Asymptotics in conjugacy classes for free groups

by

George Kenison

Thesis

Submitted to The University of Warwick

for the degree of

Doctor of Philosophy

Warwick Mathematics Institute

November 2017

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Acknowledgments

I owe a great debt of gratitude to my supervisor Professor Richard Sharp without whom this thesis would not have come to fruition. Richard, thank you for your patience and guidance.

To the staff and students at the University of Warwick whom I am fortunate to call my friends, thank you for your kindness and humour. I am especially thankful to AlJalila and Rhiannon for accompanying me on this journey.

Last, but by no means least, thank you Mum for your unwavering love, support and encouragement.

Declarations

The orbit counting results in chapter 3 of this thesis appear in the paper Kenison, G. and Sharp, R., *Orbit counting in conjugacy classes for free groups acting on trees* published in the Journal of Topology and Analysis. The statistical results discussed in chapter 5 appear in the paper Kenison, G. and Sharp, R., *Statistics in conjugacy classes in free groups* currently available as a preprint on the arXiv.

Except as noted above, I declare that the material in this thesis is, to the best of my knowledge, my own except where otherwise indicated or cited in the text, or else where the material is widely known. This material has not been submitted for any other degree or qualification.

Abstract

In this thesis we consider asymptotic counts of words in free groups. In particular, we establish results when we restrict the group elements to a non-trivial conjugacy class.

We study the action of the fundamental group of a finite metric graph on its universal covering tree. We assume the graph is finite, connected and the degree of each vertex is at least three. Our results require an irrationality condition on the edge lengths. We obtain an asymptotic formula for the number of elements in a fixed conjugacy class for which the associated displacement of a given base vertex in the universal covering tree is at most T. Under a mild extra assumption we also obtain a polynomial error term.

Related to the above orbit counting result for metric trees, we also consider the spatial distribution of the lattice points of a given conjugacy class in the universal covering tree. We show that the lattice points of a fixed conjugacy class are asymptotically spatially distributed according to a Patterson–Sullivan measure supported on the boundary of the universal cover.

For a class of functions on a free group with suitable symbolic properties we establish an asymptotic average and, subject to an appropriate normalisation, a central limit theorem when the elements of the free group are restricted to a non-trivial conjugacy class.

Chapter 1

Introduction

1.1 Counting functions

Asymptotic analysis is the study of quantification, approximation and estimation associated with long-term or limiting behaviour. In this thesis we consider asymptotics counting and spatial distribution results associated to free groups. Of particular interest are results that arise from restricting the group elements to a non-trivial conjugacy class. In this introduction we give a brief history of related asymptotic results. The introduction is structured as follows. We draw standard comparisons between the prime number theorem and the prime geodesic theorem, a geometric analogue of the prime number theorem. Then we recall the collection of geometric counting problems studied by Huber, including the prime geodesic theorem. The other counting problems in this collection concern groups of isometries acting on the hyperbolic plane. Finally, we introduce the original results that appear in later chapters.

As a thesis that discusses asymptotic analysis, it would be remiss not to mention the prime number theorem. The prime number theorem, which we recall below, is a classical asymptotic result in analytic number theory that establishes the asymptotic distribution of the prime numbers in the positive integers. The infinitude of the prime numbers has been known since the time of Euclid; however, Euclid's proof sheds no light on the question of the asymptotic distribution of the prime numbers. In order to study the distribution, we consider the limiting behaviour of the counting function $\pi : \mathbb{N} \to \mathbb{N}$ given by

$$\pi(n) = \#\{p\colon p \textit{ is prime and } p \leq n\}.$$

The prime number theorem, first conjectured by, amongst others, Legendre and Gauss and proved independently by Hadamard and de la Vallée-Poussin in 1896, gives the following asymptotic formula $\pi(n) \sim n/\log(n)$ as $n \to \infty$. Here we use the notation $f(x) \sim g(x)$ to mean $f(x)/g(x) \to 1$ as $x \to \infty$. Ongoing work on this topic focuses on improving the precision of the error terms. We refer the reader to [15] for further information on the prime number theorem.

A geometric analogue of the prime number theorem is the prime geodesic theorem. Informally, a *geodesic* is a path that locally minimises length and by a *closed geodesic* we mean a geodesic that is a closed smooth path. By a *prime geodesic* we mean a closed geodesic that is not the n-fold concatenation of another closed geodesic for an integer $n \geq 2$. The prime geodesic theorem gives the asymptotic distribution of the number of geodesics of length at most T as $T \to \infty$.

Theorem 1.1.1 (Prime Geodesic Theorem)**.** *Let M be a compact surface with constant negative curvature $\kappa < 0$. For each prime closed geodesic γ on the surface, let $l(\gamma)$ denote its length. Then, for $h = |\kappa|^{1/2}$, as $T \to \infty$,*

$$\#\{\gamma\colon l(\gamma) \leq T\} \sim \frac{\mathrm{e}^{hT}}{hT}.$$

The use of the Riemann zeta function in the early proofs of the prime number theorem encouraged the use of zeta functions in other areas mathematics. Notably Selberg's zeta function used to study the lengths of closed geodesics on surfaces of constant negative curvature. In 1959, Huber [26] established the prime geodesic theorem for compact hyperbolic surfaces with constant negative curvature and gave an error term analysis in [27] and [28].

Remark 1.1.2. In 1970, Margulis' seminal thesis (translated into English in [39]) proved many groundbreaking results. In particular, Margulis established asymptotic properties of the geodesic flow in the setting of variable negative curvature. In fact, Margulis' results hold for a class of hyperbolic flows, but we consider only the geodesic flow.

Let M be a compact manifold with variable negative curvature, then a natural flow to consider is the geodesic flow on the unit tangent bundle T_1M. Let $(x, v) \in T_1M$ and $\gamma : \mathbb{R}^+ \to M$ the unique geodesic with unit speed such that $\gamma(0) = x$ and $\gamma'(0) = v$. We define the *geodesic flow* $g_t : T_1M \to T_1M$ by $g_t(x, v) = (\gamma(t), \gamma'(t))$ for $t \in \mathbb{R}^+$. For each $t > 0$ there are finitely many closed orbits of the geodesic flow with period length at most t. Closed orbits of the geodesic flow $g_t : T_1M \to T_1M$ naturally project onto the closed geodesics in M. Further, the closed orbits of the geodesic flow are in one-to-one correspondence with the (oriented) closed geodesics in M. In this way, Margulis studied the prime geodesic theorem from a dynamical viewpoint using ergodic techniques. Margulis proved the prime geodesic theorem for compact manifolds with variable negative curvature. Here the positive constant h is the topological entropy of the geodesic flow. We note Margulis' work in this more general setting does not establish an error term.

In the same setting as his prime geodesic theorem, Huber [25] also established asymptotic results for certain groups of isometries of the hyperbolic plane. Here the connection to the prime geodesic theorem is as follows. Instead of counting prime closed geodesics on a surface, these results counted closed geodesics. We shall state Huber's results in terms of the action of a group of isometries acting on the hyperbolic plane and so we use the term orbit counting in this setting. Huber also established an orbit counting result when the elements of the group are restricted to a non-trivial conjugacy class and this corresponds to counting based closed geodesics.

We refer the reader to [1] for an introduction to the action of $\mathrm{PSL}(2, \mathbb{C})$ on the hyperbolic plane $\mathbb{H}^2$. A *Möbius map* f on the Riemann sphere has the form

$$f(z) = \frac{az + b}{cz + d}$$

with $a, b, c, d \in \mathbb{C}$ and, so that f is non-constant, $ad - bc \neq 0$. From now on we consider the set of Möbius maps such that

(M1) $a, b, c, d \in \mathbb{R}$, and

(M2) $ad - bc = 1$.

Note there is no penalty in assuming (M2).The set of Möbius maps together with the binary operation of composition form a group. In this way the group of Möbius maps is isomorphic to $\mathrm{PSL}(2, \mathbb{R}) = \mathrm{SL}(2, \mathbb{R})/\{\pm I_2\}$.

It is useful to consider Huber's counting problems, for certain groups of isometries of the hyperbolic plane, in terms of subgroups of $\mathrm{PSL}(2, \mathbb{R})$. We recall the terminology for such groups below.

A subgroup Γ of $\mathrm{PSL}(2, \mathbb{R})$ is *discrete* if each point in Γ is isolated in the topology inherited from $\mathrm{PSL}(2, \mathbb{R})$. We call a discrete subgroup of $\mathrm{PSL}(2, \mathbb{R})$ a *Fuchsian group*. Let $z \in \mathbb{H}^2$, then we call the set

$$\Gamma z = \{\gamma z \in \mathbb{H}^2 \colon \gamma \in \Gamma\}$$

the *orbit of* z, or the *lattice points* associated to z. Let Γ be a Fuchsian group, then for each $z \in \mathbb{H}^2$ the set Γz is discrete in $\mathbb{H}^2$. We denote by $\Lambda(\Gamma)$ the limit set of Γz for $z \in \mathbb{H}^2$. For a Fuchsian group Γ, the limit set is a subset of $\partial \mathbb{H}^2$. There are three cases to consider for the cardinality of $\Lambda(\Gamma)$ and we note the cardinality is independent of $z \in \mathbb{H}^2$. We have that $\Lambda(\Gamma)$ is: empty, contains one or two points, or contains infinitely many points. We are interested in the third case, Fuchsian groups in this case are called *non-elementary*, and there are two subcases: A *Fuchsian group of the first type* is a Fuchsian group whose limit set is $\mathbb{R} \cup \{\infty\}$, otherwise a Fuchsian group is of the *second type*. The limit set of a Fuchsian group of the second type is a Cantor set. We note $\Lambda(\Gamma)$ is independent of the choice of base point $z \in \mathbb{H}^2$ when Γ is non-elementary. A Fuchsian group Γ is *co-compact* if the surface $\mathbb{H}^2/\Gamma$ is compact. We note all compact surfaces of constant negative curvature are obtained by taking the quotient $\mathbb{H}^2/\Gamma$ where Γ is a co-compact Fuchsian group (with the appropriate scaling for curvature).

We say a subgroup Γ of $\mathrm{PSL}(2,\mathbb{R})$ acts *discontinuously* on $\mathbb{H}^2$ if for each compact subset $K \subset \mathbb{H}^2$ there are only finitely many elements $g \in \Gamma$ for which $gK \cap K$ is non-empty. We shall assume that Γ acts freely on $\mathbb{H}^2$.

Delsarte [12] (in 1942) and Huber [25] (in 1956) considered the orbit counting problem for co-compact Fuchsian groups acting on $\mathbb{H}^2$. For a chosen base point $p \in \mathbb{H}^2$ and a Fuchsian group Γ, Delsarte and Huber established the asymptotic behaviour of the following counting function $N : \mathbb{R}^+ \to \mathbb{R}^+$ given by

$$N(T) = \#\{\gamma \in \Gamma \colon d_{\mathbb{H}^2}(p, \gamma p) \leq T\}.$$

They each established the asymptotic equality

$$N(T) \sim \frac{1}{4(g-1)} \mathrm{e}^T,$$

as $T \to \infty$ where $g > 1$ is the genus of the surface $\mathbb{H}^2/\Gamma$. Broadly speaking, both authors connected the spectrum of the Laplacian of a hyperbolic surface to the set of lengths of closed geodesics in the surface. We shall refer to the set of lengths of closed geodesics in the surface as the surface's *length spectrum*. Delsarte [12] wrote $N(T)$ as a series of hypergeometric functions related to the Laplacian, whilst Huber [25] used a Tauberian theorem to establish the asymptotic formula from the complex analytic properties of a Dirichlet series. We note Huber's use of the spectral theory of the Laplacian is independent of Selberg's work in [61]. Patterson [45], [46] established an asymptotic result in the more general setting of all Fuchsian groups of the first type without parabolic elements. Patterson's proof used the spectral techniques developed by Huber and gave an error term approximation.

Remark 1.1.3. We briefly recall Selberg's work in [61]. Let Γ be a group of isometries of a Riemannian manifold M. In [61], Selberg considered spectral problems associated to groups of isometries of certain Riemannian manifolds M. A linear operator L is *invariant* with respect to a group of isometries Γ if it commutes with the operator $f(x) \mapsto f(gx)$ for each function f on M and $g \in G$. The Selberg trace formula relates spectral data for invariant linear operators to geometric properties of the manifold. Selberg gave a general trace formula in the case that the quotient is compact and also considered certain special cases when the quotient is not compact. For our discussion of Huber's work, we are interested in the case $M = \mathbb{H}^2$ and Γ is a co-compact Fuchsian group (cf. [22]).

Orbit counting results have also been established for manifolds with variable negative curvature. Suppose that M is a compact Riemannian manifold with negative sectional curvatures and universal cover $\widetilde{M}$ (so that $M = \widetilde{M}/\Gamma$ where Γ, a group of isometries acting on $\widetilde{M}$, is isomorphic to the fundamental group of $\widetilde{M}$). Choose a point $\tilde{x} \in \widetilde{M}$ and define a counting function $N(T) = \#\{\gamma \in \Gamma \colon d_{\widetilde{M}}(\tilde{x}, \gamma\tilde{x}) \leq T\}$. In his thesis [39], Margulis proved that there exists a positive constant C depending on Γ such that $N(T) \sim C\mathrm{e}^{hT}$ as $T \to \infty$. Here $h > 0$ is the topological entropy of the geodesic flow $g_t : T_1M \to T_1M$ on the unit tangent bundle.

Let $x \in M$ be the image of the point $\tilde{x} \in \widetilde{M}$ under the natural projection. The problem of counting the number $N(T)$ of elements in $\Gamma\tilde{x}$ within a ball of radius T centred at the point $\tilde{x} \in \widetilde{M}$ can be interpreted as counting the number $\rho(T)$ of geodesic loops based at the point $x \in M$ with length at most T. In [50], Pollicott gave a symbolic proof that $\rho(T) \sim C\mathrm{e}^{hT}$ as $T \to \infty$ and, in addition, an asymptotic formula for more general hyperbolic flows on compact manifolds.

Huber also considered a third problem: the conjugacy counting problem in this setting. Let $\mathfrak{C}$ be a non-trivial conjugacy class in F. Let $N_{\mathfrak{C}}(T)$ be the counting function given by

$$N_{\mathfrak{C}}(T) = \#\{g \in \mathfrak{C} \colon d_{\mathbb{H}^2}(p, gp) \leq T\}.$$

Huber [25] showed there exists a positive constant $C(\mathfrak{C}) > 0$ such that $N_{\mathfrak{C}}(T) \sim C\mathrm{e}^{T/2}$ as $T \to \infty$. Note the exponential growth rate of the function $N_{\mathfrak{C}}(T)$ is precisely half of the exponential growth rate of the counting function $N(T)$. This intriguing phenomenon forms the foundation for the research in this thesis.

1.2 Orbit counting

In analogy to Huber's third problem, we consider the orbit counting problem in conjugacy classes in the setting of metric graphs. Rather than compact hyperbolic surfaces with constant negative curvature -1, we consider finite connected metric graphs such that the degree of each vertex is at least 3. Briefly, the closed geodesics on a metric graph G are non-backtracking paths whose initial and terminal vertices coincide (modulo the cyclic permutation of edges in the path). The fundamental group of G, a finitely generated free group F, acts freely on the graph's universal covering tree $\mathcal{T}$. Indeed, when we equip the universal cover with the metric lifted from G the action is also isometric.

We make one further assumption: an irrationality condition on the lengths of the closed geodesics in G. We require that there are a pair of closed geodesics in G such that the ratio of their metric lengths is irrational. Guillopé [20] considered the asymptotic behaviour of the orbit counting function

$$N(T) = \#\{x \in F \colon d_{\mathcal{T}}(o, xo) \leq T\}$$

where $o \in \mathcal{T}$ is a prescribed base vertex. Guillopé showed that $N(T) \sim C\mathrm{e}^{hT}$ as $T \to \infty$ for positive constants C and h; here h is the volume entropy of the tree $\mathcal{T}$.

We consider the effect on Guillopé's asymptotic formula when the counting function counts only those free group elements in a non-trivial conjugacy class.

Theorem 3.1.1. *Suppose that G is a finite connected metric graph such that the degree of each vertex is at least 3 and there exists a pair of closed geodesics in G such that the ratio of their metric lengths is irrational. Let $\mathfrak{C}$ be a non-trivial conjugacy class in F. Then, for some constant $C > 0$, depending on $\mathfrak{C}$,*

$$N_{\mathfrak{C}}(T) \sim Ce^{hT/2}, \quad \text{as } T \to \infty.$$

We summarise recent results for counting problems in conjugacy classes (cf. [14], [42], [5]) below.

In [14] Douma, using spectral results for the discrete Laplacian reminiscent of Huber's spectral approach [26], established an orbit counting result in the setting of finite simple $(q+1)$-regular graphs G with fundamental group F. Let $\mathcal{T}$ be the universal covering tree of G and assign to each edge in $\mathcal{T}$ metric length 1. Fix a non-trivial conjugacy class $\mathfrak{C}$ in F and base vertex $o \in \mathcal{T}$. Douma studied the counting function

$$N_{\mathfrak{C}}(n) = \#\{x \in \mathfrak{C} \colon d_{\mathcal{T}}(o, xo) \leq n\}$$

and showed there exists a positive constant $C(\mathfrak{C})$ such that

$$N_{\mathfrak{C}}(n) \sim C q^{\lfloor (n - \mu(\mathfrak{C}))/2 \rfloor}$$

for $n \in \mathbb{Z}^+$ as $n \to \infty$. Here $\mu(\mathfrak{C})$ is the infimum distance $d_{\mathcal{T}}(o, xo)$ obtained by an element $x \in \mathfrak{C}$.

Margulis [39] did not consider orbit counting in conjugacy classes; however, Parkkonen and Paulin considered orbit counting in conjugacy classes in the setting of variable negative curvature. In [42], Parkkonen and Paulin established asymptotic formulae for orbit counting in conjugacy classes in the setting of compact manifolds with pinched variable negative curvature. By *pinched* we mean the manifold's sectional curvatures are bounded between two negative constants. They proved an exponential growth rate for the conjugacy counting function and gave an asymptotic formula for finitely generated discrete groups of isometries of the hyperbolic plane. Indeed, in the special case that the group has co-finite volume Parkkonen and Paulin's result is analogous to the result in Theorem 3.1.1 for metric graphs. Let Γ be a finitely generated non-elementary Fuchsian group and $\mathfrak{C}$ a non-trivial conjugacy class in Γ. Parkkonen and Paulin showed that the associated conjugacy counting function $N_{\mathfrak{C}}(T)$ has an asymptotic formula $N_{\mathfrak{C}}(T) \sim C e^{hT/2}$ as $T \to \infty$ and subject to further technical assumptions gave an error term approximation. Here $C(\mathfrak{C}) > 0$ and $h > 0$ is the critical exponent of Γ.

In [5], Broise-Alamichel, Parkkonen and Paulin proved results in the more general situation of graphs of groups in the sense of Bass–Serre theory. They gave asymptotic formulae for orbit counting in conjugacy classes of the form $\sim C(\mathfrak{C})e^{hT/2}$ as $T \to \infty$ and in certain specialisations an error term estimate. One of these special cases includes the fundamental group of a finite connected metric graph acting on the graph's universal covering tree. Broise-Alamichel, Parkkonen and Paulin attribute the conjugacy counting asymptotic and error term in the metric graph case to Sharp and the author for their earlier work in [33] (work that appears in Chapter 3 of this thesis). In [5], an ergodic-geometric approach using mixing properties of the geodesic flow of the Bowen–Margulis measure was used in contrast to the spectral analysis of the transfer operator for subshifts of finite type used by Sharp and the author.

1.3 Spatial distribution

The orbit counting asymptotic formula discussed in the previous section concerns the counting function $N_{\mathfrak{C}}(T) = \#\{x \in \mathfrak{C}\colon d_{\mathcal{T}}(o, xo) \leq T\}$. In chapter 4, we consider the asymptotic spatial distribution of the lattice points $\{xo\colon x \in \mathfrak{C}\}$ of distance at most T from $o \in \mathcal{T}$ as $T \to \infty$.

Subject to the same assumptions on the metric graph in the previous section, we establish the following result. Given a non-trivial conjugacy class $\mathfrak{C}$ in the free group and vertex $o \in \mathcal{T}$, the lattice points $\{xo \in \mathcal{T}\colon x \in \mathfrak{C}\}$ are asymptotically uniformly distributed with respect to a Patterson–Sullivan measure μ_{PS} on the boundary of $\mathcal{T}$. Specifically, let B be a Borel set in the boundary of $\mathcal{T}$, written $\partial\mathcal{T}$. We write $[o, \xi]$ for the geodesic ray, an infinite non-backtracking path in $\mathcal{T}$, originating from $o \in \mathcal{T}$ with endpoint $\xi \in \partial\mathcal{T}$. Let $\mathcal{S}(B) \subseteq \mathcal{T}$ denote the set of points $y \in \mathcal{T}$ such that if $[o, \xi]$ passes through y then $\xi \in B$. Let $N_{\mathfrak{C}}^B(T)$ be the counting function given by

$$N_{\mathfrak{C}}^B(T) = \#\{x \in \mathfrak{C}\colon d_{\mathcal{T}}(o, xo) \leq T, xo \in \mathcal{S}(B)\}.$$

Then

$$\lim_{T \to \infty} \frac{N_{\mathfrak{C}}^B(T)}{N_{\mathfrak{C}}(T)} = \mu_{\mathrm{PS}}(B).$$

For reference, we state the main theorem of Chapter 4 in full.

Theorem 4.1.1. *Suppose that G is a finite connected metric graph such that the degree of each vertex is at least 3 and the set of closed geodesics has lengths not contained within a discrete subgroup of $\mathbb{R}$. Let $\mathfrak{C}$ be a non-trivial conjugacy class in F. Let $\mathcal{T}$ be the universal cover of G. Then given B a Borel subset in the boundary of the tree*

$$\lim_{T\to\infty} \frac{N_{\mathfrak{C}}^{B}(T)}{N_{\mathfrak{C}}(T)} = \mu_{\mathrm{PS}}(B).$$

The proof of this spatial distribution result uses techniques from symbolic dynamics and the thermodynamic formalism; in particular, our method follows work by Sharp in [63] for Kleinian groups acting on n-dimensional hyperbolic space.

The asymptotic spatial distribution of lattice points in the setting of n-dimensional hyperbolic space is discussed in both [40] and [63]. We note the results within do not restrict elements to a non-trivial conjugacy class. In [40], Nicholls, generalising earlier work by Patterson for co-compact Fuchsian groups acting on the hyperbolic plane, showed that each orbit of a co-compact discrete group of isometries acting on n-dimensional hyperbolic space is asymptotically uniformly distributed in all directions. In [63], Sharp proved that asymptotically the lattice points of certain convex co-compact Kleinian groups acting on n-dimensional hyperbolic space are uniformly distributed with respect to the class of Patterson–Sullivan measures on the boundary.

1.4 Asymptotic statistics

Let F be a free group on $l \geq 2$ generators acting convex co-compactly on a CAT(-1) space (X, d). There has been considerable work in trying to understand the statistics of this action. For example, fix a free generating set $\mathcal{A} = \{a_1, a_2, \ldots, a_l\}$ and for each $x \in F$ let $|x|$ denote the word length of x with respect to the set $\mathcal{A}$. Let $W_n' = \{x \in F\colon |x| = n\}$ and o an arbitrarily chosen basepoint in X. Then it is known that the averages

$$\frac{1}{W'_n}\sum_{x\in W'_n}\frac{d(o,xo)}{n}$$

converge in the limit as $n\to\infty$ to a positive constant λ that is independent of the choice of basepoint $o\in X$ [8], [54], [56]. Under the additional assumption that the set $\{d(o,xo)-\lambda|x|\colon x\in F\}$ is unbounded a central limit theorem is also known: the distribution of $(d(o,xo)-\lambda|x|)/\sqrt{n}$ with respect to the counting measure on W'_n converges to the distribution function of a Gaussian $N(0,\sigma^2)$ as $n\to\infty$.

Remark 1.4.1. The number $\lambda>0$ may also be characterised in the following way. Let Σ be the space of infinite reduced words on $\mathcal{A}\cup\mathcal{A}^{-1}$ and let μ_0 be the measure of maximal entropy for the shift map $\sigma:\Sigma\to\Sigma$. Then, for μ_0-a.e. $(x_i)_{i=0}^{\infty}\in\Sigma$,

$$\lim_{n\to\infty}\frac{d(o,x_0x_1\cdots x_{n-1}o)}{n}=\lambda.$$

This follows from the representation of $d(o,xo)$ as a Birkhoff sum of a Hölder continuous function on Σ^* and the ergodic theorem. (See, for example, Lemma 4.4 and Corollary 4.5 of [57].)

It is interesting to consider whether analogous results hold when we restrict the group elements to a non-trivial conjugacy class $\mathfrak{C}$ in F. Let $k=\min\{|x|\colon x\in\mathfrak{C}\}$ and let $\mathfrak{C}_n=\{x\in\mathfrak{C}\colon |x|=n\}$. Observe that $\mathfrak{C}_n$ is non-empty if and only if $n=k+2m$ for some $m\in\mathbb{Z}^+$.

Theorem 1.4.2. *We have*

$$\lim_{m\to\infty}\frac{1}{\#\mathfrak{C}_{k+2m}}\sum_{x\in\mathfrak{C}_{k+2m}}\frac{d(o,xo)}{k+2m}=\lambda.$$

Subject to an additional non-degeneracy condition we have the following.

Theorem 1.4.3. *Suppose that the set $\{d(o,xo)-\lambda|x|\colon x\in F\}$ is unbounded then*

$$\lim_{n\to\infty}\frac{1}{\#\mathfrak{C}_{k+2m}}\left\{x\in\mathfrak{C}_{k+2m}\colon\frac{d(o,xo)-\lambda(k+2m)}{\sqrt{k+2m}}\le a\right\}=\frac{1}{2\sqrt{\pi}\sigma}\int_{-\infty}^{a}\mathrm{e}^{-t^2/4\sigma^2}\,dt.$$

Hence we have a central limit asymptotic in this setting: the distribution of $(d(o, xo) - \lambda(k+2m))/\sqrt{k+2m}$ normalised by the counting measure on $\mathfrak{C}_{k+2m}$ converges to a Gaussian distribution $N(0, 2\sigma^2)$ as $m \to \infty$. We note that the constant $\sigma > 0$ is independent of the conjugacy class $\mathfrak{C}$ and that the variance witnessed here is twice the variance in the unrestricted case.

We remark that the limiting distribution function is independent of the choice of non-trivial conjugacy class. Interestingly, the variance of the limit function in Theorem 1.4.3 has twice the variance of the limit function when we do not restrict elements $x \in F$ to a non-trivial conjugacy class.

In Chapter 5 we state and prove Theorems 5.1.1 and 5.1.3 from which Theorems 1.4.2 and 1.4.3 follow, respectively.

Chapter 2

Preliminaries

2.1 Dynamical systems and ergodic theory

By a *dynamical system* we mean a pair (X, T) with X a compact metric space and a continuous map $T : X \to X$. We shall use the notation T^k with $k \in \mathbb{Z}^+$ (or in the case of an invertible map $k \in \mathbb{Z}$) for the transformation given by the k-fold composition of T. Briefly, ergodic theory studies the statistical properties of dynamical systems. In particular, quantities that are invariant under the transformation. In this section we recall standard concepts from the literature.

