Brauer Relations, Induction Theorems and Applications

by

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Contents

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Declarations

Chapters three, four and five represent original work on the part of the author. Chapters three and five are modified forms of joint papers between my supervisor Alex Bartel and myself; chapter three is based on [4] while chapter five is a modified version of a preprint by Alex Bartel and myself which, at the time of writing, remains to be completed. The fourth chapter is solely my work. The background and application chapters are summaries of well known results and are thus not original, references are given for these chapters as well as for any results stated in the introduction.

Abstract

Let G be a finite group and F a field, then to any finite G-set X we may associate a $F[G]$-permutation module whose F-basis is indexed by elements of X. We seek to describe when two non-isomorphic G-sets give rise isomorphic permutation modules. This amounts to describing the kernel $K_F(G)$ of a map between the Burnside Ring of G and the ring of representation ring of $F[G]$-representations of G. Elements of this kernel are known as Brauer Relations and have extensive applications in Number Theory, for example giving relationships between class numbers of the intermediate number fields of a Galois extension. In characteristic 0, the generators of $K_F(G)$ have been classified in [2]. We extend this classification to characteristic $p > 0$ for all finite groups G save for groups which admit a subquotient which is an extension of a non-elementary p-quasi-elementary group by a p-group. Our approach initially mimics that in characteristic 0, and so we give a much more general description of these steps in terms of Green functors.

Notation and conventions

- The letters G and H will always denote finite groups.
- For a finite group G and subgroups $H, K \leqslant G$ the symbol $K\backslash G/H$ denotes a set of (K, H)-double coset representatives $\{g_i\}$ where $g_i \in G$, $Kg_iH \cap Kg_jH = \emptyset$ for $i \neq j$, and $\coprod_{g_i} Kg_iH = G$.
- We adopt the convention that the statement $H \leqslant G$ means H is a subgroup of G while $H < G$ will mean that H is a proper subgroup. A subscript G on a containment will mean that we are concerned with objects up to G-conjugacy for instance $H \leqslant_G G$ means a representative H of a G-conjugacy class of subgroups of G, similarly $g \in_G G$ will mean a representative g of a G-conjugacy class of elements.
- We will write $h^g = g^{-1}hg$ and $H^g = g^{-1}Hg$ for the right action by conjugation on elements and subgroups respectively. We will use ${}^g h$ and ${}^g H$ for the corresponding left actions.
- If M is an $R[H]$-module for some $H \leqslant G$ we write M^g for the corresponding $R[H^g]$-module with multiplication $h^g(m)^g = (hm)^g$ for $h \in H$ and $(m)^g \in M^g$. Similarly we write ${}^g M$ for the corresponding $R[{}^g H]$-module.
- For a finite set S we will use $\#S$ to denote its size.
- The letters R, and S will denote unital associative rings.
- The letters p, q, and l will denote rational primes.
- Gothic letters such as $\mathfrak{p}$, $\mathfrak{q}$, and $\mathfrak{l}$ will denote primes in an arbitrary ring different from $\mathbb{Z}$. When $\mathbb{Z}$ is a subring we will adopt the convention that the gothic letter divides the corresponding rational prime e.g. $\mathfrak{p} \mid p$.
- The symbol $\mathrm{Ind}_{G/H}(-)$, will denote the induction map.

- The symbol $\mathrm{Res}_{G/H}(-)$ will denote the restriction map.
- The symbol $\mathrm{Inf}_{G/N}(-)$ will denote the inflation map.
- We will use $a(R[G])$ to denote the representation ring associated to $R[G]$ see definition 2.2.2. We will write $a(G, \mathrm{triv})$ for the subring of trivial source modules see definition 2.2.14. Furthermore we will use the following shorthands: $a_S(-) = S \otimes_{\mathbb{Z}} a(-)$ and $a_p(-) = \mathbb{Z}[1/p] \otimes_{\mathbb{Z}} a(-)$.
- The Grothendieck ring associated to $R[G]$ will be denoted by $G_0(R[G])$ see definition 2.2.18.
- Unless otherwise stated, a script letter will denote a Mackey functor. The letters $\mathcal{F}$ and $\mathcal{G}$ will always denote a Mackey functor.
- Let G be a finite group, and let $U, H \leqslant G$ be subgroups such that $H \leqslant N_G(U)$. Then we denote by $[U, H]$ the commutator subgroup generated by commutators $[u.h]$ for $u \in U, h \in H$.
- Let S be a set of primes and G a finite group. An S-Hall subgroup is a subgroup of G whose order is divisible only by primes in S and whose index is coprime to every prime in S. When S is equal to the set of all primes dividing the order of G that are coprime to p we will call a S-Hall subgroup a $(-p)$-Hall subgroup, and denote it by $G_{p'}$.
- Given a finite group G and a prime number p, we denote the largest normal p-subgroup of G by $O_p(G)$, and the smallest normal subgroup of p-power index by $O^p(G)$. We also define $O^0(G)$ to be G. If π is a set of prime numbers, and n is an integer, then we denote by $n_{\pi'}$ the largest positive integer dividing n that is coprime to all $p \in \pi$.
- For a Mackey functor $\mathcal{F}$, $\mathcal{C}(F)$ will denote the family of coprimordial groups for $\mathcal{F}$ and $\mathcal{P}(\mathcal{F})$ will denote the family of primordial groups for $\mathcal{F}$. See definition 3.3.2.

Chapter 1

Introduction

1.1 Motivation and Introduction to the Problem

Given a finite set X together with an action by some finite group G, and a ring R we may form a permutation $R[G]$-module $R[X]$. As an R-module, $R[X]$ is free with basis indexed by $x \in X$, the action of G on $R[X]$ is then by permuting this basis. A natural question to ask is to what extent does $R[X]$ determine X? For instance one may ask, for a fixed G and R, does the isomorphism class of $R[X]$ uniquely determine X? If not, can we describe all X which give isomorphic permutation modules? How does this change as we vary R?
It turns out that the answer to first question is negative, and that non-isomorphic G-sets which give rise to isomorphic $\mathbb{Q}[G]$-permutation modules can be used to prove rich results. A nice result exploiting this is due to Brauer in 1951.

Theorem 1.1.1. *[12, Satz 5] Let K/F be a Galois extension of number fields with Galois group S_3, let L_1, L_2, L_3 denote the three Galois conjugate degree 3 subfields and let J denote the unique degree 2 subextension. Then writing $h(-)$ for the class number the ratio:*

$$h(K)^2 h(F)/h(L_1)^2 h(J),$$

takes finitely many values as K varies.

It important to note that this theorem is explicitly dependent upon the Galois group of the extension but otherwise there are no restrictions on the extensions of F considered. While this result requires some use of number theory we can retrieve a weaker result, which says that the valuation of this ratio is only non-trivial at $p = 3$, using strictly algebraic methods. Such results are not only common in Galois theory. Another example comes from manifold theory, where one may ask the question: are there 2-manifolds which are non-isometric but have the same spectrum? In the case of bounded two manifolds this question is more colloquially phrased as 'Can one hear the shape of a drum?' and was posed in [25]. The following result due to Sunada, gives a construction of isospectral manifolds.

Theorem 1.1.2. *[32, Theorem 1] Let M_1 be a finite Riemannian covering of M_0 with covering group G, suppose that there are two non-conjugate subgroups $H, K < G$ such that as complex representations $\mathbb{C}[G/H] \cong \mathbb{C}[G/K]$ then their associated coverings M_H and M_K are isospectral.*

This construction is however not strong enough to ensure that the two manifolds M_H and M_K are isometric, although it is possible, and Sunada [32, Corollary 1] proves that for any Riemann surface of genus at least 2 there exists isospectral covers M_H and M_K which are not isometric.

In both these theorems, while significant amount of specialist theory is used to get the strongest possible result, the first step amounts to describing a pair of non-isomorphic G-sets for some finite group G whose associated permutation representations are isomorphic. We will see in the later sections of chapter 2 that there are a plethora of examples where such pairs of G-sets give non-obvious structure.

Our aim is to compute and classify which G-sets give rise to isomorphic $\mathbb{F}_p[G]$-permutation modules. We will see that such a classification with $\mathbb{F}_p$ replaced by a general field F is only dependent upon its characteristic and thus this is the natural extension of the work in [2] in characteristic 0.

1.2 Structure of the Thesis and Statement of the Main Results

This thesis is divided into chapters of independent interest, which are not necessarily independent mathematically. The basic outline is as follows:

1. In the *Background* chapter we present the necessary theory of G-sets, representations, and Mackey functors which underpins the results of the later sections. In particular in this section we formally define Brauer relations and discuss the results already known. In addition we introduce an algebraic frame work, that of the cohomological Mackey functor, which describes the situations these relations are most useful in. This chapter contains material which is well known. We do not always include proof although we have provided proofs when we feel that they are in the spirit of the thesis as a whole.

2. The third chapter *'Mackey and Green Functors with Inflation'* is used to develop abstract machinery, which will formalise our approach to problems considered in chapters 4 and 5.

3. Chapter 4 *'Brauer Relations in Positive Characteristic Semisimplified'* is our first attempt at a classification of Brauer relations in positive characteristic, but we consider when two G sets give rise to the same modular representation up to semisimplification. This is a much weaker requirement and our analysis is similarly much easier in this situation than in the modular non-semisimple case. Despite the simplification this chapter establishes the blueprint for the method we will use when tackling the full problem.

4. In chapter 5 *'Brauer Relations in Positive Characteristic'*, we put all of these techniques together and tackle our main problem with substantial success. We are able to describe all primitive relations save for one class of groups and at the end of the chapter we summarise what is known and what we conjecture in this remaining case.

5. The final chapter, chapter 6, is devoted towards giving some brief applications of our results.

The main result of this thesis is a classification of Brauer relations in positive characteristic and appears as Theorem 5.3.3. Along the way, particularly in chapter 3, we prove several important results which may be used to tackle problems of this type in great generality. Our main results are summarised in the following theorem:

Theorem 1.2.1. *Let G be a finite group and let k be a field of characteristic $p > 0$. Then all Brauer relations over k, and over k after semisimplification, are linear combinations of those inflated or induced from a finite, explicit, list of families of subquotients of G in each case.*

We are able to give a very precise description of these families, and in the case that G is soluble give 'generating' relations in most cases.

Chapter 2

Background Material

This section is intended to develop theory which is already present in the literature, and known to specialists, but which will be required in the subsequent chapters. The reader is invited to skip this section entirely, and treat it instead as a reference when reading the later chapters. In the first three sections we will roughly follow [15, Chapter 11] and [5, Chapter 5].

2.1 G-Sets and the Burnside Ring

We hope to give a concise introductory account of the theory of G-sets and the Burnside ring. These objects are an integral part of the problem we consider in this thesis. Throughout we shall assume, for convenience, that G is finite.

Definition 2.1.1 (G-sets)**.** Let G be a finite group. A left *G-set* is a finite set X on which G acts on the left by permutations. Thus a G-set is a pair (X, ϕ) where $\phi : G \times X \to X$ satisfies $\phi(1, x) = x$ for all $x \in X$ and $\phi(g_2, \phi(g_1, x)) = \phi(g_2 g_1, x)$ for all $g_1, g_2 \in G$ and $x \in X$.
A *morphism* of left G-sets (X_1, ϕ_1) and (X_2, ϕ_2) is a map $f : X_1 \to X_2$ such that $f(\phi_1(g, x)) = \phi_2(g)(f(x))$ for all $(g, x) \in G \times X_1$.

Throughout this thesis all G-sets will be left G sets unless otherwise specified so we will omit the left. Where the action is clear, we will suppress ϕ and write $\phi(g, x)$ as gx. A G-set X is called transitive if for each any two elements $x, y \in X$ there exists a $g \in G$ such that $x = gy$.

Example 2.1.2. Let G be a finite group and $H \leqslant G$ be a subgroup. Then the set of left cosets $G/H = \{g_1 H, \cdots, g_n H\}$ carries a left action of G by $g(g_1 H) = g_2 H$ where $g g_1 = g_2 h$ for some $h \in H$. It is easy to verify that this is a group action.

We shall now show that every transitive G-set is isomorphic to G/H for an appropriate choice of subgroup H.

Lemma 2.1.3. *There exists a bijection between isomorphism classes transitive G-sets and conjugacy classes of subgroups $H \leqslant G$. This bijection is given by $\phi_x : X \mapsto G/\operatorname{Stab}_G(x)$ for any choice of basepoint $x \in X$.*

Proof. Let X be a transitive G-set, and for $x \in X$ let $H_x = \operatorname{Stab}_G(x)$. The action of G is transitive so H_x and H_y are conjugate for any $x, y \in X$. Furthemore H_x^g stabilises $z = gx$ and so all conjugates of H_x occur as point stabilisers. Thus we may assign to X a well defined conjugacy class of subgroup $[H_x]$. The map $f : gx \mapsto gH_x$ gives a morphism of G-sets from X to G/H_x which is clearly bijective. Thus it is an isomorphism of G-sets. By the previous argument this choice is unique up to conjugacy. As G/K is itself a transitive G-set we have the stated bijection. $\square$

Example 2.1.4. Consider $G = S_3$, there are four conjugacy classes of subgroups namely $[\{e\}], [\langle(1,2)\rangle], [\langle(123)\rangle], [S_3]$. These correspond to the isomorphism classes of transitive S_3-sets, for instance $S_3/\langle(1,2)\rangle = \{e\langle(1,2)\rangle, (123)\langle(1,2)\rangle, (132)\langle(1,2)\rangle\}$ is a transitive left S_3-set.

Example 2.1.5. Let X and Y be G-sets and let $Z = X \coprod Y$ be their disjoint union as sets. Then Z is canonically G-set determined uniquely up to isomorphism.

Any G-set X is isomorphic to a disjoint union of transitive G-sets, $X = \coprod_{x \in G \backslash X} \operatorname{Orb}_G(x)$ where $\operatorname{Orb}_G(x) = \{gx : g \in G\}$ has transitive G-action.

Example 2.1.6. Let S and T be G-sets. The cartesian product $S \times T$ with a G-action $g(s,t) = (gs, gt)$ giving it the structure of a G-set.

The two operations, disjoint union and cartesian product, are analogues of addition and multiplication for G-sets and allow us to form a ring.

Definition 2.1.7. The *Burnside ring* $b(G)$ of a finite group G is, as a group, the free abelian group generated by isomorphism classes of G-sets modulo expressions of the form $[S \coprod T] - [S] - [T]$. Multiplication is given by $[S][T] = [S \times T]$.

The additive identity in this ring is the empty set and the multiplicative identity is given by the trivial G-set $1 = [G/G]$. The ring is commutative as $S \times T \cong T \times S$. The following lemma checks that $b(G)$ is no smaller than we expected.

Lemma 2.1.8. $[S] = [T]$ *in* $b(G)$ *if and only if* $S \cong T$ *as G-sets.*

Proof. Suppose that $S \cong T$ then it follows from the definition of $b(G)$ that they give rise to the same class in $b(G)$. Otherwise suppose that $[S] = [T]$ then either S and T are isomorphic or there exists a G-set X such that $S \coprod X \cong T \coprod X$. Now since every G-set uniquely decomposes into transitive G-sets decomposing both sides gives $S \cong T$.

□

Remark 2.1.9. The previous lemma may seem trivial, however we explicitly had to use unique decomposition. This assumption is not always true, for instance $\mathbb{Z}[G]$-modules need not have unique decomposition and so when we perform a similar construction in defining the representation ring the classes may be larger than expected.

Example 2.1.10. Again returning to our example of S_3 we see that

$$b(S_3) = \langle [S_3/\{e\}], [S_3/\langle(1,2)\rangle], [S_3/\langle(123)\rangle], [S_3/S_3]\rangle_{\mathbb{Z}}.$$

To save on notation, when G is clear from the context we will write $[H]$ for the class in $b(G)$ corresponding to the transitive G-set G/H.

We would like to fined a way to determine when two G-sets are isomorphic. We study the ring homomorphisms from $b(G)$ to $\mathbb{Z}$ and show that any element of $b(G)$ determined by its image under all of these homomorphisms. The next theorem says that any element $X \in b(G)$ is determine by the values of the functions $f_H(X) := \#X^H$ as H ranges over representatives of conjugacy classes of subgroups of G.

Theorem 2.1.11. *There exists ring homomorphisms from $b(G) \mapsto \mathbb{C}$ of the form $f_H(X) = \#(X)^H$. After tensoring with $\mathbb{C}$ we have an isomorphism of rings:*

$$\sum_{H \leqslant_G G} f_H : \mathbb{C} \otimes b(G) \xrightarrow{\sim} \bigoplus_{H \leqslant_G G} \mathbb{C}$$

where the sum runs over conjugacy classes of subgroups.

Proof. First note that $f_{H_i} = f_{H_j}$ if and only if H_i and H_j are conjugate, so the f_H in the sum are linearly independent (see [5] lemma 5.2.2). Since there is no linear dependence the image must have full rank thus the map is surjective. Counting the $\mathbb{C}$-dimension of both sides gives injectivity. It remains to show that f_H is a ring homomorphism but this is clear. □

We finish this section by defining induction, restriction, inflation and deflation for G-sets.

Definition 2.1.12. Let G be a finite group $H \leqslant G$ a subgroup and $N \leqslant G$ a normal subgroup. We define induction, restriction, and inflation of H-sets, G-sets and G/N-sets respectively.

- Let X be a H-set we define the induction of X to G by; $\mathrm{Ind}_{G/H}(X) = G \times_H X := G \times X/\{(gh^{-1}, hx) \sim (g,x) | g \in G, h \in H, x \in X\}$ with a G-action given by $g[(g_1, x_1)] = [(gg_1, x_1)]$.
- Let $\alpha : H \hookrightarrow G$ be the inclusion of H into G then given a G-set Y we define the restriction to H, $\mathrm{Res}_{G/H}(Y) = Y$ where H acts by first embedding in G by α.
- Let $\phi : G \twoheadrightarrow G/N$ and let Z be a G/N-set, we define the inflation of Z to G by $\mathrm{Inf}_{G/N}(Z) = Z$ where G acts by first applying ϕ.

2.2 Representation Rings

We now move on to the second important object needed to define Brauer relations, the representation ring.

Definition 2.2.1. Let R be a commutative ring and let G be a finite group then an $R[G]$-lattice is an $R[G]$-module which is finitely generated and projective as an R-module.

Definition 2.2.2. Let R be a commutative ring, and let G be a finite group then we define the *representation ring* (also Green Ring) $a(G) = a(R[G])$ to be, as a group, the free abelian group on isomorphism classes of finitely generated R-projective $R[G]$-modules ($R[G]$-lattices) modulo relations of the form $[M \oplus N] - [M] - [N]$. We equip this with multiplication defined by $[M] \cdot [N] = [M \otimes N]$.

We follow [5] and [15] and denote various rings of coefficients by: $A(R[G]) = \mathbb{C} \otimes_{\mathbb{Z}} a(R[G])$, $a(G)_{\mathbb{Q}} = \mathbb{Q} \otimes_{\mathbb{Z}} a(R[G])$, $a(G)_p = \mathbb{Z}[\frac{1}{p}] \otimes_{\mathbb{Z}} a(G)$ and finally $a(R[G])_{(S)}$ where S is a set of primes and we allow denominators coprime to those primes. There is an important thing to note here, these coefficient rings control the coefficients with which the $R[G]$-lattices can appear, they do not affect R.

This ring is in general very hard to work with. We will restrict our attention to the trivial source ring, a proper subring of the representation ring. Before proceeding we define maps on $R[G]$-modules which are analogues of those for G-sets defined in definition 2.1.12.

Definition 2.2.3. Let R be a commutative ring, G be a finite group, $H \leqslant G$ a subgroup, and $Q = G/L$ a quotient of G. Furthermore let M, N, S be $R[G]$, $R[H]$, and $R[Q]$-modules respectively. We define the following maps:

- The induction map $\mathrm{Ind}_{G/H}(N) = R[G] \otimes_{R[H]} N$, where G acts by left multiplication. This defines a group homomorphism from $a(H)$ to $a(G)$.
- The restriction map $\mathrm{Res}_{G/H}(M) = M$ viewed as a $R(H)$-module. This defines a ring homomorphism from $a(G)$ to $a(H)$.
- The inflation map $\mathrm{Inf}_{G/L}(S) = S$ viewed as a $R[G]$-module where the action factors through Q. This defines a ring homomorphism from $a(Q)$ to $a(G)$.
- Let $g \in G$ then the conjugation by g map $c_g(N) = N^g$ which is an $R[H^g]$-module. This gives a ring isomorphism between $a(H)$ and $a(H^g)$

Lemma 2.2.4. *The above maps satisfy the following properties:*

- *Transitivity:* $\mathrm{Res}_{G/H} \mathrm{Res}_{H/K} = \mathrm{Res}_{G/K}$, $\mathrm{Ind}_{G/H} \mathrm{Ind}_{H/K} = \mathrm{Ind}\, G/K$ *and* $\mathrm{Inf}_{G/N=Q} \mathrm{Inf}_{Q/N_2} = \mathrm{Inf}_{G\phi^{-1}(N_2)}$ *where* $\phi : Q \to Q/N_2$.
- *Commutativity of induction and inflation* $\mathrm{Ind}_{G/H} \mathrm{Inf}_{H/(N\cap H)} = \mathrm{Inf}_{G/N} \mathrm{Ind}_{(G/N)/(H/(N\cap H))}$ *where* $N \lhd G$.
- *The Mackey decomposition formula:*

$$\mathrm{Res}_{G/H} \mathrm{Ind}_{G/K} = \sum_{g \in K\backslash G/H} \mathrm{Ind}_{H/(H\cap K^g)} \mathrm{Res}_{K^g/(H\cap K^g)} c_g.$$

- *Frobenius reciprocity:* $\mathrm{Hom}_{R[G]}(\mathrm{Ind}_{G/H}(-), -) = \mathrm{Hom}_{R[H]}(-, \mathrm{Res}_{G/H}(-))$ *and* $\mathrm{Hom}_{R[G]}(-, \mathrm{Ind}_{G/H}(-)) = \mathrm{Hom}_{R[H]}(\mathrm{Res}_{G/H}(-), -)$.

Proof. These results are standard. See for instance [16, Chapter 1 Section 10]. □

2.2.1 Trivial Source Modules

When R is not a field of characteristic zero, there may exist non-projective $R[G]$-lattices and as a result the representation theory need not be semisimple, that is not every $R[G]$-lattice decomposes into a direct sum of simple $R[G]$-lattices. To help understand this situation better we introduce relative projectivity.

Definition 2.2.5. Let R be a ring, G a finite group an $R[G]$-module M is indecomposable if it has no proper direct summands.

Remark 2.2.6. When the representation theory of $R[G]$ is semisimple indecomposable modules are simple.

Definition 2.2.7. An $R[G]$-module M is relatively H*-projective* (also (G, H)-projective) if any short exact sequence $0 \to A \to B \to M \to 0$ which splits on restriction to H also splits for G. We will write $a(R[G], H)$ for the ideal of relatively H-projective $R[G]$-lattices.

Remark 2.2.8. To show that $a(R[G], H)$ is an ideal one needs to prove that if A is K-projective and B is H-projective then $A \otimes_{\mathbb{R}} B$ is H-projective, this is proven in [5, Corollary 3.6.7] .

Remark 2.2.9. Clearly every $R[G]$-module is relatively G-projective and the modules which are $\{e\}$-projective are precisely those which are projective.

Assumption 2.2.10. For the rest of section 2.2 we fix a prime p and will adopt the following restrictions on R:

1. R is a commutative ring,

2. for all finite groups G, the prime to p part $|G|_{p'}$ is invertible in R, and

3. any finitely generated $R[G]$ module decomposes uniquely into indecomposable modules.

This third assumption is satisfied for instance if $R[G]$ is Artinian, or $R = \mathbb{Z}_p$ the p-adic integers [5, Theorems 1.4.6, 1.9.3].

We now use relative projectivity to define the concepts of the source and vertex of an $R[G]$-module:

Definition 2.2.11. A *vertex* of an indecomposable $R[G]$-module M is a subgroup D of G such that M is projective relative to D but not to any proper subgroup of D. A *source* of M is then an indecomposable $R[D]$-module M_0 such that M is a summand of $\mathrm{Ind}_{G/D}(M_0)$.

Proposition 2.2.12. *Let M be an indecomposable $R[G]$-module. Then:*

1. *All vertices of M are conjugate in G.*

2. *Let D be a vertex of M and let M_1, M_2 be two $R[D]$-modules which are both sources of M. Then there exists an element $g \in N_G(D)$ such that $M_1 \cong M_2^g$.*

3. *If the p' part of G is invertible in R, then the vertices of M are p-subgroups.*

Proof. The first two parts follow from the Mackey decomposition formula:

1. Suppose that M has vertices H and K. Then M is a summand of $Ind_{G/H} \operatorname{Res}_{G/H} M$ and of $Ind_{G/K} \operatorname{Res}_{G/K} M$. It follows that M is a summand of $Ind_{G/K}(\operatorname{Res}_{G/K} \operatorname{Ind}_{G/H}) \operatorname{Res}_{G/K} M$. We apply the Mackey decomposition formula to the bracketed part and use transitivity of restriction and induction M is a summand of $\sum_{g \in H\backslash G/K}(\operatorname{Ind}_{G/(K\cap H^g)} \operatorname{Res}_{G/(K\cap H^g)} c_g M)$. Thus any vertex of M is contained in $K \cap H^g$ for some g, from the minimality of H it follows H and K are conjugate.

2. Suppose that M_1 and M_2 are both sources for M it follows that M_i are summands of $\operatorname{Res}_{G/D}(M)$. Now as M is a summand of $\operatorname{Ind}_{G/D}(M_1)$ it follows that M_2 is a summand of $\operatorname{Res}_{G/D} \operatorname{Ind}_{G/D}(M_1) = \sum_{D\backslash G/D} \operatorname{Ind}_{G/(D\cap D^g)} \operatorname{Res}_{D^g/(D\cap D^g)}(M_1^g)$. Since M_2 is a source it must be an indecomposable summand of M_1^g for some $g \in N_G(D)$, otherwise D would not be a vertex. Since M_1 and thus M_1^g is indecomposable we have an isomorphism.

3. Higman's criterion [5, Proposition 3.6.4] and [5, Corollary 3.6.9] show that that if $[G : H]$ is invertible in R then every $R[G]$-module is projective relative to H. The result follows.

