ON THE EXISTENCE OF HARMONIC MAPS.

Luc Lemaire.

Submitted for the degree of
Doctor of Philosophy.
University of Warwick.
May 1977.

ABSTRACT.

We study different properties of harmonic maps between two compact Riemannian manifolds M and M' and in particular the following existence question : do there exist harmonic elements in the different homotopy classes of maps from M to M' ?

Our first result is an affirmative answer to that question when M is a surface and the second homotopy group of M' is zero. This result is then extended to certain products of manifolds and to φ-harmonic maps.

When M and M' are orientable surfaces, this solves the existence question as long as M' is not a sphere . When M is a surface of genus p and M' a sphere, the question was answered by J. Eells and J. Wood for all classes of maps of degree greater or equal to p . For all remaining cases (degree $\leq$ p - 1), we obtain existence results for particular metrics on M and M'.

We also study the question of existence for non-orientable surfaces, and obtain complete results for maps between spheres and projective planes.

A second type of results is a finiteness theorem for harmonic maps : we show that if the sectional curvature of M' is negative, there is only a finite number of non-constant harmonic maps from M to M' of dilatation bounded by a fixed constant . This implies a similar result on the number of almost complex maps between almost Kaehlerian manifolds.

Other results include an example of a continuous family of harmonic maps, a remark on the derivatives of the harmonicity equations and an answer to a question of H. Eliasson concerning a higher order energy.

TABLE OF CONTENTS.

It is a great pleasure to express my thanks to James Eells for all his help during my whole stay in England.

I also wish to acknowledge the financial support of the Fonds National Belge de la Recherche Scientifique, the Royal Society and the British Council.

Declaration.

As indicated in the text, a special case of theorem 2.8, lemma 2.19 and propositions 3.11, 3.16 and 3.17 have been used in a thesis at the Université Libre de Bruxelles.

INTRODUCTION.

Let M,g and M',g' be compact C^∞ Riemannian manifolds without boundary . A map $f \in C^\infty(M,M')$ is said to be harmonic iff it is a critical point of the energy functional

$$E(f) = \frac{1}{2}\int_M |df|^2 v_g \quad [3].$$

In this thesis, we shall study different problems related to the following question : does a given homotopy class of maps from M to M' contain a harmonic representative ? This question appeared when J. Eells and J. Sampson showed that its answer is affirmative for every homotopy class when the sectional curvature of M' is non-positive [3] . On the other hand, J. Eells and J. Wood have exhibited some homotopy classes between compact surfaces containing no harmonic element [4] . It is that question of existence that we shall first study.

In chapter 1, we recall the basic definitions and properties of harmonic maps.

In chapter 2, we prove that if M is a surface and if the second homotopy group of M' is zero, there exists a harmonic element in every homotopy class of maps from M to M' . This result is obtained by means of a direct method of the calculus of variations and § a presents the basic results which are used . In § b, we prove the existence theorem, in the case of a slightly more general integral, so that we can apply it to the cases of harmonic maps between certain products of manifolds (§ d) and of φ-harmonic maps (§ e) . The regularity result required in the proof is a variation of a theorem of C. Morrey [20] and is proven in § c.

In chapter 3, we restrict ourselves to the case of two compact orientable surfaces, in order to get a more detailed picture of the situation . This will come, on one hand from existence results related to the direct method introduced above, and on the other from the relations between harmonic and holomorphic maps in dimension 2, which we recall in § a.

Let p and p' denote the genera of M and M' . When $p' > 0$, we get a complete existence result as an immediate consequence of theorem 2.8.

When $p' = 0$, the situation is more complicated . The homotopy classes of maps are parametrized by the degree ω, and, by changing the orientation of M if necessary, we can always suppose ω positive or zero.

For $\omega \geq p$, J. Eells and J. Wood have shown that every harmonic map of degree ω is holomorphic and deduced existence and non-existence results solving the existence question in all cases . We recall their results in § c.

In § d, we consider the only remaining case, namely $p' = 0$, $\omega \leq p - 1$. No complete results are then available, but we prove that for all p and $\omega < p$, there exist a surface M of genus p and a harmonic non-holomorphic map of degree ω from M to S^2. This provides a partial existence result and shows that the theorem of J. Eells and J. Wood does not extend to these degrees.

Finally we exhibit for all $p \geq 3$ a surface M of genus p such that a certain non trivial homotopy class of maps from M to S^2 contains 2 harmonic representatives, one holomorphic and one non-holomorphic.

In chapter 4, we turn to a different type of problems and try to generalize to harmonic maps a class of results stating that under certain conditions, there is only a finite number of surjective holomorphic maps between two complex spaces (cf. [12,§ 8]). For that, we have to introduce a new assumption on the maps, namely that their dilatation is bounded by a fixed constant K (§ a) . We then prove that if M and M' are compact manifolds and if the curvature of M' is negative, there is only a finite number of non-constant harmonic maps from M to M' verifying that assumption (§ b) . When M and M' are almost Kaehlerian and M' of negative curvature, this implies a finiteness result on the number of almost complex maps between M and M'.

In chapter 5, we present 3 separate results.

In § a, we exhibit a continuous family of harmonic maps between surfaces which are not isometrically or conformally equivalent.

In § b, we extend a result of K. Yano and S. Ishihara [28] by proving that, for any positive integer k, a map is harmonic as soon as the k^{th} derivative of its harmonicity equation is zero.

Finally (§ c), we answer a question of H. Eliasson by showing that the second order energy defined in [5] does not always reach its infimum in a homotopy class of maps between surfaces, and hence does not satisfy condition (C).

In the last chapter, which results from very recent joint work with James Eells, we study the existence problem for harmonic maps between not necessarily orientable surfaces M and M'.

When M' is not the sphere S^2 or the projective plane P^2, theorem 2.8 implies the existence of a harmonic map in every homotopy class, so that we then restrict ourselves to the cases $M' = S^2$ or P^2.

In § a, we recall P. Olum's classification of the homotopy classes of maps from a surface to P^2.

We can then, in § b, solve entirely the existence question for maps between spheres and projective planes, obtaining existence and non-existence results.

When M is another surface, we have as yet only partial results, which are summarized in § c.

Some of the results of this thesis will appear in [14], [15] and [16].

CHAPTER 1.

HARMONIC MAPS.

§ a : DEFINITIONS AND NOTATIONS.

In all this work, M,g and M',g' will represent compact connected C^∞ Riemannian manifolds without boundary, of dimensions n and n' . $\{x^i\}$ and $\{u^\alpha\}$ will denote coordinate systems around the points m of M and m' of M' . If $n = n' = 2$, these systems will be denoted $\{x,y\}$ and $\{u,v\}$. Γ and Γ' will be the coefficients of the Riemannian connections on M and M' and R and R' their curvature.

Let $f : M \longrightarrow M'$ be a C^∞ map . Its covariant derivatives will be denoted as follows :

$$f_i^\alpha = \frac{\partial f^\alpha}{\partial x^i}$$

$$f_{ij}^\alpha = \frac{\partial^2 f^\alpha}{\partial x^i \partial x^j} - \Gamma_{ij}^k\ f_k^\alpha.$$

We define the harmonic maps after [3].

Definition 1.1 : The energy density of f at $m \in M$ is defined by

$$e(f)(m) = \frac{1}{2} |df(m)|^2 = \frac{1}{2} g^{ij}(m)\, f^{\alpha}_{i}(m)\, f^{\beta}_{j}(m)\, g'_{\alpha\beta}(f(m)).$$

Definition 1.2 : The energy of f is the integral

$$E(f) = \int_M e(f)\, v_g$$

where v_g is the volume element associated with g.

By definition, the function $E : C^{\infty}(M,M') \longrightarrow \mathbb{R}$ is positive or zero.

Definition 1.3 : A map $f \in C^{\infty}(M,M')$ is harmonic iff it is a critical point of the energy.

Proposition 1.4 [3,§I,2] : A map f is harmonic iff it verifies the Euler-Lagrange equations $\tau(f) = 0$ where, in local coordinates,

$$\tau(f)^{\alpha} = g^{ij}\,(f^{\alpha}_{ij} + \Gamma'^{\alpha}_{\beta\gamma}(f)\, f^{\beta}_{i}\, f^{\gamma}_{j}).$$

Definition 1.5 : $\tau(f)$ is the tension of f.

Note that τ is the trace of the second fundamental form of f, defined by

$$f^{\alpha}_{;ij} = f^{\alpha}_{ij} + \Gamma'^{\alpha}_{\beta\gamma}(f)\, f^{\beta}_{i}\, f^{\gamma}_{j}$$

and can also be written

$$\tau(f)^{\alpha} = \Delta f^{\alpha} + g^{ij}\, \Gamma'^{\alpha}_{\beta\gamma}(f)\, f^{\beta}_{i}\, f^{\gamma}_{j}$$

where Δf^{α} is the Laplacian of f^{α}.

If the dimension of M is 1, a harmonic map is a geodesic of M'. If $M' = \mathbb{R}$, we have the usual definition of a harmonic function.

We define the harmonic maps after [3].

Definition 1.1 : The energy density of f at $m \in M$ is defined by

$$e(f)(m) = \frac{1}{2} |df(m)|^2 = \frac{1}{2} g^{ij}(m)\, f^\alpha_i(m)\, f^\beta_j(m)\, g'_{\alpha\beta}(f(m)).$$

Definition 1.2 : The energy of f is the integral

$$E(f) = \int_M e(f)\, v_g$$

where v_g is the volume element associated with g.

By definition, the function $E : C^\infty(M,M') \longrightarrow \mathbb{R}$ is positive or zero.

Definition 1.3 : A map $f \in C^\infty(M,M')$ is harmonic iff it is a critical point of the energy.

Proposition 1.4 [3, §I, 2] : A map f is harmonic iff it verifies the Euler-Lagrange equations $\tau(f) = 0$ where, in local coordinates,

$$\tau(f)^\alpha = g^{ij}\,(f^\alpha_{ij} + \Gamma'^\alpha_{\beta\gamma}(f)\, f^\beta_i\, f^\gamma_j).$$

Definition 1.5 : $\tau(f)$ is the tension of f.

Note that τ is the trace of the second fundamental form of f, defined by

$$f^\alpha_{;ij} = f^\alpha_{ij} + \Gamma'^\alpha_{\beta\gamma}(f)\, f^\beta_i\, f^\gamma_j$$

and can also be written

$$\tau(f)^\alpha = \Delta f^\alpha + g^{ij}\, \Gamma'^\alpha_{\beta\gamma}(f)\, f^\beta_i\, f^\gamma_j$$

where Δf^α is the Laplacian of f^α.

If the dimension of M is 1, a harmonic map is a geodesic of M'. If $M' = \mathbb{R}$, we have the usual definition of a harmonic function.

§ b : COMPOSITION LAW.

We present the results of [3;§I,5] concerning the composition of harmonic maps . Let M,M' and M'' be three manifolds.

Proposition 1.6 : Let $f \in C^\infty(M,M')$ and $h \in C^\infty(M',M'')$. Then

$$(h\circ f)^a_{;ij} = f^\alpha_{;ij}\, h^a_\alpha + h^a_{;\alpha\beta}\, f^\alpha_i\, f^\beta_j$$

$$\tau(h\circ f)^a = \tau(f)^\alpha h^a_\alpha + g^{ij}\, h^a_{;\alpha\beta}\, f^\alpha_i\, f^\beta_j .$$

Corollary 1.7 : If f is harmonic and h totally geodesic (i.e. $h^a_{;\alpha\beta} = 0$), then $h\circ f$ is harmonic.

In general, however, the composition of two harmonic maps is not harmonic.

CHAPTER 2.

EXISTENCE RESULT.

§ a : DIRECT METHODS AND SOBOLEV SPACE L_1^2.

In order to obtain different existence results for harmonic and φ-harmonic maps, it will be useful to consider, besides the energy, the slightly more general variational integral

$$I(f) = \int_M \left[\frac{1}{2} A(x,f(x))\ |df(x)|^2 + B(x,f(x))\right].v_g \qquad (1)$$

where A and B are C^∞ functions on MxM' and A is positive . The energy is of course a special case of the function I, with A = 1 and B = 0.

Since MxM' is compact, there exist constants A_o, A_1, B_o and B_1 such that

$$\begin{aligned} &0 < A_o \leq A(x,y) \leq A_1 \text{ and} \\ &B_o \leq B(x,y) \leq B_1 \end{aligned} \qquad (2).$$

For a C^∞ domain D of M, put $E_D(f) = \frac{1}{2}\int_D |df|^2.v_g$ and define $I_D(f)$ similarly . If V(D) denotes the volume of D, we deduce from (2) that

$$\begin{aligned} I_D(f) &\leq A_1\, E_D(f) + B_1\, V(D) \\ E_D(f) &\leq \frac{1}{A_o}\, (I_D(f) - B_o\, V(D)) \end{aligned} \qquad (3).$$

These relations show that I will, in many respects, have the same behaviour as E.

We now recall the definition of the space $L_1^2(M,M')$, which is closely related to the study of E and I . We follow the exposition of [21] and refer to that text and [8] for more details and other definitions.

Definition 2.1 : The space $L_1^2(M,R^N)$ is the completion of $C^\infty(M,R^N)$ for the norm $|f|^2_{L_1^2} = \int_M \left(|f|^2 + |df|^2\right).v_g$.

To define the space $L_1^2(M,M')$, we then consider a Riemannian embedding of M' in R^N (which always exists if N is large enough [22]).

Definition 2.2 : f : M $\longrightarrow$ M' belongs to $L_1^2(M,M')$ iff its composition with the embedding of M' in R^N is in $L_1^2(M,R^N)$.

One can check that the space $L_1^2(M,M')$ does not depend on the choice of the embedding (although of course the norm of the composed maps does) . As usual, we call map of class L_1^2 an equivalence class of maps, at zero L_1^2-distance from one another, and we call such a map continuous if an element of that class is continuous.

The space L_1^2 appears naturally in the study of the function I (and in particular of the energy E) . Indeed, from the definition of I and from the fact that M and M' are compact, we conclude that a set of maps on which I is bounded is precisely a bounded subset of $L_1^2(M,M')$.

In order to study the problem of existence of a minimum of I in a homotopy class H of maps from M to M', we shall use a direct method of the calculus of variations. Since I is bounded from below on H, we can choose a minimizing sequence for I in H, and we obtain easily the following result :

<u>Proposition 2.3</u> : Every minimizing sequence for I contains a subsequence (f_r) which converges weakly to a L_1^2 map f . The limit verifies

$I(f) \leq \lim\inf I(f_r)$.

<u>Proof</u>.

A minimizing sequence is a bounded set of L_1^2 . By [1;(12,15,10)] such a set contains a weakly converging subsequence . From [1;(12,15,8)] and the existence of a L^2 converging subsequence (Rellich lemma), we deduce that $I(f) \leq \lim\inf I(f_r)$.

In order to prove that I admits a minimum in H, it would then be sufficient to show that the limit $f \in C^\infty(M,M')$ and that $f \in H$. This last point would automatically be satisfied if the convergence of the sequence were uniform.

If the dimension of M is one, this will be the case, by virtue of the theorems of Sobolev and Rellich . The limit will thus be in H and by theorem 1.10.1 of [21], it will be C^∞ . So we have proven the following result :

Proposition 2.4 : If $n = 1$, I reaches its minimum in every homotopy class.

If $n \geq 2$, those theorems don't apply : an L_1^2 map is not always continuous and the weak L_1^2 convergence is not always uniform . As a result, the direct methods of the calculus of variations will not lead to a general existence theorem, as is shown by the non-existence result of §3.c (th. 3.10).

As we will see, however, different existence results can be obtained in the case $n = 2$. This is due to the fact that a L_1^2 map in dimension 2 is "not too far" from being continuous. This allowed C.Morrey to prove a regularity result for minima of E which is particular to this dimension [20] . Still, it does not imply that the limit of a minimizing sequence in H will be in H, and we shall have to introduce other assumptions.

We conclude this paragraph by a proposition that limits the "non-continuity" of a L_1^2 map in dimension 2.

Definition 2.5 : A map $f(x)$ from an interval to M' is absolutely continuous iff $\forall\, \varepsilon > 0$, $\exists\, \delta$ such that $\sum_\sigma |f(x''_\sigma)-f(x'_\sigma)| \leq \varepsilon$ whenever the intervals (x'_σ, x''_σ) are disjoint and $\sum_\sigma (x''_\sigma - x'_\sigma) \leq \delta$. The notation $| - |$ represents here the Riemannian distance on M'.

Proposition 2.6 [20] : If f is absolutely continuous, its derivative exists almost everywhere and the length of the curve $f(x)$ is equal to the integral of $|f_x|$.

Proposition 2.7 : Let M be a surface and $f \in L_1^2(M,M')$. Let (x,y) be local coordinates on a chart of M . The class f can be represented by a map - also denoted by f - such that for almost every x (resp. y), f is absolutely continuous in y (resp. x) . f can be defined almost everywhere as the Lebesgue derivative of $\int_{\square} f\, dx\, dy$.

Proof.

cf. [21, lemma 9.4.10].

Remark : In what follows, f will always denote that representative of the map.

---

§ b : EXISTENCE RESULT.

In order to deduce some existence results for harmonic and φ-harmonic maps (§ d and e), we prove the following theorem :

Theorem 2.8 : Let M,g be a compact surface and M',g' a compact manifold whose second homotopy group $\Pi_2(M')$ is zero . Every homotopy class of maps from M to M' contains a map realizing the minimum of I.

Remark 2.9 : This theorem does not involve any assumption on the metrics g and g' . In particular, the curvature of M' need not be negative or zero.

Remark : A special case of this theorem (when I = E and dim M'=2) was proven in the thesis [13] . The case I = E was also announced by K. Uhlenbeck.

Proof.

Let U and U' denote the universal coverings of M and M', and let Π and Π' be the projections . We endow U and U' with the metrics $\Pi^{*}g$ and $\Pi'^{*}g'$.

