Chow Hypersurfaces and Realizability Problems in Tropical Geometry

by

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Thesis

Submitted to the University of Warwick

for the degree of

Doctor of Philosophy

Warwick Mathematics Institute

September 2017

THE UNIVERSITY OF
WARWICK

Contents

Acknowledgments **iii**

Declaration **iv**

Abstract **v**

Chapter 1 Introduction **1**

Chapter 2 Background **5**

2.1 Fields 5

2.2 Initial Ideals 6

2.3 Polyhedral Geometry 8

2.4 Tropical Varieties 11

2.4.1 Tropical Grassmannians 13

Chapter 3 Tropical Chow Hypersurfaces **15**

3.1 Tropical Chow Hypersurfaces 16

3.2 From Tropical Chow Hypersurface to Tropical Variety 23

3.2.1 Curves in space 28

3.3 Tropicalization of Families 32

Chapter 4 Secants, Bitangents and their Congruences **36**

4.1 The Degree of a Subvariety in $\mathrm{Gr}(1, \mathbb{P}^3)$ 36

4.2 Secants of Space Curves 38

4.3 Bitangents and Inflections of a Surface 43

4.4 Projective Duality 46

4.5 Intersection Theory on $\mathrm{Gr}(1, \mathbb{P}^3)$ 52

4.6 Singular Loci of Congruences 56

Chapter 5 Relative Realizability **62**

5.1 Local Set-Theoretical Computation . . . 63

5.2 Local Weighted Computation . . . 70

5.2.1 Higher dimensional fibers . . . 71

5.2.2 Hilbert functions . . . 76

5.2.3 The main algorithm . . . 79

5.3 Change of Coordinates . . . 81

5.4 The Global Algorithm . . . 82

Bibliography **85**

Acknowledgments

I am greatly thankful to my advisor Diane Maclagan for her tireless support. She guided my growth as a mathematician with knowledge and enthusiasm.

I would like to thank everyone who contributed to make the Mathematics Institute at the University of Warwick a stimulating and friendly environment during my PhD years, in particular the algebraic geometry group and the postgraduate community.

Declaration

This thesis is submitted to the University of Warwick in support of my application for the degree of Doctor of Philosophy. It has been composed by myself, except where stated otherwise, and has not been submitted in any previous application for any degree or award.

The work presented was carried out by the author, except for the work of Chapter 4 which was carried out jointly with Kathlén Kohn and Bernt Ivar Nødland.

Abstract

Tropical geometry is an area of mathematics between algebraic geometry, polyhedral geometry and combinatorics. The basic principle of tropical geometry is to associate to an algebraic variety X a polyhedral complex $\mathrm{Trop}(X)$ called the tropicalization of X. The tropicalization of X can be studied by means of polyhedral geometry and combinatorics and reflects many properties of the original variety X.

Given a projective variety $X \subset \mathbb{P}^n$ of codimension $k+1$, the Chow hypersurface Z_X is the hypersurface of the Grassmannian $\mathrm{Gr}(k, n)$ parametrizing k-dimensional linear subspaces of $\mathbb{P}^n$ that intersect X. In Chapter 3 we introduce and describe a tropical Chow hypersurface $\mathrm{Trop}(Z_X)$. This object only depends on the tropical variety $\mathrm{Trop}(X)$ and we provide an explicit way to obtain $\mathrm{Trop}(Z_X)$ from $\mathrm{Trop}(X)$. Finally we prove that, when X is a curve in $\mathbb{P}^3$, $\mathrm{Trop}(X)$ can be reconstructed from $\mathrm{Trop}(Z_X)$.

In Chapter 4 we study the geometric properties of Chow hypersurfaces of space curves and other special varieties in the Grassmannian $\mathrm{Gr}(1, \mathbb{P}^3)$, such as the Hurwitz hypersurface and the bitangent congruence of a space surface. We give new proofs for the bidegrees of the secant, bitangent and inflectional congruences, using geometric techniques such as duality, polar loci and projections. We also study the singularities of these congruences.

In Chapter 5 we start with a family of projective curves and we study the geometric properties of the fibers by looking at their tropicalization. Given a tropical curve Σ and a family of algebraic curves $\mathcal{X}$ we produce an algorithm to describe the locus of fibers of $\mathcal{X}$ whose tropicalization contains Σ.

Chapter 1

Introduction

Tropical geometry is a new vibrant area of mathematics that combines results and ideas from algebraic geometry, polyhedral geometry and combinatorics. Tropical geometry replaces algebraic varieties with their shadows in the tropical world, namely tropical varieties. Many of the features of an algebraic variety, such as its dimension and degree, can be read directly off its tropicalization. The advantage of this procedure is that tropical varieties have, usually, a simpler geometry than their original algebraic variety and they can be studied by means of polyhedral geometry and combinatorics. In Chapter 2 we review some of the basic definitions and results of tropical geometry.

Given a variety $X \subset \mathbb{P}^n$ of codimension $k+1$ its associated Chow hypersurface is the hypersurface of the Grassmannian $\mathrm{Gr}(k,n)$ of k-dimensional linear subspaces of $\mathbb{P}^n$ defined by:

$$Z_X \coloneqq \{L \in \mathrm{Gr}(k,n) \mid L \cap X \neq \varnothing\}.$$

Chow hypersurfaces were first introduced by Cayley in [5] for curves in $\mathbb{P}^3$ and then generalized by Chow and Van der Waerden in [6]. The main feature of the Chow hypersurface is that it is the vanishing locus of a single polynomial equation, called the *Chow form*, which uniquely determines the original variety X. This property allows the construction of Chow varieties, which are moduli spaces of algebraic cycles of given degree and dimension. In Chapter 3 we explore how these ideas can be translated into tropical geometry. In particular we introduce a *tropical Chow hypersurface* and we study some of its properties. This can be seen as a generalization of the construction proposed by Fink in [13]. We define, by analogy with the classical case, the tropical Chow hypersurface in the tropical Grassmannian $\mathrm{TrGr}(k,n)$:

$$Z_{\mathrm{Trop}(X)} \coloneqq \{\Lambda \in \mathrm{TrGr}(k,n) \mid \Lambda \cap \mathrm{Trop}(X) \neq \varnothing\}.$$

If we embed $\mathrm{Gr}(k,n)$ in the projective space $\mathbb{P}^{\binom{n+1}{k+1}-1}$ via the Plücker embedding we can also construct the tropical variety $\mathrm{Trop}(Z_X)$.

Theorem 1. *We have the following equality of sets:*

$$\mathrm{Trop}(Z_X) = Z_{\mathrm{Trop}\,X}.$$

The main result of Chapter 3 concerns the structure of $\mathrm{Trop}(Z_X)$. We define φ to be the linear map $\varphi : \mathbb{R}^{n+1}/\mathbb{R} \to \mathbb{R}^N/\mathbb{R}$ defined by $\varphi(a_0,\ldots,a_n) = (\sum_{i\in I} a_i)_I$, where the homogeneous coordinates of $\mathbb{R}^N/\mathbb{R}$ are indexed by cardinality $k+1$ sets $I \subset \{0,\ldots,n\}$. Let ψ be the toric morphism associated to φ. This is the monomial morphism defined by $\psi([x_0 : \ldots : x_n]) = (\prod_{i\in I} x_i)_I$. We denote by $\star$ the Hadamard product and by $+$ the Minkowski sum (for their definitions see Section 3.1). We also denote by $\mathscr{G}_u$ the variety $\mathscr{G}_u := \{L \in \mathrm{Gr}(k,n) \mid [1 : \ldots : 1] \in L\}$ and by Γ_0 the tropical variety $\Gamma_0 := \{\Lambda \in \mathrm{TrGr}(k,n) \mid (0,\ldots,0) \in \Lambda\}$.

Theorem 2. *Let $X \subset \mathbb{P}^n$ be an algebraic variety that intersects the torus T^n. Then we have the following equalities:*

$$Z_X = \psi(X) \star \mathscr{G}_u,$$
$$\mathrm{Trop}(Z_X) = \varphi(X) + \Gamma_0.$$

We prove that the map that sends $\mathrm{Trop}(X)$ to $\mathrm{Trop}(Z_X)$ is injective in the case that X is a curve in $\mathbb{P}^3$. We conjecture that this holds for any variety X. In particular, the argument used in [13] to show the non-injectivity of their construction does not work here. At the end of Chapter 3 we describe an application of Chow hypersurfaces to the study of tropicalization of families of varieties.

The Chow hypersurface of a curve $C \subset \mathbb{P}^3$ is an example of a threefold in the Grassmannian $\mathrm{Gr}(1,\mathbb{P}^3)$. Another notable example is the the *Hurwitz hypersurface* $\mathrm{CH}_1(S) \subset \mathrm{Gr}(1,\mathbb{P}^3)$ of a surface $S \subset \mathbb{P}^3$: this is the Zariski closure of the set of all lines in $\mathbb{P}^3$ that are tangent to S at a smooth point.

In Chapter 4 we study the geometry of certain threefolds and surfaces in $\mathrm{Gr}(1,\mathbb{P}^3)$. These are classically known as *line complexes* and *congruences*. We determine their classes in the Chow ring of $\mathrm{Gr}(1,\mathbb{P}^3)$ and their singular loci. Throughout Chapter 4, we use the phrase 'singular points of a congruence' to simply refer to its singularities as a subvariety of the Grassmannian $\mathrm{Gr}(1,\mathbb{P}^3)$. In older literature, this phrase refers to points in $\mathbb{P}^3$ lying on infinitely many lines of the congruence; nowadays, these are called *fundamental points*.

The main results of Chapter 4 are consolidated in the following theorem.

Theorem 3. *Let $C \subset \mathbb{P}^3$ be a nondegenerate curve of degree d and geometric genus g having only ordinary singularities $x_1, x_2, \dots, x_s$ with multiplicities $r_1, r_2, \dots, r_s$. If $\mathrm{Sec}(C)$ denotes the locus of secant lines to C, then the singular locus of Z_C is $\mathrm{Sec}(C) \cup \bigcup_{i=1}^{s} \{L \in \mathrm{Gr}(1, \mathbb{P}^3) : x_i \in L\}$, the bidegree (see Section 4.1 for the definition) of $\mathrm{Sec}(C)$ is*

$$\left(\tfrac{1}{2}(d-1)(d-2) - g - \sum_{i=1}^{s} \tfrac{1}{2} r_i(r_i - 1), \tfrac{1}{2} d(d-1)\right),$$

and the singular locus of $\mathrm{Sec}(C)$, when C is smooth, consists of all lines that intersect C with total multiplicity at least 3.

Let $S \subset \mathbb{P}^3$ be a general surface of degree d with $d \geq 4$. If $\mathrm{Bit}(S)$ denotes the locus of bitangents to S and $\mathrm{Infl}(S)$ denotes the locus of inflectional tangents to S, then the singular locus of $\mathrm{CH}_1(S)$ is $\mathrm{Bit}(S) \cup \mathrm{Infl}(S)$, the bidegree of $\mathrm{Bit}(S)$ is

$$\left(\tfrac{1}{2} d(d-1)(d-2)(d-3), \tfrac{1}{2} d(d-2)(d-3)(d+3)\right),$$

the bidegree of $\mathrm{Infl}(S)$ is $\left(d(d-1)(d-2), 3d(d-2)\right)$, and the singular locus of $\mathrm{Infl}(S)$ consists of all lines that are inflectional tangents at at least two points of S or intersect S with multiplicity at least 4 at some point.

The bidegree of $\mathrm{Infl}(S)$ also appears in [31, Prop. 4.1]. The bidegrees of $\mathrm{Bit}(S)$, $\mathrm{Infl}(S)$, and $\mathrm{Sec}(C)$, for smooth C, already appear in [2]. Nevertheless, we give new, more geometric, proofs not relying on Chern class techniques. The singular loci of $\mathrm{Sec}(C)$, $\mathrm{Bit}(S)$, and $\mathrm{Infl}(S)$ are partially described in Lemma 2.3, Lemma 4.3, and Lemma 4.6 in [2].

Using duality, we establish some relationships of the varieties in Theorem 3.

Theorem 4. *If C is a nondegenerate smooth space curve, then the secant lines of C are dual to the bitangent lines of the dual surface $C^\vee$ and the tangent lines of C are dual to the inflectional tangent lines of $C^\vee$.*

Congruences and line complexes have been actively studied both in the 19th century and in modern times. The study of congruences goes back to Kummer [28], who classified those of order 1; the order of a congruence is the number of lines in the congruence that pass through a general point in $\mathbb{P}^3$. Many results from the second half of the 19th century are detailed in Jessop's monograph [22]. Hurwitz hypersurfaces and further generalizations known as higher associated or coisotropic hypersurfaces are studied in [15, 26, 41]. Catanese [4] shows that Chow hypersurfaces of space curves and Hurwitz hypersurfaces of surfaces are exactly the self-dual hypersurfaces in the Grassmannian $\mathrm{Gr}(1, \mathbb{P}^3)$. Ran [34] studies surfaces of order 1 in general Grassmannians and gives a modern proof of Kummer's classification.

Congruences play a role in algebraic vision and multi-view geometry, where cameras are modeled as maps from $\mathbb{P}^3$ to congruences [33]. The multidegree of the image of several of those cameras is computed by Escobar and Knutson in [12].

In Section 4.1, we collect basic facts about the Grassmannian $\mathrm{Gr}(1,\mathbb{P}^3)$ and its subvarieties. Section 4.2 studies the singular locus of the Chow hypersurface of a space curve and computes its bidegree. Section 4.3 describes the singular locus of a Hurwitz hypersurface and Section 4.4 uses projective duality to calculate the bidegree of its components. In Section 4.5, we connect the intersection theory in $\mathrm{Gr}(1,\mathbb{P}^3)$ to Chow and Hurwitz hypersurfaces. Finally, Section 4.6 analyzes the singular loci of secant, bitangent, and inflectional congruences.

In Chapter 5 we present an algorithmic approach to the problem of realizability for families of tropical varieties. We start with a (not necessarily flat) family of affine algebraic varieties

$$\begin{array}{l} \mathcal{X} \subset \mathbb{A}^k \times \mathbb{A}^n \\ \downarrow \\ \mathbb{A}^k, \end{array}$$

and a tropical curve Σ. The tropical curve Σ is a list of rays with multiplicities $\Sigma = \{(\rho_1, m_1), \ldots, (\rho_l, m_l)\}$. We describe the set $\mathrm{Real}_\Sigma \subset \mathbb{A}^k$ of parameters a such that the tropicalization of the fiber X_a is a tropical curve that contains Σ, i.e. it contains every ray ρ_i with multiplicity at least m_i. The main result of Chapter 5 is an algorithm to compute the Zariski closure of Real_Σ.

Algorithm 1.

- *Input:*

  $\mathcal{X}$, *variety in* $\mathbb{A}^k \times \mathbb{A}^n$,

  Σ, *a list of rays with multiplicities.*

- *Output:*

  Y, *variety in* $\mathbb{A}^k$, *the Zariski closure of* Real_Σ.

Algorithm 1 produces the ideal $\mathrm{Ideal}(\mathrm{Real}_\Sigma)$ by imposing conditions for each ray ρ_i. By a change of coordinates described in Section 5.3 the ray ρ_i is transformed to the ray $\mathrm{pos}(e_1)$, and by the algorithms described in Sections 5.1 and 5.2 we impose conditions on the fibers for the ray ρ_i to be contained in the tropicalization of the fiber X_a with at least multiplicity m_i. Finally in Section 5.4 we combine the information coming from the different rays, and we describe Algorithm 1.

Chapter 2

Background

In this chapter we review some background material that will be assumed throughout the rest of the thesis. In particular we introduce some of the basic ideas of tropical geometry. A complete introduction to the subject can be found in [29]. We make no claim of originality for the content of this chapter.

2.1 Fields

A valuation val on a field $\mathbb{K}$ is a map

$$\mathrm{val} : \mathbb{K} \to \mathbb{R} \cup \{\infty\}$$

satisfying the following properties:

- $\mathrm{val}(x) = \infty$ if and only if $x = 0$,
- $\mathrm{val}(xy) = \mathrm{val}(x) + \mathrm{val}(y)$ for every $x, y \in \mathbb{K}$,
- $\mathrm{val}(x + y) \geq \min(\mathrm{val}(x), \mathrm{val}(y))$ for every $x, y \in \mathbb{K}$.

Moreover it can be deduced from these properties that, whenever $\mathrm{val}(x) \neq \mathrm{val}(y)$, we have $\mathrm{val}(x + y) = \min(\mathrm{val}(x), \mathrm{val}(y))$, see [29, Lemma 2.1.1]. We often identify the valuation val with its restriction $\mathrm{val} : \mathbb{K}^* \to \mathbb{R}$. We denote by Γ_{val} the image in $\mathbb{R}$ of the valuation val, Γ_{val} is an additive subgroup of $\mathbb{R}$.

Throughout this thesis $\mathbb{K}$ will be assumed to be an algebraically closed field with a valuation val.

We will mostly work over the field of complex numbers $\mathbb{C}$, which is equipped with the trivial valuation defined by

$$\mathrm{val}(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ \infty & \text{if } x = 0 \end{cases}.$$

Valuations have an important role in tropical geometry and many results, such as the Fundamental Theorem, require the valuation to be non-trivial. For this reason it is often convenient to regard $\mathbb{C}$ as a subfield of the field of complex Puiseux series $\mathbb{C}\{\{t\}\}$. The elements of this field are formal power series with complex coefficients of the form

$$\sum_{i \geq m} a_i t^{\frac{i}{r}}$$

for some $m \in \mathbb{Z}$ and $r \in \mathbb{Z}_{>0}$. In other words, as sets, we have $\mathbb{C}\{\{t\}\} = \bigcup_{r=1}^{\infty} \mathbb{C}((t^{\frac{1}{r}}))$, where $\mathbb{C}((t^{\frac{1}{r}}))$ denotes the field of Laurent series. The field of complex Puiseux series is an algebraically closed field and it is equipped with the valuation defined by $\mathrm{val}(\sum_{i \geq m} a_i t^{\frac{i}{r}}) = \frac{m}{r}$, if $a_m \neq 0$. The image Γ_{val} of this valuation equals the set of rational numbers $\mathbb{Q}$. Moreover there exist a group homomorphism $\psi : \Gamma_{\mathrm{val}} \to \mathbb{C}\{\{t\}\}$ that sends the rational number $\frac{p}{q}$ to the Puiseux series $\psi(\frac{p}{q}) = t^{\frac{p}{q}}$.

This is actually a general fact: for any algebraically closed valued field $\mathbb{K}$ there exists a section of the valuation, i.e. a group homomorphism $\psi : \Gamma_{\mathrm{val}} \to \mathbb{K}$ such that $a = \mathrm{val}(\psi(a))$ for every $a \in \Gamma_{\mathrm{val}}$ (see [29, Lemma 2.1.15]). We always assume to have fixed such a ψ and, on any valued field $\mathbb{K}$, we use the suggestive notation $t^a := \psi(a)$.

The set of elements of non-negative valuation of a valued field $\mathbb{K}$ forms a local ring

$$R := \{x \in \mathbb{K} \mid \mathrm{val}(x) \geq 0\},$$

with maximal ideal

$$\mathfrak{m} := \{x \in \mathbb{K} \mid \mathrm{val}(x) > 0\}.$$

The quotient field $\mathbf{k} := R/\mathfrak{m}$ is called the *residue field* of $\mathbb{K}$.

Example 2.1.1. The residue field of the field of Puiseux series $\mathbb{C}\{\{t\}\}$ is the field of complex numbers $\mathbb{C}$.

2.2 Initial Ideals

Let $S := \mathbb{K}[x_1^{\pm 1}, \dots x_n^{\pm 1}]$ be the ring of Laurent polynomials over $\mathbb{K}$. We use multi-index notation: given $u = (u_1, \dots, u_n) \in \mathbb{Z}^n$ we denote by x^u the monomial

$x^u := x_1^{u_1} \cdots x_n^{u_n}$. A *monomial order* on S is a total order $\prec$ on the set of monomials $x^u \in S$ that satisfy the following property: for every $x^u, x^v, x^w \in S$ if $x^u \prec x^v$ then $x^u x^w \prec x^v x^w$. Equivalently, if we identify a monomial x^u with its exponent $u \in \mathbb{Z}^n$, it is a total order over $\mathbb{Z}^n$ with the property that for every $u, v, w \in \mathbb{Z}^n$ if $u \prec v$ then $u + w \prec v + w$.

