \ , \\ -\mathcal{L}_N H_A(n) = 1 & \text{for } n \notin A \ . \end{cases} \tag{6.13}$$

See [47] for a simple proof using the total law of expectation and the Markov property. The following proposition gives a solution to the system of equations (6.13) [47], which allows us to calculate sharp bounds for the hitting times and mixing times of birth death processes.

Proposition 6.2.2. *For a birth-death process on $\mathbb{Z}_N$ given by the generator* (6.1) *with death rates α_n and birth rates β_n, the expected hitting time of a set $A = \{k_L, k_L+1, \ldots, k_R-1, k_R\} \subseteq \mathbb{Z}_N$ is equal to*

$$H_A(n) = \sum_{k=n}^{k_L-1} \frac{1}{\beta_k} \sum_{j=0}^{k} \frac{\mu_N[j]}{\mu_N[k]} \tag{6.14}$$

for all initial conditions $n < k_L$, and

$$H_A(n) = \sum_{k=0}^{k_R-1} \frac{1}{\alpha_{N-k}} \sum_{j=N-k}^{N} \frac{\mu_N[N-j]}{\mu_N[k]} \tag{6.15}$$

for all initial conditions $n > k_R$.

The proof is completely standard and follows from (6.13) by first writing the hitting $H_A(n)$ as a telescopic series, $H_A(n) = \sum_{i=1}^{n} T_A(i) - T_A(i-1)$, and then solving for $T_A(i)$ with an appropriate boundary condition.

As a first example, we can use Proposition 6.2.2 and (6.12) to derive an N^2 bound for the mixing time of a symmetric random walk on $\mathbb{Z}_N$. First, note that the stationary measure $\mu_N[n] = 1/(N+1)$ is the uniform measure on $\mathbb{Z}_N$ and due to the symmetry of the process the hitting times are symmetric, *i.e.* $H_N(0) = H_0(N)$. Therefore,

$$H_N(0) = \sum_{n=0}^{N-1} \frac{1}{\beta_n \mu_N[n]} \sum_{k=0}^{n} \mu_N[N-k] = \sum_{n=0}^{N-1} (n+1) = \frac{1}{2} N(N+1) \ ,$$

which combined with (6.12) implies that the mixing time of the symmetric random walk on $\mathbb{Z}_N$ is of order N^2. This bound can be shown to be sharp, in the sense that there exists a constant $c > 0$ such that $T_N^{\mathrm{mix}} \geq cN^2$ which can be shown by use of a distinguishing statistic and we refer to [44] for more details.

In general, the mixing time can be computed by calculating the maximal expected hitting time a of a 'large' set $A \subset \Omega$ [57, Theorem 1] [58].

Theorem 6.2.3 (Peres, Sousi)**.** *For every irreducible and reversible Markov process on a finite state space Ω with stationary measure μ, and for each $\alpha < 1/2$ there exists constants c_α, c'_α, depending only on α, so that*

$$c_\alpha \max_{x, A: \mu[A] > \alpha} H_A(x) \leq T_{mix}\left(\frac{1}{4}\right) \leq c'_\alpha \max_{x, A: \mu[A] > \alpha} H_A(x) \ . \tag{6.16}$$

Consider the symmetric homogeneous zero-range processes on $\Omega_L = \mathbb{N}^\Lambda$, where $\Lambda = \{1, \ldots, L\}$ with periodic boundary conditions, defined by the generator

$$\mathcal{L}^{\mathrm{ZRP}} f(\eta) = \frac{1}{2} \sum_{x \in \Lambda} g(\eta_x) \left(f(\eta^{x,x+1}) + f(\eta^{x,x-1}) - 2f(\eta) \right) \ . \tag{6.17}$$

In this chapter, we consider two zero-range processes, which are known to exhibit condensation in the thermodynamic limit [7], with rates

$$g_1(n) = \left(\frac{n}{n+1} \right)^{-b} \ \text{ for } \ n \in \{1, 2, \ldots\} \ \text{ and } g_1(0) = 0 \ , \tag{6.18}$$

for $b > 3$, and

$$g_2(n) = e^{-b((n-1)^\gamma - n^\gamma)} \ \text{ for } \ n \in \{1, 2, \ldots\} \ \text{ and } g_2(0) = 0 \ , \tag{6.19}$$

for $b > 0$ and $\gamma \in (0,1)$. The zero-range dynamics with rates $g_1(\cdot)$ and $g_2(\cdot)$ exhibits a family of stationary product measures (2.11) on $\Omega_L = \mathbb{N}^\Lambda$ as given in Section 2.3.1

$$\nu_\phi^L[\eta] = \prod_{x\in\Lambda} \nu_\phi[\eta_x] \ \text{ with } \ \nu_\phi[n] = \frac{\phi^n w(n)}{z(\phi)} \ , \tag{6.20}$$

with stationary weights $w_1(n) = (n+1)^{-b}$ and $w_2(n) = e^{-b\,n^\gamma}$ respectively. The process preserves the number of particles and is irreducible on the state space $\Omega_{L,N}$ with unique invariant measure (2.18) with mass function

$$\pi_{L,N}[\eta] = \frac{1}{Z_{L,N}} \prod_{x\in\Lambda} w(\eta_x) \mathbb{1}\left(\eta \in \Omega_{L,N}\right) \ . \tag{6.21}$$

Consider the projection chain generated from the zero-range dynamics by conditioning on the value η_1 as derived for the defect site zero-range process in Chapter 5 Section 5.5, noting that the projection chain is the same for both symmetric and complete graph dynamics. The projection chain is a birth death process on $\mathbb{Z}_N = \{0,\dots,N\}$ with rates

$$\alpha_n = g(n) \quad \text{ and } \quad \beta_n = \pi_{L-1,N-n}\left(g(\cdot)\right) =: \Phi_{L,N}(n) \ . \tag{6.22}$$

This projection chain is derived by applying the Dirichlet form to a test function $h(\eta) : \Omega_L \to \mathbb{R}$, which is a function of the first site occupation number only and where the Dirichlet form is given by

$$\mathcal{D}_{L,N}(f) = -\pi_{L,N}\left(f\mathcal{L}^{\text{ZRP}} f\right) \ . \tag{6.23}$$

In this chapter, we also consider the approximation of this projection chain where the jump rate is replaced by the expected jump rate given by the grand canonical measure ν_{ϕ_c} at criticality, *i.e.* the birth death process with rates

$$\alpha_n = g(n) \quad \text{ and } \quad \beta_n = \nu_{\phi_c}\left(g(\cdot)\right)\mathbb{1}(n < N) = \phi_c \mathbb{1}(n < N) \ . \tag{6.24}$$

Let $d_N(t)$, $T_N^{\text{mix}}(\varepsilon)$, and T_N^{rel} be the distance from stationarity, ε-mixing time, and relaxation time of the birth death process with rates (6.24). Let $d_{L,N}(t)$, $T_{L,N}^{\text{mix}}(\varepsilon)$, and $T_{L,N}^{\text{rel}}$ be the distance from stationarity, ε-mixing time, and relaxation time of the birth death process with rates (6.22).

Sharp estimates of the quantities T_N^{rel} and T_N^{mix} are required to calculate metastable time-scales of condensing interacting particle systems [17, 18, 23]. Zero-range processes with jump rates given by (6.18) and (6.19) exhibit condensation

[81], and in the former case are known to be metastable [17, 18]. For these condensing zero-range processes the slow time-scale is given by the time it takes for the condensate to relocate from sites $x \to y \in \Lambda$, and the fast time scale is associated to the dynamics of the background particles outside of the condensate.

Our main results are summarised in the following theorems.

Theorem 6.2.4. *Consider the birth death process with rates* (6.24) *where* $\phi = 1$.

(i) Let the death rate be equal to $g_1(n)$ *given by* (6.18)*, then there exist constants* $c_1, c_1', c_2, c_2' > 0$ *such that the relaxation time satisfies*

$$c_1(N+1)^2 \leq T_N^{rel} \leq c_1'(N+1)^2 \ , \tag{6.25}$$

and the ε*-mixing time satisfies*

$$c_2 \log\left(\frac{1}{2\varepsilon}\right)(N+1)^2 \leq T_N^{mix}(\varepsilon) \leq c_2' \left\lceil \log_2\left(\frac{1}{\varepsilon}\right)\right\rceil (N+1)^2 \ . \tag{6.26}$$

Note that the constants c_1, c_1', c_2, c_2' *can depend on the model parameter* $b > 3$.

(ii) Let the death rate be equal to $g_2(n)$ (6.19)*, then there exists constants* $c_\gamma, c_\gamma', C_\gamma, C_\gamma' > 0$ *such that the relaxation time satisfies*

$$c_\gamma N^{2(1-\gamma)} \leq T_N^{rel} \leq c_\gamma' N^{2(1-\gamma)} \ , \tag{6.27}$$

and the mixing time satisfies

$$T_N^{mix}(\varepsilon) \leq C_\gamma' \left\lceil \log_2\left(\frac{1}{\varepsilon}\right)\right\rceil N^{2-\gamma} \ . \tag{6.28}$$

The $1/4$*-mixing time is bounded below by*

$$T_N^{mix}(1/4) \geq C_\gamma N^{2-\gamma} \ . \tag{6.29}$$

Theorem 6.2.5. *Consider the birth death process with rates* (6.22)*.*

(i) Consider the birth death process defined by the projection of the zero-range process with rates $g_1(n)$ (6.18)*. Then there exist constants* $c_b, c_b' > 0$ *such that*

$$T_{L,N}^{mix}(\varepsilon) \leq 4c_b' \left\lceil \log_2\left(\frac{1}{\varepsilon}\right)\right\rceil (N+1)^{1+b}\left(1 + o_N(1)\right) \ . \tag{6.30}$$

(ii) Consider the birth death process defined by the projection of the zero-range pro-

