SUBNORMALITY, ASCENDANCY AND PROJECTIVITIES

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CONTENTS.

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Declaration.

All the results presented in this thesis are original, except where otherwise explicitly stated.

CHAPTER 0.

§0.0 Introduction.

In 1939, Wielandt introduced the concept of subnormality ([W1]) and proved that in a finite group, the join of the two (and hence any number of) subnormal subgroups is again subnormal. This result does not hold for arbitrary groups (see [ZH], [RS]). After much work by various authors, Williams [WS] gave necessary and sufficient conditions for the join of two subgroups to be subnormal in any group in which they are each subnormally embedded; a sufficient condition ([R4]) is that the two subgroups permute (i.e. their join is their product).

This present work arises from considering what in some sense is the dual situation to the above, namely: given a group G with subgroups H and K, both of which contain X as a subnormal subgroup, we ask under what conditions is X subnormal in the join $\langle H,K\rangle$ of H and K? It makes sense here to assume that $G = \langle H,K\rangle$, so we do. We will say that G is a $\mathcal{J}$-group if whenever $G = \langle H,K\rangle$ and X are as posed, it is true that X is subnormal in G. Unfortunately, apart from obvious classes such as nilpotent groups, $\mathcal{J}$-groups do not seem to exist in abundance: Example 1.1 (due to Wielandt) shows that not even all finite groups are $\mathcal{J}$-groups. Even worse, this example has the finite group G being soluble (of derived length 3) with X central in H (in fact H is cyclic). All this does not seem to bode well for trying to find many infinite $\mathcal{J}$-groups (although whether metabelian groups are

$\mathcal{J}$-groups is an open problem). However, in [W4], Wielandt shows that, if we require that the $\mathcal{J}$-group criteria for a group G is satisfied only when H and K permute — in which case we say that G is a $\mathcal{W}$-group — then every finite group is indeed a $\mathcal{W}$-group (Theorem 1.3 here). The soluble case of this result is due to Maier ([MR]).

Our aim in this work is to develop Theorem 1.3 in (principally) three directions, a chapter being devoted to each. We give a general outline of the themes of each chapter here, insofar as they relate to Theorem 1.3, giving more details at the beginning of each chapter.

In Chapter 1 we try to find classes of groups $\mathcal{X}$ such that every $\mathcal{X}$-group is a $\mathcal{W}$-group. Stonehewer ([S4]) has shown that periodic nilpotent-by-finite groups are $\mathcal{W}$-groups (as well as other classes: see Theorem 1.9), utilising a description of triply factorised groups given by Sysak ([SY]) : such triple factorisations may, in many cases, be assumed to hold for a $\mathcal{W}$-candidate G by virtue of a useful reduction lemma ([S4]) which is Lemma 1.6 here. Using this, and other, reductions we are able to show that nilpotent-by-abelian-by-finite groups of finite (Prüfer) rank are $\mathcal{W}$-groups (Theorem 1.26). In particular, soluble linear groups of finite rank and finitely generated soluble groups of finite rank are $\mathcal{W}$-groups. The last section (§1.5) of Chapter 1 considers ascendancy, using which the ascendant counterpart $\dot{\mathcal{W}}$ of $\mathcal{W}$ is defined. Again using reductions, we show that locally soluble groups of finite rank are $\dot{\mathcal{W}}$-groups (Theorem 1.41).

In Chapter 2 we look at projectivities (i.e. isomorphisms of subgroup lattices) and consider the effect of projectivities on subnormal and ascendant subgroups (see [SH], [Z2]). Corollary 2.20 shows, in particular, that the projective image of a subnormal subgroup of a finite group has a subnormaliser. The term "subnormaliser" used here is open to several definitions, from which we have chosen, for better or worse, the following one: we say that a subgroup X of a group G *has* a subnormaliser if there exists a unique largest subgroup S of G such that X is subnormal in S. The problem with this definition is that not every subgroup has a subnormaliser, because not every group is a $\mathcal{J}$-group. Alternative definitions usually define some subgroup S_1 (containing X) which has the distinct advantage of actually existing, but X will not necessarily be subnormal in S_1 (see [S1] for a discussion of possibilities). Using Corollary 2.20 and results of [SZ], we can relax the permutability hypothesis of Theorem 1.3 by requiring that the subgroup lattice of the finite group $G = \langle H,K \rangle$ admits a projectivity σ under which H^σ and K^σ permute (Theorem 2.21). We then identify some other classes of groups contained in for which this relaxed permutability hypothesis still works. One of these classes is the class of metabelian groups, which supports the conjecture that metabelian groups are $\mathcal{J}$-groups. Other identified classes are those of polycyclic-by-finite groups and Černikov groups (Theorems 2.22, 2.26).

In Chapter 3 we consider $\mathcal{K}$-subnormality (termed $\mathcal{C}$-subnormality when introduced by Kegel ([K2])), which is a generalisation of subnormality.

Here K denotes a class of groups which is closed with respect to forming extensions, homomorphic images and subgroups. A subgroup X of a group G is K-subnormal in G if there is a chain of finite length from X to G, each step of which is either normal (as for subnormality) or a K-step (by a K-step $A \leq B$, we mean that B/A_B is a K-group). Kegel ([K2]) shows that in a finite group the K-subnormal subgroups (K fixed) form a sublattice of the subgroup lattice (Theorem 3.2 here). We consider whether Theorem 1.3 holds with "K-subnormal" in place of "subnormal"; for (finite) soluble groups it does (Theorem 3.12) and we give counter-examples in some non-soluble cases. We define the subclass W_K of W (in such a way that Theorem 3.12 says that W_K contains all finite soluble groups) and identify some non-finite W_K-groups, such as polycyclic groups and soluble Černikov groups (Theorems 3.15, 3.17).

We use well-known results and definitions without reference.

§0.1 Notation and Terminology.

Our notation and terminology is fairly standard (e.g. as in [R1], [R3]), but we include this section for convenience and just in case there are any ambiguities. To save excessive use of brackets, we use the convention that (given there is a choice) a subscript is read before a superscript (e.g. X_1^σ means $(X_1)^\sigma$).

$\mathbb{P}, \mathbb{N}, \mathbb{Z}, \mathbb{Q}$	denote (respectively) the set of prime numbers, non-negative integers, integers, rational numbers.
$\subseteq$	subset or subclass.
$\lvert S \rvert$	denotes the cardinality of the set S .

$\aleph_0$	the cardinality of $\mathbb{N}$.
ω	the first infinite ordinal
$S_1 \setminus S_2$	consists of those elements of S_1 which are not in S_2 .
π'	the set $\mathbb{P} \setminus \pi$. If $\pi = \{p\}$ then we often use p, p' in place of π, π' .
C_∞	infinite cyclic group.
Σ_n , A_n	the symmetric and alternating groups of degree n , respectively.
$GL(n,R)$	the (multiplicative) group of invertible $n \times n$ matrices over the ring R (which has an identity).
$GL(n,p)$	$GL(n,R)$ in the case R is a field of p elements (p prime).

Let G be a group with subgroups H,K .

$<$, $\leq$, $\triangleleft$	denote (respectively) proper subgroup, subgroup, normal subgroup.
$X \cong Y$	means that the groups X and Y are isomorphic.
$\mathrm{Aut}(G)$	the automorphism group of G .
$\pi(G)$	the set of primes occurring in the orders of the periodic elements of G . If G is periodic we say that G is a π-group if $\pi(G) \subseteq \pi \subseteq \mathbb{P}$.
$O_\pi(G)$	the largest normal π-subgroup of G ($\pi \subseteq \mathbb{P}$) .
$\exp(G)$	(when G is periodic) denotes the least common multiple (if it exists) of the orders of the elements of G ; otherwise $\exp(G)$ is infinite.
$\mathrm{Hall}_\pi(G)$	denotes the collection of all maximal π-subgroups of G , which are called Hall (π-)subgroups of G .

$Syl_p(G)$	denotes $Hall_p(G)$, members being called Sylow (p-)subgroups of G .
x^y	the element $y^{-1}xy$ $(x,y \in G)$.
$[x,y]$	the element $x^{-1}y^{-1}xy$ $(x,y \in G)$.
H^g	the group $\{h^g : h \in H\}$.
$\langle S_\lambda : \lambda \in \Lambda \rangle$	the subgroup of G which is generated by the subsets S_λ of G , which is called the join of the S_λ's .
H^K	denotes the group $\langle H^k : k \in K \rangle$ and is called the normal closure of H in $\langle H,K \rangle$.
H_K	denotes the group $\bigcap_{k \in K} H^k$ and is called the core of H in K . If $H_K = 1$, we say that H is core-free in K.
$[S_1,S_2]$	the subgroup $\langle [s_1,s_2] : s_1 \in S_1 , s_2 \in S_2 \rangle$ (where S_1,S_2 are subsets of G).
G' , G''	denote the subgroups $[G,G]$, $[G',G']$ respectively.
$N_K(H)$	the normaliser of H in K , viz. $\{k \in K : H^k = H\}$.
$C_K(H)$	the centraliser of H in K , viz. $\{k \in K : [H,k] = 1\}$.
$Z(G)$	the centre of G .
$\mathcal{L}(G)$	the subgroup lattice of G , i.e. the collection of all subgroups of G together with the operations intersection and join.
$[G/H]$	the sublattice of $\mathcal{L}(G)$ consisting of the subgroups which lie between H and G .
$\mathrm{Dr}_{\lambda \in \Lambda} H_\lambda$	the restricted direct product of the groups H_λ $(\lambda \in \Lambda)$.
$X] Y$	the semi-direct product of groups X and Y , with a suitably defined action of Y on X .
$X \wr Y$	the standard restricted wreath product of groups X and

	Y, i.e. $X \wr Y = (\mathrm{Dr}_{y \in Y} X_y)] Y$ where $X \cong X_y$ via $x \mapsto x_y$ and the action of Y is $x_y^{y_1} = x_{yy_1} \in X_{yy_1}$ $(y, y_1 \in Y, \; x \in X)$.
$A \otimes B$	the tensor product (over $\mathbb{Z}$) of abelian groups A and B.
Rank(G)	the (Prüfer) *rank of G*, i.e. the least integer r (if it exists) such that any finitely generated subgroup of G can be generated by at most r elements; otherwise we say that G has infinite rank.
S_1S_2	the *product* of subsets S_1, S_2 of G, viz. the set $\{S_1S_2 : S_1 \in S_1 , S_2 \in S_2\}$. H and K are said to *permute* if $\langle H,K \rangle = HK$.

Series.

Let H be a subgroup of a group G and let ν be an ordinal. An *ascending series* from H to G of length ν is a series of subgroups

$$H = G_0 \leq G_1 \leq \ldots \leq G_\nu = G \qquad (1)$$

such that $G_\beta \triangleleft G_{\beta+1}$ $(0 \leq \beta < \nu)$ and such that if $\beta \leq \nu$ is a limit ordinal, then $G_\beta = \bigcup_{\alpha<\beta} G_\alpha$. The groups $G_{\beta+1}/G_\beta$ are the *factors* of the series. An ascending series (1) is *normal* if each term G_β is normal in G.

H is *ascendant* in G, written *H asc G* or *$H \triangleleft^\nu G$*, if there exists an ascending series (1) from H to G. If, moreover, this series has finite length n, then we say that H is *subnormal* in G and write *H sn G* or *$H \triangleleft^n G$*. If H sn G, the *defect* of H in G is the least integer d such that $H \triangleleft^d G$. The *normal closure series* of H in G is defined recursively by: $H_0 = G$, $H_{i+1} = H^{H_i}$ $(i \in \mathbb{N})$. Then $H = H_n$

if and only if $H \trianglelefteq^n G$. Also, $H_n = H[G,{}_nH]$ where $[G,{}_nH] = [\ldots[[G,\overbrace{H],H]\ldots,H}^{n}]$.

We say that H has a <u>subnormaliser</u> (resp. <u>ascendiser</u>) in G if there exists a unique largest subgroup of G in which H is subnormal (ascendant).

Classes of groups.

A <u>class of groups $\mathcal{X}$</u> is a collection of groups which contains every trivial group and every isomorphic image of its members. Members of $\mathcal{X}$ are called $\mathcal{X}$-groups. We always use script capitals to denote classes of groups. The <u>product</u> of classes of groups $\mathcal{X},\mathcal{Y}$ is written $\mathcal{X}\mathcal{Y}$ and consists of all groups G which posess a normal subgroup $N \in \mathcal{X}$ such that $G/N \in \mathcal{Y}$. We write $\mathcal{X}_1\mathcal{X}_2\ldots\mathcal{X}_n$ for the product $(\ldots((\mathcal{X}_1\mathcal{X}_2)\mathcal{X}_3)\ldots)\mathcal{X}_n$ of classes of groups $\mathcal{X}_1,\ldots,\mathcal{X}_n$. If $n \in \mathbb{N}$, $\mathcal{X}^n$ denotes the class $\overbrace{\mathcal{X}\mathcal{X}\ldots\mathcal{X}}^{n}$. We use the following classes:

$\mathcal{U}$	all groups	$\mathcal{C}$	cyclic groups
$\mathcal{I}$	trivial groups	$\mathcal{A}$	abelian groups
$\mathcal{F}$	finite groups	$\mathcal{S}$	soluble groups
$\mathcal{N}$	nilpotent groups		
$\mathcal{N}_c$	nilpotent groups of nilpotency class at most $c \in \mathbb{N}$.		
$\mathcal{F}_\pi,\mathcal{S}_\pi$	finite (resp. soluble) Π-groups $(\Pi \subseteq \mathbb{P})$.		
(G)	the class consisting of all trivial groups and all groups isomorphic to the group G.		
$\mathcal{J}$	the class consisting of groups G which satisfy: whenever $G = \langle H,K\rangle$ and X sn H, X sn K then X sn G (H,K,X subgroups of G).		
$\check{\mathcal{M}},\hat{\mathcal{M}}$	groups which satisfy the minimal (resp. maximal) condition for subgroups.		
$\hat{\check{\mathcal{M}}}$	groups G with a subnormal series whose factors are $\check{\mathcal{M}}$-groups or $\hat{\mathcal{M}}$-groups. $\hat{\check{\mathcal{M}}}$-groups are called <u>minimax</u> groups.		

Hyperabelian groups are groups G which possess an ascending normal series (from 1 to G) with abelian factors. *Hypercentral* groups are groups that have an ascending series with central factors. *Černikov* groups are groups which are a finite extension of a $(S\cap \check{M})$-group; such groups are well-known to be a finite extension of a direct product of finitely many quasicyclic p-groups (various primes p).

Operations.

An *operation* A assigns to each class of groups X a class of groups AX in such a way that $AI = I$ and if Y is a class containing X, then $X \subseteq AX \subseteq AY$. A *closure operation* is an operation A such that $A^2 = A$. We say that A is *unary* if $AX = \bigcup_{G\in X} A(G)$ for each class X .

We use the following closure operations:

S	SX consists of all subgroups of X-groups.
P	PX consists of all groups G which possess a series of finite length whose factors are X-groups.
Q	QX consists of all homomorphic images of X-groups.
L	LX consists of all groups G , every finite subset of which is contained in an X-subgroup (of G).
N_0,N	X is N_0-closed (N-closed) if the product of any pair (any collection) of normal X-subgroups is an X-group.
R	X is R-closed if it is closed with respect to forming subcartesian products.

P,L and R are read as "poly", "locally" and "residually" respectively.

Let G be a group, X a class of groups and A a unary closure operation.

Then:

X^A	denotes the (unique) largest A-closed subclass of X ,
G^X	denotes the *X-residual* of G , i.e. the intersection of all normal subgroups N of G such that $G/N \in X$,
G_X	denotes the *X-radical* of G , i.e. the product of all normal X-subgroups of G .

CHAPTER 1. SUBNORMALITY AND ASCENDANCY.

§1.0 Introduction.

In this chapter we identify certain subclasses of the Wielandt class $\mathcal{W}$ (defined in §1.1) and its ascendant analogue $\bar{\mathcal{W}}$ (§1.5). We include a proof of Wielandt's theorem (Theorem 1.3) which says that contains all finite groups, and also his example which shows that $\mathcal{J}$ does not contain F . §1.2 contains reduction results ([S4]) which are useful in the sequel. Also useful is the fact that $F\mathcal{W} = \mathcal{W}$ (Proposition 1.10). Theorem 1.9 lists some subclasses of $\mathcal{W}$ which appear in [S4].

§1.3 considers classes of groups related to $\hat{A}$ and $\check{M}$. Our main results here are $(S\cap\hat{M})\mathcal{W}^{S} = \mathcal{W}^{S}$ (Theorem 1.19) and $N\check{M} \leq \mathcal{W}$ (Proposition 1.15). §1.4 considers (mainly) nilpotent-by-abelian-by-finite groups (NAF-groups). We prove that NAF-groups of finite rank are $\mathcal{W}$-groups (Theorem 1.26); this result is improved (at the expense of the bounds obtained) in Theorem 1.32 by using results of §1.3, which also give us partial results about soluble groups of finite rank.

§1.5 considers ascendancy and the class $\bar{\mathcal{W}}$. Our main result here is that locally soluble groups of finite rank are $\bar{\mathcal{W}}$-groups (Theorem 1.41): such groups are hypercentral-by-abelian-by-finite of finite rank, and we reduce this to the metabelian-by-finite case (Lemma 1.40) to prove they are $\bar{\mathcal{W}}$-groups.

§1.1 $\mathcal{W}$ and Co.

That a subgroup X of a group G does not, in general, have a subnormaliser (even if G is finite) can be seen in the following example of Wielandt ([W4]).

1.1 EXAMPLE.

Let p be an odd prime and define subgroups of $GL(3,p)$ by $G = \langle h,x,k\rangle$, $H = \langle x,h\rangle$, $K = \langle x,k\rangle$ and $X = \langle x\rangle$, where

$$x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad k = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Then X has order 2, H is cyclic of order $2p$ and K is a dihedral group of order 8. Hence $X \leq Z(H)$ and $X \lhd^2 K$. But X is not subnormal in $G = \langle H,K\rangle$. For, suppose X sn G . Then $Y = X^k$ sn G so that Y sn $\langle Y,h\rangle$. Now Y normalises $\langle h\rangle$ and so $\langle Y,h\rangle = Y\langle h\rangle$ has order $2p$. If Y sn $Y\langle h\rangle$ then (since $p \neq 2$) we must have $[Y,\langle h\rangle] = 1$. But $[x^k,h] = h^2 \neq 1$, a contradiction. Therefore X has no subnormaliser in G . Also, it is not hard to see that $G = (\langle h\rangle \times \langle h^k\rangle) \rtimes K$ has order $8p^2$ and is soluble of derived length 3.

□

The above example is particularly good because X is central in H. As mentioned in Chapter 0, it is an open problem whether metabelian groups are $\mathcal{J}$-groups. We can see, however, that not every subgroup of a metabelian group has a subnormaliser from the following example.

1.2 EXAMPLE.

For $n \geq 2$, let H_n be a dihedral group of order 2^{n+1}, say $H_n = C_n] X_n$ where C_n is cyclic of order 2^n and X_n has order 2 (so X_n acts on C_n by inversion). Let $G = \mathrm{Dr}_{n\geq 2} H_n$ with subgroup $X = \mathrm{Dr}_{n\geq 2} X_n$. Now X_n has defect n in H_n, so that X is subnormal in XH_n with defect n for each $n \geq 2$. But $\langle XH_n : n \geq 2 \rangle = G$ and X is not subnormal in G; for otherwise $X \triangleleft^r G$ for some $r \in \mathbb{N}$ and the defect of X in each XH_n is less than $r+1$, which is a contradiction if $n > r$. Therefore X has no subnormaliser in the metabelian group G. It is worth noting, however, that $X \triangleleft^{\omega} G$ ($X \triangleleft^2 XH_2$ and for $n \geq 2$, $XH_2 \ldots H_n \triangleleft^{n+1} XH_2 \ldots H_n H_{n+1}$).

□

In Example 1.1, the subgroups H and K do not permute, because $|HK| = 8p \nmid |G|$. This fact is not incidental, as we see from the following theorem. The soluble case was first proved by Maier ([MR]).

1.3 THEOREM (Wielandt [W4]).

Let G be a finite group, generated as the product of subgroups

H and K, both of which contain X as a subnormal subgroup. Then X is subnormal in G.

Proof.

Suppose the Theorem is false and choose a counter-example G of minimal order such that $|G:H| + |X|$ is also minimal. It is not hard to see that these minimality conditions imply that $H_G = 1$ and H is maximal in G. Now a subnormal subgroup A of a finite group is contained in the core of any maximal subgroup which contains A ([W3]). Therefore H contains no subnormal subgroups of G. Hence, by the minimality of $|G:H| + |X|$, X must be a simple group.

Case (i) $|X| = p \in \mathbb{P}$.

Since $G = HK$, there exists $H_p \in Syl_p(H)$ and $K_p \in Syl_p(K)$ such that $H_pK_p \in Syl_p(G)$ (see [HU] VI 4.7). Then $X^H \leq H_p$ and $X^K \leq K_p$. Therefore $\langle X, X^{kh^{-1}} \rangle = \langle X^h, X^k \rangle^{h^{-1}}$ is a p-group for all $h \in H$, $k \in K$. Hence $\langle X, X^g \rangle$ is a p-group for all $g \in G$, which is a sufficient condition to ensure X sn G ([AL] or [W3]).

Case (ii) X non-abelian simple.

Let $M = \{m \in G : X^m \leq H \cap K$ and X^m sn H, X^m sn $K\}$ and put $Y = \langle X^m : m \in M \rangle$. Then Y sn H, Y sn K ([W1]) and X sn Y. Therefore Y is not subnormal in G and so there exists $g \in G$

such that $g \in \langle Y^g, Y\rangle \backslash Y$ ([W3]). Write $g = hk$ ($h \in H$, $k \in K$) . Then $kh \in \langle Y^h, Y^{k^{-1}}\rangle = \langle X^{mh}, X^{mk^{-1}} : m \in M\rangle \le N_G(Y)$ because a non-abelian simple subnormal subgroup of a group normalises every subnormal subgroup ([W2]). Therefore $Y^k = Y^{h^{-1}}$, whence $mk \in M$ (all $m \in M$). But this means that $Y^k = Y$ and hence $Y^g = Y$, contradicting the existence of g .

□

As mentioned in Chapter 0, it is not known if Theorem 1.3 holds when G is an arbitrary group. We wish to show that Theorem 1.3 does hold when G belongs to certain classes of groups other than F . Accordingly, we define the Wielandt class of groups (and its derivatives) as follows:

1.4 Definitions.

(i) $\mathcal{W}$ is the class of groups consisting of groups G which satisfy (*) :

$$\left.\begin{array}{l}\text{Whenever } G \text{ is generated as the product of subgroups}\\ H \text{ and } K \text{, both of which contain } X \text{ as a subnormal}\\ \text{subgroup, then } X \text{ is subnormal in } G \text{ .}\end{array}\right\}(*)$$

It may happen that if $G \in \mathcal{W}$, then in (*) we can always bound (above) the defect of X in G by some function of the defects of X in H and of X in K . With this is mind, we define subclasses $\mathcal{W}_f$ of $\mathcal{W}$ by:

(ii) Let $f : \mathbb{N} \to \mathbb{N}$ be a function. Then $\mathcal{W}_f$ is the class consisting of groups $G \in \mathcal{W}$ such that whenever $X \triangleleft^n H$ and $X \triangleleft^n K$ in (*) $(n \in \mathbb{N})$, then $X \triangleleft^{f(n)} G$. (Whenever we specify the function $f = f(n)$, it will be implicit that the variable n plays the same role as in this definition).

