New special cases of the Quadratic Assignment Problem with diagonally structured coefficient matrices

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Abstract

We consider new polynomially solvable cases of the well-known Quadratic Assignment Problem involving coefficient matrices with a special diagonal structure. By combining the new special cases with polynomially solvable special cases known in the literature we obtain a new and larger class of polynomially solvable special cases of the QAP where one of the two coefficient matrices involved is a Robinson matrix with an additional structural property: this matrix can be represented as a conic combination of cut matrices in a certain normal form. The other matrix is a conic combination of a monotone anti-Monge matrix and a down-benevolent Toeplitz matrix. We consider the recognition problem for the special class of Robinson matrices mentioned above and show that it can be solved in polynomial time.

Keywords. combinatorial optimization; quadratic assignment; Robinsonian; cut matrix; Monge matrix; Kalmanson matrix.

1 Introduction

In this paper we investigate the Quadratic Assignment Problem (QAP), which is a well-known problem in combinatorial optimization; we refer the reader to the book [9] by Çela and the book [6] by Burkard, Dell'Amico & Martello for comprehensive surveys on the QAP. The QAP in Koopmans-Beckmann form [25] takes as input two $n \times n$ square matrices $A = (a_{ij})$ and $B = (b_{ij})$ with real entries and will be denoted by $QAP(A, B)$. The goal is to find a permutation π that minimizes the objective function

$$Z_\pi(A, B) \ := \ \sum_{i=1}^{n} \sum_{j=1}^{n} a_{\pi(i)\pi(j)} \, b_{ij}. \tag{1}$$

Equivalently the objective function can be written as $\langle A^\pi, B\rangle = Tr(A^\pi B^T)$, where $A^\pi = (a^\pi_{ij}) = (a_{\pi(i)\pi(j)})$ is the matrix which results from matrix A after permuting both its rows and

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its columns according to permutation π. Tr is the trace operator and $Tr(A^\pi B^T)$ is the trace of the product of the matrices A^π and B. The formulation of the goal function of the QAP by means of the trace operator is used in many applications, see eg. [9, 19]. Here π ranges over the set S_n of all permutations of $\{1, 2, \ldots, n\}$.

The QAP is a notoriously difficult problem both from the practical and from the theoretical point of view. In spite of the amazing development of computer and software technology nowadays it is still considered a challenge to exactly solve moderate size instances of size more than 30, i.e. $n \geq 30$ [30]. From the theoretical point of view Sahni and Gonzalez [40] have shown that no constant-factor approximation algorithm exists for the QAP unless $P = NP$. Queyranne [36] has shown that the existence of a constant-factor approximation algorithm for $QAP(A, B)$ implies $P = NP$ even in the case where A is the distance matrix of a set of points in the Euclidean line and B is a block-diagonal symmetric 0-1-matrix with zeroes on the main diagonal.

One branch of research on the QAP concentrates on the algorithmic behavior of strongly structured special cases; see for instance Burkard & al [5], Deineko & Woeginger [18], Çela & al [13], Çela, Deineko & Woeginger [10], and Laurent and Seminaroti [27] for typical results in this direction.

In our paper we follow recent developments and present several new results in this exciting area of research. In particular we discuss two new polynomially solvable special cases (p.s.s. cases) of the QAP involving diagonally structured matrices, see Definition 2.2. The new p.s.s. cases are *the down-benevolent QAP* and *the up-benevolent QAP*. The down-benevolent QAP is a $QAP(A, B)$ with A being both a Kalmanson and a Robinson matrix (see Definition 2.6 and 2.1, respectively) and B being a down-benevolent Toeplitz matrix (see Definition 2.2). This problem is solved to optimality by the identity permutation which will be denoted by id in the sequel. This new p.s.s. case is related to two other p.s.s. cases of the QAP known in the literature: (a) $QAP(A, B)$ with A being a Kalmanson matrix and B being a DW Toeplitz matrix (see [18] and Definition 2.6), and (b) $QAP(A, B)$ with A being a Robinson matrix and B being a simple Toeplitz matrix [27]. In the new p.s.s. case matrix A is more special and matrix B is more general than in the previous two p.s.s. cases. The up-benevolent QAP is a QAP(A,B) with A being a PS anti-Monge matrix (see Definition 3.7) and B being an up-benevolent Toeplitz matrix (see Definition 2.2). This problem is solved to optimality by the identity permutation. This new p.s.s. case is a generalization of another p.s.s. case of the QAP known in the literature, namely $QAP(A, B)$ where A is a symmetric monotone anti-Monge matrix and B is an up-benevolent Toeplitz matrix [5].

Further we focus on the so-called combined p.s.s. cases of the problem. They arise as a combination of different p.s.s. cases of the QAP which involve matrices of the same type[1]. This approach is interesting because it allows the identification of new and more complex p.s.s. cases of the QAP. In particular we show that a $QAP(A, B)$ with A being a conic combination of a symmetric anti-Monge matrix (see Definition 2.6) and a down-benevolent Toeplitz matrix (see Definition 2.2) and B being a conic combination of cut matrices in CDW normal form (see Definition 2.4) is solved by the identity permutation. Further we tackle a relevant question in this context, namely the recognition of cut matrices in CDW normal form, and show that it

[1]A short discussion on one of the combined special cases presented in this paper was published in [12]

can be decided efficiently.

Motivation. A direct, intrinsic and purely theoretical motivation is the identification of further p.s.s. cases and the better delimitation of the border between "simple" and "hard" cases of the QAP. From a more practical point of view, while being indeed very special and highly structured some of the p.s.s. cases discussed in this paper appear when modeling the *seriation problem*, which in turn has practical applications, for example in archaeology, gene sequencing, or order recovering for disordered Markov chains, see Fogel et al. [19] and the references therein.

Moreover it would be interesting to investigate whether the combined special cases of the QAP can be used to compute good lower bounds and/or heuristic solutions for the general problem. The idea is to "approximate" the coefficient matrices A and B of a given instance $QAP(A,B)$ by some matrices A' and B', respectively, such that $QAP(A',B')$ is an instance of a combined p.s.s. special case. Then, if A' and B' are chosen "appropriately", the optimal solution of $QAP(A',B')$ and its optimal value could serve as a heuristic solution and/or a lower bound for $QAP(A,B)$, respectively. Clearly, the crucial part is to find out what "approximate" and "appropriately" should mean, and this is definitely a challenging issue.

Outline of the paper. The paper is organized as follows. In Section 2 we define the matrix classes which play a role in the p.s.s. cases discussed in this paper and review relevant results from the literature. In the Section 3 we introduce two new p.s.s. cases of the QAP, the so-called down-benevolent QAP in Section 3.1, and the up-benevolent QAP in Section 3.2. Then in Section 3.3 we extend the variety of known p.s.s. cases of the QAP by introducing the so-called combined p.s.s. cases. Section 4 deals with conic representations of specially structured matrices. In Section 4.1 Kalmanson matrices and matrices which are both Kalmanson and Robinson matrices are characterized in terms of conic combinations of particular cut matrices. These results are then used in Section 4.2 to give a characterization of conic combinations of cut-matrices in CDW normal form. This characterization allows the efficient recognition of conic combinations of cut matrices in CDW normal form. Notice that this recognition problem is relevant because the conic combinations of cut matrices in CDW normal form are involved in the first combined p.s.s. case described in Section 3.3. We conclude with a summary of results and some issues for further research in Section 5.

2 Preliminaries and definitions

There are already quite a number of results known on p.s.s. cases of the QAP where the coefficient matrices A and B possess specific structural properties. We provide an overview of this kind of p.s.s. cases by introducing a classification of the involved coefficient matrices with particular properties and distinguish the following matrix classes.

Matrices with monotonicity properties including Robinson dissimilarities (called Robinson matrices in the following), Robinson similarities, monotone matrices, see Definition 2.1 later in this section.

Matrices with diagonal structural properties including Toeplitz matrices, circulant matrices, simple Toeplitz matrices, DW-Toeplitz matrices, up-benevolent Toeplitz matrices and down-benevolent Toeplitz matrices, see Definition 2.2.

Matrices with block structural properties including block matrices, cut matrices and cut matrices in CDW normal form, see Definition 2.4.

Matrices defined in terms of four-point conditions including Monge matrices, anti-Monge matrices and Kalmanson matrices, see Definition 2.6.

Sum matrices and constant matrices including sum matrices, weak sum matrices and weak constant matrices, see Definition 2.8.

Further we recall some existing results on p.s.s. cases where these matrix classes are involved.

Definition 2.1 (Matrices with monotonicity properties)
A symmetric matrix $A = (a_{ij})$ is a Robinson dissimilarity *or briefly a* Robinson matrix*, if for all $i < j < k$ it satisfies the conditions $a_{ik} \geq \max\{a_{ij}, a_{jk}\}$; in words, the entries in the matrix are placed in non-decreasing order in each row and column when moving away from the main diagonal.*

A symmetric matrix $A = (a_{ij})$ is a Robinson similarity*, if for all $i < j < k$ it satisfies the conditions $a_{ik} \leq \min\{a_{ij}, a_{jk}\}$.* [2]

An $n \times n$ matrix $B = (b_{ij})$ is called monotone*, if $b_{ij} \leq b_{i,j+1}$ and $b_{ij} \leq b_{i+1,j}$ holds for all $i, j \in \{1, 2, \ldots, n\}$, that is, if the entries in every row and column are sorted non-decreasingly from the left to the right and from the top to the bottom, respectively.*

In some QAP special cases considered in this paper the diagonal elements of the coefficient matrices do not impact the optimal solution. In these cases we assume them to be zero and set $a_{ii} = 0$, for all i.

The Robinson matrices were first introduced by Robinson [38] in 1951 in the context of an analysis of archaeological data. Since then they have been widely used in combinatorial data analysis; see the books [21, 22, 31, 32] and the surveys [3, 8] for examples of various applications of Robinson structures in quantitative psychology, analysis of DNA sequences, cluster analysis, etc. Special cases of the QAP involving Robinson matrices are discussed in Laurent and Seminaroti [27] and in Fogel et al. [19].

Definition 2.2 (Matrices with diagonal structural properties)
An $n \times n$ matrix $B = (b_{ij})$ is called a Toeplitz matrix *if it has constant entries along each of its diagonals; in other words, there exists a function $f\colon \{-n+1, -n+2, \ldots, -1, 0, 1, \ldots, n-1\} \to \mathbb{R}$ such that $b_{ij} = f(i-j)$, for all $1 \leq i, j \leq n$. The Toeplitz matrix B is fully determined by the function f and therefore f will be called the generating function of B. If $f(i) = f(i-n)$ holds for every $i \in \{1, 2, \ldots, n-1\}$, the Toeplitz matrix B is called a* circulant matrix.

A symmetric Toeplitz matrix whose generating function f fulfills $f(0) = 0$ and $f(1) \geq f(2) \geq \ldots \geq f(n-1)$ will be called a simple Toeplitz matrix*. (Notice that a simple Toeplitz*

[2] We define a Robinson matrix to be a dissimilarity. Notice that besides this definition also the definition of a Robinson matrix as a similarity is encountered in the literature.

matrix is a Toeplitz Robinson similarity with zeroes on the main diagonal.) These matrices were introduced by Laurent and Seminaroti [27].

A symmetric $n \times n$ circulant matrix B whose generating function f fulfills $f(0) = 0$, $f(1) \geq f(2) \geq \ldots \geq f(\lceil \frac{n-1}{2} \rceil)$ is called a DW-Toeplitz matrix*. (Notice that in a symmetric circulant matrix $f(i) = f(n-i)$ holds for all $i > \lceil \frac{n-1}{2} \rceil$). These matrices were introduced in Deineko and Woeginger [18].*

A symmetric $n \times n$ Toeplitz matrix B whose generating function f fulfills $f(0) = 0$, $f(1) \leq f(2) \leq \ldots \leq f(\lceil \frac{n-1}{2} \rceil)$ and $f(i) \leq f(n-i)$, for all $i \leq \lceil \frac{n-1}{2} \rceil$, is called an up-benevolent Toeplitz matrix*. These matrices where introduced in [5] as* benevolent Toeplitz matrices*.*

Analogously a symmetric $n \times n$ Toeplitz matrix B whose generating function f fulfills $f(0) = 0$, $f(1) \geq f(2) \geq \ldots \geq f(\lceil \frac{n-1}{2} \rceil)$ and $f(i) \geq f(n-i)$, for all $i \leq \lceil \frac{n-1}{2} \rceil$, is called a down-benevolent Toeplitz matrix*.*

Finally the attributes down-benevolent and up-benevolent will be also used for the generating functions of the Toeplitz matrices having the corresponding properties, respectively. So we will talk about down-benevolent functions and up-benevolent functions defined over $\{-n+1, \ldots, -1, 0, 1, \ldots, n-1\}$.

The structures introduced above appeared in several special cases of the QAP dealt with in the papers already cited in Introduction. One of the most recent results was presented by Laurent & Seminaroti in [27], and will be of special interest in the context of the paper at hand.

Theorem 2.3 *(Laurent & Seminaroti [27])*
$QAP(A, B)$, where A is a Robinson matrix and B is a simple Toeplitz matrix is solved to optimality by the identity permutation.

To help readers to better understand structures involved in various QAP special cases, we use here a color coding to visualize these structures. Figure 1 illustrates Robinson matrices and simple Toeplitz matrices - the darker the color the larger the value of the corresponding matrix entries; the white color corresponds to zero entries. The instances of matrices used for the illustrations can be found in Appendix.

Definition 2.4 (Matrices with block structural properties)
Let a $q \times q$ matrix $P = (p_{ij})$ be fixed. An $n \times n$ matrix $B = (b_{ij})$ is called a block matrix with block pattern *P if the following holds*

(i) there exists a partition of the set of row and column indices $\{1, \ldots, n\}$ into q (possibly empty) sets $I_1, \ldots, I_q$ such that for $1 \leq k \leq q-1$ all elements of I_k are smaller than all elements of I_{k+1},

(ii) for all pairs of indices (i, j) with $i \in I_k$ and $j \in I_\ell$ the equality $b_{ij} = p_{k\ell}$ holds, for all $k, \ell \in \{1, 2, \ldots, q\}$

The sets $I_1, \ldots, I_q$ are called row and column blocks of matrix *B. If it is clear from context we will sometimes refer to these sets as the* blocks *of matrix B.*

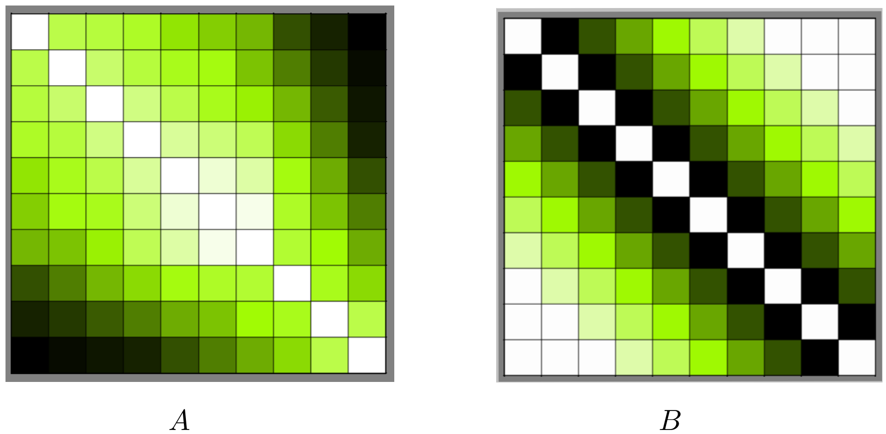


Figure 1: Illustration of the QAP instance considered in Laurent & Seminaroti QAP[27]: A - Robinson dissimilarity, B - simple Toeplitz matrix; the darker the color the larger the entries of the matrix.