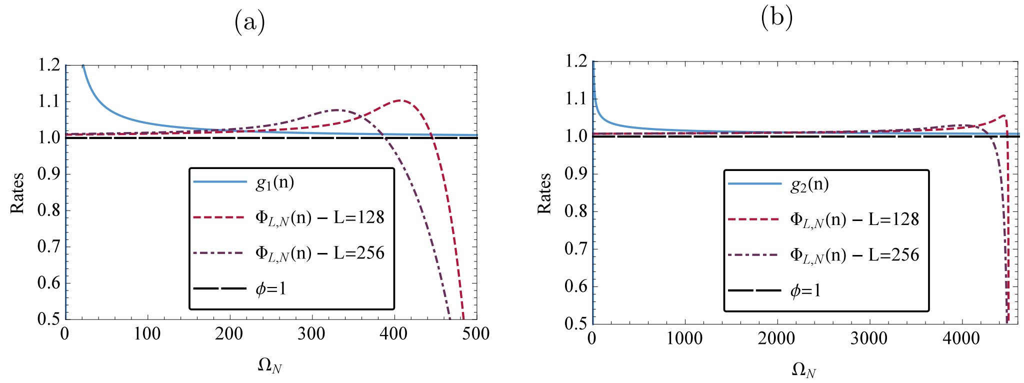

Figure 6.1: Birth and death rates for the processes (6.24) and (6.22). Figure 6.1a: Rates with $g_1(n)$ (6.18) with $b = 4$ and $N = 500$. Figure 6.1b: Rates with $g_2(n)$ (6.19) with $b = 1$, $\gamma = 0.5$, and $N = 4500$. Both processes show non-monotone behaviour, which gives rise to a change in the scaling of the relaxation and mixing times when compared with the rates given by (6.24).

cess with rate $g_2(n)$ (6.19)*, then there exists constants* $c_{b,\gamma}, c'_{b,\gamma} > 0$ *such that*

$$T^{mix}_{L,N}(\varepsilon) \leq 4c'_{b,\gamma} \left\lceil \log_2 \left(\frac{1}{\varepsilon} \right) \right\rceil N \exp \left\{ b \left(2^{1-\gamma} - 1 \right) (N+1)^\gamma \right\} . \tag{6.31}$$

It is known that the birth rates $\Phi_{L,N}(n) \to \phi_c = 1$ as $N \to \infty$, and therefore for large N the processes (6.22) and (6.24) could expect similar behaviour. However, convergence is only pointwise in n, and from Theorems 6.2.4 and 6.2.5 we see that the mixing and relaxation times exhibit different scaling forms. In Figure 6.1, we plot the birth and death rates for the processes (6.22) and (6.24) with rates (6.18) and (6.19). As discussed in Chapter 4, the canonical current $\Phi_{L,N}(n)$ is non-monotone, this gives rise to and effective potential well and metastable behaviour for the associated birth-death chain. This is not the case in the birth death processes (6.24), where the birth rate is constant. In this case, the process is only weakly driven[1] towards the right most boundary.

To calculate the bounds appearing in Theorem 6.2.5, we calculate the expected time $H_N(0)$ of the chain starting in site 0 and hitting site N. Heuristically, the birth death chain defined by projecting on the occupation of site one under the measure $\pi_{L,N}[\cdot]$ exhibits two 'wells' at $\rho_c L$ and $(\rho - \rho_c)L$ associated with the condensate being located on one of the $L-1$ sites and the condensate at site 1 (or the site we project onto) respectively. Then the time scale $H_N(0)$ can be thought of as

[1]A birth death process is weakly driven to the the right most boundary if $\alpha_n > \beta_n$ for all n sufficiently large and $a_n - \beta_n \to 0$ as $n \to \infty$.

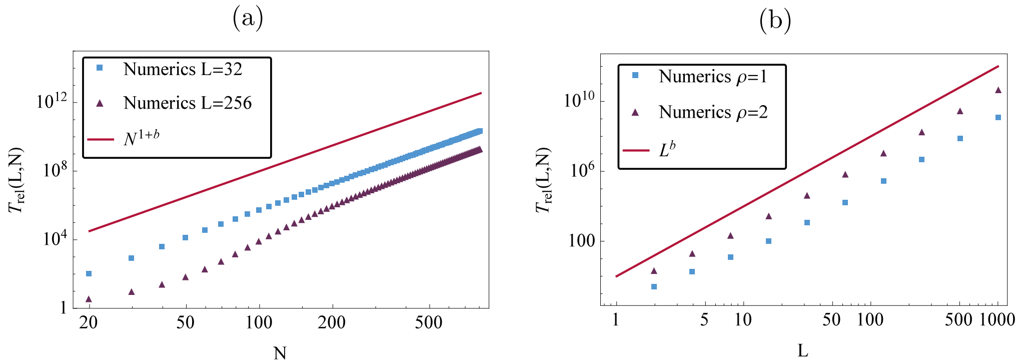

Figure 6.2: The relaxation time for the projection chain (6.22) with rates (6.18) with $b = 5$. Figure 6.2a: Fixed system size $L = 32$ and $L = 256$, and the predicted scaling form N^{1+b}. Figure 6.2b: Fixed density $\rho = 1 > \rho_c$ and $\rho = 2 > \rho_c$, and the scaling form L^b.

the time it takes for a condensate to transition to site 1 from one of the remaining sites. However, the well at $\rho_c L$ should be deeper than the well at site $(\rho - \rho_c)L$ since there is a larger probability, by a factor proportional to L, of having the condensate outside of site 1. The time scale of condensate leaving site 1 and moving into the background should be smaller then the reverse process. Therefore, we expect that $H_0(N)$ should give rise to the smaller hitting time and better bounds on the relaxation and mixing times. In Figure 6.2 we plot the relaxation times of the projection chain for fixed L varying N and for fixed ρ varying L, which suggests this intuition on the dynamics of the condensate is correct. However, for fixed L we appear to have the scaling N^{1+b} appearing in Theorem 6.2.5 correct up to a constant which depends on L. The correct scaling is expected to be given by N^{1+b}/L, which is consistent with the thermodynamic scaling showing in Figure 6.2b.

6.3 Proof of Theorem 6.2.4

6.3.1 Relaxation times and the spectral gap

First, consider case (i) of Theorem 6.2.4 where the death rates are given by (6.18). Define a basic coupling $(Z_t)_{t\geq 0} = (X_t, Y_t)_{t\geq 0}$ where particles move independently until the first hitting time, thereafter particles jump together according to the original generator (6.1). The coupled generator is given by

$$\begin{aligned}\widehat{\mathcal{L}}_N f(n,m) =& \alpha_n \left(f(n-1,m) - f(n,m)\right) + \beta_n \left(f(n+1,m) - f(n,m)\right) \\ &+ \alpha_m \left(f(n,m-1) - f(n,m)\right) + \beta_m \left(f(n,m+1) - f(n,m)\right)\end{aligned}$$

for $n \neq m$ and

$$\widehat{\mathcal{L}}_N f(n,n) = \alpha_n \left(f(n-1,n-1) - f(n,n)\right) + \beta_n \left(f(n+1,n+1) - f(n,n)\right) \ .$$

Let $F : \mathbb{Z}_N \times \mathbb{Z}_N \to \mathbb{R}$ be given by

$$F(n,m) = \begin{cases} (n+1)(m+1) & \text{if } n \neq m \\ 0 & \text{otherwise} \end{cases} .$$

With out loss of generality assume $n < m$, then the ratio (6.6) is given by

$$-\frac{\widehat{\mathcal{L}}_N F(n,m)}{F(n,m)} = \frac{1}{n+1}\left(\alpha_n - \beta_n\right) + \frac{1}{m+1}\left(\alpha_m - \beta_m\right) \tag{6.32}$$

if $|m-n| > 1$, and

$$-\frac{\widehat{\mathcal{L}}_N F(n,m)}{F(n,m)} = \frac{1}{n+1}\alpha_n + \beta_n + \alpha_m - \frac{1}{m+1}\beta_m \tag{6.33}$$

if $|m-n| = 1$. For $\alpha_n = g_1(n)$, whenever $n \geq 1$ we have $\alpha_n > \beta_n = 1$, and since α_n is decreasing for all $n > 0$ if follows that (6.32) is decreasing in n and m. Moreover, since $\alpha_n \to 1$ as $n \to \infty$, it is easy to see (6.32) tends to 0 as $n, m \to \infty$. In addition, since $0 < n < m$ we have $\frac{1}{n+1}\alpha_n - \frac{1}{m+1}\beta_m > 0$ which implies, (6.33) is strictly bounded above by $\alpha_n + \beta_m > 1$. Therefore, the minimum appearing in (6.6) is obtained when $|n-m| > 1$. Since $\alpha_m - \beta_m > 0$ for all $m \neq 0$, we can bound (6.32) below by

$$-\frac{\widehat{\mathcal{L}}_N F(n,m)}{F(n,m)} \geq \frac{1}{n+1}\left(\alpha_n - \beta_n\right) \ \text{ for all } \ n, m \in \mathbb{Z}_N \ ,$$

which implies

$$\min_{\substack{(n,m)\in\mathbb{Z}_N\times\mathbb{Z}_N \\ n\neq m}} \frac{-\widehat{\mathcal{L}}_N F(n,m)}{F(n,m)} \geq \min_{n \leq N-1} \frac{1}{n+1}(\alpha_n - \beta_n) \ . \tag{6.34}$$

To complete this part of the proof, we need the following lower bounds on the death rate $\alpha_n = g_1(n)$

$$\alpha_n = g_1(n) = \left(1 - \frac{1}{n}\right)^{-b} > 1 + \frac{b}{n} \ ,$$

and therefore by Theorem 6.2.1

$$\lambda_N \geq \frac{b}{N(N-1)} > \frac{b}{N^2} \ .$$

We need to find a matching upper bound for the spectral gap, which can be found by substituting a good test function into the variational principle (6.4). Let $f(n) = (n+1)^b$, the variance is given by

$$\begin{aligned}\mathrm{Var}_N(f) &= \frac{1}{Z_N}\sum_{n=0}^{N}(n+1)^{2b}(n+1)^{-b} - \left(\frac{1}{Z_N}\sum_{n=0}^{N}(n+1)^b(n+1)^{-b}\right)^2 \\ &= \frac{1}{Z_N}\sum_{n=0}^{N}(n+1)^b - \left(\frac{1}{Z_N}\sum_{n=0}^{N}1\right)^2 \ .\end{aligned}$$

