MODULES OVER GROUP ALGEBRAS WHICH ARE FREE ON RESTRICTION TO A MAXIMAL SUBGROUP

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CONTENTS

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DEDICATION To Annie, Tone, Chris and Nick. Thank you.

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SUMMARY

Consider the following situation: k will be an algebraically closed field of characteristic p and G will be a finite p-group. V will be a non-projective, indecomposable kG-module which is free on restriction to some maximal subgroup of G. Our purpose in doing this is to investigate Chouinard's theorem - all the proofs of which have been cohomological in nature - in a representation-theoretic way. This theorem may be shown to be equivalent to saying that, if G is not elementary abelian, V cannot be free on restriction to *all* the maximal subgroups of G.

It is shown how to construct an exact sequence:

$$0 \longrightarrow V \longrightarrow P \longrightarrow P \longrightarrow V \longrightarrow 0$$

with P projective. From this an almost split sequence,

$$0 \longrightarrow V \longrightarrow X \longrightarrow V \longrightarrow 0,$$

is constructed. It is shown that X can have at most two indecomposable summands.

If Φ denotes the Frattini subgroup of G, then V is free on restriction to Φ. We may regard the set of Φ-fixed points of V, $\bar{V}$, as a module for $\bar{G} = G/\Phi$. But $\bar{G}$ is elementary abelian, so we may consider the Carlson variety, $Y(\bar{V})$ - this may be regarded as a subset of J/J^2 where J denotes the augmentation ideal of kG. It is shown that $Y(\bar{V})$ is always a line.

We define Y_G to be the union of all the lines $Y(\bar{V})$ as V runs over all the kG-modules with the properties above. It is shown that Y_G is the whole of J/J^2 if and only if G is elementary abelian. It is also shown that, when G is one of a particular class of p-groups - the *pseudo-special* groups - which form the minimal counter-examples to Chouinard's theorem, that Y_G is the set of zeros of a sequence of homogeneous polynomials with coefficients in the field of p elements. Indeed, a specific construction for these polynomials is given.

INTRODUCTION

Consider the following result, due to Chouinard:

Let k be a field of characteristic p and G be a finite group. A kG-module is projective if and only if it is free on restriction to all the elementary abelian p-subgroups of G.

Chouinard's proof of this, and all subsequent proofs (see [Ch], [A&E] etc.), have relied heavily on cohomological techniques and, in particular, on a proposition of Serre (Proposition (4) of [Ser]) which we may state as follows:

Write $E^{m}(G)$ for the cohomology group, $\mathrm{Ext}^{m}_{\mathbb{F}_pG}(\mathbb{F}_p, \mathbb{F}_p)$, and let:

$$B : E^{1}(G) \longrightarrow E^{2}(G)$$

be the Bockstein operator. Then, if G is not elementary abelian, there exist nonzero elements, $z_1, z_2, \ldots, z_m$, of $E^{1}(G)$ such that the cup-product

$$B(z_1)B(z_2)\ldots B(z_m) \in E^{2m}(G)$$

equals 0.

The proof of this result involves, among other things, algebraic varieties and Steenrod operators; certainly it seems to have very little to do with the original problem . The motivation behind the research leading to this thesis was to try to make progress towards a proof of Chouinard's theorem (which is, after all, a simple representation-theorical result) by simple representation-theoretical means. In Chapter 1 we show how the deduction of Chouinard's theorem from Serre's proposition may be stripped of most of its cohomology, however the latter result remains an obstacle.

A straight-forward reduction shows that we may consider the following situation: k will be an algebraically closed field of nonzero characteristic, p, and G will be a finite p-group; V will be supposed to be a non-projective, indecomposable kG-module which is free on restriction to some maximal subgroup of G. Chouinard's theorem is then equivalent to saying that, if G is not elementary abelian, V cannot be free on restriction to *all* the maximal subgroups of G. Thus, as a subsidiary question, we ask: what properties does V have in the situation above ?

In Chapter 1 it is shown how to construct an exact sequence:

$$0 \longrightarrow V \longrightarrow P \longrightarrow P \longrightarrow V \longrightarrow 0$$

with P projective. Thus V is periodic of period 1 or 2. This two-step projective resolution automatically brings to mind the construction theorem for almost split sequences; this is investigated in Chapter 2. It is shown there how to construct, for each maximal subgroup, H, of G, an exact sequence:

$$0 \longrightarrow V \longrightarrow X_H \longrightarrow V \longrightarrow 0$$

which is (1) almost split if V is free on restriction to H,

(2) split otherwise.

Furthermore, the connected component of the Auslander-Reiten quiver containing V is investigated. This is shown to have the form:

 .... ,

i.e. there exists a sequence, V_n $(n = 1,2,\ldots)$, of non-projective, indecomposable kG-modules which are free on restriction to some maximal subgroup of G, such that there are almost split sequences:

$$0 \longrightarrow V_1 \longrightarrow V_2 \longrightarrow V_1 \longrightarrow 0$$

$$0 \longrightarrow V_n \longrightarrow V_{n-1} \oplus V_{n+1} \longrightarrow V_n \longrightarrow 0 \qquad (n > 1)$$

with $V \cong V_N$ for some N. It is also shown how, given V_1, to construct $V_2, V_3, \ldots$.

Chapters 3 and 4 are concerned with the following observation: let Φ be the Frattini subgroup of G, that is, the intersection of all the maximal subgroups of G, then V is free on restriction to Φ. We may regard the set of Φ-fixed points of V, $\bar{V}$, as a module for $\bar{G} = G/\Phi$. But $\bar{G}$ is elementary abelian, so there is a Carlson variety, $Y(\bar{V})$ – this may be regarded as a subset of J/J^2 where J denotes the augmentation ideal of kG. The main result proved in Chapter 3 is that $Y(\bar{V})$ is always a line in J/J^2.

Each subgroup, H, of G determines a subspace, S_H, of J/J^2. When H is a maximal subgroup, S_H is a hyperplane; V is free on restriction to H if and only if the line $Y(\bar{V})$ intersects S_H trivially. Thus we define Y_G to be the union of all the varieties $Y(\bar{V})$ as V runs over all the indecomposable kG-modules which are free on restriction to a maximal subgroup of G. Chouinard's theorem is then equivalent to showing that, if G is not elementary abelian,

$$Y_G \subseteq \bigcup_H S_H$$

– where the union is over all the maximal subgroups of G.

In Chapter 4 the constraints, if any, on Y_G are investigated. It is shown that the minimal counter-example to Chouinard's theorem is one of a particular class of p-groups, which we call *pseudo-special* – these being defined by the fact that the Frattini subgroup $\Phi(G)$ is the unique minimal normal subgroup of G. The structure of these groups may be very accurately described, so specific calculations are possible.

If $g_1, \ldots, g_n$ is a minimal set of generators for G then $(g_1 - 1) + J^2, \ldots, (g_n - 1) + J^2$ is a k-basis for J/J^2. Using this basis, any polynomial in n variables with coefficients in the field of p elements, $f(X_1, \ldots, X_n)$, may be used to define a hypersurface, $S(f)$, of J/J^2. It is shown that if G is pseudo-special then $Y_G \subseteq S(f)$ for some nonzero polynomial, f. This result is shown to be extendible to general groups to give:

Y_G is the whole of J/J^2 if and only if G is elementary abelian.

It is further shown for pseudo-special groups that $Y_G = S(f_1) \cap \ldots \cap S(f_s)$ for some polynomials, $f_1,\ldots,f_s$; indeed a specific construction is given for these polynomials. The result just mentioned implies that not all these polynomials are zero. But it may be shown that the hyperplanes S_H for H a maximal subgroup of G are precisely the subspaces of the form S(f) where f is a nonzero linear polynomial; thus it is readily seen that G satisfies Chouinard's theorem if and only if the ideal of $\mathbb{F}_p[X_1,\ldots,X_n]$ generated by $f_1,\ldots,f_s$ contains a product of nonzero linear polynomials. Therefore we have a method whereby, with sufficient patience, we may determine whether a given pseudo-special group satisfies Chouinard's theorem; however a general approach is, at the moment, elusive.

There are several examples in the text. Chapter 5 is devoted to constructing examples of non-projective, indecomposable kG-modules which are free on restriction to a maximal subgroup of G. By means of these examples we are able to show that:

$$Y_G \supseteq \bigcup_H S_H$$

– the union being over all the elementary abelian subgroups of G; in fact, Chouinard's theorem shows that equality holds. Appendix A is devoted to an extended example: it is shown how a theorem of Benson and Carlson may be used to prove the well-known classification theorem for the indecomposable modules of the Klein 4-group over an algebraically closed field of characteristic 2.

NOTATION

Throughout, k will be a field of characteristic p. For a finite-dimensional k-algebra, A (which will generally be the group algebra, kG, for some finite group, G) we let mod A denote the category of all finitely-generated, left, unital A-modules and all A-linear maps between such modules. We shall use the term *A-module* to designate an object of mod A.

If U and V are A-modules then we let $(U,V)_A$, or simply (U,V) when there can be no confusion about which algebra is meant, denote the set of all morphisms $U \longrightarrow V$ in mod A. All maps will be written on the left. The identity morphism of U will be denoted by 1_U.

Note that there are induced k-linear functors:

$$(U,-)_A : \text{mod } A \longrightarrow \text{mod } k\,,$$

$$(-,V)_A : \text{mod } A \longrightarrow \text{mod } k\,,$$

which are covariant and contravariant respectively.

We shall also use the following notation:

(a) A will also be used to denote the A-module given by the left regular representation,

(b) The Jacobson radical of A will be denoted by J(A),

(c) The direct sum of the A-modules U and V will be denoted by either $U + V$ or $U \oplus V$,

(d) $U|V$ will signify that U is a direct summand of V,

(e) $[U|V]$ will denote the multiplicity of an indecomposable A-module, U, in V - this is well-defined by the Krull-Schmidt theorem,

(f) The socle of U - that is to say, the sum of all the minimal submodules of U - will be denoted by soc (U),

(g) Similarly, the radical of U (the intersection of the maximal submodules) will be denoted by Rad (U),

(h) $\dim_k U$ will denote the dimension of U as a k-space.

In the case when A is the group algebra, kG, we shall also use the following:

(i) The trivial, one-dimensional kG-module will be denoted by k_G.

(j) Aug (kG) will denote the augmentation ideal of kG.

(k) The submodule of G-fixed points of U will be denoted by U^G.

(l) The minimal projective cover of U will be written as P_U.

(m) Ω^n $(n \in \mathbb{Z})$ will denote the Heller operators (see [Ben] p8)
a kG-module, U, is said to be *periodic* if $\Omega^n U \cong U$ for some $n \neq 0$.

(n) $\mathrm{Ext}^n_{kG}(U,V)$ will denote the cohomology groups (see [C&E] or [McL]).

If U and V are kG-modules then we may regard the tensor product, $U \otimes V$, as a kG-module by using the diagonal action of G:

$$g.(u \otimes v) = gu \otimes gv \qquad (u \in U,\ v \in V,\ g \in G)$$

and extending k-linearly to the whole of kG. Similarly the space of all k-linear maps $U \longrightarrow V$, $(U,V)_k$, may be regarded as a kG-module by using the following action of G:

$$g.f : u \longmapsto gf(g^{-1}u) \qquad (f \in (U,V)_k,\ u \in U,\ g \in G).$$

The dual module of U, U^*, then equals $(U,k_G)_k$.

Recall that we define the complex representation ring (or Green ring) of kG to be the complex vector space with the set of isomorphism classes of indecomposable kG-modules as a basis. If V is a kG-module then the isomorphism class containing V, which will be denoted by [V], may be identified with the following element of the space just defined:

$$\sum_{[U]} [U|V].[U]$$

– the sum being over all isomorphism classes of indecomposable kG-modules. The tensor product then induces a $\mathbb{C}$-algebra structure on the representation ring. We shall denote this algebra by $A_k(G)$.

The dual space operator induces an algebra automorphism of $A_k(G)$ which we write as $x \longmapsto x^*$.

If H is a subgroup of G then we have covariant functors:

$$\text{res} : \text{mod } kG \longrightarrow \text{mod } kH\,,$$

$$\text{ind} : \text{mod } kH \longrightarrow \text{mod } kG\,,$$

where $\text{res}\,(U) = U_{\downarrow H}$ is the restriction of U to H, and $\text{ind}\,(V) = V^{\uparrow G}$ is the induced module, $kG \otimes_{kH} V$.

As regards groups, we shall use the following notation:

(o) $|G|$ will denote the order of G,

(p) $|G:H|$ will denote the index of H in G,

(q) The Frattini subgroup of G – that is, the intersection of all the maximal subgroups of G – will be denoted by $\Phi(G)$,

(r) The centre of G will be denoted by $Z(G)$,

(s) The subgroup generated by $X \subseteq G$ will be denoted by $\langle X \rangle$,

(t) The direct product of G and H will be denoted by $G \times H$.

We shall also adopt the *bar convention* when talking about factor groups. Suppose that N is a normal subgroup of G, then we shall write the natural map $G \longrightarrow G/N$ as $g \longmapsto \bar{g}$. In particular, $\bar{G} = G/N$.

Other notation that we accept as standard:

$\mathbb{N}$ is the set of positive integers $1,2,3,\ldots,$

$\mathbb{Z}$ is the set of integers $\ldots,-2,-1,0,1,2,\ldots,$

$\mathbb{C}$ is the field of complex numbers,

$\mathbb{F}_p$ is the field of p elements.

Other notation will be introduced in the text. However, for convenience, we give here an index of the more commonly used terms:

CHAPTER 0

PRELIMINARY RESULTS

Introduction

This first chapter contains most of the preliminary work that we shall require and a simple reduction of Chouinard's theorem. Inevitably, not all the standard results that we shall use are proved here; many will be quoted in the text without comment. [C&R] is the classic work for the results on representation theory, although [Ben] and [Lan] both cover the ground in a fairly concise manner; the results on group theory may be found in any standard text, for example, [Hu].

§0.0 Projective modules

In this first section, we shall prove a number of preliminary results concerned mostly with the properties of the projective modules of some group algebra, kG. Most of the results are well known and we shall not give specific references. G will be an arbitrary finite group and k will be any field of characteristic p. The case $p = 0$ is not initially excluded, but we shall observe the convention that the trivial group, 1, is the unique 0-subgroup of G.

Firstly we introduce some useful notation: if X is a finite subset of the group algebra, kG, then write:

$$\mu_X = \sum_{x \in X} x .$$

With this notation, the following is easily proved:

Lemma 0.0.1 (a) If H is a subgroup of G and $g \in G$, then:

$$g\mu_H = \mu_{(gHg^{-1})}g .$$

(b) If $h \in H$ then $h\mu_H = \mu_H$.

(c) If T is a left transversal for H in G then $\mu_T\mu_H = \mu_G$.

(d) If x is an element of G of order q, where q is a power of p, then:

$$\mu_{\langle x \rangle} = (x-1)^{q-1} . \qquad \square$$

Note that, in particular, if H is a normal subgroup of G and V is a kG-module then (a) gives that $\mu_H V$ is a submodule of V. Part (b) further shows that $\mu_H V \leq V^H$ so that we may regard $\mu_H V$ as a k(G/H)-module.

A simple calculation shows that $(kG)^G = k\mu_G$ and that:

$$\mathrm{Aug}\,(kG) = \left\{ \xi \in kG \;\middle|\; \mu_G \xi = 0 \right\} .$$

Lemma 0.0.2 If P_1 denotes the projective cover of the trivial module, k_G, then $P_1 \cong P_1^*$. Furthermore, the multiplicity of P_1 in an arbitrary kG-module, V, equals $\dim_k \mu_G V$.

<u>Proof</u> Considering the augmentation map $kG \longrightarrow k_G$, we see that we may regard P_1 as a submodule of kG with $P_1 \not\leq \mathrm{Aug}(kG)$. Thus $\mu_G P_1 \neq 0$ so that $k\mu_G$ is contained within P_1. Therefore we have a monomorphism $k_G \longrightarrow P_1$; this induces an epimorphism $P_1^* \longrightarrow k_G$, thus, by the uniqueness of projective covers, $P_1 \cong P_1^*$. Note also that, by standard results, P_1 has a unique minimal submodule. This must therefore be $k\mu_G$.

For the second part of the result it suffices to assume that V is indecomposable. If $\mu_G V \neq 0$ then choose $v \in V$ with $\mu_G v \neq 0$. Define a kG-morphism $\phi : P_1 \longrightarrow V$ by $\phi(\xi) = \xi v$ (recall that $P_1 \leq kG$). By the choice of v, the unique minimal submodule of P_1, $k\mu_G$, is not contained within $\ker\phi$. Thus ϕ is a monomorphism. But P_1 is injective and V is indecomposable, thus ϕ must be an isomorphism. Using this and the fact, proved above, that $\mu_G P_1 = k\mu_G$ we have:

$$\dim_k \mu_G V = \begin{cases} 1 & \text{if } V \cong P_1, \\ 0 & \text{otherwise.} \end{cases}$$

Thus the result follows. $\square$

<u>Proposition 0.0.3</u> The following are equivalent for $f \in (U,V)$:

(a) f factors through a projective module,

(b) $f \in \mu_G(U,V)_k$.

(c) $f \in \mathrm{Im}(U,g)$ where $g : P \longrightarrow V$ is any fixed epimorphism with P projective,

(d) $f \in \mathrm{Im}(h,V)$ where $h : U \longrightarrow P'$ is any fixed monomorphism with P' projective.

<u>Proof</u> We firstly show that (a) implies (b). So suppose that f factors as the

composite $U \xrightarrow{\alpha} P \xrightarrow{\beta} V$ where P is projective. Note that P is a direct summand of $X^{\uparrow G}$ for some k-space, X. Thus f factors as:

$$U \xrightarrow{\alpha} P \xrightarrow{\iota} X^{\uparrow G} \xrightarrow{\pi} P \xrightarrow{\beta} V$$

where $\pi\iota = 1_P$. Thus it suffices to assume that $P = X^{\uparrow G}$. There exist k-linear maps $\alpha_1 : U \longrightarrow X$ and $\beta_1 : X \longrightarrow V$ such that:

$$\alpha(u) = \sum_{g\in G} g \otimes \alpha_1(g^{-1}u) ,$$

$$\beta(g \otimes x) = g\beta_1(x).$$

Thus:

$$\begin{aligned} \mu_G(\beta_1\alpha_1) : u \longmapsto & \sum_{g\in G} g\beta_1\alpha_1(g^{-1}u) \\ = & \sum_{g\in G} \beta(g \otimes \alpha_1(g^{-1}u)) \\ = & \ \beta\alpha(u) = f(u). \end{aligned}$$

Hence $f \in \mu_G(U,V)_k$, as required.

We now show that (b) implies (c). Suppose that $f = \mu_G\alpha$ for some $\alpha \in (U,V)_k$. Define $\beta : U \longrightarrow (V_{\downarrow 1})^{\uparrow G}$ by:

$$\beta(u) = \sum_{g\in G} g \otimes \alpha(g^{-1}u)$$

and $\delta : (V_{\downarrow 1})^{\uparrow G} \longrightarrow V$ by:

$$\delta(g \otimes v) = gv,$$

then $f = \delta\beta$. But $(V_{\downarrow 1})^{\uparrow G}$ is projective, thus δ factors as $g\delta'$ for some $\delta' : (V_{\downarrow 1})^{\uparrow G} \longrightarrow P$. Thus $f = g\delta'\beta$ is an element of $\mathrm{Im}(U,g)$, as required.

Similarly (b) implies (d) and trivially (c) or (d) implies (a), thus the result follows. □

<u>Corollary 0.0.4</u> The following are equivalent for a kG-module, V:

(a) V is projective,

(b) $\mu_G(V,V)_k = (V,V)$,

(c) $\mu_G(V,V)_k$ contains an automorphism of V.

<u>Proof</u> To show that (a) implies (b), take $g = 1_V$ in (0.0.3)(c) to give that $\mu_G(V,V)_k$ equals $\mathrm{Im}(V,1_V)$ – that is to say, (V,V). (b) trivially implies (c). Suppose that (c) holds, then (0.0.3) shows that there exists a projective module, P, and maps $V \longrightarrow P \longrightarrow V$ the composite of which is an automorphism of V. Thus V is a direct summand of P and so is projective. Hence (c) implies (a) and the result is proved. $\square$

For any kG-modules U, V we know that $\mu_G(U,V)_k \leq (U,V)_k^G = (U,V)$. We shall write:

$$[U,V] = (U,V)/\mu_G(U,V)_k .$$

(0.0.3) makes it clear that $\mathrm{Ext}^1_{kG}(U,V) \cong [\Omega U,V]$.

<u>Lemma 0.0.5</u> Let U be a kG-module with no projective summands. Then:

(a) the only map $U \longrightarrow k_G$ which factors through a projective module is the zero map,

(b) $[U,k_G] \cong (P_U,k_G)$ where P_U denotes the projective cover of U,

(c) $[k_G,\Omega U] \cong (k_G,P_U)$.

<u>Proof</u> Note that $\mu_G(U,k_G)_k \cong \mu_G U^*$. But U^* has no projective summands, thus, by (0.0.2), $\mu_G U^* = 0$. Hence (a) follows from (0.0.3). Moreover we have that $[U,k_G] = (U,k_G)$. To prove (b), note that any element of (U,k_G) gives an element of (P_U,k_G) by composition with the epimorphism $P_U \longrightarrow U$; but ΩU is projective-free, thus any map $P_U \longrightarrow k_G$ is zero on ΩU (by (a)) and so arises

from a map $U \longrightarrow k_G$ in this way. Hence we have an isomorphism $(U,k_G) \cong (P_U,k_G)$ - thus (b) is also proved.

For (c), note that $[k_G,\Omega U] \cong [(\Omega U)^*,k_G] \cong (P_U^*,k_G) \cong (k_G,P_U)$ by (b) applied to the projective-free module $(\Omega U)^*$ of which P_U^* is a projective cover. □

The next result shows that we may generally reduce to the case when G is a p-group.

Lemma 0.0.6 Let P be a Sylow p-subgroup of G. Then a kG-module, V, is projective if and only if $V_{\downarrow P}$ is projective.

Proof If V is projective then so, trivially, is $V_{\downarrow P}$. Conversely, suppose that $V_{\downarrow P}$ is projective, then so is $(V_{\downarrow P})^{\uparrow G}$. But we have maps $V \longrightarrow (V_{\downarrow P})^{\uparrow G} \longrightarrow V$ given by:

$$v \longmapsto \frac{1}{|G:P|}\Big(\sum_{g\in T} g \otimes g^{-1}v\Big),$$

$$g \otimes v \longmapsto gv,$$

(where T is a left transversal for P in G) the composite of which is 1_V. Thus V is a direct summand of the projective module $(V_{\downarrow P})^{\uparrow G}$, and hence is itself projective. □

Note that in the case $p=0$ we have $P=1$ so that every kG-module is projective. Hence we assume for the rest of this section that $p \neq 0$ and that G is a p-group. The following lemma summarises the important elementary facts about this situation.

Lemma 0.0.7 (a) For any nonzero kG-module, V, we have $V^G \neq 0$,

(b) k_G is the unique simple kG-module,

(c) $J(kG) = \mathrm{Aug}(kG)$,

(d) kG is indecomposable,

(e) A kG-module is projective if and only if it is free,

(f) For any kG-module, V, $\mathrm{soc}(V) = V^G$.

Proof (a) We may regard V as a (not necessarily finitely-generated) $\mathbb{F}_pG$-module. Let V_1 be a nonzero, finitely-generated $\mathbb{F}_pG$-submodule of V, then, in particular, V_1 is a finite-dimensional $\mathbb{F}_p$-space and so is a finite set with p^n elements for some $n > 0$. Consider the action of G on V_1; because G is a p-group all the orbits will have length a power of p. Thus the number of orbits of length one (i.e. the number of G-fixed points of V_1) is divisible by p. Hence $0 \neq V_1^G \leq V^G$.

(b) follows trivially from (a), and (c) and (f) follow from (b). (b) also implies that $k\mu_G$ is the unique minimal submodule of kG, so (d) follows by noting that a decomposable module will have at least two minimal submodules. (e) follows from (d) and the Krull-Schmidt theorem when we recall that any projective module is a direct summand of a free module. $\square$

Proposition 0.0.8 For any kG-module, V:

$$\dim_k \mu_G V \leq \frac{1}{|G|}(\dim_k V) \leq \dim_k V^G$$

with either equality holding if and only if V is free.

Proof We can write $V = F \oplus V_0$ where F is free and V_0 has no free direct summands. The multiplicity of kG in V is then:

$$\frac{1}{|G|}(\dim_k F)$$

but it is also equal to $\dim_k \mu_G V$ by (0.0.2) ($P_1 \cong kG$ by (0.0.7)(d)).

Now $\dim_k F \leq \dim_k V$ with equality holding iff V is free, thus the result is established for the first inequality.

For the second inequality, there exists an exact sequence:

$$0 \longrightarrow V \xrightarrow{\theta} F' \longrightarrow W \longrightarrow 0$$

where F' is free and W has no free direct summand. By (0.0.2), $\mu_G W = 0$; thus $\mu_G F' \leq \mathrm{Im}\,\theta$. Hence we see that:

$$\mu_G F' = (F')^G = \theta(V^G)$$

so that:

$$\dim_k V^G = \dim_k \mu_G F' = \frac{1}{|G|}(\dim_k F').$$

But $\dim_k V \leq \dim_k F'$ with equality holding iff V is free. Thus the result is also proved for the second inequality. □

Corollary 0.0.9 $\mu_G V \leq V^G$ with equality holding if and only if V is free. □

Corollary 0.0.10 Let $\hat{k}$ be an extension field of k. For a kG-module, V, we regard $\hat{V} = \hat{k} \otimes_k V$ as a $\hat{k}$G-module in the obvious way. Then $\hat{V}$ is a free $\hat{k}$G-module if and only if V is a free kG-module.

Proof The rank of μ_G as a linear transformation is unchanged by extending the scalars to $\hat{k}$. Thus $\dim_{\hat{k}} \mu_G \hat{V} = \dim_k \mu_G V$. But trivially $\dim_{\hat{k}} \hat{V} = \dim_k V$ so the result follows using the first inequality in (0.0.8). □

We may easily extend (0.0.10) to general groups: let P be a Sylow p-subgroup of G then (0.0.6) gives that V is projective iff $V_{\downarrow P}$ is projective. But a kP-module is projective iff it is free, by (0.0.7). Thus V is projective iff $V_{\downarrow P}$ is free . A similar result holds for $\hat{V}$. (0.0.10) gives that $\hat{V}_{\downarrow P}$ is free iff $V_{\downarrow P}$ is free. Thus:

$\hat{V}$ is a projective $\hat{k}$G-module iff V is projective.

Lemma 0.0.11 (a) V is free if and only if $V^* \otimes V$ is free.

(b) $U \otimes V$ is free if and only if $U^* \otimes V$ is free.

(c) $U \otimes V$ is free if and only if:

$$\dim_k (U,V) = \frac{1}{|G|}(\dim_k U)(\dim_k V).$$

Proof (a) If V is free then so, trivially, is $V^* \otimes V$. Conversely if $V^* \otimes V \cong (V,V)_k$ is free then $\mu_G(V,V)_k = (V,V)_k^G = (V,V)$. Thus V is free by (0.0.4).

(b) follows from (a) and the isomorphism:

$$(U \otimes V)^* \otimes (U \otimes V) \cong (U^* \otimes V)^* \otimes (U^* \otimes V).$$

Thus $U \otimes V$ is free iff $(U,V)_k$ is free iff:

$$|G| \dim_k (U,V)_k^G = \dim_k (U,V)_k$$

– by (0.0.8). But $(U,V)_k^G = (U,V)$ and:

$$\dim_k (U,V)_k = (\dim_k U)(\dim_k V)$$

so (c) is proved. □

As with the previous result, (0.0.11)(a) and (b) may be extended to general groups by replacing the word "free" by "projective".

The following construction was mentioned above and will prove useful on a number of occasions:

Theorem 0.0.12 Let N be a normal subgroup of G and write $\bar{G} = G/N$. For any kG-module, V, we may regard $\mu_N V$ as a $k\bar{G}$-module which we shall denote $\bar{V}$.

Suppose that V is free on restriction to N, then $\bar{V}$ is a free $k\bar{G}$-module if and only if V is a free kG-module.

Proof Since $V_{\downarrow N}$ is free we have $\dim_k \bar{V} = \frac{1}{|N|}\dim_k V$. Note that, using (0.0.1)(c),

$\mu_{\bar G}\bar V = \mu_G V$. Thus apply (0.0.8): $\bar V$ is a free $k\bar G$-module iff:

$$\dim_k \mu_{\bar G}\bar V = \frac{1}{|G/N|}\dim_k \bar V$$

that is to say, iff:

$$\dim_k \mu_G V = \frac{1}{|G|}\dim_k V$$

– precisely the condition (0.0.8) gives for V to be free. $\square$

We shall also need the following simple result. The notation is as in (0.0.12).

<u>Lemma 0.0.13</u> Let V be a kG-module which is free on restriction to N and U be a $k\bar G$-module which is regarded as a kG-module by letting N act trivially. $U \otimes V$ is free on restriction to N and:

$$(\overline{U \otimes V}) = U \otimes \bar V.$$

<u>Proof</u> For $g \in N$, $g(u \otimes v) = gu \otimes gv = u \otimes gv$. Thus $\mu_N(u \otimes v) = u \otimes \mu_N v$ and the result easily follows. $\square$

§0.1 Chouinard's theorem

Chouinard, in [Ch], proved a result that will be of particular interest. Although he proved it in a more general form, we may state the result as follows:

> Chouinard's theorem A kG-module is projective if and only if it is free on restriction to all the elementary abelian p-subgroups of G.

Note that any projective module is projective, and hence free (by (0.0.7)), on restriction to any p-subgroup of G. Thus one implication is trivial. For the converse, we may apply a number of elementary reductions:

(1) It suffices to assume that G is a p-group. For let P be a Sylow p-subgroup of G, then any kG-module which is free on restriction to all the elementary abelian p-subgroups of G is also free on restriction to all the elementary abelian subgroups of P. Thus, if Chouinard's theorem holds for p-groups, it is projective as a kP-module, and hence, by (0.0.6), as a kG-module.

(2) It suffices to assume that k is algebraically closed. If V is a kG-module which is free on restriction to all the elementary abelian p-subgroups of G then consider the $\hat{k}$G-module $\hat{V} = \hat{k} \otimes_k V$ where $\hat{k}$ is an algebraic closure of k; this is free on restriction to all the elementary abelian subgroups of G, so if Chouinard's theorem holds for algebraically closed fields then $\hat{V}$ is a free $\hat{k}$G-module. Hence (0.0.10) gives that V is a free kG-module.

So now suppose that we try to prove Chouinard's theorem for p-groups by induction on the order of G. The result is trivial if G is elementary abelian, so we may assume otherwise. Suppose that V is a non-projective kG-module which is free on restriction to all the elementary abelian subgroups of G, then the same conditions hold with respect to one of the non-projective indecomposable summands of V; thus it suffices to assume that V is indecomposable. Inductively, V is free on restriction to all the proper subgroups of G. Thus it suffices to prove:

<u>Chouinard's theorem</u> – <u>Second version</u> Let k be an algebraically closed field of characteristic p. We call a finite p-group, G, a *<u>Chouinard group</u>* if there do not exist non-projective, indecomposable kG-modules which are free on restriction to all the maximal subgroups of G. Then any group which is not elementary abelian is a Chouinard group.

Moreover, this version is implied by the first, so the two formulations are equivalent.

As an example, let us prove that any cyclic p-group of order at least p^2 is a Chouinard group. Let G be such a group of order p^n $(n > 1)$ and choose a generator x. $H = \langle x^p \rangle$ is the unique maximal subgroup of G, it has order $p^{n-1} = q$, say. Suppose, for a contradiction, that V is a non-projective, indecomposable kG-module which is free on restriction to H.

V is the direct sum of a number of copies of kH, thus $q \mid \dim_k V$. Since V is not projective, (0.0.2) gives that $\mu_G V = 0$, that is to say, using (0.0.1)(d),

$$(x-1)^{(p^n-1)} V = 0 .$$

Thus the minimum polynomial of x on V divides $(X-1)^{(p^n-1)}$ and hence is $(X-1)^r$ for some $r \leq p^n - 1$. Using the fact that V is indecomposable, the matrix representing x on V must be the $(r \times r)$ Jordan block with 1's on the diagonal. In particular, $r = \dim_k V$, so that $q \mid r$. Hence we can write $r = qs$ for some $s \leq p-1$. Thus $q(p-1) \geq r$ so that:

$$(x-1)^{q(p-1)} V = (x^q - 1)^{(p-1)} V = 0 .$$

But $E = \langle x^q \rangle$ is a subgroup of G of order p, so the equation above says that $\mu_E V = 0$. Hence V is not free on restriction to E – a contradiction since $E \leq H$. Thus:

Proposition 0.1.1 Any cyclic p-group of order at least p^2 is a Chouinard group. □

Later we shall give at least two further proofs of this proposition. Meanwhile we prove an elementary reduction theorem. The methods used are essentially those of [Cal] slightly extended.

Theorem 0.1.2 Any p-group which has a Chouinard factor group is itself a Chouinard group.

Proof Let G be a p-group with a normal subgroup, N, such that $\bar{G} = G/N$ is a Chouinard group. Suppose that V is an indecomposable kG-module which is free on restriction to all the maximal subgroups of G. We cannot have $G = N$ so N is contained within some maximal subgroup of G; in particular, V is free on restriction to N. Thus we consider the $k\bar{G}$-module $\bar{V} = \mu_N V$ as in (0.0.12). The maximal subgroups of $\bar{G}$ correspond to the maximal subgroups of G containing N; thus $\bar{V}$ is free on restiction to all the maximal subgroups of $\bar{G}$. Since $\bar{G}$ is a Chouinard group this implies that $\bar{V}$ is projective. Thus V is projective and the result follows. □

Corollary 0.1.3 If G is a p-group for which the derived subgroup is properly contained within the Frattini subgroup, then G is a Chouinard group.

Corollary 0.1.4 Any abelian p-group which is not elementary abelian is a Chouinard group.

Proofs Before proving (0.1.3), we remark that the derived subgroup of G, G', is always contained in the Frattini subgroup. If the inclusion is strict then G/G' is an abelian group which is not elementary abelian. Thus (0.1.2) shows that it suffices to prove (0.1.4).

Any abelian group is the direct product of cyclic groups. If it is not elementary abelian then one of these factors will have order at least p^2. Thus (0.1.4) follows from (0.1.2) and (0.1.1). □

<u>Remark</u> There is a slight ambiguity in the definition above as to whether the trivial group is a Chouinard group. On the one hand there are no non-projective k1-modules; on the other, it makes no sense to talk about maximal subgroups of 1. We make the convention (tacitly assumed in the proof of (0.1.2)) that 1 is not a Chouinard group. Since 1 is certainly elementary abelian, this makes no difference to the validity of Chouinard's theorem.

CHAPTER 1

PERIODIC RESOLUTIONS AND BOCKSTEIN OPERATORS

Introduction

Suppose that V is a kG-module which is free on restriction to some maximal subgroup of the p-group, G. In this chapter we show how to construct an exact sequence:

$$0 \longrightarrow V \longrightarrow P \longrightarrow P \longrightarrow V \longrightarrow 0$$

with P projective. We also discuss how this construction is related to the Bockstein map:

$$B : \mathrm{Ext}^1_{\mathbb{F}_pG}(\mathbb{F}_p, \mathbb{F}_p) \longrightarrow \mathrm{Ext}^2_{\mathbb{F}_pG}(\mathbb{F}_p, \mathbb{F}_p)$$

and show how the result of Serre concerning the cup-product of certain elements of $\mathrm{Im}\, B$ may be used to prove Chouinard's theorem.

§1.0 Cyclic groups

Let C be a cyclic p-group. There is a well-known periodic projective resolution for k_C described, for example, in [C&E]. Choose a generator, x, for C and consider the sequence:

$$0 \longrightarrow k_C \xrightarrow{\alpha} kC \xrightarrow{\beta} kC \xrightarrow{\delta} k_C \longrightarrow 0 \qquad -(1)$$

where the maps are given by:

$$\alpha : 1 \longmapsto \mu_C ,$$
$$\beta : 1 \longmapsto x - 1 ,$$
$$\delta : 1 \longmapsto 1 ,$$

and are extended kC-linearly. Note that $\operatorname{Aug}(kC) = (x-1)kC$, that is to say $\ker \delta = \operatorname{Im} \beta$. Thus, using dimensions, the sequence is exact.

Recall that kC is indecomposable, so this must be a minimal resolution. Thus k_C is periodic of period one or two. It has period one iff $\operatorname{Im} \beta$ is isomorphic to k_C, that is to say, iff

$$1 = \dim_k \operatorname{Im} \beta = \dim_k kC - \dim_k k_C = |C| - 1,$$

i.e. C has order 2. In this case $\mu_C = x - 1$ so the sequence above is the join of two copies of the sequence:

$$0 \longrightarrow k_C \xrightarrow{\alpha} kC \xrightarrow{\delta} k_C \longrightarrow 0 .$$

In general, the sequence (1) depends on the choice of the generator, x. Suppose that we choose another generator, x', and construct the corresponding sequence. The end maps, α and δ, will be unchanged, but the central map $\beta' : kC \longrightarrow kC$ will be given by $1 \longmapsto x' - 1$. Now, we may complete the commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & k_C \arrow[r, "\alpha"] & kC \arrow[r, "\beta'"] & kC \arrow[r, "\epsilon"] & k_C \arrow[r] & 0 \\
0 \arrow[r] & k_C \arrow[r, "\alpha"] \arrow[u, dashed, "\varepsilon"] & kC \arrow[r, "\beta"] \arrow[u, dashed, "\gamma'"] & kC \arrow[r, "\epsilon"] \arrow[u, dashed, "\gamma"] & k_C \arrow[r] \arrow[u, equal] & 0 .
\end{tikzcd}$$

<u>Lemma 1.0.1</u> There exists a with $x = (x')^a$. With this notation, the map ε in the diagram equals $a1_k$.

<u>Proof</u> Note that ε is uniquely determined up to the addition of a map factoring through kC. Hence, by (0.0.5), ε is determined uniquely. We shall construct a particular diagram.

We may take $\gamma = 1_{kC}$; then:

$$\begin{aligned} \gamma\beta(1) &= x - 1 = (x')^a - 1 \\ &= (x' - 1)(1 + x' + \ldots + (x')^{a-1}) \end{aligned}$$

so that we may define γ' by:

$$1 \longmapsto 1 + x' + \ldots + (x')^{a-1}.$$

Hence:

$$\gamma'\alpha(1) = \mu_C(1 + x' + \ldots + (x')^{a-1}) = a\mu_C$$

and the result is seen to follow. $\square$

§ 1.1 Periodic Resolutions

Suppose that H is a maximal subgroup of a p-group, G. H is then normal of index p, so consider the cyclic group $C = G/H$. The sequence of kC-modules, (1), given in the previous section may be regarded as a sequence of kG-modules by letting H act trivially. This takes the form:

$$0 \longrightarrow k_G \xrightarrow{\alpha} k(G/H) \xrightarrow{\beta} k(G/H) \xrightarrow{\delta} k_G \longrightarrow 0 \qquad -(2)$$

where the maps are given by:

$$\alpha : 1 \longmapsto P_{G/H},$$
$$\beta : H \longmapsto (g-1)H,$$
$$\delta : H \longmapsto 1.$$

Here gH is the choice of generator for G/H, so g can be chosen from any of the elements of $G - H$. If g' is another choice then (1.0.1) gives that there is a commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & k_G \arrow[r] & k(G/H) \arrow[r, "\beta'"] & k(G/H) \arrow[r] & k_G \arrow[r] & 0 \\
0 \arrow[r] & k_G \arrow[r] \arrow[u, dashed, "a1_k"] & k(G/H) \arrow[r, "\beta"] \arrow[u, dashed] & k(G/H) \arrow[r] \arrow[u, dashed] & k_G \arrow[r] \arrow[u, equal] & 0
\end{tikzcd}$$

where a is defined by $g(g')^{-a} \in H$.

The main usefulness of the sequence (2) lies in the following result:

<u>Theorem 1.1.1</u> Suppose that V is a kG-module which is free on restriction to H, then $k(G/H) \otimes V$ is free.

More concretely, if $v_1, \ldots, v_m$ is a free kH-basis for V, then:

$$H \otimes v_1, \ldots\ldots, H \otimes v_m$$

is a free kG-basis for $k(G/H) \otimes V$.

Proof If we just wish to prove the first statement then note that $k(G/H)$ is isomorphic to $(k_H)^{\uparrow G}$. Thus $k(G/H) \otimes V \cong (k_H \otimes V_{\downarrow H})^{\uparrow G} \cong (V_{\downarrow H})^{\uparrow G}$ is free, as required.

For the concrete version, it suffices, by dimensions, to show that $H \otimes v_1$, ..., $H \otimes v_m$ are kG-linearly independent. So suppose that:

$$\sum_{i=1}^{m} \xi_i (H \otimes v_i) = 0$$

for some $\xi_i \in kG$. Write each ξ_i as:

$$\xi_i = \sum_{j=0}^{p-1} g^j \varrho_{ij} \qquad \text{for some } g \in G-H,\ \varrho_{ij} \in kH,$$

then:

$$0 = \sum_{i=1}^{m} \sum_{j=0}^{p-1} g^j \varrho_{ij}(H \otimes v_i) = \sum_{i,j} g^j H \otimes g^j \varrho_{ij} v_i .$$

Thus, for each j,

$$g^j\Big(\sum_{i=1}^{m} \varrho_{ij} v_i\Big) = 0$$

but the v_i's are kH-linearly independent, so each $\varrho_{ij} = 0$. Thus each $\xi_i = 0$, as required. $\square$

Tensor the sequence, (2), by V to obtain:

$$0 \longrightarrow V \longrightarrow k(G/H) \otimes V \longrightarrow k(G/H) \otimes V \longrightarrow V \longrightarrow 0 .$$

By (1.1.1) this is a two-step projective resolution for V. We can improve on this in the case $p = 2$ by taking the sequence:

$$0 \longrightarrow k_G \longrightarrow k(G/H) \longrightarrow k_G \longrightarrow 0$$

and tensoring this by V to get a one-step projective resolution. Thus:

<u>Theorem 1.1.2</u> Let V be a non-projective, indecomposable kG-module which is free on restriction to some maximal subgroup of G. Then V is periodic of period one or two. In the case $p = 2$, the period is always one. $\square$

This method is used to prove the same result, Lemma 2.5, in [Ca2], and seems to be generally well known.