A dynamical system (X, T) is *topologically transitive* if for each pair of non-empty open sets $U, V \subseteq X$ there exists $k \in \mathbb{N}$ such that $T^{-k}U \cap V \neq \emptyset$. We call a dynamical system *topologically mixing* if for each pair of non-empty open sets $U, V \subseteq X$ there exists $n \in \mathbb{N}$ such that $T^{-k}U \cap V \neq \emptyset$ for each $k > n$. Clearly if a system is topologically mixing then it is topologically transitive.

We impose additional structure on our system as follows. Let X be a probability space with σ-algebra $\mathscr{B}$ and probability measure $\mu : \mathscr{B}(X) \to [0, 1]$. We shall assume that the map T is *measurable*; that is, for each $B \in \mathscr{B}(X)$ we assume that $T^{-1}B \in \mathscr{B}(X)$. We say that T is *measure-preserving* if $\mu(T^{-1}B) = \mu(B)$ for each $B \in \mathscr{B}(X)$.

We consider a class of measure-preserving transformations, which we call the *ergodic* transformations, that satisfy an additional criterion: if $T^{-1}B = B$

then either $\mu(B) = 0$ or $\mu(B) = 1$. Intuitively, ergodicity is an indecomposibility condition, by which we mean an ergodic dynamical system does not decompose into a disjoint union of two invariant sub-systems $T : X_1 \to X_1$ and $T : X_2 \to X_2$ (with $X = X_1 \cup X_2$) with $\mu(X_1) > 0$ and $\mu(X_2) > 0$.

We denote by h the topological entropy for a transformation $T : (X, d) \to (X, d)$ on a compact metric space (X, d). Conceptually, topological entropy of a dynamical system is a quantification of the complexity of the system's orbit structure. We refer the reader to [67] for the details of the construction.

Suppose that $T : X \to X$ is a continuous map on a compact metric space. Let $\mathcal{M}(X)$ denote the set of Borel probability measures on X and let $\mathcal{M}(X, T)$ be the subset of T-invariant probability measures on X. For a compact metric space X, the space $\mathcal{M}(X)$ is weak* compact by the Riesz representation theorem. By the Krylov–Bogolyubov theorem, there is a T-invariant probability measure on X and so $\mathcal{M}(X, T)$ is non-empty. In addition, $\mathcal{M}(X, T)$ is convex and weak* closed. Thus because $\mathcal{M}(X, T)$ is a closed subset of a compact set, $\mathcal{M}(X, T)$ is weak* compact.

Topological entropy is analogous to the concept of measure-theoretic entropy for measurable dynamical systems. We refer the reader to [67] for the construction of measure-theoretic entropy $h_\mu(T)$ of the system (X, T, μ) where $\mu \in \mathcal{M}(X, T)$. It is interesting to consider the entropy map $\mu \mapsto h_\mu(T)$ for $\mu \in \mathcal{M}(X, T)$. For a large class of measurable dynamical systems the image of this map is bounded and a family of measures attains the supremum. We call a measure that attains this supremum a *measure of maximal entropy*. The variational principle (cf. [44]), which relates topological and measure-theoretic entropies, states that for a continuous transformation of a compact metric space $h(T) = \sup\{h_\mu(T) : \mu \in \mathcal{M}(X, T)\}$. Thus if μ is a measure of maximal entropy $h_\mu(T) = h(T)$. It can be shown that if a dynamical system has a unique measure of maximal entropy μ, then μ is ergodic.

A *symbolic dynamical system* is a dynamical system whose state space consists of sequences with terms from an alphabet or symbol set. Examples of symbolic dynamical systems are subshifts of finite type, sometimes called topological Markov chains. Subshifts of finite type are used extensively in this thesis. The introductory material we present below can be found in a number of sources, for instance [44] and [67].

Let $\mathcal{S} = \{s_1, \ldots, s_n\}$ be a finite set of symbols. A *word* $x_0 x_1 \cdots x_{n-1}$ is a concatenation of elements $x_k \in \mathcal{S}$. We shall determine the admissible words according to the entries of an $(n \times n)$-matrix A whose rows and columns are indexed by the $\mathcal{S}$. The matrix A, which we call a *transition matrix*, has entries in $\{0, 1\}$ and we say a word $x_0 x_1 \cdots x_{n-1}$ is *admissible* if $A(x_k, x_{k+1}) = 1$ for each $k \in \{0, 1, \ldots, n-2\}$.

We extend the definition of finite admissible words to infinite admissible sequences. Given a symbol set $\mathcal{S}$, we define a shift space Σ_A as the set of infinite admissible sequences determined by a given matrix A as follows:

$$\Sigma_A = \{(x_k)_{k=0}^{\infty} \in \mathcal{S}^{\mathbb{Z}^+} : A(x_k, x_{k+1}) = 1, \forall k \in \mathbb{Z}^+\}.$$

A *subshift of finite type* is a dynamical system consisting of a shift space Σ_A and a *shift map* $\sigma : \Sigma_A \to \Sigma_A$ given by $\sigma(x_n)_{n=0}^{\infty} = (x_{n+1})_{n=0}^{\infty}$. We endow Σ_A with the following metric. Fix $0 < \theta < 1$, for each $x \in \Sigma_A$ we set $d_\theta(x, x) = 0$. For $x, y \in \Sigma_A$ with $x \neq y$ we set $d_\theta(x, y) = \theta^k$ with $k = \min\{n \in \mathbb{Z}^+ : x_n \neq y_n\}$. Endowed with this metric Σ_A is compact, perfect and totally disconnected. In addition, the shift map is continuous.

There are two standard assumptions which may be imposed on transition matrices, or more generally, square matrices with non-negative entries. We say that a non-negative (square) matrix A is *irreducible* if, for each pair of indices (i, j), there exists $n \geq 1$ such that $A^n(i, j) > 0$ and that A is *aperiodic* if there exists $n \geq 1$ such that, for each pair of indices (i, j), $A^n(i, j) > 0$. An aperiodic matrix is necessarily irreducible. In the case that a transition matrix A is irreducible the associated subshift of finite type on Σ_A is topologically transitive. If, in addition, A is aperiodic, then Σ_A is topologically mixing.

2.2 Free groups

In this section we introduce the notation and terminology for free groups which we will use throughout this thesis.

Let F be a free group with free generating set $\mathcal{A} = \{a_1, \ldots, a_l\}$, $l \geq 2$. Write $\mathcal{A}^{-1} = \{a_1^{-1}, \ldots, a_l^{-1}\}$. A word $x_0 \cdots x_{n-1}$, with letters $x_k \in \mathcal{A} \cup \mathcal{A}^{-1}$, is said to be *reduced* if $x_{k+1} \neq x_k^{-1}$ for each $k \in \{0, \ldots, n-2\}$ and *cyclically reduced* if, in addition, $x_0 \neq x_{n-1}^{-1}$. Every non-identity element $x \in F$ has a unique representation as a reduced word $x = x_0 x_1 \cdots x_{n-1}$ and we set the *word length* $|x|$ of x, by $|x| = n$. We associate to the identity element the empty word and take $|1| = 0$. For each $m \geq 0$, let W_m denote the set of reduced words of length at most m and let W'_m denote the set of reduced words of length precisely m. We let $W^* = \bigcup_{n=0}^{\infty} W'_n$ denote the set of all finite reduced words.

Let $\mathfrak{C}$ be a non-trivial conjugacy class in F and let $k = \inf\{|x| \colon x \in \mathfrak{C}\} > 0$. The set of elements with minimal word length in the conjugacy class is precisely the set of elements with cyclically reduced word representations. In fact, if $g = g_1 \cdots g_k \in \mathfrak{C}$ is cyclically reduced then all cyclically reduced words in $\mathfrak{C}$ are given by cyclic permutations of the letters in $g_1 \cdots g_k$. Let $\mathfrak{C}_n = \{x \in \mathfrak{C} \colon |x| = n\}$ and note that $\mathfrak{C}_n$ is non-empty if and only if $n = k + 2m$. If $x \in \mathfrak{C}_{k+2m}$ then its reduced word representation is of the form $w_m^{-1} \cdots w_1^{-1} g_1 \cdots g_k w_1 \cdots w_m$, for some cyclically reduced $g = g_1 \cdots g_k \in \mathfrak{C}_k$ and $w = w_1 \cdots w_m \in W'_m$ with $w_1 \neq g_1, g_k^{-1}$ in order that no pairwise cancellation occurs when concatenating w^{-1}, g and w. Hence it is convenient to introduce the notation $W'_m(g) = \{w \in W'_m \colon w_1 \neq g_1, g_k^{-1}\}$. A simple calculation shows that the number of elements in $\mathfrak{C}_{k+2m}$ is given by $\#\mathfrak{C}_{k+2m} = (2l-2)(2l-1)^{m-1} \#\mathfrak{C}_k$.

It is useful to consider infinite sequences and dynamics on them. In particular, we associate to each closed geodesics in the graph a periodic orbit of a shift map. We can define a function on both finite and infinite reduced words which will encode the lengths $L(x)$ and which will also give the lengths of closed geodesics by summing the function around the corresponding periodic orbits. Introducing the shift map on infinite sequences also has the advantage of allowing us to use thermodynamic concepts from ergodic theory, for example pressure and equilibrium states defined below, and standard results about differentiating pressure.

We associate to the free group F a subshift of finite type. This subshift of finite type is formed from the space of infinite reduced words (with the obvious definition) adjoined to the elements of W^* together with the dynamics given by the action of the shift map. It will be convenient to describe this space by means of a transition matrix. Define an $l \times l$ matrix A, with rows and columns indexed by $\mathcal{A} \cup \mathcal{A}^{-1}$, by $A(a,b) = 0$ if $b = a^{-1}$ and $A(a,b) = 1$ otherwise. We then define

$$\Sigma = \{(x_n)_{n=0}^{\infty} \in (\mathcal{A} \cup \mathcal{A}^{-1})^{\mathbb{Z}^+} : A(x_n, x_{n+1}) = 1, \forall n \in \mathbb{Z}^+\}.$$

The shift map $\sigma : \Sigma \to \Sigma$ is defined by $(\sigma(x_n)_{n=0}^{\infty}) = (x_{n+1})_{n=0}^{\infty}$. We give $\mathcal{A} \cup \mathcal{A}^{-1}$ the discrete topology, $(\mathcal{A} \cup \mathcal{A}^{-1})^{\mathbb{Z}^+}$ the product topology and Σ the subspace topology; then σ is continuous. Since the matrix A is aperiodic, $\sigma : \Sigma \to \Sigma$ is mixing.

We augment Σ by defining $\Sigma^* = \Sigma \cup W^*$. The shift map naturally extends to a map $\sigma : \Sigma^* \to \Sigma^*$. For a finite reduced word $x_0 x_1 \cdots x_{n-1} \in W^*$, we set $\sigma(x_0 x_1 \cdots x_{n-1}) = x_1 \cdots x_{n-1}$; and for the empty word $\sigma 1 = 1$. It is sometimes useful to think of an element of W^* as an infinite sequence ending in an infinite string of 1s.

We endow Σ^* with the following metric, consistent with the topololgy on Σ. Fix $0 < \theta < 1$ then let $d_\theta(x,x) = 0$ and, for $x \neq y$, let $d_\theta(x,y) = \theta^k$, where $k = \min\{n \in \mathbb{Z}^+ : x_n \neq y_n\}$. For a finite word $x = x_0 x_1 \cdots x_{m-1} \in W'_m$ we take $x_n = 1$ (the empty symbol) for each $n \geq m$. Then $\sigma : \Sigma^* \to \Sigma^*$ is continuous and W^* is a dense subset of Σ^*.

The topological entropy of the system $\sigma : \Sigma \to \Sigma$ is given by $h = \log(2l-1)$ (the logarithm of the largest eigenvalue of the matrix A). Further, μ_0 is characterised by $\mu_0([w]) = (2l)^{-1}(2l-1)^{-(n-1)}$, where, for a reduced word $w = w_0 w_1 \cdots w_{n-1} \in W'_n$, $[w]$ is the associated cylinder set $[w] \subset \Sigma^*$ given by $[w] = \{(x_j)_{j=0}^{\infty} : x_j = w_j,\ j = 0, \ldots, n-1\}$. (Technically, this defines μ_0 as a measure on Σ^* with support equal to Σ.)

Remark 2.2.1. Another subshift of finite type naturally associated to the graph is obtained by taking infinite paths, i.e. infinite sequences of oriented edges with the restriction that e' can follow e only if e terminates at the initial vertex of e'. The advantage of our approach is that is makes it easier to systematically enumerate the elements of a given conjugacy class. On the other hand, it requires more work to represent the edge lengths and we introduce a function that does this below.

2.3 Thermodynamic formalism

Suppose that A is an aperiodic transition matrix. It was shown by Parry (cf. Theorem 8.10, [67]) that there is a unique measure $\mu_0 \in \mathcal{M}(\Sigma_A, \sigma)$ that satisfies the supremum $h_{\mu_0} = \sup\{h_m \colon m \in \mathcal{M}_\sigma\}$. We call μ_0 the *measure of maximal entropy*. Further, the value h_{μ_0} coincides with h the topological entropy of the shift map $\sigma : \Sigma_A \to \Sigma_A$.

For $0 < \theta < 1$, we write $\mathcal{F}_\theta(\Sigma_A, \mathbb{C})$ for the space of complex-valued functions $f : \Sigma_A \to \mathbb{C}$ that are Lipschitz with respect to the metric d_θ. By *Lipschitz* we mean there exists $\kappa > 0$ so that for each $x, y \in \Sigma_A$ we have $|f(x) - f(y)| \leq \kappa d_\theta(x, y)$. We endow $\mathcal{F}_\theta(\Sigma_A, \mathbb{C})$ with the norm $\|\cdot\|_\theta = \|\cdot\|_\infty + |\cdot|_\theta$ where $\|\cdot\|_\infty$ is the usual supremum norm and $|\cdot|_\theta$ is a seminorm such that $|f|_\theta$ is given by

$$|f|_\theta = \sup_{x \neq y} \left\{ \frac{|f(x) - f(y)|}{d_\theta(x, y)} \right\}.$$

The space $\mathcal{F}_\theta(\Sigma_A, \mathbb{C})$ together with the norm $\|\cdot\|_\theta$ is a Banach space. Further, for $0 < \theta < \theta' < 1$ we have the inclusion $\mathcal{F}_\theta(\Sigma_A, \mathbb{C}) \subseteq \mathcal{F}_{\theta'}(\Sigma_A, \mathbb{C})$.

Let $\alpha > 0$, a function $f : \Sigma_A \to \mathbb{C}$ is *α-Hölder* with respect to d_θ if there exists a positive constant $\kappa > 0$ such that for each $x, y \in \Sigma_A$ we have $|f(x) - f(y)| \leq \kappa d_\theta(x, y)^\alpha$. There is no loss of generality in restricting our discussion to Lipschitz functions since if the function $f : \Sigma_A \to \mathbb{C}$ is α-Hölder then $f \in \mathcal{F}_{\theta^\alpha}(\Sigma_A, \mathbb{C})$.

A function $f : \Sigma_A \to \mathbb{C}$ is *locally constant* if there exists $n \geq 1$ such that for all pairs $x, y \in \Sigma_A$ with $x_k = y_k$ for $0 \leq k \leq n$, $f(x) = f(y)$. It is clear that if $f : \Sigma_A \to \mathbb{C}$ is a locally constant function then for all $\theta \in (0, 1)$, $f \in \mathcal{F}_\theta(\Sigma_A, \mathbb{C})$.

We say two functions $f, g \in \mathcal{F}_\theta(\Sigma_A, \mathbb{C})$ are *cohomologous* if there exists a continuous function $u : \Sigma_A \to \mathbb{R}$ such that $f = g + u \circ \sigma - u$. We call a function $f \in \mathcal{F}_\theta(\Sigma_A, \mathbb{C})$ a *coboundary* if f is cohomologous to the zero function. Clearly cohomology is an equivalence relation on $\mathcal{F}_\theta(\Sigma_A, \mathbb{C})$. Let $\mu \in \mathcal{M}(\Sigma_A, \sigma)$ and suppose that $f, g \in \mathcal{F}_\theta(\Sigma_A, \mathbb{C})$ are cohomologous then $\int f \, d\mu = \int g \, d\mu$. We use the following notation for a Birkhoff sum $f^n := f + f \circ \sigma + \cdots + f \circ \sigma^{n-1}$.

The quantity pressure generalises the concept of entropy. Suppose that $f : \Sigma_A \to \mathbb{R}$ is continuous then the *pressure* of f, written $P(f)$, is defined by

$$P(f) = \sup\left\{ h_\mu + \int f \, d\mu \colon \mu \in \mathcal{M}(\Sigma_A, \sigma) \right\}.$$

Any such measure $\mu \in \mathcal{M}(\Sigma_A, \sigma)$ for which the above supremum is attained is called an *equilibrium state* of the function f. If $f \in \mathcal{F}_\theta(\Sigma_A, \mathbb{C})$ then f has a unique equilibrium state μ_f. We see that the measure of maximal entropy μ_0 is the equilibrium state of the zero function. Additionally, $P(0) = h$. It can be shown that two real-valued Hölder functions have the same equilibrium state if and only if they differ by a coboundary and a constant. Thus we can recover a Hölder continuous function (up to the addition of a coboundary and a constant) from its equilibrium state. Let $\mathrm{Fix}_n = \{x \in \Sigma_A \colon \sigma^n x = x\}$. Livsic gave the following criterion for cohomologous functions: two functions $f, g \in \mathcal{F}_\theta(\Sigma_A, \mathbb{C})$ are cohomologous if and only if $f^n(x) = g^n(x)$, for each $x \in \mathrm{Fix}_n$. Suppose that $f, g \in \mathcal{F}_\theta(\Sigma_A, \mathbb{C})$ are real-valued functions and there exists constant c such that f and $g + c$ are cohomologous then $P(f) = P(g) + c$.

We have the following result [44], [60].

Proposition 2.3.1. *Suppose that $f \in \mathcal{F}_\theta(\Sigma_A, \mathbb{C})$ is real-valued. Then, for $t \in \mathbb{R}$, $t \mapsto P(tf)$ is real analytic,*

$$\left.\frac{dP(tf)}{dt}\right|_{t=0} = \int f \, d\mu_0$$

and
$$\left.\frac{d^2P(tf)}{dt^2}\right|_{t=0} = \sigma_f^2 := \lim_{n\to\infty} \frac{1}{n} \int \left(f^n(x) - n \int f \, d\mu_0 \right)^2 d\mu_0.$$
Furthermore, $\sigma_f^2 = 0$ *if and only if* f *is cohomologous to a constant.*

2.4 The transfer operator

In this section we recall an operator, first introduced by Ruelle in [59], that plays a key role in the theory of the thermodynamic formalism. This operator's spectral properties are paramount in this thesis. After introducing the operator, we continue recall further concepts from the thermodynamic formalism.

The operator we recall in this section is called the transfer operator. Suppose that $0 < \theta < 1$ and $f \in \mathcal{F}_\theta(\Sigma_A, \mathbb{C})$. The *transfer operator* $L_f : \mathcal{F}_\theta(\Sigma_A, \mathbb{C}) \to \mathcal{F}_\theta(\Sigma_A, \mathbb{C})$, or more generally $L_f : C(\Sigma_A, \mathbb{C}) \to C(\Sigma_A, \mathbb{C})$, is defined pointwise by

$$L_f\varphi(x) = \sum_{\sigma y = x} \mathrm{e}^{f(y)}\varphi(y).$$

It is straightforward to show that the transfer operator is a bounded linear operator. We recall the following spectral properties of the transfer operator (cf. [44], Theorem 2.2).

Theorem 2.4.1 (Ruelle–Perron–Frobenius Theorem)**.** *Suppose that* $f \in \mathcal{F}_\theta(\Sigma_A, \mathbb{C})$ *is real-valued. Then the operator* $L_f : \mathcal{F}_\theta(\Sigma_A, C) \to \mathcal{F}_\theta(\Sigma_A, \mathbb{C})$ *has a simple positive eigenvalue* β*, associated real-valued eigenfunction* $\psi \in \mathcal{F}_\theta(\Sigma_A, \mathbb{C})$ *that is strictly positive and eigenmeasure* m*. That is,* $L_f\psi = \beta\psi$ *and* $L_f^* m = \beta m$*. The eigenfunction and eigenmeasure are normalised so that* m *is a probability measure and* $\int \psi \, dm = 1$*. The remainder of the spectrum of* L_f *is contained in a disk centred at the origin with radius strictly smaller than* β*. Furthermore, for each* $v \in C(\Sigma_A, \mathbb{R})$ *we have* $\beta^{-n} L_f^n v \to \psi \int v \, dm$ *uniformly as* $n \to \infty$*.*

For $f \in \mathcal{F}_\theta(\Sigma_A, \mathbb{C})$, the next theorem gives upper bounds for the spectral radius of L_f, denoted by $\mathrm{spr}(L_f)$.

Theorem 2.4.2 (cf. Theorem 4.5, [44])**.** *Suppose that* $f = u + \mathrm{i}v$ *where* $u, v \in \mathcal{F}_\theta(\Sigma_A, \mathbb{C})$ *are real-valued. Then* $\mathrm{spr}(L_f) \leq \mathrm{e}^{P(u)}$*. If* L_f *has a simple eigenvalue of modulus* $\mathrm{e}^{P(u)}$ *then* $L_f = \alpha M L_u M^{-1}$*. Here* M *is the multiplication operator* $M(v) = \psi v$ *and* $\alpha \in \mathbb{C}$ *with* $|\alpha| = 1$*. Further, the remainder of the spectrum is contained within a disk of radius strictly smaller than* $\mathrm{e}^{P(u)}$ *centred at the origin. If* L_f *does not have an eigenvalue of modulus* $\mathrm{e}^{P(u)}$ *then* $\mathrm{spr}(L_f) < \mathrm{e}^{P(u)}$.

Let $f \in \mathcal{F}_\theta(\Sigma_A, \mathbb{C})$ be real-valued, in this thesis we frequently use the following operator decomposition that follows as a consequence of the spectral gap exhibited by the quasi-compact operator L_f. Let R_f denote the projection of L_f onto the eigenspace spanned by ψ, the eigenfunction associated to the maximal eigenvalue β, and $Q_f = L_f - \beta R_f$. Recall from the Ruelle–Perron–Frobenius Theorem that, given $v \in C(\Sigma_A, \mathbb{R})$ and m the normalised eigenmeasure associated to β, the convergence $\beta^{-n} L_f^n v \to \psi \int v\, dm$ is uniform as $n \to \infty$. Under the additional assumption that $v \in \mathcal{F}_\theta(\Sigma_A, \mathbb{C})$ is real-valued we have $\|\beta^{-n} Q_f^n v\| \to 0$ as $n \to \infty$. Thus for real-valued $v \in \mathcal{F}_\theta(\Sigma_A, \mathbb{C})$ we have

$$\beta^{-n} L_f^n v = R_f^n v + \beta^{-n} Q_f^n v \to R_f v$$

as $n \to \infty$. It follows that $R_f v = \psi \int v\, dm$. Since the set of functions $\mathcal{F}_\theta(\Sigma_A, \mathbb{C})$ is dense in the set of continuous functions, we have $R_f v = \psi \int v\, dm$ for each $v \in C(\Sigma_A, \mathbb{R})$.

Suppose that $f \in \mathcal{F}_\theta(\Sigma_A, \mathbb{C})$ is real-valued and let β be the simple maximal eigenvalue of the operator L_f. By the variational principle ([44], Theorem 3.5) we have $\beta = \mathrm{e}^{P(f)}$.

Let ψ be the strictly positive eigenfunction of L_f associated to the eigenvalue $\mathrm{e}^{P(f)}$. Consider $f' = f - P(f) + u - u \circ \sigma$ where $u = \log \psi$. Then f' is *normalised* so that $L_{f'} 1 = 1$, which consequently means $P(f') = 0$. Clearly f and f' have the same equilibrium state m.

Suppose that $f, g \in \mathcal{F}_\theta(\Sigma_A, \mathbb{C})$ are real-valued functions. We shall consider small perturbations of the operator $L_g : \mathcal{F}_\theta(\Sigma_A, \mathbb{C}) \to \mathcal{F}_\theta(\Sigma_A, \mathbb{C})$ of the form L_{g+sf}

for values of $s \in \mathbb{C}$. Importantly, for small perturbations, where the parameter $s \in \mathbb{C}$ is in a neighbourhood of the origin, the simple maximal eigenvalue of the operator L_g persists. We make the following spectral deductions for the operator L_{g+sf}, with $s \in \mathbb{C}$, from the perturbation theory for linear operators (we refer the reader to [32] for the general theory). For small perturbations, by which we mean for s close to the origin, the simple maximal eigenvalue persists. That is, the operator L_{g+sf} has a simple eigenvalue $\beta(s)$ and corresponding eigenfunction ψ_s that vary analytically with s so that $\beta(0) = \beta$ and $\psi_0 = \psi$. Furthermore, due to the upper semi-continuity of the spectral radius, L_{g+sf} has a spectral gap: there is an $\varepsilon > 0$ such that, for $s \in \mathbb{C}$ in a neighbourhood of the origin, the remainder of the spectrum of L_{g+sf} is contained in a disk of radius $\mathrm{e}^{P(g)-\varepsilon}$ centred at the origin.

We extend the definition of the pressure function for complex valued functions of the form $g + sf$. We define $P(g + sf) = \log \beta(s)$ with the requirement that $P(g + sf)$ is real-valued when $s \in \mathbb{R}$. We note $P(g + sf)$ varies analytically in s. In summary,

Proposition 2.4.3 (Proposition 4.8, [44]). *The domain of the pressure function in $\mathcal{F}_\theta(\Sigma_A, \mathbb{C})$ is open and the function $f \mapsto P(f)$ is analytic from this domain into $\mathbb{C}$.*

We find it useful to consider $\sigma : \Sigma^* \to \Sigma^*$ as a subshift of finite type and will use the previous notation and concepts introduced for Σ in this setting. We modify the definition of the transfer operator $L_{sf} : \mathcal{F}_\theta(\Sigma^*, \mathbb{C}) \to \mathcal{F}_\theta(\Sigma^*, \mathbb{C})$ as follows:

$$L_{sf}\omega(x) = \sum_{\substack{\sigma y = x \\ y \neq 1}} e^{sf(y)}\omega(y).$$

Here 1 denotes the identity element in F, considered as an infinite word $(1, 1, \ldots)$. We note the transfer operator we use differs from the usual definition by excluding the preimage $y = 1$ from the summation over the set $\{y \in \Sigma^* : \sigma y = x\}$; however, the definition of this transfer operator agrees with our previous definition for each $x \neq 1$. Following Lemma 2 of [54], $L_{sf} : \mathcal{F}_\theta(\Sigma^*, \mathbb{C}) \to \mathcal{F}_\theta(\Sigma^*, \mathbb{C})$ has the same isolated eigenvalues as $L_{sf} : \mathcal{F}_\theta(\Sigma \cup \{1\}, \mathbb{C}) \to \mathcal{F}_\theta(\Sigma \cup \{1\}, \mathbb{C})$. Since the modified definition of L_{sf} excludes the eigenvalue $e^{sf(1)}$ associated to the eigenfunction

$\chi_{\{1\}}$ (the indicator function of the set $\{1\}$), $L_{sf} : \mathcal{F}_\theta(\Sigma^*, \mathbb{C}) \to \mathcal{F}_\theta(\Sigma^*, \mathbb{C})$ therefore has the same isolated eigenvalues as $L_{sf} : \mathcal{F}_\theta(\Sigma, \mathbb{C}) \to \mathcal{F}_\theta(\Sigma, \mathbb{C})$. Furthermore, again by Lemma 2 of [54], $L_{sf} : \mathcal{F}_\theta(\Sigma^*, \mathbb{C}) \to \mathcal{F}_\theta(\Sigma^*, \mathbb{C})$ is quasi-compact with essential spectral radius at most $\theta e^{P(\mathrm{Re}(s)f)}$, and so it suffices to consider the spectral theory of L_{sf} on $\mathcal{F}_\theta(\Sigma, \mathbb{C})$.