Since $(n+1)^b$ is increasing, the variance of f is bounded below by

$$\begin{aligned}\mathrm{Var}_N(f) &\geq \frac{1}{Z_N}\left(1 + \int_0^N (1+x)^b dx\right) - \left(\frac{N+1}{Z_N}\right)^2 \\ &= \frac{1}{Z_N}\frac{1}{b+1}\left(b + (N+1)^{b+1}\right) - \left(\frac{N+1}{Z_N}\right)^2 \ . \end{aligned} \tag{6.35}$$

The Dirichlet form is given by

$$\begin{aligned}\mathcal{D}_N(f) &= \frac{1}{2}\sum_{n=1}^{N} g_1(n)\mu_N[n]\left(n^b - (n+1)^b\right)^2 \\ &\leq \frac{1}{2}\sum_{n=1}^{N} g_1(n)\mu_N[n]\left(2b(n+1)^{b-1}\right)^2 \ ,\end{aligned}$$

where we have bounded the term appearing inside the square as follows

$$\begin{aligned}(n+1)^b - n^b &= (n+1)^b\left(1 - \left(\frac{n}{n+1}\right)^b\right) = (n+1)^b\left(1 - \left(1 + \frac{1}{n}\right)^{-b}\right) \\ &\leq (n+1)^b\left(1 - \left(1 - \frac{b}{n}\right)\right) = b\frac{(n+1)^b}{n} \leq 2b(n+1)^{b-1} \ .\end{aligned}$$

Substituting the form of the invariant measure into the mass function (6.2), we have

$$\mathcal{D}_N(f) \leq \frac{4b^2}{Z_N} \sum_{n=1}^{N} g_1(n)(n+1)^{b-2} \ .$$

Now since $g_1(n) = \left(\frac{n}{n+1}\right)^{-b}$ is decreasing for $n \geq 1$ with a maximum of $g(1) = 2^b$, we get the following upper bound

$$\mathcal{D}_N(f) \leq 2^b \frac{4b^2}{Z_N} \int_1^{N+1} (x+1)^{b-2} dx = 2^b \frac{4b^2}{Z_N} \frac{1}{b-1} \left((N+2)^{b-1} - 2^{b-1}\right) \ . \quad (6.36)$$

Combining (6.35), (6.36), and knowing Z_N has a finite limit as $N \to \infty$, the spectral gap is bounded above by

$$\lambda_N \leq c_b \frac{1}{(N+1)^2} (1 + o(1)) \ , \quad (6.37)$$

for some constant $c_b > 0$.

For case (ii) of Theorem 6.2.4, we follow a different approach since the lower bound (6.34) fails to give the correct scaling form for the spectral gap. Recall the functions $B_+(m)$ and $B_-(m)$ appearing in Theorem 5.3.3

$$B_+(m) = \sup_{x>m} \left(\sum_{y=m+1}^{x} \frac{1}{\mu_N[y] a_y} \right) \left(\sum_{y \geq x} \mu_N[y] \right) \ ,$$
$$B_-(m) = \sup_{x<m} \left(\sum_{y=x}^{m-1} \frac{1}{\mu_N[y] b_y} \right) \left(\sum_{y \leq x} \mu_N[y] \right) \ ,$$

where m is the median of the invariant measure μ_N. First, we approximate the function $B_+(m)$. Define

$$\begin{aligned} B_+(m,x) &:= \left(\sum_{y=m+1}^{x} \frac{1}{\mu_N[y] a_y} \right) \left(\sum_{y \geq x} \mu_N[y] \right) \\ &= \left(\sum_{y=m}^{x-1} e^{y^\gamma} \right) \left(\sum_{y \geq x} e^{-y^\gamma} \right) \ , \end{aligned}$$

where in the last step we simplified the equation by cancelling the partition functions Z_N and used the reversibility of the measure μ_N. Approximating the series by an integral, we have

$$\sum_{y=x}^{N} e^{-y^\gamma} \asymp N \int_{\hat{x}}^{1} e^{-N^\gamma \hat{y}^\gamma} d\hat{y}$$

where $\hat{x} = x/N$. Taylor expanding $\hat{y}^\gamma$ about $\hat{y} = 1$ we have

$$N \int_{\hat{x}}^{1} e^{-N^\gamma \hat{y}^\gamma} d\hat{y} \asymp N \int_{\hat{x}}^{1} e^{-N^\gamma (1+\gamma(\hat{y}-1))} d\hat{y} = \frac{1}{\gamma} N^{1-\gamma} e^{-N^\gamma} \left(e^{-\gamma N^\gamma (\hat{x}-1)} - 1 \right) .$$

By the same method, we have

$$\sum_{y=m}^{x-1} e^{y^\gamma} \asymp N \int_{m/N}^{\hat{x}-1/N} e^{N^\gamma (1+\gamma(\hat{y}-1))} d\hat{y} = \frac{1}{\gamma} N^{1-\gamma} \left(e^{N^\gamma \left(1+\left(x-1-\frac{1}{N}\right)\gamma\right)} - e^{N^\gamma \left(\left(\frac{m}{N}-1\right)\gamma-1\right)} \right) .$$

The function $B_+(m,x)$ is then approximated by

$$B_+(m,x) \asymp \frac{N^{2(1-\gamma)}}{\gamma^2} e^{-N^\gamma} \left(e^{N^\gamma \left(1+\left(x-1-\frac{1}{N}\right)\gamma\right)} - e^{N^\gamma \left(\left(\frac{m}{N}-1\right)\gamma-1\right)} \right) \left(e^{-\gamma N^\gamma (\hat{x}-1)} - 1 \right) ,$$

which is maximised when $\hat{x} = \frac{1+m+n}{2n}$. Therefore,

$$B_+(m) \asymp \frac{N^{2(1-\gamma)}}{\gamma^2} C_{N,m} ,$$

where

$$C_{N,m} = e^{-\gamma N^{\gamma-1}} \left(1 - e^{\frac{\gamma}{2}(1+m-N)N^{\gamma-1}} \right)^2 .$$

We have that $C_{N,m} \to 1$ as $N \to \infty$ since $\gamma \in (0,1)$ and $m < N$, *i.e.* $e^{-\gamma N^{\gamma-1}} \to 1$ and $e^{\frac{\gamma}{2}(1+m-N)N^{\gamma-1}} \to 0$ as $N \to \infty$. Therefore, the function $B_+(m)$ has the approximate form

$$B_+(m) \asymp \frac{N^{2(1-\gamma)}}{\gamma^2} .$$

To complete the proof, we must show that the scaling form of $B_-(m)$ is not larger than $N^{2(1-\gamma)}$. Since the measure μ_N is sub-exponential, the median is bounded above by a constant $C_\gamma > 0$ which depends on the parameter γ. Hence, the maximum appearing in the definition of $B_-(m)$ is over a finite interval and the function being maximised is independent of the system size N. Therefore, $B_-(m)$ is bounded above by a constant independent of N and by Theorem 5.3.3 there exist constants $c_\gamma, c'_\gamma > 0$ such that the spectral gap is bounded above and below by

$$c_\gamma N^{-2(1-\gamma)} \leq \lambda_N \leq c'_\gamma N^{-2(1-\gamma)} . \tag{6.38}$$

This completes the proof of the relaxation time statements in Theorem 6.2.4.

6.3.2 Mixing times via hitting times

Heuristically, since the birth death processes (6.24) with rates (6.18) and (6.19) exhibit drift towards the left boundary (see Figure 6.1) the expected time to hit site N from 0 should be larger than the expected time to hit 0 from N. This can be made rigorous by the coupling argument constructed in Section 6.2

First, consider the process with death rates (6.18), where the stationary measure μ_N has power law tails. The expected hitting time of the chain starting in site N and hitting site 0 is given by

$$\begin{aligned} H_0(N) &= \sum_{n=0}^{N-1} \frac{1}{\alpha_{N-n}\mu_N[N-n]} \sum_{k=0}^{n} \mu_N[N-k] \\ &= \sum_{n=0}^{N-1} \frac{1}{w(N-n-1)} \sum_{k=0}^{n} w(N-k) \ , \end{aligned}$$

where in the last step we used detailed balance and simplified the equation by cancelling the partition functions. Since the stationary weights satisfy $w(n) = (n+1)^{-b}$ we have

$$\begin{aligned} H_0(N) &= \sum_{n=0}^{N-1} (N-n)^b \sum_{k=0}^{n} (N-k+1)^{-b} \\ &= \sum_{n=0}^{N-1} (N-n)^b \sum_{k=N-n}^{N} (k+1)^{-b} \ . \end{aligned}$$

The term in the second summation is decreasing, so we have the upper bound

$$\begin{aligned} H_0(N) &\leq \sum_{n=0}^{N-1} (N-n)^b (N-n+1)^{-b} (n+1) \\ &\leq \sum_{n=0}^{N-1} (n+1) = \frac{1}{2} N(N+1) \leq \frac{1}{2}(N+1)^2 \ . \end{aligned}$$

Therefore, by (6.12) we have

$$d_N\left(\frac{\beta}{2}(N+1)^2\right) \leq \frac{1}{\beta}$$

for all $\beta > 0$, which implies

$$T_N^{\text{mix}}(\varepsilon) \le \frac{1}{2\varepsilon}(N+1)^2 \ .$$

The ε dependence can be improved by the inequality

$$T_N^{\text{mix}}(\varepsilon) \le \left\lceil \log_2\left(\frac{1}{\varepsilon}\right)\right\rceil T_N^{\text{mix}}(1/4) \tag{6.39}$$

as derived in [44, Section 4.5]. This bound is sharp in N since for all $\varepsilon > 0$

$$T_N^{\text{mix}}(\varepsilon) \ge \log\left(\frac{1}{2\varepsilon}\right)\left(T_N^{\text{rel}} - 1\right) \ , \tag{6.40}$$

see for example [44, Theorem 12.4], where a lower bound for T_N^{rel} is give by (6.37).