§1.2 Serre groups

Let us for the moment restrict our attention to the field of p elements, $\mathbb{F}_p$. For a fixed p-group, G, take a minimal $\mathbb{F}_pG$-projective resolution:

$$\cdots\cdot \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow \mathbb{F}_p \longrightarrow 0.$$

If $(H_1,\ldots,H_m)$ is an m-tuple of maximal subgroups of G then, for each i, we choose $g_i \in G - H_i$ and consider the sequence:

$$0 \longrightarrow \mathbb{F}_p \longrightarrow \mathbb{F}_p(G/H_i) \longrightarrow \mathbb{F}_p(G/H_i) \longrightarrow \mathbb{F}_p \longrightarrow 0$$

where the central map is given by $H_i \longmapsto (g_i - 1)H_i$. We may string these m sequences together by identifying the left-hand $\mathbb{F}_p$ of one sequence with the right-hand $\mathbb{F}_p$ of the next. Complete the commutative DIAGRAM 1 (see over). This determines a map $\alpha : \Omega^{2m}\mathbb{F}_p \longrightarrow \mathbb{F}_p$ depending, up to multiplication by a nonzero scalar, only on $(H_1,\ldots,H_m)$.

Call G a *Serre group* if there exists an m-tuple, $(H_1,\ldots,H_m)$, for some m such that the corresponding map $\Omega^{2m}\mathbb{F}_p \longrightarrow \mathbb{F}_p$ is zero.

Theorem 1.2.1 Any Serre group is a Chouinard group.

Proof Let G be a Serre group and suppose that Diagram 1 is such that $\alpha = 0$. Let V be a non-projective, indecomposable kG-module which is free on restriction to all the maximal subgroups of G. Firstly apply $k \otimes_{\mathbb{F}_p} -$ to Diagram 1 to obtain a diagram:

$$\begin{tikzcd}
0 \arrow[r] & k_G \arrow[r] & k(G/H_m) \arrow[r] & \cdots\cdot \arrow[r] & k(G/H_1) \arrow[r] & k_G \arrow[r] & 0 \\
0 \arrow[r] & \Omega^{2m}k_G \arrow[r] \arrow[u, "0"] & P'_{m'} \arrow[r] \arrow[u] & \cdots\cdot \arrow[r] & P'_0 \arrow[r] \arrow[u] & k_G \arrow[r] \arrow[u, equal] & 0
\end{tikzcd}$$

(here $m' = 2m - 1$)

<u>DIAGRAM 1</u>

α is uniquely determined by this diagram, using (0.0.5)(a). We may gauge the effect of changing the choice of the elements, g_i , of $G - H_i$ by using the extension of (1.0.1) mentioned in §1.1 : it is to multiply α by a nonzero scalar. Thus whether or not $\alpha = 0$ is independent of this choice.

$$\begin{tikzcd}
0 \arrow[r] & F_p \arrow[r] & F_p(G/H_m) \arrow[r] & F_p(G/H_m) \arrow[r] & \cdots \arrow[r] & F_p(G/H_1) \arrow[r] & F_p(G/H_1) \arrow[r] & F_p \arrow[r] & 0 \\
0 \arrow[r] & \Omega^{2m} F_p \arrow[r] \arrow[u, dashed, "\alpha"] & P_{2m-1} \arrow[r] \arrow[u, dashed] & P_{2m-2} \arrow[r] \arrow[u, dashed] & \cdots \arrow[r] & P_1 \arrow[r] \arrow[u, dashed] & P_0 \arrow[r] \arrow[u, dashed] & F_p \arrow[r] \arrow[u, equal] & 0
\end{tikzcd}$$

where the lower sequence is a minimal projective resolution of k_G (use (0.0.10) to show that it is minimal). Now tensor this diagram by V. V is free on restriction to each of $H_1,\dots,H_m$ so, as in the previous section, the top row is a 2m-step projective resolution of V. Thus we may complete the diagram:

$$\begin{tikzcd}
0 \arrow[r] & V \arrow[r] & k(G/H_m)\otimes V \arrow[r] & \dots \arrow[r] & k(G/H_1)\otimes V \arrow[r] & V \arrow[r] & 0 \\
0 \arrow[r] & \Omega^{2m}k_G\otimes V \arrow[r] \arrow[u, "0"] & P'_m\otimes V \arrow[r] \arrow[u] & \dots \arrow[r] & P'_0\otimes V \arrow[r] \arrow[u] & V \arrow[r] \arrow[u, equal] & 0 \\
0 \arrow[r] & V \arrow[r] \arrow[u, dashed] & k(G/H_m)\otimes V \arrow[r] \arrow[u, dashed] & \dots \arrow[r] & k(G/H_1)\otimes V \arrow[r] \arrow[u] & V \arrow[r] \arrow[u, equal] & 0 .
\end{tikzcd}$$

Therefore 1_V factors through the projective module $k(G/H_1)\otimes V$. Hence V is projective - a contradiction. $\square$

As an example, we again take the case when G is a cyclic group of order at least p^2, generated by x, say. We have a minimal projective resolution:

$$0 \longrightarrow \mathbb{F}_p \xrightarrow{\alpha} \mathbb{F}_pG \xrightarrow{\beta} \mathbb{F}_pG \xrightarrow{\varepsilon} \mathbb{F}_p \longrightarrow 0$$

as in §1.0, where the maps are given by:

$$\alpha : 1 \longmapsto \mu_G ,$$
$$\beta : 1 \longmapsto x-1 ,$$
$$\varepsilon : 1 \longmapsto 1 .$$

Let $H = \langle x^p \rangle$ be the unique maximal subgroup of G and complete the diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{F}_p \arrow[r] & \mathbb{F}_p(G/H) \arrow[r] & \mathbb{F}_p(G/H) \arrow[r] & \mathbb{F}_p \arrow[r] & 0 \\
0 \arrow[r] & \mathbb{F}_p \arrow[r, "\alpha"] \arrow[u, dashed] & \mathbb{F}_pG \arrow[r, "\beta"] \arrow[u, dashed, "\gamma'"] & \mathbb{F}_pG \arrow[r, "\varepsilon"] \arrow[u, "\gamma"] & \mathbb{F}_p \arrow[r] \arrow[u, equal] & 0
\end{tikzcd}$$

where the central map in the upper sequence is given by $H \longmapsto (x-1)H$.

We may define γ by $1 \longmapsto H$ and let $\gamma' = \gamma$. Then:

$$\gamma'\alpha(1) = \gamma'(\mu_G) = \mu_G H = 0$$

so that $\epsilon = 0$. Thus G is a Serre group, and hence a Chouinard group, thereby giving another proof of (0.1.1).

The next section deals with a slightly less trivial example.

§1.3 Example - Q_8

In this section we deal with what seems to be a fairly difficult case. Let:

$$G = \left\langle x,y \;\middle|\; x^2 = y^2 = (xy)^2 \right\rangle$$

be the quaternion group, Q_8. G has three maximal subgroups, $\langle x\rangle$, $\langle y\rangle$ and $\langle xy\rangle$, each of which is cyclic of order 4. The subgroups of order 2 of these three groups coincide, thus (0.1.1) implies that if a kG-module is free on restriction to one of the maximal subgroups of G, then it is free on restriction to them all. Thus we may concentrate on a single maximal subgroup, H; say $H = \langle x\rangle$.

We give two proofs of the fact that G is a Chouinard group. The first method is brute force. Write:

$$\xi = x - 1 \;,\; \eta = y - 1$$

for the elements of kG. Now:

$$\eta^2 \;=\; y^2 - 1 \;=\; x^2 - 1 \;=\; \xi^2$$

so that, in particular, $\eta\xi^2 = \xi^2\eta$. Also:

$$\begin{aligned}\eta\xi \;&=\; yx - x - y + 1 \;=\; x^3y - x - y + 1 \;=\; (x^3 - 1)y - (x - 1)\\ &=\; (\xi + \xi^2 + \xi^3)(1 + \eta) - \xi \;=\; (\xi + \xi^2)\eta + \xi^2 + \xi^3 + \mu_G\end{aligned}$$

– for note that $\xi^3\eta = \mu_G$.

Now suppose that k is an algebraically closed field of characteristic 2, and that V is a non-projective, indecomposable kG-module which is free on restriction to H (and thus to all the maximal subgroups of G). Take a free kH-basis, $v_1,\ldots,v_m$, for V. Note that $\mu_G V = 0$ so $\mu_H \eta V = 0$, hence:

$$\eta V \subseteq \mathrm{Aug}\,(kH).V \;=\; \xi V\,.$$

Thus we may write:

$$\eta v_i = \sum_{j=1}^{m} (a_{ij}\xi v_j + b_{ij}\xi^2 v_j + c_{ij}\xi^3 v_j)$$

for some constants, a_{ij}, b_{ij}, c_{ij} in k. Now:

$$\xi^2 v_i = \eta^2 v_i = \sum_j (a_{ij}\eta\xi v_j + b_{ij}\eta\xi^2 v_j + c_{ij}\mu_G v_j)$$

$$= \sum_j (a_{ij}((\xi + \xi^2)\eta v_j + \xi^2 v_j + \xi^3 v_j + \mu_G v_j) + b_{ij}\xi^2 \eta v_j)$$

$$= \sum_j (a_{ij}(\xi^2 + \xi^3)v_j + ((a_{ij} + b_{ij})\xi^2 + a_{ij}\xi)\eta v_j)$$

$$= \sum_j (a_{ij}(\xi^2 + \xi^3)v_j + ((a_{ij} + b_{ij})\xi^2 + a_{ij}\xi)(\sum_r (a_{jr}\xi + b_{jr}\xi^2 + c_{jr}\xi^3)v_r)))$$

$$= \sum_j a_{ij}(\xi^2 + \xi^3)v_j + \sum_{j,r} (a_{ij}a_{jr}\xi^2 + a_{ij}b_{jr}\xi^3 + (a_{ij} + b_{ij})a_{jr}\xi^3)v_r .$$

Thus if A denotes the matrix (a_{ij}) and B the matrix (b_{ij}) then this implies that $I = A + A^2$ and $0 = A + AB + (A + B)A$. Hence there exist $(m \times m)$ matrices A, B with:

$$A^2 + A = I$$
$$AB + BA = I$$

– we will show that these equations give a contradiction.

By conjugating suitably, we may assume that A is in its Jordan canonical form. If A_1 is a constituent Jordan block then $A_1^2 + A_1 = I$. Write A_1 as:

$$\begin{vmatrix} a & 1 & & & \\ & a & 1 & & \\ & & \cdots & & \\ & & & a & 1 \\ & & & & a \end{vmatrix}$$

then:

$$A_1^2 + A_1 = \begin{vmatrix} b & 1 & 1 & & & \\ & b & 1 & 1 & & \\ & & \cdots & \cdots & & \\ & & & b & 1 & 1 \\ & & & & b & 1 \\ & & & & & b \end{vmatrix}$$

where $b = a^2 + a$. The only way that this can be the identity matrix is if A_1 is a (1×1) matrix and $a^2 + a = 1$.

Thus A is a diagonal matrix with the diagonal entries satisfying $X^2 + X = 1$. Thus we may assume that:

$$A = \begin{bmatrix} a_1 I_1 & \\ & a_2 I_2 \end{bmatrix}$$

where a_i $(i = 1,2)$ are the roots of $X^2 + X - 1$ and I_i are identity matrices of various sizes. Write:

$$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

in compatible block form. Then the equation $AB + BA = I$ implies that:

$$\begin{bmatrix} a_1 B_{11} & a_1 B_{12} \\ a_2 B_{21} & a_2 B_{22} \end{bmatrix} + \begin{bmatrix} a_1 B_{11} & a_2 B_{12} \\ a_1 B_{21} & a_2 B_{22} \end{bmatrix} = \begin{bmatrix} I_1 & \\ & I_2 \end{bmatrix}$$

so $I_1 = I_2 = 0$. Thus $A = 0$. Hence $a = 0$ – a contradiction.

Thus G is a Chouinard group.

The second approach to proving this fact is to show that G is a Serre group. There is a 4-step periodic projective resolution of k_G given, for example, in [C&E]. The details are given in DIAGRAM 2.

<u>DIAGRAM 2</u>

In the projective resolution, P denotes the free $\mathbb{F}_2G$-module on two symbols, a and b. The maps are given by:

$\delta_0 : 1 \longmapsto 1$,

$\delta_1 : a \longmapsto x-1$, $b \longmapsto y-1$,

$\delta_2 : a \longmapsto (x-1)a + (y-1)b$,

$\qquad b \longmapsto (xy-1)a + (x-1)b$,

$\delta_3 : 1 \longmapsto (x-1)a + (xy-1)b$,

$\delta_4 : 1 \longmapsto \mu_G$.

The maps in the upper sequence are given by:

$\alpha : 1 \longmapsto \mu_{G/H} = (y-1)H$,

$\beta : H \longmapsto (y-1)H$,

$\gamma : H \longmapsto 1$.

Thus we may define α_0 by $1 \longmapsto H$. Then:

$\alpha_0\delta_1 : a \longmapsto (x-1)H = 0$,

$\qquad b \longmapsto (y-1)H$,

so define α_1 by $a \longmapsto 0$, $b \longmapsto H$. Now:

$\alpha_1\delta_2 : a \longmapsto (y-1)H$,

$\qquad b \longmapsto (x-1)H = 0$,

so define α_2 by $a \longmapsto H$, $b \longmapsto 0$. Then:

$\alpha_2\delta_3 : 1 \longmapsto (x-1)H = 0$.

Thus α_3, and hence α_4, may be taken to be zero maps. Therefore G is a Serre group.

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{F}_2 \arrow[r, "\alpha"] & \mathbb{F}_2(G/H) \arrow[r, "\beta"] & \mathbb{F}_2(G/H) \arrow[r, "\alpha"] & \mathbb{F}_2(G/H) \arrow[r, "\beta"] & \mathbb{F}_2(G/H) \arrow[r, "\gamma"] & \mathbb{F}_2 \arrow[r] & 0 \\
0 \arrow[r] & \mathbb{F}_2 \arrow[r, "\delta_4"] \arrow[u, dashed, "\alpha_4"] & \mathbb{F}_2G \arrow[r, "\delta_3"] \arrow[u, dashed, "\alpha_3"] & P \arrow[r, "\delta_2"] \arrow[u, dashed, "\alpha_2"] & P \arrow[r, "\delta_1"] \arrow[u, dashed, "\alpha_1"] & \mathbb{F}_2G \arrow[r, "\delta_0"] \arrow[u, dashed, "\alpha_0"] & \mathbb{F}_2 \arrow[r] \arrow[u, equal] & 0
\end{tikzcd}$$

Remark The fact that Q_8 is a Chouinard group is proved in [Ca1]. The method used there is similar to the "brute force" method given above, but appeals to the classification theorem for the indecomposable modules of the Klein 4-group (see Appendix A).

§1.4 Connection with Bockstein operators

Write $E^m(G)$ for the cohomology group:

$$E^m(G) = \mathrm{Ext}^m_{\mathbb{F}_pG}(\mathbb{F}_p, \mathbb{F}_p).$$

We shall think of $E^m(G)$ in two ways: the details may be found in [McL].

(1) Using (0.0.5), there is an isomorphism $E^m(G) \cong (\Omega^m\mathbb{F}_p, \mathbb{F}_p)$ – this method is particularly useful when we are considering the $\mathbb{F}_p$-space structure of the cohomology ring:

$$E^*(G) = \bigoplus_{m=0}^{\infty} E^m(G).$$

(2) We may also think of $E^m(G)$ as the set of all equivalence classes of exact sequences of $\mathbb{F}_pG$-modules:

$$0 \longrightarrow \mathbb{F}_p \longrightarrow X_m \longrightarrow \dots\dots \longrightarrow X_1 \longrightarrow \mathbb{F}_p \longrightarrow 0 \qquad -(1)$$

– each such sequence being assigned a (well-defined) map $\Omega^m\mathbb{F}_p \longrightarrow \mathbb{F}_p$ by means of the diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{F}_p \arrow[r] & X_m \arrow[r] & \dots\dots \arrow[r] & X_1 \arrow[r] & \mathbb{F}_p \arrow[r] & 0 \\
0 \arrow[r] & \Omega^m\mathbb{F}_p \arrow[r] \arrow[u, dashed] & P_m \arrow[r] \arrow[u, dashed] & \dots\dots \arrow[r] & P_1 \arrow[r] \arrow[u, dashed] & \mathbb{F}_p \arrow[r] \arrow[u, equal] & 0
\end{tikzcd}$$

where the lower sequence is a minimal projective resolution – two sequences are then equivalent if they are assigned the same map. This is particularly useful when we are considering the cup-product in $E^*(G)$. The product of the image of sequence (1) with the image of the sequence:

$$0 \longrightarrow \mathbb{F}_p \longrightarrow Y_n \longrightarrow \dots\dots \longrightarrow Y_1 \longrightarrow \mathbb{F}_p \longrightarrow 0$$

is just the image of the join of the two sequences:

$$0 \longrightarrow \mathbb{F}_p \longrightarrow Y_n \longrightarrow \cdots\cdot \longrightarrow Y_1 \longrightarrow X_m \longrightarrow \cdots\cdot$$
$$\cdots\cdot \longrightarrow X_1 \longrightarrow \mathbb{F}_p \longrightarrow 0\,.$$

With this notation, we derive a characterisation of the image of the Bockstein map:

$$B : E^1(G) \longrightarrow E^2(G)\,.$$

See [Ca4] or [Pic] for the same result, although not necessarily expressed in the same terms.

Firstly note that $E^1(G)$ is isomorphic to $(\mathrm{Aug}\,(\mathbb{F}_pG)\,,\,\mathbb{F}_p)$. But $\mathbb{F}_pG$-linear maps $f : \mathrm{Aug}\,(\mathbb{F}_pG) \longrightarrow \mathbb{F}_p$ are in 1-1 correspondence with group homomorphisms $f' : G \longrightarrow \mathbb{F}_p^+$ via the rule:

$$f'(x) \;=\; f(x-1).$$

– To check the details: given f we have:

$$\begin{aligned} f'(xy) &= f(xy-1) \;=\; f((x-1) + (y-1) + (x-1)(y-1)) \\ &= f(x-1) + f(y-1) + (x-1)f(y-1) \\ &= f'(x) + f'(y) \end{aligned}$$

so that f' is a group homomorphism. Similarly, given f', f is $\mathbb{F}_p$-linear.

So let z be a nonzero element of $E^1(G)$. We regard z as a nonzero group homomorphism $G \longrightarrow \mathbb{F}_p^+$ so that $H = \ker z$ is a maximal subgroup of G. Write $\bar{G} = G/H$ and let ϕ be the natural homomorphism $G \longrightarrow \bar{G}$. z factors as:

$$G \xrightarrow{\;\phi\;} \bar{G} \xrightarrow{\;z'\;} \mathbb{F}_p^+$$

for some $z' \in E^1(\bar{G})$; that is to say, z is in the image of the induced map $\phi_1^* : E^1(\bar{G}) \longrightarrow E^1(G)$.

The Bockstein operator is natural, thus there is a commutative diagram:

$$\begin{array}{ccc} E^1(G) & \xrightarrow{B} & E^2(G) \\ {\scriptstyle \phi_1^*}\uparrow & & \uparrow{\scriptstyle \phi_2^*} \\ E^1(\bar{G}) & \xrightarrow{\bar{B}} & E^2(\bar{G}) \end{array}$$

where the horizontal maps are the Bockstein operators and the vertical maps are those induced by ϕ. Thus $B(z) = \phi_2^*\bar{B}(z')$.

Using the properties of cyclic groups we have:

$$E^1(\bar{G}) \cong (\mathrm{Aug}(\mathbb{F}_p\bar{G}), \mathbb{F}_p) \cong \mathbb{F}_p$$

$$E^2(\bar{G}) \cong (\Omega^2\mathbb{F}_p, \mathbb{F}_p) \cong (\mathbb{F}_p, \mathbb{F}_p) \cong \mathbb{F}_p$$

and $\bar{B} \neq 0$. Thus, choosing any nonzero element, e, of $E^2(\bar{G})$ there exists a non-zero $c \in \mathbb{F}_p$ with $\bar{B}(z') = ce$.

But we may take e to be the image of the sequence:

$$0 \longrightarrow \mathbb{F}_p \longrightarrow \mathbb{F}_p\bar{G} \longrightarrow \mathbb{F}_p\bar{G} \longrightarrow \mathbb{F}_p \longrightarrow 0$$

so that $\phi_2^*(e)$ is the image of:

$$0 \longrightarrow \mathbb{F}_p \longrightarrow \mathbb{F}_p(G/H) \longrightarrow \mathbb{F}_p(G/H) \longrightarrow \mathbb{F}_p \longrightarrow 0\,.$$

Now $B(z) = \phi_2^*\bar{B}(z') = c\phi_2^*(e)$. To conclude:

<u>Theorem 1.4.1</u> If z is any nonzero element of $E^1(G)$ then $B(z)$ is a nonzero scalar multiple of the image in $E^2(G)$ of the sequence:

$$0 \longrightarrow \mathbb{F}_p \longrightarrow \mathbb{F}_p(G/H) \longrightarrow \mathbb{F}_p(G/H) \longrightarrow \mathbb{F}_p \longrightarrow 0$$

for some maximal subgroup, H, of G. $\square$

In view of this, let us re-examine our definition of Serre groups. G is a Serre group if and only if there exist nonzero elements, $z_1,\ldots,z_m$ of $E^1(G)$ such that the cup-product, $B(z_1)B(z_2)\ldots B(z_m)$, equals 0. Thus Proposition (4) of [Ser] says that:

<u>Theorem 1.4.2</u> Every group which is not elementary abelian is a Serre group.

Hence this and (1.2.1) imply that every group which is not elementary abelian is a Chouinard group. That is to say Chouinard's theorem holds.

This is, broadly speaking, Chouinard's proof stripped of its cohomological guise. Serre's result remains a stumbling-block however.

CHAPTER 2

ALMOST SPLIT SEQUENCES

Introduction

We have shown, given a non-projective, indecomposable kG-module, V, which is free on restriction to a maximal subgroup of G, how to construct a two-step projective resolution:

$$0 \longrightarrow V \longrightarrow P \longrightarrow P \longrightarrow V \longrightarrow 0.$$

But the construction of almost split sequences (sketched, for completeness, in §2.0) requires just this - a two-step projective resolution. Therefore in this chapter we apply this to V to obtain an almost split sequence of the form:

$$0 \longrightarrow V \longrightarrow X \longrightarrow V \longrightarrow 0.$$

We also investigate the decomposition of X into indecomposable summands, and the irreducible maps involving V.

§2.0 The construction of almost split sequences

We begin with the definition of an almost split sequence that we shall use: this is the dual of the one usually given, but, as is well known, the two are equivalent (see §2.2 for details). So, an exact sequence:

$$0 \longrightarrow U \xrightarrow{\rho} X \longrightarrow V \longrightarrow 0$$

is said to be *almost split* if the following conditions hold:

(1) U and V are indecomposable,

(2) the sequence does not split,

(3) any map $f : U \longrightarrow W$ which is not a split monomorphism factors as $f'\rho$ for some $f' : X \longrightarrow W$.

The existence of such sequences was first proved in a general context by Auslander and Reiten; see [A&R] or, for a more succinct construction, [Gab]. We shall follow the existence proof given, for example, in [Ben] or [Lan], adapted slightly. Although basically the same, this avoids the general category-theory arguments of the original. We shall skip over many of the details of the proof.

The first step is to prove that, for general kG-modules, U and V, there is a natural k-isomorphism:

$$D[U,V] \cong [V,\Omega U] \qquad -(1)$$

where D denotes the dual space. Applying this twice we have that $[U,V]$ is naturally isomorphic to $[\Omega U,\Omega V]$; indeed we may demonstrate a particular isomorphism – for $f \in (U,V)$ define f' by the diagram:

$$\begin{tikzcd}
0 \arrow[r] & \Omega V \arrow[r] & P_V \arrow[r] & V \arrow[r] & 0 \\
0 \arrow[r] & \Omega U \arrow[r] \arrow[u, dotted, "f'"] & P_U \arrow[r] \arrow[u, dotted] & U \arrow[r] \arrow[u, "f"'] & 0 .
\end{tikzcd}$$

f' is determined up to the addition of a map that factors through a projective module; thus the map $f \longmapsto f'$ induces a well-defined map $[U,V] \longrightarrow [\Omega U, \Omega V]$ which is readily checked to be an isomorphism. In the case $U = V$ this clearly gives a ring homomorphism:

$$[U,U] \cong [\Omega U, \Omega U].$$

Thus we may regard both sides in (1) as $[U,U]$-$[V,V]$-bimodules. The question then arises: can we assume that the isomorphism in (1) is a bimodule isomorphism? Not surprisingly, a little work shows that we can.

Now let V be a non-projective, indecomposable kG-module, and take an exact sequence:

$$0 \longrightarrow \Omega^2 V \xrightarrow{\alpha} P_1 \xrightarrow{\beta} P_0 \xrightarrow{\delta} V \longrightarrow 0$$

with P_0 and P_1 projective. Let $U = \operatorname{Im}\beta$ so that U is isomorphic to the direct sum of ΩV and a projective module. By the result above, there is a $[V,V]$-$[V,V]$-bimodule isomorphism:

$$[V,U] \cong [V, \Omega V] \cong D[V,V]. \qquad -(2)$$

Since V is indecomposable, (V,V) is a local ring. Thus $[V,V]$ has a unique maximal left $[V,V]$-submodule. Therefore (2) gives that $[V,U]$ has a unique minimal right $[V,V]$-submodule. Equivalently, the socle of $[V,U]$ as a right $[V,V]$-module, $\operatorname{soc}[V,U]$, is simple.

Choose $\theta \in (V,U)$ such that the image of θ in $[V,U]$ generates the socle – i.e. the image of θ is a nonzero element of $\operatorname{soc}[V,U]$. Form the pull-back:

$$\begin{tikzcd}
0 \arrow[r] & \Omega^2 V \arrow[r, "\alpha"] & P_1 \arrow[r, "\beta"] & U \arrow[r] & 0 \\
0 \arrow[r] & \Omega^2 V \arrow[u, equal] \arrow[r, dashed, "\gamma"] & X \arrow[u, dashed] \arrow[r, dashed, "\sigma"] & V \arrow[u, "\theta"'] \arrow[r] & 0.
\end{tikzcd} \qquad -(3)$$

Note that, because the image of θ in $[V,U]$ is nonzero, the lower sequence does not split; furthermore, both its end-terms, V and Ω^2V, are indecomposable. To show that this sequence is actually almost split, let $f : \Omega^2V \longrightarrow W$ be any map. Form the diagram:

$$\begin{tikzcd}
0 \arrow[r] & W \arrow[r] & \text{Proj.} \arrow[r, "\pi"] & \Omega^{-1}W \arrow[r] & 0 \\
0 \arrow[r] & \Omega^2V \arrow[r] \arrow[u, "f"] & P_1 \arrow[r] \arrow[u, dotted] & U \arrow[r] \arrow[u, dotted, "f'"'] & 0 \\
0 \arrow[r] & \Omega^2V \arrow[r, "\rho"] \arrow[u, equal] & X \arrow[r] \arrow[u] & V \arrow[r] \arrow[u, "\theta"'] & 0
\end{tikzcd}$$

then f is a split monomorphism

$\iff$ the induced map $(W,\Omega^2V) \longrightarrow (\Omega^2V,\Omega^2V)$, $g \longmapsto gf$ is surjective

$\iff$ the induced map $[W,\Omega^2V] \longrightarrow [\Omega^2V,\Omega^2V] \cong [V,V]$ is surjective

$\iff$ the induced map:

$$f^* : [V,U] \cong D[V,V] \longrightarrow D[W,\Omega^2V] \cong D[\Omega^{-1}W,\Omega V] \cong [V,\Omega^{-1}W]$$

is injective

$\iff$ f^* is nonzero on $\operatorname{soc}[V,U]$ – note that f^* is a right $[V,V]$-module homomorphism

$\iff$ the image of $f'\theta$ is nonzero in $[V,\Omega^{-1}W]$ (calculating f^* explicitly)

$\iff$ $f'\theta$ does not factor through π

$\iff$ f does not factor through ρ.

Thus, if f is not a split monomorphism, it factors through ρ. Hence the sequence $0 \longrightarrow \Omega^2V \longrightarrow X \longrightarrow V \longrightarrow 0$ is almost split. To conclude:

<u>Theorem 2.0.1</u> For any non-projective, indecomposable kG-module, V, there is an almost split sequence:

$$0 \longrightarrow \Omega^2V \longrightarrow X \longrightarrow V \longrightarrow 0 .$$

$\square$

Remark The reason for our somewhat perverse choice of the definition of almost split sequences may now be revealed. If we use the normal definition then the pull-back diagram, (3), needs to be replaced by a push-out diagram; however, I for one find the former easier to calculate.

PAGINATION
ERROR

§2.1 The almost split sequence of a module which is free on restriction to some maximal subgroup

Let us now apply the construction given in the previous section to the case when V is a non-projective, indecomposable kG-module which is free on restriction to some maximal subgroup of G. By (1.1.2) there is a two-step projective resolution:

$$0 \longrightarrow V \longrightarrow P \longrightarrow P \longrightarrow V \longrightarrow 0$$

for some projective module, P.

Note that the Jacobson radical of the endomorphism ring of V, $J(V,V)$, is nilpotent. Thus we cannot have that $J(V,V)V = V$; for if this were so then we would have $J(V,V)^n V = V$ for all n, so that, taking n sufficiently large, $V = 0$ – a contradiction. Hence we may choose an epimorphism:

$$\phi : V \longrightarrow k_G$$

such that:

$$J(V,V)V \leq \ker \phi . \qquad -(1)$$

We also fix an element, v_0, of V such that $\phi(v_0) \neq 0$.

Now let H be any maximal subgroup of G, not necessarily one with V being free on restriction to H. Choose $g \in G - H$ so that, as in §1.1, we have an exact sequence:

$$0 \longrightarrow k_G \xrightarrow{\alpha} k(G/H) \xrightarrow{\beta} k(G/H) \xrightarrow{\gamma} k_G \longrightarrow 0 \qquad -(2)$$

where the maps are given by:

$$\alpha : 1 \longmapsto \mu_{G/H} ,$$
$$\beta : H \longmapsto (g-1)H ,$$
$$\gamma : H \longmapsto 1 .$$

Define a map $\theta_H : V \longrightarrow k(G/H) \otimes V$ to be the composite of ϕ and the map $k_G \longrightarrow k(G/H) \otimes V$ given by $1 \longmapsto \mu_G(H \otimes v_0)$. We can, thinking of $\phi(v)$ as a scalar, write:

$$\theta_H(v) = \phi(v)\,\mu_G(H \otimes v_0)\,.$$

Now we may form the pull-back:

$$\begin{tikzcd}
0 \arrow[r] & V \arrow[r] \arrow[d, equal] & k(G/H) \otimes V \arrow[r] & k(G/H) \otimes V \arrow[r] & V \arrow[r] & 0 \\
0 \arrow[r] & V \arrow[r, dashed] & X_H \arrow[u, dashed] \arrow[rr, dashed] & & V \arrow[lu, "\theta_H"'] \arrow[r] & 0
\end{tikzcd} \qquad -(3)$$

where the upper sequence is the result of tensoring (2) by V.

<u>Lemma 2.1.1</u> The sequence $0 \longrightarrow V \longrightarrow X_H \longrightarrow V \longrightarrow 0$ is either split or almost split.

<u>Proof</u> Since P is injective as well as projective, we may form a commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & V \arrow[r] & P \arrow[r] & P \arrow[r] & V \arrow[r] & 0 \\
0 \arrow[r] & V \arrow[r] \arrow[u, equal] & k(G/H) \otimes V \arrow[r] \arrow[u, dashed] & k(G/H) \otimes V \arrow[r] \arrow[u, dashed, "\xi"'] & V \arrow[r] \arrow[u, dashed] & 0\,.
\end{tikzcd}$$

The sequence we are interested in is then given by another pull-back diagram:

$$\begin{tikzcd}
0 \arrow[r] & V \arrow[r] & P \arrow[r] & P \arrow[r] & V \arrow[r] & 0 \\
0 \arrow[r] & V \arrow[r, dashed] \arrow[u, equal] & X_H \arrow[r, dashed] \arrow[u, dashed] & V \arrow[r] \arrow[u, "\xi\theta_H"'] & 0
\end{tikzcd}$$

We claim that the image of $\xi\theta_H$ is in $\mathrm{soc}\,[V,U]$. If it is zero then the pull-back splits. Otherwise, as in the previous section, it is an almost split sequence.

$\Sigma\theta_H$ is given by $v \longmapsto \phi(v)\mu_G e$ where $e = \Sigma(H \otimes v_0)$. In particular, this implies that $\Sigma\theta_H$ maps V into $U = \ker\alpha$ - for $\mu_G V = 0$. Hence we may regard $\Sigma\theta_H$ as an element of (V,U) and consider its image in $[V,U]$ which we denote by θ.

To show that θ is in $\operatorname{soc}[V,U]$, let f be an element of (V,V). If f is in $J(V,V)$ - i.e. if f is a non-automorphism - then:

$$\Sigma\theta_H f : v \longmapsto \phi(fv)\mu_G e = 0 \qquad \text{since } fv \in J(V,V)V \leq \ker\phi .$$

Hence the $[V,V]$-submodule of $[V,U]$ generated by θ consists of 0 and the images of all $\Sigma\theta_H f$ for f an automorphism of V. Any nonzero submodule of this contains one of the images, $\Sigma\theta_H f$; but f is an automorphism, thus this submodule also contains the image of $(\Sigma\theta_H f)f^{-1}$ - that is to say, θ.

Hence we have shown that $\theta[V,V]$ is either simple or zero. Thus it is contained within $\operatorname{soc}[V,U]$, as required. □

We have explicit formulae for all the maps in the pull-back diagram, (3), hence we now proceed to calculate a concrete realisation of the pull-back.

$$X_H = \{ (e,v) \in (k(G/H) \otimes V) \times V \mid (\alpha \otimes 1_V)e = \theta_H(v) \} .$$

$$\rho : V \longrightarrow X_H \text{ is given by } v \longmapsto ((\alpha \otimes 1_V)v, 0) = (\mu_{G/H} \otimes v, 0) .$$

$$\sigma : X_H \longrightarrow V \text{ is given by } (e,v) \longmapsto v .$$

We may write a general element of $k(G/H) \otimes V$ uniquely in the form:

$$e = \sum_{i=0}^{p-1} g^i H \otimes w_i \qquad \text{with } w_i \in V .$$

We now calculate that:

$$(\delta \otimes 1_V)e = \sum_{i=0}^{p-1} (g-1)g^i H \otimes w_i$$

$$= \sum_{i=0}^{p-1} g^i H \otimes (w_{i-1} - w_i)$$

where $w_{-1} = w_{p-1}$. Also:

$$\theta_H(v) = \phi(v)\mu_G(H \otimes v_0) = \phi(v)(1 + g + \ldots + g^{p-1})\mu_H(H \otimes v_0)$$

$$= \sum_{i=0}^{p-1} g^i H \otimes \phi(v) g^i \mu_H v_0 .$$

Thus $(\delta \otimes 1_V)e = \theta_H(v)$ iff:

$$w_{i-1} - w_i = \phi(v) g^i \mu_H v_0 \quad \text{for all } i,$$

iff $$w_i = w_0 - \phi(v)(g + g^2 + \ldots\ldots + g^i)\mu_H v_0 \quad \text{for } i = 1,2,\ldots,p-1.$$

Hence X_H is the set of all points (e,v) where e has the form:

$$e = \sum_{i=0}^{p-1} g^i H \otimes (w_0 - \phi(v)(g + g^2 + \ldots\ldots + g^i)\mu_H v_0)$$

$$= \mu_{G/H} \otimes w_0 - \phi(v) e_0 \qquad \text{for some } w_0 \in V,$$

where:

$$e_0 = \sum_{i=1}^{p-1} g^i H \otimes (g + g^2 + \ldots\ldots + g^i)\mu_H v_0 .$$

Note that e_0 is a constant element of $k(G/H) \otimes V$. We readily calculate that:

$$he_0 = e_0 \quad \text{for all } h \in H.$$

Furthermore:

$$ge_0 = \sum_{i=1}^{p-1} g^{i+1}H\otimes(g^2+g^3+\dots+g^{i+1})\mu_H v_0$$

$$= \sum_{i=2}^{p-1} g^iH\otimes(g+g^2+\dots+g^i-g)\mu_H v_0 \quad + \quad H\otimes(g^2+g^3+\dots+g^p)\mu_H v_0$$

$$= e_0 - gH\otimes g\mu_H v_0 - \left(\sum_{i=2}^{p-1} g^iH\otimes g\mu_H v_0\right) + H\otimes(\mu_G - g\mu_H)v_0$$

$$= e_0 - \mu_{G/H}\otimes g\mu_H v_0 .$$

The map $\phi\colon V\oplus V \longrightarrow X_H$ given by:

$$\phi(w,v) = (\mu_{G/H}\otimes w - \phi(v)e_0 , v)$$

is a k-isomorphism by what was proved above. Let us calculate the action of G on X_H in terms of this map.

$$\begin{aligned}
h.\phi(w,v) &= h(\mu_{G/H}\otimes w - \phi(v)e_0 , v)\\
&= (\mu_{G/H}\otimes hw - \phi(v)he_0 , hv)\\
&= (\mu_{G/H}\otimes hw - \phi(hv)e_0 , hv) = \phi(hw,hv) \qquad \text{for } h\in H.\\
g.\phi(w,v) &= g(\mu_{G/H}\otimes w - \phi(v)e_0 , v)\\
&= (\mu_{G/H}\otimes gw - \phi(v)ge_0 , gv)\\
&= (\mu_{G/H}\otimes gw - \phi(gv)(e_0 - \mu_{G/H}\otimes g\mu_H v_0), gv)\\
&= \phi(gw + \phi(v)g\mu_H v_0 , gv).
\end{aligned}$$

Finally the maps $\rho : V \longrightarrow X_H$, $\sigma : X_H \longrightarrow V$ are given by:

$$\rho(v) = (\mu_{G/H}\otimes v , 0) = \phi(v,0) ,$$

$$\sigma(\phi(w,v)) = \sigma(\mu_{G/H}\otimes w - \phi(v)e_0 , v) = v .$$

We may state all this as:

<u>Theorem 2.1.2</u> We may define a kG-module, X_H, by means of a kH-isomorphism:

$$\phi : V \oplus V \longrightarrow X_H$$

with the action of g being given by:

$$g.\phi(w,v) \;=\; \phi(gw + \phi(v)g\mu_H v_0 \,,\, gv).$$

The maps $v \longmapsto \phi(v,0)$, $\phi(w,v) \longmapsto v$ give an exact sequence:

$$0 \longrightarrow V \longrightarrow X_H \longrightarrow V \longrightarrow 0$$

which is either split or almost split. $\square$

Now assume that H is so chosen that V is free on restriction to H. We shall show that, in this case, the sequence in (2.1.2) is almost split.

If we restrict the epimorphism $\phi : V \longrightarrow k_G$ to H then it gives a projective cover of k_H. Thus there is a free kH-basis, $v_1, \ldots, v_n$, for V such that:

$$\ker\phi \;=\; \mathrm{Aug}\,(kH)v_1 \oplus kHv_2 \oplus \ldots\ldots \oplus kHv_n \,.$$

In particular, v_0 can be written as $\xi_1 v_1 + \ldots\ldots + \xi_n v_n$ where $\xi_1 \notin \mathrm{Aug}(kH)$; thus $v_0, v_2, \ldots, v_n$ is also a free kH-basis for V. Thus we may assume that $v_1 = v_0$.

By (1.1.1), $F = k(G/H) \otimes V$ is a free kG-module with free basis $e_1, \ldots, e_n$ where $e_i = H \otimes v_i$. Let π_i denote the projection map $F \longrightarrow kG$ which maps an element to its e_i-coordinate. Also let $\chi : kG \longrightarrow k$ be the map that sends the identity element to 1 and the other elements of G to 0 and is extended k-linearly. For $f \in (F,F)$ we write:

$$t(f) \;=\; \sum_i \chi(\pi_i f(e_i)).$$

<u>Lemma 2.1.3</u> If f_1, f_2 are elements of (F,F) then $t(f_1f_2) = t(f_2f_1)$.