Example 2.2.1. The *lexicographic* order is defined by $(u_1, \dots, u_n) \prec_{\text{Lex}} (v_1, \dots, v_n)$ if there exists some i such that $u_i < v_i$ and $u_1 = v_1, \ \dots, \ u_{i-1} = v_{i-1}$. The *graded lexicographic* order is defined by $(u_1, \dots, u_n) \prec_{\text{GLex}} (v_1, \dots, v_n)$ if $\sum u_i < \sum v_i$ or if $\sum u_i = \sum v_i$ and $(u_1, \dots, u_n) \prec_{\text{Lex}} (v_1, \dots, v_n)$.

Example 2.2.2. The *reverse lexicographic* order is defined by $(u_1, \dots, u_n) \prec_{\text{RevLex}} (v_1, \dots, v_n)$ if there exists some i such that $u_i > v_i$ and $u_1 = v_1, \ \dots, \ u_{i-1} = v_{i-1}$. The *graded reverse lexicographic* order is defined by $(u_1, \dots, u_n) \prec_{\text{GRevLex}} (v_1, \dots, v_n)$ if $\sum u_i < \sum v_i$ or if $\sum u_i = \sum v_i$ and $(u_1, \dots, u_n) \prec_{\text{RevLex}} (v_1, \dots, v_n)$.

Let $\prec$ be a monomial order over S and let $f = \sum a_u x^u \in S$ be a Laurent polynomial. The *initial form* of f is defined as:

$$\text{in}_\prec(f) = a_v x^v, \quad \text{where } v = \min_\prec \{u \mid a_u \neq 0\}.$$

Given an ideal $I \subset S$ the initial ideal $\text{in}_\prec(I)$ is the ideal generated by the initial forms of elements of I:

$$\text{in}_\prec(I) := (\text{in}_\prec(f) \mid f \in I).$$

Let $\{f_1, \dots, f_r\}$ be a set of generators for the ideal I. The set of their initial forms $\{\text{in}_\prec(f_1), \dots, \text{in}_\prec(f_r)\}$ need not be a generating set for the initial ideal $\text{in}_\prec(I)$. When they are we say that $\{f_1, \dots, f_r\}$ is a *Gröbner basis*. Gröbner bases were first introduced by Bruno Buchberger in [3] and are an essential tool in computational commutative algebra.

We now consider a variation of this construction that takes into account the valuation val. As in the end of the previous section we denote by $\mathbf{k}$ the residue field of $\mathbb{K}$, and by t^a the image of $a \in \Gamma_{\text{val}}$ under a fixed section $\psi : \Gamma_{\text{val}} \to \mathbb{K}$.

Fix a vector $w \in \mathbb{R}^n$ and a Laurent polynomial $f = \sum a_u x^u \in S$, and let $M = \min_u \{\text{val}(a_u) + u \cdot w\}$. The initial form of f with respect to $\prec_w$ is defined as:

$$\text{in}_w(f) := \sum_{\substack{v \ \mid \\ \text{val}(a_v) + v \cdot w = M}} [t^{-\text{val}(a_v)} a_v] x^v \in \mathbf{k}[x_1^{\pm 1}, \dots, x_n^{\pm 1}],$$

where we denote by $[y]$ the class in the residue field $\mathbf{k}$ of the field element $y \in \mathbb{K}$.

Similarly, given an ideal $I \subset S$, we define the initial ideal $\mathrm{in_w}(I)$ as:

$$\mathrm{in}_w(I) = (\mathrm{in}_w(f) \mid f \in I).$$

As before, the initial ideal $\mathrm{in}_w(I)$ is not always generated by the initial forms of a set of generators of I. The following is an example.

Example 2.2.3. Consider the ideal $I = (x+y+z^2+t^3, x+y)$ in $S = \mathbb{C}\{\{t\}\}[x^{\pm 1}, y^{\pm 1}, z^{\pm 1}]$, and let $w = (1,1,1)$. We have $\mathrm{in}_w(x+y+z^2+t^3) = x+y$ and $\mathrm{in}_w(x+y) = x+y$. However $z^2+t^3 \in I$, and hence $\mathrm{in}_w(z^2+t^3) = z^2 \in \mathrm{in}_w(I)$. In particular, since z^2 is invertible in S, we have $\mathrm{in}_w(I) = (1)$.

2.3 Polyhedral Geometry

We now introduce the basic notions of polyhedral geometry. A full introduction to these ideas can be found in [48].

A *polyhedron* $P \subset \mathbb{R}^n$ is a closed subset defined by linear inequalities

$$P = \{x \in \mathbb{R}^n \mid Ax \leq b\},$$

for some matrix A and some vector b. The polyhedron P is *rational* if A and b can be chosen with rational entries. More generally, given an additive subgroup $\Gamma \subset \mathbb{R}$, P is Γ*-rational* if A and b can be chosen with entries in Γ.

A bounded polyhedron is called a *polytope*. Given a finite subset $A = \{v_1, \dots v_l\} \subset \mathbb{R}^n$, we define its *convex hull* $\mathrm{conv}(A)$ as:

$$\mathrm{conv}(A) := \{\lambda_1 v_1 + \ldots + \lambda_l v_l \mid \lambda_1, \dots, \lambda_l \geq 0,\ \lambda_1 + \ldots + \lambda_l = 1\}.$$

A subset $P \subset \mathbb{R}^n$ is a polytope if and only if it is the convex hull of a finite set (see [48, Theorem 1.1]).

Given a finite subset $A = \{v_1, \dots v_l\} \subset \mathbb{R}^n$, we define its *positive hull* $\mathrm{pos}(A)$ as:

$$\mathrm{pos}(A) := \{\lambda_1 v_1 + \ldots + \lambda_l v_l \mid \lambda_1, \dots, \lambda_l \geq 0\}.$$

A *polyhedral cone* is the positive hull of a finite subset of $\mathbb{R}^n$, a *ray* is the positive hull of a single element of $\mathbb{R}^n$. Throughout this thesis the expression “cone” will always mean “polyhedral cone”.

Given a polyhedron P and a subset $Q \subset P$ we say that Q is a face of P if

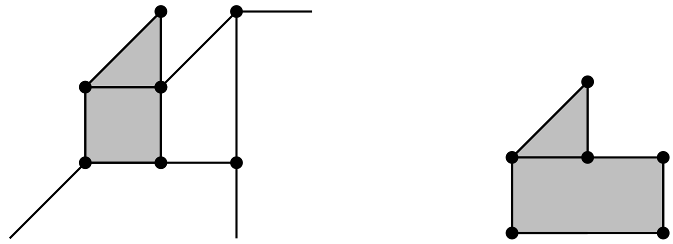

Figure 2.1: A polyhedral complex and not a polyhedral complex

there exist some $y \in \mathbb{R}^n$ such that

$$Q = Q_y := \{x \in P \mid x \cdot y \geq z \cdot y \;\; \forall z \in P\}. \tag{2.1}$$

The *relative interior* $\operatorname{relint}(P)$ of a polyhedron P is the interior of P in the smallest affine subspace of $\mathbb{R}^n$ that contains it.

A collection Σ of polyhedra is called a *polyhedral complex* if it satisfies the following two properties:

1. given a polyhedron $P \in \Sigma$ and a face F of P we have $F \in \Sigma$,
2. given two polyhedra $P, Q \in \Sigma$ their intersection $P \cap Q$ is either empty or a face of both.

A polyhedral complex is *rational* if all its polyhedra are rational. Throughout this thesis all polyhedral complexes will be assumed to be rational. A *polyhedral fan* is a polyhedral complex whose polyhedra are all cones. Throughout this thesis the expression “fan” will always mean “polyhedral fan”.

A polyhedral complex has pure dimension d if all its maximal polyhedra have dimension d. A *weight* w on a pure d-dimensional polyhedral complex Σ is a map $w : \Sigma(d) \to \mathbb{Z}_{>0}$ from the set of maximal polyhedra $\Sigma(d)$ to the set of positive integer numbers. We will use the words weight and multiplicity interchangeably. From time to time we will allow negative weights, and we will use the convention that a polyhedron P has weight 0 in a weighted polyhedral complex Σ if $P \notin \Sigma$.

A one-dimensional (rational) fan Σ is a finite collection of rational rays $\rho_1, \ldots, \rho_k$. For $i = 1, \ldots, k$ we denote by v_i the first integer point of the ray ρ_i: the point v_i is the unique point in $\rho_i \cap \mathbb{Z}^n$ whose coordinates are relatively prime. Suppose Σ is equipped with a weight w. We say that Σ is *balanced* if

$$\sum_{i=1}^{k} w(\rho_i) v_i = 0.$$

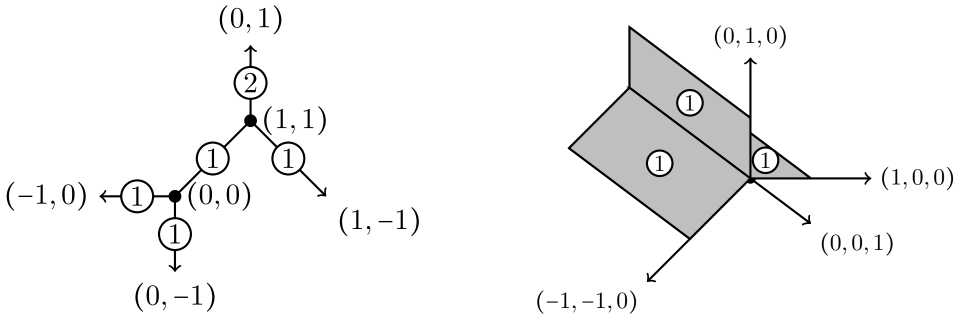

Figure 2.2: Balanced polyhedral complexes

We now extend the definition of balancing to any (rational) pure d-dimensional polyhedral complex. Let Σ be a pure d-dimensional polyhedral complex in $\mathbb{R}^n$, and let P be a $(d-1)$-dimensional polyhedron in Σ. Up to a translation, we can assume that $0 \in P$. Let $L_1, \dots L_k$ be the set of d-dimensional polyhedra of Σ that contain P as a face. The star fan of Σ at P is the one-dimensional fan $\mathrm{star}_P(\Sigma)$ in $\mathbb{R}^{n-d-1} = \mathbb{R}^n/\mathrm{span}(P)$ whose rays $\rho_1, \dots, \rho_k$ are defined by the property that, if v_i denotes the first integer point of ρ_i, then $\epsilon v_i + P \subset L_i$ for some $\epsilon > 0$, for every $1 \le i \le k$. Here by integer point of $\mathbb{R}^n/\mathrm{span}(P)$ we mean an element of $\mathbb{Z}^n/(\mathbb{Z}^n \cap \mathrm{span}(P))$. Let m_i be the weight of L_i in Σ. We consider the weight w on $\mathrm{star}_P(\Sigma)$ defined by $w(\rho_i) = m_i$. We say that Σ is balanced at P if $\mathrm{star}_P(\Sigma)$ is balanced. We say that Σ is *balanced* if it is balanced at every $(d-1)$-dimensional polyhedron $P \in \Sigma$.

An important class of examples of balanced fans is given by normal fans of polytopes. To a polytope $P \subset \mathbb{R}^n$ we can associate its (inner) *normal fan* $\mathcal{N}(P)$ in $\mathbb{R}^n$, whose cones correspond to faces of P. In particular, for a face $F \subset P$ the corresponding cone σ_F is the set of $y \in \mathbb{R}^n$ such that, with the notation of Equation (2.1), $F \subset Q_y$. More explicitly, when P is full dimensional, rays of $\mathcal{N}(P)$ are the positive hull of (inner) normal vectors of facets of P, and the rays $\rho_1, \dots, \rho_k$ corresponding to the facets $F_1, \dots, F_k$ of $\mathcal{N}(P)$ form a cone if and only if $F_1 \cap \dots \cap F_k$ is non-empty.

We will often be interested in the codimension-one skeleton $\mathcal{N}^1(P)$ of the normal fan of a polytope P. This is the fan formed by all cones of dimension at most $n-1$ in $\mathcal{N}^1(P)$. This fan has a natural choice of weights on it. As maximal cones in $\mathcal{N}^1(P)$ correspond to 1-dimensional faces (edges) of P, we associate to the maximal cone $\sigma \in \mathcal{N}^1(P)$ corresponding to the edge $L \subset P$ the lattice length $\#(L \cap \mathbb{Z}^n) - 1$ of L. This choice of weights turns $\mathcal{N}^1(P)$ into a balanced fan.

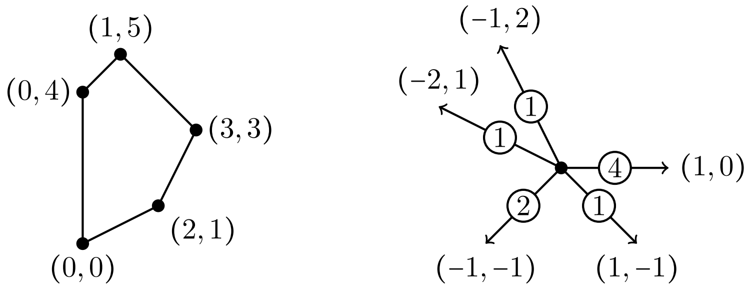

Figure 2.3: The polytope P and the fan $\mathcal{N}^1(P)$ of Example 2.3.1

Example 2.3.1. Let $P \subset \mathbb{R}^2$ be the polytope with vertices

$$\{(0,0),(2,1),(3,3),(1,5),(0,4)\}.$$

The codimension-one skeleton of the normal fan of P is the weighted fan with cones

$$(\mathrm{pos}(-1,2),\mathrm{pos}(-2,1),\mathrm{pos}(-1,-1),\mathrm{pos}(1,-1),\mathrm{pos}(1,0))$$

and weights $(1,1,2,1,4)$. The balancing condition for this fan consists in the assertion

$$1\cdot(-1,2)+1\cdot(-2,1)+2\cdot(-1,-1)+1\cdot(1,-1)+4\cdot(1,0)=(0,0).$$

The polytope P and the fan $\mathcal{N}^1(P)$ are depicted in Figure 2.3.

A pure d-dimensional polyhedral complex Σ is *connected through codimension one* if, for any two maximal polyhedra $P, P' \in \Sigma$, there exist a sequence of maximal polyhedra $P = P_0, P_1, \dots, P_k = P' \in \Sigma$ such that $P_i \cap P_{i+1}$ has dimension $d-1$ for every $0 \le i < k$.

2.4 Tropical Varieties

Let $f = \sum a_u x^u \in S = \mathbb{K}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ be a Laurent polynomial. The tropical polynomial $\mathrm{Trop}(f)(v)$ is the piecewise linear function $\mathrm{Trop}(f): \mathbb{R}^n \to \mathbb{R}$ defined by $\mathrm{Trop}(f)(v) = \min_u(\mathrm{val}(a_u) + v \cdot u)$.

The tropical hypersurface $\mathrm{Trop}(\mathrm{V}(f)) \subset \mathbb{R}^n$ is defined as

$$\mathrm{Trop}(\mathrm{V}(f)) := \{v \in \mathbb{R}^n \mid \text{ the minimum in } \mathrm{Trop}(f)(v) \text{ is achieved at least twice}\}.$$

Let $I \subset S$ be an ideal. The tropical variety $\mathrm{Trop}(\mathrm{V}(I))$ is defined as

$$\mathrm{Trop}(\mathrm{V}(I)) := \bigcap_{f \in I} \mathrm{Trop}(\mathrm{V}(f)). \tag{2.2}$$

We stress that it is not, in general, sufficient to take the intersection in (2.2) over a set of generators of I.

Theorem 2.4.1 (Fundamental Theorem, see Theorem 3.2.5 [29])**.** *Let $\mathbb{K}$ be an algebraically closed field and suppose that the image $\Gamma_{\mathrm{val}} = \mathrm{val}(\mathbb{K})$ of the valuation* val *is dense in $\mathbb{R}$. Let $X = \mathrm{V}(I) \subset \mathrm{T}^n = (\mathbb{K}^*)^n$ be an algebraic variety. Then the following sets are the same:*

1. *the tropical variety $\mathrm{Trop}(X)$,*
2. *the closure in the Euclidean topology of $\mathrm{val}(X)$, where by $\mathrm{val}(X)$ we mean the image of X under the map $\mathrm{val} : (\mathbb{K}^*)^n \to \mathbb{R}^n$ defined by*

$$\mathrm{val}(x_1, \dots, x_n) = (\mathrm{val}(x_1), \dots, \mathrm{val}(x_n)),$$

3. *the closure in the Euclidean topology of the set of $w \in \Gamma_{\mathrm{val}}^n$ such that $in_w(I) \neq \boldsymbol{k}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$.*

Given an algebraic variety $X = \mathrm{V}(I)$, the tropical variety $\mathrm{Trop}(X)$ can be given the structure of a polyhedral complex. When X is irreducible $\mathrm{Trop}(X)$ is pure dimensional of dimension $d = \dim X$. Moreover, it is possible to turn it into a weighted polyhedral complex. To do so we first recall some definitions from commutative algebra (see, for example, [11, Chapter 3]).

An ideal $Q \subset S$ is *primary* if $fg \in Q$ implies that $f \in Q$ or $g^m \in Q$ for some $m > 0$. An ideal is prime if and only if it is primary and radical. Any ideal $I \subset S$ admits a *primary decomposition* $I = \bigcap_{i=1}^r Q_i$, where the Q_i's are primary ideals. This decomposition is not unique. However, the set of *associated primes* $\mathrm{AssPrim}(I) = \{\mathrm{Rad}(Q_1), \dots, \mathrm{Rad}(Q_r)\}$, which is the set of prime ideals involved in the decomposition, is unique. We denote by $\mathrm{AssPrim}^{\min}(I)$ the subset of primes of $\mathrm{AssPrim}(I)$ that do not contain any other prime of $\mathrm{AssPrim}(I)$. Given two ideals $J, K \in S$ the saturation ideal $(J : K^\infty)$ is defined as $(J : K^\infty) = \{f \in S \mid fg^k \in J, \text{ for some } g \in K,\ k > 0\}$. The multiplicity of $P \in \mathrm{AssPrim}^{\min}(I)$ is defined as

$$\mathrm{mult}(P, I) := \ell(((I : P^\infty)/I)_P),$$

where $\ell(A)$ denotes the length of the S_P-module A, which is the maximum size of a chain of submodules of A.

Let σ be a cone of $\mathrm{Trop}(X)$ of dimension d. We can assume that $\mathrm{in}_w(I)$ is constant for $w \in \mathrm{relint}(\sigma)$. The multiplicity of σ in $\mathrm{Trop}(X)$ is defined as

$$\mathrm{mult}(\sigma, \mathrm{Trop}(X)) := \sum_{J \in \mathrm{AssPrim}^{\min}(\mathrm{in}_w(I))} \mathrm{mult}(J, \mathrm{in}_w(I)), \tag{2.3}$$

where w is any point in the relative interior of σ.

The geometric intuition behind this definition is the following (see [29, Remark 3.4.4]). If σ is a maximal cone of $\mathrm{Trop}(X)$, where $X = \mathrm{V}(I)$ is irreducible of dimension d, and $w \in \mathrm{relint}(\sigma)$, the variety $\mathrm{V}(\mathrm{in}_w(I))$ is a union of d-dimensional torus orbits. The multiplicity $\mathrm{mult}(\sigma, \mathrm{Trop}(X))$ is the number (counted with multiplicities) of such orbits.

Theorem 2.4.2. *Let $X \subset \mathrm{T}^n$ be an irreducible d-dimensional variety. Then* $\mathrm{Trop}(X)$ *is the support of a balanced Γ_{val}-rational pure d-dimensional polyhedral complex that is connected through codimension one.*

Let $f = \sum a_u x^u \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ be a Laurent polynomial. The Newton polytope of f is the polytope $\mathrm{Newt}(f) = \mathrm{convHull}\{u \mid a_u \neq 0\} \subset \mathbb{R}^n$. The tropicalization of the hypersurface $\mathrm{V}(f)$ can be computed as $\mathrm{Trop}(\mathrm{V}(f)) = \mathcal{N}^1(\mathrm{Newt}(f))$.