Now, consider the process with death rates (6.19), where the stationary measure μ_N has stretched exponential tails. The expected hitting time of the chain starting in site N and hitting 0 is given by

$$H_0(N) = \sum_{n=0}^{N-1} e^{(N-n)^\gamma} \sum_{k=0}^{n} e^{-(N-k)^\gamma} \ . \tag{6.41}$$

Setting $x = n/N$, we can approximate the first series as the integral

$$\sum_{k=0}^{n} e^{-(N-k)^\gamma} \asymp N \int_0^x e^{-N^\gamma(1-y)^\gamma} dy \ .$$

Expanding $-(1-y)^\gamma$ about $y = 0$ to first order we have

$$N \int_0^x e^{-N^\gamma(1-y)^\gamma} dy \asymp N \int_0^x e^{N^\gamma(\gamma y - 1)} dy = \frac{e^{-N^\gamma} N^{1-\gamma}}{\gamma}\left(e^{\gamma x N^\gamma} - 1\right) \ .$$

We can now approximate the expected hitting time as the integral

$$H_0(N) \asymp N \frac{e^{-N^\gamma} N^{1-\gamma}}{\gamma} \int_0^{1-1/N} e^{N^\gamma(1-x)^\gamma}\left(e^{\gamma x N^\gamma} - 1\right) dx \ .$$

Again, expanding $(1-x)^\gamma$ about $x = 0$ to first order and integrating we have

$$H_0(N) \asymp \frac{N^{2-\gamma}}{\gamma} - \frac{N^{2-2\gamma}}{\gamma^2} - \frac{N^{1-\gamma}}{\gamma} + \frac{e^{N^{\gamma-1}(\gamma - \gamma N + N) - N^\gamma} N^{2-2\gamma}}{\gamma^2} \ . \tag{6.42}$$

For $\gamma \in (0, 1)$ the leading order term is $N^{2-\gamma}$, which implies that there exists a constant $c'_\gamma > 0$ such that

$$H_0(N) \leq c'_\gamma N^{2-\gamma} \ , \tag{6.43}$$

and therefore

$$T_N^{\text{mix}}(\varepsilon) \leq \frac{c'_\gamma}{\varepsilon} N^{2-\gamma} \ . \tag{6.44}$$

The ε dependence can be improved by (6.39). To complete the proof, we must bound the 1/4-mixing time below by a function which scales like $N^{2-\gamma}$. By Theorem 6.2.3, the 1/4-mixing time is bounded below by

$$T_N^{\text{mix}}(1/4) \geq \max_{x \in \mathbb{Z}_N} H_A(x) \tag{6.45}$$

for some set A such that $\mu_N[A] \geq 1/2$. Since the measure μ_N is sub-exponential and we are trying to bound the maximum over sets $A \subseteq \mathbb{Z}_N$, we can choose the set A to be of the form $\{0, \ldots, m_\gamma\}$ for some $m_\gamma \in \mathbb{Z}_N$, which depends on the parameter γ. Clearly, the maximum appearing in (6.45) is given when $x = N$, so we have

$$T_N^{\text{mix}}(1/4) \geq H_{m_\gamma}(N) \ . \tag{6.46}$$

Repeating the same computations as before, it is clear that there exists a constant $C_\gamma > 0$ such that

$$T_N^{\text{mix}}(1/4) \geq C_\gamma N^{2-\gamma}$$

completing the proof of Theorem 6.2.4.

6.4 Proof of Theorem 6.2.5

6.4.1 Upper bounds via hitting times

First, consider the birth death processes (6.22), the single site projection of the (homogeneous) zero-range process with rates

$$g(n) = \left(\frac{n}{n+1}\right)^{-b} \text{ for } n \in \{1, 2, \ldots\} \ . \tag{6.47}$$

The expected hitting time of the birth death process starting in site 0 and ending at N is given by

$$H_N(0) = \sum_{n=0}^{N-1} \frac{\sum_{k=0}^n \mu_{L,N}[k]}{\Phi_{L,N}(n)\mu_{L,N}[n]} = \sum_{n=0}^{N-1} \frac{\sum_{k=0}^n w(k) Z_{L-1,N-k}}{\Phi_{L,N}(n) w(n) Z_{L-1,N-n}} \ .$$

The inner summation can be bounded above by the partition function $Z_{L,N}$, so we have

$$H_N(0) \le \sum_{n=0}^{N-1} \frac{Z_{L,N}}{\Phi_{L,N}(n)w(n)Z_{L-1,N-n}} = \sum_{n=1}^{N} \frac{Z_{L,N}}{g(n)w(n)Z_{L-1,N-n}} ,$$

where in the last step we used detailed balance. Since $g(n) > 1$ for all $n \ge 1$, we find

$$\sum_{n=1}^{N} \frac{Z_{L,N}}{g(n)w(n)Z_{L-1,N-n}} \le Z_{L,N} \sum_{n=1}^{N} \frac{1}{w(n)Z_{L-1,N-n}} .$$

Now, taking the maximum in the above summation we get

$$\begin{aligned} Z_{L,N} \sum_{n=1}^{N} \frac{1}{w(n)Z_{L-1,N-n}} &\le N Z_{L,N} \max_{1\le n\le N} \frac{1}{w(n)Z_{L-1,N-n}} \\ &\le N Z_{L,N} \max_{1\le n\le N} \frac{1}{w(n)} \max_{1\le n\le N} \frac{1}{Z_{L-1,N-n}} \\ &= N Z_{L,N}(N+1)^b \max_{1\le n\le N} \frac{1}{Z_{L-1,N-n}} . \end{aligned}$$

Since we know $Z_{L,N}/w(N) \to Lz(1)^{L-1}$ as $N \to \infty$ for each $L \ge 2$, see Section 3.4.1 for more details, there exists a constant $C_b > 0$ such that

$$\begin{aligned} N Z_{L,N}(N+1)^b \max_{1\le n\le N} \frac{1}{Z_{L-1,N-n}} \le& C_b N(N+1)^b Lz(1)^{L-1}(N+1)^{-b}(1+o_N(1)) \\ &\times \left(\frac{1}{(L-1)z(1)^{L-2}(N+1)^{-b}(1+o_N(1))} \right) , \end{aligned}$$

which is bounded above by

$$\frac{L}{L-1} z(1)C_b(N+1)^{1+b}(1+o_N(1)) \le c_b'(N+1)^{1+b}(1+o_N(1)) .$$

Therefore, by (6.12) we have

$$d_{L,N}\left(\beta c_b'(N+1)^{1+b}(1+o_N(1))\right) \le \frac{1}{\beta} \tag{6.48}$$

for all $\beta > 0$, which implies

$$T^{\mathrm{mix}}_{L,N}(\varepsilon) \le \frac{1}{\varepsilon} c_b'(N+1)^{1+b}(1+o_N(1)) . \tag{6.49}$$

Following the same approach we have the following bound

$$T^{\mathrm{mix}}_{L,N}(\varepsilon) \leq \frac{1}{\varepsilon} c'_{b,\gamma} N \exp\left\{ b\left(2^{1-\gamma} - 1\right)(N+1)^{\gamma} \right\} \ , \tag{6.50}$$

for the projection chain with rates (6.19).

The ε dependence in both bounds can be improved by (6.39).

6.5 Conclusion

In this chapter, we have calculated the relaxation and mixing times for a class of projection chains arising from homogeneous condensing zero-range processes. From Theorem 6.2.5, the mixing times (and therefore relaxation times) of the projection chain are consistent with the metastable time scales of the motion of the condensate [17, 18]. Since this metastable motion of the condensate is heuristically the slowest mode in the system, this time scale is expected to give the scaling form of the relaxation and mixing times. To find the relaxation time for the full zero-range process we would need to follow the analysis outlined in Chapter 5. This method is not expected to work here since it typically relies on having the restriction process reach a stationary state on a time scale which is faster than the dynamics of the projection $\eta_1(t)$, which is not necessarily true in this case. Alternatively, one could consider projections which are more natural to the underlying process. For example, the decomposition of the state space into a disjoint union of metastable wells given in [18]. However, this method would require sharp estimates of the relaxation time of the zero-range process restricted to a metastable well, and current known estimates are not expected to be sharp.

The replacement of the canonical current with the grand canonical current (6.24) in Theorem 6.2.4 leads to a different scaling of the relaxation time and mixing times. Whilst the dynamics of this birth-death chain are not consistent with the full zero-range process they are used to give crude bounds for the mixing and relaxation times of the zero-range process restricted to a metastable well [18]. In the metastable well the location of the condensate is known and the distribution of background sites is well approximated by the maximal invariant measure ν_{ϕ_c}.

CHAPTER 7

Conclusion

In this chapter, we summarise the main results contained in this thesis and give a short summary of possible future research.

In Chapter 3, we discuss various definitions of condensation in terms of the stationary measure of the underlying process in both the thermodynamic limit and on finite lattices. Whilst most rigorous results on condensation require the stationary measure to be a (conditional) product measure, the definitions we present are general, up to spatial homogeneity of the process. In [64] it was shown that condensation can occur on finite lattices if the tails of the single site marginals decay as a power law. The main result of Chapter 3 is to generalise this to prove the equivalence with a class of measures with a sub-exponential tail, which includes single site marginals that have a stretched exponential tail, that has been well studied in the literature on condensing particle systems. One interesting feature of the class of sub-exponential tails is that they do not require the stationary measure to have a finite first moment, which is a necessary condition for condensation in the thermodynamic limit. This implies that processes can condense on finite lattices but not in the thermodynamic limit. Condensation can also occur in the thermodynamic limit but not on finite lattices if the tails of the single site marginal do not satisfy certain regularity conditions, and we present such an example in Section 3.5. For processes that do not exhibit stationary (conditional) product measures, rigours results are difficult to obtain and what is known is obtained from heuristic methods and simulation. In Section 3.8, we study the chipping model which was first introduced in [84]. By mapping the process on two sites to a 1-D random walk with resetting, we prove that condensation occurs on fixed finite lattices according to Definition 3.2.3, and furthermore, we calculate the critical density which is different to the predicted critical density in the thermodynamic limit. This heuristically implies that the critical density can be dependent on the system size, which is not the case for processes that exhibit stationary (conditional) product measures.