<u>Proof</u> We may write:

$$f_m(e_i) = \sum_j \xi_{ij}^{(m)} e_j \qquad \text{for some } \xi_{ij}^{(m)} \in kG.$$

Then:

$$f_1f_2(e_i) = \sum_j \xi_{ij}^{(2)} f_1(e_j) = \sum_{j,m} \xi_{ij}^{(2)}\xi_{jm}^{(1)} e_m$$

so that:

$$t(f_1f_2) = \sum_{i,j} \varkappa(\xi_{ij}^{(2)}\xi_{ji}^{(1)}).$$

Similarly $t(f_2f_1)$ is given by a corresponding equation with $\xi_{ij}^{(1)}$ and $\xi_{ij}^{(2)}$ interchanged. But $\varkappa(\xi_1\xi_2) = \varkappa(\xi_2\xi_1)$, thus the result is seen to follow. □

Considering the pull-back diagram, (3), we see that the sequence in (2.1.2) splits if and only if $\theta_H = (\partial \otimes 1_V)f$ for some $f : V \longrightarrow F$. Assume that this happens, then:

$$\begin{aligned} t(\theta_H(\delta \otimes 1_V)) &= t((\partial \otimes 1_V).f(\delta \otimes 1_V)) \\ &= t(f(\delta \otimes 1_V).(\partial \otimes 1_V)) && \text{by (2.1.3),} \\ &= 0 && \text{since } \delta\partial = 0. \end{aligned}$$

But:

$$\begin{aligned} t(\theta_H(\delta \otimes 1_V)) &= \sum_i \varkappa(w_i\theta_H(\delta \otimes 1_V)(H \otimes v_i)) \\ &= \sum_i \varkappa(w_i\theta_H(v_i)) \\ &= \varkappa w_1(\phi(v_1)\mu_G e_1) \\ &= \phi(v_1)\varkappa(\mu_G) = \phi(v_1) \neq 0. \end{aligned}$$

Thus we have a contradiction.

<u>Theorem 2.1.4</u> If V is free on restriction to H then the sequence given in (2.1.2) is almost split. $\square$

We will later show that the converse also holds when V is absolutely indecomposable: that the sequence is split if V is not free on restriction to H.

§2.2 Further results on almost split sequences

We now demonstrate various results on almost split sequences. Most can be found in [A&R].

Firstly suppose that $0 \longrightarrow U \xrightarrow{\rho} X \xrightarrow{\sigma} V \longrightarrow 0$ is an almost split sequence and that $f : W \longrightarrow V$ is not a split epimorphism. Form the pull-back diagram:

$$\begin{tikzcd}
0 \arrow[r] & U \arrow[r, "\rho"] & X \arrow[r, "\sigma"] & V \arrow[r] & 0 \\
0 \arrow[r] & U \arrow[u, equal] \arrow[r, dashed, "g"] & Y \arrow[u, dashed] \arrow[r, dashed] & W \arrow[u, "f"'] \arrow[r] & 0 .
\end{tikzcd}$$

If g is a split monomorphism then f factors through σ. Otherwise, $g = g'\rho$ for some $g' : X \longrightarrow Y$. Complete the diagram:

$$\begin{tikzcd}
0 \arrow[r] & U \arrow[r, "g"] & Y \arrow[r] & W \arrow[r] & 0 \\
0 \arrow[r] & U \arrow[u, equal] \arrow[r] & X \arrow[u, "g'"'] \arrow[r] & V \arrow[u, dashed, "f'"'] \arrow[r] & 0 .
\end{tikzcd}$$

Since f is not a split epimorphism then ff' is not an automorphism of V; thus it is nilpotent, so that we have a commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & U \arrow[r] & X \arrow[r] & V \arrow[r] & 0 \\
0 \arrow[r] & U \arrow[u, equal] \arrow[r, "\rho"] & X \arrow[u] \arrow[ul, dashed] \arrow[r, "\sigma"] & V \arrow[u, "(ff')^m = 0"'] \arrow[r] & 0 .
\end{tikzcd}$$

Hence 1_U factors through ρ - contradicting the fact that ρ is not split.

To conclude: if f is not a split epimorphism then it factors through σ. This establishes the equivalence of the definition that we have used and that usually given. It also proves:

<u>Lemma 2.2.1</u> If $0 \longrightarrow U \longrightarrow X \longrightarrow V \longrightarrow 0$ is an almost split sequence then so is the dual sequence, $0 \longrightarrow V^* \longrightarrow X^* \longrightarrow U^* \longrightarrow 0$. □

<u>Theorem 2.2.2</u> For any non-projective, indecomposable kG-module, V, there is a unique (up to isomorphism) almost split sequence:

$$0 \longrightarrow U \longrightarrow X \longrightarrow V \longrightarrow 0\,.$$

In fact, $U \cong \Omega^2 V$.

<u>Proof</u> Existence follows from (2.0.1). To prove uniqueness, suppose that $0 \longrightarrow U_i \longrightarrow X_i \xrightarrow{\sigma_i} V \longrightarrow 0$ is an almost split sequence for $i = 1,2$. Since the σ_i are not split epimorphisms we may construct a commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & U_1 \arrow[r] & X_1 \arrow[r] & V \arrow[r] & 0 \\
0 \arrow[r] & U_2 \arrow[r] \arrow[u, dashed, "\alpha_1"] & X_2 \arrow[r] \arrow[u, dashed] & V \arrow[r] \arrow[u, equal] & 0 \\
0 \arrow[r] & U_1 \arrow[r] \arrow[u, dashed, "\alpha_2"] & X_1 \arrow[r] \arrow[u, dashed] & V \arrow[r] \arrow[u, equal] & 0\,.
\end{tikzcd}$$

As above, $\alpha_1\alpha_2$ is not nilpotent, thus it is an automorphism of U_1. Therefore α_1 and α_2 are both isomorphisms, so that, by the five-lemma, the two almost split sequences are isomorphic. □

It is sometimes convenient to think of almost split sequences in terms of an inner product on the complex representation ring, $A_k(G)$. This was first done by Benson in [B&P]. Define the bilinear form by using the equation:

$$([U],[V]) = \dim_k(U,V)$$

for kG-modules, U and V, and extending $\mathbb{C}$-bilinearly. Also, for a non-projective, indecomposable kG-module, V, write:

$$g(V) = [X_V] - [V] - [\Omega^2 V] \in A_k(G)$$

where $0 \longrightarrow \Omega^2 V \longrightarrow X_V \xrightarrow{\sigma} V \longrightarrow 0$ is the (unique) almost split sequence with V as its right-hand term.

Note also that for any indecomposable kG-module, V, $(V,V)/J(V,V)$ is a division algebra over k. We write $a(V)$ for the dimension of this algebra. In the case when k is algebraically closed we always have that $a(V) = 1$.

Proposition 2.2.3 Let U,V be indecomposable kG-modules with V non-projective, then:

(a) $([U],g(V)) = \begin{cases} -a(V) & \text{if } U \cong V, \\ 0 & \text{otherwise.} \end{cases}$

(b) $(g(V),[U]) = \begin{cases} -a(\Omega^2 V) & \text{if } U \cong \Omega^2 V, \\ 0 & \text{otherwise.} \end{cases}$

Proof (a) We have an exact sequence:

$$0 \longrightarrow (U,\Omega^2 V) \longrightarrow (U,X_V) \xrightarrow{(U,\sigma)} (U,V).$$

Thus, taking dimensions:

$$\begin{aligned} ([U],g(V)) &= \dim_k (U,X_V) - \dim_k (U,V) - \dim_k (U,\Omega^2 V) \\ &= \dim_k \operatorname{Im}(U,\sigma) - \dim_k (U,V). \end{aligned}$$

But $\operatorname{Im}(U,\sigma)$ is the set of all maps $U \longrightarrow V$ which factor through σ, that is to say, all such maps which are not split epimorphisms.

If U is not isomorphic to V then there are no split epimorphisms $U \longrightarrow V$ so that $\operatorname{Im}(U,\sigma)$ is the whole of (U,V). However if U is isomorphic to V then $\operatorname{Im}(U,\sigma)$ is (isomorphic to) the space of all non-automorphisms of V, $J(V,V)$. Thus in either case the result follows from the equation above.

(b) Using (2.2.1) we have:

$$(g(V),[U]) = ([U^*],g(V)^*) = ([U^*],g((\Omega^2V)^*))$$

so that the result is an easy consequence of (a). $\square$

A striking corollary of this is that $(-,-)$ is a nonsingular bilinear form on $A_k(G)$. We shall now indicate a proof of this.

Suppose that $x \in A_k(G)$ is such that $(x,y) = 0$ for all $y \in A_k(G)$. We can write x in the form:

$$x = \sum_{[U]} c_U[U] \qquad\qquad (c_U \in \mathbb{C})$$

where the sum is over all isomorphism classes of indecomposable kG-modules. If V is non-projective and indecomposable then:

$$0 = (x,g(V)) = \sum_{[U]} c_U([U],g(V)) = -c_V a(V)$$

so that $c_V = 0$. Thus x is a linear combination of the isomorphism classes of the projective indecomposable kG-modules. Note that if U,V are such modules then $(U,\mathrm{soc}\,(V)) = 0$ unless $U \cong V$. Thus:

$$0 = (x,[\mathrm{soc}\,(V)]) = \sum_{[U]} c_U([U],[\mathrm{soc}\,(V)]) = c_V \dim_k(V,\mathrm{soc}\,(V))$$

so that $c_V = 0$ in this case too. Thus $x = 0$.

If $x \in A_k(G)$ is such that $(y,x) = 0$ for all $y \in A_k(G)$ then $(x^*,y^*) = 0$ for all y so that, by what we have just proved, $x^* = 0$. Thus $x = 0$. This completes the proof that $(-,-)$ is nonsingular.

A concept closely related to that of almost split sequences is now defined: if U,V are indecomposable kG-modules then a map $f\colon U \longrightarrow V$ is said to be *irreducible* if f is not an isomorphism but if it factors as the composite

$U \xrightarrow{g} W \xrightarrow{h} V$ then either g is a split monomorphism or h is a split epimorphism. The connection between the two concepts is given by:

Proposition 2.2.4 Let V be a non-projective, indecomposable kG-module and:

$$0 \longrightarrow \Omega^2 V \xrightarrow{\rho} X_V \xrightarrow{\sigma} V \longrightarrow 0$$

be an almost split sequence. If U is any indecomposable kG-module then:

(a) $f : U \longrightarrow V$ is irreducible if and only if $f = \sigma i$ for some split monomorphism $i : U \longrightarrow X_V$.

(b) $f : \Omega^2 V \longrightarrow U$ is irreducible if and only if $f = \pi\rho$ for some split epimorphism $\pi : X_V \longrightarrow U$.

In particular, there exists an irreducible map $U \longrightarrow V$ if and only if there exists one $\Omega^2 V \longrightarrow U$ – both conditions are equivalent to $U | X_V$.

Proof We shall prove (b); the dual statement, (a), will then follow from (2.2.1). Suppose first that $f : \Omega^2 V \longrightarrow U$ is irreducible. Because f is not a split monomorphism, it factors as $\pi\rho$ for some $\pi : X_V \longrightarrow U$. ρ is not a split monomorphism, thus π must be a split epimorphism by the definition of irreducible maps. Conversely let $\pi : X_V \longrightarrow U$ be a split epimorphism and define $f = \pi\rho$. Because ρ is not a split monomorphism, f is not an isomorphism. Suppose that f factors as $\Omega^2 V \xrightarrow{g} W \xrightarrow{h} U$ where g is not a split monomorphism. Then g factors as $g'\rho$ for some $g' : X_V \longrightarrow W$. Thus:

$$\pi\rho = f = hg = hg'\rho$$

so that $hg' = \pi + f'\sigma$ for some $f' : V \longrightarrow U$. But π is a split epimorphism, so choose $i : U \longrightarrow X_V$ with $\pi i = 1_U$. Then:

$$hg'i = 1_U + f'\sigma i;$$

but $f'\sigma i$ is a non-automorphism of the indecomposable module, U; thus $hg'i$ is an automorphism of U. Hence h is a split epimorphism. Thus f is irreducible. $\square$

Lemma 2.2.5 An irreducible map is either an epimorphism with an indecomposable kernel or a monomorphism with an indecomposable cokernel.

Proof Let $f : U \longrightarrow V$ be an irreducible map. The factorisation:

$$U \longrightarrow U/\ker f \cong \operatorname{Im} f \longrightarrow V$$

shows that f is either a monomorphism or an epimorphism. If f is an epimorphism then write $\ker f = U_1 \oplus U_2$; then f factors as:

$$U \xrightarrow{g} U/U_1 \xrightarrow{h} U/(U_1 \oplus U_2) \cong V$$

thus either g is a split monomorphism or h is a split epimorphism. Since U is indecomposable we see that either $U_1 = 0$ or $U_2 = 0$; thus $\ker f$ is indecomposable. The case when f is a monomorphism may be similarly dealt with. □

Proposition 2.2.6 (a) If $P | X_V$ for some projective, indecomposable kG-module, P, then $V \cong P/\operatorname{soc}(P)$.

(b) X_V has no indecomposable direct summands of dimension $\dim_k V$ or $\dim_k \Omega^2 V$.

Proof (a) By (2.2.4), there exists an irreducible map $f : P \longrightarrow V$. If this is a monomorphism then, since P is projective, it splits - a contradiction. Thus f is an epimorphism with a nonzero kernel. Hence the unique minimal submodule of P, $\operatorname{soc}(P)$, is contained in $\ker f$. Thus f factors as:

$$P \longrightarrow P/\operatorname{soc}(P) \xrightarrow{h} V.$$

h is then an isomorphism (using the fact that f is irreducible).

(b) If U is an indecomposable summand of X_V then there exist irreducible maps $\Omega^2 V \longrightarrow U$, $U \longrightarrow V$ by (2.2.4). These are either monomorphisms or epimorphisms but not isomorphisms. Thus U cannot have dimension $\dim_k V$ or $\dim_k \Omega^2 V$. □

§2.3 The structure of X_V when V is free on restriction to a maximal subgroup

We have shown how to construct an almost split sequence:

$$0 \longrightarrow V \longrightarrow X_V \longrightarrow V \longrightarrow 0$$

when V is a non-projective, indecomposable kG-module which is free on restriction to a maximal subgroup of G. We now consider the decomposition of X_V into indecomposable modules. To do this we apply the analysis of Webb, [Web], of the Auslander-Reiten quiver, which, in this special case, may be considerably simplified. However, we do assume that k is algebraically closed and exclude the case when G is cyclic of order p - for then all the kG-modules are free on restriction to the only maximal subgroup of G - namely, the identity subgroup.

Let $\underset{\sim}{A}$ denote the set of all isomorphism classes, [V], of non-projective, indecomposable kG-modules which are free on restriction to some maximal subgroup of G.

Lemma 2.3.1 If $[V] \in \underset{\sim}{A}$ and U is an indecomposable direct summand of X_V then $[U] \in \underset{\sim}{A}$.

Proof Clearly U is free on restriction to the same maximal subgroups as V is. The only problem, therefore, is if U is projective. Because we are dealing with p-groups, this implies that $U \cong kG$. (2.2.6)(a) implies that $V \cong kG/k\mu_G$, so that k_G is free on restriction to a maximal subgroup of G. The only way that this can happen is if G is cyclic of order p - the case we have already excluded. □

The next lemma is when the fact that k is algebraically closed becomes vital. It does not hold for general fields.

<u>Lemma 2.3.2</u> If $[U], [V] \in \underline{A}$ then $[U|X_V] = [V|X_U]$.

<u>Proof</u> This is trivial when $U \cong V$. Otherwise:

$$\begin{aligned}(g(U), g(V)) &= ([X_U] - 2[U], g(V)) = -[V|X_U] \\ &= (g(U), [X_V] - 2[V]) = -[U|X_V]\end{aligned}$$

using (2.2.3). □

Hence we may define a graph structure on $\underline{A}$ by joining [U] and [V] by $[U|X_V] = [V|X_U]$ undirected edges. The relationship of this with the Auslander -Reiten quiver is clear. We also define a map:

$$d : \underline{A} \longrightarrow \mathbb{N} \quad \text{by} \quad d[V] = \dim_k V.$$

This then satisfies:

(1) $2d[V] = \sum d[U]$, the sum being over all edges $[V] \circ\!\!-\!\!\!-\!\!\circ [U]$.

(2) If we have an edge $[V] \circ\!\!-\!\!\!-\!\!\circ [U]$ then $d[V] \neq d[U]$.

The first result is proved by taking dimensions in the almost split sequence $0 \longrightarrow V \longrightarrow X_V \longrightarrow V \longrightarrow 0$, bearing (2.3.1) in mind. The second result follows from (2.2.6)(b), for we have an irreducible map $V \longrightarrow U$.

<u>Lemma 2.3.3</u> For any subgraph, $\underline{B}$, of $\underline{A}$, the vertex of $\underline{B}$ with minimal d-value is joined to the rest of $\underline{B}$ by at most one edge.

<u>Proof</u> Let $[V] \in \underline{B}$ be a vertex of minimal d-value. We have:

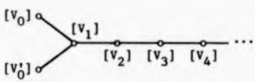
$$2d[V] \geq \sum d[U],$$

the sum being over all edges $[V] \circ\!\!-\!\!\!-\!\!\circ [U]$ in $\underline{B}$. By minimality, $d[V] \leq d[U]$

for such edges; moreover the inequality is strict by property (2) above. Thus if $[V]$ is joined to the rest of $\underline{B}$ by n edges then $2d[V] > nd[V]$. Thus $n < 2$, as required. □

<u>Theorem 2.3.4</u> Any connected component, $\underline{B}$, of $\underline{A}$ is either finite or has the form o——o——o——....

<u>Proof</u> Applying (2.3.3) to the various two-vertex subgraphs of $\underline{B}$, we see that $\underline{B}$ contains no multiple edges. Similarly, it contains no closed loops. Thus if $\underline{B}$ is infinite then either every vertex of $\underline{B}$ has at most two edges attached to it, or $\underline{B}$ has a subgraph of the form:

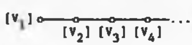


In the former case, (2.3.3) shows that some vertex of $\underline{B}$ will have only one edge attached to it; moreover for $\underline{B}$ to be infinite, all the other vertices must have two edges. Thus $\underline{B}$ must be of the form o——o——o——....

In the latter case, property (1) above gives:

$$2d[V_0] \geq d[V_1]$$
$$2d[V_0'] \geq d[V_1]$$
$$2d[V_1] \geq d[V_0] + d[V_0'] + d[V_2]$$

thus we have that $d[V_1] \geq d[V_2]$, the inequality being in fact strict by property (2). But apply (2.3.3) to the subgraph:



;

the vertex of minimal d-value is attached by at most one edge, thus it must be

$[V_1]$. This contradicts $d[V_1] > d[V_2]$. ☐

<u>Proposition 2.3.5</u> $\underline{A}$ has no non-empty, finite, connected component, $\underline{B}$.

<u>Proof</u> Let B denote the $\mathbb{C}$-linear span of $\underline{B}$ in $A_k(G)$. By (2.3.1), if $[V] \in \underline{B}$ then $g(V) \in B$ (because $\underline{B}$ is a component, any indecomposable summand of X_V is connected to $[V]$ and is thus in $\underline{B}$). Define:

$$x = \sum_{[V]\in\underline{B}} -(\dim_k V^G)g(V) \in B.$$

Then for $[U] \in \underline{B}$,

$$\begin{aligned}(x,[U]) &= \sum_{[V]} -(\dim_k V^G)(g(V),[U]) \\ &= \dim_k U^G = \dim_k (k_G,[U]) \\ &= ([k_G],[U]).\end{aligned}$$

Thus $(x,y) = ([k_G],y)$ for all $y \in B$. Note that, for any U, $(x,[U]) \neq 0$, thus $x \neq 0$. We can write:

$$x = \sum_{[V]} c_V[V] \quad \text{- where } c_V \in \mathbb{C} \text{ and } [V] \text{ runs over } \underline{B}.$$

choose U such that $c_U \neq 0$, then:

$$0 \neq -c_U = (x,g(U)) = ([k_G],g(U)) = -[U|k_G].$$

Hence $U \cong k_G$. But k_G is not free on restriction to any maximal subgroup of G so we have a contradiction. (See [Ben] p161 for this proof.) ☐

Thus we may translate (2.3.4) and (2.3.5) back into terms of almost split sequences to obtain:

<u>Theorem 2.3.6</u> Let k be an algebraically closed field of characteristic p and G be a finite p-group of order at least p^2. The non-projective, indecomposable kG-modules which are free on restriction to a maximal subgroup of G fall, up to isomorphism, into disjoint sequences, V_n $(n = 1,2,\ldots.)$ such that we have almost split sequences:

$$0 \longrightarrow V_1 \longrightarrow V_2 \longrightarrow V_1 \longrightarrow 0\,,$$

$$0 \longrightarrow V_n \longrightarrow V_{n-1} \oplus V_{n+1} \longrightarrow V_n \longrightarrow 0 \quad (n > 1). \qquad \square$$

We remark that, with this notation, $\dim_k V_n = n.\dim_k V_1$. Thus if $\underline{A} \neq \emptyset$ then there are indecomposable kG-modules of arbitrarily large dimension. In particular, G cannot be cyclic, for then kG is of finite-representation type. Thus if G is a cyclic p-group of order at least p^2 then there do not exist any non-projective, indecomposable kG-modules which are free on restriction to the unique maximal subgroup of G. This gives our third (and final) proof of (0.1.1).

Let us now consider irreducible maps $U \longrightarrow V$ where either U or V is a non-projective, indecomposable kG-module which is free on restriction to some maximal subgroup of G. Of course, (2.2.4) gives that $V \mid X_U$ and $U \mid X_V$ so, as in (2.3.1), both U and V have this same property. Furthermore [U] and [V] are connected by an edge in $\underline{A}$, so they belong to the same connected component of $\underline{A}$. (2.3.6) gives that there exists a sequence V_n $(n = 1,2,\ldots.)$ such that either:

(1) $U \cong V_n$, $V \cong V_{n+1}$ for some n,

or (2) $U \cong V_{n+1}$, $V \cong V_n$ for some n.

By dimensions, in case (1) the irreducible map will be a monomorphism, and in case (2), an epimorphism (bear (2.2.5) in mind).

Let us consider irreducible monomorphisms $V_n \longrightarrow V_{n+1}$. We know that these have indecomposable cokernels, but we claim that the cokernels are actually isomorphic to V_1.

The proof is by induction on n. For $n = 1$ we have an almost split sequence:

$$0 \longrightarrow V_1 \xrightarrow{\rho} V_2 \xrightarrow{\sigma} V_1 \longrightarrow 0,$$

thus any irreducible map $V_1 \longrightarrow V_2$ has the form $\alpha\rho$ for some automorphism, α, of V_2. So consider the sequence:

$$0 \longrightarrow V_1 \xrightarrow{\alpha\rho} V_2 \xrightarrow{\sigma\alpha^{-1}} V_1 \longrightarrow 0.$$

So assume that $n > 1$ and that $f : V_n \longrightarrow V_{n+1}$ is an irreducible map.
Let:

$$0 \longrightarrow V_n \xrightarrow{\rho} X \xrightarrow{\sigma} V_n \longrightarrow 0$$

be an almost split sequence, then $f = \pi\rho$ for some split epimorphism $\pi : X \longrightarrow V_{n+1}$. Note that $X \cong V_{n-1} \oplus V_{n+1}$ so there exist maps i, i', π' as shown in the diagram below, with:

$$\pi i = 1_{V_{n+1}} \quad , \quad \pi' i' = 1_{V_{n-1}} \quad , \quad i\pi + i'\pi' = 1_X .$$

$$\begin{tikzcd}
 & & V_{n+1} \arrow[rr, dashed] \arrow[d, shift left, "i"] \arrow[dr, dashed, "g"] & & V_1 \\
0 \arrow[r] & V_n \arrow[ur, "f"] \arrow[r, "\rho"] \arrow[dr, dashed, "g'"'] & X \arrow[r, "\sigma"] \arrow[u, shift left, "\pi"] \arrow[d, shift right, "\pi'"'] & V_n \arrow[r] \arrow[ur, dashed, "h"] & 0 \\
 & & V_{n-1} \arrow[u, shift right, "i'"'] \arrow[ur, dashed, "f'"'] & &
\end{tikzcd}$$

Define $f' = \sigma i'$, $g = \sigma i$, $g' = \pi'\rho$ then, by (2.2.4), these are all irreducible maps. By dimensions, f' is a monomorphism and g, g' are epimorphisms. Also:

$$gf + f'g' = \sigma(i\pi + i'\pi')\rho = \sigma 1_X \rho = 0.$$

Inductively, we have an exact sequence:

$$0 \longrightarrow V_{n-1} \xrightarrow{f'} V_n \xrightarrow{h} V_1 \longrightarrow 0 .$$

But g and h are epimorphisms, thus so is $hg : V_{n+1} \longrightarrow V_1$; also:

$$(hg)f = -hf'g' = 0.$$

Thus $\mathrm{Im}\, f \leq \ker hg$. But both of these submodules of V_{n+1} have dimension $n.\dim_k V_1$, so they are the same. Thus we have an exact sequence:

$$0 \longrightarrow V_n \xrightarrow{f} V_{n+1} \xrightarrow{hg} V_1 \longrightarrow 0 .$$

So the result is proved.

We may summarise this and the dual result as:

<u>Theorem 2.3.7</u> The only irreducible maps involving non-projective, indecomposable kG-modules which are free on restriction to some maximal subgroup of G, are given, up to isomorphism, as follows:

Let V_n $(n = 1,2,\ldots.)$ be as in (2.3.6), then, for each n, there are irreducible monomorphisms $V_n \longrightarrow V_{n+1}$ with cokernel V_1 , and irreducible epimorphisms $V_{n+1} \longrightarrow V_n$ with kernel V_1 . $\square$

§2.4 Terminal modules

If V is a non-projective, indecomposable kG-module then there is a unique almost split sequence:

$$0 \longrightarrow \Omega^2 V \longrightarrow X_V \longrightarrow V \longrightarrow 0;$$

let us call V a *terminal module* if X_V is indecomposable. In the case dealt with in the previous section, namely when k is algebraically closed and V is free on restriction to a maximal subgroup of G, (2.3.6) gives that $V \cong V_N$ for some N, where V_n $(n = 1, 2, \ldots.)$ is a sequence of such modules such that there are almost split sequences:

$$0 \longrightarrow V_1 \longrightarrow V_2 \longrightarrow V_1 \longrightarrow 0,$$

$$0 \longrightarrow V_n \longrightarrow V_{n-1} \oplus V_{n+1} \longrightarrow V_n \longrightarrow 0 \quad (n > 1).$$

Now, clearly, V is terminal if and only if $N = 1$. (2.3.7) thus gives that the terminal modules are classified as the cokernels of irreducible monomorphisms or as the kernels of irreducible epimorphisms.

We now use the construction of §2.1 to show how, given a terminal module, V, which is free on restriction to some maximal subgroup, H, of G, we may find a sequence V_n $(n = 1, 2, \ldots.)$ with the properties above such that $V_1 \cong V$. The result may be summarised as:

Theorem 2.4.1 There exists a sequence of epimorphisms:

$$\phi_n : V \longrightarrow k_G \quad \text{with} \quad J(V,V)V \leq \ker \phi_n$$

and elements $v_n \in V$ with $\phi_n(v_n) \neq 0$ such that, if we define a kG-module, V_n, by means of a kH-isomorphism:

$$\psi_n : \underbrace{V \oplus \ldots \oplus V}_{n \text{ copies}} \longrightarrow V_n$$

and the action of g:

$$g\cdot\phi_n(w_1,\dots,w_n)$$
$$= \phi_n(gw_1 + \phi_1(w_2)g\mu_H v_1, \dots, gw_{n-1} + \phi_{n-1}(w_n)g\mu_H v_{n-1}, gw_n),$$

then V_n is an indecomposable kG-module which is free on restriction to H, such that there are almost split sequences as above.

Moreover, the maps $\alpha_n : V_n \longrightarrow V_{n+1}$ given by:

$$\phi_n(w_1,\dots,w_n) \longmapsto \phi_{n+1}(w_1,\dots,w_n,0)$$

are irreducible monomorphisms.

<u>Proof</u> The proof is by induction on n. For $n=1$ we may take $V_1 = V$, $\phi_1 = 1_V$. For $n=2$ the result follows from (2.1.2) and (2.1.4); moreover since V is terminal, V_2 is indecomposable. So assume that $n>2$ and that we have already defined $\phi_1,\dots,\phi_{n-1}$; $v_1,\dots,v_{n-1}$; $\phi_1,\dots,\phi_n$; $V_1,\dots,V_n$; $\alpha_1,\dots,\alpha_{n-1}$ as above with the required properties.

<u>Lemma</u> $J(V_n,V_n)V_n \supseteq \{\phi_n(w_1,\dots,w_n) \mid w_1,\dots,w_{n-1} \in V,\ w_n \in J(V,V)V\}$.

<u>Proof</u> We know that there is an irreducible map $\alpha_{n-1} : V_{n-1} \longrightarrow V_n$, thus (2.2.4) gives that there is another irreducible map $\beta : V_n \longrightarrow V_{n-1}$ which, by dimensions, is an epimorphism. The composite $\alpha_{n-1}\beta$ cannot be an automorphism of V_n (since it factors through V_{n-1}), thus:

$$J(V_n,V_n)V_n \supseteq \operatorname{Im}\alpha_{n-1}\beta = \operatorname{Im}\alpha_{n-1}$$
$$= \{\phi_n(w_1,\dots,w_{n-1},0) \mid w_1,\dots,w_{n-1} \in V\}. \qquad -(1)$$

For $f \in J(V,V)$ we define $f' : V_n \longrightarrow V_n$ by:

$$f' : \psi_n(w_1,\ldots,w_n) \longmapsto \psi_n(0,\ldots,0,f(w_n)).$$

We claim that f' is kG-linear and hence a non-automorphism of V_n. This would imply that:

$$J(V_n,V_n)V_n \supseteq \left\{\psi_n(0,\ldots,0,w) \;\middle|\; w \in J(V,V)V\right\}$$

which, combined with (1), would give the result of the lemma.

The only problem in showing that f' is kG-linear is the action of g. But:

$$\begin{aligned} g.f'(\psi_n(w_1,\ldots,w_n)) &= g.\psi_n(0,\ldots,0,f(w_n)) \\ &= \psi_n(0,\ldots,0,\phi_{n-1}(f(w_n))g\mu_H v_{n-1}, gf(w_n)) \\ &= \psi_n(0,\ldots,0,0,f(gw_n)) \end{aligned}$$

– for note that $f(w_n) \in J(V,V)V \leq \ker \phi_{n-1}$. However:

$$\begin{aligned} f'(g.\psi_n(w_1,\ldots,w_n)) &= f'(\psi_n(?,\ldots,?,gw_n)) \\ &= \psi_n(0,\ldots,0,f(gw_n)) \end{aligned}$$

so the claim is established, and the lemma proved. $\square$

We are now in a position to apply the construction of (2.1.2) to V_n. This will, by (2.1.4), give us an almost split sequence. We are required to choose:

(1) an epimorphism $\phi : V_n \longrightarrow k_G$ with $J(V_n,V_n)V_n \leq \ker \phi$,

and (2) an element, v_0, of V_n with $\phi(v_0) \neq 0$.

The lemma above makes it clear that there exists an epimorphism $\phi_n : V \longrightarrow k_G$ with $J(V,V)V \leq \ker \phi_n$ such that:

$$\phi(\psi_n(w_1,\ldots,w_n)) \;=\; \phi_n(w_n).$$

Thus we may assume that v_0 has the form $\psi_n(0,\ldots,0,v_n)$ for some $v_n \in V$ with $\phi_n(v_n) \neq 0$.

Define a kG-module, X, by means of a kH-isomorphism:

$$\phi : V_n \oplus V_n \longrightarrow X$$

with the action of g being given by:

$$g \cdot \phi(w,v) = \phi(gw + \phi(v) g\mu_H v_0 , gv).$$

then the maps $\rho : v \longmapsto \phi(v,0)$, $\sigma : \phi(w,v) \longmapsto v$ give an almost split sequence:

$$0 \longrightarrow V_n \xrightarrow{\rho} X \xrightarrow{\sigma} V_n \longrightarrow 0.$$

Note that
$$\begin{aligned} g\mu_H v_0 &= g\phi_n(0,\ldots,0,\mu_H v_n) \\ &= \phi_n(0,\ldots,0,\phi_{n-1}(\mu_H v_n) g\mu_H v_{n-1} , g\mu_H v_n) \\ &= \phi_n(0,\ldots,0,g\mu_H v_n). \end{aligned}$$

Define $i : V_{n-1} \longrightarrow X$ by:

$$i(v) = \phi(0,\alpha_{n-1}(v)).$$

We readily check that i is kG-linear (again, the only problem is the action of g). Note that σi equals the irreducible map α_{n-1} and that σ is not a split epimorphism; thus i is a split monomorphism. Define $V_{n+1} = \operatorname{coker} i$ then:

$$X \cong V_{n-1} \oplus V_{n+1} .$$

But (2.3.6) implies that X has at most two indecomposable summands. Thus V_{n+1} must be a non-projective, indecomposable kG-module which is free on restriction to H. Write:

$$\phi_{n+1}(w_1,\ldots,w_{n+1}) = \phi(\phi_n(w_1,\ldots,w_n) , \phi_n(0,\ldots,0,w_{n+1})) + \operatorname{Im} i$$

then ψ_{n+1} is a kH-linear isomorphism between V_{n+1} and $(n+1)$ copies of V. The action of g is given by:

$$g \cdot \psi_{n+1}(w_1, \ldots, w_{n+1})$$

$$= \phi(g\psi_n(w_1, \ldots, w_n) + \phi(\psi_n(0, \ldots, 0, w_{n+1}))g\mu_H v_0 \, , \, g\psi_n(0, \ldots, 0, w_{n+1})) + \operatorname{Im} i$$

$$= \phi(\psi_n(gw_1 + \phi_1(w_2)g\mu_H v_1, \ldots, gw_n) + \phi_n(w_{n+1})\psi_n(0, \ldots, 0, g\mu_H v_n) \, ,$$

$$\psi_n(0, \ldots, 0, gw_{n+1})) + \operatorname{Im} i$$

$$= \psi_{n+1}(gw_1 + \phi_1(w_2)g\mu_H v_1, \ldots, gw_n + \phi_n(w_{n+1})g\mu_H v_n \, , \, gw_{n+1}).$$

Finally $\alpha_n = \pi\rho$ where $\pi : X \longrightarrow V_{n+1}$ is the natural map. But π is a split epimorphism, thus α_n is irreducible.

Hence we have constructed $\phi_n, v_n, \psi_{n+1}, V_{n+1}, \alpha_n$ with the required properties. Thus the result follows. □

The following lemma gives a practical way of identifying at least some terminal modules:

<u>Lemma 2.4.2</u> Let V be a non-projective kG-module which is free on restriction to some maximal subgroup of G. Suppose that V has a unique maximal submodule, then V is indecomposable and terminal.

<u>Proof</u> It is trivial that V is indecomposable (a decomposable module will have at least two maximal submodules). To see that it is terminal, let V_n $(n = 1, 2, \ldots)$ be a sequence of kG-modules with the properties outlined in (2.3.6) such that $V \cong V_N$ for some N. No V_n is isomorphic to k_G, thus (2.2.3) implies that $(g(V_n), [k_G]) = 0$. But:

$$g(V_n) = \begin{cases} [V_2] - 2[V_1] & \text{if } n = 1, \\ [V_{n+1}] + [V_{n-1}] - 2[V_n] & \text{otherwise,} \end{cases}$$

so that, inductively, $([V_n],[k_G]) = n([V_1],[k_G])$. Since V has a unique maximal submodule we have:

$$([V_N],[k_G]) = \dim_k(V,k_G) = 1.$$

Thus $N([V_1],[k_G]) = 1$. Hence $N = 1$, as required. □

We may also mention:

Lemma 2.4.3 Let V be a non-projective kG-module which is free on restriction to some maximal subgroup, H, of G. Then V has a unique maximal submodule if and only if $\mu_H V$ is indecomposable as a k(G/H)-module.

Proof The map $v \longmapsto \mu_H v$ gives a kG-epimorphism $V \longrightarrow \mu_H V$ with kernel:

$$V_0 = \{ v \in V \mid \mu_H v = 0 \} = \mathrm{Aug}(kH)\, V.$$

If M is any maximal submodule of V then G acts trivially on V/M. Thus $V_0 \leq M$. Thus the maximal submodules of V are in 1-1 correspondence with those of $V/V_0 \cong \mu_H V$. Because G/H is cyclic, all the indecomposable k(G/H)-modules have a unique maximal submodule. Thus $\dim_k(V,k_G)$ equals the number of indecomposable summands of $\mu_H V$. Hence the result follows. □

In the case when V has a unique maximal submodule, (2.4.1) may be simplified. Let M be the maximal submodule, then we may take all the ϕ_n's to be the same, namely a fixed epimorphism with kernel M, and all the v_n's to be the same element of $V - M$.

CHAPTER 3

CARLSON VARIETIES

Introduction

Let V be a non-projective, indecomposable kG-module which is free on restriction to some maximal subgroup of G. Then, in particular, V is free on restriction to the Frattini subgroup, Φ, of G – for Φ is, by definition, just the intersection of all the maximal subgroups of G. We regard $\bar{V} = \mu_{\Phi}V$ as a module for $\bar{G} = G/\Phi$. But $\bar{G}$ is elementary abelian, so we may, at least when k is algebraically closed, consider the variety, $Y(\bar{V})$, defined in [Ca3]. The main result of this chapter may be stated as:

$$Y(\bar{V}) \text{ is a line.}$$

Our development of this material incorporates most of Carlson's fundamental results. These are to be found in §3.5, which runs parallel with the original. The main difference is the constants, $\lambda_H(V)$, introduced in §3.1; these enable us to avoid any cohomological arguments.

§3.0 Essential subgroups

Let A be a finite-dimensional k-algebra. We call a subgroup, G, of the group of units of A an *essential subgroup* if the elements of G form a k-basis for A. Note that, if G is an essential subgroup then we can identify A with the group algebra, kG; indeed an essential subgroup exists if and only if A is a group algebra.

In this section we shall investigate the properties of essential subgroups in the special case when $\dim_k A$ is a power of p; i.e. when any essential subgroup is a p-group. Most of the results are to be found in [Ca3].

If U and V are A-modules and G is an essential subgroup of A, then we may regard $U \otimes V$ as an A-module by letting G act diagonally and using the identific--ation of A with kG. Taking different essential subgroups does not in general give isomorphic tensor products; however:

Lemma 3.0.1 Let G_1 and G_2 be essential subgroups of A. Then $U \otimes V$ is free with respect to the diagonal action of G_1 if and only if it is free with respect to the diagonal action of G_2.

Proof Since each G_i is a p-group we may apply (0.0.11): $U \otimes V$ is free with respect to the diagonal action of G_i iff:

$$\dim_k (U,V) = \frac{(\dim_k U)(\dim_k V)}{\dim_k A}$$

- for note that $\dim_k A = |G_i|$. But this criterion is independent of i, so the result follows. $\square$

Write J for the Jacobson radical of A. If G is an essential subgroup of

A then, under the identification of A with kG, J corresponds to the augmentation ideal of kG (see (0.0.7)(c)). In particular, if $g \in G$ then $g-1 \in J$; hence any essential subgroup is a subgroup of the subgroup, $1+J$, of the full group of units of A. We also have the following well-known result:

<u>Proposition 3.0.2</u> Let Φ be the Frattini subgroup of G and write $\bar{G} = G/\Phi$. Regard $\bar{G}$ as an $\mathbb{F}_p$-vector space, then we may consider the k-space:

$$S = k \otimes_{\mathbb{F}_p} \bar{G}.$$

The map $g \longmapsto g-1$, $G \longrightarrow J$ then induces a k-isomorphism $S \longrightarrow J/J^2$.

<u>Proof</u> We show firstly that the given map induces a group homomorphism, α, $G \longrightarrow (J/J^2)^+$. Because the image of this map is elementary abelian, Φ will be contained in $\ker\alpha$. Thus we have a group homomorphism ($\mathbb{F}_p$-linear map) $\bar{G} \longrightarrow J/J^2$ which we may readily extend to a k-linear map:

$$\bar{\alpha} : S \longrightarrow J/J^2.$$

To show that α is a group homomorphism:

$$\begin{aligned} \alpha(gh) &= (gh-1) + J^2 \\ &= (g-1)(h-1) + (g-1) + (h-1) + J^2 \\ &= \alpha(g) + \alpha(h) \end{aligned}$$

– for note that $(g-1)(h-1) \in J^2$.

Thus it only remains to show that $\bar{\alpha}$ is an isomorphism. To do this we will construct an inverse map. $J = \mathrm{Aug}(kG)$ is a k-space with basis $\{ g-1 \mid g \in G-1 \}$. So define $\delta : J \longrightarrow S$ by $g-1 \longmapsto \bar{g}$ and extending k-linearly. Note that:

$$\begin{aligned} (g-1)(h-1) &= (gh-1) - (g-1) - (h-1) \\ &\longmapsto (\overline{gh})\bar{g}^{-1}\bar{h}^{-1} = \bar{1}, \end{aligned}$$

thus γ induces a k-linear map $J/J^2 \longrightarrow S$ which is clearly an inverse to α. $\square$

We now assume that A has a fixed essential subgroup, G_0, which is elementary abelian of order p^n. Using the identification of J with $\mathrm{Aug}(kG_0)$, we readily see that:

(a) if $\xi \in J$ then $(1+\xi)^p = 1+\xi^p = 1$,

(b) $0 \neq \gamma_{G_0} \in J^{n(p-1)}$,

(c) J is nilpotent, indeed $J^{n(p-1)+1} = 0$.

In this case we may completely determine the essential subgroups of A:

<u>Theorem 3.0.3</u> Let $\xi_1, \ldots, \xi_n$ be a k-basis for J modulo J^2, then:

$$G = \langle 1+\xi_1, \ldots\ldots, 1+\xi_n \rangle$$

is an essential subgroup of A. Moreover every essential subgroup is of this form.

<u>Proof</u> Suppose firstly that G is an essential subgroup of A. Take a minimal set of generators, $g_1, \ldots, g_m$ for G, then, in the notation of (3.0.2), $\bar{g}_1, \ldots, \bar{g}_m$ is an $\mathbb{F}_p$-basis for $\bar{G}$ and hence a k-basis for S. Thus $g_1 - 1, \ldots, g_m - 1$ is a k-basis for J modulo J^2. In particular, $m = \dim_k (J/J^2)$; but taking $G = G_0$, we see that this dimension also equals n. Thus $m = n$ and G is indeed of the form stated.

