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Slow north-south dynamics on $\mathcal{PML}$

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Abstract. We consider the action of a pseudo-Anosov mapping class on $\mathcal{PML}(S)$. This action has north-south dynamics and so, under iteration, laminations converge exponentially to the stable lamination.

We study the rate of this convergence and give examples of families of pseudo-Anosov mapping classes where the rate goes to one, decaying exponentially with the word length. Furthermore we prove that this behaviour is the worst possible.

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1. Introduction

A pseudo-Anosov mapping class $h \in \mathrm{Mod}^+(S)$ acts on $\mathcal{PML}(S)$ with north-south dynamics. Therefore its action has a pair of fixed points $\mathcal{L}^{\pm}(h) \in \mathcal{PML}(S)$ and under iteration laminations (other than $\mathcal{L}^-(h)$) converge to $\mathcal{L}^+(h)$. A pants decomposition, collection of train tracks or ideal triangulation gives a coordinate system on $\mathcal{PML}(S)$ [5, Exposé 6]. In any such system the convergence to $\mathcal{L}^+(h)$ is exponential.

Thurston suggested that under iteration laminations always converge to $\mathcal{L}^+(h)$ "rather quickly" [9, page 427]. If this were true for all pseudo-Anosov mapping classes then iteration would give an efficient algorithm to find $\mathcal{L}^+(h)$. However it is false:

Theorem 1.1. *Suppose that $3g - 3 + p \geq 4$ and fix a finite generating set for $\mathrm{Mod}^+(S_{g,p})$. There is an infinite family of pseudo-Anosov mapping classes where the rate of convergence goes to one, and decays exponentially with respect to word length.*

As usual, we use $S_{g,p}$ to denote the surface of genus g with p punctures.

In Section 2 we show how to construct such a family on $S_{3,0}$ and in Section 3 we show how to generalise this construction to other surfaces. Furthermore, in Proposition 4.1 we show that this type of convergence is the worst possible. Finally in Section 5 we describe how these examples can be rigorously verified using `flipper` [2].

In order to bound the rate of convergence, we use the following definition.

Definition 1.2. Suppose that $f \in \mathbb{Z}[x]$ is a polynomial with roots $\lambda_1, \ldots, \lambda_m$, ordered such that $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_m|$. The *spectral ratio* of f is $\omega(f) := |\lambda_1/\lambda_2|$.

This is motivated by the equivalent problem of $\mathrm{GL}(N, \mathbb{Z})$ acting on $\mathbb{RP}^{N-1}$. Under iteration of a matrix $M \in \mathrm{GL}(N, \mathbb{Z})$, generic vectors in $\mathbb{RP}^{N-1}$ converge exponentially to the dominant eigenvector of M. The rate of this convergence is bounded above by $\omega(M) := \omega(\chi_M)$, the spectral ratio of the characteristic polynomial of M [11, Section 4.1].

Definition 1.3. Suppose that $h \in \mathrm{Mod}^+(S)$ is a pseudo-Anosov mapping class. Let $\mu_{\lambda(h)} \in \mathbb{Z}[x]$ denote the minimal polynomial of its *dilatation* $\lambda(h)$. The *spectral ratio* of h is $\omega(h) := \omega(\mu_{\lambda(h)})$.

Choose one of the above coordinate systems on $\mathcal{PML}(S)$ and a pseudo-Anosov mapping class h. On a suitable neighbourhood of $\mathcal{L}^+(h)$, the action of h is given by an integer matrix M. Hence, for a generic lamination $\mathcal{L}$ the rate of convergence of $h^n(\mathcal{L})$ to $\mathcal{L}^+(h)$ is bounded above by $\omega(M)$. However, the dominant eigenvalue of M is $\lambda(h)$ and so $\omega(M) \leq \omega(h)$. Thus we achieve Theorem 1.1 by producing mapping classes with spectral ratio exponentially close to one.

Finally, we conjecture that the surface complexity condition in Theorem 1.1 is not only sufficient but also necessary. If so then this problem is subtly different from the equivalent problem for matrices. In $\mathrm{GL}(N, \mathbb{Z})$ exponentially slow convergence occurs even when $N = 3$. One such family is given by the matrices

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 2^k \\ 0 & 1 & 0 \end{pmatrix}.$$

These have word length only $O(k)$, due to distorted subgroups [8, Theorem 4.1].

2. An upper bound by example

We start by constructing an explicit family of pseudo-Anosov mapping classes on $S_{3,0}$ whose spectral ratio goes to one exponentially with the word length. To do this we use:

- $\varphi := \frac{1+\sqrt{5}}{2}$ to denote the *golden ratio*,
- F_n to denote the n^{th} *Fibonacci number*, and
- $x \approx_t y$ to denote that $|x - y| \leq t$.