A cut matrix *B is a block matrix whose block pattern has* 0*'s along the main diagonal and* 1*'s everywhere else. A cut matrix is in* CDW normal form*, if its block sizes are in non-decreasing order, i.e.* $|I_1| \leq |I_2| \leq \cdots \leq |I_q|$ *holds. (These matrices were introduced in [11].)*

It is easy to see that any cut matrix is a Robinson matrix. Theorem 2.3 implies that if A is a cut matrix and B is a simple Toeplitz matrix, then the QAP is solved by the identity permutation. Another p.s.s. case of the QAP involving cut matrices which will be of special interest in the context of this paper was identified by Çela, Deineko & Woeginger [11].

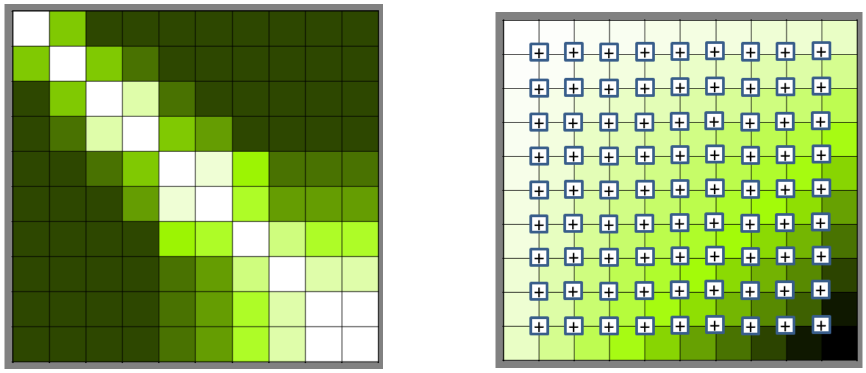
Theorem 2.5 *(Çela, Deineko & Woeginger [11])*
$QAP(A, B)$, where A is a cut matrix in CDW normal form and B is a monotone anti-Monge matrix (see Definition 2.6) is solved to optimality by the identity permutation.

As a consequence of this result $QAP(A, B)$ where A is a Robinson matrix obtained as a *conic* combination of cut matrices in CDW normal form, (i.e., A is a linear combination of such matrices with non-negative weight coefficients) and B is a monotone anti-Monge matrix is solved by the identity permutation. This special case is illustrated in Figure 2. The fulfillment of the anti-Monge inequalities is illustrated by the symbol "+". Notice that the block structure of matrix A is not that obvious any more in the picture.

In the context of the special case mentioned in Theorem 2.5 the recognition of Robinson matrices, which can be represented as conic combinations of cut matrices in CDW normal form, becomes relevant:

> Given an $n \times n$ Robinson matrix, can it be represented as a conic combination of cut matrices in CDW normal form?

The solution of this non-trivial problem is discussed in Section 4.2. In general *the recognition problem for a special classes $\mathcal{K}$ of matrices* asks whether a given a matrix A belongs to the



A B

Figure 2: Illustration of a generalisation of the Cela, Deineko & Woeginger QAP [11]: A - a conic combination of Block matrices in CDW normal form, B - anti-Monge monotone matrix; the darker the color the larger the entries of the matrix.

class $\mathcal{K}$ or not. Recognition problems can be highly non-trivial. There are a number of papers dealing with recognition problems for different (permuted) classes of matrices, especially also for Robinson matrices [1, 15, 26, 28].

As an illustrative example for the recognition of a conic combination of cut matrices in CDW normal form we consider the following Robinson matrix:

$$C = \begin{pmatrix} 0 & 1 & 2 & 3 & 3 & 3 \\ 1 & 0 & 2 & 3 & 3 & 3 \\ 2 & 2 & 0 & 2 & 3 & 3 \\ 3 & 3 & 2 & 0 & 2 & 2 \\ 3 & 3 & 3 & 2 & 0 & 1 \\ 3 & 3 & 3 & 2 & 1 & 0 \end{pmatrix}$$

which is obtained as a sum of three cut-matrices $C = C_1 + C_2 + C_3$; here matrix C_1 has the three blocks $\{1, 2, 3\}$, $\{4\}$, $\{5, 6\}$, matrix C_2 has three blocks $\{1, 2\}$, $\{3\}$, $\{4, 5, 6\}$, and matrix C_3 has five blocks $\{1\}$, $\{2\}$, $\{3, 4\}$, $\{5\}$, and $\{6\}$. As none of these matrices above is a cut matrix in CDW normal form, there are no reasons to assume that the QAP with C and a monotone anti-Monge matrix B is solved by the identity permutation. In Section 4.2 we will revisit this example after the proof of Theorem 4.6 and will show that C can indeed be represented as a conic combination of cut matrices in CDW normal form. Hence the corresponding QAP is solved by the identity permutation.

Definition 2.6 (Matrices defined in terms of four point conditions)
An $n \times n$ matrix B is a Monge matrix*, if its entries are non-negative and satisfy the Monge inequalities*

$$b_{ij} + b_{rs} \ \leq \ b_{is} + b_{rj} \qquad \text{for } 1 \leq i < r \leq n \text{ and } 1 \leq j < s \leq n. \tag{2}$$

In other words, in every 2×2 submatrix the sum of the entries on the main diagonal is smaller than the sum of the entries on the other diagonal. (The Monge property essentially dates back to the work of Gaspard Monge [33] in the 18th century.)

Analogously, an $n \times n$ matrix B is an anti-Monge matrix*, if its entries are non-negative and satisfy the anti-Monge inequalities*

$$b_{ij} + b_{rs} \;\geq\; b_{is} + b_{rj} \qquad \text{for } 1 \leq i < r \leq n \text{ and } 1 \leq j < s \leq n. \tag{3}$$

A symmetric $n \times n$ matrix (c_{ij}) is called a Kalmanson matrix*, if it satisfies the conditions*

$$c_{ij} + c_{kl} \leq c_{ik} + c_{jl} \tag{4}$$

$$c_{ik} + c_{jl} \geq c_{il} + c_{jk} \tag{5}$$

for all i,j,k and l with $1 \leq i < j < k < l \leq n$. (These matrices were introduced in 1975 by Kenneth Kalmanson [23].)

Kalmanson matrices can be also defined as matrices which fulfill the following inequalities:

$$c_{i,j+1} + c_{i+1,j} \leq c_{ij} + c_{i+1,j+1} \qquad \forall i,j : 1 \leq i \leq n-3,\ i+2 \leq j \leq n-1, \tag{6}$$

$$c_{i,1} + c_{i+1,n} \leq c_{in} + c_{i+1,1} \qquad \forall i :\ 2 \leq i \leq n-2. \tag{7}$$

In this equivalent characterisation (proved for example in [16, 17]), the fulfillment of (4)-(5) is required just for $O(n^2)$ quadruples of entries.

Much research has been done on the role played by these classes of matrices in combinatorial optimization, in particular with respect to p.s.s. cases of hard combinatorial optimization problems arising when (one of) the input matrices belongs to some of those classes. Probably the first reference in this context is due to Supnik [41], while the term "four point conditions" was independently introduced by Quintas & Supnick [37] and Buneman [4].

Monge structures play a special role in p.s.s. cases of the QAP [5, 11, 13]. We refer the reader to the survey [7] by Burkard, Klinz & Rudolf for more general information on Monge and anti-Monge structures.

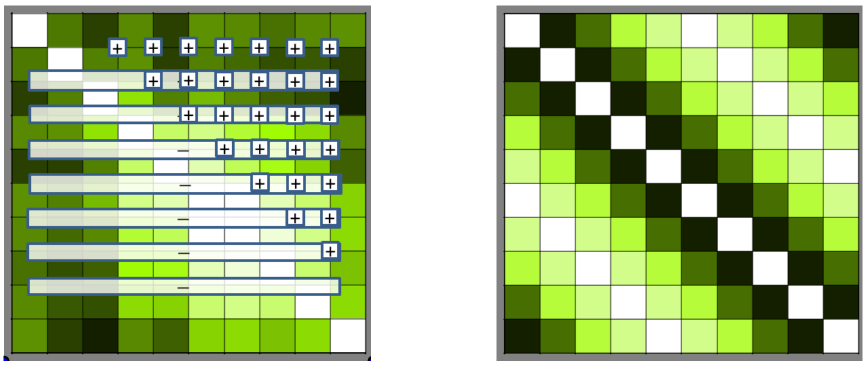
Kalmanson matrices play a role in p.s.s. cases of the QAP [18] and also in special cases of a number of other combinatorial optimization problems as the travelling salesman problem [23], the prize-collecting TSP [14], the master tour problem [17], the Steiner tree problem [24], the three-dimensional matching problem [35].

A special case of the QAP involving a Kalmanson matrix and relevant also in the context of this paper was considered by Deineko & Woeginger [18].

Theorem 2.7 *(Deineko & Woeginger [18])*
The QAP(A,B) where A is a Kalmanson matrix and B is a DW-Toeplitz matrix is solved to optimality by the identity permutation.

This special case is illustrated in Figure 3. The inequalities (4) and (5) fulfilled by the entries of the Kalmanson matrix C are illustrated by the "+" and "-", respectively.

Finally we formally define (weak) sum and constant matrices.



A $\qquad\qquad$ B

Figure 3: Illustration of the Deineko & Woeginger QAP[18] : A - a Kalmanson matrix, B - a DW-Toeplitz matrix.

Definition 2.8 (Sum matrices and constant matrices)
An $n \times n$ matrix $A = (a_{ij})$ is called a sum matrix*, iff there exist real numbers $\alpha_1, \dots, \alpha_n$ and $\beta_1, \dots, \beta_n$ such that*

$$a_{ij} = \alpha_i + \beta_j \qquad \text{for } 1 \le i, j \le n. \tag{8}$$

An $n \times n$ matrix $A = (a_{ij})$ is called a constant matrix*, if all elements in the matrix are the same. Notice that a constant matrix is just a special case of a sum matrix.*

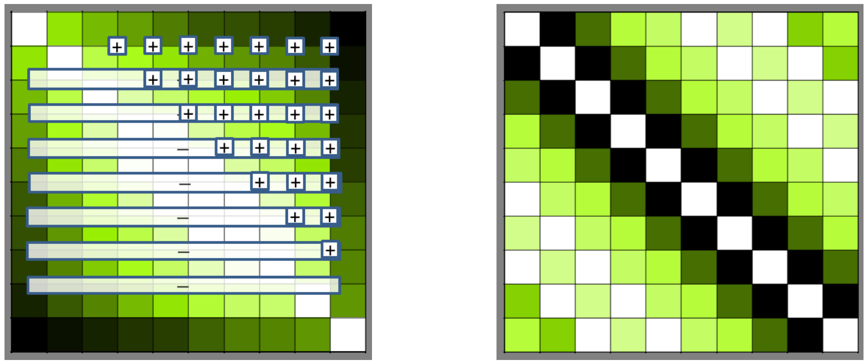
An $n \times n$ matrix $A = (a_{ij})$ is called a weak sum matrix*, if A can be turned into a sum matrix by appropriately changing the entries on its main diagonal. Equivalently, a matrix A is a weak sum matrix iff there exist real numbers $\alpha_1, \dots, \alpha_n$ and $\beta_1, \dots, \beta_n$ such that $a_{ij} = \alpha_i + \beta_j$ for $1 \le i, j \le n$, $i \ne j$.*

An $n \times n$ matrix $A = (a_{ij})$ is a weak constant matrix*, if A can be turned into a constant matrix by appropriately changing the entries on its main diagonal, or equivalently, if all its off-diagonal entries have a common value.*

Notice that a constant matrix A fulfills all matrix properties introduced in this section, with exception of the properties of cut matrices (in CDW normal form); A is a cut matrix (in CDW normal form) only iff its entries equal 1.

We close this session with a well known and easily proved observation which formalizes the relationship between the optimal solutions of two QAP instances of the same size, where the input matrices of one of them are obtained by permuting the input matrices of the other instance, respectively.

Observation 2.9 *Let A and B be two $n \times n$ matrices, and let $\pi, \psi \in \mathcal{S}_n$, where $\mathcal{S}_n$ is the set of permutations of $\{1, 2, \dots, n\}$. Let $A^\pi := (a^\pi_{ij})$ ($B^\psi := (b^\psi_{ij})$) be the matrix obtained from A (B) by permuting its rows and columns according to the permutations π (ψ). Then $Z_\phi(A^\pi, B^\psi) = Z_{\phi\circ\pi\circ\psi^{-1}}(A, B)$, for all $\phi \in \mathcal{S}_n$. Moreover, if $\phi^* \in \mathcal{S}_n$ is an optimal solution of $QAP(A, B)$ then $\phi^* \circ \psi \circ \pi^{-1}$ is an optimal solution of $QAP(A^\pi, B^\psi)$. Finally, the optimal objective function values of the two problems $QAP(A, B)$ and $QAP(A^\pi, B^\psi)$ coincide.*



A B

Figure 4: Illustration of Theorem 3.4: A - a Kalmanson and Robinson matrix, B - a down-benevolent Toeplitz matrix.

3 New special cases of the QAP solved by the identity permutation

3.1 The down-benevolent QAP

In this section we consider a new p.s.s. case $QAP(A, B)$ which we call *down-benevolent QAP*: A is both a Robinson matrix and a Kalmanson matrix, and B is a down-benevolent Toeplitz matrix. We show that this special case, illustrated in Figure 4, is solved by the identity permutation.

Notice that a simple Toeplitz matrix is a special case of a down-benevolent Toeplitz matrix. Analogously a DW-Toeplitz matrix is also a special case of a down-benevolent Toeplitz matrix. Thus, the QAP p.s.s. case considered here is related to the QAP p.s.s. cases considered in [27] and in [18]. In [27] it was shown that $QAP(A, B)$ with A being a Robinson matrix and B being a simple Toeplitz matrix is solved by the identity permutation. In [18] it was shown that $QAP(A, B)$ with A being a Kalmanson matrix and B being a DW-Toeplitz matrix is solved by the identity permutation. The new special case involves a matrix B from a class which is strictly larger than both classes of matrices B considered in [18, 27]. However the matrix A is required to have more restrictive properties than in [18, 27]: A is both a Robinson and a Kalmanson matrix.