Now let $\xi_1, \ldots, \xi_n$ and G be as in the statement of the theorem. Take a minimal set of generators $h_1, \ldots, h_n$ for G_0 then we can write:

$$(h_i - 1) = \lambda_{i1}\xi_1 + \ldots + \lambda_{in}\xi_n + \zeta_i$$

where $\zeta_i \in J^2$ and (λ_{ij}) is a nonsingular $(n \times n)$ matrix with coefficients in k.

Define $\phi : A \longrightarrow A$ by $h_i \longmapsto 1+\xi_i$, this then extends to an algebra homomorphism since $1+\xi_1, \ldots, 1+\xi_n$ are units of order p which commute with each other. Write:

$$\varrho_i = \lambda_{i1}(h_1 - 1) + \ldots + \lambda_{in}(h_n - 1)$$

so that $\phi(\varrho_i) \equiv h_i - 1 \pmod{J^2}$. Then:

$$\phi(\varrho_1^{p-1} \ldots \varrho_n^{p-1}) \equiv (h_1 - 1)^{p-1} \ldots (h_n - 1)^{p-1} \equiv \mu_{G_0} \pmod{J^{n(p-1)+1}}$$

– however, $J^{n(p-1)+1} = 0$ so this congruence is actually an equality. Note that $\varrho_1^{p-1} \ldots \varrho_n^{p-1}$ is an element of $J^{n(p-1)} = k\mu_{G_0}$, but its image under ϕ is nonzero so it must equal $\lambda\mu_{G_0}$ for some $\lambda \neq 0$. Thus:

$$\phi(\mu_{G_0}) = \frac{1}{\lambda}\mu_{G_0}$$

so that ϕ is injective ($k\mu_{G_0}$ is the unique minimal submodule of A) and hence, by dimensions, an algebra automorphism of A. But $\phi(G_0) = G$, thus G is an essential subgroup of A. □

Choose a subspace, L, of J with $J = L \oplus J^2$ then define, for an A-module V:

$$Y_L(V) = \{0\} \cup \{\, \xi \in L \mid V_{\downarrow\langle 1+\xi \rangle} \text{ is } \textit{not free} \,\}$$

(see [Ca3]). We refer to $Y_L(V)$ as the <u>*Carlson variety*</u> of V (with respect to L).

<u>Lemma 3.0.4</u> For $0 \neq \xi \in L$ we have:

(a) $\dim_k \xi^{p-1}V \leq \frac{1}{p}\dim_k V$,

(b) $\dim_k \xi V \leq (1 - \frac{1}{p})\dim_k V$,

strict inequality holding in either case if and only if $\xi \in Y_L(V)$. In particular, therefore, $Y_L(V)$ is closed under scalar multiplication.

<u>Proof</u> (a) is immediate from (0.0.8). For (b), we may write $V_{\downarrow\langle 1+\xi \rangle}$ as a direct

sum of indecomposable modules, $V_1 \oplus \ldots\ldots \oplus V_r$. Each V_i has dimension at most p, thus $r \geq \frac{1}{p}\dim_k V$, equality holding iff each V_i has dimension p, that is, iff V is free on restriction to $\langle 1 + X \rangle$. But XV_i has dimension one less than V_i , so:

$$\dim_k XV \quad = \quad \dim_k V - r.$$

Now, substituting for r gives the result. $\square$

§3.1 The constants $\lambda_H(V)$

We now introduce the main tool that we shall use in this chapter. Let k be a field of characteristic p and G be a finite p-group. We suppose that V is a non-projective, absolutely indecomposable kG-module which is free on restriction to some maximal subgroup, H_0, of G. The situation that we shall be normally concerned with is when k is algebraically closed, when "absolutely indecomposable" may be replaced by just "indecomposable".

For each maximal subgroup, H, of G we choose a fixed element $g \in G - H$ and consider the sequence:

$$0 \longrightarrow k_G \longrightarrow k(G/H) \longrightarrow k(G/H) \longrightarrow k_G \longrightarrow 0 \qquad -(1)$$

where the central map is given by $H \longmapsto (g-1)H$ (see §1.1).

We tensor this sequence by V. When V is free on restriction to H (for example when $H = H_0$) this gives a two-step projective resolution of V. In any case we may complete the commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & V \arrow[r] & k(G/H) \otimes V \arrow[r] & k(G/H) \otimes V \arrow[r] & V \arrow[r] & 0 \\
0 \arrow[r] & V \arrow[r] \arrow[u, dashed, "\alpha"] & k(G/H_0) \otimes V \arrow[r] \arrow[u, dashed] & k(G/H_0) \otimes V \arrow[r] \arrow[u, dashed] & V \arrow[r] \arrow[u, equal] & 0.
\end{tikzcd}$$

α is uniquely determined up to the addition of a map which factors through the projective module $k(G/H_0) \otimes V$, which, since V is not projective, is a non--automorphism of V. Thus the scalar $\lambda \in k$ such that $\alpha - \lambda 1_V$ is a non-automorphism of V is uniquely determined by this diagram (recall that $(V,V)/J(V,V) \cong k$). We write λ_H or $\lambda_H(V)$ for this scalar.

Note that the scalars $\lambda_H(V)$ depend on:

(1) the choice of H_0 from those maximal subgroups of G restricted to which V is free,

(2) the choice of the elements $g \in G-H$,

thus care will be taken to emphasise when any change in these parameters has occurred.

If V is free on restriction to H then both sequences in the defining diagram are projective resolutions, thus α is an automorphism. That is to say, $\lambda_H(V)$ is nonzero. In fact the converse also holds: if $\lambda_H(V) \neq 0$ then V is free on restriction to H. The proof of this is straight-forward when $p=2$:

As in §1.1, sequence (1) is the join of two copies of the sequence:

$$0 \longrightarrow k_G \longrightarrow k(G/H) \longrightarrow k_G \longrightarrow 0$$

thus we may complete the diagram:

$$\begin{tikzcd}
0 \arrow[r] & V \arrow[r] & k(G/H)\otimes V \arrow[r] & V \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & V \arrow[r] \arrow[u, dashed, "\alpha"] & k(G/H_0)\otimes V \arrow[r] \arrow[u] & V \arrow[r] & 0.
\end{tikzcd}$$

as above, the scalar $\mu \in k$ such that $\alpha-\mu 1_V$ is a non-automorphism of V is uniquely determined. We have:

$$\begin{tikzcd}
0 \arrow[r] & V \arrow[r] & k(G/H)\otimes V \arrow[r] & V \arrow[r] \arrow[d, equal] & 0 & & \\
0 \arrow[r] & V \arrow[r] \arrow[u, dashed, "\alpha"] & k(G/H_0)\otimes V \arrow[r] \arrow[u] & V \arrow[r] & k(G/H)\otimes V \arrow[r] & V \arrow[r] \arrow[d, equal] & 0 \\
 & & & V \arrow[r] \arrow[u, dashed, "\alpha"] \arrow[d, equal] & k(G/H_0)\otimes V \arrow[r] \arrow[u] & V \arrow[r] & 0 \\
0 \arrow[r] & V \arrow[r] \arrow[uu, dashed, "\alpha"] & k(G/H_0)\otimes V \arrow[r] \arrow[uu, dashed, "1\otimes\alpha"] & V \arrow[r] & 0 & &
\end{tikzcd}$$

thus $\lambda_H(V)=\mu^2$. Hence, if $\lambda_H(V)\neq 0$ then $\mu \neq 0$; thus α is an automorphism of V. Thus the five-lemma gives that $(V_{\downarrow H})^{\uparrow G} \cong k(G/H) \otimes V \cong k(G/H_0) \otimes V$ is free. But $V_{\downarrow H} \mid ((V_{\downarrow H})^{\uparrow G})_{\downarrow H}$ thus V is free on restriction to H.

This completes the proof for $p = 2$. A general proof will be given in §3.3. However, we may remark here that it suffices to assume that k is algebraically closed. For let $\hat{k}$ be an algebraic closure of k and write $\hat{V}$ for the $\hat{k}G$-module $\hat{k} \otimes_k V$. Because V is absolutely indecomposable, $\hat{V}$ is indecomposable and (0.0.10) gives that $\hat{V}$ is non-projective but is free on restriction to H_0. Applying $\hat{k} \otimes_k -$ to the diagram defining $\lambda_H(V)$ we obtain a commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \hat{V} \arrow[r] & \hat{k}(G/H) \otimes \hat{V} \arrow[r] & \hat{k}(G/H) \otimes \hat{V} \arrow[r] & \hat{V} \arrow[r] & 0 \\
0 \arrow[r] & \hat{V} \arrow[r] \arrow[u, dashed, "1\otimes\alpha"] & \hat{k}(G/H_0) \otimes \hat{V} \arrow[r] \arrow[u] & \hat{k}(G/H_0) \otimes \hat{V} \arrow[r] \arrow[u, dashed] & \hat{V} \arrow[r] \arrow[u, equal] & 0 .
\end{tikzcd}$$

Now $(1_{\hat{k}} \otimes \alpha) - \lambda_H(V) 1_{\hat{V}} = 1_{\hat{k}} \otimes (\alpha - \lambda_H(V) 1_V)$ is nilpotent. Therefore:

$$\lambda_H(\hat{V}) = \lambda_H(V).$$

Thus, if we can prove the result for algebraically closed fields, we will have:

$$\begin{aligned}
\lambda_H(V) \neq 0 \quad &\Rightarrow \quad \lambda_H(\hat{V}) \neq 0 \\
&\Rightarrow \quad \hat{V} \text{ is free on restriction to } H \\
&\Rightarrow \quad V \text{ is free on restriction to } H. \qquad \blacksquare
\end{aligned}$$

§3.2 The Elementary abelian group of order p^2

We now investigate what will turn out to be the most important special case. Let k be an algebraically closed field of characteristic p and:

$$E = \langle x,y \mid x^p = y^p = 1, xy = yx \rangle$$

be an elementary abelian group of order p^2. Write X and Y for the subgroups generated by x and y respectively. The first two results are lifted directly from [Ca2] and our (3.2.5) should be compared with Corollary 3.3 of that paper. Carlson appeals to the classification of indecomposable kE-modules in the case $p = 2$, although the proof he gives for p odd also seems to work in this case. (See Appendix A if I am wrong about this.)

Lemma 3.2.1 Let W be the direct sum of a projective and a periodic kE-module. If $w \in \mathrm{soc}(W)$ then we can write:

$$w = (x-1)^{p-1}w' + (y-1)^{p-1}w''$$

for some $w', w'' \in W$ such that $(x-1)^{p-1}w'$ and $(y-1)^{p-1}w''$ are in $\mathrm{soc}(W)$.

Proof We have an exact sequence:

$$\cdots \xrightarrow{\delta} F_n \longrightarrow \cdots \xrightarrow{\delta} F_1 \xrightarrow{\epsilon} k_E \longrightarrow 0$$

where F_n is the free kE-module on $a_1, \ldots, a_n$; ϵ is the map defined by $a_1 \longmapsto 1$ and $\delta_n : F_{n+1} \longrightarrow F_n$ is given by:

(1) $n = 2m+1$

$$\begin{aligned} a_1 &\longmapsto (x-1)a_1 & \\ a_{2j} &\longmapsto (y-1)a_{2j-1} + (x-1)^{p-1}a_{2j} & (j = 1,2,\ldots,m) \\ a_{2j+1} &\longmapsto (y-1)^{p-1}a_{2j} + (x-1)a_{2j+1} & (j = 1,2,\ldots,m) \\ a_{2m+2} &\longmapsto (y-1)a_{2m+1} & \end{aligned}$$

(2) <u>n = 2m</u>

$$\begin{aligned}
a_1 &\longmapsto (x-1)^{p-1}a_1 \\
a_{2j} &\longmapsto (y-1)a_{2j-1} - (x-1)a_{2j} && (j = 1,2,\dots,m) \\
a_{2j+1} &\longmapsto (y-1)^{p-1}a_{2j} - (x-1)^{p-1}a_{2j+1} && (j = 1,2,\dots,m-1) \\
a_{2m+1} &\longmapsto (y-1)^{p-1}a_{2m}
\end{aligned}$$

We further write $K_n = \operatorname{Im}\delta_n$.

With this notation, define:

$$\theta_j : F_{2m+1} \longrightarrow W \qquad (j = 1,2,\dots,m+1)$$

by:

$$a_i \longmapsto \begin{cases} w & \text{if } i = 2j-1, \\ 0 & \text{otherwise,} \end{cases}$$

then, since E acts trivially on w, θ_j is seen to be zero on K_{2m+1} and thus to factor as $\phi_j\delta_{2m}$ for some $\phi_j : K_{2m} \longrightarrow W$.

But:

$$\begin{aligned}
[K_{2m}, W] &\cong [K_{2m} \otimes W^*, k_E] \cong [\Omega^{2m}W^* \oplus \text{projective}, k_E] \\
&\cong [\Omega^{2m}W^*, k_E]
\end{aligned}$$

(see §0.0 for notation), and, by the definition of W, $\dim_k[\Omega^{2m}W^*, k_E]$ is bounded independently of m. Thus, if we take m sufficiently large, the images of the $m+1$ elements $\phi_1, \dots, \phi_{m+1}$ are linearly dependent in $[K_{2m}, W]$; that is to say, there exist elements $\lambda_1, \dots, \lambda_{m+1}$ of k, not all zero, such that

$$\phi = \lambda_1\phi_1 + \dots + \lambda_{m+1}\phi_{m+1}$$

factors as $K_{2m} \xrightarrow{\text{incl.}} F_{2m} \xrightarrow{f} W$ for some f. Thus:

$$f\delta_{2m} = \phi\delta_{2m} = \lambda_1\theta_1 + \dots + \lambda_{m+1}\theta_{m+1}.$$

Choose j with $\lambda_j \neq 0$ then:

$$\begin{aligned} f\delta_{2m}(a_{2j-1}) &= (y-1)^{p-1}f(a_{2j}) - (x-1)^{p-1}f(a_{2j+1}) \text{ or } (x-1)^{p-1}f(a_1) \\ &= \lambda_1\theta_1(a_{2j-1}) + \ldots + \lambda_{m+1}\theta_{m+1}(a_{2j-1}) \\ &= \lambda_j w. \end{aligned}$$

Thus we have expressed w in the form $(x-1)^{p-1}w' + (y-1)^{p-1}w''$. Now:

$$\begin{aligned} 0 = (y-1)w &= (y-1)(x-1)^{p-1}w' + (y-1)^p w'' \\ &= (y-1).(x-1)^{p-1}w' \end{aligned}$$

also: $$(x-1).(x-1)^{p-1}w' = (x^p-1)w' = 0$$

so that $(x-1)^{p-1}w'$, and hence $(y-1)^{p-1}w''$, is an element of $W^G = \mathrm{soc}(W)$. □

Theorem 3.2.2 If V is an indecomposable periodic kE-module then V is free on restriction to either X or Y.

Proof Let $W = (V,V)_k$ then W satisfies the conditions of (3.2.1) and soc (W) is just (V,V). Thus take $w = 1_V$ in the lemma to show that:

$$1_V = f' + f'' \quad \text{for some } f', f'' \in (V,V) \text{ with } f' \in \mu_X W,\ f'' \in \mu_Y W.$$

Since (V,V) is a local ring, either f' or f'' is an automorphism; thus by (0.0.4) the result follows. □

With this notation, we suppose, without loss of generality, that V is free on restriction to Y. Thus we consider the sequence:

$$0 \longrightarrow k_E \longrightarrow k(E/Y) \longrightarrow k(E/Y) \longrightarrow k_E \longrightarrow 0$$

where the central map is given by $Y \longmapsto (x-1)Y$. When tensored by V, this gives a two-step projective resolution of V.

We now, in the notation of §3.0, take $A = kE$ and $L = k(x-1) \oplus k(y-1)$ and calculate $Y_L(V)$.

Let (λ_{ij}) be a nonsingular 2×2 matrix with coefficients in k, then we define $\xi, \eta \in L$ by means of the equations:

$$\begin{aligned} x-1 &= \lambda_{11}\xi + \lambda_{12}\eta \\ y-1 &= \lambda_{21}\xi + \lambda_{22}\eta \end{aligned}$$

then $\bar{E} = \langle 1+\xi\, ,\, 1+\eta \rangle$ is an essential subgroup of A by (3.0.3). Also write $\bar{Y} = \langle 1+\eta \rangle$, then we have an exact sequence of $k\bar{E}$-modules, which are also A-modules under the identification of A with $k\bar{E}$:

$$0 \longrightarrow k_E \longrightarrow k(\bar{E}/\bar{Y}) \longrightarrow k(\bar{E}/\bar{Y}) \longrightarrow k_E \longrightarrow 0$$

in which the maps are given by:

$$1 \longmapsto \mu_{\bar{E}/\bar{Y}} \quad , \quad \bar{Y} \longmapsto \xi\bar{Y} \quad , \quad \bar{Y} \longmapsto 1 \, .$$

We tensor this sequence by V (using the diagonal action of E) and use the projective resolution above to obtain a commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & V \arrow[r] & k(\bar{E}/\bar{Y}) \otimes V \arrow[r] & k(\bar{E}/\bar{Y}) \otimes V \arrow[r] & V \arrow[r] & 0 \\ 0 \arrow[r] & V \arrow[r] \arrow[u, dotted, "\alpha"] & k(E/Y) \otimes V \arrow[r] \arrow[u, dotted] & k(E/Y) \otimes V \arrow[r] \arrow[u, dotted] & V \arrow[r] \arrow[u, equal] & 0 \, ; \end{tikzcd} \qquad -(1)$$

then, as in §3.1, the scalar $\lambda \in k$ such that $\alpha - \lambda 1_V$ is a non-automorphism of V depends only on the matrix (λ_{ij}). Thus we have a well-defined map:

$$c : GL(2,k) \longrightarrow k.$$

For example, if I is the 2×2 identity matrix then $c(I) = 1$, for the two sequences in the diagram (1) are identical in this case. We also consider the matrix:

$$R = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

In this case $\bar{E} = E$, $\bar{Y} = X$ and the map $k(\bar{E}/\bar{Y}) \longrightarrow k(\bar{E}/\bar{Y})$ is given by

$X \longmapsto (y-1)X$. Thus $c(R) = \lambda_X(V)$ where λ_X is calculated using Y as the maximal subgroup restricted to which V is free, $x \in E - Y$, $y \in E - X$.

Returning to the general case we have:

<u>Lemma 3.2.3</u> If V is free on restriction to $\bar{Y}$ then $c(\lambda_{ij}) \neq 0$.

<u>Proof</u> With respect to the diagonal action of $\bar{E}$, $k(\bar{E}/\bar{Y}) \otimes V \cong (V_{\downarrow \bar{Y}})^{\uparrow \bar{E}}$ is free. Thus by (3.0.1), $k(\bar{E}/\bar{Y}) \otimes V$ is also free with respect to the diagonal action of E. Hence the upper sequence in (1) is a projective resolution of V. Thus α is an automorphism, and the result follows. $\square$

We now consider the projective resolution of k_E given in the proof of (3.2.1) and, in particular, the first few terms. We shall construct a particular commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & k_E \arrow[r] & k(\bar{E}/\bar{Y}) \arrow[r] & k(\bar{E}/\bar{Y}) \arrow[r] & k_E \arrow[r] & 0 \\
0 \arrow[r] & K_2 \arrow[r] \arrow[u, dashed, "\alpha_2"] & F_2 \arrow[r, "\lambda"] \arrow[u, dashed, "\alpha_1"] & F_1 \arrow[r] \arrow[u, dashed, "\alpha_0"] & k_E \arrow[r] \arrow[u, equal] & 0.
\end{tikzcd} \qquad -(2)$$

Clearly we may take $\alpha_0 : a_1 \longmapsto \bar{Y}$; now calculate $\delta_1 \alpha_0$:

$$a_1 \longmapsto (x-1)\bar{Y} = (\lambda_{11}\xi + \lambda_{12}\eta)\bar{Y} = \lambda_{11}\xi\bar{Y}$$
$$a_2 \longmapsto (y-1)\bar{Y} = (\lambda_{21}\xi + \lambda_{22}\eta)Y = \lambda_{21}\xi\bar{Y}$$

hence we may define α_1 by:

$$a_1 \longmapsto \lambda_{11}\bar{Y} \quad , \quad a_2 \longmapsto \lambda_{21}\bar{Y} .$$

Write $b_i = \delta_2(a_i)$ for $i = 1,2,3$ then:

$$
\begin{aligned}
a_1(b_1) &= \lambda_{11}(x-1)^{p-1}\bar{Y} = \lambda_{11}(\lambda_{11}^{p-1}\xi^{p-1} + 2(\ldots))\bar{Y} \\
&= \lambda_{11}^{p}\mu_{E/\bar{Y}} \\
a_1(b_2) &= \lambda_{11}(y-1)\bar{Y} - \lambda_{21}(x-1)\bar{Y} = (\lambda_{11}\lambda_{21} - \lambda_{21}\lambda_{11})\xi\bar{Y} \\
&= 0 \\
a_1(b_3) &= \lambda_{21}(y-1)^{p-1}\bar{Y} = \lambda_{21}(\lambda_{21}^{p-1}\xi^{p-1} + 2(\ldots))\bar{Y} \\
&= \lambda_{21}^{p}\mu_{E/\bar{Y}}
\end{aligned}
$$

hence:

$$a_2 = \lambda_{11}^{p}\phi_1 + \lambda_{21}^{p}\phi_2$$

where ϕ_j $(j=1,2)$ are as in the proof of (3.2.1) if we identify k_E with kw.

Now we also have a fixed commutative diagram:

$$
\begin{tikzcd}
0 \arrow[r] & K_2 \otimes V \arrow[r] & F_2 \otimes V \arrow[r] & F_1 \otimes V \arrow[r] & V \arrow[r] & 0 \\
0 \arrow[r] & V \arrow[r] \arrow[u, "\iota"] & k(E/Y) \otimes V \arrow[r] \arrow[u] & k(E/Y) \otimes V \arrow[r] \arrow[u] & V \arrow[r] \arrow[u, equal] & 0 .
\end{tikzcd}
\qquad -(3)
$$

Write $\psi_j = (\phi_j \otimes 1_V)\iota$ and tensor the diagram (2) by V; attaching the diagram (3) to this we see that we may take:

$$\alpha = \lambda_{11}^{p}\psi_1 + \lambda_{21}^{p}\psi_2$$

in the diagram (1).

Taking (λ_{ij}) to be the identity matrix, I, we see that $\psi_1 - 1_V$ is a non-automorphism of V. Similarly, taking the matrix R defined above, $\psi_2 - \lambda_\chi 1_V$ is a non-automorphism. Therefore:

<u>Lemma 3.2.4</u> $c(\lambda_{ij}) = \lambda_{11}^{p} + \lambda_{21}^{p}\lambda_\chi(V)$. □

So let $\lambda = \lambda_\chi(V)^{1/p}$ be the (unique) pth root of λ_χ in k, and write:

$$\varrho_0 = (x-1) + \lambda(y-1).$$

If $\lambda_X = 0$ then by (3.2.3), V is not free on restriction to X. Thus $\varrho_0 = x-1$ is in $Y_L(V)$. If $\lambda_X \neq 0$ then consider the matrix:

$$\begin{bmatrix} \lambda & 0 \\ -1 & \lambda \end{bmatrix}$$

then $\xi = \frac{1}{\lambda}(x-1)$, $\varrho = \frac{1}{\lambda}\varrho_0$. Using (3.2.4), the value of c at this matrix is $\lambda^p + (-1)^p\lambda_X = 0$. Hence by (3.2.3), V is not free on restriction to $\langle 1+\varrho\rangle$: thus ϱ, and hence $\varrho_0 = \lambda\varrho$, is in $Y_L(V)$.

Thus we have shown that in all cases that $Y_L(V)$ contains the line $k\varrho_0$. If ϱ_1 is any point in L outside this line then ϱ_0 and ϱ_1 form a basis for L. Let $\bar{E} = \langle 1+\varrho_0, 1+\varrho_1\rangle$ so that $\bar{E}$ is an essential subgroup of kE. Now V is an indecomposable, periodic $k\bar{E}$-module, thus by the reasoning of (3.2.2) it is free on restriction to either $\langle 1+\varrho_0\rangle$ or $\langle 1+\varrho_1\rangle$. But V is not free on restriction to $\langle 1+\varrho_0\rangle$, thus it must be free on restriction to $\langle 1+\varrho_1\rangle$: that is to say $\varrho_1 \notin Y_L(V)$. Thus we have that $Y_L(V) = k\varrho_0$.

We summarise this as:

<u>Theorem 3.2.5</u> Let V be a non-projective, indecomposable kE-module which is free on restriction to Y. We calculate $\lambda_X(V)$ using the elements $x \in E-Y$, $y \in E-X$. Let $L = k(x-1) \oplus k(y-1)$ then:

$$Y_L(V) = k((x-1) + \lambda_X(V)^{1/p}(y-1)). \qquad \square$$

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§3.3 Application to the general case

Now let k be an algebraically closed field of characteristic p and G be a finite p-group. Assume that V is a non-projective, indecomposable kG-module which is free on restriction to some maximal subgroup, H_0, of G as in §3.1. In calculating the constants $\lambda_H(V)$ it suffices to assume that $H \neq H_0$, for clearly $\lambda_{H_0}(V) = 1$. Let $N = H \cap H_0$ and $E = G/N$, so that E is an elementary abelian group of order p^2. As in (0.0.12), we may consider the kE-module, $U = \mu_N V$; this is free on restriction to H_0/E and has no projective summands, for $\mu_E U = \mu_G V = 0$.

Lemma 3.3.1 Let $g_0 \in G - H_0$, $g \in G - H$ be the elements chosen to calculate $\lambda_H(V)$. There exist $g' \in (gH) \cap H_0$, $g_0' \in (g_0 H_0) \cap H$; let $x = g_0' N$, $y = g' N$ so that x and y generate E. If λ_1, λ_2 are elements of k, not both zero, then U is free on restriction to:

$$\langle 1 + \lambda_1(x-1) + \lambda_2(y-1) \rangle$$

if and only if $\lambda_2^p \neq \lambda_H(V)\lambda_1^p$.

Proof Choose $g'' \in H_0 - H$, then $g \in (g'')^i H$ for some i; thus $(g'')^i \in (gH) \cap H_0$. Hence g' and, similarly, g_0' do exist and x and y are seen to have the property claimed.

Let W be an indecomposable summand of U, then W is free on restriction to $H_0/E = \langle y \rangle$. Hence we are in the situation of (3.2.5). We claim that $\lambda_x(W)$ equals $\lambda_H(V)$ so that:

$$Y_L(W) = k((x-1) + \lambda_H(V)^{1/p}(y-1)).$$

But then $Y_L(W)$ is this same line for all summands, W, of U. Hence $Y_L(U)$ also equals this line. Thus the result follows.

Thus we must prove the claim. There exist maps $W \xrightarrow{i} U \xrightarrow{\pi} W$ with $\pi i = 1_W$. Multiply the defining diagram for $\lambda_H(V)$ by μ_N, noting that N acts trivially on $k(G/H_0)$ and $k(G/H)$ so that we may apply (0.0.13) to show that there is a commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & U \arrow[r] & k(E/X)\otimes U \arrow[r] & k(E/X)\otimes U \arrow[r] & U \arrow[r] & 0 \\
0 \arrow[r] & U \arrow[r] \arrow[u, "\bar{\alpha}"] & k(E/Y)\otimes U \arrow[r] \arrow[u] & k(E/Y)\otimes U \arrow[r] \arrow[u] & U \arrow[r] \arrow[u, equal] & 0
\end{tikzcd}$$

where $\bar{\alpha}$ is the restriction of α to U and the central maps are 1_U tensored by

$$X \longmapsto (g-1)H/E = (g'-1)H/E = (y-1)X$$

and
$$Y \longmapsto (g_0-1)H_0/E = (g_0'-1)H_0/E = (x-1)Y$$

respectively. We can write $\bar{\alpha} = \lambda_H(V)1_U + \beta$ where β is nilpotent.

Now we have:

$$\begin{tikzcd}
0 \arrow[r] & W \arrow[r] & k(E/X)\otimes W \arrow[r] & k(E/X)\otimes W \arrow[r] & W \arrow[r] & 0 \\
0 \arrow[r] & U \arrow[r] \arrow[u, "\pi"] & k(E/X)\otimes U \arrow[r] \arrow[u] & k(E/X)\otimes U \arrow[r] \arrow[u] & U \arrow[r] \arrow[u, "\pi"'] & 0 \\
0 \arrow[r] & U \arrow[r] \arrow[u, "\bar{\alpha}"] & k(E/Y)\otimes U \arrow[r] \arrow[u] & k(E/Y)\otimes U \arrow[r] \arrow[u] & U \arrow[r] \arrow[u, equal] & 0 \\
0 \arrow[r] & W \arrow[r] \arrow[u, "i"] & k(E/Y)\otimes W \arrow[r] \arrow[u] & k(E/Y)\otimes W \arrow[r] \arrow[u] & W \arrow[r] \arrow[u, "i"'] & 0
\end{tikzcd}$$

and $\pi\bar{\alpha}i = \lambda_H(V)1_W + \pi\beta i$. Now $i\pi\beta$ is nilpotent – for the nilpotent elements of (U,U) form an ideal (the radical). Thus, for m sufficiently large,

$$(\pi\beta i)^{m+1} = \pi\beta(i\pi\beta)^m i = 0.$$

Hence $\pi\beta i$ is a non-automorphism of W and the result follows. $\square$

Theorem 3.3.2 V is free on restriction to H if and only if $\lambda_H(V) \neq 0$.

Proof The result is trivial for $H = H_0$. Otherwise, in the notation of (3.3.1), V is free on restriction to H iff U is free on restriction to $\langle x \rangle$. But, taking $\lambda_1 = 1$, $\lambda_2 = 0$ in (3.3.1), this happens iff $\lambda_H(V) \neq 0$. □

As mentioned in §3.1, this result extends to the case when k is not necessarily algebraically closed but V is absolutely indecomposable.

§3.4 An alternative description of $\lambda_H(V)$

Let V be as in §3.1. Choose an epimorphism $\phi : V \longrightarrow k_G$ such that $J(V,V)V \leq \ker\phi$ and an element $v_0 \in V$ with $\phi(v_0) \neq 0$. For each maximal subgroup, H, of G we may construct the exact sequence:

$$0 \longrightarrow V \longrightarrow X_H \longrightarrow V \longrightarrow 0$$

as in (2.1.2), the element $g \in G - H$ in the statement of this theorem being the same as is used to calculate $\lambda_H(V)$. This sequence is either split or almost split. It is almost split when V is free is free on restriction to H, for example when $H = H_0$. In either case we may construct a commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & V \arrow[r, "\rho"] & X_H \arrow[r, "\sigma"] & V \arrow[r] & 0 \\
0 \arrow[r] & V \arrow[r, "\rho_0"] \arrow[u, dashed, "\beta"] & X_{H_0} \arrow[r, "\sigma_0"] \arrow[u, dashed, "\alpha"] & V \arrow[r] \arrow[u, equal] & 0 .
\end{tikzcd}$$

Write $\beta = \lambda 1_V + \delta$ where $\lambda \in k$ and δ is a non-automorphism of V. Because the lower sequence is almost split we can write $\delta = \bar{\delta}\rho_0$ for some $\bar{\delta} : X_{H_0} \longrightarrow V$. Now let $\alpha' = \alpha - \rho\bar{\delta}$ then:

$$\sigma\alpha' = \sigma\alpha - \sigma\rho\bar{\delta} = \sigma_0$$

$$\alpha'\rho_0 = \alpha\rho_0 - \rho\bar{\delta}\rho_0 = \rho(\beta - \delta) = \rho.\lambda 1_V$$

thus we have a commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & V \arrow[r] & X_H \arrow[r] & V \arrow[r] & 0 \\
0 \arrow[r] & V \arrow[r] \arrow[u, "\lambda 1_V"] & X_{H_0} \arrow[r] \arrow[u, "\alpha'"] & V \arrow[r] \arrow[u, equal] & 0 .
\end{tikzcd}$$

The scalar λ is uniquely determined: if λ' were another possible value then $(\lambda - \lambda')1_V$ would factor through ρ_0 and hence be a non-automorphism of V

(since ϱ_0 is not split). The only way this can happen is if $\lambda = \lambda'$.

Theorem 3.4.1 $\lambda = \lambda_H(V)$.

Corollary 3.4.2 The sequence $0 \longrightarrow V \longrightarrow X_H \longrightarrow V \longrightarrow 0$ is:

(a) almost split if V is free on restriction to H,

(b) split otherwise.

Proof of Corollary (a) follows from (2.1.4). Conversely if the sequence is almost split then the map α in the diagram above will be an automorphism of V (cf. (2.2.2)). Thus $\lambda \neq 0$. Hence by (3.4.1) and (3.3.2), V is free on restriction to H. Part (b) now follows using (2.1.2): if V is not free on restriction to H then the sequence is not almost split, therefore it must be split. □

Proof of Theorem (refer to §2.1) We may choose a free kH_0-basis $v_1, \ldots, v_n$ for V such that $v_1 = v_0$. $k(G/H_0) \otimes V$ is then the free kG-module on the elements $H_0 \otimes v_1, \ldots, H_0 \otimes v_n$. Thus we may construct a commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & V \arrow[r] & k(G/H) \otimes V \arrow[r] & k(G/H) \otimes V \arrow[r] & V \arrow[r] & 0 \\
0 \arrow[r] & V \arrow[r] \arrow[u, dashed, "\varrho"] & k(G/H_0) \otimes V \arrow[r] \arrow[u, dashed] & k(G/H_0) \otimes V \arrow[r] \arrow[u, dashed, "\xi"] & V \arrow[r] \arrow[u, equal] & 0
\end{tikzcd}$$

where ξ is given by:

$$H_0 \otimes v_i \longmapsto H \otimes v_i \qquad (i = 1, 2, \ldots, n).$$

Note that, by construction, $\xi\theta_{H_0} = \theta_H$.

Hence we have two commutative diagrams:

(1)

$$\begin{tikzcd}
0 \arrow[r] & V \arrow[r] & k(G/H)\otimes V \arrow[r] & k(G/H)\otimes V \arrow[r] & V \arrow[r] & 0 \\
0 \arrow[r] & V \arrow[r] \arrow[u, equal] & X_H \arrow[r] \arrow[u] & V \arrow[r] \arrow[u, "\theta_H"'] & 0 \\
0 \arrow[r] & V \arrow[r] \arrow[u, "\lambda 1_V"] & X_{H_0} \arrow[r] \arrow[u] & V \arrow[r] \arrow[u, equal] & 0
\end{tikzcd}$$

(2)

$$\begin{tikzcd}
0 \arrow[r] & V \arrow[r] & k(G/H)\otimes V \arrow[r] & k(G/H)\otimes V \arrow[r] & V \arrow[r] & 0 \\
0 \arrow[r] & V \arrow[r] \arrow[u, "\rho"] & k(G/H_0)\otimes V \arrow[r] \arrow[u] & k(G/H_0)\otimes V \arrow[r] \arrow[u, "\xi"'] & V \arrow[r] \arrow[u, equal] & 0 \\
0 \arrow[r] & V \arrow[r] \arrow[u, equal] & X_{H_0} \arrow[r] \arrow[u] & V \arrow[r] \arrow[u, "\theta_{H_0}"'] & 0
\end{tikzcd}$$

so that we have a diagram:

$$\begin{tikzcd}
0 \arrow[r] & V \arrow[r] & k(G/H)\otimes V \arrow[r] & k(G/H)\otimes V \arrow[r] & V \arrow[r] & 0 \\
0 \arrow[r] & V \arrow[r] \arrow[u, "\rho-\lambda 1_V"] & X_{H_0} \arrow[r] \arrow[u] & V \arrow[r] \arrow[u, "0"'] & 0\,.
\end{tikzcd}$$

Thus $\rho - \lambda 1_V$ factors through ρ_0 and therefore is a non-automorphism of V. But this gives the result simply by the definition of $\lambda_H(V)$. $\square$

<u>Note</u> The only point we need to be slightly careful about in the proof above is that we need to be sure that the central maps in the four-module sequences are consistent. However this is ensured by using the same elements $g \in G - H$ in the calculation of $\lambda_H(V)$ as in the sequence $0 \longrightarrow V \longrightarrow X_H \longrightarrow V \longrightarrow 0$.

Let us examine (3.4.2): V is not free on restriction to H if and only if the sequence:

$$0 \longrightarrow V \longrightarrow X_H \xrightarrow{\sigma} V \longrightarrow 0$$

splits; that is to say, if and only if there exists a kG-homomorphism $\tau : V \longrightarrow X_H$ with $\sigma\tau = 1_V$.

The sequence certainly splits on restriction to H, so we may choose a kH-homomorphism $\tau : V \longrightarrow X_H$ with $\sigma\tau = 1_V$. Clearly τ has the form:

$$\tau(v) = \phi(s(v), v)$$

for some kH-automorphism, s, of V; conversely, any map of this form will be a kH-linear right inverse for σ. τ is kG-linear if and only if:

$$g\tau(v) = \tau(gv) \qquad \text{for all } v \in V.$$

But:

$$g\tau(v) = g.\phi(s(v), v) = \phi(gs(v) + \phi(v)g\mu_H v_0 , gv),$$
$$\tau(gv) = \phi(s(gv) , gv).$$

Thus:

<u>Corollary 3.4.3</u> V is not free on restriction to H if and only if there exists a kH-linear map, $s : V \longrightarrow V$, satisfying:

$$s(gv) = gs(v) + \phi(v)g\mu_H v_0 \qquad \text{for all } v \in V. \qquad \square$$

It is possible, analysing the map $\alpha' : X_{H_0} \longrightarrow X_H$ in the defining diagram for λ, to obtain a condition on $k(H_0 \cap H)$-linear maps $V \longrightarrow V$ that determines λ. This does not however seem any more enlightening than (3.4.3).

§3.5 Dade's lemma

Let us now return to the notation of §3.0: let A be the group algebra of an elementary abelian p-group over an algebraically closed field of characteristic p, k. Write J for the radical of A and let L be a subspace of J such that $J = L \oplus J^2$. We shall follow the general development of [Ca3], taking over some proofs entirely and adapting others slightly. The first result falls into the former category.

Lemma 3.5.1 (i) $Y_L(U \oplus V) = Y_L(U) \cup Y_L(V)$.

(ii) $Y_L(U \otimes V) \subseteq Y_L(U) \cap Y_L(V)$.

Proof (i) is trivial. For (ii), let $\xi \in L$ and write $C = \langle 1 + \xi \rangle$. If V is free on restriction to C then we prove, by induction on $\dim_k U$, that $U \otimes V$ is also free on restriction to C. This is trivial if $U = 0$; otherwise, we may choose a maximal submodule, U_0, of U. We then have an exact sequence:

$$0 \longrightarrow U_0 \otimes V \longrightarrow U \otimes V \longrightarrow V \longrightarrow 0.$$

Inductively, both end terms of this sequence are free on restriction to C, thus so is the middle term.

This implies that $Y_L(U \otimes V) \subseteq Y_L(V)$. But similarly $Y_L(U \otimes V) \subseteq Y_L(U)$, so the result follows. □

The next result was first proved in [Da] using essentially elementary techniques. Our proof has more in common with that given in [Ca3]. Most of the spade-work has already been done in §3.2.

Theorem 3.5.2 *(Dade's lemma)* V is free if and only if $Y_L(V) = \{0\}$.

Proof Suppose firstly that V is free. Let ξ be a nonzero element of L, then, by (3.0.3), there exists an essential subgroup, G, of A containing $1+\zeta$. V is a free kG-module, thus it is free on restriction to the subgroup, $\langle 1+\zeta\rangle$, of G. Hence $\zeta \notin Y_L(V)$ and the result follows.

The converse is proved by induction on n, where $p^n = \dim_k A$. The case $n=1$ is trivial: choose $0 \neq \zeta \in L$, then V is free on restriction to $\langle 1+\zeta\rangle$, but this is an essential subgroup of A. So suppose that $n \geq 2$ and that $Y_L(V) = \{0\}$. Using (3.5.1)(i), it suffices to assume that V is indecomposable. Take a k-basis $\zeta_1, \ldots, \zeta_n$ for L and define:

$$N = \langle 1+\zeta_3, \ldots, 1+\zeta_n\rangle,$$
$$g' = 1+\zeta_1, \quad g_0' = 1+\zeta_2,$$
$$H = \langle g_0', N\rangle, \quad H_0 = \langle g', N\rangle,$$
$$G = \langle g_0', g', N\rangle.$$

Then G is an essential subgroup of A; also, inductively, V is free on restriction to H_0. Assume that V is not free; then we are in the situation of (3.3.1). Write:

$$\zeta = \lambda_H(V)^{1/p}\zeta_1 + \zeta_2$$

and let $H' = \langle 1+\zeta, N\rangle$. Inductively V is free on restriction to H', thus $\mu_N V$ is free on restriction to $H'/N = \langle (1+\zeta)N\rangle$ – contradicting the result of (3.3.1). Thus V must be free. □

<u>Proposition 3.5.3</u> $U \otimes V$ is free if and only if $Y_L(U) \cap Y_L(V) = \{0\}$.

<u>Remark</u> Note that, by (3.0.1), it does not matter which essential subgroup of A we choose to act diagonally on the tensor product. The proof of this result is lifted directly from [Ca3].