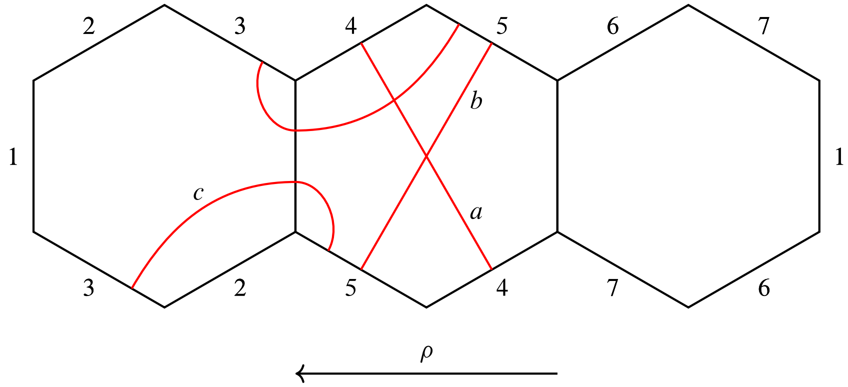


Figure 1. Curves on the surface of genus 3.

Let S be the surface of genus 3 as shown in Figure 1 in which sides with the same label are identified. Fix $k \geq 7$ such that $k \equiv 2 \pmod 8$. Consider the mapping class

$$h := \rho \circ T_c \circ (T_a^{-1} \circ T_b)^k$$

where T_x denotes a right Dehn twist about x and ρ is the order three mapping class which cycles these hexagons to the left.

Theorem 2.1. *The mapping class h is pseudo-Anosov and $\omega(h) \leq 1 + 14\varphi^{-k}$.*

Proof. First note that as in [6, page 448] it follows immediately from [7, Theorem 3.1] that h^3 is pseudo-Anosov and so h is too.

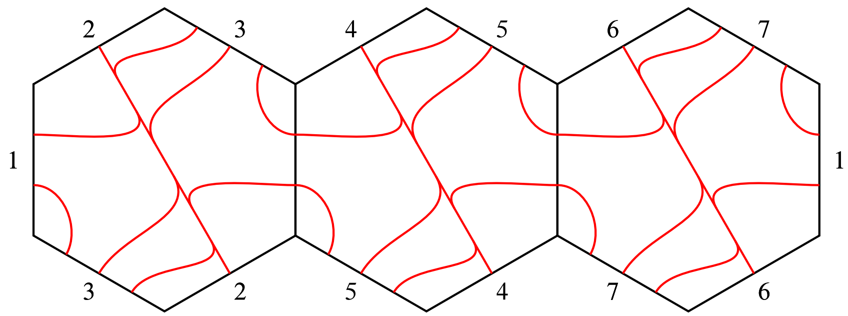


Figure 2. The invariant train track τ of h [6, Figure 3b].

Now the train track τ, shown in Figure 2, is invariant under h. Direct calculation show that the hitting matrix of τ under h with respect to the basis $a, b, c, \rho(a), \rho(b), \rho(c), \rho^2(a), \rho^2(b), \rho^2(c)$ is

$$M := \left(\begin{array}{ccc|ccc|ccc} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \hline F_{2k+1} & F_{2k} & F_{2k} & 0 & 0 & 0 & 0 & 0 & F_{2k} \\ F_{2k} & F_{2k-1} & F_{2k-1}-1 & 0 & 0 & 0 & 0 & 0 & F_{2k-1}-1 \\ F_{2k+1} & F_{2k} & F_{2k}+1 & 1 & 0 & 0 & 0 & 0 & F_{2k} \\ \hline 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{array}\right)$$

and that the characteristic polynomial of M is

$$(x^3 - 1)(x^6 - F_{2k}x^4 - F_{2k+3}x^3 - F_{2k}x^2 + 1).$$

Using the fact that $k \equiv 2 \pmod 8$, reducing the right-hand factor of this modulo 7 we obtain $x^6 + 4x^4 + x^3 + 4x^2 + 1 \in \mathbb{F}_7[x]$. This is irreducible in $\mathbb{F}_7[x]$ and so the minimal polynomial of $\lambda(h)$ is

$$\mu_{\lambda(h)}(x) = x^6 - F_{2k}x^4 - F_{2k+3}x^3 - F_{2k}x^2 + 1.$$

To find the roots of $\mu_{\lambda(h)}$, we divide by x^3 and substitute $y := x + x^{-1}$ to obtain:

$$y^3 - (F_{2k} + 3)y - F_{2k+3}.$$

Let y_{-1}, y_0 and y_1 be the three roots of this cubic. These are all real numbers as $\Delta = 4(F_{2k} + 3)^3 - 27F_{2k+3}^2 > 0$ and using the cubic Viète formula are given by

$$y_j := \frac{2}{\sqrt{3}}\sqrt{F_{2k} + 3}\cos\left(\frac{1}{3}\arccos\left(\frac{3\sqrt{3}F_{2k+3}}{(F_{2k} + 3)\sqrt{F_{2k} + 3}}\right) - (j - 1)\frac{2\pi}{3}\right).$$