The spectral properties of the transfer operator discussed in this section are infinite dimensional analogues of the following spectral results for matrix operators acting on finite dimensional vector spaces.

Theorem 2.4.4 (Perron–Frobenius Theorem [17]). *Suppose that A is an aperiodic matrix with non-negative entries. Then A has a simple and positive eigenvalue β such that β is strictly greater in modulus than all the remaining eigenvalues of A. The left and right eigenvectors associated to the eigenvalue β have strictly positive entries. Moreover, β is the only eigenvalue of A that has an eigenvector whose entries are all non-negative.*

Theorem 2.4.5 (Wielandt's Theorem [17]). *Suppose that A is a square matrix with complex entries and let $|A|$ be the matrix whose entries are given by $|A|(s, s') = |A(s, s')|$. Suppose further that $|A|$ is aperiodic and let β be its maximal eigenvalue guaranteed by the Perron–Frobenius theorem. Then the moduli of the eigenvalues of A are bounded above by β. Moreover, A has an eigenvalue of the form $\beta \mathrm{e}^{i\theta}$ (with $\theta \in [0, 2\pi]$) if and only if $A = \mathrm{e}^{i\theta} D|A|D^{-1}$ where D is a diagonal matrix whose entries along the main diagonal all have modulus one.*

2.5 Graph theory

In this section we introduce the terminology for graphs and also recall related fundamental concepts from algebraic topology.

A *graph* G consists of a set of edges E and a set of vertices V. Each edge has two endpoints, which can coincide, and the vertex set V is given by the set of these endpoints. A graph is *finite* if the cardinalities of both E and V are finite.

We say an edge *joins* the vertices associated to its endpoints. Two vertices joined by an edge are said to be *adjacent.* An edge that joins a vertex to itself is called a *loop.* We say an edge and vertex are *incident* if the vertex is an endpoint of the edge. For $v \in V$ the *degree* of v, denoted by $\deg(v)$, is given by the number of edges incident to v and note that we count each loop incident to v twice.

We draw a graph in the plane by plotting a point to represent each vertex and for each edge we draw a line or curve that connects its endpoints. We *traverse* an edge by tracing out the line associated to the edge. We can traverse each edge in two directions and thus to each edge we associate a pair of oriented edges e and $\overline{e}$. Here we use the notation $\overline{e}$ for the edge e with orientation reversed. We denote by E^o the set of oriented edges. When traversing $e \in E^o$, we call the vertices we travel to and from the *terminal* and *initial* vertices, respectively.

A *path* $e_0, e_1, \ldots, e_{n-1}$ is a sequence of oriented edges such that for each $k \in \{0, 1, \ldots, n-2\}$ the terminal vertex of e_k and initial vertex of e_{k+1} coincide. A path $e_0, e_1, \ldots, e_{n-1}$ is *non-backtracking* if $e_{k+1} \neq \overline{e_k}$ for each $k \in \{0, 1, \ldots, n-2\}$. A path $e_0, e_1, \ldots, e_{n-1}$ is *closed* if the initial vertex of e_0 and the terminal vertex of e_{n-1} coincide. A graph G is *connected* if between each pair of vertices $v, w \in V$ there exists a path $e_0, e_1, \ldots, e_{n-1}$ such that v is the initial vertex of e_0 and w is the terminal vertex of e_{n-1}.

Let $e_0, e_1, \ldots, e_{n-1}$ be a closed non-backtracking path with the additional condition $e_{n-1} \neq \overline{e_0}$. We call the set of paths given by cyclic permutations of the edges in $e_0, e_1, \ldots, e_{n-1}$ a *closed geodesic.* In the work that follows when referring to a closed geodesic we often implicitly mean a representative path in the set of cyclic permutations. It is important we distinguish between *closed non-backtracking paths* and *closed geodesics.* Explicitly, let $\gamma = e_0, e_1, \ldots, e_{n-1}$ be a closed path and write $\gamma^m = (e_0, e_1, \ldots, e_{n-1})^m$ with $m \in \mathbb{Z}^+$ denote the m-fold concatenation of γ. If γ is a closed geodesic then γ^m is too. However, if γ is only a closed non-backtracking path then γ^m with $m \geq 2$ is closed but no longer non-backtracking.

We make G into a metric graph by assigning to each edge $e \in E^o$ a positive length, sometimes *metric length,* $l(e)$ with the requirement that $l(e) = l(\overline{e})$. Thus we identify to each edge in the graph an isometric copy of a real interval. The length of a path $l(e_0, e_1, \ldots, e_{n-1})$ is given by the sum of the lengths of its constituent edges.

We recall basic notions from algebraic topology in the context of graphs. Two paths that share common endpoints in a topological space are *homotopic* if there is continuous deformation between the paths. In fact, homotopy defines a natural equivalence class on the set of paths. We write $\pi_1(G, x_0)$ for the set of all homotopy classes of closed paths in a graph G based at x_0. The set $\pi_1(G, x_0)$ together with the binary operation given by concatenating closed paths based at x_0 forms a group. Let G be a connected graph and suppose that $x_0, x_1 \in G$, then the groups $\pi_1(G, x_0)$ and $\pi_1(G, x_1)$ are isomorphic. Thus for a connected graph G we refer to the *fundamental group* $\pi_1(G)$ of G without reference to a basepoint. A *tree* is an acyclic connected graph. An *acyclic* graph has neither loops nor closed non-backtracking paths. Thus a tree is a contractible graph. It follows that the fundamental group of any tree is trivial.

A *subgraph* $H = (V_H, E_H)$ of a graph $G = (V, E)$ is a graph such that E_H is a subset of E and V_H is the set of incident vertices to the edges in E_H. A subgraph T of G is a *maximal tree* (of G) if T is a tree and each vertex in G is a vertex in T. Every connected graph contains a maximal tree (cf. Proposition 1A.1, [21]).

Proposition 2.5.1 (Proposition 1A.2, [21]). *Let G be a finite connected graph with maximal tree T. Then the fundamental group $\pi_1(G)$ of G is a free group. A generating set of $\pi_1(G)$ corresponds to the unoriented edges in $G \setminus T$.*

It is immediate from Proposition 2.5.1 that if we assume the degree of each vertex in a finite connected graph G is at least 3 then the generating set of the fundamental group of G contains at least 2 elements.

A group F is said to act on the space X if there is a map $\varphi : F \times X \to X$ such that for each $x \in X$ the following two conditions hold: $\varphi(e, x) = x$ where e is the identity element of F and, a compatibility condition, for each $g_1, g_2 \in G$ we have $\varphi(g_1 g_2, x) = \varphi(g_1, \varphi(g_2, x))$. We say a group action $\varphi : F \times X \to X$ is *free*, alternatively the *group acts freely* on X, if for each $x \in X$, $\varphi(g, x) = x$ implies that g is the identity element in F. Clearly, when F acts freely on a space X the *isotropy subgroup* of x, the subgroup of F that fixes x, is trivial. If X is a space endowed with a metric d_X then the group F acts *isometrically* on X if for each pair $x, y \in X$ and $g \in F$ we have $d_X(\varphi(g,x), \varphi(g,y)) = d_X(x,y)$. We use the notation $\varphi(g,x) = gx$ for the action of the group F.

In the chapters that follow, we concern ourselves with graphs that are finite and connected such that the degree of each vertex is at least 3. This condition on the vertices ensures that the fundamental group of G is a free group F on $l \geq 2$ generators and that the universal cover is an infinite tree $\mathcal{T}$. We put a metric on G by assigning a positive length to each edge and this lifts to a metric on $\mathcal{T}$. We define the function $L : F \to \mathbb{R}$ given by $L(x) = d_{\mathcal{T}}(o, xo)$ for each $x \in F$ and a prescribed base point $o \in \mathcal{T}$. We define a *Gromov product* (cf. pg.89 [19]) $(\cdot,\cdot)_L : F \times F \to \mathbb{R}$ as follows. For the pair $x, y \in F$ the value $(x,y)_L$ is given by

$$(x,y)_L = (L(x) + L(y) - L(x^{-1}y))/2.$$

The function $L : F \to \mathbb{R}$ satisfies the following axioms [38]. Suppose that $x, y, z \in F$ we have

(A1) $L(x) = 0$ if and only if $x = 1$;

(A2) $L(x) = L(x^{-1})$;

(A3) $(x,y)_L \geq 0$;

(A4) if $(x,y)_L < (x,z)_L$ then $(y,z)_L = (x,y)_L$; and

(A5) if $(x,y)_L + (x^{-1},y^{-1})_L > L(x) = L(y)$ then $x = y$.

A function that satisfies the above list of axioms is called a Lyndon length function. Since the path between any two distinct points on $\mathcal{T}$ is unique the function $L : F \to \mathbb{R}$ satisfies the additional *Archimedean property*

(A6) if $x \neq 1$ then $L(x^2) > L(x)$.

It is easily shown that there exist positive constants $a, A \in \mathbb{R}$ such that for each $x \in F$, $a|x| \leq L(x) \leq A|x|$. We define a second Gromov product as a measure of discrepancy between pairs of elements in F. For each pair $x, y \in F$ we define a product (x,y) such that $(x,y) = (|x| + |y| - |x^{-1}y|)/2$.

A *quasi-isometry* (cf. [18]) is a map $f : X \to Y$ between metric spaces

(X, d_X) and (Y, d_Y) such that for each pair $x_1, x_2 \in X$ there exist positive constants $\lambda, c \in \mathbb{R}$ with

$$\lambda^{-1} d_X(x_1, x_2) - c \leq d_Y(f(x_1), f(x_2)) \leq \lambda d_X(x_1, x_2) + c.$$

Let us equip F with the word metric $d_F(x, y) = |x^{-1}y|$. The map $f : F \to \mathcal{T}$ given by $f(x) = xo$ for a prescribed basepoint $o \in \mathcal{T}$ is a quasi-isometry. The following inequality, which follows from the fact that $f : F \to \mathcal{T}$ is a quasi-isometry (see Lemma 2.6.1) and Proposition 15 in [18], describes the bounded distortion between the two elements. For each pair of elements $x, y \in F$ there exist positive constants $b, B, K \in \mathbb{R}$ such that

$$b(x, y) - K \leq (x, y)_L \leq B(x, y) + K. \tag{2.5.1}$$

In our symbolic approach for the asymptotics results related to metric graphs we will frequently use the following lemma from [65] (Lemma 3.1) to construct a function that, when summed over periodic orbits in a subshift of finite type, encodes the set of lengths of closed geodesic in a metric graph. We state the lemma and reproduce the proof.

Lemma 2.5.2. *Let F be a free group on $l \geq 2$ generators and recall the function $L : F \to \mathbb{R}$. Then there exists $N \in \mathbb{N}$ such that if $x_0 x_1 \cdots x_{n-1}$ is a reduced word with $n \geq N$ we have*

$$L(x_0 x_1 \cdots x_{n-1}) - L(x_1 \cdots x_{n-1}) = L(x_0 x_1 \cdots x_{N-1}) - L(x_1 \cdots x_{N-1}).$$

Proof. For brevity, let us take $x = x_0 x_1 \cdots x_{n-1}$ and $y = x_0 x_1 \cdots x_{N-1}$. Then the required equation becomes $L(x) - L(x_0^{-1} x) = L(y) - L(x_0^{-1} y)$. Our first aim will be to rewrite this equation in terms of the Gromov product $(\cdot, \cdot)_L$. We add $L(x_0)$ to each side of the equation to obtain

$$L(x_0) + L(x) - L(x_0^{-1} x) = L(x_0) + L(y) - L(x_0^{-1} y).$$

By (A2), this is equivalent to the equation

$$L(x_0^{-1}) + L(x^{-1}) - L(x^{-1}(x_0^{-1})^{-1}) = L(x_0^{-1}) + L(y^{-1}) - L(y^{-1}(x_0^{-1})^{-1}).$$

Then, from the definition of the Gromov product $(\cdot,\cdot)_L$, we require

$$(x^{-1}, x_0^{-1})_L = (y^{-1}, x_0^{-1})_L.$$

Axiom (A4) gives a sufficient condition to prove the lemma: there exists $N \in \mathbb{N}$ such that if $n > N$ then $(x^{-1}, x_0^{-1})_L < (x^{-1}, y^{-1})_L$.

We calculate the values of $2(x^{-1}, y^{-1}) = n + N - (n - N) = 2N$ and $2(x^{-1}, x_0^{-1}) = n + 1 - (n - 1) = 2$. Then, by inequality 2.5.1, $(x^{-1}, y^{-1})_L \geq b(x^{-1}, y^{-1}) - K = bN - K$. Hence to prove the lemma, we need only choose N sufficiently large in order that $bN - K \geq (x^{-1}, x_0^{-1})_L$. Rearranging this inequality and substituting for $(x^{-1}, x_0^{-1})_L$ and (x^{-1}, x_0^{-1}) we have

$$\begin{aligned} N = (x^{-1}, y^{-1}) &> b^{-1}(x^{-1}, x_0^{-1})_L + b^{-1}K \\ &\geq b^{-1}(B(x^{-1}, x_0^{-1}) + K) + b^{-1}K \\ &= b^{-1}B + 2b^{-1}K \end{aligned}$$

and we are done. □

In the orbit counting asymptotics we establish we shall assume that there exists a pair of closed geodesics $\gamma_1, \gamma_2 \in G$ such that the ratio of the lengths $l(\gamma_1)/l(\gamma_2)$ is irrational. We note that this assumption is equivalent to the condition that the set of lengths of closed geodesics is not contained in a discrete subgroup of $\mathbb{R}$.

Remark 2.5.3. This irrationality assumption is critical in our proof of Theorem 3.1.1; moreover, in the next paragraph we explain why it is not possible to establish asymptotic orbit counting results such as $N(T) = \{x \in F \colon L(x) \leq T\}$ of the form $N(T) \sim Ce^{hT}$ as $T \to \infty$ if the lengths of the closed geodesics are contained in a discrete subgroup of $\mathbb{R}$.

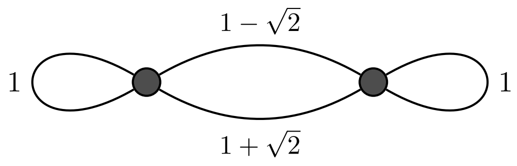

Figure 2.1: metric graph with edge lengths in $\{1, 1-\sqrt{2}, 1+\sqrt{2}\}$

Recall that every discrete subgroup of $\mathbb{R}$ is either trivial, or an infinite cyclic group, i.e. a group of the form $m\mathbb{Z}$ where m is the smallest positive real number in the group. Clearly the set of lengths is not contained in the trivial group. Let $N(T)$ count the number of closed geodesics in the graph with metric length at most T. If the set of lengths of closed geodesics is contained in the group $m\mathbb{Z}$ then $N(T)$ is constant on intervals $[mn, m(n+1))$ for $n \in \mathbb{Z}$ and so we cannot have $N(T) \sim C\mathrm{e}^{hT}$ as $T \to \infty$.

We note we cannot replace the irrationality condition for closed geodesics with the assumption that there exists a pair of edges $e, e' \in G$ such that the ratio of their lengths $l(e)/l(e')$ is irrational. For example, consider the graph pictured in Figure 2.1 with metric edge lengths in $\{1, 1-\sqrt{2}, 1+\sqrt{2}\}$. Although we can choose a pair of edges such that the ratio of their lengths is irrational, the set of closed geodesics has lengths in $\mathbb{Z}$ and so we cannot establish an asymptotic result with the desired form.

2.6 Geometric group theory

Geometric group theory studies groups as automorphisms of geometric spaces. For example, the action of a group on its Cayley graph. Gromov's seminal paper [19] brought much attention to this research area. In this section we recall concepts from geometric group theory.

Let us fix a base point o in the metric space (X, d_X). We define $(x, y)_o$, the *Gromov product with respect to o*, for two points $x, y \in X$ by

$$(x, y)_o = (d_X(o, x) + d_X(o, y) - d_X(x, y))/2.$$

A metric space (X, d_X) is called *δ-hyperbolic with respect to o* with $\delta \geq 0$ if for each $x, y, z \in X$

$$(x, y)_o \geq \min((x, z)_o, (y, z)_o) - \delta.$$

A metric space is called *δ-hyperbolic* if it is δ-hyperbolic with respect to each point $x \in X$.

Let (X, d_X) be a metric space. A *geodesic segment between two points $x, y \in X$*, written $[x, y]$, is the image of an isometry $\gamma : [0, d_X(x, y)] \to (X, d_X)$ so that $\gamma(0) = x$ and $\gamma(d_X(x, y)) = y$. We note geodesic segments in metric spaces are not necessarily unique. For $t \in [0, 1]$, we denote by $t[x, y]$ the image of $td_X(x, y)$ under the map γ. A metric space (X, d_X) is called *geodesic* if between any two points $x, y \in X$ there is a geodesic segment $[x, y]$. A metric space is *proper* if all closed balls in X are compact.

Gromov introduced the concept of hyperbolicity for geodesic metric spaces in [19]. Perhaps the most intuitive feature of a hyperbolic space is that its geodesic triangles are all 'thin'. Hyperbolic metric spaces establish attributes of negatively curved manifolds in the less restrictive setting of metric spaces. Indeed, it is well known that every complete Riemannian manifold is geodesic.

In a geodesic metric space a *geodesic triangle* Δ is the union of three geodesic segments such that each pairwise intersection contains a single common endpoint. We refer to these geodesic segments as the *sides* of Δ. We call Δ a *δ-thin* triangle if each side of Δ is entirely contained in the δ-neighbourhoods of the other two sides. If each geodesic triangle in Δ is δ-thin then X is called *δ-hyperbolic.* A *hyperbolic metric space* is any metric space that is δ-hyperbolic for some $\delta \geq 0$. Examples of δ-hyperbolic spaces are metric trees. We see that any metric tree is 0-hyperbolic since each geodesic triangle is a tripod.

For a hyperbolic metric space X we consider its boundary set. A sequence $(x_k)_{k=1}^{\infty}$ of points in X is *convergent at infinity* if $(x_m, x_n)_o \to \infty$ as $m, n \to \infty$. We note that this convergence is independent of the choice of base point $o \in X$ and there is a natural equivalence relation on the sequences that are convergent at infinity; we say two sequences $(x_k)_{k=1}^{\infty}, (y_k)_{k=1}^{\infty}$ convergent at infinity are equivalent if

$$\liminf_{m,n\to\infty}(x_m, y_n)_o = \infty.$$

The *hyperbolic boundary* ∂X of a hyperbolic space X is the set of equivalence classes of sequences in X convergent at infinity. For a proper geodesic metric space, this notion of boundary coincides with the *geodesic boundary* (cf. [30])

$$\partial^g X = \{[\gamma]\colon \gamma : [0,\infty) \to X \text{ is a geodesic ray in } X\}.$$

Suppose that $\mathcal{T}$ is the universal covering tree of a finite connected metric graph. To describe the boundary $\partial\mathcal{T}$ it is sufficient to consider the endpoints of the geodesic rays emanating from a single prescribed vertex.

For a geodesic triangle Δ in a geodesic space X, a *comparison triangle* Δ' is a geodesic triangle in a complete simply connected surface M with constant curvature $\kappa \le 0$ together with an isometry $f_\Delta : \Delta \to \Delta'$ that sends each side of Δ onto a side of Δ'. We say triangle Δ is CAT(κ) if for each $x, y \in \Delta$ we have the CAT(κ)-*inequality*

$$d_X(x, y) \le d_M(f_\Delta(x), f_\Delta(y)).$$

Thus triangle Δ is at least as 'thin' as its comparison triangle Δ'. Every CAT(κ) space with $\kappa < 0$ is hyperbolic. Clearly if a metric space M is CAT(κ) then M is also CAT(κ') for $\kappa' > \kappa$. We note the model surface M is the Euclidean plane when $\kappa = 0$ and for $\kappa < 0$ the surface is the hyperbolic plane with the appropriate scaling. Because the geodesic triangles in any metric tree are degenerate, we have that every metric tree is CAT(κ) for each $\kappa < 0$.

Suppose that F is a finitely generated group with a finite generating set S. The *Cayley graph* X_S of F with respect to the generating set S is the graph with vertex set F and the set E consists of edges connecting x to xs for each $x \in F$ and $s \in S$. We assign to each edge in the graph metric length 1 (isometric copies of the Euclidean unit interval) and let the distance between two points in X_S be given by the length of the shortest path between the two points. Endowed with this metric X_S is a geodesic metric space. The group F inherits a subspace metric from X_S, a *word metric*, that induces a norm given by $|x|_S = d_S(1, x)$ (in agreement with the previously defined *word length*) for each $x \in F$. Thus for $x, y \in F$, $d_S(x, y) = |y^{-1}x|_S = |x^{-1}y|_S$.

Lemma 2.6.1 (Schwarz Lemma (cf. [7]))**.** *Let G be a group of isometries acting properly discontinuously and co-compactly on a proper geodesic metric space (X, d_X). Then G has a finite generating set S, and the map $f : (G, d_S) \to (X, d_X)$ given by $f_x : g \mapsto gx$ for each $x \in X$ is a quasi-isometry.*

We note the above construction of a word metric on F depends on the choice of generating set S. However, some local structure is preserved. Let S, T be finite generating sets of a group F. Then the identity map from X_S to X_T is a quasi-isometry. Moreover, suppose that X_S is δ-hyperbolic for some $\delta \geq 0$ then X_T is δ'-hyperbolic for some $\delta' \geq 0$ because hyperbolicity is a quasi-isometric invariant. Thus there is a natural definition of hyperbolicity for a finitely generated group. A finitely generated group F is a *hyperbolic group* if X_S is δ-hyperbolic for some $\delta \geq 0$ and finite generating set S (and therefore any finite generating set). For example, free groups with finite rank are hyperbolic because their Cayley graphs with respect to free generating sets are trees, which are δ-hyperbolic. In addition, all finite groups are hyperbolic. If F is hyperbolic, then X_S is a proper geodesic metric space. For a hyperbolic group F, we define the group's *boundary*, or *limit set*, by $\partial F = \partial X_S$. We note that the topology on ∂F does not depend on the choice of finite generating set since any two Cayley graphs associated to finite generating sets are quasi-isometric.

Suppose that the action of a group F on a geodesic space X is isometric, co-compact and properly discontinuous. Then we say the action of F on X is *geometric*. We attribute the next theorem, which follows from Lemma 2.6.1, to Gromov [19].

Theorem 2.6.2. *A group F is hyperbolic if and only if it has a geometric action on a proper geodesic metric space (X, d_X). In addition, in this case ∂X and ∂F are homeomorphic.*

Chapter 3

Orbit counting in conjugacy classes for free groups acting on trees

3.1 Introduction

Let G be a finite connected graph. We always assume that each vertex of G has degree at least 3, in which case the fundamental group of G is a free group F on $k \geq 2$ generators. We make G into a metric graph by assigning to each edge e a positive real length $l(e)$. The length of a path in G is given by the sum of the lengths of its edges. We assume the set of closed geodesics in G (i.e. closed paths without backtracking) has lengths not contained in a discrete subgroup of $\mathbb{R}$.

The universal cover of G is an infinite tree $\mathcal{T}$ and the metric on G lifts to a metric $d_{\mathcal{T}}$ on $\mathcal{T}$. We consider each edge in $\mathcal{T}$ as an isometric copy of a real interval. Then the ball of radius T centred at $o \in \mathcal{T}$ is the set

$$B(o,T) = \{y \in \mathcal{T} \colon d_{\mathcal{T}}(o,y) < T\}.$$

The *volume* of $B(o,T)$ is the sum of the metric edge lengths in $B(o,T)$. Let $h > 0$

denote the *volume entropy* of $\mathcal{T}$ given by

$$\lim_{T\to\infty} \frac{1}{T} \log \mathrm{vol}(B(o,T)).$$

We note the volume entropy is independent of the choice of $o \in \mathcal{T}$.

Let $x \in F$ and fix a base vertex $o \in \mathcal{T}$. We define $L : F \to \mathbb{R}^+$ by $L(x) = d_{\mathcal{T}}(o, xo)$. Guillopé [20] showed that $\#\{x \in F \colon L(x) \leq T\} \sim c\mathrm{e}^{hT}$ as $T \to \infty$, for some $c > 0$. Here $f(T) \sim g(T)$ means that $f(T)/g(T) \to 1$ as $T \to \infty$.

Let $\mathfrak{C}$ be a non-trivial conjugacy class in F. Then $\mathfrak{C}$ is infinite and it is interesting to study the restriction of the above counting problem to this conjugacy class, i.e. to study the asymptotic behaviour of $N_{\mathfrak{C}}(T) := \#\{x \in \mathfrak{C} \colon L(x) \leq T\}$. The following is our main result in this chapter.

Theorem 3.1.1. *Suppose that G is a finite connected metric graph such that the degree of each vertex is at least 3 and the set of lengths of closed geodesics in G is not contained in a discrete subgroup of $\mathbb{R}$. Let $\mathfrak{C}$ be a non-trivial conjugacy class in F. Then, for some constant $C > 0$, depending on $\mathfrak{C}$, as $T \to \infty$ we have $N_{\mathfrak{C}}(T) \sim C\mathrm{e}^{hT/2}$.*

We can also obtain a polynomial error term in our approximation to $N_{\mathfrak{C}}(T)$ subject to a mild additional condition on the lengths of closed geodesics (see Theorem 3.4.2).

In the case of a co-compact group of isometries of the hyperbolic plane, an analogue of Theorem 3.1.1 was obtained by Huber [26] in the 1960s. If Γ is a co-compact Fuchsian group and $\mathfrak{C}$ is a non-trivial conjugacy class, he showed that as $T \to \infty$

$$\#\{g \in \mathfrak{C} :\ d_{\mathbb{H}^2}(o, go) \leq T\} \sim C\mathrm{e}^{T/2},$$

for some $C > 0$ depending on Γ and $\mathfrak{C}$ (while the unrestricted counting function $\#\{g \in \Gamma : d_{\mathbb{H}^2}(o, go) \leq T\}$ is asymptotic to a constant times e^T). Very recently, Parkkonen and Paulin [42] have studied the same problem in higher dimensions and variable curvature, obtaining many results. In particular, they have shown that for the fundamental group of a compact negatively curved manifold acting on its universal cover, the conjugacy counting function has exponential growth rate equal to $h/2$, where h is the topological entropy of the geodesic flow. They have an ergodic-geometric approach using, in particular, the mixing properties of the Bowen–Margulis measure.

More closely related to our situation, suppose that G is a $(q+1)$-regular graph (i.e. each vertex has degree $q+1$) with each edge given length 1. Then Douma [14] showed that $N_{\mathfrak{C}}(n) \sim Cq^{\lfloor (n-l(\mathfrak{C}))/2 \rfloor}$ as $n \to \infty$ (for $n \in \mathbb{Z}^+$), for some $C > 0$, where $l(\mathfrak{C})$ is the length of the closed geodesic in the conjugacy class $\mathfrak{C}$.