$\square$

Lemma 2.2.13. *Let G be a finite group, and let R be a field of characteristic p. Then the trivial $R[G]$-module 1 has vertex $\operatorname{Syl}_p(G)$.*

Proof. By the previous Lemma the vertex D of 1 is contained in a Sylow subgroup. Now suppose that D is not a Sylow subgroup of G. Then restricting to a Sylow subgroup P containing D it follows that 1_P is a summand of $M = \operatorname{Ind}_{P/(P\cap D^g)}(1)$ for some $g \in G$. Now we claim that there only exists one simple $R[P]$-module 1 and by Frobenius reciprocity $\operatorname{Hom}_{R[P]}(\operatorname{Ind}_{P/(P\cap D^g)}(1), 1) = \operatorname{Hom}_{R[(P\cap D^g)]}(1, 1) = 1$ so there is only one trivial quotient and M is indecomposable of dimension greater than 1 by assumption a contradiction. It remains to show that the only simple $R[P]$-module is R. Suppose that M is a simple $R[P]$-module, then $\#M = p^n$ for some n, P fixes the zero element and all orbits are of p-power size so it follows that P fixes some non-zero x. Then the subgroup generated by x is a trivial submodule of M and thus as M is simple $M = 1$. $\square$

We now define the trivial source modules

Definition 2.2.14. An $R[G]$-module is a *trivial source module* if each indecomposable summand has the trivial module $R = 1$ as its source. A trivial source module of the form $\sum_{H\leqslant G} \operatorname{Ind}_{G/H}(1)^{\oplus a_h}$ will be called a permutation module.

Lemma 2.2.15. *An indecomposable $R[G]$-module has trivial source if and only if it is a direct summand of a permutation module.*

Proof. Any trivial source module is a summand of a permutation module by definition. Assume that M has vertex P and is a summand of $\mathrm{Ind}_{G/H}(1)$, then $\mathrm{Res}_{G/P}\mathrm{Ind}_{G/H}(1) = \sum_{H\backslash G/P} \mathrm{Ind}_{P/(P\cap H^g)} \mathrm{Res}_{H^g/(P\cap H^g)}(1)$ so the only indecomposable summand with vertex P is $M_0 = 1$.

□

Definition 2.2.16. The *trivial source ring* $a(R[G], triv)$ is the sub-ring of $a(R[G])$ spanned by $\mathbb{Z}$-linear combinations of trivial source modules.

Remark 2.2.17. One may easily verify that the product of two trivial source modules is again trivial source so the $a(R[G], triv)$ is genuinely a ring.

Definition 2.2.18. For any subgroup $H \leqslant G$ let $a_0(R[G], H)$ be the ideal generated by elements of the form $M_3 - M_1 - M_2$ where the short exact sequence $1 \to M_1 \to M_3 \to M_2 \to 1$ splits on restriction to H. Similarly for $a_0(G, \mathcal{H})$ where $\mathcal{H}$ is some family of subgroups. We define the *Grothendieck ring* to be $G_0(R[G]) = a(R[G])/a_0(R[G], 1)$.

Remark 2.2.19. The image of an $R[G]$-module in the Grothendieck ring is often called its *semisimplification.*

Both the trivial source ring and the Grothendieck ring are easier to work with than $a(R[G])$ when R is a field of positive characteristic.

Definition 2.2.20. A p-modular system is a triple $(K, \mathcal{O}, k)$ where $\mathcal{O}$ is a complete rank 1 discrete valuation ring (d.v.r) with field of fractions K , maximal ideal $\mathfrak{p}$ and quotient field $k = \mathcal{O}/\mathfrak{p}$ of characteristic p.

Lemma 2.2.21. *Let $(K, \mathcal{O}, k)$ be a p-modular system then the reduction mod $\mathfrak{p}$ map gives an isomorphism:*

$$a(\mathcal{O}[G], triv) \cong a(k[G], triv)$$

Proof. We first note that given any $M \in a((\mathcal{O})[G], triv)$ then $M/\mathfrak{p}M$ is a trivial source $k[G]$-module. It remains to exhibit a unique lift of each $k[G]$-module. Let $\mathcal{O}[X]$ be a permutation module. Then $\mathrm{Hom}_{\mathcal{O}[G]}(\mathcal{O}[X], \mathcal{O}[X]) \to \mathrm{Hom}_{k[G]}(k[X], k[X])$ is a surjection. To see this note that by definition $\mathrm{Hom}_{R[G]}(R[G/H], R[G/H]) = \mathrm{Hom}_{R[G]}(\mathrm{Ind}_{G/H}(1), \mathrm{Ind}_{G/H}(1))$. Applying Frobenius reciprocity and the Mackey decomposition formula on each side yields $\mathrm{Hom}_{R[G]}(R[G/H], R[G/H]) = \sum_{g\in H\backslash G/H} 1_H$.

Let N be a trivial source $k[G]$-module then it is a direct summand of some permutation module $k[X]$. Then there is a surjection of endomorphism rings $\mathrm{End}(\mathcal{O}[X]) \to \mathrm{End}(k[X])$.
Let e_N be the idempotent of $\mathrm{End}(k[X])$ corresponding to N then we may lift this idempotent to $\mathrm{End}(\mathcal{O}[X])$ by the idempotent refinement theorem [5, Theorem 1.9.4]. This then gives a lift of N to $\mathcal{O}[G]$ which is again trivial source.
We now show this lift is unique. Suppose otherwise. Then there are two possible lifts M_1 and M_2. Since $\mathrm{Hom}_{\mathcal{O}[G]}(\mathcal{O}[X], \mathcal{O}[X]) \to \mathrm{Hom}_{k[G]}(k[X], k[X])$ is a surjection the identity automorphism of N lifts a pair of maps $\phi_1 : M_1 \to M_2$ and $\phi_2 : M_2 \to M_1$ whose composite reduces to the identity. Their composition differs from the identity by something in the kernel of reduction modulo p say a so $\mathrm{End}(M_i) = (1 - a_i)\,\mathrm{End}(M_i) + a_i\,\mathrm{End}(M_i)$. If $1 - a_i$ is not invertible, then $(1 - a_i)\,\mathrm{End}(M_i) \subsetneq \mathrm{End}(M_i)$. But then $(1 - a_i)\,\mathrm{End}(M_i)$ is contained in a maximal ideal, which does not contain $a_i\,\mathrm{End}(M_i)$. But $a_i \in \mathfrak{p}\,\mathrm{End}(M_i)$ which is equal to the Jacobson radical (since $\mathrm{End}(M_i)$ is local). This is a contradiction, so $1 - a_i$ is invertible and the maps ϕ_i are isomorphisms $\square$

2.2.2 Species

We describe ring homomorphisms from the various rings introduced into $\mathbb{C}$ and the degree to which they separate out elements. Recall that in the previous section we defined fixed points maps on the Burnside ring and these exactly separated the elements of $b(G)$.

Definition 2.2.22. Let A be a sub-algebra or ideal of $a(R[G])$ then an element of $\mathrm{Hom}(A, \mathbb{C})$ (as rings) is called a *species* of A.

Remark 2.2.23. We will primarily be interested in the study of species of the trivial source ring, and we will simply refer to these as species.

Definition 2.2.24 (The Character)**.** Let G be a finite group and let k be a field containing all $|G|$th roots of unity, by which we mean all roots of unity in the algebraic closure of k, $\bar{k}$ whose order divides $|G|$ lie in k. Fix an embedding α of $\mu(k)$ into $\mathbb{C}$ then for a $k[G]$-module M we define $Tr(g, M)$ to be the sum of the images of the eigenvalues of g acting on M under this embedding, $T(g, M) = \sum_i \alpha(\lambda_{g,i})$. The tuple $(T(g, M))_{g \in G} G$ is called the character of M.

Remark 2.2.25. Over fields k of characteristic p there are no non-trivial pth roots of unity in $\bar{k}$. In this case our definition will only require that the $|G|_{p'}$th primitive roots of unity are in k.

Theorem 2.2.26. *Let G be a finite group and let k be a field of characteristic 0 containing the $|G|$th roots of unity. For each $g \in G$ let $t_g : a(k[G]) \to \mathbb{Z}$ denote the trace function. Then summing over conjugacy class representatives of $g \in G$ we have the following isomorphism:*

$$\sum_{g \in_G G} t_g : \mathbb{C} \otimes a(k[G]) \xrightarrow{\sim} \sum_{g \in_G G} \mathbb{C}.$$

Proof. Clearly the t_g are ring homomorphisms. We claim that $\sum_{g \in_G G} t_g$ is injective, since the t_g are distinct they are linearly independent by [5, Lemma 5.2.2] and so the map is surjective. The map $\sum_{g \in_G G} t_g$ is represented by $(\sum_{g \in_G G} t_g)_{i,j} = t_{g_i}(S_j)$, with respect to the basis of $\mathbb{C} \otimes a(k[G])$ consisting of the classes of simple modules $B = \{[S_i]\}$, and the standard basis of $\sum_{g \in_G G} \mathbb{C}$ indexed by $g_j \in_G G$.
We show that the pairing $1/|G| \sum_{g \in_G G} t_g(S_i) t_{g^{-1}}(S_j) = \delta_{i,j}$, and thus $\sum_{g \in_G G} t_g$ is injective. If $T : S_1 \to S_2$ is a map of k-vector spaces then $A = 1/|G| \sum_{g \in G} gTg^{-1}$ is a $k[G]$-homomorphism. If the S_i are distinct then this must be zero for all T by Schur's lemma, and so, after choosing a basis, the entry $A_{i,j} = g_{i,k} t_{k,l} g^{-1}_{l,j}$ must be zero for all T. In particular we may choose T so that $T_{(i,j)} = 1$ if $i = a, j = b$ and 0 otherwise this then implies $1/|G| \sum_{g \in G} t_g(S_1) t_{g^{-1}}(S_2) = 0$ and similarly if $S_1 = S_2$ then $1/|G| \sum_{g \in G} t_g(S_1) t_{g^{-1}}(S_1) = 1$. $\square$

An immediate consequence of this theorem is that the species of $a(k[G])$ uniquely determine an element of $a(k[G])$ when k has characteristic zero. This phenomenon is fortunately repeated in several important cases. It is easy to see that an element of any finite dimensional commutative semisimple complex algebra is determined by its species.

In general $a(k[G])$ need not be finite dimensional when k is a field of positive characteristic, and so it is clear that taking traces can't possibly identify elements of $a(k[G])$ up to isomorphism. There are two approaches to this problem. The first, and most versatile, is to identify species of $a(k[G])$ which determine elements of a finite dimensional semisimple quotient of $a(k[G])$ for instance the set of species which vanish on $a_0(k[G])$ determine the images of elements in $G_0(k[G])$. The second is only to consider modules in a semisimple finite dimensional subalgebra such as $a(G, triv)$.

Definition 2.2.27. Let G be a finite group and let n be an integer such that $p \mid |G|$ then let $G_{n'} = \{g \in G : \gcd(\mathrm{ord}(g), n) = 1\}$. We adopt the convention that $G_{0'} = G$. Note that G acts on $G_{n'}$ by conjugation.

Theorem 2.2.28. *Let G be a finite group and let k be a field containing the $|G|$th roots of unity. For each $g \in G$ let $t_g : a(k[G]) \to \mathbb{Z}$ denote the function $\operatorname{Tr}(g, -)$. Let p be the characteristic of k. Then summing over G-conjugacy class representatives of $g \in G_{p'}$. We have the following isomorphism:*

$$\sum_{g \in_G G_{p'}} t_g : \mathbb{C} \otimes G_0(k[G]) \xrightarrow{\sim} \sum_{g \in_G G_{p'}} \mathbb{C}.$$

Proof. If k has characteristic 0 we are done by Theorem 2.2.26. So assume k has characteristic $p > 0$. We know that t_g is a species of $a(k[G])$.

We wish to show that these species vanish on $a_0(k[G], 1)$. Let $\alpha = [M] - [N] - [L]$ be any element of $a_0(G, 1)$ and let $g \in G$. Then the restriction to $a(k[\langle g \rangle])$ is split for all such sequences if $g \in G_{p'}$. It follows that $t_g(\alpha) = 0$ for any $\alpha \in a_0(k[G], 1)$ and $g \in G_{p'}$. For $g \notin G_{p'}$ we have $t_g(M) = t_e(M) = \dim_k(M)$ as all (generalised) eigenvalues of g are equal to 1 as this is the only p-power root of unity. So these species factor to the quotient.

As in the proof of Theorem 2.2.26 $\sum_{g \in G} t_g$ is injective. It follows that $\sum_{g \in_G G_{p'}} t_g$ is also injective. It remains to show surjectivity; since the t_g for $g \in_G G_{p'}$ are distinct it follows from [5, Lemma 5.2.2] that they are linearly independent and hence the map is surjective.

□

Remark 2.2.29. The above isomorphism is usually denoted by the Brauer character table. The Brauer character table is a table whose columns are labelled by conjugacy classes of p' elements and whose rows are labelled by simple $k[G]$-modules. The theorem is then equivalent to saying that the columns in the Brauer character table are linearly independent and the table is square.

Theorem 2.2.30 (The Green Correspondence)**.** *[5, Theorem 3.12.2] Let G be a finite group, p be a fixed prime and let R be a ring satisfying assumption 2.2.10. Let P be a p-subgroup of G, suppose that $N_G(P) \leqslant H \leqslant G$. Then there is a one to one correspondence between indecomposable $R[G]$-modules with vertex P and indecomposable $R[H]$-modules with vertex P given as follows:*

1. *If M is an indecomposable $R[G]$-module with vertex P, then $\operatorname{Res}_{G/H}(M)$ has a unique indecomposable summand $f(M)$ with vertex P with all other summands having vertex strictly less than P.*

2. *If N is an indecomposable $R[H]$-module with vertex P then $\operatorname{Ind}_{G/H}(N)$ has a unique indecomposable summand $g(N)$ with vertex P and the remaining terms*

have vertex which are contained in the intersection of P and a conjugate.

3. $f(g(N)) = N$ *and* $g(f(M)) = M$.

4. *This correspondence takes trivial source modules to trivial source modules.*

Proof. 1. Suppose that M is an indecomposable $R[G]$-module with vertex P and source S then $\mathrm{Ind}_{H/P}(S) = S_1 \oplus S_2$ with S_1 indecomposable such that M is a summand of $\mathrm{Ind}_{G/H}(S_1)$ and consequently $\mathrm{Res}_{G/H}(M)$ is a summand of $\mathrm{Res}_{G/H} \mathrm{Ind}_{G/H}(S_1)$.
Now $\mathrm{Res}_{G/H} \mathrm{Ind}_{G/H}(S_1) = \sum_{g \in H\backslash G/H} \mathrm{Ind}_{H/H\cap H^g} \mathrm{Res}_{H^g/H\cap H^g}(S_1^g)$ but since $H \geqslant N_G(P)$ exactly one of these summands, S_1 when $g = e$, has vertex P. But the restriction of M to H has an indecomposable summand of vertex P (as M is a summand of $\mathrm{Ind}_{G/H} \mathrm{Res}_{G/H}(M)$). We therefore take $f(M) = S_1$.

2. Let N be an indecomposable $R[H]$-module with vertex P then letting L be any indecomposable summand of $\mathrm{Ind}_{G/H}(N)$, and restricting, gives $\mathrm{Res}_{G/H}(L)$ is a summand of $\sum_{g \in H\backslash G/H} \mathrm{Ind}_{H/H\cap H^g} \mathrm{Res}_{H^g/H\cap H^g}(N^g)$. This shows that the vertex of L is contained in $P \cap P^g$ for some g. Now there exists some indecomposable summand L_1 of $\mathrm{Ind}_{G/H}(N)$ such that N is a summand of $\mathrm{Res}_{G/H}(L_1)$ and it follows L_1 has vertex containing and thus equal to P. Since N only occurs once in $\mathrm{Res}_{G/H} \mathrm{Ind}_{G/H} N$ showing uniqueness. Let $g(N) = L_1$.

3. Is clear from the definitions of f and g, and Mackey's decomposition formula.

4. Is clear from the construction, since the restriction and induction of a trivial source module is again trivial source.

$\square$

Corollary 2.2.31. *There is a one to one correspondence between isomorphism classes of indecomposable trivial source $R[G]$-modules of vertex $D \leqslant G$ and isomorphism classes of projective indecomposable $R[N_G(D)/D]$-modules.*

Proof. The Green correspondence, Theorem 2.2.30, states that we have a bijection between isomorphism classes of indecomposable trivial source modules with vertex D and isomorphism classes of indecomposable $N_G(D)$ trivial source modules with vertex D. But $D \triangleleft N_G(D)$ must act trivially so all such modules are inflated from projective indecomposable $N_G(D)/D$-modules, and similarly inflating such a module gives an indecomposable trivial source module with vertex D. $\square$

Definition 2.2.32. Let G be a finite group and let R be a discrete valuation ring of residue characteristic $p > 0$ or a field of characteristic $p > 0$. Then for every p-subgroup P of G and every $g \in N_G(P)$ of order coprime to p we may define a ring homomorphism $S_{P,g} : a(G, triv) \to \mathbb{C}$, as follows. Let M be a trivial source module and let N be its vertex P summand, and let N' be the projective $R[N_G(P)/P]$-module corresponding to N by the correspondence in Corollary 2.2.31. Then $S_{P,g}(M) := t_g(N')$.

Lemma 2.2.33. *Let G be a finite group and R be a field of characteristic $p > 0$ or a d.v.r of residue characteristic p then for all pairs P, g $S_{P,g}$ is a ring homomorphism $\mathbb{C} \otimes_{\mathbb{Z}} a(R[G], triv) \to \mathbb{C}$. Furthermore $S_{P_1,g_1} = S_{P_2,g_2}$ if and only if there exists $h \in G$ such that $P_1 = P_2^h, g_1 = g_2^h$. Finally let M_1 and M_2 be trivial source modules then $S_{P,g}(M_1) = S_{P,g}(M_2)$ for all pairs (P, g) if and only if $M_1 \cong M_2$.*

Proof. The fact that $S_{P,g}$ are ring homomorphisms is clear; the correspondence in Corollary 2.2.31 is a ring homomorphism and t_g is again a ring homomorphism.
Now we wish to show that $S_{P_1,g_1} = S_{P_2,g_2}$ if and only if there exists $h \in G$ such that $P_1 = P_2^h, g_1 = g_2^h$.
First suppose $P_1 = P_2^h, g_1 = g_2^h$ for some h in G. Suppose that a trivial source module has vertex P then by proposition 2.2.12 any two vertices are conjugate, that any conjugate is a vertex follows from the isomorphism $\mathrm{Ind}_{G/H} M \cong \mathrm{Ind}_{G/H^g} M^g$. Conjugation by h then gives an isomorphism between the submodule with vertex P and the submodule with vertex P^h. We have that $S_{P,g}(M) = t_g(N) = t_{g^h}(N^h) = S_{P^h,g^h}(M)$ for all M.
Now suppose that $S_{P_1,g_1} = S_{P_2,g_2}$, then letting N be an indecomposable module with vertex P_1 we see that $P_1 = P_2^h$ for some $h \in G$ as P_2 must also be a vertex for N. Now by Corollary 2.2.31 these modules are in bijection with projective $N_G(P_i)/P_i$ modules. These modules are determined by their Brauer characters t_g, and the conjugation isomorphism $P_1 = P_2^h$ takes the column of the Brauer character table corresponding to g to the one corresponding to g^h. Thus $t_{g_2^h}$ must agree with t_{g_2} as they both determine the same column of the Brauer character table.
Finally it is clear that if two trivial source modules are isomorphic then species agree on them. For the converse assume that we have two trivial source modules M_1 and M_2 with no common summands and let P be the maximal vertex among all vertices of M_1 and M_2. Since $S_{P,g}$ agrees for all G the vertex P part of M_i define the same projective $R[N_G(P)/P]$-module and hence by Corollary 2.2.31 the same vertex P summand of M_i a contradiction. □

Definition 2.2.34. Let p and q be prime numbers.

- A finite group is called *p-quasi-elementary* if it has a normal cyclic subgroup of p-power index, equivalently if it is a split extension of a p-group by a cyclic group of order co-prime to p.
- A finite group G is called *p-hypo-elementary* if $G/O_p(G)$ is cyclic, equivalently if G is a split extension of a cyclic group of order co prime to p by a p-group.
- Let q be a prime. A group G is called a (p,q)-Dress group if $G/O_p(G)$ is q-quasi-elementary.

Lemma 2.2.35. *Let G be a finite group. If G is p-quasi-elementary, or p-hypo-elementary, or (p,q)-Dress then all its subquotients are p-quasi-elementary, p-hypo-elementary, or (p,q)-Dress respectively.*

Proof. Consider the case of a proper subgroup and proper quotient separately The proof then follows immediately by combining these cases. □

Lemma 2.2.36. *Let G be a finite group and $H = P \rtimes C \leqslant G$ where P is a p-group and C is cyclic. Let $S_{H,g}$ for $g \in C$ a generator, denote the map which restricts a trivial source module to H, takes the summand N with vertex exactly P and computes $t_g(N)$. Then $S_{H.g}$ coincides with $S_{P,g}$.*

Proof. This amounts to the check that the vertex of an indecomposable trivial source $N_G(P)$-module of vertex P remains the same when we restrict it to P and thus to any intermediate subgroup $P \leqslant H \leqslant N_G(P)$. But this is immediate since its restriction to P is a direct sum of trivial modules, by Mackey's formula, which all have vertex P by Lemma 2.2.13. □

Remark 2.2.37. This lemma provides an easy classification of the species of the trivial source ring.

Theorem 2.2.38. *Let G be a finite group and let k be a field of characteristic $p > 0$ then after summing over pairs H, g up to conjugacy we have an isomorphism:*

$$\sum_{(H,g)_G} S_{H,g} : \mathbb{C} \otimes_{\mathbb{Z}} a(k[G], triv) \xrightarrow{\sim} \sum_{(H,g)_G} \mathbb{C}$$

Proof. That this map is an injective ring homomorphism and the $S_{H,g}$ are distinct follows from Lemmas 2.2.36 and 2.2.33. Surjectivity then follows from [5, Lemma 5.2.2]. □

Lemma 2.2.39. *Let $M = \mathrm{Ind}_{G/H}(1)$ then $S_{K,g}(M) = f_K(H)$.*

Proof. We simply evaluate the species on M. Suppose $K = P \rtimes C$. When we restrict to K by Mackey the restriction splits into a direct sum of terms $\mathrm{Ind}_{K/K\cap H^g}(1)$. Only terms where $P \leqslant K \cap H^g$ can possibly have summands of vertex P, and furthermore by restricting to P we see that each such summand does have vertex P by Clifford's theorem [16, Theorem 11.1] and 2.2.13. The vertex P part of $\mathrm{Res}_{G/K}(M)$ is a direct sum of terms of the form $\mathrm{Ind}_{K/P\rtimes C'}$ for some $C' \leqslant C$. We have $\mathrm{Ind}_{K/P\rtimes C'} = \mathrm{Inf}_{H/P}(\mathrm{Ind}_{C/C'}(1))$, and as $\langle g \rangle = C$ we have $t_g(\mathrm{Ind}_{C/C'}(1)) = \delta_{C,C'}$. It follows that $S_{K,g}$ is equal to the number of times $K \cap H^g = K$ in the Mackey decomposition, but this is exactly $f_K(H)$.

□

2.3 Induction Theorems

The study of induction maps is of interest in representation theory because the induction map $\mathrm{Ind}_{G/H} : a(R[H]) \to a(R[G])$ is not, in general, a unital ring homomorphism. By Frobenius reciprocity the image of induction is an ideal in $a(R[G])$ and thus is equal to all of $a(R[G])$ if and only if it contains 1_G. The purpose of induction theorems, is to classify when $\sum_{H<G} \mathrm{Ind}_{G/H}$ is surjective. Our study of induction theorems will follow the following pattern, fix R and a sub-ring or quotient of $a(R[G])$ say a and try to prove an induction theorem for $\mathbb{Q} \otimes_{\mathbb{Z}} a$ and then use this result as a stepping stone for an induction theorem for a. We will restrict ourselves to induction theorems concerning permutation modules as this leads into the study of Brauer relations later. The prototypical example of an induction theorem is Artin's induction theorem.

Theorem 2.3.1 (Artin's Induction Theorem)**.** *[5, Theorem 5.6.1]. Let G be a finite group, k a field, then there exists unique $\alpha_H \in \mathbb{Q}$ such that:*

$$1_G = \sum_{H=C\leqslant_G G} \alpha_H \,\mathrm{Ind}_{G/H}(1)$$

in $\mathbb{Q} \otimes_{\mathbb{Z}} G_0(k[G])$ where the sum runs over conjugacy classes of cyclic subgroups of order coprime to p if k has characteristic $p > 0$ or all cyclic subgroups otherwise. Furthermore we have an equality:

$$\mathbb{Q} \otimes_{\mathbb{Z}} G_0(k[G]) = \sum_{H=C\leqslant_G G} \mathrm{Ind}_{G/H}(\mathbb{Q} \otimes_{\mathbb{Z}} G_0(k[H])),$$

where we sum over the same set as previously.

Proof. If G is an element of the set we sum over then the statement is a tautology, so we assume only proper subgroups of G appear on this list. Since the image of induction is an ideal the second statement is an immediate consequence of the first. To prove the first statement consider the natural map $\mathbb{Q} \otimes_{\mathbb{Z}} b(G) \to \mathbb{Q} \otimes_{\mathbb{Z}} G_0(k[G])$, given by $[H] \mapsto \mathrm{Ind}_{G/H}(1)$. Two elements of the image are isomorphic if and only if the species of $G_0(k[G])$ identified in 2.2.28 vanish on them. The image is spanned by $\mathrm{Ind}_{G/H}(1)$ and we have $t_g(\mathrm{Ind}_{G/H}(1)) = t_g(\mathrm{Res}_{G/\langle g \rangle}(\mathrm{Ind}_{G/H}(1))) = f_{\langle g \rangle}(H)$. It follows that a $\mathbb{Q}$-basis for the image is $\mathrm{Ind}_{G/\langle g \rangle}(1)$ where $\langle g \rangle$ ranges over representatives of conjugacy classes of cyclic subgroups of order coprime to the characteristic. Since by assumption G is not of this form the claim follows by writing 1_G in this basis.

□

Remark 2.3.2. Note that if k has characteristic 0 then $G_0(k[G]) = a(k[G])$.

Remark 2.3.3. Since $f_H(G) = 1$ for all $H \leqslant G$ and $f_C(H) = [N_G(H) : H]$ we see the denominators of α_H divide $[N_G(H) : H]$.

We now state Solomon's induction theorem.

Theorem 2.3.4. *[5, Proposition 5.6.3]. Let G be a finite group and k be a field of characteristic 0, then there exist $\beta_H \in \mathbb{Z}_{(q)}$ such that:*

$$1_G = \sum_{H \leqslant_G G} \beta_H \,\mathrm{Ind}_{G/H}(1)$$

in $\mathbb{Z}_{(q)} \otimes_{\mathbb{Z}} a(k[G])$ where the sum runs over conjugacy classes of q-quasi-elementary groups. Furthermore we have an equality:

$$\mathbb{Z}_{(q)} \otimes_{\mathbb{Z}} a(k[G]) = \sum_{H \leqslant G} \mathrm{Ind}_{G/H}(\mathbb{Z}_{(q)} \otimes_{\mathbb{Z}} a(k[H])),$$

where the indexing set is as above.

Proof. See [5, Proposition 5.6.3.] an the discussion preceding it. In particular note that the image of the natural map from the Burnside ring to the representation ring is contained in the subring generated by permutation modules.