In the following we will work with some particular symmetric 0-1 Toeplitz matrices.

Definition 3.1 *For $n \in \mathbb{N}$ and $i \in \mathbb{N}$, $\lceil \frac{n-1}{2} \rceil < i \leq n-1$, let $T^{(i)}$ be the* 0-1 *Toeplitz matrix with entries fulfilling $T^{(i)}_{kl} = 1$ iff $|k-l| = i$.*

It can be easily seen that every $n \times n$ down-benevolent Toeplitz matrix can be obtained from a DW-Toeplitz matrix by subtracting from it a conic combination of Toeplitz $T^{(i)}$. More precisely the following lemma holds.

Lemma 3.2 *Let B be an $n \times n$ down-benevolent Toeplitz matrix. Then there exists an $n \times n$ DW-Toeplitz matrix B' and nonnegative numbers β_i, $\lceil \frac{n-1}{2} \rceil < i \le n-1$, such that $B = B' - \sum_{i=\lceil \frac{n-1}{2} \rceil +1}^{n-1} \beta_i T^{(i)}$, where $T^{(i)}$ is defined as in Definition 3.1.*

Proof. For a given down-benevolent Toeplitz matrix $B = (b_{kl})$ define a DW-Toeplitz matrix $B' = (b'_{kl})$ as follows: $b'_{kl} = b_{kl}$ for $k, l \in \{1, 2, \ldots, n\}$ with $|k-l| \le \lceil \frac{n-1}{2} \rceil$ and $b'_{kl} = b_{1,\alpha_{kl}}$ for $k, l \in \{1, 2, \ldots, n\}$ with $|k-l| > \lceil \frac{n-1}{2} \rceil$, where $\alpha_{kl} = n+1-|k-l|$. It can be easily seen that B' is a DW-Toeplitz matrix and that $B = B' - \sum_{i=\lceil \frac{n-1}{2} \rceil +1}^{n-1} \beta_i T^{(i)}$, where $\beta_i = b'_{1,i+1} - b_{1,i+1}$ for all $i \in \{1, 2, \ldots, n-1\}$, $\lceil \frac{n-1}{2} \rceil \le i \le n-1$. $\square$

Let n be an arbitrary but fixed natural number and $i \in \mathbb{N}$, $\lceil \frac{n-1}{2} \rceil < i \le n-1$. Consider the maximization version of $QAP(A, T^{(i)})$ with an $n \times n$ Kalmanson matrix A which is also a Robinson matrix, and a $T^{(i)}$ is a Toeplitz matrix as above. Thus we deal with the optimization problem $\max\{Z_\pi(A, T^{(i)}) \colon \pi \in \mathcal{S}_n\}$, where $\mathcal{S}_n$ is the set of permutations of $\{1, 2, \ldots, n\}$. Observe that $T^{(i)}$ contains exactly $2(n-i)$ ones placed in pairwise symmetric positions with respect to the diagonal. The 1-entries above the diagonal lie in the rows with indices $\{1, 2, \ldots, n-i\}$ and in the columns with indices $\{i+1, i+2, \ldots n\}$ with exactly one 1-entry per row and column. Notice that since $i > \lceil \frac{n-1}{2} \rceil$ the sets of row indices and column indices above do not intersect. The objective function value of $QAP(A, T^{(i)})$ corresponding to permutation $\pi \in \mathcal{S}_n$ is given as

$$Z_\pi(A, T^{(i)}) = \sum_{k=1}^{n} \sum_{j=1}^{n} a_{\pi(k)\pi(j)} T^{(i)}_{kj} = \sum_{k=1}^{n-i} a_{\pi(k)\pi(k+i)} + \sum_{k=i+1}^{n} a_{\pi(k)\pi(k-i)} = 2 \sum_{k=1}^{n-i} a_{\pi(k)\pi(k+i)} ,$$

where the last equality holds because A is by definition a symmetric matrix. Thus $Z_\pi(A, T^{(i)})$ is just the sum of $2(n-i)$ pairwise symmetric (non-diagonal) entries selected from A, such that in every row and column there is at most one selected entry. Notice that if each pair of symmetric entries is represented by the above-diagonal entry than the goal function $QAP(A, T^{(i)})$ can be seen as twice the sum of $n-i$ above-diagonal entries selected in A such that the row indices of selected entries build a set R, the column indices of selected entries build a set C, and $R \cap C = \emptyset$ as well as $|R| = |C| = n-i$ hold.

Vice versa, consider a set of row indices R and a set of column indices C with $R \cap C = \emptyset$, $|R| = |C| = n-i$ and a bijection $\phi \colon R \to C$. Now select in A the entries $a_{i\phi(i)}$, for $i \in R$, together with their symmetric counterparts. It can be easily seen that the overall sum of these selected entries equals $Z_\pi(A, T^{(i)})$ for any $\pi \in \mathcal{S}_n$ with $\{\pi(1), \pi(2), \pi(n-i)\} = R$, $\pi(i+j) = \phi(\pi(j))$, for $1 \le j \le n-i$. Thus the maximization version of $QAP(A, T^{(i)})$ of size n with i such that $\lceil \frac{n-1}{2} \rceil < i \le n-1$ is equivalent to the following selection problem

Selection problem

Input: $n \in \mathbb{N}$, a Kalmanson and Robinson $n \times n$ matrix A, $i \in \mathbb{N}$ such that $\lceil \frac{n-1}{2} \rceil < i \le n-1$ holds.

Output: Select $(n-i)$ above-diagonal entries $a_{r_j c_j}$, $1 \le j \le n-i$, from A, such that the overall sum $\sum_{j=1}^{n-i} a_{r_j c_j}$ of the selected entries is maximized, under the condition that the set $R=\{r_j\colon 1\le j\le n-i\}$ of row indices of the selected entries and the set $C=\{c_j\colon 1\le j\le n-i\}$ of column indices of the selected entries fulfill $R\cap C=\emptyset$, $|R|=|C|=n-i$.

Since in the selection problem we select $n-i$ entries with at most one entry per row, its solution can be represented by a pair (R,ϕ), where R is the set of indices of the selected rows, $|R|=n-i$, and $\phi\colon R\to\{1,2\ldots,n\}$ is an injective mapping which maps each $r\in R$ to the column index of the entry $a_{r\phi(r)}$ selected in row r. Then, clearly, $R\cap C=\emptyset$ would hold with $C=\{\phi(r)\colon r\in R\}$. If an entry a_{jl} is selected in a solution (R,ϕ), i.e. $\phi(j)=l$, $j\in R$, we will say that row index j is matched with column index l and column index l is matched with row index j in that solution.

Next we show that the maximization version of $QAP(A,T^{(i)})$, with $n\in\mathbb{N}$ and $i\in\mathbb{N}$, $\lceil\frac{n-1}{2}\rceil<i\le n-1$, is solved by the identity permutation.

Lemma 3.3 *The maximization version of $QAP(A,T^{(i)})$ with an $n\times n$ Kalmanson and Robinson matrix A and a Toeplitz matrix $T^{(i)}$, $i>\lceil\frac{n-1}{2}\rceil$, defined in Definition 3.1 is solved to optimality by the identity permutation.*

Proof. We consider the corresponding selection problem and show that it is solved to optimality by selecting the entries $a_{1,1+i}$, $a_{2,2+i}$, ..., $a_{n-i,n}$. Clearly, this selection is feasible and corresponds to the identity permutation in $\mathcal{S}_n$ as an optimal solution of the maximization version of $QAP(A,T^{(i)})$, and this would complete the proof.

Consider an optimal solution (R,ϕ) of the selection problem where the row indices of the selected entries build the set $R=\{r_1,r_2,\ldots,r_{n-i}\}$ and the corresponding column indices are $\phi(r_j)$, for $1\le j\le n-i$. Then, clearly, $R\cap\{\phi(r_j)\colon 1\le j\le n-i\}=\emptyset$ holds. Assume w.l.o.g. that $r_1<r_2<\ldots r_{n-i}$. First we claim that there exists an optimal solution with $\max\{r_j\colon 1\le j\le n-i\}<\min\{\phi(r_j)\colon 1\le j\le n-i\}$, i.e. an optimal solution with the following property:

(P): any row index of a selected entry is smaller that any column index of a selected entry.

Assume the optimal solution (R,ϕ) above does not have Property P. Then there exist two indices $j,l\in\{1,2,\ldots,n-i\}$ such that $\phi(r_l)<r_j$ holds. Let r_j be the smallest element in R for which such a column index of a selected entry smaller than r_j exists, i.e. $\phi(R)\cap\{1,2,\ldots,r_{j-1}\}\neq\emptyset$, and let r_l be such that $\phi(r_l)$ is the smallest column index of a selected entry which is smaller than r_j, i.e. $\phi(r_l)=\min\phi(R)\cap\{1,2,\ldots,r_{j-1}\}$.

Then we clearly have $r_l<\phi(r_l)<r_j<\phi(r_j)$. Consider a pair (R',ϕ') obtained by *exchanging* r_j and $\phi(r_l)$ in the following sense:

$$R':=(R\setminus\{r_j\})\cup\{\phi(r_l)\}$$

$$\phi'(r)=\phi(r),\ \forall r\in R\setminus\{r_j,r_l\}\text{ and }\phi'(r_l)=r_j,\ \phi'(\phi(r_l))=\phi(r_j).$$

(R',ϕ') is a feasible solution of the selection problem because the two entries $a_{r_l\phi(r_l)}$, $a_{r_j\phi(r_j)}$ selected with (R,ϕ) are replaced by the entries $a_{r_lr_j}$, $a_{\phi(r_l)\phi(r_j)}$ selected with (R',ϕ') and the sets R', C' of the row and column indices of selected entries, respectively, fulfill the properties

$R' \cap C' = \emptyset$, $|R'| = |C'| = n - i$. Moreover, since A is a Kalmanson matrix inequality (4) applies and we get

$$a_{r_l r_j} + a_{\phi(r_l)\phi(r_j)} \geq a_{r_l \phi(r_l)} + a_{r_j \phi(r_j)} \, .$$

Thus the solution (R', ϕ') is not worse than the optimal solution (R, ϕ), hence it is also an optimal solution. If (R', ϕ') does not have property P, then there will be again a smallest row index r_k of a selected entry for which there exists a column index of a selected entry which is smaller than r_k. Notice that in this case r_k has to be larger than r_j because for indices in the set $(R \cup C) \cap \{1, 2, \ldots, r_{j-1}\}$ the following statement holds: any row index of an entry selected by the solution (R', ϕ') is smaller than any column index of an entry selected by (R', ϕ'). So, if (R', ϕ') does not have property P, then we could perform again an exchange to obtain a new optimal solution as described above. We would repeat this step as long as the current optimal solution does not have property P. The process would terminate because the smallest row index of a selected entry for which there is an even smaller column index of a selected entry, increases in every repetition of the exchange step described above. So the claim about the existence of an optimal solution with the property P is proven.

Let (R, ϕ) be an optimal solution with property P and let $R = \{r_1, r_2, \ldots, r_{n-i}\}$ be the row indices of the selected entries with $r_1 < r_2 < \ldots r_{n-i}$. We can assume w.l.o.g. that $r_l < r_j$ implies $\phi(r_l) < \phi(r_j)$, for all $l, j \in \{1, 2, \ldots, n-i\}$. Indeed if there exists a pair $r_l < r_j$ for which $\phi(r_l) > \phi(r_j)$, then consider the solution (R, ϕ') with $\phi'(r_k) = \phi(r_k)$ for all $k \in \{1, 2, \ldots, n-i\} \setminus \{j, l\}$ and $\phi'(j) = \phi(l)$, $\phi'(l) = \phi(j)$. Thus the entries $a_{r_l \phi(r_l)}$, $a_{r_j \phi(r_j)}$ selected with (R, ϕ) are replaced by the entries $a_{r_l \phi(r_j)}$, $a_{r_j \phi(r_l)}$ selected with (R, ϕ'). From inequality (5) in the definition of Kalmanson matrices we get

$$a_{r_l \phi(r_j)} + a_{r_j \phi(r_l)} \geq a_{r_l \phi(r_l)} + a_{r_j \phi(r_j)} \, ,$$

which implies that the solution (R, ϕ') is not worse than the optimal solution (R, ϕ). Hence (R, ϕ') is an optimal solution.

Let us denote the set of column indices of the entries selected with (R, ϕ) by $C = \{c_1, c_2, \ldots, c_{n-i}\}$ where $c_1 < c_2 < \ldots < c_{n-i}$. Then the selected entries are $a_{r_j c_j}$, $j = 1, 2, \ldots, n-i$, and $r_1 < r_2 < \ldots < r_{n-i} < c_1 < c_2 < \ldots < c_{n-i}$ holds. The last inequalities imply $j \leq r_j$ and $c_j \leq i + j$, for all $j = 1, 2, \ldots, n-i$. Since matrix A is a Robinson matrix we get $a_{r_j c_j} \leq a_{j, i+j}$, for all $j = 1, 2, \ldots, n-i$, which imply

$$\sum_{j=1}^{n-i} a_{r_j c_j} \leq \sum_{j=1}^{n-i} a_{j, i+j} \, .$$

Hence selecting the entries $a_{1,1+i}$, $a_{2,2+i}$, ..., $a_{n-i,n}$ is not worse then the optimal solution (R, ϕ), which means that $a_{1,1+i}$, $a_{2,2+i}$, ..., $a_{n-i,n}$ is an optimal selection. $\square$

Theorem 3.4 *$QAP(A, B)$ where A is both a Robinson matrix and a Kalmanson matrix, and B is a down-benevolent Toeplitz matrix, is solved to optimality by the identity permutation id.*

Proof. According to Lemma 3.2 there exists a DW-Toeplitz matrix B' and nonnegative numbers $\beta_i \geq 0$, $\lceil \frac{n-1}{2} \rceil < i \leq n-1$, such that

$$B = B' - \sum_{i=\lceil \frac{n-1}{2} \rceil + 1}^{n-1} \beta_i T^{(i)}\,, \tag{9}$$

where $T^{(i)}$ is the Toeplitz matrix defined in Definition 3.1. Equation (9) implies:

$$Z_\pi(A,B) = Z_\pi(A,B') - \sum_{i=\lceil \frac{n-1}{2} \rceil + 1}^{n-1} \beta_i Z_\pi(A, T^{(i)})\,.$$

Theorem 2.7 implies $QAP(A,B')$ is solved to optimality by the identity permutation and Lemma 3.3 implies that the maximization version of $QAP(A,T^{(i)})$ with $T^{(i)}$ as above is solved to optimality by the identity permutation for all $i > \lceil \frac{n-1}{2} \rceil$. Summarizing we get:

$$Z_\pi(A,B) = Z_\pi(A,B') - \sum_{i=\lceil \frac{n-1}{2} \rceil + 1}^{n-1} \beta_i Z_\pi(A, T^{(i)}) \geq$$

$$Z_{id}(A,B') - \sum_{i=\lceil \frac{n-1}{2} \rceil + 1}^{n-1} \beta_i Z_{id}(A, T^{(i)}) = Z_{id}(A,B)$$

for all $\pi \in \mathcal{S}_n$. □

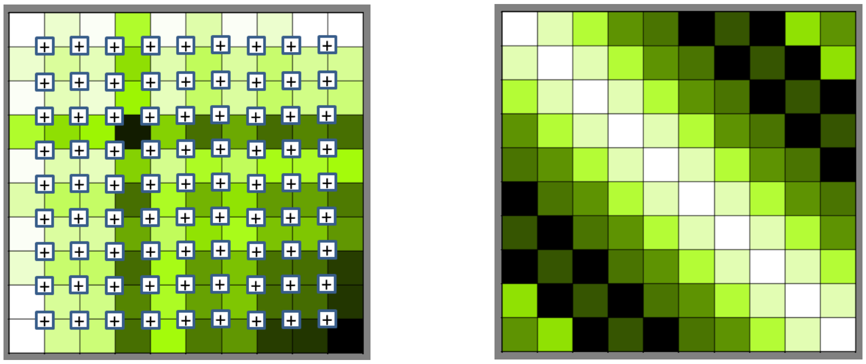
3.2 The up-benevolent QAP

Burkard & al. [5] have considered $QAP(A,B)$ with a monotone anti-Monge matrix A with nonnegative entries and an up-benevolent Toeplitz matrix B (called *benevolent* Toeplitz matrix in the original paper). They have proven the following result:

Theorem 3.5 *(Burkard & al. [5])*
$QAP(A,B)$ with a monotone anti-Monge matrix A with nonnegative entries and an up-benevolent Toeplitz matrix B is solved to optimality by the so-called Supnick permutation $\pi^* = \langle 1,3,5,\ldots,6,4,2\rangle$.