The orbit counting results in this chapter appear in the paper [33]; after this paper was submitted for publication Broise-Alamichel, Parkkonen and Paulin uploaded a preprint [5] to the arXiv that discusses orbit counting in conjugacy classes in the more general situation of graphs of groups in the sense of Bass–Serre theory. They established asymptotic formulae of the form $\sim C(\mathfrak{C})\mathrm{e}^{hT/2}$ as $T \to \infty$ and in certain specialisations an error term estimate. One of these special cases includes the fundamental group of a finite connected metric graph acting on the graph's universal covering tree.

In contrast to the ergodic-geometric approach of Parkkonen and Paulin in [42] or the use of spectral theory of the graph Laplacian in [14] (which is inspired by Huber's original spectral approach [26]), we use a method based on a symbolic coding of the group F in terms of the subshift of finite type Σ^*. We may then study a generating function via the spectra of a family of matrices. In the next section, we describe how the lengths on the graph may be encoded in terms of a function on our subshift and use this function to define a family of matrices and sketch a proof of Guillopé's result given above. Next, we introduce a generating function appropriate to our problem and carry out an analysis which leads to the proof of Theorem 3.1.1. In the final section we discuss error terms.

3.2 Length functions, matrices and spectra

We will prove Theorem 3.1.1 by studying the analytic properties of a generating function $\sum_{x\in\mathfrak{C}} \mathrm{e}^{-sL(x)}$. More precisely, we will show that the generating function is analytic in the half-plane $\mathrm{Re}(s) > h/2$, has a simple pole at $s = h/2$ and, crucially, apart from this pole has an analytic extension to a neighbourhood of $\mathrm{Re}(s) = h/2$. To do this, we will show that the generating function can essentially be written in terms of a family of weighted matrices and their eigenvectors. The required analytic properties will then follow from results about the spectra of these matrices. In turn, the key spectral property (see Lemma 3.2.7 below) is a consequence of the hypothesis that the lengths of closed geodesics in our metric graph do not lie in a discrete subgroup of the real numbers.

In this section, we will set up the machinery required to study the generating function. We first introduce a function defined on the set of (finite and infinite) reduced words which encodes information about lengths on the graph. (This is similar to the constructions in [64] and [65].) We will then introduce our weighted matrices and establish some of their properties.

Definition 3.2.1. We define a function $r : \Sigma \to \mathbb{R}$ by

$$r((x_i)_{i=0}^{\infty}) = L(x_0 x_1 \cdots x_{N-1}) - L(x_1 \cdots x_{N-1}).$$

We also define $r : W^* \to \mathbb{R}$ by $r(1) = 0$ and, for $n \geq 1$,

$$r(x_0 \cdots x_{n-1}) = L(x_0 \cdots x_{n-1}) - L(x_1 \cdots x_{n-1}).$$

Note that, by Lemma 2.5.2, if $n \geq N$ then

$$r(x_0 \cdots x_{n-1}) = L(x_0 \cdots x_{N-1}) - L(x_1 \cdots x_{N-1}).$$

We may also extend the definition of the shift map σ to finite reduced words by defining $\sigma : W^* \to W^*$ by

$$\sigma(x_0 x_1 \cdots x_{n-1}) = (x_1 \cdots x_{n-1})$$

and $\sigma 1 = 1$. We shall continue to write $r^n = r + r \circ \sigma + \cdots + r \circ \sigma^{n-1}$. The next lemma is immediate from the definition of r.

Lemma 3.2.2. *For any finite reduced word $x_0 \cdots x_{n-1}$, we have $L(x_0 \cdots x_{n-1}) = r^n(x_0 \cdots x_{n-1})$.*

The following lemma connects closed geodesics in the graph G to periodic points in the subshift of finite type $\sigma : \Sigma \to \Sigma$. The fact that the sums of r over periodic points do not lie in a discrete subgroup will be crucial in establishing that our generating function has no non-real poles on its abscissa of convergence.

Lemma 3.2.3. *Let γ be the unique closed geodesic corresponding to the periodic orbit $\{x, \sigma x, \ldots, \sigma^{n-1} x\}$ ($\sigma^n x = x$) with $x = (x_i)_{i=0}^{\infty} \in \Sigma$. Then $r^n(x) = l(\gamma)$. In particular, $\{r^n(x) \colon \sigma^n x = x, n \geq 1\}$ is not contained in a discrete subgroup of $\mathbb{R}$.*

Proof. Let $(x_0 \cdots x_{n-1})^m$ denote the m-fold concatenation of the cyclically reduced word $x_0 \cdots x_{n-1}$. By the definition of r and Lemma 3.2.2, we have

$$|r^{mn}(x) - L((x_0 \cdots x_{n-1})^m)| \leq 2N\|r\|_\infty$$

and so, since $r^{mn}(x) = mr^n(x)$, we deduce

$$r^n(x) = \lim_{m\to\infty} \frac{1}{m} L((x_0 \cdots x_{n-1})^m).$$

Now consider the closed geodesic γ in G. This lifts to a non-backtracking path in $\mathcal{T}$, from some vertex p to gp, where $g \in F$ is conjugate to $x_0 \cdots x_{n-1}$. For each $m \geq 1$, we have $ml(\gamma) = d_{\mathcal{T}}(p, g^m p)$. For any vertex $q \in \mathcal{T}$, the triangle inequality gives

$$d_{\mathcal{T}}(p, g^m p) - 2d_{\mathcal{T}}(p, q) \leq d_{\mathcal{T}}(q, g^m q) \leq d_{\mathcal{T}}(p, g^m p) + 2d_{\mathcal{T}}(p, q),$$

so that

$$l(\gamma) = \lim_{m\to\infty} \frac{1}{m} d_{\mathcal{T}}(q, g^m q).$$

Suppose that $x_0 \cdots x_{n-1} = w^{-1} g w$ and choose $q = wo$. Then $l(\gamma)$ is given by the following limit

$$\lim_{m\to\infty} \frac{d_{\mathcal{T}}(wo, g^m wo)}{m} = \lim_{m\to\infty} \frac{d_{\mathcal{T}}(o, (x_0 \cdots x_{n-1})^m o)}{m} = \lim_{m\to\infty} \frac{L((x_0 \cdots x_{n-1})^m)}{m}.$$

This completes the proof. □

Now we turn to the definition of the matrices we use to analyse the generating function. We begin by defining an unweighted transition matrix A, whose rows and columns are indexed by W_{N-1}, the set of reduced words of length at most $N-1$, where the number $N \geq 2$ is given by Lemma 2.5.2. The entries of A are defined as follows. We have $A(x,y) = 1$ if there exists $n \leq N-2$ and there exists $x_0, x_1, \ldots, x_{n-2} \in \mathcal{A} \cup \mathcal{A}^{-1}$ such that x has the reduced word representation $x = x_0 x_1 \cdots x_{n-2}$ and y has the reduced word representation $y = x_1 \cdots x_{n-2}$, or if there exists $x_0, x_1, \cdots, x_{N-2}, y_{N-2} \in \mathcal{A} \cup \mathcal{A}^{-1}$ such that x has the reduced word representation $x = x_0 x_1 \cdots x_{N-2}$ and y has the reduced word representation $y = x_1 \cdots x_{N-2} y_{N-2}$. We have $A(x,y) = 0$ in all other cases.

We next define the numbers we shall use to define weighted matrices compatible with A.

Definition 3.2.4. For each pair $(x,y) \in W_{N-1} \times W_{N-1}$ with $A(x,y) = 1$, we define a number $R(x,y)$ by

$$R(x,y) = r(x_0 x_1 \cdots x_{N-2} y_{N-2}) = L(x_0 x_1 \cdots x_{N-2} y_{N-2}) - L(x_1 \cdots x_{N-2} y_{N-2}),$$

in the case where $x = x_0 x_1 \cdots x_{N-2}$ and $y = x_1 x_2 \cdots x_{N-2} y_{N-2}$, and

$$R(x,y) = r(x_0 x_1 \cdots x_{n-2}) = L(x_0 x_1 \cdots x_{n-2}) - L(x_1 \cdots x_{n-2}),$$

in the case where $x = x_0 x_1 \cdots x_{n-2}$ and $y = x_1 x_2 \cdots x_{n-2}$, with $n \leq N$.

We now introduce the family of weighted matrices with which we encode edge lengths. The matrices A_s, with $s \in \mathbb{C}$, have rows and columns indexed by W_{N-1} with entries

$$A_s(x,y) = \begin{cases} 0 & \text{if } A(x,y) = 0, \\ \mathrm{e}^{-sR(x,y)} & \text{otherwise.} \end{cases}$$

We will be interested in the spectral properties of A_s. If $s \in \mathbb{R}$ then A_s has non-negative entries but it is not aperiodic or even irreducible and we cannot apply the Perron–Frobenius theorem directly. Similarly, we cannot apply Wielandt's theorem directly to A_s when $s \in \mathbb{C}$. Instead, we will consider the submatrix of A_s with rows and columns indexed by W'_{N-1}, which we will denote by B_s. One can easily see that A_s and B_s have the same non-zero spectrum, though A_s has additional zero eigenvalues.

When $s \in \mathbb{R}$, the matrix B_s is aperiodic and so, by the Perron–Frobenius theorem, has a simple and positive eigenvalue $\beta(s)$, which is strictly greater in modulus than all of the other eigenvalues of B_s. Let $\tilde{u}(s)$ and $\tilde{v}(s)$ denote the left and right eigenvectors of B_s corresponding to $\beta(s)$, normalised so that $\tilde{u}(s)$ is a probability vector and $\tilde{u}(s) \cdot \tilde{v}(s) = 1$.

We have the following lemma.

Lemma 3.2.5. *For $s \in \mathbb{R}$, suppose that $\beta(s)$ is the maximal eigenvalue of the matrix B_s. Then $\beta(s)$ is real analytic and is related to the integral of $r : \Sigma \to \mathbb{R}$ by the formula*

$$\beta'(s) = -\beta(s) \int r \, d\mu_{-sr},$$

where μ_{-sr} is the equilibrium state for $-sr$. Furthermore, $\int r \, d\mu_{-sr} > 0$.

Proof. The analyticity of an isolated simple eigenvalue is standard. Furthermore, $\beta(s) = \exp P(-sr)$, where P is the pressure function for the shift $\sigma : \Sigma \to \Sigma$. The formula for the derivative is then given in [44], for example. For any $n \geq 1$, r is cohomologous to $n^{-1}r^n$ and, for n sufficiently large, r^n is strictly positive. These observations prove positivity of the integral. □

Corollary 3.2.6. *There exists a unique positive real number h such that $\beta(h) = 1$.*

Proof. By comparing with the trace of B_s^n, one can easily check that $\beta(0) > 1$ and that $\beta(s) < 1$ for sufficiently large s. By Lemma 3.2.5, $\beta(s)$ is strictly decreasing, so the required number h exists and is unique. $\square$

We will also need to consider B_s for $s \in \mathbb{C}$. In particular, we have the following result.

Lemma 3.2.7. *For $t \neq 0$, the matrix B_{h+it} does not have 1 as an eigenvalue.*

Proof. Lemma 3.2.3 tells us $\{r^n(x) \colon \sigma^n x = x,\ n \geq 1\}$ is not contained in a discrete subgroup of $\mathbb{R}$. It then follows from [43] that the result holds. $\square$

For $s \in \mathbb{R}$, let $u(s)$ and $v(s)$ denote, respectively, the left and right eigenvectors of A_s associated to the eigenvalue $\beta(s)$. We choose to normalise $u(s)$ so that each entry is strictly positive and scale the entries so that $(u(s))_1 = 1$. (Note that here the subscript 1 denotes the entry index by the identity element in F, i.e. the word of length zero.) We normalise $v(s)$ so that the entries $(v(s))_y$ for $y \in W'_{N-1}$ are strictly positive, whilst all other entries are 0. We then scale the entries of $v(s)$ so that $u(s) \cdot v(s) = 1$.

We conclude this section by explaining why the number h, from Corollary 3.2.6, that gives $\beta(h) = 1$ is equal to the volume entropy. We will do this by sketching a proof of Guillopé's theorem. This will also serve as a prelude to the analysis in the next section, where the arguments will be given in more detail.

To evaluate the asymptotic behaviour of $\#\{x \in F \colon L(x) \leq T\}$ we first establish analytic properties of the complex generating function

$$\eta(s) = \sum_{n=1}^{\infty} \sum_{|x|=n} \mathrm{e}^{-sL(x)}.$$

Letting $\mathsf{1} = (1, \dots, 1)$ and e_x denote the standard unit vector associated to $x \in$

W_{N-1}, we can rewrite $\eta(s)$ in terms of the matrices A_s:

$$\eta(s) = \sum_{n=1}^{\infty} \mathtt{1} A_s^n \mathtt{e}_1.$$

This converges absolutely for $\beta(\mathrm{Re}(s)) < 1$, i.e. for $\mathrm{Re}(s) > h$.

Since $\beta(s)$ is a simple eigenvalue, it varies analytically and we can show that $\eta(s) = c_0/(s-h) + \phi_1(s)$ where $\phi_1(s)$ is an analytic function in a neighbourhood of $s = h$ and $c_0 = -(v(s) \cdot \mathtt{1})/\beta'(h) > 0$. Furthermore, using Lemma 3.2.7, we may show that $\eta(s)$ has no further poles on the line $\mathrm{Re}(s) = h$.

The generating function $\eta(s)$ is related to the counting function $N(T) := \#\{x \in F \colon L(x) \leq T\}$ by the following Stieltjes integral:

$$\eta(s) = \int_0^{\infty} \mathrm{e}^{-sT} \, dN(T).$$

This enables us to apply the Ikehara–Wiener Tauberian theorem.

Theorem 3.2.8 (Ikehara–Wiener Tauberian theorem ([44], Theorem 6.7))**.** *Suppose that the function* $\eta(s) = \int_0^\infty \mathrm{e}^{-sT} \, dN(T)$ *is analytic for* $\mathrm{Re}(s) > h$*, has a simple pole at* $s = h$*, and the function* $\eta(s) - c_0/(s-h)$ *has an analytic extension to a neighbourhood of* $\mathrm{Re}(s) \geq h$*. Then*

$$N(T) \sim \frac{c_0 \mathrm{e}^{hT}}{h}, \quad \text{as } T \to \infty.$$

As a consequence, we recover Guillopé's result that

$$\#\{x \in F \colon L(x) \leq T\} \sim c\mathrm{e}^{hT} \quad \text{as } T \to \infty,$$

for some $c > 0$, and hence that the constant h defined by $\beta(h) = 1$ is the volume entropy.

3.3 A complex generating function

Suppose that $\mathfrak{C}$ is a non-trivial conjugacy class of the free group F. Let $\mathfrak{C}_n$ denote the set of $x \in \mathfrak{C}$ such that $|x| = n$ and suppose that $k = \min_{x\in\mathfrak{C}} |x|$. The set of $g \in \mathfrak{C}_k$ are precisely those elements in $\mathfrak{C}$ whose reduced word representations are cyclically reduced. In fact, if $g = g_1 \cdots g_k \in \mathfrak{C}_k$ then all the remaining elements of $\mathfrak{C}_k$ are given by cyclic permutations of the letters in $g_1 \cdots g_k$.

The reduced word representation of each $x \in \mathfrak{C}$ takes the form

$$x = w^{-1}gw = w_m^{-1} \cdots w_1^{-1} g_1 \cdots g_k w_1 \cdots w_m$$

with $g = g_1 \cdots g_k \in \mathfrak{C}_k$; and $w = w_1 \cdots w_m \in W'_m$ subject to the restriction $w_1 \neq g_1, g_k^{-1}$ in order that no pairwise cancellation occurs when concatenating w^{-1}, g and w. For a given $g = g_1 \cdots g_k$ we say that $w \in W'_m(g)$ if $w \in W'_m$ and $w_1 \neq g_1, g_k^{-1}$. Clearly $\mathfrak{C}_n$ is non-empty if and only if $n = k + 2m$ for $m \in \mathbb{N}$.

The next Lemma, which gives a useful decomposition of $L(x)$ for $x \in \mathfrak{C}$ with sufficiently long word length, follows from Lemma 2.5.2.

Lemma 3.3.1. *Suppose that $x = w^{-1}gw \in \mathfrak{C}$ such that $g \in \mathfrak{C}_k$ and $w \in W'_m(g)$ with $m \geq N - 1$. Let $y = w_1 \cdots w_{N-1}$ then*

$$L(x) = L(w^{-1}gw) = L(y^{-1}gy) - 2L(y) + 2L(w).$$

In order to study the counting problem in the conjugacy class $\mathfrak{C}$, we define a generating function

$$\eta_{\mathfrak{C}}(s) = \sum_{x\in\mathfrak{C}} \mathrm{e}^{-sL(x)}.$$

We use the reduced word representation of elements in $\mathfrak{C}$ to rewrite $\eta_{\mathfrak{C}}(s)$ as follows:

$$\begin{aligned}\eta_{\mathfrak{C}}(s) &= \sum_{n=1}^{\infty} \sum_{x\in\mathfrak{C}_n} \mathrm{e}^{-sL(x)} \\ &= \sum_{g\in\mathfrak{C}_k} \sum_{y\in W'_{N-1}(g)} \mathrm{e}^{-s(L(y^{-1}gy)-2L(y))} \sum_{m=N-1}^{\infty} \sum_{\substack{w\in W'_m \\ y=w_1\cdots w_{N-1}}} \mathrm{e}^{-2sL(w)} + \phi(s)\end{aligned}$$

where $\phi(s)$ is an entire function. We may write $\eta_{\mathfrak{C}}(s)$ in terms of A_{2s} by using

$$\sum_{m=N-1}^{\infty} \sum_{\substack{w \in W'_m \\ y = w_1 \cdots w_{N-1}}} \mathrm{e}^{-2sL(w)} = \sum_{m=N-1}^{\infty} \mathsf{e}_y A_{2s}^m \mathsf{e}_1.$$

We will need the following classical result from linear algebra.

Lemma 3.3.2. *Let M be a $d \times d$ matrix with real entries. Suppose that M has a simple eigenvalue β, and that u and v are the associated left and right eigenvectors, normalised so that $u \cdot v = 1$. Then $w \in \mathbb{R}^d$ can be written $w = (u \cdot w)v + \overline{v}$, where $\overline{v}$ is in the span of generalised right eigenvectors of M not associated to β.*

Proof. Let $\{v\} \cup \mathcal{S}$ be a Jordan basis for M. Then $\mathbb{R}^d = \mathbb{R}v \oplus \operatorname{span}(\mathcal{S})$ and u is orthogonal to each element of $\mathcal{S}$. The result follows. $\square$

Proposition 3.3.3. *The generating function $\eta_{\mathfrak{C}}(s)$ is analytic for $\mathrm{Re}(s) > h/2$, has a simple pole at $s = h/2$ with positive residue and, apart from this, has an analytic extension to a neighbourhood of $\mathrm{Re}(s) \geq h/2$.*

Proof. For $\sigma \in \mathbb{R}$, $A_{2\sigma}$ has spectral radius $\beta(2\sigma)$. Since this is strictly decreasing and $\beta(h) = 1$, it is clear that $\sum_{m=N-1}^{\infty} \mathsf{e}_y^{\mathrm{T}} A_{2\sigma}^m \mathsf{e}_1$ converges for $\sigma > h/2$ and hence that, for $s \in \mathbb{C}$, $\eta_{\mathfrak{C}}(s)$ is analytic for $\mathrm{Re}(s) > h/2$.

We now consider the analyticity of $\eta(s)$ for s in a neighbourhood of $h/2 + it \in \mathbb{C}$, for an arbitrary $t \in \mathbb{R}$. We will let $\operatorname{spr}(M)$ denote the spectral radius of a matrix M. By Wielandt's theorem (Theorem 2.4.5), either

1. $\operatorname{spr}(A_{h+2it}) = \operatorname{spr}(B_h) = 1$, in which case A_{h+2it} has a simple eigenvalue $\beta(h+2it)$ with $|\beta(h+2it)| = \beta(h) = 1$ and such that the remaining eigenvalues are strictly smaller in modulus; or

2. $\operatorname{spr}(A_{h+2it}) < \operatorname{spr}(B_h) = 1$.

When (1) holds, standard eigenvalue perturbation theory gives that the simple eigenvalue $\beta(2s)$ persists and varies analytically for s in a neighbourhood of $h + it$, as are the corresponding left and right eigenvectors $u(2s)$ and $v(2s)$ [32]. By Lemma

3.3.2 and recalling that $(u(2s))_1 = 1$, we have $\mathsf{e}_1 = v(2s) + \overline{v}(2s)$, where $\overline{v}(2s)$ is a vector in the subspace spanned by the generalised eigenvectors associated to the non-maximal eigenvalues of A_{2s}. Thus we have

$$\sum_{m=N-1}^{\infty} \mathsf{e}_y^{\mathrm{T}} A_{2s}^m \mathsf{e}_1 = \sum_{m=N-1}^{\infty} (\mathsf{e}_y \cdot v(2s))\beta(2s)^m + \sum_{m=N-1}^{\infty} \mathsf{e}_y^{\mathrm{T}} A_{2s}^m \overline{v}(2s)$$
$$= \frac{(\mathsf{e}_y \cdot v(2s))\beta(2s)^{N-1}}{1-\beta(2s)} + \phi_2(s),$$

where $\phi_2(s)$ is analytic in a neighbourhood of $h/2 + it$. Therefore, $\eta_{\mathfrak{C}}(s)$ is analytic in a neighbourhood of $h/2 + it$ unless $\beta(h + 2it) = 1$. Lemma 3.2.7 tells us that this only occurs when $t = 0$. When (2) holds, we immediately obtain that $\sum_{m=N-1}^{\infty} \mathsf{e}_v^{\mathrm{T}} A_{2s}^m \mathsf{e}_1$ converges and hence that $\eta_{\mathfrak{C}}(s)$ converges to an analytic function for s in a neighbourhood of $h/2 + it$.

For s in a neighbourhood of $h/2$, we have that, modulo an analytic function,

$$\eta_{\mathfrak{C}}(s) = \sum_{g \in \mathfrak{C}_k} \sum_{y \in W'_{N-1}(g)} \mathrm{e}^{-s(L(y^{-1}gy)-2L(y))} \frac{(\mathsf{e}_y \cdot v(2s))\beta(2s)^{N-1}}{1-\beta(2s)}.$$

From the analyticity of β and the fact that $\beta(h) = 1$, we obtain that, in a neighbourhood of $h/2$,

$$\eta_{\mathfrak{C}}(s) = \frac{c}{s - h/2} + \phi_3(s),$$

where $\phi_3(s)$ is analytic and $c > 0$. The latter holds because $\mathsf{e}_y \cdot v(h) > 0$, for each $y \in \bigcup_{g \in \mathfrak{C}_k} W'_{N-1}(g)$, and, by Lemma 3.2.5, $-1/(2\beta'(h)) > 0$.

Combining the above observations, we have that $\eta_{\mathfrak{C}}(s)$ is analytic for $\mathrm{Re}(s) > h/2$ and, apart from a simple pole at $s = h/2$, has an analytic extension to a neighbourhood of $\mathrm{Re}(s) \geq h/2$. Furthermore, the residue at the simple pole is positive. □

Since $\eta_{\mathfrak{C}}(s)$ is related to the counting function $N_{\mathfrak{C}}(T) = \#\{x \in \mathfrak{C} \colon d_{\mathcal{T}}(o, xo) \leq T\}$ via the Stieltjes integral

$$\eta_{\mathfrak{C}}(s) = \int_0^{\infty} \mathrm{e}^{-sT} \, dN_{\mathfrak{C}}(T),$$

we may apply the Ikehara–Wiener Tauberian theorem (Theorem 3.2.8) to conclude that

$$N_{\mathfrak{C}}(T) \sim C\frac{\mathrm{e}^{hT/2}}{h/2},$$

for some $C > 0$. This completes the proof of Theorem 3.1.1.

3.4 Error terms

In this final section we discuss the error terms which may appear when estimating $N_{\mathfrak{C}}(T)$. We first note the following, which may be deduced from the analysis above and the arguments in [49] (Propositions 6 and 7).

Proposition 3.4.1. *There is never an exponential error term in Theorem 3.1.1, i.e. for no $\epsilon > 0$ do we have $N_{\mathfrak{C}}(T) = C\mathrm{e}^{hT/2} + O(\mathrm{e}^{(h-\epsilon)T/2})$.*

A more interesting problem is to ask when there is a polynomial error term for $N_{\mathfrak{C}}(T)$. Recall that an irrational number α is said to be *Diophantine* if there exist $c > 0$ and $\beta > 1$ such that $|q\alpha - p| \geq cq^{-\beta}$ for all $p, q \in \mathbb{Z}$, $q > 0$. We have the following.

Theorem 3.4.2. *Suppose that G is a finite connected metric graph such that the degree of each vertex is at least 3. Suppose also that G contains two closed geodesics γ and γ' such that $l(\gamma)/l(\gamma')$ is Diophantine. Then there exists $\delta > 0$ such that*

$$N_{\mathfrak{C}}(T) = C\mathrm{e}^{hT/2} + O(\mathrm{e}^{hT/2}T^{-\delta}).$$

Note that, since the lengths of the closed geodesics in G are not contained in a discrete subgroup of $\mathbb{R}$ if and only if there are two closed geodesics the ratio of whose lengths is irrational, the hypothesis of Theorem 3.4.2 is strictly stronger that that of Theorem 3.1.1.

The crucial new ingredient is to obtain a bound on powers of the matrix A_s, for $\mathrm{Re}(s)$ close to h and $\mathrm{Im}(s)$ away from zero. To do this, we use the work of Dolgopyat [13] on transfer operators, where it is a key ingredient in studying the mixing rate of flows (cf. also [55]). Since $r : \Sigma \to \mathbb{R}$ is locally constant, the transfer operator L_{-sr}, defined pointwise by

$$(L_{-sr}\psi)(x) = \sum_{\sigma y = x} \mathrm{e}^{-sr(y)}\psi(y),$$

acts on the space of complex valued Hölder continuous functions with exponent α, for any $\alpha > 0$. For definiteness, we will consider the action on the Banach space of Lipschitz functions, $L_{-sr} : \mathcal{F}_\theta(\Sigma, \mathbb{C}) \to \mathcal{F}_\theta(\Sigma, \mathbb{C})$. Recall the norm on $\mathcal{F}_\theta(\Sigma, \mathbb{C})$ given by $\|f\| = \|f\|_\infty + |f|_\theta$, where

$$|f|_\theta = \sup_{x \neq y} \frac{|f(x) - f(y)|}{d_\theta(x, y)}.$$

Let $V_{N-1} \subset \mathcal{F}_\theta(\Sigma, \mathbb{C})$ denote the finite dimensional subspace consisting of locally constant functions depending on the first $N - 1$ coordinates. This has dimension $D := \#W'_{N-1}$. Then $L_{-sr} : V_{N-1} \to V_{N-1}$ and $\|L_{-sr}|_{V_{N-1}}\| \leq \|L_{-sr}\|$. Furthermore, the restriction of L_{-sr} to V_{N-1} can be identified with the action of B_s on $\mathbb{C}^D$. We can therefore use the results of [13] on bounding the norm of iterates of transfer operators to give a bound on the norm of the iterates of B_s and hence A_s.

For the remainder of this section, it will be convenient to write $s = \varsigma + it$.

