Amenable covers for surfaces and growth of closed geodesics

Mark Pollicott

July 27, 2015

1 Introduction

In the study of surfaces and closed geodesics an important characteristic is the topological entropy. Let M be a compact surface with a smooth Riemannian metric and denote by $\pi(M,T)$ the number of closed geodesics of length at most T. A dynamical perspective comes from considering the geodesic flow $\phi_t : SM \to SM$ on the three dimensional unit tangent bundle SM for M. For compact surfaces of negative curvature the topological entropy $h(\phi)$ of the associated geodesics flow corresponds to the growth rate of the number of closed geodesics $\pi(M,T)$ with length at most T:

Theorem 1.1 (Sinai [10]). *If M has strictly negative curvature then*

$$\lim_{T\to+\infty} \frac{1}{T}\log \pi(M,T) = h(\phi),$$

where $h(\phi)$ denotes the topological entropy of the associated geodesic flow $\phi_t : SM \to SM$.

This was extended to an asymptotic formula for $\pi(M,T)$ by Margulis [6].

For non-compact surfaces with strictly negative curvature the situation is a little more complicated [7] and even more so for surfaces of infinite area. Consider a non-compact infinite area surfaces $\widehat{M}$ which occurs as a cover for a compact surface M of negative curvature. Given any closed geodesic on $\widehat{M}$ there will be infinitely many of the same length by translating by an element of the covering group $\Gamma = \pi_1(M)/\pi_1(\widehat{M})$. Therefore it is natural to count only the closed geodesics on $\widehat{M}$ up to translation. Since every such closed geodesic on $\widehat{M}$ projects to (possibly shorter) closed geodesic on M we see that the growth of the number $\pi(\widehat{M},T)$ of closed geodesics on $\widehat{M}$ is less than or equal to that for M, i.e., $\pi(\widehat{M},T) \leq \pi(M,T)$. We call the following definition.

Definition 1.2. *We say a group Γ is amenable if it has a Folner sequence (i.e., a sequence of finite sets $F_n \subset \Gamma$ which exhaust the group and for any $g \in \Gamma$ we have $\#gF_n\Delta F_n/F_n \to 0$).*

Examples of amenable groups include infinite abelian groups (such as $\mathbb{Z}^d$) and nilpotent groups (such as the discrete Heisenberg group with entries in $\mathbb{Z}$) are amenable.

In this case, it follows from work of Roblin on critical exponents and recent work of Dougall-Sharp that providing the covering group G is amenable then $h(\phi)$ still gives the growth rate of closed geodesics (up to translation by G) on the surface $\widehat{M}$:

Theorem 1.3 (Roblin, Dougall-Sharp). *If M has strictly negative curvature an the covering group for $\widehat{M}$ is amenable then*

$$\lim_{T\to+\infty} \frac{1}{T} \log \pi(\widehat{M}, T) = h(\phi).$$

This naturally prompts the question of what can be said when we relax the assumption of negative curvature. Let M be any compact surface with a smooth Riemannian metric. We recall the following well known result of Katok. [1]

Theorem 1.4 (Katok [3]). *We can bound*

$$\limsup_{T\to+\infty} \frac{1}{T} \log \pi(M, T) \geq h(\phi),$$

where $h(\phi)$ denotes the topological entropy of the associated geodesic flow $\phi_t : SM \to SM$.

At this level of generality, we can expect to generally have an inequality in Theorem 1.4. For example, the surface may have uncountably many closed geodesics if it contains an embedded flat cylinder. Let us again consider a covering surface $\widehat{M}$ for M with covering group Γ. To prove an analogue of Theorem 1.3, we need to impose an extra assumption.

Hypothesis 1.5. *Assume that there exists a transitive geodesic on the non-wandering set for $\widehat{M}$.*

By a result of Eberlein, this hold for example, if $\widehat{M}$ is a uniform visibility manifold (e.g., if M has non-positive curvature) [2]. Our main result is that for transitive amenable covers we still have the same lower bound on the growth rate as in Theorem 1.4.