<u>Proof</u> If the intersection of the two varieties is zero then $U \otimes V$ is free by (3.5.1)(ii) and (3.5.2). Conversely suppose that ζ is a nonzero element of $Y_L(U) \cap Y_L(V)$ and write $C = \langle 1+\zeta\rangle$. Then there exist non-projective, indecomposable

kC-summands U_0, V_0 of U,V respectively. Note that because C is cyclic of order p, U_0 and V_0 have dimension less than p. By (3.0.3) there exists an essential subgroup, G, of A containing C. If we let G act diagonally on $U \otimes V$ then $U_0 \otimes V_0$ is a direct summand of $(U \otimes V)_{\downarrow C}$. But this summand is not free (for its dimension is not divisible by p), thus $U \otimes V$ is not free with respect to the diagonal action of G, or, therefore, any other essential subgroup of A. $\square$

We now investigate how $Y_L(V)$ varies with L. For $\gamma \in J - J^2$ define a module $U_\gamma = \gamma^{p-1}A$.

<u>Lemma 3.5.4</u> If $\gamma, \gamma' \in J - J^2$ then U_γ is free on restriction to $\langle 1 + \gamma' \rangle$ if and only if γ and γ' are linearly independent modulo J^2.

<u>Proof</u> If they are linearly independent then (3.0.3) shows that there exists an essential subgroup, G, of A containing both $C = \langle 1 + \gamma \rangle$ and $C' = \langle 1 + \gamma' \rangle$. Then $U_\gamma = \mu_C(kG)$ has dimension $\frac{|G|}{p}$ and $\mu_{C'}(U_\gamma) = \mu_{CC'}(kG)$ has dimension $\frac{|G|}{pp}$. Thus U_γ is free on restriction to C' by (0.0.12).

Conversely, suppose that γ and γ' are linearly dependent modulo J^2. Choose L' with $J = L' \oplus J^2$ and $\gamma' \in L'$. Then any element of $L' - k\gamma'$ is linearly independent of γ modulo J^2 and so, by the first part of the proof, is not in $Y_{L'}(U_\gamma)$. Thus $Y_{L'}(U_\gamma)$ is contained within the line $k\gamma'$; but U_γ is not free, so (3.5.2) shows that $Y_{L'}(U_\gamma)$ is not zero. Thus it must be the whole line. In particular, U_γ is not free on restriction to $\langle 1 + \gamma' \rangle$. $\square$

<u>Theorem 3.5.5</u> Let $\gamma, \gamma' \in J - J^2$ with $\gamma \equiv \gamma'$ (mod J^2). Then V is free on restriction to $\langle 1 + \gamma \rangle$ if and only if it is free on restriction to $\langle 1 + \gamma' \rangle$.

<u>Proof</u> Choose a subspace of dimension $n-1$, L'', such that:

$$J = (k\xi \oplus L'') \oplus J^2$$

and write $L = k\xi \oplus L''$, $L' = k\xi' \oplus L''$ so that L and L' are both complements to J^2 in J.

Using (3.5.4), $Y_L(U_\xi) = k\xi$, $Y_{L'}(U_\xi) = k\xi'$. Hence use (3.5.3):

$$\begin{aligned} \xi \in Y_L(V) &\Leftrightarrow Y_L(U_\xi) \cap Y_L(V) \text{ is nonzero} \\ &\Leftrightarrow U_\xi \otimes V \text{ is not free} \\ &\Leftrightarrow Y_{L'}(U_\xi) \cap Y_{L'}(V) \text{ is nonzero} \quad \Leftrightarrow \quad \xi' \in Y_{L'}(V). \end{aligned}$$

Thus the result follows. □

Hence, if we regard $Y_L(V)$ as a subset of J/J^2 by identifying L with J/J^2 in the obvious way, then the result is independent of L. We shall denote this set by $Y(V)$. Thus:

$$Y(V) = \{0\} \cup \{\xi + J^2 \in J/J^2 \mid V_{\downarrow\langle 1+\xi\rangle} \text{ is not free}\}$$

– this being well-defined by (3.5.5).

We shall need the following result, the proof of which is virtually the same as that of (3.5.5):

<u>Lemma 3.5.6</u> Let $\xi \in J - J^2$ then:

$$Y(U_\xi \otimes V) = \begin{cases} k(\xi + J^2) & \text{if } \xi + J^2 \in Y(V), \\ \{0\} & \text{otherwise.} \end{cases}$$ □

§3.6 Subspaces determined by subgroups

Let us introduce the notation that we shall be using for the remainder of this chapter. k will be an algebraically closed field of characteristic p and G will be a finite p-group. Φ will denote the Frattini subgroup of G and we shall write $\bar{G} = G/\Phi$. Take a minimal set of generators $g_1, \ldots, g_n$ for G and write:

$$H_i = \langle g_1, \ldots, g_{i-1}, \Phi, g_{i+1}, \ldots, g_n \rangle$$

$(i = 1, 2, \ldots, n)$ so that each H_i is a maximal subgroup of G. We may assume that when choosing the representatives of $G - H$ as in §3.1 we choose g_i to represent $G - H_i$.

<u>Lemma 3.6.1</u> Let $J, \bar{J}$ denote the augmentation ideals of kG and $k\bar{G}$ respectively. The natural map $G \longrightarrow \bar{G}$ induces a k-isomorphism $J/J^2 \longrightarrow \bar{J}/\bar{J}^2$.

<u>Proof</u> Apply (3.0.2) to G: the map $k \otimes \bar{G} \longrightarrow J/J^2$ given by $\bar{g} \longmapsto (g-1) + J^2$ is a k-isomorphism. But we may also apply (3.0.2) to $\bar{G}$: the Frattini subgroup of $\bar{G}$ is trivial, so we have a k-isomorphism $k \otimes \bar{G} \longrightarrow \bar{J}/\bar{J}^2$ given by $\bar{g} \longmapsto (\bar{g} - \bar{1}) + \bar{J}^2$. Thus the map $(g-1) + J^2 \longmapsto (\bar{g} - \bar{1}) + \bar{J}^2$ is a k-isomorphism $J/J^2 \longrightarrow \bar{J}/\bar{J}^2$, as required. $\square$

Suppose that V is a kG-module which is free on restriction to Φ. We may consider the $k\bar{G}$-module, $\bar{V} = \mu_\Phi V$, as in (0.0.12); but $\bar{G}$ is elementary abelian so there is a Carlson variety, $Y(\bar{V})$. This is a subset of $\bar{J}/\bar{J}^2$, but by (3.6.1) we may regard it as a subset of J/J^2 in a natural way. We shall alternate between these two viewpoints without giving the matter too much thought.

For any subgroup, H, of G we let S_H denote the k-subspace of J/J^2 spanned by all elements of the form $(h-1) + J^2$ with $h \in H$.

__Lemma 3.6.2__ Let $h_1, \ldots, h_r$ be a minimal set of generators for H modulo $H \cap \Phi$. Then $(h_1 - 1) + J^2, \ldots, (h_r - 1) + J^2$ is a k-basis for S_H.

__Proof__ Note that $S_H = S_{H\Phi}$ is the image of the subspace $k \otimes \overline{H\Phi}$ under the isomorphism $\alpha : k \otimes \bar{G} \longrightarrow J/J^2$ given in (3.0.2). The isomorphism $H/(H \cap \Phi) \cong H\Phi/\Phi$ shows that $\bar{h}_1, \ldots, \bar{h}_r$ is a minimal set of generators of $\overline{H\Phi}$, i.e. if we think of $\bar{G}$ as an $\mathbb{F}_p$-space, $\bar{h}_1, \ldots, \bar{h}_r$ is an $\mathbb{F}_p$-basis for $\overline{H\Phi}$. Thus $\alpha(\bar{h}_1), \ldots, \alpha(\bar{h}_r)$ is a k-basis for S_H, and the result follows. □

In particular this implies that $(g_1 - 1) + J^2, \ldots, (g_n - 1) + J^2$ is a k-basis for $S_G = J/J^2$.

The following is the extension of Dade's lemma to this situation:

__Theorem 3.6.3__ Let V be a kG-module which is free on restriction to Φ and H be any subgroup of G. V is free on restriction to $H\Phi$ if and only if:

$$Y(\bar{V}) \cap S_H = \{0\}.$$

__Proof__ Let $h_1, \ldots, h_r$ be as in (3.6.2). $Y(\bar{V}) \cap S_H = \{0\}$ iff $\bar{V}$ is free on restriction to all nonzero elements of

$$k(\bar{h}_1 - \bar{1}) \oplus \ldots \oplus k(\bar{h}_r - \bar{1}).$$

But Dade's lemma applied to $\overline{H\Phi}$ shows that this happens iff $\bar{V}$ is free on restriction to $\overline{H\Phi}$. But (0.0.12) gives that this occurs iff V is free on restriction to $H\Phi$. Thus the result follows. □

A case that will be of particular interest is that when H is a maximal subgroup of G. (3.6.2) shows that S_H is a hyperplane (i.e. a subspace of co-dimension one) of J/J^2, but we also have the following description:

<u>Theorem 3.6.4</u> If $\underline{c} = (c_1, \dots, c_n)$ is a nonzero element of $\mathbb{F}_p^n$ then let $S(\underline{c})$ denote the hyperplane:

$$\left\{ \sum_i \lambda_i(g_i - 1) + J^2 \;\middle|\; \sum_i c_i\lambda_i = 0 \right\}$$

of J/J^2.

(a) Let H be a maximal subgroup of G and choose $g \in G - H$. Define $c_i \in \mathbb{F}_p$ by $g_i \in (gH)^{c_i}$ $(i = 1,2,\dots,n)$ then $S_H = S(c_1, \dots, c_n)$.

(b) For any nonzero element, $\underline{c}$, of $\mathbb{F}_p^n$ there exists a maximal subgroup, H, of G with $S(\underline{c}) = S_H$.

<u>Proof</u> (a) Write $g_i = (g^{c_i})h_i$ $(h_i \in H)$ then:

$$\sum_i \lambda_i(g_i - 1) + J^2 = \sum_i c_i\lambda_i(g - 1) + \sum_i \lambda_i(h_i - 1) + J^2$$

is an element of S_H iff $\sum_i c_i\lambda_i = 0$.

(b) Write $\underline{c} = (c_1, \dots, c_n)$ and let r be such that $c_r \neq 0$. Considering $c_r^{-1}\underline{c}$, we may assume that $c_r = 1$. Define:

$$H = \langle g_i g_r^{-c_i}, \Phi \mid i \neq r \rangle$$

then H is a maximal subgroup of G. Take $g = g_r$ in (a) then, by construction, $S_H = S(\underline{c})$. □

<u>Corollary 3.6.5</u> If V is free on restriction to Φ and $Y(\bar{V})$ is a line then V is free on restriction to one of the subgroups $H_1, \dots, H_n$.

<u>Proof</u> Let $\sum \lambda_i(g_i - 1) + J^2$ be a nonzero element of $Y(\bar{V})$ then $\lambda_i = 0$ iff this element is in $S_{H_i} = S(0,\dots,0,1,0,\dots,0)$. This happens iff V is not free on restriction to H_i, by (3.6.3). Choosing i with $\lambda_i \neq 0$ gives the result. □

§3.7 Modules which are free on restriction to a maximal subgroup

Now let V be a non-projective indecomposable kG-module which is free on restriction to some maximal subgroup, H_0, of G as in §3.1. The following may be considered to be the main theorem of this chapter:

Theorem 3.7.1 (a) V is free on restriction to one of $H_1, \dots, H_n$.

(b) $Y(\bar{V})$ is the line:

$$k\left(\sum_{i=1}^{n} \lambda_{H_i}(V)^{1/p}(g_i - 1) + J^2\right).$$

Proof Firstly suppose that $g_1', \dots, g_n'$ is a minimal set of generators for G such that, if H_i' is the maximal subgroup

$$\langle g_1', \dots, g_{i-1}', \Phi, g_{i+1}', \dots, g_n' \rangle$$

$(i = 1,2,\dots,n)$, V is free on restriction to one of $H_1', \dots, H_n'$. Without loss of generality, V is free on restriction to H_1'. We make a choice, possibly different from that made in §3.1, of representatives for $G - H$ for H a maximal subgroup of G, in which g_i' is the representative of $G - H_i'$. Proceed as in §3.1 with this set of representatives and H_1' in place of H_0 to obtain constants $\lambda_H' = \lambda_H'(V)$.

Note that, because it is free on restriction to a maximal subgroup, V is free on restriction to Φ. Because V is non-projective, $\bar{V}$ is not free – that is to say, by Dade's lemma, $Y(\bar{V}) \neq \{0\}$. Let ξ be any nonzero element of $Y(\bar{V})$ then we can write ξ in the form $\xi' + J^2$ where:

$$\xi' = \lambda_1(\bar{g}_1' - \bar{1}) + \dots + \lambda_n(\bar{g}_n' - \bar{1})$$

for some $\lambda_i \in k$. We claim that $\lambda_i = \lambda_1(\lambda_{H_i'}')^{1/p}$ for all i.

Note that $\lambda_{H_1'}' = 1$ so that the result is trivial for $i = 1$. It is also trivial if $\lambda_1 = \lambda_i = 0$. So assume otherwise and apply (3.3.1) with H_i' in place of H_0 and H_i' in place of H. Then

$$N = H_1' \cap H_i' = \langle g_2', \ldots, g_{i-1}', \hat{\ }, g_{i+1}', \ldots, g_n' \rangle$$

and we may take $x = g_1'N$, $y = g_i'N$. By (3.0.3), $\langle 1+\Gamma', N\rangle$ is a maximal subgroup of an essential subgroup of $k\bar{G}$. $\bar{V}$ is not free on restriction to $\langle 1+\Gamma'\rangle$ and hence not to $\langle 1+\Gamma', N\rangle$. Thus the image of $\langle 1+\Gamma'\rangle$ in $k(\bar{G}/\bar{N})$ does not act freely on $\mu_{\bar{N}}V = \mu_N V$. But the image of Γ' is $\lambda_1(x-1) + \lambda_i(y-1)$, thus (3.3.1) gives that $\lambda_i^p = \lambda_{H_i'}\lambda_1^p$ and the claim is proved.

Hence in these circumstances we have that:

$$Y(\bar{V}) = k\Big(\sum_{i=1}^{n} (\lambda_{H_i'})^{1/p}(g_i' - 1) + J^2\Big). \qquad -(1)$$

Now let us prove the result:

(a) By choosing $g_1', \ldots, g_n'$ suitably, we may assume that $H_1' = H_0$; thus the result above shows that $Y(\bar{V})$ is a line. Hence (3.6.5) gives that V is free on restriction to one of $H_1, \ldots, H_n$.

(b) Apply the result above with $g_1, \ldots, g_n$ as the set of generators. By (a), V is free on restriction to one of $H_1, \ldots, H_n$; without loss of generality, H_1. We do not need to change the representatives for $G - H$; the only difference in calculating the constants $\lambda_H(V)$ and $\lambda_H'(V)$ is that H_0 is the base in one, H_1 in the other. Thus consider the diagram:

$$\begin{tikzcd}
0 \arrow[r] & V \arrow[r] & k(G/H)\otimes V \arrow[r] & k(G/H)\otimes V \arrow[r] & V \arrow[r] & 0 \\
0 \arrow[r] & V \arrow[r] \arrow[u, dashed, "\alpha'"] & k(G/H_1)\otimes V \arrow[r] \arrow[u, dashed] & k(G/H_1)\otimes V \arrow[r] \arrow[u, dashed] & V \arrow[r] \arrow[u, equal] & 0 \\
0 \arrow[r] & V \arrow[r] \arrow[u, dashed, "\alpha"] & k(G/H_0)\otimes V \arrow[r] \arrow[u, dashed] & k(G/H_0)\otimes V \arrow[r] \arrow[u, dashed] & V \arrow[r] \arrow[u, equal] & 0 .
\end{tikzcd}$$

By definition, $\alpha - \lambda_{H_1} 1_V$ and $\alpha' - \lambda'_H 1_V$ are non-automorphisms of V, thus so is

$$\alpha'\alpha - \lambda_{H_1}\lambda'_H 1_V ;$$

But $\alpha'\alpha$ is a suitable map to calculate λ_H ; hence:

$$\lambda_H(V) = \lambda_{H_1}(V).\lambda'_H(V).$$

But, because V is free on restriction to H_1 , $\lambda_{H_1}(V)$ is a nonzero constant. Thus the result is seen to follow by substituting into (1). □

Corollary 3.7.2 Let H be a maximal subgroup of G and choose $g \in G - H$. Define $c_i \in \mathbb{F}_p$ by $g_i \in (gH)^{c_i}$. Then V is free on restriction to H if and only if:

$$c_1\lambda_{H_1}(V) + \dots + c_n\lambda_{H_n}(V) \neq 0.$$

Corollary 3.7.3 V is free on restriction to all the maximal subgroups of G if and only if $\lambda_{H_1}(V), \dots, \lambda_{H_n}(V)$ are $\mathbb{F}_p$-linearly independent.

Proofs Using (3.6.4)(a), (3.6.3) and (3.7.1)(b), V is free on restriction to H iff $c_1\lambda_{H_1}^{1/p} + \dots + c_n\lambda_{H_n}^{1/p} \neq 0$. But this is zero iff its pth power is zero, thus the result follows on noting that $c_i^p = c_i$.

(3.7.3) follows from (3.7.2) and (3.6.4)(b). □

In fact we can prove an improved version of (3.7.2): if g is the element of $G - H$ used in calculating $\lambda_H(V)$ then:

$$\lambda_H(V) = c_1\lambda_{H_1}(V) + \dots + c_n\lambda_{H_n}(V).$$

The result of (3.7.2) follows trivially from this and (3.3.2).

Proof We may assume, without loss of generality, that V is free on restriction to H_1. The result is trivial if $H = H_1$ so assume otherwise. As in (3.3.1) there exist $g_1' \in (g_1H_1)\cap H$, $g_2' \in (gH)\cap H_1$; also let $g_3', \ldots, g_n'$ be a minimal set of generators for $H\cap H_1$ modulo Φ. Write:

$$H_i' = \langle g_1', \ldots, g_{i-1}', \Phi, g_{i+1}', \ldots, g_n' \rangle$$

$(i = 1,2,\ldots,n)$ so that each H_i' is a maximal subgroup of G and $H_1' = H_1$, $H_2' = H$. We may change the representatives of $G-N$ for N a maximal subgroup of G so that g_i' represents $G-H_i'$. Also let H_1 be taken in place of H_0 to calculate the constants $\lambda_N'(V)$. (3.7.1)(b) with $g_1', \ldots, g_n'$ in place of $g_1, \ldots, g_n$ and H_1 in place of H_0 gives:

$$Y(\bar{V}) = k(\mathfrak{Z} + J^2) \quad \text{where} \quad \mathfrak{Z} = \sum_{i=1}^{n} (\lambda_{H_i'})^{1/p}(g_i' - 1).$$

If we think of $\bar{g}_1', \ldots, \bar{g}_n'$ as an $\mathbb{F}_p$-basis for $\bar{G}$ we see that there is a nonsingular $n\times n$ matrix (c_{ij}) with entries in $\mathbb{F}_p$ such that:

$$\bar{g}_i = \prod_{j=1}^{n} (\bar{g}_j')^{c_{ij}} \qquad -(1)$$

$(i = 1,2,\ldots,n)$. The isomorphism of (3.0.2) gives that:

$$(g_i - 1) + J^2 = \sum_{j=1}^{n} c_{ij}(g_j' - 1) + J^2$$

so by (3.7.1)(b) the following is a nonzero element of $Y(\bar{V})$:

$$\sum_{i,j} c_{ij}\lambda_{H_i}^{1/p}(g_j' - 1) + J^2$$

so this equals $c(\mathfrak{Z} + J^2)$ for some $0 \neq c \in k$. Since $(g_1' - 1) + J^2, \ldots, (g_n' - 1) + J^2$ are k-linearly independent, this gives for $j = 1,2,\ldots,n$:

$$c(\lambda_{H_j'})^{1/p} = \sum_{i=1}^{n} c_{ij}\lambda_{H_i}^{1/p}. \qquad -(2)$$

(1) gives that $g_iH_1 = (g_1'H_1)^{c_{i1}} = (g_1H_1)^{c_{i1}}$ so that c_{i1} equals 1 if $i = 1$ and 0 otherwise. Substituting this into (2) and noting that $\lambda_{H_1'} = 1$ gives that $c = \lambda_{H_1}^{1/p}$.

(1) also gives that $g_iH = (g_2'H)^{c_{i2}} = (gH)^{c_{i2}}$ so that, in the notation of (3.7.2), $c_{i2} = c_i$. Substituting into (2) gives:

$$\lambda_{H_1}^{1/p}(\lambda_{H_2'})^{1/p} = \sum_{i=1}^{n} c_i\lambda_{H_i}^{1/p}$$

but $c_i^p = c_i$ so:

$$\lambda_{H_1}\lambda_{H_2'} = \sum_{i=1}^{n} c_i\lambda_{H_i}.$$

But $H_1' = H_1$, $g_1'H_1' = g_1H_1$, $H_2' = H$, $g_2'H_2' = gH$ so we are in precisely the same situation as given by the diagram in the proof of (3.7.1)(b). Thus, as there, we have that $\lambda_{H_1}\lambda_{H_2'} = \lambda_H$. Hence:

$$\lambda_H(V) = \sum_{i=1}^{n} c_i\lambda_{H_i}(V) \quad \text{– as required.}$$ □

<u>Remark</u> Corollary 3.7.2, its improved version, and Corollary 3.7.3 also hold in the more general case when k is not necessarily algebraically closed but V is absolutely indecomposable. As in §3.1, we may extend the scalars to an algebraic closure of k, $\bar{k}$, to get a $\bar{k}G$-module, $\bar{V}$, with $\lambda_H(\bar{V}) = \lambda_H(V)$. The results in question all hold for $\bar{V}$, thus they must also hold for V.

The scalars $\lambda_{H_1}(V), \ldots, \lambda_{H_n}(V)$ all lie in k, so for V to be free on restriction to all the maximal subgroups of G, (3.7.3) shows that the dimension of k as an $\mathbb{F}_p$-space must be at least n. Thus we get the following fairly feeble result:

<u>Corollary 3.7.4</u> There cannot be an absolutely indecomposable kG-module which is free on restriction to all the maximal subgroups of G but is not itself projective, unless k has at least $|\bar{G}|$ elements. $\square$

§3.8 The elementary abelian case

We may follow [Ca2] in extending (3.2.2) to general elementary abelian groups:

Proposition 3.8.1 Let G be elementary abelian and V be the direct sum of a projective and a periodic kG-module. Then V is free on restriction to a maximal subgroup of some essential subgroup of kG.

Proof The proof is by induction on $|G|$. The result is trivial when G has order p. So assume that $|G| \geq p^2$ and let H be any maximal subgroup of G. Inductively, V is free on restriction to some maximal subgroup, N, of an essential subgroup of kH. We may choose an essential subgroup, G', of kG containing N. Let $U = p_N V$ be regarded as a $k(G'/N)$-module, then U is readily seen to be the direct sum of a projective and a periodic module. G'/N has order p^2 so, applying (3.2.2) and (3.2.5), the Carlson variety of U is the union of a finite number of lines, one for each indecomposable, periodic summand. Because k is infinite, we may choose a line which intersects the Carlson variety of U trivially. This line corresponds to a maximal subgroup of an essential subgroup of $k(G'/N)$ restricted to which U is free. Hence, lifting back to G', there is a maximal subgroup of an essential subgroup of kG restricted to which V is free. □

If V is in fact an indecomposable, periodic kG-module then (3.8.1) shows that we can apply (3.7.1)(b) to V to show that $Y(V)$ is a line - it does not matter that, in the statement of the theorem, G is replaced by another essential subgroup. Hence (3.6.5) shows that in fact V is free on restriction to a maximal subgroup of G. Thus we have the following result, also due to Carlson, [Ca3]:

<u>Theorem 3.8.2</u> Let G be elementary abelian and V be an indecomposable, periodic kG-module. Then:

(a) V is free on restriction to some maximal subgroup of G,

(b) the Carlson variety of V is a line.

In particular, V must have period 1 or 2. □

Suppose that we tried to use this result to prove (3.7.1) in the general case. We would certainly have that $\bar{V}$ is a module for the elementary abelian group, $\bar{G}$, which is free on restriction to some maximal subgroup. However we do not in general have that $\bar{V}$ is indecomposable, so the best we can say is $Y(\bar{V})$ is the union of a finite number of lines. The proof that these lines coincide would involve at least as much work as our original one.

To see that $\bar{V}$ is not always indecomposable we need the following lemma:

<u>Lemma 3.8.3</u> Suppose that G is not elementary abelian. Let:

$$0 \longrightarrow V \longrightarrow X \xrightarrow{\sigma} V \longrightarrow 0$$

be the almost split sequence with V as its right-hand term, then the induced sequence:

$$0 \longrightarrow V \longrightarrow \bar{X} \longrightarrow \bar{V} \longrightarrow 0$$

splits.

<u>Proof</u> Because $\Phi \neq 1$, the inclusion map $i : \mu_{\Phi}V \longrightarrow V$ is not a split epimorphism. Thus i factors as $\sigma i'$ for some $i' : \mu_{\Phi}V \longrightarrow X$. In fact i' maps $\mu_{\Phi}V$ into $\mu_{\Phi}X$. The composite of i' with the restriction:

$$\bar{\sigma} : \mu_{\Phi}X \longrightarrow \mu_{\Phi}V$$

is the identity map on $\mu_{\Phi}V$. Thus:

$$0 \longrightarrow \mu_g V \longrightarrow \mu_g X \longrightarrow \mu_g V \longrightarrow 0$$

splits as a sequence of kG-modules, and hence also splits as a sequence of $k\bar{G}$-modules. □

So let V_n $(n = 1,2,\ldots.)$ be as in (2.3.6). The result above readily shows that $\bar{V}_n$ is isomorphic to the direct sum of n copies of $\bar{V}_1$.

Therefore, when G is not elementary abelian, $\bar{V}$ is indecomposable only if V is terminal. Whether the converse holds is an open question. If it did then it would help us out of our difficulties in trying to deduce (3.7.1) from (3.8.2): we would have $Y(\bar{V}_n) = Y(\bar{V}_1)$ for all n, and, because $\bar{V}_1$ is indecomposable, that $Y(\bar{V}_1)$ is a line. Thus we would have that $Y(\bar{V})$ is always a line. Precisely which line is not important in most applications – the constants $\lambda_H(V)$ are really only a relic of the approach we used to the proof.

CHAPTER 4

THE SET OF PERMITTED VARIETIES

Introduction

We have shown that $Y(\bar{V})$ is always a line in J/J^2. We now let Y_G denote the union of all these lines as V runs over the various non-projective, indecomposable kG-modules which are free on restriction to a maximal subgroup, and ask what constraints, if any, are there on Y_G ?

A reduction is made to a particular class of p-groups, the pseudo-special groups (a rather unfortunate name of mixed Latin and Greek elements, so-called because the concept of a pseudo-special group is a generalisation of that of an extra-special group, but is different from that of a special group). The structure of these groups may be very accurately defined, so specific calculations involving them are possible. Using this reduction we are able to prove that Y_G is the whole of J/J^2 if and only if G is elementary abelian.

It is also shown that, if G is pseudo-special, Y_G is an $\mathbb{F}_p$-variety - that is, the solution set of a series of polynomials with coefficients in the field of p elements.

§4.0 The set Y_G

We continue with the notation of §3.6. Define Y_G to be the subset of J/J^2 given by the union of all the varieties, $Y(\bar{V})$, as V runs over the indecomposable kG-modules which are free on restriction to a maximal subgroup of G. Note that $Y(\bar{V})$ equals $\{0\}$ if V is projective and is a line otherwise.

<u>Theorem 4.0.1</u> If W is a kG-module which is free on restriction to Φ then, regarding $\bar{W} = \mu_{\Phi}W$ as a $k\bar{G}$-module, $Y(\bar{W}) \subseteq Y_G$.

<u>Proof</u> For $\alpha \in J-J^2$ we may define the $k\bar{G}$-module, U_α, to be $\alpha^{p-1}k\bar{G}$, as in §3.5. We may regard U_α as a kG-module by letting Φ act trivially. Consider $U_\alpha \otimes W$: this is free on restriction to Φ and (0.0.13) gives that:

$$\overline{U_\alpha \otimes W} = U_\alpha \otimes \bar{W}.$$

Thus, by (3.5.6),

$$Y(\overline{U_\alpha \otimes W}) = \begin{cases} k(\alpha + J^2) & \text{if } \alpha + J^2 \in Y(\bar{W}), \\ \{0\} & \text{otherwise.} \end{cases}$$

Thus, if V is an indecomposable direct summand of $U_\alpha \otimes W$, V is free on restriction to Φ and $Y(\bar{V})$ is either $\{0\}$ or $k(\alpha + J^2)$. Hence V is free on restriction to some maximal subgroup of G – it is free if $Y(\bar{V}) = \{0\}$, otherwise apply (3.6.5). Thus $Y(\bar{V}) \subseteq Y_G$. But $Y(\overline{U_\alpha \otimes W})$ is the union of the $Y(\bar{V})$'s as V runs over all the indecomposable summands, thus this too is contained within Y_G.

Now the result above makes it clear that $Y(\bar{W})$ is the union of the varieties $Y(\overline{U_\alpha \otimes W})$ as α runs over $J-J^2$. Thus $Y(\bar{W}) \subseteq Y_G$, as required. □

<u>Corollary 4.0.2</u> If G is elementary abelian then Y_G is the whole of J/J^2.

<u>Proof</u> Put $W = k_G$ in (4.0.1). In this case $\Phi = 1$ so W is indeed free on restriction to Φ. □

This notation enables us to express conditions for projectivity in a fairly economical form. For example:

<u>Proposition 4.0.3</u> Let $\underline{H}$ be a set of subgroups of G. Then:

$$Y_G \subseteq \bigcup_{H \in \underline{H}} S_H$$

if and only if there is no non-projective kG-module which is free on restriction to $H\Phi$ for all $H \in \underline{H}$.

<u>Proof</u> If the condition on Y_G holds, let V be a kG-module which is free on restriction to all the $H\Phi$'s. Then V is free on restriction to Φ and $Y(\bar{V}) \cap S_H = \{0\}$ (by (3.6.3)). Thus $Y(\bar{V})$ intersects Y_G trivially and so must itself be trivial. Thus V is free, as required.

Conversely, let V be a non-projective indecomposable kG-module which is free on restriction to some maximal subgroup of G. V must be non-projective on restriction to $H\Phi$ for some $H \in \underline{H}$, so that the line, $Y(\bar{V})$, is contained within S_H. Thus the condition on Y_G holds. □

As an example of the applications of this result, we have that G is a Chouinard group if and only if:

$$Y_G \subseteq \bigcup_{H} S_H$$

– the union being over all the maximal subgroups of G.

§4.1 Pseudo-special groups

Suppose that we attempted to prove Chouinard's theorem by induction on the order of G: let G be a group of minimal order which is neither elementary abelian nor a Chouinard group. If N is any non-trivial normal subgroup of G then, by minimality, G/N is either elementary abelian or a Chouinard group. The latter possibility is excluded by (0.1.2), so we must have that G/N is elementary abelian and, consequently, that $N \geq \Phi$. Therefore, because G is not elementary abelian, Φ is a non-trivial normal subgroup of G which is contained within all the other non-trivial normal subgroups. Thus we are motivated to make the following definition:

A finite p-group, G, is said to be *pseudo-special* if the Frattini subgroup, $\Phi = \Phi(G)$, is the unique minimal normal subgroup of G.

Now turn to the general case. Any non-trivial normal subgroup of a p-group intersects the centre of the group non-trivially. Thus any minimal normal subgroup is contained within the centre. Hence we have:

Lemma 4.1.1 The minimal normal subgroups of a p-group, G, are precisely the subgroups of the centre of G, Z(G), of order p. In particular, G has a unique minimal normal subgroup if and only if Z(G) is cyclic. □

Lemma 4.1.2 If G is not elementary abelian then there exists a normal subgroup, N, of G with $N < \Phi$ such that the Frattini subgroup of G/N is a minimal normal subgroup. In fact, $\Phi(G/N) = \Phi/N$.

Proof Because G is not elementary abelian, we have that $\Phi \neq 1$. Let $\underline{M}$ be the set of all maximal subgroups of Φ. Because Φ is a normal subgroup of G, G acts by conjugation to permute the elements of $\underline{M}$. $\underline{M}$ is in 1-1 correspondence with the set of $\mathbb{F}_p$-hyperplanes of the elementary abelian group, $\Phi/\Phi(\Phi)$. Thus the size of $\underline{M}$ is congruent to 1 modulo p. The size of each of the orbits of G on $\underline{M}$ divides the order of G, and hence is a power of p. $\underline{M}$ is the disjoint union of these orbits, so there must be an orbit of size one. That is, there is a maximal subgroup, N, of Φ which is normal in G. The maximal subgroups of G/N are precisely the H/N's for H a maximal subgroup of G; thus $\Phi(G/N) = \Phi/N$ and the result easily follows. □

<u>Theorem 4.1.3</u> Let G be a p-group which is not elementary abelian. There exist normal subgroups, P,E,N, of G with $N \leqslant \Phi \leqslant P$, $N \leqslant E$ such that:

P/N is pseudo-special with Frattini subgroup Φ/N,

E/N is elementary abelian,

G/N is the direct product of P/N and E/N.

<u>Proof</u> Let N be as in the previous lemma. By considering G/N, it suffices to assume that $\Phi(G)$ is a minimal normal subgroup of G. Φ is contained within Z(G); we readily see that we may write $Z(G) = E \times C$ where C is cyclic and contains Φ. E is isomorphic to $E\Phi/\Phi$ and is thus elementary abelian.

We may choose $P \geqslant C$ with $G/\Phi = (E\Phi/\Phi) \times (P/\Phi)$. Then $G = EP$ and, by considering orders, $E \cap P = 1$. Thus G is the direct product, $E \times P$. Now:

$$Z(G) = Z(E) \times Z(P) = E \times Z(P)$$

and
$$\Phi(G) = \Phi(E) \times \Phi(P) = \Phi(P)$$

so that $Z(P) = C$, $\Phi(P) = \Phi$. P has a cyclic centre so, by (4.1.1), it has a unique minimal normal subgroup - namely Φ. Thus P is pseudo-special, and the result follows. □

With the notation of this theorem, G/Φ is the direct product of P/Φ and $E\Phi/\Phi$. Thus (3.6.2) makes it clear that:

$$J/J^2 = S_P \oplus S_E .$$

Write J' for the augmentation ideal of $k(P/N)$. (3.0.2) shows that the map $x \longmapsto x-1$ induces a k-isomorphism $k \otimes Q \longrightarrow J'/J'^2$ where

$$Q = (P/N)/\Phi(P/N) = (P/N)/(\Phi/N) \cong P/\Phi = \bar{P}.$$

But $k \otimes \bar{P}$ may be identified with S_P using the k-isomorphism $k \otimes \bar{G} \longrightarrow J/J^2$. Thus there is a natural k-isomorphism between J'/J'^2 and S_P. Hence we may regard the subset, $Y_{P/N}$, of J'/J'^2 as a subset of S_P. If this is done then:

<u>Theorem 4.1.4</u> $Y_G \subseteq Y_{P/N} \oplus S_E$.

<u>Proof</u> Let V be a non-projective, indecomposable kG-module which is free on restriction to some maximal subgroup of G. We may write:

$$Y(V) = k(\mathfrak{x}_1 + \mathfrak{x}_2 + J^2)$$

where $\mathfrak{x}_1 \in \mathrm{Aug}(kP)$, $\mathfrak{x}_2 \in \mathrm{Aug}(kE)$. It suffices to show that the image of $\mathfrak{x}_1$ in J'/J'^2 is in $Y_{P/N}$. This is trivial if $\mathfrak{x}_1 + J^2 = 0$, so assume otherwise. Then $Y(V) \cap S_E = \{0\}$ so, by (3.6.3), V is free on restriction to $E\Phi$, and hence to E.

Let $U = \mu_E V$ be regarded as a $k(G/E)$-module. The Frattini subgroup of G/E equals $E\Phi/E$ so:

$$\mu_{\Phi(G/E)}U = \mu_{E\Phi}V.$$

This implies that U is free on restriction to $\Phi(G/E)$. Thus we consider the Carlson variety of $\mu_{\Phi(G/E)}U$.

Because $\mathfrak{x}_1 + J^2 \neq 0$, the group generated by $E\Phi$ and $\overline{1 + \mathfrak{x}_1 + \mathfrak{x}_2}$ is contained within some essential subgroup of $k\bar{G}$ (see (3.0.3)). This group does not act freely on $\bar{V}$ since $\langle \overline{1 + \mathfrak{X}_1 + \mathfrak{X}_2} \rangle$ does not. Thus the image of $\langle 1 + \mathfrak{X}_1 + \mathfrak{X}_2 \rangle$ does not act freely on $\mu_{\bar{E}*}\bar{V} = \mu_{E*}V = \mu_{*(G/E)}U$. But $\mathfrak{X}_2 U = 0$ so we must have that the image of $\mathfrak{X}_1$ is in the Carlson variety of $\mu_{*(G/E)}U$, and hence in $Y_{G/E}$.

Now, we may identify P/N with $(G/N)/(E/N) \cong G/E$, so the result is seen to follow. □

This theorem enables us to reduce to the case when G is pseudo-special. We proceed in the following sections to investigate Y_G in this case. This is done by considering the modules, $U_{\mathfrak{X}}$, introduced in the proof of (4.0.1).

§4.2 The modules U_{ξ} when $\Phi(G)$ is cyclic

We assume in this section that G is such that the Frattini subgroup of G is a non-trivial cyclic group. In particular, this includes the case when G is pseudo-special.

Fix $\xi \in J-J^2$ and let $U = U_{\xi}$ be as in §4.0. (3.5.4) gives that, when considered as a $k\bar{G}$-module, U has Carlson variety $k(\xi + J^2)$. Thus, as in (3.6.5), U is free on restriction to some maximal subgroup, $\bar{H}$, of $\bar{G}$ – indeed, by dimensions, $U_{\downarrow \bar{H}} \cong k\bar{H}$. Therefore:

$$U_{\downarrow H} \cong k\bar{H} \cong k_{\Phi}^{\uparrow H}.$$

Choose $u_0 \in U$ such that $U = k\bar{H}u_0$.

Let e be a generator for Φ then, as in §1.0, there is an exact sequence of $k\Phi$-modules:

$$0 \longrightarrow k_{\Phi} \longrightarrow k\Phi \longrightarrow k\Phi \longrightarrow k_{\Phi} \longrightarrow 0$$

in which the central map is given by $1 \longmapsto e-1$. Inducing this sequence up to H gives an exact sequence of kH-modules:

$$0 \longrightarrow U_{\downarrow H} \longrightarrow kH \longrightarrow kH \longrightarrow U_{\downarrow H} \longrightarrow 0$$

in which the maps are given by:

$$u_0 \longmapsto \mu_{\Phi}\ ,\ 1 \longmapsto e-1\ ,\ 1 \longmapsto u_0.$$

Note also that, because $\Phi \neq 1$, U is not free on restriction to H. Thus $U_{\downarrow H}$ is periodic.

Now take the sequence:

$$0 \longrightarrow k_G \longrightarrow k(G/H) \longrightarrow k(G/H) \longrightarrow k_G \longrightarrow 0$$

and tensor it by U to obtain:

$$0 \longrightarrow U \longrightarrow P \longrightarrow P \longrightarrow U \longrightarrow 0 \qquad -(1)$$

where $P = k(G/H) \otimes U \cong (U_{\downarrow H})^{\uparrow G} \cong k_{\Phi}^{\uparrow G}$.

Restricting (1) to H and using the projective resolution of $U_{\downarrow H}$ above, we may form the commutative diagram of kH-modules:

$$\begin{tikzcd}
0 \arrow[r] & U_{\downarrow H} \arrow[r] & P_{\downarrow H} \arrow[r] & P_{\downarrow H} \arrow[r] & U_{\downarrow H} \arrow[r] & 0 \\
0 \arrow[r] & U_{\downarrow H} \arrow[r] \arrow[u, dashed, "\gamma"] & kH \arrow[r, "\mu"] \arrow[u, dashed, "\beta"] & kH \arrow[r] \arrow[u, dashed, "\alpha"] & U_{\downarrow H} \arrow[r] \arrow[u, equal] & 0 .
\end{tikzcd} \qquad -(2)$$

We may define α by $1 \longmapsto H \otimes u_0$. Then $\alpha\mu(1) = (e-1)(H \otimes u_0) = 0$ so that we may take β and γ to be the zero maps.

We are now in a position to prove the main result of this section.

<u>Theorem 4.2.1</u> Suppose that G has a non-trivial, cyclic Frattini subgroup and that $\xi \in J-J^2$ is such that U_ξ is periodic when regarded as a kG-module. Then $\xi + J^2 \notin Y_G$.

<u>Proof</u> We may form the commutative diagram of kG-modules:

$$\begin{tikzcd}
0 \arrow[r] & U \arrow[r] & P \arrow[r] & \cdots\cdots \arrow[r] & P \arrow[r] & U \arrow[r] & 0 \\
0 \arrow[r] & U \arrow[r] \arrow[u, dashed, "\varepsilon"] & F_{2r} \arrow[r] \arrow[u, dashed] & \cdots\cdots \arrow[r] & F_1 \arrow[r] \arrow[u, dashed] & U \arrow[r] \arrow[u, equal] & 0
\end{tikzcd} \qquad -(3)$$

where the lower sequence is a projective resolution of U and the upper sequence is the join of r copies of the sequence, (1). Restricting to H and comparing with (2) shows that ε factors through a projective kH-module. Thus ε is a non-automorphism of the indecomposable module, $U_{\downarrow H}$, and is hence nilpotent.

If we form a similar diagram to (3), but m times as long, then the effect on the left-hand map is to replace ε by ε^m. But, taking m sufficiently large, $\varepsilon^m = 0$. Thus it suffices to assume that $\varepsilon = 0$ in (3).