Any map h in the given homotopy class H can be lifted to a map $\bar{h} : U \longrightarrow U'$, which is determined by the choice of the image $\bar{h}(\bar{P})$ of a single point within $\Pi'^{-1}(h(\Pi(\bar{P})))$.

Let $\Pi_1(M)$ and $\Pi_1(M')$ denote the fundamental groups of M and M' . A map h induces a conjugacy class of homomorphisms from $\Pi_1(M)$ to $\Pi_1(M')$, which we denote h_* . An element γ of $\Pi_1(M)$ can be seen as an automorphism of U and the lift $\bar{h}$ verifies the relations $\bar{h}\circ\gamma = \bar{h}_*(\gamma)\circ\bar{h}$ where $\bar{h}_*$ is an element of the conjugacy class h_*, depending on the chosen lift.

Since $\Pi_2(M') = 0$, the proof of theorem 8.1.11 of [26] implies that there is a one-to-one correspondence between the homotopy classes of maps from M to M' and the induced conjugacy classes of homomorphisms . Therefore, a continuous map f belongs to H iff $\forall\gamma$ its lift verifies

$$\bar{f}\circ\gamma = \alpha.\bar{h}_x(\gamma).\alpha^{-1}\circ\bar{f} \tag{4}$$

for a certain $\alpha \in \Pi_1(M')$.

Consider a map $\bar{f}$ verifying (4) . In general, U is not compact so that $\bar{f}$ is not in $L_1^2(U,U')$, but of course we can give sense to a Sobolev condition on $\bar{f}$, for instance by saying that $\bar{f} \in \mathcal{L}_1^2$ iff its restriction to any fundamental domain of U is in L_1^2 . In the same spirit, we define the "energy" $\mathfrak{J}$ of $\bar{h}$ as the "energy" of its projection h : $\mathfrak{J}(\bar{h}) = I(h)$.

Consider now a minimizing sequence $(\bar{f}_r)$ for $\mathfrak{J}$ in the class of maps verifying (4) . We shall see that, after composition with automorphisms of U', a subsequence converges weakly in $\mathcal{L}_1^2$.

Since the $(\bar{f}_r)$ are a minimizing sequence for $\mathfrak{J}$, it is sufficient to show that (after composition with elements of $\Pi_1(M')$) they are uniformly bounded in $\mathcal{L}^2$.

For that, it is convenient to consider a single compact rectangular chart Σ of U covering a fundamental domain $\mathfrak{F}$ (this is possible except in the case $M = S^2$, where we can take two charts). For two subsets of $\mathfrak{F}$ with Euclidean coordinates (x,y) and (X,Y), we obtain by integrating formula (9.4.22) of [21] :

$$\int_{(x,y)} \int_{(X,Y)} |\bar{f}_r(x,y) - \bar{f}_r(X,Y)|^2 \, dx\, dy\, dX\, dY$$

$$\leq \int (|\bar{f}_r(x,y) - \bar{f}_r(X,y)|^2 + |\bar{f}_r(X,y) - \bar{f}_r(X,Y)|^2) \, dx\, dy\, dX\, dY$$

$$\leq c_1 \int_\Sigma (|\bar{f}_x|^2 + |\bar{f}_y|^2) \, dx\, dy \leq c_2\, \mathcal{E}(\bar{f}_r) \leq c_3\, \mathfrak{J}(\bar{f}_r) + c_4 \leq c_5 \quad (5).$$

where $\mathcal{E}(\bar{f}_r) = E(f_r)$, f_r being the projection of $\bar{f}_r$.

This implies that for a certain positive η, there exists for all r a subset of $\mathfrak{F}$ of area η whose image is in a single fundamental domain of U' . Indeed, if it where not the case, the sets $\bar{f}_r(\mathfrak{F})$ would intersect more and more fundamental domains in a way contradicting (5) . By composing $\bar{f}_r$ with an automorphism of U', we can suppose that this fundamental domain is always the same . The $\bar{f}_r$ are then uniformly bounded by (5) and hence admit a weak limit $\bar{f}$.

Furthermore, only a finite number of conjugations appears in the relations $\bar{f}_{r*} = \alpha\, \bar{h}_* \,\alpha^{-1}$. If not, a domain of $\mathfrak{F}$ and its image would drift apart when r increases, contradicting an extension of (5) to that situation . Therefore, we can take a subsequence of the $\bar{f}_r$'s such that $\bar{f}_{r*} = \bar{h}_*$ for all r . The limit $\bar{f}$ will then also verify

$$\bar{f} \circ \gamma = \bar{h}_*(\gamma) \circ \bar{f} \qquad (6)$$

and minimize $\mathfrak{J}$ among all maps verifying the same relations.

Let $\bar{P}$ be a point of U and c a positive constant . By

Since the $(\bar f_r)$ are a minimizing sequence for $\mathfrak{J}$, it is sufficient to show that (after composition with elements of $\Pi_1(M'))$ they are uniformly bounded in $\mathcal{L}^2$.

For that, it is convenient to consider a single compact rectangular chart Σ of U covering a fundamental domain $\mathfrak{F}$ (this is possible except in the case $M = S^2$, where we can take two charts). For two subsets of $\mathfrak{F}$ with Euclidean coordinates (x,y) and (X,Y), we obtain by integrating formula (9.4.22) of [21] :

$$\int_{(x,y)} \int_{(X,Y)} |\bar f_r(x,y) - \bar f_r(X,Y)|^2 \, dx\, dy\, dX\, dY$$

$$\leq \int (|\bar f_r(x,y) - \bar f_r(X,y)|^2 + |\bar f_r(X,y) - \bar f_r(X,Y)|^2) \, dx\, dy\, dX\, dY$$

$$\leq C_1 \int_\Sigma (|\bar f_x|^2 + |\bar f_y|^2) \, dx\, dy \leq C_2\, \mathcal{E}(\bar f_r) \leq C_3\, \mathfrak{J}(\bar f_r) + C_4 \leq C_5 \quad (5).$$

where $\mathcal{E}(\bar f_r) = E(f_r)$, f_r being the projection of $\bar f_r$.

This implies that for a certain positive η, there exists for all r a subset of $\mathfrak{F}$ of area η whose image is in a single fundamental domain of U' . Indeed, if it where not the case, the sets $\bar f_r(\mathfrak{F})$ would intersect more and more fundamental domains in a way contradicting (5) . By composing $\bar f_r$ with an automorphism of U', we can suppose that this fundamental domain is always the same . The $\bar f_r$ are then uniformly bounded by (5) and hence admit a weak limit $\bar f$.

Furthermore, only a finite number of conjugations appears in the relations $\bar f_{r*} = \alpha\, \bar h_* \,\alpha^{-1}$. If not, a domain of $\mathfrak{F}$ and its image would drift apart when r increases, contradicting an extension of (5) to that situation . Therefore, we can take a subsequence of the $\bar f_r$'s such that $\bar f_{r*} = \bar h_*$ for all r . The limit $\bar f$ will then also verify

$$\bar f \circ \gamma = \bar h_*(\gamma) \circ \bar f \qquad (6)$$

and minimize $\mathfrak{J}$ among all maps verifying the same relations.

Let $\bar P$ be a point of U and c a positive constant . By proposition 2.7, $\bar{f}$ is absolutely continuous on the boundary of the disk D centred in $\bar{P}$ and of radius r, for almost all $r \leq c$.

Let us take r so small that $D \cap \gamma D = \emptyset$ for all $\gamma \in \Pi_1(M)$. $\bar{f}_{|D}$ is then a minimum of I among all maps on D having the same value on the boundary . Indeed, if it were not the case, we could build by translations a map verifying (6) and of smaller $\mathcal{E}$ than $\bar{f}$.

In the next paragraph, we shall see that $\bar{f}$ is then C^∞ . By (6), it has a projection $f \in C^\infty(M,M')$ which is in the homotopy class H.

Remark 2.10 : As a by-product of this proof, we see that when M is a surface and M' a manifold, there exists a harmonic map from M to M' inducing any given conjugacy class of homomorphisms from $\Pi_1(M)$ to $\Pi_1(M')$.

---

§ c : REGULARITY.

Following closely a method of C. Morrey [20], we shall prove the regularity result used in the last paragraph.

Definition 2.11 : Hölder-continuity . If D is a disk of M, a map $f : D \longrightarrow M'$ belongs to $C^0_\lambda(D,M')$, $\lambda \in (0,1)$ iff $\forall\, m_1, m_2 \in D$, $|f(m_1) - f(m_2)| \leq C.|m_1 - m_2|^\lambda$.

Proposition 2.12 : If D is a disk of M of sufficiently small radius R and if $f \in L^2_1(D,M')$ minimizes I among all maps in $L^2_1(D,M')$ coinciding with f on ∂D, then f is of class C^0_λ in the interior of D.

Proposition 2.13 : If $f \in C^{0}_{\lambda}(D,M')$ is a critical point of I, then $f \in C^{\infty}(D,M')$.

Proposition 2.12 is due to C. Morrey in the case of the energy [20] [21], and it will be an easy matter to adapt his proof to the case of I. For completeness, we shall do it in details in the remainder of this paragraph.

Proposition 2.13 is an immediate consequence of theorem 1.10.4 iii of [21].

Proof of proposition 2.12.

We consider on M an atlas composed of exponential disks of radius R. We note D such a disk and $\tilde{D} = \exp^{-1} D$. For any map $h : D \longrightarrow M'$, we put $\tilde{h} = h \circ \exp$. $\tilde{h}$ is a function of the Euclidean coordinates (x,y) and if (r,θ) are polar coordinates on $\tilde{D}$, we put $\tilde{H}(r,\theta) = \tilde{h}(x,y)$. We define $E_{\tilde{D}}(\tilde{h})$ and $I_{\tilde{D}}(\tilde{h})$ by

$$E_{\tilde{D}}(\tilde{h}) = \frac{1}{2} \int_{\tilde{D}} \delta^{ij}\, \tilde{h}^{\alpha}_{i}\, \tilde{h}^{\beta}_{j}\, g'_{\alpha\beta}\, dx\, dy$$

$$I_{\tilde{D}}(\tilde{h}) = \int_{\tilde{D}} \Big[\frac{1}{2} A(\exp(x,y),\tilde{h}(x,y))\, \delta^{ij}\, \tilde{h}^{\alpha}_{i}\, \tilde{h}^{\beta}_{j}\, g'_{\alpha\beta} + B(\exp(x,y),\tilde{h}(x,y)\Big]\, dx\, dy.$$

Since M is compact, we can suppose R so small that there exists a positive number w such that for all vector field X, 1-form ξ and function h on a disk $\tilde{\Delta}$ in $\tilde{D}$, one has

$$\begin{array}{lllll} w \sum (X^{i})^{2} & \leq & g_{ij}\, X^{i}\, X^{j} & \leq & W \sum (X^{i})^{2} \\ w \sum (\xi_{i})^{2} & \leq & g^{ij}\, \xi_{i}\, \xi_{j} & \leq & W \sum (\xi_{i})^{2} \\ w \int_{\tilde{\Delta}} \tilde{h}\, dx\, dy & \leq & \int_{\exp\tilde{\Delta}} h\, v_{g} & \leq & W \int_{\tilde{\Delta}} \tilde{h}\, dx\, dy \\ w \int_{\partial\tilde{\Delta}} \tilde{H}\, d\theta & \leq & \int_{\exp\partial\tilde{\Delta}} H\, ds_{g} & \leq & W \int_{\partial\tilde{\Delta}} \tilde{H}\, d\theta \end{array} \qquad (7)$$

where $W = \frac{1}{w}$.

On M' we choose an atlas of the same type, and denote R', w' and W' the associated quantities . A disk of $\mathbb{R}^2$ of centre m_o and radius r will be denoted $\tilde{D}(m_o,r)$ or sometimes simply $\tilde{D}$.

Lemma 2.14 : Let $\tilde{F} \in L_1^2(\partial\tilde{D}(m_o,r),\mathbb{R}^{n'})$ and let $\tilde{h}$ be the harmonic function on $\tilde{D}(m_o,r)$ coinciding with $\tilde{F}$ on $\partial\tilde{D}$. Then

$$E_{\tilde{D}}(\tilde{h}) \leq \frac{1}{2}\int_0^{2\pi} |\tilde{F}_\theta(\theta)|^2 \, d\theta .$$

Proof.

If the Fourier expansion of $\tilde{F}(\theta)$ is

$\tilde{F}^\alpha(\theta) = \frac{a_o^\alpha}{2} + \sum_{\sigma=1}^{\infty} (a_\sigma^\alpha \cos\sigma\theta + b_\sigma^\alpha \sin\sigma\theta)$, the harmonic function $\tilde{H}$ can be written $\tilde{H}^\alpha(\rho,\theta) = \frac{a_o^\alpha}{2} + \sum_{\sigma=1}^{\infty} \frac{\rho^\sigma}{r^\sigma} (a_\sigma^\alpha \cos\sigma\theta + b_\sigma^\alpha \sin\sigma\theta)$.

We have then:

$$\begin{aligned} 2\, E_{\tilde{D}}(\tilde{H}) &= \sum_\alpha \int_{\tilde{D}} ((\tilde{H}_\rho^\alpha)^2 + \frac{1}{\rho^2}(\tilde{H}_\theta^\alpha)^2)\, \rho \, d\rho \, d\theta \\ &= \sum_\alpha \sum_{\sigma=1}^{\infty} \sigma\pi \, ((a_\sigma^\alpha)^2 + (b_\sigma^\alpha)^2) \\ &\leq \sum_\alpha \sum_{\sigma=1}^{\infty} \sigma^2\pi \, ((a_\sigma^\alpha)^2 + (b_\sigma^\alpha)^2) \\ &= \sum_\alpha \int_0^{2\pi} (\tilde{F}_\theta^\alpha)^2 \, d\theta . \end{aligned}$$

Lemma 2.15 : Let $\tilde{D}$ be a disk of radius r of $\mathbb{R}^2$ and $\tilde{F} : \partial\tilde{D} \to M'$ an absolutely continuous map such that

$$\int |\tilde{F}_\theta(\theta)|^2 \, d\theta \leq \frac{w'R'^2}{\pi} .$$

There exists a map $\tilde{h} \in L_1^2(\tilde{D},M')$ equal to $\tilde{F}$ on $\partial\tilde{D}$ and such that

$$I_{\tilde{D}}(\tilde{h}) \leq \frac{A_1 W'}{2} \int_{\partial\tilde{D}} |\tilde{F}_\theta(\theta)|^2 \, d\theta + B_1 \, \pi \, r^2 \qquad (8).$$

Proof.

Fix an angle θ_o . There exists a θ_1 such that $|\theta_1-\theta_o| \leq \pi$ and $|\tilde{F}(\theta) - \tilde{F}(\theta_o)| \leq |\tilde{F}(\theta_1) - \tilde{F}(\theta_o)|$ $\forall$ θ . Then

$$|\tilde{F}(\theta_1) - \tilde{F}(\theta_o)|^2 \leq \left[\int_0^{\pi} |\tilde{F}_\theta| \, d\theta\right]^2 \qquad \text{(prop. 2.6)}$$

$$\leq \pi \int_0^{\pi} |\tilde{F}_\theta|^2 \, d\theta \qquad \text{(Hölder's inequality).}$$

Hence $|\tilde{F}(\theta) - \tilde{F}(\theta_o)|^2 \leq w'R'^2$.

Consider an exponential chart of M' of radius R' centred at $\tilde{F}(\theta_o)$. For all path $u^\alpha(s)$ (where s is the Euclidean arc lenght), we have

$$\int_0^\lambda \sqrt{g'_{\alpha\beta}(u)\, u_s^\alpha\, u_s^\beta}\, ds \geq \sqrt{w'} \int_0^\lambda \left[\sum_\alpha (u_s^\alpha)^2\right]^{\frac{1}{2}} ds$$

$$\geq \lambda \sqrt{w'}$$

and every point at a distance $\leq R'\sqrt{w'}$ of $\tilde{F}(\theta_o)$ is in the chart . This is the case for the set $\{\tilde{F}(\theta) \mid \theta \in [0,2\pi]\}$.

Now define $\tilde{h} : \tilde{D} \longrightarrow M'$ as the map from $\tilde{D}$ to that chart, equal to $\tilde{F}$ on $\partial\tilde{D}$ and whose components are harmonic functions. By lemma 2.14 and (7), we have

$$E_{\tilde{D}}(\tilde{h}) = \frac{1}{2} \int_{\tilde{D}} \sum_i g'_{\alpha\beta}\, \tilde{h}_i^\alpha\, \tilde{h}_i^\beta \, dx\, dy$$

$$\leq \frac{1}{2} W' \int_{\tilde{D}} \sum_{i,\alpha} |\tilde{h}_i^\alpha|^2 \, dx\, dy$$

$$\leq \frac{1}{2} W' \int_0^{2\pi} |\tilde{F}_\theta|^2 \, d\theta.$$

(8) is then an immediate consequence of (3).

End of the proof of proposition 2.12.

Suppose now that f is the L_1^2 minimum of I that we wish to study . It minimizes I on every disk and is absolutely continuous on the boundary of almost every disk . We shall show in three steps that it is Hölder-continuous.

<u>Step 1</u> : For a point $m_1 = (x_1, y_1)$ at a distance $R - \delta$ of the origin of $T_{m_o} M$, consider in the disk $\tilde{D}(m_1, \delta)$ the Euclidean coordinates (x,y) and the polar coordinates (r,θ) (so that $(x_1, y_1) = (0,\theta)$) . Put $E_r(\tilde{f}) = E_{\tilde{D}(m_1,r)}(\tilde{f})$ and $E_r(f) = E_{\exp_{m_o} \tilde{D}(m_1,r)}(f)$ and use similar notations for I . Then there exist two constants $\lambda \in (0,1)$ and N such that

$$E_r(\tilde{f}) \leq \frac{N^2}{2} \left(\frac{r}{\delta}\right)^{2\lambda} \quad \text{for all } r \leq \delta.$$

<u>Proof</u>.