By using the Taylor series for cosine and arccosine together with the fact that $\sqrt{F_{2k} + 3} \approx_1 \sqrt[4]{5}F_k$, we deduce that

$$y_{-1} \approx_2 -\sqrt[4]{5}F_k, \ y_0 \approx_2 0 \text{ and } y_1 \approx_2 \sqrt[4]{5}F_k.$$

Since $y = x + x^{-1}$, the six roots of $\mu_{\lambda(h)}$ are given by

$$x_i^\pm := \frac{y_i \pm \sqrt{y_i^2 - 4}}{2}$$

Thus for each i either $x_i^+ \approx_1 y_i$ and $x_i^- \approx_1 0$, or vice versa. In particular

$$x_{-1}^- \approx_3 -\sqrt[4]{5}F_k \quad \text{and} \quad x_1^+ \approx_3 \sqrt[4]{5}F_k.$$

The other four roots of $\mu_{\lambda(h)}$ lie in $B(0, 3)$, the disk about 0 of radius 3. So we deduce that the spectral ratio of h is at most the ratio of $|x_{-1}^{-}|$ and $|x_1^{+}|$. As these both lie in $X := B(\sqrt[4]{5}F_k, 3)$, we therefore have that

$$\omega(h) \leq \frac{\max(X)}{\min(X)} \leq \frac{\sqrt[4]{5}F_k + 3}{\sqrt[4]{5}F_k - 3} \leq 1 + \frac{6}{F_k} \leq 1 + 14\varphi^{-k}. \qquad \square$$

3. Other surfaces

We also consider the possible spectral ratios of pseudo-Anosov mapping classes on other surfaces. We summarise the results of this section in Table 1 where

- N denotes that there are no pseudo-Anosov mapping classes,
- B denotes that the spectral ratios of pseudo-Anosov mapping classes are bounded away from one,
- $P_{\leq}$ denotes that $\omega(h) \leq 1 + \frac{1}{\mathrm{poly}(|h|)}$ for some infinite family of pseudo-Anosov mapping classes, and
- E denotes that $\omega(h) \leq 1 + \frac{1}{\exp(|h|)}$ for some infinite family of pseudo-Anosov mapping classes.

genus \ punctures	0	1	2	3	4	5	6	7
0	N				B	$P_{\leq}$		
1	B		$P_{\leq}$					
2	$P_{\leq}$				E			
3								

Table 1. Spectral ratios in other surfaces.

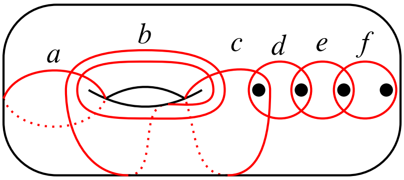
Conjecture 3.1. *In all of the $P_{\leq}$ cases, there is no family of pseudo-Anosov mapping classes whose spectral ratios converge to one exponentially. That is, none of the $P_{\leq}$ cases are actually E cases.*

Note that in all of our examples of slow convergence there is a pair of identical disjoint subsurfaces supporting much of the dynamics of the mapping class. Furthermore, some of the topology of S lies outside of these subsurfaces. If these are necessary conditions then, since Dehn twists subgroups are undistorted in $\mathrm{Mod}^{+}(S)$ [3, Theorem 1.1], the conjecture should follow.

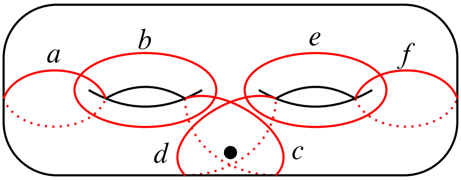
3.1. Exponential convergence. We start by considering the cases where $3g - 3 + p = 4$. Here the same argument as in Theorem 2.1 shows that a similar exponential spectral ratio bound also holds for:

- $S_{0,7}$ via the (spherical) braid $\sigma_4^{-1}(\sigma_5\sigma_6^{-1})^k(\sigma_1\sigma_2^{-1})^k\sigma_3$,
- $S_{1,4}$ via $T_d^{-1} \circ (T_e \circ T_f^{-1})^k \circ (T_a \circ T_b^{-1})^k \circ T_c$, and
- $S_{2,1}$ via $T_d^{-1} \circ (T_e \circ T_f^{-1})^k \circ (T_a \circ T_b^{-1})^k \circ T_c$.