An illustration of this special case is presented in Figure 5.

In [5] it was shown that for any $n \in \mathbb{N}$ the $n \times n$ monotone anti-Monge matrices with nonnegative entries form a cone whose extremal rays are given by the 0-1 matrices $R^{(p,q)} = \left(r_{ij}^{(p,q)}\right)$, with a $p \times q$ block of ones in the lower-right corner, for $1 \leq p,q \leq n$. Thus $r_{ij}^{(p,q)} = 1$ if $n-p+1 \leq i \leq n$ and $n-q+1 \leq j \leq n$, and $r_{ij}^{(p,q)} = 0$, otherwise. As a consequence of this fact it can be shown that the extremal rays of the cone of *symmetric* monotone anti-Monge matrices with nonnegative entries are the matrices $\bar{R}^{(p,q)} := R^{(p,q)} + R^{(q,p)} = \left(\bar{r}_{ij}^{(p,q)}\right)$, for $1 \leq p < q \leq n$,



A B

Figure 5: Illustration of the special case of Burkard & al. [5] QAP: A - an anti-Monge matrix, to be permuted with π^*; B - an up-benevolent Toeplitz matrix.

see Rudolf and Woeginger [39], Burkard et al. [5], and Çela et al. [11]. $\bar{R}^{(p,q)}$ are explicitly given as follows.

$$\bar{r}_{ij}^{(p,q)} = \begin{cases} 2, & n-p+1 \le i,j \le n \\ 1, & n-q+1 \le i \le n-p, n-p+1 \le j \le n \\ 1, & n-p+1 \le i \le n,\ n-q+1 \le j \le n-p \\ 0, & \text{otherwise} \end{cases}$$

Further let us denote $\bar{R}^{(p,p)} = R^{(p,p)}$, $1 \le p \le n$, for the sake of completeness. Thus the following lemma holds.

Lemma 3.6 *The symmetric monotone anti-Monge matrices with nonnegative entries form a cone with extremal rays given as $\bar{R}^{(p,q)}$, for $1 \le p < q \le n$, and $R^{(p,p)}$, for $1 \le p \le n$.*

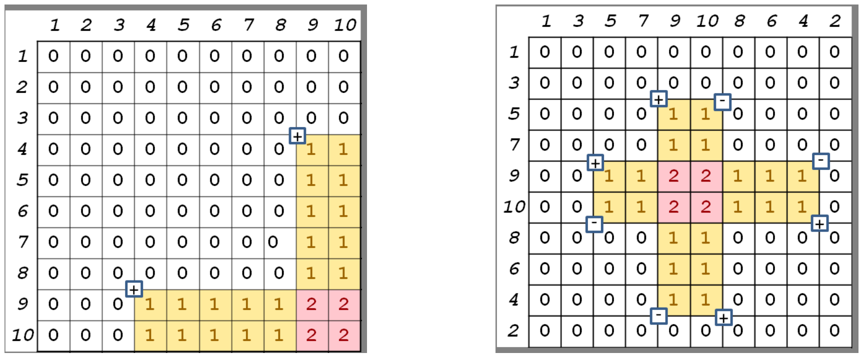
According to Observation 2.9, if π^* is an optimal solution of $QAP(A,B)$ with a symmetric monotone anti-Monge matrix A and an up-benevolent matrix B, then id is an optimal solution of $QAP(A^{\pi^*},B)$. In particular this clearly holds for $A = \bar{R}^{(p,q)}$ with $1 \le p \le q \le n$. Notice that, in general, the permuted matrix A^{π^*} is not an anti-Monge matrix any more. Figure 6 illustrates the effect of permuting the 10×10 matrix $\bar{R}^{(2,7)}$ according to permutation π^*.

Let us denote by $\bar{R}^{(p,q,\pi^*)}$ the matrix obtained by permuting $\bar{R}^{(p,q)}$ according to the Supnick permutation π^*, $1 \le p \le q \le n$.[3]

By taking a closer look at the matrices $\bar{R}^{(p,q,\pi^*)}$ we can observe that they are given as follows

$$\bar{r}_{ji}^{(p,q,\pi^*)} = \bar{r}_{ij}^{(p,q,\pi^*)} = \begin{cases} 2 & \lceil\frac{n-p}{2}\rceil + 1 \le i,j \le n - \lfloor\frac{n-p}{2}\rfloor \\ 1 & \lceil\frac{n-p}{2}\rceil - \lfloor\frac{q-p}{2}\rfloor + 1 \le i \le \lceil\frac{n-p}{2}\rceil, \lceil\frac{n-p}{2}\rceil + 1 \le j \le n - \lfloor\frac{n-p}{2}\rfloor \\ 1 & \lceil\frac{n-p}{2}\rceil + 1 \le i \le n - \lfloor\frac{n-p}{2}\rfloor, n - \lfloor\frac{n-p}{2}\rfloor + 1 \le j \le n - \lfloor\frac{n-p}{2}\rfloor + \lceil\frac{q-p}{2}\rceil \\ 0 & \text{otherwise} \end{cases}$$

[3]The consistent notation would be $(\bar{R}^{(p,q)})^{\pi^*}$ but we are using $\bar{R}^{(p,q,\pi^*)}$ instead, for the ease of presentation.



A $\qquad\qquad$ A^{π^*}

Figure 6: Illustration of permuted matrices: $A := \bar{R}^{(2,7)}$ - a 10×10 symmetric monotone anti-Monge matrix; A^{π^*} - a permuted anti-Monge matrix: four anti-Monge inequalities are violated; the quadruples of entries involved in the violated inequalities are the ones around the "minus" signs, respectively.

for $1 \le i \le j \le n$.

Thus the non-zero entries of these permuted matrices build a kind of a cross with entries equal to 2 at the center of the cross and entries equal to 1 at the arms of the cross (see also Figure 6). Now consider a transformation of the matrix $\bar{R}^{(p,q,\pi^*)}$ realized by sliding the cross of non-zero entries along the diagonal such that its arms do not wrap around the border of the matrix (they may touch the border but should not wrap around it). This transformation is done by permuting (the rows and columns of) $R^{(p,q,\pi^*)}$ according to a shift $\sigma_u \in \mathcal{S}_n$ of the form $\langle u, u+1, \dots, n, 1, \dots, u-1\rangle$ with $1 < u \le \lceil \frac{n-p}{2} \rceil - \lfloor \frac{q-p}{2} \rfloor + 1$, or $n - \lfloor \frac{n-p}{2} \rfloor + \lceil \frac{q-p}{2} \rceil + 1 \le u \le n$.

Let us denote by $C^{(p,q,u)}$ the matrix obtained from $R^{(p,q,\pi^*)}$ by permuting it according to σ_u with u as described above. Obviously $Z(C^{(p,q,u)}, B, id) = Z(\bar{R}^{(p,q,\pi^*)}, B, id)$ holds for all $1 \le p \le q \le n$, for all possible values of u as given above, and for any Toeplitz matrix B. This is due to the facts that a) the permutation σ_u shifts non-zero entries of $\bar{R}^{(p,q)}$ along lines parallel to the main diagonal and b) a Toeplitz matrix has constant entries along any line parallel to the main diagonal. Combined with the third statement of Observation 2.9 the above equation shows that id is also the optimal solution of $QAP(C^{(p,q,u)}, B)$. This observation motivates the following definition.

Definition 3.7 *Let $C^{(p,q,u)}$ be the matrix obtained from $R^{(p,q,\pi^*)}$ by permuting it according to σ_u, where $\sigma_u \in \mathcal{S}_n$ is a shift of the form $\langle u, u+1, \dots, n, 1, \dots, u-1\rangle$ with $1 < u \le \lceil \frac{n-p}{2} \rceil - \lfloor \frac{q-p}{2} \rfloor + 1$ or $n - \lfloor \frac{n-p}{2} \rfloor + \lceil \frac{q-p}{2} \rceil + 1 \le u \le n$. A symmetric $n \times n$ matrix A' is called a permuted-shifted anti-Monge matrix (PS anti-Monge matrix), if it can be obtained as a conic combination of $R^{(p,q,\pi^*)}$ and matrices $C^{(p,q,u)}$, for $1 \le p \le q \le n$, and $1 < u \le \lceil \frac{n-p}{2} \rceil - \lfloor \frac{q-p}{2} \rfloor + 1$ or $n - \lfloor \frac{n-p}{2} \rfloor + \lceil \frac{q-p}{2} \rceil + 1 \le u \le n$.*

Analogously, a symmetric $n \times n$ matrix A' is called a permuted-shifted Monge matrix (PS Monge matrix), if it can be obtained from a permuted-shifted anti-Monge A by multiplying *it by -1 and by then adding a sum matrix to it (which can also be the zero matrix, i.e. the matrix containing only entries equal to zero). See Figure 7 for a graphical illustration of PS anti-Monge and PS Monge matrices.*

Summarizing we have proved the following result (recall that a sum matrix is both Monge and anti-Monge):

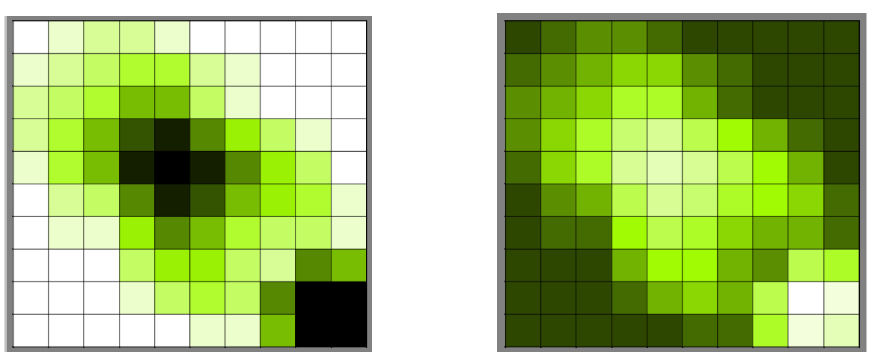
Theorem 3.8 *$QAP(A, B)$ with a PS anti-Monge matrix A and an up-benevolent Toeplitz matrix B is solved to optimality by the identity permutation.*

Notice that this result is a strict generalization of the result of Burkard & al. [5], because the permutation of a PS-anti-Monge matrix by the inverse $(\pi^*)^{-1}$ of the Supnick permutation π^* does not yield an anti-Monge matrix in general.

Finally observe that, clearly, the $n \times n$ PS anti-Monge matrices form also cone whose extremal rays are the matrices $C^{(p,q,u)}$ with $1 \leq p \leq q \leq n$, $1 < u \leq \lceil \frac{n-p}{2} \rceil - \lfloor \frac{q-p}{2} \rfloor + 1$, or $n - \lfloor \frac{n-p}{2} \rfloor + \lceil \frac{q-p}{2} \rceil + 1 \leq u \leq n$. Thus these extremal rays build a three parametric family of matrices in contrast to the extremal rays of the (symmetric) monotone anti-Monge matrices which build a two parametric family.

Since the equality $Z(A, B, \pi) = Z(-A, -B, \pi)$ trivially holds for any permutation π, $QAP(A, B)$ and $QAP(-A, -B)$ have the same set of optimal solutions. Notice, moreover, that if B is up-benevolent Toeplitz matrix than $-B$ is a down-benevolent Toeplitz matrix. Summarizing we obtain:

Corollary 3.9 *$QAP(A, B)$ with a PS Monge matrix A and a down-benevolent Toeplitz matrix B is solved to optimality by the identity permutation.*



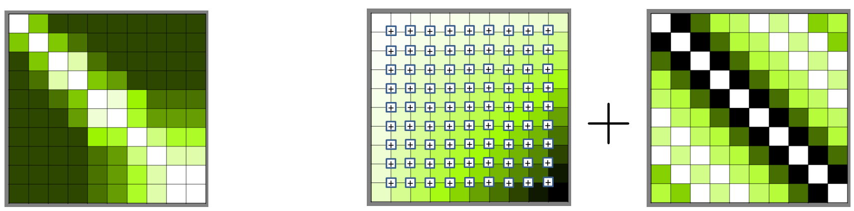
A $\qquad\qquad$ A^-

Figure 7: Illustration of PS matrices: A, a PS anti-Monge matrix; A^-, a PS Monge matrix; Monge (anti-Monge) conditions are not satisfied any more.

3.3 Combined QAPs

In the previous sections we reviewed known p.s.s. cases of the QAP and proved some new results. Some of the old and new p.s.s. cases use the same special structures, for example the cut matrices in CDW normal form are involved in the old p.s.s. case described in Theorem 2.5 and in the new p.s.s. case described in Theorem 3.4. In this section we show that such old and new p.s.s. cases can be combined into new structures and new p.s.s. cases.