For example, putting $f = 1$ (constant) we have $\mathcal{W}_1$ contained in the class of T-groups (that is, groups in which every subnormal subgroup is normal). Because $F \subseteq \mathcal{W}$ (Theorem 1.3), $\mathcal{W}_1$ contains all finite T-groups. Also, $\mathcal{W}_1$ contains all soluble T-groups because such groups are metabelian ([R3] 13.4.2) and hence they are $\mathcal{W}$-groups (Theorem 1.9). We mention that Stonehewer has shown that $F \cap N_5 \subseteq \mathcal{W}_3$ ([S5]).

If A and B are any subgroups of a group, then $A \triangleleft^n A^B$ if and only if $A \triangleleft^{n+1} \langle A,B \rangle$ $(n \in \mathbb{N})$. From this it is clear that $\mathcal{W}$ could also be defined as the class consisting of groups G which satisfy: whenever $G = HK$ and $X \text{ sn } X^H$, $X \text{ sn } X^K$ (X,H,K subgroups), then X sn G . Also, $\mathcal{J}$-groups have an analogous characterisation (recall that $\mathcal{J}$-groups are groups G which satisfy (*) in 1.4(i) even if H and K do not permute).

§1.2 Reductions.

We will repeatedly make use of certain reductions in the sequel, so we present them here (Lemma 1.6 and Theorem 1.8). Essentially, Lemma 1.5 appears in [S4] .

1.5 LEMMA.

Let G be a group with subgroups A,H,K where A is abelian and $A \triangleleft G = \langle H,K \rangle$. Define subgroups $G_1 = AH \cap AK$, $H_1 = H \cap AK$, $K_1 = K \cap AH$ and $N = (A \cap H)(A \cap K)$. Then:

(i) $G_1 = AH_1 = AK_1$. Also $G_1 = \langle H_1,K_1 \rangle$ if and only if $A \leq \langle H_1,K_1 \rangle$.

(ii) $G_1 = H_1K_1$ if and only if $G_1 \subseteq HK$ (subset).

(iii) $N = (A \cap H_1)(A \cap K_1)$ is a normal abelian subgroup of G_1 .

Let θ be the natural epimorphism from G_1 to G_1/N . Then:

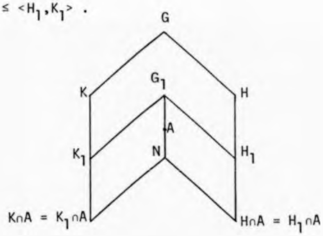
(iv) $G_1^\theta = A^\theta] H_1^\theta = A^\theta] K_1^\theta$ and H_1^θ , K_1^θ each embed in G/A .

(v) $G_1^\theta = \langle H_1^\theta, K_1^\theta \rangle$ if and only if $A \leq \langle H_1,K_1 \rangle$.

(vi) $G_1^\theta = H_1^\theta K_1^\theta$ if and only if $G_1 = H_1K_1$.

Proof.

Diagram when $A \leq \langle H_1, K_1 \rangle$.



(i) Clearly $G_1 = AH_1 = AK_1$, which also proves the second statement.

(ii) As a set, $H_1K_1 = (H \cap AK)(K \cap AH) = AH \cap AK \cap HK = G_1 \cap HK$. (ii) now follows.

(iii) Clearly N is abelian. Also $A \cap H_1 \lhd AH_1 = G_1$ by (i). Similarly $A \cap K_1 \lhd G_1$, so that $N \lhd G$.

(iv) $A \cap H_1N = (A \cap H_1)N = N$, so that $A^\theta \cap H_1^\theta = 1$. Similarly $A^\theta \cap K^\theta = 1$. Clearly $A^\theta \lhd G_1^\theta$, so that (by (i)), G_1^θ splits as required. Therefore $H_1^\theta \cong K_1^\theta \cong G_1^\theta / A^\theta \cong G_1/A \leq G/A$.

(v) This is clear from (i).

(vi) If $G_1^\theta = H_1^\theta K_1^\theta$ then $G_1 = H_1NK_1N = H_1NK_1 = H_1K_1$. The converse is clear.

□

1.6 REDUCTION LEMMA (Stonehewer [S4]).

Let G be a group with subgroups X, H, K and A such that $G = HK$, $X \triangleleft^n H$, $X \triangleleft^n K$ $(n \in \mathbb{N})$ and A is an abelian normal subgroup of G. Then, using the notation of Lemma 1.5, $G_1^\theta = H_1^\theta K_1^\theta = A^\theta] H_1^\theta = A^\theta] K_1^\theta$ and $X^\theta \triangleleft^n H_1^\theta$, $X^\theta \triangleleft^n K_1^\theta$.

Further, if $X^\theta \triangleleft^m G_1^\theta$ and $AX \triangleleft^\ell G$ $(\ell, m \in \mathbb{N})$, then $X \triangleleft^{\ell+m+2n} G$.

Suppose the hypotheses of the first part of Lemma 1.6 and also that $G/A \in \mathfrak{W}^s$. Then Lemma 1.6 says that to prove $X \text{ sn } G$ we may in many cases assume that $G = A] H = A] K = HK$.

The proof of 1.6 will use the following Lemma, which is essentially in [S4].

1.7 LEMMA.

Let G be a group with subgroups X, H, K and N such that $G = \langle H, K \rangle$, $X \triangleleft^n H$, $X \triangleleft^n K$ $(n \in \mathbb{N})$ and N is a normal abelian subgroup of G. If $G = N(H \cap K)$ then $X \triangleleft^{2n} G$. If $G = NX$ then $H \triangleleft^n G$ (and $K \triangleleft^n G$).

Proof of 1.7.

$N \cap H \triangleleft NH = G$. Also,

$$H = H \cap N(H \cap K) = (H \cap K)(N \cap H) \leq K(N \cap H) .$$

Therefore $G = K(N \cap H)$, so that

$$X \triangleleft^n X(N \cap H) \triangleleft^n G .$$

If $G = NX$ then $X(N \cap H) = H \triangleleft^n G$ and similarly $K \triangleleft^n G$.

□

Proof of 1.6.

The first part follows from Lemma 1.5. Suppose also that $X^\theta \triangleleft^m G_1^\theta$ and $AX \triangleleft^\ell G$. Then

$$NX \triangleleft^m AX \triangleleft^\ell G .$$

Since $NX = (N \cap H)X(N \cap K)X$ then by Lemma 1.7, $X \triangleleft^{2n} NX$ as required.

□

Remark.

Suppose we wish to show that a group G is a $\mathcal{J}$-group; then we suppose that $G = \langle H,K \rangle$ with X sn H , X sn K (X,H,K subgroups) and try to show that X sn G . If G contains a normal abelian subgroup A , such that $G/A \in \mathcal{J}^s$, then, in the notation of Lemma 1.5, it is enough to prove that X^θ sn $G_1^\theta = A^\theta \;] \; H_1^\theta = A^\theta \;] \; K_1^\theta$ where X^θ sn H_1^θ , X^θ sn K_1^θ . However, we need to show that $A \leq \langle H_1,K_1 \rangle$ in order to also have $G_1^\theta = \langle H_1^\theta,K_1^\theta \rangle$; the fact that this might not happen prevents us from being able to make a real reduction (there is a similar impediment to the J-analogue of Theorem 1.8). We note that $\langle H_1^\theta, K_1^\theta \rangle = (A^\theta \cap \langle H_1^\theta, K_1^\theta \rangle)] H_1^\theta = (A^\theta \cap \langle H_1^\theta, K_1^\theta \rangle)] K_1^\theta$, and if $X^\theta \text{ sn } \langle H_1^\theta, K_1^\theta \rangle$ then $X \text{ sn } \langle H_1, K_1 \rangle$ (Lemma 1.7).

The following theorem shows that if we wish to show that $NX \leq W$ for some S-closed class X $(\leq W)$, then it is enough to consider AX-groups (and then we could use Lemma 1.6). This theorem is proved in [S4] using induction on nilpotency class together with Lemma 1.6. We give an alternative proof, whose method will be of use when we consider ascendancy in §1.5.

1.8 THEOREM.

Let X, Y and Z be S-closed classes of groups with Y and Z also Q-closed. Suppose that $(A \cap Z)X \cap Y \leq W$. Then $(N \cap Z)X \cap Y \leq W^S$. If $(A \cap Z)X \cap Y \leq W_f$ then $(N_c \cap Z)X \cap Y \leq W_g$ where $g = cf + (c-1)n$.

Proof.

Let $G \in (N_c \cap Z)X \cap Y$ be generated as the product of subgroups H and K , both of which contain X as a subnormal subgroup of defect at most $n \in \mathbb{N}$. Let $B \in N_c \cap Z$ be a normal subgroup of G such that $G/B \in X$. Let Z_i $(0 \leq i \leq c)$ denote the i^{th} term of the upper central series of B. We will show that

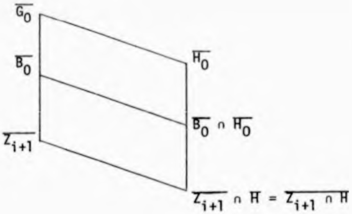
$$XZ_i \text{ sn } XZ_{i+1} \quad \text{for} \quad 0 \le i \le c-1 \,. \qquad (*)$$

Fix i in the range $0 \le i \le c-1$ and let bars denote subgroups of G modulo Z_i. Let $G_0 = Z_{i+1} H \cap Z_{i+1} K$, $B_0 = B \cap G_0$, $H_0 = H \cap G_0$ and $K_0 = K \cap G_0$. Then

$$\overline{G_0} = \overline{H_0}\,\overline{K_0} = \overline{Z_{i+1}}\,\overline{H_0} = \overline{Z_{i+1}}\,\overline{K_0} \quad \text{and} \quad \overline{B_0} = \bar{B} \cap \overline{G_0} \,.$$

Also $\overline{Z_{i+1}} \le Z(\overline{B_0})$ so that

$$\overline{B_0} \cap \overline{H_0} \lhd \overline{Z_{i+1}}\,\overline{H_0} = \overline{G_0} \,.$$



Now

$$\frac{\overline{G_0}}{\overline{B_0}} \cong \frac{\overline{G_0}\,\bar{B}}{\bar{B}} \le \frac{\bar{G}}{\bar{B}} \cong \frac{G}{B} \in X \quad \text{and}$$

$$\frac{\overline{B_0}}{\overline{B_0} \cap \overline{H_0}} = \frac{\overline{Z_{i+1}}(\overline{B_0} \cap \overline{H_0})}{\overline{B_0} \cap \overline{H_0}} \cong \frac{\overline{Z_{i+1}}}{\overline{Z_{i+1}} \cap \overline{H_0}} \in Q(A \cap Z) = A \cap Z \quad .$$

Hence $\overline{G_0}/(\overline{B_0} \cap \overline{H_0}) \in (A \cap Z)X \cap \mathcal{Y} \subseteq \mathcal{W}$ (by hypothesis), whence

$$\bar{X} \lhd^{n} \bar{X}(\overline{B_0} \cap \overline{H_0}) \text{ sn } \overline{G_0} \qquad (1)$$

Therefore $\bar{X}$ sn $\bar{X}\,\overline{Z_{i+1}} = \overline{XZ_{i+1}}$, so that XZ_i sn XZ_{i+1} and (*) is proved.

From (*), X sn $Z_cX = BX$. Also $G/B \in X \cap \mathcal{Y} \subseteq \mathcal{W}$ so that BX sn G . Hence X sn G and the first part of the theorem is true.

Suppose also that $(A \cap Z)X \cap \mathcal{Y}$ is bounded by f . Following the above proof, we see from (1) that

$$XZ_i \lhd^{n+f} XZ_{i+1} \quad \text{for} \quad 0 \leq i \leq c-1 \ .$$

Consequently $X \lhd^{(c-1)(n+f)} XZ_{c-1}$. Since $G/Z_{c-1} \in (A \cap Z)X \cap \mathcal{Y}$, we have $XZ_{c-1} \lhd^{f} G$. Therefore $X \lhd^{(c-1)n+cf} G$, as required.

$\square$

Theorem 1.8, in conjunction with Lemma 1.6 are used in [S4] to prove the following:

1.9 THEOREM.

The following classes of groups are contained in W^{s} :

NA , $N(PC)F$, $S \cap N\check{M}$ (c.f. Proposition 1.15) , $S \cap \bar{M}$ (c.f. Corollary 1.28).

□

In Lemma 1.6, suppose that G is metabelian and put $A = G'$. Then H_1^{θ} and K_1^{θ} are abelian (Lemma 1.5 (iv)) and so $X^{\theta} \triangleleft G_1^{\theta}$. Hence $X \triangleleft^{2n+2} G$. Then by Theorem 1.8 (with $X = A$, $Y = Z = U$) , we see that $N_c A \leq W_f$ where $f(n) = 3nc + 2c-n$.

1.10 PROPOSITION.

$FW = W$. In particular, $FW^{s} = W^{s}$.

Proof.

Suppose that $G \in W$ is generated as the product of subgroups H and K , both of which contain X as a subnormal subgroup of defect at most $n \in \mathbb{N}$. Let N be a finite normal subgroup of G such that $G/N \in W$. Then $NH = H(NH \cap K)$ and $|NH : H| \leq |N| = r$, say. Therefore H_{NH} has index at most $r!$ in NH . Considering the group NH/H_{NH} we see from Theorem 1.3 that $XH_{NH} \triangleleft^{r!} NH$. But $G/N \in W$ so that NX sn G . Therefore $X \triangleleft^{n} XH_{NH} \triangleleft^{r!} NX$ sn G and so $FW \leq W$.

□

§1.3 Min, Max and $\mathcal{W}$.

1.12 LEMMA.

$\check{M} \subseteq \mathcal{W}^s$.

To prove Lemma 1.12 we need the following results:

1.13 PROPOSITION (Roseblade [RB]).

Let G be a group which satisfies the minimal condition on subnormal subgroups. If X is a subnormal subgroup of G , then $|G : N_G(X)|$ is finite.

□

1.14 LEMMA (Amberg [A1], see [S4]).

Let G be a group which is generated as the product of subgroups H and K. Suppose that H_0, K_0 are subgroups of H,K respectively such that $|H : H_0| = r$ and $|K : K_0| = s$ are finite. Then $|G : \langle H_0, K_0 \rangle| \leq rs$.

Proof.

There exists $h_1, \dots, h_r \in H$ and $k_1, \dots, k_s \in K$ such that

$$G = HK = \bigcup_{i=1}^{r} \bigcup_{j=1}^{s} (h_i H_0 K_0 k_j) = \bigcup_{i,j} (h_i \langle H_0, K_0 \rangle k_j)$$

$$= \bigcup_{i,j} h_i k_j \langle H_0, K_0 \rangle^{k_j} . \qquad (*)$$

By a result of B. Neumann ([NE]) we can omit from the union (*) all the cosets such that $|G : \langle H_0, K_0 \rangle^{k_j}|$ is infinite. Therefore $|G : \langle H_0, K_0 \rangle^{k_j}|$ is finite for some j, whence $|G : \langle H_0, K_0 \rangle|$ is finite. Factoring by $\langle H_0, K_0 \rangle_G$, we may assume that G is finite and so $|G| \leq rs|\langle H_0, K_0 \rangle|$ by (*). Therefore $|G : \langle H_0, K_0 \rangle| \leq rs$.

□

Proof of 1.12.

Let $G \in \check{M}$ be generated as the product of subgroups H and K, both of which contain X as a subnormal subgroup. Then by Proposition 1.13, $|H : N_H(X)|$ and $|K : N_K(X)|$ are finite. Putting $J = \langle N_H(X), N_K(X) \rangle$, then $|G:J|$ is finite by Lemma 1.14. Hence $G/J_G \in F$ and XJ_G sn G (Theorem 1.3). Since $X \lhd XJ_G$ we have X sn G as required.

□

We note that the proof of 1.12 requires only that H and K have the minimal condition on subnormal subgroups. We use this fact in the following result, which supercedes 1.12.

1.15 PROPOSITION.

$N\check{M} \subseteq W^S$.

Proof.

By Theorem 1.8 (with $X = \check{M}$ and $Y = Z = U$) it is enough to prove $A\check{M} \subseteq W^S$. Let $G \in A\check{M}$ be generated as the product of subgroups H and K , both of which contain X as a subnormal subgroup. Let A be an abelian normal subgroup of G such that $G/A \in \check{M}$. Then AX sn G (Lemma 1.12) and so by Lemma 1.6 we may assume that $G = HK = A] H = A] K$. Thus $H, K \in \check{M}$, so by the remark above we have X sn G , as required.

□

Let X_0 denote the following class of groups: G is an X_0-group if and only if $G/G^F \in F$ and G^F is a periodic abelian group such that for each prime p , the p-component of G^F is the direct product of finitely many quasicyclic p-groups. A result of Amberg ([A3]) Cor.2.8) shows that the soluble product of X_0-groups is an X_0-group. Also, $X_0 \subseteq AF \subseteq W^S$ (Theorem 1.9) so by Lemma 1.6, we have $S \cap AX_0 \subseteq W^S$. Then by Theorem 1.8 (with $Y = S$, $Z = U$) we have

1.16 PROPOSITION.

$$N(S \cap X_0) = S \cap NX_0 \subseteq W^S \ .$$

□

Of course, proposition 1.16 is a particular case of the result that nilpotent-by-(periodic abelian)-by-finite groups lie in $\mathcal{W}^s$ ([S 4]).

We now wish to look at $(S \cap \check{M})\mathcal{W}^s$-groups. It is not clear whether or not they all lie in $\mathcal{W}^s$, but Proposition 1.18 gives us a partial result in this direction; certain restrictions are imposed which enable us to make use of the fact that periodic subgroups of $GL(n,R_p)$ are finite, where R_p denotes the ring of p-adic integers (see [R1] Cor. 3.28). These restrictions present no impediment in $\mathcal{NAF}$-groups (see Lemma 1.21). We will also need the following result ([R1] Lemma 3.13).

1.17 LEMMA.

Let A be a normal divisible abelian subgroup of a group G and suppose that X is a subgroup of G such that $[A,_sX] = 1$, where $s \in \mathbb{N}$. If X/X' is periodic, then $[A,X] = 1$.

□

Let G be a Černikov-by-$\mathcal{W}^s$ group. By Proposition 1.10, $G \in \mathcal{D}\mathcal{W}^s$ where $\mathcal{D}$ denotes the class of divisible abelian groups with min. In particular, $(S \cap \check{M})\mathcal{W}^s = \mathcal{D}\mathcal{W}^s$.

1.18 PROPOSITION.

Let G be a Černikov-by-$\mathcal{W}^s$ group which is generated as the product of subgroups H and K , both of which contain X as a subnormal subgroup of defect at most n $(n \in \mathbb{N})$. If X is periodic, then X sn G . If, moreover, $G \in \mathcal{D}(\mathcal{W}^{s} \cap \mathcal{W}_{f})$ then $X \lhd^{f(n)+1} G$.

Proof.

As mentioned above, $G \in \mathcal{D}\mathcal{W}^{s}$, so let $A \in \mathcal{D}$ be a normal subgroup of G such that $G/A \in \mathcal{W}^{s}$. Then by Lemma 1.6 (and since $Q\mathcal{D} = \mathcal{D}$) , we may assume that

$$G = HK = A] H = A] K . \qquad (*)$$

If $g \in C_G(A)$ then $g = ah$ $(a \in A , h \in H)$ and $1 = [ah,A] = [h,A]$, so that $g \in AC_H(A)$. Hence $C_G(A) = AC_H(A) = AC_K(A)$. Also, $C_H(A) \lhd AH = G$ and, if bars denote subgroups of G modulo $C_H(A)$, we have

$$\bar{G} = \bar{H}\bar{K} = \bar{A}] \bar{H} = \bar{A}\bar{K} , \quad \bar{A} \in \mathcal{D} , \quad \bar{G}/\bar{A} \in \mathcal{W}^{s} .$$

Since $[H,A] \cap C_H(A) \leq A \cap H = 1$, it follows that $C_{\bar{H}}(\bar{A}) = 1$ and so there is an embedding

$$\bar{H} \rightarrow \mathrm{Aut}(\bar{A}) .$$

Now $\bar{X}$ sn $\bar{H}$ and $\bar{X}$ is periodic, so that $\bar{X}^{\bar{H}}$ is isomorphic to a periodic group of automorphisms of $\bar{A}$. By the remark before this proposition,

$$\frac{\bar{X}^{\bar{H}}}{C_{\bar{X}^{\bar{H}}}(\bar{A}_{p_i})} \text{ is finite } (1 \leq i \leq r)$$

where $\bar{A}_{p_1}, \ldots, \bar{A}_{p_r}$ are the primary components of $\bar{A} \in \mathcal{D}$. But

$$\bigcap_{i=1}^{r} C_{\bar{X}^{\bar{H}}}(\bar{A}_{p_i}) \leq C_{\bar{H}}(\bar{A}) = 1 ,$$

so that $\bar{X}^{\bar{H}}$ is finite.

Let $N = N_H(XC_H(A))$. Then $\bar{N} = N_{\bar{H}}(\bar{X})$ and

$$|H : N| \text{ is finite.}$$

From (*), we have $H \cong K$ and we can see this isomorphism using $\phi \in \mathrm{Aut}(G)$ which is defined as follows: if $g \in G$ then (by (*)) g can be written uniquely as $g = ah = bk$ where $a, b \in A$, $h \in H$, $k \in K$; then we define $g^{\phi} = ak$. So $H^{\phi} = K$ and the fixed-point subgroup for ϕ is $A(H \cap K)$. Therefore $[g,A]^{\phi} = [g^{\phi},A]$ for any $g \in G$, and so

$$C_H(A)^{\phi} = C_K(A) . \qquad (**)$$

Also, if S is any subgroup of H then $h \in N_H(S)$ if and only if $h^\phi \in N_K(S^\phi)$. So by (**),

$$N^\phi = N_K(XC_K(A)) .$$

Since $|H : N|$ and $|K : N^\phi|$ are finite, so is $|G : J|$ where $J = \langle N, N^\phi \rangle$ (Lemma 1.14). Therefore $G/J_G \in \mathcal{F}$ and XJ_G sn G (Theorem 1.3). So it is enough (for the first part of the proposition) to show that X sn J .

$XC_H(A)C_K(A)$ is normalised by N and N^ϕ . X sn $XC_H(A)$ implies that $XC_K(A)$ sn $XC_H(A)C_K(A)$. Therefore X sn $XC_K(A)$ sn J and the first part of the proposition is proved.

Now suppose $G \in \mathcal{D}(\mathcal{W}^s \cap \mathcal{W}_f)$ and let $B \in \mathcal{D}$ be a normal subgroup of G such that $G/B \in \mathcal{W}^s \cap \mathcal{W}_f$. Then $BX \triangleleft^{f(n)} G$ and (from the first part) X sn $BX = G_0$, say. Then $B \cap X \triangleleft G_0$ and if bars denote subgroups of G_0 modulo $B \cap X$, we have $\overline{G_0} = \bar{B}] \bar{X}$ and $\bar{X} \triangleleft^s \overline{G_0}$ for some $s \in \mathbb{N}$. Hence $[\bar{B},_s \bar{X}] \leq \bar{X} \cap \bar{B} = 1$, so by Lemma 1.17 $[\bar{B},\bar{X}] = 1$ i.e. $[B,X] \leq X \cap B$. Therefore B normalises X , so that $X \triangleleft BX \triangleleft^{f(n)} G$ as required.