Example 2.4.3. Let $f = 1 + x^2y + x^3y^3 + xy^5 + y^4 \in \mathbb{C}[x_1^{\pm 1}, x_2^{\pm 1}]$. The Newton polytope $\mathrm{Newt}(f)$ and the tropical curve $\mathrm{Trop}(\mathrm{V}(f))$ equal the polytope P and the fan $\mathcal{N}^1(P)$ of Example 2.3.1.

2.4.1 Tropical Grassmannians

The projective Grassmannian $\mathrm{Gr}(k,n) = \mathrm{Gr}(k, \mathbb{P}^n)$ parametrizes projective k-dimensional planes in the projective space $\mathbb{P}^n$. The Grassmannian $\mathrm{Gr}(k,n)$ embeds via the Plücker embedding in $\mathbb{P}^{N-1}$, where $N = \binom{n+1}{k+1} - 1$. The algebraic torus T^{N-1} is embedded in $\mathbb{P}^{N-1}$ as $\mathrm{T}^{N-1} := \{[x_0 : \dots : x_n] \mid x_0 \cdot \dots \cdot x_n \neq 0\}$. We denote by $\mathrm{Gr}^\circ(k,n) = \mathrm{Gr}(k,n) \cap \mathrm{T}^{N-1}$ the intersection of the Grassmannian with the torus, and by $\mathrm{TrGr}(k,n)$ the tropicalization $\mathrm{Trop}(\mathrm{Gr}^\circ(k,n))$. A tropical variety Λ is called a *tropicalized linear space* if it is the tropicalization of some linear space $L \subset \mathbb{P}^n$, moreover, it is said *uniform* if L can be chosen so that its Plücker vector lies in $\mathrm{Gr}^\circ(k,n)$. We have the following theorem.

Theorem 2.4.4 (see Theorem 4.3.13, [29])**.** *The bijection between the Grassmannian* $\mathrm{Gr}(k,n)$ *and the set of k-dimensional subspaces of $\mathbb{P}^n$ induces a bijection*

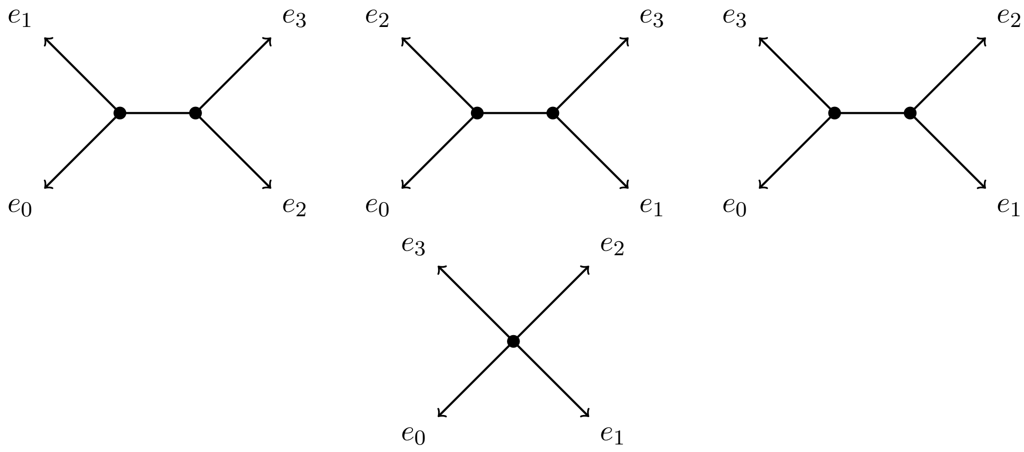

Figure 2.4: The possible combinatorial types of uniform tropicalized lines in $\mathbb{R}^3$.

$w \mapsto L_w$ between $\mathrm{TrGr}(k, n)$ *and the set of uniform tropicalized* $(r-1)$*-planes in* $\mathbb{R}^{n+1}/\mathbb{R}$.

Example 2.4.5. The tropical Grassmannian $\mathrm{TrGr}(1, 3)$ is a 4-dimensional fan in $\mathbb{R}^5 = \mathbb{R}^6/\mathbb{R}$. It has a 3-dimensional lineality space, which descends from the natural action of T^3 on $\mathrm{Gr}(1, 3)$. This lineality space is described as the image of the linear map $\varphi : \mathbb{R}^4/\mathbb{R} \to \mathbb{R}^6/\mathbb{R}$ whose matrix, in the standard bases e_0, e_1, e_2, e_3 of $\mathbb{R}^4$ and $e_{01}, e_{02}, e_{12}, e_{03}, e_{13}, e_{23}$ of $\mathbb{R}^6$ is

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

The Grassmannian $\mathrm{TrGr}(1, 3)$ modulo its lineality space is a 1-dimensional fan in $\mathbb{R}^2 = \mathbb{R}^5/\mathbb{R}^3$ with three rays. These rays and the origin correspond to the four possible combinatorial types of uniform tropicalized lines in $\mathbb{R}^4/\mathbb{R}$, see Figure 2.4. In particular, the origin corresponds to the tropizalized line with no bounded edges.

Chapter 3

Tropical Chow Hypersurfaces

Given a variety $X \subset \mathbb{P}^n$ of codimension $k+1$ its associated Chow hypersurface is the hypersurface of the Grassmannian $\mathrm{Gr}(k,n)$ of k-dimensional linear subspaces of $\mathbb{P}^n$ defined by:

$$Z_X \coloneqq \{L \in \mathrm{Gr}(k,n) \mid L \cap X \neq \varnothing\}.$$

Chow hypersurfaces were first introduced by Cayley in [5] for curves in $\mathbb{P}^3$ and then generalized by Chow and Van der Waerden in [6].

In this chapter we present a tropical analogue of Chow hypersurfaces. We define, by analogy with the classical case, the tropical Chow hypersurface in the tropical Grassmannian $\mathrm{TrGr}(k,n)$:

$$Z_{\mathrm{Trop}(X)} \coloneqq \{\Lambda \in \mathrm{TrGr}(k,n) \mid \Lambda \cap \mathrm{Trop}(X) \neq \varnothing\}.$$

If we embed $\mathrm{Gr}(k,n)$ in the projective space $\mathbb{P}^{\binom{n+1}{k+1}-1}$ via the Plücker embedding we can also construct the tropical variety $\mathrm{Trop}(Z_X)$.

Theorem 5. *We have the following equality of sets:*

$$\mathrm{Trop}(Z_X) = Z_{\mathrm{Trop}\,X}.$$

Section 3.2 concerns the structure of $\mathrm{Trop}(Z_X)$. We define φ to be the linear map $\varphi : \mathbb{R}^{n+1}/\mathbb{R} \to \mathbb{R}^N/\mathbb{R}$ defined by $\varphi(a_0,\dots,a_n) = (\sum_{i\in I} a_i)_I$. Let ψ be the toric morphism associated to φ. This is the monomial morphism defined by $\psi(x_0,\dots,x_n) = (\prod_{i\in I} x_i)_I$. We denote by $\star$ the Hadamard product and by $+$ the Minkowski sum (for their definitions see Section 3.1). We also denote by $\mathscr{G}_u$ the variety $\mathscr{G}_u \coloneqq \{L \in \mathrm{Gr}(k,n) \mid [1:\ldots:1] \in L\}$ and by Γ_0 the tropical variety $\Gamma_0 \coloneqq \{\Lambda \in \mathrm{TrGr}(k,n) \mid (0,\dots,0) \in \Lambda\}$.

Theorem 6. *Let $X \subset \mathbb{P}^n$ be an algebraic variety that intersects the torus T^n. Then we have the following equalities:*

$$Z_X = \psi(X) \star \mathscr{G}_u,$$
$$\mathrm{Trop}(Z_X) = \varphi(X) + \Gamma_0.$$

We prove that the map that associates to a tropical variety $\mathrm{Trop}(X)$ its tropical Chow hypersurface $\mathrm{Trop}(Z_X)$ is injective in the case that X is a curve in $\mathbb{P}^3$. We conjecture that this holds for any variety X. In the last section we describe an application to the study of tropicalization of families of varieties.

Notation

We set throughout the chapter $N = \binom{n+1}{k+1}$. The Grassmannian $\mathrm{Gr}(k,n)$ parametrizes k-dimensional projective linear spaces in $\mathbb{P}^n$, and it is naturally embedded in $\mathbb{P}^{N-1}$ via the Plücker embedding. The coordinates of $\mathbb{P}^{N-1}$ are indexed by the collection of all subsets of cardinality $k+1$ of $[n+1] = \{0,\dots,n\}$, we denote this collection by $\binom{[n+1]}{k+1}$.

We denote by T^n the embedded torus $\mathrm{T}^n \coloneqq \{[x_0 : \dots : x_n] \mid x_0 \cdot \ldots \cdot x_n \neq 0\} \subset \mathbb{P}^n$. Moreover, given an algebraic variety $X \subset \mathbb{P}^n$ we use the notation $X^\circ = X \cap \mathrm{T}^n$.

We write $\mathbb{R}^{n+1}/\mathbb{R}$ for the quotient of $\mathbb{R}^{n+1}$ by the one-dimensional linear space generated by the vector $(1,\dots,1)$. Given a variety $X \subset \mathbb{P}^n$ we will denote by $\mathrm{Trop}(X) \subset \mathbb{R}^{n+1}/\mathbb{R}$ the tropicalization of X°. By a slight abuse of notation, we also refer to $\mathrm{Trop}(X)$ as the tropicalization of X.

We work over an algebraically closed field K with a non-trivial valuation $\mathrm{val} : K \smallsetminus \{0\} \to \mathbb{R}$. We denote by $\Gamma_{\mathrm{val}} \subset \mathbb{R}$ the image of val and we say that a point $p \in \mathbb{R}^{n+1}/\mathbb{R}$ is Γ_{val}-rational if it has a representative in $\Gamma_{\mathrm{val}}^{n+1}$.

3.1 Tropical Chow Hypersurfaces

In this section we begin by describing the structure of the Chow hypersurface Z_X and then define a tropical Chow hypersurface.

We denote by L_z the linear space corresponding to a point $z \in \mathrm{Gr}(k,n)$. Given a variety $X \subset \mathbb{P}^n$ of dimension $n-k-1$ the Chow hypersurface associated to X is

$$Z_X \coloneqq \{z \in \mathrm{Gr}(k,n) \mid X \cap L_z \neq \varnothing\} \subset \mathrm{Gr}(k,n).$$

For a complete exposition of the main properties of Z_X see [15]. For a quick and more concrete introduction we suggest [8]. If X is irreducible of degree d, then Z_X

is an irreducible hypersurface of the Grassmannian $\mathrm{Gr}(k,n)$ of degree d (see, for example, [15, Chapter 2, Proposition 2.2]).

The torus T^n naturally embeds in T^{N-1} via the following regular map:

$$\begin{array}{rccc} \psi: & \mathrm{T}^n & \longrightarrow & \mathrm{T}^{N-1} \\ & [x_0:\ldots:x_n] & \longmapsto & [\prod_{i\in I} x_i]_I. \end{array} \tag{3.1}$$

For $x \in \mathbb{P}^n$ we denote by $\mathscr{G}_x$ the subvariety of the Grassmannian $\mathrm{Gr}(k,n)$ defined by

$$\mathscr{G}_x := \{z \in \mathrm{Gr}(k,n) \mid x \in L_z\}.$$

Given two quasi-projective varieties $X, Y \subset \mathbb{P}^n$, their *Hadamard product* $X \star Y \subset \mathbb{P}^n$ is defined to be the closure in the Zariski topology of the set

$$\{[x_0y_0,\ldots,x_ny_n] \mid [x_0,\ldots,x_n] \in X \text{ and } [y_0,\ldots,y_n] \in Y\}.$$

Given two points $x = [x_0,\ldots,x_n]$ and $y = [y_0,\ldots,y_n]$ in $\mathbb{P}^n$, whenever it is well defined, we also denote by $x \star y \in \mathbb{P}^n$ the point with coordinates $[x_0y_0,\ldots,x_ny_n]$. For a variety $X \subset \mathbb{P}^n$ we consider the set of linear spaces intersecting X in the torus $\mathrm{T}^n \subset \mathbb{P}^n$. We define $Z_{X^\circ} := \{z \in \mathrm{Gr}(k,n) \mid X^\circ \cap L_z \neq \varnothing\}$.

Lemma 3.1.1. *Let $X \subset \mathbb{P}^n$ be an irreducible variety of dimension $n-k-1$, and let $u = [1:\ldots:1] \in \mathrm{T}^n$. Then $\overline{Z}_{X^\circ} = \psi(X^\circ) \star \mathscr{G}_u$. In particular, if X intersects the torus T^n, then $Z_X = \psi(X^\circ) \star \mathscr{G}_u$.*

Proof. There is a natural action of the torus T^n on the torus T^{N-1}. For $y \in \mathrm{T}^n$ this action is defined by the multiplication map

$$\begin{array}{rccc} m_y: & \mathrm{T}^{N-1} & \longrightarrow & \mathrm{T}^{N-1} \\ & [p_I]_I & \longmapsto & [p_I \prod_{i\in I} y_i]_I. \end{array}$$

The multiplication map m_y is the Hadamard product with $\psi(y)$. It can be extended to an automorphism of $\mathbb{P}^{N-1}$ with inverse $m_{y'}$, for $y' = [y_0^{-1}:\ldots:y_n^{-1}]$. Moreover the Plücker ideal is homogeneous with respect to the grading of $K[p_I \mid I \in \binom{[n+1]}{k+1}]$ associated to this torus action, as a consequence m_y preserves the Grassmannian.

We claim that, for $x \in \mathbb{P}^n$, if we restrict m_y to $\mathscr{G}_x$ we get an isomorphism

$$m_y|_{\mathscr{G}_x}: \ \mathscr{G}_x \xrightarrow{\sim} \mathscr{G}_{x\star y}. \tag{3.2}$$

To prove this we just need to show that $m_y(\mathscr{G}_x) \subset \mathscr{G}_{x\star y}$, as the claim will then follow from the analogous statement for $m_{y'}$. Fix a point $p \in \mathrm{Gr}(k,n)$. The coordinates p_I of p arise as the determinants of the maximal minors of some $k \times n$ matrix

$$A_p = \begin{pmatrix} a_{0,0} & \dots & a_{0,n} \\ \vdots & \ddots & \vdots \\ a_{k,0} & \dots & a_{k,n} \end{pmatrix} \tag{3.3}$$

whose rows are a basis for the linear space L_p corresponding to p. The minor indexed by I of the matrix

$$A_{p\star\psi(y)} = \begin{pmatrix} a_{0,0}y_0 & \dots & a_{0,n}y_n \\ \vdots & \ddots & \vdots \\ a_{k,0}y_0 & \dots & a_{k,n}y_n \end{pmatrix} \tag{3.4}$$

has determinant $p_I \prod_{i\in I} y_i = (m_y(p))_I$ which shows that the linear space corresponding to $m_y(p) \in \mathrm{Gr}(k,n)$ is the rowspace of $A_{p\star\psi(y)}$. By assumption x lies in the linear span of the rows of A_p, so there exists some $m \in K^{k+1}$ such that $\sum_j m_j a_{j,i} = x_i$ for $i = 0,\dots,n$. The entries $a'_{i,j} = a_{i,j}y_i$ of $A_{p\star\psi(y)}$ satisfy $\sum_j m_j a'_{j,i} = \sum_j m_j a_{j,i} y_i = x_i y_i = (x \star y)_i$ for $i = 1,\dots,n$ proving the claim.

We can now conclude the proof of the first part of the statement. A point $z \in \mathrm{Gr}(k,n)$ is in Z_{X° if and only if there is a point $x \in X^\circ$ such that $x \in L_z$. The latter condition is, by definition, equivalent to $z \in \mathscr{G}_x$ so that, by Equation (3.2), we can find a $y \in \mathscr{G}_u$ with $z = m_x(y)$. As $m_x(y) = \psi(x) \star y$ we have that

$$\overline{Z}_{X^\circ} = \overline{\{\psi(x) \star y \in \mathrm{Gr}(k,n) \mid x \in X^\circ \text{ and } y \in \mathscr{G}_u\}} = \psi(X^\circ) \star \mathscr{G}_u.$$

We now prove the last statement. We have that Z_X is irreducible as X is (see [15, Chapter 3, Proposition 2.2]). Moreover Z_{X° is an open subset of Z_X, as its complement $Z_X \setminus Z_{X^\circ}$ is the Zariski closed set $\{L \mid L \cap X \cap \mathrm{V}(x_0 \cdot \ldots \cdot x_n) \neq \varnothing\}$. Finally Z_{X° is not empty as X° is not, so $Z_X = \overline{Z}_{X^\circ}$ concluding the proof. □

Remark 3.1.2. It is useful to notice that the morphism $\psi(X^\circ) \times \mathscr{G}_u^\circ \to \psi(X^\circ) \star \mathscr{G}_u^\circ$ defined by $(x,y) \mapsto x \star y$ is generically one-to-one. This is equivalent to the well-known fact that a generic linear space L_z with $z \in Z_X$ intersects X in a unique point.

We now establish a tropical version of Lemma 3.1.1. A *tropical variety* $\Sigma \subset \mathbb{R}^{n+1}/\mathbb{R}$ of dimension $n-k-1$ is a balanced weighted Γ_{val}-rational polyhedral complex of pure dimension $n-k-1$. A detailed introduction to this notion can be found in [29, Chapter 3]. There, however, the name tropical variety is restricted to polyhedral complexes that arise as $\mathrm{Trop}(X)$ for some $X \subset \mathrm{T}^n$.

We recall the notion of *Minkowski sum* for polyhedral complexes. Given two polyhedra σ_1, σ_2 of dimension d_1 and d_2 and weight m_{σ_1} and m_{σ_2}, their Minkowski sum is defined, as a set, to be

$$\sigma_1 + \sigma_2 = \{a + b \mid a \in \sigma_1,\ b \in \sigma_2\}.$$

The weight $m_{\sigma_1+\sigma_2}$ of $\sigma_1 + \sigma_2$ is defined to be

$$m_{\sigma_1+\sigma_2} = \begin{cases} 0 & \text{if } \dim(\sigma_1+\sigma_2) \neq d_1 + d_2 \\ m_{\sigma_1} m_{\sigma_2} [N_{\sigma_1+\sigma_2} : N_{\sigma_1} + N_{\sigma_2}] & \text{if } \dim(\sigma_1+\sigma_2) = d_1 + d_2 \end{cases} \tag{3.5}$$

where N_σ denotes, for a cone $\sigma \subset \mathbb{R}^{n+1}/\mathbb{R}$, the lattice generated by the integer points of σ.

The Minkowski sum $\Sigma_1 + \Sigma_2$ of two polyhedral complexes $\Sigma_1,\ \Sigma_2 \subset \mathbb{R}^{n+1}/\mathbb{R}$ of pure dimension is defined as a set to be the Minkowsi sum of the underlying sets of Σ_1 and Σ_2. The set $\Sigma_1 + \Sigma_2$ is actually a polyhedral complex, and we can give it a polyhedral complex structure so that, for any $\sigma_1 \in \Sigma_1$ and $\sigma_2 \in \Sigma_2$, the polyhedron $\sigma_1 + \sigma_2$ is a union of polyhedra of $\Sigma_1 + \Sigma_2$. If Σ_1 and Σ_2 are weighted, then we define the multiplicity of a polyhedron $\sigma \in \Sigma_1 + \Sigma_2$ to be

$$m_\sigma = \sum_{\sigma_1+\sigma_2 \supset \sigma} m_{\sigma_1+\sigma_2}.$$

Equivalently $\Sigma_1 + \Sigma_2$ is the image of $\Sigma_1 \times \Sigma_2 \subset \mathbb{R}^{n+1}/\mathbb{R} \times \mathbb{R}^{n+1}/\mathbb{R}$ under the map

$$\begin{array}{rccc} \alpha : & \mathbb{R}^{n+1}/\mathbb{R} \times \mathbb{R}^{n+1}/\mathbb{R} & \longrightarrow & \mathbb{R}^{n+1}/\mathbb{R} \\ & (a,b) & \longmapsto & a+b, \end{array}$$

where the polyhedron $\sigma_1 \times \sigma_2 \in \Sigma_1 \times \Sigma_2$ has weight $m_{\sigma_1} m_{\sigma_2}$.