Proposition 3.4.3. *Under the hypotheses of Theorem 3.4.2, there exist constants* $C_1, C_2 > 0$*,* $\tau > 0$ *and* $t_1 \geq 1$ *such that when* $|t| \geq t_1$ *and* $m \geq 1$*,*

$$\|A_{\varsigma+it}^{2\nu m}\| \leq C_1 |t| \beta(\varsigma)^{2\nu m} \left(1 - |t|^{-\tau}\right)^{m-1},$$

where $\nu = \lfloor C_2 \log |t| \rfloor$*.*

We will also use the elementary inequality (cf. Proposition 2.1 of [44]):

Lemma 3.4.4. *There exists a constant* $D_1 > 0$*, independent of* t *and uniform in* ς*, such that, for all* $n > 0$*, we have*

$$\|A_{\varsigma+it}^n\| \leq \beta(\varsigma)^n \left(D_1 |t| + 2^{-n}\right).$$

Proof. Let ϕ denote the strictly positive eigenfunction of the operator $L_{-\varsigma r}$ associated to the eigenvalue $\beta(\varsigma)$, guaranteed by the Ruelle-Perron-Frobenius Theorem for this operator (see [44], Theorem 2.2). Let

$$g = -\varsigma r + \log \phi - \log \phi \circ \sigma - \log \beta(\varsigma).$$

Then $L_g 1 = 1$ and it follows from Proposition 2.1 of [44] that

$$|L^n_{g-itr}\psi|_\theta \leq D_0|t|\|\psi\|_\infty + 2^{-n}|\psi|_\theta,$$

for some $D_0 > 0$ independent of t and uniform in ς, and

$$\|L^n_{g-itr}\psi\|_\infty \leq \|\psi\|_\infty.$$

(The only difference from the statement given in [44] is the appearance of the term $|t|$ and the uniformity of D_0 but this follows from an inspection of the proof.) In particular, we have

$$\|L^n_{g-itr}\psi\| \leq (D_1|t| + 2^{-n})\|\psi\|.$$

for some $D_1 > 0$ independent of t and uniform in σ. Since

$$L_{-(\varsigma+it)r} = \beta(\varsigma)\Delta_\phi L_{g-itr}\Delta_\phi^{-1},$$

where Δ_ϕ is the multiplication operator $\Delta_\phi(\psi) = \phi\psi$, the result follows. □

Combining Proposition 3.4.3 and Lemma 3.4.4 gives the following bound on $\|A^n_s\|$ for all $n \in \mathbb{N}$.

Proposition 3.4.5. *Let $n = 2\nu m + l$ where $m = \left\lfloor \frac{n}{2\nu} \right\rfloor$ and $l \in \{0, \ldots, 2\nu - 1\}$, then, for $|t| \geq t_1$*

$$\|A^n_{\varsigma+it}\| \leq C_3|t|^2\beta(\varsigma)^n(1 - |t|^{-\tau})^{m-1},$$

for some $C_3 > 0$ independent of t and uniform in ς.

We use this to study the analyticity of $\eta_{\mathfrak{C}}(s)$ (using a simpler version of the arguments in [55]).

Proposition 3.4.6. *There exist constants $\rho > 0$ and $t_2 \geq t_1$ such that $\eta_{\mathfrak{C}}(s)$ has an analytic extension to the region*

$$\mathcal{R}(\rho) = \{\varsigma + it \in \mathbb{C} \colon 2\varsigma > h - 1/|2t|^{\rho}, |t| \geq t_2\},$$

where it satisfies the bound $|\eta_{\mathfrak{C}}(\varsigma + it)| = O(|t|^{2+\rho})$.

Proof. Suppose that $\varsigma + it \in \mathbb{C}$ satisfies $2\varsigma > h - |2t|^{-\rho}$ and $|2t| \geq t_1$. Take $\rho > \tau$ (with τ the constant from Proposition 3.4.3). Since

$$\beta(2\varsigma) = 1 + \beta'(h)(2\varsigma - h) + O((2\varsigma - h)^2)$$

with $\beta'(h) < 0$, for $|t| \geq t_2$, where $t_2 \geq t_1$, we have $(1 - |2t|^{-\tau})^{1/2\nu} < \beta(2\varsigma)^{-1}$.

We can estimate $|\eta_{\mathfrak{C}}(s)| \leq M \sum_{m=1}^{\infty} |A_{2s}^m \mathsf{e}_1|$, for some $M > 0$. Thus, for $\varsigma + it \in \mathcal{R}(\rho)$, we have, by Proposition 3.4.5,

$$\begin{aligned}\sum_{m=1}^{\infty} |A_{2(\varsigma+it)}^m \mathsf{e}_1| &\leq \sum_{m=1}^{\infty} C_3 |2t|^2 \beta(2\varsigma)^m \left(1 - |2t|^{-\tau}\right)^{\lfloor m/2\nu \rfloor - 1} \\ &\leq \frac{C_3 |2t|^2 \beta(2\varsigma)}{(1 - |2t|^{-\tau})^{2-2/\nu} \left(1 - \beta(2\varsigma)\,(1 - |2t|^{-\tau})^{1/2\nu}\right)} \\ &\leq \frac{C_4 |2t|^2}{1 - \beta(2\varsigma)\,(1 - |2t|^{-\tau})^{1/2\nu}} = O(|t|^{2+\rho}),\end{aligned}$$

which shows that $\eta_{\mathfrak{C}}(s)$ is analytic in the desired region and gives the bound. $\square$

Let $\xi_{\mathfrak{C}}(s)$ be the normalised generating function given by

$$\xi_{\mathfrak{C}}(s) = \eta_{\mathfrak{C}}(sh/2) = \sum_{x \in \mathfrak{C}} \mathrm{e}^{-shL(x)/2}.$$

We immediately deduce that $\xi_{\mathfrak{C}}(s)$ is analytic in the half-plane $\mathrm{Re}(s) > 1$, has an analytic extension to a neighbourhood of $\mathrm{Re}(s) \geq 1$ apart from the simple pole at $s = 1$, which has positive residue. As a consequence of the the additional Diophantine condition on the lengths of closed geodesics, there exist positive constants

ρ and $t_3 = 2t_2/h$ such that $\xi_{\mathfrak{C}}(s)$ has an analytic extension to

$$\mathcal{R}_\xi(\rho) = \left\{ \varsigma + it \colon \varsigma > 1 - \frac{1}{h^{\rho+1}|t|^\rho},\ |t| > t_3 \right\};\ \text{and}$$

where it satisfies $|\xi_{\mathfrak{C}}(s)| = O(|t|^{2+\rho})$.

Let us introduce a normalised counting function

$$\psi_0(T) = \sum_{\mathrm{e}^{hL(x)/2} \leq T} 1.$$

Adapting the arguments of [55], we will establish an error term for $\psi_0(T)$, from which Theorem 3.4.2 will follow since $\psi_0(\mathrm{e}^{hT/2}) = N_{\mathfrak{C}}(T)$. We introduce the following family of auxiliary functions. Let $\psi_1(T) = \int_1^T \psi_0(u)\, du$ and continue inductively so that

$$\psi_k(T) = \int_1^T \psi_{k-1}(u)\, du = \frac{1}{k!} \sum_{\mathrm{e}^{hL(x)/2} \leq T} (T - \mathrm{e}^{hL(x)/2})^k.$$

We use the following identity ([29], Theorem B, page 31) to connect the functions $\xi_{\mathfrak{C}}(s)$ and $\psi_k(T)$. If k is a positive integer and $d > 0$ we have

$$\frac{1}{2\pi i} \int_{d-i\infty}^{d+i\infty} \frac{y^s}{s(s+1)\cdots(s+k)}\, ds = \begin{cases} 0 & \text{if } 0 < y < 1, \\ \frac{1}{k!}(1 - 1/y)^k & \text{if } y \geq 1. \end{cases}$$

This gives us the following.

Lemma 3.4.7. *For $d > 1$ we may write*

$$\psi_k(T) = \frac{1}{2\pi i} \int_{d-i\infty}^{d+i\infty} \frac{\xi_{\mathfrak{C}}(s) T^{s+k}}{s(s+1)\cdots(s+k)}\, ds.$$

We briefly outline the method to approximate the integral for $\psi_k(T)$. First, compare the integral for $\psi_k(T)$ to the truncated integral on the line segment $[d - iR, d + iR]$, where $R = (\log T)^\varepsilon$ and $0 < \varepsilon < 1/\rho$. Let us choose $d = 1 + 1/\log T$

and then since $\xi_{\mathfrak{C}}(d) = O((d-1)^{-1})$, we deduce

$$\left|\psi_k(T) - \frac{1}{2\pi i}\int_{d-iR}^{d+iR} \frac{\xi_{\mathfrak{C}}(s)T^{s+k}}{s(s+1)\cdots(s+k)}\, ds\right| = O\left(\frac{(\log T)T^{k+1}}{R^{k+1}}\right) = O\left(\frac{T}{(\log T)^{k\varepsilon}}\right).$$

We evaluate the truncated integral using Cauchy's Residue theorem. Consider a closed contour $\Gamma \cup [d-iR, d+iR]$. Here Γ is the union of the line segments $[d+iR, c(R)+iR]$, $[c(R)-iR, d-iR]$ and $[c(R)+iR, c(R)-iR]$, where

$$c(R) = 1 - \frac{1}{2h^{\rho+1}R^{\rho}},$$

so that Γ lies in $\mathcal{R}_{\xi}(\rho)$. Note that $\Gamma \cup [d-iR, d+iR]$ encloses the simple pole of $\xi_{\mathfrak{C}}(s)$ at $s = 1$. Cauchy's Residue theorem gives that, for some $c > 0$,

$$\frac{1}{2\pi i}\int_{d-iR}^{d+iR} \frac{\xi_{\mathfrak{C}}(s)T^{s+k}}{s(s+1)\cdots(s+k)}\, ds = \frac{cT^{k+1}}{(k+1)!} + \frac{1}{2\pi i}\int_{\Gamma} \frac{\xi_{\mathfrak{C}}(s)T^{s+k}}{s(s+1)\cdots(s+k)}\, ds.$$

We consider the contribution made by each of the line segments in Γ. First, integrating over the interval $[d+iR, c(R)+iR]$, we have

$$\begin{aligned}\left|\int_{d+iR}^{c(R)+iR} \frac{\xi_{\mathfrak{C}}(s)T^{s+k}}{s(s+1)\cdots(s+k)}\, ds\right| &\le \frac{T^{d+k}}{R^{k+1}}\left|\int_{d+iR}^{c(R)+iR} \xi_{\mathfrak{C}}(s)\, ds\right| \\ &= O\left(\frac{T^{d+k}}{(\log T)^{(k-\rho-1)\varepsilon}}\right)\end{aligned}$$

and similarly, for $[c(R)-iR, d-iR]$, we have

$$\left|\int_{c(R)-iR}^{d-iR} \frac{\xi_{\mathfrak{C}}(s)T^{s+k}}{s(s+1)\cdots(s+k)}\, ds\right| = O\left(\frac{T^{d+k}}{(\log T)^{(k-\rho-1)\varepsilon}}\right).$$

We estimate the modulus of the integral along $[c(R)+iR, c(R)-iR]$ by

$$\begin{aligned}T^{c(R)+k}\left|\int_{c(R)+iR}^{c(R)-iR} \frac{\xi_{\mathfrak{C}}(s)}{s(s+1)\cdots(s+k)}\, ds\right| &= O\left(T^{c(R)+k}\int_1^R t^{1+\rho-k}\, dt\right) \\ &= O(T^{c(R)+k}R^{2+\rho-k}),\end{aligned}$$

which means for any positive γ we have

$$T^{c(R)+k}R^{2+\rho-k} = T^{k+1}\mathrm{e}^{-\frac{\log T}{2h^{\rho+1}(\log T)^{\varepsilon\rho}}}(\log T)^{(2+\rho-k)\varepsilon} = O(T^{k+1}(\log T)^{-\gamma}).$$

Together the integral estimates give us an error term for $\psi_k(T)$:

$$\psi_k(T) = c'T^{k+1} + O\left(\frac{T^{k+1}}{(\log T)^{(k-\rho-1)\varepsilon}}\right)$$

where $c' > 0$. Then repeatedly applying the inequality

$$\psi_{j-1}(T-\Delta T)\Delta T \leq \psi_j(T) - \psi_j(T-\Delta T) \leq \psi_{j-1}(T)\Delta T,$$

where

$$\Delta T = T(\log T)^{-(k-\rho-1)2^{j-k-1}\varepsilon},$$

we obtain

$$\psi_0(T) = CT + O\left(T(\log T)^{-\delta}\right),$$

where $C, \delta > 0$. Thus we have the error term $N_{\mathfrak{C}}(T) = C\mathrm{e}^{hT/2} + O(\mathrm{e}^{hT/2}T^{-\delta})$, completing the proof of Theorem 3.4.2.

Chapter 4

Spatial distribution of conjugacy classes in free groups acting on trees

4.1 Introduction

We briefly recall the notation from the previous chapters. Let G be a finite connected metric graph such that the degree of each vertex is at least 3. Then the fundamental group of G is a free group F on $k \geq 2$ generators. We shall assume throughout that G has a pair of closed geodesics γ and γ' such that the ratio of their metric lengths $l(\gamma)/l(\gamma')$ is irrational. Let $\mathcal{T}$ be the universal covering tree of G endowed with the metric $d_{\mathcal{T}}$ lifted from G.

The free group F acts freely and isometrically on $\mathcal{T}$. Let $L : F \to \mathbb{R}$ be the based length function $L(x) = d_{\mathcal{T}}(o, xo)$ for a given vertex $o \in \mathcal{T}$. Then Guillopé [20] showed that there exists a positive constant C such that

$$\#\{x \in F \colon L(x) \leq T\} \sim C\mathrm{e}^{hT}$$

as $T \to \infty$. Here $f(T) \sim g(T)$ denotes $\lim_{T\to\infty} f(T)/g(T) = 1$.

We consider Guillopé's orbit counting function when the group elements are restricted to a non-trivial conjugacy class $\mathfrak{C}$ in F, which was the focus of the previous chapter. We recall the result in Theorem 3.1.1 for the counting function $N_{\mathfrak{C}}(T) = \#\{x \in \mathfrak{C}\colon L(x) \leq T\}$; there exists a positive constant C depending on $\mathfrak{C}$ such that as $T \to \infty$,

$$N_{\mathfrak{C}}(T) = \#\{x \in \mathfrak{C}\colon L(x) \leq T\} \sim C\mathrm{e}^{hT/2}.$$

In this chapter we study the spatial distribution of lattice points associated to the conjugacy class $\mathfrak{C}$ in the universal covering tree $\mathcal{T}$. We show that the lattice points are asymptotically uniformly distributed with respect to a natural measure on the boundary of $\mathcal{T}$.

Let $\partial\mathcal{T}$ denote the boundary of the tree $\mathcal{T}$. We define $\partial\mathcal{T}$ as the set of endpoints of geodesic rays in $\mathcal{T}$ originating at a prescribed base vertex $o \in \mathcal{T}$. By a *geodesic ray* we mean an infinite non-backtracking path in $\mathcal{T}$. We denote by $[x, \xi]$ a geodesic ray originating at $x \in \mathcal{T}$ with endpoint ξ. We note that $\partial\mathcal{T}$ is independent of the choice of base point (cf. §2, [30]) and additionally, we will see that $\partial\mathcal{T}$ is homeomorphic to Σ^*. Suppose that B is a Borel subset of $\partial\mathcal{T}$. We call $\mathcal{S}(B) = \{y \in \mathcal{T}\colon \text{if } y \in [o,\xi] \text{ then } \xi \in B\}$ the *sector* in $\mathcal{T}$ associated to B. We remark that $\mathcal{S}(B)$ depends on the vertex $o \in \mathcal{T}$ and is a subset of $\bigcup_{\xi \in B}[o, \xi]$. We are interested in the asymptotic behaviour of the counting function

$$N_{\mathfrak{C}}^{B}(T) = \#\{x \in \mathfrak{C}\colon L(x) \leq T, xo \in \mathcal{S}(B)\}.$$

The purpose of this chapter is to show that in the limit as $T \to \infty$ the ratio $N_{\mathfrak{C}}^{B}(T)/N_{\mathfrak{C}}(T)$ is proportional to the size of B. In fact, the ratio converges to $\mu_{\mathrm{PS}}(B)$ where μ_{PS} is a Patterson–Sullivan probability measure on $\partial\mathcal{T}$. We formally state this result in the next theorem.

Theorem 4.1.1. *Suppose that G is a finite connected metric graph such that the degree of each vertex is at least 3. Additionally, we assume that there is a pair of closed geodesics in G whose ratio of metric lengths is irrational. Let $\mathfrak{C}$ be a non-*

trivial conjugacy class in F. Let B be a Borel subset of $\partial\mathcal{T}$. Then

$$\lim_{T\to\infty}\frac{N_{\mathfrak{C}}^{B}(T)}{N_{\mathfrak{C}}(T)}=\mu_{\mathrm{PS}}(B).$$

Remark 4.1.2. The irrationality assumption on the lengths of closed geodesics in G is crucial in the proof of Theorem 4.1.1 we present. It is interesting to ask whether the irrationality assumption is necessary to establish the limiting ratio in Theorem 4.1.1. However, we do not have a solution to this question.

For the remainder of the introduction, we recall related results from the literature, which we state in terms of the Poincaré ball model of n-dimensional hyperbolic space. The ball model for n-dimensional hyperbolic space consists of the set $\mathbb{D}^n=\{x\in\mathbb{R}^n\colon |x|<1\}$ equipped with the Poincaré metric given by the differential

$$d\rho=\frac{2|dx|}{1-|x|^2}.$$

Following Patterson, we call a discrete group of isometries of n-dimensional hyperbolic space a *Kleinian group*. Suppose that Γ is a non-elementary Kleinian group acting on n-dimensional hyperbolic space. We denote by $\Lambda(\Gamma)\subseteq\partial\mathbb{D}^n$ the limit set of the group Γ. We write $C(\Gamma)\subseteq\mathbb{D}^n\cup\partial\mathbb{D}^n$ for the convex hull of the limit set $\Lambda(\Gamma)$. We say Γ is *co-compact* if $\mathbb{D}^n/\Gamma$ is compact and *convex co-compact* if $(C(\Gamma)\cap\mathbb{D}^n)/\Gamma$ is compact. Note that requiring Γ to be convex co-compact is weaker than requiring Γ to be co-compact since, by Theorem 2.6.2, if Γ is co-compact then $\Lambda(\Gamma)=\partial\mathbb{D}^n$. We say Γ is *geometrically finite* if it has a fundamental domain that can be represented as a polyhedron P with a finite number of sides. The condition that Γ is geometrically finite is weaker than than requiring Γ to be convex co-compact because a convex co-compact group cannot have parabolic elements.

In [26], Huber studied orbit counting in the setting of co-compact Fuchsian groups acting on $\mathbb{D}^2$. Fix a basepoint $o\in\mathbb{D}^2$. Then Huber showed that there exists a positive constant C depending on Γ such that as $T\to\infty$, $\#\{\gamma\in\Gamma\colon d_{\mathbb{D}^2}(o,\gamma o)\leq T\}\sim C\mathrm{e}^T$. Huber also gave an orbit counting asymptotic when the group elements of the Fuchsian group are restricted to a non-trivial conjugacy class.

Specifically, he showed there exists a positive constant $C(\mathfrak{C})$ so that as $T \to \infty$, $\#\{\gamma \in \mathfrak{C}\colon d_{\mathbb{D}^2}(o, \gamma o) \leq T\} \sim C\mathrm{e}^{T/2}$.

The *critical exponent* $\delta > 0$ of a Kleinian group Γ is the abscissa of convergence of the Dirichlet series

$$\sum_{\gamma \in \Gamma} \mathrm{e}^{-s d_{\mathbb{D}^n}(o, \gamma o)}$$

and note that δ is independent of the choice of base point $o \in \mathbb{D}^n$. In [37], Lax and Phillips proved, using spectral properties of the Laplacian, an orbit counting result for geometrically finite Kleinian groups acting on n-dimensional hyperbolic space such that $h > (n+1)/2$ and gave an error term estimate. An orbit counting result was established by Patterson in [48] for $h \leq (n+1)/2$ for convex co-compact Kleinian groups; however, Patterson does not give an error term estimate.

In [40] Nicholls gave a number of asymptotic formulae for discrete groups acting on n-dimensional hyperbolic space including a formula for the orbit counting problem in this setting. Compared to Huber's asymptotic results, not only are Nicholl's orbit counting results in higher dimensions, but the assumptions are also weaker in that they require only co-finite volume of the discrete group. Let Γ be a Kleinian group with co-finite volume acting on $\mathbb{D}^n$ and $N(T) = \#\{\gamma \in \Gamma\colon d_{\mathbb{D}^n}(o, \gamma o) < T\}$ for some prescribed base point $o \in \mathbb{D}^n$. Then there exists a positive constant $C(\Gamma)$ such that as $T \to \infty$, $N(T) \sim C\mathrm{e}^{\delta T}$.

The main result in [40] is a more delicate asymptotic spatial distribution result for the lattice points of Γ in hyperbolic space and we state a simple case here. Let B be the interesection of $\partial\mathbb{D}^n$ with the interior of a ball. Consider the sector $\mathcal{S}(B)$ formed by the union of geodesic rays originating from a prescribed base point $o \in \mathbb{D}^n$ whose endpoints lie in B. Let $N^B(T) = \#\{\gamma \in \Gamma\colon \gamma o \in \mathcal{S}(B)\}$. Then there exist positive constants $C(\Gamma)$ and $0 < \mu(B) < 1$, the normalised volume of B, such that as $T \to \infty$, $N^B(T) \sim C\mu(B)\mathrm{e}^{\delta T}$. Thus we conclude the lattice points of Γ are asymptotically uniformly distributed in all directions from the origin.

Sharp [63] showed that, asymptotically, the lattice points of certain convex co-compact Kleinian groups acting on n-dimensional hyperbolic space are uniformly distributed with respect to the Patterson–Sullivan measures on the boundary of n-dimensional hyperbolic space. In [63], Sharp assumes that $\Gamma < \mathrm{Isom}(\mathbb{D}^n)$ is a convex co-compact Kleinian group of isometries that satisfies the even corners condition. First introduced by Bowen and Series in [4] for $n = 2$ and generalised by Bourdon in [2] to $n \geq 3$, a Kleinian group Γ satisfies the *even corners condition* if Γ is geometrically finite and the group orbit of the associated polyhedron's boundary is given by a union of hyperplanes. Let B be a Borel subset on the boundary of $\mathbb{D}^n$ and define the counting functions $N(T)$ and $N^B(T)$ as before. Employing techniques from symbolic dynamics and the thermodynamic formalism, Sharp showed that

$$\lim_{T \to \infty} \frac{N^B(T)}{N(T)} = \mu_{\mathrm{PS}}(B).$$

Here μ_{PS} denotes a Patterson–Sullivan probability measure on the boundary of n-dimensional hyperbolic space.

4.2 Length spectra and matrices

We prove Theorem 4.1.1 by considering the analytic properties of a complex generating function. Our analysis of this complex generating function relies on the spectral properties of the transfer operator. We require much of the same terminology and machinery introduced in the previous chapter. Thus, we duplicate some of the definitions below and otherwise make reference to the relevant technical results.

Recall Definition 3.2.1 for the function $r : \Sigma^* \to \mathbb{R}$, which is inspired by the constructions in [64] and [65], from the previous chapter and Lemma 3.2.3 that connected the metric lengths of closed geodesics in the graph G to sums of $r : \Sigma \to \mathbb{R}$ over periodic points. The lemma established the fact that the set of values $\{r^n(x) \colon \sigma^n x = x, n \geq 1\}$ does not lie in a discrete subgroup of $\mathbb{R}$. This fact will be crucial in establishing that our generating function has no non-real poles on its abscissa of convergence.

We recall the transition matrix A, whose rows and columns are indexed by W_{N-1} where $N \geq 2$ is the constant given by Lemma 2.5.2, from the previous chapter. For $n \leq N-2$ we set $A(x,y)=1$ in the following two cases: either we have $x, y \in W_{N-1}$ with reduced word representations $x_0 x_1 \cdots x_{n-2}$ and $x_1 \cdots x_{n-2}$, respectively; or $x, y \in W'_{N-1}$ with reduced word representations $x_0 x_1 \cdots x_{N-2}$ and $x_1 \cdots x_{N-2} y_{N-2}$, respectively. For all other ordered pairs of elements $x, y \in W_{N-1}$ we set $A(x,y)=0$.

We find the following family of matrices A_s with $s \in \mathbb{C}$ useful. This family of matrices is defined in terms of the transition matrix A and the entries are weighted according to the function $R(x,y)$, defined in Definition 3.2.4, that assigns to each admissible pair of elements $x, y \in W_{N-1}$ a real value.

The family of weighted transition matrices A_s, with $s \in \mathbb{C}$, we use to encode edge lengths have rows and columns indexed by W_{N-1} with entries

$$A_s(x,y) = \begin{cases} 0 & \text{if } A(x,y) = 0, \\ e^{-sR(x,y)} & \text{otherwise.} \end{cases}$$

We are interested in the spectral properties of A_s and briefly summarise the relevant spectral results from the previous chapter. If $s \in \mathbb{R}$ then A_s has non-negative entries but it is not aperiodic nor even irreducible and we cannot apply the Perron–Frobenius theorem directly. Similarly, we cannot apply Wielandt's theorem directly to A_s when $s \in \mathbb{C}$. Instead, we will consider the submatrix of A_s with rows and columns indexed by W'_{N-1}, which we will denote by B_s. One can easily see that A_s and B_s have the same non-zero spectrum, though A_s has additional zero eigenvalues.

When $s \in \mathbb{R}$, the matrix B_s is aperiodic and so, by the Perron–Frobenius theorem, has a simple and positive eigenvalue $\beta(s)$, which is strictly greater in modulus than all of the other eigenvalues of B_s. Let $\tilde{u}(s)$ and $\tilde{v}(s)$ denote the left and right eigenvectors of B_s corresponding to $\beta(s)$, normalised so that $\tilde{u}(s)$ is a probability vector and $\tilde{u}(s) \cdot \tilde{v}(s) = 1$.

Suppose that $s \in \mathbb{R}$ then, by Lemma 3.2.5, the simple maximal eigenvalue $\beta(s)$ of the matrix B_s is real analytic and strictly decreasing in $\mathbb{R}$. Moreover, by Corollary 3.2.6 there is a unique positive real number h such that $\beta(h) = 1$. We also use the following spectral result for B_s with $s \in \mathbb{C}$; by Lemma 3.2.7, the matrix B_{h+it} does not have 1 as an eigenvalue when $t \neq 0$.

For $s \in \mathbb{R}$, let $u(s)$ and $v(s)$ denote, respectively, the left and right eigenvectors of A_s associated to the eigenvalue $\beta(s)$. We choose to normalise $u(s)$ so that each entry is strictly positive and scale the entries so that $(u(s))_1 = 1$. (Note that here the subscript 1 denotes the entry indexed by the identity element in F, i.e. the empty word.) We normalise $v(s)$ so that the entries $(v(s))_y$ for $y \in W'_{N-1}$ are strictly positive, whilst all other entries are 0. We then scale the entries of $v(s)$ so that $u(s) \cdot v(s) = 1$.