□

In §1.4 we will extend Proposition 1.18 - at the expense of another restriction on X - to the case where A (in the proof of 1.18) is a periodic divisible abelian group of finite rank (so that A may have infinitely many p-components).

Turning our attention now to the maximal condition, we see from Theorem 1.9 that $(PC)F = (S \cap \hat{M})F \subseteq W^S$ (this can be shown directly using a result of Kegel [K1] which says that a subgroup X of a $(PC)F$-group G is subnormal if X^θ is subnormal in every finite homomorphic image G^θ of G ; then use Theorem 1.3). In fact, we have

1.19 THEOREM.

$(S \cap \hat{M})W^S = W^S$.

Proof.

Let $G \in (S \cap \hat{M})W^S$ be generated as the product of subgroups H and K , both of which contain X as a subnormal subgroup. G has a series

$$1 \triangleleft B \triangleleft C \triangleleft G$$

such that $G/C \in W^S$, $C \in S \cap \hat{M}$, $B \in P(C_\infty)$ and $C/B \in F$. We prove that $G \in W^S$ by induction on the Hirsch length $h(C)$ of C . If $h(C) = 0$ then $C \in F$ and so $G \in W^S$ (Proposition 1.10). Suppose that $h(C) \geq 1$ with the usual induction hypothesis.

C is finitely generated, so $C/B_G \in \mathcal{F}$ (and $h(B_G) = h(C)$) . Therefore $B_G \in P(C_\infty)$ and $G/B_G \in \mathcal{W}^S$ (Proposition 1.10). Hence we may assume that $B = C$.

Let A be the penultimate term of the derived series of C (A is torsion-free abelian of finite rank). Then $A \triangleleft G$ and by induction,

$$AX \text{ sn } G \ . \tag{1}$$

By Lemma 1.6 (and using its notation), it is enough to show that $X^\theta \text{ sn } G_1^\theta$, where

$$G_1^\theta = A^\theta \rtimes H_1^\theta = A^\theta \rtimes K_1^\theta = H_1^\theta K_1^\theta \quad (G_1^\theta/A^\theta \in \mathcal{W}^S) \tag{2}$$

(recall $\theta : G_1 \to G_1/N$ where $N \leq A$) . Since $A^\theta X^\theta = AX/N$, it is enough (by (1)) to show that $X^\theta \text{ sn } A^\theta X^\theta$.

$A^\theta = A/N$ so we can write

$$A^\theta = A_1^\theta \times A_2^\theta$$

where A_1^θ is torsion-free and A_2^θ is finite. If $A_2^\theta \neq 1$ then $h(A^\theta) < h(A)$ and by induction, $X^\theta \text{ sn } A^\theta X^\theta$.

It remains to consider the case $A_2^\theta = 1$. Let $p \in \mathbb{P}$. Then

$G^\theta/(A^\theta)^p \in F\mathcal{W}^S$ from (2), so by Proposition 1.10,

$$(A^\theta)^p X^\theta \text{ sn } G_1^\theta . \qquad (3)$$

Since

$$|A^\theta X^\theta : (A^\theta)^p X^\theta| = |A^\theta : (A^\theta)^p| = p^r$$

where $r = \text{rank }(A^\theta)$, (3) gives $(A^\theta)^p X^\theta \lhd^r A^\theta X^\theta$. Therefore

$$\bigcap_{p \in \mathbb{P}} ((A^\theta)^p X^\theta) \lhd^r A^\theta X^\theta .$$

Since $A^\theta \cap X^\theta = 1$ (and A^θ is free abelian),

$$\bigcap_{p \in \mathbb{P}} ((A^\theta)^p X^\theta) = (\bigcap_{p \in \mathbb{P}} (A^\theta)^p) X^\theta = X^\theta .$$

Therefore $X^\theta \lhd^r A^\theta X^\theta$, as required.

□

1.20 COROLLARY.

$$(P(C \cup F))\mathcal{W}^S = \mathcal{W}^S .$$

Proof.

$((PC)F)X = (PC)(FX)$ for any class of groups X , so that

$((PC)F)W^s = W^s$ by Propositions 1.19 and 1.10. Also, $(PC)F =$ $= P(C \cup F)$ ([R1] 3.1).

□

§1.4 NAF Groups.

1.21 LEMMA.

Let Y, Z be S- and Q-closed classes of groups. Let $G \in (N \cap Z)AF \cap Y$ be generated as the product of subgroups H and K, both of which contain X as a subnormal subgroup. Let $1 \lhd A \lhd B \lhd G$ be a normal series of G such that $A \in N \cap Z$, $B/A \in A$ and $G/B \in F$. Then to prove that X sn G, we may assume the following:

(i) $A \in A \cap Z$ and $G = A] H = A] K = HK$.

(ii) $B \cap H \cap K = 1$ (assuming (i)).

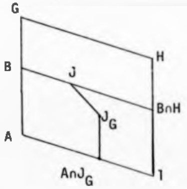
(iii) $B = \langle B \cap H , B \cap K \rangle$ (assuming (i), (ii)).

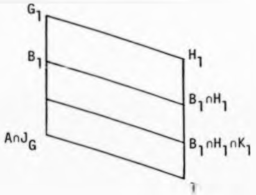
Proof.

(i) This is clear from Theorem 1.8 and Lemma 1.6.

(ii) Assume (i) and let $J = \langle B \cap H , B \cap K \rangle$. $G/J_G \in F$ (Lemma 1.14) and, since $B \cap H , B \cap K \in A$,

$$[J_G , B \cap H \cap K] = 1 . \qquad (1)$$





Apply Lemma 1.6 with $A \cap J_G$ in place of A . Then (using the notation of 1.6 and thinking of θ as the identity map),

$$G_1 = (A \cap J_G)] H_1 = (A \cap J_G)] K_1 = H_1 K_1 .$$

Put $B_1 = B \cap G_1 \triangleleft G_1$. Then by (1),

$$[A \cap J_G , B_1 \cap H_1 \cap K_1] = 1 .$$

Hence $(B_1 \cap H_1 \cap K_1)^{G_1} = (B_1 \cap H_1 \cap K_1)^{H_1} = (B_1 \cap H_1 \cap K_1)^{K_1} \leq B_1 \cap H_1 \cap K_1$, so that $B_1 \cap H_1 \cap K_1 \triangleleft G_1$. Then, if bars denote subgroups of G_1 modulo $B_1 \cap H_1 \cap K_1$,

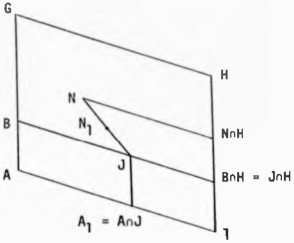
$$\bar{G}_1 = \overline{A \cap J_G}] \overline{H_1} = \overline{A \cap J_G}] \overline{K_1} = \overline{H_1}\, \overline{K_1} \in \mathcal{Y} .$$

Also $\overline{B_1} \cap \overline{H_1} \cap \overline{K_1} = 1$. Now X sn $(B_1 \cap H_1 \cap K_1)X$, so that $\bar{X}$ sn $\overline{G_1}$ implies X sn G_1 . Also, $J_G/(A \cap J_G) \in A$ so that $G/(A \cap J_G) \in AF \subseteq \mathcal{W}^S$ and $(A \cap J_G)X$ sn G . Hence (from Lemma 1.6) $\bar{X}$ sn $\overline{G_1}$ implies X sn G . So we may assume that $B_1 \cap H_1 \cap K_1 = 1$.

Since $A \cap J_G \in A \cap Z$, $B_1/(A \cap J_G) \in A$ and $G_1/B_1 \in F$, we may assume that (i) holds and $B \cap H \cap K = 1$. Then $H \cap K$ embeds in G/B and $H \cap K$ is finite, as required.

(iii) Suppose that (i) and (ii) hold. Again, let $J = \langle B \cap H, B \cap K \rangle$ and put $N = N_G(J) \geq X$. Then $G/J_G \in F$ and by Lemma 1.23 (which follows this proof),

$$N = (N \cap H)(N \cap K) .$$



Also. $A \cap J \lhd N$ and $(A \cap J)(J \cap H) = A(J \cap H) \cap J = A(B \cap H) \cap J = J$. So, writing $A_1 = A \cap J$,

$$J = A_1(J \cap H) = A_1(J \cap K) . \qquad (*)$$

Since XN_G sn G (Theorem 1.3), it is enough to show that X sn N .

Apply Lemma 1.6 to the group N with abelian normal subgroup A_1. Then (using the notation of 1.6 and thinking of θ as the identity map)

$$N_1 = A_1] H_1 = A_1] K_1 = H_1 K_1 \in \mathcal{Y} \ , \qquad (**)$$

where $H_1 = A_1(N \cap K) \cap N \cap H$ and $K_1 = A_1(N \cap H) \cap N \cap K$.

Let $B_1 = B \cap N_1$, so that $B_1 = B \cap A_1 H_1 = A_1(B \cap H_1)$. Now $B \cap H = J \cap H = A_1(J \cap K) \cap H$ by (*), so that $B \cap H \leq H_1$, and

$$B \cap H = B \cap H_1 \ , \ B \cap K = B \cap K_1 \ .$$

Thus $A_1(B \cap H_1) = A_1(J \cap H)$ so by (*), $B_1 = J$ and

$$B_1 \cap H_1 = B \cap N_1 \cap H_1 = B \cap H \ .$$

Hence

$$B_1 = \langle B_1 \cap H_1 , B_1 \cap K_1 \rangle \ .$$

We must check that (i) and (ii) hold in N_1 (and then the Lemma is proved). For (i), the series $1 \triangleleft A_1 \triangleleft B_1 \triangleleft N_1$ has the required properties ($A_1 \in A \cap Z$, $B_1/A_1 = J/A_1 \in A$, $N_1/B_1 = N_1/J \in F$) and N_1 has the triple factorisation (**). For (ii), $B_1 \cap H_1 \cap K_1 \leq B \cap H \cap K \in F$.

□

1.22 Remarks.

(a) Suppose, with the hypotheses of Lemma 1.21, that $A \in N_c \cap Z$, $|G:B| \leq m$ and $X \triangleleft^n H$, $X \triangleleft^n K$ $(c,m,n \in \mathbb{N})$. Then, in order to prove that there exists an integer $f = f(n,c,m,Y)$ such that $X \triangleleft^f G$, we may assume that (i), (ii) and (iii) of 1.21 hold and prove that there exists $g = g(n,m,Y) \in \mathbb{N}$ such that $X \triangleleft^g G$. To see this, we just follow the proof of Lemma 1.21, noting that Theorem 1.8 and Lemma 1.6 allow us to make 'bounded reductions'.

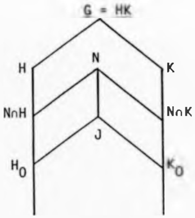
(b) Suppose the hypotheses of Lemma 1.21, except that $G \in (N \cap Z)(A \cap Y)F$ and $B/A \in A \cap Y$ (rather than $G \in Y$, so that A need not lie in Y). Then we may still assume that (i), (ii) and (iii) hold in order to prove X sn G. Also, if c, m and n are as in (a) above, we can make a similar bounded reduction from f to g as in (a) (with the same justification).

The following Lemma (used in the proof of Lemma 1.21 (iii)) is well-known.

1.23 LEMMA.

Let G be a group with subgroups H_0, H, K_0, K such that $G = HK$, $H_0 \triangleleft H$, $K_0 \triangleleft K$ and H/H_0 (or K/K_0) is periodic. Put $J = \langle H_0, K_0 \rangle$ and $N = N_G(J)$. Then $N = (N \cap H)(N \cap K)$.

Proof.



Let $g \in N$. Then $g = hk$ where $h \in H$, $k \in K$. Then $K_0 = K_0^{k^{-1}} \leq J^{k^{-1}} = J^{gk^{-1}} = J^h$ and $H_0 = H_0^h \leq J^h$. Hence $J \leq J^h$. But H/H_0 is periodic, so that $h^n \in J$ for some $n \geq 1$. Hence $J = J^h$ so that h, and hence k, lies in N. The reverse inclusion is clear. □

In order to show that *NAF*-groups of finite rank lie in $\mathfrak{W}^S$, we first have

1.24 LEMMA.

Let X be a subnormal subgroup of $H \in AF$, with defect at most $n \in \mathbb{N}$. Let A be an abelian normal subgroup of H such that $|H:A| = m$ is finite. Suppose that X has finite exponent e. Then X^H has finite exponent at most $e^{(3m)^{n-1}}$.

Proof.

We use induction on n, the result being clear if $n = 1$. So suppose $n \geq 2$ with the usual induction hypothesis. By induction, $\exp(X^{(X^H)}) \leq e^{(3m)^{n-2}}$, so it is enough to prove that $\exp(X^H) \leq e^{3m}$

when

$$X \lhd X^H \lhd H \ .$$

Then $X^A \lhd X^H$ and X^A has at most m conjugates in H, each of which is normal in X^H and all of which generate X^H. Hence it is enough to prove that $\exp(X^A) \le e^3$. Now $X^A = X[X,A] = X\langle [X,a] : a \in A\rangle = X\langle XX^a \cap A : a \in A\rangle$. Let $a \in A$. Then $\exp(XX^a \cap A) \le e^2$ and since A is abelian, $\exp(X^A) \le e.e^2$ as required.

□

1.25 LEMMA.

Suppose the hypotheses of Lemma 1.24 and also that H has finite rank r and X has finite order e. Then X^H is finite of order at most $me^{r(3m)^{n-1}}$.

Proof.

Let $B = A \cap X^H$. Then B is abelian of rank at most r and $\exp(B) \le \exp(X^H) \le e^{(3m)^{n-1}}$ (Lemma 1.24). Therefore $|B| \le e^{r(3m)^{n-1}}$. Since $|X^H : B| \le m$, we have the required bound.

□

Let ${}_m F$ $(m \in \mathbb{N})$ denote the class of finite groups of order at most m.

1.26 THEOREM.

Let $\mathcal{V}_r$ denote the class of groups of (finite) rank at most $r \in \mathbb{N}$. Then $\mathcal{N}_c\mathcal{A}(_m\mathcal{F}) \cap \mathcal{V}_r \leq \mathcal{W}_f$, where $f = f(n,c,m,r)$. In particular, nilpotent-by-abelian-by-finite groups of finite rank are $\mathcal{W}$-groups.

Proof.

Let $G \in \mathcal{N}_c\mathcal{A}(_m\mathcal{F}) \cap \mathcal{V}_r$ be generated as the product of subgroups H and K, both of which contain X as a subnormal subgroup of defect at most $n \in \mathbb{N}$. By Remark 1.22(a), it is enough to show that $X \lhd^{g} G$ where $g = g(n,m,r) \in \mathbb{N}$ and (i), (ii) of Lemma 1.21 hold.

Using the notation of 1.21, $|X| = |BX : B| \leq m$. Also $B \cap H \in \mathcal{A}$ and $|H : B \cap H| \leq m$. By Lemma 1.25, X^H (and similarly X^K) is finite of order less than a function of n,m and r. Hence X has a finite number g_1 of conjugates in H (or in K) where $g_1 = g_1(n,m,r)$. Let $J = \langle N_H(X), N_K(X)\rangle$. Then by Lemma 1.14, $|G : J| \leq 2g_1$ and so G/J_G is finite of order at most $g_2 = g_2(n,m,r) \in \mathbb{N}$. Hence $XJ_G \lhd^{g_2} G$ (Theorem 1.3). Since $X \lhd XJ_G$, we may take $g = g_2+1$.

□

Suppose (in the notation of Theorem 1.26) that $G_1 \in \mathcal{N}_c(\mathcal{A} \cap \mathcal{V}_r)(_m\mathcal{F})$. Then, following the proof of 1.26 with G_1

in place of G, we see (using Remark 1.22(b)) that $H \cong K \cong G/A \in Y_{r+m}$ (1.21(i)), so we can bound the defect of X in G_1 by $g(n,m,r+m)$. Therefore we have proved

1.27 COROLLARY.

$N_c(A \cap Y_r)({}_mF) \leq W_f \cap W^S$, where $f = f(n,c,m,r)$.

□

Also we have

1.28 COROLLARY (see Theorem 1.9)

$(S \cap \bar{M})F \leq W^S$.

Proof.

Let $B \in S \cap \bar{M}$. Then by ([R2] p.166), B has finite rank and $B \in NAF$. Therefore $(S \cap \bar{M})F \leq NAF \leq W^S$ by Theorem 1.26.

□

1.29 COROLLARY.

Let $G \in NAF$. If the abelian subgroups of G have finite rank, then $G \in W^S$.

Proof.

G contains a soluble subgroup B of finite index, and the abelian subgroups of B have finite rank. By a result of Kargapolov ([KV]), B has finite rank. Therefore G has finite rank and so $G \in \mathcal{W}^S$ by Theorem 1.26.

□

In Theorem 1.31, we will improve the second part of Theorem 1.26 by showing that, if $\mathcal{Y}$ denotes the class of groups of finite rank, then $(\mathcal{N} \cap \mathcal{Y})\mathcal{AF} \subseteq \mathcal{W}^S$, thereby removing the finite rank hypothesis from the abelian section B/A . The cost of this improvement will be any bounded result, which we will not be able to have with the proof used.

1.30 LEMMA.

Let $\mathcal{Z}$ denote the class of periodic groups of finite rank. Let $G \in (\mathcal{A} \cap \mathcal{Z})\mathcal{W}^S$ be generated as the product of subgroups H and K , both of which contain X as a subnormal subgroup. Suppose that X is periodic and that $\pi = \pi(X/X')$ finite. Then X sn G .

Proof.

Let $A \in \mathcal{A} \cap \mathcal{Z}$ be a normal subgroup of G such that $G/A \in \mathcal{W}^S$. By Lemma 1.6, we may assume $G = A \rtimes H = A \rtimes K = HK$. For $p \in \mathbb{P}$, let A_p denote the p-component of A . Then $A_p = D_p \times F_p$ where D_p is divisible and F_p is finite.

Now $A_pH = H(A_pH \cap K)$ and $H \in \mathcal{W}^S$. By Lemma 1.18, X sn A_pH. Therefore $[D_p,{}_sX] \le X \cap A = 1$ (some $s \in \mathbb{N}$), so by Lemma 1.17 we have $[D_p,X] = 1$. Therefore X is centralised by the divisible part D of A and so (factoring G by D) we may assume that $A = \mathrm{Dr}_{p \in \mathbb{P}} F_p$.

For each $p \in \mathbb{P}$ there exists an epimorphism

$$\theta_p : A_p \otimes \frac{X}{X'} \to \frac{[A_p,X]}{[A_p,X,X]}$$

which arises from the bilinear map $(a_p, xX') \to [a_p,x]\,[A_p,X,X]$.

Suppose that $p \in \pi'$. Then $A_p \otimes (X/X') = 1$, so that $[A_p,X] = [A_p,X,X]$. But X sn $A_p] X$ so we must have $[A_p,X] = 1$. Therefore (factoring G by $\mathrm{Dr}_{p\in\pi'} A_p$) we may assume that A is a π-group and hence finite. Hence $G \in F\mathcal{W}^S$ and X sn G (Proposition 1.10). □

We can remove the periodic hypothesis from the class Z of Lemma 1.30 :

1.31 LEMMA.

Let Z denote the class of groups of finite rank. Let $G \in (A \cap Z)\mathcal{W}^S$ be generated as the products of subgroups H and K, both of which contain X as a subnormal subgroup. Suppose that X is periodic and that $\pi = \pi(X/X')$ is finite. Then X sn G.

Proof.

Let $A \in \mathcal{A} \cap \mathcal{Z}$ be a normal subgroup of G such that $G/A \in \mathcal{W}^S$. We prove that X sn G by induction on the torsion-free rank r of A, the result being clear (by Lemma 1.30) if $r = 0$. So suppose $r \geq 1$ with the usual induction hypothesis. By Lemma 1.6, we may assume that $G = A] H = A] K = HK$. Let T be the torsion subgroup of A. Then

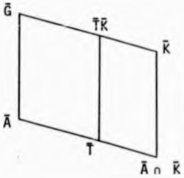
$$TH = H(K \cap TH) \in (\mathcal{A} \cap \mathcal{Y})\mathcal{W}^S ,$$

where $\mathcal{Y}$ denotes the class of periodic groups of finite rank. By Lemma 1.30, X sn TX, so (factoring G by T) we may assume that

$$G = A] H = A] K = HK \text{ and } A \text{ is torsion-free.} \qquad (*)$$

Then (as in the proof of Proposition 1.18) $C_H(A) \triangleleft G$ and if bars denote subgroups of G modulo $C_H(A)$, we have $\bar G = \bar A] \bar H = \bar A\bar K = \bar H\bar K$ and $C_{\bar H}(\bar A) = 1$. Also $C_{\bar K}(\bar A) = \bar A \cap \bar K$. Since X sn $C_H(A)X$, it is enough to prove that $\bar X$ sn $\bar G$.

Suppose that $\bar A \cap \bar K \neq 1$. Then if $\bar\theta$ is the epimorphism from $\bar G$ to $\bar G/(\bar A \cap \bar K)$, we have $\bar G^\theta = \bar A^\theta \bar H^\theta = \bar A^\theta] \bar K^\theta - \bar H^\theta \bar K^\theta$ and $\bar X^\theta$ sn $\bar H^\theta$, $\bar X^\theta$ sn $\bar K^\theta$. Let $\bar T^\theta$ be the torsion subgroup of $\bar A^\theta$. Then, repeating the argument used for T above (with H and K interchanged), we may assume that $\bar T^\theta = 1$. Therefore



$\bar{A}^{\theta}$ is torsion-free, as is $\bar{A}$ and $\bar{A} \cap \bar{K}$. Hence $\bar{A}^{\theta} = \bar{A}/(\bar{A} \cap \bar{K})$ is of torsion-free rank less than that of $\bar{A}$. By induction, $\bar{X}^{\theta}$ sn $\bar{G}^{\theta}$. Since $\bar{X}$ sn $\bar{X}(\bar{A} \cap \bar{K})$, we have $\bar{X}$ sn G (and so X sn G) . Therefore, we need only consider the case $\bar{A} \cap \bar{K} = 1$. Since $\bar{A}$ is torsion-free, we may now assume

(*) holds and $C_H(A) = C_K(A) = 1$.

Since A is torsion-free, there is an embedding $A \to A \otimes Q$ (via the mapping $a \to a \otimes 1$) . Thinking of A as a subgroup of $A \otimes Q = A^*$, say, the action of H on A extends to an action on A^* in the natural way, viz: $(a \otimes q)^h = a^h \otimes q$ $(a \in A , h \in H , q \in Q)$. Therefore $C_H(A^*) \leq C_H(A) = 1$ and H (similarly K) acts faithfully on A^* . So we have embeddings

$$H,K \to \mathrm{Aut}(A^*) \cong GL(r,Q) .$$

Since X is periodic and subnormal in H and K , X^H and X^K are periodic. But periodic subgroups of $GL(r,Q)$ are finite (see [R1] p.85), whence X^H and X^K are finite. Putting $J = \langle N_H(X), N_K(X) \rangle$ we have XJ_G sn G by Lemma 1.14 and Theorem 1.3. Since $X \triangleleft XJ_G$, this completes the proof.