The Minkowski sum of tropical varieties is the tropical analogue of the Hadamard product of algebraic varieties in the following sense. Given two varieties $X, Y \subset \mathrm{T}^n$, if the map $X \times Y \to X \star Y$ is generically one-to-one, then we have (see [29, Theorem 5.5.11]):

$$\mathrm{Trop}(X \star Y) = \mathrm{Trop}(X) + \mathrm{Trop}(Y). \tag{3.6}$$

The Plücker embedding realizes the Grassmannian $\mathrm{Gr}(k,n)$ as a subvariety of the projective space $\mathbb{P}^{N-1}$. This allows us to define the tropical Grassmannian $\mathrm{TropGr}(k,n)$ as the tropicalization $\mathrm{Trop}(\mathrm{Gr}^\circ(k,n))$ of the intersection of $\mathrm{Gr}(k,n)$ with the embedded torus $\mathrm{T}^{N-1} \subset \mathbb{P}^{N-1}$. The tropical Grassmannian is a parameter space for tropical varieties $\mathrm{Trop}(L)$ where L is a linear space whose Plücker coordinates are all different from 0. This condition comes from the fact that we are considering the tropicalization of the intersection $\mathrm{Gr}^{\circ}(k,n)$ of $\mathrm{Gr}(k,n)$ with the torus T^{N-1}. For the rest of the chapter we refer to those tropical varieties as tropicalized linear spaces.

Given a point $p \in \mathrm{TrGr}(k,n)$, we denote by $\Lambda_p \subset \mathbb{R}^{n+1}/\mathbb{R}$ the tropicalized linear space corresponding to it.

Consider a Γ_{val}-rational point $p \in \mathbb{R}^{n+1}/\mathbb{R}$ and the tropicalized linear space $\Lambda_q \subset \mathbb{R}^{n+1}/\mathbb{R}$ corresponding to the point $q \in \mathrm{TrGr}(k,n) \subset \mathbb{R}^N/\mathbb{R}$. We pick a point x with valuation $\mathrm{val}(x) = p$ and a linear space L with $\mathrm{Trop}(L) = \Lambda_q$. Equation (3.6) implies that the translation $\{p\} + \Lambda_q$ of Λ_q by p is the tropicalization of the Hadamard product $\{x\} \star L$.

Definition 3.1.3. Given a Γ_{val}-rational point $p \in \mathbb{R}^{n+1}/\mathbb{R}$ and $x \in \mathrm{T}^n$ with valuation $\mathrm{val}(x) = p$, we define $\Gamma_p := \mathrm{Trop}(\mathscr{G}_x)$.

Lemma 3.1.4.

1. *The tropical variety Γ_p does not depend on the choice of the point x.*

2. *Given two Γ_{val}-rational points $p, q \in \mathbb{R}^{n+1}/\mathbb{R}$ we have $\Gamma_p = \varphi(q-p) + \Gamma_q$.*

3. *Given a tropicalized linear space $\Lambda \subset \mathbb{R}^{n+1}/\mathbb{R}$ and $x \in \mathrm{T}^n$, we have $\mathrm{val}(x) \in \Lambda$ if and only if there exists a linear space $L \subset \mathbb{P}^n$ such that $\Lambda = \mathrm{Trop}(L)$ and $x \in L$.*

4. *We have the following equality of subsets of $\mathbb{R}^{n+1}/\mathbb{R}$*

$$\Gamma_p = \{q \in \mathrm{TrGr}(k,n) \mid p \in \Lambda_q\}.$$

Proof. Given $x, y \in \mathrm{T}^n$, we have seen in the proof of Theorem 3.1.1 that $\mathscr{G}_x = \psi(x \star y^{-1}) \star \mathscr{G}_y$ where $y^{-1} = [y_0^{-1} : \ldots : y_n^{-1}]$.

If $\mathrm{val}(x) = p$ and $\mathrm{val}(y) = q$ then $\mathrm{val}(x \star y^{-1}) = p - q$, and so, by Equation (3.6), the equality $\mathscr{G}_x = \psi(x \star y^{-1}) \star \mathscr{G}_y$ tropicalizes to $\Gamma_p = \varphi(p-q) + \Gamma_q$. This proves (2) and, as a particular case when $p = q$, (1).

We now prove (3). One implication is trivial: if $\Lambda = \mathrm{Trop}(L)$ and $x \in L$ then $\mathrm{val}(x) \in \Lambda$. On the other hand if $\Lambda = \mathrm{Trop}(L)$ and $\mathrm{val}(x) \in \Lambda$ then, by the Fundamental Theorem of Tropical Geometry ([29, Theorem 3.2.5]), there exists $y \in L$ with $\mathrm{val}(y) = \mathrm{val}(x)$. We have that $L' = (x \star y^{-1}) \star L$ is again a linear space and $x \in L'$. Moreover $\mathrm{val}(x \star y^{-1}) = (0, \ldots, 0)$ and therefore $\mathrm{Trop}(L') = 0 + \mathrm{Trop}(L) = \Lambda$.

To conclude, (4) is a consequence of (3) as, for any fixed x with $\mathrm{val}(x) = p$, we have

$$\Gamma_p = \mathrm{Trop}(\mathscr{G}_x) = \{q \in \mathrm{TrGr}(k,n) \ \mid \Lambda_q = \mathrm{Trop}(L_z) \text{ and } z \in \mathscr{G}_x\}.$$

□

Remark 3.1.5. Given $p \in \mathbb{R}^{n+1}/\mathbb{R}$ not necessarily Γ_{val}-rational, we can define Γ_p to be the weighted polyhedral complex $\varphi(p) + \Gamma_0$. This is consistent with Definition 3.1.3.

One consequence of Lemma 3.1.4, and in particular of point (3), is that the support set of $\mathrm{Trop}(Z_X)$ can be described in terms of $\mathrm{Trop}(X)$. Take a Γ_{val}-rational point $p \in \mathrm{Trop}(X)$ and suppose that a tropicalized linear space Λ intersects $\mathrm{Trop}(X)$ at p. Then there is a point $x \in X$ with valuation $\mathrm{val}(x) = p$. Using Lemma 3.1.4 we see that there exists a linear space $L \subset \mathbb{P}^n$, with $\Lambda = \mathrm{Trop}(L)$, that contains x. This shows the following equality of sets:

$$|\mathrm{Trop}(Z_X)| = \{p \in \mathrm{TrGr}(k,n) \mid \Lambda_p \cap \mathrm{Trop}(X) \neq \varnothing\}. \tag{3.7}$$

For any subset $S \subset \mathbb{R}^n/\mathbb{R}$, a tropicalized linear space Λ_p is intersects S if and only if $p \in \Gamma_s$ for some $s \in S$. By Lemma 3.1.4, $\Gamma_s = \varphi(s) + \Gamma_0$, and we get another equality of sets:

$$\{p \in \mathrm{TrGr}(k,n) \mid \Lambda_p \cap S \neq \varnothing\} = \varphi(S) + |\Gamma_0|. \tag{3.8}$$

Combining Equation (3.7) and Equation (3.8) we get that

$$|\mathrm{Trop}(Z_X)| = |\varphi(\mathrm{Trop}(X))| + |\Gamma_0|.$$

This is actually not just an equality of sets, but an equality of tropical varieties as the following Theorem shows.

Theorem 3.1.6. *Let X be of pure dimension $n-k-1$ and assume none of its irreducible components is contained in $\mathrm{V}(x_1 \cdot \ldots \cdot x_n)$. We have the following equality of tropical varieties:*

$$\mathrm{Trop}(Z_X) = \varphi(\mathrm{Trop}(X)) + \Gamma_0.$$

Proof. Let $u = [1 : \ldots : 1] \in \mathrm{T}^n$ and let ψ be the map defined by Equation (3.1).

As ψ is a monomial morphism we have that, see [29, Corollary 2.6.10], $\mathrm{Trop}(\psi(X^\circ))$ equals $\mathrm{Trop}(\psi)$ $(\mathrm{Trop}(X)) \subset \mathbb{R}^N/\mathbb{R}$ where $\mathrm{Trop}(\psi)$ is the linear map

$\mathrm{Trop}(\psi) : \mathbb{R}^{n+1}/\mathbb{R} \to \mathbb{R}^N/\mathbb{R}$ given by multiplication by the matrix $A = (a)_{i,I}$ with

$$(a)_{i,I} = \begin{cases} 0 & \text{if } x \notin I \\ 1 & \text{if } x \in I \end{cases}$$

We have $\mathrm{Trop}(\psi) = \varphi$ and hence $\mathrm{Trop}(\psi)(\mathrm{Trop}(X)) = \varphi(\mathrm{Trop}(X))$.

The result follows immediately from Lemma 3.1.1, Remark 3.1.2 and Equation (3.6). □

Theorem 3.1.6 allow us to define a notion of associated hypersurface for *any* tropical variety of pure dimension. We recall that by tropical variety we mean balanced weighted Γ_{val}-rational polyhedral complex, in the sense of [29, Theorem 3.3.5].

Definition 3.1.7. Given a tropical variety $\Sigma \subset \mathbb{R}^{n+1}/\mathbb{R}$, the tropical Chow hypersurface $Z_\Sigma \subset \mathbb{R}^N/\mathbb{R}$ associated to Σ is defined to be

$$Z_\Sigma = \varphi(\Sigma) + \Gamma_0.$$

Let $V \subset \mathbb{R}^{n+1}/\mathbb{R}$ be the support of a pure-dimensional fan. We denote by $\mathrm{A}^k_{\mathrm{unbal}}(V)$ the set of pure codimension-k $\mathbb{Q}$-weighted Γ_{val}-rational polyhedral complexes whose support is contained in V, and by $\mathrm{A}^k(V)$ the set of pure codimension-k balanced $\mathbb{Q}$-weighted Γ_{val}-rational polyhedral complexes whose support is contained in V. The support set of a weighted polyhedral complex is the union of all maximal polyhedra that have non-zero multiplicity. As usual in tropical intersection theory, two weighted polyhedral complexes (Σ_1, m_1) and (Σ_2, m_2) are identified if their polyhedral complexes Σ_1 and Σ_2 have the same support set, and if, moreover, given two maximal dimensional polyhedra $\sigma_1 \in \Sigma_1$ and $\sigma_2 \in \Sigma_2$ whose relative interiors intersect, their multiplicity $m_1(\sigma_1)$ and $m_2(\sigma_2)$ are the same. In other words we identify weighted polyhedral complexes that are the same up to the choice of polyhedral structure. This also means that we can freely add and remove polyhedra with multiplicity 0 without changing the polyhedral complex, and that a polyhedral complex is 0 if and only if all its polyhedra have multiplicity 0.

The set $\mathrm{A}^k_{\mathrm{unbal}}(V)$ is a vector space over $\mathbb{Q}$ (this is the main reason to consider rational weights rather than integer). The sum (Σ, m) of two weighted polyhedral complex (Σ_1, m_1) and (Σ_2, m_2) is defined as follows. The support set of Σ is $\Sigma_1 \cup \Sigma_2$. Up to subdividing Σ_1 and Σ_2 it can be given a polyhedral complex structure such that every polyhedron $\sigma \in \Sigma$ is a polyhedron of Σ_1 or Σ_2 (or both). The multiplicity $m(\sigma)$ of a polyhedron $\sigma \in \Sigma$ is defined to be $m_1(\sigma) + m_2(\sigma)$, where $m_1(\sigma)$ (resp.

$m_2(\sigma)$) is defined to be 0 if σ is not a polyhedron of Σ_1 (resp. Σ_2). For $x \in \mathbb{Q}$, the multiplication of (Σ, m) by k is the weighted polyhedral complex $(\Sigma, k \cdot m)$, where $k \cdot m$ is the weight defined by $(k \cdot m)(\sigma) = k \cdot m(\sigma)$ for any maximal polyhedron $\sigma \in \Sigma$.

As every polyhedral complex is a finite union of polyhedra we have that, using the operations just defined on $\mathrm{A}^k_{\mathrm{unbal}}(V)$, any weighted polyhedral complex can be written as sum of weighted polyhedral complex whose support is a single polyhedron. Moreover, as the scalar multiplication acts as multiplication on the weight, the set of polyhedral complexes made of a single codimension k polyhedron, with multiplicity one is a set of generators for $\mathrm{A}^k_{\mathrm{unbal}}(V)$.

We also have a vector space structure on $\mathrm{A}^k(V)$, inherited from the structure on $\mathrm{A}^k_{\mathrm{unbal}}(V)$. Actually $\mathrm{A}^k(V)$ is a vector subspace of $\mathrm{A}^k_{\mathrm{unbal}}(V)$.

We define the map

$$\begin{array}{cccc} \mathrm{Z}: & \mathrm{A}^k(\mathbb{R}^{n+1}/\mathbb{R}) & \longrightarrow & \mathrm{A}^1(\mathrm{TrGr}(k,n)) \\ & \Sigma & \longmapsto & Z_\Sigma = \varphi(\Sigma) + \Gamma_0. \end{array} \tag{3.9}$$

We now prove that this is a linear transformation by showing that its extension $\mathrm{Z}' : \mathrm{A}^k_{\mathrm{unbal}}(\mathbb{R}^{n+1}/\mathbb{R}) \to \mathrm{A}^1_{\mathrm{unbal}}(\mathrm{TrGr}(k,n))$ is linear. The linearity of Z' on a fan that consists of a single cone follows immediately from the linearity in m_1 of Equation (3.5). As single polyhedra span $\mathrm{A}^k_{\mathrm{unbal}}(\mathbb{R}^{n+1}/\mathbb{R})$, Z' is linear and then so is Z.

3.2 From Tropical Chow Hypersurface to Tropical Variety

In this section we address the question whether it is possible to recover the tropical variety Σ from the tropical Chow hypersurface Z_Σ or, in other words whether the map (3.9) is injective.

The hypersurface Z_X is the vanishing locus in the Grassmannian of a single polynomial in the ring of polynomials in the Plücker variables $K[p_I \mid I \in \binom{[n+1]}{k+1}]$. The coordinate ring of the Grassmannian $K[\mathrm{Gr}(k,n)]$ is the quotient of $K[p_I]$ by the Plücker ideal. The class ch_X of this polynomial in $K[\mathrm{Gr}(k,n)]$ is uniquely determined by X, and it is called *Chow form* of X.

Different lifts of $\mathrm{ch}(X)$ to $K[p_I \mid I \in \binom{[n+1]}{k+1}]$ have different Newton polytopes in $\mathbb{R}^N/\mathbb{R}$, so that there is not a natural notion of Newton polytope for the Chow form ch_X. There is, however, a polytope in $\mathrm{P}_{\mathrm{ch}_X} \subset \mathbb{R}^{n+1}/\mathbb{R}$, called *Chow polytope*. This is the weight polytope associated to the natural action of T^n on $K[\mathrm{Gr}(k,n)]$.

Explicitly, given a monomial $\prod p_I^{a_I} \in K[p_I]$, its weight is $\sum a_I e_I \in \mathbb{R}^{n+1}/\mathbb{R}$, where $e_I = \sum_{i\in I} e_i$ and $e_0, \ldots, e_n$ is the image of the standard basis of $\mathbb{R}^{n+1}$ in $\mathbb{R}^{n+1}/\mathbb{R}$. For any lift $\overline{\mathrm{ch}}_X \in K[p_I]$ of the Chow form ch_X we can write $\overline{\mathrm{ch}}_X = c_1 + \ldots + c_l$, where each c_i is a sum of monomials with the same weight p_i. The Chow polytope P_{ch_X} is the convex hull of the weights p_i, for every i such that $[c_i] \neq 0$.

Example 3.2.1. Consider the conic $C = \mathrm{V}(t, x^2 + y^2 + z^2) \subset \mathbb{P}^3$, where x, y, z, t are the coordinates of $\mathbb{P}^3$. Its Chow form can be computed (for example using the algorithm described in [8, Section 3.1]) as the class in $K[\mathrm{Gr}(k,n)]$ of the polynomial $c = p_{03}^2 + p_{13}^2 + p_{23}^2$. The Chow polytope is, in this case, the convex hull of the three weights $2(e_0 + e_3)$, $2(e_1 + e_3)$, $2(e_2 + e_3)$ of the three monomials of c. Consider the polynomial $c' = c + p_{12}p_{03} - p_{02}p_{13} + p_{01}p_{23}$, we have again that the class of c' is the Chow form of C because c and c' only differ by an element of the Plücker ideal. The weight $e_0 + e_1 + e_2 + e_3$ now appears as the weight of some monomial of c'. This is not source of ambiguity in the definition of the Chow polytope: if we sum all the monomials of c' with weight $e_0 + e_1 + e_2 + e_3$ we get the polynomial $p_{12}p_{03} - p_{02}p_{13} + p_{01}p_{23}$ whose class is 0 modulo the Plücker ideal.

Remark 3.2.2. Let $\overline{\mathrm{ch}}_X$ be a lift of ch_X, and write $\overline{\mathrm{ch}}_X = c_1 + \ldots c_l$, with each c_i being a sum of monomials with the same weight. We can always get another lift

$$\overline{\mathrm{ch}}_X - \sum_{i | [c_i] = 0} c_i,$$

where $[c_i]$ denotes the class of c_i modulo the Plücker ideal, such that the Chow Polytope P_X is the convex hull of the weights of its monomials. In other words, P_X is the projection of the Newton polygon of this lift under the linear map that sends the vector $e_I \in \mathbb{R}^N/\mathbb{R}$ to $\sum_{i\in I} e_i \in \mathbb{R}^{n+1}/\mathbb{R}$.

The codimension-one skeleton $\mathcal{N}^1(\mathrm{P}_X)$ of the (inner) normal fan of P_X was studied by Fink in [13]. In particular he proved a Minkowski sum decomposition for $\mathcal{N}^1(\mathrm{P}_X)$ which we now recall.

The linear space Λ_0 corresponding to the origin $0 \in \mathrm{TropGr}(k,n)$ is called the *standard* tropical linear k-plane. The rays of Λ_0 are $\mathrm{pos}(e_0), \ldots \mathrm{pos}(e_n)$, and every subset of k rays spans a maximal cone of multiplicity one of Λ_0.

Denote by $-\Lambda_0$ the image of Λ_0 under the map $-\mathrm{id}_{\mathbb{R}^{n+1}/\mathbb{R}} : p \mapsto -p$. Then we have (see [13, Theorem 5.1])

$$\mathcal{N}^1(\mathrm{P}_X) = X + (-\Lambda_0). \tag{3.10}$$

Proposition 3.2.3. *Let $H = \varphi(\mathbb{R}^{n+1}/\mathbb{R})$ be the image of $\mathbb{R}^{n+1}/\mathbb{R}$ in $\mathbb{R}^N/\mathbb{R}$. Then*

1. $\mathrm{Trop}(Z_X) \cap H = \varphi(\mathcal{N}^1(\mathrm{P}_X))$,

2. $(\varphi(X) + \Gamma_0) \cap H = \varphi(X + (-\Lambda_0))$.

In particular Equation (3.10) *can be obtained by intersecting the equality of Theorem 3.1.6 with* H*.*

Proof. Fix a point $z_0 \in \mathrm{Gr}(k,n)$ with $\mathrm{val}(z_0) = 0$. We consider the morphism

$$\begin{aligned} j_{z_0} : \mathrm{T}^n &\to \mathrm{Gr}(k,n) \\ x &\mapsto \psi(x) \star z_0. \end{aligned}$$

We choose a lift $\overline{\mathrm{ch}}_X \in K[p_I \mid I \in \binom{[n+1]}{k+1}]$ of ch_X with the property of Remark 3.2.2. Let A_{z_0} be the subvariety of T^n defined by $A_{z_0} = \{x \in \mathrm{T}^n \mid j_{z_0}(x) \in Z_X\}$. A point $j_{z_0}(x)$ is in Z_X if and only if $\overline{\mathrm{ch}}_X(j_{z_0}(x)) = 0$. Denote by F the polynomial defined by $F(x) \coloneqq \overline{\mathrm{ch}}_X(j_{z_0}(x))$. We have $A_{z_0} = \mathrm{V}(F) \subset \mathrm{T}^n$. We claim that $\mathrm{Newt}(F) = \mathrm{P}_X$, so that $\mathrm{Trop}(A_{z_0}) = \mathcal{N}^1(\mathrm{P}_X)$. Indeed, the monomials of F are obtained, up to coefficient, from the monomials of $\overline{\mathrm{ch}}_X$ by the ring homomorphism defined by $p_I \mapsto \prod_{i \in I} x_i$. In particular the degree of a monomial of F equals the weight of the corresponding monomial of $\overline{\mathrm{ch}}_X$. The claim follows from Remark 3.2.2.