4.3 Hyperbolic boundary of the universal cover

In this section we describe the relationship between the boundary of the universal covering tree $\mathcal{T}$ and the limit set of the lattice points $Fo = \{xo \in \mathcal{T} : x \in F\}$. Since the action of F on $\mathcal{T}$ is geometric, by Theorem 2.6.2 the limit set of Fo and the boundary of $\mathcal{T}$ are homeomorphic. However, we require an explicit homeomorphism that respects both the structure of the underlying free group and the implied symbolic structure. Indeed, the technique, coding geodesic rays by means of cutting sequences is standard for symbolic dynamical systems of hyperbolic groups. Amongst many references we draw the reader's attention to work by Series [62] for surfaces of constant negative curvature. We note in our setting that since the boundary $\partial\mathcal{T}$ is a Cantor set, rather than a connected set, we circumvent much of the difficulty involved in coding the limit set of hyperbolic surfaces.

A subset $V \subset \mathcal{T}$ is a *fundamental domain* of $\mathcal{T}$ if V is a non-empty open subset of $\mathcal{T}$ such that for distinct $x, y \in F$ we have $xV \cap yV = \emptyset$ and $\bigcup_{x \in F} \overline{xV} = \mathcal{T}$. Here $\overline{A}$ denotes the closure of the subset A in the topology of the tree $\mathcal{T}$. As before, let F be the fundamental group of the metric graph G that acts freely and isometrically on $\mathcal{T}$ with generating set $\mathcal{A}$. We shall find the following fundamental domain U useful. Let $U \subset \mathcal{T}$ be the subtree of $\mathcal{T}$ given by the interior of the union of the paths and vertices joining $o \in \mathcal{T}$ to $a_i o$ for each $a_i \in \mathcal{A}$.

We now consider the structure of $\partial\mathcal{T}$, the boundary of the tree $\mathcal{T}$. We identify $\partial\mathcal{T}$ as the set of limit points of the infinite geodesic rays in $\mathcal{T}$. We identify any two geodesic rays that remain a bounded distance apart. However, without loss of generality we can consider the boundary of the tree as the limit set of the geodesic rays originating at a given base vertex $o \in \mathcal{T}$ and note no two distinct geodesic rays emanating from $o \in \mathcal{T}$ remain a bounded distance apart.

It will be useful to consider $\partial\mathcal{T}$ as the limit set of the lattice points $Fo = \{xo \in \mathcal{T} \colon x \in F\}$ in the tree $\mathcal{T}$. To each geodesic ray emanating from $o \in \mathcal{T}$ we associate a sequence of fundamental domains: we record the translates of U (excluding the identity translation) the geodesic visits on its journey to the boundary. This *cutting sequence* has the form $(y_nU)_{n=0}^{\infty}$ with each $y_n \in F \setminus \{1\}$. We set $y_0 = x_0$ for $x_0 \in \mathcal{A} \cup \mathcal{A}^{-1}$ and then construct subsequent terms so that, for $k \geq 1$, $y_{k+1} = y_k x_{k+1}$ with $x_{k+1} \in \mathcal{A}$ such that $x_{k+1} \neq x_k^{-1}$. Note the second restriction follows because the geodesic ray is a non-backtracking path. In this way, each y_k is given by a reduced word $y_k = x_0 x_1 \cdots x_k$. Importantly, since the universal covering tree is simply connected, not only does a geodesic ray's cutting sequence record the fundamental domains it visits, but it also uniquely determines (up to identification of the origin) the sequence of lattice points $(y_k o)_{k=0}^{\infty}$ through which the ray passes.

We endow $\partial\mathcal{T}$ with the following metric: for each $\xi \in \partial\mathcal{T}$ take $d_{\partial\mathcal{T}}(\xi, \xi) = 0$, and for distinct $\xi, \eta \in \partial\mathcal{T}$ with respective cutting sequences $(a_k)_{k=0}^{\infty}$ and $(b_k)_{k=0}^{\infty}$ we take

$$d_{\partial\mathcal{T}}(\xi, \eta) = \theta^{\min\{k \in \mathbb{Z}^+ \colon a_k \neq b_k\}}$$

for a given $0 < \theta < 1$. For practical purposes we shall choose θ to coincide with our choice of constant for the metric d_θ on Σ, the space of infinite reduced words. There is an obvious bijection p between the lattice points $Fo = \{xo \in \mathcal{T} \colon x \in F\}$ and the set of finite reduced words W^*. We endow Fo with the metric that makes $p : Fo \to W^*$ an isometry. We can extend the map p to $p : Fo \cup \partial\mathcal{T} \to \Sigma^*$.

Lemma 4.3.1. *The map $p : Fo \cup \partial\mathcal{T} \to \Sigma^*$ with*

$$p|_{\partial\mathcal{T}} : \lim_{n\to\infty} x_0 x_1 \cdots x_n o \mapsto (x_n)_{n=0}^{\infty}$$

is a Hölder continuous homeomorphism.

Proof. The properties of $p|_{Fo}$ are clear and so we need only consider $p|_{\partial\mathcal{T}}$. For the remainder of the proof, we shall write p for $p|_{\partial\mathcal{T}}$. We first show that p is injective. Suppose, for a contradiction, that the map p is not an injection. Then there exist two distinct limit points $\xi, \eta \in \partial\mathcal{T}$ with $p(\xi) = p(\eta)$. We recall that the limit points correspond to two geodesic rays $[o, \xi]$ and $[o, \eta]$ that do not remain a bounded distance apart and so have distinct cutting sequences. Since the respective cutting sequences do not agree for all terms, by definition, $p(\xi)$ and $p(\eta)$ are distinct, a contradiction. It is clear that the map p is a surjection since to any $x = (x_n)_{n=0}^{\infty} \in \Sigma$ we can construct a geodesic ray in $\mathcal{T}$ by specifying the sequence $(x_0 x_1 \cdots x_n o)_{n=0}^{\infty}$ of lattice points through which the geodesic ray, emanating from $o \in \mathcal{T}$, passes. Recall the definitions for the metrics $d_{\partial\mathcal{T}}$ and d_θ. If we choose the same value of $0 < \theta < 1$ in the definitions for the metrics d_θ and $d_{\partial\mathcal{T}}$ we find that p is an isometry and so p is clearly Hölder continuous. We note that both $\partial\mathcal{T}$ and Σ are compact metric spaces, which is sufficent for the continuous bijection p to be a homeomorphism. □

4.4 Patterson–Sullivan measures

In this section we provide an introductory exposition to a family of measures intimately related to orbital counting functions. Let $\mathbb{D}^n$ be the n-dimensional unit ball equipped with the hyperbolic metric. Let Γ be a finitely generated discrete group of isometries acting properly discontinuously on $\mathbb{D}^n$. In [47], Patterson considered the structure of the limit sets of Fuchsian groups of the second kind without parabolic elements in the Poincaré disk; in particular, Patterson calculated the Hausdorff dimension of limits sets for these Fuchsian groups using a family of measures supported on the Fuchsian group's limit set.

Patterson's measure construction was generalised by Sullivan, in [66], for

Kleinian groups acting on n-dimensional hyperbolic space. We refer the reader to [41] for an account of these measures, the *Patterson–Sullivan measures*, in this setting. In [10] Coornaert extended the theory of Patterson–Sullivan measures to the case of hyperbolic groups of isometries acting properly discontinuously and co-compactly on a hyperbolic geodesic metric space. We refer the reader to [31] (Section 3), and the references therein, for an account of Patterson–Sullivan measures in the generalised setting of CAT(-1) spaces.

We restrict our exposition to the construction of a Patterson–Sullivan measure for metric trees. As before, let G be a metric graph with fundamental group F and universal covering tree $\mathcal{T}$. For a prescribed base vertex $o \in \mathcal{T}$, consider the Dirichlet series

$$\Pi_1(s) = \sum_{x \in F} \mathrm{e}^{-s d_{\mathcal{T}}(o,xo)}.$$

The *abscissa of convergence* of the series is the unique positive $h \in \mathbb{R}$ such that the series converges absolutely for $\mathrm{Re}(s) > h$ and diverges for $\mathrm{Re}(s) < h$. It is known that the series diverges for $s = h$, cf. [20], and so we say the series is of *divergence type*. In fact, h does not depend on the chosen base vertex $o \in \mathcal{T}$ and the value of h coincides with the volume entropy of the tree $\mathcal{T}$.

Let us first consider the family of measures μ_s, with $s \in \mathbb{C}$, supported on $\mathcal{T} \cup \partial\mathcal{T}$ given by

$$\mu_s = \frac{\sum_{x \in F} \mathrm{e}^{-s d_{\mathcal{T}}(o,xo)} \delta_{xo}}{\sum_{x \in F} \mathrm{e}^{-s d_{\mathcal{T}}(o,xo)}}$$

where δ_{xo} is the Dirac measure supported at $xo \in \mathcal{T}$. Take a sequence $(s_n)_{n=1}^{\infty}$ converging to h from above. Recall the Banach–Alaoglu theorem which states that the closed unit ball dual space is compact in the weak* topology. By compactness, there is a subsequence $(s_{n_k})_{k=1}^{\infty}$ such that the measures $\mu_{s_{n_k}}$ converge weakly to a limit μ_o as $k \to \infty$, a Patterson–Sullivan measure. Since the series $\Pi_1(s)$ is of divergence type, the measure μ_o is supported on the boundary $\partial\mathcal{T}$. Changing the prescribed point $o \in \mathcal{T}$ gives rise to a family of Patterson–Sullivan measures; however, we need only work with the Patterson–Sullivan μ_o. To avoid confusion with the notation μ_0 for the measure of maximal entropy, we rename the Patterson–

Sullivan measure μ_{PS}.

Because of the natural bijection between the set of finite reduced words W^* and the lattice points Fo, we shall find it useful to refer interchangeably to the spaces $Fo \cup \partial\mathcal{T}$ and Σ^*. This will permit us to avoid cumbersome notation when referring to a function $f : \Sigma^* \to \mathbb{R}$ and its pull-back $f \circ p : Fo \cup \partial\mathcal{T} \to \mathbb{R}$.

In the work that follows we establish the equidistribution limit with respect to the measure μ_{PS}. We note first that because the set of lattice points Fo are dense in the metric space $Fo \cup \partial\mathcal{T}$, we can extend a function $f : Fo \to \mathbb{R}$ continuously on the domain $Fo \cup \partial\mathcal{T}$. Suppose that $f : Fo \cup \partial\mathcal{T} \to \mathbb{R}$ is a continuous function. Take vertex $o \in \mathcal{T}$ as before. We write $\Pi_f(s)$ for the Poincaré series

$$\Pi_f(s) = \sum_{x \in F} \mathrm{e}^{-s d_{\mathcal{T}}(o,xo)} f(xo).$$

Recall the based length function $L : F \to \mathbb{R}$ given by $L(x) = d_{\mathcal{T}}(o, xo)$. Since, for each $x \in F$, $L(x)$ is given by an orbital sum of the Hölder function $r : \Sigma^* \to \mathbb{R}$, we can write the series $\Pi_f(s)$ in terms of the extended transfer operator $L_{-sr} : \mathcal{F}_\theta(\Sigma^*, \mathbb{C}) \to \mathcal{F}_\theta(\Sigma^*, \mathbb{C})$ so that

$$\Pi_f(s) = \sum_{n=0}^{\infty} L_{-sr}^n f(1).$$

Furthermore, since the family of measures μ_s converges weakly to μ_{PS} as $s \to h^+$, for each continuous function $f : \Sigma^* \to \mathbb{R}$ we have

$$\frac{\Pi_f(s)}{\Pi_1(s)} \to \int f \, \mathrm{d}\mu_{\mathrm{PS}}.$$

Recall that $L_{-sr} : \mathcal{F}_\theta(\Sigma, \mathbb{C}) \to \mathcal{F}_\theta(\Sigma, \mathbb{C})$ (with $s \in \mathbb{R}$) has a strictly positive eigenfunction $\psi_s : \Sigma \to \mathbb{R}$ associated to its maximal eigenvalue $\beta(s)$. As previously noted, we cannot apply results from the thermodynamic formalism directly for the transfer operator $L_{-sr} : \mathcal{F}_\theta(\Sigma^*, \mathbb{C}) \to C(\Sigma^*, \mathbb{C})$; nevertheless, the operator also has a continuous eigenfunction $\psi_s^* : \Sigma^* \to \mathbb{R}$ that satisfies

$$\psi_s^*(x) = \begin{cases} \psi_s(x) & \text{if } x \in \Sigma, \text{ and} \\ \dfrac{1}{\beta(s)} \displaystyle\sum_{\substack{\sigma y = x \\ y \neq 1}} \mathrm{e}^{-sr(y)} \psi_s^*(y) & \text{if } x \in W^*. \end{cases}$$

Indeed, for $s \in \mathbb{R}$ the eigenfunction ψ_s^* is the unique strictly positive solution to the above equations (Lemma 6.1, [36]).

We make clear the connection between the Patterson–Sullivan measure μ_{PS} and the eigenmeasure for the transfer operator.

Lemma 4.4.1. *Suppose that $f : \Sigma^* \to \mathbb{R}$ is an element of $\mathcal{F}_\theta(\Sigma^*, \mathbb{C})$ and $s \in \mathbb{R}$. Then as $s \to h+$,*

$$\frac{\Pi_f(s)}{\Pi_1(s)} \to \int f \, \mathrm{d}m.$$

where m is the eigenmeasure of the transfer operator L_{-hr}.

We forewarn the reader of the notation of $R_h 1(1)$ in the next proof. Here $R_h 1(1)$ denotes a projection of the function $1 : \Sigma^* \to \mathbb{R}$, where $1(x) = 1$ for each $x \in \Sigma^*$, evaluated at the empty word $1 \in W^*$.

Proof of Lemma 4.4.1. Suppose that $s \in \mathbb{R}$ and $s > h$. We decompose the transfer operator L_{-sr} into the projection R_s associated to the eigenspace spanned by the eigenfunction ψ_s^* associated to the eigenvalue $\beta(s)$ and $Q_s = L_{-sr} - \beta(s) R_s$. The spectral radius of Q_s is strictly smaller than $\beta(s)$. Then $\Pi_f(s)$ is given as follows:

$$\Pi_f(s) = \sum_{n=0}^{\infty} L_{-sr}^n f(1) = \sum_{n=1}^{\infty} \beta(s)^n R_s f(1) + \sum_{n=1}^{\infty} Q_s^n f(1) + f(1).$$

Here the series $\sum_{m=1}^{\infty} Q_s^n f(1)$ converges absolutely in the half plane $\mathrm{Re}(s) > h$. The convergence of the geometric series with terms $\beta(s)^n R_s f(1)$ gives

$$\frac{R_s f(1) \beta(s)}{1 - \beta(s)} = -\frac{R_h f(1)}{\beta'(h)(s-h)} + \phi_1(s).$$

The function $\phi_1(s)$ is analytic in a neighbourhood of h and vanishes at $s = h$, from which obtain the following ratio in the limit as $s \to h^+$

$$\frac{\Pi_f(s)}{\Pi_1(s)} \to \frac{R_h f(1)}{R_h 1(1)}.$$

We evaluate the ratio using the Ruelle–Perron–Frobenius Theorem and recover the desired limit. □

Since the family of Lipschitz functions is uniformly dense within the continuous functions, the limit in Lemma 4.4.1 holds for all continuous functions, i.e. we have weak* convergence of the family of measures μ_s, $s \in \mathbb{C}$, with respect to the eigenmeasure m. The following lemma is clear.

Lemma 4.4.2. *The eigenmeasure m of the transfer operator L_{-hr} is the pushforward of the Patterson–Sullivan measure, i.e. $p_* \mu_{PS} = m$.*

4.5 A complex generating function

In this section we give the proof of Theorem 4.1.1. For the benefit of the reader we recall the definition of the counting function $N_{\mathfrak{C}}^B(T)$ given by

$$N_{\mathfrak{C}}^B(T) = \#\{x \in \mathfrak{C} \colon L(x) \leq T,\ xo \in \mathcal{S}(B)\}$$

and the statement of the result below.

Theorem 4.1.1. *Suppose that G is a finite connected metric graph such that the degree of each vertex is at least 3. Additionally we assume that there is a pair of closed geodesics in G whose ratio of metric lengths is irrational. Let $\mathfrak{C}$ be a non-trivial conjugacy class in F. Let B be a Borel subset of $\partial \mathcal{T}$. Then*

$$\lim_{T \to \infty} \frac{N_{\mathfrak{C}}^B(T)}{N_{\mathfrak{C}}(T)} = \mu_{\mathrm{PS}}(B).$$

In order to prove Theorem 4.1.1 we shall establish the asymptotic behaviour of $N_{\mathfrak{C}}^B(T)$ as $T \to \infty$ in terms of the asymptotic result due to Kenison and Sharp in Chapter 3 for $N_{\mathfrak{C}}(T)$ as $T \to \infty$. Indeed, our approach, studying the complex analytic properties of a related complex generating function, is a refinement of the analysis in Chapter 3.

A non-trivial word $\alpha = \alpha_0 \cdots \alpha_{n-1}$ with each $\alpha_k \in \mathcal{A} \cup \mathcal{A}^{-1}$ determines a *cylinder set* $[\alpha] = \{(x_k)_{k=0}^{\infty} \in \Sigma \colon x_k = \alpha_k,\, k \in \{0, 1, \ldots, n-1\}\}$. The subset $[\alpha]$ is non-empty if and only if α is a reduced word representation of a non-trivial element in F (so $\alpha_{k+1} \neq \alpha_k^{-1}$). From here on, when considering a cylinder set $[\alpha]$ we shall always assume that α is a reduced word. The *length* of the cylinder set $[\alpha]$ is the word length of α. The algebra formed from the cylinder sets in Σ generates the Borel σ-algebra and each cylinder set is both closed and open in the topology induced by the above metric on Σ.

Recall that a function $f : \Sigma \to \mathbb{R}$ is said to be *locally constant* if there exists an $N \in \mathbb{N}$ such that for any two elements $x, y \in \Sigma$ with $x_n = y_n$ for every $n \in \{0, 1, \ldots, N-1\}$ we have $f(x) = f(y)$. Indeed, the connection between locally constant functions and cylinder sets is clear: given a cylinder set $[\alpha]$ of length N, for any pair $x, y \in [\alpha]$ we have $f(x) = f(y)$ if f considers only the first N terms in a sequence.

In this section, we shall prove Theorem 4.1.1 for the subsets on the boundary given by the pushforward of cylinder sets in Σ^* and then give an approximation argument to recover the result for a general Borel set.

In addition, to the cylinder subsets of Σ, we shall also refer to certain subsets of Σ^* as cylinder sets. A *cylinder set*, written $[\alpha] \subset \Sigma^*$ with $\alpha = \alpha_0 \cdots \alpha_{k-1} \in W'_k$, is a set of the form

$$[\alpha] = \{x \in \Sigma^* \colon |x| \geq k \text{ and } x_n = \alpha_n \, \forall n \in \{0, \ldots, k-1\}\},$$

where $|x| = \infty$ for each $x \in \Sigma$. Thus $(x_n)_{n=0}^{\infty} \in \Sigma^*$ is an element of $[\alpha]$ if $x_n = \alpha_n$ for each $n \in \{0, \ldots, k-1\}$. We define the length of the cylinder set $[\alpha]$, written $|[\alpha]|$, to be the word length $|\alpha|$.

Consider the counting function $N_{\mathfrak{C}}^{[\alpha]}(T)$ (we shall drop the square brackets and instead write $N_{\mathfrak{C}}^{\alpha}(T)$). The sector $\mathcal{S}([\alpha])$ restricts precisely to those lattice points $xo \in \mathcal{T}$ with $x \in [\alpha]$. So it is clear that

$$N_{\mathfrak{C}}^{\alpha}(T) = \{x \in \mathfrak{C} \colon L(x) \leq T, x \in [\alpha]\}.$$

In the exposition that follows we shall assume, without loss of generality, that $|[\alpha]| = N$ with N the natural number from Lemma 2.5.2. We do not lose any generality under this assumption because if $|[\alpha]| < N$ then we can decompose $[\alpha]$ into a disjoint union of cylinder sets of length N and then consider the sum of their respective counting functions in order to recover $N_{\mathfrak{C}}^{\alpha}(T)$. In the case that $|[\alpha]| > N$ we update the value of N, setting $N = |[\alpha]|$, and the proof that follows works verbatim.

We recall our earlier notation related to non-trivial conjugacy classes. Suppose that $\mathfrak{C}$ is a non-trivial conjugacy class in the free group F. Let $\mathfrak{C}_n$ denote the set of $x \in \mathfrak{C}$ such that $|x| = n$ and let $k = \min\{|x| \colon x \in \mathfrak{C}\} > 0$. Clearly $\mathfrak{C}_n$ is non-empty if and only if $n = k + 2m$ for $m \in \mathbb{N}$. The set of $g \in \mathfrak{C}_k$ are precisely those elements in $\mathfrak{C}$ whose reduced word representations are cyclically reduced. In fact, if $g = g_1 \cdots g_k \in \mathfrak{C}_k$ then all the remaining elements of $\mathfrak{C}_k$ are given by cyclic permutations of the letters in $g_1 \cdots g_k$.

The reduced word representation of each $x \in \mathfrak{C}$ takes the form

$$x = wgw^{-1} = w_1 \cdots w_m g_1 \cdots g_k w_m^{-1} \cdots w_1^{-1}$$

with $g = g_1 \cdots g_k \in \mathfrak{C}_k$; and $w = w_1 \cdots w_m \in W_m'$ subject to the restriction $w_m \neq g_k^{-1}, g_1$ in order that no pairwise cancellation occurs when concatenating w, g and w^{-1}. For a given $g = g_1 \cdots g_k$, let $W_m'(g)$ denote the set of elements $w \in W_m'$ such that $w_m \neq g_1^{-1}, g_k$.

We study the analytic properties of the generating function $\eta_{\mathfrak{C}}^{\alpha}(s)$ given by

$$\eta_{\mathfrak{C}}^{\alpha}(s) = \sum_{x \in \mathfrak{C}} e^{-sL(x)} \chi_\alpha(x) = \sum_{n=1}^{\infty} \sum_{x \in \mathfrak{C}_n} e^{-sL(x)} \chi_\alpha(x).$$

Here the function $\chi_\alpha : F \to \mathbb{R}$ is a characteristic function such that for $x \in F$ with reduced word representation $x_0x_1\cdots x_{n-1}$, $\chi_\alpha(x) = 1$ if $x_0x_1\cdots x_{n-1} \in [\alpha]$ and 0 otherwise.

Using the above decomposition of elements in the conjugacy class we have

$$\eta^\alpha_{\mathfrak{C}}(s) = \sum_{g\in\mathfrak{C}_k} \sum_{m=0}^{\infty} \sum_{w\in W'_m(g)} \mathrm{e}^{-sL(wgw^{-1})}\chi_\alpha(w).$$

Then, modulo the addition of an entire function, we use Lemma 3.3.1 to write $\eta^\alpha_{\mathfrak{C}}(s)$ as follows

$$\eta^\alpha_{\mathfrak{C}}(s) = \sum_{g\in\mathfrak{C}_k} \sum_{y\in W'_{N-1}(g)} e^{-s(L(ygy^{-1})-2L(y))} \sum_{m=N+1}^{\infty} \sum_{\substack{w\in W'_m \\ y=w_{m-N}\cdots w_m}} e^{-2sL(w)}\chi_\alpha(w).$$

We have the following equality using the family of matrices A_{2s} from Section 4.2

$$\sum_{m=N+1}^{\infty} \sum_{\substack{w\in W'_m \\ y=w_{m-N}\cdots w_m}} e^{-2sL(w)}\chi_\alpha(w) = \mathrm{e}^{-2sL(y)} \sum_{m=1}^{\infty} \mathsf{e}_\alpha^{\mathrm{T}} A_{2s}^m \mathsf{e}_y.$$

Here we write e_x for the standard unit column vector with entry 1 in the term indexed by $x \in W_{N-1}$.

We will find Lemma 3.3.2 useful in the next proposition. For convenience, we restate Lemma 3.3.2 below.

Lemma 3.3.2. *Let $M \in \mathbb{R}^{d\times d}$. Suppose that M has a simple eigenvalue β, and that u and v are the associated left and right eigenvectors, normalised so that $u \cdot v = 1$. Then for each $w \in \mathbb{R}^d$ we can write $w = (u\cdot w)v + \overline{v}$, where $\overline{v}$ lies in the span of the generalised right eigenvectors of M not associated to β.*

Proposition 4.5.1. *The generating function $\eta^\alpha_{\mathfrak{C}}(s)$ is analytic in the half-plane* $\mathrm{Re}(s) > h/2$*, has a simple pole at $s = h/2$ with positive residue, and has an analytic extension to a neighbourhood of* $\mathrm{Re}(s) \geq h/2$ *save for the simple pole at $s = h/2$.*

Proof. For $\sigma \in \mathbb{R}$, the spectral radius of the matrix $A_{2\sigma}$, given by $\beta(2\sigma)$, is strictly decreasing. Since $\beta(h) = 0$, the series $\sum_{m=1}^{\infty} \mathsf{e}_\alpha^{\mathrm{T}} A_{2\sigma}^m \mathsf{e}_y$ converges absolutely for $\sigma >$

$h/2$. It follows that the generating function $\eta_{\mathfrak{C}}^{\alpha}(s)$ is analytic in the half-plane $\mathrm{Re}(s) > h/2$.

Now consider $s \in \mathbb{C}$ with $s = h/2 + \mathrm{i}t$. The analyticity of $\eta_{\mathfrak{C}}^{\alpha}(s)$ is determined by the convergence of $\sum_{m=1}^{\infty} \mathsf{e}_{\alpha}^{\mathrm{T}} A_{2s}^{m} \mathsf{e}_{y}$. By Wielandt's Theorem, there are two cases to consider, either

1. $\mathrm{spr}(A_{h+2\mathrm{i}t}) = \mathrm{spr}(B_{h+2\mathrm{i}t}) = \mathrm{spr}(B_h) = 1$, in which case $A_{h+2\mathrm{i}t}$ has a simple eigenvalue $\beta(h + 2\mathrm{i}t)$ such that $|\beta(h + 2\mathrm{i}t)| = \beta(h)$, whilst all the other eigenvalues of $A_{h+2\mathrm{i}t}$ are strictly smaller in modulus, or

2. $\mathrm{spr}(A_{h+2\mathrm{i}t}) = \mathrm{spr}(B_{h+2\mathrm{i}t}) < 1$.

In case (1), perturbation theory ensures that the simple maximal eigenvalue $\beta(2s)$ of A_{2s} persists and varies analytically with s in a neighbourhood of $h/2 + \mathrm{i}t$; as do the associated left and right eigenvectors $u(2s)$ and $v(2s)$. From the eigenvalue decomposition in Lemma 3.3.2 we have

$$\begin{aligned}\sum_{m=1}^{\infty} \mathsf{e}_{\alpha}^{\mathrm{T}} A_{2s}^{m} \mathsf{e}_{y} &= \sum_{m=1}^{\infty} \mathsf{e}_{\alpha}^{\mathrm{T}} A_{2s}^{m} (u(2s) \cdot \mathsf{e}_{y}) v(2s) + \sum_{m=1}^{\infty} \mathsf{e}_{\alpha}^{\mathrm{T}} A_{2s}^{m} \overline{v}(2s) \\ &= (\mathsf{e}_{\alpha}^{\mathrm{T}} \cdot v(2s))(u(2s) \cdot \mathsf{e}_{y}) \frac{\beta(2s)}{1 - \beta(2s)} + \phi_2(s),\end{aligned}$$

where $\phi_2(s)$ is analytic in a neighbourhood of $h/2 + \mathrm{i}t$. Thus $\eta_{\mathfrak{C}}(s)$ is analytic in a neighbourhood of $h/2 + \mathrm{i}t$ unless $\beta(h + 2\mathrm{i}t) = 1$, which, by Lemma 3.2.7, occurs only when $t = 0$.