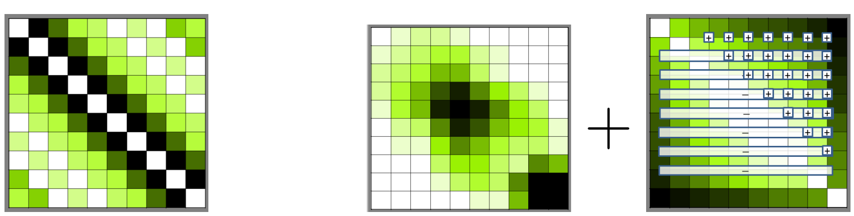
Cut matrices in CDW normal form. Consider a $QAP(A, B)$ where the matrix A is a conic combination of cut matrices in CDW normal form. Since this matrix is both a Kalmanson and a Robinson matrix, we can combine the old p.s.s. case described in Theorem 2.5 and the new p.s.s. case presented in Theorem 3.4: matrix B can now be chosen to be a conic combination of two matrices, a symmetric monotone anti-Monge matrix and a down-benevolent Toeplitz matrix (see illustration on Figure 8).



A $\qquad$ B_1 $\qquad$ B_2

Figure 8: Illustration of $QAP(A, B_1 + B_2)$ where A is a conic combination of cut matrices in CDW normal form, B_1 is a monotone anti-Monge matrix, B_2 is a down-benevolent Toeplitz matrix.

Down-benevolent Toeplitz. By combining the p.s.s. cases described in Theorem 3.4 and in Corollary 3.9 we get a new p.s.s. case $QAP(A, B_1 + B_2)$, where A is a down-benevolent Toeplitz matrix, and the second matrix B is a conic combination of matrices which are both Kalmanson and Robinson matrices and a PS Monge matrix (see illustration on Figure 9).



A $\qquad$ B_1 $\qquad$ B_2

Figure 9: Illustration of $QAP(A, B_1 + B_2)$ where A is a down-benevolent Toeplitz matrix, B_1 is a PS monotone Monge matrix, B_2 is a Kalmanson and Robinson matrix.

DW-Toeplitz. Let A be a DW-Toeplitz matrix. Clearly such a matrix A is a special down-benevolent matrix. By definition A is also a symmetric circulant matrix. Consider now a p.s.s. case $QAP(A, B)$ for which the identity is an optimal solution. Since matrix A has a circular structure, the identity is still an optimal solution of $QAP(A, B^{(u)})$, where $B^{(u)}$ is obtained from B by applying to it an arbitrary cyclic shift according to some permutation $\sigma_u = \langle u, u+1, \ldots, n, 1, \ldots, u-1\rangle$, for any $1 \le u \le n$ ($u = 1$ yields the identity permutation as a trivial cyclic shift). In particular consider a $QAP(A, B)$, where A is a DW-Toeplitz matrix and $B = -\bar{R}^{(p,q)}$, for some $1 \le p \le q \le n$. This QAP is solved to optimality by the identity permutation as stated in Theorem 3.5. But then the identity permutation is also an optimal solution of $QAP(A, -C^{(p,q,u)})$, where $-C^{(p,q,u)}$ is obtained from $B = -\bar{R}^{(p,q)}$ by permuting it according to σ_u, $1 \le u \le n$. Thus we can extend the class of PS Monge matrices defined in Section 3.2 and obtain the class of the *cyclic PS Monge matrices*, which is the class of matrices obtained by first permuting $-R^{(p,q)}$ according to π^* and then by permuting the resulting matrix according to a cyclic shift σ_u, $1 \le p \le q \le n$ and $1 \le u \le n$.

Figure 10 illustrates such cyclic PS Monge matrices.

We can define now a new combined special case of the QAP solved to optimality by the identity permutation, namely $QAP(A, B_1 + B_2)$, where A is a DW-Toeplitz matrix, and B is a conic combination of a Kalmanson matrix and a cyclic PS monotone Monge matrix (see the illustration in Figure 11).

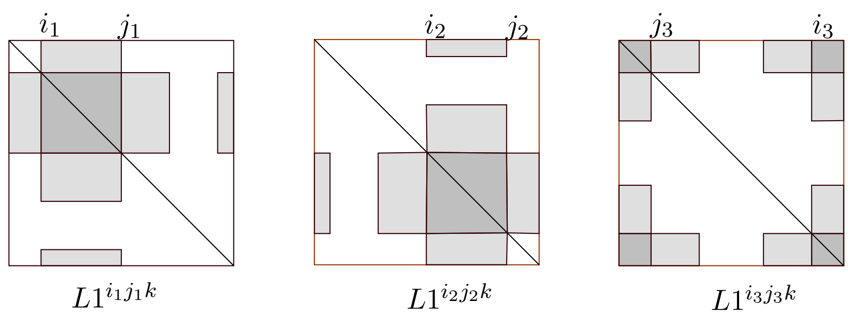


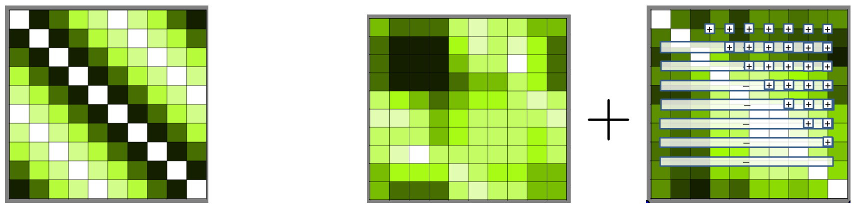
Figure 10: Illustration of extremal rays which generate the cone of Circular PS Monge matrices.

4 Conic representation of specially structured matrices

4.1 Cut weights and specially structured matrices

In this section, we investigate the structure of matrices which are both Kalmanson and Robinson matrices. We show that any matrix in this class can be represented as a sum of a constant matrix and a conic combination of cut matrices.

We use the alternative definition of Kalmanson matrices, see Definition 2.6. Consider special cut matrices $A^{(k,l)} = (a_{ij})$, $1 \le k < l \le n$, containing one block of size $(k - l + 1)$ with $a_{ij} = 0$ for $k \le i, j \le l$, and all other $n - k + l - 1$ blocks of size 1.



A $\qquad$ B_1 $\qquad$ B_2

Figure 11: Illustration of $QAP(A, B_1 + B_2)$ where A is a DW-Toeplitz matrix, B_1 is a cyclic PS monotone Monge matrix, B_2 is a Kalmanson matrix.

It can be easily observed that the matrices $A^{(k,l)}$ fulfill the inequalities (6) and (7) and are therefore Kalmanson matrices. Notice moreover that for any $n \times n$ cut matrix $A^{(k,l)}$, $1 < k < l < n$, there is only one strict inequality in (6), namely

$$a_{k-1,l} + a_{k,l+1} > a_{kl} + a_{k-1,l+1}\,,$$

whereas all inequalities (7) are fulfilled with equality. Analogously, there is only one strict inequality in (7) for the matrices $A^{(1,k-1)}$ and $A^{(k,n)}$, $2 < k < n$, namely

$$a_{k-1,1} + a_{kn} < a_{k1} + a_{k-1,n}\,,$$

whereas all inequalities (6) are fulfilled with equality [4].

The following lemma shows that any Kalmanson matrix can be represented as a linear combination of a weak sum matrix with cut matrices $A^{(k,l)}$ which can be computed explicitly in terms of simple formulas involving the entries of the considered Kalmanson matrix [5]. Similar structural properties of Kalmanson matrices in terms of cuts and cut-weights have also been studied in [2] and [15]. In both papers though the authors suggest algorithms for calculating the cut-weights while we provide simple analytical expressions for them.

Lemma 4.1 *A symmetric $n \times n$ matrix C is a Kalmanson matrix if and only if it can be represented as a linear combination of a weak sum matrix S and cut matrices $A^{(k,l)}$ as follows*

$$C = S + \sum_{i=1}^{n-3} \sum_{j=i+2}^{n-1} \delta_{i+1,j} A^{(i+1,j)} + \sum_{i=2}^{n-2} (\alpha_i A^{(1,i)} + \beta_i A^{(i+1,n)})\,. \tag{10}$$

The coefficients of the linear combination, the so-called cut weights, are given as $\delta_{i+1,j} = (-c_{i,j+1} - c_{i+1,j} + c_{ij} + c_{i+1,j+1})$, $\alpha_i = c_{i+1,1} - c_{i,1}$, $\beta_i = c_{in} - c_{i+1,n}$ and fulfill $\delta_{i+1,j} \geq 0$, $\alpha_i + \beta_i \geq 0$.

[4]Notice that we refrain from using superscripts in the entries of the matrix $A^{(k,l)}$; this yields to a slightly inconsistent but less incumbent notation.

[5]0-1-matrices analogous to $A^{(k,l)}$ have been used by Fogel et al. [19] in a similarity context.

Proof. It can easily be checked that any weak sum matrix, a cut matrix $A^{(k,l)}$, and a linear combination $\alpha_i A^{(1,i)} + \beta_i A^{(i+1,n)}$ with $\alpha_i + \beta_i \geq 0$ are Kalmanson matrices, and therefore any matrix given as in (10) is a Kalmanson matrix.

Assume now that C is a Kalmanson matrix. Let i and j, $1 \leq i < i+2 \leq j < n-1$ be two indices, such the corresponding inequality in (6) is strict, i.e. $c_{i,j+1} + c_{i+1,j} < c_{ij} + c_{i+1,j+1}$ holds. The involved matrix entries are printed in boldface in the illustration below, note that all these entries lie above the main diagonal.

$$C = \begin{pmatrix} & & & \cdots & & & \\ \cdots & c_{i,p} & \cdots & c_{i,j-1} & \mathbf{c}_{i,j} & \mathbf{c}_{i,j+1} & \cdots \\ \cdots & c_{i+1,p} & \cdots & c_{i+1,j-1} & \mathbf{c}_{i+1,j} & \mathbf{c}_{i+1,j+1} & \cdots \\ & & & \cdots & & & \\ \cdots & c_{q,p} & \cdots & c_{q,j-1} & c_{q,j} & c_{q,j+1} & \cdots \\ & & & \cdots & & & \end{pmatrix}$$

Set $\delta_{i+1,j} := -c_{i,j+1} - c_{i+1,j} + c_{ij} + c_{i+1,j+1} > 0$ and consider the matrix $C' = C - \delta_{i+1,j} A^{(i+1,j)}$, represented schematically below (to simplify the illustration we use the notation $\delta := \delta_{i+1,j}$):

$$C' = \begin{pmatrix} c_{11}-\delta & \cdots & c_{1,i}-\delta & c_{1,i+1}-\delta & \cdots & c_{1j}-\delta & c_{1,j+1}-\delta & \cdots & c_{1,n}-\delta \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ c_{i,1}-\delta & \cdots & c_{i,i}-\delta & c_{i,i+1}-\delta & \cdots & \mathbf{c}_{i,j}-\delta & \mathbf{c}_{i,j+1}-\delta & \cdots & c_{i,n}-\delta \\ c_{i+1,1}-\delta & \cdots & c_{i+1,i}-\delta & c_{i+1,i+1} & \cdots & \mathbf{c}_{i+1,j} & \mathbf{c}_{i+1,j+1}-\delta & \cdots & c_{i+1,n}-\delta \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ c_{j,1}-\delta & \cdots & c_{j,i}-\delta & c_{j,i+1} & \cdots & c_{j,j} & c_{j,j+1}-\delta & \cdots & c_{j,n}-\delta \\ c_{j+1,1}-\delta & \cdots & c_{j+1,i}-\delta & c_{j+1,i+1}-\delta & \cdots & c_{j+1,j}-\delta & c_{j+1,j+1}-\delta & \cdots & c_{j+1,n}-\delta \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ c_{n,1}-\delta & \cdots & c_{n,i}-\delta & c_{n,i+1}-\delta & \cdots & c_{n,j}-\delta & c_{n,j+1}-\delta & \cdots & c_{n,n}-\delta \end{pmatrix}$$

Notice that in matrix $C' = (c'_{ij})$ we have $c'_{i,j+1} + c'_{i+1,j} = c'_{ij} + c'_{i+1,j+1}$. Moreover the status of the other inequalities in (6) does not change, meaning that all inequalities are still fulfilled by matrix C' and only the inequalities which were strictly fulfilled by C are strictly fulfilled by C'. Finally it is also easy to see that C' fulfills inequalities (6). Hence C' is a Kalmanson matrix, and we check again whether there is a pair of indices for which the corresponding inequality in (6) is strict. If yes, we perform an analogous transformation as the one described above by defining the corresponding δ-coefficient and subtracting from C' the corresponding cut matrix multiplied by that coefficient. We repeat this process, update C' in every step, and eventually obtain a Kalmanson matrix C' which fulfills all inequalities (6) by equality.

Assume now that there exists some inequality in (7) strictly fulfilled by the entries of C'.

Let i, $2 \le i \le n-2$, be an index such that $c_{i1} + c_{i+1,n} < c_{i+1,1} + c_{in}$:

$$C = \begin{pmatrix} & & & \cdots & & & \\ \mathbf{c}_{i1} & c_{i2} & \cdots & 0 & \cdots & c_{i,n-1} & \mathbf{c}_{in} \\ \mathbf{c}_{i+1,1} & c_{i+1,2} & \cdots & 0 & \cdots & c_{i+1,n-1} & \mathbf{c}_{i+1,n} \\ & & & \cdots & & & \end{pmatrix}$$

Set $\alpha_i = c_{i+1,1} - c_{i,1}$, $\beta_i = c_{in} - c_{i+1,n}$. Clearly $\alpha_i + \beta_i > 0$ holds, due to $c_{i1} + c_{i+1,n} < c_{i+1,1} + c_{in}$. Consider the matrix $C' = C - \alpha A^{(1,i)} - \beta\, A^{(i+1,n)}$ where $\alpha := \alpha_i$, $\beta := \beta_i$:

$$C' = \begin{pmatrix} c_{1,1} - \beta & \cdots & c_{1,i} - \beta & c_{1,i+1} - \alpha - \beta & \cdots & c_{1,n} - \alpha - \beta \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \mathbf{c}_{i,1} - \beta & \cdots & c_{i,i} - \beta & c_{i,i+1} - \alpha - \beta & \cdots & \mathbf{c}_{i,n} - \alpha - \beta \\ \mathbf{c}_{i+1,1} - \alpha - \beta & \cdots & c_{i+1,i} - \alpha - \beta & c_{i+1,i+1} - \alpha & \cdots & \mathbf{c}_{i+1,n} - \alpha \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ c_{n,1} - \alpha - \beta & \cdots & c_{n,i} - \alpha - \beta & c_{n,i+1} - \alpha & \cdots & c_{n,n} - \alpha \end{pmatrix}$$

It can be easily checked that $c'_{i1} + c'_{i+1,n} = c'_{i+1,1} + c'_{in}$, that all inequalities (6) remain fulfilled with equality, and that the status of the other inequalities in (7) does not change, meaning that all these inequalities are still fulfilled by matrix C' and only those inequalities among them which were strictly fulfilled by C, are strictly fulfilled by C'. As long as there are inequalities (7) strictly fulfilled by C' we apply a transformation as above on C' and update C'. So eventually we get a transformed matrix where all inequalities (6), (7) are fulfilled with equality. Such a matrix is a weak sum matrix, as shown in Lemma 4.2 below, and this completes the proof. $\square$

Lemma 4.2 *Let C be an $n \times n$ Kalmanson matrix for which all inequalities in* (6) *and* (7) *are fulfilled with equality. Then C is a weak sum matrix.*

Proof. We show that the entries of matrix $C = (c_{ij})$ can be represented as

$$c_{ij} = \gamma_i + \gamma_j \text{ for } i, j \in \{1, 2, \ldots, n\},\ i < j, \text{ and some } \gamma = (\gamma_i) \in \mathbb{R}^n\,. \tag{11}$$

First, we show how to calculate γ_i, $i = 1, \ldots, n$, and then we prove that the above representation is valid.

