Stochastic partial differential equations with coefficients depending on VaR

by

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Contents

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Declaration

This thesis is submitted to the University of Warwick in support of my application for the degree of Doctor of Philosophy in Statistics. I declare that this thesis has not been submitted in any previous application for any degree and it contains the material which is my own original work, unless otherwise stated, cited or commonly known.

Abstract

In this paper we prove the well-posedness for a stochastic partial differential equation (SPDE) whose solution is a probability-measure-valued process. We allow the coefficients to depend on the median or, more generally, on the γ-quantile (or some its useful extensions) of the underlying distribution. Such SPDEs arise in many applications, for example, in auction system described in [2]. The well-posedness of this SPDE does not follow by standard arguments because the γ-quantile is not a continuous function on the space of probability measures equipped with the weak convergence.

1 Introduction

1.1 Statement of the problem and the main result

In 2012 Dan Crisan, Thomas G. Kurtz and Yoonjung Lee described a model of asset price determination by an infinite collection of competing traders in their article "Conditional distributions, exchangeable particle systems, and stochastic partial differential equations". In this model each trader's valuation of the assets is the solution of a stochastic differential equation (SDE). Existence of a solution for the infinite exchangeable system of SDEs is proven in [2], and it was noted that uniqueness of this solution is an interesting problem. Our goal in this paper is the well-posedness of the SPDE satisfied by the de Finetti measure of an infinite exchangeable system of valuations. We will give motivation and explanation in more detail in Section 1.3.

We start with some notions. Let us denote by ∇ the gradient with respect to the spatial variable $x \in \mathbf{R}^d$, by $\|u\|_L$ the norm of functions in $L_1\left(\mathbf{R}^d\right)$, by δ_x the Dirac measure at $x \in \mathbf{R}^d$, by $|v|$ the Euclidean norm of a vector $v \in \mathbf{R}^d$.

For any $\gamma \in (0,1)$ and any continuous positive integrable function u on $\mathbf{R}$ let us define the *quantile* $Q_\gamma(u)$ as the number satisfying

$$\int_{-\infty}^{Q_\gamma(u)} u(x)\,dx = \gamma\|u\|_L.$$

In a special case when $\gamma = 1/2$, the value $Q_\gamma(u)$ is called the *median* of u. In finances, the values of $Q_\gamma(u)$ for small γ determine the so-called *Value at Risk* (VaR) (see [6]).

More generally, for any vector $\gamma = (\gamma_1, \ldots, \gamma_d)$ with all $\gamma_k \in (0,1)$ and any continuous positive integrable function u on $\mathbf{R}^d$, let us define the quantile vector $Q_\gamma(u) = \left(Q_\gamma^1, \ldots, Q_\gamma^d\right)(u)$ with the coordinates uniquely specified by the equations

$$\int_{A_k^{Q_\gamma(u)}} u(x)\,dx = \gamma_k\|u\|_L, \qquad k = 1, \ldots, d,$$

where

$$A_k^{Q_\gamma(u)} = \left\{x \in \mathbf{R}^d \colon x^k \le Q_\gamma^k(u)\right\}.$$

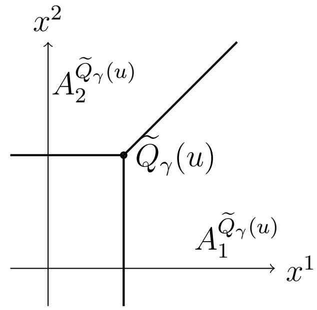
Finally, for any vector $\gamma = (\gamma_1, \dots, \gamma_d)$ with all $\gamma_k \in (0,1)$ and $\sum \gamma_k < 1$, and any continuous positive integrable function u on $\mathbf{R}^d$, let us define the vector $\widetilde{Q}_\gamma(u) = \left(\widetilde{Q}^1_\gamma, \dots, \widetilde{Q}^d_\gamma\right)(u)$ with the coordinates uniquely specified by the equations

$$\int_{A_k^{\widetilde{Q}_\gamma(u)}} u(x)\, dx = \gamma_k \|u\|_L, \qquad k = 1, \dots, d,$$

where

$$A_k^{\widetilde{Q}_\gamma(u)} = \left\{ x \in \mathbf{R}^d \colon x^k - \widetilde{Q}^k_\gamma(u) \geq 0 \vee \max_{l \neq k} \left(x^l - \widetilde{Q}^l_\gamma(u) \right) \right\}$$

and $a \vee b = \max(a, b)$. The following diagram helps to visualise sets $A_k^{\widetilde{Q}_\gamma(u)}$ when dimension $d = 2$.



Vector $\widetilde{Q}_\gamma(u)$ was introduced in [2] (the intuition behind it is given in Section 1.3), and we shall call it the *CKL-quantile*, named after the authors of [2]. In dimension $d = 1$, $\widetilde{Q}_\gamma(u) = Q_{1-\gamma}(u)$, i.e. quantile and CKL-quantile coincide.

In this paper we shall study the well-posedness of the following second order SPDE

$$\begin{aligned} u_t(x) = u_0(x) &+ \int_0^t L_s(Q_\gamma(u_s)) u_s(x)\, ds \\ &+ \int_0^t ((\beta(s,x), \nabla) u_s(x) + \alpha(s,x) u_s(x)) \circ dW_s, \qquad t \in (0, T], \end{aligned} \tag{1}$$

where L_t is an operator of the form

$$L_t(q)u = \frac{1}{2} \left(\bar{\sigma}\bar{\sigma}^T(t,x,q) \nabla, \nabla \right) u + (g(t,x,q), \nabla) u + d(t,x,q) u,$$

W_t is a standard 1-dimensional Brownian motion, functions $\alpha(t,x)$ and $\beta(t,x)$

are, respectively, 1-dimensional and $d \times 1$ vector-valued continuous functions on $[0,T] \times \mathbf{R}^d$ satisfying Condition $(E.1)_k$ below for some $k \geq 4$, functions $d(t,x,q)$, $g(t,x,q)$ and $\bar{\sigma}(t,x,q)$ are, respectively, 1-dimensional, $d \times 1$ vector-valued and $d \times d''$ matrix-valued continuous functions on $[0,T] \times \mathbf{R}^d \times \mathbf{R}^d$ satisfying Condition $(E.2)$ below. Differential $\circ dW_t$ in equation (1) is the Stratonovich differential. We will consider this equation for both quantile $Q_\gamma(u_t)$ and CKL-quantile $\widetilde{Q}_\gamma(u_t)$.

Condition $(E.1)_k$**.** *Functions $\alpha(t,x)$, $\beta^i(t,x)$ are continuous and bounded on $[0,T] \times \mathbf{R}^d$, $(k+2)$-times differentiable in x and continuously differentiable in t with bounded derivatives.*

Condition $(E.2)$**.** *A square matrix-valued function $\bar{\sigma}\bar{\sigma}^T(t,x,q)$ is uniformly elliptic, i.e. there exists some constant $m > 0$ such that*

$$m^{-1}|\xi|^2 \leq \left(\bar{\sigma}\bar{\sigma}^T(t,x,q)\xi,\xi\right) \leq m|\xi|^2$$

holds for all $\xi, x, q \in \mathbf{R}^d$ and $t \in [0,T]$. Functions $\bar{\sigma}^{il}(t,x,q)$, $g^i(t,x,q)$, $d(t,x,q)$ are continuous and bounded on $[0,T] \times \mathbf{R}^d \times \mathbf{R}^d$, twice continuously differentiable in x with bounded derivatives, twice continuously differentiable in q, and uniformly Lipschitz continuous in q.

Let $u_t \in L_1\left(\mathbf{R}^d\right) \cap C^2\left(\mathbf{R}^d\right)$ be a positive continuous C^2-semimartingale (see definitions in the fifth paragraph of Section 1.2). Suppose u_t is continuous in $L_1\left(\mathbf{R}^d\right)$. In Appendix E we show that both quantile $Q_\gamma(u_t)$ and CKL-quantile $\widetilde{Q}_\gamma(u_t)$ are continuous in t. Then it is called a *solution with the initial value* u_0 if it satisfies equation (1) for any $x \in \mathbf{R}^d$ and $t \in (0,T]$ a.s.

Throughout this paper we will use the Stratonovich integrals only. It allows us to deal with the stochastic analysis for irregular functionals of time (e.g. semimartingales) in the same way as the deterministic analysis for regular (smooth) functionals of time, due to the Itô's formula for the Stratonovich differential.

As an auxiliary problem we shall consider the equation

$$\begin{aligned} u_t(x) &= u_0(x) + \int_0^t L_s(q_s)u_s(x)\,ds \\ &\quad + \int_0^t ((\beta(s,x),\nabla)u_s(x) + \alpha(s,x)u_s(x)) \circ dW_s, \qquad t \in (0,T], \end{aligned} \tag{2}$$

with a given d-dimensional continuous semimartingale q_t. In Section 2 we will show that if $u_0 \in L_1(\mathbf{R}^d) \cap C(\mathbf{R}^d)$ is a bounded strictly positive function on $\mathbf{R}^d$, then there exists a unique bounded solution $u_t(x)$ of equation (2) such that it is a continuous C^3-process (see definitions in the fifth paragraph of Section 1.2). Moreover, by Proposition 2.8 this solution $u_t(x)$ is strictly positive. Under the following Condition $(E.3)$ and $\|u_0\|_L = 1$, we will show in motivational Section 1.3 that $u_t(x)$ is the density function of some measure $v_t \in \mathcal{P}(\mathbf{R}^d)$. (See system (14) and equation (17).)

Condition $(E.3)$**.** *Functions $\bar\sigma$, g, d, α and β satisfy*

$$\frac{1}{2}\sum_{i,j=1}^{d} \frac{\partial^2 \left(\bar\sigma\bar\sigma^T\right)^{ij}}{\partial x^i \partial x^j}(t,x,q) + d(t,x,q) = \sum_{i=1}^{d} \frac{\partial g^i}{\partial x^i}(t,x,q),$$

$$\alpha(t,x) = \sum_{i=1}^{d} \frac{\partial \beta^i}{\partial x^i}(t,x)$$

for all $x, q \in \mathbf{R}^d$ and $t \in [0,T]$.

The preservation of the L_1-norm of u_t will be used in Proposition 2.11.

For the following uniqueness theorems we need Sobolev spaces. Recall that Sobolev spaces $H_1^k(\mathbf{R}^d)$ are defined as the spaces of integrable functions on $\mathbf{R}^d$ with all derivatives up to and including order k being well defined in the distribution sense and being again integrable functions. The norm $\|u\|_{H_1^k(\mathbf{R}^d)}$ is defined as the sum of L_1-norms of u and all its partial derivatives up to and including order k.

Theorem 1.1. *For a given $T > 0$ and vector $\gamma = (\gamma_1, \ldots, \gamma_d)$ with all $\gamma_k \in (0,1)$, consider equation* (1) *in case of the quantile vector Q_γ. Assume that Conditions $(E.1)_k$, $(E.2)$ and $(E.3)$ hold for some $k \geq 4$. Then for any bounded strictly positive $u_0 \in H_1^2(\mathbf{R}^d) \cap C(\mathbf{R}^d)$, there exists a unique bounded solution $u_t(x)$ of equation* (1) *with initial condition u_0, such that it is a continuous C^3-process.*

Theorem 1.2. *For a given $T > 0$, vector $\gamma = (\gamma_1, \ldots, \gamma_d)$ with all $\gamma_k \in (0,1)$ and $\sum \gamma_k < 1$, and dimension $d \leq 3$, consider equation* (1) *in case of the CKL-quantile vector $\widetilde{Q}_\gamma$. Assume that Conditions $(E.1)_k$, $(E.2)$ and $(E.3)$*

hold for some $k \geq 4$. Then for any bounded strictly positive $u_0 \in H_1^2\left(\mathbf{R}^d\right) \cap C\left(\mathbf{R}^d\right)$, there exists a unique bounded solution $u_t(x)$ of equation (1) *with initial condition u_0, such that it is a continuous C^3-process.*

The smooth dependence of the solution of equation (1) on initial data will be considered in Section 4. Also, in Section 5, we will prove the well-posedness of equation (1) in the case of multidimensional W_t and some additional restrictions on α and β.

Remark. *The restriction on the dimension $d \leq 3$ in Theorem 1.2 is due to the inability to prove Proposition B.1 in Appendix B for $d \geq 4$.*

Some of the simplified versions of equation (1) has been considered before. The well-posedness and sensitivity analysis for equation (1) with $\alpha(x)$, $\beta(x)$, $\bar{\sigma}(x)$ depending only on x and $g(t, x, [u])$, $d(t, x, [u])$ depending in a smooth way on the underlying function u (not on its quantile) was developed in [9]. The well-posedness for equation (1) in case of the quantile vector Q_γ and $\alpha, \beta = 0$ was done in [7].

In Section 1.2 we will recall the basic definitions and notions of stochastic calculus such as stochastic integrals, differential equations and stochastic flows. These results are taken from [11] and adapted to our needs for solving equation (2). Section 1.3 is devoted to motivations for considering our equation (1). We have already mentioned before that SPDEs of this type arise in [2] as an application to finances, so in this section we will briefly review the introductory and the multiple assets sections of that article, and rewrite some of the SPDEs in the Stratonovich form. While the existence of a solution to system (13) is proven in [2], Theorem 1.2 provides the uniqueness of the solution to system (14), which is a special case of system (13). In Section 2.1 we will adapt the Hiroshi Kunita's method of stochastic characteristics for solving SPDEs to equation (2). Also we will prove some essential properties of stochastic characteristics that are used in further sections. In Section 2.2 we will generalize some of the propositions from [7] to the case of equation (1) for both quantile $Q_\gamma(u_t)$ and CKL-quantile $\widetilde{Q}_\gamma(u_t)$. Section 3 contains the proof of Theorems 1.1 and 1.2. In Section 4 we will make the sensitivity analysis of the solution of equation (1) with respect to initial data. The case of multidimensional W_t will be considered in Section 5.

1.2 Preliminaries

In this section we will recall the basic definitions and notions of stochastic calculus adapted to our needs in this paper.

Quadratic variational processes

Let X_t, $t \in [0, T]$ be a continuous stochastic process, $\Delta = \{0 = t_0 < \ldots < t_l = T\}$ be a partition of the interval $[0, T]$ and let $|\Delta| = \max_k(t_{k+1} - t_k)$. Define a continuous process $\langle X \rangle_t^\Delta$ associated with the partition Δ by

$$\langle X \rangle_t^\Delta = \sum_{k=0}^{l-1} \left(X_{t \wedge t_{k+1}} - X_{t \wedge t_k} \right)^2 ,$$

where $t \wedge s = \min(t, s)$. This process is called the *quadratic variational process* or simply the *quadratic variation of X_t associated with the partition Δ.*

Let $\{\Delta_n\}$ be a sequence of partitions with $|\Delta_n| \to 0$. Suppose that the limit of $\left\{ \langle X \rangle_t^{\Delta_n} \right\}$ exists in probability for every t and it is independent of the choice of sequences $\{\Delta_n\}$ a.s. Then it is called the *quadratic variational process* or simply the *quadratic variation* of X_t and is denoted by $\langle X \rangle_t$ or by $\langle X_t \rangle$.

The quadratic variation does not exist for any continuous stochastic process.

Theorem 1.3 (Theorem 2.2.5 in [11])**.** *Let M_t be a continuous localmartingale. Then there exists a continuous increasing process $\langle M \rangle_t$ such that $\langle M \rangle_t^\Delta$ converges uniformly to $\langle M \rangle_t$ in probability.*

Theorem 1.4 (Theorem 2.2.10 in [11])**.** *Let X_t be a continuous semimartingale. Then $\langle X \rangle_t^\Delta$ converges uniformly to $\langle M \rangle_t$ in probability as $|\Delta| \to 0$, where M_t is the localmartingale part of X_t.*

Let M_t and N_t be continuous localmartingales. Define the *joint quadratic variation of M_t and N_t associated with the partition $\Delta = \{0 = t_0 < \ldots < t_l = T\}$* by

$$\langle M, N \rangle_t^\Delta = \sum_{k=0}^{l-1} \left(M_{t \wedge t_{k+1}} - M_{t \wedge t_k} \right) \left(N_{t \wedge t_{k+1}} - N_{t \wedge t_k} \right) . \tag{3}$$

Theorem 1.5 (Theorem 2.2.11 in [11]). *$\langle M, N\rangle_t^\Delta$ converges uniformly to a continuous process of bounded variation in probability as $|\Delta| \to 0$.*

The limit is denoted by $\langle M, N\rangle_t$ or $\langle M_t, N_t\rangle$ and is called the *joint quadratic variation* of M_t and N_t.

Finally, the joint quadratic variation $\langle X, Y\rangle_t^\Delta$ of continuous semimartingales X_t and Y_t associated with the partition Δ is defined in the same way as (3).

Theorem 1.6 (Theorem 2.2.14 in [11]). *$\langle X, Y\rangle_t^\Delta$ converges uniformly in probability to a continuous process of bounded variation $\langle X, Y\rangle_t$. If M_t and N_t are the parts of localmartingales of X_t and Y_t, respectively, then $\langle X, Y\rangle_t$ coincides with $\langle M, N\rangle_t$.*

The above process $\langle X, Y\rangle_t$ is called the *joint quadratic variation of X_t and Y_t*.

Stochastic integrals and Itô's formula

Let M_t and f_t be a continuous localmartingale and a continuous $(\mathscr{F}_t)$-adapted process respectively, and let $\Delta = \{0 = t_0 < \ldots < t_l = T\}$ be a partition of $[0, T]$. Define

$$L_t^\Delta \equiv \sum_{k=0}^{l-1} f_{t\wedge t_k}\left(M_{t\wedge t_{k+1}} - M_{t\wedge t_k}\right).$$

It is shown in [11] that L_t^Δ converges uniformly in probability to a continuous localmartingale as $|\Delta| \to 0$. This limit is denoted by $\int_0^t f_s\, dM_s$ and is called the *Itô's integral of f_t by dM_t*.

Let X_t be a continuous semimartingale decomposed as $X_t = M_t + A_t$, where M_t is a continuous localmartingale and A_t is a continuous process of bounded variation. For an arbitrary continuous $(\mathscr{F}_t)$-adapted process f_t we define the *Itô's integral by dX_t*:

$$\int_0^t f_s\, dX_s \equiv \int_0^t f_s\, dM_s + \int_0^t f_s\, dA_s.$$

It is a continuous semimartingale.

Let us define another stochastic integral by the differential $\circ dX_t$:

$$\int_0^t f_s \circ dX_s = \lim_{|\Delta|\to 0} \sum_{k=0}^{l-1} \frac{1}{2} \left(f_{t\wedge t_{k+1}} + f_{t\wedge t_k} \right) \left(X_{t\wedge t_{k+1}} - X_{t\wedge t_k} \right).$$

If this limit converges in probability, it is called the *Stratonovich integral of f_s by $\circ dX_s$.*

Theorem 1.7 (Theorem 2.3.5 in [11])**.** *If f_t is a continuous semimartingale, the Stratonovich integral is well defined. Further it satisfies*

$$\int_0^t f_s \circ dX_s = \int_0^t f_s \, dX_s + \frac{1}{2} \langle f, X \rangle_t.$$

Finally, let us recall the celebrated *Itô's formula.*

Theorem 1.8 (Theorem 2.3.11 in [11])**.** *Let $X_t = \left(X_t^1, \ldots, X_t^d \right)$ be a continuous semimartingale. If $F\left(x^1, \ldots, x^d\right)$ is a C^2-function, then $F(X_t)$ is a continuous semimartingale and satisfies the formula*

$$F(X_t) - F(X_0) = \sum_{i=1}^d \int_0^t F_{x^i}(X_s) \, dX_s^i + \frac{1}{2} \sum_{i,j=1}^d \int_0^t F_{x^i x^j}(X_s) \, d\left\langle X^i, X^j \right\rangle_s.$$

Furthermore if F is a C^3-function, then we have

$$F(X_t) - F(X_0) = \sum_{i=1}^d \int_0^t F_{x^i}(X_s) \circ dX_s^i.$$

Stochastic differential equations

Let $\beta(x,t) = \left(\beta^1(x,t), \ldots, \beta^d(x,t) \right)$ be a continuous function on $\mathbf{R}^d \times [0,T]$, continuously differentiable in t with bounded derivatives. Let W_t be a standard 1-dimensional Brownian motion. Then $\int_0^t \beta(x,s) \, dW_s$ is a continuous martingale for every $x \in \mathbf{R}^d$. In the third paragraph of Section 2.1.1 we will show that $\int_0^t \beta(x,s) \, dW_s$ can be rewritten in an explicit form

$$\int_0^t \beta(x,s) \, dW_s = -\int_0^t \frac{\partial \beta}{\partial s}(x,s) W_s \, ds + \beta(x,t) W_t.$$

Therefore, we see that $\int_0^t \beta(x,s)\,dW_s$ is a continuous martingale with values in $C = C\left(\mathbf{R}^d \colon \mathbf{R}^d\right)$.

First of all let us recall Itô's stochastic differential equation based on $\int_0^t \beta(x,s)\,dW_s$. The term "based on" comes from [11], where stochastic integrals and stochastic differential equations are defined for a more general class of continuous semimartingales with spatial parameters. Assume that $\beta^i(x,t)$ are differentiable in x, $\beta^i(x,t)$ and all their derivatives are bounded functions on $\mathbf{R}^d \times [0,T]$. Let $t_0 \in [0,T]$ and $x_0 \in \mathbf{R}^d$ be given. A continuous $\mathbf{R}^d$-valued process φ_t, $t \in [t_0, T]$ adapted to $(\mathscr{F}_t)$ is called a *solution of Itô's stochastic differential equation based on* $\int_0^t \beta(x,s)\,dW_s$ *starting at* x_0 *at time* t_0 if it satisfies

$$\varphi_t = x_0 + \int_{t_0}^t \beta(\varphi_s, s)\,dW_s. \tag{4}$$

By Theorem 3.4.1 in [11], for each t_0 and x_0, Itô's equation (4) has a unique solution.

Next let us recall the notion of Stratonovich stochastic differential equation based on $\int_0^t \beta(x,s)\,dW_s$. Assume that $\beta^i(x,t)$ are 3-times differentiable in x, $\beta^i(x,t)$ and all their derivatives are bounded functions on $\mathbf{R}^d \times [0,T]$. Let $t_0 \in [0,T]$ and $x_0 \in \mathbf{R}^d$ be given. A continuous $\mathbf{R}^d$-valued semimartingale φ_t, $t_0 \le t \le T$ is called a *solution of Stratonovich stochastic differential equation based on* $\int_0^t \beta(x,s)\,dW_s$ *starting at* x_0 *at time* t_0 if it satisfies

$$\varphi_t = x_0 + \int_{t_0}^t \beta(\varphi_s, s)\circ dW_s. \tag{5}$$

By Theorem F.3 in Appendix F, for each t_0 and x_0, Stratonovich equation (5) has a unique solution.

Backward integrals and backward equations

In this paragraph we shall recall backward stochastic integrals and backward stochastic differential equations. The arguments are completely parallel to those of (forward) stochastic integrals and stochastic differential equations. The only difference is that these are defined to the backward direction.

A family $\{\mathscr{F}_{s,t} \colon 0 \le s \le t \le T\}$ of sub σ-fields of $\mathscr{F}$ is called a *filtration with two parameters* if it contains all null sets and satisfies $\mathscr{F}_{s,t} \subset \mathscr{F}_{s',t'}$,

$\bigcap_{\varepsilon>0}\mathscr{F}_{s,t+\varepsilon}=\mathscr{F}_{s,t}$ and $\bigcap_{\varepsilon>0}\mathscr{F}_{s-\varepsilon,t}=\mathscr{F}_{s,t}$ for all $s'\leq s\leq t\leq t'$. A continuous process $\widehat{M}_t$ is called a *backward martingale* adapted to $(\mathscr{F}_{s,t})$ if it is integrable, $\widehat{M}_t-\widehat{M}_s$ is $\mathscr{F}_{s,t}$-measurable and satisfies $E\left[\widehat{M}_r-\widehat{M}_t\middle|\mathscr{F}_{s,t}\right]=\widehat{M}_s-\widehat{M}_t$ for any $r\leq s\leq t$. A *backward localmartingale* is defined similarly to the (forward) localmartingale. Let $\widehat{X}_t$ be a continuous process such that $\widehat{X}_t-\widehat{X}_s$ is $(\mathscr{F}_{s,t})$-adapted. If $\widehat{X}_t$ can be written as the sum of a continuous backward localmartingale and a process of bounded variation, then it is called a *backward semimartingale*.

For example, let W_t be a standard 1-dimensional Brownian motion. For $s<t$, let $\mathscr{F}_{s,t}$ be the least sub σ-field of $\mathscr{F}$ containing all null sets and $\bigcap_{\varepsilon>0}\sigma(W_u-W_v\colon s-\varepsilon\leq u,v\leq t+\varepsilon)$. Then W_t is a continuous backward martingale adapted to $(\mathscr{F}_{s,t})$.

Now let X_t be a continuous backward semimartingale. Let us fix an arbitrary time t, and let f_s, $s\in[0,t]$ be a continuous $(\mathscr{F}_{s,t})$-adapted process. If the right hand side of the following formula converges in probability, then it is called the *backward Itô's integral of f_s by $\hat{d}X_s$*

$$\int_s^t f_r\,\hat{d}X_r=\lim_{|\Delta|\to 0}\sum_{k=0}^{n-1}f_{t_{k+1}\vee s}\big(X_{t_{k+1}\vee s}-X_{t_k\vee s}\big)\,,$$

where $\Delta=\{0=t_0<\ldots<t_n=t\}$, $t\vee s=\max(t,s)$. It is a continuous backward semimartingale with respect to s. Suppose that f_s is a continuous backward semimartingale. Since the right hand side of the following formula converges in probability, it is called the *backward Stratonovich integral of f_s by $\hat{d}X_s$*.

$$\int_s^t f_r\circ\hat{d}X_r=\lim_{|\Delta|\to 0}\sum_{k=0}^{n-1}\frac{1}{2}\left(f_{t_{k+1}\vee s}+f_{t_k\vee s}\right)\big(X_{t_{k+1}\vee s}-X_{t_k\vee s}\big)\,.$$

The backward Itô's and Stratonovich integrals are related by

$$\int_s^t f_r\circ\hat{d}X_r=\int_s^t f_r\,\hat{d}X_r-\frac{1}{2}\left\langle f,X\right\rangle_{s,t}\,,$$

where $\langle\ ,\ \rangle_{s,t}$ denotes the joint quadratic variation for the time interval $[s,t]$.

Let f_t and X_t be continuous forward-backward semimartingales. While the forward Stratonovich integral $\int_0^t f_s \circ dX_s$ is a continuous forward semimartingale with respect to t and the backward Stratonovich integral $\int_s^t f_r \circ \hat{d}X_r$ is a continuous backward semimartingale with respect to s, these two integrals coincide when $s = 0$.

Next, a continuous $(\mathscr{F}_{s,t_0})$-adapted process φ_s, $s \in [0, t_0]$ with values in $\mathbf{R}^d$ is called the solution of the *backward Itô's stochastic differential equation based on* $\int_0^t \beta(x,s)\, dW_s$ *starting at* x_0 *at time* t_0 if it satisfies

$$\varphi_s = x_0 - \int_s^{t_0} \beta(\varphi_r, r)\, \hat{d}W_r.$$

The solution of the *backward Stratonovich stochastic differential equation* is defined similarly.

Semimartingales with spatial parameters

In this paragraph let us recall the definition of continuous semimartingales according to their regularity with respect to the spatial parameter.

We can regard a family of real valued processes $F(x,t)$ with parameter $x \in \mathbf{R}^d$ as a random field with double parameters x and t. If $F(x,t,\omega)$ is a continuous function of x a.s. for any t, we can regard $F(\cdot,t)$ as a *stochastic process with values in* $C = C\left(\mathbf{R}^d \colon \mathbf{R}\right)$ or a *C-valued process.* If $F(x,t,\omega)$ is m-times continuously differentiable with respect to x a.s. for any t, it can be regarded as a *stochastic process with values in* C^m or a *C^m-valued process.* In the case where $F(x,t)$ is a continuous process with values in C^m, it is called a *continuous C^m-process.*

Let $\alpha = (\alpha_1, \ldots, \alpha_d)$ be a multi index of non-negative integers and $|\alpha| = \alpha_1 + \ldots + \alpha_d$. Set

$$D_x^\alpha f = \frac{\partial^{|\alpha|} f}{(\partial x^1)^{\alpha_1} \ldots (\partial x^d)^{\alpha_d}}.$$

A family of continuous localmartingales $M(x,t), x \in \mathbf{R}^d$ is said to be a *continuous localmartingale with values in* C^m or a *continuous C^m-localmartingale* if $M(x,t)$ is a continuous C^m-process, and for each α with $|\alpha| \le m$, $D_x^\alpha M(x,t), x \in \mathbf{R}^d$ is a family of continuous localmartingales.