We can now prove 1 by showing that the two sets have the same Γ_{val}-rational points. Let $v \in \mathcal{N}^1(P_X)$ be a Γ_{val}-rational point and fix any point $z_0 \in \mathrm{Gr}(k,n)$ with $\mathrm{val}(z_0) = 0$. Then there exists some $x \in A_{z_0}$ with $\mathrm{val}(x) = v$, in particular $j_{z_0}(x) \in Z_X$ and, as $\mathrm{val}(j_{z_0}(x)) = \mathrm{val}(\psi(x) \star z_0) = \varphi(v) + 0 = \varphi(v)$, we have $\varphi(v) \in \mathrm{Trop}(Z_X)$. Conversely given a Γ_{val}-rational point $\varphi(v) \in \mathrm{Trop}(Z_X) \cap H$ we can find $z \in Z_X$ with $\mathrm{val}(z) = \varphi(v)$. Let $x \in \mathrm{T}^n$ be any point with valuation v, and let $z_0 = \psi(x)^{-1} \star z$, where by $\psi(x)^{-1}$ we mean the point in $\mathbb{P}^{N-1}$ whose i-th coordinate is the inverse of the i-th coordinate of $\psi(x)$. Then $x \in A_{z_0}$ as $j_{z_0}(x) = z \in Z_X$, so that $v = \mathrm{val}(x) \in \mathcal{N}^1(\mathrm{P}_X)$. This concludes the proof of 1.

A point $\varphi(v) \in \varphi(\mathbb{R}^{n+1}/\mathbb{R})$ lies in Γ_0 if and only if the linear space $\Lambda_{\varphi(v)} = v + \Lambda_0$ contains the origin, and this happens if and only if $-v \in \Lambda_0$. This implies that $\varphi(-\Lambda_0) = \Gamma_0 \cap H$. On the other hand $\varphi(X)$ is already contained in H. Therefore $(\varphi(X) + \Gamma_0) \cap H = \varphi(X) + (\Gamma_0 \cap H) = \varphi(X) + \varphi(-\Lambda_0) = \varphi(X + (-\Lambda_0))$. This completes the proof of 2. □

The tropical variety $\mathrm{Trop}(X)$ is contained in $\mathcal{N}^1(\mathrm{P}_X)$. Unfortunately, it is impossible to recover $\mathrm{Trop}(X)$ from $\mathcal{N}^1(\mathrm{P}_X)$. In [13] Fink gave an example of two distinct tropical surfaces Σ_1, Σ_2 in $\mathbb{R}^5/\mathbb{R}$, such that $\Sigma_1 + (-\Lambda_0) = \Sigma_2 + (-\Lambda_0)$. They are depicted in Figure 3.1. Computing $Z_{\Sigma_1} = \varphi(\Sigma_1) + \Gamma_0$ and $Z_{\Sigma_2} =$

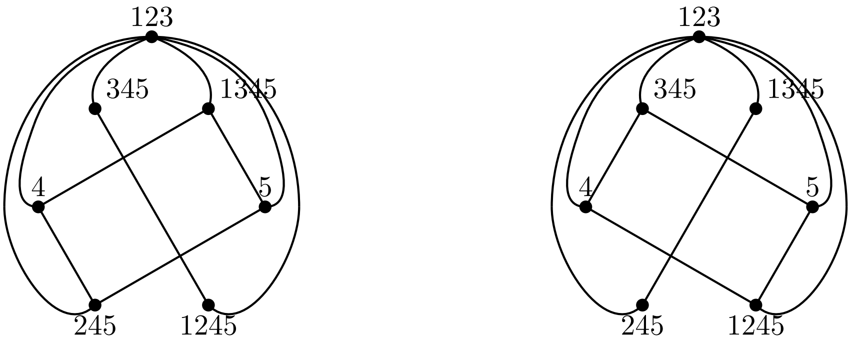

Figure 3.1: The tropical varieties of Σ_1 and Σ_2 described in [13]. A point labeled by $i_1 \dots i_k$ represents a ray generated by $e_{i_1} + \dots + e_{i_k}$. Edges represent 2-dimensional cones.

$\varphi(\Sigma_2) + \Gamma_0$, for example with the package `Polyhedra` in `Macaulay2` ([16]), we see that $Z_{\Sigma_1} \neq Z_{\Sigma_2}$. For example, if we give where $\mathbb{R}^{10}/\mathbb{R}$ homogeneous coordinates $(p_{01}, p_{02}, p_{12}, p_{03}, p_{13}, p_{23}, p_{04}, p_{14}, p_{24}, p_{34})$, for

$$p_1 = (14, 6, 8, 11, 13, 20, 18, 16, 8, 13),\ \ p_2 = (17, 12, 11, 14, 13, 23, 24, 19, 14, 16),$$

we have that $p_1 \in Z_{\Sigma_1}$ but $p_1 \notin Z_{\Sigma_2}$, and $p_2 \in Z_{\Sigma_1}$ but $p_2 \notin Z_{\Sigma_2}$. Equivalently, consider the tropical line Λ_1 depicted in Figure 3.2. Let $L \subset \mathbb{P}^4$ be an algebraic line such that $\Lambda_1 = \mathrm{Trop}(L)$ and consider the points $u = (10, 8, 0, 5, 13)$ and $v = (6, 8, 15, 21, 8)$ in Λ_1. By the Fundamental Theorem of Tropical Geometry ([29, Theorem 3.2.5]), there exist $x, y \in L$ such that $\mathrm{val}(x) = u, \mathrm{val}(y) = v$. As L is spanned by x, y, the Plücker coordinates of L are $q_{ij} = x_i y_j - x_j y_i$, for $0 \le j < i \le 4$. The valuation $\mathrm{val}(q_{ij})$ of q_{ij} satisfies $\mathrm{val}(q_{ij}) = \min\{\mathrm{val}(x_i y_j), \mathrm{val}(x_j y_i)\} = \min\{u_i + v_j, u_j + v_i\}$ because we have $u_i + v_j \neq u_j + v_i$ for all $0 \le j < i \le 4$. This allows us to compute that Λ_1 is the tropical line corresponding to the point p_1. Similarly, the tropical line $\Lambda_2 = \{(3, 0, 3, 0, 3)\} + \Lambda_1$ that is the translation of Λ_1 by $(3, 0, 3, 0, 3)$ corresponds to the point $p_2 = p_1 + \varphi(3, 0, 3, 0, 3) = p_1 + (3, 6, 3, 3, 0, 3, 6, 3, 6, 3)$. One can check that Λ_1 intersects Σ_1 at the point $(6, 8, 6, 11, 8)$ but it does not intersect Σ_2, while Λ_2 intersects Σ_2 at the point $(9, 8, 9, 11, 11)$ but it does not intersect Σ_1. This fact suggests the following conjecture.

Conjecture 3.2.4. *Let $\Sigma_1, \Sigma_2 \subset \mathbb{R}^{n+1}/\mathbb{R}$ be pure $(n - k - 1)$-dimensional tropical varieties. Suppose $Z_{\Sigma_1} = Z_{\Sigma_2}$ then $\Sigma_1 = \Sigma_2$.*

The conjecture is false if we state it in term of equalities of sets rather than tropical varieties. The following example shows two different tropical varieties Σ_1

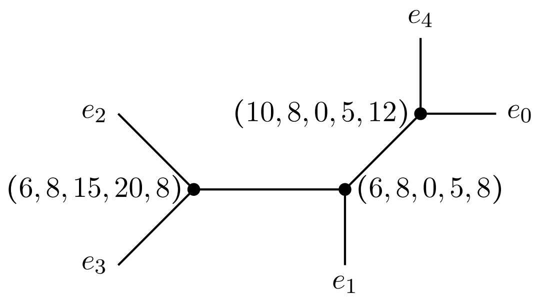

Figure 3.2: The tropical line Λ_1 corresponding to the point $p_1 = (14,6,8,11,13,20,18,16,8,13) \in \mathrm{TrGr}(1,4)$.

and Σ_2 with different support sets $|\Sigma_1| \neq |\Sigma_2|$, such that Z_{Σ_1} and Z_{Σ_2} have the same support set and only differ in the multiplicities.

Example 3.2.5. Let Σ_1 be the tropical curve in $\mathbb{R}^4/\mathbb{R}$ with rays

$$\begin{array}{lll}\{\rho_1 = \mathrm{pos}(1,-1,-1,1), & \rho_2 = \mathrm{pos}(1,-1,1,-1), & \rho_3 = \mathrm{pos}(-1,-1,1,1),\\ \rho_4 = \mathrm{pos}(-1,1,0,0), & \rho_5 = \mathrm{pos}(0,1,-1,0), & \rho_6 = \mathrm{pos}(0,1,0,-1),\\ \rho_7 = \mathrm{pos}(0,-1,0,0), & \rho_8 = \mathrm{pos}(1,1,0,0), & \rho_9 = \mathrm{pos}(0,1,1,1)\},\end{array}$$

and multiplicities

$$\{m_1 = 1, m_2 = 1, m_3 = 1, m_4 = 1, m_5 = 1, m_6 = 1, m_7 = 1, m_8 = 1, m_9 = 1\}.$$

Let Σ_2 be the tropical curve with rays

$$\{\rho_1, \rho_2, \rho_3, \rho_4, \rho_5, \rho_6, \rho_6, \rho_7, \rho_8, \rho_9, \rho_{10} = \mathrm{pos}(0,1,0,0)\},$$

and multiplicities

$$\{m_1 = 1, m_2 = 1, m_3 = 1, m_4 = 1, m_5 = 1, m_6 = 1, m_7 = 2, m_8 = 1, m_9 = 1, m_{10} = 1\}.$$

Then the support sets of $Z_{\Sigma_1} = \varphi(\Sigma_1) + \Gamma_0$ and $Z_{\Sigma_2} = \varphi(\Sigma_2) + \Gamma_0$ are equal, despite Σ_1 and Σ_2 have different support.

Remark 3.2.6. The difference in the support of Σ_1 and Σ_2 is in the ray $\mathrm{pos}(0,1,0,0)$. This is not a coincidence and we will show in Remark 3.2.10 that, in the case of curves in $\mathbb{R}^4/\mathbb{R}$, the support of Z_Σ determines the support of Σ outside $\Lambda_0 \cup (-\Lambda_0)$.

In classical Algebraic Geometry the variety X can be deduced from Z_X as a set via the equality

$$X = \{x \in \mathbb{P}^n \mid \mathscr{G}_x \subset Z_X\}.$$

Example 3.2.5 also shows that this does not happen in Tropical Geometry. Set $p = (0,1,0,0)$. We have that $p \in \Sigma_2$, so that $\Gamma_p \subset |Z_{\Sigma_2}| = |Z_{\Sigma_1}|$. However $p \notin \Sigma_1$.

3.2.1 Curves in space

In the last part of this section we prove Conjecture 3.2.4 for space curves whose support is a fan. This property is not uncommon: if K is a field extension of a field with trivial valuation $\mathbf{k}$, then all the tropicalizations $\mathrm{Trop}(X)$ of varieties X defined over $\mathbf{k}$ are supported on a fan. This is the case, for example, of varieties defined over $\mathbb{C}$, when we consider the field $K = \mathbb{C}\{\{t\}\}$.

We denote by e_0, e_1, e_2, e_3 the images of the vectors of the standard basis of $\mathbb{R}^4$ in $\mathbb{R}^4/\mathbb{R}$. The Grassmannian $\mathrm{TrGr}(1,3)$ is embedded in $\mathbb{R}^6/\mathbb{R}$. We denote by $e_{01}, e_{02}, e_{12}, e_{03}, e_{13}, e_{23}$ the images of the standard basis of $\mathbb{R}^6$, and finally we write $f_i = \varphi(e_i) = \sum_j e_{i,j}$ for $i = 0,1,2,3$. We denote by Λ_0 the tropical standard line in $\mathbb{R}^4/\mathbb{R}$. This is the one-dimensional fan with rays

$$\{\mathrm{pos}(e_0), \mathrm{pos}(e_1), \mathrm{pos}(e_2), \mathrm{pos}(e_3)\}.$$

We denote by Λ the set $\Lambda = \Lambda_0 \cup (-\Lambda_0)$.

The tropical variety $\Gamma_0 \subset \mathrm{TrGr}(1,3)$ is depicted in Figure 3.2.1. It can be computed, for example with `gfan` ([21]) as the tropicalization of the variety $\mathscr{G}_u \subset \mathrm{Gr}(1,3)$ of lines through the origin:

$$\mathscr{G}_u = \mathrm{V}(p_{03}p_{12} - p_{02}p_{13} + p_{01}p_{23}, p_{12} - p_{13} + p_{23}, p_{02} - p_{03} + p_{23}, p_{01} - p_{03} + p_{13}, \\ p_{01} - p_{02} + p_{12}).$$

Lemma 3.2.7. *Let H_{ij} be the plane in $\mathbb{R}^4/\mathbb{R}$ generated by e_i and e_j, $0 \le i < j \le 3$. Then*

$$\Lambda = (H_{01} \cup H_{23}) \cap (H_{02} \cup H_{13}) \cap (H_{03} \cup H_{12}).$$

Proof. We have $H_{ij} \cap H_{kl} = \mathrm{span}(e_i + e_j) = \mathrm{span}(e_k + e_l)$ if $\{i,j\}$ and $\{k,l\}$ are disjoint, and $H_{ij} \cap H_{ik} = \mathrm{span}\, e_i$ otherwise, moreover any two such lines only intersect in the origin. If we expand the right hand side we obtain

$$(0) \cup \mathrm{span}(e_0) \cup \mathrm{span}(e_1) \cup \mathrm{span}(e_2) \cup \mathrm{span}(e_3) = \Lambda.$$

□

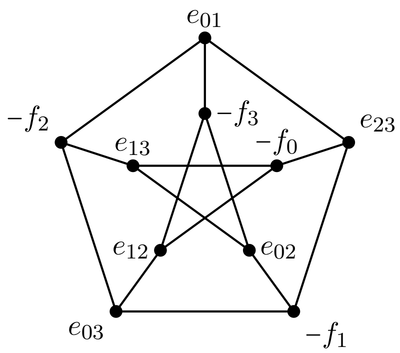

Figure 3.3: The tropical variety Γ_0 in $\mathbb{R}^6/\mathbb{R}$. Points in the picture represent rays in Γ_0 and edges in the picture represent 2-dimensional cones in Γ_0. Each point is labeled with a generator of the corresponding ray. All maximal cones have multiplicity one.

As in the last part of Section 3.1, we denote by $\mathrm{A}^0(\Lambda)$ the set of one-dimensional balanced $\mathbb{Q}$-weighted Γ_{val}-rational polyhedral complexes in Λ and by $\mathrm{A}^1(\mathrm{TrGr}(k,n))$ the set of pure codimension-one balanced $\mathbb{Q}$-weighted Γ_{val}-rational polyhedral complexes in $\mathrm{TrGr}(1,3)$. These are vector spaces over $\mathbb{Q}$, and we have a linear map

$$\begin{array}{rccc} \mathrm{Z}: & \mathrm{A}^0(\Lambda) & \longrightarrow & \mathrm{A}^1(\mathrm{TrGr}(1,3)) \\ & \Sigma & \longmapsto & Z_\Sigma = \varphi(\Sigma) + \Gamma_0. \end{array} \tag{3.11}$$

Lemma 3.2.8. *The linear map* (3.11) *is injective.*

Proof. The space $\hat{\mathrm{A}}^0_{\mathrm{unbal}}(\Lambda)$ of (possibly not balanced) Γ_{val}-rational fans in Λ of dimension one is an eight-dimensional vector space, whose elements are given by a choice of weights $m(\rho)$ on each of the eight rays ρ of Λ. As every balanced polyhedral complex contained in Λ is a fan, $\hat{\mathrm{A}}^0_{\mathrm{unbal}}(\Lambda)$ contains $\mathrm{A}^0(\Lambda)$ as a subspace. We denote the rays of Λ as $\rho_1 = \mathrm{pos}(e_0)$, $\rho_2 = \mathrm{pos}(e_1), \dots, \rho_8 = \mathrm{pos}(-e_4)$. A natural basis of $\hat{\mathrm{A}}^0_{\mathrm{unbal}}(\Lambda)$ is given by the eight fans $F_1, \dots, F_8$, where F_i is the fan that has weight one on ρ_i and 0 on every other ray ρ_j, $j \neq i$. An element $\Sigma \in \hat{\mathrm{A}}^0_{\mathrm{unbal}}(\Lambda)$ can be written as $\Sigma = \sum a_i F_i$, where a_i is the multiplicity of the ray ρ_i in Σ. We denote by $\mathrm{A}^1_{\mathrm{unbal}}(\mathrm{TrGr}(1,3))$ the space of (possibly not balanced) $\mathbb{Q}$-weighted Γ_{val}-rational polyhedral complexes of dimension 3 contained in $\mathrm{TrGr}(1,3)$. The vector space $\mathrm{A}^1_{\mathrm{unbal}}(\mathrm{TrGr}(1,3))$ contains $\mathrm{A}^1(\mathrm{TrGr}(1,3))$ as a subspace. We can extend the map (3.11) to the following linear map,

$$\begin{array}{rccc} \mathrm{Z}': & \hat{\mathrm{A}}^0_{\mathrm{unbal}}(\Lambda) & \longrightarrow & \mathrm{A}^1_{\mathrm{unbal}}(\mathrm{TrGr}(1,3)) \\ & \Sigma & \longmapsto & \varphi(\Sigma) + \Gamma_0. \end{array}$$

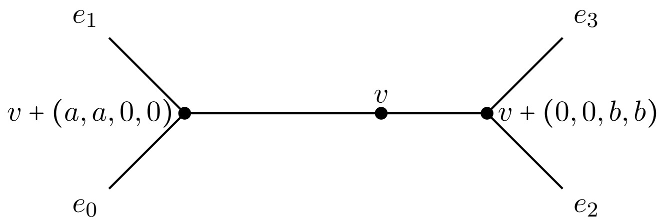

Figure 3.4: The tropical line corresponding to the point $\varphi(v)+ae_{01}+be_{23}$.

We claim that Z' is injective. By computing all the relevant Minkowski sums, we can see that for every fan F_i in the given basis of $\hat{\mathrm{A}}^0_{\mathrm{unbal}}(\Lambda)$ we can find a cone $\sigma_i \in \Gamma_0$ such that $\varphi(\rho_i)+\sigma_i$ is a three-dimensional cone whose interior does not intersect the support of any other fan $\varphi(\rho_j)+\Gamma_0$ for $j \neq i$. For example we can choose

$$\sigma_1=\sigma_2=\sigma_5=\sigma_6=\mathrm{pos}(e_{01},-f_3),\ \sigma_3=\sigma_4=\sigma_7=\sigma_8=\mathrm{pos}(e_{01},-f_2).$$

Now consider an element $\Sigma=\sum a_iF_i \in \hat{\mathrm{A}}^0_{\mathrm{unbal}}(\Lambda)$, and suppose that $\mathrm{Z}'(\Sigma)=0$. For $i=1,\dots 8$, the multiplicity of $\varphi(\rho_i)+\sigma_i$ in $\mathrm{Z}'(\Sigma)$ is given by Formula (3.5) as $a_i[N_{\rho_i+\sigma_i}:N_{\rho_i}+N_{\sigma_i}]$, as every cone in Γ_0 has multiplicity one. As $\mathrm{Z}'(\Sigma)=0$ the multiplicity of $\varphi(\rho_i)+\sigma_i$ in $\mathrm{Z}'(\Sigma)$ must be 0, and this forces $a_i=0$ as desired. □

Theorem 3.2.9. *Let $\Sigma,\Sigma' \in \mathbb{R}^4/\mathbb{R}$ be tropical curves whose support is a fan. Suppose $Z_\Sigma = Z_{\Sigma'}$. Then $\Sigma=\Sigma'$.*

Proof. We may assume that Σ and Σ' have the same set of rays $\{\rho_1,\dots,\rho_m\}$ by allowing multiplicities to be 0. For any i we denote by m_i the multiplicity of ρ_i in Σ and by m_i' the multiplicity of ρ_i in Σ'. We need to prove $m_i=m_i'$ for any i.