In Chapter 4, we study the monotonicity properties of condensing stochastic particle systems. Couplings, which require monotonicity, are powerful tools for deriving hydrodynamic limits, relaxation times, and mixing times of Markov processes. Previous results on condensing zero-range processes show non-monotonicity by an overshoot of the canonical current, which results in a metastable hysteresis as the process switches from condensed to non-condensed regimes [67, 16]. We prove non-monotonicity for all condensing processes that exhibit stationary product measures with a finite critical density, where we rely on properties of sub-exponential distributions discussed in Chapter 3. If the critical density is infinite, we are not able to prove that all processes with stationary weights of the form $w(n) \asymp n^{-b}$ for $b \in (1, 3/2)$ are non-monotone and in fact there exists a monotone example recently found in [54]. For processes that do not exhibit stationary product measures condensing processes can be monotone, and the chipping model introduced in Section 3.8, is such an example.

In Chapter 5, we study a zero-range process that exhibits condensation due to single site defects under complete graph dynamics. By a decomposition of the state space first developed in [32] for the Kawasaki Ising model and for a zero-range process in [87], we calculate the relaxation time for two condensing zero-range processes. We decompose the state space by conditioning on the value of a defect site, which allows us to compare the relaxation time of the full zero-range process with a constant rate zero-range process and a projection chain that jumps between the disjoint sets in the partition under some average rate of the dynamics. For the zero-range process with one defect, below the critical density, the projection chain exhibits drift towards the left most boundary resulting in a size independent relaxation time. Above the critical density, the projection is driven towards a value which scales as the system size and we show the relaxation time scales linearly with the system size. This transition in the relaxation time for the projection chain leads to a transition in the relaxation time of the full zero-range process. For the zero-range process with two defects, above the critical density, the process behaves like a symmetric random walk where the relaxation time scales like the square of the system size resulting in a different transition in the relaxation time of the full process. Heuristically, the dynamics of the defect site (the typical maximum) is the slowest mode in the system and therefore, the mixing time of the projection chain should be similar to the mixing time of the zero-range process. We validate this heuristic by numerically calculating by simulation the coupling time of the full zero-range process and the projection chain. The process with one defect appears to show a cutoff at $c_\rho L \log(L)$ with a window of order L for some constant $c_\rho > 0$.

The process with two defects does not appear to show a cutoff and the mixing time scales like the relaxation time. A possible approach to prove this claim would be to follow the method outlined in the sequence of papers [38, 40] by first showing that before the mixing time the transition kernal of the process can be well approximated by the kernel of the projection chain.

In Chapter 6, we calculate the mixing and relaxation times for the projection chain given in Chapter 5 for condensing homogeneous zero-range processes. The bounds we compute are consistent with the metastable time scales of the motion of the condensate given in [17, 18], which is heuristically the slowest mode in the system.

There exist several open and interesting problems which follow from the work done in this thesis. In Chapter 3, we focus on processes that exhibit stationary product measures, however, there has been recent work on condensing processes where the stationary measure is pair-factorised or cluster factorised [72, 97, 99, 98]. It would be interesting to adapt Definition 3.2.3 of condensation for pair and cluster factorised measures, similar to the equivalence of the definition with sub-exponential measures for processes with stationary product measures. Difficulties may arise since the condensate can be spatially extended (see for example [99]), which is typically not the case for processes with stationary product measures. The sharp bounds for the relaxation time of the projection chain of a homogeneous zero-range process we give in Chapter 6 could be used to calculate a sharp bound for the relaxation time of the full process. However, it is clear that the method outlined in Chapter 5 is not enough and better bounds are required when comparing Dirichlet forms. An alternative approach is to consider a different partition by conditioning on the location of the condensate, which is more natural to the process. So far, only suboptimal bounds on the relaxation time for the zero-range process restricted to a metastable well are known and given in [18]. Further difficulties may arise when calculating the relaxation time restricted outside the metastable wells and when comparing Dirichlet forms.

APPENDIX A

Numerical methods

A.1 Numerics

In the following, we summarise methods of numerically calculating expectations of relevant test functions with respect to a conditional product measure with mass function

$$\pi_{L,N}[\eta] = \frac{1}{Z_{L,N}} \prod_{x\in\Lambda} w(\eta_x) \mathbb{1}(\eta \in \Omega_{L,N}) \ , \tag{A.1}$$

where $\Lambda = \{1, \ldots, L\}$ and

$$\Omega_{L,N} = \{\eta \in \mathbb{N}^\Lambda \, : \, \sum_{x\in\Lambda} \eta_x = N\} \ .$$

In Chapter 4, we provide numerics for the expectations of the test functions

$$f_1(\eta) = \mathbb{1}\left(\eta_1 = \ldots = \eta_{L-1} = 0\right) \tag{A.2}$$

and the background density

$$f_2(\eta) = \frac{N - M_L(\eta)}{L-1} \ , \tag{A.3}$$

where

$$M_L(\eta) = \max_{x\in\Lambda} \eta_x$$

is the maximum occupation number.

A simple computation shows that the expectation of (A.2) with respect to $\pi_{L,N}$ is given by

$$\pi_{L,N}\left(f_1(\eta)\right) = \frac{w(0)^{L-1} w(N)}{Z_{L,N}} \ . \tag{A.4}$$

The partition function (normalisation) $Z_{L,N}$ satisfies

$$Z_{L,N} = \sum_{n=0}^{N} w(n) Z_{L-1,N-n} \ .$$

Therefore, we can calculate the partition function recursively with an initial condition $Z_{1,n} = w(n)$ for all $n \in \mathbb{N}$. For systems of size $L = 2^n$, for some $n \in \mathbb{N}$ the partition function can be written as

$$Z_{L,N} = \sum_{n=0}^{N} Z_{L/2,n} Z_{L/2,N-n} \ ,$$

which gives rise to an efficient method of numerically calculating the partition function and (A.4) for large systems.

Recursive methods can also be used to numerically calculate the expectation of the maximum occupation number and therefore, the background density

$$R_L^{\mathrm{bg}}(N) := \pi_{L,N}\left(f_2(\eta)\right) = \frac{1}{L-1} \pi_{L,N}\left(N - M_L(\eta)\right) \ .$$

First, we consider the cumulative distribution function of the maximum occupation number

$$\pi_{L,N}\left[M_L \leq M\right] = \frac{1}{Z_{L,N}} Q_{L,N,M} \ .$$

As for the partition function, the function $Q_{L,N,M}$ can be computed recursively since

$$Q_{L,N,M} = \sum_{n=0}^{M} w(n) Q_{L-1,N-n,M} \ ,$$

and again for $L = 2^n$ for some $n \in \mathbb{N}$ we can write

$$Q_{L,N,M} = \sum_{n=0}^{M} Q_{L/2,n,M} Q_{L/2,N-n,M} \ .$$

The initial condition of the recursion is given by $Q_{1,n,M} = w(n)$ for $n \leq M$ and $Q_{1,n,M} = 0$ for all $n > M$. The expectation of $M_L(\eta)$ with respect to $\pi_{L,N}$ is given by

$$\pi_{L,N}\left(M_L(\eta)\right) = \sum_{n=0}^{N} n\pi_{L,N}\left[\{M_L(\eta) = n\}\right] = \frac{1}{Z_{L,N}} \sum_{n=0}^{N} n\left[Q_{L,N,n} - Q_{L,N,n-1}\right]$$
$$= N - \frac{\sum_{n=0}^{N-1} Q_{L,N,n}}{Z_{L,N}} .$$

A.2 Sampling from $\pi^{\Delta}_{\Lambda,N}[\eta]$

Consider $(\eta(t), \zeta(t))_{t\geq 0}$ a coupling of the defect site zero-range process introduced in Chapter 5. To calculate the coupling time, we must initialise ζ according to the stationary measure $\pi^{\Delta}_{\Lambda,N}$. In this section, we review a method of sampling perfectly from $\pi^{\Delta}_{\Lambda,N}$ without simulating the zero-range dynamics.

First, consider the case when $\Delta = \emptyset$, *i.e.* no defect sites and $\pi^{\emptyset}_{\Lambda,N}$ is the uniform measure on $\Omega_{L,N}$. To sample from $\pi^{\emptyset}_{\Lambda,N}$, first initialise a configuration $\zeta = (0, \ldots, 0)$ the zero vector of length L. Then, whilst the number of particles is less than N, add a particle to site $x \in \Lambda$ with probability proportional to $\zeta_x + 1$. It is easy to show, after N steps the configuration ζ will be uniformly distributed on $\Omega_{L,N}$. This algorithm can be interpreted as a system of L independent continuous time pure-birth chains $\zeta(t) = (\zeta_x(t))_{x\in\Lambda}$, each corresponding to a site $x \in \Lambda$ with birth rate $\zeta_x + 1$. Starting from the configuration $\zeta(0) = (0, \ldots, 0)$ and stopping when $\sum_{x\in\Lambda} \zeta_x(t) = N$ gives a uniform sample from $\Omega_{L,N}$.