□

Applying the Chinese remainder theorem we have the usual phrasing of Solomon's theorem.

Corollary 2.3.5 (Solomon's Induction Theorem)**.** *[22, Theorem 8.10]. Let G be a finite group and k a field of characteristic 0 then we have there exist $\gamma_H \in \mathbb{Z}$ such that:*

$$1_G = \sum_{H \leqslant_G G} \gamma_H \operatorname{Ind}_{G/H}(1)$$

in $a(k[G])$ where the sum runs over conjugacy classes of subgroups of G q-quasi-elementary subgroups for at least one prime q. We have the corresponding equality:

$$a(k[G]) = \sum_{H \leqslant_G G} \operatorname{Ind}_{G/H}(a(k[H])).$$

Remark 2.3.6. The proof in [5, section 5.6] uses a classification of idempotents in the Burnside ring which we have not discussed here. There are several other proofs for instance in [22] of this theorem, one may retrieve it from Artin's Induction theorem and the claim, that there exists an isomorphism $q1 = \sum_{H<G} a_H \operatorname{Ind}_{G/H}(1)$ with $a_H \in \mathbb{Z}$ but not with coefficient 1 whenever G is q-quasi-elementary which is shown in [18, Theorem 1].

Theorem 2.3.7. *[18, Theorem 1] Let k be a field of characteristic 0, and let G be a non-cyclic q-quasi-elementary group. Then there exist integers a_H such that:*

$$q1 = \sum_{H<G} a_H \operatorname{Ind}_{G/H}(1).$$

Furthermore if $c1_G = \sum_{H<G} c_H \operatorname{Ind}_{G/H}(1)$ for some integers c_H then $q \mid c$.

When k has characteristic p, the first result is the characteristic p analogue of Artin's induction theorem due to Conlon.

Theorem 2.3.8 (Conlon's Induction Theorem)**.** *[15, Theorem 80.51]. Let G be a finite group and let k be a field of characteristic p then there exist $\alpha_H \in \mathbb{Q}$ such that:*

$$1_G = \sum_{H \leqslant_G G} \alpha_H \operatorname{Ind}_{G/H}(1),$$

in $\mathbb{Q} \otimes_{\mathbb{Z}} a(G, triv) \subseteq \mathbb{Q} \otimes_{\mathbb{Z}} a(k[G])$ where the sum runs over conjugacy classes of p-hypo-elementary subgroups. We have the corresponding equality:

$$\mathbb{Q} \otimes_{\mathbb{Z}} a(k[G]) = \sum_{H \leqslant_G G} \operatorname{Ind}_{G/H}(\mathbb{Q} \otimes_{\mathbb{Z}} a(k[H])),$$

where the sum runs over the same set. This remains an isomorphism upon restriction to the trivial source ring.

Proof. The second two statements are immediate consequences of the first. Consider the natural map $m_k \otimes \mathbb{Q} : \mathbb{Q} \otimes_{\mathbb{Z}} b(G) \to \mathbb{Q} \otimes_{\mathbb{Z}} a(k[G], triv)$ which is the $\mathbb{Q}$-linear extension of $[H] \mapsto \mathrm{Ind}_{G/H}(1)$. The elements of the kernel of this map by lemma 2.2.39, are precisely those elements for which the fixed points under p-hypo-elementary subgroups of G vanish. It follows therefore that if $S = \{H \leqslant_G G | H \text{ is } p\text{-hypo-elementary}\}$ then $\{\mathrm{Ind}_{G/H}(1) | H \in S\}$ gives a $\mathbb{Q}$-basis of the image. $\square$

Remark 2.3.9. By comparing ranks we see that the kernel of $m_k \otimes \mathbb{Q}$ is of rank equal to the number of conjugacy classes of subgroups of G which are not p-hypo-elementary. Since both $b(G)$ and its image in $a(k[G])$ are free abelian groups of finite rank it follows that the $\mathbb{Z}$-rank of the kernel of the map restricted to $b(G)$ is also the number of conjugacy classes of non-p-hypo-elementary subgroups of G.

We now present the characteristic p analogue of Solomon's induction theorem due to Dress.

Theorem 2.3.10 (Dress' Induction Theorem)**.** *[2, Theorem 9.4]. Let G be a finite group and k a field of characteristic p, then there exist $\beta_H \in \mathbb{Z}_{(q)}$ such that:*

$$1_G = \sum_{H \in D_{p,q}(G)} \beta_H \, \mathrm{Ind}_{G/H}(1),$$

where $D_{p,q}(G)$ is a set of conjugacy class representatives of (p,q)-Dress subgroups of G. Furthermore there exist $\gamma_H \in \mathbb{Z}$ such that:

$$1_G = \sum_{H \in D_p(G)} \gamma_H \, \mathrm{Ind}_{G/H}(1),$$

where $D_p(G)$ is a set of conjugacy class representatives of subgroups of G which are (p,q)-Dress for at least one prime q. Finally we have isomorphisms: $\mathbb{Z}_{(q)} \otimes_{\mathbb{Z}} a(k[G]) \cong \sum_{H \in D_{p,q}(G)} \mathrm{Ind}_{G/H}(\mathbb{Z}_{(q)} \otimes_{\mathbb{Z}} a(k[H]))$ and $a(k[G]) \cong \sum_{H \in D_p(G)} \mathrm{Ind}_{G/H}(a(k[H]))$. These isomorphisms remain isomorphism when restricted to trivial source rings.

Proof. The proof of the first statement is given [2, Theorem 9.4]. The subsequent claims are a consequence of the first. $\square$

Remark 2.3.11. In Theorem 2.3.8 one can show that the denominator of α_H divides $[N_G(H) : H]$ This combined with Theorem 3.5.1 gives an alternative proof of Dress' induction theorem.

Remark 2.3.12. The reader may suspect that there is some hidden machinery here which allows us to translate an induction theorem with rational coefficients to one with integral ones by first passing to local coefficients. It follows from proposition 3.3.4 in the next chapter that given a rational induction theorem we can prove an integral induction theorem.

2.4 Some Group Theory

We will have cause to make regular use of Frattini subgroups and Hall l subgroups, in the sequel so for convenience we recall their definitions and properties. We do not provide proofs but a good reference is [23].

Definition 2.4.1. Let G be a finite group, the Frattini subgroup of G is the intersection of the maximal subgroups of G. Recall that a subgroup of G is maximal if it is a proper subgroup which is contained in no other proper subgroup of G.

Lemma 2.4.2. *The Frattini subgroup has the following properties*

- *The Frattini subgroup is a characteristic subgroup of G, that is it is invariant under all automorphisms of G.*
- *The Frattini subgroup of a finite p-group is the minimal subgroup such that the resulting quotient group is elementary abelian, in particular this quotient group is cyclic if and only if the original p-group is cyclic.*
- *The Frattini subgroup is the set of non-generators for G, i.e. elements which may be excluded from any generating set.*

We will need to use the second property, a proof of which can be found in [23, Lemma 4.5]. We now define Hall subgroups which are a generalisation of the Sylow subgroups.

Definition 2.4.3. Let π be an arbitrary set of primes, a π-Hall subgroup H of G is a subgroup whose order is a product of primes in π and such that $[G : H]$ is not divisible by any prime in π.

Remark 2.4.4. When we want π to be the set of all prime divisors of G save for a fixed l we will call this a Hall l' or $(-l)$-Hall subgroup.

Hall subgroups are a vast generalisation of the Sylow subgroups, the downside to this is that we can't always guarantee existence. The following theorem gives us existence in the cases we will need.

Theorem 2.4.5 (Hall's Theorem)**.** *[23, Theorem 3.13]. If G is soluble then for every possible choice of π, π-Hall subgroups exist, and are conjugate in G.*

Finally we state the Schur-Zassenhaus theorem concerning the splitting of group extensions.

Theorem 2.4.6 (Schur-Zassenhaus)**.** *Let G be a finite groups and let $N \lhd G$ be a normal subgroup whose order is coprime to it's index then $G = N \rtimes H$ for some subgroup $H \leqslant G$ such that $H \cong G/N$. Moreover H is unique up to G-conjugacy.*

For a proof see [23, Chapter 3B] specifically Theorems 3.8 and 3.12.

2.5 Brauer Relations

In this section we will define Brauer relations and describe the steps which have already been taken in their classification.

2.5.1 Brauer Relations: Definition and Motivation

Earlier in section 2.3 we showed that we could use the map between the Burnside ring of a finite group and it's representation ring over some field to prove induction theorems. In fact these theorems were really descriptions of an element of the kernel of the canonical map from the Burnside ring to the representation ring. This map takes a representative X of a class in the Burnside ring to the class of the $k[G]$-permutation module $k[X]$ in the representation ring over k. Here we denote by $k[X]$ the k-vector space whose basis is indexed by elements of X on which G acts by permuting the basis elements as it would elements of X.

Definition 2.5.1. A Brauer relation for a finite group G over a ring R is an element of the kernel of the canonical map $b(G) \to a(R[G])$ given by $X \mapsto R[X]$. When R is a field, since this kernel is only dependent on the characteristic p of R, we will call elements of this kernel Brauer relations for G in characteristic p.

Remark 2.5.2. In later chapters, starting with chapter 3 we will find it useful to generalise this definition to include the elements of any kernel of the canonical map $b(-) \to \mathcal{G}(-)$ where we consider both objects as Green functors which we will define in the next chapter.

The primary motivation to study Brauer relations is their applications to various number theoretic problems. Brauer relations have been heavily exploited in the works of [11], [19], [1] and [3] for example.

2.5.2 Brauer Relations over $\mathbb{Q}$

Brauer relations in characteristic 0 have been completely classified in [2]. This paper has been the starting point for formulating for our attempts to classify Brauer relations in positive characteristic and we will recall its main result here.

Theorem 2.5.3. *[2, Theorem A] Let G be a finite group and let k be a field of characteristic 0 then all Brauer relations for G over k are linear combinations of those inflated then induced from subquotients of the following types:*

1. *A dihedral group of order $2^n \geqslant 8$, or*
2. *a Heisenberg group of order p^3 with $p \geqslant 3$, or*
3. *an extension of the form $1 \to S^d \to H \to Q \to 1$ with S simple, Q quasi-elementary and $Q \hookrightarrow \mathrm{Out}(S^d)$ and either:*
   (a) *S^d a minimal normal subgroup of H or,*
   (b) *$H = (C_l \rtimes P_1) \times (C_l \rtimes P_2)$ with cyclic p-groups acting faithfully and $l \neq p$, or*
4. *$H = C \rtimes P$ is a quasi-elementary group $|C| = \prod_{[1,t]} l_i > 1$ with the l_i distinct primes $l_i \neq p$ and the kernel K of the action of P on C is trivial, D_8 or has normal p rank one. Furthermore writing $K_j = \cap_{i\neq j} \ker(P \to \mathrm{Aut}(C_{l_i}))$ either:*
   (a) *$K = \{1\}$ and $t > 1$ and all K_j have the same image in the Frattini quotient of P ,or*
   (b) *$K = C_p$, $P \cong K \times (P/K)$, and all K_j have the same two dimensional image in the Frattini quotient of P, or*
   (c) *$|K| > p$ or P is not a direct product by K and the graph Γ associated to G by [2, Theorem 7.30] is disconnected.*

2.6 Mackey and Green Functors

In this section we will review the common formulation of Mackey and Green functors in the literature, we will revisit this with a slightly more specialised definition, more suited to our goals in the next chapter. We will start with some motivational examples of Mackey and Green functors before giving a formal definition. We will follow [34].

2.6.1 Mackey and Green Functors: Motivating Examples

In this section we describe some examples whose structure we will later axiomatise as Mackey functors. Our first examples will be familiar from earlier in the Background and will be our main object of study.

Example 2.6.1. Let G be a finite group. The following are our motivating examples of Mackey functors. The important data we wish to convey is that each subgroup has a module attached to it with induction and restriction maps between them.

1. We may assign to every subgroup $H \leqslant G$ its Burnside ring $b(H)$, and for all $K \leqslant H \leqslant G$ we have maps $\mathrm{Ind}_{H/K} : b(K) \to b(H)$ and $\mathrm{Res}_{H/K} : b(H) \to b(K)$ between them.

2. We can assign to each $H \leqslant G$ its representation ring $a(R[G])$ for some ring R. Between each of these rings we have the usual induction and restriction maps.

Other more interesting examples, with wide applications to number theory and elsewhere include.

Example 2.6.2. Let G be a finite group.

1. Let R be a ring and M be a $R[G]$-module then to each subgroup $H \leqslant G$ we can associate the R-module $M^H = \{m \in M : hm = m \forall h \in h\}$. For any $K \leqslant H \leqslant G$ we have an inclusion $\mathrm{Res}_{H/K} : M^H \hookrightarrow M^K$. We also have a map $\mathrm{Ind}_{H/K} : M^K \to M^H$ which is given by $\mathrm{Ind}_{H/K}(m) = \sum_{g \in H/K} gm$.

2. Let K/F be a Galois extension of number fields and let E/F be an elliptic curve defined over F. Then to each subgroup $H \leqslant G$ we may associate the abelian group $E(K^H)$. Here the induction map is the trace map and the restriction map is just inclusion.

3. We have an identical situation whenever we have a Galois extension K/F and an Abelian variety A defined over the base field.

4. Let M/M_0 be a finite Galois covering of Riemannian manifolds with covering group G and let $H_i(M(H))$ be the ith homology of the intermediate manifold associated to $H \leqslant G$. For $K \leqslant H \leqslant G$ we have maps $\mathrm{Ind}_{H/K} : H_i(M(K)) \to H_i(M(H))$ given by corestriction of homology and $\mathrm{Res}_{H/K} : H_i(M(H)) \to H_i(M(K))$ given by restriction of homology.

All of the previous examples in example 2.6.2 are special cases of a cohomological Mackey functor and we will see later in Yoshida's thereom that Brauer relations give non-obvious isomorphisms for these functors.

2.6.2 Mackey and Green Functors: Formal Definitions and Basic Results

We begin by giving a definition of a (local) Mackey functor for a finite group G over a ring R. There are many equivalent definitions of Mackey functors, we shall follow [34] a detailed treatment can also be found in [10].

Definition 2.6.3. A Mackey functor for a finite group G over a ring R is a function $\mathcal{M} : \{H \leqslant G\} \to R$**-mod** with R-module morphisms:

1. For all $H \leqslant K \leqslant G$ we have induction $\mathrm{Ind}_{K/H} : \mathcal{M}(H) \to \mathcal{M}(K)$ and restriction $\mathrm{Res}_{K/H} : \mathcal{M}(K) \to \mathcal{M}(H)$.
2. For all $g \in G$ and $H \leqslant G$ we have $c_g : \mathcal{M}(H) \to \mathcal{M}(H^g)$ which are the identity if $h \in H$.
3. We have the following relations:
   (a) Transitivity of restriction and induction; for all $H \leqslant K \leqslant L \leqslant G$ we have $\mathrm{Res}_{K/H}(\mathrm{Res}_{L/K}) = \mathrm{Res}_{L/H}$ and $\mathrm{Ind}_{H/K}(\mathrm{Ind}_{K/L}) = \mathrm{Ind}_{H/L}$.
   (b) Decomposition of conjugation; $c_{gh} = c_g c_h$.
   (c) Commutativity of conjugation; for all $H \leqslant K \leqslant G$ and for all $g \in G$ $\mathrm{Res}_{K^g/H^g} c_g = c_g \mathrm{Res}_{K/H}$, $\mathrm{Ind}_{K^g/H^g} c_g = c_g \mathrm{Ind}_{K/H}$.
   (d) The Mackey axiom; for all $H, K \leqslant L \leqslant G$ we have $\mathrm{Res}_{L/H} \mathrm{Ind}_{L/K} = \sum_{g \in K \backslash L/H} \mathrm{Ind}_{H/H \cap K^g} \mathrm{Res}_{K^g/H \cap K^g} c_g$.

Remark 2.6.4. If a Mackey functor $\mathcal{M}$ satisfies $\mathrm{Ind}_{K/H} \mathrm{Res}_{K/H} = [K : H]$ for all $H \leqslant K \leqslant G$, then we call $\mathcal{M}$ cohomological.

Now we define a Green functor, which is essentially a Mackey functor taking values in R-algebras.

Definition 2.6.5. A Green functor $\mathcal{G}$ for a finite group G over an algebra A is a Mackey functor over A for G satisfying:

1. The morphisms $\mathrm{Res}_{K/H}$ and c_g are unitary A-algebra homomorphisms for all $H \leqslant K \leqslant G$ and $g \in G$.
2. The Frobenius axiom. The image of $\mathrm{Ind}_{K/H}$ is a two sided ideal of $\mathcal{G}(K)$ for all $H \leqslant K \leqslant G$. Moreover we have $a \,\mathrm{Ind}_{K/H}(b) = \mathrm{Ind}_{K/H}(\mathrm{Res}_{K/H}(a) b)$ and $\mathrm{Ind}_{K/H}(b) a = \mathrm{Ind}_{K/H}(b \,\mathrm{Res}_{K/H}(a))$ for all $H \leqslant K \leqslant G$ and $a \in \mathcal{G}(K)$ and $b \in \mathcal{G}(H)$.

Remark 2.6.6. Mackey and Green functors generalise properties we are already familiar with from representation rings. From this point of view these are simply an precise abstraction of the statement X 'looks' representation theoretic. Similarly a functor being cohomological can be taken to mean that it 'looks like' cohomology.

We now give an important example of a Green functor:

Example 2.6.7. The Burnside functor $b(-)$ which takes a finite group to its Burnside ring is a Green functor with the usual induction and restriction of G-sets. This functor takes values in $\mathbb{Z}$-algebras but after extending scalars we retrieve an A-algebra valued functor $b_A(-)$.

Definition 2.6.8. A morphism of Mackey (resp. Green) functors is a natural transformation, that is R-module (resp. A-algebra homomorphisms) for each $H \leqslant G$ which commute with the morphisms in the definition.We may form categories $\mathrm{Mack}_R(G)$ (resp $\mathrm{Green}_A(G)$)whose objects are R Mackey functors for a finite group G (resp. Green functors over R and A for G). Furthermore cohomological Mackey functors over R for G denoted by $\mathrm{Mack}_{R,\mathrm{coh}}(G)$ form a full subcategory of $\mathrm{Mack}_R(G)$.

Lemma 2.6.9. *The Burnside functor* b_A *is the initial object in* $\mathrm{Green}_A(G)$*.*

Proof. For each $H \leqslant G$ and each $\mathcal{G} \in \mathrm{Green}_A(G)$ there is an A algebra homomorphism $m_{\mathcal{G}}(H) : b(H) \to \mathcal{G}(H)$ defined on transitive H-sets by $[H/K] \mapsto \mathrm{Ind}_{H/K}(1_{\mathcal{G}(K)})$. One readily sees that the $m_{\mathcal{G}}(H)$ are compatible with induction, restriction and conjugation. It is also clear that this morphism is unique, suppose not and we have some other $\eta_H : b(H) \to \mathcal{G}(H)$ then since in $b(H)$ we have $H/K = \mathrm{Ind}_{H/K} \mathrm{Res}_{H/K}(H/H)$, it follows that $\eta_H(H/K) = \mathrm{Ind}_{H/K} \mathrm{Res}_{H/K}(1_{\mathcal{G}(K)})$ which uniquely determines η. $\square$

Definition 2.6.10. A (left) module over a Green functor $\mathcal{G}$ is a Mackey functor M such that for all $H \leqslant G$, $\mathcal{M}(H)$ is a $\mathcal{G}(H)$-module and that this multiplication is compatible with induction, restriction and conjugation in the obvious way. If $\mathcal{M}$ is a Green functor then we call $\mathcal{M}$ an algebra over a Green functor $\mathcal{G}$ if there is a ring homomorphism $\mathcal{G}(H) \to Z(\mathcal{M}(H))$ for each $H \leqslant G$ compatible with induction, restriction and conjugation.

Remark 2.6.11. This makes every Green functor for G over A into an algebra over b_A. In particular, one may show that;

$$\mathcal{H}(\mathcal{M},\mathcal{M})(H) := \mathrm{Hom}_{\mathrm{Mack}_R(H)}(\mathcal{M}(\mathrm{Ind}_{G/H}), \mathcal{M}(\mathrm{Ind}_{G/H})),$$

is a Green functor for any Mackey functor $\mathcal{M}$. Thus any $\mathcal{M}$ is a b-module (see for instance [10, Section 3.4.3]). This means that idempotents of the Burnside ring play an important role in the theory of Mackey functors.

2.6.3 The Mackey and Yoshida Algebras

An alternative perspective on Mackey functors is to view them as modules over the Mackey algebra which we define below. Again our main reference for this section is [34].

Definition 2.6.12. Let R be a commutative unital ring and let G be a finite group then we define the Mackey algebra $\mu_R(G)$ to be the quotient of the free algebra on symbols $\mathrm{Res}_{H/K}$, $\mathrm{Ind}_{H/K}$ and $c_{g,H}$ for all subgroups $K \leqslant H \leqslant G$ and elements $g \in G$ by the relations:

1. $\mathrm{Ind}_{H/H} = \mathrm{Res}_{H/H} = c_{h,H}$ for all subgroups $H \leqslant G$ and elements $h \in H$.
2. Transitivity of restriction and induction; for all $H \leqslant K \leqslant L \leqslant G$ we have $\mathrm{Res}_{K/H}(\mathrm{Res}_{L/K}) = \mathrm{Res}_{L/H}$ and $\mathrm{Ind}_{H/K}(\mathrm{Ind}_{K/L}) = \mathrm{Ind}_{H/L}$.
3. Decomposition of conjugation; $c_{gh,K} = c_{g,K^h} c_{h,K}$ for all $g, h \in G$ and $K \leqslant G$.
4. Commutativity of conjugation; for all $H \leqslant K \leqslant G$ and for all $g \in G$ $\mathrm{Res}_{K^g/H^g}\, c_{g,H} = c_{g,K}\, \mathrm{Res}_{K/H}$, $\mathrm{Ind}_{K^g/H^g}\, c_{g,H} = c_{g,K}\, \mathrm{Ind}_{K/H}$.
5. The Mackey axiom; for all $H, K \leqslant L \leqslant G$ we have: $\mathrm{Res}_{L/H}\,\mathrm{Ind}_{L/K} = \sum_{g \in K\backslash L/H} \mathrm{Ind}_{H/H\cap K^g}\, \mathrm{Res}_{K^g/H\cap K^g}\, c_{g,K}$.
6. $\sum_{H \leqslant G} \mathrm{Ind}_{H/H} = 1$.
7. all other products are zero.

We may identify a Mackey functor M for G over R with the $\mu_R(G)$-module $\sum_{H \leqslant G} M(H)$ with the obvious action of $\mu_R(G)$. From this perspective one can develop the representation theory of the Mackey functors and we have analogues of all of the major concepts and results described in the previous background chapter. Of particular interest to us will be the Yoshida algebra, and Yoshida's theorem as this result gives the most immediate source of applications for Brauer relations.

Definition 2.6.13. Let R be a commutative unital ring and let G be a finite group. The Yoshida algebra $\gamma_R(G)$ is defined analogously to the Mackey algebra except we impose the additional relation $\mathrm{Ind}_{H/K}\,\mathrm{Res}_{H/K} = [H : K]\,\mathrm{Ind}_{H/H}$.

The modules over the Yoshida algebra are precisely the cohomological Mackey functors. The following theorem due to Yoshida is key to our applications.

Theorem 2.6.14 (Yoshida's Theorem)**.** *[36, Theorem 4.3]. The Yoshida Algebra $\gamma_R(G)$ is isomorphic to the Hecke algebra $\mathcal{E} = \mathrm{End}_{R[G]}(\sum_{H \leqslant G} \mathrm{Ind}_{G/H}(1))$.*

Corollary 2.6.15. *[6, Theorem 3.1]. Let $\mathcal{M}$ be a cohomological Mackey functor over R for G. An isomorphism of permutation modules $\sum_{i \in I} R[G/H_i]^{n_i} \cong \sum_{j \in J} R[G/K_j]^{m_j}$ induces an isomorphism $\sum_{i \in I} \mathcal{M}(H_i)^{n_i} \cong \sum_{j \in J} \mathcal{M}(K_j)^{m_j}$.*

Chapter 3

Mackey and Green Functors with Inflation

3.1 Introduction

This chapter is a modified version of my paper, joint with Alex Bartel [4]. The later chapters will depend heavily on the techniques developed in this chapter. The purpose of this chapter therefore, is to develop these tools in full generality, but we keep in mind our ultimate goal. This is, informally, to describe the kernel of the map from the Burnside ring to a representation ring or similar object. To do this we define global Green functors with inflation maps, which are an analogue of the local Green functors described in the background but defined on all groups and with additional maps. Once we set up this structure we shall develop techniques to describe the kernel of maps between two such functors. We will do this in as general a setting as we can to get the desired results, but the reader should always think of our motivation and view the maps as maps from the Burnside functor to the representation ring functor.

Let k be a field of positive characteristic. For every finite group G there is a natural homomorphism $b(G) \to a(k[G])$, which sends the isomorphism class represented by a G-set X to the isomorphism class of the $k[G]$-module $k[X]$ with a canonical k-basis given by the elements of X, and with G acting by permutations on this basis. Let $\mathrm{K}_k(G)$ denote the kernel of this homomorphism. It is easy to see, for instance using the theory of species in Chapter 2, that $\mathrm{K}_k(G)$, as a subgroup of $b(G)$, only depends on the characteristic p of k, and we refer to elements of $\mathrm{K}_k(G)$ as *Brauer relations of G in characteristic p.*

We would like to determine the structure of $\mathrm{K}_k(G)$, not just as an abstract group, but with an explicit description of generators. In order to arrive at such a description we will find it useful to view $\mathrm{K}_k(G)$ as a Mackey functor with inflation. We will give an informal definition here, and refer to Section 3.2 for the formal discussion.

If H is a subgroup of a finite group G, then Brauer relations of H can be induced to Brauer relations of G. Moreover, if $\bar{G}$ is a quotient of a finite group G, then Brauer relations of $\bar{G}$ can be lifted to Brauer relations of G. Let $\mathrm{Imprim}_{\mathrm{K}_k}(G)$ be the subgroup of $\mathrm{K}_k(G)$ generated by all relations that are induced from proper subgroups or lifted from proper quotients, and let $\mathrm{Prim}_{\mathrm{K}_k}(G)$ be the quotient $\mathrm{K}_k(G)/\mathrm{Imprim}_{\mathrm{K}_k}(G)$. If we can give, for every finite group G, generators of $\mathrm{Prim}_{\mathrm{K}_k}(G)$, then we obtain a list of Brauer relations with the property that all Brauer relations in all finite groups are $\mathbb{Z}$-linear combinations of inductions and lifts of relations in this list.

In [2] the structure of $\mathrm{Prim}_{\mathrm{K}_k}(G)$ has been completely determined, in the above sense, in the case when k has characteristic 0. In the process of classifying this kernel Theorem 3.1.1 was a crucial step and is a special case of the result we prove in this chapter.