Let $\rho_i=\mathrm{pos}(v)$ be a ray of Σ and Σ'. We need to show that ρ_i has the same multiplicity in Σ and Σ'. It will be enough to prove it in the case when $v \notin \Lambda$. Suppose that Σ and Σ' have the same multiplicities on all rays not contained in Λ, and consider the fan $\Sigma-\Sigma'$: this is the balanced weighted fan $\{\rho_1,\dots,\rho_m\}$ where the ray ρ_i has multiplicity m_i-m_i'. All the rays not contained in Λ have multiplicity 0 in $\Sigma-\Sigma'$, so that $\Sigma-\Sigma'$ is an element of $\mathrm{A}^0(\Lambda)$. In particular $\Sigma=\Sigma'$ now follows from Lemma 3.2.8. Thus we can assume $v \notin \Lambda$.

We claim that it will be enough to find a cone $\sigma \in \Gamma_0$ and a full dimensional subcone $\tau \subset \varphi(\rho_i)+\sigma$, such that for any ray $\rho \in \Sigma$ and for any cone $\sigma' \in \Gamma_0$ with $(\rho,\sigma') \neq (\rho_i,\sigma)$, the cone $\varphi(\rho)+\sigma'$ does not intersect τ in full dimension. Given such σ and such τ, we could find a full dimensional subcone τ' of τ that is, for some choice of fan structure, a cone of Z_Σ and, moreover, does not intersect any other

$\varphi(\rho) + \sigma'$. The multiplicity of this τ' in Z_Σ would be computed by Formula (3.5) as $m_i[N_{\rho_i+\sigma_0} : N_{\rho_i} + N_{\sigma_0}]$. As $Z_\Sigma = Z_{\Sigma'}$ this number would equal $m'_i[N_{\rho_i+\sigma_0} : N_{\rho_i} + N_{\sigma_0}]$, giving $m_i = m'_i$ as desired.

We assumed that $v \notin \Lambda$ and Λ equals, by Lemma 3.2.7, $(H_{01} \cup H_{23}) \cap (H_{02} \cup H_{13}) \cap (H_{03} \cup H_{12})$. Without loss of generality we may assume $v \notin H_{01} \cup H_{23}$. Let σ be the cone of Γ_0 with rays generated by e_{01} and e_{23}, and let $\tau_{ab} = \mathrm{pos}(\varphi(v), \varphi(v) + ae_{01}, \varphi(v) + be_{23})$, for $a, b > 0$. The cone τ_{ab} is a full dimensional subcone of $\varphi(\rho_i) + \sigma = \mathrm{pos}(\varphi(v), e_{01}, e_{23})$. We claim that, for some positive α and β, the cones σ and $\tau_{\alpha\beta}$ have the desired property. In other words, we need to prove that, for $\sigma' \in \Gamma_0$, $\rho \in \Sigma$, the dimension of $\tau_{\alpha\beta} \cap (\varphi(\rho) + \sigma')$ is less than $3 = \dim \tau_{\alpha\beta}$, whenever $(\sigma, \rho) \neq (\sigma', \rho_i)$.

We first prove that, for $\sigma' \in \Gamma_0$ with $\sigma \neq \sigma'$, the dimension of $(\varphi(\rho_i) + \sigma) \cap (\varphi(\rho_i) + \sigma')$ is at most two and so, a fortiori, so is the dimension of $\tau_{\alpha\beta} \cap (\varphi(\rho_i) + \sigma')$ for any α and β. A necessary condition for $\dim((\varphi(\rho_i) + \sigma) \cap (\varphi(\rho_i) + \sigma')) = 3$ is that $\mathrm{span}(\varphi(\rho_i) + \sigma) = \mathrm{span}(\varphi(\rho_i) + \sigma')$, which implies that $\mathrm{span}(\varphi(\mathbb{R}^4/\mathbb{R}) + \sigma) = \mathrm{span}(\varphi(\mathbb{R}^4/\mathbb{R}) + \sigma')$. The fan Γ_0 is depicted in Figure 3.3, and we can check that this last condition is only satisfied by four cones: $\sigma_0 = \mathrm{pos}(e_{01}, -f_2)$, $\sigma_1 = \mathrm{pos}(e_{01}, -f_3)$, $\sigma_2 = \mathrm{pos}(e_{23}, -f_0)$ and $\sigma_3 = \mathrm{pos}(e_{23}, -f_1)$. Let us now consider the cone σ_0. It is enough to show that $\mathrm{span}(\varphi(\rho_i) + \sigma_0) \neq \mathrm{span}(\varphi(\rho_i) + \sigma)$, and equivalently we can show that the dimension of $\mathrm{span}(\varphi(\rho_i) + \sigma_0) + \mathrm{span}(\varphi(\rho_i) + \sigma) = \mathrm{span}(\varphi(v), e_{01}, e_{23}, -f_2)$ is at least four. A simple computation in $\mathbb{R}^6$ is now enough. The matrix with rows e_{01}, e_{23}, $\varphi(v_0, v_1, v_2, v_3)$, $-f_2$, $(1,1,1,1,1,1)$ has rank less than five if and only if $v_0 - v_1 = 0$, which is false, as we assumed $v \notin H_{01}$. Similar computations work for the cones σ_1, σ_2, σ_3.

We are left to prove that, for some α and β, $\tau_{\alpha\beta}$ intersects $\varphi(\rho) + \sigma'$ in dimension at most two for any $\rho_i \neq \rho \in \Sigma$ and any $\sigma \in \Gamma_0$. Equivalently we have to show that, for any $\rho_i \neq \rho \in \Sigma$, the fan $\tau_{\alpha\beta} \cap (\varphi(\rho) + \Gamma_0)$ has no cone of dimension more than two.

Let $P_{\alpha\beta}$ be the triangle with vertices $\varphi(v), \varphi(v) + \alpha e_{01}, \varphi(v) + \beta e_{23}$. As $\tau_{\alpha\beta}$ is the cone over $P_{\alpha\beta}$, it will be enough to show that every $\varphi(\rho) + \Gamma_0$ intersects $P_{\alpha\beta}$ in dimension at most one.

By Equation (3.8) the fan $\varphi(\rho) + \Gamma_0$ parametrizes tropical lines intersecting the ray ρ. As a result a point $v + ae_{01} + be_{23} \in P_{\alpha\beta}$ is in $\varphi(\rho) + \Gamma_0$ if and only if the line Λ_{ab} associated to it intersects ρ. We thus need to prove that, for some α and β, the set of points $\varphi(v) + ae_{01} + be_{23}$, with $0 \le a < \alpha$, $0 \le b < \beta$ and such that the line Λ_{ab} intersect a ray ρ of Σ different from ρ_i, has dimension at most one.

The line Λ_{ab} corresponding to a point $\varphi(v) + ae_{01} + be_{23} \in \varphi(\rho_i) + \sigma$ is depicted in Figure 3.4. To see this just consider the two points $p_1 = \varphi(v) + (a+1, a, 0, 0)$ and

$p_2 = \varphi(v) + (0, 0, b+1, b)$ on it, fix a realization L of Λ_{ab} and take two points x_1, x_2 with valuation p_1 and p_2. The coordinate of Λ_{ab} corresponding to the basis element e_{ij} can now be computed as the valuation of $x_i y_j - x_j y_i$. This equals $\min\{(p_1)_i + (p_2)_j, (p_2)_i + (p_1)_j\}$ because $(p_1)_i + (p_2)_j \neq (p_2)_i + (p_1)_j$ for every i, j, and so one can compute that Λ_{ab} corresponds to the point $\varphi(v) + ae_{01} + be_{23}$.

We denote by δ the union of all the tropicalized lines Λ_{ab} for $a, b \geq 0$, δ is the union of two cones pointed at v, $\delta = (v + \mathrm{pos}(e_0, e_1)) \cup (v + \mathrm{pos}(e_2, e_3))$. We have that δ is contained in the union of the affine planes $(v + H_{01}) \cup (v + H_{23})$. As we assumed that $v \notin H_{01} \cup H_{23}$ the two affine planes $v + H_{01}$ and $v + H_{23}$ do not pass through the origin. As a result any ray ρ_j intersects δ in at most two points because the line it spans intersects the two affine planes $v + H_{01}$ and $v + H_{23}$ that contain δ in at most one point each.

This gives us a finite set $S \subset \delta$ made of the intersection points of all the rays $\rho \neq \rho_i$ with δ. A line Λ_{ab} intersects some ray ρ if and only if it contains a point $p \in S$, and we just have to prove that, for each $p \in S$, the set of points $\varphi(v) + ae_{01} + be_{23}$, with $0 \leq a < \alpha$, $0 \leq b < \beta$ and such that the line Λ_{ab} contains p, has dimension at most one.

Let $p \in \delta$ be a point not contained in the line $v + \mathrm{span}(1, 1, 0, 0)$. We can assume, without loss of generality, that $p \in v + H_{01}$ so that $p = v + c_0 e_0 + c_1 e_1$ for some $c_0, c_1 \geq 0$. A tropical line Λ_{ab} contains p if and only if $a = \min\{c_0, c_1\}$. It follows that the set of points $\varphi(v) + ae_{01} + be_{23}$, with $0 \leq a < \alpha$ and $0 \leq b < \beta$ and such that the line Λ_{ab} contains the point p, is the line segment $\{\varphi(v) + \min\{c_0, c_1\} e_{01} + be_{23} \mid 0 \leq b < \beta\}$.

It remains to prove that the points $p \in S$ contained in the line $v + \mathrm{span}(1, 1, 0, 0)$ are only contained in finitely many tropicalized lines Λ_{ab} with $a \leq \alpha$ and $b \leq \beta$ for some α and β. We can finally define α and β: α is the minimum positive t such that there is a point $p \in S$ of the form $p = v - t(1, 1, 0, 0)$, and β the minimum positive t such that there is a point $p \in l$ of the form $p = v + t(1, 1, 0, 0)$. By definition no line Λ_{ab} with $a < \alpha$ and $b < \beta$ contain any point $p \in S$ contained in $v + \mathrm{span}(1, 1, 0, 0)$, and this concludes the proof. $\square$

Remark 3.2.10. The proof of Theorem 3.2.9 also shows that if Z_Σ and $Z_{\Sigma'}$ have the same support set, then the support set of Σ and Σ' is the same outside Λ. This is consistent with Example 3.2.5.

3.3 Tropicalization of Families

We now give an application of tropical Chow hypersurfaces to the study of tropicalization of families of algebraic varieties. In this section we will work with trivial valuation.

Consider a family of projective algebraic curves $\mathcal{X} \subset \mathbb{A}^k \times \mathbb{P}^3$ and denote, for $a \in \mathbb{A}^k$, its fiber by X_a. We will show how to use the theory of tropical Chow hypersurfaces to answer the following question.

Question 1. *What are all the possible tropical varieties* $\mathrm{Trop}(X_a)$ *as* a *varies in* $\mathbb{A}^k$*?*

We can associate to $\mathcal{X}$ a lift of the Chow form $\mathrm{ch}_{\mathcal{X}} \in K[c_1, \dots, c_k, p_I \mid I \in \binom{[4]}{2}]$. This form can be computed, for example, with the algorithm described in [8, 3.1]. It has the property that, for any $a \in \mathbb{A}^k$ such that X_a is a curve, a lift of the Chow form ch_{X_a} can be obtained by substituting the c_i's with the a_i's in $\mathrm{ch}_{\mathcal{X}}$. By a slight abuse of notation we will again denote this lift as ch_{X_a}.

Proposition 3.3.1. *Let* $\mathcal{X} \subset \mathbb{A}^k \times \mathbb{P}^3$ *be a family of projective algebraic curves and let* $a, b \in \mathbb{A}^k$*. Suppose that* $\mathcal{N}^1(\mathrm{Newt}(\mathrm{ch}_{X_a}))$ *and* $\mathcal{N}^1(\mathrm{Newt}(\mathrm{ch}_{X_b}))$ *intersect transversely the tropical Grassmannian* $\mathrm{TrGr}(1,3)$*. Then* $\mathrm{Trop}(X_a) = \mathrm{Trop}(X_b)$ *if and only if* $\mathcal{N}^1(\mathrm{Newt}(\mathrm{ch}_{X_a})) \cap \mathrm{TrGr}(1,3) = \mathcal{N}^1(\mathrm{Newt}(\mathrm{ch}_{X_b})) \cap \mathrm{TrGr}(1,3)$*. In particular if* $\mathrm{Newt}(\mathrm{ch}_{X_a}) = \mathrm{Newt}(\mathrm{ch}_{X_b})$ *then* $\mathrm{Trop}(X_a) = \mathrm{Trop}(X_b)$*.*

Proof. The Chow hypersurface Z_{X_a} is the intersection of $\mathrm{V}(\mathrm{ch}_{X_a})$ with the Grassmannian $\mathrm{Gr}(1,3)$. The tropicalization of $\mathrm{V}(\mathrm{ch}_{X_a})$ is the codimension-one skeleton $\mathcal{N}^1(\mathrm{Newt}(\mathrm{ch}_{X_a}))$ of the normal fan of the Newton polytope of ch_{X_a}. As this intersection is transverse, we have that $\mathrm{Trop}(Z_{X_a}) = \mathrm{TrGr}(1,3) \cap \mathrm{Trop}(\mathrm{V}(\mathrm{ch}_{X_a}))$. The same argument shows that $\mathrm{Trop}(Z_{X_b}) = \mathrm{TrGr}(1,3) \cap \mathrm{Trop}(\mathrm{V}(\mathrm{ch}_{X_b}))$. The statement now follows from Theorem 3.2.9. □

Remark 3.3.2. The assumption on the dimension of the varieties X_a is only due to the same assumption being made in Theorem 3.2.9. A proof of Conjecture 3.2.4 would automatically extend Proposition 3.3.1 to families of projective varieties of arbitrary dimension.

Consider the stratification of $\mathbb{A}^k$ defined by the coefficients of the lift of the Chow form $\mathrm{ch}_{\mathcal{X}} \in K[c_1, \dots, c_k, p_I \mid I \in \binom{[4]}{2}]$. This stratification has as closed strata the vanishing loci of subsets of coefficients of $\mathrm{ch}_{\mathcal{X}}$. Two points a, b in the same stratum have the same Newton polytope $\mathrm{Newt}(\mathrm{ch}_{X_a}) = \mathrm{Newt}(\mathrm{ch}_{X_b})$ and hence, if the transversality condition of Proposition 3.3.1 holds, we also have $\mathrm{Trop}(X_a) = \mathrm{Trop}(X_b)$. The transversality condition holds in many cases, making this argument an useful tool to approach Question 1. This is shown in the following example.

Example 3.3.3. Let $\tilde{I} \subset K[c_1, c_2][x, y, z, t]$ be the ideal generated by the polynomials

$$f = x^2 + y^2 + zt, \;\; g = c_1 z^2 + c_2 zt + xy + t^2.$$

The ideal $\tilde{I}$ defines a family of quartic curves in $\mathbb{P}^3$ parametrized by $\mathbb{A}^2$. The Chow form of $\tilde{I}$ is a polynomial of degree four in the Plücker coordinates with 47 monomials:

$$\begin{aligned}
\mathrm{ch}_{\mathcal{X}} = &-c_1 p_{01}^4 + c_2 p_{01}^2 p_{02}^2 - p_{02}^4 + p_{01}^2 p_{02} p_{12} + c_2 p_{01}^2 p_{1,2}^2 - 2 p_{02}^2 p_{1,2}^2 - p_{1,2}^4 - 4c_1 p_{01}^2 p_{02} p_{03} + 2c_2 p_{02}^3 p_{03} \\
&+ c_1 c_2 p_{01}^2 p_{03}^2 + (-c_2^2 - 2c_1) p_{02}^2 p_{03}^2 + 2c_1 c_2 p_{02} p_{03}^3 - c_1^2 p_{03}^4 + 2 p_{02}^3 p_{13} - 4c_1 p_{01}^2 p_{1,2} p_{13} + 4c_2 p_{02}^2 p_{1,2} p_{13} \\
&+ 2 p_{02} p_{1,2}^2 p_{13} + 2c_2 p_{1,2}^3 p_{13} + c_1 p_{01}^2 p_{03} p_{13} - 2c_2 p_{02}^2 p_{03} p_{13} + 2c_1 p_{02} p_{03}^2 p_{13} + c_1 c_2 p_{01}^2 p_{13}^2 + (-2c_2^2 - 4c_1 - 1) p_{02}^2 p_{13}^2 \\
&- 2c_2 p_{02} p_{1,2} p_{13}^2 + (-c_2^2 - 2c_1) p_{1,2}^2 p_{13}^2 + 4c_1 c_2 p_{02} p_{03} p_{13}^2 - 2c_1^2 p_{03}^2 p_{13}^2 + 2c_1 p_{02} p_{13}^3 + 2c_1 c_2 p_{1,2} p_{13}^3 - c_1^2 p_{13}^4 \\
&- 3 p_{01} p_{02}^2 p_{23} - 2c_2 p_{01} p_{02} p_{1,2} p_{23} + p_{01} p_{1,2}^2 p_{23} + c_2 p_{01} p_{02} p_{03} p_{23} + c_1 p_{01} p_{03}^2 p_{23} + (2c_2^2 + 4c_1 + 1) p_{01} p_{02} p_{13} p_{23} \\
&+ c_2 p_{01} p_{1,2} p_{13} p_{23} - 2c_1 c_2 p_{01} p_{03} p_{13} p_{23} - 3c_1 p_{01} p_{13}^2 p_{23} + (-c_2^2 + 2c_1) p_{01}^2 p_{23}^2 - 4 p_{02} p_{1,2} p_{23}^2 + p_{02} p_{03} p_{23}^2 \\
&+ 4c_2 p_{02} p_{13} p_{23}^2 + p_{1,2} p_{13} p_{23}^2 - 4c_1 p_{03} p_{13} p_{23}^2 - 2c_2 p_{01} p_{23}^3 - p_{23}^4.
\end{aligned}$$

The coefficients of $\mathrm{ch}_{\mathcal{X}}$ are products of the following polynomials:

$$c_1, \;\; c_2, \;\; c_2^2 - 2c_1, \;\; c_2^2 + 2c_1, \;\; 2c_2^2 + 4c_1 + 1.$$

This defines a stratification of $\mathbb{A}^2$ in seven strata: in each stratum the Newton polytope of the Chow form is constant. The closure of these strata are:

$$\begin{gathered}
V_0 = \mathbb{A}^2, V_1 = \mathrm{V}(c_1), \;\; V_2 = \mathrm{V}(c_2), \;\; V_3 = \mathrm{V}(c_2^2 - 2c_1), \;\; V_4 = \mathrm{V}(c_2^2 + 2c_1), \\
V_5 = \mathrm{V}(2c_2^2 + 4c_1 + 1), \;\; V_6 = \mathrm{V}(c_1, c_2), \;\; V_7 = \mathrm{V}(c_1, 2c_2^2 + 4c_1 + 1), \\
V_8 = \mathrm{V}(c_2, 2c_2^2 + 4c_1 + 1), \;\; V_9 = \mathrm{V}(c_2^2 - 2c_1, 2c_2^2 + 4c_1 + 1), \;\; V_{10} = \varnothing.
\end{gathered}$$

For the strata $V_0, V_2, V_3, V_4, V_5, V_7, V_8, V_9, V_{10}$ the corresponding Newton polytope of ch_{X_a} defines a tropical hypersurfaces in $\mathbb{R}^6/\mathbb{R}$ that is transverse to the Grassmannian $\mathrm{TrGr}(1,3)$. As a result, by Proposition 3.3.1, the tropicalization of $\mathrm{V}(\tilde{I})$ is constant within these strata. The tropicalization within the stratum V_1 is constant too, as, for $c_1 = 0$ and $c_2 \neq 0$ the tropical hypersurfaces $\mathrm{Trop}(\mathrm{V}(f))$ and $\mathrm{Trop}(\mathrm{V}(g))$ intersect transversely, and thus determine the tropical curve $\mathrm{Trop}(\mathrm{V}(f,g))$. Finally the tropicalization is trivially constant within the stratum $V_6 = \{(0,0)\}$.