<u>1^{st} case</u> : If $\int_0^{2\Pi} |\tilde{F}_\theta|^2 \, d\theta \leq \frac{w'R'^2}{\Pi}$, we know by lemma 2.15 that there exists a map $\tilde{h} \in L_1^2(\tilde{D}, M')$, equal to $\tilde{f}$ on $\partial\tilde{D}$ and verifying (8) . Since f minimizes I in exp $\tilde{D}$, we have, using (7) :

$$\begin{aligned}
E_r(\tilde{f}) &\leq W^2 E_r(f) \leq W^2 A_o^{-1} (I_r(f) - B_o V(\exp \tilde{D})) \\
&\leq W^2 A_o^{-1} (I_r(h) - B_o V(\exp \tilde{D})) \\
&\leq W^4 A_o^{-1} (I_r(\tilde{h}) - B_o \Pi r^2) \\
&\leq W^4 A_o^{-1} \left[\frac{A_1 W'}{2} \int_{\partial\tilde{D}} |\tilde{F}_\theta(\theta)|^2 \, d\theta + (B_1 - B_o) \Pi r^2\right].
\end{aligned}$$

<u>2^d case</u> : If $\int_0^{2\Pi} |\tilde{F}_\theta|^2 \, d\theta > \frac{w'R'^2}{\Pi}$, we have

$$\begin{aligned}
E_r(\tilde{f}) &\leq A_o^{-1} (I_r(\tilde{f}) - B_o \Pi r^2) \\
&\leq W^2 A_o^{-1} (I_r(f) - B_o \Pi r^2) \\
&\leq W^2 A_o^{-1} (\inf I_H - B_o \Pi r^2) \\
&\leq W^2 A_o^{-1} (\inf I + |B_o| \Pi R^2) \frac{\Pi}{w'R'^2} \int_0^{2\Pi} |\tilde{F}_\theta|^2 \, d\theta.
\end{aligned}$$

So in both cases we have :

$$E_r(\tilde{f}) \leq \frac{1}{4\lambda} \left[\int_0^{2\Pi} |\tilde{F}_\theta|^2 \, d\theta + r^2 \right]$$

where $\lambda \in (0,1)$ is chosen so that

$$\frac{1}{4\lambda} \geq \left(\frac{W^4 W' A_1}{2A_o} , W^4 (B_1 - B_o)\Pi , \frac{W^2 W'\Pi}{A_o R'^2} (\inf I + |B_o| \Pi R^2) \right).$$

λ will be kept fixed for the remainder of the proof.

Since $2\, E_r(\tilde{f}) = \int_0^r \int_0^{2\Pi} \left(|\tilde{F}_r|^2 + \frac{1}{\rho^2} |\tilde{F}_\theta|^2 \right) \rho \, d\rho \, d\theta$,

we have for almost all r

$$2r\, E_r'(\tilde{f}) = \int_0^{2\Pi} \left(|\tilde{F}_r|^2 + \frac{1}{r^2} |\tilde{F}_\theta|^2 \right) r^2 \, d\theta$$

$$\geq \int_0^{2\Pi} |\tilde{F}_\theta|^2 \, d\theta \geq 4\lambda\, E_r(\tilde{f}) - r^2$$

i.e. $E_r'(\tilde{f}) \geq \frac{2\lambda}{r} E_r(\tilde{f}) - \frac{r}{2}$.

Integrating this inequality from r to the fixed radius δ, we get

$$E_r(\tilde{f}) \leq E_\delta(\tilde{f}) \left(\frac{r}{\delta}\right)^{2\lambda} + \frac{\delta^2}{4-4\lambda} \left[\left(\frac{r}{\delta}\right)^{2\lambda} - \left(\frac{r}{\delta}\right)^2 \right]$$

$$\leq \left[E_\delta(\tilde{f}) + \frac{\delta^2}{4-4\lambda} \right] \left(\frac{r}{\delta}\right)^{2\lambda}.$$

<u>Step 2</u> : $\tilde{f}(m_1)$ is well defined and $\tilde{F}$ is absolutely continuous in r for almost all θ . If $l(x,y) = \sqrt{2\, e(\tilde{f})(x,y)}$, one has almost everywhere :

$$|\tilde{f}(x,y) - \tilde{f}(x_1,y_1)| \leq r \int_0^1 l((1-t)x_1 + tx \, , \, (1-t)y_1 + ty) \, dt \qquad (9).$$

<u>Proof</u>.

Let $L(r,\theta)$ be the function l in polar coordinates :

$L^2(r,\theta) = |\tilde{F}_r|^2 + r^{-2} |\tilde{F}_\theta|^2$.

So in both cases we have :

$$E_r(\tilde{f}) \leq \frac{1}{4\lambda}\left(\int_0^{2\Pi} |\tilde{F}_\theta|^2 \, d\theta + r^2\right)$$

where $\lambda \in (0,1)$ is chosen so that

$$\frac{1}{4\lambda} \geq \left(\frac{W^4 W' A_1}{2A_o}, \ W^4(B_1 - B_o)\Pi, \ \frac{W^2 W'\Pi}{A_o R'^2}(\inf I + |B_o|\Pi R^2)\right).$$

λ will be kept fixed for the remainder of the proof.

Since $2\, E_r(\tilde{f}) = \int_0^r \int_0^{2\Pi}\left(|\tilde{F}_r|^2 + \frac{1}{\rho^2}|\tilde{F}_\theta|^2\right) \rho \, d\rho \, d\theta$,

we have for almost all r

$$2r\, E_r'(\tilde{f}) = \int_0^{2\Pi}\left(|\tilde{F}_r|^2 + \frac{1}{r^2}|\tilde{F}_\theta|^2\right) r^2 \, d\theta$$

$$\geq \int_0^{2\Pi} |\tilde{F}_\theta|^2 \, d\theta \geq 4\lambda\, E_r(\tilde{f}) - r^2$$

i.e. $E_r'(\tilde{f}) \geq \frac{2\lambda}{r} E_r(\tilde{f}) - \frac{r}{2}$.

Integrating this inequality from r to the fixed radius δ, we get

$$E_r(\tilde{f}) \leq E_\delta(\tilde{f}) \left(\frac{r}{\delta}\right)^{2\lambda} + \frac{\delta^2}{4-4\lambda}\left(\left(\frac{r}{\delta}\right)^{2\lambda} - \left(\frac{r}{\delta}\right)^2\right)$$

$$\leq \left(E_\delta(\tilde{f}) + \frac{\delta^2}{4-4\lambda}\right)\left(\frac{r}{\delta}\right)^{2\lambda}.$$

<u>Step 2</u> : $\tilde{f}(m_1)$ is well defined and $\tilde{F}$ is absolutely continuous in r for almost all θ . If $l(x,y) = \sqrt{2\, e(\tilde{f})(x,y)}$, one has almost everywhere :

$$|\tilde{f}(x,y) - \tilde{f}(x_1,y_1)| \leq r \int_0^1 l((1-t)x_1 + tx \, , \, (1-t)y_1 + ty)\, dt \qquad (9).$$

<u>Proof</u>.

Let $L(r,\theta)$ be the function l in polar coordinates :

$L^2(r,\theta) = |\tilde{F}_r|^2 + r^{-2}\,|\tilde{F}_\theta|^2$.

Put $\Psi(r) = \int_0^r \int_0^{2\Pi} \rho^{\frac{1}{2}} L(\rho,\theta)\, d\rho\, d\theta$

By Hölder's inequality and step one

$$\Psi(r) \leq \sqrt{2\Pi r}\,\sqrt{\int_{\tilde{D}} l^2\, dx\, dy}$$

$$\leq \sqrt{2\Pi r}\, N\, \frac{r^\lambda}{\delta^\lambda}\,.$$

Since $\Psi'(\rho) = \int_0^{2\Pi} \rho^{\frac{1}{2}} L(\rho,\theta)\, d\theta$ a.e.,

$$\int_0^r\int_0^{2\Pi} L(\rho,\theta)d\rho\, d\theta = \int_0^r \rho^{-\frac{1}{2}}\, \Psi'(\rho)\, d\rho$$

$$= r^{-\frac{1}{2}}\, \Psi(r) + \frac{1}{2}\int_0^r \rho^{-\frac{3}{2}}\, \Psi(\rho)\, d\rho$$

$$\leq \sqrt{2\Pi}\,\frac{N}{\delta^\lambda}\left(r^\lambda + \frac{1}{2}\int_0^r \rho^{-1+\lambda}\, d\rho\right)$$

$$\leq \text{cst. } r^\lambda \tag{10}$$

which tends to zero with r.

$\tilde{F}(r,\theta)$ is therefore absolutely continuous for almost all θ down to $r = 0$.

By prop. 2.6, for $0 < r_1 < r_2$

$$\int_0^{2\Pi} |\tilde{F}(r_2,\theta) - \tilde{F}(r_1,\theta)|\, d\theta \leq \int_0^{2\Pi}\int_{r_1}^{r_2}\left(|\tilde{F}_r|^2 + r^{-2}|\tilde{F}_\theta|^2\right)^{\frac{1}{2}} dr\, d\theta.$$

Hence $\lim\limits_{\substack{r_2\to 0\\ 0<r_1<r_2}} \int_0^{2\Pi} |\tilde{F}(r_2,\theta) - \tilde{F}(r_1,\theta)|\, d\theta = 0$

and there exists a map $\tilde{F}(0,\theta)$ such that

$$\lim_{r\to 0}\int |\tilde{F}(r,\theta) - \tilde{F}(0,\theta)|\, d\theta = 0.$$

From (10) we deduce that $\underline{\lim}_{r\to 0} \int_0^{2\Pi} |\tilde{F}_\theta|\, d\theta = 0$ so that $\tilde{F}(0,\theta)$ is

Put $\Psi(r) = \int_0^r \int_0^{2\Pi} \rho^{\frac{1}{2}} L(\rho,\theta)\, d\rho\, d\theta$

By Hölder's inequality and step one

$$\Psi(r) \leq \sqrt{2\Pi r} \sqrt{\int_{\tilde{D}} 1^2\, dx\, dy}$$

$$\leq \sqrt{2\Pi r}\, N \frac{r^{\lambda}}{\delta^{\lambda}}.$$

Since $\Psi'(\rho) = \int_0^{2\Pi} \rho^{\frac{1}{2}} L(\rho,\theta)\, d\theta$ a.e.,

$$\int_0^r \int_0^{2\Pi} L(\rho,\theta) d\rho\, d\theta = \int_0^r \rho^{-\frac{1}{2}} \Psi'(\rho)\, d\rho$$

$$= r^{-\frac{1}{2}} \Psi(r) + \frac{1}{2} \int_0^r \rho^{-\frac{3}{2}} \Psi(\rho)\, d\rho$$

$$\leq \sqrt{2\Pi} \frac{N}{\delta^{\lambda}} \left[r^{\lambda} + \frac{1}{2} \int_0^r \rho^{-1+\lambda}\, d\rho \right]$$

$$\leq \text{cst. } r^{\lambda} \tag{10}$$

which tends to zero with r.

$\tilde{F}(r,\theta)$ is therefore absolutely continuous for almost all θ down to $r = 0$.

By prop. 2.5, for $0 < r_1 < r_2$

$$\int_0^{2\Pi} |\tilde{F}(r_2,\theta) - \tilde{F}(r_1,\theta)|\, d\theta \leq \int_0^{2\Pi} \int_{r_1}^{r_2} \left(|\tilde{F}_r|^2 + r^{-2} |\tilde{F}_\theta|^2 \right)^{\frac{1}{2}} dr\, d\theta.$$

Hence $\lim\limits_{\substack{r_2 \to 0 \\ 0<r_1<r_2}} \int_0^{2\Pi} |\tilde{F}(r_2,\theta) - \tilde{F}(r_1,\theta)|\, d\theta = 0$

and there exists a map $\tilde{F}(0,\theta)$ such that

$$\lim_{r\to 0} \int |\tilde{F}(r,\theta) - \tilde{F}(0,\theta)|\, d\theta = 0.$$

From (10) we deduce that $\underline{\lim}_{r\to 0} \int_0^{2\Pi} |\tilde{F}_\theta|\, d\theta = 0$ so that $\tilde{F}(0,\theta)$ is almost everywhere independent of θ . $\tilde{f}(m_1)$ is thus defined.

(9) is then proven as follows :

$$|\tilde{F}(r,\theta) - \tilde{F}(0,\theta)| \leq \int_0^r |\tilde{F}_r| \, d\rho$$
$$\leq \int_0^r L(\rho,\theta) \, d\rho$$
$$= r \int_0^1 L(t.r,\theta) \, dt.$$

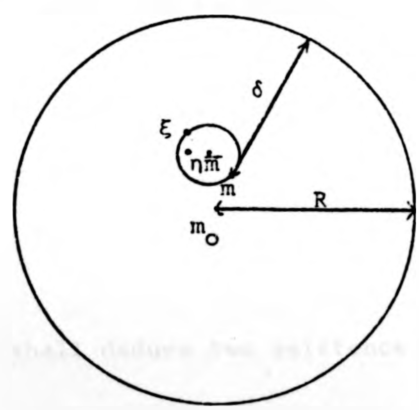
Step 3 (Dirichlet Growth lemma) : Let $\tilde{f} \in L_1^2(\tilde{D}(m_o,R),M')$ be such that $\forall\, m \in \tilde{D}(m_o,R)$ and for $0 \leq r \leq \delta = R - |m - m_o|$

$$\int_{\tilde{D}(m,r)} (|\tilde{F}_r|^2 + \rho^{-2} |\tilde{F}_\theta|^2) \, \rho \, d\rho \, d\theta \leq N^2 \left|\frac{r}{\delta}\right|^{2\lambda} \quad (11).$$

Then $\tilde{f} \in C_\lambda^o(\tilde{D}(m_o,\rho))$ $\forall \rho < R$ and $|\tilde{f}(\xi) - \tilde{f}(m)| \leq C.N \left|\frac{|\xi - m|}{\delta}\right|^\lambda$ for $|\xi - m| \leq \frac{\delta}{2}$.

Proof.

Let m and $\xi \in \tilde{D}(m_o,R)$, with $|m - m_o| = R - \delta$ and $|m - \xi| \leq \frac{\delta}{2}$. Let $\bar{m} = \frac{m+\xi}{2}$ and $\chi = \frac{|m-\xi|}{2}$. Consider finally a point $\eta \in \tilde{D}(\bar{m},\chi)$.



By step 2, for almost every η,

$$|\tilde{f}(\eta) - \tilde{f}(\xi)| \leq 2\,\chi \int_0^1 L(\xi + t(\eta - \xi)) \, dt.$$

We now integrate on $\tilde{D}(\bar{m},\chi)$ and make the substitution $\kappa = \xi + t(\eta - \xi)$. Putting $\bar{m}_t = (1 - t)\xi + t\bar{m}$, we get

$$\int_{\tilde{D}(\bar{m},\chi)} |\tilde{f}(\eta) - \tilde{f}(\xi)|\, d\eta \leq 2\chi \int_{\tilde{D}(\bar{m},\chi)} d\eta \int_0^1 L(\xi + t(\eta-\xi))\, dt$$

$$\leq 2\chi \int_0^1 dt \int_{\tilde{D}(\bar{m}_t,t\chi)} L(\kappa)\, d\kappa.$$

By Hölder's inequality and (10),

$$\int_{\tilde{D}(\bar{m}_t,t\chi)} L(\kappa)\, d\kappa \leq \left[\int_{\tilde{D}(\bar{m}_t,t\chi)} L^2(\kappa)\, d\kappa\right]^{\frac{1}{2}} \left[V(\tilde{D}(\bar{m}_t,t\chi))\right]^{\frac{1}{2}}$$

$$\leq N\, (t\chi)^{\lambda} \left(\frac{\delta}{2}\right)^{-\lambda} C'\, t\, \chi.$$

Therefore, the average of $|\tilde{f}(\eta) - \tilde{f}(\xi)|$ on $\tilde{D}(m,\chi)$ verifies :

$$V(\tilde{D}(\bar{m},\chi))^{-1} \int_{\tilde{D}(\bar{m},\chi)} |\tilde{f}(\eta) - \tilde{f}(\xi)|\, d\eta$$

$$\leq C'\, 2\, \chi\, \chi^{-2}\, N\, \chi^{1+\lambda} \left(\frac{\delta}{2}\right)^{-\lambda} \int_0^1 t^{1+\lambda}\, dt$$

$$\leq C''\, N \left(\frac{\chi}{\delta}\right)^{\lambda}.$$

The same inequality applies to the average of $|\tilde{f}(\eta) - \tilde{f}(m)|$. By averaging on η the two sides of $|\tilde{f}(m) - \tilde{f}(\xi)| \leq |\tilde{f}(m) - \tilde{f}(\eta)| + |\tilde{f}(\eta) - \tilde{f}(\xi)|$, we obtain the result since the left-hand side does not depend on η.

---

§ d : HARMONIC MAPS.

From theorem 2.8, we shall deduce two existence results for harmonic maps.

First, by putting A = 1 and B = 0, we obtain immediately the following :

Theorem 2.16 : Let M be a compact surface and M' a compact manifold whose second homotopy group is zero . Every homotopy class of maps from M to M' contains a harmonic representative which realizes the minimum of the energy in the class.

Consider then the following situation . Let $M = N \times K$ and $M' = N' \times K'$ be differentiable products of compact Riemannian manifolds . We denote by ${}_N g$, ${}_N\Gamma$ and ${}_N R$ the components of the metric, the connection and the curvature on N, and use similar notations on K, N' and K' . Let us choose on M coordinate systems $(x^1,\dots,x^r;x^{r+1},\dots,x^n) = (x^i;x^a)$ such that the (x^i) are coordinates on N and the (x^a), on K . Similarly, on $N' \times K'$, we use the coordinates $(u^1,\dots,u^{r'};u^{r'+1},\dots,u^{n'}) = (u^\alpha;u^A)$.