□

Remark.

The reduction afforded by Theorem 1.8, when used on some group

G , is not hindered by the additional hypothesis that X (where X sn H , X sn K etc.) belongs to some Q-closed class. Hence Lemma 1.31 remains true if $G \in (N \cap Z)W^S$.

We can now prove

1.32 THEOREM.

Let Z denote the class of groups of finite rank. Then $(N \cap Z)AF \subseteq W^S$.

Proof.

Let $G \in (N \cap Z)AF$ be generated as the product of subgroups H and K , both of which contain X as a subnormal subgroup. By Lemma 1.21 (with $V = U$) we may assume that $G \in (A \cap Z)AF$ and that X is finite. Since $AF \subseteq W^S$ (Theorem 1.9), X sn G by Lemma 1.31. Hence $G \in W^S$.

□

It would be interesting to know if W^S contains the class of soluble groups of finite rank. Since the proof of Lemma 1.31 (and Lemmas 1.18, 1.30) work under the (weaker) hypothesis that AX sn G (rather than $G/A \in W^S$) , a simple induction yields the following partial result:

1.33 PROPOSITION.

Let G be a soluble group of finite rank, generated as the product of subgroups of H and K, both of which contain X as a subnormal subgroup. If X is periodic and $\pi(X/X')$ is finite (so that X/X' is a Černikov group), then $X \text{ sn } G$.

$\square$

We shall see in §1.5 (Theorem 1.41) that if we remove the restrictions imposed on X in Proposition 1.33, then we can at least conclude that X is ascendant in G (even if G is *locally* soluble of finite rank). We finish this section by identifying two subclasses of the class of soluble groups of finite rank which will lie in $\mathfrak{W}^S$, using well-known results of Mal'cev ([MV]).

DEFINITION.

Let A_1 denote the class of abelian groups A which satisfy: if $T(A)$ denotes the torsion subgroup of A then $T(A) \in \check{M}$ and $A/T(A)$ has finite rank. (A_1-groups are otherwise known as abelian groups of finite total rank. Mal'cev calls them A_3-groups). Then define the class S_1 by $S_1 = PA_1$. (S_1 can also be defined as the class of hyperabelian groups of finite abelian section rank, which contains elements of only finitely many distinct prime orders: see [R2] 9.3).

Soluble linear groups are nilpotent-by-abelian-by-finite and S_1-groups are nilpotent-by-abelian-by-finite of finite rank (Mal'cev, see [R1] 3.2). Then by Theorem 1.26, we have:

1.34 COROLLARY.

(i) $S_1 \leq \mathfrak{W}^s$

(ii) Soluble linear groups of finite rank are $\mathfrak{W}^s$-groups.

□

§1.5 Ascendancy and $\dot{\mathfrak{W}}$.

By considering ascendancy, rather than subnormality, we can define the class of groups $\dot{\mathfrak{W}}$ in an analogous way to $\mathfrak{W}$:

1.35 DEFINITION.

$\dot{\mathfrak{W}}$ is the class of groups consisting of groups G which satisfy (*) :

> Whenever G is generated as the product of subgroups H and K , both of which contain X as an ascendant subgroup, then X is ascendant in G . $\Bigg\}$(*)

Whilst we might expect that it will in general be more difficult to find $\dot{\mathfrak{W}}$-groups than to find $\mathfrak{W}$-groups, ascendancy can sometimes allow us more freedom than subnormality. For example, the ascendant analogue of Theorem 1.8 (Theorem 1.37) allows us to make a reduction from *hypercentral*-by-$\dot{\mathfrak{W}}^s$ groups to abelian-by-$\dot{\mathfrak{W}}^s$ groups. Theorem 1.37 is used to prove the main result of this section (Theorem 1.41) which says that locally soluble groups of finite rank are $\dot{\mathfrak{W}}^s$-groups.

Obvious subclasses of $\bar{W}$ are F and $(PC)F$; more generally, because ascendancy is equivalent to subnormality in $\hat{M}$-groups, $W \cap \hat{M} = \bar{W} \cap \hat{M} \subseteq \bar{W}$. In fact, $(ZA)\hat{M} \cap W \subseteq \bar{W}$ (Proposition 1.38(i), ZA = hypercentral groups).

1.36 LEMMA (c.f. Lemma 1.6)

Let G be a group with subgroups X,H,K and A such that $G = HK$, $X \lhd^{\nu} H$, $X \lhd^{\nu} K$ (ν ordinal) and A is an abelian normal subgroup of G . Then, in the notation of Lemma 1.5, $G_1^\theta = H_1^\theta K_1^\theta = A_1^\theta \,]\, H_1^\theta = A_1^\theta \,]\, K_1^\theta$ and $X^\theta \lhd^{\nu} H_1^\theta$, $X^\theta \lhd^{\nu} K_1^\theta$. Also, H_1^θ and K_1^θ embed in G/A . Further, if $X^\theta \lhd^{\mu} G_1^\theta$ and $AX \lhd^{\lambda} G$ (λ,μ ordinals), then $X \lhd^{\nu.2+\mu+\lambda} G$.

Proof.

The proof of Lemma 1.6 (and Lemma 1.7) works here (with λ,μ,ν in place of ℓ,m,n respectively).

□

Let ZA denote the class of hypercentral groups and $(ZA)_\alpha$ the class of hypercentral groups with hypercentral series of length at most α (ordinal).

1.37 THEOREM.

Let X,Y and Z be S-closed classes of groups with Y and Z also Q-closed. Suppose that $(A \cap Z)X \cap Y \subseteq \bar{W}$. Then $(ZA \cap Z)X \cap Y \subseteq \bar{W}^S$.

Proof.

The proof is essentially that of Theorem 1.8. Let $G \in ((ZA)_\alpha \cap Z)X \cap Y$ be generated as the product of subgroups H and K, both of which contain X as an ascendant subgroup. Let $B \in (ZA)_\alpha \cap Z$ be a normal subgroup of G such that $G/B \in X$. Let Z_β $(0 \le \beta \le \alpha)$ denote the $\beta^{\underline{th}}$ term of the upper central series of B. Then, as in the proof of 1.8,

$$XZ_\beta \text{ asc } XZ_{\beta+1} \quad (0 \le \beta \le \alpha) .$$

Therefore $X \text{ asc } \bigcup_{0\le\beta\le\alpha} (XZ_{\beta+1}) = XB$. Since $G/B \in \tilde{W}$, then $XB \text{ asc } G$. This completes the proof.

$\square$

1.38 PROPOSITION.

(i) $(ZA)\hat{M} \cap W^S \le \tilde{W}^S$. In particular, $N(PC)F \le \tilde{W}^S$.

(ii) $(ZA)A \le \tilde{W}^S$.

(iii) $F\tilde{W} = \tilde{W}$. In particular $F\tilde{W}^S = \tilde{W}^S$.

Proof.

Let G be a group which is generated as the product of subgroups H and K, both of which contain X as an ascendant subgroup. We wish to prove that $X \text{ asc } G$.

(i) Let $G \in (ZA)\hat{M} \cap \mathfrak{W}^S$. $\mathfrak{W}^S$ is Q-closed, so, by Theorem 1.37 and Lemma 1.36, we may assume that $G \in A\hat{M} \cap \mathfrak{W}^S$ and that $H,K \in \hat{M}$. Therefore X sn H and X sn K , so that X sn $G \in \mathfrak{W}^S$. Hence $G \in \bar{\mathfrak{W}}^S$.

(ii) Let $G \in (ZA)A$. Again by Theorem 1.37 and Lemma 1.36, we may assume that $G \in A^2$ and that $H,K \in A$. Hence $X \triangleleft G$. This proves (ii).

(iii) Let $G \in F\bar{\mathfrak{W}}$ and let N be a finite normal subgroup of G such that $G/N \in \bar{\mathfrak{W}}$. Then, as in the proof of Proposition 1.10, $|NH : H_{NN}|$ is finite and XH_{NH} sn NH . Therefore X asc NX asc G , proving (iii).

□

Parts (i), (ii) and (iii) of the next lemma are the $\bar{\mathfrak{W}}$-analogues of Lemmas 1.18, 1.30 and 1.31 (with the hypothesis on X/X' removed). Note that, since the class of periodic groups is N-closed and the union of periodic subgroups is periodic, then the normal closure of an ascendant periodic subgroup is periodic ([R1] 1.31).

1.39 LEMMA.

Let G be generated as the product of subgroups H and K , both of which contain X as an ascendant subgroup. Suppose that X is periodic. Then each of the following imply that X asc G .

(i) G is a Černikov-by-$\hat{\mathfrak{W}}^S$ group.

(ii) G is a $\hat{\mathfrak{W}}^S$-group modulo a periodic abelian subgroup of finite rank.

(iii) G is a $\hat{\mathfrak{W}}^S$-group modulo a hypercentral subgroup of finite rank.

(iv) G is a hyperabelian group of finite rank.

Note that for groups of finite rank, the conditions hyperabelian and locally soluble are equivalent (see [R2] 10.38 Cor.1). Hence, by (iv) above, $\hat{\mathfrak{W}}$ contains the class of periodic locally soluble groups of finite rank (we will remove the periodic restriction in Theorem 1.41). In particular, $\hat{\mathfrak{W}}$ contains the class of Černikov groups (this is implied directly by (i) above).

<u>Proof</u>.

(i) Let $A \in \mathcal{D}$ be a normal subgroup of G such that $G/A \in \hat{\mathfrak{W}}^S$ (recall that $\mathcal{D}$ denotes the class of divisible abelian groups with min. Such an $A \in \mathcal{D}$ exists by Proposition 1.38(iii)). By Lemma 1.36, we may assume $G = HK = A] H = A] K$. Then, if bars denote subgroups of G modulo $C_H(A)$, we have $\bar{X}^{\bar{H}}$ periodic and, following the proof of Lemma 1.18, $\bar{X}^{\bar{H}}$ is finite and X asc $XC_K(A)$ asc $XC_H(A)C_K(A)$ sn G.

(ii) Let A be a normal periodic abelian subgroup of G such that A has finite rank and $G/A \in \mathfrak{W}^S$. By Lemma 1.36, we may assume

$G = A] H = A] K = HK$. For $p \in \mathbb{P}$, let A_p denote the p-component of A . Then

$$A_pH = H(A_pH \cap K) \in \mathfrak{D}\tilde{\mathfrak{W}}^S$$

and by (i) of this lemma, $X \text{ asc } XA_p$. Writing $\mathbb{P} = \{p_1, p_2, \ldots\}$ and putting $B_r = \langle A_{p_i} : 1 \le i \le r \rangle$ for $1 \le r < \omega$, we have

$$XB_r \text{ asc } XB_{r+1} \quad (1 \le r < \omega) .$$

Therefore $X \text{ asc } \bigcup_{1 \le r < \omega} (XB_{r+1}) = XA$. Finally, $XA \text{ asc } G$ (since $G/A \in \tilde{\mathfrak{W}}^S$) , which proves (ii).

(iii) By Theorem 1.37 (with $\mathcal{X} = \tilde{\mathfrak{W}}^S$, $\mathcal{Z}$ = finite rank groups, $\mathcal{Y} = \mathcal{U}$) , through which we can carry the hypothesis that X is periodic, we may assume that there exists a normal abelian subgroup A of G such that A has finite rank, and $G/A \in \tilde{\mathfrak{W}}^S$. We can use the same argument as in the proof of Lemma 1.31 (using (ii) above in place of Lemma 1.30 to get rid of the torsion subgroup) to embed H and K in $GL(r,Q)$ where $r \le \text{rank}(A)$. Since, in our case, X^H and X^K are still periodic, we can repeat the remainder of the proof of 1.31, thus proving (iii).

(iv) Let $\{G_\beta\}_{0 \le \beta \le \alpha}$ be an ascending normal series of G with abelian factors (α ordinal). For $\beta \leq \alpha$, consider the group

$$\bar{G}_1 = \overline{G_{\beta+1}}\,\bar{H} \cap \overline{G_{\beta+1}}\,\bar{K} = (\bar{H} \cap \overline{G_{\beta+1}}\,\bar{K})(\bar{K} \cap \overline{G_{\beta+1}}\,\bar{H}) ,$$

where bars denote subgroups of G modulo G_β. Since $\bar{X}$ asc $\bar{H}$ then $\bar{X}\,\overline{G_{\beta+1}}$ asc $\bar{G}_1$ $(\leq \bar{H}\,\overline{G_{\beta+1}})$. Part (iii) (and (ii) and (i)) of this lemma remains true if (in its proof) AX asc G (rather than $G/A \in \acute{\omega}^{S}$); $\overline{G_{\beta+1}}$ is abelian of finite rank, therefore $\bar{X}$ asc $\overline{G_{\beta+1}}\,\bar{X}$ $(\leq \bar{G}_1)$. Therefore

$$G_\beta X \text{ asc } G_{\beta+1} X \quad \text{for} \quad 0 \leq \beta < \alpha .$$

Hence X asc $\bigcup_{\beta<\alpha} (G_{\beta+1} X) = G$, as required.

□

Hypercentral groups form an N_0-closed class ([HA]), so that $(ZA)AF = ZA(AF)$.

1.40 LEMMA (c.f. Lemma 1.21)

Let $\mathcal{Y}$ be an S-closed and Q-closed class of groups. Let $G \in (ZA)AF \cap \mathcal{Y}$ be generated as the product of subgroups H and K, both of which contain X as a subnormal subgroup. Let $1 \triangleleft A \triangleleft B \triangleleft G$ be a normal series of G such that $A \in ZA$, $B/A \in A$, $G/B \in F$.

Then in order to prove X asc G , we may assume the following conditions hold:

(i) $A \in \mathcal{A}$ and $G = A \,]\, H = A \,]\, K = HK$.

(ii) $B \cap H \cap K = 1$ (assuming (i)).

(iii) $B = \langle B \cap H, B \cap H \rangle$ (assuming (i), (ii)).

Proof.

(i) follows from Theorem 1.37 (with $\mathcal{X} = AF$, $\mathcal{Z} = \mathfrak{U}$) and Lemma 1.36. For (ii) and (iii), we can use the proofs of Lemma 1.21 (ii), (iii) (with the obvious modifications).

□

Putting Lemmas 1.39 and 1.40 together, we now have:

1.41 THEOREM.

Let $\mathcal{Y}$ denote the class of groups of finite rank. Then $L\mathcal{S} \cap \mathcal{Y} \subseteq \mathfrak{W}^S$.

Proof.

Let $G \in L\mathcal{S} \cap \mathcal{Y}$. Then $G \in (ZA)AF$ ([R2] 8.16). Let G be generated as the product of subgroups H and K , both of which contain X as an ascendant subgroup. Then by Lemma 1.40 (and using its notation) we may assume that X is finite and $G = A \,]\, H = A \,]\, K = HK$. Then $G/A \in AF \subseteq \mathfrak{W}^S$ (Proposition 1.38(i)), so by Lemma 1.39 (iii) we have X asc G , as required.

□

Remark.

Let $\mathcal{Y}$ denote the class of groups of finite rank. As noted in the above proof, $LS \cap \mathcal{Y} \subseteq (ZA)AF$. Also, $(ZA)A \cap \mathcal{Y} \subseteq LS$ by ([R2] 10.38 Corollary 1). Therefore $(LS)F \cap \mathcal{Y} = (ZA)AF \cap \mathcal{Y}$ and, using the above proof, we have $(LS)F \cap \mathcal{Y} \subseteq \hat{\mathfrak{W}}^s$.

CHAPTER 2. PROJECTIVITIES.

§2.0 Introduction.

In this chapter we consider projectivities (i.e. isomorphisms of subgroup lattices) and their effect on subnormal and ascendant subgroups. §§2.1 and 2.2 are preliminary, in which the unary closure operation u and the class of $\mathcal{R}$-groups (often called P-groups) are defined. In §2.3 we consider subnormality and projectivities. Using results of Schmidt ([SH]) and Suzuki ([SZ]), we give necessary and sufficient conditions for a projectivity of a finite group lattice to preserve subnormality (Theorem 2.12). Theorem 2.17 shows that projective images of subnormal subgroups of Černikov groups behave in a similar way to the finite case. Using a result of Zacher ([Z2]), we show that the projective image of an ascendant subgroup (of any group) has an ascendiser (Corollary 2.19).

In §2.4 we generalise Theorem 1.3 to include the case where H and K do not permute, but they are such that $\mathcal{L}(\langle H,K\rangle)$ admits a projectivity σ for which H^{σ} and K^{σ} permute (Theorem 2.21). We define the class $\mathcal{W}_{u}$ so that Theorem 2.21 says that $F \subseteq \mathcal{W}_{u}$ and so that $\mathcal{W}_{u}$ lies between $\mathcal{J}$ and $\mathcal{W}$. We then identify some other subclasses of $\mathcal{W}_{u}$: the classes of Černikov groups (Proposition 2.22), metabelian groups (Theorem 2.24) and polycyclic-by-finite groups (Theorem 2.26) are all contained in $\mathcal{W}_{u}$.

§2.1 Preliminaries.

A projectivity is defined to be an isomorphism of subgroup lattices; that is, if G and $\bar{G}$ are groups, then a map

$$\sigma : L(G) \to L(\bar{G})$$

is called a projectivity if and only if σ is a bijection and whenever $\{A_\lambda : \lambda \in \Lambda\}$ is a collection of subgroups of G, then

$$\Big(\bigcap_{\lambda\in\Lambda} A_\lambda\Big)^\sigma = \bigcap_{\lambda\in\Lambda} (A_\lambda^\sigma) \quad \text{and} \quad \langle A_\lambda : \lambda \in \Lambda\rangle^\sigma = \langle A_\lambda^\sigma : \lambda \in \Lambda\rangle \ . \qquad (*)$$

Note that, in this definition, it is sufficient to require that σ is a bijection and that $(*)$ holds whenever $|\Lambda| = 2$. This is because these (seemingly) weaker conditions are equivalent to the conditions

$$\sigma \text{ is a bijection and } \sigma, \sigma^{-1} \text{ preserve subgroup inclusion,} \qquad (**)$$

and $(**)$ holds if and only if σ is a projectivity; for, suppose $(**)$ holds and let $\{A_\lambda : \lambda \in \Lambda\}$ be a collection of subgroups of G. Let $A^\sigma = \bigcap_{\lambda\in\Lambda} (A_\lambda^\sigma)$. Then for $\lambda \in \Lambda$, $A^\sigma \leq A_\lambda^\sigma$ so that $A \leq \bigcap_{\lambda\in\Lambda} A_\lambda$, whence $A^\sigma \leq (\bigcap_{\lambda\in\Lambda} A_\lambda)^\sigma$. Also, for $\mu \in \Lambda$, $\bigcap_{\lambda\in\Lambda} A_\lambda \leq A_\mu$ so that $(\bigcap_{\lambda\in\Lambda} A_\lambda)^\sigma \leq A^\sigma$. The remainder of $(*)$ is proved similarly, and clearly $(**)$ holds if σ is a projectivity.

Clearly, if σ is a projectivity as above, then $G^\sigma = \bar{G}$, $1_G^\sigma = 1_{\bar{G}}$ and σ^{-1} is a projectivity. Obvious examples of projectivities are those induced by (any) group isomorphisms, but not every projectivity is so induced (groups of different prime orders have isomorphic lattices). So we define the <u>closure operation u</u> as follows.

If X is a class of groups, uX is the class of groups given by

$\bar{G} \in uX$ if and only if there exists $G \in X$ and a projectivity

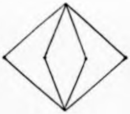
$\sigma : L(G) \to L(\bar{G})$.

That is, uX consists of the projective images of X-groups. When we write $G^\sigma \in uX$, we will mean that $G \in X$ and σ is a projectivity of $L(G)$. Note that $uX = \bigcup_{G \in X} u(G)$ i.e. u is unary.

Since an infinite group has an infinite number of subgroups, $uF = F$. Moreover, if G is a finite group and σ a projectivity of $L(G)$, then the number of primes (including multiplicities) dividing the orders of G and G^σ are equal ([SZ]). Other u-closed classes of groups are the classes of soluble groups ([YV]), simple groups ([Z1]) and perfect groups ([NA]). Also, $uA \subseteq A^2$ ([SZ]) and, more generally, given $n \in \mathbb{N}$, there exists $f = f(n) \in \mathbb{N}$ such that $uA^n \subseteq A^f$ ([YV]). However, $uA \nsubseteq A$ and $uN \nsubseteq N$ as we see from:

2.1 EXAMPLE.

Let G be an elementary abelian 3-group of order 9 and let $\bar{G}$ be a non-abelian group of order 6 . Clearly, G and $\bar{G}$ have isomorphic lattices.



The above example, although a simple one, provides a very good model of how a projectivity can fail to map either a normal or ascendant subgroup to the same (in 2.1, any Sylow 2-subgroup of $\bar{G}$ is self-normalising). If G is a group and $G^\sigma \in u(G)$, then we say that σ preserves ascendancy if whenever X is an ascendant subgroup of G then X^σ is ascendant in G^σ . Preservation of subnormality and normality are defined similarly, in the obvious way.

Let G be a group and let $G^\sigma \in u(G)$. Then we say that σ is index-preserving if and only if

$$|U:V| = |U^\sigma:V^\sigma| \quad \text{for all subgroups } V \le U \text{ of } G . \qquad (*)$$

Suzuki ([SZ] II.6) calls such a projectivity strictly index-preserving and shows that when G is finite, (*) is equivalent to the condition that (*) holds when U is cyclic. These conditions are equivalent even if G is not finite; this was proved by Zacher in [Z 1], using the following important result.

2.2 THEOREM (Rips [RP], Zacher [Z1])

Let G be a group and $G^\sigma \in u(G)$. Then if H is a subgroup of finite index in G , H^σ has finite index in G^σ .

□

Let X be a subgroup of a group G and let $G^\sigma \in u(G)$. Then we will write

$$X_{G^\sigma}, X^{G^\sigma}$$

to denote the pre-image (under σ) of the core $(X^\sigma)_{G^\sigma}$ and normal closure $X^{\sigma(G^\sigma)}$ respectively. In [BU], Busetto proves the following result (the finite case is due to Schmidt [SH]):

2.3 THEOREM.

Let N be a normal subgroup of a group G and let $G^\sigma \in u(G)$. Then N_{G^σ} and N^{G^σ} are normal in G .

□

2.4 REMARK.

If H is a subgroup of finite index in a group G and $G^\sigma \in u(G)$, then Theorems 2.2 and 2.3 imply that H contains a normal subgroup N of finite index in G such that $N^\sigma \triangleleft G^\sigma$ (and $|G^\sigma : N^\sigma|$ is finite). Hence σ induces a projectivity of finite lattices $\hat{\sigma} : L(G/N) \to L(G^\sigma/N^\sigma)$.

An index-preserving projectivity of a finite subgroup lattice will map a maximal normal subgroup to a normal subgroup ([SZ] II, 2.6) which is therefore also maximal normal. So for finite group lattices, an index-preserving projectivity will map a composition series to a composition series. Thus we have

2.5 LEMMA. ([SH] Lemma 4.1)

Let G be a finite group and let $G^\sigma \in u(G)$. If σ is index-preserving, then σ preserves subnormality.