Let $F(x,t)$, $x \in \mathbf{R}^d$ be a family of continuous semimartingales decomposed as $F(x,t) = M(x,t) + B(x,t)$, where $M(x,t)$ is a continuous localmartingale and $B(x,t)$ is a continuous process of bounded variation. A family of continuous semimartingales $F(x,t)$, $x \in \mathbf{R}^d$ is said to *belong to the class* C^m or simply to be a C^m*-semimartingale* if $M(x,t)$ is a continuous C^m-localmartingale, $B(x,t)$ is a continuous C^m-process and $D_x^\alpha B(x,t)$, $x \in \mathbf{R}^d$, $|\alpha| \leq m$ are all processes of bounded variation. A continuous *backward* C^m*-semimartingale* is defined similarly.

Stochastic flows

In this paragraph let us recall the definition of the stochastic flow of homeomorphisms and diffeomorphisms on the Euclidean space. Let $\varphi_{s,t}(x,\omega)$, $s,t \in [0,T]$, $x \in \mathbf{R}^d$ be a continuous $\mathbf{R}^d$-valued random field. Then for any s,t and almost all ω, $\varphi_{s,t}(\omega) \equiv \varphi_{s,t}(\cdot,\omega)$ defines a continuous map from $\mathbf{R}^d$ into itself. It is called a *stochastic flow of homeomorphisms* if there exists a subset $N \subset \Omega$ of probability 1 such that for any $\omega \in N$, the family of continuous maps $\{\varphi_{s,t}(\omega)\colon s,t \in [0,T]\}$ defines a *flow of homeomorphisms*, i.e. it satisfies the following properties:

1. $\varphi_{s,u}(\omega) = \varphi_{t,u}(\omega) \circ \varphi_{s,t}(\omega)$ holds for all s,t,u where $\circ$ denotes the composition of maps,

2. $\varphi_{s,s}(\omega) =$ identity map for all s,

3. the map $\varphi_{s,t}(\omega)\colon \mathbf{R}^d \to \mathbf{R}^d$ is an onto homeomorphism for all s,t.

Further $\varphi_{s,t}(\omega)$ is called a *stochastic flow of* C^k*-diffeomorphisms*, if it satisfies the following condition

4. $\varphi_{s,t}(x,\omega)$ is k-times differentiable with respect to x for all s,t and the derivatives are continuous in (s,t,x).

1.3 Motivation

In mathematical finance the price process for an asset is modeled by a stochastic process $\{S_t, t \geq 0\}$, and an important objective is to find a good model for asset prices. A famous example is Brownian motion introduced by Bachelier in 1900 (see [1]) as a model for the price fluctuations on the Paris stock market. In 1964, Samuelson (see [18]) suggested the use of geometric Brownian motion as a suitable model. Since then a number of other stochastic processes have been used to model price processes.

Let us consider an asset pricing model, where the price of a single asset is determined through a continuous-time *auction system*. Assume that N traders compete for n units of the asset, where $n < N$. Each trader owns either one share or no shares. At any point in time, the traders submit their bid prices, and those who submit the highest n bids each own a share. Denote the log of the bid price (or valuation) of the i-th trader at time t by X_t^i, and the log of the stock price by S_t^N. Throughout this section, by bid price or stock price we will understand the log of these prices omitting the word "log". Based on the rules of this auction system, the market clearing condition for the equilibrium stock price S_t^N is given by:

$$\sum_{i=1}^{N} \mathbf{1}_{\left\{X_t^i \geq S_t^N\right\}} = n$$
$$\text{(Demand)} = \text{(Supply)}.$$

If we denote the empirical measure of $\left\{X_t^1, \ldots, X_t^N\right\}$ by

$$v_t^N = \frac{1}{N} \sum_{i=1}^{N} \delta_{X_t^i},$$

then the market clearing condition rewrites as

$$v_t^N \left[S_t^N, \infty\right) = \frac{n}{N}.$$

As N tends to infinity and $\frac{n}{N} \to 1 - \alpha$, for some $\alpha \in (0, 1)$, the stock price

$$S_t = \lim_{N \to \infty} S_t^N$$

becomes the *α-quantile process* V_t^α of the measure

$$v_t = \lim_{N \to \infty} v_t^N$$

that is the limit of the empirical distribution v_t^N of the bid prices.

Let us consider the following geometric mean-reverting process as a model for X_t^i:

$$X_t^i = X_0^i + \beta \int_0^t \left(S_s - X_s^i\right) ds + \sigma W_t + \bar{\sigma} B_t^i,$$

where β, σ and $\bar{\sigma}$ are some positive constants. In this example, each investor takes the stock price as a signal for the value of the asset and adjusts his or her valuation upward if it is below the stock price and downward if it is above. The parameter β evaluates the mean reversion rate toward S_t. The higher this parameter value is, the faster the positions tend to mean-revert. The Brownian motion W_t models the common market noise, while the Brownian motion B_t^i models the trader's own uncertainty.

More generally, let us consider the following system

$$X_t^i = X_0^i + \int_0^t f\left(X_s^i, V_s^\alpha\right) ds + \int_0^t \sigma\left(X_s^i, V_s^\alpha\right) dW_s + \int_0^t \bar{\sigma}\left(X_s^i, V_s^\alpha\right) dB_s^i, \quad (6)$$

where, asymptotically (as the number of traders N tends to infinity), the stock price V_t^α should be determined by

$$V_t^\alpha = \inf\left\{x \in \mathbf{R} \colon v_t(-\infty, x] \geq \alpha\right\} \quad (7)$$

and

$$v_t = \lim_{m\to\infty} \frac{1}{m} \sum_{i=1}^m \delta_{X_t^i}. \quad (8)$$

We assume that $\{X_0^i\}$ is exchangeable and require the solutions $\{X_t^i\}$ to be exchangeable so that the limit in formula (8) exists by de Finetti's theorem (see Theorem 2.2 in [4]). The process W_t is a d'-dimensional standard Brownian motion, common to all diffusions, while the processes $\{B_t^i\}_{i\geq 1}$ are mutually independent 1-dimensional standard Brownian motions.

Systems of this type in the case when the coefficients are Lipschitz functions of v in the Wasserstein metric on $\mathcal{P}\left(\mathbf{R}^d\right)$ have been considered in [12]–[14].

It is shown in [2] that v is a solution of the stochastic partial differential equation

$$\langle \varphi, v_t \rangle = \langle \varphi, v_0 \rangle + \int_0^t \langle L(S_s)\varphi, v_s \rangle\, ds + \int_0^t \langle \sigma(\cdot, S_s)\varphi', v_s \rangle\, dW_s, \quad (9)$$

where $\langle \varphi, v_t \rangle$ denotes

$$\langle \varphi, v_t \rangle = \int_{\mathbf{R}} \varphi(x) v_t(dx)$$

and

$$L(S)\varphi = \frac{1}{2}\left(\sigma(x,S)^2 + \bar{\sigma}(x,S)^2\right)\frac{d^2\varphi}{dx^2} + f(x,S)\frac{d\varphi}{dx}.$$

Conditions. *Assume that the coefficients* $f\colon \mathbf{R}\times\mathbf{R} \to \mathbf{R}, \sigma\colon \mathbf{R}\times\mathbf{R} \to \mathbf{R}^{d'}, \bar{\sigma}\colon \mathbf{R}\times\mathbf{R} \to \mathbf{R}$ *satisfy the following*

1. $f, \sigma, \bar{\sigma}$ *are continuous functions, uniformly Lipschitz in the first argument*[1]*,*

2. $\bar{\sigma}(x,y)$ *is positive,*

3. *There exists a constant* K *such that* $f, \sigma, \bar{\sigma}$ *are bounded by* $K(1+|x|+|y|)$.

The following existence theorem is proven in [2].

Theorem 1.9. *Suppose that Conditions above hold, that* $\{X_0^i\}$ *is exchangeable*[2]*, and that* $E(|X_0^i|) < \infty$. *Then there exists a weak solution for the system* (6)–(8) *such that* $\{X^i\}$ *is exchangeable and* v *satisfies the stochastic partial differential equation* (9).

However, the authors do not have a general uniqueness theorem for equation (6) (it is not covered by the theory developed in this paper either) and they explicitly state that this is an interesting and important problem.

Now we extend the asset price model (6)–(8) to a market with multiple assets. In order to specify the model, we need to identify an appropriate market clearing condition. Suppose there are d assets and N competing traders. Each trader owns at most one unit of one of the assets. If the prices of the assets are $s_1, \ldots, s_d$ and the value that the i-th trader places on the k-th asset is $x_{i,k}$, then the i-th trader will buy the k-th asset provided

$$x_{i,k} - s_k \geq 0 \vee \max_{l \neq k}(x_{i,l} - s_l). \tag{10}$$

[1]There exists a constant c_1 such that $|f(x_1,y) - f(x_2,y)| \leq c_1|x_1 - x_2|$ holds for all $x_1, x_2, y \in \mathbf{R}$ with a similar inequality holding for σ and $\bar{\sigma}$.

[2]Let $X = \{X^i\}_{i=1}^{\infty}$ be a sequence of real-valued random variables. We say that X is exchangeable if, for every finite set $\{k_1, \ldots, k_j\}$ of distinct indices, $\left(X^{k_1}, \ldots, X^{k_j}\right)$ is equal in distribution to $\left(X^1, \ldots, X^j\right)$. (See Definition 2.1 in [4].)

Suppose there are n_k units of the k-th asset and $\sum_{k=1}^{d} n_k < N$. Then the prices should be set so that the assets can be allocated to the traders in such a way that each unit of the k-th asset goes to a trader whose valuations satisfy (10) and each trader with valuations satisfying $x_{i,k} - s_k > 0 \vee \max_{l \neq k}(x_{i,l} - s_l)$ receives a unit of asset k. Define

$$A_k^s = \left\{ i \colon x_{i,k} - s_k \geq 0 \vee \max_{l \neq k}(x_{i,l} - s_l) \right\}$$

and

$$A_0^s = \{ i \colon x_{i,k} \leq s_k, k = 1, \ldots, d \} .$$

Each trader who receives asset k must have index in A_k^s, and each trader who does not receive any asset must have index in A_0^s. Denote $n_0 = N - \sum_{k=1}^{d} n_k$, then the classical marriage theorem states that this allocation can be performed if and only if for each $I \subset \{0, \ldots, d\}$,

$$\# \bigcup_{k \in I} A_k^s \geq \sum_{k \in I} n_k. \tag{11}$$

Assume that N tends to infinity, $\frac{n_k}{N} \to a_k$ and

$$\frac{1}{N} \sum_{i=1}^{N} \delta_{x_i} \Rightarrow v \in \mathcal{P}\left(\mathbf{R}^d\right) .$$

Now for each $s \in \mathbf{R}^d$, let

$$A_k^s = \left\{ x \in \mathbf{R}^d \colon x^k - s^k \geq 0 \vee \max_{l \neq k}\left(x^l - s^l\right) \right\}, \quad k = 1, \ldots, d,$$

and

$$A_0^s = \left\{ x \in \mathbf{R}^d \colon x^k \leq s^k, k = 1, \ldots, d \right\} .$$

The continuous version of the market clearing condition (11) becomes

$$v\left(\bigcup_{k \in I} A_k^s \right) \geq \sum_{k \in I} a_k. \tag{12}$$

Lemma 3.1 in [2] states that for each $v \in \mathcal{P}\left(\mathbf{R}^d\right)$, there exists $s \in \mathbf{R}^d$ such that for each $I \subset \{0, \ldots, d\}$, inequality (12) holds.

Let us consider the following infinite system

$$X_t^i = X_0^i + \int_0^t f\left(X_s^i, S_s\right) ds + \int_0^t \sigma\left(X_s^i, S_s\right) dW_s + \int_0^t \bar{\sigma}\left(X_s^i, S_s\right) dB_s^i, \quad (13)$$

where each X_t^i is a d-dimensional stochastic process and S_t is the vector of prices determined by the requirement that

$$v_t\left(\bigcup_{k\in I} A_k^{S_t}\right) \geq \sum_{k\in I} a_k.$$

That is, for the i-th trader, X_t^i gives the valuations at time t of the d assets, and

$$v_t = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^n \delta_{X_t^i}$$

gives the distribution of valuations by the infinite collection of traders. In system (13), W_s and B_s^i are d' and d''-dimensional standard Brownian motions.

Existence of a solution is proven in [2]. Also it is shown that v_t has a strictly positive density u_t which ensures that S_t is uniquely determined by v_t (S_t is the CKL-quantile $\widetilde{Q}_a(u_t)$ of the density function u_t), and v_t satisfies the stochastic partial differential equation

$$\langle \varphi, v_t \rangle = \langle \varphi, v_0 \rangle + \int_0^t \langle L(S_s)\varphi, v_s \rangle\, ds + \int_0^t \left\langle \nabla\varphi^T \sigma(\cdot, S_s), v_s \right\rangle dW_s,$$

where

$$L(S)\varphi = \frac{1}{2}\left(\left(\sigma\sigma^T + \bar{\sigma}\bar{\sigma}^T\right)(x, S)\nabla, \nabla\right)\varphi + (f(x, S), \nabla)\varphi.$$

However, the authors do not have a general uniqueness theorem for system (13), it is not covered by the theory developed in this paper either.

Finally, let us consider the following special case of system (13)

$$X_t^i = X_0^i + \int_0^t \left(\sum_{j=1}^d \frac{\partial}{\partial x^j} \left(\sum_{l=1}^{d''} \bar{\sigma}^{\cdot l} \bar{\sigma}^{jl} \right) \left(X_s^i, S_s \right) + \frac{1}{2} \sum_{j=1}^d \sum_{k=1}^{d'} \beta^{jk} \frac{\partial \beta^{\cdot k}}{\partial x^j} \left(X_s^i \right) \right.$$
$$\left. - g\left(X_s^i, S_s \right) \right) ds - \int_0^t \beta\left(X_s^i \right) dW_s + \int_0^t \bar{\sigma}\left(X_s^i, S_s \right) dB_s^i. \tag{14}$$

The corresponding equation on the measure v_t becomes

$$\langle \varphi, v_t \rangle = \langle \varphi, v_0 \rangle + \int_0^t \langle L(S_s)\varphi, v_s \rangle \, ds - \int_0^t \left\langle \nabla \varphi^T \beta, v_s \right\rangle dW_s, \tag{15}$$

where

$$L(S)\varphi = \frac{1}{2} \left(\left(\beta\beta^T(x) + \bar{\sigma}\bar{\sigma}^T(x,S) \right) \nabla, \nabla \right) \varphi + \left(\sum_{j=1}^d \frac{\partial}{\partial x^j} \left(\sum_{l=1}^{d''} \bar{\sigma}^{\cdot l} \bar{\sigma}^{jl} \right) (x,S) \right.$$
$$\left. + \frac{1}{2} \sum_{j=1}^d \sum_{k=1}^{d'} \beta^{jk} \frac{\partial \beta^{\cdot k}}{\partial x^j}(x) - g(x,S), \nabla \right) \varphi.$$

The evolution of density u_t satisfies the dual equation to (15)

$$\begin{aligned} u_t(x) = u_0(x) &+ \int_0^t L\left(\widetilde{Q}_a(u_s) \right)^* u_s(x) \, ds \\ &+ \int_0^t \left(\sum_{i=1}^d \beta^{i\cdot}(x) \frac{\partial u_s}{\partial x^i}(x) + \sum_{i=1}^d \frac{\partial \beta^{i\cdot}}{\partial x^i}(x) u_s(x) \right) dW_s, \end{aligned} \tag{16}$$

where $L(S)^*$ is the dual operator to $L(S)$

$$L(S)^* u = \frac{1}{2} \sum_{i,j=1}^d \left(\sum_{l=1}^{d''} \bar{\sigma}^{il} \bar{\sigma}^{jl}(x,S) + \sum_{k=1}^{d'} \beta^{ik} \beta^{jk}(x) \right) \frac{\partial^2 u}{\partial x^i \partial x^j}$$
$$+ \sum_{i=1}^d \left(\sum_{j=1}^d \sum_{k=1}^{d'} \left(\beta^{ik} \frac{\partial \beta^{jk}}{\partial x^j}(x) + \frac{1}{2} \beta^{jk} \frac{\partial \beta^{ik}}{\partial x^j}(x) \right) + g^i(x,S) \right) \frac{\partial u}{\partial x^i}$$
$$+ \left(-\frac{1}{2} \sum_{i,j=1}^d \frac{\partial^2}{\partial x^i \partial x^j} \left(\sum_{l=1}^{d''} \bar{\sigma}^{il} \bar{\sigma}^{jl} \right) (x,S) + \frac{1}{2} \sum_{i,j=1}^d \sum_{k=1}^{d'} \frac{\partial}{\partial x^i} \left(\beta^{ik} \frac{\partial \beta^{jk}}{\partial x^j} \right) (x) \right.$$

$$+\sum_{i=1}^{d}\frac{\partial g^i}{\partial x^i}(x,S)\Bigg)u.$$

Assume that u_t is a continuous C^1-semimartingale. In Appendix D we show that using the Stratonovich integral, equation (16) can be written as

$$\begin{aligned} u_t(x) = u_0(x) &+ \int_0^t \left(L\left(\widetilde{Q}_a(u_s)\right)^* - \widetilde{L}\right) u_s(x)\, ds \\ &+ \int_0^t \left(\sum_{i=1}^{d} \beta^{i\cdot}(x)\frac{\partial u_s}{\partial x^i}(x) + \sum_{i=1}^{d}\frac{\partial \beta^{i\cdot}}{\partial x^i}(x) u_s(x)\right) \circ dW_s, \end{aligned} \tag{17}$$

where

$$\begin{aligned} \widetilde{L}u = &\frac{1}{2}\sum_{i,j=1}^{d}\left(\sum_{k=1}^{d'}\beta^{ik}\beta^{jk}(x)\right)\frac{\partial^2 u}{\partial x^i \partial x^j} \\ &+ \sum_{i,j=1}^{d}\sum_{k=1}^{d'}\left(\beta^{ik}\frac{\partial \beta^{jk}}{\partial x^j}(x) + \frac{1}{2}\beta^{jk}\frac{\partial \beta^{ik}}{\partial x^j}(x)\right)\frac{\partial u}{\partial x^i} \\ &+ \frac{1}{2}\sum_{i,j=1}^{d}\sum_{k=1}^{d'}\frac{\partial}{\partial x^i}\left(\beta^{ik}\frac{\partial \beta^{jk}}{\partial x^j}(x)\right)u. \end{aligned}$$

2 Auxiliary results

2.1 Method of stochastic characteristics

In this section we will adapt the Hiroshi Kunita's method of stochastic characteristics for solving SPDEs to equation (2). Also we will make analysis of stochastic characteristics that will be extensively used in further sections.

2.1.1 First order stochastic partial differential equations

Preliminaries

Let us first recall the theory of the deterministic partial differential equation of the first order. Consider the initial value problem of the following linear equation of the first order

$$\frac{\partial u_t}{\partial t}(x) = (\beta(x,t), \nabla) u_t(x) + \alpha(x,t) u_t(x), \qquad u_0 = f, \tag{18}$$

where $\alpha(x,t)$ and $\beta^i(x,t)$ are continuous bounded functions on $\mathbf{R}^d \times [0,T]$, continuously differentiable in x with bounded derivatives. The theory of linear partial differential equations of the first order tells us that the problem of integrating equation (18) can be reduced to the characteristic system of ordinary differential equations

$$\frac{dx^i}{dt} = -\beta^i(x,t), \qquad \frac{du}{dt} = \alpha(x,t)u. \tag{19}$$

Indeed, let

$$(x,u) = \left(\varphi_t(c_1), c_2 \exp\left\{ \int_0^t \alpha(\varphi_s(c_1), s)\, ds \right\} \right)$$

be the solution of equation (19) starting from (c_1, c_2) at $t = 0$. Then the solution of equation (18) is represented by means of the solution of the associated characteristic equation in form $c_2 = f(c_1)$ or

$$u_t(x) = f\left(\varphi_t^{-1}(x)\right) \exp\left\{ \int_0^t \alpha\left(\varphi_s\left(\varphi_t^{-1}(x)\right), s\right) ds \right\},$$

where φ_t^{-1} is the inverse map of the map $\varphi_t \colon \mathbf{R}^d \to \mathbf{R}^d$.

In this section we shall study the initial value problem of the linear SPDE of the first order given by

$$u_t(x) = f(x) + \int_0^t ((\beta(x,s),\nabla)u_s(x) + \alpha(x,s)u_s(x)) \circ dW_s, \qquad t \in (0,T], \tag{20}$$

where $\alpha(x,t)$ and $\beta^i(x,t)$ are continuous functions on $\mathbf{R}^d \times [0,T]$, and W_t is a standard 1-dimensional Brownian motion. A continuous C^1-semimartingale $u_t(x)$, $x \in \mathbf{R}^d$, $t \in [0,T]$ is called a *(global) solution up to time T with the initial value f* if it satisfies equation (20) for any x and t a.s. Hiroshi Kunita developed a general theory with *local solutions* for nonlinear equations, but we will be interested only in global solutions for linear equations.

Assume that the coefficients $\alpha(x,t)$ and $\beta^i(x,t)$ satisfy Condition $(E.1)_k$ for some $k \geq 3$. Using the characteristic system of stochastic differential equations developed in [11], we will solve linear equation (20).

Existence and uniqueness of solutions

The *stochastic characteristic system associated with* equation (20) is defined by a system of Stratonovich stochastic differential equations of the form

$$\varphi_{s,t}(x) = x - \int_s^t \beta(\varphi_{s,r}(x), r) \circ dW_r, \tag{21}$$

where $\beta(x,t) = \left(\beta^1(x,t), \ldots, \beta^d(x,t)\right)^T$ and

$$\eta_{s,t}(x,u) = u + \int_s^t \eta_{s,r}(x,u)\alpha(\varphi_{s,r}(x), r) \circ dW_r. \tag{22}$$

By Theorem F.3 in Appendix F, for each $s \in [0,T]$ and $(x,u) \in \mathbf{R}^d \times \mathbf{R}$ this system of Stratonovich stochastic differential equations has a unique solution $(\varphi_{s,t}(x), \eta_{s,t}(x,u))$, $t \in [s,T]$ starting at (x,u) at time s. By Theorem 4.6.5[3] in [11] this solution has a modification $(\varphi_{s,t}(x), \eta_{s,t}(x,u))$, $0 \leq s \leq t \leq T$ such that it is a forward stochastic flow of C^k-diffeomorphisms, and it is also a C^k-semimartingale.

Since $\eta_{s,t}$ satisfies the linear equation (22), we can find its solution straight

[3]We first rewrite the stochastic characteristic system using the Itô's integral (rewritten system is presented in Appendix D), and then apply the theorem.

away

$$\eta_{s,t}(x,u) = u \exp\left\{ \int_s^t \alpha(\varphi_{s,r}(x), r) \circ dW_r \right\}.$$

Let $f(x)$ be a function of the class C^{k+1} corresponding to the initial value of equation (20). We denote $\varphi_{0,t}(x)$ by $\varphi_t(x)$. The Jacobian matrix $\partial\varphi_t(x)$ is always non-singular and the inverse $\psi_t(x) \equiv \varphi_t^{-1}(x)$ is defined for all $t \in [0,T], x \in \mathbf{R}^d$. Then by Theorem 6.1.9 in [11] equation (20) has a unique global solution which is a continuous C^k-semimartingale, and it is represented by

$$u_t(x) = f(\psi_t(x)) \exp\left\{ \int_0^t \alpha(\varphi_s(y), s) \circ dW_s \Big|_{y=\psi_t(x)} \right\}. \tag{23}$$

Note that $\alpha(\varphi_s(y), s)$ is a continuous semimartingale by the Itô's formula (Theorem 1.8), and therefore the Stratonovich integral in formula (23) is well defined. By Theorem 4.4.4 in [11] the inverse $\varphi_{t,s} = \varphi_{s,t}^{-1}$ satisfies the backward Stratonovich equation

$$\varphi_{t,s}(x) = x + \int_s^t \beta(\varphi_{t,r}(x), r) \circ \hat{d}W_r, \qquad s < t. \tag{24}$$

Note that $\psi_t(x) = \varphi_{t,0}(x)$. Then formula (23) can be rewritten by

$$u_t(x) = f(\varphi_{t,0}(x)) \exp\left\{ \int_0^t \alpha(\varphi_{t,s}(x), s) \circ \hat{d}W_s \right\}.$$

Explicit solution of the stochastic characteristic system

First of all let us denote

$$\mu_{s,t}(x) = \int_s^t \alpha(\varphi_{s,r}(x), r) \circ dW_r, \qquad s < t,$$
$$\mu_{t,s}(x) = -\int_s^t \alpha(\varphi_{t,r}(x), r) \circ \hat{d}W_r, \quad s < t.$$

In this paragraph we will find $\varphi_{s,t}$, $\varphi_{t,s}$ and rewrite $\mu_{s,t}$, $\mu_{t,s}$ in an explicit form using Doss–Sussman method (see [3], [16], [19]). Also we will find some useful properties of these stochastic characteristics. Note that in this paragraph we are dealing only with the standard 1-dimensional Brownian motion W_t.

For every $z \in [0,T]$, consider the following autonomous system of ordinary differential equations (ODEs) of the first order

$$\frac{\partial h}{\partial t}(y,z,t) = -\beta(h(y,z,t),z), \qquad h(y,z,0) = y. \tag{25}$$

For the well-posedness of this Cauchy problem we need the following condition on function $\beta(x,s)$

1. for every $s \in [0,T]$, $\beta(\cdot,s)$ is Lipschitz continuous in x, i.e. there exists some constant $L(s) > 0$ such that $|\beta(x_1,s) - \beta(x_2,s)| \leq L(s)|x_1 - x_2|$ holds for all $x_i \in \mathbf{R}^d$.

This condition follows from the boundedness of functions β^i and their first order derivatives $\frac{\partial \beta^i}{\partial x^j}$ on $\mathbf{R}^d \times [0,T]$. Then for every $(y,z) \in \mathbf{R}^d \times [0,T]$ there exists a unique solution $h(y,z,t)$, $t \in \mathbf{R}$ of system (25).

Proposition 2.1. *Let functions $\beta^i(x,s), x \in \mathbf{R}^d, s \in [0,T]$ satisfy Condition $(E.1)_k$ for some $k \geq 3$, and let $h(y,z,t)$ be the unique solution of equation* (25)*. Then*

1. *$h(y,z,t)$ is continuously differentiable in z,*

2. *for every $z \in [0,T]$, $h(\cdot,z,t)$ is a flow of C^{k+1}-diffeomorphisms on $\mathbf{R}^d$,*

3. *for arbitrary $T_1 > 0$, all partial derivatives of function $h(y,z,t)$ with respect to (y,z) and function $h(y,z,t) - y$ are bounded on $\mathbf{R}^d \times [0,T] \times [-T_1,T_1]$,*

4. *there exists $C > 0$ such that*

$$|h(y,z,t_2) - h(y,z,t_1)| \leq C|t_2 - t_1|$$

   holds for all $(y,z,t_1,t_2) \in \mathbf{R}^d \times [0,T] \times \mathbf{R}^2$,

5. *for arbitrary $T_1 > 0$, there exists $C > 0$ such that*

$$\left| \frac{\partial h^i}{\partial y^k}(y,z,t_2) - \frac{\partial h^i}{\partial y^k}(y,z,t_1) \right| \leq C|t_2 - t_1|$$

   holds for all $(y,z,t_1,t_2) \in \mathbf{R}^d \times [0,T] \times [-T_1,T_1]^2$.

Proof. We know from the theory of ODEs (see Theorem 1.5.3 in [10]) that if β^i, $\frac{\partial \beta^i}{\partial x^j}$ and $\frac{\partial \beta^i}{\partial s}$ are continuous on $\mathbf{R}^d \times [0,T]$, then $h(y,z,t)$ is continuously differentiable in z, and the derivatives $\frac{\partial h^i}{\partial z}$ satisfy

$$\begin{aligned}
&\frac{\partial}{\partial t}\frac{\partial h^i}{\partial z}(y,z,t) = -\sum_{j=1}^{d} \frac{\partial \beta^i}{\partial x^j}(h(y,z,t),z)\frac{\partial h^j}{\partial z}(y,z,t) - \frac{\partial \beta^i}{\partial s}(h(y,z,t),z), \\
&\quad \frac{\partial h^i}{\partial z}(y,z,0) = 0.
\end{aligned} \tag{26}$$

Let $C > 0$ be the bound from Condition $(E.1)_k$. Then the coefficients in linear equation (26) are uniformly in $(y,z) \in \mathbf{R}^d \times [0,T]$ bounded functions of $t \in \mathbf{R}$

$$\max\left(\left|-\frac{\partial \beta^i}{\partial x^j}(h(y,z,t),z)\right|, \left|-\frac{\partial \beta^i}{\partial s}(h(y,z,t),z)\right|\right) \le C.$$

Therefore, the solution $\frac{\partial h}{\partial z}(y,z,t)$ has no more than uniform exponential growth in t

$$\left|\frac{\partial h^i}{\partial z}(y,z,t)\right| \le \frac{e^{dC|t|}-1}{d},$$

yielding the boundedness on $\mathbf{R}^d \times [0,T] \times [-T_1, T_1]$.