Theorem 3.1.1. *[2, Theorem 4.3] Let G be a finite group that is not quasi-elementary. Then:*

(a) *if all proper quotients of G are cyclic, then* $\mathrm{Prim}_{\mathrm{K}_\mathbb{Q}}(G) \cong \mathbb{Z}$*;*

(b) *if q is a prime number such that all proper quotients of G are q-quasi-elementary, and at least one of them is not cyclic, then* $\mathrm{Prim}_{\mathrm{K}_\mathbb{Q}}(G) \cong \mathbb{Z}/q\mathbb{Z}$*;*

(c) *if there exists a proper quotient of G that is not quasi-elementary, or if there exist distinct prime numbers q_1 and q_2 and, for $i = 1$ and 2, a proper quotient of G that is non-cyclic q_i-quasi-elementary, then* $\mathrm{Prim}_{\mathrm{K}_\mathbb{Q}}(G)$ *is trivial.*

Moreover, in all cases, $\mathrm{Prim}_{\mathrm{K}_\mathbb{Q}}(G)$ *is generated by any element of* $\mathrm{K}_\mathbb{Q}(G) \subseteq b(G)$ *of the form* $[G/G] + \sum_{H \lneq G} a_H[G/H]$, $a_H \in \mathbb{Z}$.

The main motivation for this chapter is to understand $\mathrm{Prim}_{\mathrm{K}_k}(G)$ when k has positive characteristic. To that end, we prove the following characteristic p analogue of Theorem 3.1.1, which will be used in the next chapter to give a characterisation of $\mathrm{Prim}_{\mathbb{F}_p}(G)$.

Theorem 3.1.2. *Let G be a finite group that is not a (p,q)-Dress group for any prime number q. Then:*

(a) *if all proper quotients of G are p-hypo-elementary, then* $\mathrm{Prim}_{\mathrm{K}_{\mathbb{F}_p}}(G) \cong \mathbb{Z}$*;*

(b) *if q is a prime number such that all proper quotients of G are (p,q)-Dress groups, and at least one of them is not p-hypo-elementary, then* $\mathrm{Prim}_{\mathrm{K}_{\mathbb{F}_p}}(G) \cong \mathbb{Z}/q\mathbb{Z}$*;*

(c) *if there exists a proper quotient of G that is not a (p,q)-Dress group for any prime number q, or if there exist distinct prime numbers q_1 and q_2 and, for $i = 1$ and 2, a proper quotient of G that is a non-p-hypo-elementary (p,q_i)-Dress group, then* $\mathrm{Prim}_{\mathrm{K}_{\mathbb{F}_p}}(G)$ *is trivial.*

Moreover, in all cases, $\mathrm{Prim}_{\mathrm{K}_{\mathbb{F}_p}}(G)$ *is generated by any element of* $\mathrm{K}_{\mathbb{F}_p}(G) \subseteq b(G)$ *of the form* $[G/G] + \sum_{H \lneq G} a_H[G/H]$*,* $a_H \in \mathbb{Z}$*.*

To prove parts (b) and (c) of Theorem 3.1.2, we prove an induction theorem for (p,q)-Dress groups, which we believe to be of independent interest. It is a characteristic p analogue of Theorem 2.3.7.

Theorem 3.1.3. *Let p and q be prime numbers, let G be a (p,q)-Dress group that is not p-hypo-elementary, and let a be an integer. Then there exists an element in* $\mathrm{K}_{\mathbb{F}_p}(G)$ *of the form* $a[G/G] + \sum_{H \lneq G} a_H[G/H]$*,* $a_H \in \mathbb{Z}$ *if and only if* $q|a$*.*

In fact, we deduce Theorems 3.1.1 and 3.1.2, as special cases of a general result on kernels of morphisms between Green functors with inflation. This formalism, which is a mix of axiomatisations that have appeared in the literature many times before, see e.g. [34] , [8] and chapter 2 of this thesis, will be introduced in Section 3.2. In Section 3.3 we recall the concept of primordial groups for a Mackey functor. Our main theorems on kernels of morphisms of Green functors will be proven in Section 3.4. Section 3.5 is devoted to concrete applications, and it is there that we prove Theorems 3.1.1, 3.1.2, and 3.1.3.

Our rings are always assumed to be associative, with a unit element. In particular this means all homomorphisms of rings are unital, and so the 0-map is not a homomorphism. Let R be a commutative ring. By an R-algebra we mean a ring A equipped with a map $R \to \mathrm{Z}(A)$, where $\mathrm{Z}(A)$ denotes the centre of A. If $\mathfrak{p}$ is a prime ideal of R, then $R_{\mathfrak{p}}$ denotes the localisation of R at $\mathfrak{p}$. In this chapter, R will always denote a domain.

3.2 Mackey and Green Functors with Inflation

As noted in the background section there are several different formulations of Mackey and Green functors in the literature. For our purposes it is much more helpful to consider global Mackey functors rather than the more commonly used local Mackey functors defined previously. We will use an axiomatisation that is very similar to those of [8, 34] except that we will introduce an additional morphism, inflation.

Definition 3.2.1. A *global Mackey functor with inflation* (MFI) over R is a collection $\mathcal{F}$ of the following data.

- For every finite group G, $\mathcal{F}(G)$ is an R-module;
- for every injection $\alpha\colon H \hookrightarrow G$ of finite groups, $\mathcal{F}_*(\alpha)\colon \mathcal{F}(H) \to \mathcal{F}(G)$ is a covariant R-module homomorphism (which we think of as induction and will write as $\mathrm{Ind}_{G/H}$);
- for every homomorphism $\epsilon\colon H \to G$ of finite groups, $\mathcal{F}^*(\epsilon)\colon \mathcal{F}(G) \to \mathcal{F}(H)$ is a contravariant R-module homomorphism (which we think of as restriction when ϵ is a injection which we will denote by $\mathrm{Res}_{G/H}$, and as inflation when ϵ is an surjection which we will write as $\mathrm{Inf}_{G/N}$);

with the following structure.

(MFI 1) Transitivity of induction: for all group injections $U \overset{\beta}{\hookrightarrow} H \overset{\alpha}{\hookrightarrow} G$, we have $\mathcal{F}_*(\alpha\beta) = \mathcal{F}_*(\alpha)\mathcal{F}_*(\beta)$.

(MFI 2) Transitivity of restriction/inflation: for all group homomorphisms $U \overset{\beta}{\to} H \overset{\alpha}{\to} G$, we have $\mathcal{F}^*(\alpha\beta) = \mathcal{F}^*(\beta)\mathcal{F}^*(\alpha)$.

(MFI 3) For all inner automorphisms $\alpha\colon G \to G$, we have $\mathcal{F}^*(\alpha) = \mathcal{F}_*(\alpha) = 1$.

(MFI 4) For all automorphisms α, we have $\mathcal{F}_*(\alpha) = \mathcal{F}^*(\alpha^{-1})$.

(MFI 5) The Mackey condition: for all pairs of injections $\alpha\colon H \hookrightarrow G$ and $\beta\colon K \hookrightarrow G$, we have

$$\mathcal{F}^*(\beta)\mathcal{F}_*(\alpha) = \sum_{g \in \alpha(H)\backslash G/\beta(K)} \mathcal{F}_*(\phi_g)\mathcal{F}^*(\psi_g),$$

where ϕ_g is the composition

$$\phi_g\colon \beta(K)^g \cap \alpha(H) \overset{c_g}{\to} \beta(K) \cap {}^g\alpha(H) \hookrightarrow \beta(K) \overset{\beta^{-1}}{\to} K,$$

c_g denoting conjugation by g, and ψ_g is the composition

$$\psi_g\colon \alpha(H) \cap \beta(K)^g \hookrightarrow \alpha(H) \overset{\alpha^{-1}}{\to} H.$$

(MFI 6) Commutativity of induction and inflation: whenever there is a commutative diagram

$$\begin{tikzcd} H \arrow[r, "\alpha"] \arrow[d, "\epsilon"'] & G \arrow[d, "\delta"] \\ \bar{H} \arrow[r, "\beta"] & \bar{G}, \end{tikzcd}$$

where ϵ, δ are surjections, and α, β are injections, we have $\mathcal{F}^*(\delta)\mathcal{F}_*(\beta) = \mathcal{F}_*(\alpha)\mathcal{F}^*(\epsilon)$.

We will often use the following more intuitive notation: if $\mathcal{F}$ is an MFI, and $\alpha\colon H \hookrightarrow G$ is a injection, we will write $\mathrm{Res}_{G/H}$ for $\mathcal{F}^*(\alpha)$, and $\mathrm{Ind}_{G/H}$ for $\mathcal{F}_*(\alpha)$. The suppressed dependence on α and $\mathcal{F}$ will not cause any confusion. Similarly, if $\epsilon\colon G \to \bar{G}$ is a surjection with kernel N, we will write $\mathrm{Inf}_{G/N}$ for $\mathcal{F}^*(\epsilon)$.

Definition 3.2.2. A *Green functor with inflation* (GFI) over R is an MFI $\mathcal{F}$ over R, satisfying the following additional conditions.

(GFI 1) For every finite group G, $\mathcal{F}(G)$ is an R-algebra.

(GFI 2) For every homomorphism $\alpha\colon H \to G$ of finite groups, $\mathcal{F}^*(\alpha)$ is a homomorphism of R-algebras.

(GFI 3) Frobenius reciprocity: for every injection $\alpha\colon H \hookrightarrow G$ and for all $x \in \mathcal{F}(H)$, $y \in \mathcal{F}(G)$, we have

$$\begin{aligned} \mathrm{Ind}_{G/H}(x) \cdot y &= \mathrm{Ind}_{G/H}(x \cdot \mathrm{Res}_{G/H}(y)), \\ y \cdot \mathrm{Ind}_{G/H}(x) &= \mathrm{Ind}_{G/H}(\mathrm{Res}_{G/H}(y) \cdot x). \end{aligned}$$

Many of the examples of Green functors we have already seen in Chapter 2, are in fact GFIs.

Example 3.2.3. The following are examples of GFIs over $\mathbb{Z}$.

(a) The Burnside ring functor b: recall that for a finite group G, $b(G)$ is the free abelian group on isomorphism classes $[X]$ of finite G-sets, modulo the relations $[X \sqcup Y] - [X] - [Y]$ for all G-sets X, Y, and with multiplication defined by $[X] \cdot [Y] = [X \times Y]$. Here, b_* is the usual induction of G-sets, and b^* is inflation/restriction of G-sets.

(b) The representation ring functor $a(F[-])$ over a given field F: for a finite group G, recall that $a(F[G])$ is the free abelian group on isomorphism classes $[V]$ of finitely generated $F[G]$-modules, modulo the relations $[U \oplus V] - [V] - [U]$, and with multiplication defined by $[U] \cdot [V] = [U \otimes_F V]$, with diagonal G-action on the tensor product. As in the previous example, $(a(F[-]))_*$ is induction of modules, and $(a(F[-]))^*$ is inflation/restriction.

(c) The Grothendieck ring functor $G_0(F[-])$ over a field F. For a field F and finite group G, $G_0(F[G])$ is the free abelian group on isomorphism classes of finitely generated $F[G]$-modules modulo relations $[L]-[M]-[N]$ for every short exact sequence $[M] \to [L] \to [N]$ and with multiplication given by $[U]\cdot[V] = [U \otimes_F V]$. Induction, restriction and inflation are defined as in the previous example.

(d) The monomial ring functor M: for a finite group G, $\mathrm{M}(G)$ is the free abelian group on conjugacy classes of symbols $[H, \lambda]$, where H runs over subgroups of G, and λ runs over complex 1-dimensional representations of H, and with multiplication defined by

$$[H,\lambda] \cdot [K,\chi] = \\ \textstyle\sum_{g \in H \backslash G/K} [H^g \cap K, \mathrm{Res}_{H^g/(H^g \cap K)} \lambda^g \cdot \mathrm{Res}_{K/(H^g \cap K)} \chi].$$

If $\alpha : U \hookrightarrow G$ is a injection, $[H, \lambda] \in \mathrm{M}(U)$, and $[K, \chi] \in \mathrm{M}(G)$, then

$$\begin{aligned} &\mathrm{M}_*(\alpha)([H,\lambda]) = [\alpha(H), \lambda \circ \alpha^{-1}], \\ &\mathrm{M}^*(\alpha)([K,\chi]) = \\ &\sum_{g \in \alpha(U) \backslash G/K} [\alpha^{-1}(\alpha(U) \cap K^g), \mathrm{Res}_{K^g/(\alpha(U) \cap K^g)} \chi^g \circ \alpha]. \end{aligned}$$

We will now define morphisms, kernels, images and quotients of these functors and in Lemma 3.2.8 we show that these satisfy the expected properties.

Definition 3.2.4. A *morphism* from an MFI (respectively GFI) $\mathcal{F}$ to an MFI (respectively GFI) $\mathcal{G}$ is a collection r of R-module (respectively R-algebra) homomorphisms $r_G : \mathcal{F}(G) \to \mathcal{G}(G)$ for each finite group G, commuting in the obvious way with $\mathcal{F}_*, \mathcal{F}^*, \mathcal{G}_*, \mathcal{G}^*$.

Definition 3.2.5. Let $\mathcal{F}$ be a GFI over R. A (left) *module* under $\mathcal{F}$ is an MFI $\mathcal{M}$ over R, satisfying the following conditions.

(MOD 1) For every group G, $\mathcal{M}(G)$ is an R-linear (left) $\mathcal{F}(G)$-module, i.e. there is a map $\mathcal{F}(G) \times \mathcal{M}(G) \to \mathcal{M}(G)$ factoring through $\mathcal{F}(G) \otimes_R \mathcal{M}(G)$.

(MOD 2) For every homomorphism $\epsilon \colon H \to G$, and for all $x \in \mathcal{F}(G)$, $y \in \mathcal{M}(G)$, we have

$$\mathcal{M}^*(\epsilon)(x \cdot y) = \mathcal{F}^*(\epsilon)(x) \cdot \mathcal{M}^*(\epsilon)(y).$$

(MOD 3) For every injection $\alpha\colon H \hookrightarrow G$ and for all $x \in \mathcal{F}(H)$, $y \in \mathcal{M}(G)$, we have

$$\mathcal{F}_*(\alpha)(x) \cdot y = \mathcal{F}_*(\alpha)(x \cdot \mathcal{M}^*(\alpha)(y)).$$

Every GFI is a module under itself, called the (left) *regular module.* We also have the obvious notions of sub-MFIs, sub-GFIs, and submodules. We will wish to form quotient GFIs and so we will define ideals of GFIs.

Definition 3.2.6. A *left ideal of a GFI* is a sub-MFI that is also a submodule of the left regular module.

Definition 3.2.7. Let $r : \mathcal{F} \to \mathcal{G}$ be a morphism of MFIs over R. Its *kernel* $\mathcal{K}$ is defined as follows: for every finite group G, we define $\mathcal{K}(G) = \ker(r(G)\colon \mathcal{F}(G) \to \mathcal{G}(G))$; for every homomorphism $\epsilon\colon H \to G$ of groups, we define $\mathcal{K}^*(\epsilon) = \mathcal{F}^*(\epsilon)|_{\mathcal{F}(G)}$; and for every injection $\alpha\colon H \to G$ of groups, we define $\mathcal{K}_*(\alpha) = \mathcal{F}_*(\alpha)|_{\mathcal{F}(H)}$. The *image* of a morphism is defined analogously. Let $\mathcal{F}$ be a sub-MFI (respectively an ideal) of the MFI (respectively GFI) $\mathcal{G}$. The quotient $\mathcal{Q} = \mathcal{G}/\mathcal{F}$ is defined as follows: for every finite group G, we define $\mathcal{Q}(G) = \mathcal{G}(G)/\mathcal{F}(G)$; for every homomorphism $\epsilon\colon H \to G$, we define $\mathcal{Q}^*(\epsilon) = \mathcal{G}^*(\epsilon) \pmod{\mathcal{F}(H)}$; and for every injection $\alpha\colon H \to G$, we define $\mathcal{Q}_*(\alpha) = \mathcal{G}_*(\alpha) \pmod{\mathcal{F}(G)}$.

Lemma 3.2.8. (a) *Let $r\colon \mathcal{F} \to \mathcal{G}$ be a morphism of MFIs over R. Then its kernel is a sub-MFI of $\mathcal{F}$, and its image is a sub-MFI of $\mathcal{G}$.*

(b) *Let $r\colon \mathcal{F} \to \mathcal{G}$ be a morphism of GFIs over R. Then its kernel is an ideal of $\mathcal{F}$, and its image is a sub-GFI of $\mathcal{G}$.*

(c) *Let $\mathcal{F}$ be a sub-MFI of an MFI $\mathcal{G}$. Then the quotient $\mathcal{G}/\mathcal{F}$ is an MFI.*

(d) *Let $\mathcal{F}$ be an ideal of a GFI $\mathcal{G}$. Then $\mathcal{G}/\mathcal{F}$ is a GFI.*

Proof. The first two statements are an easy consequence of morphisms of MFIs (resp GFIs) commuting with induction, restriction and inflation maps. The final two statements follow from first two. □

Example 3.2.9. The following are some motivating examples for this work.

(a) There is a GFI morphism $m'_{\mathbb{C}}\colon \mathrm{M} \to a(\mathbb{C}[-])$, sending, for every finite group G, a symbol $[H, \lambda] \in \mathrm{M}(G)$ to $\mathrm{Ind}_{G/H}\,\lambda \in a(\mathbb{C}[G])$. The kernel of $m'_{\mathbb{C}}$ was investigated by, among many others, Langlands [27], Deligne [17], Snaith [30], Boltje [7], and Boltje–Snaith–Symonds [9].

(b) Let F be a field. There is a GFI morphism $m_F\colon b \to a(F[-])$, which maps, for every finite group G, a G-set X to the permutation module $F[X]$ over F. Its kernel K_F is the MFI of *Brauer relations over F*. In [2], an explicit description of generators of this MFI is given in the case when F is a field of characteristic 0. The primary motivation here is to give a similarly explicit description when F is a field of positive characteristic.

3.3 Primordial Groups and Coprimordial Groups

Recall that R denotes a domain. If S is a commutative R-algebra, and $\mathcal{F}$ an MFI (respectively GFI) over R, then $S \otimes_R \mathcal{F}$, defined in the obvious way, is an MFI (respectively GFI) over S. If $R = \mathbb{Z}$, then we will suppress any mention of R, and will just say "$\mathcal{F}$ is a MFI (respectively GFI)". From here onwards, Q will denote the field of fractions of R. For a prime ideal $\mathfrak{p}$ of R, we will write $\mathcal{F}_\mathfrak{p}$ for $R_\mathfrak{p} \otimes_R \mathcal{F}$, and $\mathcal{F}_Q$ for $Q \otimes_R \mathcal{F}$.

Notation 3.3.1. Let $\mathcal{F}$ be an MFI, and let $\mathcal{X}$ be a class of groups closed under isomorphisms. For every finite group G, we define the following R-submodules of $\mathcal{F}(G)$:

$$\begin{aligned}
\mathcal{I}_{\mathcal{F},\mathcal{X}}(G) &= \sum_{H \leq G,\ H \in \mathcal{X}} \operatorname{Ind}_{G/H} \mathcal{F}(H), \\
\mathcal{I}_{\mathcal{F}}(G) &= \sum_{H \lneq G} \operatorname{Ind}_{G/H} \mathcal{F}(H), \\
\mathcal{K}_{\mathcal{F},\mathcal{X}}(G) &= \bigcap_{H \leq G,\ H \in \mathcal{X}} \ker(\operatorname{Res}_{G/H} \mathcal{F}(G)), \\
\mathcal{K}_{\mathcal{F}}(G) &= \bigcap_{H \lneq G} \ker(\operatorname{Res}_{G/H} \mathcal{F}(G)).
\end{aligned}$$

Definition 3.3.2. Let $\mathcal{F}$ be an MFI and let G be a finite group. We say that G is *primordial* for $\mathcal{F}$ if either G is trivial, or $\mathcal{F}(G) \neq \mathcal{I}_{\mathcal{F}}(G)$. We denote the class of all primordial groups for $\mathcal{F}$ by $\mathcal{P}(\mathcal{F})$.

We say that G is *coprimordial* for $\mathcal{F}$ if either G is trivial, or $\mathcal{K}_{\mathcal{F}}(G) \neq 0$. We denote the class of all coprimordial groups for $\mathcal{F}$ by $\mathcal{C}(\mathcal{F})$.

Remark 3.3.3. Let $\mathcal{F}$ be an MFI.

(a) Suppose that $\mathcal{X}$ is a class of finite groups that is closed under isomorphisms and under taking subgroups, with the property that for every finite group G, we have $\mathcal{F}(G) = \mathcal{I}_{\mathcal{F},\mathcal{X}}(G)$. Then it is shown in [33, Theorem 2.1] that $\mathcal{X}$ contains the closure of $\mathcal{P}(\mathcal{F})$ under taking all subgroups.

(b) Suppose that $\mathcal{F}$ is a GFI. Then it follows from axiom (GFI 3) that G is primordial for $\mathcal{F}$ if and only if $1_{\mathcal{F}(G)} \notin \mathcal{I}_{\mathcal{F}}(G)$. It easily follows from this and from axioms (GFI 2) and (MFI 6) that $\mathcal{P}(\mathcal{F})$ is closed under quotients.

We will make use of the following straightforward result, which follows from the work of Yoshida [35] and Boltje [8].

Proposition 3.3.4. *Let $\mathcal{F}$ be a GFI over a Euclidean domain R, and assume that $\mathcal{F}(G)$ is R-torsion free for all finite groups G. Let Q denote the field of fractions of R. Then:*

1. *if Q has characteristic 0, then $\mathcal{P}(\mathcal{F} \otimes Q) = \mathcal{C}(\mathcal{F} \otimes Q)$;*
2. *for any prime ideal $\mathfrak{p}$ of R, we have $\mathcal{C}(\mathcal{F}) = \mathcal{C}(\mathcal{F} \otimes R_{\mathfrak{p}})$, and in particular $\mathcal{C}(\mathcal{F}) = \mathcal{C}(\mathcal{F} \otimes Q)$;*
3. *we have $\mathcal{P}(\mathcal{F}) \subseteq \cup_{\mathfrak{p}} \mathcal{P}(\mathcal{F} \otimes R_{\mathfrak{p}})$;*
4. *if Q has characteristic 0, then for any prime ideal $\mathfrak{p}$ of R, we have $\mathcal{P}(\mathcal{F} \otimes R_{\mathfrak{p}}) \subseteq \{H : O^p(H) \in \mathcal{C}(\mathcal{F})\}$, where $(p) = \mathfrak{p} \cap \mathbb{Z}$.*

Proof. 1. If Q has characteristic 0, then by [8, Proposition 6.2], we have a decomposition of Q-modules $\mathcal{F}(G) \otimes Q = \mathcal{K}_{\mathcal{F}\otimes Q}(G) \oplus \mathcal{I}_{\mathcal{F}\otimes Q}(G)$. The result follows.

2. Since $\mathcal{F}(G)$ is R-torsion free for all finite groups G, $\mathcal{F}(G)$ naturally injects into $\mathcal{F}(G) \otimes R_{\mathfrak{p}}$ and generates $\mathcal{F}(G) \otimes R_{\mathfrak{p}}$ over $R_{\mathfrak{p}}$ and similarly over Q. Moreover, this inclusion is functorial with respect to restriction. It follows that we have a natural isomorphism $\mathcal{K}_{\mathcal{F}\otimes R\mathfrak{p}}(G) = \mathcal{K}_{\mathcal{F}}(G) \otimes R_{\mathfrak{p}}$, and in particular one of these kernels is non-trivial if and only if both are, as claimed.

3. Suppose that $G \notin \cup_{\mathfrak{p}} \mathcal{P}(\mathcal{F} \otimes R_{\mathfrak{p}})$ then in particular $1_{\mathcal{F}\otimes R_{\mathfrak{p}(G)}} \in \mathcal{I}_{\mathcal{F}\otimes R_{\mathfrak{p}}}(G)$ for all $\mathfrak{p}$. Since R is Euclidean it follows that $1_{\mathcal{F}(G)} \in \mathcal{I}_{\mathcal{F}}(G)$ and as $\mathcal{I}_{\mathcal{F}}(G)$ is an ideal in $\mathcal{F}(G)$ they coincide so $G \notin \mathcal{P}(\mathcal{F})$.

4. Let G be a finite group, and let $p \in \mathbb{Z}$ be such that $\mathfrak{p} \mid p$. Let $\mathcal{H}_p(\mathcal{C}(\mathcal{F})) = \{H : O^p(H) \in \mathcal{C}(\mathcal{F})\}$. Let $(\#G)_{p'}$ denote the maximal divisor of $\#G$ which is coprime to p. Since $(\#G)_{p'}$ is invertible in $R_{\mathfrak{p}}$, [35, Theorem 4.1] applied with $\mathscr{X} = \{H \leq G : H \in \mathcal{C}(\mathcal{F})\}$ implies that $\mathcal{F}(G) \otimes R_{\mathfrak{p}} = \mathcal{I}_{\mathcal{F}\otimes R_{\mathfrak{p}},\mathcal{H}_p(\mathcal{C}(\mathcal{F}))}(G) + \mathcal{K}_{\mathcal{F}\otimes R_{\mathfrak{p}},\mathcal{C}(\mathcal{F})}(G)$. By part (2) of the present lemma, we have $\mathcal{C}(\mathcal{F}) = \mathcal{C}(\mathcal{F} \otimes R_{\mathfrak{p}})$. It then follows that $\mathcal{K}_{\mathcal{F}_{\mathfrak{p}},\mathcal{C}(\mathcal{F})}(G) = \mathcal{K}_{\mathcal{F}_{\mathfrak{p}},\mathcal{C}(\mathcal{F}_{\mathfrak{p}})}(G) = 0$ by definition, and therefore that
$\mathcal{F}_{\mathfrak{p}}(G) = \mathcal{I}_{\mathcal{F}_{\mathfrak{p}},\mathcal{H}_p(\mathcal{C}(\mathcal{F}))}(G)$. So $\mathcal{P}(\mathcal{F} \otimes R_{\mathfrak{p}}) \subseteq \mathcal{H}_p(\mathcal{C}(\mathcal{F}))$, as claimed.

□

Example 3.3.5. (a) Every finite group is primordial for the Burnside ring functor b, and also for $b_{\mathbb{Q}}$. Indeed, no non-zero multiple of the identity element of $b(G)$ can be contained in the image of induction from proper subgroups. Similarly, every finite group is primordial for the monomial ring functor M, and also for $\mathrm{M}_{\mathbb{Q}}$.

(b) Recall from Example 3.2.3 (b) the representation ring functor $a(\mathbb{C}[-])$. It follows from Brauer's induction Theorem [5, Theorem 5.6.4] that $\mathcal{P}(a(\mathbb{C}[-]))$ is contained in the class of elementary groups, i.e. of direct products of finite cyclic groups by p-groups. Moreover, it is a theorem of Green [20] that in fact $\mathcal{P}(a(\mathbb{C}[-]))$ consists precisely of the elementary groups.