In case (2), the result follows immediately: the series $\sum_{m=1}^{\infty} \mathsf{e}_{\alpha}^{\mathrm{T}} A_{2s}^{m} \mathsf{e}_{y}$ converges; hence $\eta_{\mathfrak{C}}^{\alpha}(s)$ is analytic for s in a neighbourhood of $h/2 + \mathrm{i}t$.

We now show that the pole at $s = h/2$ is simple and has positive residue. The component of $\eta_{\mathfrak{C}}^{\alpha}(s)$ that determines the analyticity at $s = h/2$ is the infinite series $\sum_{m=1}^{\infty} \mathsf{e}_{\alpha}^{\mathrm{T}} A_{2s}^{m} \mathsf{e}_{y}$ that, modulo the addition of a function that is analytic in a neighbourhood of $s = h/2$, we write as

$$(\mathsf{e}_{\alpha}^{\mathrm{T}} \cdot v(2s))(u(2s) \cdot \mathsf{e}_{y}) \frac{\beta(2s)}{1 - \beta(2s)} = \frac{(\mathsf{e}_{\alpha}^{\mathrm{T}} \cdot v(h))(u(h) \cdot \mathsf{e}_{y})}{-2\beta'(h)(s - h/2)}.$$

The terms $\mathsf{e}_\alpha^{\mathrm{T}} \cdot v(h)$ and $u(h) \cdot \mathsf{e}_y$ are both positive by definition. We also know that $\beta'(h) < 0$ since β is strictly decreasing in $\mathbb{R}$. We conclude that $\eta_{\mathfrak{C}}^\alpha(s)$ has positive residue at $s = h/2$. □

For the required asymptotic result in Theorem 4.1.1 we interpret the residue of the simple pole of $\eta_{\mathfrak{C}}^\alpha(s)$ at $s = h/2$, given by $(\mathsf{e}_\alpha^{\mathrm{T}} \cdot v(h))(u(h) \cdot \mathsf{e}_y)/(-2\beta'(h))$ in terms of eigenfunctions of the transfer operator. Thus it makes sense that we first make clear the connection between the family of weighted transition matrices B_s and the transfer operator L_{-sr}.

We write $V_{N-1} \subset \mathcal{F}_\theta(\Sigma^*, \mathbb{C})$ for the finite dimensional subspace consisting of locally constant functions depending on the first $N-1$ coordinates. This has dimension $D := \#W'_{N-1}$. Then $L_{-sr} : V_{N-1} \to V_{N-1}$ and the restriction of L_{-sr} to V_{N-1} can be identified with the action of B_s on $\mathbb{C}^D$.

By the Ruelle–Perron–Frobenius Theorem we have, as $n \to \infty$,

$$\frac{1}{\beta(h)^n} L_{-hr}^n \chi_\alpha(y) \to \psi_h(y) \int \chi_\alpha \, dm = \psi_h(y) \mu_{\mathrm{PS}}([\alpha]).$$

Since $\chi_\alpha \in V_{N-1}$, we write the above limit in terms of the family of weighted transition matrices as follows

$$\frac{1}{\beta(h)^n} \mathsf{e}_\alpha^{\mathrm{T}} A_h^n \mathsf{e}_y \to (\mathsf{e}_\alpha^{\mathrm{T}} \cdot v(h))(u(h) \cdot \mathsf{e}_y).$$

Here we note $\mathsf{e}_\alpha^{\mathrm{T}} A_h^n \mathsf{e}_y = \mathsf{e}_\alpha^{\mathrm{T}} B_h^n \mathsf{e}_y$. Thus we equate the limits $(\mathsf{e}_\alpha^{\mathrm{T}} \cdot v(h))(u(h) \cdot \mathsf{e}_y) = \psi_h(y) \mu_{\mathrm{PS}}([\alpha])$ and so for s in a neighbourhood of $h/2$ we have, modulo the addition of an analytic function,

$$\eta_{\mathfrak{C}}^\alpha(s) = \frac{\mu_{\mathrm{PS}}([\alpha])}{-2\beta'(h)} \sum_{g \in \mathfrak{C}_k} \sum_{y \in W'_{N-1}(g)} \mathrm{e}^{-hL(ygy^{-1})} \psi_h(y) \frac{1}{s - h/2}.$$

Since $\eta_{\mathfrak{C}}^\alpha(s)$ is related to the counting function $N_{\mathfrak{C}}^\alpha(T)$ via the Stieltjes integral

$$\eta_{\mathfrak{C}}^\alpha(s) = \int_0^\infty \mathrm{e}^{-sT} \, dN_{\mathfrak{C}}^\alpha(T),$$

the asymptotic formula for $N_{\mathfrak{C}}^{\alpha}(T)$ follows from the Ikehara–Wiener Tauberian theorem (Theorem 3.2.8). We conclude that, as $T \to \infty$,

$$N_{\mathfrak{C}}^{\alpha}(T) \sim \mu_{\mathrm{PS}}([\alpha]) C \mathrm{e}^{hT/2}.$$

Here $C(\mathfrak{C}) > 0$ independent of $[\alpha]$. We obtain the main result for $N_{\mathfrak{C}}(T)$ from the previous chapter: we have $N_{\mathfrak{C}}(T) \sim C\mathrm{e}^{hT/2}$ as $T \to \infty$.

We now extend the asymptotic formula to all Borel subsets with the following argument. Since the family of cylinder sets generate the Borel σ-algebra on the probability space (Σ^*, m), given a Borel set B we ε-approximate B (in measure) by collections of cylinder sets.

Suppose that $\mathcal{U}$ is a countable collection of disjoint cylinder sets then clearly we have

$$\sum_{[\alpha]\in\mathcal{U}} N_{\mathfrak{C}}^{\alpha}(T) \sim \mu_{\mathrm{PS}}\left(\bigcup_{[\alpha]\in\mathcal{U}} [\alpha]\right) C\mathrm{e}^{hT/2}.$$

as $T \to \infty$. We can find $\mathcal{U}, \mathcal{U}'$ collections of disjoint cylinder sets such that $\bigcup_{[\alpha]\in\mathcal{U}}[\alpha] \subseteq B \subseteq \bigcup_{[\alpha]\in\mathcal{U}'}[\alpha]$ and by regularity

$$\mu_{\mathrm{PS}}(B) - \varepsilon \leq \mu_{\mathrm{PS}}\left(\bigcup_{[\alpha]\in\mathcal{U}} [\alpha]\right) \leq \mu_{\mathrm{PS}}\left(\bigcup_{[\alpha]\in\mathcal{U}'} [\alpha]\right) \leq \mu_{\mathrm{PS}}(B) + \varepsilon.$$

Clearly $\sum_{[\alpha]\in\mathcal{U}} N_{\mathfrak{C}}^{\alpha}(T) \leq N_{\mathfrak{C}}^{B}(T) \leq \sum_{[\alpha]\in\mathcal{U}'} N_{\mathfrak{C}}^{\alpha}(T)$ and so, since $\varepsilon > 0$ was arbitrarily chosen, we have the asymptotic formula $N_{\mathfrak{C}}^{B}(T) \sim \mu_{\mathrm{PS}}(B) C\mathrm{e}^{hT/2}$ as $T \to \infty$. Thus

$$\lim_{T\to\infty} \frac{N_{\mathfrak{C}}^{B}(T)}{N_{\mathfrak{C}}(T)} = \mu_{\mathrm{PS}}(B),$$

the desired limit for Theorem 4.1.1.

Chapter 5

Asymptotic statistics in conjugacy classes for free groups

5.1 Introduction

In the introduction to this thesis we stated two results, Theorems 1.4.2 and 1.4.3, for a convex compact action of a free group on a CAT(-1) space where we consider only those elements in a non-trivial conjugacy class of the group. We now state state the technical result from which Theorem 1.4.2 follows. We consider functions $\Theta : F \to \mathbb{R}$ which satisfy the following two asssumptions.

(A1) There exists a Hölder continuous function $f : \Sigma^* \to \mathbb{R}$ so that $\Theta(x) = f^n(x)$ for each $x \in W_n'$ with $n \geq 0$, and

(A2) $\Theta(x) = \Theta(x^{-1})$.

We will prove the following theorem.

Theorem 5.1.1. *Suppose that $\Theta : F \to \mathbb{R}$ satisfies assumptions (A1) and (A2). There exists $\overline{\Theta} \in \mathbb{R}$ such that*

$$\lim_{m\to\infty} \frac{1}{\#\mathfrak{C}_{k+2m}} \sum_{x\in\mathfrak{C}_{k+2m}} \frac{\Theta(x)}{k+2m} = \overline{\Theta}.$$

Furthermore, $\overline{\Theta} = \int f\, d\mu_0$. *Here* μ_0 *is the measure of maximal entropy for the subshift of finite type* Σ^* *supported on* Σ.

We remark that, without the restriction to the conjugacy class, the analogous result

$$\lim_{m\to\infty} \frac{1}{\# W'_m} \sum_{x\in W'_m} \frac{\Theta(x)}{m} = \overline{\Theta}$$

holds subject only to assumption (A1). This follows from the analysis in [54] or from a large deviations argument following the ideas of Kifer [34] as employed in [52]. Thus the measure of maximal entropy μ_0 describes the asymptotic distribution of pre-images. Specifically, for each $m \geq 1$, define a measure μ_m by

$$\frac{1}{\# W'_m} \sum_{x\in W'_m} \frac{f^m(x)}{m} = \int f\, d\mu_m$$

then we have weak* convergence of μ_m to μ_0 as $m \to \infty$: for each continuous function $f : \Sigma^* \to \mathbb{R}$ we have

$$\lim_{m\to\infty} \int f\, d\mu_m = \int f\, d\mu_0.$$

This distribution result for pre-images in Σ^* is analogous to the asymptotic distribution of periodic points for mixing subshifts of finite type (cf. Theorem 8.17, [67]). By the asymptotic distribution of periodic points for mixing subshifts of finite type, we mean the weak* convergence of the orbital measures

$$\frac{1}{\# \mathrm{Fix}_n} \sum_{x\in \mathrm{Fix}_n} \delta_x$$

to the measure of maximal entropy μ_0 as $n \to \infty$.

We also establish a central limit theorem over the elements of a non-trivial conjugacy class in the free group. In addition to assumptions (A1) and (A2), we require a third assumption.

(A3) The function $\Theta(\cdot) - \overline{\Theta}|\cdot| : F \to \mathbb{R}$ is unbounded.

Before we prove the next lemma, let us recall the constant σ_f^2, defined in Proposition 2.3.1, given by

$$\frac{d^2P(tf)}{dt^2}\bigg|_{t=0} = \sigma_f^2 := \lim_{n\to\infty}\frac{1}{n}\int\left(f^n(x) - n\int f\,d\mu_0\right)^2 d\mu_0.$$

Lemma 5.1.2. *Let Θ and f be as in (A1). Then $\Theta(\cdot) - \overline{\Theta}|\cdot|$ is bounded if and only if $f|_\Sigma$ is cohomologous to a constant.*

Proof. For simplicity, we will write $f|_\Sigma = f$. If $\Theta(\cdot) - \overline{\Theta}|\cdot|$ is bounded then $\left\{f^n(x) - n\int f\,d\mu_0 \colon x \in W_n',\ n \geq 1\right\}$ is a bounded set. Since f is Hölder continuous, we also have that $\left\{f^n(x) - n\int f\,d\mu_0 \colon x \in \Sigma,\ n \geq 1\right\}$ is bounded too. In particular, $\left(f^n - n\int f\,d\mu_0\right)^2/n$ converges uniformly to zero and it is easy to deduce that $\sigma_f^2 = 0$. Therefore, by Proposition 2.3.1, f is cohomologous to a constant.

On the other hand, if f is cohomologous to a constant then, again by Hölder continuity, $\{\Theta(x) - \overline{\Theta}|x| : x \in F\} = \left\{f^n(x) - n\int f\,d\mu_0 : x \in W_n',\ n \geq 1\right\}$ is bounded. □

By Lemma 5.1.2, assumption (A3) is necessary and sufficient for $\sigma_f^2 > 0$. It is well-known (cf. [44], Chapter 4) that if $f : \Sigma \to \mathbb{R}$ is not cohomologous to a constant then the deterministic process $f \circ \sigma^n$, $n \geq 1$ satisfies a central limit theorem with respect to μ_0. More precisely, we mean that the sequence $(f^n(x) - n\int f\,d\mu_0)/\sqrt{n}$ converges in distribution to a Gaussian random variable $N(0, \sigma_f^2)$ [9], i.e., for $a \in \mathbb{R}$

$$\lim_{n\to\infty}\mu_0\left\{x \in \Sigma \colon \left(f^n(x) - n\int f\,d\mu_0\right)/\sqrt{n} \leq a\right\} = \frac{1}{\sqrt{2\pi}\sigma_f}\int_{-\infty}^{a} \mathrm{e}^{-t^2/2\sigma^2}\,dt.$$

Analogous results hold periodic points of $\sigma : \Sigma \to \Sigma$ [9] and, by adapting the proof, for pre-images of a given point.

Restricting to a non-trivial conjugacy class we have the following result.

Theorem 5.1.3. *Suppose that $\Theta : F \to \mathbb{R}$ satisfies assumptions (A1), (A2) and (A3). Then the sequence of real-valued functions $(\mathscr{G}_m)_{m=1}^\infty$ defined by*

$$\mathscr{G}_m(a) = \frac{1}{\#\mathfrak{C}_{k+2m}}\#\left\{x \in \mathfrak{C}_{k+2m} \colon \left(\Theta(x) - \overline{\Theta}(k+2m)\right)/\sqrt{k+2m} \leq a\right\}$$

converges to the distribution function of a normal random variable with mean 0 *and positive variance* $2\sigma_f^2$.

We note the limiting distribution function is independent of the choice of non-trivial conjugacy class. Further, it is interesting that the variance of the limiting function in Theorem 5.1.3 is twice the variance of the limiting function when we do not restrict elements $x \in F$ to a non-trivial conjugacy class.

Let F be a free group on $l \geq 2$ generators acting convex co-compactly on a CAT(-1) space (X, d) and a non-trivial conjugacy class $\mathfrak{C}$ as before. Recall Theorems 1.4.2 and 1.4.3 from the introductory chapter, which we repeat below for convenience. Given a free generating set $\mathcal{A} = \{a_1, \ldots a_l\}$ of the free group F, there is a finite positive constant $\lambda \in \mathbb{R}$ such that for any arbitrarily chosen basepoint $o \in X$ the following holds.

Theorem 1.4.2. *We have the limit*

$$\lim_{m\to\infty} \frac{1}{\#\mathfrak{C}_{k+2m}} \sum_{x\in\mathfrak{C}_{k+2m}} \frac{d(o, xo)}{k+2m} = \lambda.$$

Subject to an additional non-degeneracy condition we have a central limit theorem.

Theorem 1.4.3. *Suppose that the set* $\{d(o, xo) - \lambda|x| \colon x \in F\}$ *is unbounded then*

$$\lim_{n\to\infty} \frac{1}{\#\mathfrak{C}_{k+2m}} \left\{ x \in \mathfrak{C}_{k+2m} \colon \frac{d(o, xo) - \lambda(k+2m)}{\sqrt{k+2m}} \leq a \right\} = \frac{1}{2\sqrt{\pi}\sigma} \int_{-\infty}^{a} \mathrm{e}^{-t^2/4\sigma^2}\, dt.$$

Proof of Theorems 1.4.2 and 1.4.3. Suppose that the free group F act convex co-compactly on a CAT(-1) space (X, d). Then it was shown in [56] that $\Theta(x) := d(o, xo)$ satisfies (A1). (In fact, the result in [56] is stated when X is a simply connected manifold with bounded negative curvatures, but the proof only requires we assume (X, d) is CAT(-1).) Assumption (A2) is clearly satisfied. Therefore, Theorem 1.4.2 follows from Theorem 5.1.1. Furthermore, the additional non-degeneracy assumption on $d(o, xo)$ in Theorem 1.4.3 is equivalent to assumption (A3) by Lemma 5.1.2. Thus Theorem 1.4.3 similarly follows from Theorem 5.1.3. □

In [9], Coelho and Parry established a central limit theorem for Hölder continuous functions on a subshift of finite type Σ_A with aperiodic transition matrix A. Let $f, g : \Sigma_A \to \mathbb{R}$ be real-valued Hölder continuous functions and m the equilibrium state of g. We suppose that g is normalised, $\sigma_f^2 > 0$ and $\int f\, dm = 0$. The authors' analysis used perturbations of the transfer operator L_g of the form $L_{g+\mathrm{i}tf/\sqrt{n}}$. For small perturbations, the operator $L_{g+\mathrm{i}tf/\sqrt{n}}$ has a simple maximal eigenvalue $\mathrm{e}^{P(g+\mathrm{i}tf/\sqrt{n})}$ and crucially

$$\lim_{n\to\infty} \mathrm{e}^{nP(g+\mathrm{i}tf/\sqrt{n})} = \mathrm{e}^{-t^2\sigma_f^2/2}.$$

Coelho and Parry deduced that the distribution function

$$\mathscr{H}_n(a) = m\{y \in \Sigma_A \colon f^n(y)/\sqrt{n} < a\}$$

converges as $n \to \infty$ to the distribution of a normal random variable with mean 0 and variance σ_f^2. The authors also gave the error term $O(1/\sqrt{n})$ for the speed of convergence. The authors' deduction followed from the convergence of the sequence of Fourier transformations associated to the distribution functions (see Theorem 5.3.1). The authors found it useful to write the sequence of Fourier transformations in terms of $\mathrm{e}^{nP(g+\mathrm{i}tf/\sqrt{n})}$ since $\mathrm{e}^{-t^2\sigma_f^2/2}$ is the Fourier transform of a normal distribution with mean 0 and variance σ_f^2. The transfer operator method employed by the Coelho and Parry is convenient in this setting: the Fourier transform of $\mathscr{H}_n$ is given by

$$\int \mathrm{e}^{\mathrm{i}tf^n(x)/\sqrt{n}}\, dm = \int L^n_{g+\mathrm{i}tf/\sqrt{n}} 1\, dm.$$

We say $f : \Sigma_A \to \mathbb{R}$ is *non-lattice* if f is not cohomologous to a function $a + b\psi$ where $a, b \in \mathbb{R}$ and $\psi : \Sigma_A \to \mathbb{Z}$. Under the additional assumption that f is non-lattice Coelho and Parry achieved a more precise error term estimate $o(1/\sqrt{n})$.

Coelho and Parry [9] also gave a central limit theorem for orbital measures (Theorem 5, [9]) that is close to the result in Theorem 5.1.3. We give a brief account of their analysis: let us assume that $f, g : \Sigma_A \to \mathbb{R}$ are Hölder continuous functions such that g is normalised with equilibrium state m. We also assume that $\sigma_f^2 > 0$ and $\int f\, dm = 0$ as before. Let $\mathrm{Fix}_n = \{x \in \Sigma_A \colon \sigma^n x = x\}$ and let $(m_n)_{n=1}^\infty$ be the sequence of orbital measures so that

$$\int k\, dm_n = \frac{\sum_{x \in \mathrm{Fix}_n} k(x) \mathrm{e}^{g^n(x)}}{\sum_{x \in \mathrm{Fix}_n} \mathrm{e}^{g^n(x)}}.$$

The sequence $(m_n)_{n=1}^\infty$ converges weak* to the equilibrium state m (cf. [60]). The authors showed that the sequence of distribution functions $m_n\{y \in \Sigma \colon f^n(y)/\sqrt{n} < a\}$ converges as $n \to \infty$ to the distribution of a normal random variable with mean 0 and variance σ_f^2. Again, by Theorem 5.3.1 it sufficed to show that as $n \to \infty$ we have the following convergence of the Fourier transformations

$$\frac{\sum_{x \in \mathrm{Fix}_n} \mathrm{e}^{g^n(x) + \mathrm{i}t f^n(x)/\sqrt{n}}}{\sum_{x \in \mathrm{Fix}_n} \mathrm{e}^{g^n(x)}} \to \mathrm{e}^{-\sigma_f^2 t^2/2}.$$

Much work has been done establishing central limit theorems for functions acting on hyperbolic groups. For example, let F be a free group generated by $l \geq 2$ elements and consider the abelianization homomorphism $[\cdot] : F \to F/[F,F] \cong \mathbb{Z}^l$. It is interesting to consider the distribution of the images under the mapping $[\cdot]$ of the elements W_n' in $\mathbb{Z}^l$. Indeed, Rivin proved a central limit theorem in this setting [58].

Horsham and Sharp [24] (see also [23]) proved central limit results for quasimorphisms on free groups. A function $\varphi : F \to \mathbb{R}$ is a *quasimorphism* if there exists a constant $D \geq 0$ so that for each $x, y \in F$, $|\varphi(xy) - \varphi(x) - \varphi(y)| \leq D$. It is clear that if a function φ is either a homomorphism or a bounded function then φ is a quasimorphism and it is easily shown that quasimorphisms are bounded on conjugacy classes. The authors used the thermodynamic formalism for subshifts of finite type to establish central limit theorems for quasimorphisms. We note Rivin's combinatorial method, involving cyclic counting properties, for homomorphisms does not generalise for maps such as quasimorphisms. Calegari and Fujiwara [8] prove a central limit theorem for quasimorphisms on Gromov hyperbolic groups, but have more restrictions on the regularity of the quasimorphism.

Example 5.1.4. Brooks gave the following construction, in [6], for a family of quasimorphisms on free groups that are not generally homomorphisms. Let ς be a reduced word in W^* and $\varphi_\varsigma : F \to \mathbb{Z}$ the map such the $\varphi_\varsigma(x)$ is given by the difference between the number of occurances of ς and ς^{-1} as a subword in the reduced word representation of x. Then φ_ς is a quasimorphism [6].

Remark 5.1.5. We briefly comment on the asymptotic statistics in Theorems 5.1.1 and 5.1.3 and discuss the inclusion of assumption (A2) when we restrict the group elements to a non-trivial conjugacy class. Indeed, assumption (A2) is crucial in our proofs of Theorems 5.1.1 and 5.1.3 and so it is worth considering the necessity and sufficiency of this assumption.

By definition, a homomorphism $\Theta : F \to \mathbb{R}$ of the free group satisfies $\Theta(x) = -\Theta(x^{-1})$ for each $x \in F$ and so certainly does not satisfy (A2). We make two observations for homomorphisms related to Theorems 5.1.1 and 5.1.3.

We first observe that in the case that $\Theta : F \to \mathbb{R}$ is a homomorphism or, more generally, a quasimorphism the limit

$$\lim_{m\to\infty} \frac{1}{\#W'_m} \sum_{x \in W'_m} \frac{\Theta(x)}{m} = 0$$

follows from the trivial observation that $x \in W'_m$ if and only if $x^{-1} \in W'_m$. Moreover, the limit is still 0 when we restrict the group elements to a non-trivial conjugacy class; however, in this instance the limit holds because quasimorphisms are bounded on conjugacy classes. We cannot comment in general on the existence of the limit when we do not assume (A2).

From our first observation, we note that homomorphisms, and more generally quasimorphisms, that satisfy (A1) have $\int f \, d\mu_0 = 0$. Thus when considering central limit theorems related to homomorphisms we do not need to normalise to obtain a mean zero function.

A central limit theorem holds for non-zero homomorphisms on F and, more generally, for unbounded Hölder quasimorphisms (cf. [23]); however, the same cannot be said when we restrict the group elements to a non-trivial conjugacy class $\mathfrak{C}$. This observation follows from the fact that homomorphisms are constant on conjugacy classes (and, in the more general case, quasimorphisms are bounded on conjugacy classes). If $\Theta : F \to \mathbb{R}$ is a homomorphism, or quasimorphism, we have the following limit:

$$\lim_{m\to\infty} \frac{1}{\#\mathfrak{C}_{k+2m}} \#\{x \in \mathfrak{C}_{k+2m} \colon \Theta(x)/\sqrt{k+2m} \le a\} = \begin{cases} 0 & \text{if } a < 0, \\ 1 & \text{if } a > 0. \end{cases}$$

For a homomorphism $\Theta : F \to \mathbb{R}$ the limit when $a = 0$ depends on the sign of Θ when restricted to the conjugacy class $\mathfrak{C}$.

5.2 Proof of Theorem 5.1.1

In this section we prove the following asymptotic result.

Theorem 5.1.1. *Let $\mathfrak{C}$ be a non-trivial conjugacy class in the free group F such that the cyclically reduced words in $\mathfrak{C}$ have word length $k > 0$. Suppose that the function $\Theta : F \to \mathbb{R}$ satisfies assumptions (A1) and (A2). Then we have the limit*

$$\lim_{m\to\infty} \frac{1}{\#\mathfrak{C}_{k+2m}} \sum_{x\in\mathfrak{C}_{k+2m}} \frac{\Theta(x)}{k+2m} = \overline{\Theta}.$$

Moreover, $\overline{\Theta} = \int f \, d\mu_0$ where μ_0 is the measure of maximal entropy for the subshift of finite type Σ^* supported on Σ and $f : \Sigma^* \to \mathbb{R}$ is the Hölder continuous function from assumption (A1).

In order to prove Theorem 5.1.1, we introduce a generating function $\eta_{\mathfrak{C}}(s, z)$ on two complex variables given by

$$\eta_{\mathfrak{C}}(s,z) = \sum_{m=0}^{\infty} z^{k+2m} \sum_{x\in\mathfrak{C}_{k+2m}} \mathrm{e}^{s\Theta(x)} = \sum_{m=0}^{\infty} z^{k+2m} \sum_{g\in\mathfrak{C}_k} \sum_{w\in W_m'(g)} \mathrm{e}^{sf^{k+2m}(w^{-1}gw)}.$$

We prove the theorem by studying the asymptotic behaviour, as $m \to \infty$, of the coefficients of z^{k+2m} in the power series

$$\frac{\partial}{\partial s}\eta_{\mathfrak{C}}(s,z)\Big|_{s=0} = \sum_{m=0}^{\infty} z^{k+2m} \sum_{x\in\mathfrak{C}_{k+2m}} \Theta(x).$$

For convenience, in the work that follows we shall interchangeably refer to elements $x \in F$ and the associated element of the sequence space $x \in \Sigma^*$. We will find the following bound useful.