Also we know from the theory of ODEs (see Theorem 1.5.3 in [10]) that if $\beta^i(\cdot,s)$ and $\frac{\partial \beta^i}{\partial x^j}(\cdot,s)$ are continuous on $\mathbf{R}^d$, then $h(y,z,t)$ is continuously differentiable in y, and the derivatives $\frac{\partial h^i}{\partial y^{k_1}}$ satisfy

$$\begin{aligned}
&\frac{\partial}{\partial t}\frac{\partial h^i}{\partial y^{k_1}}(y,z,t) = -\sum_{j=1}^{d} \frac{\partial \beta^i}{\partial x^j}(h(y,z,t),z)\frac{\partial h^j}{\partial y^{k_1}}(y,z,t), \\
&\frac{\partial h^i}{\partial y^{k_1}}(y,z,0) = \begin{cases} 0, & \text{if } i \neq k_1, \\ 1, & \text{if } i = k_1. \end{cases}
\end{aligned} \tag{27}$$

The chain property

$$h(y,z,t+s) = h(h(y,z,t),z,s) \tag{28}$$

follows from the existence and uniqueness of a solution to equation (25), and the inverse $h^{-1}(y,z,t) = h(y,z,-t)$. Thus for every $z \in [0,T]$, $h(\cdot,z,t)$ is a flow of C^1-diffeomorphisms on $\mathbf{R}^d$.

Similarly to equation (26), the solution $\frac{\partial h}{\partial y^{k_1}}(y,z,t)$ of linear equation (27) has no more than uniform exponential growth in t

$$\left|\frac{\partial h^i}{\partial y^{k_1}}(y,z,t)\right| \le \frac{e^{dC|t|}-1}{d}+1, \tag{29}$$

yielding the boundedness on $\mathbf{R}^d \times [0,T] \times [-T_1, T_1]$.

The existence of higher order derivatives of function $h(y,z,t)$ and their boundedness on $\mathbf{R}^d \times [0,T] \times [-T_1,T_1]$ can be proven in the same way.

Finally,

$$|h(y,z,t_2) - h(y,z,t_1)| = \left| -\int_{t_1}^{t_2} \beta(h(y,z,s),z)\,ds \right| \le C|t_2 - t_1|$$

holds for all $(y,z,t_1,t_2) \in \mathbf{R}^d \times [0,T] \times \mathbf{R}^2$, and by (29)

$$\begin{aligned}\left|\frac{\partial h^i}{\partial y^k}(y,z,t_2) - \frac{\partial h^i}{\partial y^k}(y,z,t_1)\right| &= \left| -\int_{t_1}^{t_2} \sum_{j=1}^{d} \frac{\partial \beta^i}{\partial x^j}(h(y,z,r),z)\frac{\partial h^j}{\partial y^k}(y,z,r)\,dr \right| \\ &\le dC\left(\frac{e^{dCT_1}-1}{d}+1\right)|t_2 - t_1|\end{aligned}$$

holds for all $(y,z,t_1,t_2) \in \mathbf{R}^d \times [0,T] \times [-T_1,T_1]^2$. □

Note that by Proposition 2.1(2) the Jacobian matrix $\frac{\partial h}{\partial y}$ is always non-singular.

Corollary 2.2. *1. Assume the same conditions as in Proposition 2.1. Then function*

$$\widetilde{h}(y,z,t) = \left(\frac{\partial h}{\partial y}\right)^{-1}\left(\frac{\partial h}{\partial z}\right)(y,z,t) \tag{30}$$

is k-times continuously differentiable in y, and for arbitrary $T_1 > 0$, $\widetilde{h}(y,z,t)$ and all its partial derivatives are bounded on $\mathbf{R}^d \times [0,T] \times [-T_1,T_1]$.

2. In addition to (1), assume that function $\alpha(x,t), x \in \mathbf{R}^d, t \in [0,T]$ satisfies Condition $(E.1)_k$ for some $k \ge 3$. Then function

$$\bar{h}(y,z,t) = \int_0^t \alpha(h(y,z,r),z)\,dr \tag{31}$$

is $(k+1)$-times continuously differentiable in y, continuously differentiable in z, and for arbitrary $T_1 > 0$, $\bar{h}(y,z,t)$ and all its partial derivatives are bounded on $\mathbf{R}^d \times [0,T] \times [-T_1, T_1]$.

3. *Under the assumptions above, the function*

$$\hat{h}(y,z,t) = \left(\left(\frac{\partial \bar{h}}{\partial y}\right)\widetilde{h} - \frac{\partial \bar{h}}{\partial z}\right)(y,z,t) \tag{32}$$

is k-times continuously differentiable in y, and for arbitrary $T_1 > 0$, $\hat{h}(y,z,t)$ and all its partial derivatives are bounded on $\mathbf{R}^d \times [0,T] \times [-T_1, T_1]$.

Proof. 1. The proof becomes straightforward if we notice the following formula for the inverse of the Jacobian matrix $\frac{\partial h}{\partial y}$

$$\left(\frac{\partial h}{\partial y}\right)^{-1}(y,z,t) = \left(\frac{\partial h}{\partial y}\right)(h(y,z,t), z, -t),$$

which can be proven by differentiation with respect to y of equation (28) for $s = -t$ and equation (25).

2. The second statement about $\bar{h}(y,z,t)$ follows directly from Proposition 2.1.

3. The third statement about $\hat{h}(y,z,t)$ follows directly from 1 and 2.

□

For every $s \in [0,T]$ and almost all ω (when $W_t(\omega)$ is continuous), let us consider the following system of ODEs

$$\begin{aligned} \frac{\partial}{\partial t} D_{s,t}(x) &= -\widetilde{h}(D_{s,t}(x), t, W_t - W_s), \qquad t \in [0,T], \\ D_{s,s}(x) &= x, \end{aligned} \tag{33}$$

where the function $\widetilde{h}(y,z,t)$ is given by formula (30). Note that $W_t - W_s \in \left[-2\overline{W}_T, 2\overline{W}_T\right]$ for all $s,t \in [0,T]$, where $\overline{W}_T = \max_{t\in[0,T]} |W_t|$. Then by Corollary 2.2(1) the function

$$f(t,y) = -\widetilde{h}(y, t, W_t - W_s) \tag{34}$$

and its first order derivatives with respect to y are bounded on $[0,T] \times \mathbf{R}^d$. Thus there exists a unique solution $D_{s,t}(x)$, $t \in [0,T]$ of system (33). Moreover, this solution is a process of bounded variation.

Let us extend the solution $D_{s,t}(x)$ by adding

$$D_{s,t}^{d+1}(x) = \int_s^t \hat{h}(D_{s,r}(x), r, W_r - W_s)\, dr, \tag{35}$$

where function $\hat{h}(y,z,t)$ is given by formula (32).

Proposition 2.3. *1. Let functions $\beta^i(x,t), x \in \mathbf{R}^d, t \in [0,T]$ satisfy Condition $(E.1)_k$ for some $k \geq 3$, and let $D_{s,t}(x)$ be the unique solution of equation (33). Then $D_{s,t}(x)$ is k-times continuously differentiable in x, function $D_{s,t}(x) - x$ and all partial derivatives of $D_{s,t}(x)$ are bounded on $[0,T]^2 \times \mathbf{R}^d$. Moreover, there exists a positive constant C such that*

$$|D_{s,t_2}(x) - D_{s,t_1}(x)| \leq C|t_2 - t_1|, \quad \left| \frac{\partial D^i_{s,t_2}}{\partial x^k}(x) - \frac{\partial D^i_{s,t_1}}{\partial x^k}(x) \right| \leq C|t_2 - t_1|$$

hold for all $(s,t_1,t_2,x) \in [0,T]^3 \times \mathbf{R}^d$.

2. In addition to (1), assume that function $\alpha(x,t), x \in \mathbf{R}^d, t \in [0,T]$ satisfies Condition $(E.1)_k$ for some $k \geq 3$. Then $D^{d+1}_{s,t}(x)$ is k-times continuously differentiable in x, $D^{d+1}_{s,t}(x)$ and all its partial derivatives are bounded on $[0,T]^2 \times \mathbf{R}^d$. Moreover, there exists a positive constant C such that

$$\left| D^{d+1}_{s,t_2}(x) - D^{d+1}_{s,t_1}(x) \right| \leq C|t_2 - t_1|$$

holds for all $(s,t_1,t_2,x) \in [0,T]^3 \times \mathbf{R}^d$.

Proof. 1. We know from the theory of ODEs (see Theorem 1.5.3 in [10]) that if $f^i(t,y)$ defined by (34) and $\frac{\partial f^i}{\partial y^j}(t,y)$ are continuous on $[0,T] \times \mathbf{R}^d$, then $D_{s,t}(x)$ is continuously differentiable in x, and the derivatives $\frac{\partial D^i_{s,t}}{\partial x^{k_1}}$

satisfy

$$\frac{\partial}{\partial t}\frac{\partial D^i_{s,t}}{\partial x^{k_1}}(x) = -\sum_{j=1}^{d}\frac{\partial \widetilde{h}^i}{\partial y^j}(D_{s,t}(x), t, W_t - W_s)\frac{\partial D^j_{s,t}}{\partial x^{k_1}}(x),$$
$$\frac{\partial D^i_{s,s}}{\partial x^{k_1}}(x) = \begin{cases} 0, & \text{if } i \neq k_1, \\ 1, & \text{if } i = k_1. \end{cases} \tag{36}$$

Let $C > 0$ be the bound from Corollary 2.2(1). Then the coefficients in linear equation (36) are uniformly in (s, x) bounded functions of $t \in [0, T]$

$$\left| -\frac{\partial \widetilde{h}^i}{\partial y^j}(D_{s,t}(x), t, W_t - W_s) \right| \leq C.$$

Therefore, the solution $\frac{\partial D_{s,t}}{\partial x^{k_1}}(x)$ has no more than uniform exponential growth in t

$$\left| \frac{\partial D^i_{s,t}}{\partial x^{k_1}}(x) \right| \leq \frac{e^{dC|t-s|} - 1}{d} + 1,$$

yielding the boundedness on $[0, T]^2 \times \mathbf{R}^d$.

The existence of higher order derivatives of function $D_{s,t}(x)$ and their boundedness on $[0, T]^2 \times \mathbf{R}^d$ can be proven in the same way.

Finally,

$$|D_{s,t_2}(x) - D_{s,t_1}(x)| = \left| -\int_{t_1}^{t_2} \widetilde{h}(D_{s,r}(x), r, W_r - W_s)\, dr \right| \leq C|t_2 - t_1|,$$
$$\left| \frac{\partial D^i_{s,t_2}}{\partial x^k}(x) - \frac{\partial D^i_{s,t_1}}{\partial x^k}(x) \right| = \left| -\int_{t_1}^{t_2} \sum_{j=1}^{d} \frac{\partial \widetilde{h}^i}{\partial y^j}(D_{s,r}(x), r, W_r - W_s)\frac{\partial D^j_{s,r}}{\partial x^k}(x)\, dr \right|$$
$$\leq dC\left(\frac{e^{dCT} - 1}{d} + 1\right)|t_2 - t_1|$$

hold for all $(s, t_1, t_2, x) \in [0, T]^3 \times \mathbf{R}^d$.

2. Let $C > 0$ be the bound from Corollary 2.2(3). Then

$$\left|D^{d+1}_{s,t_2}(x) - D^{d+1}_{s,t_1}(x)\right| = \left| \int_{t_1}^{t_2} \hat{h}(D_{s,r}(x), r, W_r - W_s)\, dr \right| \leq C|t_2 - t_1|$$

holds for all $(s,t_1,t_2,x)\in[0,T]^3\times\mathbf{R}^d$. The rest of the proposition follows directly from Corollary 2.2(3).

□

Let us denote

$$\varphi_{s,t}(x)=h(D_{s,t}(x),t,W_t-W_s),\qquad s<t. \tag{37}$$

By a direct computation of differentials we can prove that it is the solution of equation (21). Indeed, by the Itô's formula (Theorem 1.8) and (33), (30), (25) we have

$$\begin{aligned}
\varphi^j_{s,t}(x)-x^j &= h^j(D_{s,t}(x),t,W_t-W_s)-h^j(D_{s,s}(x),s,W_s-W_s)\\
&=\int_s^t\sum_{i=1}^d\frac{\partial h^j}{\partial y^i}(D_{s,r}(x),r,W_r-W_s)\,dD^i_{s,r}(x)\\
&\quad+\int_s^t\frac{\partial h^j}{\partial z}(D_{s,r}(x),r,W_r-W_s)\,dr\\
&\quad+\int_s^t\frac{\partial h^j}{\partial t}(D_{s,r}(x),r,W_r-W_s)\circ dW_r\\
&=-\int_s^t\beta^j(\varphi_{s,r}(x),r)\circ dW_r.
\end{aligned}$$

Similarly, we can prove that

$$\varphi_{t,s}(x)=h(D_{t,s}(x),s,W_s-W_t),\qquad s<t \tag{38}$$

is the solution of equation (24).

Next, let us prove the following new formula of $\mu_{s,t}$

$$\mu_{s,t}(x)=D^{d+1}_{s,t}(x)+\bar{h}(D_{s,t}(x),t,W_t-W_s),\qquad s<t. \tag{39}$$

By (35), (32), the Itô's formula (Theorem 1.8) and (31) we have for $s<t$

$$\begin{aligned}
&D^{d+1}_{s,t}(x)+\bar{h}(D_{s,t}(x),t,W_t-W_s)\\
&\quad=\int_s^t\left(\left(\frac{\partial\bar{h}}{\partial y}\right)\widetilde{h}-\frac{\partial\bar{h}}{\partial z}\right)(D_{s,r}(x),r,W_r-W_s)\,dr
\end{aligned}$$

$$
\begin{aligned}
&+\int_s^t \sum_{i=1}^d \frac{\partial \bar{h}}{\partial y^i}(D_{s,r}(x), r, W_r - W_s)\, dD^i_{s,r}(x) \\
&+\int_s^t \frac{\partial \bar{h}}{\partial z}(D_{s,r}(x), r, W_r - W_s)\, dr + \int_s^t \frac{\partial \bar{h}}{\partial t}(D_{s,r}(x), r, W_r - W_s) \circ dW_r \\
= &\int_s^t \alpha(\varphi_{s,r}(x), r) \circ dW_r = \mu_{s,t}(x).
\end{aligned}
$$

Similarly, we can prove the following new formula of $\mu_{t,s}$

$$
\mu_{t,s}(x) = D^{d+1}_{t,s}(x) + \bar{h}(D_{t,s}(x), s, W_s - W_t), \qquad s < t. \tag{40}
$$

The following two propositions are important results which will be used in further section.

Proposition 2.4. *Let functions $\alpha(x,t)$, $\beta^i(x,t)$, $x \in \mathbf{R}^d$, $t \in [0,T]$ satisfy Condition $(E.1)_k$ for some $k \geq 3$. Then the functions $\varphi_{s,t}(x) - x$, $\mu_{s,t}(x)$ and all partial derivatives up to and including order k of $\varphi_{s,t}(x)$ and $\mu_{s,t}(x)$ are bounded on $[0,T]^2 \times \mathbf{R}^d$. Moreover, there exists a positive constant C such that*

$$
\begin{aligned}
|\varphi_{s,t_2}(x) - \varphi_{s,t_1}(x)| &\leq C(|W_{t_2} - W_{t_1}| + |t_2 - t_1|), \\
\left| \frac{\partial \varphi^i_{s,t_2}}{\partial x^k}(x) - \frac{\partial \varphi^i_{s,t_1}}{\partial x^k}(x) \right| &\leq C(|W_{t_2} - W_{t_1}| + |t_2 - t_1|)
\end{aligned}
$$

hold for all $(s, t_1, t_2, x) \in [0,T]^3 \times \mathbf{R}^d$. As a consequence, the Jacobian

$$
\det \left(\frac{\partial \varphi_{s,t}(x)}{\partial x} \right)
$$

is bounded on $[0,T]^2 \times \mathbf{R}^d$, and moreover, there exists a positive constant C such that

$$
\left| \det \left(\frac{\partial \varphi_{s,t_2}}{\partial x}(x) \right) - \det \left(\frac{\partial \varphi_{s,t_1}}{\partial x}(x) \right) \right| \leq C(|W_{t_2} - W_{t_1}| + |t_2 - t_1|)
$$

holds for all $(s, t_1, t_2, x) \in [0,T]^3 \times \mathbf{R}^d$.

Proof. By Propositions 2.1 and 2.3(1), there exists $C > 0$ such that for all

$(s,t_1,t_2,x)\in[0,T]^3\times\mathbf{R}^d$

$$
\begin{aligned}
|\varphi_{s,t_2}(x)-\varphi_{s,t_1}(x)| &\le |h(D_{s,t_2}(x),t_2,W_{t_2}-W_s)-h(D_{s,t_2}(x),t_2,W_{t_1}-W_s)|\\
&\quad+|h(D_{s,t_2}(x),t_2,W_{t_1}-W_s)-h(D_{s,t_2}(x),t_1,W_{t_1}-W_s)|\\
&\quad+|h(D_{s,t_2}(x),t_1,W_{t_1}-W_s)-h(D_{s,t_1}(x),t_1,W_{t_1}-W_s)|\\
&\le C|W_{t_2}-W_{t_1}|+C|t_2-t_1|+C|t_2-t_1|.
\end{aligned}
$$

Then the function $\varphi_{s,t}(x)-x$ is bounded on $[0,T]^2\times\mathbf{R}^d$.

Using formulas (37), (38) we can represent all partial derivatives of $\varphi_{s,t}(x)$ in terms of partial derivatives of $h(y,z,t)$ with respect to y and partial derivatives of $D_{s,t}(x)$. Partial derivatives of $h(y,z,t)$ are bounded by Proposition 2.1(3), and partial derivatives of $D_{s,t}(x)$ are bounded by Proposition 2.3(1). Therefore, all partial derivatives up to and including order k of $\varphi_{s,t}(x)$ are bounded on $[0,T]^2\times\mathbf{R}^d$.

The boundedness of the function $\mu_{s,t}(x)$ and its partial derivatives follows directly from formulas (39), (40), Proposition 2.3 and Corollary 2.2(2).

Also, by Propositions 2.1 and 2.3(1), there exists $C>0$ such that for all $(s,t_1,t_2,x)\in[0,T]^3\times\mathbf{R}^d$ we have

$$
\begin{aligned}
&\left|\frac{\partial\varphi^i_{s,t_2}}{\partial x^k}(x)-\frac{\partial\varphi^i_{s,t_1}}{\partial x^k}(x)\right|\\
&\le\left|\frac{\partial h^i}{\partial y}(D_{s,t_2}(x),t_2,W_{t_2}-W_s)\frac{\partial D_{s,t_2}}{\partial x^k}(x)-\frac{\partial h^i}{\partial y}(D_{s,t_2}(x),t_2,W_{t_2}-W_s)\frac{\partial D_{s,t_1}}{\partial x^k}(x)\right|\\
&\quad+\left|\frac{\partial h^i}{\partial y}(D_{s,t_2}(x),t_2,W_{t_2}-W_s)\frac{\partial D_{s,t_1}}{\partial x^k}(x)-\frac{\partial h^i}{\partial y}(D_{s,t_1}(x),t_1,W_{t_1}-W_s)\frac{\partial D_{s,t_1}}{\partial x^k}(x)\right|\\
&\le C|t_2-t_1|+C\left|\frac{\partial h^i}{\partial y}(D_{s,t_2}(x),t_2,W_{t_2}-W_s)-\frac{\partial h^i}{\partial y}(D_{s,t_1}(x),t_1,W_{t_1}-W_s)\right|,
\end{aligned}
$$

where

$$
\begin{aligned}
&\left|\frac{\partial h^i}{\partial y}(D_{s,t_2}(x),t_2,W_{t_2}-W_s)-\frac{\partial h^i}{\partial y}(D_{s,t_1}(x),t_1,W_{t_1}-W_s)\right|\\
&\quad\le\left|\frac{\partial h^i}{\partial y}(D_{s,t_2}(x),t_2,W_{t_2}-W_s)-\frac{\partial h^i}{\partial y}(D_{s,t_2}(x),t_2,W_{t_1}-W_s)\right|\\
&\quad\quad+\left|\frac{\partial h^i}{\partial y}(D_{s,t_2}(x),t_2,W_{t_1}-W_s)-\frac{\partial h^i}{\partial y}(D_{s,t_2}(x),t_1,W_{t_1}-W_s)\right|
\end{aligned}
$$

$$
\begin{aligned}
&+\left|\frac{\partial h^i}{\partial y}(D_{s,t_2}(x),t_1,W_{t_1}-W_s)-\frac{\partial h^i}{\partial y}(D_{s,t_1}(x),t_1,W_{t_1}-W_s)\right| \\
&\le C|W_{t_2}-W_{t_1}|+C|t_2-t_1|+C|t_2-t_1|.
\end{aligned}
$$

$\square$

Proposition 2.5. *Let functions $\alpha(x,t)$, $\beta^i(x,t)$, $x\in\mathbf{R}^d$, $t\in[0,T]$ satisfy Condition $(E.1)_k$ for some $k\ge 3$. Then*

$$
\sup_{x\in\mathbf{R}^d}|\varphi_{t,0}(x)-\varphi_{s,0}(x)|\xrightarrow[t\to s]{}0,\qquad \sup_{x\in\mathbf{R}^d}|\mu_{t,0}(x)-\mu_{s,0}(x)|\xrightarrow[t\to s]{}0.
$$

Proof. By Propositions 2.1, 2.3(1) and 2.4 we have

$$
\begin{aligned}
\sup_{x\in\mathbf{R}^d}|\varphi_{t,0}(x)-\varphi_{s,0}(x)|&=\sup_{y\in\mathbf{R}^d}|\varphi_{s,0}(y)-\varphi_{s,0}(\varphi_{s,t}(y))| \\
&=\sup_{y\in\mathbf{R}^d}|h(D_{s,0}(y),0,-W_s)-h(D_{s,0}(\varphi_{s,t}(y)),0,-W_s)| \\
&\le\sup_{y\in\mathbf{R}^d}\left|\frac{\partial h}{\partial y}(y,0,-W_s)\right|\sup_{x\in\mathbf{R}^d}\left|\frac{\partial D_{s,0}}{\partial x}(x)\right|\sup_{y\in\mathbf{R}^d}|y-\varphi_{s,t}(y)|\xrightarrow[t\to s]{}0.
\end{aligned}
\tag{41}
$$

Next, using formula

$$
\mu_{t,0}(x)=-\mu_{0,t}(\varphi_{t,0}(x))=-D^{d+1}_{0,t}(\varphi_{t,0}(x))-\bar h(D_{0,t}(\varphi_{t,0}(x)),t,W_t),
$$

we estimate

$$
\begin{aligned}
\sup_{x\in\mathbf{R}^d}|\mu_{t,0}(x)-\mu_{s,0}(x)|&\le\sup_{x\in\mathbf{R}^d}\left|\bar h(D_{0,t}(\varphi_{t,0}(x)),t,W_t)-\bar h(D_{0,s}(\varphi_{s,0}(x)),s,W_s)\right| \\
&\quad+\sup_{x\in\mathbf{R}^d}\left|D^{d+1}_{0,t}(\varphi_{t,0}(x))-D^{d+1}_{0,s}(\varphi_{s,0}(x))\right|.
\end{aligned}
$$

Then, by Corollary 2.2(2), Proposition 2.3 and (41), the right hand side of inequality above goes to 0 when $t\to s$

$$
\begin{aligned}
&\sup_{x\in\mathbf{R}^d}\left|D^{d+1}_{0,t}(\varphi_{t,0}(x))-D^{d+1}_{0,s}(\varphi_{s,0}(x))\right| \\
&\quad\le\sup_{x\in\mathbf{R}^d}\Big(\left|D^{d+1}_{0,t}(\varphi_{t,0}(x))-D^{d+1}_{0,s}(\varphi_{t,0}(x))\right|+\left|D^{d+1}_{0,s}(\varphi_{t,0}(x))-D^{d+1}_{0,s}(\varphi_{s,0}(x))\right|\Big)
\end{aligned}
$$

$$
\begin{aligned}
&\le C|t-s| + C\sup_{x\in\mathbf{R}^d} |\varphi_{t,0}(x) - \varphi_{s,0}(x)| \xrightarrow[t\to s]{} 0,\\
&\sup_{x\in\mathbf{R}^d} \left|\bar h(D_{0,t}(\varphi_{t,0}(x)), t, W_t) - \bar h(D_{0,s}(\varphi_{s,0}(x)), s, W_s)\right|\\
&\quad\le \sup_{x\in\mathbf{R}^d} \left|\bar h(D_{0,t}(\varphi_{t,0}(x)), t, W_t) - \bar h(D_{0,t}(\varphi_{t,0}(x)), t, W_s)\right|\\
&\qquad+ \sup_{x\in\mathbf{R}^d} \left|\bar h(D_{0,t}(\varphi_{t,0}(x)), t, W_s) - \bar h(D_{0,t}(\varphi_{s,0}(x)), t, W_s)\right|\\
&\qquad+ \sup_{x\in\mathbf{R}^d} \left|\bar h(D_{0,t}(\varphi_{s,0}(x)), t, W_s) - \bar h(D_{0,s}(\varphi_{s,0}(x)), t, W_s)\right|\\
&\qquad+ \sup_{x\in\mathbf{R}^d} \left|\bar h(D_{0,s}(\varphi_{s,0}(x)), t, W_s) - \bar h(D_{0,s}(\varphi_{s,0}(x)), s, W_s)\right|\\
&\quad\le C|W_t - W_s| + C\sup_{x\in\mathbf{R}^d} |\varphi_{t,0}(x) - \varphi_{s,0}(x)| + C|t-s| + C|t-s| \xrightarrow[t\to s]{} 0.
\end{aligned}
$$

□

2.1.2 Second order stochastic partial differential equations

Preliminaries

In this section we shall study the initial value problem of the second order linear SPDE

$$
\begin{aligned}
u_t(x) = f(x) &+ \int_0^t L_s(q_s)u_s(x)\,ds\\
&+ \int_0^t ((\beta(x,s),\nabla)u_s(x) + \alpha(x,s)u_s(x)) \circ dW_s, \qquad t\in(0,T],
\end{aligned}
\tag{42}
$$

where L_t is an operator of the form

$$
L_t(q)u = \frac{1}{2}\left(\bar\sigma\bar\sigma^T(x,t,q)\nabla,\nabla\right)u + (g(x,t,q),\nabla)u + d(x,t,q)u, \tag{43}
$$

with a given d-dimensional continuous semimartingale q_t.

We assume that the coefficients $\alpha(x,t)$ and $\beta^i(x,t)$ in equation (42) satisfy Condition $(E.1)_k$ for some $k\ge 4$, and the coefficients of the operator L_t satisfy Condition $(E.2)$. A continuous C^2-semimartingale $u_t(x)$ is then called a *solution with the initial value* f if it satisfies equation (42) for any x and t a.s.

Remark. *Condition* $(E.1)_k$ *for some* $k\ge 4$ *provides four times continuous* *differentiability of the functions $\varphi_{s,t}(x)$, $\mu_{s,t}(x)$ defined in the previous section (see Proposition 2.4). This will allow us to prove twice continuous differentiability of the functions $a(x,t,q)$, $b(x,t,q)$, $c(x,t,q)$ in formula* (51) *(see Lemma 2.6). Meanwhile, Lemma 2.6 will be used in the proof of Theorems 1.1 and 1.2 in Section 3.*

We shall rewrite equation (42) using the Itô's integral. Define a second order operator $\widetilde{L}_t$ by

$$\begin{aligned}\widetilde{L}_t u &= \frac{1}{2}\sum_{i,j=1}^{d} \beta^i(x,t)\beta^j(x,t)\frac{\partial^2 u}{\partial x^i \partial x^j} \\ &+ \sum_{i=1}^{d}\left(\frac{1}{2}\sum_{j=1}^{d}\beta^j(x,t)\frac{\partial \beta^i}{\partial x^j}(x,t) + \beta^i(x,t)\alpha(x,t)\right)\frac{\partial u}{\partial x^i} \\ &+ \frac{1}{2}\left(\sum_{i=1}^{d}\beta^i(x,t)\frac{\partial \alpha}{\partial x^i}(x,t) + (\alpha(x,t))^2\right)u.\end{aligned} \tag{44}$$

In Appendix D we show that using the Itô's integral, equation (42) can be written as

$$\begin{aligned}u_t(x) = f(x) &+ \int_0^t \left(L_s(q_s) + \widetilde{L}_s\right) u_s(x)\, ds \\ &+ \int_0^t ((\beta(x,s),\nabla)u_s(x) + \alpha(x,s)u_s(x))\, dW_s.\end{aligned} \tag{45}$$

Conversely let $u_t(x)$ be a continuous C^2-process satisfying the equation represented by the Itô's integrals:

$$\begin{aligned}u_t(x) = f(x) &+ \int_0^t A_s(q_s)u_s(x)\, ds \\ &+ \int_0^t ((\beta(x,s),\nabla)u_s(x) + \alpha(x,s)u_s(x))\, dW_s.\end{aligned} \tag{46}$$

If $u_t(x)$ is a C^2-semimartingale, it is represented by equation (42), replacing $L_s(q_s)$ by $A_s(q_s) - \widetilde{L}_s$ where $\widetilde{L}_s$ is defined by equation (44).