Now consider the case when $\Delta \neq \emptyset$. Let $\zeta(t) = (\zeta_x(t))_{x\in\Lambda}$ be a sequence of pure-birth chains time dependent rates

$$\zeta_x \to \zeta_x + 1 \text{ at rate } \begin{cases} \zeta_x + 1 & \text{where } x \notin \Delta \\ h(t)(\zeta_x + 1) & \text{where } x \in \Delta \end{cases} .$$

Since we have a sequence of pure-birth chains, the distribution of $\zeta_x(t)$ at time t can be computed explicitly to find

$$\mathbb{P}[\zeta_x(t) = n] = \begin{cases} e^{-t}\left(1 - e^{-t}\right)^n & \text{for } x \notin \Delta \\ e^{-H(t)}\left(1 - e^{-H(t)}\right)^n & \text{for } x \in \Delta \end{cases} \tag{A.5}$$

where $H(t) = \int_0^t h(s)ds$. To ensure that we have a sample from $\pi^{\Delta}_{\Lambda,N}$ when $\sum_{x\in\Lambda} \zeta_x(t) = N$ we can compare (A.5) with the grand-canonical single site marginal given by (5.4), *i.e.*

$$\nu_\phi^x[n] = \begin{cases} (1-\phi)\phi^n & \text{for } x \notin \Delta \\ \left(1-\frac{\phi}{r}\right)\left(\frac{\phi}{r}\right)^n & \text{for } x \in \Delta \end{cases} . \tag{A.6}$$

This gives the solution

$$H(t) = -\log\left(1 - \frac{1-e^{-t}}{r}\right) \implies h(t) = \frac{1}{1+(r-1)e^t} . \tag{A.7}$$

Notice that $h(t)$ and $H(t)$ diverge as $t \to -\log(1-r)$, which implies that the number of particles added to the defect site will diverge whilst the number in the background will remain finite in the time window $[0, -\log(1-r))$. In Algorithm 1, we describe how to simulate the sequence of independent pure-birth chains with time dependent rates. For simplicity, we consider the case when $\Delta = \{L\}$, *i.e.* one defect placed at the last site.

A.3 Simulation methods

All simulations throughout this thesis are performed with a Gillespie type algorithm [108], which produces statistically correct trajectories of the Markov process. Here, we describe how to calculate the background density or the coupling time for the chipping model and zero-range process respectively.

A.3.1 The chipping model

In Algorithm 2, we describe the Gillespie algorithm for calculating the background density for the chipping model defined in Section 3.8 with L sites, N particles on the complete graph, *i.e.* $p(x,y) = \frac{1}{L-1}$ for all $n \neq y$.

Each update in the algorithm has complexity $\mathcal{O}(L)$, since in each time step we must update the partial sums C_n for $n \in \{\min\{i,j\},\dots,L\}$ and calculate the maximum occupation $M_L(\eta) = \max\{\eta_1,\dots,\eta_L\}$. The complexity can be reduced to $\mathcal{O}(\log(L))$, at the expense of memory, by storing a binary tree. For simplicity, consider lattices of length $L = 2^n$ for some $n \in \mathbb{N}$, then the binary tree is defined recursively by $C_{i,j} = C_{i-1,2j-1} + C_{i-1,2j}$ for $i \in \{0,\dots n\}$ and $j \in \{1,\dots,2^{n-i}\}$ and initial conditions $C_{0,j} = c_j$ for $j \in \{1,\dots,L\}$. Now the updates to the binary tree can be performed by retracing the path of the binary search to select the transition site, which has complexity $\mathcal{O}(\log(L))$. Furthermore, the maximum value can be updated by comparing the prior maximum and the occupation of the entry site

Algorithm 1 Growth algorithm to sample from $\pi^{\Delta}_{\Lambda,N}$ the stationary measure of the defect site zero-range process on the complete graph with L sites, N particles, and Δ a set of defect sites. The Algorithm outputs a configuration sampled from $\pi^{\Delta}_{\Lambda,N}$.

Require: A vector ζ of length L
Require: List of $L-1$ jump rates for each site in the current state $(c_i)_{i=1}^{L-1}$
Require: List of partial sums $C_n = \sum_{i=1}^{n} c_i$ with $C_0 = 0$
  {Initialise ζ, the jump rates, partial sums, and time}
  $\zeta \leftarrow (0, \dots 0)$
  $c_i = 1$ for all $i \in \{1, \dots, L-1\}$
  $C_n = n$ for all $n \in \{1, \dots L-1\}$
  $t \leftarrow 0$
  {Draw two exponential random variables with distribution $\text{Exp}(C_{L-1})$ and $H^{-1}(\text{Exp}(\zeta_L + 1))$}
  $E_1 \leftarrow$ Exponential random variable with mean $1/C_{L-1}$
  $p \leftarrow$ Exponential random variable with mean $1/(\zeta_L + 1)$
  $E_2 \leftarrow H^{-1}(p)$
  for each $i \in \{1, \dots, N\}$ **do**
    if $E_1 < E_2$ **then**
      {Add a particle to a non defect site with probability c_i/C_{L-1}}
      $p \leftarrow$ Uniform random number on $[0, C_{L-1})$
      Perform binary search for i such that $C_{i-1} \leq p < C_i$
      $\zeta_i = \zeta_i + 1$
      {Update birth-rates, partial sums, time, and draw time of next event}
      $c_i \leftarrow c_i + 1$
      Update C_n for $n \in \{i, \dots L-1\}$
      $t \leftarrow E_1$
      {Draw an exponential random variable with distribution $\text{Exp}(C_{L-1})$}
      $p \leftarrow$Exponential random variable with mean $1/C_{L-1}$
      $E_1 \leftarrow t + p$
    else
      {Add a particle to the defect site}
      $\zeta_L \leftarrow \zeta_L + 1$
      {Update time and draw time of next event}
      $t \leftarrow E_2$
      $p \leftarrow$ Exponential random variable with mean $1/(\zeta_L + 1)$
      $\tilde{E} \leftarrow H(E_2)$
      $\tilde{E} \leftarrow \tilde{E} + p$
      $E_2 \leftarrow H^{-1}(\tilde{E})$
    end if
  end for
  return ζ

Algorithm 2 Gillespie update algorithm for the chipping model with N particles on a lattice of L sites. Algorithm calculates the background density $R_L^{bg}(N)$ for the process.

Require: A vector η of length L
Require: List of L jump rates for each site in the current state $(c_i)_{i=1}^L$ where $c_i = (1+w)\delta(\eta_i > 0)$
Require: List of partial sums $C_n = \sum_{i=1}^n c_i$ with $C_0 = 0$
  {Sample time increment from $\mathrm{Exp}(C_L)$}
  $dt \leftarrow$ Exponential random variable with mean $1/C_L$
  $t \leftarrow t + dt$
  {Choose exit site with probability c_i/C_L}
  $p \leftarrow$ Uniform random number of $[0, C_L)$
  Perform binary search for i such that $C_{i-1} \leq p < C_i$
  {Choose entry site uniformly on remaining $L-1$ sites}
  $j \leftarrow$ Uniform integer on $\{1, \ldots L-1\}$
  if $j < i$ **then**
    $j \leftarrow j$
  else
    $j \leftarrow j + 1$
  end if
  {Decide if one particle jumps or all particles jump}
  $p \leftarrow$ Uniform random number on $[0, 1+w]$
  if $p < w$ **then**
    $\eta_i \leftarrow \eta_i - 1$
    $\eta_j \leftarrow \eta_j + 1$
  else
    $\eta_i \leftarrow 0$
    $\eta_j \leftarrow \eta_j + \eta_i$
  end if
  {Update transition rates and partial sums}
  Update c_i and c_j
  Update C_n for $n \in \{\min\{i,j\}, \ldots, L\}$
  {Calculate the maximum occupation}
  $M_L(\eta) \leftarrow \max\{\eta_1, \ldots, \eta_L\}$
  return $\frac{N - M_L(\eta)}{L-1}$

given by the dynamics of the underlying random walk.

A.3.2 The zero-range process

In Algorithm 3, we describe the Gillespie algorithm for calculating the coupling time of the defect site zero-range process on the complete graph with L sites, N particles, and a set Δ of defect sites.

The complexity of the algorithm is $\mathcal{O}(L)$, since at each time step we must update the partial sums C_n for $n \in \{\min\{i, j\}, \ldots, L\}$, which can be reduced to $\mathcal{O}(\log(L))$ by considering the binary tree discussed in Section A.3.1. Furthermore, the performance of the algorithm can be improved by utilising the simplicity of the coupled generator. For coupled processes $(\eta(t))_{t\geq 0}$ and $(\zeta(t))_{t\geq 0}$, the exit rate only changes when $\max\{\eta_i, \zeta_i\}$ jumps from $1 \to 0$ or $0 \to 1$, since the jump rates $g_x(n)$ are positive and constant for $n > 0$ and 0 if and only if $n = 0$.

Algorithm 3 Gillespie update algorithm for the coupled defect site zero-range process with N particles on a lattice of L sites with a set Δ of defect sites. Algorithm calculates the coupling time for the process.

Require: Vectors η and ζ of length L
Require: List of L jump rates for each site in the current state $(c_i)_{i=1}^L$
Require: List of partial sums $C_n = \sum_{i=1}^n c_i$ with $C_0 = 0$
  {Initialise η with N particles on a non defect site and ζ sampled according to $\pi_{\Lambda,N}^{\Delta}$}
  $\eta \leftarrow N\delta_x$ for some $x \notin \Delta$
  $\zeta \leftarrow$ Random sample from $\pi_{\Lambda,N}^{\Delta}$
  {Initialise list of jump rates}
  $c_i = \max\{g_i(\eta_i), g_i(\zeta_i)\}$
  Fill list of partial sums C_n
  $t \leftarrow 0$
  while $|\{i : \eta_i \neq \zeta_i\}| > 0$ **do**
    {Sample time increment from $\text{Exp}(C_L)$}
    $dt \leftarrow$ Exponential random variable with mean $1/C_L$
    $t \leftarrow t + dt$
    {Choose exit site with probability c_i/C_L}
    $p \leftarrow$ Uniform random number of $[0, C_L)$
    Perform binary search for i such that $C_{i-1} \leq p < C_i$
    {Choose entry site uniformly on remaining $L - 1$ sites}
    $j \leftarrow$ Uniform integer on $\{1, \dots L - 1\}$
    if $j < i$ **then**
      $j \leftarrow j$
    else
      $j \leftarrow j + 1$
    end if
    {Update configurations according to coupled dynamics}
    if $\eta_i > 0$ and $\zeta_i > 0$ **then**
      $\eta_i \leftarrow \eta_i - 1$ and $\eta_j \leftarrow \eta_j + 1$
      $\zeta_i \leftarrow \zeta_i - 1$ and $\zeta_j \leftarrow \zeta_j + 1$
    else if $\eta_i = 0$ and $\zeta_i > 0$ **then**
      $\zeta_i \leftarrow \zeta_i - 1$ and $\zeta_j \leftarrow \zeta_j + 1$
    else if $\eta_i > 0$ and $\zeta_i = 0$ **then**
      $\eta_i \leftarrow \eta_i - 1$ and $\eta_j \leftarrow \eta_j + 1$
    end if{Update transition rates and partial sums}
    Update c_i and c_j
    Update C_n for $n \in \{\min\{i, j\}, \dots, L\}$
  end while
  return t

Bibliography

[1] A. Einstein. Quantentheorie des einatomigen idealen Gases. *Sitzungs- berichte der Preuss. Akad. der Wissenschaften*, 6:261–267, 1924.