(c) Recall from Example 3.2.9 (a) the GFI morphism $m'_{\mathbb{C}} \colon \mathrm{M} \to a(\mathbb{C}[-])$ from the monomial ring functor to the complex representation ring functor. It follows from Brauer's induction theorem that $(m'_{\mathbb{C}})_G$ is surjective for every finite group G, so by the previous example, $\mathcal{P}(\operatorname{Im} m'_{\mathbb{C}})$ consists precisely of the elementary groups.

(d) Recall from Example 3.2.9 (b) the GFI morphism $m_{\mathbb{Q}} \colon b \to a(\mathbb{Q}[-])$. Let q be a prime number. Solomon's induction theorem (corollary 2.3.5) implies that $\mathcal{P}(\operatorname{Im}(m_{\mathbb{Q}})_q)$ is contained in the class of q-quasi-elementary groups, i.e. of semidirect products $C \rtimes U$, with C finite cyclic and U a q-group. Moreover, it is a theorem of Dokchitser [18, Theorem 1] that if G is q-quasi-elementary, then the trivial character of G is not in the image of induction of trivial characters from proper subgroups, so $\mathcal{P}(\operatorname{Im}(m_{\mathbb{Q}})_q)$ is precisely the class of all q-quasi-elementary groups.

(e) Let $m_{\mathbb{Q}}$ be as above. It follows from Artin's induction Theorem [5, Theorem 5.6.1] that $\mathcal{P}(\operatorname{Im}(m_{\mathbb{Q}})_{\mathbb{Q}})$ is the class of finite cyclic groups.

(f) Let p be a prime number, and let $m_{\mathbb{F}_p} \colon b \to a(\mathbb{F}_p[-])$ be as in Example 3.2.9 (b). Dress' induction theorem (Theorem 2.3.10) implies that $\mathcal{P}(\operatorname{Im} m_{\mathbb{F}_p})$ is contained in the class of all groups that are (p,q)-Dress groups for some prime number q. We will show in Theorem 3.5.1 that the trivial representation of a (p,q)-Dress group is not in the image of induction of trivial representations from proper subgroups, so in fact, $\mathcal{P}(\operatorname{Im} m_{\mathbb{F}_p})$ is precisely the class of all finite groups that are (p,q)-Dress groups for some prime number q.

3.4 The Primitive Quotient

In this section, we prove our main theorems on kernels of morphisms of GFIs. The main results of the section are Theorem 3.4.6, 3.4.7, and 3.4.9. Recall that throughout R denotes a domain.

Lemma 3.4.1. *Let $m\colon \mathcal{F} \to \mathcal{G}$ be a morphism of GFIs over a ring R with kernel $\mathcal{K}$, and let G be a finite group. Then the following are equivalent:*

(i) *the group G is not primordial for* $\operatorname{Im} m$*;*

(ii) *for each proper subgroup H of G, there exists an element $x_H \in \mathcal{F}(H)$ such that $x = 1_{\mathcal{F}(G)} + \sum_{H \lneq G} \operatorname{Ind}_{G/H}(x_H) \in \mathcal{K}(G)$.*

Proof. By Remark 3.3.3 (b), G is not primordial for $\operatorname{Im} m$ if and only if

$$m_G(1_{\mathcal{F}(G)}) \in \sum_{H \lneq G} \operatorname{Ind}_{G/H}(m_H(\mathcal{F}(H))) = m_G \sum_{H \lneq G} \operatorname{Ind}_{G/H}(\mathcal{F}(H)).$$

This is equivalent to the existence of elements $x_H \in \mathcal{F}(H)$ for $H \lneq G$ such that $x = 1_{\mathcal{F}(G)} + \sum_{H \lneq G} \operatorname{Ind}_{G/H}(x_H) \in \mathcal{K}(G)$. $\square$

Definition 3.4.2. Let G be a finite group, let $\mathcal{F}$ be a GFI over R, and let $\mathcal{M}$ be a module under $\mathcal{F}$. Let $D(G)$ be an R-subalgebra of the centre of $\mathcal{F}(G)$. Define the set of *D-imprimitive elements* of $\mathcal{M}(G)$ by

$$\operatorname{Imprim}_{\mathcal{M},D}(G) = D(G) \cdot \left(\sum_{H \lneq G} \operatorname{Ind}_{G/H} \mathcal{M}(H) + \sum_{1 \neq N \triangleleft G} \operatorname{Inf}_{G/N} \mathcal{M}(G/N) \right).$$

This is an R-submodule of $\mathcal{M}(G)$. Define the *D-primitive quotient* of $\mathcal{M}(G)$ to be the quotient of R-modules

$$\operatorname{Prim}_{\mathcal{M},D}(G) = \mathcal{M}(G)/\operatorname{Imprim}_{\mathcal{M},D}(G).$$

When $D(G)$ is generated by $1_{\mathcal{F}(G)}$ over R, we will drop it from the notation.

Notation 3.4.3. For the rest of the section, we put ourselves in the following situation. We fix a morphism $m : \mathcal{F} \to \mathcal{G}$ of GFIs over a domain R with the property that $\mathcal{F}(H)$ is R-torsion free for all finite groups H, and we let $\mathcal{K}$ denote its kernel. Recall from Lemma 3.2.8 that $\mathcal{K}$ is an ideal of $\mathcal{F}$. Further, we fix a finite group G, and an R-subalgebra $D(G)$ of the centre of $\mathcal{F}(G)$. Assume for the rest of the section that the R-module $\mathcal{F}(G)$ is generated by $\mathcal{I}_{\mathcal{F}}(G)$ and $D(G)$.

Lemma 3.4.4. *Under the hypotheses of Notation 3.4.3, let $\mathcal{M}$ be any module under $\mathcal{F}$, and let x be any element of $\mathcal{M}(G)$. Then the R-submodule of $\mathcal{M}(G)$ generated by $D(G) \cdot \mathcal{I}_{\mathcal{M}}(G)$ and $D(G) \cdot x$ is an $\mathcal{F}(G)$-submodule.*

Proof. Let Θ be an element of the R-module $D(G) \cdot \mathcal{I}_{\mathcal{M}}(G) + D(G) \cdot x$, and let $\alpha \in \mathcal{F}(G)$. If $\alpha = \operatorname{Ind}_{G/H} y$ for some $y \in \mathcal{F}(H)$, where H is a proper subgroup of G, then by property (MOD 3), $\alpha \cdot \Theta = \operatorname{Ind}_{G/H}(y \cdot \operatorname{Res}_{G/H} \Theta) \in \mathcal{I}_{\mathcal{M}}(G)$. If, on the other hand, $\alpha \in D(G)$, then $\alpha \cdot \Theta \in D(G) \cdot \mathcal{I}_{\mathcal{M}}(G) + D(G) \cdot x$ by definition. Since $\mathcal{F}(G)$ is assumed to be generated by $\mathcal{I}_{\mathcal{F}}(G)$ and by $D(G)$, it follows that $\alpha \cdot \Theta \in D(G) \cdot \mathcal{I}_{\mathcal{M}}(G) + D(G) \cdot x$ for all $\alpha \in \mathcal{F}(G)$. ☐

Lemma 3.4.5. *Under the hypotheses of Notation 3.4.3, suppose that the equivalent conditions of Lemma 3.4.1 are satisfied for m and G, and let $x \in \mathcal{K}(G)$ be an element of the form $x = 1_{\mathcal{F}(G)} + \sum_{H \lneq G} \operatorname{Ind}_{G/H}(x_H)$, where $x_H \in \mathcal{F}(H)$. Then*

$$\mathcal{K}(G) = D(G) \cdot \mathcal{I}_{\mathcal{K}}(G) + D(G) \cdot x.$$

Proof. Let $I = D(G) \cdot \mathcal{I}_{\mathcal{K}}(G) + D(G) \cdot x \subseteq \mathcal{K}(G)$. We claim that $\mathcal{K}(G) \subseteq I$. Let $y \in \mathcal{K}(G)$. Lemma 3.4.4 implies that I is an ideal of $\mathcal{F}(G)$. Since we have $x \in D(G) \cdot x \subseteq I$, it follows that $y \cdot x \in I$. Also,

$$y \cdot x - y = \sum_{H \lneq G} y \cdot \operatorname{Ind}_{G/H}(x_H) = \sum_{H \lneq G} \operatorname{Ind}_{G/H}(\operatorname{Res}_{G/H}(y) \cdot x_H)$$

is in $\mathcal{I}_{\mathcal{K}}(G)$, and therefore in I. It follows that $y = y \cdot x + (y - y \cdot x) \in I$. Thus $\mathcal{K}(G) \subseteq I$, and the proof is complete. ☐

Theorem 3.4.6. *Under the hypotheses of Notation 3.4.3, suppose that there is a non-trivial normal subgroup N of G such that G/N is not primordial for* $\operatorname{Im} m$. *Then* $\operatorname{Prim}_{\mathcal{K},D}(G)$ *is trivial.*

Proof. By Lemma 3.4.1, applied to the quotient G/N, there exists an element $z = 1_{\mathcal{F}(G/N)} + \sum_{H/N \lneq G/N} \operatorname{Ind}_{(G/N)/(H/N)}(x_H) \in \mathcal{K}(G/N)$. Since N is non-trivial, the inflation $x = \operatorname{Inf}_{G/N} z$ is contained in $\operatorname{Imprim}_{\mathcal{K},D}(G)$. It follows from Lemma 3.4.5 that $\mathcal{K}(G) = D(G) \cdot \mathcal{I}_{\mathcal{K}}(G) + D(G) \cdot x \subseteq \operatorname{Imprim}_{\mathcal{K},D}(G)$, as claimed. ☐

Theorem 3.4.7. *Under the hypotheses of Notation 3.4.3, suppose that G is non-trivial, and that* $\operatorname{Prim}_{\mathcal{K},D}(G)$ *is non-trivial. Then G is an extension of the form $1 \to S^d \to G \to H \to 1$, where S is a finite simple group, and H is primordial for* $\operatorname{Im} m$.

Proof. By the existence of chief series, there exists a normal subgroup of G that is isomorphic to S^d, where S is a finite simple group, and $d \geq 1$ is an integer. By Theorem 3.4.6, the quotient G/S^d is primordial for $\operatorname{Im} m$. $\square$

Assumption 3.4.8. In addition to the assumptions of Notation 3.4.3, we now assume that:

- the ring R is a Notherian domain;
- for every normal subgroup N of G, the inflation map $\operatorname{Inf}_{G/N} \colon \mathcal{F}(G/N) \to \mathcal{F}(G)$ is injective;
- for every quotient G/N, the R-module $\mathcal{F}(G/N)$ is generated by $\mathcal{I}_{\mathcal{F}}(G/N)$ and 1. In particular, the subalgebra $D(G)$ will be assumed to be generated by $1_{\mathcal{F}(G)}$ over R, and will now be dropped from the notation.

Theorem 3.4.9. *Under the hypotheses of Notation 3.4.3 and Assumption 3.4.8, suppose that G is primordial for $\mathcal{F}_Q$ and not primordial for* $\operatorname{Im} m$. *Let $\mathfrak{a}$ be the ideal of R generated by all those $a \in R$ for which there exists a proper quotient G/N and an element $a1_{\mathcal{F}(G/N)} + y \in \mathcal{K}(G/N)$ with $y \in \mathcal{I}_{\mathcal{F}}(G/N)$. Then* $\operatorname{Prim}_{\mathcal{K}}(G)$ *is isomorphic to $R/\mathfrak{a}$ and is generated by the image of any element of the form $x = 1_{\mathcal{F}(G)} + \sum_{H \lneq G} \operatorname{Ind}_{G/H} x_H \in \mathcal{K}(G)$.*

Proof. By Lemma 3.4.5, the quotient $\operatorname{Prim}_{\mathcal{K}}(G)$ is generated by any $x \in \mathcal{K}(G)$ of the form $x = 1_{\mathcal{F}(G)} + \sum_{H \lneq G} \operatorname{Ind}_{G/H} x_H$, where $x_H \in \mathcal{F}(H)$. Since by assumption G is primordial for $\mathcal{F}_Q$, Remark 3.3.3 (b) implies that $ax \notin \mathcal{I}_{\mathcal{K}}(G)$ for any non-zero $a \in R$. It also follows from the same remark and from the assumptions 3.4.3 and 3.4.8 that any element of $\mathcal{K}(G)$ can be uniquely written as $a1_{\mathcal{F}(G)} + y$, where $a \in R$ and $y \in \mathcal{I}_{\mathcal{F}}(G)$, and analogously for any element of $\mathcal{K}(G/N)$ for every normal subgroup N of G. We deduce that the annihilator $\mathfrak{a} \subseteq R$ of $x + \operatorname{Imprim}_{\mathcal{K}}(G) \in \operatorname{Prim}_{\mathcal{K}}(G)$ is generated, as an R-module, by all those $a \in R$ for which there exists a non-trivial normal subgroup N of G and an element $a1_{\mathcal{F}(G/N)} + y \in \mathcal{K}(G/N)$, where $y \in \mathcal{I}_{\mathcal{F}}(G/N)$. Moreover, we then have $\operatorname{Prim}_{\mathcal{K}}(G) \cong R/\mathfrak{a}$, as claimed. $\square$

Corollary 3.4.10. *Under the hypotheses of Theorem 3.4.9, if all proper quotients of G are primordial for $(\operatorname{Im} m)_Q$, then $\operatorname{Prim}_{\mathcal{K}}(G)$ is isomorphic to R.*

Proof. Since all proper quotients G/N are primordial for $(\operatorname{Im} m)_Q$, Remark 3.3.3 (b) implies that the ideal $\mathfrak{a}$ of Theorem 3.4.9 is zero. $\square$

Corollary 3.4.11. *Under the hypotheses of Theorem 3.4.9, suppose that there exists a prime ideal $\mathfrak{p}$ of R such that for every prime ideal $\mathfrak{q} \neq \mathfrak{p}$ there exists a proper* *quotient of G that is not primordial for $(\operatorname{Im} m)_{\mathfrak{q}}$. Then $\operatorname{Prim}_{\mathcal{K}}(G) \cong R/I$, where I is a $\mathfrak{p}$-primary ideal.*

Proof. Let $\mathfrak{q} \neq \mathfrak{p}$ be a prime ideal of R. By Lemma 3.4.1, applied to the map $\mathcal{F}_{\mathfrak{q}} \to \mathcal{G}_{\mathfrak{q}}$ and to a proper quotient $G/N \notin \mathcal{P}((\operatorname{Im} m)_{\mathfrak{q}})$, there exists $a \in \mathfrak{a}$ that is not in $\mathfrak{q}$, where $\mathfrak{a}$ is the ideal of Theorem 3.4.9. Since R is a Notherean domain, this implies that $\mathfrak{a} = I$ for some primary ideal I. $\square$

Remark 3.4.12. In the case that R is Dedekind all $\mathfrak{p}$-primary ideals are of the form $\mathfrak{p}^n$. In this case corollary 3.4.11 states that $\operatorname{Prim}_{\mathcal{K}}(G) \cong R/\mathfrak{p}^n$ where n is the smallest non-negative integer for which there exists a proper quotient G/N and an element $a1_{\mathcal{F}(G/N)} + y \in \mathcal{K}(G/N)$ with $a \in \mathfrak{p}^n \setminus \{0\}$ and $\operatorname{Inf}_{G/N} y \in \mathcal{I}_{\mathcal{F}}(G)$.

Corollary 3.4.13. *Under the hypotheses of Theorem 3.4.9, suppose that for every non-zero prime ideal $\mathfrak{p}$ of R there exists a proper quotient of G that is not primordial for $(\operatorname{Im} m)_{\mathfrak{p}}$. Then $\operatorname{Prim}_{\mathcal{K}}(G)$ is trivial.*

Proof. Let $\mathfrak{p}$ be any non-zero prime ideal. By Lemma 3.4.1, applied to the map $\mathcal{F}_{\mathfrak{p}} \to \mathcal{G}_{\mathfrak{p}}$ and to a proper quotient $G/N \notin \mathcal{P}((\operatorname{Im} m)_{\mathfrak{p}})$, there exists $a \in \mathfrak{a}$ that is not in $\mathfrak{p}$, where $\mathfrak{a}$ is the ideal of Theorem 3.4.9. Since R is a Notherean domain, either $1 \in \mathfrak{a}$ or $\mathfrak{a}$ is contained in some maximal hence prime ideal, a contradiction. $\square$

3.5 Applications

Throughout this section we fix a prime p and consider the map $m_{\mathbb{F}_p}$ as in example 3.2.9 (b).

Theorem 3.5.1. *Let q be a prime number, let G be a (p,q)-Dress group that is not p-hypo-elementary, and let a be an integer. Then $a[G/G] \in \mathcal{I}_{\operatorname{Im} m_{\mathbb{F}_p}}(G)$ if and only if $q|a$.*

Proof. Since G is a (p,q)-Dress group, it is an extension of a q-group U by a normal p-hypo-elementary subgroup $N = P \rtimes C$, where P is a p-group and C is cyclic of order coprime to pq.

First we prove that if $a[G/G] \in \mathcal{I}_{\operatorname{Im} m_{\mathbb{F}_p}}(G)$, then $q|a$. Suppose that there exist integers a_H for $H \lneq G$ such that

$$a\mathbb{F}_p[G/G] = \sum_{H \lneq G} a_H \mathbb{F}_p[G/H] \in a(\mathbb{F}_p[G]),$$

where the sum runs over representatives of conjugacy classes of subgroups of G, and where $\mathbb{F}_p[G/H] \in a(\mathbb{F}_p[G])$ denotes the linear permutation module $\operatorname{Ind}_{G/H} \mathbf{1}_H$ over

$\mathbb{F}_p$. By restricting to the normal p-hypo-elementary subgroup N, we find that

$$a\mathbb{F}_p[N/N] = \sum_{H \lneq G} a_H \sum_{g \in G/HN} \mathbb{F}_p[N/N \cap gHg^{-1}]. \tag{3.5.2}$$

By Conlon's Induction (Theorem 2.3.8), p-hypo-elementary groups are primordial for $\operatorname{Im} m_{\mathbb{F}_p}$, so the coefficient of $\mathbb{F}_p[N/N]$ on the right hand side of equation 3.5.2 must be equal to a:

$$a = \sum_{N \leq H \lneq G} a_H \cdot \#(G/H).$$

But for every $H \leq G$ that contains N, the quantity $\#(G/H)$ is divisible by q, so a is divisible by q, as claimed.

Now we show that $q[G/G] \in \mathcal{I}_{\operatorname{Im} m_{\mathbb{F}_p}}(G)$. First, we treat a special case: assume that P is the trivial group, so that $G \cong C \rtimes U$ is non-cyclic q-quasi-elementary, where C is cyclic of order coprime to pq. Assume further that either $p \neq q$, or U acts faithfully on C. By Theorem 2.3.7, there exists an element $x = q[G/G] + \sum_{H \lneq G} a_H[G/H] \in \mathrm{K}_{\mathbb{Q}}(G)$. By Artin's Induction Theorem (Theorem 2.3.1), this is equivalent to the statement that there exists an $x \in \mathrm{K}_{\mathbb{Q}}(G)$ as above such that for all cyclic subgroups $H \leq G$, we have $f_H(x) = 0$, where $f_H \colon b(G) \to \mathbb{Z}$ is defined on a G-set X as the number of fixed points $\#X^H$. But under the hypotheses on G, the cyclic subgroups of G are precisely the p-hypo-elementary subgroups of G. By Conlon's Induction Theorem [15, Lemma 81.2], the above statements are therefore equivalent to the existence of an element $x = q[G/G] + \sum_{H \lneq G} a_H[G/H] \in \mathrm{K}_{\mathbb{F}_p}(G)$, as required.

Now, we deduce the general case. Given a non-p-hypo-elementary (p, q)-Dress group G, let $\tilde{G} = G/P$. This is a non-cyclic q-quasi-elementary group, $\tilde{G} = C \rtimes U$, where U is a q-group, and C is cyclic of order coprime to pq. Let K be the kernel of the action of U on C. If $K = U$ and $p = q$, then $\tilde{G} \cong C \times U$, and G is p-hypo-elementary, contradicting the assumptions. Otherwise, $\bar{G} = \tilde{G}/K$ is as in the special case above, so there exists an element $x = q[\bar{G}/\bar{G}] + \sum_{H \lneq \bar{G}} a_H[\bar{G}/H] \in \mathrm{K}_{\mathbb{F}_p}(\bar{G})$. Taking the inflation of x to G yields the desired element of $\mathrm{K}_{\mathbb{F}_p}(G)$, and the proof is complete. □

Corollary 3.5.3. *Let q be a prime number. Then $\mathcal{P}((\operatorname{Im} m_{\mathbb{F}_p})_q)$ is the class of (p, q)-Dress groups.*

Proof. By Dress' Induction Theorem in the version as stated in [2, Theorem 9.4], and by Remark 3.3.3 (a), all primordial groups for $(\operatorname{Im} m_{\mathbb{F}_p})_q$ are (p, q)-Dress groups. The reverse inclusion follows from Theorem 3.5.1. □

Theorem 3.5.4. *Let G be a finite group that is not a (p,q)-Dress group for any prime number q. Then:*

(a) *if all proper quotients of G are p-hypo-elementary, then $\mathrm{Prim}_{\mathrm{K}_{\mathbb{F}_p}}(G) \cong \mathbb{Z}$;*

(b) *if q is a prime number such that all proper quotients of G are (p,q)-Dress groups, and at least one of them is not p-hypo-elementary, then $\mathrm{Prim}_{\mathrm{K}_{\mathbb{F}_p}}(G) \cong \mathbb{Z}/q\mathbb{Z}$;*

(c) *if there exists a proper quotient of G that is not a (p,q)-Dress group for any prime number q, or if there exist distinct prime numbers q_1 and q_2 and, for $i = 1$ and 2, a proper quotient of G that is a non-p-hypo-elementary (p,q_i)-Dress group, then $\mathrm{Prim}_{\mathrm{K}_{\mathbb{F}_p}}(G)$ is trivial.*

Moreover, in all cases, $\mathrm{Prim}_{\mathrm{K}_{F_p}}(G)$ is generated by any element of $\mathrm{K}_{\mathbb{F}_p}(G) \subseteq b(G)$ of the form $[G/G] + \sum_{H \lneq G} a_H[G/H]$, $a_H \in \mathbb{Z}$.

Proof. By Conlon's Induction Theorem 2.3.8, $\mathcal{P}((\mathrm{Im}\, m_{\mathbb{F}_p})_{\mathbb{Q}})$ is the class of p-hypo-elementary groups. Let q be a prime number. By Corollary 3.5.3, $\mathcal{P}((\mathrm{Im}\, m_{\mathbb{F}_p})_q)$ is the class of (p,q)-Dress groups, and $\mathcal{P}(\mathrm{Im}\, m_{\mathbb{F}_p})$ is the class of all groups that are (p,q')-Dress groups for some prime number q'. Moreover, if U is a non-p-hypo-elementary (p,q)-Dress group, then by Theorem 3.5.1, there exists an element of $\mathrm{K}_{\mathbb{F}_p}(U) \subseteq b(U)$ of the form $q[U/U] + \sum_{H \lneq U} a_H[U/H]$. Part (a) of the theorem follows from Corollary 3.4.10. Finally, note that if q_1 and q_2 are distinct prime numbers, then a finite group is both a (p,q_1)-Dress group and a (p,q_2)-Dress group if and only if it is p-hypo-elementary. Parts (b) and (c) of the theorem therefore follow from Corollaries 3.4.11 and 3.4.13, respectively. □

Chapter 4

Brauer Relations, in Positive Characteristic Semisimplified

4.1 Introduction

In this chapter we consider a simplified version of our main problem; the question of when two G sets give rise to two $\mathbb{F}_p[G]$-permutation modules which are not necessarily isomorphic, but have the same composition factors. It is strictly weaker for two $\mathbb{F}_p[G]$-permutation modules to have the same composition factors than be isomorphic as $\mathbb{F}_p[G]$-modules, this is made precise in Corollary 4.2.3 below. In fact this requirement is also strictly weaker than requiring that the associated $\mathbb{Q}[G]$-permutation modules are isomorphic see Remark 4.2.5. To study this we will consider the kernel between the map of GFIs $m_{\mathbb{F}_p,\mathrm{ss}} : b(-) \to G_0(\mathbb{F}_p[-])$ and we will refer to elements of this kernel as Brauer relations in positive characteristic semisimplified.
The main goal of this chapter is to prove Theorem 4.3.1 in Section 4.3.
This will provide a template for our analysis on the final case considered next chapter.

4.2 Primordial and Coprimordial Groups for $\mathrm{Im}(m_{\mathbb{F}_p,\mathrm{ss}})$

As stated in the introduction, our approach will be to view $b(-)$ and $G_0(\mathbb{F}_p[-])$ as Green functors with inflation with a GFI map $m_{\mathbb{F}_p,\mathrm{ss}}$ between them, and to utilise the machinery developed in the previous chapter to describe the elements of $\ker(m_{\mathbb{F}_p,\mathrm{ss}})(-) = \mathrm{K}_{\mathbb{F}_p,\mathrm{ss}}(-)$ and its structure as an MFI.

The functors taking a finite group G to $b(G)$ and to $G_0(\mathbb{F}_p[G])$ for any prime

p are both GFIs over $\mathbb{Z}$, furthermore the map $m_{\mathbb{F}_p,\mathrm{ss}}$ defined by:

$$\begin{aligned} m_{\mathbb{F}_p,\mathrm{ss}}(G) : b(G) &\longrightarrow G_0(\mathbb{F}_p[G]) \\ [H] &\longmapsto \mathrm{Ind}_{G/H}(1) \end{aligned}$$

is a morphism of GFIs over $\mathbb{Z}$. By Lemma 3.2.8 the kernel of this map $\mathrm{K}_{\mathbb{F}_p,\mathrm{ss}}(-)$ is an ideal of $b(-)$. We will refer to elements of $\mathrm{K}_{\mathbb{F}_p,\mathrm{ss}}(G)$ as *Brauer relations* for G over $\mathbb{F}_p$ semisimplified. We now hope to exploit the machinery of chapter 3 to classify elements of the kernel. We note that in our situation the assumptions of 3.4.3 with $D(G) = 1$ and 3.4.8 are satisfied so we may proceed to use the results of the previous chapter. That is to say that the Burnside ring of a finite group is a free $\mathbb{Z}$-module and is generated by the image of induction plus the trivial G-set, and the inflation map is injective. We fix the following notation in line with that of chapter 3.