□

The converse of Lemma 2.5 is clearly false (G and G^σ could have different prime orders); see Theorem 2.12 for necessary and sufficient conditions for the converse to hold. Lemma 2.5 does not hold if G is an arbitrary group, even if G is abelian as the following example shows. However, Rips ([RP]) has shown that for an arbitrary group G , an index-preserving projectivity will preserve ascendancy - in fact, if N is a normal subgroup of G then $N^\sigma \triangleleft^\omega G^\sigma$ (see also Theorem 2.8).

2.6 EXAMPLE. (See [I2]).

Let $G = A \times H$ where A is a quasicyclic p-group ($p \in \mathbb{P}$) and $H = \langle h \rangle$ is an infinite cyclic group. Let $\alpha \neq 1$ be a p-adic integer such that $\alpha \equiv 1 \pmod p$ ($\alpha \equiv 1 \pmod 4$ if $p = 2$) . Let $\bar{A}$, $\bar{H}$ be isomorphic to A, H respectively and define an action of $\bar{H} = \langle \bar{h} \rangle$ on $\bar{A}$ by $\bar{a}^{\bar{h}} = \bar{a}^\alpha$ ($\bar{a} \in \bar{A}$) . Put $\bar{G} = \bar{A}] \bar{H}$. Then there is an index-preserving projectivity $\sigma : L(G) \to L(\bar{G})$ for which $A^\sigma = \bar{A}$

and $H^{\sigma} = \bar{H}$. Then $H \lhd G$ but $\bar{H}$ is not subnormal in $\bar{G}$ ($\alpha-1$ is a non-zero endomorphism of $\bar{A}$, so that $[\bar{A},\bar{H}] = \bar{A}$ and $\bar{H}^{G} = \bar{H}[\bar{A},\bar{H}] = \bar{G}$) . Note that Rips' result mentioned above implies that $\bar{H} \lhd^{\omega} \bar{G}$.

The following results will be useful in the sequel.

2.7 *THEOREM* ([SZ] I, Theorem 4)

Let G be a periodic group which is the direct product of Hall subgroups G_{λ} $(\lambda \in \Lambda)$. If $G^{\sigma} \in u(G)$ then $G^{\sigma} = \underset{\lambda \in \Lambda}{\mathrm{Dr}}\, G_{\lambda}^{\sigma}$ is a Hall decomposition of G^{σ} .

□

A subgroup M of a group G is said to be *modular in G* , written *M mod G* , if and only if given any subgroup X of G the map $\sigma_X : [\langle X,M\rangle/M] \to [X/X \cap M]$ defined by $Y^{\sigma_X} = Y \cap X$ is a lattice isomorphism. Equivalent conditions are that M satisfies the modular identities

$$X \cap \langle Y,M\rangle = \langle Y, X \cap M\rangle \quad \text{for all subgroups} \quad Y \leq X \quad \text{and}$$

$$A \cap \langle B,M\rangle = \langle A \cap B, M\rangle \quad \text{for all subgroups} \quad A,B \quad \text{with} \quad M \leq A .$$

A subgroup X of a group G is said to be *permutable* in G , written *X per G* if and only if $XU = UX$ for all subgroups U of G .

The concepts of modularity, ascendancy and permutability are linked by the following result.

2.8 <u>THEOREM</u> (Stonehewer [S 3])

Let X be a subgroup of a group G; then X per G if and only if both X asc G and X mod G. (In fact, if X per G then $X \triangleleft^{\omega+1} G$ ([S 5].)

□

An example of an ascendant non-modular subgroup can be found in a dihedral group of order 8. A 2-subgroup of Σ_3 is modular but not ascendant. Clearly, a normal subgroup N of a group G is modular (and permutable) in G, and therefore N^σ mod G^σ for any $G^\sigma \in u(G)$.

§2.2 Singular Projectivities.

Following Suzuki ([SZ] p.42), we say that a projectivity is <u>singular</u> if it is not index-preserving. Suppose that G is a group, $G^\sigma \in u(G)$ and σ is singular. Then there exists subgroups $V \leq U$ of G such that $|U^\sigma : V^\sigma| \neq |U:V| < \infty$. By Remark 2.4, V contains a normal subgroup N of U such that $N^\sigma \triangleleft G^\sigma$ and $|U:N|$, $|U^\sigma:N^\sigma|$ are finite. Hence the induced projectivity

$$\hat{\sigma} : L(\frac{U}{N}) \rightarrow L(\frac{U^\sigma}{N^\sigma})$$

is a singular projectivity of finite group lattices. Now $\bar{\sigma}$ induces a projectivity on the subgroup lattice of each Sylow subgroup of U/N ; if all such projectivities are index-preserving, so is $\bar{\sigma}$. Hence there is a prime p and a Sylow p-subgroup S/N of U/N such that $\bar{\sigma}/_{L(S/N)}$ is singular. Then we say that σ is *singular at p* , σ is *p-singular*. Further, the Sylow p-subgroups of U/N are cyclic or elementary abelian ([SZ] I, Theorem 12).

If, in the above, G is a finite group, then we will take $U = G$ and $N = 1$, so that the Sylow p-subgroups (when σ is p-singular) of G are cyclic or elementary abelian. Note here that σ need not be singular on every Sylow p-subgroup; for example, if $G \cong \Sigma_3$ and σ is a 3-singular auto-projectivity of $L(G)$, then σ is 2-singular but σ is index-preserving on two Sylow 2-subgroups of G . Also, we note that if G is any p-group (p prime) and $G^{\sigma} \in u(G)$, then G^{σ} is a p-group if and only if σ is index-preserving.

Given a prime p , we define the *class of groups R_p* by: a group G is an R_p-group if and only if either (a) or (b) hold:

(a) G is an elementary abelian p-group.

(b) $G = P \,]\, Q$ where P is a subgroup of type (a) and Q is a subgroup of prime order $q < p$ whose generator acts faithfully on P by raising each element of P to one and the same power $r \equiv 1 (\mathrm{mod}\ p)$.

We define the class of groups R as $R = \bigcup_{p \in \mathbf{P}} R_p$. (R-groups are often called P-groups in the literature). The smallest example of a non-abelian R-group is the R_3-group Σ_3 . R is the u-closure of the class of elementary abelian groups, and if G is a non-simple R_p-group so is any projective image of G (see [SZ] I.3). Note that if $G \in R$ then every subgroup of G is either normal or self-normalising in G (if G is of type (b) above, the self-normalising subgroups are those that contain a subgroup of order q).

Let G be a finite group and suppose that σ is a p-singular projectivity of $L(G)$ (p prime). Let S be a Sylow p-subgroup of G such that $\sigma/_{L(S)}$ is singular. We say that σ is p-singular of the first kind if there does *not* exist a (non-abelian) R_p-subgroup of G which contains S as a proper normal subgroup. If there does exist such an R_p-subgroup, we say that σ is p-singular of the second kind. (These definitions are independent of our choice of S.)

We collect together some results of [SZ] in the following theorem, which we use in the sequel.

2.9 THEOREM. (Suzuki [SZ] Propositions 2.7, 2.8, 2.9)

Let G be a finite group and let σ be a p-singular projectivity of $L(G)$ (p prime). Then

(a) σ is q-singular of the first kind for some prime q ;

(b) if σ is p-singular of the first kind, then G contains a normal Sylow p-complement;

(c) if σ is p-singular of the second kind, then $G = R \times T$ where R is a non-abelian Hall R_p-subgroup of G containing a Sylow p-subgroup S of G as a proper normal subgroup (so that $S \lhd G$) .

□

2.10 LEMMA.

Let G be a finite group and suppose that σ is a projectivity of $\mathcal{L}(G)$. If σ is p-singular of the second kind, then so is σ^{-1} (and conversely).

Proof.

Suppose that σ is p-singular of the second kind. Then by Theorem 2.9(c), we can write $G = R \times T$ as a Hall decomposition of G with R a non-abelian R_p-group. Write $R = P] Q$ where P is a p-subgroup and Q has prime order $q < p$. If R^σ were abelian, it would be a p-group, contradicting the fact that P^σ is not a p-group. So R^σ is a non-abelian R_p-group of order $p^\alpha r$ (where $p^\alpha = |P|$, $p > r \in \mathbb{P}$) . Let $Q_1 , \ldots , Q_{p^\alpha}$ be the conjugates of Q in G . P^σ contains a subgroup of order r , and $P^\sigma \cap Q_i^\sigma = 1$ $(1 \leq i \leq p^\alpha)$. Therefore some $Q_i^\sigma = Q_1^\sigma$, say, has order p , so we may assume that Q^σ has order p (i.e. σ^{-1} is p-singular). R^σ is non-abelian, so σ^{-1} is p-singular of the second kind.

□

2.11 LEMMA.

Let G be a group and let $G^\sigma \in u(G)$. Suppose that N is a normal subgroup of G with $|G:N| = p \in \mathbb{P}$. If σ is not p-singular, then $N^\sigma \triangleleft G^\sigma$ and $|G^\sigma:N^\sigma| = p$.

Proof.

By Theorems 2.2 and 2.3, $N_{G^\sigma} \triangleleft G$ and $|G:N_{G^\sigma}|$ is finite. Hence σ induces a projectivity of finite lattices $\hat{\sigma} : L(G/N_{G^\sigma}) \rightarrow L(G^\sigma/(N^\sigma)_{G^\sigma})$ and clearly, if $\hat{\sigma}$ is p-singular, so is σ. Therefore we may assume that G is finite; that $N^\sigma \triangleleft G^\sigma$ now follows by ([SZ] Proposition 2.11). Let P be a Sylow p-subgroup of G, so that $PN = G$. Then $G^\sigma = P^\sigma N^\sigma$ and $p = |G:N| = |P:P \cap N| = |P^\sigma:P^\sigma \cap N^\sigma| = |G^\sigma:N^\sigma|$, as required.

□

The fact that an index-preserving projectivity will map a normal subgroup of prime index to a normal subgroup (which is implied by 2.11) is used by Rips [RP] to prove that an index-preserving projectivity preserves ascendancy.

§2.3 Subnormality and Projectivities.

As we saw in Lemma 2.5, an index-preserving projectivity σ of a finite group lattice will preserve subnormality. The following result tells us when the converse holds.

2.12 THEOREM.

Let G be a finite group and let $G^{\sigma} \in u(G)$. Let $\pi = \{p_1, \ldots, p_r\}$ denote the set of primes $p \in \pi(G)$ for which σ is p-singular. Then σ preserves subnormality if and only if the following conditions hold.

(i) $G = H] K$ where $H = O_{\pi'}(G)$ and K is a cyclic π-group.

(ii) $\sigma/_{L(H)}$ is index-preserving and every singularity of σ is of the first kind.

(iii) $G^{\sigma} = H^{\sigma}] K^{\sigma}$ and, given $p \in \pi$, σ is p-singular on every Sylow p-subgroup of G, their images under σ being isomorphic.

REMARKS.

If $\pi = \emptyset$ in the above then the theorem is true $(G = H)$ by Lemma 2.5. Also, we note that if σ preserves subnormality, then in (iii), H^{σ} need not be a Hall subgroup of G^{σ} (for example, if G and G^{σ} are R_3-groups of order 6 and 9 respectively, then in the above notation, $\pi = \{2\}$ and $|H^{\sigma}| = |K^{\sigma}| = 3$); if H^{σ} *is* a Hall subgroup of G^{σ}, then σ^{-1} also preserves subnormality (see Corollary 2.14).

Proof.

By the first remark above, we may assume that π is non-empty.

We first prove the necessity of conditions (i)-(iii), so suppose σ preserves subnormality.

Let $p \in \pi$ and suppose (for a contradiction) that σ is p-singular of the second kind. Then by Theorem 2.9(c), $G = R \times T$ where R, T are Hall-subgroups of G and R is a non-abelian $\mathcal{R}_p$-group. Then by Theorem 2.7, $G^\sigma = R^\sigma \times T^\sigma$ is a Hall decomposition of G^σ. Let P be the unique Sylow p-subgroup of G. P^σ sn R^σ (by hypothesis) and $R^\sigma \in \mathcal{R}_p$. Hence P^σ is a p-group, contradicting the p-singularity of σ. Therefore every singularity is of the first kind. Because $H = \mathcal{O}_{\pi'}(G)$, $\sigma/_{\mathcal{L}(H)}$ is index-preserving and (ii) holds.

Pick subgroups $S_i \in \mathrm{Syl}_{p_i}(G)$ $(1 \leq i \leq r)$ such that $\sigma/_{\mathcal{L}(S_{p_i})}$ is singular. Then by Theorem 2.9(b) (and conjugating by a suitable $g_i \in G$) there exists normal p_i'-subgroups A_i of G such that

$$G = A_i \rtimes S_i \qquad (1 \leq i \leq r) . \qquad (*)$$

Clearly $H = \bigcap_{i=1}^{r} A_i$ and $S_j \leq A_i$ for $1 \leq j \leq r$, $j \neq i$.

We prove, by induction on $r = |\pi|$, that

$$G = (\ldots((H \rtimes S_{\tau(1)}) \rtimes S_{\tau(2)})\ldots) \rtimes S_{\tau(r)} \qquad (1)$$

where $H = \mathcal{O}_{\pi'}(G)$, $\sigma/_{\mathcal{L}(H)}$ is index-preserving and τ is any permutation of $1, 2, \ldots, r$.

If $r = 1$, (1) is true by (*), so suppose $r \geq 2$. Let $i \in \{1,\dots r\}$ and consider $\sigma_i = \sigma/_{L(A_i)}$. σ_i preserves subnormality and if $\Pi_i = \Pi\backslash\{p_i\}$, then by (*), σ_i is p-singular if and only if $p \in \Pi_i$. Put $H_i = O_{\pi_i'}(A_i)$. Then $H_i = O_{\pi'}(G) = H$ and $\sigma_i/_{L(S_j)}$ is singular for $p_j \in \Pi_i$. By induction,

$$A_i = (\dots(H \mathbin{]} S_{\tau_i(i_1)}) \mathbin{]} \dots)S_{\tau_i(i_{r-1})} \qquad (**)$$

where $\sigma_i/_{L(H)}$ is index-preserving and τ_i is any permutation of $\{i_1,\dots,i_{r-1}\}=\{1,\dots,i-1,i+1,\dots,r\}$. Combining (*) and (**) gives the desired result (1).

By (1), $HS_i \lhd G$ for $1 \leq i \leq r$. Also, $HS_i \cap HS_j = H(S_i \cap HS_j) = H$ for $1 \leq i \neq j \leq r$. Hence

$$\frac{G}{H} = \frac{S_1H}{H} \times\dots\times \frac{S_rH}{H} . \qquad (2)$$

Let $i \in \{1,\dots,r\}$. Suppose, for a contradiction, that S_i is not cyclic. Then S_i is elementary abelian and S_i^σ is a non-abelian R_{p_i}-group. Put $S_i^\sigma = P_i^\sigma \mathbin{]} Q_i^\sigma$ where $P_i^\sigma = O_{p_i}(S_i^\sigma)$ and Q_i^σ has prime order $q_i < p_i$. Then $Q_i \lhd S_i$ so that $Q_iA_i \lhd S_iA_i = G$, whence $(Q_iA_i)^\sigma$ sn G^σ . Therefore Q_i^σ is contained in the subnormal subgroup

$S_i^\sigma \cap (Q_iA_i)^\sigma$ of S_i^σ, which implies that $S_i^\sigma \leq (Q_iA_i)^\sigma$. But now $S_i = Q_i(S_i \cap A_i) = Q_i$, a contradiction. Hence $S_i H/H \cong S_i$ is cyclic so that G/H is cyclic (from (2)). By the Schur-Zassenhaus Theorem there exists a Π-subgroup K of G such that $G = H \rtimes K$, and (i) holds.

$H \lhd G$ so that H^σ mod G^σ and H^σ sn G^σ, whence H^σ per G^σ by Theorem 2.8. Let $p \in \Pi$ and let $S_1, S_2 \in \mathrm{Syl}_p(G)$. Since G/H is abelian, $HS_1 = HS_2 \lhd G$. Therefore $|S_1^\sigma| = |H^\sigma S_1^\sigma : H^\sigma| = |H^\sigma S_2^\sigma : H^\sigma| = |S_2^\sigma|$. Since S_1^σ and S_2^σ are cyclic, they must be isomorphic. It remains to be shown that $H^\sigma \lhd G^\sigma$ (and then (iii) holds). Since $G^\sigma = H^\sigma K^\sigma$ and $H^\sigma \cap K^\sigma = 1$, then $\Pi(G^\sigma) = \Pi(H^\sigma) \cup \Pi(K^\sigma) = \Pi' \cup \Pi(K^\sigma)$. Now $K^\sigma = S_1^\sigma \times \ldots \times S_r^\sigma$ where S_i is a cyclic Sylow p_i-subgroup of K (Theorem 2.7) and S_i^σ is a cyclic q_i-group ($p_i \neq q_i \in \mathbb{P}$, $1 \leq i \leq r$). If some $q_i \notin \Pi'$, then S_i^σ normalises the subnormal subgroup H^σ of $H^\sigma S_i^\sigma$. Alternatively, suppose $q_i \in \Pi'$. Let Q_i^σ be a Sylow q_i-subgroup of G^σ containing S_i^σ. Then $1 \neq Q_i^\sigma \cap H^\sigma \in \mathrm{Syl}_{q_i}(H^\sigma)$, so that $Q_i \cap H$ is a non-trivial q_i-group and $|Q_i^\sigma| \geq q_i^2$. If Q_i^σ is cyclic, then Q_i is a cyclic q_i-group containing S_i. But S_i is a p_i-group, a contradiction. Therefore Q_i is elementary abelian and $|S_i^\sigma| = q_i$. Now H^σ sn $H^\sigma S_i^\sigma$ and since $|H^\sigma S_i^\sigma : H^\sigma| = q_i$, S_i^σ must normalise H^σ. Therefore $H^\sigma \lhd G^\sigma$ and we have proved the necessity of (i), (ii) and (iii). In order to prove the sufficiency of the conditions (i)-(iii), we will use the following result, due to Schmidt ([SH] Theorem 4.2).

2.13 THEOREM.

Let $G \in F$ and $G^\sigma \in u(G)$. Suppose X is a subnormal subgroup of G such that X^σ is not subnormal in G^σ . Let N be the maximal normal subgroup of G such that $N \leq X$ and $N^\sigma \trianglelefteq G^\sigma$. Then there exists non-abelian R_{p_i}-groups R_i^σ/N^σ of order $p_i^{n_i}q_i$ $(p_i, q_i \in \mathbb{P}, n_i \geq 1, 1 \leq i \leq \ell, \ell \geq 1)$ such that

(a) $\dfrac{G^\sigma}{N^\sigma} = \dfrac{R_1^\sigma}{N^\sigma} \times \ldots \times \dfrac{R_\ell^\sigma}{N^\sigma} \times \dfrac{T^\sigma}{N^\sigma}$ is a Hall decomposition of $\dfrac{G^\sigma}{N^\sigma}$,

(b) $\dfrac{G}{N} = \dfrac{R_1}{N} \times \ldots \times \dfrac{R_\ell}{N} \times \dfrac{T}{N}$ is a Hall decomposition of $\dfrac{G}{N}$,

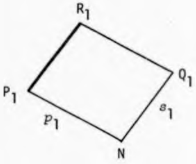
(c) $X^\sigma \cap T^\sigma \text{ sn } G^\sigma$, $\left|\dfrac{X^\sigma \cap R_i^\sigma}{N^\sigma}\right| = q_i < p_i = \left|\dfrac{X \cap R_i}{N}\right|$ $(1 \leq i \leq \ell)$.

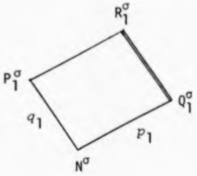
$\square$

Proof of 2.12 ctd.

Suppose that conditions (i)-(iii) of Theorem 2.12 hold, but that σ does not preserve subnormality. Let X be a subnormal subgroup of G such that X^σ is not subnormal in G^σ. Then (a), (b), (c) of Theorem 2.13 hold and (using the notation of 2.13) the induced projectivity $\hat{\sigma}$ on $L(G/N)$ is p_i-singular for $1 \leq i \leq \ell$; hence, if P_i/N is a (unique) Sylow p_i-subgroup of G/N , then

$P_i/N = T_1N/N$ for any $T_1 \in \mathrm{Syl}_{p_i}(G)$, $P_i^\sigma/N^\sigma \cong T_i^\sigma/(T_i^\sigma \cap N^\sigma)$ is not a p_i-group and so σ is p_i-singular (so that $\{p_1,\dots,p_\ell\} \subseteq \pi$ and there is no conflict of notation). P_1 is cyclic (by (i)) and P_1/N is elementary abelian, so that $|P_1/N| = p_1$. Therefore $|R_1/N| = p_1s_1$ where $p_1 > s_1 \in \mathbb{P}$ ($P_1 \ntrianglelefteq R_1$ because $|P_1^\sigma : N^\sigma| = q_1 < |R_1^\sigma : N^\sigma|$). Let $Q_1^\sigma/N^\sigma \in \mathrm{Syl}_{p_1}(R_1^\sigma/N^\sigma)$.





Then $|Q_1/N| = s_1$ so that $\bar{\sigma}$ is s_1-singular and, as above, so is σ . Therefore $p_1, s_1 \in \pi$. Since $Q_1/N \in \mathrm{Syl}_{s_1}(G/N)$ we can pick $S_1 \in \mathrm{Syl}_{s_1}(G)$ such that $Q_1/N = S_1N/N$. Since K is a cyclic Hall π-subgroup of G , there exists $g_1 \in G$ such that $[S_1^{g_1}, T_1] = 1$. But then $[Q_1^{g_1}, P_1] = [S_1^{g_1}N, T_1N] \leq N$, so that $P_1^{g_1^{-1}} = P_1$ normalises Q_1 , a contradiction. Hence σ preserves subnormality and Theorem 2.12 is proved.

$\square$

2.14 COROLLARY.

Let G be a finite group and let $G^\sigma \in u(G)$, where σ preserves subnormality. Then, in the notation of Theorem 2.12, σ^{-1} preserves subnormality if and only if $\Pi(H^\sigma) \cap \Pi(K^\sigma) = \emptyset$ (that is, $H^\sigma = O_{\pi'}(G^\sigma)$). If, however, there exists $q \in \Pi(H^\sigma) \cap \Pi(K^\sigma)$ then a Sylow q-subgroup of G^σ is elementary abelian of rank at least two and a Sylow q-subgroup of K^σ has order q.

Proof.

We use the notation of Theorem 2.12 throughout. We prove the second part of the corollary first. Let $q \in \Pi(H^\sigma) \cap \Pi(K^\sigma)$ and pick $Q^\sigma \in \mathrm{Syl}_q(G^\sigma)$ such that Q^σ contains the cyclic Sylow q-subgroup K_q^σ of K^σ. Suppose, if possible, that Q^σ is cyclic. Since $1 \neq Q^\sigma \cap H^\sigma \in \mathrm{Syl}_q(H^\sigma)$ and $\sigma^{-1}/_{L(H^\sigma)}$ is index-preserving, Q must be a cyclic q-group. But K_q is a p-group for some $p \neq q$ $(p \in \Pi)$ which is a contradiction. Therefore Q^σ is elementary abelian of rank at least 2. Since K_q^σ is also cyclic, K_q^σ has order q, which proves the second part of the corollary.