Let W be any kG-module which is free on restriction to Φ. We tensor the diagram (3) by W to obtain:

$$\begin{tikzcd}
0 \arrow[r] & U \otimes W \arrow[r] & P \otimes W \arrow[r] & \cdots \arrow[r] & P \otimes W \arrow[r] & U \otimes W \arrow[r] & 0 \\
0 \arrow[r] & U \otimes W \arrow[r] \arrow[u, "0"] & F_{2r} \otimes W \arrow[r] \arrow[u] & \cdots \arrow[r] & F_1 \otimes W \arrow[r] \arrow[u] & U \otimes W \arrow[r] \arrow[u, equal] & 0 .
\end{tikzcd}$$

But each $F_i \otimes W$ is projective, as is $P \otimes W$ - for this is isomorphic to $k_\Phi^{\uparrow G} \otimes W \cong (W_{\downarrow\Phi})^{\uparrow G}$ - thus both sequences in this diagram are projective resolutions of $U \otimes W$. The only way this can happen is if $U \otimes W$ is projective. Therefore:

$$Y(\overline{U_\Phi \otimes W}) = \{0\}$$

so that, as in the proof of (4.0.1), $I + J^2 \notin Y(\bar{W})$. Hence we have the required result. □

<u>Corollary 4.2.2</u> Suppose that G is either a cyclic group of order at least p^2 or the quaternion group, Q_8. Then $Y_G = \{0\}$. Therefore a kG-module is free if and only if it is free on restriction to the Frattini subgroup of G.

<u>Proof</u> We readily see that G satisfies the required conditions in these cases. Furthermore, the trivial module, k_G, is periodic (see §1.0 and §1.3). Hence every kG-module is periodic, so the result easily follows from (4.2.1). □

There is not much hope of extending Theorem 4.2.1 to any more general class of groups. For, if U_Φ is to be periodic, it must be periodic or projective on restriction to Φ. But Φ acts trivially on U_Φ, so we must have that k_Φ is either periodic or projective. This happens only if Φ is cyclic or generalised quaternion (see [C&E]). The only case not dealt with in the former case is that with $\Phi = 1$, i.e. with G elementary abelian. But (4.0.2) then gives that $I + J^2$ is always an element of Y_G. The latter case - the class of groups with a generalised quaternion Frattini subgroup - does not seem a promising object for study.

In fact, (4.2.2) does not give a different proof of Chouinard's theorem for cyclic groups from those already given. For when G is cyclic, $U_\xi \cong k_G$ so that, in this case, the proof is equivalent to showing that G is a Serre group (see §1.2).

Obviously (4.2.1) means that we are going to be interested in projective resolutions of the modules, U_ξ. The following lemma gives a small step in this direction.

<u>Lemma 4.2.3</u> (a) The epimorphism $kG \longrightarrow U$, given by $1 \longmapsto u_0$, has kernel $kG(e-1) + kG\xi$.

(b) The element, $u_1 = \mu_\xi \xi^{p-1}$, of kG generates a submodule isomorphic to U.

<u>Proof</u> Inducing the projective resolution for k_ξ above up to G gives us an exact sequence:

$$0 \longrightarrow k\bar{G} \longrightarrow kG \longrightarrow kG \longrightarrow k\bar{G} \longrightarrow 0$$

in which the maps are given by:

$$\bar{1} \longmapsto \mu_\xi, \quad 1 \longmapsto e-1, \quad 1 \longmapsto \bar{1}.$$

(a) The given map is the composite of the map $kG \longrightarrow k\bar{G}$ above – which has kernel $kG(e-1)$ – and the $k\bar{G}$-epimorphism, $k\bar{G} \longrightarrow U$ given by $\bar{1} \longmapsto u_0$. Because $k\bar{G}$ is free as a $k\langle\overline{1+\xi}\rangle$-module, the latter is readily seen to have kernel $k\overline{G\xi}$. Thus the kernel of the given map is $kG(e-1) + kG\xi$, as claimed.

(b) We may regard U as the submodule of $k\bar{G}$ generated by ξ^{p-1}. The result follows on mapping by the monomorphism $k\bar{G} \longrightarrow kG$ given above. □

§4.3 The structure of pseudo-special groups

Before we can attempt to apply (4.2.1), we need to know a bit more about the structure of pseudo-special groups. If G is pseudo-special then we have the following facts:

(1) Φ and G/Φ both have exponent p, so G must have exponent dividing p^2.

(2) Z(G) is cyclic with Φ as its (unique) subgroup of order p.

(3) The derived subgroup, G', of G is a normal subgroup which is contained within Φ. Thus, by minimality, either $G' = 1$ or $G' = \Phi$. In the former case G is abelian and thus, using (1) and (2), cyclic of order p^2.

Thus G falls into one of two classes:

(a) $Z(G) = G' = \Phi$, i.e. G is *extra-special*.

or (b) Z(G) is cyclic of order p^2.

We now proceed to extend the well-known classification theorem of extra-special groups given, for example, in [Gor], to pseudo-special groups. The trick needed for this is given as part (a) of the following result:

Lemma 4.3.1 (a) Regard $\bar{G}$ as an $\mathbb{F}_p$-space and fix a generator, e, for Φ. Define $\langle -,- \rangle : \bar{G} \times \bar{G} \longrightarrow \mathbb{F}_p$ by:

$$\langle \bar{x}, \bar{y} \rangle = a \quad \text{where } 0 \leq a < p \text{ is such that } [x,y] = e^a.$$

(Here [x,y] denotes the commutator, $x^{-1}y^{-1}xy$.) Then $\langle -,- \rangle$ is a symplectic bilinear form on $\bar{G}$.

(b) For p odd, the map $G \longrightarrow \Phi$ given by $x \longmapsto x^p$ is a group homomorphism. Thus either G has exponent p or the elements of order p form a maximal subgroup of G.

(c) For $p = 2$, G always has exponent 4.

<u>Proof</u> (a) Note firstly that a exists because $[x,y] \in G' \leqslant \Phi$; it depends only on the cosets, $\bar{x}$ and $\bar{y}$, because $\Phi \leqslant Z(G)$. Thus $\langle -,- \rangle$ is well-defined. Now:

$$[xy,z] = y^{-1}(x^{-1}z^{-1}xz)z^{-1}yz = [x,z][y,z]$$

– using the fact that $G' \leqslant Z(G)$. Thus $\langle -,- \rangle$ is seen to be linear in the first variable. Similarly it is linear in the second. Finally, for all $x \in G$, $[x,x] = 1$ implies that $\langle \bar{x},\bar{x} \rangle = 0$ – that is to say, $\langle -,- \rangle$ is symplectic.

(b) For $x,y \in G$ we prove inductively that:

$$(xy)^i = x^i y^i [y,x]^s \quad \text{where } s = 1+2+\ldots.+(i-1).$$

The inductive step is as follows:

$$\begin{aligned}(xy)^{i+1} &= xy.x^i y^i [y,x]^s \\ &= x^{i+1} y(y^{-1}x^{-i}yx^i)y^{-1}.y^{i+1}[y,x]^s \\ &= x^{i+1}[y,x^i]y^{i+1}[y,x]^s \\ &= x^{i+1}y^{i+1}[y,x]^{s+i}\end{aligned}$$

– using (a) and the fact that $G' \leqslant Z(G)$. Thus this result is established.

Substituting $i = p$, we have that $s = 1+2+\ldots.+(p-1) = \frac{1}{2}p(p-1)$ is divisible by p (since p is odd). Since G' has exponent p, this implies that $(xy)^p = x^p y^p$. Thus the map $x \longmapsto x^p$ is indeed a group homomorphism $G \longrightarrow \Phi$. The kernel of this map, namely the set of all elements of G of order p (plus the identity element), is then a subgroup with index dividing $|\Phi| = p$. Thus the result of part (b) follows.

(c) Suppose that G has exponent 2. For $x,y \in G$ we have:

$$[x,y] = x^{-1}y^{-1}xy = xyxy = (xy)^2 = 1.$$

Thus G must be elementary abelian – a contradiction. Hence G must have exponent 4. $\square$

We need to know a little of the general theory of symplectic forms. Let V be a vector space over some field, K, and $\langle -,- \rangle$ be a symplectic form on V. If U is a subspace of V then we write:

$$U^{\perp} = \left\{ v \in V \;\middle|\; \langle v,u \rangle = 0 \text{ for all } u \in U \right\},$$

$$\mathrm{Rad}\, U = U \cap U^{\perp}$$

so that $U^{\perp}$ and $\mathrm{Rad}\, U$ are both subspaces of V. If $\mathrm{Rad}\, U = 0$ then we say that $\langle -,- \rangle$ is *nonsingular* on U. A two-dimensional subspace of V is called a *hyperbolic plane* if it has a K-basis, v_1, v_2, with $\langle v_1, v_2 \rangle = 1$. Clearly $\langle -,- \rangle$ is nonsingular on any hyperbolic plane.

Lemma 4.3.2 For any subspace, U, of V we have:

$$\dim_K U + \dim_K U^{\perp} \geqslant \dim_K V.$$

Thus, if $\langle -,- \rangle$ is nonsingular on U, $V = U \oplus U^{\perp}$.

Proof Take a K-basis, $u_1, \ldots, u_r$, for U ($r = \dim_K U$). Then:

$$U^{\perp} = \bigcap_{i=1}^{r} (Ku_i)^{\perp}.$$

But $(Ku_i)^{\perp}$ is the kernel of the K-linear map $V \longrightarrow K$ given by $v \longmapsto \langle v, u_i \rangle$ and so has dimension at least $\dim_K V - 1$. Thus $U^{\perp}$ has dimension at least $\dim_K V - r$, and the result follows. □

Theorem 4.3.3 There exist hyperbolic planes, H_i ($i = 1,2,\ldots,m$), with $\langle H_i, H_j \rangle = 0$ ($i \neq j$) such that:

$$V = (\mathrm{Rad}\, V) \oplus H_1 \oplus H_2 \oplus \cdots \oplus H_m .$$

Proof The proof is by induction on $\dim_K V$. The result is trivial if $\mathrm{Rad}\, V$ is the whole of V, so assume otherwise. There then exist $u, v \in V$ with

$\langle u,v\rangle \neq 0$. Thus u and $\frac{1}{\langle u,v\rangle}v$ span a hyperbolic plane, H_1. (4.3.2) gives that $V = H_1 \oplus H_1^\perp$ and, inductively, we can write:

$$H_1^\perp = (\mathrm{Rad}\ H_1^\perp) \oplus H_2 \oplus \ldots\ldots \oplus H_m$$

for some mutually perpendicular hyperbolic planes, $H_2,\ldots,H_m$. Because H_i is contained within $H_1^\perp$ for $i \neq 1$, we further have that $\langle H_1,H_i\rangle = 0$. Thus $H_1,H_2,\ldots,H_m$ are mutually perpendicular hyperbolic planes with:

$$V = (\mathrm{Rad}\ H_1^\perp) \oplus H_1 \oplus H_2 \oplus \ldots\ldots \oplus H_m .$$

But:

$$\mathrm{Rad}\ V = V^\perp = (H_1 \oplus H_1^\perp)^\perp = H_1^\perp \cap (H_1^\perp)^\perp = \mathrm{Rad}\ H_1^\perp$$

so that we have in fact written V in the required form. □

<u>Corollary 4.3.4</u> Let G be a pseudo-special p-group. There exist subgroups, P_i $(i=1,2,\ldots,m)$, of G satisfying:

(a) Each P_i is extra-special of order p^3 with $\Phi(P_i) = \Phi$,

(b) $[P_i,P_j] = 1$ for $i \neq j$,

(c) $Z(G) \cap (P_1P_2\ldots.P_m) = \Phi$,

(d) $P_i \cap (Z(G)P_1\ldots.P_{i-1}P_{i+1}\ldots.P_m) = \Phi$ for each i,

such that $G = Z(G)P_1P_2\ldots.P_m$.

<u>Proof</u> This is simply a matter of applying (4.3.3) to the symplectic form given in (4.3.1)(a). Rad $\bar{G}$ is seen to be just $\overline{Z(G)}$. Each of the hyperbolic planes, H_i, is the subgroup of $\bar{G}$ generated by $\bar{x}$ and $\bar{y}$ for some elements, x and y, of G, satisfying $[x,y] = e$. That is to say, $H_i = \bar{P}_i$ where P_i has the properties given in (a). $\langle H_i,H_j\rangle = 0$ implies that $[P_i,P_j] = 1$.

The fact that $\bar{G}$ is the direct sum of Rad $\bar{G}$ and the H_i's implies that:

$$\bar{G} = \overline{Z(G)} \times \bar{P}_1 \times \bar{P}_2 \times \ldots\ldots \times \bar{P}_m$$

from which the remaining statements are clear. □

This result enables us to introduce some notation that will be used in the following sections. G will be a pseudo-special group with Frattini subgroup, Φ, generated by e, say. We may write G in the form given in (4.3.4). For each $i = 1,2,\dots,m$, we may choose generators, x_i and y_i, for P_i satisfying:

$$[x_i, y_i] = e.$$

Note that x_i and y_i commute with x_j and y_j for $i \neq j$. Write:

$$g_{2i-1} = x_i , \; g_{2i} = y_i .$$

We then have two cases:

(a) If G is extra-special, write $n = 2m$. Then $g_1,\dots,g_n$ is a minimal set of generators for G.

(b) When Z(G) is cyclic of order p^2, $g_1,\dots,g_{2m}$ only generate $P_1P_2\dots P_m$. Thus choose a generator, z, for Z(G) such that $z^p = e$. Write $n = 2m+1$ and $g_n = z$. Then $g_1,\dots,g_n$ is a minimal set of generators for G.

Hence we have chosen a particularly well-behaved minimal set of generators for G; indeed the only thing to prevent us writing down a presentation for G in terms of these generators is that we have no information at present about the pth powers of the g_i's. This may be rectified as follows:

Define a map $a : G \longrightarrow \mathbb{F}_p$ by $a(x) = c$ where $x^p = e^c$ (note that x^p is always an element of Φ since G/Φ has exponent p).

Now write:

$$a_i = a(x_i) \; , \; b_i = a(y_i).$$

Then G is in fact completely determined by n and the constants, a_i and b_i $(i = 1,2,\dots,m)$.

Every nonzero element of J/J^2 may be written in the form $\xi + J^2$ where:

$$\xi = \sum_{i=1}^{m} (\alpha_i(x_i - 1) + \beta_i(y_i - 1)) + \gamma(z - 1)$$

(the final term is to be ignored in the case when G is extra-special). Here α_i, β_i $(i = 1,2,\ldots,m)$ and γ are elements of k. In the following sections we shall investigate what conditions on these elements suffice for U_ξ to be periodic. This will, using (4.2.1), give us conditions which must be satisfied for $\xi + J^2$ to be an element of Y_G.

§4.4 Some calculations

Before we go any further we need to do a couple of moderately unpleasant calculations. The first two lemmata are taken almost without change from [Ca5]. Let P be an extra-special group of order p^3 with Frattini subgroup E. Take generators, x and y, for P and write $e = [x,y]$. We shall consider the element:

$$\xi = \alpha(x-1) + \beta(y-1)$$

of kP.

We need one further piece of notation. Recall that, for $i = 1,2,\ldots,p-1$, the binomial coefficient, $\binom{p}{i}$, is divisible by p. Write:

$$R_i = \frac{1}{p}\binom{p}{i}.$$

Lemma 4.4.1 $\xi^p = \alpha^p(x^p-1) + \beta^p(y^p-1) + \mu_E t$ where:

$$t = \sum_{i=1}^{p-1} R_i \alpha^i \beta^{p-i} x^i y^{p-i}.$$

Proof $\xi = (\alpha x + \beta y) - (\alpha + \beta)1$ so $\xi^p = (\alpha x + \beta y)^p - (\alpha^p + \beta^p)1$. The coefficient of $\alpha^i\beta^{p-i}$ in the expansion of $(\alpha x + \beta y)^p$, w_i, equals the sum of all $\binom{p}{i}$ possible words containing i x's and $(p-i)$ y's. Note that each such word may be expressed in the form $e^j x^i y^{p-i}$ for some $j = 0,1,\ldots,p-1$.

Let us consider the effect on a constituent word of w_i of moving the left-most letter, step by step, to the right-hand end:

If the first letter is "x" then passing each of the $(p-i)$ y's multiplies by a factor of $[x,y]$; passing each of the $(i-1)$ other x's has no effect. Thus the total effect is to multiply by $[x,y]^{p-i}$.

Similarly, if the first letter is "y", the effect is to multiply by $[y,x]^i$.

Thus in both cases the effect is to multiply by e^{-i}. But this operation clearly just permutes the constituent words of w_i. This shows that the number of words equal to $e^j x^i y^{p-i}$ equals the number equal to $e^{j-1} x^i y^{p-i}$ for each j. Thus for $i = 1,2,\dots,p-1$ there must be an equal number of the $\binom{p}{i}$ words equal to $e^j x^i y^{p-i}$ for each j. Therefore:

$$w_i = R_i(1 + e + \dots + e^{p-1})x^i y^{p-i} = R_i p_E x^i y^{p-i}.$$

Finally, $w_0 = y^p$ and $w_p = x^p$, so the result follows. $\square$

<u>Lemma 4.4.2</u> If t is as in the previous lemma then there exists $q \in kP$ such that $\xi t - t\xi = (e-1)q$. If $\varepsilon\colon kP \longrightarrow k$ denotes the augmentation map then:

$$\varepsilon(q) = \alpha^p \beta - \alpha\beta^p.$$

<u>Proof</u>

$$\begin{aligned}
\xi t - t\xi &= \alpha(xt - tx) + \beta(yt - ty) \\
&= \sum_{i=1}^{p-1} R_i \alpha^{i+1}\beta^{p-i} x^{i+1} y^{p-i}(1 - e^i) \\
&\qquad + \sum_{i=1}^{p-1} R_i \alpha^i \beta^{p-i+1} x^i y^{p-i+1}(e^{p-i} - 1) \\
&= (e-1)q
\end{aligned}$$

where:

$$\begin{aligned}
q = &-\sum_{i=1}^{p-1} R_i \alpha^{i+1}\beta^{p-i} x^{i+1} y^{p-i}(1 + e + \dots + e^{i-1}) \\
&+ \sum_{i=1}^{p-1} R_i \alpha^i \beta^{p-i+1} x^i y^{p-i+1}(1 + e + \dots + e^{p-i-1}).
\end{aligned}$$

Now:

$$\begin{aligned}
\varepsilon(q) &= -\sum_{i=1}^{p-1} R_i \alpha^{i+1}\beta^{p-i} i + \sum_{i=1}^{p-1} R_i \alpha^i \beta^{p-i+1}(p - i) \\
&= -(\alpha + \beta)\Big(\sum_{i=1}^{p-1} i R_i \alpha^i \beta^{p-i}\Big).
\end{aligned}$$

But $iR_i = \frac{i}{p}\binom{p}{i} = \binom{p-1}{i-1}$ so:

$$\begin{aligned}\varepsilon(q) &= -\alpha(\alpha+\beta)\Big(\sum_{i=1}^{p-1}\binom{p-1}{i-1}\alpha^{i-1}\beta^{(p-1)-(i-1)}\Big)\\ &= -\alpha(\alpha+\beta)((\alpha+\beta)^{p-1}-\alpha^{p-1})\\ &= \alpha^{p}(\alpha+\beta) - \alpha(\alpha+\beta)^{p}\\ &= \alpha^{p}\beta - \alpha\beta^{p}.\end{aligned}$$

$\square$

We now apply these results to the situation described at the end of the previous section. With this notation we have:

<u>Theorem 4.4.3</u> (a) $\zeta^p = \sum_{i=1}^{m}(\alpha_i^p(x_i^p - 1) + \beta_i^p(y_i^p - 1)) + \gamma^p(e-1) + \mu_\beta t$

where:

$$t = \sum_{i=1}^{m}\sum_{j=1}^{p-1} R_j \alpha_i^j \beta_i^{p-j} x_i^j y_i^{p-j}.$$

(b) There exists $q \in kG$ such that $\zeta t - t\zeta = (e-1)q$. If $\varepsilon : kG \longrightarrow k$ denotes the augmentation map then:

$$\varepsilon(q) = \sum_{i=1}^{m}(\alpha_i^p\beta_i - \alpha_i\beta_i^p).$$

<u>Proof</u> Write $\xi_i = \alpha_i(x_i - 1) + \beta_i(y_i - 1)$ so that ξ_i is in the form to which (4.4.1) and (4.4.2) apply. We have that:

$$\zeta = \xi_1 + \dots. + \xi_m + \delta(z-1)$$

and that each of the terms in this sum commutes with each of the others. Thus:

$$\zeta^p = \xi_1^p + \dots. + \xi_m^p + \delta^p(z^p - 1).$$

But we can write:

$$\xi_i^p = \alpha_i^p(x_i^p - 1) + \beta_i^p(y_i^p - 1) + \mu_\beta t_i$$

where:

$$t_i = \sum_{j=1}^{p-1} R_j \alpha_i^j \beta_i^{p-j} x_i^j y_i^{p-j}.$$

Thus, noting that $t = t_1 + \ldots + t_m$, the result of part (a) follows.

Using the commutativity relations we have that:

$$\xi t - t\xi = \sum_{i=1}^{m} (\xi_i t_i - t_i \xi_i) = \sum_{i=1}^{m} (e-1)q_i$$

where q_i is as in (4.4.2). Thus, letting $q = q_1 + \ldots + q_m$, part (b) is also proved. □

Note that $e^a - 1 = (e-1)(1+e+\ldots+e^{a-1})$ and that $\varepsilon(1+e+\ldots+e^{a-1})$ equals a. Thus (4.4.3)(a) shows that ξ^p can be written in the form:

$$(e-1)s + \mu_\xi t$$

where:

$$\varepsilon(s) = \sum_{i=1}^{m} (a_i \alpha_i^p + b_i \beta_i^p) + \gamma^p.$$

But $\mu_\xi t = (e-1).(e-1)^{p-2} t$ and:

$$\varepsilon((e-1)^{p-2}t) = \begin{cases} \varepsilon(t) = \sum_{i=1}^{m} \alpha_i \beta_i & \text{if } p = 2, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore:

<u>Corollary 4.4.4</u> $\xi^p = (e-1)u$ where $\varepsilon(u)$ equals:

(a) $\sum_{i=1}^{m} (a_i \alpha_i^2 + \alpha_i \beta_i + b_i \beta_i^2) + \gamma^2$ if $p = 2$,

(b) $\sum_{i=1}^{m} (a_i \alpha_i^p + b_i \beta_i^p) + \gamma^p$ if p is odd. □

<u>Remark</u> The terms involving γ in (4.4.3) and (4.4.4) are to be ignored in the case when G is extra-special.

§4.5 A condition for U_ξ to be periodic

Let us continue with the notation of the previous section. We write $\xi^p = (e-1)u$ as in (4.4.4). Suppose that u is a unit in kG. Then:

$$e - 1 = u^{-1}\xi^p$$

so that $kG(e-1) \leq kG\xi$. Thus (4.2.3)(a) implies that the epimorphism:

$$kG \longrightarrow U = U_\xi \quad \text{given by} \quad 1 \longmapsto u_0$$

has kernel $kG\xi$. Hence we have an exact sequence:

$$kG \xrightarrow{f} kG \longrightarrow U \longrightarrow 0$$

where f is given by $1 \longmapsto \xi$. Note that, if u_1 is as in (4.2.3)(b),

$$f(u_1) = \mu_\xi \xi^p = 0\,.$$

Thus $\ker f$ contains a submodule isomorphic to U. But, by dimensions, this is the whole of $\ker f$. Hence we have constructed an exact sequence:

$$0 \longrightarrow U \longrightarrow kG \longrightarrow kG \longrightarrow U \longrightarrow 0.$$

To conclude:

<u>Theorem 4.5.1</u> If u is a unit then U_ξ is periodic. □

<u>Corollary 4.5.2</u> If $\xi + J^2 \in Y_G$ then $s(u) = 0$.

<u>Proof</u> This follows from (4.2.1), (4.5.1) and the fact that $s(u) \neq 0$ if and only if u is a unit. □

<u>Alternative proof</u> We may give a second proof of (4.5.2) which is independent of most of the work done in §4.2 etc. Consider the element, $w = 1 + \xi$, of kG.

Assume that u is a unit. Then:

$$w^p = 1 + \mathfrak{z}^p = 1 + (e-1)u \neq 1$$

and
$$w^{p^2} = 1 + (e-1)^p u^p = 1 .$$

Hence w is a unit of order p^2. Let V be any kG-module which is free on restriction to $\mathfrak{Z}$. We have that:

$$(w^p - 1)^{p-1} V = (e-1)^{p-1} u^{p-1} V = \mu_{\mathfrak{Z}} V$$

– the last equality holding because u is a unit. This has dimension $\frac{1}{p}\dim_k V$, so (0.0.8) implies that V is free on restriction to $\langle w^p \rangle$. But the cyclic group of order p^2 is a Chouinard group, so V must be free on restriction to $\langle w \rangle$. Thus:

$$\dim_k \mu_{\langle w \rangle} V = (1/p^2) \dim_k V.$$

But $\mu_{\langle w \rangle} V = (w-1)^{p-1}(w^p - 1)^{p-1} V = \mathfrak{z}^{p-1} \mu_{\mathfrak{Z}} V$ so the result above implies that:

$$\dim_k \mathfrak{z}^{p-1} \bar{V} = (1/p) \dim_k \bar{V}$$

so that $\mathfrak{z} + J^2 \notin Y(\bar{V})$. Thus the result follows. $\square$

Define a polynomial in n variables with coefficients in $\mathbb{F}_p$ by setting $F(X_1, \ldots, X_m, Y_1, \ldots, Y_m, Z)$ equal to:

$$\sum_{i=1}^{m} (a_i X_i^2 + X_i Y_i + b_i Y_i^2) + Z^2 \qquad \text{if } p = 2,$$

$$\sum_{i=1}^{m} (a_i X_i + b_i Y_i) + Z \qquad \text{if } p \text{ is odd.}$$

(The term involving Z is to be ignored when G is extra-special.) Writing $c = F(\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_m, \gamma)$, (4.4.4) gives that:

$$s(u) = \begin{cases} c & \text{if } p = 2, \\ c^p & \text{if } p \text{ is odd.} \end{cases}$$

Thus we may express (4.5.2) as:

<u>Corollary 4.5.3</u> If $\mathfrak{x} + J^2 \in Y_G$ then $F(\alpha_1, \dots, \alpha_m, \beta_1, \dots, \beta_m, \mathfrak{x}) = 0$. $\square$

This result is obviously only of interest when F is not the zero polynomial. Suppose that $F = 0$. We must have that p is odd and that G is extra-special. Furthermore, $x_i^p = y_i^p = 1$ for all i, so each P_i must be isomorphic to the extra-special group of order p^3 and exponent p. Thus G has exponent p. This case will be considered in the next section. We assume in the rest of this section that G has exponent p^2, so that $F \neq 0$.

When p is odd, (4.5.3) and (3.6.4)(b) imply that:

$$Y_G \subseteq S_H \qquad -(1)$$

for some maximal subgroup, H, of G. We claim that H is in fact the subgroup consisting of all the elements of G of order p (see (4.3.1)(b)). If not, there is $g \in G - H$ of order p. Let C be the subgroup generated by g, and consider $k_C^{\uparrow G}$. There is only one double coset, $HgC = gHC = G$, and $C \cap H = 1$, so the Mackey decomposition gives:

$$(k_C^{\uparrow G})_{\downarrow H} \cong k_{C \cap H}^{\uparrow H} \cong kH.$$

Thus $k_C^{\uparrow G}$ is a non-projective kG-module which is free on restriction to H. This contradicts (1). Hence we have proved:

<u>Theorem 4.5.4</u> Let p be odd and G be a pseudo-special group of exponent p^2. The elements of G of order p form a maximal subgroup, H, with the property that a kG-module is free if and only if it is free on restriction to H. $\square$

(4.5.3) does not imply any such pleasant result in the case $p = 2$. The following example does however add another Chouinard group to our list.

<u>Example</u> Let G be isomorphic to the dihedral group of order 8,

$$D_8 = \langle x, y \mid x^4 = y^2 = 1,\ y^{-1}xy = x^{-1} \rangle .$$

This is extra-special and we may take $a = x^2$, $x_1 = x$, $y_1 = y$ in §4.3. F is then given by:

$$F(X,Y) = X^2 + XY = X(X+Y).$$

Thus, if $\xi + J^2 \in Y_G$, either $\alpha_1 = 0$ or $\alpha_1 + \beta_1 = 0$ – that is to say, $\xi + J^2$ is an element of S_H for either $H = \langle x^2, y \rangle$ or $H = \langle x^2, xy \rangle$ (see (3.6.4)). Thus a kG-module is free if and only if it is free on restriction to both $\langle x^2, y \rangle$ and $\langle x^2, xy \rangle$. In particular, G is a Chouinard group.

§4.6 The exponent p case

Now assume that p is odd and that G is extra-special of exponent p. Let q be as in (4.4.3)(b). We shall prove:

Theorem 4.6.1 If q is a unit then U_ξ is periodic.

Using this, we will be able to prove the analogous statement to (4.5.3), but with the zero polynomial, F, being replaced by the nonzero polynomial, F', given by:

$$F'(X_1,\dots,X_m,Y_1,\dots,Y_m) = \sum_{i=1}^{m} (X_i^p Y_i - X_i Y_i^p).$$

The proof of (4.6.1) is adapted from [Ca5], where the result is proved in the case $m = 1$. Before embarking on the proof, let us note that, in this case, F' splits as a product of linear factors:

$$F'(X,Y) = X(X-Y)(X-2Y)\dots(X-(p-1)Y)Y$$

so that, as in the example, D_8, dealt with in the previous section, G is a Chouinard group.

Proof of (4.6.1) The proof consists of actually constructing a projective resolution of $U = U_\xi$. Let us firstly prove a lemma:

Lemma 4.6.2 With the notation of (4.4.3) we have that, for $i = 1,2,\dots,p-1$,

$$(\xi^i t - t\xi^i)(e-1)^{p-2} = i\xi^{i-1} q\mu_\xi .$$

Proof $\xi^i t - t\xi^i = \sum_{j=0}^{i-1} \xi^j(\xi t - t\xi)\xi^{i-1-j}$

$$= \sum_{j=0}^{i-1} \xi^j q(e-1)\xi^{i-1-j}.$$

Thus the result follows on multiplying by $(e-1)^{p-2}$. (Note that $\mu_\Phi kG$ is isomorphic to $k\bar{G}$ and is hence commutative.) □

Now let F be the free kG-module on two generators, a and b. For $i = 1,2,\dots,p-1$, let U_i be the submodule of F generated by:

$$u_{i1} = \xi^i a - (e-1)b$$

and $$u_{i2} = t(e-1)^{p-2}a - \xi^{p-i}b.$$

Lemma 4.6.3 $\dim_k U_i \geq (1+p^{-2})|G|$.

Proof Let the maximal subgroup H be as in §4.2. Then the elements $\mu_H \xi^j$ $(j=0,1,\dots,p-1)$ are k-linearly independent.

Let W_1 be the kH-submodule of U_i generated by the elements $\xi^j u_{i1}$ $(j=0,1,\dots,p-i-1)$ and $\xi^j u_{i2}$ $(j=0,1,\dots,i-1)$. Note that:

$$\mu_H(\xi^j u_{i1}) = \mu_H \xi^{i+j} a$$

$$\mu_H(\xi^j u_{i2}) = -\mu_H \xi^{p-i+j} b$$

so that $\mu_H W_1$ has dimension p. Thus W_1 is a free kH-module of dimension $p|H| = |G|$.

Let
$$\begin{aligned} w &= t(e-1)^{p-2}u_{i1} - \xi^i u_{i2} \\ &= (t\xi^i - \xi^i t)(e-1)^{p-2}a - (t\mu_\Phi - \xi^p)b \\ &= -i\xi^{i-1}q\mu_\Phi a. \end{aligned}$$

Consider $q^{-1}w$: this is an element of U_i on which Φ acts trivially. Thus, if W_2 denotes the kH-submodule of U_i generated by $q^{-1}w$, we have:

$$W_2 = k\bar{H}(q^{-1}w) \quad \text{and} \quad \mu_{\bar{H}}(q^{-1}w) = -i\mu_H \xi^{i-1} a \neq 0$$

so that $W_2 \cong k\bar{H}$.

Now we claim that W_1 and W_2 intersect trivially. If not, W_1^H and W_2^H would intersect non-trivially, implying that:

$$\mu_{\bar{H}}(g^{-1}w) \in \mu_H W_1$$

– a contradiction. Thus $W_1 + W_2$ is a kH-submodule of U_i of dimension $|G| + |\bar{H}| = (1 + p^{-2})|G|$. Hence we have the result. $\square$

(4.2.3)(a) implies that we have an exact sequence:

$$F \xrightarrow{f} kG \longrightarrow U \longrightarrow 0$$

where f is given by $a \longmapsto e - 1$, $b \longmapsto \xi$. But:

$$f(u_{11}) = \xi(e-1) - (e-1)\xi = 0$$
$$f(u_{12}) = t\mu_{\xi} - \xi^p = 0$$

so that U_1 is contained within $\ker f$. But $\ker f$ has dimension:

$$2|G| - (|G| - \dim_k U) = (1 + p^{-2})|G|$$

so (4.6.3) implies that U_1 must be the whole of the kernel. Thus:

<u>Lemma 4.6.4</u> There is an exact sequence:

$$0 \longrightarrow U_1 \longrightarrow F \longrightarrow kG \longrightarrow U \longrightarrow 0.$$

Thus $\dim_k U_1 = (1 + p^{-2})|G|$. $\square$

<u>Lemma 4.6.5</u> For each $i = 1,2,\ldots,p-2$ there is an exact sequence:

$$0 \longrightarrow U_{i+1} \longrightarrow F \longrightarrow F \longrightarrow U_i \longrightarrow 0.$$

Moreover, $\dim_k U_i = (1 + p^{-2})|G|$ $(i = 1,2,\ldots,p-1)$.

<u>Proof</u> We may assume inductively that $\dim_k U_i = (1+p^{-2})|G|$. Define an epimorphism $f : F \longrightarrow U_i$ by $a \longmapsto u_{i1}$, $b \longmapsto u_{i2}$. We readily check that $\ker f$ contains the elements:

$$w_1 = \xi^{p-i}a - (e-1)b$$

$$\text{and} \qquad w_2 = (\xi t + iq(e-1))(e-1)^{p-2}a - \xi^{i+1}b.$$

Let W be the submodule of F generated by w_1 and w_2.

Let W_1 be the kH-submodule of W generated by $\xi^j w_1$ $(j = 0, 1, \ldots, i-1)$ and $\xi^j w_2$ $(j = 0, 1, \ldots, p-i-2)$ then, by a similar argument to that employed in (4.6.3), W_1 is free of dimension $(p-1)|H|$.

Let $w = \xi^i w_1$. Because the cyclic group, Ξ, acts freely on F, the fact that $\mu_\Xi w = 0$ implies that $w = (e-1)w'$ for some $w' \in F$. Now:

$$\mu_H w' = \mu_H (e-1)^{p-2} w = -\mu_H \xi^i b$$

– this is not an element of $\mu_H W_1$, so w' generates a free kH-submodule of F which intersects W_1 trivially. Ξ acts freely on kHw' so:

$$(e-1)kHw' = kHw$$

is a submodule of W of dimension $(1-p^{-1})|H|$. But this submodule of W intersects W_1 trivially, thus W must have dimension at least:

$$(p-1)|H| + (1-p^{-1})|H| = (1-p^{-2})|G|.$$

Hence, by a dimensions argument, W is the whole of $\ker f$. Thus we have an exact sequence:

$$F \xrightarrow{f'} F \longrightarrow U_i \longrightarrow 0$$

where f' is given by $a \longmapsto w_1$, $b \longmapsto w_2$. A simple calculation shows that $U_{i+1} \leqslant \ker f'$. Another dimensions argument using (4.6.3) shows that equality must hold. Hence the result follows. $\square$

<u>Lemma 4.6.6</u> There is an exact sequence:

$$0 \longrightarrow U \longrightarrow kG \longrightarrow F \longrightarrow U_{p-1} \longrightarrow 0.$$

<u>Proof</u> We may define an epimorphism $f : F \longrightarrow U_{p-1}$ by $a \longmapsto u_{p-1,1}$, $b \longmapsto u_{p-1,2}$. Let:

$$w = \xi a - (e-1)b,$$

then $f(w) = 0$. By a similar argument to that used in (4.6.5) we see that the elements $\xi^j w$ $(j = 0,1,\dots,p-2)$ generate a free kH-submodule of kGw of dimension $(p-1)|H|$, and that $\xi^{p-1}w$ generates a kH-submodule of dimension $(1-p^{-1})|H|$ which intersects the first submodule trivially. Thus, by another dimensions argument, $\ker f = kGw$.

Define an epimorphism $f' : kG \longrightarrow kGw$ by $1 \longmapsto w$. $\ker f'$ contains the element $\mu_* \xi^{p-1}$ and hence has a submodule isomorphic to U (see (4.2.3)(b)). By dimensions, this submodule is the whole of $\ker f'$. Thus the result follows. □

Combining the results of (4.6.4), (4.6.5) and (4.6.6), we see that there is an exact sequence:

$$0 \longrightarrow U \longrightarrow kG \longrightarrow F \longrightarrow \dots \longrightarrow F \longrightarrow kG \longrightarrow U \longrightarrow 0$$

which is a 2p-step projective resolution of U. Hence the result of (4.6.1) is established. □

§4.7 Conclusions

Let us summarise (4.5.3) and the analogous result proved in the previous section. Let G be a pseudo-special p-group and pick the minimal set of generators, $g_1,\ldots,g_n$, as in §4.3. We have shown that there is a nonzero homogeneous polynomial, $F(X_1,\ldots,X_n)$, with coefficients in $\mathbb{F}_p$, such that:

$$\sum_{i=1}^{n} \lambda_i(g_i - 1) + J^2 \in Y_G \quad \Leftrightarrow \quad F(\lambda_1,\ldots,\lambda_n) = 0. \qquad -(1)$$

The degree of F equals:

2	if $p = 2$,
$p+1$	if p is odd and G has exponent p,
1	if p is odd and G has exponent p^2.

We may extend this result to the case when G is any p-group which is not elementary abelian. Let P,E,N be as in (4.1.3). Choose $g_1,\ldots,g_r \in G$ such that $g_1N,\ldots,g_rN$ is a minimal set of generators for the pseudo-special group, P/N, of the required type. Let $g_{r+1},\ldots,g_n$ be a minimal set of generators for E modulo N. Then $g_1,\ldots,g_n$ is a minimal set of generators for G. Let $F'(X_1,\ldots,X_r)$ be the polynomial corresponding to P/N and define:

$$F(X_1,\ldots,X_n) = F'(X_1,\ldots,X_r).$$

(4.1.4) then implies that (1) holds for G with this F.

Note that, because F is nonzero, Y_G cannot be the whole of J/J^2. Thus we may add the converse statement to (4.0.2):

<u>Theorem 4.7.1</u> Let G be a general p-group. Then Y_G is the whole of J/J^2 if and only if G is elementary abelian. □

Although we have only proved (1) for a specific set of generators for G, it actually holds with $g_1,\dots,g_n$ replaced by a general minimal set of generators – say, $g'_1,\dots,g'_n$. By (3.0.2), the change of basis of J/J^2 from $(g'_1-1)+J^2,\dots,(g'_n-1)+J^2$ to $(g_1-1)+J^2,\dots,(g_n-1)+J^2$ corresponds to a non-singular $n \times n$ matrix, $C=(c_{ij})$, with entries in $\mathbb{F}_p$. Thus:

$$\xi = \sum_{i=1}^{n} \lambda'_i(g'_i-1)+J^2 = \sum_{i=1}^{n} \lambda_i(g_i-1)+J^2$$

where:

$$\lambda_i = \sum_{j=1}^{n} c_{ij}\lambda'_j .$$

If ξ is an element of Y_G then $F(\lambda_1,\dots,\lambda_n)=0$ – but we may naturally write this equation as $FC(\lambda'_1,\dots,\lambda'_n)=0$. $FC(X_1,\dots,X_n)$ is a nonzero homogeneous polynomial with coefficients in $\mathbb{F}_p$ of the same degree as F.

We may state this as:

Theorem 4.7.2 Let G be a p-group which is not elementary abelian. Take a minimal set of generators, $g_1,\dots,g_n$, for G. There exists a nonzero homogeneous polynomial, $F(X_1,\dots,X_n)$, with coefficients in $\mathbb{F}_p$ of degree at most:

$$\begin{cases} 2 & \text{if } p=2, \\ p+1 & \text{if } p \text{ is odd,} \end{cases}$$

such that:

$$\sum_{i=1}^{n} \lambda_i(g_i-1)+J^2 \in Y_G \Leftrightarrow F(\lambda_1,\dots,\lambda_n)=0. \qquad \square$$

§4.8 Defining equations for Y_G

Assume that G is pseudo-special. We have shown that Y_G is contained within some hypersurface defined by a polynomial with coefficients in $\mathbb{F}_p$ (Theorem 4.7.2); now we show that more can be said on this subject. The key to our approach is the following surprising result:

Theorem 4.8.1 Suppose that $f : \Omega^r k_G \longrightarrow k_G$ is an epimorphism for some $r \neq 0$ such that $\ker f$ is free on restriction to E. Then:

$$Y(\overline{\ker f}) = Y_G .$$

Proof Let $\xi \in J - J^2$ and write $U = U_\xi$. We have an exact sequence:

$$0 \longrightarrow (\ker f) \otimes U \longrightarrow \Omega^r k_G \otimes U \longrightarrow U \longrightarrow 0 . \qquad -(1)$$

Now:

$$\begin{array}{llll}
\xi + J^2 \notin Y_G & \Leftrightarrow & \xi + J^2 \notin Y(\overline{\ker f}) & \text{by } (4.0.1), \\
 & \Leftrightarrow & Y(\overline{(\ker f) \otimes U}) = \{0\} & \text{as in } (4.0.1), \\
 & \Leftrightarrow & (\ker f) \otimes U \text{ is free} & \text{by } (3.5.2), \\
 & \Leftrightarrow & \Omega^r k_G \otimes U \cong U \oplus \text{projective} & \text{using } (1), \\
 & \Leftrightarrow & \Omega^r U \cong U & \\
 & \Leftrightarrow & U \text{ is periodic} & \\
 & \Leftrightarrow & \xi + J^2 \notin Y_G & \text{by } (4.2.1),
\end{array}$$

thus all the statements in this chain of implications are equivalent. □

The obvious question is therefore, does such a map, f, exist ? The answer is given by the following result. The proof is fairly long and involves some vaguely cohomological material, it is postponed until the end of the section.