By solving the simple system of linear equations $\gamma_1 + \gamma_2 = c_{12}$, $\gamma_1 + \gamma_3 = c_{13}$, and $\gamma_2 + \gamma_3 = c_{23}$ we get

$$\gamma_1 = (c_{12} + c_{13} - c_{23})/2\,, \gamma_2 = (c_{12} + c_{23} - c_{13})/2\,, \gamma_3 = (c_{13} + c_{23} - c_{12})/2\,. \tag{12}$$

The remaining γ_i, $i \in \{4, 5, \ldots, n\}$ are calculated as $\gamma_i := c_{1i} - \gamma_1$. We have now $c_{1j} = c_{j1} = \gamma_1 + \gamma_j$, for $j = 2, \ldots, n$, and $c_{23} = c_{32} = \gamma_2 + \gamma_3$.

Next we show that $c_{ij} = \gamma_i + \gamma_j$ holds for the remaining pairs of indices (i, j), i.e. for $(i, j) \in \{1, 2, \ldots, n\}^2$, $i \ne 1$, $i < j$, and $(i, j) \ne (2, 3)$. We consider those pairs (i, j) of indices in the following order $(2,4),(2,5),\ldots,(2,n),(3,n),(3,n-1),\ldots,(3,4),\ldots,(4,n),\ldots,(4,5),(n-1,n)$. For $(2,4)$ we obtain by applying (in)equality (6) and equalities (12) $c_{24}=c_{23}+c_{14}-c_{13}=\gamma_2+\gamma_3+\gamma_1+\gamma_4-\gamma_1-\gamma_3=\gamma_2+\gamma_4$. Observe that for each of the remaining entries c_{jl} there is always one of the (in)equalities (6) or (7) which involves c_{jl} and three other entries c_{ik}, c_{il}, c_{jk} which have been considered before to c_{jl} in the above specified order. Thus for those three entries the corresponding equalities (11) hold: $c_{ik}=\gamma_i+\gamma_k$, $c_{jk}=\gamma_j+\gamma_k$, and $c_{il}=\gamma_i+\gamma_l$. From $c_{ik}+c_{jl}=c_{il}+c_{jk}$ and the above three equalities we get $c_{jl}=c_{jk}+c_{il}-c_{ik}=\gamma_j+\gamma_k+\gamma_i+\gamma_l-\gamma_i-\gamma_k=\gamma_j+\gamma_l$, and this completes the proof. □

Next we give a characterization of Kalmanson matrices which are also Robinson matrices. This characterization involves again the special cut matrices $A^{(k,l)}$.

Lemma 4.3 *A symmetric $n\times n$ matrix C is both a Kalmanson and a Robinson matrix if and only if it can be represented as a conic combination of a weak constant matrix Z and cut matrices $A^{(k,l)}$ as follows*

$$C=Z+\sum_{i=1}^{n-3}\sum_{j=i+2}^{n-1}\delta_{i+1,j}A^{(i+1,j)}+\sum_{i=2}^{n-1}\alpha_i A^{(1,i)}+\sum_{i=1}^{n-2}\beta_i A^{(i+1,n)}\,, \tag{13}$$

where $\delta_{i+1,j}:=(-c_{i,j+1}-c_{i+1,j}+c_{ij}+c_{i+1,j+1})$, for $1\le i\le n-3$, $i+2\le j\le n-1$, $\alpha_i:=c_{i+1,1}-c_{i,1}$, for $2\le i\le n-1$, $\beta_i:=c_{in}-c_{i+1,n}$, for $1\le i\le n-2$, and $\delta_{i+1,j}\ge 0$, $\alpha_i\ge 0$, $\beta_i\ge 0$.

Proof. The proof of the "if"-part of the lemma is straightforward; just observe that all matrices in the conic combination are Kalmanson and Robinson matrices and that a conic combination preserves the Kalmanson and Robinson properties because all of them are defined in terms of inequalities involving the entries of the matrix.

We prove now the "only if"-part. Since C is a Kalmanson matrix it has a representation as stated by Lemma 4.1 in (10). Observe that (10) and (13) differ on the first summand, which is a weak sum matrix in (10) and constant matrix in (13), and on the range of summation for the third and the fourth summand (combined in one single summand in (10)). We go through the procedure applied in Lemma 4.1 and show the matrix C' resulting after each transformation step is again a Robinson matrix. The non-negativity of the coefficients α_i and β_i in (10) would then follow directly from the definition of a Robinson matrix.

Consider first a transformation of the type $C'=C-\Delta A^{(i+1,j)}$, where $\Delta=\delta_{i+1,j}$. We claim that $c_{ip}-\Delta\ge c_{i+1,p}$ for all $p=i+2,\ldots,j$. Let $\Lambda_p=c_{ip}-c_{i+1,p}$, $p=i+2,\ldots,j+1$.

Since C is a Robinson matrix, we have $\Lambda_p\ge 0$. Since C is a Kalmanson matrix, we have $c_{ip}+c_{i+1,j}-c_{i+1,p}-c_{i,j}=\Lambda_p-\Lambda_j\ge 0$ and $\Lambda_p\ge\Lambda_j$. Clearly $\Delta=\Lambda_j-\Lambda_{j+1}$ and therefore $c_{ip}-\Delta-c_{i+1,p}=\Lambda_p-\Lambda_j+\Lambda_{j+1}\ge 0$, which proves the claim. The claim that $c_{q,j+1}-\Delta\ge c_{qj}$ for all $q=i+1,\ldots,j-1$ can be proved in a similar way. So the new matrix C' is a Robinson matrix.

Consider now a transformation of the type $C'=C-\alpha_i A^{(1,i)}-\beta_i A^{(i+1,n)}$, for $2\le i\le n-2$. The Kalmanson inequalities (6) ensure that C' is a Robinson matrix. So, what is left to prove is that $Z = S - \alpha_{n-1}A^{(1,n-1)} - \beta_1 A^{(2,n)}$ is a weak constant matrix, where S is the weak sum matrix in the presentation (10).

Since every transformation step results in a Robinson matrix, as shown above, the weak sum matrix S resulting after the last transformation in the proof of Lemma 4.1 is a Robinson matrix, too. It is easily seen that a symmetric weak sum matrix $S = (s_{ij})$ with $s_{ij} = \gamma_i + \gamma_j$, for $1 \le i < j \le n$, is a Robinson matrix, if and only if $\gamma_1 \ge \gamma_2 = \ldots = \gamma_{n-1} \le \gamma_n$. Indeed $s_{1j} = \gamma_1 + \gamma_j \le s_{1,j+1} = \gamma_1 + \gamma_{j+1}$ implies $\gamma_j \le \gamma_{j+1}$, for $j \in \{2, 3, \ldots, n-1\}$, and $s_{i-1,n} = \gamma_{i-1} + \gamma_n \ge s_{i,n} = \gamma_i + \gamma_n$, implies $\gamma_{i-1} \ge \gamma_i$, for $i \in \{2, 3, \ldots, n-1\}$.

After the last transformation the equalities $s_{n,1} - s_{n-1,1} = \gamma_n - \gamma_{n-1} = c_{n,1} - c_{n-1,1} = \alpha_{n-1}$ and $s_{1,n} - s_{2,n} = \gamma_1 - \gamma_2 = c_{1,n} - c_{2,n} = \beta_1$ clearly hold. Observe finally that

$$S - (\gamma_n - \gamma_{n-1}) \times A^{(1,n-1)} - (\gamma_1 - \gamma_2) \times A^{(2,n)} = S - \alpha_{n-1}A^{(1,n-1)} - \beta_1 A^{(2,n)}$$

is a weak constant matrix (with all non-diagonal elements equal to $2\gamma_2$), which completes the proof. □

By applying the above lemma to compute the coefficients of the conic combination for a cut matrix (which is a Kalmanson and a Robinson matrix) we obtain

Corollary 4.4 *Let C be a cut matrix with m blocks such that k of them ($k \le m$) contain more than one element. Let the corresponding k row and column blocks $I_1, I_2, \ldots, I_k$, $|I_j| > 1$, $\forall j$, of C be given as $I_1 = \{i_1 = 1, \ldots, j_1\}$, $I_2 = \{i_2, \ldots, j_2\}$, $\ldots$, $I_k = \{i_k, \ldots, j_k\}$, where $i_l \ge j_{l-1} + 1$ and $i_l < j_l$, for $1 \le l \le k$. Then C can be represented as $C = Z + \sum_{l=1}^{l=k} A^{(i_l,j_l)}$, where $Z = (z_{ij})$ with $z_{ij} = -(k-1)$ for $i \ne j$.*

4.2 Recognizing conic combinations of cut matrices in CDW normal form

As mentioned in Section 3.3 a combined p.s.s. case of the QAP arises if one of the coefficient matrices is a conic combination of cut matrices in CDW normal form and the other one is a conic combination of a symmetric anti-Monge matrix and a down-benevolent Toeplitz matrix. Thus, given a matrix C which is both a Kalmanson matrix and a Robinson matrix, it is a question of interest whether the matrix can be represented as a conic combination of cut matrices in CDW normal form. Notice that every cut matrix in CDW normal form is both a Robinson and a Kalmanson matrix but not vice versa. Notice moreover that a weakly constant matrix with zeroes on the diagonal and constant $K \in \mathbb{R}$ elsewhere can be obtained by multiplying with K a special cut matrix in CDW normal form with all blocks of length one. In order to formulate a simple rule for recognizing this special subclass of Kalmanson (and Robinson) matrices, we will associate to every (Kalmanson and Robinson) matrix C an $n \times n$ symmetric *cut-weight matrix* $D(C) = (d_{ij})$ with $d_{ij} := \delta_{ij} = c_{i-1,j} + c_{i,j+1} - c_{ij} - c_{i-1,j+1}$ for $2 \le i < j \le n-1$, and $d_{1i} := \alpha_i = c_{i+1,1} - c_{i1}$, $i = 2, \ldots, n-1$, $d_{in} := \beta_{i-1} = c_{i-1,n} - c_{in}$, $i = 2, \ldots, n-1$. Observe that the coefficients δ_{ij}, α_i and β_{i-1} are as defined in Lemma 4.3. The elements which are not defined are irrelevant for further considerations, and are set to be zeros.

Consider a cut matrix in CDW normal form. Let $I_l = \{i_l, \ldots, j_l\}$, $1 \le l \le k$, be its k blocks with more than one element, involved in the representation described in Corollary 4.4. These blocks have the following properties: $2 \le |I_1| \le |I_2| \le \ldots \le |I_k|$, $\cup_{l=1}^{l=k} I_l = \{i_1, i_1 + 1, \ldots, n\}$

and $i_l = j_{l-1}+1$, for $2 \le l \le k$. Clearly, the corresponding cut-weight matrix contains only k non-zero elements $d_{i_l,j_l} = 1$, $1 \le l \le k$, exactly one for each block.

Next we will represent an $n \times n$ cut matrix in CDW normal form by a directed graph with $n+1$ nodes on a line, and edges determined in terms of the cut-weight matrix, as follows. Let the nodes be labeled by $\{1, 2, \ldots, n+1\}$, increasing from the left to the right. For each non-zero entry $d_{i_1,j_1} = 1$, $1 \le l \le k$, of the cut-weight matrix we introduce an edge that connects nodes i_1 and j_1+1 and is directed from i_1 to j_1+1, hence from the left to the right; see Figure 12 for an illustration. Let k be the vertex with the smallest index having a positive degree. Then the degree of k equals 1, i.e. $deg(k) = 1$ and every node $i \in \{k+1, \ldots, n\}$ has degree 0 or 2, whereas the degree of node $n+1$ equals 1. Furthermore notice that for every directed edge (i,k) in this graph $i+1 < k$ holds. For such an edge we will say that it enters node k and leaves node i. Finally notice that if there is an edge entering a node $k \le n-1$, then the edge leaving node k is at least as long as the edge entering k, where the length of an edge (i,k) is given as $k-i$, for $i+1<k$. A directed graph with all these properties is called a *multi-cut graph* and is formally defined as follows.

Definition 4.5 *A directed graph $G = (V,E)$ with node set $V = \{1, 2, \ldots, n+1\}$ and edge set $E = \{(i_p, i_{p+1}) \colon 1 \le p \le |E|\}$, where all edges are directed from the node with the smaller index to the node with larger index, with the following properties*

(a) Let $i_1 := k$ be the vertex with the smallest index such that $deg(k) > 0$. Then $deg(k) = 1$, $deg(v) \in \{0,2\}$, for $v \in \{k+1, k+2, \ldots, n\}$ and $deg(n+1) = 1$, where $deg(v)$ denotes the degree of v in G.

(b) The length of every edge $(i_p, i_{p+1}) \in E$ is not smaller than 2, i.e. $i_{p+1} - i_p \ge 2$, for all $1 \le p \le |E|$.

(c) The inequalities $i_{p+1} - i_p \le i_{p+2} - i_{p+i}$ hold for all $1 \le p \le |E|-1$.

is called a cut-weight graph.

It is straightforward to see that the symmetric matrix $D = (d_{ij})$ constructed in relationship with an arbitrary cut-weight graph G by setting its entries $d_{i_p,i_{p+1}} = 1$ for any edge index p, $1 \le p \le |E|$ and $d_{ij} = 0$ otherwise, for all $i<j$, is the cut-vertex matrix $D(C) =: D$ of an $n \times n$ block matrix C in CDW normal form. C has $k-1+|E|$ blocks which are given as $I_j = \{j\}$, for $j < k = i_1$, and $I_{k-1+p} = \{i_p, i_p+1, \ldots, i_{p+1}-1\}$, for $1 \le p \le |E|$.

So to every block matrix in CDW normal form a cut-weight graph can be associated, and vice versa, to every cut-weight graph a block matrix in CDW normal form can be associated, as above.