On M, we define the metric

$$g = \begin{pmatrix} {}_N g_{ij}(x^k) & 0 \\ 0 & \lambda(x^i)\ {}_K g_{ab}(x^c) \end{pmatrix}$$ where λ is a positive function on N.

On M', the metric g' is defined by

$$g' = \begin{pmatrix} {}_{N'} g_{\alpha\beta}(u^\gamma) & 0 \\ 0 & \mu(u^\alpha)\ {}_{K'} g_{AB}(u^C) \end{pmatrix},$$ μ being positive on N'.

Remark 2.17 : Even when N' and K' are flat (for instance when they are circles), the manifold M' is not necessarily of non-positive curvature . So, even simple cases of this situation can bring outside the scope of the Eells-Sampson theorem [3].

Theorem 2.18 : Suppose that there exists a harmonic map

$\mathfrak{J} : K \longrightarrow K'$ of constant energy density, i.e. a map such that

$\tau(\mathfrak{J}) = 0$

$e(\mathfrak{J}) = \text{cst}.$

If N is a surface and if $\Pi_2(N') = 0$, then for all homotopy class of maps from N to N', there exists a harmonic map $U : N \times K \longrightarrow N' \times K'$ of the form
$U : (x^1,\dots,x^n) \longmapsto (f^\alpha(x^1,x^2); \mathfrak{J}^A(x^3,\dots,x^n))$ such that f is in the given homotopy class.

Proof.

By direct calculations in local coordinates, we can obtain the expression of the energy and the tension of U :

$$E(U) = V(K) \int_N e(f)\, \lambda^{\frac{n-r}{2}}\, v_{N^g} + E(\mathfrak{J}) \int_N \mu(f)\, \lambda^{\frac{n-r}{2}-1}\, v_{N^g} \qquad (12).$$

$$\tau(U)^\alpha = \tau(f)^\alpha + \frac{n-r}{2}\, g^{ik} \frac{\partial\lambda}{\partial x^i} \frac{1}{\lambda} \frac{\partial f^\alpha}{\partial x^k} - e(\mathfrak{J})\, {}_{N'}g^{\alpha\beta} . \frac{1}{\lambda} \frac{\partial\mu}{\partial u^\beta}(f)$$

$$\tau(U)^A = \frac{1}{\lambda}\, \tau(\mathfrak{J})^A.$$

Since $\tau(\mathfrak{J}) = 0$ and $e(\mathfrak{J}) = \text{cst.}$, $\tau(U)^A = 0$ and $\tau(U)^\alpha$ depends only on the coordinates on N and their image by f. The following lemma shows then that the variational problem in U is equivalent to the study of the problem (12) in f. Since (12) is of the form (1), theorem 2.18 becomes a consequence of theorem 2.8.

Lemma 2.19 [13] : Every map from M to M' extremizing E among the maps of the form

$(x^1,\dots,x^n) \longrightarrow (f^\alpha(x^1,\dots,x^r); \mathfrak{J}^A(x^{r+1},\dots,x^n))$

is harmonic.

Proof.

For such an extremal U and for each vector field along U

$\mathfrak{J} : K \longrightarrow K'$ of constant energy density, i.e. a map such that

$$\tau(\mathfrak{J}) = 0$$

$$e(\mathfrak{J}) = \text{cst.}$$

If N is a surface and if $\Pi_2(N') = 0$, then for all homotopy class of maps from N to N', there exists a harmonic map $U : N \times K \longrightarrow N' \times K'$ of the form $U : (x^1,\dots,x^n) \longmapsto (f^\alpha(x^1,x^2); \mathfrak{J}^A(x^3,\dots,x^n))$ such that f is in the given homotopy class.

Proof.

By direct calculations in local coordinates, we can obtain the expression of the energy and the tension of U :

$$E(U) = V(K) \int_N e(f)\, \lambda^{\frac{n-r}{2}}\, v_{N^g} + E(\mathfrak{J}) \int_N \mu(f)\, \lambda^{\frac{n-r}{2}-1}\, v_{N^g} \qquad (12).$$

$$\tau(U)^\alpha = \tau(f)^\alpha + \frac{n-r}{2}\, g^{ik} \frac{\partial\lambda}{\partial x^i} \frac{1}{\lambda} \frac{\partial f^\alpha}{\partial x^k} - e(\mathfrak{J})\, {}_{N'}g^{\alpha\beta} \cdot \frac{1}{\lambda} \frac{\partial\mu}{\partial u^\beta}(f)$$

$$\tau(U)^A = \frac{1}{\lambda}\, \tau(\mathfrak{J})^A.$$

Since $\tau(\mathfrak{J}) = 0$ and $e(\mathfrak{J}) = \text{cst.}$, $\tau(U)^A = 0$ and $\tau(U)^\alpha$ depends only on the coordinates on N and their image by f. The following lemma shows then that the variational problem in U is equivalent to the study of the problem (12) in f. Since (12) is of the form (1), theorem 2.18 becomes a consequence of theorem 2.8.

Lemma 2.19 [13] : Every map from M to M' extremizing E among the maps of the form

$$(x^1,\dots,x^n) \longrightarrow (f^\alpha(x^1,\dots,x^r); \mathfrak{J}^A(x^{r+1},\dots,x^n))$$

is harmonic.

Proof.

For such an extremal U and for each vector field along U of the form

$$X^\alpha = X^\alpha(x^1,\dots,x^r)$$

$$X^A = 0,$$

we have $-\langle\tau(U),X\rangle = D_X E(U) = 0$,

i.e. $\displaystyle\int_N \int_K g'_{\alpha\beta}(f)\ X^\alpha(x^1,\dots,x^r)\ \tau(U)^\beta\ V_g = 0.$

Since $\tau(U)^\beta$ does not depend on $(x^{r+1},\dots,x^n)$, this implies

$$\int_N \lambda^{\frac{n-r}{2}}\ g'_{\alpha\beta}(f)\ X^\alpha(x^1,\dots,x^r)\ \tau(U)^\beta\ V_{N^g} = 0 \text{ and } \tau(U)^\beta = 0.$$

We have noted already that $\tau(U)^A = 0$.

Remark 2.20 : One can give different examples of maps $\mathfrak{F}$ such that $\tau(\mathfrak{F}) = 0$ and $e(\mathfrak{F}) = \text{cst}$.. The simplest ones are of course the identity from a manifold to itself, or a harmonic (that is minimal [3;§I,2,D]) Riemannian immersion . The harmonic maps between spheres or products of spheres obtained in [24,examples 1.3 and 3.3] also have constant energy density.

---

§ e : φ-HARMONIC MAPS.

In [17], A. Lichnerowicz introduces the following notion :

Definition 2.21 : Let φ be a positive function defined on the Riemannian manifold M,g . A C^∞ map $f : M,g \longrightarrow M',g'$ is φ-harmonic iff it is a critical point of the function

$$E_\varphi(f) = \frac{1}{2}\int |df|^2\ \varphi\ V_g$$

Clearly this integral is of the form (1), and therefore we have the existence result :

Proposition 2.22 : Let M,g be a compact surface and M',g' a compact manifold such that $\pi_2(M') = 0$. Let φ be a positive function on M . Then every homotopy class of maps from M to M' contains a φ-harmonic representative.

It turns out that such a result, with $\dim M = 2$, has a special interest in the case of φ-harmonic maps . Indeed, it is observed in [17] that when $\dim M \neq 2$, a φ-harmonic map from M,g to M',g' is precisely a harmonic map from $M,\varphi^{2/(n-2)}.g$ to M',g' . Therefore, in these dimensions, the study of φ-harmonic maps reduces for certain problems to the study of harmonic ones . For instance, if the sectional curvature R' of M' is negative or zero, a homotopy class of maps from M to M' always contains a φ-harmonic map, since it is harmonic for another g . This is used extensively in § 14 and 15 of [17]. If $\dim M = 2$, this reasoning is not valid, since a conformal transformation of g will preserve φ-harmonic maps, and not make them harmonic . However, proposition 2.22 applies in this case and we have :

Theorem 2.23 : Let φ be a positive function on M and suppose $R' \leq 0$. Then every homotopy class of maps from M to M' contains a φ-harmonic representative.

Proof.

As explained above, if $\dim M \neq 2$, this result is obtained in [17] as a consequence of [3] . If $\dim M = 2$, it is a corollary of proposition 2.22 . Indeed, since $R' \leq 0$, the Cartan-Hadamard theorem implies that $\pi_2(M') = 0$.

The consequences of theorem 2.23 appearing in § 14 and § 15 of [17] are thus valid in dimension 2 also.

CHAPTER 3.

HARMONIC MAPS BETWEEN ORIENTABLE SURFACES.

In all this chapter, M and M' will be compact orientable surfaces, whose genera will be denoted by p and p'.

§ a : HARMONIC AND HOLOMORPHIC MAPS.

We first recall the relations between harmonic and holomorphic maps in dimension 2 [3] [27] [13] . To simplify, we do not always present the results in complete generality.

If M,g is an oriented surface, we can define on M a complex structure J such that (M,J,g) is Hermitian . In a tangent plane T_mM, the action of J is defined as the rotation of 90° in a sense determined by the orientation . Then $J^2 = -I$ and $g(JX,JY) = g(X,Y)$. The almost complex structure J is automatically complex since M is a surface.

Around every point of M, we can then consider an isothermal chart, in which the metric is written $ds^2 = \rho^2 (dx^2 + dy^2)$. In the same way, the metric on M' can be denoted by $ds'^2 = \sigma^2 (du^2 + dv^2)$. We put $z = x + iy$, $w = u + iv$ and define as usual the complex derivatives f_z and $f_{\bar{z}}$.

<u>Proposition 3.1</u> [2] : If $f \in C^\infty(M,M')$, then

$$E(f) = \frac{1}{2} \int_M \sigma^2(f) (|f_z|^2 + |f_{\bar{z}}|^2) \, dx \, dy$$

$$\tau(f) = \frac{4}{\rho^2} \left(w_{z\bar{z}} + 2 \frac{\sigma_w}{\sigma} w_z w_{\bar{z}} \right).$$

<u>Corollary 3.2</u> : If $f \in C^\infty(M,M')$ is holomorphic or anti-holomorphic, then it is harmonic.

<u>Corollary 3.3</u> : If $f \in C^\infty(M,M')$, the energy $E(f)$ and the equation $\tau(f) = 0$ are independent of the function ρ.

In other words, the harmonicity of the maps does not depend on the choice of the Hermitian metric on M,J . We can therefore talk of a harmonic map from a Riemann surface to a Riemannian one.

As in [13], these properties can also be presented in a real framework, by using the following notion :

<u>Definition 3.4</u> : A map $f \in C^\infty(M,M')$ is conformal iff there exists a function $\nu \geq 0$ on M such that $f^*g' = \nu g$.

Remark that we admit in this definition that ν could have some zeros . This definition is related with the preceding discussion by the fact that a map between orientable surfaces is conformal iff it is either holomorphic or antiholomorphic.

From the composition law (proposition 1.6) we then deduce :

<u>Proposition 3.5</u> : If M, M' and M'' are surfaces, if $f \in C^{\infty}(M,M')$ is holomorphic or antiholomorphic and if $h \in C^{\infty}(M',M'')$ is harmonic, then $h \circ f$ is harmonic.

We now proceed with a discussion of the existence of harmonic maps, according to the genera p and p'.

<u>§ b : $p' \geq 1$</u>.

As a special case of theorem 2.16, we get :

<u>Proposition 3.6</u> : If the genus of M' is non zero, then every homotopy class of maps from M to M' contains a harmonic representative, which realizes the infimum of the energy in the class.

<u>§ c : $p' = 0$, $d \geq p$</u>.

We now consider the only case left out by proposition 3.6, namely the case where M' is a sphere . We then have a simple parametrization of the homotopy classes :

<u>Definition 3.7</u> : Let f be a C^{∞} map between surfaces . The degree of f at a regular value Q is defined as

$$\mathfrak{d} = \sum_{P \in f^{-1}(Q)} \text{sign det}(df)\,(P).$$

$\mathfrak{d}$ is then independent of the point Q and of the map f within a homotopy class [19] . Moreover :

<u>Theorem 3.8</u> (Hopf) : If M' is a sphere, the homotopy classes of maps from M to M' are parametrized by their degree.

<u>Remark 3.9</u> : By definition, the degree is an integer, and its sign depends on the chosen orientations on M and M' . Therefore, we shall in the sequel suppose that $\mathfrak{d}$ is positive . The negative case reduces to the positive one by a change of orientation on one of the surfaces, which would for instance transform a antiholomorphic map of negative degree in a holomorphic one of positive degree.

For all $\mathfrak{d} \geq p$, the existence problem for harmonic maps was solved by J. Eells and J. Wood in [4] . (The case $p = 0$ had previously been obtained in [27] and [13]) . We summarize their results :

<u>Theorem 3.10</u> : Let M and M' be orientable surfaces with arbitrary metrics g and g' . If $p' = 0$ and $\mathfrak{d} \geq p$, every harmonic map of degree $\mathfrak{d}$ from M to M' is holomorphic.

Such a holomorphic map exists in the following cases :

i) $\mathfrak{d} \geq p + 1$

ii) $\mathfrak{d} = p$ and M non hyperelliptic

iii) $\mathfrak{d} = p$, p even and M hyperelliptic.

As we shall see in the following paragraph, those holomorphic maps are minima of the energy in their class.

On the other hand, when p is odd and M hyperelliptic, there is no harmonic map of degree p from M to M'.

In particular, there is no harmonic map of degree 1 from the torus to the sphere, whatever metrics are chosen on those surfaces . For higher odd genera and $d = p$, the existence will depend on the metric on M, or more precisely on the hyperellipticity of the induced complex structure.

---

§ d : $p' = 0$, $d \leq p - 1$.

When $d \leq p - 1$, no complete existence results are available, but we can obtain different partial results.

Proposition 3.11 : Let $p' = 0$ and suppose that $f : M \longrightarrow M'$ minimizes the energy among all maps of degree d . Then f is holomorphic.

This proposition is proven in the thesis [13] . As its proof is similar to that of proposition 5.5 of § 5, c, we shall not repeat it here.

Corollary 3.12 : If $p \geq 1$ and $p' = 0$, the infimum of the energy is not reached in the class of maps of degree 1.

Indeed, a holomorphic map of degree 1 would have to be a diffeomorphism, which is impossible.

Remark 3.13 : For a d such that $2 \leq d \leq p - 1$, the existence of a holomorphic map of degree d from M to the sphere depends on the conformal structure of M.

So we see that in general, we cannot expect to find a minimum of E in a class, and therefore a direct method of the calculus of variations could lead to a sequence converging weakly in L_1^2 to a map contained in another homotopy class. Nevertheless, we shall now show how to modify this method in certain cases to obtain harmonic maps which are not minima of the energy . We shall prove the following result :

Theorem 3.14 : For every $p \geq 1$ and $0 \leq d \leq p - 1$, there exists a Riemann surface M of genus p and a metric on the sphere M' such that there is a harmonic non-holomorphic map of degree d from M to M' . That map is not a minimum of the energy in the class.

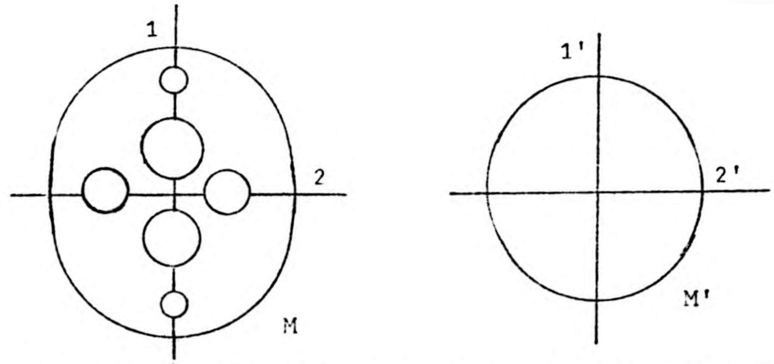
Remark 3.15 : This shows that the result of J. Eells and J. Wood (theorem 3.10) does not extend to smaller degrees.

This theorem is a corollary of propositions 3.16, 3.17, 3.20 and 3.23, which concern different values of d.

First case : $d = 1$.

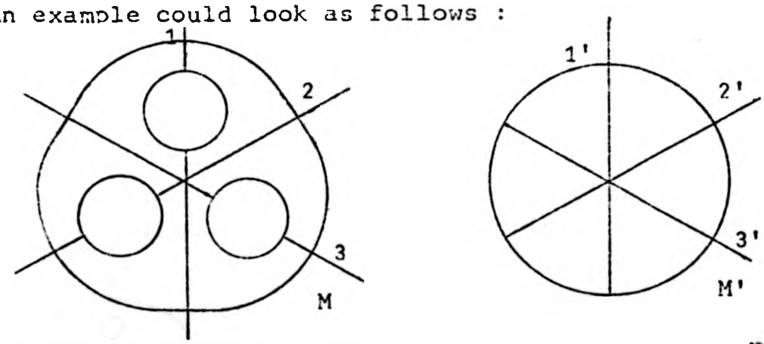
Proposition 3.16 : Let M be a surface of even genus ≥ 2, symmetric with respect to three orthogonal lines, and M' a sphere with the same symmetries . Then there is a non-holomorphic harmonic map of degree 1 from M to M'.

Such surfaces could for instance look as follows :



<u>Proposition 3.17</u> : Let M be a surface of genus $p \geq 2$ which can be represented as a sphere with handles symmetric with respect to an equator and p meridians forming angles Π/p . Let M' be a surface of revolution of genus zero, symmetric with respect to its equator . Then there is a harmonic map from M to M' of all degrees $\leq p/2$.

An example could look as follows :



These two propositions were proven in the thesis [13] . For completeness, we recall briefly the argument, in the case $\Omega = 1$. The case $2 \leq \Omega \leq \frac{p}{2}$ is treated similarly, and is not needed for the proof of theorem 3.14.