Theorem 4.8.2 There exists a map, f, with the properties outlined in (4.8.1) for $r = 2|G:E|$.

<u>Corollary 4.8.3</u> U_Σ is periodic if and only if $\Sigma + J^2 \not\subseteq Y_G$. If it is periodic then its period divides $2|G:\Phi|$.

<u>Proof</u> This follows from the chain of implications in the proof of (4.8.1). Note that we may take $r = 2|G:\Phi|$ by (4.8.2). $\square$

We may even get rid of the mysterious map, f. Note that μ_Φ acts trivially on k_G, thus $\mu_\Phi(\Omega^r k_G) \subseteq (\ker f)^\Phi = \mu_\Phi(\ker f)$. But the reverse inclusion is trivial, so we have:

<u>Corollary 4.8.4</u> Regard $\mu_\Phi(\Omega^{2|G:\Phi|} k_G)$ as a $k\bar{G}$-module, then the Carlson variety of this is Y_G. $\square$

Note how this construction differs from that of (0.0.12) – $\Omega^r k_G$ is not free on restriction to Φ.

Let V be an $\mathbb{F}_p G$-module isomorphic to the direct sum of $\Omega^{2|G:\Phi|}\mathbb{F}_p$ and a free $\mathbb{F}_p G$-module; for example, we may take:

$$V = \mathrm{Aug}(\mathbb{F}_p G) \otimes \ldots. \otimes \mathrm{Aug}(\mathbb{F}_p G) \qquad (2|G:\Phi| \text{ factors}).$$

Define W to be the $\mathbb{F}_p \bar{G}$-module $\mu_\Phi V$ and W' to be the $k\bar{G}$-module $k \otimes_{\mathbb{F}_p} W$, then W' is isomorphic to the direct sum of $\mu_\Phi(\Omega^{2|G:\Phi|} k_G)$ and a free $k\bar{G}$-module, thus (4.8.4) gives:

$$Y(W') = Y_G.$$

Take an $\mathbb{F}_p$-basis for W, and let A_i be the matrix representing $\bar{g}_i - \bar{1}$ with respect to this basis. Then this basis is also a k-basis for W', so A_i also represents the action of $\bar{g}_i - \bar{1}$ on W'.

Let:

$$\mathfrak{X} = \sum_{i=1}^{n} \lambda_i(\bar{g}_i - \bar{1})$$

where the λ_i's are elements of k, not all zero. (3.0.4) gives that $\mathfrak{X} + J^2$ is an element of $Y(W') = Y_G$ if and only if $\mathfrak{X}W'$ has dimension less than d, where $d = (1 - \frac{1}{p}) \dim_k W'$, that is to say, if and only if all the $d \times d$ minors of any matrix representing the action of $\mathfrak{X}$ on W' vanish. But with respect to the basis above, $\mathfrak{X}$ is represented by:

$$\lambda_1 A_1 + \cdots + \lambda_n A_n :$$

recall that each A_i has entries in $\mathbb{F}_p$, so that any fixed $d \times d$ minor of this matrix is given by $F(\lambda_1, \ldots, \lambda_n)$ where $F(X_1, \ldots, X_n)$ is a homogeneous polynomial of degree d (unles $F = 0$) with coefficients in $\mathbb{F}_p$ which is independent of $\lambda_1, \ldots, \lambda_n$. Let $F_1, \ldots, F_s$ be all the nonzero polynomials arising from $d \times d$ minors in this way, then $\mathfrak{X} + J^2 \in Y_G$ if and only if

$$F_i(\lambda_1, \ldots, \lambda_n) = 0 \quad \text{for all } i.$$

Note that, because Y_G is not the whole of J/J^2, there must be at least one nonzero polynomial, F_i.

Let us state the result just proved as:

Theorem 4.8.5 If G is pseudo-special then there exist nonzero homogeneous polynomials with coefficients in $\mathbb{F}_p$, $F_i(X_1, \ldots, X_n)$ $(i = 1, 2, \ldots, s)$, such that Y_G is the subset of J/J^2 given by all points

$$\sum_{j=1}^{n} \lambda_j(g_j - 1) + J^2$$

with $F_i(\lambda_1, \ldots, \lambda_n) = 0$ for all i. □

We may express this succinctly by saying that Y_G is a homogeneous variety defined by a series of polynomials with coefficients in $\mathbb{F}_p$.

<u>Proposition 4.8.6</u> Let $F_1,\ldots,F_m$ be as in (4.8.5); define I to be the ideal of $\mathbb{F}_p[X_1,\ldots,X_n]$ generated by $F_1,\ldots,F_m$. Suppose that I contains a product of nonzero linear polynomials, then G is a Chouinard group.

<u>Proof</u> Suppose that $f_1f_2\ldots f_N \in I$ where each f_i is a nonzero linear polynomial in $X_1,\ldots,X_n$. Then $\xi + J^2 \in Y_G \Leftrightarrow f_i(\lambda_1,\ldots,\lambda_n) = 0$ for some $i \Leftrightarrow \xi + J^2 \in S_H$ for some maximal subgroup, H, of G (by (3.6.4)). Thus:

$$Y_G \subseteq \bigcup_H S_H \quad \text{(the union being over all maximal subgroups of G)}$$

that is to say, G is a Chouinard group.

We may remark that the converse also holds. Let f be the product of all the nonzero linear polynomials in $X_1,\ldots,X_n$ with coefficients in $\mathbb{F}_p$, then the fact that Y_G is contained within the union of all the S_H's implies that:

$$f(\lambda_1,\ldots,\lambda_n) = 0 \quad \text{for all } \xi + J^2 \in Y_G .$$

Thus, by Hilbert's Nullstellensatz, some power of f is an element of I. The converse statement is thus proved. □

Thus we have a very concrete way to prove Chouinard's theorem:

(1) It suffices to assume that G is pseudo-special (see §4.1).

(2) Let $V = \mathrm{Aug}(\mathbb{F}_pG) \oplus \ldots\ldots \oplus \mathrm{Aug}(\mathbb{F}_pG)$ $(2|G:\Phi|$ factors), the structure of G is well understood (see §4.3) so we write down a concrete matrix representation for V.

(3) Calculate the polynomials $F_1,\ldots,F_m$ as above.

(4) Show that the ideal of $\mathbb{F}_p[X_1,\ldots,X_n]$ generated by $F_1,\ldots,F_m$ contains a product of nonzero linear products.

Proof of Theorem 4.8.2

We now come to the postponed proof of (4.8.2); it will be recalled that this theorem stated that there is an epimorphism:

$$f : \Omega^{2|G:\Phi|} k_G \longrightarrow k_G$$

such that $\ker f$ is free on restriction to Φ.

We shall prove that if N is a subgroup of G containing Φ then there is an exact sequence of kN-modules:

$$0 \longrightarrow k_N \longrightarrow Q_{r-1} \longrightarrow \dots \longrightarrow Q_0 \longrightarrow k_N \longrightarrow 0$$

where $r = 2|N:\Phi|$, such that each Q_i is free on restriction to Φ. This will prove the result: take the sequence above for $N = G$ and form the diagram:

$$\begin{tikzcd}
0 \arrow[r] & k_G \arrow[r] & Q_{r-1} \arrow[r] & \dots \arrow[r] & Q_0 \arrow[r] & k_G \arrow[r] & 0 \\
0 \arrow[r] & \Omega^r k_G \arrow[r] \arrow[u, dashed, "f"] & P_{r-1} \arrow[r] \arrow[u, dashed] & \dots \arrow[r] & P_0 \arrow[r] \arrow[u, dashed] & k_G \arrow[r] \arrow[u, equal] & 0
\end{tikzcd}$$

where the lower sequence is a minimal projective resolution of k_G; then both sequences in this diagram are, on restriction to Φ, projective resolutions of k_Φ, thus f is a split $k\Phi$-epimorphism and $(\ker f)_{\downarrow\Phi}$ is free. Thus it suffices to prove the existence of the sequence above.

The proof is by induction on $|N|$. For $N = \Phi$ we may take the sequence:

$$0 \longrightarrow k_\Phi \longrightarrow k\Phi \longrightarrow k\Phi \longrightarrow k_\Phi \longrightarrow 0$$

(recall that Φ is cyclic).

So assume that $N \neq \Phi$. We may choose H to be a maximal subgroup of N containing Φ. Inductively, there is an exact sequence of kH-modules:

$$0 \longrightarrow k_H \longrightarrow \tilde{Q}_{r-1} \longrightarrow \dots \longrightarrow Q_0 \longrightarrow k_H \longrightarrow 0$$

where $r = 2|H:\Phi|$ and each Q_i is free on restriction to Φ. Define $Q_r = k_H$, $Q_i = 0$ $(i > r)$ so that there is an exact sequence of kH-modules:

$$\underline{Q} : \quad \ldots\ldots \longrightarrow Q_2 \longrightarrow Q_1 \longrightarrow Q_0 \longrightarrow k_H \longrightarrow 0 .$$

Form the complex $\underline{X} = \underline{Q} \otimes \underline{Q} \otimes \ldots\ldots \otimes \underline{Q}$ (p factors), this may be regarded as a sequence of $k(H \times H \times \ldots\ldots \times H)$-modules (see [C&E] for this construction). Let C be a cyclic group of order p, generated by z, say. C acts on $H \times H \times \ldots\ldots \times H$ by letting z permute the factors on step cyclically right to left. Form the corresponding semi-direct product:

$$S = (H \times H \times \ldots\ldots \times H)C$$

(so that S is the *wreath product*, $H \wr C$). We make $\underline{X}$ into a sequence of kS-modules by defining the action of z as:

$$z(x_{n_1} \otimes x_{n_2} \otimes \ldots\ldots \otimes x_{n_p}) = (-1)^d (x_{n_2} \otimes \ldots\ldots \otimes x_{n_p} \otimes x_{n_1})$$

where $x_{n_i} \in Q_{n_i}$, $d = n_1(n_2 + \ldots\ldots + n_p)$. This may be checked to make $\underline{X}$ into an exact sequence of kS-modules:

$$\ldots\ldots \longrightarrow X_2 \longrightarrow X_1 \longrightarrow X_0 \longrightarrow k_S \longrightarrow 0 .$$

We may regard N as a subgroup of S as follows: fix $g \in N - H$ and identify $x \in N$ with:

$$(h_0, \ldots\ldots, h_{p-1}, z^a) \in S$$

where $x \in (Hg)^a$ and $h_i \in H$ is defined by the equation $g^i x = h_i g^j$ $(0 \leq j < p)$. Note that, because Φ is contained within both H and Z(G), $x \in \Phi$ is identified with $(x, \ldots\ldots, x, 1) \in S$.

Thus we may, by restriction, regard $\underline{X}$ as a sequence of kN-modules. (For this construction see [Ev].) Note that the action of Φ is just the diagonal action on the factors of $\underline{X} = \underline{Q} \otimes \ldots\ldots \otimes \underline{Q}$.

As a k-space, X_i is the direct sum of all terms of the form:

$$Q_{n_1} \otimes Q_{n_2} \otimes \dots \otimes Q_{n_p} \qquad \text{with } n_1 + n_2 + \dots + n_p = i.$$

Thus, using the definition of the Q_i's and writing $s = pr = 2p|H:\Phi| = 2|N:\Phi|$, we have:

(a) $X_i = 0$ for $i > s$.

(b) $X_s \cong k_N$.

(c) each of the summands above of X_i when $i < s$ has a factor Q_j for some $j < r$, which is therefore free on restriction to Φ; this shows that this summand is a free $k\Phi$-submodule of X_i (since Φ acts diagonally), and hence that X_i is free on restriction to Φ.

Hence we have an exact sequence of kN-modules:

$$0 \longrightarrow k_N \longrightarrow X_{s-1} \longrightarrow \dots \longrightarrow X_0 \longrightarrow k_N \longrightarrow 0$$

where each X_i is free on restriction to Φ. This has the required form, so the result is proved. $\square$

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CHAPTER 5

EXAMPLES

Introduction

This chapter deals mainly with examples of non-projective, indecomposable kG-modules which are free on restriction to some maximal subgroup of G. Several results relating to the set Y_G but not properly belonging in Chapter 4 are also proved. These may be used to construct further examples from those already given.

§5.0 Split products

Suppose that G is a p-group which has a maximal subgroup, H, such that the group extension:

$$1 \longrightarrow H \longrightarrow G \longrightarrow Z_p \longrightarrow 1$$

splits; i.e. there is a subgroup, C, of G of order p such that $G = HC$, $H \cap C = 1$. Choose a generator, g, for C. Assume that C is not the whole of G.

Let U be one of the $p-1$ non-projective, indecomposable kC-modules. Let $V = U^{\uparrow G}$. We have:

(1) V is not projective – for $U | V_{\downarrow C}$.

(2) V has a unique maximal submodule – for $(V, k_G)_{kG} \cong (U, k_C)_{kC}$, but U has a unique maximal submodule (indeed, it is uniserial), so this has dimension one,

(3) V is free on restriction to H – there is only one double coset, $HgC = gHC = G$, so the Mackey decomposition gives:

$$V_{\downarrow H} \cong (U_{\downarrow H \cap C})^{\uparrow H} = (U_{\downarrow 1})^{\uparrow H}.$$

Thus V is a terminal module - see (2.4.2). Let us apply (2.4.1) to V. There exists $u \in U$ such that:

$$u,\ (g-1)u,\ \dots\dots,\ (g-1)^{d-1}u$$

is a k-basis for U, and $(g-1)^d u = 0$ (here $d = \dim_k U$). The elements $1 \otimes (g-1)^i u$ $(i = 0, 1, \dots, d-1)$ form a free kH-basis for V.

We may take all the ϕ_n's to be equal to the map $\phi : V \longrightarrow k_G$ given by:

$$1 \otimes (g-1)^i u \longmapsto \begin{cases} 1 & \text{if } i = 0, \\ 0 & \text{otherwise,} \end{cases}$$

and all the v_n's to be equal to $g^{-1}(1 \otimes u)$.

V_n is then the free kH-module on the elements e_{ij} $(i=1,2,\dots,n;$ $j=1,2,\dots,d)$, where:

$$e_{ij} = \psi_n(0,\dots,0,1\otimes(g-1)^{j-1}u,0,\dots,0)$$

– the nonzero entry being in the ith coordinate. Now:

$$\begin{aligned}(g-1)e_{ij} &= \psi_n(0,\dots,0,\phi(1\otimes(g-1)^{j-1}u)\,g\mu_H g^{-1}(1\otimes u),\\ &\qquad (g-1)(1\otimes(g-1)^{j-1}u),0,\dots,0)\\ &= \psi_n(0,\dots,0,\phi(1\otimes(g-1)^{j-1}u)\,\mu_H(1\otimes u),\\ &\qquad 1\otimes(g-1)^{j}u,0,\dots,0)\,.\end{aligned}$$

The irreducible map $\alpha_n : V_n \longrightarrow V_{n+1}$ is given by $e_{ij} \longmapsto e_{ij}$ $(i=1,2,\dots,n;\ j=1,2,\dots,d)$, so it is just the inclusion.

To conclude:

Let V_∞ be the (infinite-dimensional) kG-module with free kH-basis e_{ij} $(i=1,2,\dots;\ j=1,2,\dots,d)$ with the action of g being given by:

<u>$d=1$</u>

$$(g-1)e_{i1} = \begin{cases} \mu_H e_{i-1,1} & \text{if } i>1,\\ 0 & \text{if } i=1.\end{cases}$$

<u>$d>1$</u>

$$(g-1)e_{ij} = \begin{cases} e_{i2} + \mu_H e_{i-1,1} & \text{if } j=1,\ i>1,\\ e_{12} & \text{if } j=1,\ i=1,\\ e_{i,j+1} & \text{if } 1<j<d,\\ 0 & \text{if } j=d.\end{cases}$$

Let V_n be the submodule generated by e_{ij} $(i=1,2,\dots,n;\ j=1,2,\dots,d)$. The modules V_n then have the properties outlined in (2.3.6). The inclusion maps $V_n \longrightarrow V_{n+1}$ are irreducible.

Using the almost split sequences involving the V_n's, we see that:

$$Y(\bar{V}_1) = Y(\bar{V}_2) = \dots\dots = Y(\bar{V}_n) = \dots\dots$$

Thus it suffices to calculate $Y(\bar{V})$. V is not free on restriction to $C\Phi$ – it is not free on restriction to C – so $\bar{V}$ is not free on restriction to $\overline{C\Phi} = \langle\bar{g}\rangle$. Thus $(g-1)+J^2 \in Y(\bar{V})$. Note that, because $g \notin \Phi$, $(g-1)+J^2 \neq 0$ (by (3.0.2)). Thus we have constructed a nonzero element of the line, $Y(\bar{V})$. Hence:

$$Y(\bar{V}) = k((g-1)+J^2) = S_C .$$

The initial constraint on G, that it should be a split product, is not as artificial as it might at first seem. Suppose that G is any p-group such that there is a non-projective kG-module which is free on restriction to some maximal subgroup, H, of G. Chouinard's theorem gives that this module is non-projective on restriction to some elementary abelian subgroup, E, of G. Clearly we cannot have $E \leq H$, so choose $g \in E - H$. g then has order p. By the maximality of H, $G = H\langle g\rangle$. This, considering orders, implies that $H \cap \langle g\rangle = 1$. Thus G is the split product of H by $\langle g\rangle$.

This enables us to state:

<u>Theorem 5.0.1</u> Let G be a p-group and H be a maximal subgroup of G. There exists a non-projective kG-module which is free on restriction to H if and only if the group extension:

$$1 \longrightarrow H \longrightarrow G \longrightarrow Z_p \longrightarrow 1$$

splits. $\square$

§5.1 Elementary abelian groups

Now assume that G is elementary abelian. Take $\xi \in J - J^2$ and let $C = \langle 1+\xi \rangle$. There is an essential subgroup, G', of kG containing C - indeed G' splits as the direct product of C and a maximal subgroup. Thus we are in a position to apply the construction of the previous section. We let U be one of the $p-1$ non-projective, indecomposable kC-modules, i.e.

$$U \cong \xi^i kC$$

for some $i = 1,2,\ldots,p-1$, and consider $V = U^{\uparrow G'}$. Clearly:

$$V \cong \xi^i kG' = \xi^i kG$$

(cf. the modules U_ξ).

Let us summarise some of the properties of these modules:

(1) $Y(\xi^i kG) = k(\xi + J^2)$ – thus $\xi^i kG$ is free on restriction to some maximal subgroup of G.

(2) $\xi^i kG$ is terminal and, using the calculation in §5.0, we may write down a sequence of modules with the properties outlined in (2.3.6) with its first member isomorphic to $\xi^i kG$.

(3) $\dim_k \xi^i kG = (1 - \frac{i}{p})|G|$.

(4) the exact sequence $0 \longrightarrow \xi^{p-i}kC \longrightarrow kC \longrightarrow \xi^i kC \longrightarrow 0$, when induced up to G', shows that $\Omega(\xi^i kG) \cong \xi^{p-i}kG$.

Thus we have a large number of examples. The question arises as to when $\xi_1^i kG$ is isomorphic to $\xi_2^j kG$. By dimensions, we must have that $i = j$. Also:

Lemma 5.1.1 $\xi_1^i kG \cong \xi_2^i kG$ implies that $\xi_1^i = \xi_2^i u$ for some unit $u \in kG$.

Proof The isomorphism implies that $\xi_1^{p-i}\xi_2^i = 0$. Hence we can write

$\xi_2^i = \xi_1^i u_1$ for some $u_1 \in kG$. Similarly, we can write $\xi_1^i = \xi_2^i u_2$. Now

$$\xi_1^i(u_1 u_2 - 1) = 0$$

so that $u_1 u_2 - 1 \in \xi_1^{p-i} kG \leq \mathrm{Aug}(kG)$. Thus $u_1 u_2$, and hence u_2, is a unit. This proves the result. $\square$

Let $P(i)$ be the improved statement:

$$\text{"}\xi_1^i kG \cong \xi_2^i kG \quad \Rightarrow \quad \xi_1 = \xi_2 u \text{ for some unit } u \in kG\text{"}.$$

Then:

<u>Lemma 5.1.2</u> (a) $P(1)$ holds.

(b) If $P(i)$ holds then so does $P(p-i)$.

(c) If $P(i)$ holds and $j | i$ then $P(j)$ holds.

<u>Proof</u> (a) follows immediately from (5.1.1). To prove (b):

$$\begin{aligned} \xi_1^{p-i} kG \cong \xi_2^{p-i} kG \quad &\Rightarrow \quad \xi_1^i kG \cong \xi_2^i kG \qquad \text{(by (4) above)} \\ &\Rightarrow \quad \xi_1 = \xi_2 u \quad \text{for some unit, } u \quad \text{(by } P(i)) \end{aligned}$$

thus $P(p-i)$ holds. To prove (c):

$$\begin{aligned} \xi_1^j kG \cong \xi_2^j kG \quad &\Rightarrow \quad \xi_1^j = \xi_2^j u \quad \text{for some unit, } u \quad \text{(by (5.1.1))} \\ &\Rightarrow \quad \xi_1^i = \xi_2^i u^{(i/j)} \\ &\Rightarrow \quad \xi_1^i kG = \xi_2^i kG \\ &\Rightarrow \quad \xi_1 = \xi_2 u' \quad \text{for some unit, } u' \quad \text{(by } P(i)) \end{aligned}$$

so $P(j)$ holds. $\square$

<u>Theorem 5.1.3</u> $\xi_1^i kG \cong \xi_2^j kG$ if and only if $i = j$ and $\xi_1 = \xi_2 u$ for some unit $u \in kG$.

<u>Proof</u> One implication is trivial. For the other it suffices to show that $P(i)$ holds for all i. Assume otherwise, and pick i minimal such that $P(i)$ does not hold. Let m be maximal with respect to $mi < p$.

$P(mi)$ does not hold by part (c) of the previous lemma. Thus, by part (b), $P(p-mi)$ does not hold. So, by the minimality of i, $p-mi \geqslant i$. But the maximality of m gives $(m+1)i \geqslant p$. Thus:

$$p = (m+1)i$$

so that $i=1$, contradicting (5.1.2)(a). $\square$

Note that $\xi_1 = \xi_2 u$ for some unit, u, implies that $k(\xi_1 + J^2)$ equals $k(\xi_2 + J^2)$. However the converse does not hold in general:

$$\{\xi u \mid u \text{ is a unit}\} = \{c\xi + \zeta \mid 0 \neq c \in k,\ \zeta \in \xi J\}$$

thus we require that ξJ is the whole of J^2.

<u>Lemma 5.1.4</u> $\xi J = J^2$ if and only if G has order p or 4.

<u>Proof</u> By (3.0.2), $\dim_k J^2 = p^n - 1 - n$ where $p^n = |G|$. The map $J \longrightarrow \xi J$ given by $\zeta \longmapsto \xi\zeta$ is a kG-epimorphism with kernel,

$$\xi^{p-1}kG \cap J = \xi^{p-1}kG,$$

of dimension $\frac{1}{p}|G|$. Thus $\dim_k \xi J = p^n - 1 - p^{n-1}$.

We know that $\xi J \leqslant J^2$; by dimensions, equality holds iff $p^{n-1} = n$. We readily check that the only solutions to this are $n=1$ and $n=2$, $p=2$. Thus the result follows. $\square$

If we are not in the exceptional cases, let ξ_1 be any element of J/J^2. We can choose ξ_2 with $k(\xi_2 + J^2) = k(\xi_1 + J^2)$ but with ξ_2 not being of the form $\xi_1 u$ for a unit, u. Consider the modules $\xi_1^i kG$ and $\xi_2^i kG$:

- they are both terminal,
- they have the same dimension,
- they have the same Carlson variety,

but they are not isomorphic.

Of the exceptional cases; the first, when G is cyclic of order p, is too simple to be of any real interest (indeed the construction of (2.4.1) does not apply in this case); the second, when G is the Klein 4-group, is dealt with in Appendix A. It is shown that the terminal kG-modules are in 1-1 correspondence with the set of lines in J/J^2 – the correspondence being given by the Carlson variety. By what we have shown above, this result does not extend to more general elementary abelian groups. Again, the result that all the terminal modules are of the form $\zeta^i kG$ for some ζ, i, and hence that any non-projective, indecomposable kG-module which is free on restriction to some maximal subgroup may be obtained from one of these modules using the construction given in §5.0, does not extend to more general groups. This problem is addressed in the following section.

§5.2 The terminal modules of elementary abelian groups

Let G be any non-cyclic elementary abelian group. The problem of classifying all the non-projective, indecomposable kG-modules which are free on restriction to some maximal subgroup obvious reduces to that of classifying all such modules which are terminal. We have already constructed a large class of such terminal modules, $\xi^i kG$ $(i = 1,2,\ldots,p-1,\ \xi \in J - J^2)$. In the case when G is the Klein 4-group, every terminal module belongs to this class; thus we ask, does this result extend to general elementary abelian groups? The following result gives a moderately promising start:

Lemma 5.2.1 Let V be a kG-module such that $V_{\downarrow H} \cong kH$ for some maximal subgroup, H, of G. Then $V \cong \xi^{p-1}kG$ for some $\xi \in J - J^2$.

Proof Choose $v \in V$ with $V = kHv$. Let g be an element of $G - H$ then we can write:

$$(g-1).v = \gamma v \qquad \text{for some } \gamma \in kH.$$

Now $0 = (g-1)^p v = \gamma^p v$, so that $\gamma^p = 0$. Hence $\gamma \in \mathrm{Aug}\,(kH)$.

Write $\xi = (g-1) - \gamma \in J$; note that J/J^2 is the direct sum of S_H and $k((g-1)+J^2)$, so ξ cannot be an element of J^2.

Define a map $f : kG \longrightarrow V$ by $1 \longmapsto v$. This is an epimorphism, and the fact that $\xi v = 0$ implies that $\xi kG \leqslant \ker f$. By dimensions, we must have:

$$\Omega V \cong \ker f = \xi kG \cong \Omega(\xi^{p-1}kG)$$

so the result follows. □

Proposition 5.2.2 Let V be a non-projective kG-module which is free on restriction to some maximal subgroup, H, of G, and suppose that V has a unique maximal submodule. If $p = 2$ or 3 then $V \cong \xi^i kG$ for some $i = 1,2,\ldots,p-1$, $\xi \in J - J^2$.

Proof We have an epimorphism $kG \longrightarrow V$ given by mapping 1 to any element outside the maximal submodule of V. The kernel of this map, W, is free on restriction to H. We have an exact sequence of k(G/H)-modules:

$$0 \longrightarrow \mu_H W \longrightarrow k(G/H) \longrightarrow \mu_H V \longrightarrow 0,$$

thus $\dim_k \mu_H V$ and $\dim_k \mu_H W$ are positive integers with sum p. If $p = 2$ or 3 this implies that either $\mu_H V$ or $\mu_H W$ has dimension one – that is to say, either V or W satisfies the conditions of (5.2.1) and hence is isomorphic to $\xi^{p-1}kG$ for some ξ. If $V \cong \xi^{p-1}kG$ then we are done; otherwise:

$$\Omega V \cong W \cong \xi^{p-1}kG$$

so $V \cong \xi kG$. Thus the result follows. $\square$

Thus when $p = 2,3$ we have proved that all the terminal modules of a particular sort – those with a unique maximal submodule (see (2.4.2)) – are of the required form. For all other values of p, we do not even have this, as the following example shows.

Suppose that $p \geq 5$. Let H be a maximal subgroup of G and g be any element of $G - H$. Write $\xi = g - 1 \in kG$. Let V be the free kH-module on two generators, v_1 and v_2. We make V into a kG-module by defining the action of g as follows:

$$\xi . v_1 = v_2 ,$$
$$\xi . v_2 = \mu_H v_1 .$$

To check that this is a valid action, we need only check that $\xi^p v_1 = 0$. But:

$$\xi^3 v_1 = \xi^2 v_2 = \xi \mu_H v_1 = \mu_H v_2 ,$$

so $$\xi^4 v_1 = \xi^3 v_2 = \xi \mu_H v_2 = \mu_H^2 v_1 = 0$$

– the result then follows because $p > 4$. Indeed we have $\xi^{p-1} V = 0$.

Clearly $\mu_H V$ is isomorphic to the 2-dimensional indecomposable

$k(G/H)$-module. Thus (2.4.3) gives that V has a unique maximal submodule. Hence, by (2.4.2), V is indecomposable and terminal. Because $\xi^{p-1}V = 0$, the line $Y(V)$ must be equal to $k(\xi + J^2)$.

Suppose that $V \cong \zeta^i kG$ for some $i = 1, 2, \ldots, p-1$, $\zeta \in J - J^2$. By dimensions, $i = p-2$, so we must have $\zeta^2 V = 0$. We can write ζ in the form:

$$\zeta = \sum_{j=0}^{p-1} \rho_j \xi^j \qquad \text{for some } \rho_j \in kH.$$

We have that $k(\xi + J^2) = Y(V) = Y(\zeta^i kG) = k(\zeta + J^2)$ thus:

$$\zeta \equiv c\xi \pmod{J^2} \quad \text{for some } 0 \neq c \in k.$$

Hence $\rho_0 \in \mathrm{Aug}(kH)^2$, $\rho_1 - c1 \in \mathrm{Aug}(kH)$. Note that:

$$\zeta^2 = \rho_0^2 + 2\rho_0(\rho_1 + \rho_2\xi^2 + \rho_3\xi^3) + \rho_1^2\xi^2 + 2\rho_1\rho_2\xi^3$$
$$+ \text{(terms involving } \xi^4 \text{ and higher)}$$

so:

$$0 = \zeta^2 v_1 = \rho_0^2 v_1 + 2\rho_0\rho_1 v_2 + (2\rho_0\rho_2 + \rho_1^2)\mu_H v_1 + (2\rho_0\rho_3 + 2\rho_1\rho_2)\mu_H v_2$$
$$= (\rho_0^2 + c^2\mu_H)v_1 + 2\rho_1(\rho_0 + \rho_2\mu_H)v_2$$

thus:

$$\rho_0^2 + c^2\mu_H = 0 \ , \quad 2\rho_1(\rho_0 + \rho_2\mu_H) = 0.$$

But ρ_1 is a unit, so the latter equation implies that $\rho_0 = -\rho_2\mu_H$. Thus:

$$c^2\mu_H = -\rho_0^2 = -\rho_2^2\mu_H^2 = 0$$

– contradicting the fact that $c \neq 0$.

Hence V is a non-projective kG-module which is free on restriction to H

and has a unique maximal submodule, but is not isomorphic to $\xi^i kG$ for any $\xi \in J - J^2$, $i = 1,2,\ldots,p-1$.

§5.3 Restricted and induced modules

Now let G be a general p-group and N be a subgroup of G. The inclusion map $N \longrightarrow G$ gives rise to a k-algebra homomorphism $kN \longrightarrow kG$ which, in turn, induces a k-linear map:

$$i_N : J'/J'^2 \longrightarrow J/J^2$$

where J' denotes the augmentation ideal of kN. The image of i_N is just the subspace S_N introduced in §3.6.

If U is a kN-module which is free on restriction to the Frattini subgroup of N, $\Phi(N)$, we consider the $k(N/\Phi(N))$-module, $\mu_{\Phi(N)}U$. This has a Carlson variety, which we shall denote by $Y'(\mu_{\Phi(N)}U)$, which is a subset of J'/J'^2.

<u>Theorem 5.3.1</u> Let V be a kG-module which is free on restriction to some maximal subgroup of G. Then $V_{\downarrow N}$ is free on restriction to some maximal subgroup of N, and:

$$i_N(Y'(\mu_{\Phi(N)}V)) \subseteq Y(\bar{V}).$$

<u>Proof</u> Note that if V is free on restriction to the maximal subgroup H then $V_{\downarrow N}$ is free on restriction to $H \cap N$. But $H \cap N$ is either the whole of N or a maximal subgroup of it. Thus the first statement is trivial.

For the second statement, it suffices to assume that V is indecomposable and is not free on restriction to N - in the latter case we would have that $Y'(\mu_{\Phi(N)}V) = \{0\}$, so the result would be trivial. In this case we have an explicit expression for the line, $Y(\bar{V})$, in terms of the constants $\lambda_H(V)$ (see (3.7.1)(b)).

Let U be any non-projective, indecomposable direct summand of $V_{\downarrow N}$. There are maps $U \xrightarrow{i} V_{\downarrow N} \xrightarrow{\pi} U$ with composite 1_U. (3.7.1)(b) also gives an explicit expression for the line, $Y'(\mu_{\Phi(N)}U)$; we will show that this line is mapped onto $Y(\bar{V})$ by i_N. This will suffice to prove the result.

We trivially see that $\Phi(N) \leqslant \Phi \cap N$. Thus we may choose a minimal set of generators, $h_1, \dots, h_r$, for N such that $h_{s+1}, \dots, h_r$ generate $\Phi \cap N$ modulo $\Phi(N)$. We may extend $h_1, \dots, h_s$ to a minimal set of generators for G, $g_1, \dots, g_n$ ($g_i = h_i$ for $i = 1,2,\dots,s$). Now define the maximal subgroups, H_i $(i = 1,2,\dots,n)$, as in §3.6.

Note that $H_i \geqslant N$ for $i > s$. Thus V cannot be free on restriction to H_i in this case. Thus (3.7.1)(a) gives that V is free on restriction to one of $H_1, \dots, H_s$; without loss of generality, we may assume that V is free on restriction to H_1. Let us use H_1 as the base in calculating the constants, $\lambda_{H_i}(V)$ $(i = 1,2,\dots,n)$. By (3.3.2), we know that $\lambda_{H_i}(V) = 0$ for $i > s$.

If we take the sequence:

$$0 \longrightarrow k_G \longrightarrow k(G/H_i) \longrightarrow k(G/H_i) \longrightarrow k_G \longrightarrow 0 \qquad (i \leqslant s),$$

in which the central map is given by $H_i \longmapsto (g_i - 1)H_i$, and restrict it to N we obtain:

$$0 \longrightarrow k_N \longrightarrow k(N/M_i) \longrightarrow k(N/M_i) \longrightarrow k_N \longrightarrow 0$$

where the central map is given by $M_i \longmapsto (g_i - 1)M_i$. Here:

$$\begin{aligned} M_i &= N \cap H_i = \langle g_1, \dots, g_{i-1}, \Phi \cap N, g_{i+1}, \dots, g_s \rangle \\ &= \langle h_1, \dots, h_{i-1}, \Phi(N), h_{i+1}, \dots, h_r \rangle . \end{aligned}$$

Note that U is free on restriction to M_1 and that the M_i's are in precisely the right form to apply (3.7.1) to U.

Take the defining diagram for $\lambda_{H_i}(V)$:

$$\begin{tikzcd} 0 \arrow[r] & V \arrow[r] & k(G/H_i) \otimes V \arrow[r] & k(G/H_i) \otimes V \arrow[r] & V \arrow[r] & 0 \\ 0 \arrow[r] & V \arrow[r] \arrow[u, "\alpha"] & k(G/H_1) \otimes V \arrow[r] \arrow[u] & k(G/H_1) \otimes V \arrow[r] \arrow[u] & V \arrow[r] \arrow[u, equal] & 0 . \end{tikzcd}$$

Restricting to H, we may form the diagram:

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & U & \longrightarrow & k(N/M_1)\otimes U & \longrightarrow & k(N/M_1)\otimes U & \longrightarrow & U & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle\pi} & & \uparrow & & \uparrow & & \uparrow{\scriptstyle\pi} & & \\
0 & \longrightarrow & V_{\downarrow H} & \longrightarrow & k(N/M_1)\otimes V_{\downarrow H} & \longrightarrow & k(N/M_1)\otimes V_{\downarrow H} & \longrightarrow & V_{\downarrow H} & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle\alpha} & & \uparrow & & \uparrow & & \| & & \\
0 & \longrightarrow & V_{\downarrow H} & \longrightarrow & k(N/M_1)\otimes V_{\downarrow H} & \longrightarrow & k(N/M_1)\otimes V_{\downarrow H} & \longrightarrow & V_{\downarrow H} & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle\iota} & & \uparrow & & \uparrow & & \uparrow{\scriptstyle\iota} & & \\
0 & \longrightarrow & U & \longrightarrow & k(N/M_1)\otimes U & \longrightarrow & k(N/M_1)\otimes U & \longrightarrow & U & \longrightarrow & 0
\end{array}$$

$\pi\alpha\iota - \lambda_{H_1}(V)1_U$ is nilpotent (cf. the proof of (3.3.1)), thus:

$$\lambda_{M_1}(U) = \lambda_{H_1}(V).$$

But (3.7.1)(b) gives that:

$$Y(\bar{V}) = k\Big(\sum_{i=1}^{s} \lambda_{H_i}^{1/p}(g_i - 1) + J^2\Big)$$

and $$Y'(\mu_{g(N)}U) = k\Big(\sum_{i=1}^{s} \lambda_{M_i}^{1/p}(h_i - 1) + (\text{terms in } (h_i - 1) \text{ for } i > s) + J'^2\Big).$$

Thus the result follows on noting that:

$$i_N(h_i) = \begin{cases} g_i & \text{for } i \leq s, \\ 1 & \text{for } i > s. \end{cases}$$ □

Suppose now that U is a non-projective, indecomposable kN-module which is free on restriction to some maximal subgroup of N. With the notation above, $i_N(Y'(\mu_{g(N)}U))$ is just $\{0\}$ unless $\lambda_{M_i}(U) \neq 0$ for some $i \leq s$. So assume that we are in the latter case. Then U is free on restriction to $M_i = N \cap H_i$. There is only one double coset $H_i g N = gH_i N = G$, so the Mackey decomposition gives:

$$(U^{\uparrow G})_{\downarrow H_i} \cong (U_{\downarrow M_i})^{\uparrow H_i}$$

so $U^{\uparrow G}$ is free on restriction to H_i. We have that $U \mid (U^{\uparrow G})_{\downarrow N}$ so that we may choose an indecomposable direct summand, V, of $U^{\uparrow G}$ with $U \mid V_{\downarrow N}$ (in fact $U^{\uparrow G}$ is indecomposable – see [Gr]). (5.3.1) then gives:

$$i_N(Y'(\mu_{E(N)}U)) = Y(\bar{V}).$$

Hence we have shown that in all cases $i_N(Y'(\mu_{E(N)}U)) \subseteq Y_G$. Thus:

<u>Theorem 5.3.2</u> $i_N(Y_N) \subseteq Y_G$. □

<u>Corollary 5.3.3</u> $Y_G \supseteq \bigcup_N S_N$ where the union is over all the elementary abelian subgroups of G.

<u>Proof</u> This follows easily from (5.3.2) and (4.0.2). □

In fact, Chouinard's theorem gives that a kG–module is projective if and only if it is free on restriction to N for all elementary abelian subgroups, N, of G. Thus (4.0.3) shows that the opposite inclusion to that in (5.3.2) holds. Hence we have equality.

The method of inducing modules up from subgroups, particularly elementary abelian subgroups, may be used to construct many more examples for general groups than those already given.

§5.4 Group and field automorphisms

Let α be either an automorphism of the group G, or of the field k. We may extend α to a ring automorphism of kG by defining:

$$\alpha\Big(\sum_{g\in G} \lambda_g g\Big) = \begin{cases} \sum_{g\in G} \lambda_g \alpha(g) & \text{if } \alpha \text{ acts on } G, \\ \sum_{g\in G} \alpha(\lambda_g) g & \text{if } \alpha \text{ acts on } k. \end{cases}$$

Given a kG–module, W, we may define another module, which will be denoted by W^α, to have the same underlying abelian group as W but with the action of kG being given by:

$$\xi . w = \alpha(\xi) w \qquad (\xi \in kG,\ w \in W).$$

It is readily checked that this does make W^α into a kG–module of the same dimension as W.