As a result these ten strata correspond to ten (non-empty) potentially different tropical varieties

$$\Sigma_0, \;\; \Sigma_1, \;\; \Sigma_2, \;\; \Sigma_3, \;\; \Sigma_4, \;\; \Sigma_5, \;\; \Sigma_6, \;\; \Sigma_7, \;\; \Sigma_8, \;\; \Sigma_9.$$

The tropical varieties can be computed, for example with `gfan` ([21]), as $\mathrm{Trop}(X_a)$

for arbitrary parameters a in the correct stratum. We have that

$$\Sigma_0 = \{\mathrm{pos}(1,0,0,0),\ \mathrm{pos}(0,1,0,0),\ \mathrm{pos}(0,0,1,0),\ \mathrm{pos}(0,0,0,1)\},$$
$$\text{with multiplicities } \{4,4,4,4\},$$

$$\Sigma_1 = \{\mathrm{pos}(1,0,0,0),\ \mathrm{pos}(0,1,0,0),\ \mathrm{pos}(0,0,-1,1),\ \mathrm{pos}(0,0,1,0)\},$$
$$\text{with multiplicities } \{2,2,2,4\},$$

$$\Sigma_6 = \{\mathrm{pos}(3,-1,-3,1),\ \mathrm{pos}(0,0,1,0),\ \mathrm{pos}(-1,3,-3,1)\},$$
$$\text{with multiplicities } \{1,4,1\}.$$

Moreover

$$\Sigma_0 = \Sigma_2 = \Sigma_3 = \Sigma_4 = \Sigma_5 = \Sigma_8 = \Sigma_9 \text{ and } \ \Sigma_1 = \Sigma_7,$$

so that there are actually only three possible tropical varieties arising as $\mathrm{Trop}(X_a)$ for some $a \in \mathbb{C}^2$. All those identifications, with the exception of $\Sigma_1 = \Sigma_7$, are already visible looking at the Newton polytope of ch_{X_a}. This happens because each strata correspond to a different set of exponents of the specialized polynomial ch_{X_a}, but to the same Newton polytope as they have the same convex hull.

Chapter 4

Secants, Bitangents and their Congruences

The aim of this Chapter is to study subvarieties of Grassmannians which arise naturally from subvarieties of complex projective 3-space $\mathbb{P}^3$. We are mostly interested in threefolds and surfaces in $\mathrm{Gr}(1,\mathbb{P}^3)$. These are classically known as *line complexes* and *congruences*.

In Section 4.1, we collect basic facts about the Grassmannian $\mathrm{Gr}(1,\mathbb{P}^3)$ and its subvarieties. Section 4.2 studies the singular locus of the Chow hypersurface of a space curve and computes the bidegree of its secant congruence. The bidegree of the secant congruence, in the case of a smooth curve, appears in [2]. Section 4.3 describes the singular locus of the Hurwitz hypersurface of a surface and Section 4.4 uses projective duality to calculate the bidegree of its bitangent and inflectional congruences. The computation of the bidegree of bitangent and inflectional congruences already appears in [2] and [31, Prop. 4.1]. Nevertheless, we give new, more geometric, proofs not relying on Chern class techniques. In Section 4.5, we connect the intersection theory in $\mathrm{Gr}(1,\mathbb{P}^3)$ to Chow and Hurwitz hypersurfaces. Finally, Section 4.6 analyzes the singular loci of secant, bitangent, and inflectional congruences; these are partially described in Lemma 2.3, Lemma 4.3, and Lemma 4.6 in [2].

4.1 The Degree of a Subvariety in $\mathrm{Gr}(1,\mathbb{P}^3)$

In this section, we provide the geometric definition for the degree of a subvariety in $\mathrm{Gr}(1,\mathbb{P}^3)$. An alternative approach, using coefficients of classes in the Chow ring, can be found in Section 4.5. For information about subvarieties of more general Grassmannians, we recommend [1].

The Grassmannian $\mathrm{Gr}(1, \mathbb{P}^3)$ of lines in $\mathbb{P}^3$ is a 4-dimensional variety that embeds into $\mathbb{P}^5$ via the Plücker embedding. In particular, the line in 3-space spanned by the distinct points $(x_0 : x_1 : x_2 : x_3), (y_0 : y_1 : y_2 : y_3) \in \mathbb{P}^3$ is identified with the point $(p_{0,1} : p_{0,2} : p_{0,3} : p_{1,2} : p_{1,3} : p_{2,3}) \in \mathbb{P}^5$, where $p_{i,j}$ is the minor formed of ith and jth columns of the matrix $\left[\begin{smallmatrix} x_0 & x_1 & x_2 & x_3 \\ y_0 & y_1 & y_2 & y_3 \end{smallmatrix}\right]$. The Plücker coordinates $p_{i,j}$ satisfy the relation $p_{0,1}p_{2,3} - p_{0,2}p_{1,3} + p_{0,3}p_{1,2} = 0$. Moreover, every point in $\mathbb{P}^5$ satisfying this relation is the Plücker coordinates of some line. Dually, a line in $\mathbb{P}^3$ is the intersection of two distinct planes. If the planes are given by the equations $a_0x_0 + a_1x_1 + a_2x_2 + a_3x_3 = 0$ and $b_0x_0 + b_1x_1 + b_2x_2 + b_3x_3 = 0$, then the minors $q_{i,j}$ of the matrix $\left[\begin{smallmatrix} a_0 & a_1 & a_2 & a_3 \\ b_0 & b_1 & b_2 & b_3 \end{smallmatrix}\right]$ are the dual Plücker coordinates and also satisfy $q_{0,1}q_{2,3} - q_{0,2}q_{1,3} + q_{0,3}q_{1,2} = 0$. The map given by $p_{0,1} \mapsto q_{2,3}$, $p_{0,2} \mapsto -q_{1,3}$, $p_{0,3} \mapsto q_{1,2}$, $p_{1,2} \mapsto q_{0,3}$, $p_{1,3} \mapsto -q_{0,2}$, and $p_{2,3} \mapsto q_{0,1}$ allows one to conveniently pass between these two coordinate systems.

A *line complex* is a threefold $\Sigma \subset \mathrm{Gr}(1, \mathbb{P}^3)$. For a general plane $H \subset \mathbb{P}^3$ and a general point $v \in H$, the degree of Σ is the number of points in Σ corresponding to a line $L \subset \mathbb{P}^3$ such that $v \in L \subset H$. For instance, if $C \subset \mathbb{P}^3$ is a curve, then the Chow hypersurface $Z_C := \{L \in \mathrm{Gr}(1, \mathbb{P}^3) : C \cap L \neq \varnothing\}$ is a line complex. A general plane H intersects C in $\deg(C)$ many points, so there are $\deg(C)$ many lines in H that pass through a general point $v \in H$ and intersect C; see Fig. 4.1. Hence, the

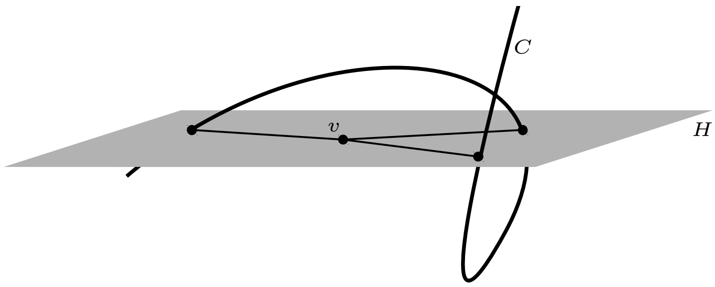

Figure 4.1: The degree of the Chow hypersurface

degree of the Chow hypersurface is equal to the degree of the curve.

A *congruence* is a surface $\Sigma \subset \mathrm{Gr}(1, \mathbb{P}^3)$. For a general point $v \in \mathbb{P}^3$ and a general plane $H \subset \mathbb{P}^3$, the bidegree of a congruence is a pair (α, β), where the *order* α is the number of points in Σ corresponding to a line $L \subset \mathbb{P}^3$ such that $v \in L$ and the *class* β is the number of points in Σ corresponding to lines $L \subset \mathbb{P}^3$ such that $L \subset H$. For instance, consider the congruence of all lines passing through a fixed point x. Given a general point v, this congruence contains a unique line passing through v, namely the line spanned by x and v. Given a general plane H, we have $x \notin H$, so this congruence does not contain any line that lies in H. Hence, the set of lines passing through a fixed point is a congruence with bidegree $(1,0)$. A similar argument shows that the congruence of lines lying in a fixed plan has bidegree $(0,1)$.

The degree of a curve $\Sigma \subset \mathrm{Gr}(1,\mathbb{P}^3)$ is the number of points in Σ corresponding to a line $L \subset \mathbb{P}^3$ that intersects a general line in $\mathbb{P}^3$. Equivalently, it is the number of points in the intersection of Σ with the Chow hypersurface of a general line. For instance, the set of all lines in $\mathbb{P}^3$ that lie in a fixed plane $H \subset \mathbb{P}^3$ and contain a fixed point $v \in H$ forms a curve in $\mathrm{Gr}(1,\mathbb{P}^3)$. This curve has degree 1, because a general line has a unique intersection point with H and there is a unique line passing through this point and v. In other words, this curve is a line in $\mathrm{Gr}(1,\mathbb{P}^3)$.

Finally, the degree of a zero-dimensional subvariety is simply the number of points in the variety.

4.2 Secants of Space Curves

This section describes the singular locus of the Chow hypersurface for a space curve. For a curve with mild singularities, we also compute the bidegree of its secant congruence. This generalizes the computation in [2], which assumes the space curve to be smooth.

A curve $C \subset \mathbb{P}^3$ is defined by at least two homogeneous polynomials in the coordinate ring of $\mathbb{P}^3$, and these polynomials are not uniquely determined. However, there is a single equation that encodes the curve C. Specifically, its Chow hypersurface $Z_C := \{L \in \mathrm{Gr}(1,\mathbb{P}^3) : C \cap L \neq \varnothing\}$ is determined by a single polynomial in the Plücker coordinates on $\mathrm{Gr}(1,\mathbb{P}^3)$. This equation, known as the *Chow form* of C, is unique up to rescaling and the Plücker relation. For more on Chow forms, see [8].

Example 4.2.1 ([8, Prop. 1.2])**.** The twisted cubic is a smooth rational curve of degree 3 in $\mathbb{P}^3$. Parametrically, this curve is the image of the map $\nu_3 \colon \mathbb{P}^1 \to \mathbb{P}^3$ defined by $(s:t) \mapsto (s^3 : s^2t : st^2 : t^3)$. The line L, which is determined by the two equations $a_0x_0 + a_1x_1 + a_2x_2 + a_3x_3 = 0$ and $b_0x_0 + b_1x_1 + b_2x_2 + b_3x_3 = 0$, intersects the twisted cubic if and only if there exists a point $(s:t) \in \mathbb{P}^1$ such that

$$a_0s^3 + a_1s^2t + a_2st^2 + a_3t^3 = 0 = b_0s^3 + b_1s^2t + b_2st^2 + b_3t^3 .$$

The resultant for these two cubic polynomials, which can be expressed as a determinant of an appropriate matrix with entries in $\mathbb{Z}[a_0,a_1,a_2,a_3,b_0,b_1,b_2,b_3]$, vanishes exactly when they have a common root. It follows that the line L meets the twisted cubic if and only if

$$0 = \det \begin{bmatrix} a_0 & a_1 & a_2 & a_3 & 0 & 0 \\ 0 & a_0 & a_1 & a_2 & a_3 & 0 \\ 0 & 0 & a_0 & a_1 & a_2 & a_3 \\ b_0 & b_1 & b_2 & b_3 & 0 & 0 \\ 0 & b_0 & b_1 & b_2 & b_3 & 0 \\ 0 & 0 & b_0 & b_1 & b_2 & b_3 \end{bmatrix} = -\det \begin{bmatrix} q_{0,1} & q_{0,2} & q_{0,3} \\ q_{0,2} & q_{0,3} + q_{1,2} & q_{1,3} \\ q_{0,3} & q_{1,3} & q_{2,3} \end{bmatrix},$$

where $q_{i,j}$ are the dual Plücker coordinates. Hence, the Chow form of the twisted cubic is $q_{0,3}^3 + q_{0,3}^2 q_{1,2} - 2q_{0,2}q_{0,3}q_{1,3} + q_{0,1}q_{1,3}^2 + q_{0,2}^2 q_{2,3} - q_{0,1}q_{0,3}q_{2,3} - q_{0,1}q_{1,2}q_{2,3}$.

We next record a technical lemma. If I_X is the saturated homogeneous ideal defining the subvariety $X \subset \mathbb{P}^n$, then the tangent space $T_x(X)$ at the point $x \in X$ can be identified with $\left\{y \in \mathbb{P}^n : \sum_{i=0}^n \frac{\partial f}{\partial x_i}(x) y_i = 0 \text{ for all } f(x_0, x_1, \dots, x_n) \in I_X\right\}$.

Lemma 4.2.2. *Let $f : X \to Y$ be a birational finite surjective morphism between irreducible projective varieties and let $y \in Y$. The variety Y is smooth at the point y if and only if the fibre $f^{-1}(y)$ contains exactly one point $x \in X$, the variety X is smooth at the point x, and the differential $d_x f : T_x(X) \to T_y(Y)$ is an injection.*

Proof. First, suppose that Y is smooth at the point y. Since Y is normal at the point y, the Zariski Connectedness Theorem [30, Section III.9.V] proves that the fibre $f^{-1}(y)$ is a connected set in the Zariski topology. As f is a finite morphism, its fibres are finite and we deduce that $f^{-1}(y) = \{x\}$. If Y_0 is the open set of smooth points in Y and let $X_0 := f^{-1}(Y_0)$, then Zariski's Main Theorem [30, Section III.9.I] implies that the restriction of f to X_0 is an isomorphism of X_0 with Y_0. In particular, we have that $x \in X_0 \subset X$ is a smooth point. Moreover, Theorem 14.9 in [18] shows that the differential $d_x f$ is injective.

For the other direction, suppose that $f^{-1}(y) = \{x\}$ for some smooth point $x \in X$ with injective differential $d_x f$. Let Y_1 be an open neighbourhood of y containing points in Y with one-element fibres and injective differentials. Combining Lemma 14.8 and Theorem 14.9 in [18] produces an isomorphism of $X_1 := f^{-1}(Y_1)$ with Y_1. Since $x \in X_1$ is smooth, we conclude that $y \in Y_1 \subset Y$ is smooth. □

When the curve C has degree at least two, the set of lines that meet it in two points forms a surface $\mathrm{Sec}(C) \subset \mathrm{Gr}(1, \mathbb{P}^3)$ called the *secant congruence* of C. More precisely, $\mathrm{Sec}(C)$ is the closure in $\mathrm{Gr}(1, \mathbb{P}^3)$ of the set of points corresponding to a line in $\mathbb{P}^3$ which intersects the curve C at two smooth points. A line meeting

C at a singular point might not belong to $\mathrm{Sec}(C)$, even though it has intersection multiplicity at least two with the curve; see Remark 4.2.4.

The following theorem is the main result in this section.

Theorem 4.2.3. *Let $C \subset \mathbb{P}^3$ be an irreducible curve of degree at least 2. If $\mathrm{Sing}(C)$ denotes the singular locus of the curve C, then the singular locus of the Chow hypersurface for C is* $\mathrm{Sec}(C) \cup \left(\bigcup_{x \in \mathrm{Sing}(C)} \{L \in \mathrm{Gr}(1, \mathbb{P}^3) : x \in L\}\right)$.

Proof. We first show that the incidence variety $\Phi_C := \{(v, L) : v \in L\} \subset C \times \mathrm{Gr}(1, \mathbb{P}^3)$ is smooth at the point (v, L) if and only if the curve C is smooth at the point $v \in C$. Let $f_1, f_2, \ldots, f_k \in \mathbb{C}[x_0, x_1, x_2, x_3]$ be generators for the saturated homogeneous ideal of C in $\mathbb{P}^3$. Consider the affine chart of $\mathbb{P}^3 \times \mathrm{Gr}(1, \mathbb{P}^3)$ where $x_0 \neq 0$ and $p_{0,1} \neq 0$. We may assume that $v = (1 : \alpha : \beta : \gamma)$ and the line L is spanned by the points $(1 : 0 : a : b)$ and $(0 : 1 : c : d)$. We have that $v \in L$ if and only if the line L is given by the row space of matrix

$$\begin{bmatrix} 1 & \alpha & \beta & \gamma \\ 0 & 1 & c & d \end{bmatrix} = \begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & \beta - \alpha c & \gamma - \alpha d \\ 0 & 1 & c & d \end{bmatrix},$$

which is equivalent to $a = \beta - \alpha c$ and $b = \gamma - \alpha d$. Hence, in the chosen affine chart, Φ_C can be written as

$$\left\{(\alpha, \beta, \gamma, a, b, c, d) : f_i(1, \alpha, \beta, \gamma) = 0 \text{ for } 1 \le i \le k,\ a = \beta - \alpha c,\ b = \gamma - \alpha d\right\}.$$

As $\dim \Phi_C = 3$, it is smooth at the point (v, L) if and only if its tangent space has dimension three or, equivalently, the Jacobian matrix

$$\begin{bmatrix}
\frac{\partial f_1}{\partial x_1}(1,\alpha,\beta,\gamma) & \frac{\partial f_1}{\partial x_2}(1,\alpha,\beta,\gamma) & \frac{\partial f_1}{\partial x_3}(1,\alpha,\beta,\gamma) & 0 & 0 & 0 & 0 \\
\frac{\partial f_2}{\partial x_1}(1,\alpha,\beta,\gamma) & \frac{\partial f_2}{\partial x_2}(1,\alpha,\beta,\gamma) & \frac{\partial f_2}{\partial x_3}(1,\alpha,\beta,\gamma) & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\frac{\partial f_k}{\partial x_1}(1,\alpha,\beta,\gamma) & \frac{\partial f_k}{\partial x_2}(1,\alpha,\beta,\gamma) & \frac{\partial f_k}{\partial x_3}(1,\alpha,\beta,\gamma) & 0 & 0 & 0 & 0 \\
-c & 1 & 0 & -1 & 0 & -\alpha & 0 \\
-d & 0 & 1 & 0 & -1 & 0 & -\alpha
\end{bmatrix}$$

has rank four. We see that this Jacobian matrix has rank four if and only if the Jacobian matrix of C has rank two, in which case $v \in C$ is smooth. Therefore, we deduce that Φ_C is smooth at the point (v, L) exactly when C is smooth at the point v.

By Lemma 14.8 in [18], the projection $\pi : \Phi_C \to Z_C$ defined by $(v, L) \mapsto L$ is finite; otherwise C would contain a line contradicting our assumptions. Moreover, the general fibre of π has cardinality 1 because the general line $L \in Z_C$ intersects C in a single point. Hence, π is birational. Applying Lemma 4.2.2 shows that Z_C is smooth at L if and only if $\pi^{-1}(L) = \{(v, L)\}$ where $v \in C$ is a smooth point and the differential $d_{(v,L)}\pi$ is injective. Using our chosen affine chart, we see that the differential $d_{(v,L)}\pi$ sends every element in the kernel of the Jacobian matrix to its last four coordinates. This map is not injective if and only if the kernel contains an element of the form $\begin{bmatrix} * & * & * & 0 & 0 & 0 & 0 \end{bmatrix}^\top \neq 0$. Such an element belongs to the kernel if and only if it is equal to $\begin{bmatrix} \lambda & c\lambda & d\lambda & 0 & 0 & 0 & 0 \end{bmatrix}^\top$ for some $\lambda \in \mathbb{C} \setminus \{0\}$ and

$$\frac{\partial f_i}{\partial x_1}(1, \alpha, \beta, \gamma) + c\frac{\partial f_i}{\partial x_2}(1, \alpha, \beta, \gamma) + d\frac{\partial f_i}{\partial x_3}(1, \alpha, \beta, \gamma) = 0$$

for all $1 \leq i \leq k$. Hence, for a smooth point $v \in C$, the differential $d_{(v,L)}\pi$ is not injective if and only if L is the tangent line of C at v. Since we have that $l\pi^{-1}(L)| = 1$ if and only if L is not a secant line and all tangent lines to C are contained in $\operatorname{Sec}(C)$, we conclude that Z_C is smooth at L if and only if $L \notin \operatorname{Sec}(C)$ and L meets C at a smooth point. □

Remark 4.2.4. Local computations show that the secant congruence of C generally does not contain all lines through singular points of C. To be more explicit, let $x \in C$ be an *ordinary singularity*; the point x is the intersection of r branches of C with $r \geq 2$, and the r tangent lines of the branches at x are pairwise different. We claim that a line L intersecting C only at the point x is contained in $\operatorname{Sec}(C)$ if and only if L lies in a plane spanned by two of the r tangent lines at x. The union of all those lines forms the tangent star of C at x; see [23, 37].