Indeed, a holomorphic map of degree 1 would have to be a diffeomorphism, which is impossible.

<u>Remark 3.13</u> : For a $\mathscr{D}$ such that $2 \leq \mathscr{D} \leq p - 1$, the existence of a holomorphic map of degree $\mathscr{D}$ from M to the sphere depends on the conformal structure of M.

So we see that in general, we cannot expect to find a minimum of E in a class, and therefore a direct method of the calculus of variations could lead to a sequence converging weakly in L^2_1 to a map contained in another homotopy class. Nevertheless, we shall now show how to modify this method in certain cases to obtain harmonic maps which are not minima of the energy . We shall prove the following result :

<u>Theorem 3.14</u> : For every $p \geq 1$ and $0 \leq \mathscr{D} \leq p - 1$, there exists a Riemann surface M of genus p and a metric on the sphere M' such that there is a harmonic non-holomorphic map of degree $\mathscr{D}$ from M to M' . That map is not a minimum of the energy in the class.

<u>Remark 3.15</u> : This shows that the result of J. Eells and J. Wood (theorem 3.10) does not extend to smaller degrees.

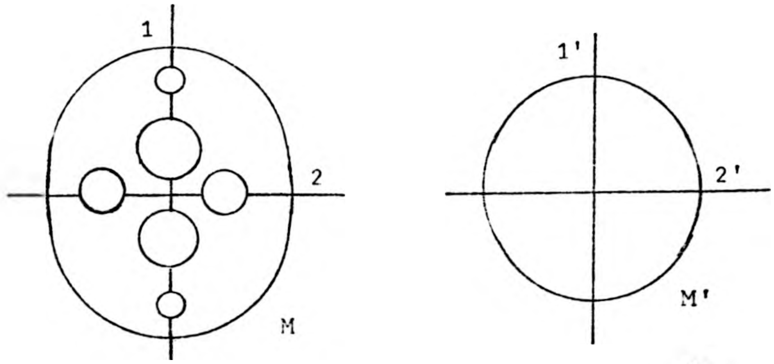
This theorem is a corollary of propositions 3.16, 3.17, 3.20 and 3.23, which concern different values of $\mathscr{D}$.

<u>First case : $\mathscr{D} = 1$</u>.

<u>Proposition 3.16</u> : Let M be a surface of even genus ≥ 2, symmetric with respect to three orthogonal lines, and M' a sphere with the same symmetries . Then there is a non-holomorphic

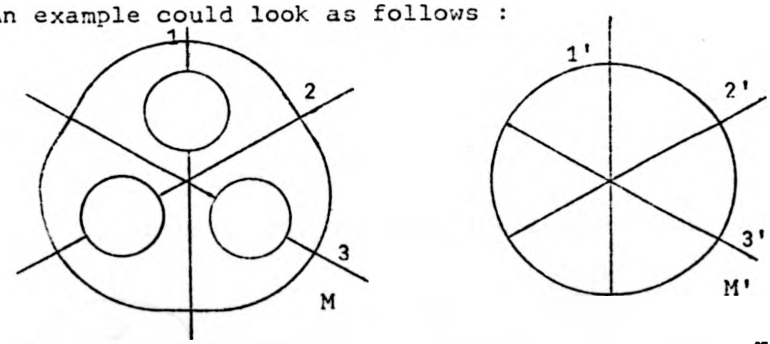
harmonic map of degree 1 from M to M'.

Such surfaces could for instance look as follows :



<u>Proposition 3.17</u> : Let M be a surface of genus $p \geq 2$ which can be represented as a sphere with handles symmetric with respect to an equator and p meridians forming angles Π/p . Let M' be a surface of revolution of genus zero, symmetric with respect to its equator . Then there is a harmonic map from M to M' of all degrees $\leq p/2$.

An example could look as follows :



These two propositions were proven in the thesis [13] . For completeness, we recall briefly the argument, in the case $\Omega = 1$. The case $2 \leq \Omega \leq \frac{p}{2}$ is treated similarly, and is not needed for the proof of theorem 3.14.

Proof of propositions 3.16 and 3.17.

Step 1 : For proposition 3.16, we call 0,1,2 the lines of symmetry on M and 0',1',2' the lines on M' . For prop. 3.17, we call 0,...,p the lines on M, 0' the equator of M' and 1',...,p' a set of meridians of M' put in correspondence with the symmetry lines on M (see figures above) . We call S_i and S_i' the symmetries with respect to i and i'.

We then choose a minimizing sequence (f_r) among the maps $h : M \longrightarrow M'$ which commute with the symmetries, i.e. which verify $h \circ S_i = S_i' \circ h$.

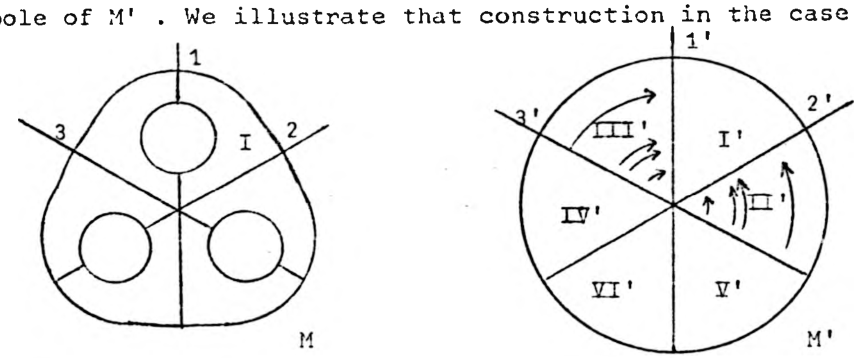
Step 2 : In order to insure that the limit of a converging subsequence is of degree 1, we shall build a new sequence (f_r'), which also commute with the S_i's, such that $E(f_r') \leq E(f_r)$ (so that it is a minimizing sequence in the same class) but with the additional property that the image by f_r' of a portion of M limited by the lines 0,i and i+1 (mod.p) is contained in the corresponding portion of M', limited by the lines 0',i' and (i+1)' (mod.p).

The map f_r' is obtained as follows :

For proposition 3.16, consider the image of the region I by f_r.

The line $I \cap i$ has its image in i', but the image of I could possibly go out of I' . By applying in succession the symmetries S_o', S_1' and S_2' to the part of $f_r(I)$ not contained in I', we bring it within that region . The image of the line i is still in i', and therefore, when we apply the same construction to f_r restricted to each portion of M, we obtain a map $f'_r \in L^2_1$. We also have $E(f'_r) = E(f_r)$ and $f'_r \circ S_i = S'_i \circ f'_r$.

For proposition 3.17, we use a similar construction . But to insure that the image of the line i by f'_r is contained in i', we have to use, as well as the symmetries, certain projections of a portion on its boundary along the cercles centred at the pole of M' . We illustrate that construction in the case p = 3.



II' and III' are projected on the lines 2' and 1'.

IV', V' and VI' are then folded on I'.

The lines 1' and 2' are thus preserved.

Again, by applying that construction to each portion of M, we obtain a map $f'_r \in L^2_1$, commuting with the symmetries and such that $E(f'_r) \leq E(f_r)$.

<u>Step 3</u> : We now consider a subsequence of (f'_r) converging weakly to a map $f \in L^2_1$. By construction, f minimizes E in the class of maps commuting with the symmetries . One can check that the proof of proposition 2.12 applies to this case, so that f is Hölder-continuous . For that, one shows that it is possible to consider only coordinate disks symmetric with respect to

any line they intersect . In lemmas 2.14 and 2.15, the map $\tilde{h}$ which is constructed has then the same symmetry properties as the given map $\tilde{F}$.

Step 4 : As f does not minimize E in a homotopy class, we have to show that it is harmonic, which we do chart by chart.

In a chart of M which does not meet any of the line i, the symmetries don't induce any condition and f is harmonic.

Consider now a chart symmetric with respect to one line and without intersection with the others . Call S and S' the associated symmetries on M and M' . Let X be a vector field along f in that chart, null on the boundary . We want to show that $D_X E(f) = 0$. Call $\mathcal{S}$ the symmetry induced on X by S' . X can be written in the form

$$X = X_1 + X_2 = \frac{X + \mathcal{S}X}{2} + \frac{X - \mathcal{S}X}{2}$$

where by definition $\mathcal{S}X_1 = X_1$ and $\mathcal{S}X_2 = -X_2$.

Since X_1 is symmetric, $D_{X_1}E(f) = 0$, as f minimizes E in the class of maps commuting with S and S'.

On the other hand,

$$D_{X_2}E(f) = D_{\mathcal{S}X_2}E(S' \circ f \circ S) = -D_{X_2}E(f)$$

and therefore $D_{X_2}E(f) = 0$.

In a chart symmetric with respect to more than one line, the relation $D_X E(f) = 0$ is obtained by symmetrizing X with respect to all the lines.

Step 5 : f being harmonic and Hölder-continuous is C^∞ by proposition 2.13.

Step 6 : f is then harmonic, C^∞, commutes with the symmetries and sends a portion of M on the corresponding portion of M'. This implies that it is of degree 1 . Since $p \geq 2$ and $p' = 0$, f cannot be holomorphic.

Remark 3.18 : In the space $L_1^2(M,M')$, this construction can be interpreted as follows : the solution is a minimum of the energy in a subspace of L_1^2, but not in the whole space . The graph of E on $L_1^2(M,M')$ is "symmetric" with respect to that subspace and therefore the solution is a critical point in the whole space.

Remark 3.19 : Since f is not an absolute minimum of E, it would be interesting to know whether it's a local minimum, degenerate or not, or whether it has a positive Morse index . It seems that it is not possible to answer those questions, because one lacks the necessary information on f.

For instance, let us consider a deformation of f obtained by composition with an infinitesimal conformal transformation of M', perpendicular to the equator 0' . (This choice is motivated by some of the results of [25]).

We endow M' with isothermal coordinates $(u,v) \in R \times [0,2\Pi)$ such that the equator 0' is defined by u = 0 . The canonical metric on M' is then written as :

$$g'_{11} = g'_{22} = \left(\frac{2\, e^u}{1 + e^{2u}}\right)^2 \qquad g'_{12} = 0$$

and the vector field $\frac{\partial}{\partial u}$ is an infinitesimal conformal transformation.

Using the second variation formula of [2], we get :

$$D^2_{\frac{\partial}{\partial u}} E(f) = \int_M 4g^{ij} \frac{e^{2u}}{(1+e^{2u})^2} \left[\left(1 - \frac{4\ e^{2u}}{(1+e^{2u})^2} \right) f^1_i\ f^1_j \ + \right.$$

$$\left. \left(1 - \frac{8\ e^{2u}}{(1+e^{2u})^2} \right) f^2_i\ f^2_j \right] V_g \qquad (1).$$

The function $e^{2u} \ / \ (1+e^{2u})^2$ takes its maximum value $\frac{1}{4}$ in 0, and is greater or equal to $\frac{1}{8}$ between $-\frac{1}{2}\ln(3+2\sqrt{2})$ and $\frac{1}{2}\ln(3+2\sqrt{2})$. The terms involving $f^1_i\ f^1_j$ in (1) bring therefore only a positive contribution, but the same is not true for the terms involving $f^2_i\ f^2_j$. Without more information on f, it does not seem possible to determine the sign of (1).

Second case : $\omega \geq 2$.

We now turn to the case $2 \leq \omega \leq p - 1$ and prove :

Proposition 3.20 : Let M' be a surface of genus zero symmetric with respect to three orthogonal lines . For all $p \geq 3$ and ω such that $2 \leq \omega \leq p - 1$, there exist a surface M of genus p and a harmonic non-holomorphic map of degree ω from M to M'.

Proof.

By proposition 3.16, we know that there is a surface $\overline{M}$ of genus 2 for which there exists a harmonic non-holomorphic map of degree 1 on the sphere and by proposition 3.5, we know that the composition of a holomorphic map with a harmonic one is harmonic . We shall show that there exists a Riemann surface of genus p which is a ω-sheeted branched covering of $\overline{M}$ with holomorphic projection . The composition of that projection with the harmonic map of proposition 3.16 will then be of degree ω, harmonic and non-holomorphic.

In fact, it will be sufficient to build a topological surface M which is a branched covering of $\overline{M}$. Indeed, M will then admit a unique complex structure for which the projection is holomorphic [7].

Let us recall that if M is a ω-sheeted covering of $\overline{M}$, their genera p and $\overline{p}$ are related by Hurwitz' formula :

$$2 - 2p + r = \omega(2 - 2\overline{p}) \qquad (2)$$

where r is the ramification index, i.e. the sum on all points at which the derivative of the projection is 0 of the local degree minus one.

Here we have $\overline{p} = 2$, and (2) becomes

$$r = -2\omega - 2 + 2p \qquad (3).$$

Since r has to be positive or zero, a necessary condition for the existence of a covering is therefore $\omega \leq p - 1$, which is what we have supposed.

p and ω being given, we shall now build the covering.

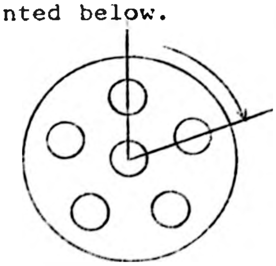
We consider ω copies of $\overline{M}$. With any two of these copies, we can form a two-sheeted covering by joining them crosswize along a line (by this, we mean that we choose on both copies the "same" line -i.e. 2 lines such that one is the copy of the other-, make a slit in each surface along the line and join them crosswise along the slit) . That line could be closed, but not homotopic to zero, in which case there is no ramification or it could be a segment with two endpoints, and in that case we have a ramification index of two . We can also join two copies by more than one segment, which gives a higher ramification index . Having joined two copies, we can then add the others in a similar way . The idea is to choose the number of segments in

order to obtain a ramification index r that will impose the right genus p, through formula (3).

If for the fixed p, $\frac{r}{2} = \omega - 1$, we use exactly $\frac{r}{2}$ segments to join the ω copies . If $\frac{r}{2} > \omega - 1$, we join two of the copies along $\frac{r}{2} - \omega + 2$ segments, and add the others using 1 segment at a time . If, finally, $\frac{r}{2} < \omega - 1$, we use $\frac{r}{2}$ segments and $\omega - 1 - \frac{r}{2}$ closed lines . In all cases, we obtain a ω sheeted covering with ramification index r and hence genus p.

Remark 3.21 : As we have noted in the proof, such a covering cannot exist for $\omega \geq p$.

Remark 3.22 : In the case $\omega = p - 1$, we can give a more geometric representation of M, suggested by E. Calabi . Suppose that M is a surface of genus p, invariant by a rotation of order p - 1, as represented below.



The quotient of M by the rotation is a surface $\overline{M}$ of genus 2 . We can then endow $\overline{M}$ with the right complex structure and lift it to the original surface . We have here an unbranched covering (r = 0).

Third case : $\omega = 0$.

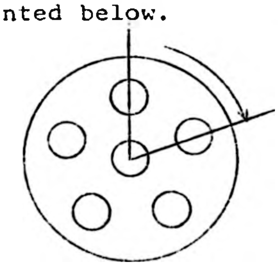
Let us finally consider the case $\omega = 0$. We have a stronger result :

order to obtain a ramification index r that will impose the right genus p, through formula (3).

If for the fixed p, $\frac{r}{2} = \omega - 1$, we use exactly $\frac{r}{2}$ segments to join the ω copies . If $\frac{r}{2} > \omega - 1$, we join two of the copies along $\frac{r}{2} - \omega + 2$ segments, and add the others using 1 segment at a time . If, finally, $\frac{r}{2} < \omega - 1$, we use $\frac{r}{2}$ segments and $\omega - 1 - \frac{r}{2}$ closed lines . In all cases, we obtain a ω sheeted covering with ramification index r and hence genus p.

Remark 3.21 : As we have noted in the proof, such a covering cannot exist for $\omega \geq p$.

Remark 3.22 : In the case $\omega = p - 1$, we can give a more geometric representation of M, suggested by E. Calabi . Suppose that M is a surface of genus p, invariant by a rotation of order p - 1, as represented below.



The quotient of M by the rotation is a surface $\bar{M}$ of genus 2 . We can then endow $\bar{M}$ with the right complex structure and lift it to the original surface . We have here an unbranched covering (r = 0).

Third case : $\omega = 0$.

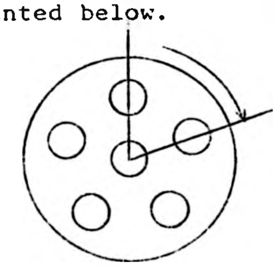
Let us finally consider the case $\omega = 0$. We have a stronger result :

order to obtain a ramification index r that will impose the right genus p, through formula (3).

If for the fixed p, $\frac{r}{2} = \omega - 1$, we use exactly $\frac{r}{2}$ segments to join the ω copies . If $\frac{r}{2} > \omega - 1$, we join two of the copies along $\frac{r}{2} - \omega + 2$ segments, and add the others using 1 segment at a time . If, finally, $\frac{r}{2} < \omega - 1$, we use $\frac{r}{2}$ segments and $\omega - 1 - \frac{r}{2}$ closed lines . In all cases, we obtain a ω sheeted covering with ramification index r and hence genus p.

<u>Remark 3.21</u> : As we have noted in the proof, such a covering cannot exist for $\omega \geq p$.

<u>Remark 3.22</u> : In the case $\omega = p - 1$, we can give a more geometric representation of M, suggested by E. Calabi . Suppose that M is a surface of genus p, invariant by a rotation of order p - 1, as represented below.



The quotient of M by the rotation is a surface $\bar{M}$ of genus 2 . We can then endow $\bar{M}$ with the right complex structure and lift it to the original surface . We have here an unbranched covering (r = 0).

<u>Third case : $\omega = 0$.</u>

Let us finally consider the case $\omega = 0$. We have a stronger result :

<u>Proposition 3.23</u> : Let M be a surface of genus ≥ 1 and M' a surface of genus 0, with arbitrary metrics . Then there exists a harmonic non-holomorphic map of degree 0 from M to M'.