Note that the second order part of the operator L_t in equation (42) and that of the operator $A_t = L_t + \widetilde{L}_t$ in equation (45) are different because $\beta^i(x,t)\beta^j(x,t)$ are not identically 0. Conversely, solutions of the equations with the common second order part and the common random first order part can have different properties if one is written by the Stratonovich integral and the other is written by the Itô's integral. In the case $A_t = \widetilde{L}_t$, equation (46) represented by the Itô's integrals is a second order equation, but the same equation represented by the Stratonovich integrals is just a first order equation. Moreover if the coefficient $a^{ij}(x,t)$ of the operator A_t is less than $\beta^i(x,t)\beta^j(x,t)$, equation (46) does not have a solution, since the second order part of the same equation represented by the Stratonovich integrals is no longer non-negative definite.

For this reason, we will solve the equation represented by the Stratonovich integrals.

Existence and uniqueness of solutions

In this paragraph we will show how to construct a solution of linear equation (42) under Condition $(E.1)_k$ for some $k \geq 4$ and Condition $(E.2)$.

Note that the right hand side of equation (42) consists of the random first order part and the deterministic second order part $\int_0^t L_s(q_s)u_s(x)\,ds$. The first order part can be regarded as a perturbation term adjoined to the second order part. We will show that the well-posedness of equation (42) can be reduced to the well-posedness of a certain deterministic second order equation, which is a modification of the equation $\partial u_t/\partial t = L_t(q_t)u_t$ affected by the perturbation term.

We first consider the first order part. Let $w_t(f)$ be the solution of the first order equation (42) where $L_t(q_t) \equiv 0$. Then by Theorem 6.1.9 in [11] it is represented by

$$w_t(f)(x) = \xi_{t,0}(x) f(\varphi_{t,0}(x)), \tag{47}$$

where $\varphi_{t,0}$ is the inverse of the stochastic flow $\varphi_{0,t}$ which was defined in the previous section, and

$$\xi_{t,0}(x) = \exp\left\{\int_0^t \alpha(\varphi_{t,s}(x), s) \circ \hat{d}W_s\right\} = \exp\left\{-\mu_{t,0}(x)\right\}. \tag{48}$$

We may consider that for almost all ω, w_t is a linear map on $C^k\left(\mathbf{R}^d \colon \mathbf{R}\right)$. It is one to one and onto. The inverse map is given by

$$w_t^{-1}(f)(x) = \xi_{t,0}(\varphi_{0,t}(x))^{-1} f(\varphi_{0,t}(x)) = \xi_{0,t}(x)^{-1} f(\varphi_{0,t}(x)), \tag{49}$$

where

$$\xi_{0,t}(x) = \exp\left\{ \int_0^t \alpha(\varphi_{0,s}(x), s) \circ dW_s \right\}.$$

Define the operator $L_t^w(q)$ by $w_t^{-1} L_t(q) w_t$. Then we can prove by a direct computation that $L_t^w(q)$ is a second order differential operator represented by

$$L_t^w(q) f = \frac{1}{2} (a(x,t,q)\nabla, \nabla) f + (b(x,t,q), \nabla) f + c(x,t,q) f. \tag{50}$$

Here a^{ij}, b^i, c are smooth functions with random parameter ω defined by

$$\begin{aligned}
a^{ij}(x,t,q) &= \left(\sum_{k,l=1}^d \left(\bar\sigma\bar\sigma^T\right)^{kl}(y,t,q) \partial_k\left(\varphi_{t,0}^i\right)(y) \partial_l\left(\varphi_{t,0}^j\right)(y) \right)\Bigg|_{y=\varphi_{0,t}(x)}, \\
b^i(x,t,q) &= \Bigg(\frac{1}{2} \sum_{k,l=1}^d \left(\bar\sigma\bar\sigma^T\right)^{kl}(y,t,q) \Big(\partial_k\partial_l\left(\varphi_{t,0}^i\right)(y) - 2\partial_k(\mu_{t,0})(y) \partial_l\left(\varphi_{t,0}^i\right)(y) \Big) \\
&\qquad + \sum_{k=1}^d g^k(y,t,q) \partial_k\left(\varphi_{t,0}^i\right)(y) \Bigg)\Bigg|_{y=\varphi_{0,t}(x)}, \\
c(x,t,q) &= \Bigg(\frac{1}{2} \sum_{k,l=1}^d \left(\bar\sigma\bar\sigma^T\right)^{kl}(y,t,q) \Big(\partial_k(\mu_{t,0})(y) \partial_l(\mu_{t,0})(y) - \partial_k\partial_l(\mu_{t,0})(y) \Big) \\
&\qquad - \sum_{k=1}^d g^k(y,t,q) \partial_k(\mu_{t,0})(y) + d(y,t,q) \Bigg)\Bigg|_{y=\varphi_{0,t}(x)},
\end{aligned} \tag{51}$$

where $\partial_k = \partial/\partial x^k$. Indeed, by equations (47), (49) and (43) we have

$$L_t^w(q) f(x) = w_t^{-1} L_t(q) w_t f(x) = \xi_{t,0}(\varphi_{0,t}(x))^{-1} L_t(q) w_t f(\varphi_{0,t}(x)), \tag{52}$$

$$
\begin{aligned}
L_t(q)w_t f(x) = \frac{1}{2}\sum_{k,l=1}^{d} &\left(\bar\sigma\bar\sigma^T\right)^{kl}(x,t,q)\partial_k\partial_l(\xi_{t,0}(x)f(\varphi_{t,0}(x))) \\
&+ \sum_{k=1}^{d} g^k(x,t,q)\partial_k(\xi_{t,0}(x)f(\varphi_{t,0}(x))) \\
&+ d(x,t,q)\xi_{t,0}(x)f(\varphi_{t,0}(x)).
\end{aligned} \tag{53}
$$

Partial derivatives in equation (53) are calculated by

$$
\begin{aligned}
\partial_k(\xi_{t,0}(x)f(\varphi_{t,0}(x))) = \xi_{t,0}(x)&\sum_{i=1}^{d}\partial_i(f)(\varphi_{t,0}(x))\partial_k\left(\varphi^i_{t,0}\right)(x) \\
&+ \partial_k(\xi_{t,0})(x)f(\varphi_{t,0}(x)),
\end{aligned} \tag{54}
$$

$$
\begin{aligned}
\partial_k\partial_l(\xi_{t,0}(x)f(\varphi_{t,0}(x))) &= \xi_{t,0}(x)\sum_{i,j=1}^{d}\partial_i\partial_j(f)(\varphi_{t,0}(x))\partial_k\left(\varphi^i_{t,0}\right)(x)\partial_l\left(\varphi^j_{t,0}\right)(x) \\
&+ \sum_{i=1}^{d}\partial_i(f)(\varphi_{t,0}(x))\Big(\xi_{t,0}(x)\partial_k\partial_l\left(\varphi^i_{t,0}\right)(x) + \partial_l(\xi_{t,0})(x)\partial_k\left(\varphi^i_{t,0}\right)(x) \\
&+ \partial_k(\xi_{t,0})(x)\partial_l\left(\varphi^i_{t,0}\right)(x)\Big) + \partial_k\partial_l(\xi_{t,0})(x)f(\varphi_{t,0}(x)).
\end{aligned} \tag{55}
$$

Substituting equations (54), (55) into (53), and equation (53) into (52), we obtain formulas (51).

Lemma 2.6. *A square matrix-valued function $a(x,t,q)$ is uniformly elliptic, functions $a^{ij}(x,t,q)$, $b^i(x,t,q)$, $c(x,t,q)$ are continuous and bounded on $\mathbf{R}^d\times[0,T]\times\mathbf{R}^d$, twice continuously differentiable in x with bounded derivatives, and uniformly Lipschitz continuous in q. (If we denote by M,χ the bound and the Lipschitz constant, then we may choose $\chi\geq dM/3$.)*

Proof. First of all, let us prove the uniform ellipticity of $a(x,t,q)$. Denote

$$
C(x,t) = \left(\frac{\partial\varphi_{t,0}}{\partial x}\right)^T(\varphi_{0,t}(x)).
$$

Then we can rewrite formula (51) for a as

$$
a(x,t,q) = C^T(x,t)\left(\bar\sigma\bar\sigma^T\right)(\varphi_{0,t}(x),t,q)C(x,t).
$$

By Proposition 2.4, all elements of matrices $C(x,t)$ and its inverse

$$C^{-1}(x,t) = \left(\left(\frac{\partial\varphi_{t,0}}{\partial x}\right)^T\right)^{-1}(\varphi_{0,t}(x)) = \left(\left(\frac{\partial\varphi_{t,0}}{\partial x}\right)^{-1}\right)^T(\varphi_{0,t}(x)) = \left(\frac{\partial\varphi_{0,t}}{\partial x}\right)^T(x)$$

are bounded functions on $\mathbf{R}^d \times [0,T]$. Then there exists some constant $M_\varphi > 0$ such that the norms

$$\|C(x,t)\|_{\mathbf{R}^d\to\mathbf{R}^d} \le M_\varphi, \qquad \left\|C^{-1}(x,t)\right\|_{\mathbf{R}^d\to\mathbf{R}^d} \le M_\varphi$$

for all $(x,t) \in \mathbf{R}^d \times [0,T]$. Therefore

$$\begin{aligned}
(a(x,t,q)\xi,\xi) &\le m|C(x,t)\xi|^2 \le m\|C(x,t)\|^2|\xi|^2 \le mM_\varphi^2|\xi|^2,\\
(a(x,t,q)\xi,\xi) &\ge m^{-1}|C(x,t)\xi|^2 \ge m^{-1}\frac{|\xi|^2}{\|C^{-1}(x,t)\|^2} \ge m^{-1}\frac{|\xi|^2}{M_\varphi^2}
\end{aligned}$$

hold for all $\xi, x, q \in \mathbf{R}^d$ and $t \in [0,T]$, i.e. $a(x,t,q)$ is uniformly elliptic.

The rest of the Lemma follows directly from Condition $(E.2)$ and Proposition 2.4. □

Consider the following deterministic second order equation

$$u_t(x) = f(x) + \int_0^t L_s^w(q_s)u_s(x)\,ds, \qquad t \in (0,T]. \tag{56}$$

A continuous C^2-process $u_t(x)$ is then called a *solution with the initial value* f if it satisfies equation (56) for any x and t a.s. By Theorems 1.1 and 1.3 in [15], if $f \in C\left(\mathbf{R}^d\right)$ is a bounded function on $\mathbf{R}^d$, then equation (56) has a unique bounded solution $u_t(x)$. Moreover by Theorem 1.2 in [15], this solution is a continuous C^3-process.

The following lemma shows the relationship between the solutions of equations (42) and (56).

Lemma 2.7 (Lemma 6.2.3 in [11])**.** *Let $u_t(x)$ be a continuous C^3-process and C^2-semimartingale. It is a solution of equation* (42) *if and only if $u_t'(x) \equiv w_t^{-1}(u_t)(x)$ is a solution of equation* (56)*.*

Let $f \in C\left(\mathbf{R}^d\right)$ be a bounded function on $\mathbf{R}^d$ and let $u_t(x)$ be the unique bounded solution of equation (56), then by Lemma 6.2.3 in [11], $w_t(u_t)(x)$ is the unique bounded solution of equation (42) such that it is a continuous C^3-process.

2.2 Supporting propositions

The well-posedness for equation (1) in case of the quantile vector Q_γ and $\alpha, \beta = 0$ was done in [7]. In this section we will present a number of propositions that are generalizations of the propositions from [7] to the case of both the quantile vector Q_γ and the CKL-quantile vector $\widetilde{Q}_\gamma$, and arbitrary α, β satisfying Condition $(E.1)_k$.

Recall that the heat kernel of the standard heat equation in $\mathbf{R}^d$ with a diffusion coefficient $\sigma > 0$ is defined by

$$G_\sigma(t,x) = \left(2\pi t\sigma^2\right)^{-d/2} \exp\left\{-\frac{x^2}{2t\sigma^2}\right\}.$$

Note that $\int_{\mathbf{R}^d} G_\sigma(t,x)\,dx = 1$ for all $\sigma, t > 0$.

Consider the general heat equation

$$\frac{\partial u_t}{\partial t}(x) = L_t u_t(x), \tag{57}$$

where

$$L_t u(x) = \frac{1}{2}(a(t,x)\nabla, \nabla)u(x) + (b(t,x), \nabla)u(x) + c(t,x)u(x).$$

The following fact about two-sided estimates for heat kernels is well known (see [7], [15]).

Proposition 2.8. *Suppose that a is uniformly elliptic with ellipticity constant m, a, b, c are continuous in t, a is twice continuously differentiable in x and b, c are continuously differentiable in x, a, b, c and all their derivatives are bounded by some constant M. Then there exist positive constants $\sigma_i, C_i, i = \overline{1,3}$, depending only on m, M, T such that the Green function (or heat kernel) of equation (57) (i.e. its solution $G(t,x,s,\xi)$ with the initial condition δ_ξ at* *time s) is well defined, differentiable in x, ξ and satisfies*

$$C_1 G_{\sigma_1}(t-s, x-\xi) \le G(t,x,s,\xi) \le C_2 G_{\sigma_2}(t-s, x-\xi), \tag{58}$$

$$\max\left(\left|\frac{\partial}{\partial \xi} G(t,x,s,\xi)\right|, \left|\frac{\partial}{\partial x} G(t,x,s,\xi)\right|\right) \le C_3 (t-s)^{-1/2} G(t,x,s,\xi) \tag{59}$$

for all $x, \xi \in \mathbf{R}^d$ and $0 \le s < t \le T$.

Let positive m, M, T be given. For every $t \in [0,T]$, denote by $U_{m,M,t}(u_0)$ the set of solutions $u_t(x)$ of the Cauchy problems of all equations (57) satisfying the conditions of Proposition 2.8 with a given initial condition u_0. Also denote

$$U_{m,M,(0,T]}(u_0) = \bigcup_{t \in (0,T]} U_{m,M,t}(u_0).$$

The following two propositions are generalized versions of Propositions 3.2 and 3.3 in [7].

Proposition 2.9. *Let positive m, M, T be given. Under the assumptions of Proposition 2.4, for any $u_0, v_0 \in L_1\left(\mathbf{R}^d\right)$ and $\varepsilon > 0$ there exists $K > 0$ such that for all $u_t \in U_{m,M,[0,T]}(u_0 + h v_0)$, $h \in [0,1]$*

$$\int_{|x| \ge K} |w_t(u_t)(x)|\, dx \le \varepsilon,$$

where w_t is defined by (47)*.*

Proof. Let C_w be the bound from Proposition 2.4. By Proposition 3.2 in [7] there exists $K_1 > 0$ such that for all $u_t^1 \in U_{m,M,[0,T]}(u_0)$

$$\int_{|x| \ge K_1} \left|u_t^1(x)\right|\, dx \le \frac{\varepsilon}{2} \exp\{-C_w\} C_w^{-1}.$$

Note that $|x| \ge K_1 + C_w$ yields $|\varphi_{t,0}(x)| \ge K_1$. Therefore

$$\begin{aligned} \int_{|x| \ge K_1 + C_w} \left|w_t\left(u_t^1\right)(x)\right|\, dx &\le \int_{|\varphi_{t,0}(x)| \ge K_1} \xi_{t,0}(x) \left|u_t^1(\varphi_{t,0}(x))\right|\, dx \\ &\le \exp\{C_w\} \int_{|y| \ge K_1} \left|u_t^1(y)\right| |\det(\partial \varphi_{0,t}(y))|\, dy \le \frac{\varepsilon}{2}. \end{aligned}$$

Similarly, there exists $K_2 > 0$ such that for all $u_t^2 \in U_{m,M,[0,T]}(v_0)$

$$\int_{|x|\geq K_2+C_w} \left|w_t\left(u_t^2\right)(x)\right| \, dx \leq \frac{\varepsilon}{2}.$$

For any $u_t \in U_{m,M,[0,T]}(u_0 + hv_0)$, $h \in [0,1]$ let u_t^1 and u_t^2 be the solutions of the corresponding Cauchy problem starting with u_0 and v_0 respectively. Then

$$\begin{aligned}\int_{|x|\geq \max(K_1,K_2)+C_w} \left|w_t\left(u_t\right)(x)\right| \, dx \leq & \int_{|x|\geq K_1+C_w} \left|w_t\left(u_t^1\right)(x)\right| \, dx \\ & + h\int_{|x|\geq K_2+C_w} \left|w_t\left(u_t^2\right)(x)\right| \, dx \leq \varepsilon.\end{aligned}$$

□

Proposition 2.10. *Let positive m, M, T be given. Under the assumptions of Proposition 2.4, for any strictly positive $u_0 \in C\left(\mathbf{R}^d\right)$ and $K > 0$ there exists $\theta > 0$ such that for all $u_t \in U_{m,M,[0,T]}(u_0)$ and $|x| \leq K$*

$$w_t(u_t)(x) \geq \theta,$$

where w_t is defined by (47).

Proof. Let C_w be the bound from Proposition 2.4. By Proposition 3.3 in [7] there exists $\theta_0 > 0$ such that for all $u_t \in U_{m,M,[0,T]}(u_0)$ and $|x| \leq K + C_w$

$$u_t(x) \geq \theta_0.$$

Note that $|x| \leq K$ yields $|\varphi_{t,0}(x)| \leq K + C_w$. Therefore, for all $u_t \in U_{m,M,[0,T]}(u_0)$ and $|x| \leq K$

$$w_t(u_t)(x) = \xi_{t,0}(x)u_t(\varphi_{t,0}(x)) \geq \exp\{-C_w\}\theta_0.$$

□

Combining these two facts we obtain the following proposition which is a generalization of Proposition 3.4 in [7].

Proposition 2.11. *Let $T > 0$ and $\gamma = (\gamma_1, \ldots, \gamma_d)$ with all $\gamma_j \in (0,1)$ be given. Assume that conditions of Proposition 2.4 hold. Consider equation* (57) *under the assumptions of Proposition 2.8 with given m, M and the additional condition that $w_t(u_t)$ preserves the L_1-norm, where mapping w_t is given by* (47)*. Then for any strictly positive $u_0 \in C\left(\mathbf{R}^d\right) \cap L_1\left(\mathbf{R}^d\right)$ and any non-negative $v_0 \in C\left(\mathbf{R}^d\right) \cap L_1\left(\mathbf{R}^d\right)$, there exist $K, \theta > 0$ such that for all $u_t \in U_{m,M,(0,T]}(u_0 + hv_0)$, $h \in [0,1]$*

$$\max_j \left|Q^j_\gamma(w_t(u_t))\right| \le K, \tag{60}$$

$$\inf\left\{w_t(u_t)(x) \colon \max_j \left|x^j\right| \le K\right\} \ge \theta. \tag{61}$$

The set of functions satisfying conditions (60)–(61) *is convex. Moreover, if $\sum \gamma_j < 1$, then for any strictly positive $u_0 \in C\left(\mathbf{R}^d\right) \cap L_1\left(\mathbf{R}^d\right)$ and any non-negative $v_0 \in C\left(\mathbf{R}^d\right) \cap L_1\left(\mathbf{R}^d\right)$, there exist $K, \theta > 0$ such that for all $u_t \in U_{m,M,(0,T]}(u_0 + hv_0)$, $h \in [0,1]$*

$$\max_j \left|\widetilde{Q}^j_\gamma(w_t(u_t))\right| \le K, \tag{62}$$

$$\inf\left\{w_t(u_t)(x) \colon \max_j \left|x^j\right| \le 2dK\right\} \ge \theta. \tag{63}$$

Proof. Let us pick up an $\varepsilon > 0$ such that

$$\varepsilon < \min\left(\gamma_1, \dots, \gamma_d, 1-\gamma_1, \dots, 1-\gamma_d\right) \|u_0\|_L.$$

By Proposition 2.9 there exists $K > 0$ such that for all $u_t \in U_{m,M,(0,T]}(u_0+hv_0)$, $h \in [0,1]$

$$\int_{|x|\ge K} w_t(u_t)(x)\,dx \le \varepsilon. \tag{64}$$

The following inequalities prove (60).

$$\int_{|x|\ge K} w_t(u_t)(x)\,dx \le \varepsilon < \gamma_i(\|u_0\|_L + h\|v_0\|_L) = \int_{A_i^{Q_\gamma(w_t(u_t))}} w_t(u_t)(x)\,dx,$$
$$\int_{|x|\ge K} w_t(u_t)(x)\,dx \le \varepsilon < (1-\gamma_i)(\|u_0\|_L + h\|v_0\|_L)$$
$$= \int_{\mathbf{R}^d \setminus A_i^{Q_\gamma(w_t(u_t))}} w_t(u_t)(x)\,dx.$$

Existence of $\theta > 0$ such that estimate (61) holds follows directly from Proposition 2.10. Obviously, the set of functions satisfying conditions (60)–(61) is convex.

Suppose $\sum \gamma_j < 1$. Let us pick up an $\varepsilon > 0$ such that

$$\varepsilon < \min\left(\gamma_1, \ldots, \gamma_d, 1 - \sum \gamma_j\right) \|u_0\|_L,$$

and then choose $K > 0$ such that (64) holds for all $u_t \in U_{m,M,(0,T]}(u_0 + hv_0)$, $h \in [0, 1]$. The following inequalities prove (62).

$$\begin{aligned}\int_{|x|\geq K} w_t(u_t)(x)\, dx &\leq \varepsilon < \Big(1 - \sum \gamma_j\Big)(\|u_0\|_L + h\|v_0\|_L)\\ &= \int_{A_0^{\tilde{Q}_\gamma(w_t(u_t))}} w_t(u_t)(x)\, dx,\\ \int_{|x|\geq K} w_t(u_t)(x)\, dx &\leq \varepsilon < \gamma_i(\|u_0\|_L + h\|v_0\|_L) = \int_{A_i^{\tilde{Q}_\gamma(w_t(u_t))}} w_t(u_t)(x)\, dx.\end{aligned}$$

Existence of $\theta > 0$ such that estimate (63) holds follows directly from Proposition 2.10. $\square$

The next proposition shows the Lipschitz continuity of both the quantile Q_γ and the CKL-quantile $\widetilde{Q}_\gamma$ in L_1-norm.

Proposition 2.12. *Let $T > 0$ and $\gamma = (\gamma_1, \ldots, \gamma_d)$ with all $\gamma_j \in (0, 1)$ be given. Assume that conditions of Proposition 2.4 hold. Consider equation* (57) *under the assumptions of Proposition 2.8 with given m, M and the additional condition that $w_t(u_t)$ preserves the L_1-norm, where mapping w_t is given by* (47)*. Then for any strictly positive $u_0 \in C\left(\mathbf{R}^d\right) \cap L_1\left(\mathbf{R}^d\right)$ and any non-negative $v_0 \in C\left(\mathbf{R}^d\right) \cap L_1\left(\mathbf{R}^d\right)$, there exists $C_4 > 0$ such that for all $u_t^1, u_t^2 \in U_{m,M,(0,T]}(u_0 + hv_0)$, $h \in [0, 1]$ (u_t^1, u_t^2 must be chosen for the same h and at the same time t)*

$$\left|Q_\gamma\big(w_t(u_t^2)\big) - Q_\gamma\big(w_t(u_t^1)\big)\right| \leq C_4 \left\|u_t^2 - u_t^1\right\|_L.$$

Moreover, if the dimension $d \leq 3$ and $\sum \gamma_j < 1$, then for any strictly positive $u_0 \in C\left(\mathbf{R}^d\right) \cap L_1\left(\mathbf{R}^d\right)$ and any non-negative $v_0 \in C\left(\mathbf{R}^d\right) \cap L_1\left(\mathbf{R}^d\right)$, there *exists $C_4 > 0$ such that for all $u_t^1, u_t^2 \in U_{m,M,(0,T]}(u_0 + hv_0)$, $h \in [0,1]$*

$$\left|\widetilde{Q}_\gamma\big(w_t\big(u_t^2\big)\big) - \widetilde{Q}_\gamma\big(w_t\big(u_t^1\big)\big)\right| \leq C_4 \left\|u_t^2 - u_t^1\right\|_L .$$

Proof. Let C_w be the bound from Proposition 2.4. Then for all $u_t^1, u_t^2 \in U_{m,M,(0,T]}(u_0 + hv_0)$, $h \in [0,1]$

$$\left\|w_t\big(u_t^2\big) - w_t\big(u_t^1\big)\right\|_L \leq C_w \exp\{C_w\} \left\|u_t^2 - u_t^1\right\|_L . \tag{65}$$

Let $u_t^1, u_t^2 \in U_{m,M,(0,T]}(u_0 + hv_0)$, $h \in [0,1]$. Consider the following continuous positive integrable function

$$v_s(x) = \big(w_t\big(u_t^1\big) + s\left(w_t\big(u_t^2\big) - w_t\big(u_t^1\big)\right)\big)(x), \qquad s \in [0,1].$$

In Appendix E we prove that the function $q(s) = Q_\gamma\left(v_s\right)$ is continuous in s. Also it satisfies the equation

$$\int_{\left\{x \in \mathbf{R}^d :\, x^j \leq q_j(s)\right\}} \big(w_t\big(u_t^1\big) + s\big(w_t\big(u_t^2\big) - w_t\big(u_t^1\big)\big)\big)(x)\, dx = \gamma_j(\|u_0\|_L + h\|v_0\|_L).$$

Considering the difference of the equation above at points $s + \Delta s$ and s, we see that $q_j(s)$ is differentiable and satisfies

$$\begin{aligned} q_j'(s) = &- \left(\int_{\mathbf{R}^{d-1}} \big(w_t\big(u_t^1\big) + s\big(w_t\big(u_t^2\big) - w_t\big(u_t^1\big)\big)\big)(x)\big|_{x^j = q_j(s)} \prod_{k \neq j} dx^k\right)^{-1} \\ &\times \int_{\left\{x \in \mathbf{R}^d :\, x^j \leq q_j(s)\right\}} \big(w_t\big(u_t^2\big) - w_t\big(u_t^1\big)\big)(x)\, dx. \end{aligned}$$

Let us choose $K, \theta > 0$ from Proposition 2.11. Then by (60) we deduce that point $x^j = q_j(s)$ lies in $[-K, K]$, and therefore, by (61) and (65), we have

$$\begin{aligned} \left|Q_\gamma^j\big(w_t\big(u_t^2\big)\big) - Q_\gamma^j\big(w_t\big(u_t^1\big)\big)\right| &= |q_j(1) - q_j(0)| \leq \sup_s \left|q_j'(s)\right| \\ &\leq \frac{C_w \exp\{C_w\}}{\theta K^{d-1}} \left\|u_t^2 - u_t^1\right\|_L . \end{aligned}$$

Now assume that the dimension $d \leq 3$ and $\sum \gamma_j < 1$, and let $u_t^1, u_t^2 \in$

$U_{m,M,(0,T]}(u_0 + hv_0)$, $h \in [0,1]$. Then the function

$$q(s) = \widetilde{Q}_\gamma \left(w_t(u_t^1) + s\left(w_t(u_t^2) - w_t(u_t^1) \right) \right), \qquad s \in [0,1]$$

is continuous in s by the same arguments as above, and satisfies the equation

$$\begin{aligned} &\gamma_j(\|u_0\|_L + h\|v_0\|_L) \\ &\quad = \int_{\left\{x \in \mathbf{R}^d :\, x^j - q_j(s) \geq 0 \vee \max_{i \neq j}(x^i - q_i(s))\right\}} \big(w_t(u_t^1) + s\big(w_t(u_t^2) - w_t(u_t^1)\big)\big)(x)\, dx. \end{aligned}$$