Consider now a conic combination $A = \sum_{p=1}^{q} \alpha_p A_p$ of (Kalmanson and Robinson) matrices A_p, for $1 \le p \le q$. Clearly the cut-weight matrix of the linear (conic) combination can be obtained as a conic combination of the cut-weight matrices of the summands, i.e. $D(A) = \sum_{p=1}^{q} \alpha_p D(A_p)$. If A_p are block matrices in CDW normal form and $\alpha_p \in \mathbb{N}$, for all $1 \le p \le q$, then the entries of $D(A)$ are natural numbers and A can be represented as a directed multigraph with $n+1$ nodes on a line with edges determined in terms of the cut-weight matrix $D(A)$.

More precisely for every two nodes i and $k+1$, $1 \le i \le n-2$ and $i+2 \le k+1 \le n+1$, the number of directed edges $(i, k+1)$ equals $\sum_{p=1}^{q} \alpha_p d_{i,k}^{(p)}$ where $d_{i,k}^{(p)}$ is the corresponding entry of $D(A_p)$, for $1 \le p \le q$. Consequently for each edge of length x entering a node k, there is one edge of length at least x leaving node k. Let us denote by $E^-(k+1, x)$ and $E^+(k+1, x)$ the number of edges of length at least x entering or leaving the node $k+1$, respectively. Then, clearly the number of edges of length at least x entering node $k+1$ does not exceed the number of edges of length at least x leaving node $k+1$, thus $E^-(k+1, x) \le E^+(k+1, x)$, holds for $3 \le k+1 \le n-1$, $2 \le x \le k$. By considering that $E^-(k+1, x) = \sum_{i=1}^{k+1-x} d_{ik}$ and $E^+(k+1, x) = \sum_{j=k+1+x-1}^{n} d_{k+1,j}$, we get:

$$\sum_{i=1}^{k+1-x} d_{ik} \le \sum_{j=k+1+x-1}^{n} d_{k+1,j}\,, \text{ for } 2 \le k+1 \le n-2 \text{ and } 2 \le x \le k. \tag{14}$$

Notice that by setting $l := k+1-x$ inequalities (14) can be equivalently written as

$$\sum_{i=1}^{l} d_{ik} \le \sum_{j=2k+1-l}^{n} d_{k+1,j} \tag{15}$$

for $k = 2, \ldots, n-2$ and $l = 1, \ldots, k-1$. The right-hand sum in (15) is considered to be zero if $2k+1-l > n$. This in particular means that $d_{ik} = 0$ for $k = \lceil n/2 \rceil, \ldots, n-1$ and $i = 1, \ldots, 2k-n$.

It turns out that the later inequalities fulfilled by the entries of the cut-weight matrix $D := D(A)$ of some matrix A are not only necessary, but also sufficient for A to be a conic combination of cut matrices in CDW normal form, i.e. for A to be representable as $A = \sum_{p=1}^{q} \alpha_p A_p$, with cut matrices A_p, $1 \le p \le q$, in CDW normal form.

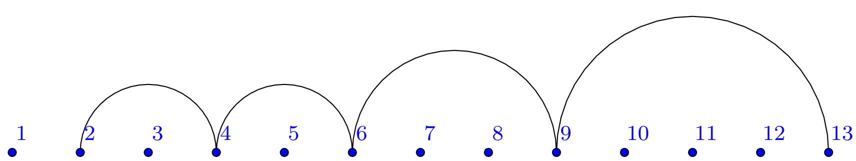


Figure 12: Illustration of the graph representation of a 12×12 cut matrix in CDW normal form with the blocks $\{2,3\}$, $\{4,5\}$, $\{6,7,8\}$, and $\{9,10,11,12\}$. All edges are directed from left to right.

Theorem 4.6 *A symmetric $n \times n$ Kalmanson matrix C which is also a Robinson matrix with cut-weight matrix $D(C) = (d_{ij})$ can be represented as the sum of a weak constant matrix and a conic combination of cut matrices in CDW normal form if and only if inequalities (15) hold for $k = 2, \ldots, n-2$ and $l = 1, \ldots, k-1$.*

Proof. The proof of the necessary condition is trivial. Let $A = \sum_{p=1}^{q} \alpha_p A_p$ be an $n \times n$ matrix which is a conic combination of $n \times n$ cut matrices A_p in CDW normal form, $1 \le p \le q$. Let $D(A) = (d_{ij})$ be the cut-weight matrix of A. Assume for simplicity (and without loss of generality) that the weight coefficients α_p, $1 \le p \le q$, are natural numbers. Then, since $D(A) = \sum_{p=1}^{q} \alpha_p D(A_p)$, the entries d_{lk}, $1 \le l, k \le n$, of $D(A)$ are also natural numbers. Inequalities (14), and consequently also inequalities (15), are obviously fulfilled due to the construction of the directed multigraph corresponding to A.

Assume now that the entries of the cut-matrix $D(C)$ of an integer Kalmanson and symmetric matrix C satisfy the inequalities (15). Similarly as in the discussion preceding the theorem we build an auxiliary directed multigraph with $n+1$ nodes $\{1, 2, \ldots, n+1\}$ and d_{ij} edges directed from i to node $j+1$ for each $d_{ij} > 0$ and $j > i$. We refer to this multigraph as the *cut-weight multigraph* of matrix C.

We build a conic representation of C as follows. We start with node $n+1$ in the cut-weight multigraph, and build a path from the right to the left by choosing in the first step an (reversed) edge $(i_{k-1}, i_k = n+1)$ with the largest length. Then in each following step $p > 1$ we construct a longest edge (i_{k-p}, i_{k-p+1}) with length $i_{k-p+1} - i_{k-p} \le i_{k-p+2} - i_{k-p+1}$, as long as such an edge exists. Let the constructed path be $P_1 := (i_1, i_2, \ldots, i_k = n+1)$ where $1 \le i_1 < i_2 < \ldots < i_k = n+1$. Consider now the graph $G = (V, E)$ with vertex set $V = \{1, 2, \ldots, n+1\}$ and edge set $E = \{(i_p, i_{p+1})\colon 1 \le p \le k-1\}$. This is clearly a cut-weight graph and hence it can be associated to a cut matrix in CDW normal form as described before the theorem. We denote this matrix by A_1. Let α_1 be the minimum multiplicity of the edges in path P_1. We remove α_1 copies of P_1 from the cut-weight multigraph of C and set $d_{i_p, i_{p+1}-1} := d_{i_p, i_{p+1}-1} - \alpha_i$, for $1 \le p \le k-1$. We show that (14), and equivalently also (15), remain fulfilled after this update. The update of the coefficients d_{ij} can only affect inequality (15) for indices k such that $k+1$ is an endpoint of an edge in P_1, i.e. $k+1 \in \{i_1, i_2, \ldots, i_{k-1}\}$. For $k+1 \in \{i_2, \ldots, i_{k-1}\}$ the update of d_{ij} results in subtracting α_i from both sides of (15), and hence it does not affect the validity of the inequality. It remains the case $k+1 = i_1$ for $i_1 > 2$. Let $\bar{E}^-(i_1, x)$, $\bar{E}^+(i_1, x)$ be the values of $E^-(i_1, x)$ and $E^+(i_1, x)$, after the update of the coefficients d_{ij}, respectively. Thus we have to show that $\bar{E}^-(i_1, x) \le \bar{E}^+(i_1, x)$ holds. Let $l_1 := i_2 - i_1$ be the length of the first edge in P_1. If $x > l_1$ than $\bar{E}^+(i_1, x) = E^+(i_1, x)$ and $\bar{E}^-(i_1, x) = E^-(i_1, x)$, and since $E^-(i_1, x) \le E^+(i_1, x)$ holds, there is nothing to show in this case. If $x \le l_1$, $\bar{E}^+(i_1, x) = E^+(i_1, x) - \alpha_1$ and $\bar{E}^-(i_1, x) = E^-(i_1, x)$. Notice that $E^-(i_1, x) = E^-(i_1, x+1)$, otherwise P_1 could have been prolonged beyond i_1. Moreover, $E^-(i_1, x+1) \le E^+(i_1, x+1)$ due to the assumption of the theorem, and $E^+(i_1, x+1) \le E^+(i_1, x) - \alpha_1 = \bar{E}^+(i_1, x)$ because there are at least α_1 edges of length x leaving i_1. By putting things together we get the required inequality $\bar{E}^-(i_1, x) \le \bar{E}^+(i_1, x)$.

The path construction and the corresponding update of the coefficient d_{ij} can be then inductively repeated as long as possible, while (15) remains an invariant during this process and in every step i, $i \in \mathbb{N}$, a cut matrix A_i in CDW normal form is identified. A_i corresponds to the path P_i constructed in the i-th step. If α_i is the minimum multiplicity of the edges in P_i then $\alpha_i P_i$ is a summand of the required conic combination. This process is finite because in every step at least one edge is removed from the original cut-edge multigraph. The process terminates when there are no edges entering node $n+1$ any more, say after t steps. We claim that after the t-th step, there are no more edges in the cut-weight multigraph at all. This means that the actual coefficients d_{ij} fulfill $d_{ij}=0$ for all $i<j$, which implies that the matrix C is transformed into a weak constant matrix Z by subtracting $\sum_{p=1}^{t}\alpha_p A_p$, and thus $C = Z + \sum_{p=1}^{t}\alpha_p A_p$ holds for the original matrix C.

Now let us prove the claim. Assume by contradiction that after t transformation steps there is no edge entering node $n+1$ in the cut-weight multigraph while there is still at least one edge in the graph. Let j be the largest node index such that there is an edge entering j. Then $j \le n$ according to our assumption. The inequalities (15) have to be fulfilled because they are an invariant of the transformation process. In particular, $1 \le E^-(j,1) \le E^+(j,1)$ must hold. This implies the existence of an edge leaving j, and hence entering some node with an index strictly larger than j. This contradicts the choice of j and completes the proof of the claim. $\square$

An illustrative example. Consider the matrix C which is in the class of Kalmanson and Robinson matrices:

$$C = \begin{pmatrix} 0 & 1 & 2 & 3 & 3 & 3 \\ 1 & 0 & 2 & 3 & 3 & 3 \\ 2 & 2 & 0 & 2 & 3 & 3 \\ 3 & 3 & 2 & 0 & 2 & 2 \\ 3 & 3 & 3 & 2 & 0 & 1 \\ 3 & 3 & 3 & 2 & 1 & 0 \end{pmatrix}$$

We first illustrate the proofs of Lemmas 4.1 and 4.3 and show how to represent C as sum of a conic combination of cut matrices A^{kl} with a weak constant matrix.

Note that for matrix C there is only one strict inequality in system (6) $c_{25}+c_{34} < c_{24}+c_{35}$, and three strict inequalities in system (7): $c_{i1}+c_{i+1,6} < c_{i6}+c_{i+1,1}$ with $i=2,3,4$.

We first eliminate the strict inequality $c_{25}+c_{34} < c_{24}+c_{35}$ by subtracting from C the cut matrix A^{34} multiplied by $\delta_{34} = c_{24}+c_{35}-c_{25}-c_{34} = d_{24} = 1$. The transformed matrix $C' = C - A^{34}$ is given as follows.

$$C' = C - A^{34} = \begin{pmatrix} 0 & 0 & 1 & 2 & 2 & 2 \\ 0 & 0 & 1 & 2 & 2 & 2 \\ 1 & 1 & 0 & 2 & 2 & 2 \\ 2 & 2 & 2 & 0 & 1 & 1 \\ 2 & 2 & 2 & 1 & 0 & 0 \\ 2 & 2 & 2 & 1 & 0 & 0 \end{pmatrix}$$

Next we set $\alpha_2 = c'_{31} - c'_{21} = 1$, $\beta_2 = c'_{26} - c'_{36} = 0$, therefore in the next transformation step the cut matrix A^{12} is subtracted and we get:

$$C - A^{34} - A^{12} = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & -1 \\ 1 & 1 & 1 & 0 & -1 & 0 \end{pmatrix}$$

In the next step we set $\alpha_3 = 1$, $\beta_3 = 1$, and subtract $A^{13} + A^{46}$ from the current matrix C to obtain

$$C - A^{34} - A^{12} - A^{13} - A^{46} = \begin{pmatrix} 0 & -1 & -1 & -1 & -1 & -1 \\ -1 & 0 & -1 & -1 & -1 & -1 \\ -1 & -1 & 0 & -1 & -1 & -1 \\ -1 & -1 & -1 & 0 & -1 & -1 \\ -1 & -1 & -1 & -1 & 0 & -2 \\ -1 & -1 & -1 & -1 & -2 & 0 \end{pmatrix}$$

Finally we set $\alpha_4 = c'_{51} - c'_{41} = 0$ and $\beta_4 = c'_{4,6} - c'_{56} = 1$ and subtract A^{56} from the actual matrix C to obtain a weak constant matrix

$$C - A^{34} - A^{12} - A^{13} - A^{46} - A^{56} = \begin{pmatrix} 0 & -2 & -2 & -2 & -2 & -2 \\ -2 & 0 & -2 & -2 & -2 & -2 \\ -2 & -2 & 0 & -2 & -2 & -2 \\ -2 & -2 & -2 & 0 & -2 & -2 \\ -2 & -2 & -2 & -2 & 0 & -2 \\ -2 & -2 & -2 & -2 & -2 & 0 \end{pmatrix}$$

The cut-weight matrix $D(C)$ contains five non-zero entries corresponding to the coefficients δ_{34}, α_2, α_3, β_3 and β_4 above:

$$D(C) = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The corresponding cut-weight multigraph is depicted in Figure 13. It can be easily seen that the entries of $D(C)$ fulfill the inequalities 15 and hence C can be represented as the sum of weak constant matrix with a conic combination of block matrices in CDW normal form. The block matrices A_1 and A_2 in CDW normal form correspond to the paths $P_1 = (1, 4, 7)$ and $P_2 = (1, 3, 5, 7)$ and hence, A_1 has two blocks $\{1, 2, 3\}$ and $\{4, 5, 6\}$, and A_2 has three blocks $\{1, 2\}$, $\{3, 4\}$, and $\{5, 6\}$.

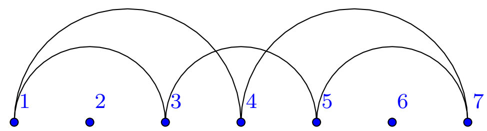


Figure 13: Cut weight multigraph for the illustrative example

5 Conclusions

Summary of results. In this paper we introduced two new polynomially solvable special (p.s.s.) cases of the QAP. We call the first one *the down-benevolent QAP*; this is a $QAP(A, B)$ where A is both a Kalmanson and a Robinson matrix and B is a down-benevolent Toeplitz matrix, and it is solved to optimality by the identity permutation. This new p.s.s. case is related to two other p.s.s. cases of the QAP known in the literature: (a) $QAP(A, B)$ with A being a Kalmanson matrix and B being a DW Toeplitz matrix [18], and (b) $QAP(A, B)$ with A being a Robinson matrix and B being a simple Toeplitz matrix [27]. In the new p.s.s. case matrix A is more special and matrix B is more general than in the previous two p.s.s. cases.