Suppose that σ^{-1} preserves subnormality. Then by Theorem 2.12 applied to σ^{-1} we must have $H^\sigma = O_{\pi'}(G^\sigma)$. Conversely, suppose that $\Pi(H^\sigma) \cap \Pi(K^\sigma) = \emptyset$. We show that conditions (i)-(iii) of Theorem 2.12 are satisfied for σ^{-1}, which then implies that σ^{-1} preserves subnormality; $H^\sigma = O_{\pi'}(G)$ so that (i) holds. Clearly $\sigma^{-1}/_{L(H^\sigma)}$ is index-preserving, and by Lemma 2.10, every singularity of σ^{-1} is of the first kind; thus (ii) holds. Since the Sylow subgroups of K and K^{σ} are mapped to each other, (iii) holds. This completes the proof.

□

Lack of a Sylow structure in an arbitrary periodic group G means that we cannot immediately say anything about the p-subgroups of G if $L(G)$ admits a p-singular projectivity. For locally finite groups, however, we have the following Lemma, which shows that (as in the finite case) an abelian p-group must be of a specific type in order to admit a singular projectivity.

2.15 LEMMA.

Let G be a locally finite group and let $G^{\sigma} \in u(G)$. Suppose that σ is p-singular (p prime). Then one of the following holds:

(i) every p-subgroup of G is elementary abelian;

(ii) every p-subgroup of G is cyclic or quasicyclic.

Proof.

There exists subgroups N,S,U of G such that $N \triangleleft U$, $N^{\sigma} \triangleleft U^{\sigma}$, $|U:N| \neq |U^{\sigma}:N^{\sigma}| < \infty$, $S/N \in Syl_p(G/N)$ and S^{σ}/N^{σ} is not a p-group. Let T be a finite subgroup of S such that $S = NT$. Suppose, if possible, that $\sigma/_{L(T)}$ is not p-singular. Then $|S:N| = |T:T\cap N| = |T^\sigma:T^\sigma \cap N^\sigma| = |S^\sigma:N^\sigma|$, a contradiction. Hence $\sigma/_{L(T)}$ is p-singular and there exists $T_1 \in Syl_p(T)$ such that T_1^σ is not a p-group.

Let P be a finite p-subgroup of G . If G contains no elements of order p^2 then, since σ is p-singular on the finite lattice $L(<P,T_1>)$, P must be elementary abelian. If G contains an element g of order p^2 then, since σ is p-singular on the finite lattice $L(<P,T_1,g>)$, P must be cyclic. Therefore either every p-subgroup of G is elementary abelian (and (i) holds) or every such subgroup is cyclic (and (ii) holds). This completes the proof.

□

2.16 LEMMA.

Let G be a group and let $G^\sigma \in u(G)$. Then $(G^\sigma)^F = (G^F)^\sigma$. If G is a Černikov group, so is G^σ .

Proof.

The subgroups of finite index in G are in bijective correspondence (via σ) with the subgroups of finite index in G^σ , by Theorem 2.2. Hence $(G^\sigma)^F = (G^F)^\sigma$. If G is a Černikov group, then $G^F \in S \cap \check{M}$. Since $uS = S$ ([YV]) and $u\check{M} = \check{M}$ (clearly) then $(G^\sigma)^F = (G^F)^\sigma \in S \cap \check{M}$ and so G^σ is a Černikov group.

□

We can exploit the structure of a Černikov group to say something about the projective image of a subnormal subgroup; the relation of the following result to Theorem 2.13 is made explicit in the remark after the proof.

2.17 THEOREM.

Let X be a subnormal subgroup of a Černikov group G and let $G^\sigma \in u(G)$. Then X^σ is normalised by $(G^\sigma)^F$ and X^σ sn $X^\sigma(G^\sigma)^F(G^\sigma)^*$.

Proof.

G^σ is a Černikov group by Lemma 2.16. Let $G_1^\sigma = X^\sigma(G^\sigma)^F$. Then $G_1^F = G^F$, $X \cap G_1^F \triangleleft XG_1^F = G_1$ and $X^\sigma \cap (G_1^\sigma)^F \triangleleft G_1^\sigma$. Therefore, in order to prove that X^σ is normalised by $(G^\sigma)^F$, we may assume that $G = G_1$ and $X \cap G^F = 1$. By Theorem 1.13, $|G:N_G(X)|$ is finite so that $X \triangleleft G^F X = G$. Therefore

$$G = G^F \times X \text{ and } G^\sigma = (G^\sigma)^F \,]\, X .$$

Define sets of primes $\pi_1 = \pi(G^F) \setminus \pi(X)$ and $\pi_2 = \pi(G^F) \cap \pi(X)$. Then $G^F = O_{\pi_1}(G^F) \times O_{\pi_2}(G^F)$ and by Theorem 2.7,

$$\left.\begin{array}{l} (G^{\sigma})^{F} = (O_{\pi_1}(G^{F}))^{\sigma} \times (O_{\pi_2}(G^{F}))^{\sigma} \quad \text{and} \\ (X \times O_{\pi_1}(G^{F}))^{\sigma} = X^{\sigma} \times (O_{\pi_1}(G^{F}))^{\sigma} \quad . \end{array}\right\} \qquad (*)$$

If $\pi_2 = \emptyset$, then $X^{\sigma} \triangleleft G^{\sigma}$ as required, so assume $\pi_2 \neq \emptyset$. Suppose we have proved that X^{σ} asc G^{σ} . Then, since X^{σ} mod G^{σ} , we have X^{σ} per G^{σ} by Theorem 2.8. But a finite permutable subgroup is subnormal ([S2] Theorem F) and then by Theorem 1.13, $X^{\sigma} \triangleleft G^{\sigma}$ as required. Therefore it is enough to prove X^{σ} asc G^{σ} . By (*), we may assume that $O_{\pi_1}(G^{F}) = 1$, so that

$$G^{F} \text{ is a } \pi_2\text{-group.}$$

Suppose, if possible, that σ is p-singular for some $p \in \pi_2$. Then by Lemma 2.15, every p-subgroup of G is cyclic or quasicyclic. But if $X_p \in Syl_p(X)$, then $X_pO_p(G^{F})$ is a quasicyclic p-group containing a proper infinite subgroup $O_p(G^{F})$, which is impossible. Therefore

$$\sigma \text{ is not p-singular for } p \in \pi_2 .$$

For $i \geq 0$, define subgroups $\Omega_i(G^{F}) = \langle g \in G^{F} : g^{(p^i)} = 1, p \in \pi_2 \rangle$ and $X_i = X\,\Omega_i(G^{F})$. Then $X = X_0 \leq X_1 \leq \ldots$ is an ascending chain of normal subgroups of G and $\bigcup_{i \geq 0} X_i = G$. Further, for $i \geq 0$, X_{i+1} is generated modulo X_i by elements of order $p \in \Pi_2$. Therefore (by Lemma 2.11) $X_i^\sigma \lhd X_{i+1}^\sigma$. Hence X^σ asc $\bigcup_{i \geq 0} (X_i^\sigma) = G^\sigma$ and the first part of the proposition is proved.

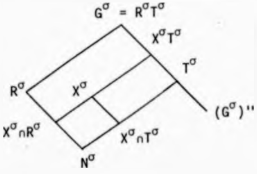
For the second part of the proposition; if $X^\sigma(G^F)^\sigma$ sn G^σ there is nothing to prove, and otherwise we can use Theorem 2.13, applied to the group XG^F/G^F sn G/G^F and (induced) projectivity $\bar{\sigma} : L(G/G^F) \to L(G^\sigma/(G^F)^\sigma)$, to give the desired result (by the following remark).

$\square$

Remark.

In [NZ], Napolitani and Zacher prove a similar result to Theorem 2.13 in the case that $X \lhd G$ and X^σ is not ascendant in G^σ (even if G is infinite; the finite case is due to Schmidt [SH]). A consequence of this result (and Theorem 2.13) is that X^σ , whilst not being ascendant in G^σ , is not far off, in the sense that X^σ asc $X^\sigma(G^\sigma)''$: in Theorem 2.13 (using its notation), if we put $R^\sigma = \langle R_1^\sigma, \ldots, R_\ell^\sigma \rangle$, then $X^\sigma = (X^\sigma \cap R^\sigma)(X^\sigma \cap T^\sigma)$ asc $X^\sigma T^\sigma$ and $G^\sigma/T^\sigma \cong R^\sigma/N^\sigma \in A^2$.

We see from the following theorem that the above consequence also holds for the projective image of an ascendant subgroup.



We say that a subgroup L of a group G is *$\mathcal{L}$-invariant* in G if and only if $L^\tau = L$ for any auto-projectivity τ of $\mathcal{L}(G)$. Clearly, an $\mathcal{L}$-invariant subgroup is characteristic and any projective image of it is also $\mathcal{L}$-invariant.

2.18 **THEOREM** (Zacher [Z 2] p.66).

Let X be an ascendant subgroup of a group G and suppose that X^σ is not ascendant in G^σ for some $G^\sigma \in u(G)$. Then there exists an $\mathcal{L}$-invariant subgroup L of G and a set of primes π such that X^σ asc $X^\sigma L^\sigma$ and $\frac{G}{L} = \underset{p \in \pi}{\mathrm{Dr}} \frac{R_p}{L}$, where R_p/L is a Hall R_p-subgroup of G/L and the induced projectivity on $\mathcal{L}(R_p/L)$ is p-singular. In particular, X^σ asc $X^\sigma(G^\sigma)^*$, and σ cannot be index-preserving.

□

2.19 **THEOREM**.

Let G be a group generated by subgroups H_λ $(\lambda \in \Lambda)$, each of which contains X as an ascendant subgroup. Suppose that $G^\sigma \in u(G)$ where X^σ asc G^σ. Then X asc G.

Proof.

Suppose, for a contradiction, that X is not ascendant in G. Then by Theorem 2.18 (applied to σ^{-1}), there exists an $\mathcal{L}$-invariant subgroup L^σ of G^σ and Hall R_p-subgroups R_p^σ/L^σ of G^σ/L^σ $(p \in \Pi)$ such that X asc $X L$,

$$\frac{G^\sigma}{L^\sigma} = \mathop{\mathrm{Dr}}_{p\in\Pi} \frac{R_p^\sigma}{L^\sigma} \quad \text{and} \quad \sigma^{-1} \quad \text{is p-singular on} \quad \mathcal{L}(R_p^\sigma/L^\sigma) \ .$$

By Theorem 2.7, $\frac{G}{L} = \mathop{\mathrm{Dr}}_{p\in\Pi} \frac{R_p}{L}$ is a Hall decomposition of G/L.

For $\lambda \in \Lambda$, $\frac{XL}{L} = \mathop{\mathrm{Dr}}_{p\in\Pi} \frac{XL\cap R_p}{L}$ asc $\frac{H_\lambda L}{L} = \mathop{\mathrm{Dr}}_{p\in\Pi} \frac{H_\lambda L\cap R_p}{L}$.

Since R_p/L is a non-abelian R_p-group, we must have $XL \cap R_p \lhd H_\lambda L\cap R_p$ for all $p \in \Pi$, $\lambda \in \Lambda$. Therefore $XL \lhd \langle H_\lambda L : \lambda \in \Lambda\rangle = G$ and so X asc G, which is a contradiction.

□

2.20 COROLLARY.

The projective image of an ascendant subgroup of a group has an ascendiser. In a finite group, the projective image of a subnormal subgroup has a subnormaliser.

□

Of course, the second part of Corollary 2.20 also follows from Theorem 2.13 by using a similar proof to that of Theorem 2.19.

§2.4 The Class $\mathcal{W}_u$.

As we saw in Chapter 1(§1.1), $\mathcal{J} \le \mathcal{W}$, $\mathcal{F} \le \mathcal{W}$ but $\mathcal{F} \not\le \mathcal{J}$. On the basis of the following result, we will define an intermediate class $\mathcal{W}_u$ that still catches $\mathcal{F}$; that is, $\mathcal{F} \le \mathcal{W}_u$ and $\mathcal{J} \le \mathcal{W}_u \le \mathcal{W}$.

2.21 THEOREM.

Let $G \in F$ be generated by subgroups H and K , both of which contain X as a subnormal subgroup. Suppose there exists $G^\sigma \in u(G)$ such that $G^\sigma = H^\sigma K^\sigma$. Then X sn G .

Proof.

Suppose that the Theorem is false and pick a counter-example such that firstly $n = |G|$ and then $d = |G:H| + |G:K| + |X|$ is minimal.

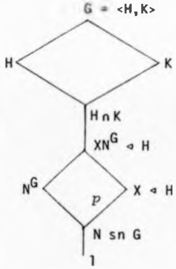
Suppose, if possible, that H is not a maximal subgroup of G and let M be a proper subgroup of G which properly contains H . Then $M^\sigma = H^\sigma(M^\sigma \cap K^\sigma)$, $M = \langle H, M \cap K\rangle$ and X sn H , X sn $M \cap K$. Hence X sn M by minimality of n . But now $G^\sigma = M^\sigma K^\sigma$, $G = \langle M, K\rangle$ and X sn M , X sn K . Hence X sn G by minimality of d , a contradiction. Therefore H and (similarly) K are maximal subgroups of G .

If X^σ is subnormal in both H^σ and K^σ then X^σ sn G^σ by

Theorem 1.3, whence by Corollary 2.20 we have $X \text{ sn } G$, a contradiction. So we may assume that X^σ is not subnormal in H^σ. Apply Theorem 2.13 to the subnormal subgroup X of H, with projectivity $\sigma/_{L(H)}$: Then (in the notation of 2.13) $H/N = R_i/N \times \ldots \times R_\ell/N \times T/N \quad (R_i/N \in R_{p_i})$ and

$$\frac{X}{N} = \frac{X \cap R_1}{N} \times \ldots \times \frac{X \cap R_\ell}{N} \times \frac{X \cap T}{N} \qquad (\ell \geq 1)$$

where $(X \cap R_i)/N$ has prime order p_i $(i = 1, \ldots, \ell)$ and $p_i \neq p_j$ if $i \neq j$. N is (defined as) the largest normal subgroup of H such that $N \leq X$ and $N^\sigma \triangleleft H^\sigma$. Also $X^\sigma \cap T^\sigma \text{sn } H^\sigma$. Suppose that this decomposition of X/N contains more than one direct factor. Then at least two of the groups $X \cap R_1, \ldots, X \cap R_\ell$, $X \cap T$ are proper (non-trivial) normal subgroups of X and each such subgroup will be subnormal in both H and K; by minimality of d, each will be subnormal in G and therefore their join X is subnormal in G ([W1]), a contradiction. Therefore $X/N = (X \cap R_1)/N$ and has order p, say. Further, $X^H = (X \cap R_1)^{R_1 \ldots R_\ell T} = X$. Also, by the minimality of d, we have $N \text{ sn } G$. Since H and K are maximal in G, then $N^G \leq H \cap K$. Clearly, $N^G \cap X \neq X$, so we must have $N^G \cap X = N$. Because $|X:N| = p \neq q = |X^\sigma : N^\sigma|$, $\sigma/_{L(X)}$, and hence $\sigma/_{L(K)}$, is p-singular. We consider the two different types of singularity:



(a) $\sigma/_{L(K)}$ is p-singular of the first kind.

By Theorem 2.9(b), $K = A_p \rtimes S_p$ where $A_p = O_{p'}(K)$ and $S_p \in Syl_p(K)$. Therefore

$$\frac{XN^G}{N^G} \text{ sn } \frac{K}{N^G} = \frac{A_pN^G}{N^G} \rtimes \frac{S_pN^G}{N^G} .$$

Therefore XN^G/N^G (order p) is normalised by the p'-group A_pN^G/N^G and so (choosing S_p such that $XN^G \leq S_pN^G$) $XN^G \lhd K$. But $XN^G \lhd H$, which implies $X \lhd^2 G$, a contradiction.

(b) $\sigma/_{L(K)}$ is p-singular of the second kind.

By Theorem 2.9(c), $K = R \times B$ where R is a non-abelian R_p-subgroup of K and R,B have co-prime orders. Then

$$\frac{K}{N^G} = \frac{RN^G}{N^G} \times \frac{BN^G}{N^G}$$

and every p-subgroup of K/N^G is contained in RN^G/N^G and is normal in K/N^G . Therefore $XN^G \lhd K$ which, as in case (a), gives a contradiction. This completes the proof.

□

We define the <u>class of groups $\mathfrak{W}_u$</u> as follows: a group G is a $\mathfrak{W}_u$-group if and only if (*) holds:

$$\left.\begin{array}{l}\text{Whenever } G \text{ is generated by subgroups } H \text{ and } K\text{,}\\ \text{both of which contain } X \text{ as a subnormal subgroup, and}\\ G^{\sigma} \in u(G) \text{ with } G^{\sigma} = H^{\sigma}K^{\sigma}\text{, then } X \text{ sn } G\text{.}\end{array}\right\} (*)$$

Theorem 2.21 says that $\mathfrak{F} \leq \mathfrak{W}_u$, and clearly $\mathfrak{J} \leq \mathfrak{W}_u \leq \mathfrak{W}$.

2.22 <u>PROPOSITION</u>.

$\mathfrak{W}_u$ contains the class of Černikov groups.

<u>Proof</u>.

Let G be a Černikov group generated by subgroups H and K, both of which contain X as a subnormal subgroup, and suppose that $G^{\sigma} \in u(G)$ where $G^{\sigma} = H^{\sigma}K^{\sigma}$. Since $H, K \in \check{\mathfrak{M}}$, X is normalised by $H^{\mathfrak{F}}$ and by $K^{\mathfrak{F}}$. But $G^{\sigma} = H^{\sigma}K^{\sigma}$ is a Černikov group (Lemma 2.16), so by Lemma 1.14 we have $(G^{\sigma})^{\mathfrak{F}} = (H^{\sigma})^{\mathfrak{F}}(K^{\sigma})^{\mathfrak{F}}$. Therefore $G^{\mathfrak{F}} = H^{\mathfrak{F}}K^{\mathfrak{F}}$ normalises X. So to prove that X sn G, we may assume that $G^{\mathfrak{F}} = 1$. Now G is finite and X sn G by Theorem 2.21.

□

Proposition 2.22 will be superceeded by Proposition 2.25, which says that $(\mathfrak{A}\mathfrak{F})^{u} \leq \mathfrak{W}_u$ (Cernikov groups form a u-closed class by Lemma 2.16). Proposition 2.25 is proved using a reduction lemma akin to Lemma 1.6:

2.23 LEMMA.

Let G be a group generated by subgroups H and K, both of which contain X as a subnormal subgroup. Suppose that $G^\sigma \in u(G)$ and $G^\sigma = H^\sigma K^\sigma$. Let A be an abelian normal subgroup of G and put $G_1 = AH \cap AK$, $H_1 = H \cap AK$, $K_1 = AK \cap H$ and $N = (A \cap H_1)(A \cap K_1)$. Then

(i) $N \lhd G_1 = AH_1 = AK_1 = \langle H_1, K_1 \rangle$ and $G_1^\sigma = \langle A^\sigma, H_1^\sigma \rangle = \langle A^\sigma, K_1^\sigma \rangle = H_1^\sigma K_1^\sigma$.

Let bars denote subgroups of G_1 modulo N. Then $\bar{X}$ sn $\overline{H_1}$, $\bar{X}$ sn $\overline{K_1}$ and

(ii) $\overline{G_1} = A \,]\, \overline{H_1} = A \,]\, \overline{K_1} = \langle \overline{H_1}, \overline{K_1} \rangle$ and $\overline{H_1}, \overline{K_1}$ embed in G/A;

(iii) if $\bar{X}$ sn $\bar{G_1}$ and AX sn G, then X sn G.

Suppose that A^σ is an abelian normal subgroup of G_1^σ. Then $N^\sigma \lhd G^\sigma$ and, if bars denote subgroups of G_1^σ modulo N^σ,

(iv) $\overline{G_1}^\sigma = \bar{A}^\sigma \,]\, \overline{H_1}^\sigma = \bar{A}^\sigma \,]\, \overline{K_1}^\sigma = \overline{H_1}^\sigma \overline{K_1}^\sigma$ and $\overline{H_1}^\sigma, \overline{K_1}^\sigma$ embed in G^σ/A^σ.

Proof.

(i) By Lemma 1.5 (i), $N \lhd G_1 = AH_1 - AK_1$. Therefore $G_1^\sigma = \langle A^\sigma, H_1^\sigma \rangle = \langle A^\sigma, K_1^\sigma \rangle$. Also, $H_1^\sigma K_1^\sigma = (H^\sigma \cap \langle A^\sigma, K^\sigma \rangle)(\langle A^\sigma, H^\sigma \rangle \cap K^\sigma) = \langle A^\sigma, H^\sigma \rangle \cap H^\sigma K^\sigma \cap \langle A^\sigma, K^\sigma \rangle = G_1^\sigma$. Therefore $G_1 = \langle H_1, K_1 \rangle$.

(ii) From (i), $A \leq \langle H_1, K_1 \rangle$ so, by Lemma 1.5(iv),(v), (ii) follows.

(iii) As in the proof of Lemma 1.6, the hypotheses of (iii) imply that NX sn AK sn G and X sn NK by Lemma 1.7.

(iv) This follows from Lemma 1.5(iv),(vi).

□

Lemma 2.23 is limited in application by the fact that A^σ has to be abelian and normal in G^σ. This need not bother us when G is metabelian, as we see from:

2.24 THEOREM.

$A^2 \leq \omega_u$.

Proof.

Let $G \in A^2$ be generated by subgroups H and K , both of which contain K as a subnormal subgroup. Suppose that $G^\sigma \in u(G)$ and $G^\sigma = H^\sigma K^\sigma$. Let $A = G'$. Then, by Lemma 2.23 (ii) (and using its notation), $\overline{G_1} = \langle \overline{H_1}, \overline{K_1} \rangle$ and $\overline{H_1}, \overline{K_1}$ embed in $G/A \in A$. Therefore $\bar{X} \triangleleft \overline{G_1}$. Since $AX \triangleleft G$, 2.23(iii) gives X sn G , as required.

□

2.25 PROPOSITION.

$(AF)^u \leq \omega_u$.

Proof.

We prove something stronger, that if $G \in AF$ and $G^\sigma \in u(G) \cap AF$, with G generated by subgroups H and K , both of which contain X as a subnormal subgroup, and $G^\sigma = H^\sigma K^\sigma$, then X sn G . (We do not require $u(G) \leq AF$.) Let G, H, K, X and G^σ be as posed. Let A_1, A_2^σ be abelian normal subgroups of G, G^σ (respectively) such that $|G:A_1|$ and $|G^\sigma:A_2^\sigma|$ are finite. Let $B = A_1 \cap A_2$. Then $B_G \in A$ and $|G:B_G|$ is finite. $|G^\sigma:(B_G^\sigma)_{G^\sigma}|$ is finite (by Theorem 2.2) and $(B_G^\sigma)_{G^\sigma} \leq A_2^\sigma \in A$. By Theorems 2.2 and 2.3, $(B_G)_{G^\sigma}$ is normal in G and has finite index in G . Therefore, we may assume $A_1 = A_2 = A$, say.