Considering the difference of the equation above at points $s + \Delta s$ and s, we will get a system of linear equations on $\Delta q_i / \Delta s$

$$\begin{aligned} &\int_{q_j(s)}^{+\infty} \int_{-\infty}^{x^j - q_j(s) + q_k(s)} \big(w_t(u_t^2) - w_t(u_t^1)\big)(x) \prod_{k \neq j} dx^k dx^j + \bar{\bar{o}}(1) \\ &\quad = \sum_{\substack{i=1,\dots,d \\ i \neq j}} \left(\frac{\Delta q_j}{\Delta s} - \frac{\Delta q_i}{\Delta s} \right) \Bigg(\int_{q_j(s)}^{+\infty} \int_{-\infty}^{x^j - q_j(s) + q_k(s)} \\ &\qquad\qquad \big(w_t(u_t^1) + s\big(w_t(u_t^2) - w_t(u_t^1)\big)\big)(x)\big|_{x^i = x^j - q_j(s) + q_i(s)} \prod_{\substack{k \neq j \\ k \neq i}} dx^k dx^j + \bar{\bar{o}}(1) \Bigg) \\ &\qquad + \frac{\Delta q_j}{\Delta s} \Bigg(\int_{-\infty}^{q_k(s)} \big(w_t(u_t^1) + s\big(w_t(u_t^2) - w_t(u_t^1)\big)\big)(x)\big|_{x^j = q_j(s)} \prod_{k \neq j} dx^k + \bar{\bar{o}}(1) \Bigg). \end{aligned} \tag{66}$$

Note that, all coefficients converge as $\Delta s \to 0$. Then by Proposition A.2, for small enough Δs, this system has a unique solution that also converges as $\Delta s \to 0$. Let us choose $K, \theta > 0$ from Proposition 2.11. Then by (62) and Proposition B.1, all $q_i(s)$ lie in $[-(2d-1)K, (2d-1)K]$, and therefore, by (63), we have

$$\int_{-\infty}^{q_k(s)} \big(w_t(u_t^1) + s\big(w_t(u_t^2) - w_t(u_t^1)\big)\big)(x)\big|_{x^j = q_j(s)} \prod_{k \neq j} dx^k \geq \theta K^{d-1}.$$

We do not have to solve the limiting equation for $q_j'(s)$ explicitly, but instead we can apply Proposition A.2. It gives us

$$\left|q_j'(s)\right| \le \frac{1}{\theta K^{d-1}} \left\|w_t\big(u_t^2\big) - w_t\big(u_t^1\big)\right\|_L ,$$

and therefore, by (65), we have

$$\begin{aligned}\left|\widetilde{Q}_\gamma^j\big(w_t\big(u_t^2\big)\big) - \widetilde{Q}_\gamma^j\big(w_t\big(u_t^1\big)\big)\right| &= |q_j(1) - q_j(0)| \le \sup_s \left|q_j'(s)\right| \\ &\le \frac{C_w \exp\{C_w\}}{\theta K^{d-1}} \left\|u_t^2 - u_t^1\right\|_L .\end{aligned}$$

□

Proposition 2.13. *In terms and assumptions of Proposition 2.12, $w_t(u_t)$ is continuous in $L_1\left(\mathbf{R}^d\right)$. Therefore (see Appendix E), both quantile $Q_\gamma(w_t(u_t))$ and CKL-quantile $\widetilde{Q}_\gamma(w_t(u_t))$ are continuous in t.*

Proof. Let us fix time s. By Proposition 2.4, there exists $C > 0$ such that

$$\begin{aligned}&\|w_t(u_t) - w_s(u_s)\|_L \\ &\quad\le \|\xi_{t,0}u_t(\varphi_{t,0}) - \xi_{t,0}u_s(\varphi_{s,0})\|_L + \|\xi_{t,0}u_s(\varphi_{s,0}) - \xi_{s,0}u_s(\varphi_{s,0})\|_L \\ &\quad\le \sup_{x\in\mathbf{R}^d} \xi_{t,0}(x)\|u_t(\varphi_{t,0}) - u_s(\varphi_{t,0})\|_L + \sup_{x\in\mathbf{R}^d} \xi_{t,0}(x)\|u_s(\varphi_{t,0}) - u_s(\varphi_{s,0})\|_L \\ &\qquad + \sup_{x\in\mathbf{R}^d} |\xi_{t,0}(x) - \xi_{s,0}(x)|\|u_s(\varphi_{s,0})\|_L \\ &\quad\le C\|u_t - u_s\|_L + C\|u_s(\varphi_{t,0}(\varphi_{0,s})) - u_s\|_L + C \sup_{x\in\mathbf{R}^d} |\xi_{t,0}(x) - \xi_{s,0}(x)|.\end{aligned}$$

We will show that the right hand side of inequality above goes to 0 when $t \to s$.

By Propositions 2.4 and 2.5 we have

$$\begin{aligned}\sup_{x\in\mathbf{R}^d} |\xi_{t,0}(x) - \xi_{s,0}(x)| &= \sup_{x\in\mathbf{R}^d} \exp\{-\mu_{s,0}(x)\} \left|\exp\{-\mu_{t,0}(x) + \mu_{s,0}(x)\} - 1\right| \\ &\le C\left(\exp\left\{\sup_{x\in\mathbf{R}^d} |-\mu_{t,0}(x) + \mu_{s,0}(x)|\right\} - 1\right) \xrightarrow[t\to s]{} 0.\end{aligned}$$

Next, by the strong continuity of the propagators solving equation (56) in L_1 we have

$$\|u_t - u_s\|_L \xrightarrow[t\to s]{} 0.$$

Finally, let an arbitrary $\varepsilon > 0$ be given. By Proposition 3.2 in [7], there exists $K > 1$ such that

$$\int_{|x|\geq K-1} |u_s(x)|\,dx \leq \frac{\varepsilon}{C_w^2+2},$$

where C_w is the bound from Proposition 2.4. By Proposition 2.5 there exists $\delta > 0$ such that

$$\sup_{x\in\mathbf{R}^d} |\varphi_{t,0}(\varphi_{0,s}(x)) - x| \sum_{k=1}^{d} \sup_{|y|\leq K+1} \left|\frac{\partial u_s}{\partial y^k}(y)\right| \int_{|x|\leq K} 1\,dx \leq \frac{\varepsilon}{C_w^2+2},$$

$$\varphi_{t,0}(\varphi_{0,s}(x)) \in U_1(x) = \left\{y \in \mathbf{R}^d \colon |y-x| < 1\right\}$$

hold for all $t \in (s-\delta, s+\delta)$ and $x \in \mathbf{R}^d$. Then for all $t \in (s-\delta, s+\delta)$ we have

$$\begin{aligned}
&\|u_s(\varphi_{t,0}(\varphi_{0,s})) - u_s\|_L \\
&\quad= \int_{|x|\geq K} |u_s(\varphi_{t,0}(\varphi_{0,s}(x))) - u_s(x)|\,dx + \int_{|x|\leq K} |u_s(\varphi_{t,0}(\varphi_{0,s}(x))) - u_s(x)|\,dx \\
&\quad\leq C_w^2 \frac{\varepsilon}{C_w^2+2} + \frac{\varepsilon}{C_w^2+2} + \int_{|x|\leq K} \sum_{k=1}^{d} \sup_{y\in U_1(x)} \left|\frac{\partial u_s}{\partial y^k}(y)\right| |\varphi_{t,0}(\varphi_{0,s}(x)) - x|\,dx \\
&\quad\leq \frac{\varepsilon\,(C_w^2+1)}{C_w^2+2} + \sup_{x\in\mathbf{R}^d} |\varphi_{t,0}(\varphi_{0,s}(x)) - x| \sum_{k=1}^{d} \sup_{|y|\leq K+1} \left|\frac{\partial u_s}{\partial y^k}(y)\right| \int_{|x|\leq K} 1\,dx \leq \varepsilon.
\end{aligned}$$

Therefore

$$\|u_s(\varphi_{t,0}(\varphi_{0,s})) - u_s\|_L \xrightarrow[t\to s]{} 0.$$

□

Let us recall a proposition from [7] that shows Lipschitz continuity, in L_1-norm, of the solutions to diffusion equations with respect to functional parameters.

Proposition 2.14. *Consider two equations* (57)*, specified by two families of operators* L_t^1, L_t^2 *with the coefficients* a_1, b_1, c_1 *and* a_2, b_2, c_2 *respectively, each satisfying the assumptions of Proposition 2.8 with given* m, M*. Assume that all coefficients* a_i, b_i, c_i *are twice continuously differentiable in* x *(with all derivatives bounded). Let* $T > 0$ *be given. Then there exists* $C_5 > 0$ *depending on* *the bounds for the derivatives and m, M, T such that, for any $t_0 \in [0, T]$ and any $u_{t_0} \in H_1^2\left(\mathbf{R}^d\right)$, the solutions $u_t^1, u_t^2, t \in [t_0, T]$ of the corresponding Cauchy problems satisfy the estimate*

$$\left\|u_t^1 - u_t^2\right\|_L \leq C_5(t - t_0) \sup_{\substack{x \in \mathbf{R}^d \\ r \in [t_0, t]}} (|a_1(r,x) - a_2(r,x)| + |b_1(r,x) - b_2(r,x)|$$
$$+ |c_1(r,x) - c_2(r,x)|) \|u_{t_0}\|_{H_1^2\left(\mathbf{R}^d\right)}.$$

Finally, we give the last proposition that yields the bounds for $H_1^2\left(\mathbf{R}^d\right)$ norms of the solutions to diffusion equations.

Proposition 2.15. *Let $T > 0$ be given. Consider equation* (57) *under the assumptions of Proposition 2.8 with given m, M. Assume that coefficients a, b, c are twice continuously differentiable in x (with all derivatives bounded). Then there exists $C > 0$ depending on the bounds for the derivatives and m, M, T such that, for any $u_0 \in H_1^2\left(\mathbf{R}^d\right)$, solution $u_t \in U_{m,M,(0,T]}(u_0)$ satisfies*

$$\|u_t\|_{H_1^2\left(\mathbf{R}^d\right)} \leq C \|u_0\|_{H_1^2\left(\mathbf{R}^d\right)}.$$

For the proof, see Appendix C.

3 Proof of Theorems 1.1 and 1.2

Proof of the theorems. Let us pick up a u_0. For almost every ω (when $W_t(\omega)$ is continuous) consider equations (56) on $[0,T]\times\mathbf{R}^d$ with different d-dimensional continuous stochastic processes q_t. By Lemma 2.6 all these equations (56) satisfy assumptions of Proposition 2.8. Let us choose $K>0$ from Proposition 2.11.

Let $C\left([0,T],\mathbf{R}^d\right)$ denote the Banach space of $\mathbf{R}^d$-valued continuous functions on $[0,T]$ with the usual norm $\|q\|=\sup_t|q_t|$, and let $\mathcal{C}(K)$ be the cube of side $2K$ centred at the origin. For any $q_0\in\mathcal{C}(K)$ let $C_{q_0}\left([0,T],\mathcal{C}(K)\right)$ denote the convex subset of $C\left([0,T],\mathbf{R}^d\right)$ consisting of curves with q_0 given and with values in $\mathcal{C}(K)$. Let

$$q_0=Q_\gamma(u_0).$$

For a given curve $q\in C_{q_0}\left([0,T],\mathcal{C}(K)\right)$ let $u_t[q](x)$ denote the solution at time t of equation (56) with the initial data $u_0(x)$. Let us define

$$\Phi_t[q]=Q_\gamma(w_t(u_t[q])).$$

By Proposition 2.13, we conclude that $\Phi_t[q]$ depends continuously on t. Consequently, applying Proposition 2.11 we deduce that the mapping $q\to\Phi[q]$ is a mapping from $C_{q_0}\left([0,T],\mathcal{C}(K)\right)$ to itself. It is clear that bounded C^3-process $u_t(x)$ solves equation (1) if and only if $q_t=Q_\gamma(u_t)$ is a fixed point of this mapping and is a continuous semimartingale. Thus well posedness of equation (1) is reduced to the problem of existence and uniqueness of this fixed point.

Denote

$$\tau_0=\frac{1}{2}\left(3\chi C_4C_5\|u_0\|_{H_1^2}\right)^{-1},$$

where χ is the Lipschitz constant from Lemma 2.6, and C_4,C_5 are the constants from Propositions 2.12, 2.14 respectively. Consider mapping Φ on $C_{q_0}\left([0,\tau_0],\mathcal{C}(K)\right)$. Let q^1,q^2 be two curves in $C_{q_0}\left([0,\tau_0],\mathcal{C}(K)\right)$ and $\Phi[q^1],\Phi[q^2]$ their respective images. Then by Propositions 2.12, 2.14 we get

$$\begin{aligned}\left|\Phi_t\left[q^1\right]-\Phi_t\left[q^2\right]\right|&\le C_4\left\|u_t\left[q^1\right]-u_t\left[q^2\right]\right\|_L\\&\le 3\chi C_4C_5t\left\|u_0\right\|_{H_1^2}\sup_{r\in[0,t]}\left|q_r^1-q_r^2\right|\le\frac{1}{2}\sup_{r\in[0,t]}\left|q_r^1-q_r^2\right|,\end{aligned}$$

i.e. Φ is a contraction on a closed set, and therefore, there exists a unique fixed point $\widetilde{q}$ in $C_{q_0}([0,\tau_0],\mathcal{C}(K))$.

Denote

$$C_0 = C_2\left(1 + dC_3\tau_0^{-1/2} + d^2C_3\tau_0^{-1}\right)\|u_0\|_L, \qquad \tau = \frac{1}{2}\left(3\chi C_4C_5C_0\right)^{-1}.$$

Using induction by n, we will show that Φ has a unique fixed point in $B_n = C_{q_0}([0,\tau_0+n\tau],\mathcal{C}(K))$. We will see below that this uniform jump τ is possible, because by Proposition 2.8, we have the uniform upper bound

$$\|u_t\|_{H_1^2} \leq C_0 \tag{67}$$

for all $u_t \in U_{m,M,[\tau_0,T]}(u_0)$.

Suppose for some $n \geq 1$, mapping Φ has a unique fixed point $\widetilde{q}$ on B_{n-1}. We need to prove the existence and uniqueness of a fixed point of Φ on B_n. Consider restriction of mapping Φ on invariant $C_{\widetilde{q}}([0,\tau_0+n\tau],\mathcal{C}(K))$, the convex subset of B_n consisting of curves that coincide with $\widetilde{q}$ on $[0,\tau_0+(n-1)\tau]$. It is easy to deduct that any fixed point in B_n must be an element of $C_{\widetilde{q}}([0,\tau_0+n\tau],\mathcal{C}(K))$. Let q^1, q^2 be two curves in $C_{\widetilde{q}}([0,\tau_0+n\tau],\mathcal{C}(K))$ and $\Phi[q^1], \Phi[q^2]$ their respective images. Then by Propositions 2.12, 2.14 and (67) we get

$$\begin{aligned}
\left|\Phi_t\left[q^1\right] - \Phi_t\left[q^2\right]\right| &\leq C_4\left\|u_t\left[q^1\right] - u_t\left[q^2\right]\right\|_L \\
&\leq 3\chi C_4C_5C_0(t-\tau_0-(n-1)\tau)\sup_{r\in[\tau_0+(n-1)\tau,t]}\left|q_r^1 - q_r^2\right| \\
&\leq \frac{1}{2}\sup_{r\in[0,t]}\left|q_r^1 - q_r^2\right|,
\end{aligned}$$

i.e. restricted Φ is a contraction on a closed set, and therefore, there exists a unique fixed point $\widetilde{\widetilde{q}}$ in $C_{\widetilde{q}}([0,\tau_0+n\tau],\mathcal{C}(K))$, which is also a unique fixed point in B_n.

Let us recursively define a sequence $\{q^n\}_{n=1}^{\infty}$ of elements in $C_{q_0}([0,T],\mathcal{C}(K))$ as

$$q_t^{n+1} = \Phi_t[q^n]$$

with the first element $q_t^1 \equiv q_0$. Then the limit

$$\widetilde{q} = \lim_{n \to \infty} q^n$$

is the unique fixed point of the mapping Φ. By Proposition 4.1 in [2], all q_t^n and $\widetilde{q}_t$ are continuous semimartingales.

The proof is complete. $\square$

4 Sensitivity analysis

Consider equation (1) under the assumptions of Theorem 1.1 and its perturbed version

$$u_t(x) = u_0(x) + \int_0^t L_s^w(Q_\gamma(w_s(u_s)))u_s(x)\,ds, \qquad t \in (0,T]. \tag{68}$$

Let $u_t \in L_1\left(\mathbf{R}^d\right) \cap C^2\left(\mathbf{R}^d\right)$ be a positive continuous C^2-process, such that $w_t(u_t)$ is continuous in $L_1\left(\mathbf{R}^d\right)$. Then it is called a *solution with the initial value* u_0 if it satisfies equation (68) for any $x \in \mathbf{R}^d$ and $t \in (0,T]$ a.s.

Let us fix a bounded strictly positive $u_0 \in H_1^2\left(\mathbf{R}^d\right) \cap C\left(\mathbf{R}^d\right)$ and a bounded non-negative $v_0 \in H_1^2\left(\mathbf{R}^d\right) \cap C\left(\mathbf{R}^d\right)$. In previous sections we proved that for all $h \in [0,1]$ equation (68) has a unique bounded solution $u_t[u_0 + hv_0](x)$ with the initial value $u_0 + hv_0$.

The objective of this section is to study the sensitivity of the solutions u_t and $w_t(u_t)$ of equations (68) and (1) with respect to initial data, that is

$$\frac{\delta u_t[u_0]}{\delta u_0(v_0)}(x) = \left.\frac{\partial}{\partial h}\right|_{h=0+} u_t[u_0 + hv_0](x), \tag{69}$$

$$\begin{aligned}\frac{\delta w_t(u_t[u_0])}{\delta u_0(v_0)}(x) &= \left.\frac{\partial}{\partial h}\right|_{h=0+} w_t(u_t[u_0 + hv_0])(x) \\ &= \xi_{t,0}(x)\left.\frac{\partial}{\partial h}\right|_{h=0+} u_t[u_0 + hv_0](\varphi_{t,0}(x)) = w_t\left(\frac{\delta u_t[u_0]}{\delta u_0(v_0)}\right)(x).\end{aligned} \tag{70}$$

Basic definitions for the variational derivatives of functionals on measures and some of their elementary properties can be found in [8].

By formula (70) we deduce that the existence of the variational derivative of the solution of equation (1) is equal to the existence of the variational derivative of the solution of equation (68).

Theorem 4.1. *Consider equation* (1) *under the assumptions of Theorem 1.1. Assume that the first order derivatives with respect to q of the coefficients of the operator L_t are bounded. Then for any bounded strictly positive $u_0 \in H_1^2\left(\mathbf{R}^d\right) \cap C\left(\mathbf{R}^d\right)$ and any bounded non-negative $v_0 \in H_1^2\left(\mathbf{R}^d\right) \cap C\left(\mathbf{R}^d\right)$, the variational derivative* (70) *exists.*

Let positive C_w, C_2, K, θ, C_4, C_5, C be the constants from Propositions 2.4, 2.8, 2.11, 2.12, 2.14, 2.15. Note that we may choose $C_5 \geq C_2 C$ and $C_4 \geq C_w \exp\{C_w\}/\left(\theta K^{d-1}\right)$. Recall how for every $h \in [0,1]$ we construct a unique bounded solution of equation (68) with the initial value $u_0 + hv_0$. Denote

$$\tau_0 = \frac{1}{2}\left(3\chi C_4 C_5 \left(\|u_0\|_{H_1^2} + \|v_0\|_{H_1^2}\right)\right)^{-1}, \qquad q_0(h) = Q_\gamma(u_0 + hv_0),$$

where χ is the Lipschitz constant from Lemma 2.6. Let us recursively define a sequence $\{q^n(h)\}_{n=1}^\infty$ of elements in the set $C_{q_0(h)}\left([0,\tau_0], \mathbf{R}^d\right)$ with the first element $q_t^1(h) \equiv q_0(h)$. Denote by $u_{t,h}^n$ the solution of

$$u_{t,h}^n(x) = u_0(x) + hv_0(x) + \int_0^t L_s^w(q_s^n(h)) u_{s,h}^n(x)\, ds, \qquad t \in (0, \tau_0]$$

and define

$$q_t^{n+1}(h) = Q_\gamma\left(w_t\left(u_{t,h}^n\right)\right).$$

In Section 3 we showed that $q^n(h)$ converges to some element $q(h)$, which corresponds to the solution $u_t[u_0 + hv_0]$, $t \in [0, \tau_0]$ of equation (68) with the initial value $u_0 + hv_0$.

Let us find the recursive formula for $\frac{\partial}{\partial h} q^n(h)$. We start with the first element $q^1(h)$. Differentiation with respect to h of equation

$$\int_{\left\{x \in \mathbf{R}^d :\, x^k \leq q_{0,k}(h)\right\}} (u_0(x) + hv_0(x))\, dx = \gamma_k(\|u_0\|_L + h\|v_0\|_L)$$

gives us

$$\begin{aligned} \frac{\partial}{\partial h} q_{0,k}(h) = &\left(\gamma_k \|v_0\|_L - \int_{\left\{x \in \mathbf{R}^d :\, x^k \leq q_{0,k}(h)\right\}} v_0(x)\, dx\right) \\ &\times \left(\int_{\mathbf{R}^{d-1}} (u_0(x) + hv_0(x))\Big|_{x^k = q_{0,k}(h)} \prod_{i \neq k} dx^i\right)^{-1}. \end{aligned}$$

Next, we will express $\frac{\partial}{\partial h} q^{n+1}(h)$ in terms of $\frac{\partial}{\partial h} q^n(h)$. Assume that coefficients a^{ij}, b^i, c in operator L_t^w are continuously differentiable in q with bounded derivatives. Then $u^n_{t,h}(x)$ is continuously differentiable in h, and

$$\frac{\partial}{\partial t}\frac{\partial}{\partial h}u^n_{t,h}(x) = L^w_t(q^n_t(h))\frac{\partial}{\partial h}u^n_{t,h}(x) + \frac{1}{2}\sum_{i,j=1}^{d}\frac{\partial}{\partial h}a^{ij}(t,x,q^n_t(h))\frac{\partial^2}{\partial x^i\partial x^j}u^n_{t,h}(x)$$
$$+\sum_{i=1}^{d}\frac{\partial}{\partial h}b^i(t,x,q^n_t(h))\frac{\partial}{\partial x^i}u^n_{t,h}(x) + \frac{\partial}{\partial h}c(t,x,q^n_t(h))u^n_{t,h}(x).$$

Applying Duhamel's principle, we get

$$\begin{aligned}\frac{\partial}{\partial h}u^n_{t,h}(x) &= \int_{\mathbf{R}^d} G(t,x,0,\xi)[q^n(h)]v_0(\xi)\,d\xi + \int_0^t\sum_{k=1}^{d}\frac{\partial}{\partial h}q^n_{r,k}(h)\\ &\times\int_{\mathbf{R}^d} G(t,x,r,\xi)[q^n(h)]\left(\frac{1}{2}\sum_{i,j=1}^{d}\frac{\partial a^{ij}}{\partial q^k}(r,\xi,q^n_r(h))\frac{\partial^2 u^n_{r,h}}{\partial\xi^i\partial\xi^j}(\xi)\right.\\ &\left.+\sum_{i=1}^{d}\frac{\partial b^i}{\partial q^k}(r,\xi,q^n_r(h))\frac{\partial u^n_{r,h}}{\partial\xi^i}(\xi) + \frac{\partial c}{\partial q^k}(r,\xi,q^n_r(h))u^n_{r,h}(\xi)\right)d\xi dr.\end{aligned} \tag{71}$$

By formula (47) we deduce

$$\frac{\partial}{\partial h}w_t\big(u^n_{t,h}\big)(x) = w_t\left(\frac{\partial}{\partial h}u^n_{t,h}\right)(x).$$

Differentiation with respect to h of equation

$$\int_{\left\{x\in\mathbf{R}^d:\, x^k\le q^{n+1}_{t,k}(h)\right\}} w_t\big(u^n_{t,h}\big)(x)\,dx = \gamma_k\left\|w_t\big(u^n_{t,h}\big)\right\|_L = \gamma_k(\|u_0\|_L + h\|v_0\|_L)$$

gives us

$$\begin{aligned}\frac{\partial}{\partial h}q^{n+1}_{t,k}(h) &= \left(\gamma_k\|v_0\|_L - \int_{\left\{x\in\mathbf{R}^d:\, x^k\le q^{n+1}_{t,k}(h)\right\}} w_t\left(\frac{\partial}{\partial h}u^n_{t,h}\right)(x)\,dx\right)\\ &\times\left(\int_{\mathbf{R}^{d-1}} w_t\big(u^n_{t,h}\big)(x)\Big|_{x^k=q^{n+1}_{t,k}(h)}\prod_{i\neq k}dx^i\right)^{-1}.\end{aligned} \tag{72}$$

Denote

$$B = \frac{2\left(1 + C_w\exp\{C_w\}C_2\right)\|v_0\|_L}{\theta K^{d-1}}.$$

Using induction by n we will show that for all $t \in [0, \tau_0]$, $h \in [0, 1]$

$$\max_k \left| \frac{\partial}{\partial h} q^n_{t,k}(h) \right| \leq B. \tag{73}$$

The basis is obvious. Suppose (73) holds for some $n \geq 1$. From (71) we deduce

$$\begin{aligned}
\int_{\mathbf{R}^d} \left| \frac{\partial}{\partial h} u^n_{t,h}(x) \right| dx &\leq C_2 \|v_0\|_L + dMBC_2 \int_0^t \left\| u^n_{r,h} \right\|_{H^2_1} dr \\
&\leq C_2 \|v_0\|_L + dMBC_2 \tau_0 C \left(\|u_0\|_{H^2_1} + \|v_0\|_{H^2_1} \right) \\
&= C_2 \|v_0\|_L + \frac{B}{2C_4} \frac{dM}{3\chi} \frac{C_2 C}{C_5} \leq C_2 \|v_0\|_L + \frac{B}{2} \frac{\theta K^{d-1}}{C_w \exp\{C_w\}}.
\end{aligned}$$

Then from (72) we deduce

$$\begin{aligned}
\left| \frac{\partial}{\partial h} q^{n+1}_{t,k}(h) \right| &\leq \frac{1}{\theta K^{d-1}} \left(\|v_0\|_L + C_w \exp\{C_w\} \int_{\mathbf{R}^d} \left| \frac{\partial}{\partial h} u^n_{t,h}(x) \right| dx \right) \\
&\leq \frac{1}{\theta K^{d-1}} \left(\|v_0\|_L + C_w \exp\{C_w\} \left(C_2 \|v_0\|_L + \frac{B}{2} \frac{\theta K^{d-1}}{C_w \exp\{C_w\}} \right) \right) \\
&= B.
\end{aligned}$$

We are going to show that $\frac{\partial}{\partial h} q^n_{t,k}(h)$ converges uniformly in $h \in [0, 1]$ for all t, k. Then by Theorem 7.17 in [17], we will deduce that $q_{t,k}(h)$ is differentiable in h, and therefore, variational derivative (69) exists.