The curves used for $S_{1,4}$ and $S_{2,1}$ are shown in Figure 3a and Figure 3b respectively.



(a) Curves on $S_{1,4}$.



(b) Curves on $S_{2,1}$.

Figure 3. Surfaces with exponential convergence.

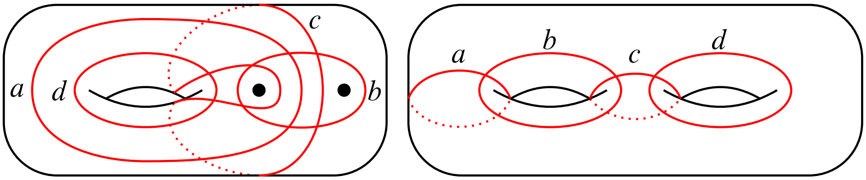
We now deal with surfaces with more punctures. If we have an exponential family on $S_{g,p}$ then by taking a common power we can obtain an additional fixed point. Removing this point gives an exponential family on $S_{g,p+1}$. Conversely, if we have an exponential family on $S_{g,p+1}$ and one of the punctures is a singularity with order at least two (for each mapping class in the family) then we may fill it and obtain an exponential family on $S_{g,p}$.

We now deal with surfaces of higher genus. Note that having an exponential family is preserved under taking covers. Thus if $g > 3$ is odd then $S_{g,0}$ is a cover of $S_{3,0}$ and so we may lift our exponential example from Section 2 to it. On the other hand, if $g > 3$ is even then $S_{g,2}$ is a cover of $S_{2,2}$. Therefore, after first adding an additional puncture to our $S_{2,1}$ example by the preceding paragraph, we can lift this exponential example to $S_{g,2}$. Now note that for this lifted family the punctures are both singularities of order at least $g/2 \geq 2$ and so can be filled. Hence we can construct an exponential family on $S_{g,0}$ in this case too.

3.2. Polynomial convergence. When $3g - 3 + p$ is even lower, a polynomial spectral ratio bound still holds for:

- $S_{0,5}$ via the (spherical) braid $\sigma_3\sigma_1^k\sigma_4^{-k}\sigma_2^{-1}$,
- $S_{1,2}$ via $T_c \circ T_a^k \circ T_d^{-k} \circ T_b^{-1}$, and
- $S_{2,0}$ via $T_c \circ T_a^k \circ T_d^{-k} \circ T_b^{-1}$.

Again, the curves used for $S_{1,2}$ and $S_{2,0}$ are shown in Figure 4a and Figure 4b respectively.



(a) Curves on $S_{1,2}$.

(b) Curves on $S_{2,0}$.

Figure 4. Surfaces with polynomial convergence.

In the case of $S_{0,5}$, for example, if $k > 4$ and $4k + 1$ is not a square then the minimal polynomial of the dilatation of the pseudo-Anosov (spherical) braid $\sigma_3\sigma_1^k\sigma_4^{-k}\sigma_2^{-1}$ is

$$1 - (2k + 5)x + (k^2 + 4k + 8)x^2 - (2k + 5)x^3 + x^4.$$

The same substitution trick allows us to explicitly compute the roots of this polynomial and so determine that the spectral ratio of this mapping classes is at most $1 + \frac{1}{\sqrt{k}}$.

Again, by taking a common power of these families we can obtain additional fixed points. By puncturing these out we can then also obtain a family of pseudo-Anosov mapping classes on $S_{0,6}$, $S_{1,3}$ and $S_{1,4}$ whose spectral ratios converge to one polynomially.

3.3. No convergence. For $S_{0,4}$, $S_{1,0}$ and $S_{1,1}$ the dilatation λ of a pseudo-Anosov h is a quadratic irrational. Thus λ has a single Galois conjugate, its reciprocal, and so $\omega(h) = \lambda^2$. However, since $\lambda + 1/\lambda \geq 3$ [4, Section 5.1.3] it follows that $\omega(h) \geq \varphi^4 \approx 6.854101\cdots$.

Finally, there are no pseudo-Anosov mapping classes on $S_{0,0}$, $S_{0,1}$, $S_{0,2}$ or $S_{0,3}$.

4. Lower bounds

In this section we show that the behaviour seen in the previous examples, where the spectral ratio converges to one exponentially with the word length of h, is actually the worst possible.