Notation 4.2.1. Let G be a finite group, and let $\mathrm{K}_{\mathbb{F}_p,\mathrm{ss}}(G)$ denote the kernel of the map $m_{\mathbb{F}_p,\mathrm{ss}} : b(G) \to G_0(\mathbb{F}_p[G])$. Let $\mathrm{Imprim}(G) = \{\sum_{H<G} \mathrm{Ind}_{G/H}(\mathrm{K}_{\mathbb{F}_p,\mathrm{ss}}(H)) + \sum_{1\neq N \lhd G} \mathrm{Inf}_{G/N}(\mathrm{K}_{\mathbb{F}_p,\mathrm{ss}}(G/N))\}$ and finally let $\mathrm{Prim}_{\mathrm{K}_{\mathbb{F}_p,\mathrm{ss}}}(G) = \mathrm{K}_{\mathbb{F}_p,\mathrm{ss}}(G)/\,\mathrm{Imprim}(G)$.

Lemma 4.2.2. *The coprimordial groups for* $\mathrm{Im}(m_{\mathbb{F}_p,ss}) \subseteq G_0(\mathbb{F}_p[-])$ *are precisely the cyclic groups of order coprime to* p.

Proof. Theorem 2.3.1 combined with Proposition 3.3.4 show that $\mathcal{C}(G_0(\mathbb{F}_p[-])) = \mathcal{P}(G_0(\mathbb{F}_p[-]) \otimes \mathbb{Q})$ is contained in the class of cyclic groups of order coprime to p. It remains to show the reverse inclusion. Let C_m be a cyclic group of order coprime to p, we will exhibit an element in $\mathrm{Im}(m_{\mathbb{F}_p,ss})$ which is in the kernel of every proper restriction map. One may check that the element $m_{(\mathbb{F}_p,\mathrm{ss})}(\sum_{n|m} \mu(n)n[C_n])$, where μ is the Möbius function, is a non-zero element of $\mathrm{Im}(m_{(\mathbb{F}_p,\mathrm{ss})})(C_m)$ which restricts to zero on every proper subgroup. It follows that C_m is coprimordial.

□

Corollary 4.2.3. *For cyclic groups* C *of order coprime to* p *the map* $m_{\mathbb{F}_p,ss}(C)$ *is injective.*

Remark 4.2.4. The previous Corollary and preceding lemma follow immediately from Theorem 2.3.1 and the observation that the rank of $\mathrm{Im}(m_{\mathbb{F}_p,\mathrm{ss}})$ is precisely the number of conjugacy classes of subgroups which are not cyclic of order coprime to p.

Remark 4.2.5. Corollary 4.2.3 combined with the analogous statement for $m_{\mathbb{Q}}$ (see [2], [5]) shows that $\ker(m_{\mathbb{Q}})(G) \subset \ker(m_{\mathbb{F}_p,\mathrm{ss}})(G)$ for all G with equality if $p \nmid |G|$.

We will make extensive use of this later. We will refer to elements of $\ker(m_{\mathbb{Q}})$ as Brauer relations over $\mathbb{Q}$.

Example 4.2.6. The inclusion in Remark 4.2.5 is in general strict. Over fields of characteristic 0 cyclic groups admit no Brauer relations, but the kernel of $m_{\mathbb{F}_p,\mathrm{ss}}$ need not be trivial. For example we have a relation we have $2[C_2] - [e] \in \mathrm{K}_{\mathbb{F}_2,\mathrm{ss}}(C_2)$ for C_2 over $G_0(\mathbb{F}_2[C_2])$. Indeed the regular representation of C_2 is indecomposable as an $\mathbb{F}_2[C_2]$-module, and has as its composition factors two copies of the trivial representation.

Lemma 4.2.7. *Let p be a prime, and let C_p be the cyclic group of order p then $K_{\mathbb{F}_p,\mathrm{ss}}(C_p)$ is generated by the relation $p[C_p] - [\{e\}]$.*

Proof. It is easy to verify that the claimed element of $b(C_p)$ is in $K_{\mathbb{F}_p,\mathrm{ss}}(C_p)$. Furthermore since $\mathrm{Im}(m_{\mathbb{F}_p,\mathrm{ss}}) \supseteq \langle 1 \rangle_{\mathbb{Z}}$ the kernel has rank 1. Clearly no integral relation divides $p[C_p] - [\{e\}]$ so we are finished. $\square$

Having identified the coprimordial groups for $\mathrm{Im}(m_{\mathbb{F}_p,\mathrm{ss}})$ Proposition 3.3.4 states that the primordial groups for $\mathrm{Im}(m_{\mathbb{F}_p,\mathrm{ss}})$ are a subclass of q-quasi-elementary groups with cyclic part of order coprime to p. Note that we must allow $p = q$.

Lemma 4.2.8. *The primordial groups for $\mathrm{Im}(m_{\mathbb{F}_p,\mathrm{ss}})$, are precisely the groups H such that for some prime q, $O^q(H)$ is cyclic of order prime to p.*

Proof. As previously stated it is a consequence of Lemma 4.2.2 and Proposition 3.3.4 that every primordial group is of this form. In the case $p \nmid \#H$ the observation in Remark 4.2.5 shows that H is primordial for $\mathrm{Im}(m_{\mathbb{F}_p,\mathrm{ss}})$ if and only if it is for $\mathrm{Im}(m_{\mathbb{Q}})$. Theorem 4.3.6 and Theorem 2.3.7 show that the quasi-elementary groups are primordial for $\mathrm{Im}(m_{\mathbb{Q}})$.
So we may assume H is p-quasi-elementary. If H were not primordial for $\mathrm{Im}(m_{\mathbb{F}_p,\mathrm{ss}})$, then in particular $1_{G_0(\mathbb{F}_p[H])} = 1_{\mathrm{Im}(m_{\mathbb{F}_p,\mathrm{ss}})(H)} \in \mathcal{I}_{\mathrm{Im}(m_{\mathbb{F}_p,\mathrm{ss}})}(H)$. It follows that for non-primordial groups there is a non-zero element $[H] + \sum_{K<H}[K]$ in $K_{\mathbb{F}_p,\mathrm{ss}}(H)$. We split into two cases, that where H is not a p-group and the case where it is.

1. If H has a non-trivial coprime to p cyclic subgroup C then restriction of any element of $\mathrm{K}_{\mathbb{F}_p,\mathrm{ss}}(H)$ to this subgroup must vanish by Corollary 4.2.3. Since $p \mid [H : C]$ it follows from direct calculation that any relation must have coefficient of $1_{G_0(\mathbb{F}_p[H])}$ divisible by p. It follows that in this case H must be primordial.

2. Otherwise H is a p-group and upon restriction to a central cyclic subgroup of order p any relation must be of the form $a(p[C_p]-[e])$ by Lemma 4.2.7. Direct calculation shows that for any K such that $C_p < K < H$ the restriction of $[K]$ to $b(C_p)$ is $[K : C_p][C_p]$. By an identical argument to the previous case, it follows the coefficient of $1_{G_0(\mathbb{F}_p[H])}$ is divisible by p. This completes the proof.

$\square$

Thus we have identified the primordial groups for $\mathrm{Im}(m_{\mathbb{F}_p,ss})$ as the set of quasi-elementary groups, for which the cyclic part C is coprime to p.

We can phrase this as an induction theorem.

Corollary 4.2.9. *Let G be a finite group and let p be a prime. Let T be the set of conjugacy classes of primordial for* $\mathrm{Im}(m_{\mathbb{F}_p,ss})$ *subgroups of G, that is subgroups which are quasi-elementary with cyclic part coprime to p then:*

$$1_{G_0(\mathbb{F}_p[G])} = \sum_{H \in T} a_H \,\mathrm{Ind}_{G/H}(1_{G_0(\mathbb{F}_p[H])})$$

where a_H are integers.

Proof. This follows immediately from having identified the primordial groups for $\mathrm{Im}(m_{\mathbb{F}_p,\mathrm{ss}})$ in Lemma 4.2.8. $\square$

4.3 Classification of $\mathrm{Prim}_{\mathrm{K}_{\mathbb{F}_p,\mathrm{ss}}}(G)$ for Soluble G

Our aim in this section is to prove the following theorem which gives a necessary condition on a soluble group G for $\mathrm{Prim}_{\mathrm{K}_{\mathbb{F}_p,\mathrm{ss}}}(G)$ to be non-trivial.

Theorem 4.3.1. *Let p be a prime, and G be a finite group of order divisible by p. All Brauer relations for $G_0(\mathbb{F}_p[G])$ are linear combinations of relations induced and inflated from subquotients of the following forms:*

1. *A cyclic group C_p of order p,*
2. *non-cyclic q-quasi-elementary groups with order coprime to p,*
3. *$(C_l \rtimes C_{q^r}) \times (C_l \rtimes C_{q^s})$ for primes q, l with $l \neq p$ and the action faithful,*
4. *an extension $1 \to S^d \to E \to H \to 1$ with S simple, d a positive integer, $H = C \rtimes Q$ is a quasi-elementary group whose cyclic part C is of order coprime to p, and S^d is a unique minimal normal subgroup of E.*

We now characterise which finite groups G may have non-trivial $\mathrm{Prim}_{\mathrm{K}_{\mathbb{F}_p,\mathrm{ss}}}(G)$, and explicitly write down all such groups in the soluble case.

Lemma 4.3.2. *Let G be a finite group which admits primitive relations over $\mathbb{F}_{p,\mathrm{ss}}$ then G is an extension of the following form:*

$$1 \to S^d \to G \to H \to 1 \tag{4.3.3}$$

where S is a finite simple group, H is quasi-elementary with cyclic part coprime to p. Furthermore if G is not primordial for $\mathrm{Im}(m_{\mathbb{F}_p}, \mathrm{ss})$ then we may describe $\mathrm{Prim}_{\mathrm{K}_{\mathbb{F}_p,\mathrm{ss}}}(G)$ as follows:

1. *If all quotients of G are cyclic of order coprime to p then $\mathrm{Prim}_{\mathrm{K}_{\mathbb{F}_p,\mathrm{ss}}}(G)$ is isomorphic to $\mathbb{Z}$,*
2. *if all quotients of G are q-quasi-elementary with cyclic part coprime to p and at least one of them is not cyclic then $\mathrm{Prim}_{\mathrm{K}_{\mathbb{F}_p,\mathrm{ss}}}(G) = \mathbb{Z}/q\mathbb{Z}$ and,*
3. *otherwise it is trivial.*

Furthermore in all cases Prim(G) is generated by any Brauer relation of the form $[G] + \sum_{H<G} n_H[H]$.

Proof. Theorem 3.4.7 along with the identification of the primordial groups for $\mathrm{Im}(m_{\mathbb{F}_p,\mathrm{ss}})$ in Lemma 4.2.8 shows that G is an extension of the claimed form and that the generator of $\mathrm{Prim}_{\mathrm{K}_{\mathbb{F}_p,\mathrm{ss}}}(G)$ is as claimed if it exists. The existence of such an element follows from Corollary 4.2.9.
If all proper quotients are cyclic of order coprime to p then they are coprimordial for $\mathrm{Im}(m_{\mathbb{F}_p,\mathrm{ss}})$ and so primordial for $\mathrm{Im}(m_{\mathbb{F}_p,\mathrm{ss}})_{\mathbb{Q}}$ by Proposition 3.3.4. Corollary 3.4.10 then shows that $\mathrm{Prim}_{\mathrm{K}_{\mathbb{F}_p,\mathrm{ss}}}(G) = \mathbb{Z}$ in this case.
It remains to consider the case where there exists a quotient which is q-quasi-elementary with cyclic part prime to p but not cyclic of order prime to p. First we note that for non-cyclic q-quasi-elementary groups [18, Theorem 1] shows that there exists a relation over $\mathbb{Q}$ of the form $qG + \sum_{H \lneq G} a_H \operatorname{Ind}_{G/H}(1)$ with the a_H in $\mathbb{Z}$ and hence by Remark 4.2.5 over $G_0(\mathbb{F}_p[-])$. It follows from Lemma 4.2.8 that G can't appear with coefficient 1. Furthermore Lemma 4.2.7, combined with inflation shows that there is a relation $p[C_{p^r}] - [C_{p^{r-1}}]$ for any C_{p^r}. Corollaries 3.4.11 and 3.4.13 of Theorem 3.4.9 then give the claimed result. □

We now classify which groups G of the form (4.3.3) have $\mathrm{Prim}_{\mathrm{K}_{\mathbb{F}_p,\mathrm{ss}}}(G)$ non-trivial.

Corollary 4.3.4. *Let G be a finite non-soluble group which admits a primitive relation over $\mathbb{F}_{p,\mathrm{ss}}$. Then G is of the form described in Lemma 4.3.2 with S non-cyclic, H injects into $\mathrm{Out}(S^d)$ and no proper non-trivial subgroup of S^d is normal in G. Furthermore every such group admits a primitive relation.*

Proof. The Corollary follows immediately from Lemma 4.3.2 upon noting that for non-soluble G in such an extension every quotient is quasi-elementary with cyclic part coprime to p if and only if the action of G on S^d is faithful and no proper non-trivial subgroup of S^d is normal in G. Since the centre of S^d is trivial the action of G is faithful if and only if $G/(S^d) = H \hookrightarrow \mathrm{Out}(S^d)$. □

Since the inclusion in Remark 4.2.5 is an equality when we restrict to groups of order coprime to p and since there is a full classification of Brauer relations in characteristic zero in [2] we now need only consider G whose order is divisible by p. Furthermore, in light of Corollary 4.3.4 we restrict to the soluble case. We will make repeated use of the following result.

Lemma 4.3.5. *Let G be a finite group, and let W an abelian normal subgroup with quotient H. Suppose that there exists a normal subgroup K of H such that $\gcd(\#K, \#W) = 1$ and such that no non-identity element of W is fixed under the natural conjugation action of K on W. Then $G \cong W \rtimes H$.*

Proof. We may view W as a module under H. Since K and W have coprime orders, the cohomology group $H^i(K, W)$ vanishes for $i > 0$, so the Hochschild–Serre spectral sequence [13, Theorem 6.3] gives an exact sequence

$$H^2(H/K, W^K) \to H^2(H, W) \to H^2(K, W).$$

The last term in this sequence also vanishes by the coprimality assumption, while the first term vanishes, since W^K is assumed to be trivial. So $H^2(H, W) = 0$, and so the extension G of H by W splits. □

Theorem 4.3.6. *Suppose that G is a soluble group, of order divisible by p, and $\mathrm{Prim}_{\mathrm{K}_{\mathbb{F}_{p,\mathrm{ss}}}}(G)$ is non-trivial. Then G is of one of the following forms:*

1. *a p-group or,*
2. *a p-quasi-elementary group or,*
3. *$(C_l)^d \rtimes H$ with l a prime, H quasi-elementary, with cyclic part coprime to p, acting faithfully and irreducibly on $(C_l)^d$ or,*

4. $(C_l \rtimes C_{p^r}) \times (C_l \rtimes C_{p^s})$ *with faithful action and* l *a prime.*

Proof. The first two parts follow from taking trivial extensions of p-groups and quasi-elementary groups respectively in Lemma 4.3.2 since p must divide the order of G trivial extensions of cyclic groups are not included. Note that for G as in Lemma 4.3.2 to be soluble is equivalent to taking $S = C_l$. By Lemma 4.3.2 G is therefore an extension of the form:

$$1 \to W := (C_l)^d \to G \to H := C \rtimes Q \to 1.$$

Where H is quasi-elementary with cyclic part coprime to p. We wish to show that under our assumptions G is a split extension or lies in case 1 or 2 of the theorem. If $l \nmid \#H$ then by Schur-Zassenhaus the result follows. Suppose that that $l \mid \#H$, we split into the following cases C is trivial, Q is trivial, and neither C nor Q is trivial.

1. C **is trivial**. In this case G is a q-group.

2. Q **is trivial**. Either G is an l group or C admits a subgroup $C_{l'}$ of order coprime to l. Either $W^{C_{l'}}$ is trivial, in which case Lemma 4.3.5 with $K = C_{l'}$ allows us to conclude G is split or $\{e\} \neq V := W^{C_{l'}} \lhd G$. Now G/V must be quasi-elementary and $(W/V)^{C_{l'}} = \{e\}$. Since this quotient must be l-quasi-elementary, in fact $W = V$ and G was l-quasi-elementary with cyclic part coprime to p. We must have $l = p$ as $p \mid \#G$ by assumption.

3. **Both** Q **and** C **are non trivial**. If C is an l group then let L denote the l-sylow subgroup of G, clearly $L \lhd G$. Let $\Phi(L)$ be the Frattini subgroup of L, if it is trivial then $L = C_l^n$ and $G = C_l^n \rtimes Q$ and so G is a split extension as claimed. Otherwise $\Phi(L) \lhd G$ and $G/\Phi(L)$ must be q-quasi-elementary with cyclic part prime to p as $\mathrm{Prim}_{\mathrm{K}_{\mathbb{F}_p,\mathrm{ss}}}(G)$ is non-trivial. Thus $L/\Phi(L)$ must be cyclic and we see that $l \neq p$. As $L/\Phi(L)$ is cyclic, L is also cyclic. It follows that in this case that G is p-quasi-elementary.

   If C is not an l group then let $K = C_{l'}$ as before, if W^K is trivial we are done by Lemma 4.3.5, if not then letting $V = W^K$ we see that G/V must be q-quasi-elementary with cyclic part coprime to p, if $l = p$ this forces $q = p$ and $W^K = W$ so G is p-quasi-elementary. In the remaining case $l \neq p$ and we have that, in particular, $(W/V)^K$ must be trivial, it follows that this G/V and thus G must be l-quasi-elementary with cyclic part coprime to p, but we assumed $p \mid \#G$ a contradiction.

It remains to consider the split sequence

$$1 \to (C_l)^d \to G \to H \to 1.$$

We subdivide into three cases; in the first $l = p$, and H is $q \neq p$-quasi-elementary, in the second $l \neq p$ and H is p-quasi-elementary, and finally $l = p$ and H is p-quasi-elementary, in all cases the cyclic part of H is coprime in order to p.

1. In the first case $G = (C_p)^d \rtimes (C \rtimes Q)$ where Q is a q-group and $p \nmid \#C$. **Faithfulness:** Suppose $C \rtimes Q$ acts with kernel $K \neq \{e\}$ then the quotient G/K must be q-quasi-elementary with cyclic part prime to p, but this is not possible since $p \neq q$. **Irreducibility:** Suppose that the action were reducible so there exists $V = (C_p)^{d_1} \lhd G$ with $d_1 < d$. The corresponding quotient must be q-quasi-elementary of order coprime to p a contradiction. We conclude that the action is faithful and irreducible.

2. In the second case $G = (C_l)^d \rtimes (C \rtimes P)$ where $l \neq p$ and P is a p-group. **Faithfulness:** Suppose that $C \rtimes P$ acts with kernel $K \neq \{e\}$ then the quotient G/K must be p-quasi-elementary. In particular, this forces $d = 1$ and $K \geqslant C$ so G was p-quasi-elementary. **Irreducibility:** Assuming that the action is faithful, either it is irreducible in which case we find ourselves in case 3 of the theorem or it is reducible. If the action on $W := C_l^d$ were reducible, then there exists $V < W$ a normal subgroup of G, the quotient group G/V must then be p-quasi-elementary. By assumption $l \neq p$ so the l-Sylow of G/V must be cyclic, if $G = (C_l)^d \rtimes (C \rtimes P)$ with $l \mid \#C$ then this would be impossible. We conclude that if the action is reducible then $G = ((C_l) \times V) \rtimes (C \rtimes P)$ with $l \nmid \#(C \rtimes P)$ with semisimple action, now quotienting by C_l shows via an identical argument that $V \cong C_l$ and so we find ourselves in part 4 of the theorem.

3. Finally $G = (C_p)^d \rtimes (C \rtimes P)$. **Faithfulness:** We claim either this group is p-quasi-elementary or the action is faithful. If the action had a kernel K the quotient by the kernel must be p-quasi-elementary, and so $C \leqslant K$ and G was already p-quasi-elementary. **Irreducibility:** If the action is faithful then we claim that it must be irreducible. Assume otherwise then there exists $V = (C_p)^{d_1} \lhd G$ such that G/V is p-quasi-elementary, this forces C to act trivially on $(C_p)^d/V$ and so $((C_p)^d)^C \neq 0$ as C has order coprime to p and thus the action is semisimple. We now require $G/((C_p)^d)^C$ to be p-quasi-elementary, now we may assume that the complement of $((C_p)^d)^C$ in $(C_p)^d$ is

non-trivial (else C is normal and G quasi-elementary) and thus the quotient can't be p-quasi-elementary. Thus either the action is faithful and irreducible or G is quasi-elementary.

□

Theorem 4.3.7. *Let G be a p-quasi-elementary group which is not cyclic of order p, then $\mathrm{Prim}_{\mathrm{K}_{\mathbb{F}_p,\mathrm{ss}}}(G)$ is trivial.*

Proof. The rank of the space of relations of G is the number of conjugacy classes of subgroups of G which are not cyclic of order coprime to p. We will construct a sublattice of imprimitive relations which has full rank, then proceed to show it is saturated.

Let $G = C_m \rtimes P$ with P a p-group and $p \nmid m$, then subgroups which are not cyclic of order coprime to p are determined up to conjugacy by their intersection with C_m and the selection of a non-trivial subgroup of the normaliser of this intersection with P. Since the intersection with C_m is characteristic in G describing these subgroups up to conjugacy amounts to picking a subgroup of C_m and a non-trivial subgroup of P. Fix a labelling on the subgroup lattice of P up to G-conjugacy let $P_{i,j}$ be the jth subgroup of size p^i these subgroups are then characterised up to conjugacy as $C_s \rtimes P_{i,j}$ where $s \mid m$ and $i \neq 0$. Each such subgroup admits an imprimitive relation inflated from any quotient C_p namely $p[C_s \rtimes P_{i,j}] - [C_s \rtimes P_{i-1,k}]$ where $P_{i,j} > P_{i-1,k}$. Note that as every maximal subgroup of a p-group has index p we may use these relations to create the relation $[C_s \rtimes P_{i,j}] - [C_s \rtimes P_{i,k}]$ for any $P_{i,k}$.

The sublattice:

We now exhibit a full rank sublattice of imprimitive relations. Take the span of $p[C_s \rtimes P_{i,0}] - [C_s \rtimes P_{i-1,0}]$ as we range over $s \mid m$ and $i > 1$ along with relations $[C_s \rtimes P_{i,0}] - [C_s \rtimes P_{i,j}]$ for $s \mid m$, and $i, j > 0$. Clearly this set is linearly independent and of the correct size, so we have a full rank sublattice;

$$\mathcal{L} = \langle p[C_s \rtimes P_{i,0}] - [C_s \rtimes P_{i-1,0}], [C_s \rtimes P_{i,0}] - [C_s \rtimes P_{i,j}] | i \in I, j \in J \rangle_{\mathbb{Z}}.$$

Saturation:

The sublattice $\mathcal{L}$ is in fact, saturated, suppose that there exists a relation θ such that $n\theta = \sum_{s|m}(\sum_{i>0} a_{s,i}(p[C_s \rtimes P_{i,0}] - [C_s \rtimes P_{i-1,0}]) + \sum_{j>0} b_{s,i,j}([C_s \rtimes P_{i,0}] - [C_s \rtimes P_{i,j}]))$ for a relation θ we seek to show that $n \mid a_{s,i}, b_{s,i,j}$ for all s, i, j in the indexing sets. Since the coefficient of $[C_s \rtimes P_{i,j}]$ on the right hand side is $b_{s,i,j}$ we may conclude that $n \mid b_{s,i,j}$, subtracting all terms in the second sum from both sides we then have $n\theta' = \sum_{s|m} \sum_{i>0} a_{s,i}(p[C_s \rtimes P_{i,0}] - [C_s \rtimes P_{i-1,0}])$, the coefficient of $[C_s \rtimes P_{0,0}]$

on the right hand side is $-a_{s,1}$ and so $n \mid a_{s,1}$. Now the coefficient of $[C_s \rtimes P_{i,0}]$ is $pa_{s,i} - a_{s,i+1}$ and so if $n \mid a_{s,i}$ then $n \mid a_{s,i+1}$ and we may proceed inductively to show $n \mid a_{s,i}$ for all s, i in the indexing set.
Thus we have a full rank saturated sublattice of imprimitive relations, it follows that every relation is imprimitive. $\square$

Theorems 4.3.6 and 4.3.7 allow us to conclude the following.

Theorem 4.3.8. *Let G be a finite soluble group, p a prime, $\mathrm{Prim}_{\mathrm{K}_{\mathbb{F}_p,\mathrm{ss}}}(G)$ is non-trivial if and only if:*

1. *$G \cong C_p$, then $\mathrm{Prim}_{\mathrm{K}_{\mathbb{F}_p,\mathrm{ss}}}(G) \cong \mathbb{Z}$ or,*
2. *$G \cong (C_l)^d \rtimes H$ with H q-quasi-elementary acting faithfully and irreducibly on $(C_l)^d$, then $\mathrm{Prim}_{\mathrm{K}_{\mathbb{F}_p,\mathrm{ss}}}(G) \cong \mathbb{Z}/q\mathbb{Z}$ or,*
3. *for $l \neq p$ a prime $G = (C_l \rtimes C_{q^r}) \times (C_l \rtimes C_{q^s})$, $\mathrm{Prim}_{\mathrm{K}_{\mathbb{F}_p,\mathrm{ss}}}(G) \cong \mathbb{Z}/q\mathbb{Z}$ or,*
4. *G is quasi-elementary of order coprime to p with $\mathrm{Prim}_{\mathrm{K}_{\mathbb{F}_p,\mathrm{ss}}}(G)$ as over $\mathbb{Q}$.*

Combining this with Corollary 4.3.4 gives Theorem 4.3.1.

4.4 Some Explicit Relations

We now establish generators of $\mathrm{Prim}_{\mathrm{K}_{\mathbb{F}_p,\mathrm{ss}}}(G)$ for the soluble groups of order divisible by p admitting primitive relations. Since the inclusion in Remark 4.2.5 becomes an equality when $p \nmid \#G$, this combined with the classification in [2, Theorem A] completely determines all Brauer relations for soluble groups over $G_0(\mathbb{F}_p[G])$.

Lemma 4.4.1. $\mathrm{Prim}_{\mathrm{K}_{\mathbb{F}_p,\mathrm{ss}}}(C_p)$ *is generated by* $p[C_p] - [\{e\}]$. $\mathrm{Prim}_{\mathrm{K}_{\mathbb{F}_p,\mathrm{ss}}}((C_l)^d \rtimes H)$ *with H quasi-elementary acting faithfully is generated by the same relation as over $\mathbb{Q}$ for $d > 1$.*

Proof. The first claim is identical to Lemma 4.2.7.
For the second, note that the Brauer relation over $\mathbb{Q}$ for $d \geqslant 2$ given in [2, Proposition 6.4] has coefficient of $[G]$ equal to 1 and, as its still a relation in this setting (see Remark 4.2.5) it must generate $\mathrm{Prim}_{\mathrm{K}_{\mathbb{F}_p,\mathrm{ss}}}(G)$ by Lemma 4.3.2, this gives the second statement. $\square$

Note that a quasi-elementary group acting faithfully on a cyclic group must be cyclic so the only remaining case (corresponding to $d = 1$) we need to calculate is the case of a coprime to p cyclic group acting faithfully on a cyclic group of order p.