Let us fix t, k. Theorem 7.8 in [17] states that $\frac{\partial}{\partial h} q^n_{t,k}(h)$ converges uniformly in $h \in [0, 1]$ if and only if for every $\varepsilon > 0$ there exists an integer N such that $m \geq N$, $n \geq N$, $h \in [0, 1]$ implies

$$\left| \frac{\partial}{\partial h} q^m_{t,k}(h) - \frac{\partial}{\partial h} q^n_{t,k}(h) \right| \leq \varepsilon.$$

Denote by a_{n+1}, b_{n+1} the numerator and denominator of the right hand side of formula (72). Then by (73) we have

$$
\begin{aligned}
\left|\frac{\partial}{\partial h} q_{t,k}^{n+1}(h) - \frac{\partial}{\partial h} q_{t,k}^{n}(h)\right| &= \left|\frac{a_{n+1}-a_n}{b_{n+1}} - \frac{\partial}{\partial h} q_{t,k}^{n}(h)\frac{b_{n+1}-b_n}{b_{n+1}}\right| \\
\leq \frac{1}{\theta K^{d-1}} &\left(\left|\int_{q_{t,k}^{n}(h)}^{q_{t,k}^{n+1}(h)} I_3\big(x^k\big)\, dx^k\right| + I_4\right) \\
+ \frac{B}{\theta K^{d-1}} &\left(I_1\big(q_{t,k}^{n+1}(h)\big) + \left|I_2\big(q_{t,k}^{n+1}(h)\big) - I_2\big(q_{t,k}^{n}(h)\big)\right|\right),
\end{aligned}
\tag{74}
$$

where

$$
\begin{aligned}
I_1\big(x^k\big) &= \left|\int_{\mathbf{R}^{d-1}} w_t\big(u_{t,h}^{n}\big)(x) \prod_{i\neq k} dx^i - \int_{\mathbf{R}^{d-1}} w_t\big(u_{t,h}^{n-1}\big)(x) \prod_{i\neq k} dx^i\right|, \\
I_2\big(x^k\big) &= \int_{\mathbf{R}^{d-1}} w_t\big(u_{t,h}^{n-1}\big)(x) \prod_{i\neq k} dx^i, \\
I_3\big(x^k\big) &= \int_{\mathbf{R}^{d-1}} w_t\left(\frac{\partial}{\partial h} u_{t,h}^{n}\right)(x) \prod_{i\neq k} dx^i, \\
I_4 &= \left|\int_{\left\{x\in\mathbf{R}^d:\, x^k\leq q_{t,k}^{n}(h)\right\}} \left(w_t\left(\frac{\partial}{\partial h} u_{t,h}^{n}\right)(x) - w_t\left(\frac{\partial}{\partial h} u_{t,h}^{n-1}\right)(x)\right) dx\right|.
\end{aligned}
$$

To estimate the right hand side of (74) we need the following upper bounds

$$
\begin{aligned}
I_1\big(x^k\big) &\leq \int_{\mathbf{R}^{d-1}} \left|w_t\big(u_{t,h}^{n}\big)(x) - w_t\big(u_{t,h}^{n-1}\big)(x)\right| \prod_{i\neq k} dx^i \\
&\leq \exp\{C_w\} \int_{\mathbf{R}^{d-1}} \left|\int_0^t U_{n-1}^{t,s}\big(L_s^n - L_s^{n-1}\big) U_n^{s,0}(u_0+hv_0)(\varphi_{t,0}(x))\, ds\right| \prod_{i\neq k} dx^i \\
&\leq \exp\{C_w\} C_2 \int_0^t \int_{\mathbf{R}^d} \left|\big(L_s^n - L_s^{n-1}\big)\, U_n^{s,0}(u_0+hv_0)(\xi)\right| \\
&\quad \times \int_{\mathbf{R}^{d-1}} G_{\sigma_2}(t-s, \varphi_{t,0}(x)-\xi) \prod_{i\neq k} dx^i\, d\xi\, ds,
\end{aligned}
$$

$$
\begin{aligned}
\left|I_2'\big(x^k\big)\right| &\leq \exp\{C_w\} C_w \int_{\mathbf{R}^{d-1}} \left(\sum_{j=1}^{d} \left|\frac{\partial u_{t,h}^{n-1}}{\partial x^j}(\varphi_{t,0}(x))\right| + u_{t,h}^{n-1}(\varphi_{t,0}(x))\right) \prod_{i\neq k} dx^i \\
&\leq \exp\{C_w\} C_w C_2 \left(C_3 t^{-1/2} d + 1\right)
\end{aligned}
$$

$$\times \int_{\mathbf{R}^d} (u_0 + hv_0)(\xi) \int_{\mathbf{R}^{d-1}} G_{\sigma_2}(t, \varphi_{t,0}(x) - \xi) \prod_{i \neq k} dx^i \, d\xi,$$

$$\begin{aligned}
\left|I_3\left(x^k\right)\right| \leq \exp\{C_w\} C_2 \int_{\mathbf{R}^d} v_0(\xi) \int_{\mathbf{R}^{d-1}} G_{\sigma_2}(t, \varphi_{t,0}(x) - \xi) \prod_{i \neq k} dx^i \, d\xi \\
+ \exp\{C_w\} C_2 dMB \int_0^t \int_{\mathbf{R}^d} \left(\sum_{i,j=1}^d \left| \frac{\partial^2 u^n_{r,h}}{\partial \xi^i \partial \xi^j}(\xi) \right| + \sum_{i=1}^d \left| \frac{\partial u^n_{r,h}}{\partial \xi^i}(\xi) \right| \right. \\
\left. + u^n_{r,h}(\xi) \right) \int_{\mathbf{R}^{d-1}} G_{\sigma_2}(t-r, \varphi_{t,0}(x) - \xi) \prod_{i \neq k} dx^i \, d\xi \, dr,
\end{aligned}$$

$$\begin{aligned}
I_4 &\leq \exp\{C_w\} C_w \int_{\mathbf{R}^d} \left| \frac{\partial}{\partial h} u^n_{t,h}(x) - \frac{\partial}{\partial h} u^{n-1}_{t,h}(x) \right| dx \\
&\leq 3\chi \exp\{C_w\} C_w C_5 \tau_0 \|v_0\|_{H^2_1} \left\| q^n(h) - q^{n-1}(h) \right\| \\
&\quad + \tau_0 C_2 C M \exp\{C_w\} C_w \left(\|u_0\|_{H^2_1} + \|v_0\|_{H^2_1} \right) \left\| \frac{\partial}{\partial h} q^n(h) - \frac{\partial}{\partial h} q^{n-1}(h) \right\|.
\end{aligned}$$

Using these estimates we finally obtain

$$\begin{aligned}
\left\| \frac{\partial}{\partial h} q^{n+1}(h) - \frac{\partial}{\partial h} q^n(h) \right\| + \left\| q^{n+1}(h) - q^n(h) \right\| \leq \\
c_1 \tau_0 \left\| \frac{\partial}{\partial h} q^n(h) - \frac{\partial}{\partial h} q^{n-1}(h) \right\| + c_2 \tau_0 \left\| q^n(h) - q^{n-1}(h) \right\|
\end{aligned}$$

for some constants c_1, c_2. Therefore, variational derivative (69) exists for a small τ_0. The global result follows from the usual iteration procedure.

5 Multidimensional W_t

In this section we shall study the well-posedness of the following second order SPDE

$$u_t(x) = u_0(x) + \int_0^t L_s(Q_\gamma(u_s))u_s(x)\,ds + \int_0^t (\beta,\nabla)u_s(x)\circ dW_s, \quad t\in(0,T], \tag{75}$$

where L_t is an operator of the form

$$L_t(q)u = \frac{1}{2}\left(\bar\sigma\bar\sigma^T(t,x,q)\nabla,\nabla\right)u + (g(t,x,q),\nabla)u + d(t,x,q)u,$$

W_t is a standard d'-dimensional Brownian motion, β is a $d\times d'$ matrix of constants, functions $d(t,x,q)$, $g(t,x,q)$ and $\bar\sigma(t,x,q)$ are, respectively, 1-dimensional, $d\times 1$ vector-valued and $d\times d''$ matrix-valued continuous functions on $[0,T]\times\mathbf{R}^d\times\mathbf{R}^d$ satisfying Conditions $(E.2)$ and $(E.3)$.

Theorem 5.1. *For a given $T>0$ and vector $\gamma=(\gamma_1,\dots,\gamma_d)$ with all $\gamma_k\in(0,1)$, consider equation (75) in case of the quantile vector Q_γ. Assume that Conditions $(E.2)$ and $(E.3)$ hold. Then for any bounded strictly positive $u_0\in H_1^2\left(\mathbf{R}^d\right)\cap C\left(\mathbf{R}^d\right)$, there exists a unique bounded solution $u_t(x)$ of equation (75) with initial condition u_0, such that it is a continuous C^3-process.*

Theorem 5.2. *For a given $T>0$, vector $\gamma=(\gamma_1,\dots,\gamma_d)$ with all $\gamma_k\in(0,1)$ and $\sum\gamma_k<1$, and dimension $d\le 3$, consider equation (75) in case of the CKL-quantile vector $\widetilde{Q}_\gamma$. Assume that Conditions $(E.2)$ and $(E.3)$ hold. Then for any bounded strictly positive $u_0\in H_1^2\left(\mathbf{R}^d\right)\cap C\left(\mathbf{R}^d\right)$, there exists a unique bounded solution $u_t(x)$ of equation (75) with initial condition u_0, such that it is a continuous C^3-process.*

The stochastic characteristic system of equation (75) with $L_s\equiv 0$ is given by

$$\varphi_{s,t}(x) = x - \beta\times\left(W_t - W_s\right), \qquad \eta_{s,t}(x,u) = u.$$

Note that function $\varphi_{s,t}(x)-x$ is bounded on $[0,T]^2\times\mathbf{R}^d$, the Jacobian matrix $(\partial\varphi_{s,t}(x))$ is the identity matrix and

$$\varphi_{t,0}(x) - \varphi_{s,0}(x) = \beta\times\left(W_t - W_s\right) \xrightarrow[t\to s]{} 0,$$

i.e. Propositions 2.4 and 2.5 hold in multidimensional case too. Then the proofs of Theorems 5.1 and 5.2 are identical to those in the case of 1-dimensional W_t with some simplifications.

Remark. *In the general case of a function $\beta(t,x)$, we cannot prove Propositions 2.4 and 2.5 in the same way as in Section 2.1.1 because the corresponding stochastic characteristic system on $\varphi_{s,t}$ does not have an explicit solution (Doss–Sussman method is not applicable). Although, it might be possible to apply techniques developed in [5] to get similar results.*

A Auxiliary propositions of linear algebra

In this appendix we will present two auxiliary propositions. The first Proposition A.1 is used in the proof of the second Proposition A.2, while Proposition A.2 is used in the proof of Proposition 2.12 for assessing the solution of equation (66).

Definition. *If A is a square matrix, then the minor of the entry in the i-th row and j-th column (also called the (i,j)-th minor) is the determinant of the submatrix formed by deleting the i-th row and j-th column.*

Proposition A.1. *Consider a square matrix $A = (a^i_j)^d_{i,j=1}$ with $a^i_i > 0$, $a^i_j < 0$ if $i \neq j$, and $\sum_{j=1}^d a^i_j > 0$ for all $i = 1, \ldots, d$. Let $M^i_j(A)$ be the (i,j)-th minor of A. Then $(-1)^{i+j} M^i_j(A) > 0$ for all $i, j = 1 \ldots, d$.*

Proof. We will prove the statement by induction on d. Case $d = 2$ is obvious. Assuming that the statement is true for $d-1$, we need to prove it for dimension d.

Let $A = (a^i_j)^d_{i,j=1}$ be the matrix from the proposition. Denote by $B = \left(b^k_l\right)^{d-1}_{k,l=1}$ the submatrix of A formed by deleting the i-th row and j-th column, then $M^i_j(A) = \det B$.

First of all, we will prove that $M^i_i(A) > 0$. Matrix B satisfies the proposition for dimension $d-1$, in particular, $\sum_{l=1}^{d-1} b^k_l > 0$ for all $k = 1, \ldots, d-1$. Then by our induction hypothesis we have

$$(-1)^{k+1} M^k_1(B) > 0.$$

If we add to the first column of B all other columns, then Laplace's formula gives us

$$M^i_i(A) = \det B = \sum_{k=1}^{d-1} (-1)^{k+1} M^k_1(B) \sum_{l=1}^{d-1} b^k_l > 0.$$

Next we will prove that $(-1)^{i+j}M_j^i(A) > 0$ for all $i < j$.

$$
B = \begin{array}{cc}
 & \begin{array}{cccccccccc} 1 & \cdots & i-1 & i & i+1 & \cdots & j-1 & j & \cdots & d-1 \end{array} \\
\begin{array}{c} 1 \\ \vdots \\ i-1 \\ i \\ \vdots \\ j-2 \\ j-1 \\ j \\ \vdots \\ d-1 \end{array} &
\begin{pmatrix}
\mathbf{a_1^1} & \cdots & a_{i-1}^1 & a_i^1 & a_{i+1}^1 & \cdots & a_{j-1}^1 & a_{j+1}^1 & \cdots & a_d^1 \\
\vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
a_1^{i-1} & \cdots & \mathbf{a_{i-1}^{i-1}} & a_i^{i-1} & a_{i+1}^{i-1} & \cdots & a_{j-1}^{i-1} & a_{j+1}^{i-1} & \cdots & a_d^{i-1} \\
a_1^{i+1} & \cdots & a_{i-1}^{i+1} & a_i^{i+1} & \mathbf{a_{i+1}^{i+1}} & \cdots & a_{j-1}^{i+1} & a_{j+1}^{i+1} & \cdots & a_d^{i+1} \\
\vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
a_1^{j-1} & \cdots & a_{i-1}^{j-1} & a_i^{j-1} & a_{i+1}^{j-1} & \cdots & \mathbf{a_{j-1}^{j-1}} & a_{j+1}^{j-1} & \cdots & a_d^{j-1} \\
a_1^{j} & \cdots & a_{i-1}^{j} & a_i^{j} & a_{i+1}^{j} & \cdots & a_{j-1}^{j} & a_{j+1}^{j} & \cdots & a_d^{j} \\
a_1^{j+1} & \cdots & a_{i-1}^{j+1} & a_i^{j+1} & a_{i+1}^{j+1} & \cdots & a_{j-1}^{j+1} & \mathbf{a_{j+1}^{j+1}} & \cdots & a_d^{j+1} \\
\vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
a_1^{d} & \cdots & a_{i-1}^{d} & a_i^{d} & a_{i+1}^{d} & \cdots & a_{j-1}^{d} & a_{j+1}^{d} & \cdots & \mathbf{a_d^{d}}
\end{pmatrix}
\end{array}
$$

Denote by C a matrix formed by consecutive interchanging of columns i and $i+1$, $i+1$ and $i+2$, $\ldots$, $j-2$ and $j-1$ in matrix B.

$$
C = \begin{array}{cc}
 & \begin{array}{cccccccccc} 1 & \cdots & i-1 & i & \cdots & j-2 & j-1 & j & \cdots & d-1 \end{array} \\
\begin{array}{c} 1 \\ \vdots \\ i-1 \\ i \\ \vdots \\ j-2 \\ j-1 \\ j \\ \vdots \\ d-1 \end{array} &
\begin{pmatrix}
\mathbf{a_1^1} & \cdots & a_{i-1}^1 & a_{i+1}^1 & \cdots & a_{j-1}^1 & a_i^1 & a_{j+1}^1 & \cdots & a_d^1 \\
\vdots & & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
a_1^{i-1} & \cdots & \mathbf{a_{i-1}^{i-1}} & a_{i+1}^{i-1} & \cdots & a_{j-1}^{i-1} & a_i^{i-1} & a_{j+1}^{i-1} & \cdots & a_d^{i-1} \\
a_1^{i+1} & \cdots & a_{i-1}^{i+1} & \mathbf{a_{i+1}^{i+1}} & \cdots & a_{j-1}^{i+1} & a_i^{i+1} & a_{j+1}^{i+1} & \cdots & a_d^{i+1} \\
\vdots & & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
a_1^{j-1} & \cdots & a_{i-1}^{j-1} & a_{i+1}^{j-1} & \cdots & \mathbf{a_{j-1}^{j-1}} & a_i^{j-1} & a_{j+1}^{j-1} & \cdots & a_d^{j-1} \\
a_1^{j} & \cdots & a_{i-1}^{j} & a_{i+1}^{j} & \cdots & a_{j-1}^{j} & a_i^{j} & a_{j+1}^{j} & \cdots & a_d^{j} \\
a_1^{j+1} & \cdots & a_{i-1}^{j+1} & a_{i+1}^{j+1} & \cdots & a_{j-1}^{j+1} & a_i^{j+1} & \mathbf{a_{j+1}^{j+1}} & \cdots & a_d^{j+1} \\
\vdots & & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
a_1^{d} & \cdots & a_{i-1}^{d} & a_{i+1}^{d} & \cdots & a_{j-1}^{d} & a_i^{d} & a_{j+1}^{d} & \cdots & \mathbf{a_d^{d}}
\end{pmatrix}
\end{array}
$$

All elements c_l^{j-1} in $(j-1)$-st row of matrix C are less than 0, and apart from $(j-1)$-st row this matrix satisfies the proposition for dimension $d-1$, i.e. by our induction hypothesis we have

$$(-1)^{j-1+l}M_l^{j-1}(C) > 0.$$

Then Laplace's formula gives us

$$
\begin{aligned}
(-1)^{i+j} M_j^i(A) = (-1)^{i+j} \det B &= (-1)^{i+j}(-1)^{j-i-1} \det C \\
&= -\sum_{l=1}^{d-1} (-1)^{j-1+l} M_l^{j-1}(C) c_l^{j-1} > 0.
\end{aligned}
$$

Finally, we will prove that $(-1)^{i+j} M_j^i(A) > 0$ for all $i > j$.

$$
B = \begin{array}{cc}
 & \begin{array}{cccccccccc} 1 & \cdots & j-1 & j & \cdots & i-2 & i-1 & i & \cdots & d-1 \end{array} \\
\begin{array}{c} 1 \\ \vdots \\ j-1 \\ j \\ j+1 \\ \vdots \\ i-1 \\ i \\ \vdots \\ d-1 \end{array} &
\left(\begin{array}{cccccccccc}
\mathbf{a}_{\mathbf{1}}^{\mathbf{1}} & \cdots & a_{j-1}^1 & a_{j+1}^1 & \cdots & a_{i-1}^1 & a_i^1 & a_{i+1}^1 & \cdots & a_d^1 \\
\vdots & & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
a_1^{j-1} & \cdots & \mathbf{a}_{\mathbf{j-1}}^{\mathbf{j-1}} & a_{j+1}^{j-1} & \cdots & a_{i-1}^{j-1} & a_i^{j-1} & a_{i+1}^{j-1} & \cdots & a_d^{j-1} \\
a_1^{j} & \cdots & a_{j-1}^{j} & a_{j+1}^{j} & \cdots & a_{i-1}^{j} & a_i^{j} & a_{i+1}^{j} & \cdots & a_d^{j} \\
a_1^{j+1} & \cdots & a_{j-1}^{j+1} & \mathbf{a}_{\mathbf{j+1}}^{\mathbf{j+1}} & \cdots & a_{i-1}^{j+1} & a_i^{j+1} & a_{i+1}^{j+1} & \cdots & a_d^{j+1} \\
\vdots & & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
a_1^{i-1} & \cdots & a_{j-1}^{i-1} & a_{j+1}^{i-1} & \cdots & \mathbf{a}_{\mathbf{i-1}}^{\mathbf{i-1}} & a_i^{i-1} & a_{i+1}^{i-1} & \cdots & a_d^{i-1} \\
a_1^{i+1} & \cdots & a_{j-1}^{i+1} & a_{j+1}^{i+1} & \cdots & a_{i-1}^{i+1} & a_i^{i+1} & \mathbf{a}_{\mathbf{i+1}}^{\mathbf{i+1}} & \cdots & a_d^{i+1} \\
\vdots & & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
a_1^{d} & \cdots & a_{j-1}^{d} & a_{j+1}^{d} & \cdots & a_{i-1}^{d} & a_i^{d} & a_{i+1}^{d} & \cdots & \mathbf{a}_{\mathbf{d}}^{\mathbf{d}}
\end{array}\right)
\end{array}
$$

Denote by C a matrix formed by consecutive interchanging of rows j and $j+1$, $j+1$ and $j+2$, $\ldots$, $i-2$ and $i-1$ in matrix B.

$$
C = \begin{array}{cc}
 & \begin{array}{cccccccccc} 1 & \cdots & j-1 & j & \cdots & i-2 & i-1 & i & \cdots & d-1 \end{array} \\
\begin{array}{c} 1 \\ \vdots \\ j-1 \\ j \\ \vdots \\ i-2 \\ i-1 \\ i \\ \vdots \\ d-1 \end{array} &
\left(\begin{array}{cccccccccc}
\mathbf{a}_{\mathbf{1}}^{\mathbf{1}} & \cdots & a_{j-1}^1 & a_{j+1}^1 & \cdots & a_{i-1}^1 & a_i^1 & a_{i+1}^1 & \cdots & a_d^1 \\
\vdots & & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
a_1^{j-1} & \cdots & \mathbf{a}_{\mathbf{j-1}}^{\mathbf{j-1}} & a_{j+1}^{j-1} & \cdots & a_{i-1}^{j-1} & a_i^{j-1} & a_{i+1}^{j-1} & \cdots & a_d^{j-1} \\
a_1^{j+1} & \cdots & a_{j-1}^{j+1} & \mathbf{a}_{\mathbf{j+1}}^{\mathbf{j+1}} & \cdots & a_{i-1}^{j+1} & a_i^{j+1} & a_{i+1}^{j+1} & \cdots & a_d^{j+1} \\
\vdots & & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
a_1^{i-1} & \cdots & a_{j-1}^{i-1} & a_{j+1}^{i-1} & \cdots & \mathbf{a}_{\mathbf{i-1}}^{\mathbf{i-1}} & a_i^{i-1} & a_{i+1}^{i-1} & \cdots & a_d^{i-1} \\
a_1^{j} & \cdots & a_{j-1}^{j} & a_{j+1}^{j} & \cdots & a_{i-1}^{j} & a_i^{j} & a_{i+1}^{j} & \cdots & a_d^{j} \\
a_1^{i+1} & \cdots & a_{j-1}^{i+1} & a_{j+1}^{i+1} & \cdots & a_{i-1}^{i+1} & a_i^{i+1} & \mathbf{a}_{\mathbf{i+1}}^{\mathbf{i+1}} & \cdots & a_d^{i+1} \\
\vdots & & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
a_1^{d} & \cdots & a_{j-1}^{d} & a_{j+1}^{d} & \cdots & a_{i-1}^{d} & a_i^{d} & a_{i+1}^{d} & \cdots & \mathbf{a}_{\mathbf{d}}^{\mathbf{d}}
\end{array}\right)
\end{array}
$$

All elements c_l^{i-1} in $(i-1)$-st row of matrix C are less than 0, and apart from

$(i-1)$-st row this matrix satisfies the proposition for dimension $d-1$, i.e. by our induction hypothesis we have

$$(-1)^{i-1+l} M_l^{i-1}(C) > 0.$$

Then Laplace's formula gives us

$$\begin{aligned}(-1)^{i+j} M_j^i(A) &= (-1)^{i+j} \det B = (-1)^{i+j}(-1)^{i-j-1} \det C \\ &= -\sum_{l=1}^{d-1}(-1)^{i-1+l} M_l^{i-1}(C) c_l^{i-1} > 0.\end{aligned}$$

□

Proposition A.2. *Consider the following system of linear equations*

$$g_j = \sum_{\substack{i=1,\dots,d \\ i\neq j}} (x_j - x_i) b_i^j + x_j c_j, \qquad j = 1, \dots, d, \tag{76}$$

where $b_i^j, c_j > 0$. *There exists a unique solution* $(x_1, \dots, x_d)$ *of system* (76), *and for all* $j = 1, \dots, d$ *the following holds*

$$|x_j| \leq \frac{\max(|g_1|, \dots, |g_d|)}{\min(c_1, \dots, c_d)}.$$

Proof. Denote by $a_j^i = -b_j^i$ for $i \neq j$ and $a_i^i = c_i + \sum_{j\neq i} b_j^i$, matrix $A = (a_j^i)_{i,j=1}^d$, M_j^i is a (i,j)-th minor of A. Then our system becomes $Ax = g$.

If we add to the j-th column of A all other columns, then Laplace's formula gives us

$$\det A = \sum_{i=1}^d (-1)^{i+j} M_j^i c_i. \tag{77}$$

By Proposition A.1 we conclude that $\det A > 0$.

By Cramer's rule the (j,i)-th element of A^{-1} is equal to $(-1)^{i+j} M_j^i / \det A$. Then

$$x_j = \sum_{i=1}^d \left(A^{-1}\right)_i^j g_i = \sum_{i=1}^d (-1)^{i+j} M_j^i g_i / \det A.$$

Finally, applying Proposition A.1 and equation (77) we obtain our estimate

$$|x_j| = \frac{\left|\sum_{i=1}^{d}(-1)^{i+j}M_j^i g_i\right|}{\sum_{i=1}^{d}(-1)^{i+j}M_j^i c_i} \leq \frac{\sum_{i=1}^{d}(-1)^{i+j}M_j^i\,|g_i|}{\sum_{i=1}^{d}(-1)^{i+j}M_j^i c_i} \leq \frac{\max(|g_1|,\ldots,|g_d|)}{\min(c_1,\ldots,c_d)}.$$

$\square$

B Estimate for the CKL-quantile $\widetilde{Q}_\gamma$ of linear combinations of two functions

The objective of this appendix is to prove a proposition, which helps us to estimate CKL-quantiles of linear combinations of two functions. We use this proposition in the proof of Proposition 2.12.

Proposition B.1. *Let dimension $d \leq 3$ and $\gamma = (\gamma_1, \ldots, \gamma_d)$ be given, such that all $\gamma_j \in (0,1)$ and $\sum \gamma_j < 1$. Then for any strictly positive $u, v \in C\left(\mathbf{R}^d\right) \cap L_1\left(\mathbf{R}^d\right)$ with*

$$\max_j \left(\left| \widetilde{Q}^j_\gamma(u) \right|, \left| \widetilde{Q}^j_\gamma(v) \right| \right) \leq K$$

and any $h \in (0,1)$, we have

$$\max_j \left| \widetilde{Q}^j_\gamma((1-h)u + hv) \right| \leq (2d-1)K.$$

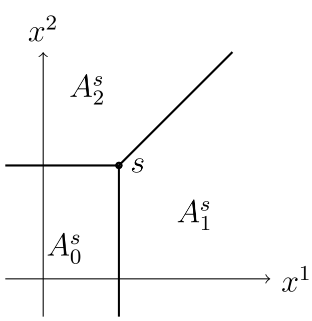
Proof. For each $s \in \mathbf{R}^d$, let

$$A^s_j = \left\{ x \in \mathbf{R}^d \colon x^j - s^j \geq 0 \vee \max_{i \neq j} (x^i - s^i) \right\}, \quad j = 1, \ldots, d,$$

and

$$A^s_0 = \left\{ x \in \mathbf{R}^d \colon x^j \leq s^j, j = 1, \ldots, d \right\}.$$

The following diagram helps to visualise sets A^s_j when dimension $d = 2$.



Note that A^s_j may intersect, for example, $s \in A^s_j$ for all $j = 0, \ldots, d$.

At first, let us make some observations on A^s_j. Let a, b be two points in $\mathbf{R}^d$.

1. If for some $j = 0, \ldots, d$, $b \in A_j^a$, then $A_j^b \subset A_j^a$.
2. If for some $j = 0, \ldots, d$, $b \in A_j^a$ and $a \in A_j^b$, then $a = b$.

For the proof, see Lemma B.2.

Note that if $\widetilde{Q}_\gamma(u) = \widetilde{Q}_\gamma(v)$, then for any $h \in (0, 1)$, CKL-quantile of a linear combination $\widetilde{Q}_\gamma((1-h)u + hv)$ coincides with $\widetilde{Q}_\gamma(u)$ and $\widetilde{Q}_\gamma(v)$.

Denote $a = \widetilde{Q}_\gamma(u)$, $b = \widetilde{Q}_\gamma(v)$ and $c = \widetilde{Q}_\gamma((1-h)u + hv)$. From the positiveness of u and v we conclude that for any $j = 0, \ldots, d$, c cannot be in A_j^a and A_j^b simultaneously, and for any $j = 0, \ldots, d$, a and b cannot be in A_j^c simultaneously, unless the case $a = b$, which has been considered before.

Let us consider the following five cases which we will be referring to in the sequel.

Case 0.1 $c \in A_0^a$, $b \in A_0^c$. In this case we have

$$c^j \le a^j, \qquad b^j \le c^j, \qquad j = 1, \ldots, d.$$

Then $\max_j |c^j| \le K$.

Case 0.2(i) $c \in A_i^b$, $a \in A_i^c$. In this case we have

$$\begin{aligned} &c^i - b^i \ge 0, \qquad && c^i - b^i \ge c^j - b^j, \qquad && j = 1, \ldots, d, \\ &a^i - c^i \ge 0, && a^i - c^i \ge a^j - c^j, && j = 1, \ldots, d. \end{aligned}$$

Then $|c^i| \le K$, and $a^j - a^i + c^i \le c^j \le b^j - b^i + c^i$, i.e. $|c^j| \le 3K$.

Case 0.3 $c \in A_1^a$, $b \in A_1^c$. In this case we have

$$\begin{aligned} &c^1 - a^1 \ge 0, \qquad && c^1 - a^1 \ge c^j - a^j, \qquad && j = 1, \ldots, d, \\ &b^1 - c^1 \ge 0, && b^1 - c^1 \ge b^j - c^j, && j = 1, \ldots, d. \end{aligned}$$

Then $|c^1| \le K$, and $b^j - b^1 + c^1 \le c^j \le a^j - a^1 + c^1$, i.e. $|c^j| \le 3K$.

Note that the three cases above hold for any dimension $d \ge 1$.