By Lemma 2.23 and Theorem 2.21, we may assume that $G = A] H = A] K = \langle H, K\rangle$ and $G^{\sigma} = A^{\sigma}] H^{\sigma} = A^{\sigma}] K^{\sigma} = H^{\sigma}K^{\sigma}$. Now $H \cong K \cong G/A \in \mathcal{F}$, so that $G^{\sigma} = H^{\sigma}K^{\sigma}$ is finite. Therefore G is finite and so X sn G by Theorem 2.21.

□

Using results of Baer ([BA]) it can be shown that $\mathcal{A}\mathcal{F} \cap \hat{\mathcal{M}}$ is a u-closed class, and hence (by Proposition 2.25) $\mathcal{A}\mathcal{F} \cap \hat{\mathcal{M}} \subseteq \mathcal{W}_u$. This is superce~~e~~ded, though, by the following result.

2.26 THEOREM.

$(PC)\mathcal{F} \subseteq \mathcal{W}_u$.

Proof.

Let X be a subgroup of a polycyclic-by-finite group G . To show that X sn G , it is enough to prove that X^{θ} sn G^{θ} for any finite homomorphic image G^{θ} of G ([K1] Satz 3.4). Suppose that G is generated by subgroups H and K , both of which contain X as a subnormal subgroup, and suppose that $G^{\sigma} \in u(G)$ with $G^{\sigma} = H^{\sigma}K^{\sigma}$. Let N be a normal subgroup of G such that G/N is finite. By Theorems 2.2 and 2.3, G and G^{σ} contain normal subgroups of finite index $N_{G^{\sigma}}$ and $(N^{\sigma})_{G^{\sigma}}$ respectively. By Theorem 2.21 (applied to $G/N_{G^{\sigma}}$) , we have $XN_{G^{\sigma}}$ sn G . Therefore $XN = XN_{G^{\sigma}}N$ sn G . By Kegel's result (above), we can conclude that X sn G .

□

Finally, we include the following result, which is a corollary of Corollary 2.20.

2.27 PROPOSITION.

$u(N \cap \hat{M}) \leq J$.

Proof.

Let $G^\sigma \in u(G)$ where $G \in N \cap \hat{M}$. Suppose that G^σ is generated by subgroups H^σ and K^σ , both of which contain X^σ as a subnormal subgroup. Then X sn G and by Corollary 2.20, X^σ asc $G^\sigma \in \hat{M}$. Therefore X^σ sn G^σ .

□

CHAPTER 3. K-SUBNORMALITY.

§3.0 Introduction.

K -subnormality in finite groups was introduced by Kegel ([K 2]) as a generalisation of subnormality. Kegel shows that in a finite group G , the K-subnormal subgroups form a sublattice of $L(G)$ (see Theorem 3.2); here K denotes a class of groups which is closed with respect to forming extensions, homomorphic images and subgroups (<P,Q,S>-closure).

This chapter is in three sections. In §3.1 we give preliminary definitions and results. In §3.2 we consider the relations n_K ("K-normality") and sn_K ("K-subnormality") for variable <P,Q,S>-closed classes K of finite groups. Distinct classes K_1 , K_2 correspond to distinct relations n_{K_1}, n_{K_2} (Proposition 3.7) and, with essentially only one exception, correspond to distinct relations sn_{K_1}, sn_{K_2} (Corollary 3.6). Also, we have two results which generalise well-known characterisations of finite nilpotent and finite Dedekind groups; if G is a finite group, then every subgroup is K-subnormal if and only if G is the direct product of a K-group and a nilpotent group (Theorem 3.4); every subgroup of G is K-normal if and only if G is either a K-group or a Dedekind group (Theorem 3.8).

In §3.3 we consider if Theorem 1.3 remains true when "subnormal" is replaced by "K-subnormal". That is, if G = HK is a finite group and X is $\mathcal{K}$-subnormal in both H and K, is X $\mathcal{K}$-subnormal in G ? This is true if G is soluble (Theorem 3.12) but false for arbitrary finite groups G, even if $X \lhd_{\mathcal{K}} H$ and $X \lhd_{\mathcal{K}} K$ (Example 3.11).

The definition of $\mathcal{K}$-subnormality need not be confined to finite groups and classes $\mathcal{K}$ of finite groups. Thus we can speak of $\mathcal{K}$-subnormality in infinite groups and define the $\mathcal{K}$-subnormal analogue, $\mathcal{W}_{\mathcal{K}}$, of the class $\mathcal{W}$ of Chapter 1. $\mathcal{W}_{\mathcal{K}}$ is contained in $\mathcal{W}$, and we identify some subclasses of $\mathcal{W}_{\mathcal{K}}$ in Theorem 3.14, Theorem 3.15 ($\mathcal{W}_{\mathcal{K}}$ contains all polycyclic groups) and Theorem 3.17 ($\mathcal{W}_{\mathcal{K}}$ contains all soluble Černikov groups).

§3.1 Preliminaries.

The classes of groups that are closed with respect to forming extensions, homomorphic images and subgroups are precisely those classes which are PQS-closed. This follows from

3.1 LEMMA.

$\langle P, Q, S\rangle = PQS$.

Proof.

Clearly $PQS \leq \langle P, Q, S\rangle$. Using the relations $SP \leq PS$, $SQ \leq QS$

and QP ≤ PQ ([R1] Lemma 1.12), we have

$$(PQS)^2 = PQSPQS \leq PQPSQS \leq P^2QSQS \leq P^2Q^2S^2 = PQS \leq (PQS)^2 .$$

Therefore PQS is a closure operation containing P,Q and S ; since <P,Q,S> is the least such closure operation, then <P,Q,S> ≤ PQS .

□

Suppose that K is a PQS-closed class of finite groups. We define L_K as the class of simple groups which occur as composition factors of K-groups. Clearly, L_K consists precisely of the simple K-groups and

$$K = PL_K \quad .$$

Conversely, if L is any class of finite simple groups which is closed with respect to taking simple sections, then any subgroup of a PL-group has composition factors which are simple sections of L-groups. Hence $SPL = PL$. Also, $QPL \leq PQL = PL$ so that $QPL = PL$. Therefore PL is PQS-closed and $L_{PL} = L$.

So we can uniquely describe any PQS-closed class K of finite groups by specifying its simple subgroups L_K . If L_K consists of finite Π-groups (some $\Pi \subseteq \mathbb{P}$) then so does K, and, defining $\Pi(K)$ $(\Pi(L_K))$ as the set of all primes that occur in the orders of K-groups (L_K-groups), then $\Pi(K) = \Pi(L_K)$ and

$$F \cap S_{\Pi(K)} \subseteq K \subseteq F_{\Pi(K)} .$$

If K_1 and K_2 are PQS-closed classes of groups, then $K_1 \cap K_2$ is PQS-closed and $L_{(K_1 \cap K_2)} = L_{K_1} \cap L_{K_2}$. Also, the smallest PQS-closed class of groups containing K_1 and K_2 is $P(K_1 \cup K_2)$, because $SP \leq PS$, $QP \leq PQ$ and both S and Q are unary.

Let K be a PQS-closed class of finite groups. Following Kegel ([K 2]), we say that a subgroup X of a finite group G is K-normal in G, written $X \lhd_K G$ or $X \; n_K \; G$, if and only if either $X \lhd G$ or $G/X_G \in K$. We say that X is K-subnormal in G, and write $X \lhd_K^n G$ or $X \; sn_K \; G$, if and only if there exists a chain of subgroups $X = X_0 \leq X_1 \leq \ldots \leq X_n = G$ $(n \in \mathbb{N})$ such that $X_i \lhd_K X_{i+1}$ for $0 \leq i < n$.

For example, taking $K = I$ in the above definition, then $X \lhd_I G$ $(X \; sn_I \; G)$ if and only if $X \lhd G$ $(X \; sn \; G)$. At the other extreme, the statements $X \triangleleft_F G$, $X \; sn_F \; G$ and $X \le G$ are equivalent. Also, if $G \in K$ (arbitrary K) then every subgroup of G is K-normal (see Theorem 3.8 for the converse).

We shall consider n_K and sn_K as relations of finite groups, and sometimes we shall write n, sn, s in place of n_I , sn_I , n_F (= sn_F) respectively. We partially order (using "≤") the relations n_K and sn_K for various K in the natural way; that is, if K_1 and K_2 are PQS-closed classes of finite groups, then we write $\underline{n_{K_1} \le n_{K_2}}$ $(\underline{sn_{K_1} \le sn_{K_2}})$ if and only if whenever $X \; n_{K_1} \; G$ $(X \; sn_{K_1} \; G)$ for a subgroup X of a finite group G , then $X \; n_{K_2} \; G$ $(X \; sn_{K_2} \; G)$. Clearly, $K_1 \subseteq K_2$ implies that $n_{K_1} \le n_{K_2}$ and $sn_{K_1} \le sn_{K_2}$.

Remark.

Let X be a subgroup of a finite group G and let K be a PQS-closed class of finite groups. Then K is N_0-closed and it is not hard to see that G_K contains all subnormal K-subgroups (in fact, G contains all K-subnormal subgroups - see below). Note that $X^G \in K \iff X \le G_K$; also $G/X^G \in K \iff G^K \le X$.

We summarise the results of [K2] in the following theorem.

3.2 THEOREM.

Let $\mathcal{K}$ be a PQS-closed class of finite groups, and let G be a finite group with subgroups X,Y and N such that $N \lhd G$ and $X \lhd^n_{\mathcal{K}} G$ $(n \in \mathbb{N})$. Then

(i) $X \cap Y \lhd^n_{\mathcal{K}} Y$;

(ii) $\frac{NX}{N} \lhd^n_{\mathcal{K}} \frac{G}{N}$ and if $N \leq Y$ then $\frac{Y}{N} \lhd^m_{\mathcal{K}} \frac{G}{N}$ implies

$Y \lhd^m_{\mathcal{K}} G$ $(m \in \mathbb{N})$;

(iii) if $X = X^{\mathcal{K}}$ then $X \lhd^n G$;

(iv) if $X \in \mathcal{K}$ then $X^G \in \mathcal{K}$;

(v) if $Y \operatorname{sn}_{\mathcal{K}} G$ then $\langle X,Y \rangle \operatorname{sn}_{\mathcal{K}} G$.

In particular, the $\mathcal{K}$-subnormal subgroups of G form a sublattice of $\mathcal{L}(G)$.

Proof.

See Kegel [K 2] (the proof of (iii) is also given here, in Lemma 3.13). We give an alternative proof of (iv) using induction on n, the result being clear if $n \leq 1$. The case $n = 2$ is the crucial one, which we prove in:

3.3 LEMMA.

In the notation of Theorem 3.2, suppose that $X \in K$ and $X \lhd_K Y \lhd_K G$. Then $X^G \in K$.

Proof.

Suppose that the lemma is false and let G be a minimal counterexample. By the remark above Theorem 3.2, X is not subnormal in G. If X is not normal in Y, then $Y/X_Y \in K$ and $Y \in PSK = K$. But $Y \lhd_K G$, so that $X^G \leq Y^G \in K$, a contradiction. Therefore $X \lhd Y \neq G$.

Let N be a non-trivial normal subgroup of G. By the minimality of G (and Theorem 3.2(ii)) we have $X^G N/N \in K$. Therefore $N \notin K$, so that $G_K = 1$ and

$$G \text{ contains no non-trivial subnormal } K\text{-subgroups.} \qquad (1)$$

By 3.2(i), $X \lhd X^G \cap Y \lhd_K X^G$. If $X^G \neq G$ then the minimality of G implies that $X^{(X^G)}$ is a K-group, contradicting (1). Therefore $X^G = G$ and so $G/N \in K$ for all non-trivial normal subgroups N of G. Hence

$$G^K \text{ is a unique minimal normal subgroup of } G. \qquad (2)$$

Therefore $G^K \leq Y_G \leq Y$, so that $X \cap G^K \lhd G^K \lhd G$ and by (1), $X \cap G^K = 1$. Therefore $X \leq C_G(G^K) \lhd G$. But $X^G = G$, so that $G^K \leq Z(G)$ and by (2), G^K has prime order p, say, and $p \notin \Pi(K)$. Thus G^K has p'-index in G, so by the Schur-Zassenhaus theorem, $G = G^K \rtimes Q$ for some p'-subgroup $Q \in K$ of G. But $G^K \leq Z(G)$ so that $G = G^K \times Q$. Then by (1), we must have $Q = 1$, so that G has order p, which is clearly a contradiction.

□

Proof of 3.2(iv).

We are using induction on n to prove that $X^G \in K$ where $X \in K$ and $X = X_0 \lhd_K X_1 \lhd_K \cdots \lhd_K X_{n-1} \lhd_K X_n = G$. The result is clear if $n = 0$ or 1. So suppose $n \geq 2$ with the usual inductive hypothesis. Then $X^{X_{n-1}} \in K$ by induction, and $X^{X_{n-1}} \lhd X_{n-1} \lhd_K G$. Therefore $X^G = (X^{X_{n-1}})^G \in K$ by Lemma 3.3.

□

§3.2 n_K and sn_K.

We know that $sn_I = sn$. However, it is not true that if $sn_K = sn$ for some PQS-closed class K of finite groups, then $K = I$; if we take $K = F_p$ (p prime), then if X is a K-normal subgroup of a finite group G, either $X \lhd G$ or $G/X_G \in \mathcal{F}_p$: in either case, $X \text{ sn } G$, so that $\text{sn}_{\mathcal{F}_p} = \text{sn}$.

More generally, we ask under what conditions does $\text{sn}_{\mathcal{K}_1} = \text{sn}_{\mathcal{K}_2}$ imply that $\mathcal{K}_1 = \mathcal{K}_2$ (where $\mathcal{K}_1, \mathcal{K}_2 \leq \mathcal{F}$ are PQS-closed classes) ?

We first have a result which, in the case $\mathcal{K} = \mathcal{I}$, gives the well-known characterisation of finite nilpotent groups as finite groups in which every subgroup is subnormal.

3.4 THEOREM.

Let $\mathcal{K}$ be a PQS-closed class of finite groups. For a finite group G, the following conditions are equivalent:

(i) Every subgroup of G is $\mathcal{K}$-subnormal.

(ii) $G = K \times N$ where $K \in \mathcal{K}$ and N is nilpotent.

(iii) $G = G_{\mathcal{K}} \times G^{\mathcal{K}}$ where $G^{\mathcal{K}}$ is nilpotent and has co-prime order to $G_{\mathcal{K}}$.

Proof.

(i) => (iii). Suppose that (i) holds and let $\pi_1 = \pi(G_{\mathcal{K}})$ and $\pi_2 = \pi(G/G_{\mathcal{K}})$. Then for $p \in \pi(\mathcal{K}) \cap \pi(G)$ and $P \in \text{Syl}_p(G)$, we have $P \in \mathcal{F}_p \subseteq \mathcal{K}$ and, by Theorem 3.2(iv), $P \leq G_{\mathcal{K}}$. Therefore

$$\pi_1 = \pi(\mathcal{K}) \cap \pi(G) \quad \text{and} \quad \pi_1 \cap \pi_2 = \emptyset . \qquad (1)$$

Let $q \in \Pi_2$ and $Q \in \mathrm{Syl}_q(G)$. If $Q^{\mathcal{K}} \neq Q$ then $\mathcal{K}$ contains a non-trivial q-group, whence $Q \in \mathcal{F}_q \subseteq \mathcal{K}$. By Theorem 3.2(iv), $q \in \Pi_1$, which contradicts (1). Hence $Q^{\mathcal{K}} = Q$ and (by Theorem 3.2(iii)) $Q \text{ sn } G$; therefore $Q \triangleleft G$. Let

$$N = \langle Q : Q \in \mathrm{Syl}_q(G) \text{ , } q \in \Pi_2 \rangle \text{ .}$$

Then $N = O_{\pi_2}(G)$ is a nilpotent normal subgroup of G and (by (1)) $N \cap G_{\mathcal{K}} = 1$. Therefore

$$G = G_{\mathcal{K}} \times N = O_{\pi_1}(G) \times O_{\pi_2}(G)$$

is a Hall decomposition of G , and $G^{\mathcal{K}} \leq N$. If $G^{\mathcal{K}} \neq N$ then there exists $q \in \Pi(\mathcal{K}) \cap \Pi(N) = \Pi(\mathcal{K}) \cap \Pi(G) \cap \Pi(N)$, so that $q \in \Pi_1 \cap \Pi_2$, contradicting (1). Therefore $G^{\mathcal{K}} = N$ and (iii) holds. Clearly, (iii) implies (ii), so it only remains to prove

<u>(ii) => (i)</u>. Suppose that (ii) holds. We prove (i) by induction on the nilpotency class c of N , the result being clear if $c = 0$. So suppose $c \geq 1$ with the usual induction hypothesis. Let X be a subgroup of $G = K \times N$ and let $Z = Z(N)$. Then $Z \leq Z(G)$ and $KZ \cap N = Z(K \cap N) = Z$, so that

$$\frac{G}{Z} = \frac{KZ}{Z} \times \frac{N}{Z} \text{ .}$$

By induction, $XZ \ \mathrm{sn}_{\mathcal{K}} \ G$. Since $X \triangleleft XZ$, then $X \ \mathrm{sn}_{\mathcal{K}} \ G$ and (i) holds. This completes the proof.

□

Remark.

A finite nilpotent group can also be characterised as a finite group in which every Sylow subgroup is normal, but we cannot append to Theorem 3.4 the condition that every Sylow subgroup of G is $\mathcal{K}$-normal. For example, take $\mathcal{K} = F_{\{2,3\}}$ and $G = H \times K$ where $H \cong \Sigma_3$ and K has order 5 . Then G satisfies (i), (ii), (iii) of 3.4, but a Sylow 2-subgroup of G is not $\mathcal{K}$-normal in G (otherwise $G \in \mathcal{K}$) .

We use Theorem 3.4 to prove that, apart from the exceptions mentioned at the beginning of this section (§3.1), distinct PQS-closed classes $\mathcal{K}_1$, $\mathcal{K}_2$ of finite groups do indeed give rise to distinct relations $\mathrm{sn}_{\mathcal{K}_1}$, $\mathrm{sn}_{\mathcal{K}_2}$. This will be a corollary of

3.5 PROPOSITION.

Let $\mathcal{K}_1$, $\mathcal{K}_2$ be PQS-closed classes of finite groups and put $\pi_1 = \pi(\mathcal{K}_1)$, $\pi_2 = \pi(\mathcal{K}_2)$. Then the following are equivalent:

(i) $\mathrm{sn}_{\mathcal{K}_1} \leq \mathrm{sn}_{\mathcal{K}_2}$.

(ii) $K_1 \subseteq K_2$ or $K_1 = F_{p_1}$ for some $p_1 \in \Pi_1 \backslash \Pi_2$ (and $sn_{K_1} = sn$) .

Proof.

Clearly (ii) => (i).

(i) => (ii). Suppose that (i) holds. If $G \in K_1$, then by Theorem 3.4, $G = G_{K_2} \times G^{K_2}$ is a Hall decomposition of G and G^{K_2} is nilpotent. If also $G \in F_{\pi_2}$ then $G^{K_2} \in N \cap F_{\pi_2} \subseteq K_2$, so that $G^{K_2} = 1$ and $G \in K_2$. Therefore $K_1 \cap F_{\pi_2} \subseteq K_2$ and

$$G \in K_1 \Rightarrow G = G_{K_2} \times G^{K_2} , G_{K_2} = O_{\pi_2}(G), G^{K_2} = O_{\pi_2'}(G) \in N \quad . \quad (1)$$

Let $\Pi = \Pi(K_1 \cap F_{\pi_2'})$. By (1), $K_1 \cap F_{\pi_2'} \subseteq N$, so that $|\Pi| \leq 1$ (otherwise $K_1 \cap F_{\pi_2'}$ contains the wreath product of two groups of distinct prime orders, which is not nilpotent). We consider the two possibilities for $|\Pi|$.

If $\Pi = \{p\}$, then $p \in \Pi_1 \backslash \Pi_2$ and we claim that $K_1 = F_p$. Certainly $F_p \subseteq K_1$, and if $F_p \nsubseteq K_1$ then there exists $q \in \Pi_1 \cap \Pi_2$. Then if $H = A \wr B$, where $|A| = q$ and $|B| = p$, we have $H \in K_1$

and (by (1)) $H = O_q(H) \times O_p(H)$. But $O_p(H) = 1$, a contradiction. Therefore $K_1 = F_p$ and (ii) holds.

Finally, if $\pi = \emptyset$ then by (1), $K_1 \leq K_2$ and so (ii) holds.

□

3.6 COROLLARY.

Let K_1 and K_2 be PQS-closed classes of finite groups. Then the following conditions are equivalent:

(i) $sn_{K_1} = sn_{K_2}$.

(ii) $K_1 = K_2$ or $K_1 = F_p$, $K_2 = F_q$ for some $p,q \in \mathbb{P} \cup \{1\}$ (and $sn_{K_1} = sn_{K_2} = sn$) .

Proof.

Suppose that (i) holds. Using Proposition 3.5 twice, we have the following possibilities:

$K_1 \leq K_2$ or $K_1 = F_p$ (p prime), and

$K_2 \leq K_1$ or $K_2 = F_q$ (q prime).

Consideration of the 4 possibilities shows that (ii) holds. Conversely, (ii) => (i) is clear.

□

Putting $\mathcal{K}_1 = \mathcal{K}$ and $\mathcal{K}_2 = \mathcal{I}$ in Corollary 3.6 shows that $sn_{\mathcal{K}} = sn$ if and only if $\mathcal{K} = \mathcal{I}$ or $\mathcal{K} = \mathcal{F}_p$ (p prime). This is the only case where a PQS-closed class $\mathcal{K}$ of finite groups is not uniquely determined by the relation $sn_{\mathcal{K}}$.

It is natural to ask whether Corollary 3.6 remains true if we replace $sn_{\mathcal{K}}$ with $n_{\mathcal{K}}$. In fact, a stronger result holds:

3.7 PROPOSITION.

Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be PQS-closed classes of finite groups. Then $n_{\mathcal{K}_1} \leq n_{\mathcal{K}_2}$ if and only if $\mathcal{K}_1 \leq \mathcal{K}_2$. Hence $n_{\mathcal{K}_1} = n_{\mathcal{K}_2}$ if and only if $\mathcal{K}_1 = \mathcal{K}_2$.

Proof.

Clearly $\mathcal{K}_1 \leq \mathcal{K}_2$ implies that $n_{\mathcal{K}_1} \leq n_{\mathcal{K}_2}$. Suppose that $n_{\mathcal{K}_1} \leq n_{\mathcal{K}_2}$. Then $sn_{\mathcal{K}_1} \leq sn_{\mathcal{K}_2}$ so that by Proposition 3.5 either $\mathcal{K}_1 \leq \mathcal{K}_2$ or $\mathcal{K}_1 = \mathcal{F}_p$ (p prime). Suppose, if possible, that $\mathcal{K}_1 \nleq \mathcal{K}_2$. Let G be a finite p-group which contains a non-normal subgroup X (for example, G is the wreath product of two groups of order p). Then $G \in \mathcal{F}_p = \mathcal{K}_1$, so that $X \lhd_{\mathcal{K}_1} G$ and hence $X \lhd_{\mathcal{K}_2} G$. Therefore $\mathcal{K}_2$ contains the non-trivial p-group G/X_G , which implies that $\mathcal{K}_1 = \mathcal{F}_p \leq \mathcal{K}_2$, a contradiction. Therefore $\mathcal{K}_1 \leq \mathcal{K}_2$. This completes the proof. □

A finite Dedekind group is (defined to be) a group in which every subgroup is normal. Such a group is either abelian or is the direct product of a quaternion group of order 8 and a finite abelian group which has no elements of order 4 . Theorem 3.8 is the K-normal analogue of this result.