[2] M. H. Anderson, J. R. Ensher, M. R. Matthews, C. E. Wieman, and E. A. Cornell. Observation of Bose-Einstein Condensation in a Dilute Atomic Vapor. *Science (80-.).*, 269(5221):198–201, 1995.

[3] J. F. C. Kingman. A simple model for the balance between selection and mutation. *J. Appl. Probab.*, 15(01):1–12, 1978.

[4] S. Dereich and P. Mörters. Emergence of condensation in Kingman's model of selection and mutation. *Acta Appl. Math.*, 127(1):17–26, 2013.

[5] G. Bianconi and A-L. Barabási. Bose-Einstein Condensation in Complex Networks. *Phys. Rev. Lett.*, 86(24):5632–5635, 2001.

[6] J-P. Bouchaud and M. Mézard. Wealth condensation in a simple model of economy. *Phys. A Stat. Mech. its Appl.*, 282(3):536–545, 2000.

[7] M. R. Evans. Phase transitions in one-dimensional nonequilibrium systems. *Braz. J. Phys.*, 30(1):42–57, 2000.

[8] M. R. Evans. Bose-Einstein condensation in disordered exclusion models and relation to traffic flow. *Eur. Lett.*, 36(1):13–18, 1996.

[9] J. Krug and P. A. Ferrari. Phase Transitions in Driven Diffusive Systems With Random Rates. *J. Phys. A-Math. Gen.*, 29:L465—-L471, 1996.

[10] F. Spitzer. Interaction of Markov processes. *Adv. Math.*, 5:246–290, 1970.

[11] C. Cocozza-Thivent. Processus des misanthropes. *Z. Wahrscheinlichkeitstheorie*, 70(4):509–523, 1985.

[12] C. Kipnis and C. Landim. *Scaling Limits of Interacting Particle Systems.* Springer, Berlin, 1999.

[13] J.-M. Drouffe, C. Godrèche, and F. Camia. A simple stochastic model for the dynamics of condensation. *J. Phys. A-Math. Gen.*, 31(1):L19, 1998.

[14] M. R. Evans, S. N. Majumdar, and R. K. P. Zia. Canonical Analysis of Condensation in Factorised Steady States. *J. Stat. Phys.*, 123(2):357–390, 2006.

[15] S. N. Majumdar, M. R. Evans, and R. K. P. Zia. Nature of the Condensate in Mass Transport Models. *Phys. Rev. Lett.*, 94(18):180601, 2005.

[16] P. Chleboun and S. Grosskinsky. Finite Size Effects and Metastability in Zero-Range Condensation. *J. Stat. Phys.*, 140(5):846–872, 2010.

[17] J. Beltrán and C. Landim. Metastability of reversible condensed zero range processes on a finite set. *Probab. Theory Rel.*, 152(3-4):781–807, 2011.

[18] I. Armendáriz, S. Grosskinsky, and M. Loulakis. Metastability in a condensing zero-range process in the thermodynamic limit. *Probab. Theory Relat. Fields*, pages 1–71, 2016.

[19] C. Godrèche. Dynamics of condensation in zero-range processes. *J. Phys. A-Math. Gen.*, 36(23):6313, 2003.

[20] W. Jatuviriyapornchai and S. Grosskinsky. Coarsening dynamics in condensing zero-range processes and size-biased birth death chains. *J. Phys. A Math. Theor.*, 49(18):185005, 2016.

[21] C. Godrèche and J. M. Drouffe. Coarsening dynamics of zero-range processes. 2016.

[22] S. Grosskinsky, F. Redig, and K. Vafayi. Dynamics of condensation in the symmetric inclusion process. *Electron. J. Probab.*, 18(0), 2013.

[23] A. Bianchi, S. Dommers, and C. Giardinà. Metastability in the reversible inclusion process. 2016.

[24] J. Cao, P. Chleboun, and S. Grosskinsky. Dynamics of Condensation in the Totally Asymmetric Inclusion Process. *J. Stat. Phys.*, 155(3):523–543, 2014.

[25] R. L. Dobrushin and S. B. Shlosman. Constructive Criterion for the Uniqueness of Gibbs Field. In *Stat. Phys. Dyn. Syst.*, pages 347–370. Birkhäuser Boston, Boston, MA, 1985.

[26] D. Stroock and B. Zegarlinski. The logarithmic sobolev inequality for discrete spin systems on a lattice. *Commun. Math. Phys.*, 149(1):175–193, 1992.

[27] F. Martinelli and E. Olivieri. Approach to equilibrium of Glauber dynamics in the one phase region. I. The attractive case. *Commun. Math. Phys.*, 161(3):447–486, 1994.

[28] F. Martinelli and E. Olivieri. Approach to equilibrium of Glauber dynamics in the one phase region. II. The general case. *Commun. Math. Phys.*, 161(3):487–514, 1994.

[29] J. Quastel. Diffusion of color in the simple exclusion process. *Commun. Pure Appl. Math.*, 45(6):623–679, 1992.

[30] B. Morris. Spectral gap for the zero range process with constant rate. *Ann. Probab.*, 34(5):1645–1664, 2006.

[31] C. Landim, S. Sethuraman, and S. Varadhan. Spectral gap for zero-range dynamics. *Ann. Probab.*, 24(4):1871–1902, 1996.

[32] S. L. Lu and H-T. Yau. Spectral gap and logarithmic Sobolev inequality for Kawasaki and Glauber dynamics. *Commun. Math. Phys.*, 156(2):399–433, 1993.

[33] M. Jerrum, J. B. Son, P. Tetali, and E. Vigoda. Elementary bounds on Poincaré and log-Sobolev constants for decomposable Markov chains. *Ann. Appl. Probab.*, 14(4):1741–1765, 2004.

[34] D. Aldous and P. Diaconis. Shuffling Cards and Stopping Times. *Am. Math. Mon.*, 93(5):333, 1986.

[35] P. Diaconis. The cutoff phenomenon in finite Markov-chains. *Proc. Natl. Acad. Sci. U.S.A.*, 93:1659–1664, 1996.

[36] P. Diaconis and M. Shahshahani. Time to Reach Stationarity in the BernoulliLaplace Diffusion Model. *SIAM J. Math. Anal.*, 18(1):208–218, 1987.

[37] H. Lacoin and R. Leblond. Cutoff phenomenon for the simple exclusion process on the complete graph. 2010.

[38] H. Lacoin. The cutoff profile for the simple-exclusion process on the circle. 2015.

[39] H. Lacoin. Mixing time and cutoff for the adjacent transposition shuffle and the simple exclusion. *Ann. Probab.*, 44(2):1426–1487, 2016.

[40] H. Lacoin. The simple exclusion process on the circle has a diffusive cutoff window. 2016.

[41] H. Rost. Mixing of the symmetric exclusion processes in terms of the corresponding single-particle random walk. *Z. Wahr. Ver. Geb*, 58:41–53, 1981.

[42] T. M. Liggett. *Interacting Particle Systems.* Springer, 2006.

[43] T. M. Liggett. *Continuous Time Markov Processes: An Introduction*, volume 113. American Mathematical Society, 2010.

[44] D. A. Levin, Y. Peres, and E. L. Wilmer. *Markov Chains and Mixing Times.* American Mathematical Society, 2009.

[45] T. Gobron and E. Saada. Couplings, attractiveness and hydrodynamics for conservative particle systems. *Ann. I. H. Poincare-PR*, 46(4):1132–1177, 2010.

[46] L. Saloff-Coste. Lectures on finite Markov chains. pages 301–413. Springer Berlin Heidelberg, 1997.

[47] J. R. Norris. *Markov chains.* Cambridge University Press, 1998.

[48] E. D. Andjel. Invariant Measures for the Zero Range Process. *Ann. Probab.*, 10(3):525–547, 1982.

[49] T. E. Harris. Nearest-Neighbor Markov Interaction Processes on Multidimensional Lattices. *Adv. Math.*, 9:66–89, 1972.

[50] T. M. Liggett. An Infinite Particle System with Zero Range Interactions. *Ann. Probab.*, 1(2):240–253, 1973.

[51] C. Giardinà, J. Kurchan, F. Redig, and K. Vafayi. Duality and Hidden Symmetries in Interacting Particle Systems. *J. Stat. Phys.*, 135(1):25–55, 2009.

[52] C. Giardinà, F. Redig, and K. Vafayi. Correlation Inequalities for Interacting Particle Systems with Duality. *J. Stat. Phys.*, 141(2):242–263, 2010.

[53] B. Waclaw and M. R. Evans. Explosive Condensation in a Mass Transport Model. *Phys. Rev. Lett.*, 108(7):70601, 2012.

[54] L. Fajfrova, T. Gobron, and E. Saada. Invariant measures of Mass Migration Processes. 2015.

[55] M. R. Evans, S. N. Majumdar, and R. K. P. Zia. Factorized steady states in mass transport models. *J. Phys. A-Math. Gen.*, 37(25):L275–L280, 2004.