Case 1.1 $d = 3$, $c \in A_0^a$, $c \in A_1^b$, $a \in A_2^c$, $b \in A_3^c$. In this case we have for all $j = 1, \ldots, d$

$$c^j \le a^j, \qquad a^2 - c^2 \ge a^j - c^j, \qquad b^3 - c^3 \ge b^j - c^j.$$

Then $a^j - a^2 + c^2 \le c^j \le a^j$ and $b^j - b^3 + c^3 \le c^j \le a^j$. Sufficient condition for $|c^j| \le 5K$ is either $-3K \le c^2$, or $-3K \le c^3$. Our objective in the sequel is to prove one of these lower bounds. There are three possible cases:

1. $A_3^a A_1^b A_1^c \ne \emptyset$. Let x be any point in $A_3^a A_1^b A_1^c$. In this case we have

$$x^3 - a^3 \ge x^1 - a^1, \qquad x^1 - c^1 \ge x^3 - c^3, \qquad c^1 - b^1 \ge 0.$$

   Then $-3K \le a^3 - a^1 + b^1 \le a^3 - a^1 + c^1 \le x^3 - x^1 + c^1 \le c^3$.

2. $A_0^a A_2^b A_0^c \ne \emptyset$. Let x be any point in $A_0^a A_2^b A_0^c$. In this case we have

$$x^2 - b^2 \ge 0, \qquad x^2 \le c^2.$$

   Then $-K \le b^2 \le x^2 \le c^2$.

3. $A_3^a A_1^b A_1^c = \emptyset$ and $A_0^a A_2^b A_0^c = \emptyset$. Denote $A_i^a A_j^b A_k^c$ by B_{ijk}, and

$$x_{ijk} = \int_{B_{ijk}} u(x)\,dx, \qquad y_{ijk} = \int_{B_{ijk}} v(x)\,dx.$$

   By Lemma B.2 we have

$$\begin{aligned} A_0^c &\subset A_0^a, \qquad & A_1^c &\subset A_1^b, \\ A_2^a &\subset A_2^c, \qquad & A_3^b &\subset A_3^c. \end{aligned}$$

   This means that the following sets are subsets of hyperplanes in $\mathbf{R}^3$, therefore, they have Lebesgue measure zero

$$\begin{aligned} &A_i^a A_0^c, \quad i \ne 0, \qquad & &A_j^b A_1^c, \quad j \ne 1, \\ &A_2^a A_k^c, \quad k \ne 2, \qquad & &A_3^b A_k^c, \quad k \ne 3. \end{aligned}$$

   We can deduce that among 64 sets B_{ijk} at most 28 can be of positive Lebesgue measure (for example, all sets B_{1j0} have Lebesgue measure zero because they lie in the set $A_1^a A_0^c$ which has Lebesgue measure zero). These 28 sets have indices

$$\begin{bmatrix}
\mathbf{000} & \cancel{020} & \cancel{100} & \cancel{120} & \cancel{200} & \cancel{220} & \cancel{300} & \cancel{320} \\
\cancel{001} & \cancel{021} & \cancel{101} & \cancel{121} & \cancel{201} & \cancel{221} & \cancel{301} & \cancel{321} \\
\mathbf{002} & \mathbf{022} & \mathbf{102} & \mathbf{122} & \mathbf{202} & \mathbf{222} & \mathbf{302} & \mathbf{322} \\
\mathbf{003} & \mathbf{023} & \mathbf{103} & \mathbf{123} & \cancel{203} & \cancel{223} & \mathbf{303} & \mathbf{323} \\
\mathbf{010} & \cancel{030} & \cancel{110} & \cancel{130} & \cancel{210} & \cancel{230} & \cancel{310} & \cancel{330} \\
\mathbf{011} & \cancel{031} & \mathbf{111} & \cancel{131} & \cancel{211} & \cancel{231} & \cancel{311} & \cancel{331} \\
\mathbf{012} & \cancel{032} & \mathbf{112} & \cancel{132} & \mathbf{212} & \cancel{232} & \mathbf{312} & \cancel{332} \\
\mathbf{013} & \mathbf{033} & \mathbf{113} & \mathbf{133} & \cancel{213} & \cancel{233} & \mathbf{313} & \mathbf{333}
\end{bmatrix}$$

Let us group all x_{ijk} and y_{ijk} with the indices above into the following

$$\begin{aligned}
x_1 &= x_{000} + x_{010}, & y_1 &= y_{000}, \\
x_2 &= x_{011}, & y_2 &= y_{002} + y_{102} + y_{202} + y_{302}, \\
x_3 &= x_{002} + x_{012} + x_{022}, & y_3 &= y_{003} + y_{103} + y_{303}, \\
x_4 &= x_{003} + x_{013} + x_{023} + x_{033}, & y_4 &= y_{010}, \\
x_5 &= x_{111}, & y_5 &= y_{011} + y_{111}, \\
x_6 &= x_{102} + x_{112} + x_{122}, & y_6 &= y_{012} + y_{112} + y_{212} + y_{312}, \\
x_7 &= x_{103} + x_{113} + x_{123} + x_{133}, & y_7 &= y_{013} + y_{113} + y_{313}, \\
x_8 &= x_{202} + x_{212} + x_{222}, & y_8 &= y_{022} + y_{122} + y_{222} + y_{322}, \\
x_9 &= x_{302} + x_{312} + x_{322}, & y_9 &= y_{023} + y_{123} + y_{323}, \\
x_{10} &= x_{303} + x_{313} + x_{323} + x_{333}, & y_{10} &= y_{033} + y_{133} + y_{333}.
\end{aligned}$$

From the definition of points a, b, c these variables x_i, y_i satisfy

$$\begin{cases}
x_8 = \gamma_2, \\
x_9 + x_{10} = \gamma_3, \\
y_8 + y_9 = \gamma_2, \\
y_{10} = \gamma_3, \\
(1-h)(x_3 + x_6 + x_8 + x_9) + h(y_2 + y_6 + y_8) = \gamma_2, \\
(1-h)(x_4 + x_7 + x_{10}) + h(y_3 + y_7 + y_9 + y_{10}) = \gamma_3.
\end{cases} \tag{78}$$

Using system (78) we get

$$\begin{aligned}&(1-h)(x_3+x_4+x_6+x_7)+h(y_2+y_3+y_6+y_7)\\&\quad=\gamma_2+\gamma_3-(1-h)(x_8+x_9+x_{10})-h(y_8+y_9+y_{10})=0.\end{aligned}\tag{79}$$

Because of non-negativity of x_{ijk} and y_{ijk} equation (79) yields

$$x_3=x_4=x_6=x_7=y_2=y_3=y_6=y_7=0.$$

Next let us prove that $c^2\geq -K$ by contradiction. Suppose $c^2<-K$. Denote

$$C=\left[c^2-3K,c^2-2K\right]\times\left[c^2,-K\right]\times\left[c^2-3K,c^2-2K\right].$$

It is a subset of B_{002}. Indeed, let $x\in C$, then

$$\begin{aligned}&x^1-a^1\leq c^2-2K-a^1\leq -3K-a^1\leq 0,\\&x^2-a^2\leq -K-a^2\leq 0,\\&x^3-a^3\leq c^2-2K-a^3\leq -3K-a^3\leq 0,\end{aligned}$$

i.e. $x\in A_0^a$. Similarly we have $x\in A_0^b$. Finally,

$$\begin{aligned}&x^2-c^2\geq 0\geq a^2-a^1-2K\geq c^2-c^1-2K\geq x^1-c^1,\\&x^2-c^2\geq 0\geq a^2-a^3-2K\geq c^2-c^3-2K\geq x^3-c^3,\end{aligned}$$

i.e. $x\in A_2^c$.

This inclusion of a compact set C into B_{002} leads to

$$0<\int_C u(x)\,dx\leq\int_{B_{002}}u(x)\,dx=x_{002}\leq x_3=0,$$

contradiction.

Case 1.2 $d=3$, $c\in A_1^a$, $c\in A_2^b$, $a\in A_0^c$, $b\in A_3^c$. In this case we have for all $j=1,\dots,d$

$$c^1-a^1\geq c^j-a^j,\qquad c^2-b^2\geq c^j-b^j,\qquad a^j\leq c^j.$$

Then $a^j \le c^j \le a^j - a^1 + c^1$ and $a^j \le c^j \le b^j - b^2 + c^2$. Sufficient condition for $|c^j| \le 5K$ is either $c^1 \le 3K$, or $c^2 \le 3K$. Our objective in the sequel is to prove one of these upper bounds. There are three possible cases:

1. $A_3^a A_2^b A_2^c \neq \emptyset$. Let x be any point in $A_3^a A_2^b A_2^c$. In this case we have

$$x^3 - a^3 \ge x^2 - a^2, \qquad x^2 - c^2 \ge x^3 - c^3, \qquad b^3 - c^3 \ge 0.$$

   Then $c^2 \le x^2 - x^3 + c^3 \le a^2 - a^3 + c^3 \le a^2 - a^3 + b^3 \le 3K$.

2. $A_1^a A_0^b A_1^c \neq \emptyset$. Let x be any point in $A_1^a A_0^b A_1^c$. In this case we have

$$x^1 - c^1 \ge 0, \qquad x^1 \le b^1.$$

   Then $c^1 \le x^1 \le b^1 \le K$.

3. $A_3^a A_2^b A_2^c = \emptyset$ and $A_1^a A_0^b A_1^c = \emptyset$. Denote $A_i^a A_j^b A_k^c$ by B_{ijk}, and

$$x_{ijk} = \int_{B_{ijk}} u(x)\, dx, \qquad y_{ijk} = \int_{B_{ijk}} v(x)\, dx.$$

   By Lemma B.2 we have

$$\begin{aligned} A_1^c &\subset A_1^a, & A_2^c &\subset A_2^b, \\ A_0^a &\subset A_0^c, & A_3^b &\subset A_3^c. \end{aligned}$$

   This means that the following sets are subsets of hyperplanes in $\mathbf{R}^3$, therefore, they have Lebesgue measure zero

$$\begin{aligned} &A_i^a A_1^c, \quad i \neq 1, & &A_j^b A_2^c, \quad j \neq 2, \\ &A_0^a A_k^c, \quad k \neq 0, & &A_3^b A_k^c, \quad k \neq 3. \end{aligned}$$

   We can deduce that among 64 sets B_{ijk} at most 28 can be of positive Lebesgue measure (for example, all sets B_{0j1} have Lebesgue measure zero because they lie in the set $A_0^a A_1^c$ which has Lebesgue measure zero). These 28 sets have indices

$$\begin{bmatrix}
\mathbf{000} & \mathbf{020} & \mathbf{100} & \mathbf{120} & \mathbf{200} & \mathbf{220} & \mathbf{300} & \mathbf{320} \\
\cancel{001} & \cancel{021} & \cancel{101} & \mathbf{121} & \cancel{201} & \cancel{221} & \cancel{301} & \cancel{321} \\
\cancel{002} & \cancel{022} & \cancel{102} & \mathbf{122} & \cancel{202} & \mathbf{222} & \cancel{302} & \cancel{322} \\
\cancel{003} & \cancel{023} & \mathbf{103} & \mathbf{123} & \mathbf{203} & \mathbf{223} & \mathbf{303} & \mathbf{323} \\
\mathbf{010} & \cancel{030} & \mathbf{110} & \cancel{130} & \mathbf{210} & \cancel{230} & \mathbf{310} & \cancel{330} \\
\cancel{011} & \cancel{031} & \mathbf{111} & \cancel{131} & \cancel{211} & \cancel{231} & \cancel{311} & \cancel{331} \\
\cancel{012} & \cancel{032} & \cancel{112} & \cancel{132} & \cancel{212} & \cancel{232} & \cancel{312} & \cancel{332} \\
\cancel{013} & \cancel{033} & \mathbf{113} & \mathbf{133} & \mathbf{213} & \mathbf{233} & \mathbf{313} & \mathbf{333}
\end{bmatrix}$$

Let us group all x_{ijk} and y_{ijk} with the indices above into the following

$$\begin{aligned}
x_1 &= x_{000} + x_{010} + x_{020}, & y_1 &= y_{000} + y_{100} + y_{200} + y_{300}, \\
x_2 &= x_{100} + x_{110} + x_{120}, & y_2 &= y_{103} + y_{203} + y_{303}, \\
x_3 &= x_{111} + x_{121}, & y_3 &= y_{010} + y_{110} + y_{210} + y_{310}, \\
x_4 &= x_{122}, & y_4 &= y_{111}, \\
x_5 &= x_{103} + x_{113} + x_{123} + x_{133}, & y_5 &= y_{113} + y_{213} + y_{313}, \\
x_6 &= x_{200} + x_{210} + x_{220}, & y_6 &= y_{020} + y_{120} + y_{220} + y_{320}, \\
x_7 &= x_{222}, & y_7 &= y_{121}, \\
x_8 &= x_{203} + x_{213} + x_{223} + x_{233}, & y_8 &= y_{122} + y_{222}, \\
x_9 &= x_{300} + x_{310} + x_{320}, & y_9 &= y_{123} + y_{223} + y_{323}, \\
x_{10} &= x_{303} + x_{313} + x_{323} + x_{333}, & y_{10} &= y_{133} + y_{233} + y_{333}.
\end{aligned}$$

From the definition of points a, b, c these variables x_i, y_i satisfy

$$\begin{cases}
x_1 = \gamma_0, \\
x_9 + x_{10} = \gamma_3, \\
y_1 + y_2 = \gamma_0, \\
y_{10} = \gamma_3, \\
(1-h)(x_1 + x_2 + x_6 + x_9) + h(y_1 + y_3 + y_6) = \gamma_0, \\
(1-h)(x_5 + x_8 + x_{10}) + h(y_2 + y_5 + y_9 + y_{10}) = \gamma_3.
\end{cases} \tag{80}$$

Using system (80) we get

$$\begin{aligned}(1-h)&(x_2+x_5+x_6+x_8)+h(y_3+y_5+y_6+y_9)\\ &=\gamma_0+\gamma_3-(1-h)(x_1+x_9+x_{10})-h(y_1+y_2+y_{10})=0.\end{aligned}\tag{81}$$

Because of non-negativity of x_{ijk} and y_{ijk} equation (81) yields

$$x_2=x_5=x_6=x_8=y_3=y_5=y_6=y_9=0.$$

Next let us prove that $c^1\le K$ by contradiction. Suppose $c^1>K$. Denote

$$C=\left[K,c^1\right]\times[-2K,-K]\times[-2K,-K].$$

It is a subset of B_{110}. Indeed, let $x\in C$, then

$$\begin{aligned}x^1-a^1\ge K-a^1\ge 0\ge -K-a^2\ge x^2-a^2,\\ x^1-a^1\ge K-a^1\ge 0\ge -K-a^3\ge x^3-a^3,\end{aligned}$$

i.e. $x\in A_1^a$. Similarly we have $x\in A_1^b$. Finally,

$$\begin{aligned}&x^1\le c^1,\\ &x^2\le -K\le a^2\le c^2,\\ &x^3\le -K\le a^3\le c^3,\end{aligned}$$

i.e. $x\in A_0^c$.

This inclusion of a compact set C into B_{110} leads to

$$0<\int_C u(x)\,dx\le\int_{B_{110}}u(x)\,dx=x_{110}\le x_2=0,$$

contradiction.

Assume that $a\neq b$, $a\neq c$ and $b\neq c$. Let us prove the proposition for all dimensions $d\in\{1,2,3\}$.

1. $d=1$. In this case $A_1^s=[s,+\infty)$, $A_0^s=(-\infty,s]$. Then c must be between a and b.

2. $d = 2$. There are two possible cases:

   (a) Suppose $c \in A_0^a$ or $c \in A_0^b$. By symmetry we may think $c \in A_0^a$, then $c \notin A_0^b$ and $a \notin A_0^c$. By symmetry we may think $c \in A_1^b$, then $b \notin A_1^c$. There are two possible cases:

      i. $b \in A_0^c$. See Case 0.1.

      ii. $b \in A_2^c$, then $a \in A_1^c$. See Case 0.2(1).

   (b) Suppose $c \notin A_0^a$ and $c \notin A_0^b$. By symmetry we may think $c \in A_1^a$, $c \in A_2^b$, then $a \notin A_1^c$ and $b \notin A_2^c$. There are two possible cases:

      i. $a \in A_0^c$, then $b \in A_1^c$. See Case 0.3.

      ii. $a \in A_2^c$. See Case 0.2(2).

3. $d = 3$. There are two possible cases:

   (a) Suppose $c \in A_0^a$ or $c \in A_0^b$. By symmetry we may think $c \in A_0^a$, then $c \notin A_0^b$ and $a \notin A_0^c$. By symmetry we may think $c \in A_1^b$, then $b \notin A_1^c$. There are two possible cases:

      i. $b \in A_0^c$. See Case 0.1.

      ii. $b \notin A_0^c$. By symmetry we may think $b \in A_3^c$, then $a \notin A_3^c$. There are two possible cases:

         A. $a \in A_1^c$. See Case 0.2(1).

         B. $a \notin A_1^c$. By symmetry we may think $a \in A_2^c$. See Case 1.1.

   (b) Suppose $c \notin A_0^a$ and $c \notin A_0^b$. By symmetry we may think $c \in A_1^a$, $c \in A_2^b$, then $a \notin A_1^c$ and $b \notin A_2^c$. There are three possible cases:

      i. $a \in A_0^c$, then $b \notin A_0^c$. There are two possible cases:

         A. $b \in A_1^c$. See Case 0.3.

         B. $b \notin A_1^c$. By symmetry we may think $b \in A_3^c$. See Case 1.2.

      ii. $a \in A_2^c$. See Case 0.2(2).

      iii. $a \notin A_0^c$ and $a \notin A_2^c$. By symmetry we may think $a \in A_3^c$, then $b \notin A_3^c$. There are two possible cases:

         A. $b \in A_0^c$. This case is similar to case 3(b)iB.

         B. $b \in A_1^c$. See Case 0.3.

$\square$

Remark. *If Proposition B.1 holds for some dimension $d \geq 4$, then Theorem 1.2 also holds for that d.*

Remark. *To prove Proposition B.1 for $d \geq 4$, one needs to consider additional three cases*

1. $c \in A_1^a$, $c \in A_2^b$, $a \in A_3^c$, $b \in A_4^c$,
2. $c \in A_0^a$, $c \in A_1^b$, $a \in A_2^c$, $b \in A_3^c$, $A_3^a A_1^b A_1^c = \emptyset$ *and* $A_0^a A_2^b A_0^c = \emptyset$,
3. $c \in A_1^a$, $c \in A_2^b$, $a \in A_0^c$, $b \in A_3^c$, $A_3^a A_2^b A_2^c = \emptyset$ *and* $A_1^a A_0^b A_1^c = \emptyset$.

Lemma B.2. *Let a, b be two points in $\mathbf{R}^d$.*

1. *If for some $j = 0, \ldots, d$, $b \in A_j^a$, then $A_j^b \subset A_j^a$.*
2. *If for some $j = 0, \ldots, d$, $b \in A_j^a$ and $a \in A_j^b$, then $a = b$.*

Proof. Suppose $b \in A_0^a$, and let $x \in A_0^b$, then for all $j = 1, \ldots, d$

$$x^j \leq b^j \leq a^j,$$

i.e. $x \in A_0^a$ and $A_0^b \subset A_0^a$.

Suppose for some $j = 1, \ldots, d$, $b \in A_j^a$, and let $x \in A_j^b$, then

$$x^j - a^j = x^j - b^j + b^j - a^j \geq 0,$$

and for all $i \neq j$

$$x^j - a^j = x^j - b^j + b^j - a^j \geq x^i - b^i + b^i - a^i = x^i - a^i,$$

i.e. $x \in A_j^a$ and $A_j^b \subset A_j^a$.

Suppose $b \in A_0^a$ and $a \in A_0^b$, then for all $j = 1, \ldots, d$

$$b^j \leq a^j, \qquad a^j \leq b^j,$$

i.e. $a = b$.

Suppose for some $j = 1, \ldots, d$, $b \in A_j^a$ and $a \in A_j^b$, then

$$b^j - a^j \geq 0, \qquad a^j - b^j \geq 0,$$

i.e. $a^j = b^j$. Then for all $i \neq j$

$$0 = b^j - a^j \geq b^i - a^i, \qquad 0 = a^j - b^j \geq a^i - b^i,$$

i.e. $a^i = b^i$ and $a = b$. $\square$

C Proof of Proposition 2.15

First of all, we need a lemma.

Lemma C.1. *For any $c \in \mathbf{R}$*

$$\sum_{n=0}^{\infty} \frac{c^n}{\Gamma\left(\frac{n}{2}+1\right)}$$

converges, where

$$\Gamma(z) = \int_0^{+\infty} t^{z-1} \exp\{-t\}\, dt.$$

Proof. Note that

$$\lim_{k\to\infty} \left(\frac{c^{2k+2}}{\Gamma(k+2)} \Big/ \frac{c^{2k}}{\Gamma(k+1)} \right) = \lim_{k\to\infty} \frac{c^2}{k+1} = 0,$$
$$\lim_{k\to\infty} \left(\frac{c^{2k+1}}{\Gamma\left(k+\frac{3}{2}\right)} \Big/ \frac{c^{2k-1}}{\Gamma\left(k+\frac{1}{2}\right)} \right) = \lim_{k\to\infty} \frac{c^2}{k+\frac{1}{2}} = 0.$$

Then, our series converges because it can be represented as a sum of two converging (by d'Alembert ratio test) series

$$\sum_{n=0}^{\infty} \frac{c^n}{\Gamma\left(\frac{n}{2}+1\right)} = \sum_{k=0}^{\infty} \frac{c^{2k}}{\Gamma(k+1)} + \sum_{k=1}^{\infty} \frac{c^{2k-1}}{\Gamma\left(k+\frac{1}{2}\right)}.$$

□

Proof of Proposition 2.15. Recall that the solution to the Cauchy problem for equation (57) can be written in terms of its Green function as

$$u_t(x) = \int_{\mathbf{R}^d} G(t,x,0,\xi) u_0(\xi)\, d\xi.$$

From the upper bound in (58) we get

$$\|u_t\|_L \le C_2 \int_{\mathbf{R}^d} |u_0(\xi)| \int_{\mathbf{R}^d} G_{\sigma_2}(t, x-\xi)\, dx\, d\xi = C_2 \|u_0\|_L. \tag{82}$$

Next, let $C\left([0,T], \left(L_1\left(\mathbf{R}^d\right)\right)^d\right)$ denote the Banach space of all continuous mappings $[0,T] \to \left(L_1\left(\mathbf{R}^d\right)\right)^d$ with norm $\|v\| = \sup_{t\in[0,T]} \|v_t\|_L$, and let

$C_{v_0}\left([0,T],\left(L_1\left(\mathbf{R}^d\right)\right)^d\right)$ denote its convex subset, consisting of mappings with $v_0 = \nabla u_0$ given. For a given $v \in C_{v_0}\left([0,T],\left(L_1\left(\mathbf{R}^d\right)\right)^d\right)$ denote

$$\begin{aligned}
\Phi_t^k[v](x) &= \int_{\mathbf{R}^d} G(t,x,0,\xi) v_0^k(\xi)\, d\xi + \int_0^t \int_{\mathbf{R}^d} G(t,x,r,\xi) \frac{\partial}{\partial \xi^k} c(r,\xi) u_r(\xi) d\xi dr \\
&- \int_0^t \int_{\mathbf{R}^d} \frac{1}{2} \sum_{i,j=1}^d \frac{\partial}{\partial \xi^j} \left(G(t,x,r,\xi) \frac{\partial}{\partial \xi^k} a^{ij}(r,\xi) \right) v_r^i(\xi) d\xi dr \\
&+ \int_0^t \int_{\mathbf{R}^d} G(t,x,r,\xi) \sum_{i=1}^d \frac{\partial}{\partial \xi^k} b^i(r,\xi) v_r^i(\xi) d\xi dr.
\end{aligned}$$

Note that $\Phi[v]$ is also an element of $C_{v_0}\left([0,T],\left(L_1\left(\mathbf{R}^d\right)\right)^d\right)$ with

$$\|\Phi_t[v]\|_L = \sum_{k=1}^d \left\|\Phi_t^k[v]\right\|_L \le C_4 + C_5 \int_0^t \|v_r\|_L \left((t-r)^{-1/2} + C_6\right) dr, \tag{83}$$

where

$$\begin{aligned}
C_4 &= C_2 \|v_0\|_L + d(C_2)^2 MT \|u_0\|_L, \\
C_5 &= \frac{1}{2} d^2 C_2 C_3 M, \qquad C_6 = \frac{d+2}{dC_3}.
\end{aligned}$$

We may think that the constants C_2 and C_3 in Proposition 2.8 are bigger than 1 and $\frac{d+2}{d}$ correspondingly, so that $C_4 > \|v_0\|_L$ and $C_6 < 1$.

For arbitrary $v^1, v^2 \in C_{v_0}\left([0,T],\left(L_1\left(\mathbf{R}^d\right)\right)^d\right)$ coinciding on $[0,t_0]$ we have

$$\begin{aligned}
\left\|\Phi_t\left[v^1\right] - \Phi_t\left[v^2\right]\right\|_L &\le C_5 \int_{t_0}^t \left\|v_r^1 - v_r^2\right\|_L \left((t-r)^{-1/2} + C_6\right) dr \\
&\le C_5 \left(2(t-t_0)^{1/2} + C_6(t-t_0)\right) \sup_{r \in [t_0,t]} \left\|v_r^1 - v_r^2\right\|_L.
\end{aligned}$$

When $t_0 = 0$, we see that Φ is a contraction for small enough T. Note that the constants C_5, C_6 do not depend on the initial condition v_0, therefore, the global contraction (contraction for any given $T > 0$) follows from the usual iteration procedure.

Denote the following integrals $I_n(t)$

$$I_0(t) \equiv 1,$$
$$I_{n+1}(t) = C_5 \int_0^t I_n(r) \left((t-r)^{-1/2} + C_6\right) dr.$$

There is a well known fact that the family of operators

$$T_\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t f(s)(t-s)^{\alpha-1}\, ds, \qquad \alpha > 0$$

is a semigroup in $C(\mathbf{R})$. Then we can find an upper bound for $I_n(t)$ on $[0,T]$.

$$\begin{aligned} I_n(t) &= C_5 \left(\sqrt{\pi} T_{\frac{1}{2}} + C_6 T_1\right) I_{n-1}(t) = \ldots = (C_5)^n \left(\sqrt{\pi} T_{\frac{1}{2}} + C_6 T_1\right)^n I_0(t) \\ &= (C_5)^n \sum_{k=0}^n \binom{n}{k} \left(\sqrt{\pi}\right)^k (C_6)^{n-k} \frac{t^{n-\frac{k}{2}}}{\Gamma\left(n - \frac{k}{2} + 1\right)} \le \frac{(2C_5 T \sqrt{\pi})^n}{\Gamma\left(\frac{n}{2}+1\right)}. \end{aligned}$$

Denote by Φ^n the n-th iteration of Φ applied initially on v_0. By a straightforward induction and (83) we deduce that

$$\|\Phi_t^n\|_L \le C_4 \sum_{k=0}^n I_k(t).$$

Then for a fixed point $v = \lim_{n\to\infty} \Phi^n$ we have

$$\|v_t\|_L = \lim_{n\to\infty} \|\Phi_t^n\|_L \le C_4 \sum_{n=0}^\infty I_n(t) \le C_4 \sum_{n=0}^\infty \frac{(2C_5 T\sqrt{\pi})^n}{\Gamma\left(\frac{n}{2}+1\right)}. \tag{84}$$

Finally, we are going to prove that ∇u is a fixed point of Φ. By differentiation of equation (57), we conclude that $v_t = \nabla u_t$ satisfies

$$\begin{aligned} \frac{\partial}{\partial t} v_t^k(x) &= L_t v_t^k(x) + \frac{1}{2} \sum_{i,j=1}^d \frac{\partial}{\partial x^k} a^{ij}(t,x) \frac{\partial}{\partial x^j} v_t^i(x) \\ &\quad + \sum_{i=1}^d \frac{\partial}{\partial x^k} b^i(t,x) v_t^i(x) + \frac{\partial}{\partial x^k} c(t,x) u_t(x), \qquad k = 1, \ldots, d, \end{aligned}$$

or its mild form

$$
\begin{aligned}
v_t^k(x) = \int_{\mathbf{R}^d} G(t,x,0,\xi) v_0^k(\xi)\, d\xi + \int_0^t \int_{\mathbf{R}^d} G(t,x,r,\xi) \Bigg(\frac{\partial}{\partial \xi^k} c(r,\xi) u_r(\xi) \\
+ \frac{1}{2} \sum_{i,j=1}^d \frac{\partial}{\partial \xi^k} a^{ij}(r,\xi) \frac{\partial}{\partial \xi^j} v_r^i(\xi) + \sum_{i=1}^d \frac{\partial}{\partial \xi^k} b^i(r,\xi) v_r^i(\xi) \Bigg) \, d\xi dr.
\end{aligned} \tag{85}
$$

Note that

$$
\left| G(t,x,r,\xi) \frac{\partial}{\partial \xi^k} a^{ij}(r,\xi) v_r^i(\xi) \right| \le C_2 G_{\sigma_2}(t-r, x-\xi) M C_3 r^{-1/2} u_r(\xi) \xrightarrow{\xi^j \to \infty} 0,
$$

then using integration by parts in (85), we conclude that $v = \nabla u$ is a fixed point of Φ. Therefore, estimate (84) gives us

$$
\begin{aligned}
\sum_{k=1}^d \left\| \frac{\partial u_t}{\partial x^k} \right\|_L &\le \left(C_2 \sum_{k=1}^d \left\| \frac{\partial u_0}{\partial x^k} \right\|_L + d(C_2)^2 MT \|u_0\|_L \right) \sum_{n=0}^\infty \frac{\left(d^2 C_2 C_3 M T \sqrt{\pi}\right)^n}{\Gamma\left(\frac{n}{2}+1\right)} \\
&\le C_7 \left(\sum_{k=1}^d \left\| \frac{\partial u_0}{\partial x^k} \right\|_L + \|u_0\|_L \right).
\end{aligned}
$$

Upper bound for L_1-norms of second order derivatives of $u_t(x)$ can be found in the same way.

$$
\begin{aligned}
\sum_{k,l=1}^d \left\| \frac{\partial^2 u_t}{\partial x^k \partial x^l} \right\|_L \le \Bigg(C_2 \sum_{k,l=1}^d \left\| \frac{\partial^2 u_0}{\partial x^k \partial x^l} \right\|_L + d^2 C_2 M T \Bigg(3C_7 \Bigg(\sum_{k=1}^d \left\| \frac{\partial u_0}{\partial x^k} \right\|_L \\
+ \|u_0\|_L \Bigg) + C_2 \|u_0\|_L \Bigg) \Bigg) \sum_{n=0}^\infty \frac{\left(2d^2 C_2 C_3 M T \sqrt{\pi}\right)^n}{\Gamma\left(\frac{n}{2}+1\right)} \\
\le C_8 \left(\sum_{k,l=1}^d \left\| \frac{\partial^2 u_0}{\partial x^k \partial x^l} \right\|_L + \sum_{k=1}^d \left\| \frac{\partial u_0}{\partial x^k} \right\|_L + \|u_0\|_L \right).
\end{aligned}
$$

$\square$

D Rewrite of Itô's and Stratonovich SDEs

In this appendix we will rewrite a second order Itô's SPDE using the Stratonovich integral and a system of Stratonovich SDEs using the Itô's integral.