Lemma 4.4.2. *Let $G = C_p \rtimes C_{mq^r}$ with faithful action then* $\mathrm{Prim}_{\mathrm{K}_{\mathbb{F}_p,\mathrm{ss}}}(G)$ *is generated by the same relation as presented in [2] over $\mathbb{Q}$ unless $m = 1$ then its generated by the following relation $-[C_{q^r}] + (p-1)/q^r[C_p] + [C_p \rtimes C_q^r]$.*

Proof. We explicitly construct such a relation and since the coefficient of $[G]$ is 1 it must generate $\mathrm{Prim}_{\mathrm{K}_{\mathbb{F}_p,\mathrm{ss}}}(G)$. In the case m is non-trivial we simply apply Remark 4.2.5 and use the Brauer relation over $\mathbb{Q}$ in [2, Proposition 6.5]. Otherwise $G = C_p \rtimes C_{q^r}$ we have the following relations, $p[C_p] - [e]$ induced from the subgroup C_p, and the primitive relation over $\mathbb{Q}$, $[C_{q^{r-1}}] - q[C_{q^r}] - [C_p \rtimes C_{q^{r-1}}] + q[C_p \rtimes C_{q^r}]$. Using a linear combination of the two identified relations we can produce a third which is a multiple of the relation in the statement. We proceed by induction on r.

If $r = 1$ then taking a linear combination $\alpha([\{e\}] - q[C_q] - [C_p] + q[C_p \rtimes C_q]) - \beta((p[C_p] - [e]))$. and setting $\alpha = 1, \beta = -1$ gives q times the relation $-[C_q] + (p-1)/q[C_p] + [C_p \rtimes C_q]$. If $r > 1$ we assume that statement holds for $r - 1 < r$. Taking the relation $[C_{q^{r-1}}] - q[C_{q^r}] - [C_p \rtimes C_{q^{r-1}}] + q[C_p \rtimes C_{q^r}]$ and adding the induced relation from $C_p \rtimes C_{q^{r-1}}$ gives q times the claimed relation. $\square$

Chapter 5

Brauer Relations in Positive Characteristic

5.1 Introduction

In this chapter we finally classify Brauer Relations in positive characteristic. Our approach will mimic the structure of the previous chapter, first we will study the induction theorems and primordial groups in this setting, this along with the theory developed in Chapter 3 will allow us to describe the structure of the primitive quotient, and in the soluble case we will give explicit generators. The final section will concern (p, p) Dress groups, these groups are the least amenable to analysis in the characteristic p setting and we only have partial results concerning their relations.

5.2 Basic Properties and Induction Theorems

We want to study the kernel $K_{\mathbb{F}_p}$ of the map of GFI's:

$$m_{\mathbb{F}_p} : b(-) \to a(\mathbb{F}_p[-]),$$

first described in Chapter 3. Note that this map of GFI's satisfies the hypothesis of assumption 3.4.8 and of notation 3.4.3, so we were able to apply the results of Chapter 3. Recall the following main results from Chapter 3 concerning this map.

Theorem 5.2.1. *Let q be a prime number, and let G be a (p, q)-Dress group that is not p-hypo-elementary. Then there exists an F-relation of the form $qG + \sum_{U \lneq G} a_U U \in \mathrm{K}_F(G)$. Conversely, if $\sum_{U \leq G} a_U U$ is an F-relation, then $q | a_G$.*

We also recall Theorem 3.5.4, a consequence of Theorem 3.5.1 and the main result of chapter 3, which is the basic structure result on $\mathrm{Prim}_{\mathrm{K}_{\mathbb{F}_p}}$.

Theorem 5.2.2. *Let G be a finite group that is not (p,q)-Dress for any prime q. Then the following holds:*

1. *if all proper quotients of G are p-hypo-elementary, then $\mathrm{Prim}_{\mathrm{K}_{\mathbb{F}_p}}(G) \cong \mathbb{Z}$;*
2. *if there exists a prime q such that all proper quotients of G are (p,q)-Dress, and at least one of them is not p-hypo-elementary, then $\mathrm{Prim}_{\mathrm{K}_{\mathbb{F}_p}}(G) \cong \mathbb{Z}/q\mathbb{Z}$;*
3. *if there exists a proper quotient of G that is not (p,q)-Dress for any prime q, or if there exist two quotients that are not p-hypo-elementary, and one of which is (p,q)-Dress and the other is (p,q')-Dress for distinct primes q and q', then $\mathrm{Prim}_{\mathrm{K}_{\mathbb{F}_p}}(G) = 0$.*

In cases (1) and (2), $\mathrm{Prim}_{\mathrm{K}_{\mathbb{F}_p}}(G)$ is generated by any relation in which G has coefficient 1.

Corollary 5.2.3. *Let G be a finite group, and suppose that $\mathrm{Prim}_{\mathrm{K}_{\mathbb{F}_p}}(G)$ is non-trivial. Then G is an extension of the form*

$$1 \to S^d \to G \to D \to 1,$$

where S is a finite simple group, $d \geqslant 1$, and D is a (p,q)-Dress group for some prime number q. Moreover, if S is not cyclic, then the canonical map $D \to \mathrm{Out}(S^d)$ is injective, and S^d has no proper non-trivial subgroups that are normal in G. In this case, $\mathrm{Prim}_{\mathrm{K}_{\mathbb{F}_p}}(G) \cong \mathbb{Z}$ if D is p-hypo-elementary, and $\mathrm{Prim}_{\mathrm{K}_{\mathbb{F}_p}}(G) \cong \mathbb{Z}/q\mathbb{Z}$ otherwise.

Proof. The group G has a chief series, so there exists a normal subgroup $W \cong S^d$, where S is a simple group and $d \geq 1$. By Theorem 5.2.2, the quotient G/W is (p,q)-Dress for some prime number q.

Now suppose that S is not cyclic. Let K be the kernel of the map $G \to \mathrm{Aut}(S^d)$ given by conjugation. The centre of S^d is trivial, so $K \cap S^d = \{1\}$. If K is non-trivial, then G/K is a proper quotient that is not soluble, and in particular not (p,q)-Dress, contradicting Theorem 5.2.2. So G injects into $\mathrm{Aut}(S^d)$, and thus $G/S^d = D$ injects into $\mathrm{Out}(S^d)$. Similarly, if $N \triangleleft G$ is a proper subgroup of S^d, then G/N is not soluble, and in particular not (p,q)-Dress, contradicting 5.2.2. Finally, the description of $\mathrm{Prim}_{\mathrm{K}_{\mathbb{F}_p}}(G)$ is given by Theorem 5.2.2. $\square$

5.3 Classification of $\mathrm{Prim}_{\mathrm{K}_{\mathbb{F}_p}}(G)$ for Soluble G

In this section, we derive necessary conditions on a soluble group G for $\mathrm{Prim}_{\mathrm{K}_{\mathbb{F}_p}}(G)$ to be non-trivial.

Lemma 5.3.1. *Let q be a prime number different from p, and let $G = P \rtimes (C \rtimes Q)$ be a (p, q)-Dress group, where P is a p-group, Q is a q-group, and C is a cyclic group of order coprime to pq. Let S be a full set of G-conjugacy class representatives of subgroups of P. For each $U \in S$, let N_U be a $(-p)$-Hall subgroup of $N_G(U)$, and let T_U be a full set of N_U-conjugacy class representatives of subgroups of N_U. Then,*

1. *for every $U \in S$, two subgroups of N_U are N_U-conjugate if and only if they are $N_G(U)$-conjugate;*

2. *for every subgroup H of G, there exists a unique $U \in S$ and a unique $V \in T_U$ such that H is G-conjugate to $U \rtimes V$.*

Proof. To prove the first part, let $U \in S$, and V_1, $V_2 \leq N_U$. Suppose that there exists an element g of $N_G(U)$ such that $V_1^g = V_2$. Since $N_G(U) = N_P(U) \rtimes N_U$, we may write $g = un$, where $u \in N_P(U)$ and $n \in N_U$. Let $v \in V_1$. By assumption, $v^g \in V_2 \subseteq N_U$, so $v^u \in N_U$, so $[v, u] = v(uv^{-1}u^{-1}) \in N_U$. On the other hand, $N_P(U) = N_G(U) \cap P$ is normal in $N_G(U)$, so $[v, u] = (vuv^{-1})u^{-1} \in N_P(U)$. Since $N_P(U) \cap N_U = \{1\}$, this implies that u and v commute. Since v was arbitrary, we deduce that u centralises V_1, so that $V_1^g = V_1^n = V_2$, as claimed.

Now, we prove the existence statement of part (2). Let H be a subgroup of G, and let $U = H \cap P$. After replacing H with a subgroup that is G-conjugate to it if necessary, we may assume that $U \in S$. We then have $H \leq N_G(U)$. Let V be a $(-p)$-Hall subgroup of H, which is contained in a $(-p)$-Hall subgroup of $N_G(U)$. Since all $(-p)$-Hall subgroups of $N_G(U)$ are conjugate to each other, we may assume, after possibly replacing H with a subgroup that is $N_G(U)$-conjugate to it, that V is contained in N_U, so that after possibly replacing H by a subgroup that is N_U-conjugate to it, we may assume that $V \in T_U$.

Finally, we prove uniqueness. Let U_1, $U_2 \in S$, and let $V_i \in T_{U_i}$ for $i = 1$, 2 be such that $H_1 = U_1 \rtimes V_1$ is G-conjugate to $H_2 = U_2 \rtimes V_2$. Since U_i is the unique subgroup of H_i for $i = 1$ and 2, this implies that U_1 and U_2 are G-conjugate; and since both are contained in P, and S is assumed to be a complete set of distinct conjugacy class representatives, this implies that $U_1 = U_2 = U$. It follows that H_1 and H_2 are $N_G(U)$-conjugate. Since V_i is a $(-p)$-Hall subgroup of H_i for $i = 1$ and 2, it follows that V_1 and V_2 are also $N_G(U)$-conjugate, so by the first part, they are N_U-conjugate. □

Theorem 5.3.2. *Let q be a prime number different from p, and let G be a (p,q)-Dress group with non-trivial p-core. Then* $\mathrm{Prim}_{\mathrm{K}_{\mathbb{F}_p}}(G)$ *is trivial.*

Proof. Keep the notation of Lemma 5.3.1. In particular, write $G = P \rtimes (C \rtimes Q)$, where P is a non-trivial p-group, Q is a q-group, and C a cyclic group of order coprime to pq.

For each $U \in S$, identify N_U with UN_U/U via the quotient map, and consider the map

$$f_U = \mathrm{Ind}_{G/UN_U} \mathrm{Inf}_{UN_U/U} \colon \mathrm{B}(N_U) \to \mathrm{B}(G).$$

Let $I_U = f_U(\mathrm{K}_F(N_U))$. Note that all $\Theta \in I_U$ are imprimitive, since either U is non-trivial, so that UN_U/U is a proper quotient, or N_U is a $(-p)$-Hall subgroup of G, which is proper since the p-core of G is assumed to be non-trivial. We will now show that $\sum_{U\in S} I_U = \mathrm{K}_F(G)$.

First, we claim that each f_U is injective. Inflation is always an injective map of Burnside rings, so it suffices to show that the induction map Ind_{G/UN_U} is injective on the image of $\mathrm{Inf}_{UN_U/U}$. Let H_1 and H_2 be subgroups of UN_U containing U that are G-conjugate. Since their common p-core is U, they are then $N_G(U)$-conjugate. Since each of their respective $(-p)$-Hall subgroups is contained in a $(-p)$-Hall subgroup of UN_U, and all $(-p)$-Hall subgroups of UN_U are conjugate, we may assume, replacing H_1 and H_2 by UN_U-conjugate subgroups if necessary, that $H_i = UV_i$, where $V_i \le N_U$ for $i = 1, 2$, and where V_1 is $N_G(U)$-conjugate to V_2. But then, by Lemma 5.3.1 (1), V_1 and V_2 are also N_U-conjugate, so H_1 and H_2 are UN_U-conjugate, proving injectivity.

Next, we claim that the I_U for $U \in S$ are linearly independent. Indeed, suppose that $\sum_{U\in S} \Theta_U = 0$, where $\Theta_U \in f_U$. Let U be maximal with respect to inclusion subject to the property that $\Theta_U \neq 0$. Then all terms in Θ_U contain U, while for all $U' \neq U$, all terms are contained in $U'N_{U'}$, which does not contain U. So for the sum to vanish, we must have $\Theta_U = 0$ – a contradiction.

A similar argument shows that $\sum_{U\in S} I_U$ is saturated in $\mathrm{K}_F(G)$, and it remains to compare the ranks. By linear independence and by Remark 2.3.9, we have

$$\begin{aligned} \mathrm{rank}\left(\sum_{U\in S} I_U\right) &= \sum_{U\in S} \mathrm{rank}\, I_U \\ &= \sum_{U} \#\{\text{conjugacy classes of non-cyclic subgroups of } N_U\}, \end{aligned}$$

and by Lemma 5.3.1 (2), this is equal to the rank of $\mathrm{K}_F(G)$, which completes the proof. $\square$

Recall Lemma 4.3.5, we will make extensive use of it in the proof of the following theorem.

Theorem 5.3.3. *Let G be a finite soluble group that admits a primitive F-relation. Then G is one of the following:*

(i) a (p,p)-Dress group,

(ii) a q-quasi-elementary group for some prime number $q \neq p$,

(iii) a semidirect product $G = W \rtimes D$, where $W = (C_l)^d$ for a prime number $l \neq p$, $d \geq 1$, and D is a (p,q)-Dress group acting faithfully and irreducibly on W, where q is a prime number,

(iv) $G = (C_l \rtimes D_1) \times (C_l \rtimes D_2)$ where D_1, D_2 are cyclic q-groups that act faithfully on $C_l \times C_l$, where q is a prime number.

Proof. By Corollary 5.2.3, G is an extension of the form

$$1 \to W = (C_l)^d \to G \to D \to 1, \tag{5.3.4}$$

where D a (p,q)-Dress group. If $p = l$, then G is a (p,q)-Dress group, and by Theorem 5.3.2, it is either q-quasi-elementary for $q \neq p$ or (p,p)-Dress. For the rest of the proof, assume that $p \neq l$. We now consider several cases:

Case 1: $l \nmid \#D$. By the Schur-Zassenhaus theorem, the short exact sequence (5.3.4) splits. So we have $G \cong W \rtimes D$, and we may view D as a subgroup of G. Let $N \lhd G$ be the centraliser of W in D.

Case 1(a): $N \neq \{1\}$ and D is p-hypo-elementary. The subgroup WN/N is normal in G/N. By Theorem 5.2.2, G/N is a (p,q)-Dress group for some prime q. Since $p \neq l$, D/N is also normal in G/N, so $G/N \cong WN/N \times D/N$. So the commutator $[W, D]$ is contained in $N \leq D$. But also, since W is normal in G, this commutator is contained in W, so it is trivial. It follows that W commutes with D, and G is a (p,l)-Dress group, so by Theorem 5.3.2 it is either l-quasi-elementary or (p,p)-Dress.

Case 1(b): $N \neq \{1\}$ and D is not p-hypo-elementary. By Theorem 5.2.2, G/N is (p,q)-Dress. Since $l \neq p, q$, this implies that W must be cyclic, and, by the same argument as in case 1(a), it must commute with $O^q(D)$. It follows that G is a (p,q)-Dress group, so by Theorem 5.3.2 it is either q-quasi-elementary for $q \neq p$, or (p,p)-Dress.

Case 1(c): $N = \{1\}$ and D acts reducibly on W. Let U be a proper non-trivial subgroup of W that is normal in G. Since $l \nmid \#D$, the $\mathbb{F}_l[D]$-module W is semisimple, so there exists a subgroup V of W that is normal in G and such that $UV = W$ and $U \cap V = \{1\}$. By Theorem 5.2.2, both G/U and G/V are (p,q)-Dress. Since $l \nmid pq$, this implies that $V \cong W/U \cong C_l$ and $U \cong W/V \cong C_l$. Thus, $G \cong (U \rtimes D_1) \times (V \rtimes D_2)$, where D_1 acts faithfully on U, and D_2 acts faithfully on V, and in particular both are cyclic. It follows that $O_p(G/U)$ is of the form NU/U for a p-subgroup N of D_1. For G/U to be (p,q)-Dress, the $(-q)$-Hall subgroup of G/UN must be cyclic, which forces D_2 to be a q-group, and similarly for D_1. This is case (iv) of the theorem.

Case 1(d): $N = \{1\}$ and D acts irreducibly on W. This is either case (i) , (ii) or (iii) of the theorem according to the structure of D if $d = 1$ and (iii) otherwise.

Case 2: $l|\#D$ and $G = W \rtimes D$. In this case, $N = \ker(D \to \operatorname{Aut} W)$ is again a normal subgroup of G.

Case 2(a): $N \neq \{1\}$. By Theorem 5.2.2, the quotient G/N is (p,q)-Dress. Since D/N acts faithfully on W, no non-trivial subgroup of D/N can be normal in G/N. In particular, $O_p(G/N)$ must be trivial, so N contains $O_p(D)$, and G/N is in fact quasi-elementary, $G/N \cong C \rtimes Q$, where C is cyclic and Q is a q-group. By the same argument, C is an l-group. Now, if $q = l$, then G/N is an l-group, and G is an extension of an l-group by the (p,l)-Dress group N, hence is itself (p,l)-Dress, so by Theorem 5.3.2, it is must be l-quasi-elementary. If $q \neq l$, then W must be cyclic, and must commute with $O_p(D)$. So $O_p(D)$ is normal in G, and $G/O_p(D)$ is q-quasi-elementary, whence G is a (p,q)-Dress group, so by Theorem 5.3.2 it is either q-quasi-elementary for $q \neq p$, or (p,p)-Dress.

Case 2(b): $N = \{1\}$ and D acts reducibly on W. Let $U \leq W$ be a non-zero proper $\mathbb{F}_l[D]$-sub-representation of W. By Theorem 5.2.2, the quotient G/U is (p,q)-Dress.

Case 2(b)(i): $l \neq q$. Then the l-Sylow subgroups of G/U must be cyclic. In particular, any l-Sylow subgroup C of D, which is non-trivial by assumption, acts trivially by conjugation on W/U. Since G is assumed to be a semi-direct product, the l-Sylow subgroup of G/U is a direct product of W/U and C, and therefore cannot be cyclic – a contradiction.

Case 2(b)(ii): $l = q$. Either G/U is an l-group, in which case so is G, and we are in case (ii) of the theorem; or there exists a subgroup $C \leq D$ of order coprime to l such that CU/U is normal in G/U, and in particular C is normal in D. The $\mathbb{F}_l[C]$-module W is then semisimple, so there exists a subgroup $V \leq W$

that is normalised by C, and such that $VU = W$ and $V \cap U = \{1\}$. Since CU/U is normal in G/U. Since W/U is also normal in G/U, CU/U and W/U commute, so we have $[C, V] \leq U$. But since V is normalised by C, we also have $[C, V] \leq V$, so C in fact centralises V. Thus, V is contained in W^C, which is a normal subgroup of G. If $W^C = W$, then $C \leq N$, contradicting the assumption that $N = \{1\}$. So W^C is a proper non-trivial subgroup of W. Moreover, since C is normal in D, W^C is normal in G. Since $l \nmid \#C$, there exists a non-trivial subgroup $U' \leq W$ such that $W = U'W^C$ and $U' \cap W^C = \{1\}$. In particular, $(U')^C = \{1\}$. By Theorem 5.2.2, the quotient G/W^C is (p, l)-Dress, so CW^C/W^C is contained in the normal subgroup $O^l(G/W^C) = O^l(D)W^C/W^C$. It follows that $[C, U'] \leq W^C O^l(D)$. But since U' is normalised by C, we also have $[C, U'] \leq U'$. Since $U' \cap W^C O^l(D)$ is trivial, we deduce that C centralises U' – a contradiction.

Case 2(c): $N = \{1\}$**, and** D **acts irreducibly on** W**.** This is either case (i) or (ii) of the theorem if $d \leq 1$, and case (iii) otherwise.

Case 3: $l | \#D$ **and the extension of** D **by** W **is not split.** The extension of $O_p(D)$ by W is split, so there exists a subgroup P of G that intersects W trivially and maps isomorphically onto $O_p(D)$ under the quotient map $G \to G/W$.

Case 3(a): $P = \{1\}$ **and** $l \neq q$**.** Then the l-Sylow subgroup S of G is normal in G. If it is elementary abelian, then the extension of D by S splits by the Schur-Zassenhaus theorem, and we are in Case 2 of the proof. Otherwise, the Frattini subgroup $\Phi = [S, S]S^l$ of S is non-trivial, and since it is a characteristic subgroup of S, it is normal in G. By Theorem 5.2.2, the quotient G/Φ is (p, q)-Dress, so the l-Sylow subgroup of G/Φ is cyclic. But since Φ consists of precisely the "non-generators" of S, this implies that S itself is cyclic, so G is q-quasi-elementary.

Case 3(b): $P = \{1\}$ **and** $p \neq l = q$**.** Let C be a $(-l)$-Hall subgroup of G. The assumptions on G imply that C is cyclic, and that D is of the form $C \rtimes Q$, where Q is a q-group. If $W^C = W$, then C is a normal subgroup of G, and G is q-quasi-elementary. If $W^C = \{1\}$, then Lemma 4.3.5 implies that the extension of D by W splits - a contradiction. So W^C is a non-trivial proper subgroup of W, which is normal in G, since C is normal in D. Since the order of C is coprime to l, the $\mathbb{F}_l[C]$-representation W is semisimple, so there exists a subgroup U of W that is normalised by C, and such that $UW^C = W$, $U \cap W^C = \{1\}$. By Theorem 5.2.2, the quotient G/W^C is (p, q)-Dress. But it has trivial p-Sylow, so it is q-quasi-elementary, and CW^C/W^C is normal in G/W^C. Thus $[C, U] \leq W^C$. But also, U is a C-sub-representation, so $[C, U] \leq U$, whence we deduce that C centralises U, so that $W^C = W$, a contradiction.

Case 3(c): $P \neq \{1\}$ **and** $W^P = W$**.** In this case, P is a non-trivial normal

p-subgroup of G. By Theorem 5.2.2, the quotient G/P is (p,q)-Dress, therefore so is G itself, so by Theorem 5.3.2 it is either quasi-elementary or (p,p)-Dress.

Case 3(d): $P \neq \{1\}$ and $W^P \neq W$. By Lemma 4.3.5, the subgroup W^P is non-trivial. Moreover, since P is a normal subgroup of D, W^P is a normal subgroup of G. The $\mathbb{F}_l$-representation W of P is semisimple, so there exists a subgroup $U \leq W$ that is normalised by P and such that $UW^P = W$, $U \cap W^P = \{1\}$. By Theorem 5.2.2, the quotient G/W^P is (p,q)-Dress. We claim that $O_p(G/W^P)$ must be trivial. Indeed, $O_p(G/W^P)$ is necessarily of the form NW^P/W^P where N is a subgroup of P that is normal in D. But then we have $[N,U] \leq W^P$, and also $[N,U] \leq U$, since U is a P-sub-representation of W. Thus N centralises U, whence $W^N = W$. By Lemma 4.3.5, the assumption that the extension of D by W is non-split forces $N = \{1\}$.

Case 3(d)(i): $l \neq q$. Then the l-Sylow subgroup of G/W^P must be cyclic and normal in G/W^P. Since $W^P \neq W$, and since we assume that $l|\#D$, this implies that the l-Sylow subgroup S of G is normal in G and has an element of order strictly greater than l. Thus, the Frattini subgroup $\Phi = [S,S]S^l$ of S is non-trivial, and since it is a characteristic subgroup of S, it is normal in G. By Theorem 5.2.2, the quotient G/Φ is (p,q)-Dress, so the l-Sylow subgroup of G/Φ is cyclic. But that implies that the l-Sylow subgroup of G is also cyclic, and therefore $W \cong C_l$, contradicting the assumptions that $\{1\} \neq W^P \neq W$.

Case 3(d)(ii): $l = q$. Then $p \neq q$, so the p-Sylow subgroup of the (p,q)-Dress group G/W^P must be normal in G/W^P, contradicting the observation that $O_p(G/W^P)$ is trivial.

This covers all possible cases, and concludes the proof of the theorem. □

5.4 Some Explicit Relations

Proposition 5.4.1. *Let l be a prime that is distinct from p, and let $G = C_l \rtimes C$, where C is a non-trivial cyclic group, acting faithfully on C_l. Then $\mathrm{Prim}_{\mathrm{K}_{\mathbb{F}_p}} G \cong \mathbb{Z}$, and is generated by the following relation Θ:*

1. *if $C \cong C_m C_n$, where m, $n > 1$ are coprime integers, then*

$$\Theta = [G] - [C] + \alpha([C_n] - [C_l \rtimes C_n]) + \beta([C_m] - [C_l \rtimes C_m]),$$

where α, β are any integers satisfying $\alpha m + \beta n = 1$;

2. *if* $C \cong C_{q^{k+1}}$*, where* q *is a prime, and* $k \geq 0$*, then*

$$\Theta = [C_{q^k}] - q[C] - [C_l \rtimes C_{q^k}] + q[G].$$

Proof. The hypotheses on G imply that all non-cyclic subquotients of G have trivial p-core, so a subquotient of G is cyclic if and only if it is p-hypo-elementary. It therefore follows from Artin's and Conlon's Induction Theorems (Theorems 2.3.1 and 2.3.8), that $\mathrm{B}_F(G) = \mathrm{B}_{\mathbb{Q}}(G)$, and $\mathrm{Prim}_{\mathrm{K}_{\mathbb{F}_p}}(G) = \mathrm{Prim}_{\mathbb{Q}}(G)$. The result therefore follows from [2, Theorem A, case 3a]. $\square$

The remainder of the section is devoted to the proof of the following result.