Suppose now that W is free on restriction to Φ. We know that $\alpha(\mu_\Phi) = \mu_\Phi$ – when α is a group automorphism this is because Φ is a characteristic subgroup of G, and so has its elements permuted by α; when α is a field automorphism it is simply because $\alpha(1) = 1$ – therefore:

$$\dim_k \mu_\Phi . W^\alpha = \dim_k \mu_\Phi W = \frac{1}{|\Phi|} \dim_k W = \frac{1}{|\Phi|} \dim_k W^\alpha .$$

Hence W^α is free on restriction to Φ. For $\xi \in J - J^2$:

$$\begin{aligned} \xi + J^2 \notin Y(\bar{W}^\alpha) &\Leftrightarrow \dim_k \xi^{p-1} \mu_\Phi . W^\alpha = \frac{1}{p} \dim_k \mu_\Phi . W^\alpha \\ &\Leftrightarrow \dim_k \alpha(\xi)^{p-1} \mu_\Phi W = \frac{1}{p} \dim_k \mu_\Phi W \\ &\Leftrightarrow \alpha(\xi) + J^2 \notin Y(\bar{W}). \end{aligned}$$

Thus, if $\alpha^* : J/J^2 \longrightarrow J/J^2$ is the map induced by α,

$$\alpha^*(Y(\bar{W}^\alpha)) = Y(\bar{W}).$$

Let α' denote the inverse of α. We have that $W^{\alpha'}$ is free on restriction to Φ (replace α by α' in the argument above), and clearly:

$$(W^{\alpha'})^{\alpha} = W.$$

So, replacing W by $W^{\alpha'}$ in the equation above, we have:

$$\alpha^{*}(Y(W)) = Y(W^{\alpha'}).$$

A consequence of this and the earlier equation is:

<u>Theorem 5.4.1</u> Let α be an automorphism of the group G, or of the field k. α induces a map $\alpha^{*} : J/J^2 \longrightarrow J/J^2$ such that:

$$\alpha^{*}(Y_G) = Y_G. \qquad \square$$

APPENDIX A

THE KLEIN FOUR-GROUP

A well-known almost split sequence

Let us firstly demonstrate the construction of a certain almost split sequence. We might as well do this in the most general context: let k be any field of characteristic p and G be any finite group. Let P be a projective, indecomposable kG-module and write U for the simple module, $\mathrm{soc}(P)$. We shall assume that $U \neq P$. We know that there are exact sequences:

$$0 \longrightarrow U \longrightarrow P \longrightarrow \Omega^{-1}U \longrightarrow 0$$

$$0 \longrightarrow \Omega U \longrightarrow P \longrightarrow U \longrightarrow 0$$

which we may join to form a two-step projective resolution of $\Omega^{-1}U$. Thus we have the first ingredient we need for the construction of an almost split sequence with $\Omega^{-1}U$ as its right-hand term (see §2.0). For the second, note that P, and hence $\Omega^{-1}U$, has a unique maximal submodule, the factor module by which is isomorphic to U. Thus $(\Omega^{-1}U, U) \cong (U,U)$ so that we have a bimodule isomorphism:

$$[\Omega^{-1}U, U] \cong [U,U] \cong [\Omega^{-1}U, \Omega^{-1}U].$$

Hence any nonzero element of $(\Omega^{-1}U, U)$ generates $\mathrm{soc}\,[\Omega^{-1}U, U]$. Thus we may take the map "θ" of §2.0 to be that induced by the natural epimorphism $f : P \longrightarrow U$. Now form the pull-back:

$$\begin{tikzcd}
0 \arrow[r] & \Omega U \arrow[r] & P \arrow[r, "f"] & U \arrow[r] & 0 \\
0 \arrow[r] & \Omega U \arrow[r, dashed] \arrow[u, equal] & X \arrow[r, dashed] \arrow[u, dashed] & \Omega^{-1}U \arrow[r] \arrow[u, "\theta"'] & 0
\end{tikzcd}$$

then $\quad X = \{(x,y) \in P \oplus \Omega^{-1}U \mid f(x) = \theta(y)\}$

$\quad\quad\; = \{(x, z+U) \mid x, z \in P,\ f(x) = f(z)\}.$

Thus X is the direct sum of the submodules:

$$X_1 = \{(x, x+U) \mid x \in P\} \cong P$$

and $$X_2 = \{(0, z+U) \mid z \in \ker f\} \cong \mathrm{Rad}(P)/\mathrm{soc}(P).$$

$\mathrm{Rad}(P)/\mathrm{soc}(P)$ is usually called the *heart* of P and is denoted by $H(P)$. To conclude, we have shown that there is an almost split sequence:

$$0 \longrightarrow \mathrm{Rad}(P) \longrightarrow P \oplus H(P) \longrightarrow P/\mathrm{soc}(P) \longrightarrow 0$$

– for note that the end terms are just $\Omega^{\pm 1}U$.

A result of Benson and Carlson

We now summarise some of the results proved in [B&C]. To simplify matters slightly, we shall assume that k is an algebraically closed field of characteristic p and that G is a p-group. The following result is proved using essentially elementary techniques (proof omitted):

Theorem A.1 Let U, V be indecomposable kG-modules. Then:

$$[k_G \mid U^* \otimes V] = \begin{cases} 1 & \text{if } U \cong V,\ p \nmid \dim_k U, \\ 0 & \text{otherwise.} \end{cases} \qquad \square$$

Now we have that:

$$[\Omega^{-1}k_G \mid U^* \otimes V] = [k_G \mid \Omega(U^* \otimes V)] = [k_G \mid U^* \otimes \Omega V]$$

since $U^* \otimes \Omega V \cong \Omega(U^* \otimes V) \oplus \text{projective}$ by Schanuel's lemma. Thus Theorem A.1 and (2.2.3) give:

$$([U^* \otimes V], g(\Omega^{-1}k_G)) = ([V], [U].g(\Omega^{-1}k_G))$$
$$= \begin{cases} -1 & \text{if } U \cong \Omega V,\ p \nmid \dim_k U, \\ 0 & \text{otherwise.} \end{cases}$$

Let us now consider two cases:

(1) $p \mid \dim_k U$ then $([V], [U].g(\Omega^{-1}k_G)) = 0$ for all indecomposable modules, V, and thus $(x, [U].g(\Omega^{-1}k_G)) = 0$ for all $x \in A_k(G)$. But $(-,-)$ is a nonsingular bilinear form on $A_k(G)$, therefore:

$$[U].g(\Omega^{-1}k_G) = 0.$$

(2) $p \nmid \dim_k U$ then $([V], [U].g(\Omega^{-1}k_G)) = -(\Omega^{-1}U|V] = ([V], g(\Omega^{-1}U))$. Again, because $(-,-)$ is nonsingular, this implies that:

$$[U].g(\Omega^{-1}k_G) = g(\Omega^{-1}U).$$

By what we proved above we know that there is an almost split sequence:

$$0 \longrightarrow \Omega k_G \longrightarrow kG \oplus H(kG) \longrightarrow \Omega^{-1}k_G \longrightarrow 0$$

so that we may calculate $g(\Omega^{-1}k_G)$ explicitly. Note that $\Omega^{\pm 1}k_G \otimes U \cong \Omega^{\pm 1}U$ modulo projective modules, thus:

$$[U].g(\Omega^{-1}k_G) = [H(kG) \otimes U] - [\Omega^{-1}U] - [\Omega U] + c[kG]$$

for some $c \in \mathbb{Z}$. Therefore the results above give:

<u>Theorem A.2</u> Let U be an indecomposable kG-module. Working modulo projectives, $H(kG) \otimes U$ is isomorphic to:

(1) $\Omega^{-1}U \oplus \Omega U$ if $p \mid \dim_k U$,

(2) the middle term of the almost split sequence:

$$0 \longrightarrow \Omega U \longrightarrow X \longrightarrow \Omega^{-1}U \longrightarrow 0$$

if $p \nmid \dim_k U$. $\square$

Application to the Klein four-group

Now let k be an algebraically closed field of characteristic 2 and

$$G = \langle x,y \mid x^2 = y^2 = (xy)^2 = 1 \rangle$$

be the Klein four-group. Theorem A.2 may be used to give a complete classification of the isomorphism classes of indecomposable kG-modules. This classification is of course well-known, having first been determined by Bašev in [Baš] and, independently, by Heller and Reiner in [H&R]. The approach used in both these papers is to solve the equivalent problem of classifying pairs of matrices X, Y with coefficients in k satisfying:

$$X^2 = Y^2 = 0 \ , \ XY = YX.$$

(Also see [Co].) The key to our approach lies in the following observation:

Lemma A.3 $H(kG) \cong k_G \oplus k_G$.

Proof Rad(kG) is spanned by $x-1$, $y-1$, μ_G; soc(kG) equals $k\mu_G$. Note that:

$$(x-1).(x-1) = (y-1).(y-1) = 0$$
$$(y-1).(x-1) = (x-1).(y-1) = \mu_G$$

so that G acts trivially on Rad(kG)/soc(kG) and the result follows. □

So let U be an indecomposable kG-module and apply Theorem A.2. We have two cases:

(1) $\dim_k U$ even then $U \oplus U \cong \Omega^{-1}U \oplus \Omega U \oplus P$ for some projective module, P. Hence U is either projective (in which case $U \cong kG$) or periodic of period one. We return to the case where U is periodic below.

(2) $\dim_k U$ odd then the middle term of the almost split sequence:

$$0 \longrightarrow \Omega U \longrightarrow X \longrightarrow \Omega^{-1}U \longrightarrow 0$$

is isomorphic to $U \oplus U \oplus P$ for some projective module, P. But (2.2.6)(a) gives that $P = 0$ unless $U \cong k_G$. We claim that $U \cong \Omega^n k_G$ for some $n \in \mathbb{Z}$: otherwise choose U of minimal dimension not isomorphic to any of these syzygies. We have an almost split sequence:

$$0 \longrightarrow \Omega U \longrightarrow U \oplus U \longrightarrow \Omega^{-1}U \longrightarrow 0$$

and $\dim_k \Omega^{\pm 1} U \geq \dim_k U$. Thus equality of dimensions must hold, contradicting (2.2.6)(b).

Thus the only indecomposable kG-modules of odd dimension are $\Omega^n k_G$ $(n \in \mathbb{Z})$. As a bonus we also have that the almost split sequences involving these modules are:

$$0 \longrightarrow \Omega^{n+1}k_G \longrightarrow \Omega^n k_G \oplus \Omega^n k_G \longrightarrow \Omega^{n-1}k_G \longrightarrow 0 \qquad (n \neq 0)$$

$$0 \longrightarrow \Omega k_G \longrightarrow kG \oplus k_G \oplus k_G \longrightarrow \Omega^{-1}k_G \longrightarrow 0\,.$$

The periodic kG-modules

Now return to the case above, where U is an indecomposable kG-module which is periodic of period one. We could quote (3.2.2) to show immediately that U is free on restriction to either $\langle x \rangle$ or $\langle y \rangle$; however we shall give a simplified, self-contained proof of this fact.

Lemma A.4 Let V be a kG-module with $V^* \cong \Omega V \cong V$. Then:

$$\operatorname{soc}(V) = (x-1)V + (y-1)V.$$

Proof Let $M = \operatorname{Rad}(kG) \otimes V$ noting that, modulo projective modules, $M \cong \Omega V \cong V$. Thus (0.0.2) shows that:

$$\dim_k \mu_G M = \tfrac{1}{2}(\dim_k M - \dim_k V)$$
$$= \tfrac{1}{2}\dim_k V = n \text{, say.}$$

But each element of M may be written uniquely in the form:

$$m = (x-1)\otimes v_1 + (y-1)\otimes v_2 + \mu_G \otimes v_3 \qquad \text{with } v_i \in V.$$

A simple calculation (bearing in mind that $\mu_G V = 0$) shows that:

$$\mu_G m = \mu_G \otimes ((x-1)v_1 + (y-1)v_2).$$

Thus, writing $V_0 = (x-1)V + (y-1)V$, we have that $\mu_G M = \mu_G \otimes V_0$ so that, in particular, $\dim_k V_0 = n$.

Consider the inclusion map $i : \mathrm{soc}(V^*) \longrightarrow V^*$. This induces an epimorphism $i^* : V \longrightarrow \mathrm{soc}(V^*)^*$ with kernel V_1, say. G acts trivially on V/V_1; thus $(x-1)V$ and $(y-1)V$ are contained in V_1. Hence $V_0 \leqslant V_1$. Now:

$$n = \dim_k V_0 \leqslant \dim_k V_1 = 2n - \dim_k \mathrm{soc}(V^*)^*$$

so $\qquad \dim_k \mathrm{soc}(V) = \dim_k \mathrm{soc}(V^*)^* \leqslant n = \dim_k V_0$.

But trivially $V_0 \leqslant V^G = \mathrm{soc}(V)$ so, by dimensions, the result follows. $\square$

<u>Theorem A.5</u> U is free on restriction to either $\langle x\rangle$ or $\langle y\rangle$.

<u>Proof</u> $(U,U)_k \cong U^* \otimes U$ may be written as $V \oplus P$ where P is projective and V has no projective summands. V clearly satisfies the conditions of Lemma A.4, thus:

$$(U,U) = \mathrm{soc}\,(U,U)_k = \mathrm{soc}\,(V) + \mathrm{soc}\,(P)$$
$$= (x-1)V + (y-1)V + \mu_G P.$$

Hence we can write $1_U = f + f' + f''$ where $f \in (x-1)V$, $f' \in (y-1)V$, $f'' \in \mu_G P$. Note that f, f', f'' are all elements of the local ring (U,U) and that their sum is an automorphism. Thus one of them must be an automorphism of U. This implies that $\mu_H(U,U)_k$ contains an automorphism for $H = \langle x\rangle$, $\langle y\rangle$ or G. Thus (0.0.4) gives the result. □

So assume that U is free on restriction to $\langle y\rangle$. Take a free $k\langle y\rangle$-basis $u_1, \ldots, u_n$ for U, then:

$$u_1, \ldots, u_n, (y-1)u_1, \ldots, (y-1)u_n$$

is a k-basis for U with respect to which y is represented by the matrix:

$$\begin{bmatrix} I & I \\ & I \end{bmatrix}.$$

Note that the matrix representing x must commute with that representing y and hence have the form:

$$\begin{bmatrix} A & B \\ & A \end{bmatrix}$$

for some $n \times n$ matrices, A,B. The equation $(x-1)(y-1)U = \mu_G U = 0$ shows that $A = I$; furthermore, conjugating by a matrix of the form

$$\begin{bmatrix} P & \\ & P \end{bmatrix},$$

we may assume that B is in its Jordan canonical form.

(U,U) is the set of all matrices commuting with those representing x and y, i.e. all those of the form:

$$E = \begin{bmatrix} C & D \\ & C \end{bmatrix} \qquad \text{with } BC = CB.$$

If B consists of more than one Jordan block then there exists $C \neq 0, I$ with $BC = CB$, $C^2 = C$. Thus

$$\begin{bmatrix} C & \\ & C \end{bmatrix}$$

is a nontrivial idempotent in (U,U) – contradicting the fact that U is indecomposable. Conversely, if B is a single Jordan block and the matrix E above satisfies $E^2 = E$, then $BC = CB$, $C^2 = C$, $D = CD + DC$ so that C equals 0 or I and, therefore, $D = 0$. Thus the only idempotents of U in this case are 0 and 1_U – that is to say, U is indecomposable.

Thus the indecomposable kG-modules which are free on restriction to $\langle y \rangle$ are given by representations of the form:

$$x \longmapsto \begin{bmatrix} I & J_n(\lambda) \\ & I \end{bmatrix}, \quad y \longmapsto \begin{bmatrix} I & I \\ & I \end{bmatrix}$$

where $J_n(\lambda)$ is the $n \times n$ Jordan block:

$$\begin{vmatrix} \lambda & 1 & & \\ & \cdots & & \\ & & \lambda & 1 \\ & & & \lambda \end{vmatrix}.$$

Let us denote the module given by this representation by $V_{n,\lambda}$.

Write $L = k(x-1) \oplus k(y-1)$ and let us calculate $Y_L(V_{n,\lambda})$. Let $\xi = c_1(x-1) + c_2(y-1)$ be any nonzero element of L; ξ is represented by the matrix:

$$\begin{bmatrix} 0 & Z \\ & 0 \end{bmatrix}$$

where Z is the $n \times n$ matrix with $(c_1\lambda + c_2)$'s on the diagonal, c_1's on the supra-diagonal and 0's elsewhere. (0.0.8) gives that $V_{n,\lambda}$ is free on restriction to $\langle 1+\xi \rangle$ iff Z has rank n – that is to say, iff $c_1\lambda + c_2 \neq 0$. Therefore:

$$Y_L(V_{n,\lambda}) = k((x-1) + \lambda(y-1)).$$

Similarly the indecomposable kG-modules which are free on restriction to $\langle x\rangle$ are the $V'_{n,\lambda}$'s given by the representations:

$$x \longmapsto \begin{bmatrix} I & I \\ & I \end{bmatrix}, \quad y \longmapsto \begin{bmatrix} I & J_n(\lambda) \\ & I \end{bmatrix}.$$

We have that:

$$Y(V'_{n,\lambda}) = k(\lambda(x-1) + (y-1))$$

so that if $\lambda \neq 0$, $V'_{n,\lambda}$ is also free on restriction to $\langle y\rangle$ and so appears in the list above; indeed just considering dimensions and Carlson varieties shows that $V'_{n,\lambda} \cong V_{n,1/\lambda}$. Thus the only new modules to add to the list are the $V'_{n,0}$'s which, by analogy with the isomorphism just given, we may denote by $V_{n,\infty}$.

To conclude: the periodic kG-modules are the $V_{n,\lambda}$ for $n = 1,2,\dots$ and $\lambda \in \mathbb{P}^1(k) = k \cup \{\infty\}$. The Carlson varieties are given by:

$$Y_L(V_{n,\lambda}) = \left\{ c_1(x-1) + c_2(y-1) \,\middle|\, c_2/c_1 = \lambda \right\}.$$

Note that, if we identify the projective line $\mathbb{P}^1(k)$ with the set of lines in L in the obvious way, then $Y_L(V_{n,\lambda})$ is just the line corresponding to λ.

The classification theorem

Theorem A.6 The indecomposable kG-modules are of three types:

(i) the projective module, kG,

(ii) the periodic modules, $V_{n,\lambda}$ $(n = 1,2,\ldots;\ \lambda \in \mathbb{P}^1(k))$,

(iii) the syzygies of the trivial module, $\Omega^n k_G$ $(n \in \mathbb{Z})$.

Let us list some of the important properties of these modules:

(a) Dimension

(i) $\dim_k kG = 4$,

(ii) $\dim_k V_{n,\lambda} = 2n$,

(iii) $\dim_k \Omega^n k_G = 2|n| + 1$.

(b) Carlson variety $(J = \mathrm{Aug}(kG))$

(i) $Y(kG) = \{0\}$,

(ii) $Y(V_{n,\lambda}) = \{c_1(x-1) + c_2(y-1) + J^2 \mid c_2/c_1 = \lambda\}$,

(iii) $Y(\Omega^n k_G) = J/J^2$.

(c) Almost split sequences

(ii) $0 \longrightarrow V_{n,\lambda} \longrightarrow V_{n-1,\lambda} \oplus V_{n+1,\lambda} \longrightarrow V_{n,\lambda} \longrightarrow 0 \qquad (n \neq 1)$

$0 \longrightarrow V_{1,\lambda} \longrightarrow V_{2,\lambda} \longrightarrow V_{1,\lambda} \longrightarrow 0,$

(iii) $0 \longrightarrow \Omega^{n+1} k_G \longrightarrow \Omega^n k_G \oplus \Omega^n k_G \longrightarrow \Omega^{n-1} k_G \longrightarrow 0 \qquad (n \neq 0)$

$0 \longrightarrow \Omega k_G \longrightarrow kG \oplus k_G \oplus k_G \longrightarrow \Omega^{-1} k_G \longrightarrow 0.$

(d) Dual module

(i) $kG^* \cong kG$,

(ii) $(V_{n,\lambda})^* \cong V_{n,\lambda}$,

(iii) $(\Omega^n k_G)^* \cong \Omega^{-n} k_G$.

__Proof__ The only items that we have yet to prove are the following:

(a)(iii) may be proved by taking dimensions in the almost split sequences (c)(iii) and using induction.

(b)(iii) is obvious: a module of odd dimension cannot be free on restriction to any subgroup of order 2.

(c)(ii) the modules which are free on restriction to some maximal subgroup of G fall into disjoint sequences V_n $(n=1,2,\ldots)$ as in (2.3.6). Clearly $Y(V_1) = Y(V_2) = \ldots$ so there exists λ with $V_n \cong V_{n,\lambda}$ for all n. Thus the result follows.

(d)(ii) is clear using Carlson varieties and dimensions. □

__Remark__ By considering their dimensions and their Carlson varieties, we see that the modules given above are pairwise non-isomorphic.

__Irreducible maps__

Theorem A.6(c) and (2.2.4) show that the irreducible maps between the indecomposable kG-modules are of five types –

(1) monomorphisms $V_{n,\lambda} \longrightarrow V_{n+1,\lambda}$,

(2) epimorphisms $V_{n+1,\lambda} \longrightarrow V_{n,\lambda}$,

(3) a monomorphism $\Omega k_G \longrightarrow kG$,

(4) an epimorphism $kG \longrightarrow \Omega^{-1}k_G$,

(5) maps $\Omega^{n+1}k_G \longrightarrow \Omega^n k_G$ which are epimorphisms for $n \geq 0$ and monomorphisms otherwise,

(we know that an irreducible map is either a monomorphism or an epimorphism; we have determined which above by using dimensions).

By (2.2.5) the cokernels or kernels of these maps (according as whether they are monomorphisms or epimorphisms) are indecomposable modules of dimension two (in cases (1),(2) and (5)) or one (in cases (3) and (4)). The only module of dimension one is the trivial module, k_G, so this is the cokernel in case (3) and the kernel in case (4). The indecomposable modules of dimension two are the $V_{1,\mu}$'s ($\mu \in \mathbb{P}^1(k)$); thus we may ask: what values of μ are such that $V_{1,\mu}$ is allowed as the cokernel/kernel in the cases (1),(2) and (5)?

<u>Cases (1) and (2)</u> (2.3.7) or a simple argument using Carlson varieties shows that the irreducible maps in these cases are:

(1) monomorphisms $V_{n,\lambda} \longrightarrow V_{n+1,\lambda}$ with cokernel $V_{1,\lambda}$,

or (2) epimorphisms $V_{n+1,\lambda} \longrightarrow V_{n,\lambda}$ with kernel $V_{1,\lambda}$.

<u>Case (5)</u> We claim that, for all $n \in \mathbb{Z}$ and all $\mu \in \mathbb{P}^1(k)$, there exists an irreducible map $\Omega^{n+1}k_G \longrightarrow \Omega^n k_G$ with cokernel/kernel isomorphic to $V_{1,\mu}$.

<u>Proof</u> The dual of an irreducible map is also irreducible, thus an irreducible epimorphism $\Omega^{n+1}k_G \longrightarrow \Omega^n k_G$ with kernel $V_{1,\mu}$ ($n \geq 0$) gives rise to an irreducible monomorphism $\Omega^{-n}k_G \longrightarrow \Omega^{-n-1}k_G$ with cokernel $V_{1,\mu}$. Thus it suffices to assume that $n \geq 0$.

Suppose that we have an exact sequence:

$$0 \longrightarrow V_{1,\mu} \longrightarrow \Omega^{n+1}k_G \xrightarrow{f} \Omega^n k_G \longrightarrow 0$$

where f is irreducible. Applying the Heller operator to this sequence is readily seen to give another exact sequence:

$$0 \longrightarrow V_{1,\mu} \longrightarrow \Omega^{n+2}k_G \xrightarrow{f'} \Omega^{n+1}k_G \longrightarrow 0$$

where f' is irreducible. Thus it suffices to assume that $n = 0$.

Hence we must consider irreducible maps $\Omega k_G \longrightarrow k_G$, that is to say maps:

$$\Omega k_G \xrightarrow{\rho} kG \oplus k_G \oplus k_G \xrightarrow{\pi} k_G$$

where ρ is the map from the almost split sequence with Ωk_G as its left-hand term and π is a split epimorphism.

The construction given above shows that, if we identify Ωk_G with Rad (kG), ρ is the direct sum of the inclusion Rad (kG) $\longrightarrow$ kG and the natural map Rad (kG) $\longrightarrow$ Rad (kG)/ soc (kG) $\cong k_G \oplus k_G$. Thus write:

$$\xi = \lambda_1(x-1) + \lambda_2(y-1) + \lambda_3\mu_G$$

for a general element of Rad (kG), then we may take:

$$\rho(\xi) = (\xi, \lambda_1, \lambda_2).$$

If c_1 and c_2 are elements of k, not both zero, then the map:

$$\pi : kG \oplus k_G \oplus k_G \longrightarrow k_G \ , \ (\xi, \lambda_1, \lambda_2) \longmapsto c_1\lambda_1 + c_2\lambda_2$$

is a split epimorphism. We have that $\pi\rho$ is an irreducible map. Write:

$$\ell = c_2(x-1) + c_1(y-1)$$

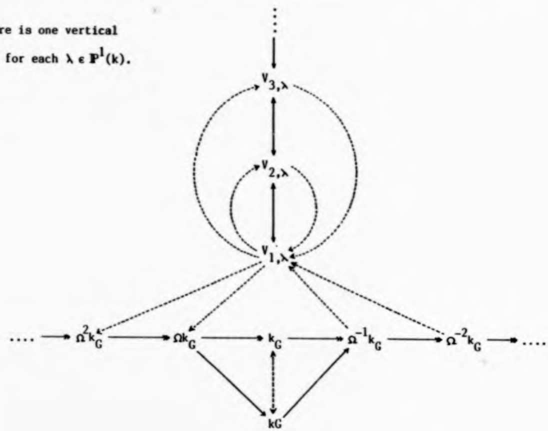
then, for $\xi \in \ker \pi\rho$, $\ell\xi = (c_1\lambda_1 + c_2\lambda_2)\mu_G = 0$. Thus $\ell + J^2 \in Y(\ker \pi\rho)$. Hence $\pi\rho$ has kernel $V_{1,\mu}$ where $\mu = c_1/c_2$.

By choosing c_1 and c_2 suitably we may assume that μ is any given element of $\mathbb{P}^1(k)$. Thus the result follows. □

We may summarise this information by use of the "Extended Auslander-Reiten quiver" given below. The solid lines between the modules indicate the irreducible maps; the dotted lines show the cokernels/kernels of the irreducible maps. Note that this diagram is connected. The general problem of investigating the cokernels/kernels of irreducible maps seems to have been very little studied. Obviously the extended AR-quiver will not always be connected – for example, kG may have more than one block – so what can we say about the modules in the various connected components ?

THE EXTENDED AUSLANDER-REITEN QUIVER OF THE KLEIN 4-GROUP OVER AN ALG. CLOSED FIELD OF CHARACTERISTIC 2

Note There is one vertical component for each $\lambda \in \mathbb{P}^1(k)$.



APPENDIX B

RELATIONS BETWEEN BOCKSTEINS

Introduction

This Appendix is devoted to sketching how the concept of the constants $\lambda_H(V)$ may be extended to give a k-algebra homomorphism:

$$\bigoplus_{m=0}^{\infty} \mathrm{Ext}_{kG}^{2m}(k_G, k_G) \longrightarrow k[X].$$

A brief discussion of this concept shows it to be related to the cohomology variety of V, first introduced by Quillen, and a variety used by Serre in his paper, [Ser].

Notation

The notation is basically that of §3.6 and §4.0. We also let $E^m(G)$ be the same cohomology group as in §1.4 and similarly write:

$$E_k^m(G) = \mathrm{Ext}_{kG}^m(k_G, k_G).$$

Let:

$$E(G) = \bigoplus_{m=0}^{\infty} E^{2m}(G) \quad \text{and} \quad E_k(G) = \bigoplus_{m=0}^{\infty} E_k^{2m}(G)$$

be the corresponding even cohomology rings. Note that the extension of scalars from $\mathbb{F}_p$ to k induces a ring homomorphism:

$$\Psi : E(G) \longrightarrow E_k(G).$$

V will be a non-projective, indecomposable kG-module which is free on restriction to some maximal subgroup, H_0, of G. We have a two-step projective resolution of V:

$$0 \longrightarrow V \longrightarrow P \longrightarrow P \longrightarrow V \longrightarrow 0 \qquad -(1)$$

obtained by tensoring the sequence:

$$0 \longrightarrow k_G \longrightarrow k(G/H_0) \longrightarrow k(G/H_0) \longrightarrow k_G \longrightarrow 0$$

by V.

The k-algebra homomorphism $\Lambda: E_k(G) \longrightarrow k[X]$

We may think of $E_k^{2m}(G)$ as being the set of equivalence classes of exact sequences of the form:

$$0 \longrightarrow k_G \longrightarrow X_{2m} \longrightarrow \dots\dots \longrightarrow X_1 \longrightarrow k_G \longrightarrow 0. \qquad -(2)$$

Given such a sequence, we may form the diagram:

$$\begin{tikzcd}
0 \arrow[r] & V \arrow[r] & X_{2m} \otimes V \arrow[r] & \cdots\cdots \arrow[r] & X_1 \otimes V \arrow[r] & V \arrow[r] & 0 \\
0 \arrow[r] & V \arrow[r] \arrow[u, dashed, "f"] & P \arrow[r] \arrow[u, dashed] & \cdots\cdots \arrow[r] & P \arrow[r] \arrow[u, dashed] & V \arrow[r] \arrow[u, equal] & 0
\end{tikzcd}$$

where the upper sequence is the result of tensoring (2) by V and the lower is the join of m copies of (1). f is determined by (2) up to the addition of a map which factors through P; thus the image of f in [V,V] is uniquely determined.

<u>Lemma B.1</u> Mapping (2) to the image of f induces a k-linear map:

$$\alpha_m : E_k^{2m}(G) \longrightarrow [V,V].$$

<u>Proof</u> The image of (2) in $E_k^{2m}(G) \cong (\Omega^{2m}k_G, k_G)$ is the map x in the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & k_G \arrow[r] & X_{2m} \arrow[r] & \cdots\cdots \arrow[r] & X_1 \arrow[r] & k_G \arrow[r] & 0 \\
0 \arrow[r] & \Omega^{2m}k_G \arrow[r] \arrow[u, dashed, "x"] & Q_{2m} \arrow[r] \arrow[u, dashed] & \cdots\cdots \arrow[r] & Q_1 \arrow[r] \arrow[u, dashed] & k_G \arrow[r] \arrow[u, equal] & 0
\end{tikzcd}$$

where the lower sequence is a minimal projective resolution of k_G.

We may also form the diagram:

$$\begin{tikzcd}
0 \arrow[r] & \Omega^{2m}k_G \otimes V \arrow[r] & Q_{2m} \otimes V \arrow[r] & \cdots\cdots \arrow[r] & Q_1 \otimes V \arrow[r] & V \arrow[r] & 0 \\
0 \arrow[r] & V \arrow[r] \arrow[u, dashed, "i"] & P \arrow[r] \arrow[u, dashed] & \cdots\cdots \arrow[r] & P \arrow[r] \arrow[u, dashed] & V \arrow[r] \arrow[u, equal] & 0.
\end{tikzcd}$$

Tensoring the first diagram by V and attaching the second, we see that we may take $f = (x \otimes 1_V)i$. But i is fixed, so the map $(2) \longmapsto f$ does induce a well

-defined k-linear map $E_k^{2m}(G) \longrightarrow [V,V]$. □

<u>Lemma B.2</u> For $x \in E_k^{2m}(G)$, $x' \in E_k^{2m'}(G)$:

$$\alpha_{m+m'}(xx') = \alpha_m(x)\alpha_{m'}(x') = \alpha_{m'}(x')\alpha_m(x).$$

<u>Proof</u> This is an easy consequence of the two diagrams (over). □

Hence, if we define $\Lambda_0 : E_k(G) \longrightarrow [V,V][X]$ by mapping $x \in E_k^{2m}(G)$ to $\alpha_m(x)X^m$, Λ_0 is a k-algebra homomorphism.

($[V,V][X]$ denotes the ring of polynomials in X with coefficients in the k-algebra, $[V,V]$.)

Define a map $tr : (V,V) \longrightarrow k$ by setting $tr(f)$ equal to the unique scalar, λ, such that $f - \lambda 1_V$ is a non-automorphism of V. tr is readily checked to be an algebra homomorphism.

Any endomorphism of V which factors through a projective module is a non-automorphism, and hence is in the kernel of tr. Thus tr induces a k-algebra homomorphism, $tr^* : [V,V] \longrightarrow k$. We may extend tr^* in the obvious way to a k-algebra homomorphism $[V,V][X] \longrightarrow k[X]$. Define:

$$\Lambda : E_k(G) \longrightarrow k[X]$$

to be the composite of this map and Λ_0.

Let b_H denote the image in $E_k^2(G)$ of the sequence:

$$0 \longrightarrow k_G \longrightarrow k(G/H) \longrightarrow k(G/H) \longrightarrow k_G \longrightarrow 0,$$

then:

$$\Lambda(b_H) = \lambda_H(V)X.$$

In particular, taking $H = H_0$, we see that $X \in \operatorname{Im} \Lambda$. Thus Λ is surjective.

(1)

$$
\begin{tikzcd}[column sep=small]
0 \arrow[r] & V \arrow[r] & X'_{m'} \otimes V \arrow[r] & \cdots \arrow[r] & X'_1 \otimes V \arrow[rr] \arrow[dr] & & X_m \otimes V \arrow[r] & \cdots \arrow[r] & X_1 \otimes V \arrow[r] & V \arrow[r] & 0 \\
& & & & & V \arrow[ur] & & & & & \\
0 \arrow[r] & V \arrow[r] \arrow[uu, "f'"] & P \arrow[r] \arrow[uu] & \cdots \arrow[r] & P \arrow[r] \arrow[uu] & V \arrow[u, equal] & & & & & \\
& & & & & V \arrow[u, "f"] \arrow[dr] & & & & & \\
0 \arrow[r] & V \arrow[r] \arrow[uu, "f"] & P \arrow[r] \arrow[uu, "g"] & \cdots \arrow[r] & P \arrow[rr] \arrow[uu, "g"] \arrow[ur] & & P \arrow[r] \arrow[uuuu] & \cdots \arrow[r] & P \arrow[r] \arrow[uuuu] & V \arrow[r, dash] \arrow[uuuu, equal] & 0
\end{tikzcd}
$$

(2)

$$
\begin{tikzcd}[column sep=small]
0 \arrow[r] & V \arrow[r] & X'_{m'} \otimes V \arrow[r] & \cdots \arrow[r] & X'_1 \otimes V \arrow[rr] \arrow[dr] & & X_m \otimes V \arrow[r] & \cdots \arrow[r] & X_1 \otimes V \arrow[r] & V \arrow[r] & 0 \\
& & & & & V \arrow[ur] & & & & & \\
0 \arrow[r] & V \arrow[r] \arrow[uu, "f"] & X'_{m'} \otimes V \arrow[r] \arrow[uu, "h_{m'}"] & \cdots \arrow[r] & X'_1 \otimes V \arrow[r] \arrow[uu, "h_1"] & V \arrow[u, "f"] & & & & & \\
& & & & & V \arrow[u, equal] \arrow[dr] & & & & & \\
0 \arrow[r] & V \arrow[r] \arrow[uu, "f'"] & P \arrow[r] \arrow[uu] & \cdots \arrow[r] & P \arrow[rr] \arrow[uu] \arrow[ur] & & P \arrow[r] \arrow[uuuu] & \cdots \arrow[r] & P \arrow[r] \arrow[uuuu] & V \arrow[r] \arrow[uuuu, equal] & 0
\end{tikzcd}
$$

where: (1) $g = 1_{k(G/H_0)} \otimes f$,

(2) $h_i = 1_{X'_i} \otimes f$.

<u>DIAGRAM ACCOMPANYING THE</u>

<u>PROOF OF LEMMA B.2</u>

those containing $\ker\Lambda$. But:

$$\ker\Lambda = \{ x \in E_k(G) \mid \Lambda_0(x) \in \ker tr^* \}.$$

We may write a general element of $E_k(G)$ in the form:

$$x = \sum_{m=0}^{\infty} x_m \quad \text{with } x_m \in E_k^{2m}(G)$$

where only finitely many x_m's are nonzero. Then:

$$\Lambda_0(x) = \sum_{m=0}^{\infty} \alpha_m(x_m)X^m$$

– this is an element of $\ker tr^*$ if and only if all the $\alpha_m(x_m)$'s are, that is if and only if all the $\alpha_m(x_m)$'s are nilpotent. Thus:

$$\begin{aligned} x \in \ker\Lambda \quad &\Leftrightarrow \quad \text{there is } s \text{ with } \alpha_m(x_m)^{p^s} = 0 \text{ for all } m \\ &\Leftrightarrow \quad \text{there is } s \text{ with} \\ &\qquad \sum_{m=0}^{\infty} \alpha_m(x_m)^{p^s}X^{mp^s} = (\sum_{m=0}^{\infty} \alpha_m(x_m)X^m)^{p^s} = 0 \\ &\qquad \text{(see Lemma B.2)} \\ &\Leftrightarrow \quad \Lambda_0(x)^{p^s} = \Lambda_0(x^{p^s}) = 0 \quad \text{for some } s. \end{aligned}$$

So if I is an ideal of $E_k(G)$ containing $\ker\Lambda_0$ and $x \in \ker\Lambda$ then, for some s, $x^{p^s} \in \ker\Lambda_0 \subseteq I$, so that $(x+I)^{p^s} = 0$ in the division ring $E_k(G)/I$. Thus $x+I = 0$. Hence we have shown that I contains $\ker\Lambda$. Thus the claim above is established.

Thus $X_G(V \otimes V^*)$ is the set of maximal ideals of $E_k(G)$ containing $\ker\Lambda$. But these are in 1-1 correspondence with the maximal ideals of $E_k(G)/\ker\Lambda$, and so with the maximal ideals of $\operatorname{Im}\Lambda = k[X]$. Now $k[X]$ is a principal ideal domain and k is algebraically closed, so the maximal ideals of $k[X]$ are just the principal ideals $(X-c)k[X]$ with $c \in k$. Thus:

$$X_G(V \otimes V^*) = \{ \ker\Lambda_c \mid c \in k \}$$

where Λ_c is the composite $E_k(G) \xrightarrow{\Lambda} k[X] \longrightarrow k$ where the latter map is given by $X \longmapsto c$. Thus we have:

<u>Theorem B.3</u> $X_G(V \otimes V^*) \cong k$. $\square$

In fact it may be shown that $X_G(V) = X_G(V \otimes V^*)$ so that this result may be thought of as being analogous to Theorem 3.7.1(b). To what extent Avrunin and Scott's work connecting the cohomology variety $X_G(W)$ with the Carlson variety $Y(W)$ for elementary abelian groups may be extended to general groups is a question for possible further research.

<u>Relations between Bocksteins</u>

Write b_i for the image of the sequence:

$$0 \longrightarrow \mathbb{F}_p \longrightarrow \mathbb{F}_p(G/H_i) \longrightarrow \mathbb{F}_p(G/H_i) \longrightarrow \mathbb{F}_p \longrightarrow 0$$

in $E^2(G)$, then we may define a ring homomorphism:

$$F : \mathbb{F}_p[X_1,\dots,X_n] \longrightarrow E(G) \quad \text{by } F(X_i) = b_i$$

so that $\ker F$ is the ideal of relations between the b_i's. This ideal may be used to define a variety in J/J^2,

$$B_G = \left\{ \sum_{i=1}^{n} \lambda_i(g_i - 1) + J^2 \;\middle|\; f(\lambda_1,\dots,\lambda_n) = 0 \text{ for all } f \in \ker F \right\}.$$

It is this variety which is considered by Serre in the proof of his proposition, [Ser], except that there it is regarded as a subset of k^n rather than of J/J^2. He is able to prove:

<u>Theorem B.4</u> (a) B_G is the union of a number of rational subspaces.

(b) If G is not elementary abelian then B_G is not the whole of J/J^2. $\square$

Corollary B.5 If G is not elementary abelian then:

$$B_G \subseteq \bigcup_H S_H$$

- the union being over all the maximal subgroups of G.

Proof The theorem implies that in this case B_G is the union of a number of proper rational subspaces. But any proper rational subspace is contained within a rational hyperplane of J/J^2. Now (3.6.4) gives that the rational hyperplanes of J/J^2 are precisely the subspaces S_H for H a maximal subgroup of G. Thus the result follows. □

Now let V be as above. Then for $f \in \ker F$ we have:

$$f(b_1,\dots,b_n) = 0 \quad \text{in } E(G).$$

$$\Rightarrow \quad f(\tau(b_1),\dots,\tau(b_n)) = 0 \quad \text{in } E_k(G).$$

$$\Rightarrow \quad \Lambda f(\tau(b_1),\dots,\tau(b_n)) = 0 \quad \text{in } k[X].$$

But:

$$\Lambda f(\tau(b_1),\dots,\tau(b_n)) = f(\Lambda\tau(b_1),\dots,\Lambda\tau(b_n))$$

and:

$$\Lambda\tau(b_i) = \lambda_{H_i}(V)X\,.$$

Therefore, for all $c \in k$, we have:

$$f(c\lambda_{H_1}(V),\dots,c\lambda_{H_n}(V)) = 0$$

so that:

$$f(c^{1/p}\lambda_{H_1}(V)^{1/p},\dots,c^{1/p}\lambda_{H_n}(V)^{1/p}) = 0\,.$$

Hence (3.7.1)(b) implies that $Y(\bar{V}) \subseteq B_G$. Hence we have proved:

Theorem B.6 $Y_G \subseteq B_G$. □

Note that Chouinard's theorem follows from this and Corollary B.5. The precise relationship between Y_G and B_G is another area for possible future research; it seems not unreasonable to conjecture that $Y_G = B_G$.

Perhaps also this leads us to consider that Serre's proof is not as unrelated to the representation theory as might at first have been thought.

Afterword

It is a pity that we have been unable to complete a non-cohomological proof of Chouinard's theorem to date. However, things are not without hope. Let us indicate three ways in which we might hope to complete the proof by showing that a general pseudo-special group, G, is a Chouinard group.

(1) We could do the calculations described in §4.8, although this does not seem a very attractive proposition.

(2) We could try to show that Y_G is the union of a number of rational subspaces of J/J^2 (possibly using (4.8.4)). This would be analogous to part (a) of Theorem B.4 in the same way as (4.7.1) is analogous to part (b). We could then complete the proof by proving Corollary B.5 with Y_G in place of B_G.

(3) We could try to prove that U_ξ is periodic for a larger range of values of ξ than in (4.5.1) and (4.6.1). (See (4.8.3).) For example, it would suffice to show that U_ξ is periodic when $\xi + J^2$ is not an element of the union of the subspaces S_H for H a maximal subgroup of G.

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