Lemma 5.2.1. *Suppose that $f \in \mathcal{F}_\theta(\Sigma^*, \mathbb{C})$, $g \in \mathfrak{C}_k$ and $w \in W'_m(g)$ then there exists a constant $K > 0$, independent of w, such that*

$$|f^{k+2m}(w^{-1}gw) - f^m(w) - f^k(g) - f^m(w^{-1})| \leq K.$$

Proof. We have $f^{k+2m}(w^{-1}gw) = f^m(w^{-1}gw) + f^k(gw) + f^m(w)$. Thus

$$\begin{aligned}|f^{k+2m}(w^{-1}gw) &- f^m(w) - f^k(g) - f^m(w^{-1})| \\ &\leq |f^m(w^{-1}gw) - f^m(w^{-1})| + |f^k(gw) - f^k(g)| \leq \frac{2|f|_\theta\theta}{1-\theta}\end{aligned}$$

and we are done. □

By Lemma 5.2.1,

$$\exp(sf^{k+2m}(w^{-1}gw)) = \exp\Big(s(f^m(w) + f^k(g) + f^m(w^{-1}))\Big) + s\kappa_w + \xi_w(s)$$

where $\kappa_w = f^{k+2m}(w^{-1}gw) - f^m(w) - f^k(g) - f^m(w^{-1})$ is uniformly bounded in W^* (by Lemma 5.2.1) and $\xi_w(s) = s^2\zeta_w(s)$, with $\zeta_w(s)$ an entire function. Combining the above approximation with assumption (A2) we have

$$\eta_{\mathfrak{C}}(s,z) = \sum_{m=0}^{\infty} z^{k+2m} \sum_{g\in\mathfrak{C}_k} \sum_{w\in W'_m(g)} \mathrm{e}^{s(f^k(g)+2f^m(w))} + \delta(s,z)$$

where

$$\delta(s,z) = \sum_{m=0}^{\infty} z^{k+2m} \sum_{g\in\mathfrak{C}_k} \sum_{w\in W'_m(g)} s\kappa_w + \xi_w(s).$$

Let $\chi_g : \Sigma^* \to \mathbb{R}$ be the locally constant function given by

$$\chi_g((w_n)_{n=0}^{\infty}) = \begin{cases} 0 & \text{if } w_0 = g_1, g_k^{-1}, \text{ and} \\ 1 & \text{otherwise.} \end{cases}$$

We introduced the function χ_g in order to write $\eta_{\mathfrak{C}}(s,z)$ in terms of the transfer operator. We have

$$\begin{aligned} \eta_{\mathfrak{C}}(s,z) &= \sum_{g \in \mathfrak{C}_k} \mathrm{e}^{sf^k(g)} \sum_{m=0}^{\infty} z^{k+2m} \sum_{w \in W'_m} \mathrm{e}^{2sf^m(w)} \chi_g(w) + \delta(s,z), \\ &= \sum_{g \in \mathfrak{C}_k} \mathrm{e}^{sf^k(g)} \sum_{m=0}^{\infty} z^{k+2m} (L_{2sf}^m \chi_g)(1) + \delta(s,z). \end{aligned}$$

Thus the power series $\sum_{m=0}^{\infty} z^{k+2m} \sum_{x \in \mathfrak{C}_{k+2m}} \Theta(x)$ can be written in terms of the transfer operator since

$$\begin{aligned} \frac{\partial}{\partial s} \eta_{\mathfrak{C}}(s,z) \bigg|_{s=0} &= \sum_{g \in \mathfrak{C}_k} \sum_{m=0}^{\infty} \frac{\partial}{\partial s} z^{k+2m} (L_{2sf}^m \chi_g)(1) \bigg|_{s=0} \\ &\quad + \sum_{g \in \mathfrak{C}_k} f^k(g) \sum_{m=0}^{\infty} z^{k+2m} (L_0^m \chi_g)(1) + \frac{\partial}{\partial s} \delta(s,z) \bigg|_{s=0}. \end{aligned}$$

We analyse the growth of the coefficients of the power series in the following sequence of lemmas.

Lemma 5.2.2. *The coefficients of z^{k+2m} in the power series $\sum_{m=0}^{\infty} z^{k+2m} (L_0^m \chi_g)(1)$ grow with order $O(\mathrm{e}^{mh})$ where $h = P(0)$.*

The coefficients in the next lemma grow with the same order.

Lemma 5.2.3. *The coefficients of z^{k+2m} in the power series $\frac{\partial}{\partial s} \delta(s,z)\big|_{s=0}$ grow with order $O(\mathrm{e}^{mh})$.*

Proof. Since, for each $w \in W^*$, $\xi'(0) = 0$,

$$\frac{\partial}{\partial s} \delta(s,z) \bigg|_{s=0} = \sum_{m=0}^{\infty} z^{k+2m} \sum_{g \in \mathfrak{C}_k} \sum_{w \in W'_m(g)} \kappa_w.$$

For each $w \in W^*$ we have $\kappa_w \leq K$. Thus the coefficient of z^{k+2m} is bounded in modulus by

$$\sum_{g \in \mathfrak{C}_k} \sum_{w \in W_m'(g)} K = K \# \mathfrak{C}_{k+2m} = K(2l-2)(2l-1)^{m-1} \# \mathfrak{C}_k = O(\mathrm{e}^{mh}),$$

from which the lemma follows. □

We decompose the transfer operator L_{sf} into the projection R_s associated to the eigenspace spanned by the eigenfunction associated to the eigenvalue $\mathrm{e}^{P(sf)}$ and $Q_s = L_{sf} - \mathrm{e}^{P(sf)} R_s$. For $s \in \mathbb{C}$ in a neighbourhood of $s = 0$, the operators R_s and Q_s are analytic. We use this operator decomposition to obtain the estimates in the next two lemmas.

Lemma 5.2.4. *The coefficients of z^{k+2m} in the power series*

$$\frac{\partial}{\partial s} \sum_{g \in \mathfrak{C}_k} \sum_{m=0}^{\infty} z^{k+2m} Q_{2s}^m \chi_g(1) \Bigg|_{s=0}$$

grow with order $O(\mathrm{e}^{m(h-\varepsilon)})$ for some $\varepsilon > 0$.

Proof. Suppose that $s \in \mathbb{C}$ such that $0 \leq |s| < \delta_1$ then, by perturbation theory, if δ_1 is sufficiently small each perturbed operator L_{2sf} has a simple maximal eigenvalue $\mathrm{e}^{P(2sf)}$; moreover, the upper semi-continuity of the spectrum ensures the spectral gap between $\mathrm{e}^{P(2sf)}$ and the remainder of the spectrum of L_{2sf} persists. Thus, for $|s| < \delta_1$, there exists $\varepsilon_1(\delta_1) > 0$ such that

$$\limsup_{m \to \infty} \|Q_{2s}^m\|^{1/m} \leq \mathrm{e}^{h-\varepsilon_1}.$$

We consider the analyticity of the series

$$\sum_{g \in \mathfrak{C}_k} \sum_{m=0}^{\infty} z^{k+2m} Q_{2s}^m \chi_g(1).$$

Suppose that we fix $z \in \mathbb{C}$ such that $|z| < \mathrm{e}^{-h+\varepsilon_1}$, then the series converges for each $s \in \mathbb{C}$ with $|s| < \delta_1$. Meanwhile, given $s \in \mathbb{C}$ such that $|s| < \delta_1$ the series converges for each $z \in \mathbb{C}$ with $|z| < \mathrm{e}^{-h+\varepsilon_1}$. Thus, by Hartogs' theorem (Theorem 1.2.5, [35]), the series converges analytically in the polydisk $\{s \in \mathbb{C}\colon |s| < \delta_1\} \times \{z \in \mathbb{C}\colon |z| < \mathrm{e}^{-h+\varepsilon_1}\}$. Thus the power series

$$\frac{\partial}{\partial s} \sum_{g \in \mathfrak{C}_k} \sum_{m=0}^{\infty} z^{k+2m} Q_{2s}^m \chi_g(1) \Big|_{s=0}$$

is analytic for $|z| < \mathrm{e}^{-h+\varepsilon_1}$ and so we estimate the coefficients of the power series by $O(\mathrm{e}^{m(h-\varepsilon)})$ with $0 < \varepsilon < \varepsilon_1$. □

There is one power series left to study.

Lemma 5.2.5. *Let $P'(0)$ denote the derivative of the function $P(sf)$ evaluated at $s = 0$. The coefficient of z^{k+2m} in the power series*

$$\frac{\partial}{\partial s} \sum_{m=0}^{\infty} z^{k+2m} \mathrm{e}^{mP(2sf)} R_{2s} \chi_g(1) \Big|_{s=0}$$

is $2m\mathrm{e}^{mh} P'(0) R_0 \chi_g(1) + O(\mathrm{e}^{mh})$.

Proof. We have

$$\frac{\partial}{\partial s} \sum_{m=0}^{\infty} z^{k+2m} \mathrm{e}^{mP(2sf)} R_{2s} \chi_g(1) \Big|_{s=0} = \sum_{m=0}^{\infty} z^{k+2m} 2m\mathrm{e}^{mh} P'(0) R_0 \chi_g(1) + \sum_{m=0}^{\infty} z^{k+2m} \mathrm{e}^{mh} \frac{\partial}{\partial s} R_{2s} \chi_g(1) \Big|_{s=0},$$

from which the result follows. □

Combining the above lemmas, we estimate the growth of the coefficients of $\frac{\partial}{\partial s} \eta_{\mathfrak{C}}(s,z)\big|_{s=0}$ by

$$\sum_{g \in \mathfrak{C}_k} 2m\mathrm{e}^{mh} P'(0) R_0 \chi_g(1) + O(\mathrm{e}^{mh}).$$

Returning to Theorem 5.1.1 we now have

$$\frac{1}{\#\mathfrak{C}_{k+2m}} \sum_{x \in \mathfrak{C}_{k+2m}} \frac{\Theta(x)}{k+2m} = \frac{2m}{k+2m} P'(0) \frac{\mathrm{e}^{mh}}{\#\mathfrak{C}_{k+2m}} \sum_{g \in \mathfrak{C}_k} R_0 \chi_g(1) + O(1/m).$$

Thus we have

$$\lim_{m\to\infty} \mathrm{e}^{-mh} \sum_{x\in\mathfrak{C}_{k+2m}} \frac{\Theta(x)}{k+2m} = \int f\, d\mu_0 \sum_{g\in\mathfrak{C}_k} R_0\chi_g(1).$$

If we substitute $f : \Sigma^* \to \mathbb{R}$ given by $f(x) = 1$ for each $x \in \Sigma^*$ into the preceding limit we obtain

$$\lim_{m\to\infty} \frac{\#\mathfrak{C}_{k+2m}}{\mathrm{e}^{mh}} = \sum_{g\in\mathfrak{C}_k} R_0\chi_g(1).$$

Hence we have the desired result

$$\lim_{m\to\infty} \frac{1}{\#\mathfrak{C}_{k+2m}} \sum_{x\in\mathfrak{C}_{k+2m}} \frac{\Theta(x)}{k+2m} = \int f\, d\mu_0.$$

5.3 Proof of Theorem 5.1.3

In this section we prove Theorem 5.1.3, which we recall below for convenience.

Theorem 5.1.3. *Suppose that the function $\Theta : F \to \mathbb{R}$ satisfies assumptions (A1), (A2) and (A3). Then the sequence $(\mathscr{G}_m)_{m=1}^{\infty}$ of distribution functions given by*

$$\mathscr{G}_m(a) = \frac{1}{\#\mathfrak{C}_{k+2m}} \#\Big\{x \in \mathfrak{C}_{k+2m} \colon (\Theta(x) - \overline{\Theta}(k+2m))/\sqrt{k+2m} \le a\Big\}$$

converges to the distribution function of a normal random variable with mean 0 *and variance* $2\sigma_f^2$.

We recall the definition for the Fourier transform (sometimes called the characteristic function) of a distribution function. The *Fourier transform* φ of a distribution function $\mathscr{F}$ is defined by

$$\varphi(t) = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}tx}\, d\mathscr{F}(x).$$

For example, the distribution function of a normal random variable with mean μ and variance σ^2 has the Fourier transform $\exp\{\mathrm{i}t\mu - t^2\sigma^2/2\}$ (cf. [16]) and the Fourier transform of the distribution function

$$\mathscr{G}_m(a) = \frac{1}{\#\mathfrak{C}_{k+2m}} \#\Big\{x \in \mathfrak{C}_{k+2m} \colon (\Theta(x) - \overline{\Theta}(k+2m))/\sqrt{k+2m} \leq a\Big\}$$

is given by

$$\varphi_m(t) = \frac{1}{\#\mathfrak{C}_{k+2m}} \sum_{x \in \mathfrak{C}_{k+2m}} \mathrm{e}^{\mathrm{i}t(\Theta(x) - \overline{\Theta}(k+2m))/\sqrt{k+2m}}.$$

The following theorem (cf. Theorem 2, Chapter XV §3, [16]), which plays a crucial role in our proof of Theorem 5.1.3, connects the convergence of a sequence of distribution functions to the convergence of the associated sequence of Fourier transforms.

Theorem 5.3.1 (Continuity Theorem)**.** *Let $(\mathscr{F}_n)_{n=1}^{\infty}$ be a sequence of distribution functions and let $(\varphi_n)_{n=1}^{\infty}$ be the sequence of their respective Fourier transforms. If $(\mathscr{F}_n)_{n=1}^{\infty}$ converges to the distribution function $\mathscr{F}$ then the sequence $(\varphi_n)_{n=1}^{\infty}$ converges pointwise to a function φ, the Fourier transform of $\mathscr{F}$. Conversely, if $(\varphi_n)_{n=1}^{\infty}$ converges pointwise to a function φ which is continuous in a neighbourhood of 0, then φ is the Fourier transform of a distribution function $\mathscr{F}$ and, in addition, the sequence of distribution functions $(\mathscr{F}_n)_{n=1}^{\infty}$ converges to $\mathscr{F}$.*

Suppose that Θ satisfies (A1), (A2) and (A3). By replacing Θ with $\Theta - \overline{\Theta}$ (which still satisfies the three assumptions) or, equivalently, f with $f - \int f \, d\mu_0$, we may assume without loss of generality that $\overline{\Theta} = \int f \, d\mu_0 = 0$. This reduction does not change the variance. We may then write

$$\varphi_m(t) = \frac{1}{\#\mathfrak{C}_{k+2m}} \sum_{x \in \mathfrak{C}_{k+2m}} \mathrm{e}^{\mathrm{i}t f^{k+2m}(x)/\sqrt{k+2m}}.$$

We recall the approximation, which we obtain from Lemma 5.2.1,

$$\exp(s f^{k+2m}(w^{-1}gw)) = \exp\Big(s(2f^m(w) + f^k(g))\Big) + s\kappa_w + \xi_w(s).$$

Here $\kappa_w = f^{k+2m}(w^{-1}gw) - 2f^m(w) - f^k(g)$ is uniformly bounded for $w \in W^*$ and $\xi_w(s)$ is an entire function such that $\xi_w(0) = 0$. Using the above approximation, we write the Fourier transform $\varphi_m(t)$ as the sum of a leading term plus an error term:

$$\frac{1}{\#\mathfrak{C}_{k+2m}} \sum_{g\in\mathfrak{C}_k} \mathrm{e}^{\tau f^k(g)/2} \sum_{w\in W_m'(g)} \mathrm{e}^{\tau f^m(w)} + \rho_m(t),$$

where $\tau = 2\mathrm{i}t/\sqrt{k+2m}$, and the error term $\rho_m(t)$ is given by

$$\rho_m(t) = \frac{1}{\#\mathfrak{C}_{k+2m}} \sum_{g\in\mathfrak{C}_k} \sum_{w\in W_m'(g)} \frac{\mathrm{i}t\kappa_w}{\sqrt{k+2m}} + \xi_w(\mathrm{i}t/\sqrt{k+2m}).$$

Since the bound on κ_w is uniform and $\xi_w(0) = 0$, we find that $\rho_m(t) \to 0$ as $m \to \infty$. Meanwhile, we rewrite the leading term using the transfer operator

$$\frac{1}{\#\mathfrak{C}_{k+2m}} \sum_{g\in\mathfrak{C}_k} \mathrm{e}^{\tau f^k(g)/2} L_{\tau f}^m \chi_g(1).$$

For sufficiently large m, the simple maximal eigenvalue $\mathrm{e}^{P(\tau f)}$ of the perturbed operator $L_{\tau f}$ persists and also plays a crucial role in determining the limit of $\varphi_m(t)$ as $m \to \infty$. Before we prove the limit, we first analyse the pressure function and establish a preliminary limit for $\mathrm{e}^{m(P(\tau f)-h)}$ as $m \to \infty$.

Recall that the pressure function $P(sf)$ is analytic in a neighbourhood of $s = 0$. For the remainder of the chapter we let $P^{(k)}(0)$ denote the kth derivative of the function $P(sf)$ evaluated at $s = 0$. By analyticity we can choose $\varepsilon > 0$ such that if $|s| < \varepsilon$ then

$$P(2sf) = \sum_{k=0}^{\infty} \frac{P^{(k)}(0)}{k!}(2s)^k = h + 2\sigma_f^2 s^2 + s^3\vartheta(s)$$

for some function $\vartheta(s)$ that is analytic in a neighbourhood of $s = 0$ and we note, by assumption, $P'(0) = \int f\, d\mu_0 = 0$. For sufficiently large $m \in \mathbb{N}$, with $\tau = 2\mathrm{i}t/\sqrt{k+2m}$ as before, we have

$$(k+2m)P(\tau f) = (k+2m)h - 2\sigma_f^2 t^2 - \frac{4\mathrm{i}t^3\vartheta(\tau)}{3\sqrt{k+2m}}$$

and so

$$\frac{\mathrm{e}^{(k+2m)P(\tau f d)}}{\mathrm{e}^{(k+2m)h}} = \mathrm{e}^{-2\sigma_f^2 t^2} \exp\left\{-\frac{4\mathrm{i}t^3\vartheta(\tau)}{3\sqrt{k+2m}}\right\},$$

from which the next proposition and corollary follow.

Proposition 5.3.2. *We have the following limit*

$$\lim_{m\to\infty} \frac{\mathrm{e}^{(k+2m)P(\tau f)}}{\mathrm{e}^{(k+2m)h}} = \mathrm{e}^{-2\sigma_f^2 t^2}.$$

We shall use the notation $\beta(\tau) = \mathrm{e}^{P(\tau f)}$ and $\beta(0) = \mathrm{e}^h$ in the proof of Proposition 5.3.4.

Corollary 5.3.3. *We have the limit*

$$\lim_{m\to\infty} \frac{\beta(\tau)^m}{\beta(0)^m} = \mathrm{e}^{-\sigma_f^2 t^2}.$$

Proposition 5.3.4. *The limit of $\varphi_m(t)$ as $m \to \infty$ is $\mathrm{e}^{-\sigma_f^2 t^2}$ where $\sigma_f^2 = P''(0)$.*

Proof. Written in terms of the transfer operator and a null sequence $(\rho_m(t))_{m=0}^{\infty}$, the transform $\varphi_m(t)$ is equal to

$$\frac{1}{\#\mathfrak{C}_{k+2m}} \sum_{g\in\mathfrak{C}_k} \mathrm{e}^{\tau f^k(g)/2} L_{\tau f}^m \chi_g(1) + \rho_m(t).$$

We recall the decomposition of the transfer operator into $L_{sf} = \beta(s)R_s + Q_s$. For sufficiently large $m \in \mathbb{N}$, with $\tau = 2it/\sqrt{k+2m}$ as before, the leading term is given by

$$\frac{\beta(\tau)^m}{\#\mathfrak{C}_{k+2m}} \sum_{g\in\mathfrak{C}_k} \mathrm{e}^{\tau f^k(g)/2} R_\tau \chi_g(1) + \frac{1}{\#\mathfrak{C}_{k+2m}} \sum_{g\in\mathfrak{C}_k} \mathrm{e}^{\tau f^k(g)/2} Q_\tau^m \chi_g(1).$$

Let $r(Q_\tau)$ denote the spectral radius of Q_τ. The bound $r(Q_\tau) < |\beta(\tau)|$ gives $\|\beta(\tau)^{-m} Q_\tau^m\| = O(\kappa^m)$ for some $\kappa \in (0,1)$ and so we have

$$\frac{1}{\#\mathfrak{C}_{k+2m}} \sum_{g\in\mathfrak{C}_k} \mathrm{e}^{\tau f^k(g)/2} Q_\tau^m \chi_g(1) = O\left(\frac{\beta(\tau)^m}{\beta(0)^m}\kappa^m\right).$$

By Corollary 5.3.3 we have $\lim_{m\to\infty} \beta(\tau)^m/\beta(0)^m = \mathrm{e}^{-\sigma_f^2 t^2}$ and so

$$\lim_{m\to\infty} \frac{1}{\#\mathfrak{C}_{k+2m}} \sum_{g\in\mathfrak{C}_k} \mathrm{e}^{\tau f^k(g)/2} Q_\tau^m \chi_g(1) = 0.$$

We now turn our attention to the asymptotic behaviour of the term

$$\frac{\beta(\tau)^m}{\#\mathfrak{C}_{k+2m}} \sum_{g\in\mathfrak{C}_k} \mathrm{e}^{\tau f^k(g)/2} R_\tau \chi_g(1).$$

In order to approximate this term, we first write the projection R_τ in terms of R_0. Since the projection is analytic in a neighbourhood of 0 we have, for sufficiently large m, $\mathrm{e}^{\tau f^k(g)/2} R_\tau \chi_g(1) = R_0\chi_g(1) + O(t/\sqrt{k+2m})$. Second, we recall that $\#\mathfrak{C}_{k+2m} = (\beta(0)-1)\beta(0)^{m-1}\#\mathfrak{C}_k$ and so

$$\frac{\beta(\tau)^m}{\#\mathfrak{C}_{k+2m}} \sum_{g\in\mathfrak{C}_k} \mathrm{e}^{\tau f^k(g)/2} R_\tau \chi_g(1) = \frac{\beta(\tau)^m}{\#\mathfrak{C}_{k+2m}} \sum_{g\in\mathfrak{C}_k} R_0\chi_g(1) + O\left(\frac{\beta(\tau)^m t}{\beta(0)^m \sqrt{k+2m}}\right).$$

We recall the limit

$$\lim_{m\to\infty} \frac{\#\mathfrak{C}_{k+2m}}{\beta(0)^m} = \sum_{g\in\mathfrak{C}_k} R_0\chi_g(1)$$

and so, together with the above approximation, we find the limit of $\varphi_m(t)$ as $m\to\infty$ is given by

$$\lim_{m\to\infty} \frac{\beta(\tau)^m}{\#\mathfrak{C}_{k+2m}} \sum_{g\in\mathfrak{C}_k} \mathrm{e}^{\tau f^k(g)/2} R_\tau \chi_g(1) = \lim_{m\to\infty} \frac{\beta(\tau)^m}{\beta(0)^m} = \mathrm{e}^{-\sigma_f^2 t^2},$$

which is the desired result. □

Chapter 6

Further Research

In this short chapter I describe avenues for further research related to the original work in this thesis.

6.1 Orbit counting in conjugacy classes

We have established orbit counting results in the setting of metric graphs. It is worthwhile pursuing orbit counting results for negatively curved convex co-compact surfaces whose fundamental groups are free. After this, a potential next step is to consider compact negatively curved surfaces. We note Huber [26] considered compact surfaces with constant negative curvature.

Let $\widetilde{M}$ be the universal cover of a negatively curved convex co-compact surface M (or even a manifold if there are closed geodesics whose lengths have irrational ratio). Let $\tilde{x} \in \widetilde{M}$ be a lift of a fixed $x \in M$. We denote by $g\tilde{x} \in \widetilde{M}$ the translation of $\tilde{x}$ by the element $g \in \pi_1(M, x)$. Let $h > 0$ be the topological entropy of the geodesic flow on T_1M. For surfaces with variable negative curvature, it was shown in [53] that there exists $c > 0$ such that as $T \to \infty$,

$$\#\{g \in \pi_1(M, x) \colon d(\tilde{x}, g\tilde{x}) \leq T\} \sim c\mathrm{e}^{hT}.$$

We note a gap in the argument was filled in [11].

Let $\mathfrak{C}$ be a non-trivial conjugacy class of elements in $\pi_1(M, x)$. Let us consider the orbit counting in conjugacy classes problem in this setting. As in the case of metric graphs, we might expect the exponential growth rate in the asymptotic to half when we restrict the group elements to a non-trivial conjugacy class, i.e. there exists a positive constant $C(\mathfrak{C})$ such that $\#\{g \in \mathfrak{C} \colon d_{\widetilde{M}}(\tilde{x}, g\tilde{x}) \leq T\} \sim C e^{hT/2}$ as $T \to \infty$.

In order to describe some of the challenges we expect to encounter, we recall our previous setting of metric graphs. Recall the based length function $L : F \to \mathbb{R}$ given by $L(g) = d_{\mathcal{T}}(o, go)$ for a prescribed vertex $o \in \mathcal{T}$. Our approach constructs a *locally constant* function $r : \Sigma^* \to \mathbb{R}$, i.e. there exists an $N \in \mathbb{N}$ such that for any pair $x, y \in \Sigma^*$ with $x_n = y_n$ for every $0 \leq n \leq N$ we have $r(x) = r(y)$, on both finite and infinite reduced words whose sums over periodic orbits encode the lengths of closed geodesics in the metric graph. We note a locally constant function is necessarily Hölder continuous. In our setting the fundamental group of each metric graph is a free group and so it is natural to approach such problems using tools from symbolic dynamics since the conjugacy classes of free groups are particularly susceptible to symbolic encoding.

It is reasonable to expect that a negatively curved convex co-compact surface M whose fundamental groups are free is also susceptible to the symbolic techniques we employ for metric trees. If we take the same approach as before, then some of the additional difficulty in this setting arises from the symbolic encoding: we would construct a Hölder continuous function $r : \Sigma^* \to \mathbb{R}$, but this function is not locally constant.

6.2 Asymptotic statistics

One goal of further research is to remove (A2) from the list of assumptions in Theorems 5.1.1 and 5.1.3. If it is possible to prove the asymptotic

$$\lim_{m \to \infty} \frac{1}{\#\mathfrak{C}_{k+2m}} \sum_{g \in \mathfrak{C}_k} \frac{\Theta(x)}{k + 2m} = \int f \, d\mu_0$$

without (A2) then equidistribution with respect to μ_0, the measure of maximal entropy, holds for each non-trivial conjugacy class in F.

We have considered statistical properties for free groups acting convex co-compactly on hyperbolic space. It is worth investigating whether these statistical properties hold in the more general setting of negatively curved convex co-compact surfaces. We first give some historical context for the problem.

Let M be a compact surface with constant negative curvature and $g_t : T_1M \to T_1M$ the geodesic flow. Let $\lambda(\gamma)$ be the least period of a closed orbit γ of the flow and by $\lambda_f(\gamma)$ the integral of a Hölder continuous function $f : T_1M \to \mathbb{R}$ on γ so that

$$\lambda_f(\gamma) = \int_0^{\lambda(\gamma)} f(g_t x)\, dt$$

for $x \in \gamma$. Bowen [3] showed that the closed orbits of the geodesic flow are equidistributed according to μ the measure of maximal entropy. By equidistribution, we mean the following limit holds

$$\lim_{T\to\infty} \frac{\sum_{\{\gamma :\, \lambda(\gamma)\le T\}} \lambda_f(\gamma)}{\sum_{\{\gamma :\, \lambda(\gamma)\le T\}} \lambda(\gamma)} \to \int f\, d\mu.$$

Since each conjugacy classes in $\pi_1(M, x)$ has an associated closed geodesic. Thus when we consider asymptotics in conjugacy classes we instead study based geodesic loops in the manifold.

Let M be a negatively curved convex co-compact surface (or, again, a manifold if there are closed geodesics whose lengths have irrational ratio) whose fundamental groups are free. For a given basepoint $x \in M$, let $\mathfrak{C}$ be a non-trivial conjugacy class in $\pi_1(M, x)$ and consider the set of geodesic loops γ_g based at x for each $g \in \mathfrak{C}$. We ask whether this restricted set of geodesics is asymptotically equidistributed with respect to the measure of maximal entropy projected to M. We can expect to use a symbolic approximation technique developed in [50] for orbit counting in the context of hyperbolic flows on compact manifolds. This approximation was adapted in [51] for orbit counting and, importantly, a refined directional asymptotic in the harder setting of $\mathbb{Z}^k$ covers of compact negatively curved manifolds. Here the results concern orbit segments of the geodesic flow on T_1M, but the asymptotic statements push down to geodesic loops in M. A potential next step is to consider compact negatively curved surfaces.

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