First of all, let us rewrite the following equation

$$du_t(x) = L_t u_t(x)\,dt + \sum_{k=1}^{d'} \left(\sum_{i=1}^{d} \beta^{ik}(t,x) \frac{\partial u_t}{\partial x^i}(x) + \alpha^k(t,x) u_t(x) \right) dW_t^k, \quad (86)$$

where L_t is an operator of the form

$$L_t u = \frac{1}{2}(a(t,x)\nabla, \nabla)u + (b(t,x), \nabla)u + c(t,x)u.$$

Denote $F_t^k(x) = \int_0^t \alpha^k(s,x)\,dW_s^k$, then by Theorem 3.2.5 in [11] the Itô's differentials in equation (86) satisfy

$$\begin{aligned}
&\alpha^k(t,x)u_t(x) \circ dW_t^k - \alpha^k(t,x)u_t(x)\,dW_t^k \\
&\quad = u_t(x) \circ dF_t^k(x) - u_t(x)\,dF_t^k(x) \\
&\quad = \frac{1}{2} d\left\langle F_t^k(x), u_t(x) \right\rangle \\
&\quad = \frac{1}{2} d\Bigg\langle \int_0^t \alpha^k(s,x)\,dW_s^k, u_0(x) + \int_0^t L_s u_s(x)\,ds \\
&\qquad + \sum_{l=1}^{d'} \int_0^t \left(\sum_{j=1}^{d} \beta^{jl}(s,x) \frac{\partial u_s}{\partial x^j}(x) + \alpha^l(s,x)u_s(x) \right) dW_s^l \Bigg\rangle \\
&\quad = \frac{1}{2}\alpha^k(t,x) \left(\sum_{j=1}^{d} \beta^{jk}(t,x) \frac{\partial u_t}{\partial x^j}(x) + \alpha^k(t,x)u_t(x) \right) dt.
\end{aligned}$$

Similarly we have

$$\begin{aligned}
&\beta^{ik}(t,x)\frac{\partial u_t}{\partial x^i}(x) \circ dW_t^k - \beta^{ik}(t,x)\frac{\partial u_t}{\partial x^i}(x)\,dW_t^k \\
&\quad = \frac{1}{2}\beta^{ik}(t,x)\frac{\partial}{\partial x^i} \left(\sum_{j=1}^{d} \beta^{jk}(t,x) \frac{\partial u_t}{\partial x^j}(x) + \alpha^k(t,x)u_t(x) \right) dt.
\end{aligned}$$

Substituting these relations into equation (86), we can rewrite it as

$$du_t(x) = \Big(L_t - \widetilde{L}_t\Big) u_t(x)\, dt + \sum_{k=1}^{d'} \left(\sum_{i=1}^{d} \beta^{ik}(t,x) \frac{\partial u_t}{\partial x^i}(x) + \alpha^k(t,x) u_t(x) \right) \circ dW_t^k,$$

where

$$\begin{aligned}
\widetilde{L}_t u = {} & \frac{1}{2} \sum_{i,j=1}^{d} \left(\sum_{k=1}^{d'} \beta^{ik}(t,x) \beta^{jk}(t,x) \right) \frac{\partial^2 u}{\partial x^i \partial x^j} \\
& + \sum_{i=1}^{d} \left(\frac{1}{2} \sum_{j=1}^{d} \sum_{k=1}^{d'} \beta^{jk}(t,x) \frac{\partial \beta^{ik}}{\partial x^j}(t,x) + \sum_{k=1}^{d'} \beta^{ik}(t,x) \alpha^k(t,x) \right) \frac{\partial u}{\partial x^i} \\
& + \frac{1}{2} \left(\sum_{i=1}^{d} \sum_{k=1}^{d'} \beta^{ik}(t,x) \frac{\partial \alpha^k}{\partial x^i}(t,x) + \sum_{k=1}^{d'} \big(\alpha^k(t,x)\big)^2 \right) u.
\end{aligned}$$

Remark. *Let us consider the following system*

$$\begin{aligned}
d\varphi_{s,t} &= -\beta(\varphi_{s,t}, t) \circ dW_t, \\
d\eta_{s,t} &= \eta_{s,t} \alpha(\varphi_{s,t}, t) \circ dW_t,
\end{aligned}$$

where $\beta(x,t) = \big(\beta^1(x,t), \ldots, \beta^d(x,t)\big)^T$. *Assume that the coefficients* $\alpha(x,t)$ *and* $\beta^i(x,t)$ *satisfy Condition* $(E.1)_k$ *for some* $k \geq 3$. *Then by Theorem 3.2.5 in [11] this system can be rewritten in*

$$d\varphi_{s,t} = -\beta(\varphi_{s,t}, t)\, dW_t + \frac{1}{2} \sum_{j=1}^{d} \frac{\partial \beta}{\partial x^j}(\varphi_{s,t}, t) \beta^j(\varphi_{s,t}, t)\, dt,$$

$$d\eta_{s,t} = \eta_{s,t} \alpha(\varphi_{s,t}, t)\, dW_t + \frac{1}{2} \eta_{s,t} \left(- \sum_{j=1}^{d} \frac{\partial \alpha}{\partial x^j}(\varphi_{s,t}, t) \beta^j(\varphi_{s,t}, t) + \alpha^2(\varphi_{s,t}, t) \right) dt.$$

E Continuity of quantiles

Let $u_t \in L_1(\mathbf{R}^d) \cap C(\mathbf{R}^d)$ be a positive continuous function of (t, x). Suppose u_t is continuous in $L_1(\mathbf{R}^d)$. In this appendix we will prove that both quantile $Q_\gamma(u_t)$ and CKL-quantile $\widetilde{Q}_\gamma(u_t)$ are continuous in t.

Note that for both quantile $Q_\gamma(u)$ and CKL-quantile $\widetilde{Q}_\gamma(u)$, for any s, t and $l \in \{1, \dots, d\}$ the following inequality holds

$$\begin{aligned}
&\frac{1}{2}\left|\int_{A_l^{Q_\gamma(u_s)}} u_s(x)\,dx - \int_{A_l^{Q_\gamma(u_t)}} u_s(x)\,dx\right| \\
&\qquad = \frac{1}{2}\left|\int_{A_l^{Q_\gamma(u_t)}} (u_t(x) - u_s(x))\,dx + \gamma_l\|u_s\|_L - \gamma_l\|u_t\|_L\right| \\
&\qquad \le \frac{1}{2}\left|\int_{A_l^{Q_\gamma(u_t)}} (u_t(x) - u_s(x))\,dx\right| + \frac{1}{2}\gamma_l\big|\|u_s\|_L - \|u_t\|_L\big| \le \|u_t - u_s\|_L.
\end{aligned}$$

Let us first prove by contradiction that the quantile $Q_\gamma(u_t)$ is continuous in t. Suppose it is not continuous at some time t_0, i.e. for some $k \in \{1, \dots, d\}$ there exists $\varepsilon_0 > 0$ such that for any $\delta > 0$

$$\left|Q_\gamma^k(u_t) - Q_\gamma^k(u_{t_0})\right| \ge \varepsilon_0$$

for some $t \in (t_0 - \delta, t_0 + \delta)$. Denote $a = Q_\gamma(u_t)$, $b = Q_\gamma(u_{t_0})$. Then we have

$$\begin{aligned}
\|u_t - u_{t_0}\|_L &\ge \frac{1}{2}\left|\int_{A_k^b} u_{t_0}(x)\,dx - \int_{A_k^a} u_{t_0}(x)\,dx\right| \\
&\ge \frac{1}{2}\min\left(\int_{\left\{x\in\mathbf{R}^d:\ x^k\in\left(b^k-\varepsilon_0,b^k\right)\right\}} u_{t_0}(x)\,dx,\right. \\
&\qquad\qquad \left.\int_{\left\{x\in\mathbf{R}^d:\ x^k\in\left(b^k,b^k+\varepsilon_0\right)\right\}} u_{t_0}(x)\,dx\right) = \varepsilon_1 > 0,
\end{aligned}$$

which is a contradiction with continuity of u_t in $L_1(\mathbf{R}^d)$.

Next, let us prove by contradiction that the CKL-quantile $\widetilde{Q}_\gamma(u_t)$ is also continuous in t. Suppose it is not continuous at some time t_0, i.e. for some

$k \in \{1, \ldots, d\}$ there exists $\varepsilon_0 > 0$ such that for any $\delta > 0$

$$\left| \widetilde{Q}^k_\gamma(u_t) - \widetilde{Q}^k_\gamma(u_{t_0}) \right| \geq \varepsilon_0$$

for some $t \in (t_0 - \delta, t_0 + \delta)$. Denote $a = \widetilde{Q}_\gamma(u_t)$, $b = \widetilde{Q}_\gamma(u_{t_0})$ and

$$\begin{aligned} X_{1l} &= \left\{ x \in \mathbf{R}^d \colon x^l \in \left(b^l, b^l + \varepsilon_0\right), x^i \leq b^i, i \neq l \right\}, \\ X_{2l} &= \left\{ x \in \mathbf{R}^d \colon x^l \in \left(b^l - \varepsilon_0, b^l\right), x^i \leq b^i - \varepsilon_0, i \neq l \right\}. \end{aligned}$$

Our goal is to show that

$$\|u_t - u_{t_0}\|_L \geq \frac{1}{2} \min_l \left(\int_{X_{1l}} u_{t_0}(x)\, dx, \int_{X_{2l}} u_{t_0}(x)\, dx \right) = \varepsilon_1 > 0,$$

which is a contradiction with continuity of u_t in $L_1\left(\mathbf{R}^d\right)$. Consider two possible cases:

1. $a^k \geq b^k + \varepsilon_0$. Let $a \in A^b_l$ for some $l \in \{1, \ldots, d\}$, then by Lemma B.2, $A^a_l \subset A^b_l$. Note that $X_{1l} \subset A^b_l$ and $A^a_l \cap X_{1l} = \emptyset$. Indeed, if $x \in X_{1l}$, then

$$x^l - b^l > 0 \geq x^i - b^i, \qquad i \neq l$$

and

$$x^l - a^l < b^l + \varepsilon_0 - a^l \leq b^k + \varepsilon_0 - a^k \leq 0.$$

Therefore

$$\|u_t - u_{t_0}\|_L \geq \frac{1}{2} \left| \int_{A^b_l} u_{t_0}(x)\, dx - \int_{A^a_l} u_{t_0}(x)\, dx \right| \geq \frac{1}{2} \int_{X_{1l}} u_{t_0}(x)\, dx.$$

2. $a^k \leq b^k - \varepsilon_0$. Let $b \in A^a_l$ for some $l \in \{1, \ldots, d\}$, then by Lemma B.2, $A^b_l \subset A^a_l$. Note that $X_{2l} \subset A^a_l$ and $A^b_l \cap X_{2l} = \emptyset$. Indeed, if $x \in X_{2l}$, then

$$\begin{aligned} x^l - a^l &> b^l - \varepsilon_0 - a^l \geq b^k - \varepsilon_0 - a^k \geq 0, \\ x^l - a^l &> b^l - \varepsilon_0 - a^l \geq b^i - \varepsilon_0 - a^i \geq x^i - a^i, \qquad i \neq l. \end{aligned}$$

Therefore

$$\|u_t - u_{t_0}\|_L \geq \frac{1}{2}\left|\int_{A_l^b} u_{t_0}(x)\,dx - \int_{A_l^a} u_{t_0}(x)\,dx\right| \geq \frac{1}{2}\int_{X_{2l}} u_{t_0}(x)\,dx.$$

F Auxiliary results from [11]

In this appendix, we will state all the results used from [11], and make sure that all the conditions under which they work are satisfied.

Theorem F.1 (Theorem 3.4.1 in [11]). *Let $F(x,t)$ be a continuous semimartingale with values in $C\left(\mathbf{R}^d\colon \mathbf{R}^d\right)$ with local characteristic belonging to the class $B_b^{0,1}$. Then for each t_0 and x_0, the stochastic differential equation*

$$\varphi_t = x_0 + \int_{t_0}^{t} F(\varphi_s, ds)$$

has a unique solution.

In terms and assumptions of the third paragraph of Section 1.2, let us check that we can apply Theorem F.1 to equation (4), where

$$F(x,t) = \int_0^t \beta(x,s)\, dW_s.$$

Here $F(x,t)$ is a continuous martingale with values in $C\left(\mathbf{R}^d\colon \mathbf{R}^d\right)$. According to the definitions in [11], the matrix

$$\left\{\beta^i(x,t)\beta^j(y,t)\right\}_{i,j=1}^{d}$$

is called the local characteristic of $F(x,t)$. It is said to belong to the class $B_b^{0,1}$ if for every $i = 1, \ldots, d$

$$\int_0^T \left(\left(\sup_{x\in\mathbf{R}^d} \frac{|\beta^i(x,t)|}{1+|x|} \right)^2 + \left(\sup_{\substack{x,x'\in\mathbf{R}^d \\ x\neq x'}} \frac{|\beta^i(x,t)-\beta^i(x',t)|}{|x-x'|} \right)^2 \right) dt < \infty,$$

which follows from the boundedness of $\beta^i(x,t)$ and all its derivatives. Therefore, for each t_0 and x_0, Itô's equation (4) has a unique solution.

Theorem F.2 (Theorem 3.2.5 in [11]). *Assume that $F(x,t)$ is a continuous C^1-semimartingale with local characteristic belonging to the class $\left(B^{2,\delta}, B^{1,0}\right)$ for some $\delta > 0$ and f_t is a continuous semimartingale. Then the Stratonovich integral of f_t based on the kernel $F(x,t)$ is well defined. It is related to the*

Itô's integral by the formula

$$\int_0^t F(f_s, \circ ds) = \int_0^t F(f_s, ds) + \frac{1}{2}\sum_{j=1}^{d}\left\langle \int_0^t \frac{\partial F}{\partial x^j}(f_s, ds), f_t^j \right\rangle.$$

In terms and assumptions of Theorem F.3 below, let us check that the continuous C^1-martingale

$$F(x,t) = \int_0^t \beta^i(x,s)\, dW_s \tag{87}$$

satisfies the conditions of Theorem F.2 for $\delta = 1$. According to the definitions in [11], $\beta^i(x,t)\beta^i(y,t)$ is called the local characteristic of $F(x,t)$. It is said to belong to the class $B^{2,1}$ if for any compact subset K of $\mathbf{R}^d$

$$\int_0^T \Bigg[\left(\sup_{x\in K} \frac{|\beta^i(x,t)|}{1+|x|} \right)^2 + \sum_{1\le|\alpha|\le 2} \left(\sup_{x\in K} \left| D_x^\alpha \beta^i(x,t) \right| \right)^2 + \sum_{|\alpha|=2} \left(\sup_{\substack{x,x'\in K\\ x\neq x'}} \frac{|D_x^\alpha \beta^i(x,t) - D_x^\alpha \beta^i(x',t)|}{|x-x'|} \right)^2 \Bigg] dt < \infty,$$

which follows from the boundedness of $\beta^i(x,t)$ and all its derivatives.

Theorem F.3 (Custom version of Theorem 3.4.7 in [11])**.** *Let $\beta(x,t) = \big(\beta^1(x,t),\ldots,\beta^d(x,t)\big)$ be a continuous bounded function on $\mathbf{R}^d\times[0,T]$, 3-times differentiable in x and continuously differentiable in t with bounded derivatives. Let W_t be a standard 1-dimensional Brownian motion. Then for each t_0 and x_0, the Stratonovich equation*

$$\varphi_t = x_0 + \int_{t_0}^t \beta(\varphi_s, s) \circ dW_s \tag{88}$$

has a unique solution. Further the solution satisfies Itô's equation

$$\varphi_t = x_0 + \int_{t_0}^t \beta(\varphi_s,s)\, dW_s + \frac{1}{2}\int_{t_0}^t \left\{ \sum_{j=1}^d \beta^j(\varphi_s,s) \frac{\partial \beta}{\partial x^j}(\varphi_s,s) \right\} ds. \tag{89}$$

Remark. *Theorem 3.4.7 in [11] considers maximal solutions, while in Theo-*

rem F.3 we deal with global solutions only.

Proof. Consider Itô's stochastic differential equation (89). Let us check that it satisfies the conditions of Theorem F.1, where

$$F(x,t)=\int_0^t \beta(x,s)\,dW_s+\frac{1}{2}\int_0^t \left\{\sum_{j=1}^d \beta^j(x,s)\frac{\partial \beta}{\partial x^j}(x,s)\right\}ds$$

is a continuous $C\left(\mathbf{R}^d\colon \mathbf{R}^d\right)$-valued semimartingale. Denote

$$b^i(x,t)=\frac{1}{2}\sum_{j=1}^d \beta^j(x,t)\frac{\partial \beta^i}{\partial x^j}(x,t).$$

According to the definitions in [11], the pair

$$\left(\left\{\beta^i(x,t)\beta^j(y,t)\right\}_{i,j=1}^d,\left\{b^i(x,t)\right\}_{i=1}^d\right)$$

is called the local characteristic of $F(x,t)$. It is said to belong to the class $B_b^{0,1}$ if for every $i=1,\dots,d$

$$\int_0^T\left(\left(\sup_{x\in\mathbf{R}^d}\frac{|\beta^i(x,t)|}{1+|x|}\right)^2+\left(\sup_{\substack{x,x'\in\mathbf{R}^d\\ x\neq x'}}\frac{|\beta^i(x,t)-\beta^i(x',t)|}{|x-x'|}\right)^2\right)dt<\infty,$$

$$\int_0^T\left(\sup_{x\in\mathbf{R}^d}\frac{|b^i(x,t)|}{1+|x|}+\sup_{\substack{x,x'\in\mathbf{R}^d\\ x\neq x'}}\frac{|b^i(x,t)-b^i(x',t)|}{|x-x'|}\right)dt<\infty,$$

which follow from the boundedness of $\beta^i(x,t)$ and all their derivatives. Therefore, for each t_0 and x_0, Itô's equation (89) has a unique solution φ_t, $t\in[t_0,T]$. Since φ_t is a continuous semimartingale, the Stratonovich integral in (88) is well defined. Applying Theorem F.2 to (87), we have

$$\begin{aligned}\int_{t_0}^t \beta(\varphi_s,s)\circ dW_s&=\int_{t_0}^t \beta(\varphi_s,s)\,dW_s+\frac{1}{2}\sum_{j=1}^d\left\langle\int_{t_0}^t\frac{\partial\beta}{\partial x^j}(\varphi_s,s)\,dW_s,\varphi_t^j\right\rangle\\&=\int_{t_0}^t \beta(\varphi_s,s)\,dW_s\end{aligned}$$

$$+\frac{1}{2}\sum_{j=1}^{d}\left\langle \int_{t_0}^{t}\frac{\partial\beta}{\partial x^j}(\varphi_s,s)\,dW_s, \int_{t_0}^{t}\beta^j(\varphi_s,s)\,dW_s\right\rangle$$
$$=\int_{t_0}^{t}\beta(\varphi_s,s)\,dW_s+\frac{1}{2}\sum_{j=1}^{d}\int_{t_0}^{t}\frac{\partial\beta}{\partial x^j}(\varphi_s,s)\beta^j(\varphi_s,s)\,ds$$
$$=\varphi_t-x_0.$$

Therefore, Stratonovich equation (88) has a solution. The uniqueness of the solution of equation (88) is also reduced to the uniqueness of the solution of equation (89). □

Theorem F.4 (Theorem 4.6.5 in [11]). *Assume that the local characteristic of the continuous C-semimartingale $F(x,t)$ belongs to the class $B_b^{k,\delta}$ for some $k\geq 1$ and $\delta>0$. Then the solution of Itô's stochastic differential equation based on F has a modification $\varphi_{s,t}$, $0\leq s\leq t\leq T$ such that it is a forward stochastic flow of C^k-diffeomorphisms. Further it is a $C^{k,\varepsilon}$-semimartingale for any $\varepsilon<\delta$.*

Let function $\beta(x,t)=\big(\beta^1(x,t),\ldots,\beta^d(x,t)\big)^T$ satisfy Condition $(E.1)_k$ for some $k\geq 3$. Consider the following Itô's equation

$$\varphi_{s,t}(x)=x-\int_s^t\beta(\varphi_{s,r}(x),r)\,dW_r+\frac{1}{2}\int_s^t\left\{\sum_{j=1}^{d}\beta^j(\varphi_{s,r}(x),r)\frac{\partial\beta}{\partial x^j}(\varphi_{s,r}(x),r)\right\}dr. \tag{90}$$

Let us check that it satisfies the conditions of Theorem F.4 for $\delta=1$, where

$$F(x,t)=-\int_0^t\beta(x,r)\,dW_r+\frac{1}{2}\int_0^t\left\{\sum_{j=1}^{d}\beta^j(x,r)\frac{\partial\beta}{\partial x^j}(x,r)\right\}dr.$$

Denote

$$b^i(x,t)=\frac{1}{2}\sum_{j=1}^{d}\beta^j(x,t)\frac{\partial\beta^i}{\partial x^j}(x,t).$$

According to the definitions in [11], the pair

$$\left(\left\{\beta^i(x,t)\beta^j(y,t)\right\}_{i,j=1}^{d},\left\{b^i(x,t)\right\}_{i=1}^{d}\right)$$

is called the local characteristic of $F(x,t)$. It is said to belong to the class $B_b^{k,1}$

if for every $i = 1, \ldots, d$

$$\int_0^T \left[\left(\sup_{x \in \mathbf{R}^d} \frac{|\beta^i(x,t)|}{1+|x|} \right)^2 + \sum_{1 \le |\alpha| \le k} \left(\sup_{x \in \mathbf{R}^d} \left| D_x^\alpha \beta^i(x,t) \right| \right)^2 \right.$$
$$\left. + \sum_{|\alpha|=k} \left(\sup_{\substack{x,x' \in \mathbf{R}^d \\ x \ne x'}} \frac{|D_x^\alpha \beta^i(x,t) - D_x^\alpha \beta^i(x',t)|}{|x-x'|} \right)^2 \right] dt < \infty,$$

$$\int_0^T \left[\sup_{x \in \mathbf{R}^d} \frac{|b^i(x,t)|}{1+|x|} + \sum_{1 \le |\alpha| \le k} \sup_{x \in \mathbf{R}^d} \left| D_x^\alpha b^i(x,t) \right| \right.$$
$$\left. + \sum_{|\alpha|=k} \sup_{\substack{x,x' \in \mathbf{R}^d \\ x \ne x'}} \frac{|D_x^\alpha b^i(x,t) - D_x^\alpha b^i(x',t)|}{|x-x'|} \right] dt < \infty,$$

which follow from the boundedness of $\beta^i(x,t)$ and all their derivatives. Therefore, the solution of Itô's equation (90) has a modification $\varphi_{s,t}$, $0 \le s \le t \le T$ such that it is a forward stochastic flow of C^k-diffeomorphisms, and it is also a C^k-semimartingale.

Condition $(D.1)_{k,\delta}$**.** $\left(F^1, \ldots, F^{d+1}\right)$ *is a continuous* C^k*-semimartingale with local characteristic belonging to the class* $\left(B_b^{k+1,\delta}, B_b^{k,\delta}\right)$.

Theorem F.5 (Theorem 6.1.9 in [11])**.** *Assume that* $\left(F^1, \ldots, F^{d+1}\right)$ *of the linear equation*

$$u(x,t) = f(x) + \sum_{i=1}^d \int_0^t F^i(x, \circ ds) \frac{\partial u}{\partial x^i}(x,s) + \int_0^t F^{d+1}(x, \circ ds) u(x,s)$$

satisfies Condition $(D.1)_{k,\delta}$ *for some* $k \ge 3$ *and* $\delta > 0$*. If the initial function is of* $C^{k,\delta}$*, the linear equation has a unique global solution which is a continuous* $C^{k,\varepsilon}$*-semimartingale for some* $\varepsilon > 0$*. It is represented by*

$$u(x,t) = f(\psi_t(x)) \exp \left\{ \int_0^t F^{d+1}(\varphi_s(y), \circ ds) \Big|_{y=\psi_t(x)} \right\},$$

where φ_t is the solution of

$$\varphi_t(x) = x - \int_0^t F(\varphi_s(x), \circ ds)$$

and ψ_t is its inverse.

Assume that the coefficients $\beta^i(x,t)$ and $\alpha(x,t)$ in linear equation (20) satisfy Condition $(E.1)_k$ for some $k \geq 3$. Let us check that the corresponding continuous C^k-martingales

$$F^i(x,t) = \int_0^t \beta^i(x,s)\, dW_s, \qquad F^{d+1}(x,t) = \int_0^t \alpha(x,s)\, dW_s$$

satisfy Condition $(D.1)_{k,1}$. According to the definitions in [11], the pair

$$\Big(\big\{ \beta^i(x,t)\beta^j(y,t) \big\}_{i,j=1}^{d}, \alpha(x,t)\alpha(y,t) \Big)$$

is called the local characteristic of $\big(F(x,t), F^{d+1}(x,t)\big)$. It is said to belong to the class $B_b^{k+1,1}$ if for every $i = 1, \dots, d$

$$\int_0^T \Bigg[\left(\sup_{x \in \mathbf{R}^d} \frac{|\beta^i(x,t)|}{1+|x|} \right)^2 + \sum_{1 \leq |\gamma| \leq k+1} \left(\sup_{x \in \mathbf{R}^d} \big| D_x^\gamma \beta^i(x,t) \big| \right)^2$$
$$+ \sum_{|\gamma| = k+1} \left(\sup_{\substack{x,x' \in \mathbf{R}^d \\ x \neq x'}} \frac{|D_x^\gamma \beta^i(x,t) - D_x^\gamma \beta^i(x',t)|}{|x - x'|} \right)^2 \Bigg] dt < \infty,$$
$$\int_0^T \Bigg[\left(\sup_{x \in \mathbf{R}^d} \frac{|\alpha(x,t)|}{1+|x|} \right)^2 + \sum_{1 \leq |\gamma| \leq k+1} \left(\sup_{x \in \mathbf{R}^d} |D_x^\gamma \alpha(x,t)| \right)^2$$
$$+ \sum_{|\gamma| = k+1} \left(\sup_{\substack{x,x' \in \mathbf{R}^d \\ x \neq x'}} \frac{|D_x^\gamma \alpha(x,t) - D_x^\gamma \alpha(x',t)|}{|x - x'|} \right)^2 \Bigg] dt < \infty,$$

which follow from the boundedness of $\beta^i(x,t)$, $\alpha(x,t)$ and all their derivatives.

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