Theorem 1.6. *If Γ is amenable and satisfies the hypothesis then*

$$\limsup_{T\to+\infty} \frac{1}{T} \log \pi(\widehat{M}, T) \geq h(\phi)$$

where $h(\phi)$ denotes the topological entropy of the geodesic flow $\phi_t : SM \to SM$.

As in the case of Theorem 1.4, at this level of generality one cannot necessarily expect to have equality in Theorem 1.6.

Remark 1.7. If we impose stronger hypotheses on Γ then we don't necessarily have to assume the transitivity hypothesis. In particular, we could assume that Γ has sub-exponential growth, in place of the transitivity hypothesis.

A property of the geodesic flow $\phi_t : SM \to SM$ which will be particularly useful to us is the following: There exists an involution $\tau : SM \to SM$ (i.e., $\tau^2 = I$) such that $\tau \circ \phi_t \circ \tau = \phi_{-t}$. A simplifying assumption that we can make without loss of generality is that $h(\phi) > 0$, since otherwise Theorem 1.6 is trivially true.

[1] The better known formulation of the theorem is for C^2 diffeomorphisms of compact manifolds, but the extension to flows is straightforward.

Our method of proof is based on using Pesin Theory and non-uniformly hyperbolic specification lemmas to orbit segments arising from an application of Kesten's result on symmetric random walks on amenable groups. Most of the arguments are very straightforward variations on corresponding results in the uniformly hyperbolic setting and so where appropriate we only sketch the proofs.

In section 2 we will explain how the hypothesis that $h(\phi) > 0$ allows us to generate suitable orbit segments and closed orbits for the geodesic flow. In section 3, we will show how Kesten's theorem on symmetric random walks on amenable groups applies. Finally, in section 4 we combine these ingredients to complete the proof of Theorem 1.6.

2 Entropy and closing lemmas

In this section we shall introduce two ingredients in the proof of Theorem 1.6. The first involves the use of the entropy to generate orbit segments for the flow and (non-uniform) hyperbolicity to create closed orbits. The second is a specification lemma to allow these orbit segments to be closed up. Both are straightforward modifications of the original proof of Theorem 1.4 in [3].

2.1 Entropy

Let $\phi_t : SM \to SM$ be a C^2 flow. We recall a convenient definition of metric entropy.

Definition 2.1 (cf. [3]). [2] *Let μ be an ergodic ϕ-invariant probability measure. For a fixed value $0 < \epsilon < 1$ and $T, \delta > 0$ we let $\mathcal{N}(T, \delta, \epsilon)$ be the smallest number of points $\{x_i\}_{i=1}^N$ required such that*

$$\mu\left(\{y \in M \,:\, \exists 1 \leq i \leq N, 0 \leq t \leq T, d(\phi_t x_i, y) < \delta\}\right) > 1 - \epsilon.$$

Then:

1. *we can then define the entropy $h(\phi, \mu)$ of the measure μ by*

$$h(\phi, \mu) = \lim_{\epsilon \to 0} \lim_{\delta \to 0} \lim_{T \to +\infty} \frac{1}{T} \log \mathcal{N}(T, \delta, \epsilon); \text{ and}$$

2. *we can define the topological entropy $h(\phi)$ using the variational principle [11], i.e.,*

$$h(\phi) = \sup\left\{h(\phi, \mu) \,:\, \mu = \phi\text{-invariant ergodic probability}\right\}.$$

In fact, it is not necessary to take the first limit in ϵ, as the next Lemma shows.

Lemma 2.2. *For any $\epsilon > 0$*

$$h(\phi, \mu) = \lim_{\delta \to 0} \lim_{T \to +\infty} \frac{1}{T} \log \mathcal{N}(T, \delta, \epsilon)$$

[2] This is a straightforward modification of the original definition for diffeomorphisms.