[56] G. Grimmett and D. Stirzaker. *Probability and Random Processes.* OUP Oxford, 2001.

[57] Y. Peres and P. Sousi. Mixing Times are Hitting Times of Large Sets. *J. Theor. Probab.*, 28(2):488–519, 2015.

[58] D. Aldous. Some inequalities for reversible markov chains. *J. London Math. Soc.*, 2(3):564–576, 1982.

[59] N. S. Pillai and A. Smith. Elementary Bounds On Mixing Times for Decomposable Markov Chains. 2015.

[60] I. Jeon, P. March, and B. Pittel. Size of the largest cluster under zero-range invariant measures. *Ann. Probab.*, 28(3):1162–1194, 2000.

[61] C. Godrèche and J. M. Luck. Condensation in the inhomogeneous zero-range process: an interplay between interaction and diffusion disorder. *J. Stat. Mech. Theor. Exp.*, (12):P12013, 2012.

[62] P. Chleboun and S. Grosskinsky. Condensation in Stochastic Particle Systems with Stationary Product Measures. *J. Stat. Phys.*, 154(1-2):432–465, 2013.

[63] M. R. Evans and B. Waclaw. Condensation in stochastic mass transport models: beyond the zero-range process. *J. Phys. A-Math. Theor.*, 47(9):095001, 2014.

[64] P. A. Ferrari, C. Landim, and V. Sisko. Condensation for a Fixed Number of Independent Random Variables. *J. Stat. Phys.*, 128(5):1153–1158, 2007.

[65] S. Grosskinsky, G. M. Schütz, and H. Spohn. Condensation in the Zero Range Process: Stationary and Dynamical Properties. *J. Stat. Phys.*, 113(3-4):389–410, 2003.

[66] I. Armendáriz and M. Loulakis. Thermodynamic limit for the invariant measures in supercritical zero range processes. *Probab. Theory Relat. Fields*, 145(1-2):175–188, 2008.

[67] I. Armendáriz, S. Grosskinsky, and M. Loulakis. Zero range condensation at criticality. *Stoch. Proc. Appl.*, 123(9):3466–3496, 2013.

[68] C. M. Goldie and C. Klüppelberg. Subexponential Distributions. In *A Pract. Guid. to Heavy Tails Stat. Tech. Anal. Heavy Tailed Distrib.*, pages 435–459. Birkhäuser, Boston, 1998.

[69] J. L. Teugels. The Class of Subexponential Distributions. *Ann. Probab.*, 3:1000–1011, 1975.

[70] E. J. G. Pitman. Subexponential distribution functions. *J. Austral. Math. Soc. Ser. A*, 29:337–347, 1980.

[71] J. Chover, P. Ney, and S. D. Wainger. Functions of probability measures. *J. d'Analyse Mathématique*, 26(Issue 1):pp 255–302, 1973.

[72] M. R. Evans, T. Hanney, and S. N. Majumdar. Interaction-driven real-space condensation. *Phys. Rev. Lett.*, 97(1):010602, 2006.

[73] B. Waclaw, J. Sopik, W. Janke, and H. Meyer-Ortmanns. Mass condensation in one dimension with pair-factorized steady states. *J. Stat. Mech. Theory Exp.*, 2009(10):P10021, 2009.

[74] P. Chleboun and S. Grosskinsky. A dynamical transition and metastability in a size-dependent zero-range process. *J. Phys. A-Math. Theor.*, 48(5):055001, 2015.

[75] S. Foss, D. Korshunov, and S. Zachary. *An Introduction to Heavy-Tailed and Subexponential Distributions*, volume 38 of *Springer Series in Operations Research and Financial Engineering*. Springer New York, New York, NY, 2011.

[76] C. Klüppelberg. Subexponential distributions and characterizations of related classes. *Probab. Theory Relat. Fields*, 82(2):259–269, 1989.

[77] A. Baltrunas and C. Klüppelberg. Subexponential Distributions - Large Deviations with Applications to Insurance and Queueing Models. *Aust N Z J Stat*, 46(1):145–154, 2004.

[78] V. P. Chistyakov. A Theorem on Sums of Independent Positive Random Variables and Its Applications to Branching Random Processes. *Theory Probab. Appl.*, 9(4):640–648, 1964.

[79] P. Embrechts and C. M. Goldie. On closure and factorization properties of subexponential and related distributions. *J. Austral. Math. Soc. Ser. A*, 29(02):243–256, 1980.

[80] D. Denisov, A. B. Dieker, and V. Shneer. Large deviations for random walks under subexponentiality: The big-jump domain. *Ann. Probab.*, 36(5):1946–1991, 2008.

[81] I. Armendáriz and M. Loulakis. Conditional distribution of heavy tailed random variables on large deviations of their sum. *Stoch. Proc. Appl.*, 121(5):1138–1147, 2011.

[82] H. O. Georgii. *Canonical Gibbs Measures*, volume 760 of *Lecture Notes in Mathematics*. Springer Berlin Heidelberg, Berlin, Heidelberg, 1979.

[83] P. Embrechts and C. M. Goldie. On convolution tails. *Stoch. Process. their Appl.*, 13(3):263–278, 1982.

[84] S. N. Majumdar, S. Krishnamurthy, and M. Barma. Nonequilibrium Phase Transition in a Model of Diffusion, Aggregation, and Fragmentation. *J. Stat. Phys.*, 99(1-2):1–29.

[85] S. N. Majumdar, S. Krishnamurthy, and M. Barma. Nonequilibrium Phase Transitions in Models of Aggregation, Adsorption, and Dissociation. *Phys. Rev. Lett.*, 81(17):3691–3694, 1998.

[86] R. Rajesh and S. N. Majumdar. Exact phase diagram of a model with aggregation and chipping. *Phys. Rev. E*, 63(3):036114, 2001.

[87] C. Landim. Hydrodynamical limit for space inhomogeneous one-dimensional totally asymmetric zero-range processes. *Ann. Probab.*, 24(2):599–638, 1996.

[88] I. Benjamini, P. A. Ferrari, and C. Landim. Asymmetric conservative processes with random rates. *Stoch. Proc. Appl.*, 61(2):181–204, 1996.

[89] E. D. Andjel, P. A. Ferrari, H. Guiol, and C. Landim. Convergence to the maximal invariant measure for a zero-range process with random rates. *Stoch. Proc. Appl.*, 90(1):67–81, 2000.

[90] P. A. Ferrari and V. Sisko. Escape of mass in zero-range processes with random rates. *IMS Lect. notes, Asymptotics Part. Process. Inverse Probl.*, 55:108–120, 2007.

[91] C. Bahadoran, T. Mountford, K. Ravishankar, and E. Saada. Supercritical behavior of asymmetric zero-range process with sitewise disorder. page 48, 2014.

[92] C. Bahadoran, T. Mountford, K. Ravishankar, and E. Saada. Supercriticality conditions for asymmetric zero-range process with sitewise disorder. *Braz. J. Probab. Stat.*, 29(2):313–335, 2015.

[93] M. G. Stamatakis. Hydrodynamic Limit of Mean Zero Condensing Zero Range Processes with Sub-Critical Initial Profiles. *J. Stat. Phys.*, 158(1):87–104, 2014.

[94] I. Jeon, P. March, and B. Pittel. Size of the Largest Cluster under Zero-Range Invariant Measures. *Ann. Probab.*, 28(3):1162–1194, 2000.

[95] M. Abramowitz and I. A. Stegun. *Handbook of mathematical functions : with formulas, graphs, and mathematical tables.* Dover Publications, 1970.

[96] A. Chatterjee, P. Pradhan, and P. K. Mohanty. Cluster-factorized steady states in finite-range processes. *Phys. Rev. E*, 92(3):032103, 2015.

[97] B. Waclaw, J. Sopik, W. Janke, and H. Meyer-Ortmanns. Mass condensation in one dimension with pair-factorized steady states. *J. Stat. Mech. Theor. Exp.*, 2009(10):P10021, 2009.

[98] M. R. Evans and B. Waclaw. Condensation in models with factorized and pair-factorized stationary states. *J. Stat. Mech. Theor. Exp.*, 2015(9):P09005, 2015.

[99] E. Ehrenpreis, H. Nagel, and W. Janke. Numerical survey of the tunable condensate shape and scaling laws in pair-factorized steady states. *J. Phys. A-Math. Theor.*, 47(12):125001, 2014.

[100] H. Touchette. The large deviation approach to statistical mechanics. *Phys. Rep.*, 478(1-3):1–69, 2009.

[101] Y. Nagahata. Spectral gap for zero-range processes with jump rate g (x) = x γ. *Stoch. Process. their Appl.*, 120(6):949–958, 2010.

[102] P. Caputo. Spectral gap inequalities in product spaces with conservation laws. In T. Funaki and H. Osada, editors, *Adv. Stud. Pure Math.*, volume 39. Math. Soc. Japan, Tokyo, 2004.

[103] A. G. Angel, M. R. Evans, and D. Mukamel. Condensation transitions in a one-dimensional zero-range process with a single defect site. *J. Stat. Mech. Theor. Exp.*, (04):P04001, 2004.

[104] W. A. Massey. Stochastic Orderings for Markov Processes on Partially Ordered Spaces. *Math. Oper. Res.*, 12(2):350–367, 1987.

[105] C. Mufa. Estimation of spectral gap for Markov chains. *Acta Math. Sin.*, 12(4):337–360, 1996.

[106] G.-Y. Chen and L. Saloff-Coste. On the mixing time and spectral gap for birth and death chains. *ALEA Lat. Am. J. Probab. Math. Stat.*, 10(1):293–321, 2013.

[107] J Beltrán and C. Landim. Tunneling and Metastability of Continuous Time Markov Chains. *J. Stat. Phys.*, 140(6):1–50, 2010.

[108] A. B. Bortz, M. H. Kalos, and J. L. Lebowitz. A new algorithm for Monte Carlo simulation of Ising spin systems. *J. Comput. Phys.*, 1975.