<u>Proof</u>.

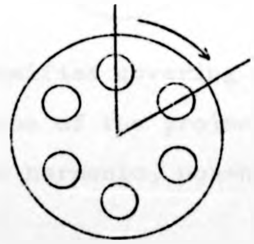
Consider a circle S^1 . By [3,I,4,D], there exists (at least) one surjective harmonic map from M to S^1 . On the other hand, the surface M' contains at least one closed geodesic [18], and therefore there is a totally geodesic map from S^1 to M' . By corollary 1.7, the composition of these two maps is harmonic. Since its image is one-dimensional, it is not holomorphic.

<u>Remark 3.24</u> : For certain metrics on M and M', one can construct more interesting maps, for instance some harmonic non-holomorphic maps of degree 0 which are surjective on the sphere.

When M is a flat orthogonal torus and M' the sphere with its canonical metric, such a map was constructed by R. T. Smith [23].

For a higher genus p of M, it is then sufficient to build a covering of the torus of genus p . (2) becomes in this case r = 2p - 2 and we obtain a covering by joining 2 or more copies of T^2 along p - 1 segments.

Again, we obtain a more geometric representation of such a covering (with p sheets) by taking the quotient with respect to the rotation of order p of the following surface :



The ramification is realized here by two branch points of index p - 1.

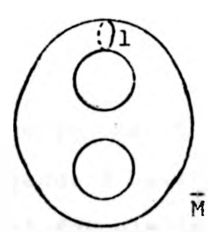
We conclude this paragraph by the following examples :

Proposition 3.25 : For every $p \geq 3$, there exist a surface M of genus p and a non-trivial homotopy class of maps from M to S^2 containing two harmonic maps, one holomorphic and one non-holomorphic . The first is a minimum of E in the class and the second has a greater energy . Such examples don't occur for $p \leq 2$.

Proof.

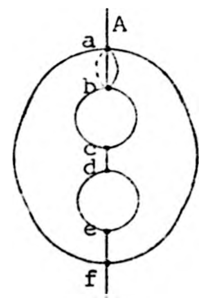
i) Suppose first $p \leq 2$. Then every harmonic map of degree greater or equal to p is holomorphic and every harmonic map of degree between 1 and $p - 1$ is non-holomorphic (since the only possible case is $p = 2$, $\mathcal{D} = 1$).

ii) Take now $p = 3$ and endow $M' = S^2$ with a metric symmetric with respect to three lines, as in proposition 3.16 . We shall obtain M as a two-sheeted covering of a surface $\bar{M}$ of genus 2, also symmetric with respect to three lines . M is defined by taking two copies of $\bar{M}$ and joining them along the closed line 1 represented on this figure



M is then a 2-sheeted unramified covering of $\bar{M}$ and is therefore of genus 3 . The composition of the projection with the harmonic map of proposition 3.16 is harmonic, non-holomorphic and of degree 2.

Consider now the surface M, represented as the 2-sheeted covering of $\bar{M}$ and define an isometry Y of M of order 2 as the rotation of 180° around the axis A which preserves each sheet, except along the line l where it is defined by continuity.



On each sheet, the copy of the points c,d,e and f is fixed by Y, but the two copies of a and b are permuted, so that Y has exactly 8 fixed points.

By Hurwitz' formula (2), the quotient M/Y is of genus 0, and is conformally equivalent to the sphere with the metric chosen above . The projection is therefore a harmonic holomorphic map of degree 2 from M to S^2.

iii) A similar example with $p = 4$ is obtained by joining the two copies of $\bar{M}$ along a line with two endpoints, and not a closed one . M is then of genus 4, and the rest of the constructio goes through.

iv) For $p \geq 5$, we simply observe that there exists a 2-sheeted covering of the surface of genus 3 built in ii) by a surface M of genus p . Indeed, Hurwitz' formula in that case reduces to

$$2 - 2p + r = 2.(2 - 6) = -8$$

or $r = 2p - 10$.

r is therefore positive or zero and we can build the covering as above, by joining 2 copies of the surface of genus 3 along a closed line (for $p = 5$) or $p - 5$ segments (for $p \geq 6$).

The composition of the projection with the maps built in ii) gives rise to 2 harmonic maps of degree 4 from M to S^2, one holomorphic and one non-holomorphic.

CHAPTER 4.

FINITENESS THEOREM.

§ a : DILATATION.

Let M,g and M',g' be Riemannian manifolds and $f : M \longrightarrow M'$ a smooth map. For each $m \in M$, the pull-back f^*g' of the metric on $T_{f(m)}M'$ is a symmetric semidefinite quadratic form on T_mM. Let $k \leq n,n'$ be its rank. We can choose an orthonormal basis $\{e_1,\dots,e_n\}$ of T_mM such that $f^*g' = \sum_{i=1}^{k} \lambda_i\, \omega_i \otimes \omega_i$, where ω_i is the dual 1-form of e_i and $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k > 0$.

Definition 4.1 [6] : $l_1 = \left(\lambda_1/\lambda_2\right)^{\frac{1}{2}}$ is the (first) dilatation of f at m.

By definition, l_1 is greater or equal to 1.

We shall use a notion of map of bounded dilatation which is slightly more general than the one in [6].

Definition 4.2 : We say that the dilatation of f is bounded by K iff at each point of M we have $l_1 \leq K$ or $df = 0$.

Remark 4.3 : We admit in this definition that the rank of f varies from a point to another, and that it takes the value 0. But it cannot take the value 1, since l_1 would then be infinite.

Remark 4.4 : When M and M' are surfaces, a map of dilatation bounded by K is a K-quasiconformal mapping . If K = 1, it is conformal.

Suppose now that the sectional curvature of M' is strictly negative . Since M and M' are compact, there exist two positive constants A and B such that -A is a lower bound for the curvature of M and -B an upper bound for that of M'.

We now observe that theorem 4.1 of [6] remains true for definition 4.2 and that we have :

Proposition 4.5 : Let M and M' be compact, $R \geq -A$, $R' \leq -B$ and call N the minimum of $\{n,n'\}$. If $f \in C^\infty(M,M')$ is a harmonic map of dilatation bounded by K, then

$$|df|^2 \leq \frac{n-1}{2} N^2 K^2 \frac{A}{B} \equiv C^2.$$

Proof.

At a point $m \in M$, the map $df : T_mM \longrightarrow T_{f(m)}M'$ induces a map $\Lambda^2 df : \Lambda^2 T_m M \longrightarrow \Lambda^2 T_{f(m)}M'$ defined by $\Lambda^2 df(X \wedge Y) = df(X) \wedge df(Y)$. At m, let us consider an orthonormal frame (e_i). The norm of $\Lambda^2 df$ is then defined by $|\Lambda^2 df|^2 = \sum_{i<j} |\Lambda^2 df(e_i, e_j)|^2$.

We first show that when f is of dilatation bounded by K, we have at every point $m \in M$

$$|df|^2 \leq N.K.|\Lambda^2 df| \qquad (1).$$

At a point where the rank of df is zero, this is trivial . We can therefore suppose that the rank k of df at m is greater or equal to two . As above, let us consider at m an orthonormal basis $\{e_i\}$ of $T_m M$ such that $f^* g' = \sum_{i=1}^{k} \lambda_i \, \omega_i \otimes \omega_i$,

$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k > 0$.

We have

$$\frac{|df|^2}{|\Lambda^2 df|} = \frac{\sum_i \lambda_i}{\left(\sum_{i<j} \lambda_i \lambda_j\right)^{\frac{1}{2}}} = \frac{\sum_i \frac{\lambda_i}{\lambda_2}}{\left[\sum_{i<j} \frac{\lambda_i \lambda_j}{\lambda_2^2}\right]^{\frac{1}{2}}} \leq \frac{k \frac{\lambda_1}{\lambda_2}}{\left(\frac{\lambda_1}{\lambda_2}\right)^{\frac{1}{2}}} \leq k\,K \leq N\,K.$$

Consider now a point $m \in M$ at which $|df|^2$ attains its maximum (M is compact) . Since f is harmonic [3,(I,3,B)] :

$$\frac{1}{2} \Delta |df|^2 = \sum_{i,j} g'_{\alpha\beta} f^\alpha_{ij} f^\beta_{ij} + \sum_{i,j} g'_{\alpha\beta} \mathrm{Ric}_{ij} f^\alpha_i f^\beta_j$$
$$- \sum_{i,j} R'_{\alpha\beta\gamma\delta} f^\alpha_i f^\beta_j f^\gamma_i f^\delta_j$$

(where Ric is the Ricci tensor of M), and we have at m

$- \sum R'_{\alpha\beta\gamma\delta} f^\alpha_i f^\beta_j f^\gamma_i f^\delta_j \leq - \sum \mathrm{Ric}_{ij} f^\alpha_i f^\beta_j$.

By the hypothesis on the curvature,

$- \sum \mathrm{Ric}_{ij} f^\alpha_i f^\beta_j \leq (n-1)\, A\, |df|^2$,

$2\, B\, |\Lambda^2 df|^2 \leq - \sum R'_{\alpha\beta\gamma\delta} f^\alpha_i f^\beta_j f^\gamma_i f^\delta_j$,

and therefore $|\Lambda^2 df|^2 \leq \frac{n-1}{2} \, \frac{A}{B} \, |df|^2$ (2).

Combining (1) and (2), we get :

$$|df|^2 \leq N^2 K^2 \frac{n-1}{2} \frac{A}{B}$$

at m . Since $|df|^2$ is maximum at m, this relation is valid everywhere.

Corollary 4.6 : With the hypothesis of proposition 4.5, f multiplies the distances by at most the constant C.

The principle of the proof of this property goes back to [11], where it is applied to the case of quasiconformal maps between surfaces.

---

§ b : FINITENESS THEOREM.

We now prove a finiteness theorem, generalizing classical results for holomorphic maps between Riemann surfaces [12].

Theorem 4.7 : Let M and M' be compact Riemannian manifolds and suppose the sectional curvature of M' strictly negative . Let $K \geq 1$. Then there is only a finite number of non-constant harmonic mappings from M to M' of dilatation bounded by K.

Proof.

Since $R' < 0$, it is an immediate consequence of assertion (I) of [9] that there can be only one non-constant harmonic map of bounded dilatation in a homotopy class of maps from M to M'. Indeed, the only condition imposed on the harmonic maps in that

Combining (1) and (2), we get :

$$|df|^2 \leq N^2\ K^2\ \frac{n-1}{2}\ \frac{A}{B}$$

at m . Since $|df|^2$ is maximum at m, this relation is valid everywhere.

Corollary 4.6 : With the hypothesis of proposition 4.5, f multiplies the distances by at most the constant C.

The principle of the proof of this property goes back to [11], where it is applied to the case of quasiconformal maps between surfaces.

---

§ b : FINITENESS THEOREM.

We now prove a finiteness theorem, generalizing classical results for holomorphic maps between Riemann surfaces [12].

Theorem 4.7 : Let M and M' be compact Riemannian manifolds and suppose the sectional curvature of M' strictly negative . Let $K \geq 1$. Then there is only a finite number of non-constant harmonic mappings from M to M' of dilatation bounded by K.

Proof.

Since $R' < 0$, it is an immediate consequence of assertion (I) of [9] that there can be only one non-constant harmonic map of bounded dilatation in a homotopy class of maps from M to M'. Indeed, the only condition imposed on the harmonic maps in that assertion is that their image is not reduced to a point or a closed geodesic, and this last possibility is excluded by the bound on the dilatation . So we just have to prove that, for a given K, only a finite number of homotopy classes can contain a non-constant harmonic map of dilatation bounded by K.

Since the sectional curvature of M' is negative, the Cartan-Hadamard theorem asserts that the homotopy groups $\Pi_i(M')$ are trivial for $i \geq 2$. By theorem 8.1.11 of [26], this implies that the homotopy classes of maps f from M to M' are parametrized by the conjugacy classes of the induced homomorphisms $f_* : \Pi_1(M) \longrightarrow \Pi_1(M')$.

As in the proof of theorem 2.8, call U and U' the universal coverings of M and M' . Choose a point P of U and a fundamental domain $\mathfrak{F}'$ of U' . Every map $f : M \longrightarrow M'$ can be lifted to a map $\overline{f} : U \longrightarrow U'$ such that $\overline{f}(P) \in \mathfrak{F}'$. For all $\gamma \in \Pi_1(M)$, $\overline{f}$ will then verify the relation $\overline{f} \circ \gamma = \overline{f}_*(\gamma) \circ \overline{f}$, where $\overline{f}_*$ is one of the conjugates of f_*, depending on the choice of $\mathfrak{F}'$.

Let $S = \{\rho_o = 1, \rho_1, \ldots, \rho_s\}$ be a set of generators of $\Pi_1(M)$ and put $P_r = \rho_r(P)$, $r = 0, \ldots, s$. The images of the P_r's by a map $\overline{f}$ are contained in $\{\alpha.\overline{f}(P) \mid \alpha \in \Pi_1(M')\}$. The set S being finite, we can find a bounded connected domain D of U containing all P_r's . Since the maps that we consider send P in $\mathfrak{F}'$ and multiply the distances by at most the fixed constant C (corollary 4.6), the images of D by these maps are contained in a bounded set D' of U' . Since D' is bounded, the set $T = \bigcup_{\overline{f}} \{\alpha \in \Pi_1(M') \mid \alpha.\overline{f}(P) \in D'\}$ is finite.

The conjugacy classes of the associated homomorphisms f_* are characterized by the restrictions $f_{*|S} : S \longrightarrow T$. Since S and

T are finite, the number of these classes is also finite, and so is the number of homotopy classes containing a non-constant harmonic map of dilatation bounded by K.

Remark 4.8 : In view of Satz 5.9 of [10] and of corollary 4.12 of next paragraph, one might ask whether theorem 4.7 can be extended to the case where M' is a product of manifolds of negative sectional curvature . We shall show by an example that it is not the case.

Let N and N' be compact Riemannian manifolds with N' of negative sectional curvature and such that there is an infinity of homotopy classes of maps from N to N' . For instance, N and N' could be Riemann surfaces with genus N $\geq$ genus N' $\geq$ 2 . (An infinite sequence of non homotopic maps can then be built by twisting a handle of N around a handle of N' an arbitrary number of times).

By [3], every homotopy class of maps from N to N' contains a harmonic element and we can choose an infinite sequence $h^{(t)}$ of distinct non-constant harmonic maps . (By theorem 4.7, the associated ratios $\lambda_1^{(t)}/\lambda_2^{(t)}$ form an unbounded set of numbers). The set of maps $h^{(t)} \times h^{(t)} : N \times N \longrightarrow N' \times N'$ is then an infinite set of distinct harmonic maps with non-constant projections on N', and their dilatation is always 1 since the eigenvalues of $(h^{(t)} \times h^{(t)})^{*} (g' \times g')$ are $(\lambda_1^{(t)}, \lambda_1^{(t)}, \ldots, \lambda_k^{(t)}, \lambda_k^{(t)})$.

---

§ c : ALMOST COMPLEX MAPS.

We shall now apply theorem 4.7 to the case of almost complex maps between almost Kaehlerian manifolds . Recall that if M,J and M',J' are almost complex manifolds, a map $f : M \longrightarrow M'$ is called almost complex if its differential verifies $df \circ J = J' \circ df$. When M and M' are complex, such an f is holomorphic.

Proposition 4.9 : Let M,J,g and M',J',g' be almost Hermitian manifolds and f an almost complex map from M to M' . For each $m \in M$, there exists an orthonormal basis $\{e_i\}$ of T_mM such that $e_{2j} = J e_{2j-1}$ and $f^*g' = \sum_{i=1}^{k} \lambda_i \; \omega_i \otimes \omega_i$, with $\lambda_{2j} = \lambda_{2j-1}$ and $\lambda_1 = \lambda_2 \geq \lambda_3 = \lambda_4 \geq \dots \geq \lambda_{k-1} = \lambda_k > 0$. In particular, the dilatation of f is bounded by 1.

Proof.

Consider an orthonormal basis $\{e_i\}$ of T_mM such that $f^*g' = \sum \lambda_i \; \omega_i \otimes \omega_i$, with $\lambda_i \geq \lambda_{i+1}$. Since f is almost complex and M' almost Hermitian, we have for $X, Y \in T_mM$:

$$f^*g'(JX,JY) = g'(df.JX,df.JY) = g'(J'.df.X,J'.df.Y)$$
$$=g'(df.X,df.Y) = f^*g'(X,Y).$$

So in particular, $f^*g'(Je_1,Je_1) = \lambda_1$. Since Je_1 is normal to e_1, it is a combination of the e_i's, $i \geq 2$. Since $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k$, λ_2 must be equal to λ_1.

If $\lambda_3 \neq \lambda_2$, then $e_2 = Je_1$ and we can consider the space generated by $e_3,\dots,e_n$ and apply the same reasoning to prove that $\lambda_3 = \lambda_4$.

If $\lambda_2 = \lambda_3 = \ldots = \lambda_p$, then Je_1 is in the space generated by $e_2,\ldots,e_p$. By a rotation of that space (which preserves the restriction of f^*g', equal to the restriction of $\lambda_2 g$), we can replace e_2 by Je_1 . We then proceed as above to prove that $\lambda_3 = \lambda_4$.

The proposition follows from a repetition of this argument.

Let F denote the fundamental 2-form of the almost Hermitian manifold M . Recall [17;IV,16,c] that M is called special if $dF^{\frac{n}{2}-1} = 0$ and special of pure type if $(dF)_{1,2} = 0$. An almost Kaehlerian manifold satisfies these conditions since its fundamental form is closed.

Proposition 4.10 : If M is a compact special almost Hermitian manifold and M' a compact special almost Hermitian manifold of pure type and of negative sectional curvature, then there is only a finite number of non-constant almost complex maps from M to M'.

Indeed, by proposition IV,16,d of [17], an almost complex map is harmonic and by proposition 4.9, its first dilatation is 1.