Proof. In [3] this result is stated and proved for discrete maps, but the generalisation to flows is immediate. □

Definition 2.3. *Given an ergodic ϕ-invariant probability measure μ we can associate the positive Lyapunov exponent given by*

$$\lambda(\mu) = \limsup_{T \geq 1} \left\{ \frac{1}{T} \int \log \|D\phi_T(x)\| d\mu(x) \right\}.$$

Remark 2.4. By the Pesin-Ruelle inequality [9] we have the inequality $h(\phi, \mu) \leq \lambda(\mu)$.

As we observed before, we only need to consider the case that $h(\phi) > 0$, since the conclusion in Theorem 1.6 is trivial in the case $h(\phi) = 0$. The following is well known.

Lemma 2.5. *Let $h(\phi) > 0$ then for any $h(\phi) > \epsilon_0 > 0$ there exists an ergodic probability measure μ such that $h(\phi, \mu) \geq h(\phi) - \epsilon_0$ and the Lyapunov exponent $\lambda = \lambda(\mu)$ is non-zero (i.e., μ is hyperbolic).*

Proof. By the variational principle we can write

$$h(\phi) = \sup\{h(\phi, \nu) : \nu \text{ is a } \phi\text{-invariant probability}\}.$$

Therefore, we can choose an ergodic measure μ such that $h(\phi, \mu) \geq h(\phi) - \epsilon_0$. Moreover, by the Pesin-Ruelle inequality we can write $\lambda(\mu) \geq h(\phi, \mu)$ [9]. □

We now introduce non-invariant hyperbolic sets,

Definition 2.6. *Given $k \geq 1$, $\lambda > 0$ and $\lambda > \epsilon > 0$ we let $N_k = N_k(\lambda, \epsilon) \subset M$ consist of points x such that there exists a splitting $T_xM = E^s_x \oplus E^0_x \oplus E^u_x$ along the orbit $\{\phi_t x\}_{t \in \mathbb{R}}$ with:*

1. *E^0 is tangent to the flow direction*
2. *$\|D\phi_t | E^s_{\phi_{t_0} x}\| \leq e^k e^{\epsilon |t_0|} e^{-\lambda t}$, for $t \geq 0$, $t_0 \in \mathbb{R}$; $\|D\phi_t | E^u_{\phi_{t_0} x}\| \leq e^k e^{\epsilon |t_0|} e^{-\lambda t}$, for $t \geq 0$, $t_0 \in \mathbb{R}$; and*
3. *$\langle (E^s_{\phi_t x}, E^u_{\phi_t x}) \geq e^{-k} e^{-|t|\epsilon}$, for $t \in \mathbb{R}$.*

Furthermore, by construction, each set $N_k \subset SM$ is compact and closed under the involution (i.e., satisfying $\iota(N_k) = N_k$).

The next lemma we need shows that the entropy can be used to give a lower bound on the number orbit segments returning to a suitable neighbourhood of one of these sets N_k.

Lemma 2.7 (Entropy Lemma)**.** *For any $\epsilon_1 > 0$ we can choose $N_k \subset SM$ closed under the involution (i.e., satisfying $\iota(N_k) = N_k$) such that:*

1. *$\mu(N_k) > 1 - \epsilon_1$; and*
2. *for any μ-density point $x \in N_k$ and sufficiently small $\delta, \epsilon_2 > 0$ there exists arbitrarily large $T > 0$ and distinct orbits segments*

$$\tau_n = \phi_{[0,T_n]}(x_n) \text{ for } n = 1, \cdots, [e^{(h(\phi) - 2\epsilon_0)T}]$$

with endpoints $x_n \in B(x, \delta) \cap N_k$, $\phi_{T_n}(x_n) \in B(\iota(x), \delta) \cap N_k$ and length $T_n \in [T - \epsilon_2, T]$.