Proposition 4.1. *Suppose that S is a surface and that X is a finite generating set for* $\mathrm{Mod}^+(S)$. *If $h \in$* $\mathrm{Mod}^+(S)$ *is pseudo-Anosov then*

$$\omega(h) \geq 1 + 2^{-O(|h|)}$$

where $|h|$ denotes the word length of h with respect to X.

To prove this result, we first recall some facts about algebraic numbers.

Definition 4.2 ([10, Section 3.4]). The *height* of a polynomial $f(x) = \sum a_i x^i \in \mathbb{Z}[x]$ is

$$\operatorname{hgt}(f) := \log(\max(|a_i|)).$$

The *height* of an algebraic number $\alpha \in \overline{\mathbb{Q}}$ is $\operatorname{hgt}(\alpha) := \operatorname{hgt}(\mu_\alpha)$ where $\mu_\alpha \in \mathbb{Z}[x]$ is its minimal polynomial.

Fact 4.3. If $\alpha, \beta \in \overline{\mathbb{Q}}$ are algebraic numbers then:

- $\operatorname{hgt}(\alpha \pm \beta) \leq \operatorname{hgt}(\alpha) + \operatorname{hgt}(\beta) + 1$ [10, Property 3.3],
- $\operatorname{hgt}(\alpha\beta) \leq \operatorname{hgt}(\alpha) + \operatorname{hgt}(\beta)$ [10, Property 3.3], and
- $\operatorname{hgt}(\alpha^{-1}) = \operatorname{hgt}(\alpha)$.

Most importantly, we observe that algebraic numbers of bounded degree and height are bounded away from zero.

Lemma 4.4 ([1, Lemma 10.3]). *If* $\alpha \neq 0$ *then*

$$-\log(|\alpha|) \leq \operatorname{hgt}(\alpha) + \deg(\alpha). \qquad \square$$

Proof of Proposition 4.1. Suppose that $h \in \operatorname{Mod}^+(S)$ is pseudo-Anosov. Let $\lambda = \lambda(h)$ be the dilatation of h and λ' a distinct Galois conjugate that maximises $|\lambda'|$. Hence $\omega(h) = |\lambda/\lambda'|$.

Since S and X are fixed, we have that

$$\operatorname{hgt}(\lambda),\ \operatorname{hgt}(\lambda') \in O(|h|) \quad \text{and} \quad \deg(\lambda),\ \deg(\lambda') \in O(1).$$

Now consider $\alpha := |\lambda/\lambda'| - 1$. It follows from Fact 4.3 that

$$\operatorname{hgt}(\alpha) \in O(|h|) \quad \text{and} \quad \deg(\alpha) \in O(1).$$

As λ is a Perron number [4, page 405], the spectral ratio $\omega(h) > 1$ and so $\alpha \neq 0$. Therefore, by Lemma 4.4 we have that

$$-\log(|\alpha|) \in O(|h|).$$

Rearranging this we obtain that

$$\omega(h) \geq 1 + 2^{-O(|h|)}$$

as required. $\square$

5. Flipper

All examples in this paper were found and verified using the Python package `flipper` [2]. For example, the following Python script uses `flipper` to recreate the examples on $S_{0,7}$ that are given in Section 3.1.

```
import flipper

S = flipper.load('SB_7')

for k in range(7, 50):
        h = S.mapping_class('S_4' + 's_5S_6s_1S_2' * k + 's_3')
        f = h.dilatation().polynomial()
        X = sorted([abs(float(x)) for x in f.real_roots()])
        print(k, f, X[-1] / X[-2])
```

Random sampling of spectral ratios can also be done using `flipper`. Such experiments suggest that these exponentially slow examples are actually very rare. One such sampled distribution is shown in Figure 5. Curiously, the distribution of $\log(\lambda_1)/\log(\lambda_2)$ is essentially flat.

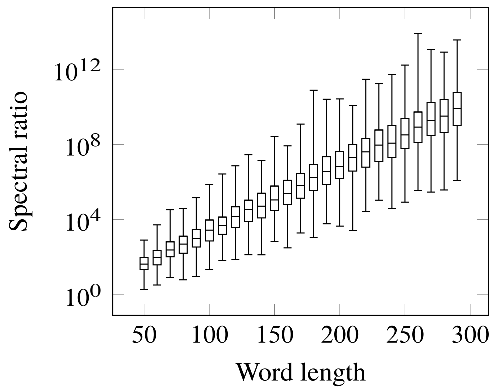


Figure 5. A sampled distribution (500 samples per word length) on $S_{0,7}$.

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