We call the second new p.s.s. case *the up-benevolent QAP*; this is a QAP(A,B) where A is a PS anti-Monge matrix and B is an up-benevolent Toeplitz matrix, and it is solved to optimality by the identity permutation. This new p.s.s. case is a generalization of another p.s.s. case of the QAP known in the literature, namely $QAP(A, B)$ where A is a symmetric monotone anti-Monge matrix and B is an up-benevolent Toeplitz matrix [5].

Further we introduce a new class of specially structured matrices. A matrix belongs to this class iff it can be represented as the sum of a weakly constant matrix and a conic combination of cut matrices in CDW normal form. The matrices of this class build a strict subclass of matrices which are both Robinson and Kalmanson matrices. It follows from a result in [11] that $QAP(A, B)$ is solved to optimality by the identity permutation if A belongs to the newly introduced class of matrices and B is a symmetric monotone anti-Monge matrix .

The new class of matrices and the down-benevolent QAP lead to another new p.s.s. case of the QAP: the combined p.s.s. case $QAP(A, B)$ where A is a conic combination of cut matrices in CDW normal form and B is a conic combination of a monotone anti-Monge matrix and a down-benevolent Toeplitz matrix, is solved to optimality by the identity permutation.

The combined p.s.s. case mentioned above gives rise to an interesting and non-trivial question related to the recognition of conic combination of cut matrices in CDW normal form: Given an $n \times n$ matrix A, $n \in \mathbb{N}$, decide whether A can be represented as the sum of a weak constant matrix and a conic combination of cut matrices in CDW normal form. We show that this decision problem can be solved efficiently by computing $O(n^2)$ numbers which we call cut-weights, and checking whether the cut-weights fulfill $O(n^2)$ linear inequalities.

Notice that both the monotone anti-Monge matrices and the down-benevolent Toeplitz matrices are defined in terms of linear inequalities. Therefore simple linear programming techniques can be used to recognize whether a given symmetric matrix B can be represented/approximated as a conic combination of two matrices B_1 and B_2 where B_1 is a monotone anti-Monge matrix and B_2 is a down-benevolent Toeplitz matrices. Thus for a given instance of QAP it can be efficiently checked whether it is an instance of the new combined special case introduced in this paper.

Questions for future research. A general and challenging question to be considered for future research is the recognition of the permuted combined p.s.s. case formulated as follows. For a given instance $QAP(A,B)$ decide whether a) there exists a permutation ϕ of the rows and the columns of A such that the matrix resulting after permuting A according to ϕ can be represented as sum of a weak constant matrix and a conic combination of cut matrices in CDW normal form, and b) there exists a permutation ψ of the rows and the columns of B such that the matrix resulting after permuting B according to ψ can be represented as the sum of a monotone anti-Monge matrix and a down-benevolent Toeplitz matrix.

Moreover it would be interesting to investigate whether the combined p.s.s. case of the QAP can be used to compute good lower bounds and/or heuristic solutions for the general problem. The idea is to "approximate" the coefficient matrices A and B of a given instance $QAP(A,B)$ by some matrices A' and B', respectively, such that $QAP(A',B')$ is an instance of the combined special case. Then, if A' and B' are chosen "appropriately", the optimal solution of $QAP(A',B')$ and its optimal value could serve as a heuristic solution and/or a lower bound for $QAP(A,B)$, respectively. Clearly, the crucial part is to find out what "approximate" and "appropriately" should exactly mean. This is definitely a challenging issue but it could well lead to a new direction of research on the QAP.

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A Instances used in illustrations

0	16	17	19	24	26	28	37	41	44
16	0	13	17	20	21	27	33	39	43
17	13	0	11	16	20	23	28	36	42
19	17	11	0	9	12	15	25	33	41
24	20	16	9	0	4	8	21	29	37
26	21	20	12	4	0	2	19	27	33
28	27	23	15	8	2	0	18	22	29
37	33	28	25	21	19	18	0	20	25
41	39	36	33	29	27	22	20	0	16
44	43	42	41	37	33	29	25	16	0

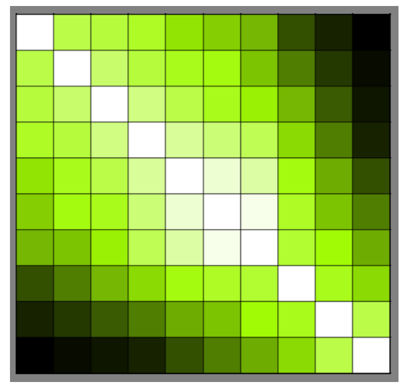

Figure 14: Robinson matrix visualised in Fig. 1 A

0	55	45	35	25	15	5	5	0	0
55	0	55	45	5	3	15	5	5	0
45	55	0	55	45	5	3	15	5	5
35	45	55	0	55	45	5	3	15	5
25	5	45	55	0	55	45	5	3	15
15	3	5	45	55	0	55	45	5	3
5	15	3	5	45	55	0	55	45	5
5	5	15	3	5	45	55	0	55	45
0	5	5	15	3	5	45	55	0	55
0	0	5	5	15	3	5	45	55	0

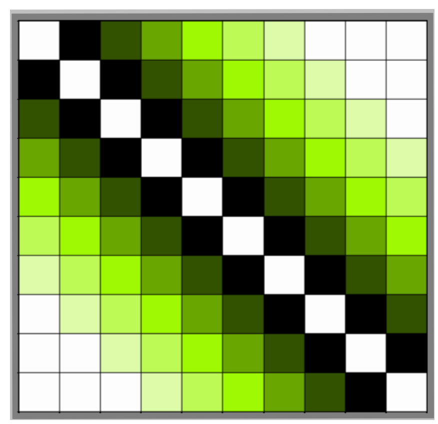

Figure 15: Simple Toeplitz matrix visualised in Fig. 1 B

0	7	10	10	10	10	10	10	10	10
7	0	7	9	10	10	10	10	10	10
10	7	0	2	9	10	10	10	10	10
10	9	2	0	7	8	10	10	10	10
10	10	9	7	0	1	6	9	9	9
10	10	10	8	1	0	5	8	8	8
10	10	10	10	6	5	0	3	5	5
10	10	10	10	9	8	3	0	2	2
10	10	10	10	9	8	5	2	0	0
10	10	10	10	9	8	5	2	0	0

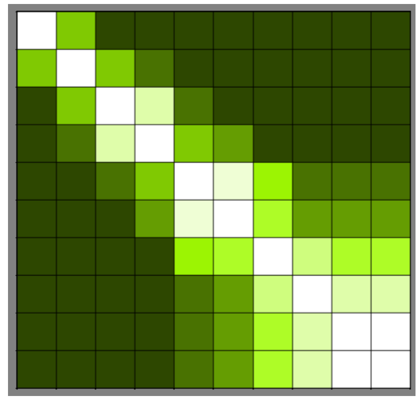

Figure 16: Conic combination of cut matrices in CDW normal form visualised in Fig. 2 A

1	3	6	7	10	13	14	18	19	24
3	5	11	14	18	23	26	33	36	45
6	11	17	24	30	36	42	51	57	69
7	14	24	31	40	49	59	69	77	93
10	18	30	40	49	60	71	83	94	114
13	23	36	49	60	71	84	97	110	134
14	26	42	59	71	84	97	112	126	152
18	33	51	69	83	97	112	127	143	170
19	36	57	77	94	110	126	143	159	187
24	45	69	93	114	134	152	170	187	215

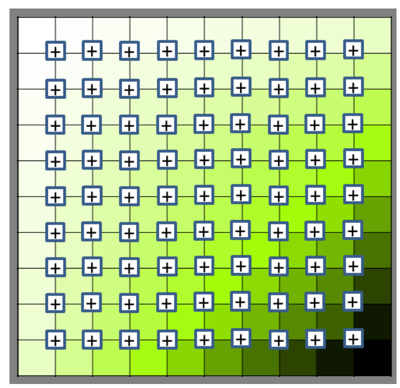

Figure 17: Anti-Monge monotone matrix visualised in Fig. 2 B

0	47	54	45	54	44	45	48	45	44
47	0	46	44	54	46	49	52	51	54
54	46	0	34	46	39	45	50	52	58
45	44	34	0	19	15	25	31	35	45
54	54	46	19	0	10	21	29	36	50
44	46	39	15	10	0	0	9	18	36
45	49	45	25	21	0	0	5	15	35
48	52	50	31	29	9	5	0	18	38
45	51	52	35	36	18	15	18	0	35
44	54	58	45	50	36	35	38	35	0

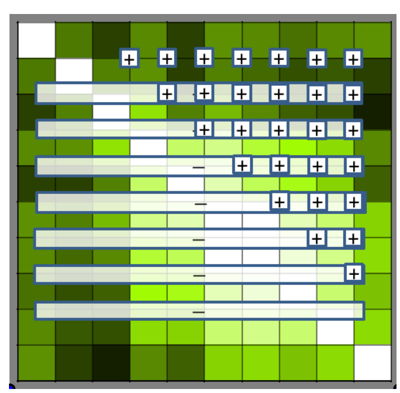

Figure 18: Kalmanson matrix visualised in Fig. 3 A

0	12	10	5	3	0	3	5	10	12
12	0	12	10	5	3	0	3	5	10
10	12	0	12	10	5	3	0	3	5
5	10	12	0	12	10	5	3	0	3
3	5	10	12	0	12	10	5	3	0
0	3	5	10	12	0	12	10	5	3
3	0	3	5	10	12	0	12	10	5
5	3	0	3	5	10	12	0	12	10
10	5	3	0	3	5	10	12	0	12
12	10	5	3	0	3	5	10	12	0

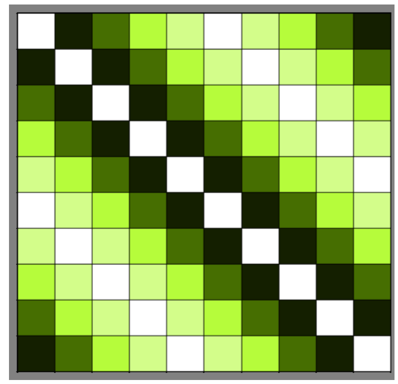

Figure 19: Toeplitz matrix visualised in Fig. 3 B

0	31	36	39	43	47	48	50	53	57
31	0	21	26	31	37	40	42	47	55
36	21	0	10	17	24	30	34	42	53
39	26	10	0	1	11	21	26	36	51
43	31	17	1	0	0	11	18	31	50
47	37	24	11	0	0	0	8	23	46
48	40	30	21	11	0	0	2	18	43
50	42	34	26	18	8	2	0	17	42
53	47	42	36	31	23	18	17	0	40
57	55	53	51	50	46	43	42	40	0

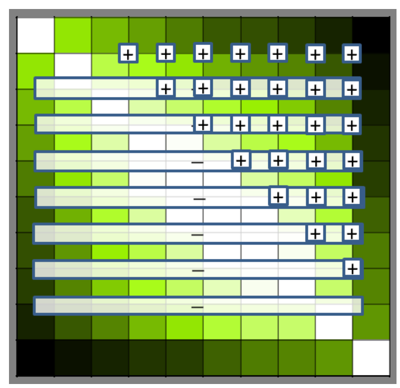

Figure 20: Kalmanson and Robinson matrix visualised in Fig. 4 A

0	28	20	10	8	0	6	0	15	10
28	0	28	20	10	8	0	6	0	15
20	28	0	28	20	10	8	0	6	0
10	20	28	0	28	20	10	8	0	6
8	10	20	28	0	28	20	10	8	0
0	8	10	20	28	0	28	20	10	8
6	0	8	10	20	28	0	28	20	10
0	6	0	8	10	20	28	0	28	20
15	0	6	0	8	10	20	28	0	28
10	15	0	6	0	8	10	20	28	0

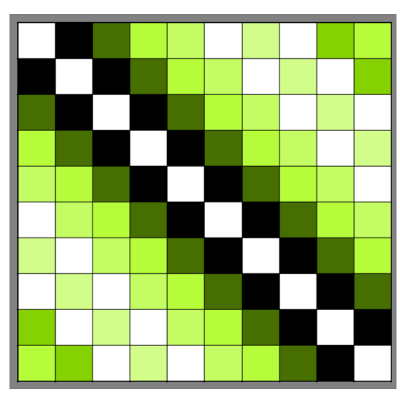

Figure 21: Down-benevolent matrix visualised in Fig. 4 B

1	15	4	60	4	25	4	15	1	1
15	29	21	79	24	47	28	42	30	31
4	21	13	73	20	44	28	44	35	39
60	79	73	133	83	110	94	111	104	110
4	24	20	83	33	62	47	66	62	69
25	47	44	110	62	91	78	98	96	107
4	28	28	94	47	78	65	87	86	99
15	42	44	111	66	98	87	109	110	124
1	30	35	104	62	96	86	110	111	126
1	31	39	110	69	107	99	124	126	141

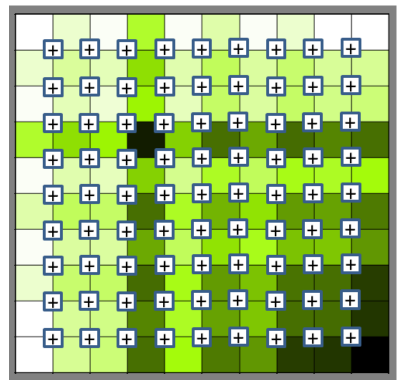

Figure 22: Anti-Monge matrix visualised in Fig. 5 A

0	5	13	23	25	33	27	33	18	23
5	0	5	13	23	25	33	27	33	18
13	5	0	5	13	23	25	33	27	33
23	13	5	0	5	13	23	25	33	27
25	23	13	5	0	5	13	23	25	33
33	25	23	13	5	0	5	13	23	25
27	33	25	23	13	5	0	5	13	23
33	27	33	25	23	13	5	0	5	13
18	33	27	33	25	23	13	5	0	5
23	18	33	27	33	25	23	13	5	0

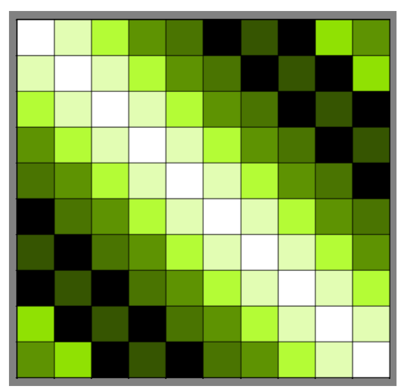

Figure 23: Up-benevolent Toeplitz matrix visualised in Fig. 5 B