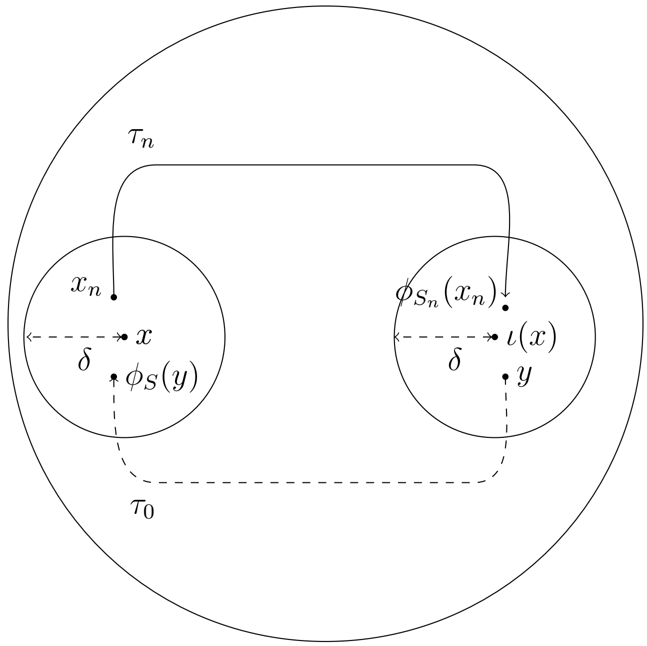

Figure 1: Orbit segments τ_n from a neighbourhood of x to a neighbourhood of $\iota(x)$ (arising from Lemma 2.7) and a single sorbet segment τ from a neighbourhood of $\iota(x)$ to a neighbourhood of x (arising from Lemma 2.8)

Proof. Using the sub additive ergodic theorem (or the more general multiplicative ergodic theorem) applied to μ, we have that $\mu(\cup_{n=1}^{\infty} N_k) = 1$. We can then choose $k \geq 1$ sufficiently large that N_k satisfies $\mu(N_k) > 0$.

Property 2 follows from the definition of $h(\phi, \mu)$ in Definition 2.1 and ergodicity (cf. [3], [1]). □

We now proceed with the construction of the orbit segments which will be used to construct closed orbits. For each of the $[e^{(h(\phi)-2\epsilon_0)T}]$ geodesic arcs τ_n, say, provided by Lemma 2.7 we can associate $[e^{(h(\phi)-2\epsilon_0)T}]$ more orbit segments using the involution, i.e., $i(\tau_n) = \phi_{[0,T_n]}(i(x_n))$ for $n = 1, \cdots, [e^{(h(\phi)-2\epsilon_0)T}]$. There exists $K \geq 1$ such that these start in $B(x, K\delta) \cap N$ and end in $B(\iota(x), K\delta) \cap N$. To simplify the notation, let us assume that $K = 1$, the more general case following by reducing the size of $\delta > 0$.

2.2 Specification lemma

We will need the following easy lemma giving an orbit segment from $B(\iota(x), \delta)$ back to $B(x, \delta)$, when x is a density point for μ.

Lemma 2.8. *We can choose $y \in B(\iota(x), \delta) \cap N_k$ and $S > 0$ such that $\phi_S(y) \in B(x, \delta) \cap N_k$.*

Proof. This follows by ergodicity. □

Later in the proof it will be convenient to assume the following additional flexibility. Given $\eta > 0$, we can assume T in Lemma 2.7 is sufficiently large that

$$S/T < \eta. \tag{2.1}$$

We denote the orbit segment arising from Lemma 2.8 by $\tau_0 = \phi_{[0,S]}(y)$. On the set N_k we have the following useful form of a specification lemma.