Corollary 4.11 : There is only a finite number of non-constant almost complex maps between two compact almost Kaehlerian manifolds if the sectional curvature of the second is negative.

In contrast with the real case (remark 4.8), we can consider a product of almost complex manifolds and obtain the following analogue of [10,Satz 5.9 (2)].

Corollary 4.12 : Let M be a compact special almost Hermitian manifold and M' a product of compact special almost Hermitian manifolds of pure type and of negative sectional curvature. Then there is only a finite number of almost complex maps from M to M' whose projections on all factors are non-constant.

Indeed, the maps followed by the projections satisfy the hypothesis of proposition 4.10.

In the case of holomorphic maps between complex manifolds, numerous strong finiteness results can be found in [12,§8].

CHAPTER 5.

REMARKS AND EXAMPLES.

§ a : A FAMILY OF HARMONIC MAPS.

In the study of surfaces, some continuous families of harmonic maps appear in a trivial way . For instance, if M' has a non-zero space of infinitesimal isometries i(M'), the composition of any harmonic map f with the associated isometries is a continuous family of harmonic maps . The same is true for isometries of M, if $df.i(M) \neq 0$.

This will always happen when $df \neq 0$ and M is a sphere or a torus, since by a conformal transformation we can always assume that they carry their canonical metric.

For that reason, when studying the second variation of the energy, R. T. Smith has introduced the reduced nullity of a harmonic map, equal to its nullity minus the dimension of span(i(M'), df i(M)) [25].

When $M = S^2$, a larger family of maps will appear, since the composition of a harmonic map with any conformal transformation of M is again harmonic . For instance, the reduced nullity of the identity on S^2 is precisely 3, the dimension of the conformal group [25].

We now give an example of a continuous family of harmonic maps which are not isometrically or conformally equivalent.

Let M be the torus $\mathbb{R}/2\pi\mathbb{Z} \times \mathbb{R}/2\pi\mathbb{Z}$ with its usual flat metric and Euclidean coordinates (x,y) . Let $0 < c < a$ and $M'_{a,c}$ be the cylinder $[-a + c , a - c] \times S^1$, equiped with the metric defined in coordinates $(u,v) \in [-a + c,a - c] \times [0,2\pi)$ by

$$g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & a^2 - u^2 \end{pmatrix} .$$

This metric is C^∞ on $M'_{a,c}$ and using a partition of unity, we can extend $M'_{a,c}$ to form a C^∞ Riemannian surface M' of any genus.

We restrict our attention to the maps from M to M' whose images are in $M'_{a,c}$ and which are of the form $f(x,y) = (F(x),y)$. A direct calculation shows that the equations $\tau(f) = 0$ reduce for these maps to $\frac{d^2F}{dx^2} + F = 0$. The solutions are therefore the maps

$$f(x,y) = (b \cos(x + d),y) \text{ with } |b| \leq a - c.$$

When b varies in $[-a + c,a - c]$ we obtain a family of harmonic maps which are not conformally equivalent . Their energy is of course independent of b and is

$$E(f) = \frac{1}{2} \int_M [b^2 \sin^2(x+d) + a^2 - b^2 \cos^2(x+d)]\, dx\, dy$$

$$= \frac{1}{2} (2 \pi a)^2.$$

As in remark 3.24, we can of course replace the torus M by a surface of higher genus, by means of a branched covering.

---

§ b : DERIVATIVES OF THE TENSION.

Let M and M' be compact Riemannian manifolds . In [28], K. Yano and S. Ishihara proved that a map f is harmonic as soon as the derivative of its tension is zero . In other words, $\tau(f) = 0 \iff \nabla\tau(f) = 0$. It is easy to obtain the following extension of that result :

Proposition 5.1 : A map $f \in C^\infty(M,M')$ is harmonic as soon as there exists an integer k such that $\nabla^k\tau(f) = 0$.

This is a consequence of the result of [28] and of the following property :

Proposition 5.2 : Let X be a vector field along the map f such that $\nabla^{k+1}X = 0$ $(k \geq 1)$. Then $\nabla^k X = 0$.

Proof.

Consider on M the vector field

$Y = g^{ip}\, g^{jq} \dots g^{ls}\, g'_{\alpha\beta}\, X^\alpha_{\underbrace{j\dots l}_{k-1}}\, X^\beta_{\underbrace{pq\dots r}_{k}} \dfrac{\partial}{\partial x^i}$ where $X^\alpha_{j\dots l}$ is the covariant derivative of X with respect to those k-1 indices.

The divergence of Y is :

$$\nabla_i Y^i = g^{ip} g^{jq} \ldots g^{ls} g'_{\alpha\beta} X^{\alpha}_{ij\ldots l} X^{\beta}_{pq\ldots s} +$$
$$g^{ip} g^{jq} \ldots g^{ls} g'_{\alpha\beta} X^{\alpha}_{j\ldots l} X^{\beta}_{ipq\ldots s} .$$

Since $\int_M \nabla_i X^i \, v_g = 0$ and $X^{\beta}_{ipq\ldots s} = 0$, we have

$\int |\nabla^k X|^2 \, v_g = 0$ and $\nabla^k X = 0$.

---

§ c : A SECOND ORDER ENERGY.

In [5], H. Eliasson introduces the integral

$$J(f) = \int_M (|\tau(f)|^2 + \lambda |df|^2) \, v_g \qquad \lambda > 0$$

and uses it to obtain a new proof of the existence of harmonic maps when dim M is at most 3 and the curvature of M' is non-positive . The principle of his proof is to show with these assumptions that J verifies condition (C) of Palais and Smale (which insures the existence of the minimum of J in every class) and that such a minimum is harmonic . The advantage of J is that, when $R' \leq 0$, it is related to the Sobolev space L^2_2 (of L^2 maps whose first and second derivatives are in L^2) and not L^2_1.

When no condition is imposed on the curvature, he asks (p.132) whether J will still verify condition (C) in dimensions 2 and 3 . (More precisely, he asks whether it is weakly proper in L^2_2 with respect to C^0, and that fact would imply condition (C) cf. [5, p.125-6 and p.130] for details)..

We shall give a negative answer to this question and show that J does not always reach its infimum in a homotopy class in dimension two . This implies in particular that it does not satisfy condition (C).

We first recall a result of [3] :

Definition 5.3 : Let $f \in C^\infty(M,M')$. The volume of f is defined by

$$V(f) = \int_M \left(\frac{\det f^* g'}{\det g}\right)^{\frac{1}{2}} v_g .$$

Proposition 5.4 [3] : Let dim M = 2 . For all $f \in C^\infty(M,M')$, $V(f) \leq E(f)$.

Equality holds when and only when f is conformal (in the weak sense of definition 3.4).

We can now build the example :

Proposition 5.5 : Let M be a flat torus containing a disk of radius 2 and M' the sphere S^2 with its canonical metric . The infimum of J is not reached in the class of maps of degree 1 from M to M'.

Remark 5.6 : By using conformal mappings, one can in fact replace M by any surface of genus $\geq$ 1.

Proof.

Let H be the homotopy class of maps of degree 1 . For all f in H, we have by proposition 5.4 :

$J(f) \geq 2\lambda E(f) \geq 2\lambda V(f) \geq 2\lambda V(M') = 8\lambda\Pi$.

Hence, the infimum of J in H satisfies

$\inf_H J \geq 8\lambda\Pi$.

We shall give a negative answer to this question and show that J does not always reach its infimum in a homotopy class in dimension two . This implies in particular that it does not satisfy condition (C).

We first recall a result of [3] :

Definition 5.3 : Let $f \in C^{\infty}(M,M')$. The volume of f is defined by

$$V(f) = \int_M \left(\frac{\det f^* g'}{\det g}\right)^{\frac{1}{2}} v_g .$$

Proposition 5.4 [3] : Let dim M = 2 . For all $f \in C^{\infty}(M,M')$, $V(f) \leq E(f)$.

Equality holds when and only when f is conformal (in the weak sense of definition 3.4).

We can now build the example :

Proposition 5.5 : Let M be a flat torus containing a disk of radius 2 and M' the sphere S^2 with its canonical metric . The infimum of J is not reached in the class of maps of degree 1 from M to M'.

Remark 5.6 : By using conformal mappings, one can in fact replace M by any surface of genus $\geq$ 1.

Proof.

Let H be the homotopy class of maps of degree 1 . For all f in H, we have by proposition 5.4 :

$J(f) \geq 2\lambda E(f) \geq 2\lambda V(f) \geq 2\lambda V(M') = 8\lambda\pi$.

Hence, the infimum of J in H satisfies

$\inf_H J \geq 8\lambda\pi$.

We now prove that we have in fact equality, by constructing a sequence of maps for which J decreases to $8 \lambda \Pi$.

Let D_1 and D_2 be concentric disks of radius 1 and 2 in M, equiped with polar coordinates (r,θ) and let $(R,\Xi) \in [0,\Pi] \times [0,2\Pi)$ be polar coordinates around a point of the sphere M' . We build a map f of the sequence as follows.

Let $\varepsilon > 0$. On D_1, f is defined by $f(r,\theta) = (F(r),\theta)$ where

$F(r) = 2 \text{ arc tg } ar$ and $a = \text{tg } \frac{\Pi-\varepsilon}{2}$.

f is then conformal and satisfies $f(0,\theta) = (0,\theta)$ and $f(1,\theta) = (\Pi-\varepsilon,\theta)$. Moreover

$$\left.\frac{dF}{dr}\right|_{r=1} = \frac{2 \text{ tg } \frac{\Pi-\varepsilon}{2}}{1 + \text{tg}^2 \frac{\Pi-\varepsilon}{2}} \equiv h(\varepsilon).$$

Observe that $\lim_{\varepsilon \to 0} h(\varepsilon) / \varepsilon = 1$.

Outside the disk D_2, we define f as constant with value (Π,θ).

Finally, we join these two maps in the annulus $D_2 \setminus D_1$ by a map also of the form $f(r,\theta) = (F(r),\theta)$, in such a way as to keep its contribution to J small . Therefore, we impose that $F(1) = \Pi - \varepsilon$, $F(2) = \Pi$, $\frac{dF}{dr}(1) = h(\varepsilon)$ and $\frac{dF}{dr}(2) = 0$ and also that F minimizes $\int_1^2 \left(\frac{d^2F}{dr^2}\right)^2 dr$.

Call G the derivative of F . We see that G will minimize $\int_1^2 (G')^2 \, dr$ under the conditions $G(1) = h(\varepsilon)$, $G(2) = 0$ and $\int_1^2 G' \, dr = \varepsilon$. Using Lagrange's multipliers method, one checks that G has then to be a second order polynomial in r . Hence, $F = a\,r^3 + b\,r^2 + c\,r + d$, and the conditions at 1 and 2 imply that

$$F(r) = (h-2\varepsilon)r^3 + (9\varepsilon-5h)r^2 + (8h-12\varepsilon)r + \Pi + 4\varepsilon - 4h \qquad (1).$$

One checks that for $r \in [1,2]$, $F \in [\Pi-\varepsilon,\Pi]$.

We can now estimate $J(f)$.

We have built f in such a way that it is C^2 except along the lines $r = 1$ and $r = 2$ where it is C^1. That last fact shows that the only thing which could happen along those lines is a discontinuity of F'', which won't bring any contribution to J. The integral $J(f)$ on M is therefore the sum of the integrals on the three regions D_1, $D_2\setminus D_1$ and $M\setminus D_2$.

In the disk D_1, f is conformal and harmonic . Therefore, $\tau = 0$ and $E(f) = V(f)$. Hence $J(f) = 8\lambda\Pi - 0(\varepsilon^2)$.

Outside the disk D_2, $J(f) = 0$.

In the annulus $D_2\setminus D_1$, we calculate J . First,

$$\int_{D_2\setminus D_1} |df|^2 \, V_g = 2\Pi \int_1^2 \left(F'^2 + \frac{1}{r^2}\sin^2 F\right) r \, dr.$$

From the expression (1) of F and the fact that $h \sim \varepsilon$, we deduce that this integral tends to zero as ε^2.

The same is true for

$$\int_{D_2\setminus D_1} |\tau(f)|^2 \, V_g = 2\Pi \int_1^2 \left(\frac{d^2F}{dr^2} + \frac{1}{r}\frac{dF}{dr} - \frac{\sin F \cos F}{r^2}\right)^2 r \, dr.$$

We see therefore that $J(f)$ tends to $8\lambda\Pi$ when ε tends to zero, which is what we wanted to show.

Suppose now that a map f minimizes J in H . We have

$$8\lambda\Pi = J(f) \geq 2\lambda E(f) \geq 2\lambda V(f) \geq 8\lambda\Pi.$$

So $E(f) = V(f)$ and f is holomorphic . Since it is of degree 1, it has to be a diffeomorphism, which is impossible.

CHAPTER 6.

HARMONIC MAPS OF NON-ORIENTABLE SURFACES.

In this chapter, which results from joint work with James Eells, we study the existence question for harmonic maps between not necessarily orientable surfaces.

As in the orientable case, a large class of surfaces is covered by theorem 2.8, which implies :

Proposition 6.1 : Let M,g and M',g' be compact surfaces and suppose that M' is not the sphere or the projective plane . Then every homotopy class of maps from M to M' contains a harmonic element, which is a minimum of E in the class.

From now on, we shall therefore suppose that M' is the sphere S^2 or the projective plane P^2.

CHAPTER 6.

HARMONIC MAPS OF NON-ORIENTABLE SURFACES.

In this chapter, which results from joint work with James Eells, we study the existence question for harmonic maps between not necessarily orientable surfaces.

As in the orientable case, a large class of surfaces is covered by theorem 2.8, which implies :

Proposition 6.1 : Let M,g and M',g' be compact surfaces and suppose that M' is not the sphere or the projective plane . Then every homotopy class of maps from M to M' contains a harmonic element, which is a minimum of E in the class.

From now on, we shall therefore suppose that M' is the sphere S^2 or the projective plane P^2.

§ a : CLASSIFICATION OF THE HOMOTOPY CLASSES.

We shall now recall the classification of the homotopy classes of maps from a surface to S^2 or P^2.

If M is a non-orientable surface and $M' = S^2$, there are two homotopy classes of maps from M to M', parametrized by their degree mod.2.

When M is a surface and $M' = P^2$, the homotopy classes were studied by P. Olum in [29] . We summarize his results, referring to [29] for some of the definitions involved.

Let $f \in C^\infty(M,P^2)$, where M is a surface . We first consider the homomorphism induced by f on the homotopy groups of M and P^2 : $f_* : \Pi_1(M) \longrightarrow \Pi_1(P^2) = Z_2 = \{e_o,e_1\}$, e_o being the neutral element.

f_* is said to be orientation-true if every loop of $\Pi_1(M)$ along which the orientation is preserved is sent to e_o and every loop along which it is reversed is sent to e_1 . If M is represented by the polygone $a_1a_1\ldots a_ha_hb_1c_1b_1^{-1}c_1^{-1}\ldots b_pc_pb_p^{-1}c_p^{-1}$ with the usual identifications, this means that the a_i's are sent to e_1 and the b_i's and c_i's, to e_o.

To a given orientation-true homomorphism is associated an infinity of homotopy classes, parametrized by the absolute value of the twisted degree of the maps, which takes either all even or all odd values . We refer to [29] for the definition of the twisted degree . As it reduces to the usual degree when M and M' are orientable, we shall also call it the degree and denote it by $\mathcal{D}$. As proven in [29], the degree of the composition of two maps is equal to the product of their degrees.

When a homomorphism of the homotopy groups is not orientation-true, there are one or two homotopy classes of maps inducing that homomorphism : one if an odd number of the a_i's goes to e_i, and two in the other case . The degree mod.2 of all the associated maps is 1 in the first case and 0 in the second.

---

§ b : MAPS OF SPHERES AND PROJECTIVE PLANES.

When M and M' are spheres or projective planes, we can solve entirely the existence question for harmonic maps.

$S^2 \longrightarrow S^2$.

As proven in [27] and [13] (see also theorem 3.10), every harmonic map from S^2 to S^2 is conformal . In order to find all harmonic maps, we can therefore suppose that both copies of the sphere are endowed with their canonical metric, since it is conformally equivalent to the given ones . In the notations of § 3.a, every harmonic map of degree $\mathcal{D}$ (resp. $-\mathcal{D}$) from S^2 to S^2 takes then the form

$$w = \frac{\sum_{i=0}^{r} a_i z^i}{\sum_{j=0}^{s} b_j z^j} \qquad \left(\text{resp. } w = \frac{\sum_{i=0}^{r} a_i \bar{z}^i}{\sum_{j=0}^{s} b_j \bar{z}^j}\right)$$

where $a_r \neq 0 \neq b_s$ and $\mathcal{D} = \max(r,s)$. Indeed, every meromorphic function on S^2 is rational.

$S^2 \longrightarrow P^2$.

For every $f : S^2 \longrightarrow P^2$, the associated homomorphism $f_* : \{e\} \longrightarrow \{e_0, e_1\}$ must of course be trivial, so that f can be lifted to a map $\bar{f}$:

$$\begin{tikzcd} & S^2 \arrow[d] \\ S^2 \arrow[ur, "\bar{f}"] \arrow[r, "f"'] & P^2 \end{tikzcd}$$

where the covering of P^2 by S^2 is Riemannian; and f_* is orientation-true.

Suppose first that S^2 and P^2 carry their canonical metrics. Since the degree of the projection is 2, the map $\bar{f}$ defined by $w = z^k$ projects to a map f of degree $2k$. The homotopy classes of maps from S^2 to P^2 must therefore be parametrized by the numbers $|\tilde{\omega}| = 2k$ and they contain harmonic representatives.

By a conformal transformation, this result extends to arbitrary metrics . We have shown :

Proposition 6.2 : Every homotopy class of maps from S^2, g to P^2, g' contains a harmonic representative.

$P^2 \longrightarrow S^2$.