Theorem 5.4.2. *Let* l *be a prime distinct from* p*, let* $G = W \rtimes Q$*, where* $W = (C_l)^d$ *with* $d > 1$*, and* Q *is a* (p, q)*-Dress group acting faithfully on* W*. Assume that either*

1. Q *acts irreducibly on* W*, or*

2. $d = 2$*, and* $G = (C_l \rtimes P_1) \times (C_l \rtimes P_2)$*, where the* P_i *are* p*-groups acting faithfully on the respective factor of* W*.*

Then $\mathrm{Prim}\, G$ *is generated by the relation*

$$\Theta = [G] - [Q] + \sum_{\substack{U \leq_G W \\ [W:U]=l}} ([UN_Q(U)] - [WN_Q(U)]),$$

where the sum runs over a full set of G*-conjugacy class representatives of index* l *subgroups of* W*.*

Lemma 5.4.3. *Let* G *be a finite group, let* l *be a prime, and let* k *be a field of characteristic* l*. Suppose that there exists a normal subgroup* N *of* G *such that* $l \nmid \#N$ *and* G/N *is a cyclic* l*-group. Then for every* $k[G]$*-module* M*, we have* $\dim_k M^G = \dim_k M_G$*. Moreover, if* M *is an indecomposable* $k[G]$*-module, then this dimension is* 0 *or* 1*.*

Proof. Let M be a $k[G]$-module. We may, without loss of generality, assume that M is indecomposable. If $M^G = M_G = 0$, then there is nothing to prove, so suppose otherwise. The element $e = (1/\#N)\sum_{n\in N} n \in k[G]$ is a central idempotent, and we have $M^N = eM \neq 0$. Since $M = eM \oplus (1-e)M$, and M is indecomposable, it follows that $eM = M$, so that M is an indecomposable $k[G/N]$-module. Since G/N is a cyclic l-group, it follows from [24, 26] that the maximal semisimple submodule and the maximal semisimple quotient module of M are both simple. But the only simple $k[G/N]$-module is the trivial one, which completes the proof. $\square$

Lemma 5.4.4. *Let $G = W \rtimes Q$ be a soluble group where $W = (C_l)^d$ for some prime $l \neq p$ and let $K \leqslant G$ be a p-hypoelementary subgroup. Then after replacing K with a conjugate we may write $K = K_{l'} \cdot \langle \gamma \rangle$ where $K_{l'}$ is a normal subgroup of K with order coprime to l, $K_{l'} \leqslant Q$ and $\langle \gamma \rangle$ is a cyclic group of l-power order.*

Proof. As G is soluble there exists a $(-l)$-Hall subgroup of G, unique up to conjugation, which may be chosen to lie in Q. After replacing K with a conjugate we may therefore assume that its $(-l)$-Hall subgroup $K_{l'}$ lay in Q. Since K is p-hypoelementary, and $p \neq l$ it follows that K_l' must be a normal subgroup of K and that the l-Sylow subgroup of K is cyclic. This completes the proof. □

Lemma 5.4.5. *Let G, and $K \leqslant G$ be as in 5.4.4 then writing $\gamma = yh$ with $y \in W$ and $h \in Q$ and viewing W as an $\mathbb{F}_l[K]$-module of K under the natural conjugation action, there exists a codimension 1 submodule U not containing y. Furthermore W/U is the trivial $\mathbb{F}_l[K]$-module.*

Proof. We restrict W to $K_{l'}$, the resulting module is semisimple and the trivial isotypical component W_1 is a summand of W as an $\mathbb{F}_l[K_{l'}]$-module. Since $K_{l'}$ is a normal subgroup of order coprime to l and index a power of l then Lemma 5.4.3 shows that W_1 is a submodule of W as an $\mathbb{F}_l[K]$-module.
We now show that $\dim(W_1) \geqslant 1$ and that $y \in W_1$. Since $K_{l'} \triangleleft K$ we have for all $k \in K_{l'}$ that $yhkh^{-1}y^{-1} = [y, hkh^{-1}]hkh^{-1} \in K_{l'} \subset Q$. Since $K_{l'}^h \leqslant Q$ it follows that $[y, hkh^{-1}] \in W \cap Q = \{e\}$, and so $y^h \in W_1$. Furthermore we have $y = \gamma y^h \gamma^{-1}$ is in W_1 as claimed.
If K acts semisimply on W, as is the case when $h = \{e\}$, then $\langle y \rangle$ is a summand of W and we are done. Otherwise let $N \leqslant W_1$ be an indecomposable summand containing y. Since $K_{l'}$ acts trivially on N we may view it as an $\mathbb{F}_l[\langle \gamma \rangle]$-module, all composition factors of which are trivial. Let $\{e_1, \ell, e_k\}$ be a basis of N such that γ acts in Jordan normal form. Now suppose that y is contained in the proper submodule L generated by $\{e_1, \ell e_{k-1}\}$ so that $y = e_1^{\alpha_1} \ell e_{k-1}^{\alpha_{k-1}}$ then the element $\beta = e_2^{-\alpha_1} \ell e_k^{-\alpha_{k-1}}$ conjugates yh to h and commutes with $k_{l'}$ a contradiction. Taking L direct sum the complement of N gives an $\mathbb{F}_l[K]$ submodule of W of codimension 1 not containing y. Since K acts trivially on y, W/U is the trivial module. □

Recall from Theorem 2.1.11 that if X is a G-set, and U is a subgroup of G, then $f_U(X)$ denotes the number of fixed points in X under U, and that this extends linearly to a ring homomorphism $f_U \colon \mathrm{B}(G) \to \mathbb{Z}$.

Lemma 5.4.6. *Let G be a finite group, and let H and K be subgroups. Then $f_K(H) = \#\{g \in G/H : {}^gK \subseteq H\}$.*

Proof. By Mackey's formula for G-sets, we have

$$\mathrm{Res}_K(G/H) = \bigsqcup_{g \in K\backslash G/H} H/H \cap {}^gK.$$

By definition, $f_K(H)$ is the number of singleton orbits under the action of K on G/H, so $f_K(H) = \#\{g \in K\backslash G/H : {}^gK \subseteq H\}$. An explicit calculation shows that the map $G/H \to K\backslash G/H$, $gH \mapsto KgH$ defines a bijection between $\{g \in G/H : {}^gK \subseteq H\}$ and $\{g \in K\backslash G/H : {}^gK \subseteq H\}$. $\square$

Lemma 5.4.7. *Let l be a prime number, let $d \geq 1$ be an integer, let $G = W \rtimes Q$, where $W = (C_l)^d$, and where Q is regarded as a subgroup of G. Let Θ be the element of $b(G)$ given by*

$$\Theta = [G] - [Q] + \sum_{\substack{U \leq_G W \\ [W:U]=l}} ([UN_Q(U)] - [WN_Q(U)]),$$

where the sum runs over a full set of G-conjugacy class representatives of index l subgroups of W. Then for every subgroup K of Q, we have $f_K(\Theta) = \#(W_K) - \#(W^K)$.

Proof. For $w \in W$, we have that ${}^wK \leq Q$ if and only if $(wkw^{-1}k^{-1})k \in Q$ for all $k \in K$. Since the bracketed term is in W, this is equivalent to $wkw^{-1}k^{-1} = 1$ for all $k \in K$, i.e. to $w \in W^K$. Since W forms a transversal for G/Q, it follows by Lemma 5.4.6 that

$$f_K(G) = 1, \tag{5.4.8}$$
$$f_K(Q) = \#\{w \in W : {}^wK \leqslant Q\} = \#(W^K). \tag{5.4.9}$$

We now calculate the remaining terms in $f_K(\Theta)$. Let $U \leq W$ be a subgroup of index l. Let $T \subseteq Q$ be a transversal for $G/WN_Q(U)$, so that

$$f_K(WN_Q(U)) = \#\{t \in T : {}^tK \leqslant N_Q(U)\}. \tag{5.4.10}$$

Let $x \in W \setminus U$. Then a transversal for $G/UN_Q(U)$ is given by $\{x^mt : t \in T, 0 \leq m \leq l-1\}$.

To count the number of elements $y = x^mt$ in this transversal for which ${}^yK \leqslant UN_Q(U)$, we note that for all $k \in K$, and for $y = x^mt$ as above, we have ${}^yk = (x^mtkt^{-1}x^{-m}tk^{-1}t^{-1})(tkt^{-1})$, and of the two bracketed terms the first is in W, and is equal to $[x^m, {}^tk]$, while the second is in Q. It follows that we have

${}^{y}K \leqslant UN_Q(U)$ if and only if $[x^m, {}^{t}K] \leqslant U$ and ${}^{t}K \leqslant N_Q(U)$. If $m \neq 0$, then these conditions are equivalent to $[\langle x\rangle, {}^{t}K] \leqslant U$ and ${}^{t}K \leqslant N_Q(U)$, and in particular are independent of m. Partitioning the transversal $\{x^m t : t \in T, 0 \leq m \leq l-1\} = T \sqcup \{x^m t : t \in T, 1 \leq m \leq l-1\}$, we find that

$$\begin{aligned}
&f_K(UN_Q(U) - WN_Q(U)) \\
&\quad = (l-1)\#\{t \in T : {}^{t}K \leqslant N_Q(U), [\langle x\rangle, {}^{t}K] \leqslant U\} \\
&\quad = (l-1)\#\{t \in T : K \leq N_Q({}^{t}U), K \text{ acts trivially on } W/{}^{t}U\}.
\end{aligned}$$

As t runs over T, ${}^{t}U$ runs once over the G-orbit of U, since T is a transversal for $G/WN_Q(U)$. It follows that if we take the sum of the above expression over a full set of representatives U of G-conjugacy classes of index l subgroups of W, we obtain

$$\begin{aligned}
\sum_{\substack{U \leq_G W \\ [W:U]=l}} f_K(UN_Q(U) - WN_Q(U)) &= (l-1)\#\{\text{quotients of } W_K \text{ of order } l\} \\
&= \#(W_K) - 1. \qquad (5.4.11)
\end{aligned}$$

The result follows by combining equations (5.4.8), (5.4.9), and (5.4.11). $\square$

We now prove our main theorem.

Theorem 5.4.12. *Let l be a prime distinct from p, let $G = W \rtimes Q$, where $W = (C_l)^d$ with $d > 1$, and Q is a (p,q)-Dress group acting faithfully on W. Assume that either*

1. *Q acts irreducibly on W, or*

2. *$d = 2$, and $G = (C_l \rtimes P_1) \times (C_l \rtimes P_2)$, where the P_i are p-groups acting faithfully on the respective factor of W.*

Then $\operatorname{Prim} G$ *is generated by the relation*

$$\Theta = [G] - [Q] + \sum_{\substack{U \leq_G W \\ [W:U]=l}} ([UN_Q(U)] - [WN_Q(U)]),$$

where the sum runs over a full set of G-conjugacy class representatives of index l subgroups of W.

Proof. We prove the proposition by counting the fixed points under p-hypo-elementary subgroups K of G. The case where these subgroups are cyclic follows from simply taking inner products with the irreducible characters of G which are easily described, for instance see [29], the calculation proceeds identically as in [2] so we omit it here.

We split the remaining subgroups into two classes, those K for which after conjugation may be chosen to lie in Q and those which cannot. In the reducible case Q is the unique complement of W in G by Schur-Zassenhaus. Otherwise Q is soluble, and it also acts faithfully and irreducibly on W, so it follows from [31, Theorem A] that Q is the unique complement of W in G up to conjugacy.

1. Let K be a p-hypo-elementary subgroup of Q then by Lemma 5.4.6 we see that $f_K(\theta) = \#W_K - \#W^K$. By Lemma 5.4.3 this number is zero.

2. Now assume that $K \nleqslant Q$. By Lemma 5.4.4 we may write, possibly after replacing K by a conjugate, $K = K_{l'} \cdot \langle \gamma \rangle$, where $K_{l'} \leqslant Q$ is the $(-l)$-Hall subgroup of K and $\gamma = \langle yh \rangle$ with $y \in W$ and $h \in Q$. We now calculate $f_K(\theta)$ termwise. We have, using the notation of Lemma 5.4.6,

$$\begin{aligned}
f_K(G) &= 1,\\
f_K(Q) &= 0,\\
f_K(WN_Q(U)) &= \#\{t \in T : {}^tK \leqslant WN_Q(U)\},\\
f_K(UN_Q(U)) &= \#\{t \in T : {}^tK \leqslant UN_Q(U)\}\\
&\quad + \#\{x^m t \in T, 1 \leqslant m \leqslant l-1 : {}^{x^m t}K \leqslant UN_Q(U)\}.
\end{aligned}$$

Suppose that for some $t \in T$ we have ${}^tK \subseteq WN_Q(U)$ but ${}^tK \nsubseteq UN_Q(U)$. Since ${}^tK \leqslant H$ if and only if ${}^tK_{l'} \leqslant H$ and ${}^t\gamma \in H$ it follows that ${}^ty \notin U$. With this observation we have,

$$\begin{aligned}
f_K(\theta) = 1 +& \\
\sum_{\substack{U \leqslant_G W \\ [W:U]=l}} &(\#\{x^m t \in T, 1 \leqslant m \leqslant l-1 : {}^{x^m t}K \leqslant UN_Q(U)\}\\
&- \#\{t \in T : {}^tK \leqslant N_G(U), {}^ty \notin U\}).
\end{aligned}$$

We see that ${}^{x^m t}K \leqslant UN_Q(U)$ if and only if $[\langle x \rangle, {}^tK] \leqslant U$ and ${}^tK \leqslant UN_Q(U)$. The second condition is then equivalent to ${}^tK \leqslant N_G(U)$ and ${}^ty \in U$ as before. We now have,

$$
\begin{aligned}
f_K(\theta) = 1+ & \\
& \sum_{\substack{U \leqslant_G W \\ [W:U]=l}} ((l-1)\#\{t \in T : {}^tK \leqslant N_G(U), {}^ty \in U, [\langle x \rangle, {}^tK] \leqslant U\} \\
& - \#\{t \in T : {}^tK \leqslant N_G(U), {}^ty \notin U\}) \\
& = 1 + (l-1)\#\{U < W, [W:U] = l : y \in U, K \leqslant N_G(U), (W/U)^K = W/U\} \\
& - \#\{U < W, [W:U] = l : y \notin U\, K \leqslant N_G(U)\}.
\end{aligned}
$$

By Lemma 5.4.5 there exists a $U_1 < W$ of index l which is normalised by K and does not contain y. We now wish to show that for each index l subgroup U of W normalised by K, containing y, and for which the quotient W/U is the trivial representation there are $(l-1)$ distinct index l subgroups not containing y determined uniquely by U and different form U_1.

Suppose we have such a U then, let $A = U \cap U_1$, this is a codimension 1 subrepresentation of both U and U_1 so there exists $v \in U_1$, such that $v \notin A$. Since K acts trivially on W/U and $v \notin U$ any element of K acts on v by $k : v \to vu_k$ for some $u_k \in U$. We must also have $vu_k \in U_1$ so in particular $u_k \in U \cap U_1 = A$. The corresponding argument also holds swapping y for v and exchanging U and U_1. Thus $\langle vy^\alpha \rangle A$ is a hyperplane normalised by K not containing y which is distinct from U_1 provided $\alpha \neq 0$. So for each hyperplane U containing y with W/U trivial, there are $(l-1)$ distinct hyperplanes uniquely determined by U, not containing y normalised by K, and different from U_1.

Now assume we have a hyperplane normalised by K not containing y and different from U_1; let A be their intersection. Now, A is codimension 2 in W and $W = \langle y \rangle \langle v \rangle A$ where $v \in U_1 \notin A$. The hyperplane $\langle y \rangle A$ contains y and is normalised by K; since $v \in U_1$ its quotient W/U is trivial. Since $\langle v^\alpha \rangle = \langle v \rangle$, there are $(l-1)$ hyperplanes not containing y normalised by K and distinct from U_1, each of which gives the same A.

We have established an $(l-1) : 1$ correspondence between $\{U < W, [W : U] = l : y \in U, K \leqslant N_G(U), (W/U)^K = W/U\}$ and $\#\{U < W, [W : U] = l : y \notin U\, K \leqslant N_G(U)\}\backslash U_1$, it follows that $f_K(\theta) = 0$ completing the proof.

□

Theorem 5.4.13. *Let G be a finite group which is not (p, q)-Dress G admits a primitive F-relation if and only if:*

1. *G is not soluble and is an extension of the form:*

$$1 \to S^d \to G \to Q \to 1$$

where Q is a (p,q)-Dress group, no non-trivial subgroup of S^d is normal in G and also $Q \hookrightarrow Out(S^d)$.

2. *$G = W \rtimes Q$ where $W = C_l^d$ and Q is a (p,q)-Dress group acting faithfully and irreducibly on W.*

3. *$G = (C_l \rtimes Q_1) \times (C_l \rtimes Q_2)$ where the Q_i are abelian q-groups acting faithfully on the cyclic groups.*

In the final two cases we have that the previous relations generate $Prim(G)$ and we can describe its shape: in case ii)$Prim(G) = \mathbb{Z}/q\mathbb{Z}$ unless Q is p-Hypo-Elementary then $Prim(G) = \mathbb{Z}$, in case iii) $Prim(G) = \mathbb{Z}/q\mathbb{Z}$ in all cases.

5.5 Dress Groups

We have already seen in Theorem 5.3.2 that (p,q)-Dress groups which are not quasi-elementary admit no primitive relations for $p \neq q$. The case where $p = q$ is much harder to analyse as the subgroup structure of a (p,p)-Dress group is much harder to understand and describe. A key step of Theorem 5.3.2 was an explicit construction of enough subquotients to contribute a full rank imprimitive sublattice of relations. Through direct computations in the computer package `MAGMA` we have some evidence to suggest the following conjecture.

Conjecture 5.5.1. *Let G be a (p,p)-Dress group. Then G admits a primitive relation only if G is quasi-elementary.*

This conjecture is the strongest possible, we know that there exist relations for p-quasi-elementary groups in characteristic p as we shall establish there existence in the following proposition.

Proposition 5.5.2. *Let $G = C_l \rtimes C_{p^r}$ with C_{p^r} acting faithfully. Then* $\mathrm{Prim}_{\mathrm{K}_{\mathbb{F}_p}}(G)$ *is isomorphic to $\mathbb{Z}$.*

Proof. This is a special case of Proposition 5.4.1. $\square$

Chapter 6

Applications

6.1 Introduction

The purpose of this chapter is to present some easy applications of the work in this thesis. The first section is devoted to investigating the application of Brauer relations in positive characteristic to arithmetic problems. There are many such applications as cohomological Mackey functors arise frequently in Number theory when one studies Galois extensions. The second, more algebraic application is to use the results of chapter three to try to describe when a group has a Brauer relation modulo p for all primes p.

6.2 Cohomological Mackey Functors in Number Theory

This was the topic of a very nice article by Bley and Boltje [6], in which they give examples of Cohomological Mackey functors which occur naturally in number theory. Using Corollary 2.6.15 of Yoshida's Theorem along with explicit induction formulae such as those appearing in [7] and [8] they give relations between the evaluations of these Mackey functors. In light of the results of Chapter 5 we have a supply of Brauer Relations in positive characteristic and equivalently by Lemma 2.2.21 over $\mathbb{Z}_p$. We will recall some examples from [6] for which our relations may give interesting results.

Example 6.2.1. The following are examples of cohomological $\mathbb{Z}_p$ Mackey functors.

1. **Elliptic Curves**. Let K/F be a Galois extension of Number fields with Galois group G and let E be an elliptic curve defined over the base field F. Then E may be viewed as a cohomological Mackey functor in the following sense, for

each subgroup $H \leqslant G$ let $E(H) := E(K^H)$ be the abelian group of the K^H-points of E. Now we have induction maps corresponding to the trace map, and restriction given by the inclusion $E(K) \hookrightarrow E(H)$ for $H \leqslant K$. This data describes a cohomological $\mathbb{Z}$-Mackey functor for G. Tensoring the torsion-part by $\mathbb{Z}_p$ gives a $\mathbb{Z}_p$ cohomological Mackey functor for G.

2. **Abelian varieties**. The p-power torsion of any Abelian variety is also a $\mathbb{Z}_p$-cohomological Mackey functor by an identical construction.

3. **Class Groups** Let K/F be a Galois extension of number fields with Galois group G. Then the group of fractional ideals $I(H) := I(K^H)$, and the subgroup of principal fractional ideals $P(H) = P(K^H)$. may both be viewed as Cohomological $\mathbb{Z}$ Mackey functors for G. The for $H \leqslant J$ the restriction map takes a prime ideal $\mathscr{P}$ of $\mathcal{O}_{K^H}$ to the ideal $\mathscr{P}\mathcal{O}_{K^J}$. The induction map is then the norm map $\mathbb{N}_{K^H/K^J}$. Taking the quotient we have the $\mathbb{Z}$ cohomological Mackey functor $Cl(H)$. The p-part of the class group is then a $\mathbb{Z}_p$ cohomological Mackey functor for G.

The relations of chapter 5 can be used to prove more theorems similar to Theorem 1.1.1 by exploiting example 2 above.

Theorem 6.2.2. *Let K/F be a Galois extension of number fields with Galois group S_4. Let $H_1 = C_4$, $H_2 = D_8$, then, writing $Cl_3(K^H)$ for the 3-part of the class group of K^H we have:*

$$Cl_3(K^{S_4}) \oplus Cl_3(K^{H_1}) = Cl_3(K^{S_3}) \oplus Cl_3(K^{H_2}).$$

Proof. This follows from the previous example, Corollary 2.6.15 and Lemma 5.4.5. □

Remark 6.2.3. It follows that for such K/F the ratio of class numbers

$$h(F)h(K^{H_1})/h(K^{S_3})h(K^{H_2}),$$

can only have non-trivial valuation at 2. Indeed since S_4 is $(2, 2)$-Dress and hence primordial for $\mathrm{Im}(m_{\mathbb{F}_2})$ the relation in Theorem 5.4.12 can't hold at 2.

6.3 Everywhere Local Brauer Relations

It would be useful, where possible, to have relations for $\mathbb{Z}$-cohomological Mackey functors, but integral Brauer relations are hard to study. Integral Brauer relations

have been investigated in [14], [21] and [28]. It is in general very hard to show when two $\mathbb{Z}[G]$-permutation lattices are isomorphic, it is remarked in [21] that one may show that if for some finite group G and $H_1, H_2 \leqslant G$ we have $\mathbb{Z}_p[G/H_1] = \mathbb{Z}_p[G/H_2]$ for all p then we may conclude that there is an integer n such that $\mathbb{Z}[G/H_1]^{\oplus n} = \mathbb{Z}[G/H_2]^{\oplus n}$ but that this need not hold for $n = 1$. One way around this is to note that the evaluations of a $\mathbb{Z}$-Mackey functor $\mathcal{M}$ which only takes values in finitely generated abelian groups are entirely determined by the completed functors $\mathcal{M}_p = \mathcal{M}_p \otimes \mathbb{Z}_p$ as p ranges over all primes. Instead of Brauer relations over $\mathbb{Z}$ therefore, it is sensible to consider everywhere local Brauer relations. More formally let $\mathcal{K}_p(-)$ denote the kernel of the map from the Burnside functor to the $\mathbb{Z}_p$ Representation ring functor then an everywhere local Brauer relation for G is an element $\theta \in \cap_p \mathcal{K}_p(G)$. Such relations will result in non-trivial isomorphisms on the evaluations of $\mathbb{Z}$-cohomological Mackey functors such as the fixed point functor for a $\mathbb{Z}[G]$-module M.
We will show that using the results of the previous chapters one may rapidly get results describing this kernel.
To begin to describe everywhere local relations we will rephrase the problem in terms of Green functors with inflation and use the results of chapter three. To that end we will require the following definition

Definition 6.3.1. Let G be a finite group, and let $a(\mathbb{Z}[G])$ be the ring of integral representations, we will define the genus ring $\Gamma(G)$ of G to be the quotient of $a(\mathbb{Z}[G])$ by the ideal generated by formal differences of isomorphism classes $[A]-[B]$ whenever A and B are in the same genus, that is $\mathbb{Z}_p \otimes_{\mathbb{Z}} A \cong \mathbb{Z}_p \otimes_{\mathbb{Z}} B$ for all primes p.

Let $g(M)$ denote the genus containing a $\mathbb{Z}[G]$-module M in $\Gamma(G)$, then equipping $\Gamma(-)$ with the maps $\mathrm{Ind}_{G/H} : g(M) \mapsto g(\mathrm{Ind}_{G/H}(M))$, $\mathrm{Res}_{G/H} : g(N) \mapsto g(\mathrm{Res}_{G/H}(N))$ and $\mathrm{Inf}_{G/N} : g(L) \mapsto g(\mathrm{Inf}_{G/N}(L))$ makes it into a GFI. Our aim is then to describe the kernel GFI K_Γ of the map of GFIs:

$$\begin{aligned} m_\Gamma(G) :& b(G) \to \Gamma(G) \\ & [H] \mapsto g(\mathrm{Ind}_{G/H}(1)) \end{aligned}$$

Lemma 6.3.2. *The class of coprimordial groups $\mathcal{C}(\mathrm{Im}(m_\Gamma))$ is precisely the class of groups which are p-hypo-elementary for some prime p.*

Proof. The kernel of restriction functor $\mathcal{K}_{\mathrm{Im}(m_\Gamma)}(-)$ is equal to $\prod'_p \mathcal{K}_{\mathrm{Im}(m_p)}(-)$ whose evaluation is non-trivial whenever a group is p-hypo-elementary for some prime p. $\square$

Lemma 6.3.3. *Let q be a prime. The class of primordial groups $\mathcal{P}(\mathrm{Im}(m_\Gamma)_q)$ is the class of groups which are (p, q)-Dress for at least one prime p.*

Proof. We have the containment $\mathcal{P}(\mathrm{Im}(m_\Gamma)_q) \subseteq \cup_p \mathcal{D}_{p,q}$ by Lemma 3.3.4. Now we note that if G is not primordial of $\mathrm{Im}(m_\Gamma)_q$ then it is not primordial for $Im(m_p)_q$ for any prime p. The result follows. $\square$

Now we may apply the results of Chapter three to describe for which non-primordial finite groups G the primitive quotient $\mathrm{Prim}_{K_\Gamma}(G)$ is non-trivial.

Theorem 6.3.4. *Let G be a finite group that is not a (p, q)-Dress group for any prime numbers p and q. Then:*

(a) *if every proper quotient Q of G is p_Q-hypo-elementary for some prime p_Q, then $\mathrm{Prim}_{\mathrm{K}_\Gamma}(G) \cong \mathbb{Z}$;*

(b) *if there exists a fixed prime number q such that the following statements hold:*

- *any proper quotient Q of G is a (p_Q, q)-Dress group for at least one prime p_Q, and*
- *any Q which is (p_H, q)-Dress and (p_2, l)-Dress for $l \neq q$ is p_H-hypo-elementary, and*
- *at least one proper quotient Q them is not hypo-elementary.*

Then $\mathrm{Prim}_{\mathrm{K}_\Gamma}(G) \cong \mathbb{Z}/q^n\mathbb{Z}$ *for some natural number $n \geqslant 1$;*

(c) *if there exists a proper quotient of G that is not a (p, q)-Dress group for any prime numbers p and q, or if there exist pairs of prime numbers p_1, q_1 and p_2, q_2 with the q_i distinct and, for $i = 1$ and 2, a proper quotient of G that is a non-p_i-hypo-elementary (p_i, q_i)-Dress group, then* $\mathrm{Prim}_{\mathrm{K}_\Gamma}(G)$ *is trivial.*

Moreover, in all cases, $\mathrm{Prim}_{\mathrm{K}_\Gamma}(G)$ *is generated by any element of* $\mathrm{K}_\Gamma(G) \subseteq b(G)$ *of the form* $[G/G] + \sum_{H \lneq G} a_H[G/H]$, $a_H \in \mathbb{Z}$.

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