Lemma 2.9 (Specification). *Given $\epsilon_3, \epsilon_4 > 0$, we can choose $\delta > 0$ sufficiently small such that for any $m \geq 1$ and any orbit segments $\{\tau_i\}_{i=1}^m$ of length T_i, each starting in $B(x, \delta) \cap N$ and finishing in $B(\iota(x), \delta) \cap N_k$, we can find a single closed orbit $\tau = \tau(\tau_i)$ which remains within the ϵ_3-neighbourhood of $\tau_1 \cup \tau_0 \cup \tau_2 \cup \tau_0 \cup \cdots \cup \tau_m \cup \tau_0$ and satisfies*

$$\left| \lambda(\tau) - \left(\sum_{i=1}^m T_i + mS \right) \right| \leq \epsilon_4 m, \forall m \geq 1$$

Moreover, there exists $k' \geq 1$ such that $\tau \cap N_{k'} \neq \emptyset$

Proof. This is actually a combination of the shadowing and closing lemmas from [4], ([1] Theorem 15.1.2 and Theorem 15.2.1). The shadowing lemma ensures that the orbit segments give rise to a single orbit segment τ'', to which one can then apply the closing lemma to obtain a single closed orbit τ, with the promised bound on the length. $\square$

In particular, for $\lambda(\tau_i) \leq T$ we can deduce that the closed orbits generated by Lemma 2.9 satisfy

$$\lambda(\tau) \leq m(T + S + \epsilon_4) \tag{2.2}$$

Moreover, by the hyperbolicity of N different choices of $\tau_i, \cdots, \tau_{i_m}$ in Lemma 2.9 can be assumed to give rise to distinct choices of $\tau = \tau(\tau_{i_1}, \tau_{i_2}, \cdots, \tau_{i_m})$. Let $\tau^{(l)}$, for $l = 1, \cdots, (2[e^{(h(\phi)-2\epsilon_0)T}])^m$, enumerate these possible closed orbits.

The basic strategy of the proof is to let m tend to infinity so as to generate a sequence of times (approximating $m(T + S)$) tending to infinity, without changing the value T arising in Lemma 2.7.

Finally, for this section, it is convenient to state separately a specification theorem for closed orbits which is clearly a corollary of Lemma 2.9 (by taking the orbit segments there to be closed orbits).

Corollary 2.10. *Given $\epsilon_5, \epsilon_6 > 0$, we can choose $\delta > 0$ sufficiently small such that for all $m \geq 1$ any closed orbits $\{\tau_i\}_{i=1}^m$ of least period $\lambda(\tau_i) = T_i'$, each passing through $B(x, \delta) \cap N_{k'}$ we can find a single closed orbit $\tau = \tau(\tau_i)$ which remains within a distance ϵ_5 of the union of $\tau_i \cup \tau_0$ and satisfies*

$$\left| \lambda(\tau) - \left(\sum_{i=1}^m T_i' + mS \right) \right| \leq \epsilon_6 m, \forall m \geq 1.$$

3 Kesten's theorem

The second main ingredient in the proof of Theorem 1.6 is a form of recurrence property for the flow on the cover. This will come from the covering group being an amenable group and Kesten's theorem on symmetric random walks. Let us fix a copy $F \subset \widehat{M}$ of M, for example the quotient of a fundamental domain in $\widetilde{M}$.

Definition 3.1. *Let τ denote an orbit segment for $\phi_t : SM \to SM$ of length T. We can associate to τ an element $g_\tau \in \Gamma$ so that the lift $\widehat{\tau}$ has $\widehat{\tau}(0)$ in F and satisfies $\widehat{\tau}(\lambda(\tau)) = g_\tau \widehat{\tau}(0)$.*

Other choices of lifts would lead to conjugate elements.

Lemma 3.2. *We can assume without loss of generality that $g_{\tau_0} = e$.*

Proof. This is a consequence of the transitivity hypothesis. [3] □

The following is immediate.