3.8 THEOREM.

Let K be a PQS-closed class of finite groups and let G be a finite group. Then every subgroup of G is K-normal in G if and only if either $G \in K$ or G is a Dedekind group

Proof.

Suppose that every subgroup of G is K-normal in G . Then by Theorem 3.4, $G = G_K \times G^K$ where G^K is nilpotent and has co-prime order to G_K . Let X be a subgroup of G^K . Suppose that $X \ntrianglelefteq G^K$. Then $X \ntrianglelefteq G$, so that $G/X_G \in K$. Therefore $G^K \leq X_G$ and $X = G^K \triangleleft G$, a contradiction. Therefore $X \triangleleft G^K$ and G^K is a Dedekind group.

If G_K is also a Dedekind group then so is G , so suppose that G_K is not Dedekind. Let Y be a non-normal subgroup of G_K Then $G/Y_G \in K$, whence $G^K \leq Y_G \leq G_K$. Therefore $G^K = 1$ and $G \in K$. The reverse implication is clear.

□

We ask whether (for a PQS-closed class $K \subseteq F$) the relations sn_K and n_K can coincide. Certainly, if $K = F$ then $sn_K = s = n_K$; Theorem 3.9 shows that this is the only case where sn_K and n_K are equal.

3.9 THEOREM.

Let K be a QS-closed class of finite groups. Then $sn_K = n_K$ if and only if $K = F$.

Proof.

If $K = F$ then $sn_K = s = n_K$. Conversely, suppose that $sn_K = n_K$. We show that $K = F$ by stages. We prove:

(i) $F \cap S \subseteq K$. Let $\pi = \pi(K)$. Suppose, if possible, that there exists $p \in \mathbb{P} \setminus \pi$. Let $G = A \wr B$ where A, B are groups of order p . Since $sn \leq sn_K$ and $B \; sn \; G$, we have $B \triangleleft_K G$ by hypothesis. But $B \ntriangleleft G$, so $G/B_G \in K$ and $p \in \pi$, a contradiction. Therefore $\pi = \mathbb{P}$ and (i) holds.

(ii) $K = F \cap \mathbf{S}$ or $K = F$. Suppose not. Then by (i), there exists non-abelian finite simple groups H and K such that $H \notin K$ and $K \in K$. Let $G = H \times K$ and consider any non-trivial proper subgroup X of K . Then $X \triangleleft_K K \triangleleft G$ so that $X \triangleleft_K G$ (by hypothesis). Since K is simple, we must have $X_G = 1$ and $G \in K$, which contradicts the supposition that $H \notin K$. Therefore (ii) holds.

<u>(iii) $K = F$</u>. By (ii), it is enough to prove that $K \neq S \cap F$. Suppose, for contradiction, that $K = S \cap F$. Consider the group $G = H \times (A \wr B)$ where A, B are groups of prime order p and H is isomorphic to A_5, the alternating group of degree 5. Then $B \text{ sn } (A \wr B) \triangleleft G$ so that $B \triangleleft_K G$. Now $B \ntriangleleft G$ so $G/B_G \in K$ and $B_G = 1$, whence $G \in K$ and $H \in K$, a contradiction. Therefore $K = F$.

□

3.10 <u>COROLLARY</u>.

Let K_1 and K_2 be PQS-closed classes of finite groups. Then $sn_{K_1} = n_{K_2}$ if and only if $K_1 = K_2 = F$.

<u>Proof</u>.

Clearly $K_1 = K_2 = F$ implies $sn_{K_1} = s = n_{K_2}$. Conversely, suppose that $sn_{K_1} = n_{K_2}$. Taking the transitive closure of both sides of this equation, we have $sn_{K_1} = sn_{K_2}$. Therefore $n_{K_2} = sn_{K_2}$, so that $K_2 = F$ by Theorem 3.9. Then by Corollary 3.6, $K_1 = K_2$, as required.

□

Let L be a class of finite simple groups which is closed under taking simple sections; there are $2^{\aleph_0}$ distinct such classes ($2^{\aleph_0}$ is certainly an upper bound, and there are $2^{\aleph_0}$ distinct classes of simple abelian groups). Hence there are $2^{\aleph_0}$ distinct PQS-closed classes K of finite groups, of which (by Corollary 3.6) only $\aleph_0$ give the same relation sn_K (= sn) . Hence, by Proposition 3.7 and Corollary 3.10, there are $2^{\aleph_0}$ distinct relations n_K or sn_K .

§3.3 The Class $\mathfrak{W}_K$.

A natural question to ask if if Theorem 1.3 remains true when "subnormal" is replaced by "K-subnormal", where K is any PQS-closed class of finite groups. The following example shows that it does not remain true. However it does remain true if we suppose that the group is soluble (Theorem 3.12).

3.11 EXAMPLE.

If G is a permutation group on the numbers 1,2,...,n and H is a (proper) transitive subgroup of G then $G = \mathrm{Stab}_G(i)H$ for

$1 \leq i \leq n$ ([SC] 13.1.9). Take $n \geq 5$ and let $G \cong A_n$.

(a): <u>n odd</u>. Let $h = (1\ 2\ \ldots\ n) \in G$ and put $H = N_G(\langle h \rangle)$ and $K = \mathrm{Stab}_G(1) \cong A_{n-1}$. Then $G = \langle h \rangle K = HK$ and $\langle h \rangle$ has $(n-2)!$ conjugates in G . Therefore $|H| = n(n-1)/2$ and $|H \cap K| = (n-1)/2$. Let L_K consist of all simple groups of finite order less than $n!/2$. Then, putting $X = H \cap K$, we have $X \lhd_K H$, $X \lhd_K K$ but X is not $\mathcal{K}$-subnormal in HK .

(b): <u>n even</u>. We partition $\{1,2,\ldots,n\}$ into pairs $P_i = \{a_i, b_i\}$, $1 \leq i \leq n/2$. Then, as in ([SC] 13.1.10), the set H of elements of G which permute the P_i's is a transitive proper subgroup of G . Then if $K = \mathrm{Stab}_G(1)$, we have $G = HK$. Put $X = H \cap K$. Since $n \geq 6$, $X \neq 1$. Therefore, if $\mathcal{K}$ is as in (a) , we have $X \lhd_K H$, $X \lhd_K K$ but X is not $\mathcal{K}$-subnormal in G .

3.12 <u>THEOREM</u>.

Let $\mathcal{K}$ be a PQS-closed class of finite groups. Suppose that G is a finite soluble group which is generated as the product of subgroups H and K , both of which contain X as a $\mathcal{K}$-subnormal subgroup. Then X is $\mathcal{K}$-subnormal in G .

Proof.

Suppose that the theorem is false, and consider a minimal counter-example G such that $|G:H| + |X| = s$ is also minimal. We proceed to derive a contradiction to the existence of such a G.

(i) *H is maximal in G*. Suppose, for a contradiction, that there exists a subgroup L lying strictly between H and G. Then $L = L \cap HK = H(L \cap K)$. By the minimality of G, we have $X \text{ sn}_K L$. But $G = LK$, so by the minimality of s, $X \text{ sn}_K G$, which is a contradiction. This proves (i), so that $|G:H| = q^{\alpha}$ (q prime, $\alpha \geq 1$).

(ii) *H is core-free*. Suppose not. The hypotheses of the theorem hold modulo the group $H_G \neq 1$, so that $XH_G \text{ sn}_K G$ by the minimality of G. But $X \text{ sn}_K XH_G$, which gives a contradiction.

(iii) *H contains no non-trivial subnormal subgroups of G*. Suppose not, and let S be a subnormal subgroup of G such that $S \leq H$. Since H is maximal in G, then $S^G \leq H$, contradicting (ii).

(iv) *$X^H, X^K \in K$*. By Theorem 3.2(iv), it is enough to prove that $X \in K$. Suppose not. If $X = X^K$ then $X \text{ sn } H$, $X \text{ sn } K$ by Theorem 3.2(iii) so that $X \text{ sn } G$ by Theorem 1.3: hence $X \text{ sn}_k G$, a contradiction. Therefore $1 < X^{\mathcal{K}} < X$. Now $X^{\mathcal{K}}$ is $\mathcal{K}$-subnormal in both H and K and, since $(X^{\mathcal{K}})^{\mathcal{K}} = X^{\mathcal{K}}$, Theorem 3.2(iii) implies that $X^{\mathcal{K}}$ is subnormal in both H and K . By Theorem 1.3, $X^{\mathcal{K}}$ sn G which contradicts (iii). Hence (iv) holds.

Let A be a minimal normal subgroup of G . Then A is an elementary abelian p-group (p prime) and $G = AH$ by (i) and (ii). Then $A \cap H \triangleleft AH = G$ and (ii) implies $A \cap H = 1$ and $p = q$. We claim that $p \notin \Pi(\mathcal{K})$. For, suppose that $p \in \Pi(\mathcal{K})$. Then $A \in \mathcal{K}$ and by (iv), $AX = A]X \in \mathcal{K}$. Therefore $X \triangleleft_{\mathcal{K}} AX$. But $AX \text{ sn}_{\mathcal{K}} AH = G$, whence $X \text{ sn}_{\mathcal{K}} G$, a contradiction; thus the claim is true.

Let $\Pi = \Pi(X^H) \cup \Pi(X^K)$. Then $p \notin \Pi$ and by (iv), $\mathcal{F} \cap \mathcal{S}_\pi \subseteq \mathcal{K}$. Let $H_\pi \in \mathrm{Hall}_\pi(H)$. Then $H_\pi \in \mathrm{Hall}_\pi(G)$ and $X^H \leq H_\pi$. Let $K_\pi \in \mathrm{Hall}_\pi(K)$. Then $X^K \leq K_\pi$ and K_π is contained in some Hall Π-subgroup H_π^g of G (where $g \in G$) . Writing $g = hk$ $(h \in H, k \in K)$ we have

$$X^K \leq K_\pi^{k^{-1}} = K_\pi^{g^{-1}h} \leq H_\pi^h \leq H .$$

Therefore $X^G = X^{KH} \leq H$ and by (ii) we have $X \leq X^G = 1$, a contradiction. Therefore the theorem is true.

□

We may still talk of $\mathcal{K}$-subnormality in infinite groups, by using the same definitions as in the finite case and allowing K to be any PQS-closed class of groups. Of course, given an infinite group G , G_K and G/G^K will not necessarily belong to K . If we impose the extra condition of R-closure on K (to ensure $G/G^K \in K$) then we will have gone too far, because this forces $K = I$ or $K = U$ (free groups are residualy F_p-groups for any prime p ([I1])). Also, we no longer have a characterisation of K in terms of L_K . If K consists of periodic groups (Π-groups, say), then K need not equal S_π - for example, K might consist of soluble Π-groups of finite rank.

If K does not consist of periodic groups, then K contains F_p for all $p \in \mathbb{P}$ and hence $S \cap \hat{M} \subseteq K$. If $K \subseteq S \cap \hat{M}$ then we must have $K = I$, $S \cap \hat{M}$ or $K = F \cap S_\pi$ for some $\pi \subseteq \mathbb{P}$ (if K consists of periodic groups then $K \subseteq F$; if K contains non-periodic groups then $K = S \cap \hat{M}$) . Therefore, if $G \in S \cap \hat{M}$ and $H \triangleleft_K G$ for some PQS-closed class K , then (because $K \cap S \cap \hat{M}$ is PQS-closed) either $G \in K$, $H \triangleleft G$ or $|G:H|$ is finite.

It is not hard to see that the basic properties of K-subnormality ("intersecting" and "factoring") given in Theorem 3.2 (i)(ii) also hold in the infinite case. Theorem 3.2 (iii) also holds in general: the proof of this in the finite case ([K2] Lemma 4) still works.

For completeness, we give a proof here.

3.13 LEMMA.

Let $\mathcal{K}$ be a PQS-closed class of groups. Let X be a subgroup of the group G and suppose that $X = X^{\mathcal{K}}\ \mathrm{sn}_{\mathcal{K}}\ G$. Then $X\ \mathrm{sn}\ G$.

Proof.

Let $X = X_0 \triangleleft_{\mathcal{K}} X_1 \triangleleft_{\mathcal{K}} \dots \triangleleft_{\mathcal{K}} X_n = G$ be a $\mathcal{K}$-subnormal series from X to G . We prove that $X\ \mathrm{sn}\ G$ by induction on n , the result being clear if $n \leq 1$. Suppose that $n \geq 2$ with the usual inductive hypothesis. Then $X\ \mathrm{sn}\ X_{n-1}$ (by induction) and we suppose, for a contradiction, that X is not subnormal in G . Let $Y = (X_{n-1})_G$. Then $G/Y \in \mathcal{K}$ and $X \nleq Y$. Therefore $X \cap Y$ is a proper normal subgroup of X and $X/(X \cap Y) \in \mathcal{K}$, which contradicts the supposition that $X = X^{\mathcal{K}}$. Therefore $X\ \mathrm{sn}\ G$.

□

We define the class $\mathcal{W}_{\mathcal{K}}$ of groups by: if G is a group then $G \in \mathcal{W}_{\mathcal{K}}$ if and only if (*) holds for any PSQ-closed class of groups $\mathcal{K}$:

$$\left.\begin{array}{l}\text{whenever } G \text{ is generated as the product of subgroups}\\ H \text{ and } K\text{, both of which contain } X \text{ as a } \mathcal{K}\text{-subnormal}\\ \text{subgroup, then } X \text{ is } \mathcal{K}\text{-subnormal in } G\,.\end{array}\right\}(*)$$

Then $\mathcal{W}_{\mathcal{K}} \leq \mathcal{W}$ and Theorem 3.12 implies that $F \cap \mathcal{S} \leq \mathcal{W}_{\mathcal{K}}$ (if

$G \in F \cap S$, then the K-subnormal subgroups of G are precisely the $(K \cap F)$-subnormal subgroups). We will identify some other subclasses of W_K ; in view of Example 3.11, we restrict ourselves to finding subclasses of $W_K \cap S$. Note that W_K is, like W , Q-closed, as is W_K^S .

It is not hard to see that Lemma 1.6 and Theorem 1.8 still hold for sn_K in place of sn (where $K = PQSK$) ; the proofs are virtually the same and even the bounds still hold. We shall refer to these K-subnormal results as <u>Lemma 1.6'</u> and <u>Theorem 1.8'</u> . Using these reductions, and similar proofs to those for W (Theorem 1.9, Proposition 1.10), we have

3.14 <u>THEOREM</u>.

W_K contains the following classes: NA , $NF \cap S$, $(F \cap S)W_K$.

□

3.15 <u>THEOREM</u>.

$S \cap \hat{M} \leq W_K$.

<u>Proof</u>.

Let K be a PQS-closed class of groups. Suppose that G is a polycyclic group which is generated as the product of subgroups

H and K, both of which contain X as a $\mathcal{K}$-subnormal subgroup. Let $\mathcal{K}_1 = \mathcal{K} \cap \mathcal{PC}$. Then, inside G, $\mathcal{K}_1$-subnormality is equivalent to $\mathcal{K}$-subnormality, so we may assume that $\mathcal{K} = \mathcal{K}_1 \subseteq \mathcal{PC}$. By the remarks after Theorem 3.12, either $\mathcal{K} = \mathcal{I}$ (and $X \text{ sn } G$ by Theorem 1.9), $\mathcal{K} = \mathcal{PC}$ (and $X \triangleleft_{\mathcal{K}} G$) or $\mathcal{K} \subseteq \mathcal{F} \cap \mathcal{S}$. Therefore we may assume that $\mathcal{K} = \mathcal{F} \cap \mathcal{S}_{\pi_1}$ for some $\pi_1 \subseteq \mathbb{P}$. Consider the $\mathcal{K}$-subnormal series

$$X = X_0 \triangleleft_{\mathcal{K}} X_1 \triangleleft_{\mathcal{K}} \dots\dots \triangleleft_{\mathcal{K}} X_n = H \quad \text{and}$$

$$X = Y_0 \triangleleft_{\mathcal{K}} Y_1 \triangleleft \quad \dots\dots \triangleleft_{\mathcal{K}} Y_n = K \quad (0 \leq n < \infty) .$$

Considering all the non-normal steps $X_i \triangleleft_{\mathcal{K}} X_{i+1}$, $Y_j \triangleleft_{\mathcal{K}} Y_{j+1}$ $(0 \leq i,j \leq n-1)$ in these series, define the set of primes π as consisting of the primes dividing the orders of the $\mathcal{K}$-groups $X_{i+1}/(X_i)_{X_{i+1}}$, $Y_{j+1}/(Y_j)_{Y_{j+1}}$. Then π is a finite subset of π_1. We show that $X \text{ sn}_{(\mathcal{F}\cap\mathcal{S}_\pi)} G$ by induction on the Hirsch length $h = h(G)$ of G.

If $h = 0$ then G is finite and the result holds by Theorem 3.12 (with $\mathcal{F} \cap \mathcal{S}_\pi$ in place of $\mathcal{K}$). So suppose $h \geq 1$ with the usual induction hypothesis. G contains a normal poly-(infinite cyclic) subgroup B of finite index. Put $\mathcal{K}_0 = \mathcal{F} \cap \mathcal{S}_\pi$ and let A be the penultimate term of the derived series of B . Let $N = (A \cap H)(A \cap K)$. Then N is a normal abelian subgroup of $G_1 = AH \cap AK$ (as in Lemma 1.5), and because $AX \; sn_{\mathcal{K}_0} \; G$ (induction) it is enough, by Lemma 1.6' , to show that $XN/N \; sn_{\mathcal{K}_0} \; G_1/N$. But if $N \neq 1$, then $h(G_1/N) < h(G)$ and the result holds by induction. So we may assume that $N = 1$ and $G = G_1$, so that

$$G = A \,] \, H = A \,] \, K = HK \, .$$

Let $p \in \mathbb{P}$. Then, by induction, $A^pX \; sn_{\mathcal{K}_0} \; G$ and if $r = \text{rank}(A)$, then $|AX : A^pX|$ divides p^r and so $A^pX \triangleleft^r_{\mathcal{K}_0} AX$. Therefore

$$A^pX \triangleleft^r AX \quad \text{for} \quad p \in \mathbb{P} \backslash \Pi \quad .$$

Now $A \cap X = 1$ and $\mathbb{P} \backslash \Pi$ is an infinite set, so

$$X = \Big(\bigcap_{p \in \mathbb{P} \backslash \Pi} A^p \Big) X = \bigcap_{p \in \mathbb{P} \backslash \Pi} (A^pX) \triangleleft^r AX \, .$$

Therefore $X \; sn \; AX \; sn_{\mathcal{K}_0} \; G$, as required.

□

In the light of the above proof, it might be hoped that for a $\mathcal{K}$-subnormal subgroup X of a polycyclic group G , there is a $\mathcal{K}$-subnormal series from X to G in which the normal steps are at at the bottom and the non-normal steps are at the top. If this were true when $h(G) = 0$, then induction would show that it is true for any polycyclic group G. However, the following is a counter-example in the case G finite (and soluble).

EXAMPLE.

Let $G = (Y \,]\, X) \,]\, Z$ where $Y = \langle y \rangle$, $X = \langle x \rangle$, $Z = \langle z \rangle$ are groups of order 7,3,5 respectively and the actions are given by $y^x = y^2$, $y^z = y^{-1}$, $x^z = yx$. Let $K = F \cap S_{(3,7)}$. Then $X \triangleleft_K XY \triangleleft G$ but X is not subnormal in G. Also, there is no subgroup V such that $X \triangleleft V \triangleleft_K G$. For otherwise $|G:V| = 7$, $V = N_G(X) = \langle x \rangle \times \langle y^5 z \rangle$ has order 6 and is core-free in G, which implies $G \in K$, a contradiction.

3.16 **COROLLARY.**

$N(S \cap \hat{M}) \leq W_K$.

Proof.

Let $G \in N(S \cap \hat{M})$ be generated as the product of subgroups H and K, both of which contain X as a K-subnormal subgroup. To show $X \ \mathrm{sn}_K \ G$ we may assume that $G \in A(S \cap \hat{M})$ (Theorem 1.8'). By

Lemma 1.6' and Theorem 3.15, we may assume $G = A] H = A] K = HK$, where A is an abelian normal subgroup of G such that $G/A \in S \cap \hat{M}$. Now G is the soluble product of polycyclic groups H and K, so by a result of Lennox and Roseblade ([LR]), G itself is polycyclic. Then $X \text{ sn}_K G$ by Theorem 3.15.

□

Theorem 3.17 deals with the dual case to 3.15 – that of $(S\cap\check{M})$-groups. A $(S\cap\check{M})$-group G has invariants $\lambda_1(G) = \text{rank } (G^F)$ and $\lambda_2(G) = |G : G^F|$. Define $\lambda(G)$ as the ordered pair $(\lambda_1(G), \lambda_2(G))$. The invariants $\lambda(G)$ (for $G \in S\cap\check{M}$) can be ordered lexicographically, so that $\lambda(L) < \lambda(G)$ for any proper subgroup L of G and $\lambda(G/N) < \lambda(G)$ for any non-trivial normal subgroup N of G.

3.17 THEOREM.

$$S \cap \check{M} \leq W_K .$$

Proof.

Suppose that the proposition is false and pick a counter-example $G \in (S\cap\check{M})\backslash W_K$ which is minimal with respect to $\lambda(G) = (\lambda_1(G), \lambda_2(G))$. So there exist subgroups X, H, K of G such that $X \text{ sn}_K H$, $X \text{ sn}_K K$ but X is not K-subnormal in G. If $H_G \neq 1$ then $\lambda(G/H_G) < \lambda(G)$ and so $XH_G \text{ sn}_K G$. But $X \text{ sn}_K XH_G$, which gives a contradiction. Therefore $H_G = 1$.

Now $XG^F = XH^F XK^F$ (using Lemma 1.14) and $X \; sn_K \; XH^F$, $X \; sn_K \; XM^F$. $G/G^F \in F \cap S$ so that $XG^F \; sn_K \; G$ (Theorem 3.12). If XG^F were a proper subgroup of G then minimality of $\lambda(G)$ would give $X \; sn_K \; G$. Hence $G = XG^F$. But now $H^F \trianglelefteq XG^F = G$ so that $H^F \leq H_G = 1$. Similarly $K^F = 1$, and therefore $G^F = H^F K^F = 1$. Hence $G = X$, a contradiction.

□

3.18 <u>COROLLARY</u>.

$$N(S \cap \check{M}) \subseteq W_K \quad .$$

<u>Proof</u>.

Let $G \in N(S \cap \check{M})$ be generated as the product of subgroups H and K, both of which contain X as a K-subnormal subgroup. Then by Theorem 1.8', Theorem 3.17 and Lemma 1.6', we may assume that $H, K \in S \cap \check{M}$. But the soluble product of Cernikov groups is again Černikov ([A2] Theorem B). Therefore $G \in S \cap \check{M}$ and $X \; sn_K \; G$ by Theorem 3.17.

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