There are two homotopy classes of maps from P^2 to S^2, one of them being the trivial one (i.e. containing the constants). We prove :

Proposition 6.3 : Every harmonic map from P^2, g to S^2, g' is constant . In particular, the non-trivial homotopy class does not contain any harmonic element.

Proof.

Let f be harmonic . It can be lifted by composition to a map $\bar{f}$:

$$\begin{tikzcd} S^2 \arrow[d] \arrow[dr, "\bar{f}"] & \\ P^2 \arrow[r, "f"] & S^2 \end{tikzcd}$$

where $S^2 \longrightarrow P^2$ is a Riemannian covering.

Since the composition of f with an orientation-reversing diffeomorphism of S^2 is homotopic to f, the same goes for $\bar{f}$ which must therefore be of degree zero . Since every harmonic map from S^2 to S^2 is conformal, $\bar{f}$ must hence be constant.

<u>$P^2 \longrightarrow P^2$.</u>

There are two homomorphisms from $\Pi_1(P^2)$ to $\Pi_1(P^2)$: the isomorphism and the zero morphism.

Suppose first that f_* is the isomorphism . f_* is then orientation-true so that there is an infinity of associated homotopy classes . The maps f inducing f_* don't lift to maps from P^2 to S^2, but we can lift them as maps $\bar{f}$ from S^2 to S^2 :

$$\begin{tikzcd} S^2 \arrow[r, "\bar{f}"] \arrow[d] & S^2 \arrow[d] \\ P^2 \arrow[r, "f"] & P^2. \end{tikzcd}$$

When both copies of P^2 and S^2 carry their canonical metrics, the maps $\bar{f}$ from S^2 to S^2 defined in complex coordinates by $w = z^k$ factorize as maps f from P^2 to P^2 iff k is odd . Since the degree of f equals the degree of $\bar{f}$, we see that the homotopy classes associated to f_* are parametrized by the numbers $|\omega| = k$, which take all positive odd values . The maps $w = z^k$ are harmonic representatives of the classes.

Since P^2,g and P^2,g' are conformally equivalent to P^2 with its canonical metric, this provides harmonic representatives for any metric.

Consider next the case of the trivial homomorphism f_*, sending e_o and e_1 to e_o .Two homotopy classes induce that f_*, and the maps f in these classes lift to maps $\bar{f}$ from P^2 to S^2 :

$$\begin{tikzcd} & S^2 \arrow[d] \\ P^2 \arrow[ur, "\bar{f}"] \arrow[r, "f"'] & P^2 . \end{tikzcd}$$

We have already seen that any harmonic map from P^2 to S^2 is constant and so is its projection . Therefore, the non-trivial among these two homotopy classes contains no harmonic representative . To summarize :

Proposition 6.4 : There are two families of homotopy classes of maps from P^2,g to P^2,g' . The first contains an infinity of classes, all of which contain a harmonic element, the second contains two classes, the trivial one (which contains the constants), and another which does not contain any harmonic map.

---

§ c : MAPS INVOLVING OTHER SURFACES.

When other surfaces than S^2 and P^2 are involved, only very partial results were obtained up to now . We shall summarize them here under their present form.

Orientable surface $\longrightarrow P^2$.

Suppose that M is an orientable surface, and consider first

Consider next the case of the trivial homomorphism f_*, sending e_o and e_1 to e_o .Two homotopy classes induce that f_*, and the maps f in these classes lift to maps $\bar{f}$ from P^2 to S^2 :

$$\begin{tikzcd} & S^2 \arrow[d] \\ P^2 \arrow[ur, "\bar{f}"] \arrow[r, "f"] & P^2 . \end{tikzcd}$$

We have already seen that any harmonic map from P^2 to S^2 is constant and so is its projection . Therefore, the non-trivial among these two homotopy classes contains no harmonic representative . To summarize :

<u>Proposition 6.4</u> : There are two families of homotopy classes of maps from P^2,g to P^2,g' . The first contains an infinity of classes, all of which contain a harmonic element, the second contains two classes, the trivial one (which contains the constants), and another which does not contain any harmonic map.

<u>§ c : MAPS INVOLVING OTHER SURFACES</u>.

When other surfaces than S^2 and P^2 are involved, only very partial results were obtained up to now . We shall summarize them here under their present form.

<u>Orientable surface $\longrightarrow P^2$</u>.

Suppose that M is an orientable surface, and consider first

Consider next the case of the trivial homomorphism f_*, sending e_o and e_1 to e_o .Two homotopy classes induce that f_*, and the maps f in these classes lift to maps $\bar{f}$ from P^2 to S^2 :

$$\begin{tikzcd} & S^2 \arrow[d] \\ P^2 \arrow[ur, "\bar{f}"] \arrow[r, "f"] & P^2 . \end{tikzcd}$$

We have already seen that any harmonic map from P^2 to S^2 is constant and so is its projection . Therefore, the non-trivial among these two homotopy classes contains no harmonic representative . To summarize :

Proposition 6.4 : There are two families of homotopy classes of maps from P^2,g to P^2,g' . The first contains an infinity of classes, all of which contain a harmonic element, the second contains two classes, the trivial one (which contains the constants), and another which does not contain any harmonic map.

---

§ c : MAPS INVOLVING OTHER SURFACES.

When other surfaces than S^2 and P^2 are involved, only very partial results were obtained up to now . We shall summarize them here under their present form.

Orientable surface $\longrightarrow P^2$.

Suppose that M is an orientable surface, and consider first

the maps f inducing the zero homomorphism $f_* : \pi_1(M) \longrightarrow Z_2$.

The maps f can then be lifted to maps $\bar{f}$:

$$\begin{tikzcd} & S^2 \arrow[d] \\ M \arrow[ur, "\bar{f}"] \arrow[r, "f"'] & P^2. \end{tikzcd}$$

Since M is orientable, the morphism f_* is orientation-true . When $\bar{f}$ is of degree $\bar{\mathcal{D}}$, f is of degree $2\bar{\mathcal{D}}$ so that the homotopy classes are parametrized by the values $|2\bar{\mathcal{D}}|$.

As there is a one-to-one correspondence between the f's and the $\bar{f}$'s, every statement made in chapter 3 on the maps $\bar{f}$ from M to S^2 implies the same statement on f, by replacing the degree $\mathcal{D}$ by $|2\mathcal{D}|$ and using the notion of weak conformality instead of holomorphy . We refer to that chapter for a complete list of results.

Let us then consider the case of a non-orientation-true (i.e. non-zero) homomorphism f_* . To such an f_* are associated two homotopy classes of maps.

From remark 2.10, we deduce that at least one of these two classes contains a harmonic element, which realizes the minimum of the energy in the two classes . Whether the second class would contain a harmonic map is not known.

Non-orientable surface $\longrightarrow P^2$.

Let M be non-orientable . We have as yet no result concerning the infinite family of homotopy classes associated to an orientation-true homomorphism.

When f_* is not orientation-true, we have one or two homotopy classes associated to f_*, and we know (remark 2.10) that each of these classes or couples of classes contains a harmonic map.

In particular, we have :

<u>Proposition 6.5</u> : Let f_* be a non-orientation-true homomorphism which sends an odd number of the a_i's to e_1 . The unique homotopy class associated to f_* contains a harmonic map.

To be complete, we finally note that we have as yet no result concerning the maps of a non-orientable surface to a sphere . All we know is that they lift to maps of degree zero from an orientable surface to S^2, and it is only when that surface is also a sphere that we can draw a conclusion.

REFERENCES.

[1] J. DIEUDONNE : Eléments d'analyse, vol.2, cahiers scientifiques, fascicule 31, Gauthier-Villars (1969).

[2] C. EARLE and J. EELLS : Deformations of Riemann surfaces, Lectures in modern analysis and applications, Lecture notes 103, Springer-Verlag (1969) 122-149.

[3] J. EELLS and J. SAMPSON : Harmonic mappings of Riemannian manifolds, Amer. J. Math. 86 (1964) 109-160.

[4] J. EELLS and J. WOOD : Restrictions on harmonic maps of surfaces, Topology 15 (1976) 263-266.

[5] H. ELIASSON : Introduction to global calculus of variations, Global Analysis and its Applications, vol.2, I.A.E.A., Vienna (1974) 113-135.

[6] S. I. GOLDBERG, T. ISHIHARA and N. C. PETRIDIS : Mappings of bounded dilatation of Riemannian manifolds, J. Diff. Geom. 10 (1975) 619-630.

[7] R. C. GUNNING : Lectures on Riemann surfaces, Math. Notes n°2, Princeton Univ. Press (1966).

[8] P. HAMILTON : Harmonic maps of manifolds with boundary, Lecture notes n°471, Springer-Verlag (1975).

[9] P. HARTMAN : On homotopic harmonic maps, Canad. J. Math. 19 (1967) 673-687.

[10] W. KAUB : Hyperbolische komplexe Raüme, Ann. Inst. Fourier (Grenoble) 18, fasc. 2 (1968) 303-330.

[11] P. KIERNAN : Quasiconformal mappings and Schwarz's lemma, Trans. Amer. Math. Soc. 148 (1970) 185-197.

[12] S. KOBAYASHI : Intrinsic distances, measures and geometric function theory, Bull. Amer. Math. Soc. 82 (1976) 357-416.

[13] L. LEMAIRE : Thèse de doctorat, Université Libre de Bruxelles (1975).

[14] ———— : Applications harmoniques de surfaces riemanniennes, to appear in J. Diff. Geom.

[15] ———— : Applications harmoniques de variétés produits, to appear in Comm. Math. Helvetici.

[16] ———— : Harmonic mappings of uniformly bounded dilatation, to appear in Topology.

[17] A. LICHNEROWICZ : Applications harmoniques et variétés Kähleriennes, Symposia Mathematica, vol. 3, Bologna (1970), 341-402.

[18] L. LUSTERNIK et L. SCHNIRELMANN : Existence de trois géodésiques fermées sur toute surface de genre 0, C. R. Acad. Sci. Paris, 188 (1929) 534-536.

[19] J. MILNOR : Topology from the differentiable viewpoint, Univ. Press of Virginia (1965).

[20] C. B. MORREY : The problem of Plateau on a Riemannian manifold, Annals of Math. 49 (1948) 807-851.

[21] ———— : Multiple integrals in the calculus of variations, Grundlehren der math. W. Band 130, Springer-Verlag (1966).

[22] J. NASH : The imbedding problem for Riemannian manifolds, Ann. of Math. 63 (1956) 20-63.

[23] R. T. SMITH : Ph. d. thesis, Warwick (1972).

[24] ———— : Harmonic mappings of spheres, Amer. J. Math. 97 (1975) 364-385.

[25] ———— : The second variation formula for harmonic mappings, Proc. Amer. Math. Soc. 47 (1975) 229-236.

[26] E. SPANIER : Algebraic topology, Mc Graw-Hill (1966).

[27] J. C. WOOD : Ph. d. thesis, Warwick (1974).

[28] K. YANO and S. ISHIHARA : Harmonic and relatively affine mappings, J. Diff. Geom. 10 (1975) 501-509.

[29] P. OLUM : Mappings of manifolds and the notion of degree, Annals of Math. 58 (1953) 458-480.

[13] L. LEMAIRE : Thèse de doctorat, Université Libre de Bruxelles (1975).

[14] ________ : Applications harmoniques de surfaces riemanniennes, to appear in J. Diff. Geom.

[15] ________ : Applications harmoniques de variétés produits, to appear in Comm. Math. Helvetici.

[16] ________ : Harmonic mappings of uniformly bounded dilatation, to appear in Topology.

[17] A. LICHNEROWICZ : Applications harmoniques et variétés Kähleriennes, Symposia Mathematica, vol. 3, Bologna (1970), 341-402.

[18] L. LUSTERNIK et L. SCHNIRELMANN : Existence de trois géodésiques fermées sur toute surface de genre 0, C. R. Acad. Sci. Paris, 188 (1929) 534-536.

[19] J. MILNOR : Topology from the differentiable viewpoint, Univ. Press of Virginia (1965).

[20] C. B. MORREY : The problem of Plateau on a Riemannian manifold, Annals of Math. 49 (1948) 807-851.

[21] ________ : Multiple integrals in the calculus of variations, Grundlehren der math. W. Band 130, Springer-Verlag (1966).

[22] J. NASH : The imbedding problem for Riemannian manifolds, Ann. of Math. 63 (1956) 20-63.

[23] R. T. SMITH : Ph. d. thesis, Warwick (1972).

[24] ________ : Harmonic mappings of spheres, Amer. J. Math. 97 (1975) 364-385.

[25] ________ : The second variation formula for harmonic mappings, Proc. Amer. Math. Soc. 47 (1975) 229-236.

[13] L. LEMAIRE : Thèse de doctorat, Université Libre de Bruxelles (1975).

[14] ________ : Applications harmoniques de surfaces riemanniennes, to appear in J. Diff. Geom.

[15] ________ : Applications harmoniques de variétés produits, to appear in Comm. Math. Helvetici.

[16] ________ : Harmonic mappings of uniformly bounded dilatation, to appear in Topology.

[17] A. LICHNEROWICZ : Applications harmoniques et variétés Kähleriennes, Symposia Mathematica, vol. 3, Bologna (1970), 341-402.

[18] L. LUSTERNIK et L. SCHNIRELMANN : Existence de trois géodésiques fermées sur toute surface de genre 0, C. R. Acad. Sci. Paris, 188 (1929) 534-535.

[19] J. MILNOR : Topology from the differentiable viewpoint, Univ. Press of Virginia (1965).

[20] C. B. MORREY : The problem of Plateau on a Riemannian manifold, Annals of Math. 49 (1948) 807-851.

[21] ________ : Multiple integrals in the calculus of variations, Grundlehren der math. W. Band 130, Springer-Verlag (1966).

[22] J. NASH : The imbedding problem for Riemannian manifolds, Ann. of Math. 63 (1956) 20-63.

[23] R. T. SMITH : Ph. d. thesis, Warwick (1972).

[24] ________ : Harmonic mappings of spheres, Amer. J. Math. 97 (1975) 364-385.

[25] ________ : The second variation formula for harmonic mappings, Proc. Amer. Math. Soc. 47 (1975) 229-236.

[26] E. SPANIER : Algebraic topology, Mc Graw-Hill (1966).

[27] J. C. WOOD : Ph. d. thesis, Warwick (1974).

[28] K. YANO and S. ISHIHARA : Harmonic and relatively affine mappings, J. Diff. Geom. 10 (1975) 501-509.

[29] P. OLUM : Mappings of manifolds and the notion of degree, Annals of Math. 58 (1953) 458-480.

[13] L. LEMAIRE : Thèse de doctorat, Université Libre de Bruxelles (1975).

[14] ________ : Applications harmoniques de surfaces riemanniennes, to appear in J. Diff. Geom.

[15] ________ : Applications harmoniques de variétés produits, to appear in Comm. Math. Helvetici.

[16] ________ : Harmonic mappings of uniformly bounded dilatation, to appear in Topology.

[17] A. LICHNEROWICZ : Applications harmoniques et variétés Kähleriennes, Symposia Mathematica, vol. 3, Bologna (1970), 341-402.

[18] L. LUSTERNIK et L. SCHNIRELMANN : Existence de trois géodésiques fermées sur toute surface de genre 0, C. R. Acad. Sci. Paris, 188 (1929) 534-535.

[19] J. MILNOR : Topology from the differentiable viewpoint, Univ. Press of Virginia (1965).

[20] C. B. MORREY : The problem of Plateau on a Riemannian manifold, Annals of Math. 49 (1948) 807-851.

[21] ________ : Multiple integrals in the calculus of variations, Grundlehren der math. W. Band 130, Springer-Verlag (1966).

[22] J. NASH : The imbedding problem for Riemannian manifolds, Ann. of Math. 63 (1956) 20-63.

[23] R. T. SMITH : Ph. d. thesis, Warwick (1972).

[24] ________ : Harmonic mappings of spheres, Amer. J. Math. 97 (1975) 364-385.

[25] ________ : The second variation formula for harmonic mappings, Proc. Amer. Math. Soc. 47 (1975) 229-236.

[13] L. LEMAIRE : Thèse de doctorat, Université Libre de Bruxelles (1975).

[14] ________ : Applications harmoniques de surfaces riemanniennes, to appear in J. Diff. Geom.

[15] ________ : Applications harmoniques de variétés produits, to appear in Comm. Math. Helvetici.

[16] ________ : Harmonic mappings of uniformly bounded dilatation, to appear in Topology.

[17] A. LICHNEROWICZ : Applications harmoniques et variétés Kähleriennes, Symposia Mathematica, vol. 3, Bologna (1970), 341-402.

[18] L. LUSTERNIK et L. SCHNIRELMANN : Existence de trois géodésiques fermées sur toute surface de genre 0, C. R. Acad. Sci. Paris, 188 (1929) 534-536.

[19] J. MILNOR : Topology from the differentiable viewpoint, Univ. Press of Virginia (1965).

[20] C. B. MORREY : The problem of Plateau on a Riemannian manifold, Annals of Math. 49 (1948) 807-851.

[21] ________ : Multiple integrals in the calculus of variations, Grundlehren der math. W. Band 130, Springer-Verlag (1966).

[22] J. NASH : The imbedding problem for Riemannian manifolds, Ann. of Math. 63 (1956) 20-63.

[23] R. T. SMITH : Ph. d. thesis, Warwick (1972).

[24] ________ : Harmonic mappings of spheres, Amer. J. Math. 97 (1975) 364-385.

[25] ________ : The second variation formula for harmonic mappings, Proc. Amer. Math. Soc. 47 (1975) 229-236.

[26] E. SPANIER : Algebraic topology, Mc Graw-Hill (1966).

[27] J. C. WOOD : Ph. d. thesis, Warwick (1974).

[28] K. YANO and S. ISHIHARA : Harmonic and relatively affine mappings, J. Diff. Geom. 10 (1975) 501-509.

[29] P. OLUM : Mappings of manifolds and the notion of degree, Annals of Math. 58 (1953) 458-480.