Lemma 3.3. *If τ_i is an orbit segment and $\iota(\tau)$ denotes the point set but with the reverse orientation then $g_{i(\tau)} = g_\tau^{-1}$.*

We can now observe that for each of the closed orbits $\tau^{(l)}$, for $l = 1, \cdots, (2[e^{(h(\phi)-2\epsilon_0)T}])^m$ constructed in the last section there corresponds another $\tau^{(l')}$, for some $1 \leq l' \leq (2[e^{(h(\phi)-2\epsilon_0)T}])^m$, such that $g_{\tau^{(l)}} = g^{-1}_{\tau^{(l')}}$. For example, this can be achieved achieved by replacing each of the $\tau_{i_1}, \tau_{i_2}, \cdots, \tau_{i_m}$ in Lemma 2.9 by their images $\iota(\tau_{i_1}), \iota(\tau_{i_2}), \cdots, \iota(\tau_{i_m})$ (or vica versa) and using Lemma 3.3.

Definition 3.4. *For $m > 0$ sufficiently large, we can associate a probability measure on Γ given by*

$$\mathbb{P}_m(g) = \frac{\#\{1 \leq l \leq (2[e^{(h(\phi)-2\epsilon_0)T}])^m \,:\, g_{\tau^{(l)}} = g\}}{(2[e^{(h(\phi)-2\epsilon_0)T}])^m} \text{ for } g \in \Gamma,$$

where the numerator counts the orbits $\tau^{(l)}$ constructed in the previous section, subject to their lifts being equal to g (i.e., $\mathbb{P}_m(g)$ is the proportion of closed orbits constructed in the previous section which satisfy $g_{\tau^{(l)}} = g$).

There is an element of arbitrariness in the construction of the closed orbits and the definition above which arises from the different possible sequences of orbit segments. However, this is unimportant in the argument.

We next observe that as a consequence of Lemma 3.3 we have the following:

Lemma 3.5. *The probability measure $\mathbb{P}_m$ is symmetric (i.e., $\mathbb{P}_m(g) = \mathbb{P}_m(g^{-1})$).*

In order to state the result we need on random walks we need to introduce some more notation. For each $m \geq 1$ we can consider an operator $U_m : l^2(\Gamma) \to l^2(\Gamma)$ defined by

$$U_m f(g) = \sum_{h \in \Gamma} \mathbb{P}_m(h) f(gh^{-1}).$$

Definition 3.6. *Let $\Gamma_m < \Gamma$ be the subgroup generated by $\{g_{\tau^{(l)}} \,:\, 1 \leq l \leq (2[e^{(h(\phi)-2\epsilon_0)T}])^m\}$.*

Let $\delta_e : \Gamma \to \mathbb{R}$ be defined by

$$\delta_e(g) = \begin{cases} 1 & \text{if } g = e \\ 0 & \text{if } g \neq e \end{cases}$$

[3] Since we only need the existence of a single orbit segment passing through the lifts $B(x_0, \delta) \cap N$ and $B(i(x_0), \delta) \cap N$ this would also follow by a simple argument assuming sub-exponential growth in Γ and the pigeonhole principle.

Definition 3.7. *We let* $\rho_m := \lim_{k\to+\infty} \left(\langle \delta_e, U_m^k \delta_e\rangle\right)^{\frac{1}{k}}$ *denote the spectral radius of* U_m.

The significance of ρ_m lies is the following important classical result.

Proposition 3.8 (Kesten [5])**.** *If* Γ_m *is amenable then* $\rho_m = 1$.

Since we are assuming that Γ is amenable then we can assume that the subgroup $\Gamma_m < \Gamma$ is amenable too (since any subgroup of an amenable group is amenable) and thus $\rho_m = 1$.

4 Proof of Theorem 1.6

Since we are assuming that Γ in amenable (and thus each Γ_m is amenable for sufficiently large m) then by Proposition 3.8 we have that $\rho_m = 1$. Then for any given $\epsilon_7 > 0$ and $m \geq 1$ we have that for sufficiently large $k = k(m)$

$$\log \left(\langle \delta_e, U_m^k \delta_e\rangle\right)^{\frac{1}{k}} > 1 - \epsilon_7. \tag{4.1}$$

In particular, we can consider k-tuples of closed orbits $\tau^{(l_1)}, \cdots, \tau^{(l_k)}$, where $1 \leq l_1, \cdots, l_k \leq (2[e^{(h(\phi)-2\epsilon_0)}])^m$, with:

1. $\lambda(\tau^{(l_1)}), \cdots, \lambda(\tau^{(l_k)}) \leq m(T + S + \epsilon_4)$;
2. $\tau^{(l_i)} \cap B(x, \delta) \cap N_{k'} \neq \emptyset$ for $1 \leq i \leq k$; and
3. $g_{\tau^{(l_1)}} \cdots g_{\tau^{(l_k)}} = e$,

and by (4.1) the total number of k-tuples $(\tau^{(l_1)}, \cdots, \tau^{(l_k)})$ satisfying 1.-3. above can be bounded from below by

$$(1 - \epsilon_7)^k \left([2e^{(h(\phi)-2\epsilon_0)T}]\right)^{mk}.$$

Using Corollary 2.10, we can replace each k-tuple $(\tau^{(l_1)}, \cdots, \tau^{(l_k)})$ by a single closed orbit τ such that:

1. $l(\tau) \leq l(\tau^{(l_1)}) + \cdots + l(\tau^{(l_k)}) + k\epsilon_6 \leq k(m(T + S + \epsilon_4) + \epsilon_6) \leq k(m(T(1+\eta) + \epsilon_4) + \epsilon_6)$ by (2.1) and (2.2); and
2. $g_\tau = g_{\tau^{(l_1)}} \cdots g_{\tau^{(l_k)}} = e$.

Moreover, by virtue of the hyperbolicity of N we can assume that the associated τ are all distinct. Finally, we have a bound

$$\pi(\widehat{M}, km(T(1+\eta) + \epsilon_4) + \epsilon_6)) \geq (1 - \epsilon_7)^k (2[e^{(h(\phi)-2\epsilon_0)Tm}])^k.$$

Let us denote $T_m := km(T(1+\eta) + \epsilon_4) + \epsilon_6$. Given $\epsilon > 0$, providing $\epsilon_0, \epsilon_4, \epsilon_6, \epsilon_7 > 0$ and $\eta > 0$ are sufficiently small we can choose $C > 0$ such that $\pi(\widehat{M}, T_m) \geq Ce^{(h(\phi-\epsilon)T_m)}$, for all $m \geq 1$. This completes the proof.

References

[1] L. Barreira and Y. Pesin, *Nonuniform Hyperbolicity*, Cambridge University Press, Cambridge, 2007.

[2] P. Eberlein, Geodesic flows in certain manifolds without conjugacy points, Trans, Amer. Math. Soc., 167 (1972) 151-170.

[3] A. Katok, Lyapunov exponents, entropy and periodic orbits for diffeomorphisms, Publications Mathématiques de l'IHES, 51 (1980) 137-173.

[4] A. Katok and L. Mendoza, Dynamical systems with nonuniformly hyperbolic behavior, in Introduction to the Modern Theory of Dynamical Systems, by A. Katok and B. Hasselblatt, Cambridge Univ. Press, 1995.

[5] H. Kesten, Full Banach mean values on countable groups, Math. Scand. 7 (1959) 146156.

[6] G. Margulis, *On some aspects of the theory of Anosov systems*, Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2004.

[7] M. Pollicott, F. Paulin and B. Schapira, Equilibrium states in negative curvature, Asterisque vol. 373, to appear

[8] T. Roblin, Un théorème de Fatou pour les densités conformes avec applications aux revêstements galoisiens en courbure négative, Israel J. Math. 147 (2005), 333357.

[9] D. Ruelle, An inequality for the entropy of differentiable maps, Bol. Doc. Bras. Mat. 9 (1978) 83-87.

[10] Ya G. Sinai, Asymptotic behaviour of closed geodesics on compact manifolds with negative curvature, Izv. Akad. Nauk SSSR Ser. Mat. , 30 (1966) 1275-s1296.

[11] P. Walters, *Ergodic Theory*, Springer, Berlin, 1981.