Suppose that $x = (1 : 0 : 0 : 0)$ and $L \in \operatorname{Sec}(C)$ intersects the curve C only at the point x. The line L must be the limit of a family of lines L_t that intersect C at two distinct smooth points. Without loss of generality, the line L is not one of the tangent lines of the curve C at the point x and each line L_t intersects at least two distinct branches of C. Since there are only finitely many branches, we can also assume that each line L_t in the family intersects the same two branches of the curve C. These two branches are parametrized by $\big(1 : f_1(s) : f_2(s) : f_3(s)\big)$ and $\big(1 : g_1(s) : g_2(s) : g_3(s)\big)$ with $f_i(0) = 0 = g_j(0)$ for $1 \leq i, j \leq 3$. It follows that tangent lines to these branches are spanned by x and $\big(1 : f_1'(0) : f_2'(0) : f_3'(0)\big)$ or $\big(1 : g_1'(0) : g_2'(0) : g_3'(0)\big)$. Parametrizing intersection points, we see that the line L_t intersects the first branch at $\big(1 : f_1\big(\varphi(t)\big) : f_2\big(\varphi(t)\big) : f_3\big(\varphi(t)\big)\big)$ and the second branch at $\big(1 : g_1\big(\psi(t)\big) : g_2\big(\psi(t)\big) : g_3\big(\psi(t)\big)\big)$ where $\varphi(0) = 0 = \psi(0)$. Hence, the

Plücker coordinates for L_t are

$$\left(\tfrac{g_1(\psi(t))-f_1(\varphi(t))}{t} : \tfrac{g_2(\psi(t))-f_2(\varphi(t))}{t} : \cdots : \tfrac{f_2(\varphi(t))g_3(\psi(t))-f_3(\varphi(t))g_2(\psi(t))}{t}\right).$$

Taking the limit as $t \to 0$, we obtain the line L with Plücker coordinates

$$\left(g_1'(0)\psi'(0) - f_1'(0)\varphi'(0) : g_2'(0)\psi'(0) - f_2'(0)\varphi'(0) : \cdots : 0\right).$$

This line is spanned by the point x and

$$\left(1 : g_1'(0)\psi'(0) - f_1'(0)\varphi'(0) : g_2'(0)\psi'(0) - f_2'(0)\varphi'(0) : g_3'(0)\psi'(0) - f_3'(0)\varphi'(0)\right),$$

so it lies in the plane spanned by the two tangent lines. From this computation, we also see that all lines passing through x and lying in the plane spanned by the tangent lines can be approximated by lines that intersect both of the branches at points different from x. For this, one need only choose $\varphi(t) = \lambda t$ and $\psi(t) = \mu t$ for all possible $\lambda, \mu \in \mathbb{C} \setminus \{0\}$.

Using Chern classes, Proposition 2.1 in [2] calculates the bidegree of the secant congruence of a smooth curve. We give a geometric description of this bidegree and extend it to curves with ordinary singularities.

Theorem 4.2.5. *If $C \subset \mathbb{P}^3$ is a nondegenerate irreducible curve of degree d and genus g having only ordinary singularities $x_1, x_2, \ldots, x_s$ with multiplicities $r_1, r_2, \ldots, r_s$, then the bidegree of the secant congruence* $\operatorname{Sec}(C)$ *is*

$$\left(\binom{d-1}{2} - g - \sum_{i=1}^{s} \binom{r_i}{2}, \binom{d}{2}\right).$$

Proof. Let $H \subset \mathbb{P}^3$ be a general plane. The intersection of H with C consists of d points. Any two of these points define a secant line lying in H; see Fig. 4.2. Hence, there are $\binom{d}{2}$ secant lines contained in H, which gives the class of $\operatorname{Sec}(C)$.

To compute the order of $\operatorname{Sec}(C)$, let $v \in \mathbb{P}^3$ be a general point. Projecting away from v defines a rational map $\pi_v : \mathbb{P}^3 \dashrightarrow \mathbb{P}^2$. Set $C' := \pi_v(C)$. The map π_v sends a line passing through v and intersecting C at two points to a simple node of the plane curve C'; see Fig. 4.6. Moreover, every ordinary singularity of C is sent to an ordinary singularity of C' with the same multiplicity, and the plane curve C' has the same degree as the space curve C. As the geometric genus is invariant under birational transformation, it also has the same genus; see [19, Theorem II.8.19]. Thus, the genus-degree formula for plane curves [36, p. 54, Eq. (7)] shows that the

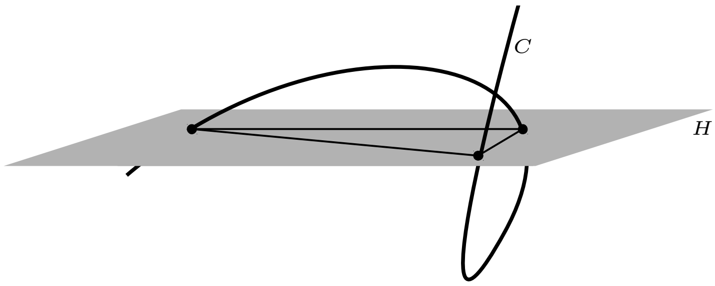

Figure 4.2: The class of the secant congruence

genus of C is equal to $\binom{d-1}{2} - \sum_{i=1}^{s} \binom{r_i}{2}$ minus the number of secants of C passing through v. □

Remark 4.2.6. If $C \subset \mathbb{P}^3$ is a curve of degree at least 2 that is contained in a plane, then its secant congruence consists of all lines in that plane and has bidegree $(0,1)$.

Problem 5 on Curves in [40] asks to compute the dimension and bidegree of $\mathrm{Sing}(Z_C)$. When C is not a line, Theorem 4.2.3 establishes that $\mathrm{Sing}(Z_C)$ is 2-dimensional. For completeness, we also state its bidegree explicitly.

Corollary 4.2.7. *If $C \subset \mathbb{P}^3$ is an irreducible curve of degree $d \geq 2$ and geometric genus g having only ordinary singularities $x_1, x_2, \ldots, x_s$ with multiplicities $r_1, r_2, \ldots, r_s$, then the bidegree of $\mathrm{Sing}(Z_C)$ equals $\left(\binom{d-1}{2} - g - \sum\limits_{i=1}^{s} \binom{r_i}{2} + s, \binom{d}{2}\right)$ if C is nondegenerate, and $(s,1)$ if C is contained in a plane.*

Proof. The bidegree of each congruence $\{L \in \mathrm{Gr}(1,\mathbb{P}^3) : x_i \in L\}$ is $(1,0)$. Hence, combining Theorem 4.2.3, Theorem 4.2.5, and Remark 4.2.6 proves the corollary. □

4.3 Bitangents and Inflections of a Surface

This section describes the singular locus of the Hurwitz hypersurface for a surface in $\mathbb{P}^3$. For a surface $S \subset \mathbb{P}^3$ that is not a plane, the Hurwitz hypersurface $\mathrm{CH}_1(S)$ is the Zariski closure of the set of all lines in $\mathbb{P}^3$ that are tangent to S at a smooth point. Its defining equation in Plücker coordinates is known as the Hurwitz form of S; see [41].

In analogy with the secant congruence of a curve, we associate two congruences to a surface $S \subset \mathbb{P}^3$. Specifically, the Zariski closure in $\mathrm{Gr}(1,\mathbb{P}^3)$ of the set of lines tangent to a surface S at two smooth points forms the *bitangent congruence*:

$$\mathrm{Bit}(S) := \overline{\left\{ L \in \mathrm{Gr}(1,\mathbb{P}^3) : \begin{array}{l} x,y \in L \subset T_x(S) \cap T_y(S) \text{ for distinct smooth} \\ \text{points } x,y \in S \end{array} \right\}}.$$

The *inflectional locus* associated to S is the Zariski closure in $\mathrm{Gr}(1,\mathbb{P}^3)$ of the set of lines that intersect the surface S at a smooth point with multiplicity at least 3:

$$\mathrm{Infl}(S) := \overline{\left\{ L \in \mathrm{Gr}(1,\mathbb{P}^3) : \begin{array}{l} L \text{ intersects } S \text{ at a smooth point with multi-} \\ \text{plicity at least } 3 \end{array} \right\}}.$$

A general surface of degree d in $\mathbb{P}^3$ is a surface defined by a polynomial corresponding to a general point in $\mathbb{P}(\mathbb{C}[x_0,x_1,x_2,x_3]_d)$. For a general surface, the inflectional locus is a congruence. However, this is not always the case, as Remark 4.4.8 demonstrates.

In parallel with Section 4.2, the main result in this section describes the singular locus of the Hurwitz hypersurface of S.

Theorem 4.3.1. *If $S \subset \mathbb{P}^3$ is an irreducible smooth surface of degree at least 4 which does not contain any lines, then we have $\mathrm{Sing}\big(\mathrm{CH}_1(S)\big) = \mathrm{Bit}(S) \cup \mathrm{Infl}(S)$.*

Proof. We first show that the incidence variety

$$\Phi_S := \{(v,L) : v \in L \subset T_v(S)\} \subset S \times \mathrm{Gr}(1,\mathbb{P}^3)$$

is smooth. Let $f \in \mathbb{C}[x_0,x_1,x_2,x_3]$ be the defining equation for S in $\mathbb{P}^3$. Consider the affine chart in $\mathbb{P}^3 \times \mathrm{Gr}(1,\mathbb{P}^3)$ where $x_0 \neq 0$ and $p_{0,1} \neq 0$. We may assume that $v = (1:\alpha:\beta:\gamma)$ and the line L is spanned by the points $(1:0:a:b)$ and $(0:1:c:d)$. In this affine chart, S is defined by $g_0(x_1,x_2,x_3) := f(1,x_1,x_2,x_3)$. As in the proof of Theorem 4.2.3, we have that $v \in L$ if and only if $a = \beta - \alpha c$ and $b = \gamma - \alpha d$. For such a pair (v,L), we also have that $L \subset T_v(S)$ if and only if $(0:1:c:d) \in T_v(S)$. Setting $g_1 := \frac{\partial g_0}{\partial x_1} + c\frac{\partial g_0}{\partial x_2} + d\frac{\partial g_0}{\partial x_3}$, we have $L \subset T_v(S)$ if and only if $g_1(\alpha,\beta,\gamma) = 0$. Hence, in the chosen affine chart, Φ_S can be written as

$$\big\{(\alpha,\beta,\gamma,a,b,c,d) : g_j(\alpha,\beta,\gamma) = 0 \text{ for } 0 \leq j \leq 1,\ a = \beta - \alpha c,\ b = \gamma - \alpha d\ \big\}.$$

As $\dim \Phi_S = 3$, it is smooth at the point (v,L) if and only if its tangent space has dimension three or, equivalently, its Jacobian matrix

$$\begin{bmatrix} \frac{\partial g_0}{\partial x_1}(\alpha,\beta,\gamma) & \frac{\partial g_0}{\partial x_2}(\alpha,\beta,\gamma) & \frac{\partial g_0}{\partial x_3}(\alpha,\beta,\gamma) & 0 & 0 & 0 & 0 \\ \frac{\partial g_1}{\partial x_1}(\alpha,\beta,\gamma) & \frac{\partial g_1}{\partial x_2}(\alpha,\beta,\gamma) & \frac{\partial g_1}{\partial x_3}(\alpha,\beta,\gamma) & 0 & 0 & \frac{\partial g_0}{\partial x_2}(\alpha,\beta,\gamma) & \frac{\partial g_0}{\partial x_3}(\alpha,\beta,\gamma) \\ -c & 1 & 0 & -1 & 0 & -\alpha & 0 \\ -d & 0 & 1 & 0 & -1 & 0 & -\alpha \end{bmatrix}$$

has rank four. Since S is smooth, we deduce that this Jacobian matrix has rank four, so Φ_S is also smooth.

Since S does not contain any lines, all fibres of the projection $\pi : \Phi_S \to \mathrm{CH}_1(S)$ defined by $(v, L) \mapsto L$ are finite, so Lemma 14.8 in [18] implies that π is finite. Moreover, the general fibre of π has cardinality 1, so π is birational. Applying Lemma 4.2.2 shows that $\mathrm{CH}_1(S)$ is smooth at the point (v, L) if and only if the fibre $\pi^{-1}(L)$ consists of one point (v, L) and the differential $d_{(x,L)}\pi$ is injective. In particular, we have $|\pi^{-1}(L)| = 1$ if and only if L is not a bitangent. It remains to show that the differential $d_{(v,L)}\pi$ is injective if and only if L is a simple tangent of S at v. Using our chosen affine chart, we see that the differential $d_{(v,L)}\pi$ projects every element in the kernel of the Jacobian matrix on its last four coordinates. This map is not injective if and only if the kernel contains an element of the form $\begin{bmatrix} * & * & * & 0 & 0 & 0 & 0 \end{bmatrix}^\top \neq 0$. Such an element belongs to the kernel if and only if it is equal to $\begin{bmatrix} \lambda & c\lambda & d\lambda & 0 & 0 & 0 & 0 \end{bmatrix}^\top$ for some $\lambda \in \mathbb{C} \smallsetminus \{0\}$ and $g_1(\alpha,\beta,\gamma) = 0 = g_2(\alpha,\beta,\gamma)$ where $g_2 := \frac{\partial g_1}{\partial x_1} + c\frac{\partial g_1}{\partial x_2} + d\frac{\partial g_1}{\partial x_3}$. Parametrizing the line L by

$$\ell(s,t) := (s : s\alpha + t : s\beta + tc : s\gamma + td)$$

for $(s:t) \in \mathbb{P}^1$ shows that the line L intersects the surface S with multiplicity at least 3 at v if and only if $f\big(\ell(s,t)\big)$ is divisible by t^3. This is equivalent to the conditions that $g_1(\alpha,\beta,\gamma) = \frac{\partial}{\partial t}\big[f\big(\ell(s,t)\big)\big]\big|_{(1,0)} = 0$ and $g_2(\alpha,\beta,\gamma) = \frac{\partial^2}{\partial^2 t}\big[f\big(\ell(s,t)\big)\big]\big|_{(1,0)} = 0$. $\square$

Remark 4.3.2. If S is a surface of degree at most 3 and the line L is bitangent to S, then L is contained in S. Indeed, if L is not contained in S, then the intersection $L \cap S$ consists of at most 3 points, counted with multiplicity, so L cannot be a bitangent. On the other hand, when the degree of S is at least four, the hypothesis that S does not contain any lines is relatively mild. For example, a general surface of degree at least 4 in $\mathbb{P}^3$ does not contain a line; see [46].

4.4 Projective Duality

This section uses projective duality to compute the bidegrees of the components of the singular locus of the Hurwitz hypersurface of a surface in $\mathbb{P}^3$, and to relate the secant congruence of a curve to the bitangent congruence of its dual surface.

Let $\mathbb{P}^n$ be the projectivization of the vector space $\mathbb{C}^{n+1}$. If $(\mathbb{P}^n)^*$ denotes the projectivization of the dual vector space $(\mathbb{C}^{n+1})^*$, then the points in $(\mathbb{P}^n)^*$ correspond to hyperplanes in $\mathbb{P}^n$. Given a projective subvariety $X \subset \mathbb{P}^n$, a hyperplane in $\mathbb{P}^n$ is tangent to X at a smooth point $x \in X$ if it contains the embedded tangent space $T_x(X) \subset \mathbb{P}^n$. The *dual variety* $X^\vee$ is the Zariski closure in $(\mathbb{P}^n)^*$ of the set of all hyperplanes in $\mathbb{P}^n$ that are tangent to X at some smooth point.

Example 4.4.1. If V is a linear subspace of $\mathbb{C}^{n+1}$ and $X := \mathbb{P}(V)$, then the dual variety $X^\vee$ is the set of all hyperplanes containing $\mathbb{P}(V)$, which is exactly the projectivization of the orthogonal complement $V^\perp \subset (\mathbb{C}^{n+1})^*$ with respect to the nondegenerate bilinear form $(x, y) \mapsto \sum_{i=0}^n x_i y_i$. In particular, $X^\vee$ is not the projectivization of V^*, and $(\mathbb{P}^n)^\vee = \varnothing$.

Remark 4.4.2. The dual of a line in $\mathbb{P}^2$ is a point, and the dual of a plane curve of degree at least 2 is again a plane curve. The dual of a line in $\mathbb{P}^3$ is a line, and the dual of a curve in $\mathbb{P}^3$ of degree at least 2 is a surface. The dual of a plane in $\mathbb{P}^3$ is a point and the dual of a surface in $\mathbb{P}^3$ of degree at least 2 can be either a curve or a surface.

From our perspective, the key properties of dual varieties are the following. If X is irreducible, then its dual $X^\vee$ is also irreducible; see [15, Proposition I.1.3]. Moreover, the Biduality Theorem shows that, if $x \in X$ is smooth and $H \in X^\vee$ is smooth, then H is tangent to X at the point x if and only if the hyperplane in $(\mathbb{P}^n)^*$ corresponding to x is tangent to $X^\vee$ at the point H; see [15, Theorem I.1.1]. In particular, any irreducible variety $X \subset \mathbb{P}^n$ is equal to its double dual $(X^\vee)^\vee \subset \mathbb{P}^n$; again see [15, Theorem I.1.1].

The next lemma, which relates the number and type of singularities for a plane curve to the degree of its dual variety, plays an important role in calculating the bidegrees of the bitangent and inflectional congruences. A point v on a planar curve C is a *simple node* or a *cusp* if the formal completion of $\mathcal{O}_{C,v}$ is isomorphic to $\mathbb{C}[[z_1, z_2]]/(z_1^2+z_2^2)$ or $\mathbb{C}[[z_1, z_2]]/(z_1^3+z_2^2)$ respectively; see Fig. 4.3. Both singularities have multiplicity 2; nodes have two distinct tangents and cusps have a single tangent.

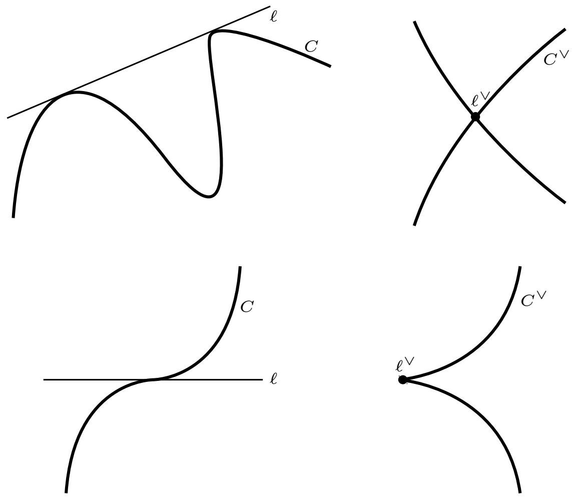

Figure 4.3: A bitangent and an inflectional line corresponding to a node and a cusp of the dual curve

Lemma 4.4.3 (Plücker's formula [9, Example 1.2.8])**.** *If $C \subset \mathbb{P}^2$ is an irreducible curve of degree d with exactly κ cusps, δ simple nodes, and no other singularities, then the degree of the dual curve $C^\vee$ is $d(d-1) - 3\kappa - 2\delta$.*

Sketch. Let $f \in \mathbb{C}[x_0, x_1, x_2]$ be the defining equation for C in $\mathbb{P}^2$, so we have $\deg(f) = d$. To begin, assume that C is smooth. The degree of its dual $C^\vee \subset (\mathbb{P}^2)^*$ is the number of points of $C^\vee$ lying on a general line $L \subset (\mathbb{P}^2)^*$. By duality, the degree equals the number of tangent lines to C passing through a general point $y \in \mathbb{P}^2$. Such a tangent line at the point $v \in C$ passes through the point y if and only if $g := y_0 \frac{\partial f}{\partial x_0}(v) + y_1 \frac{\partial f}{\partial x_1}(v) + y_2 \frac{\partial f}{\partial x_2}(v) = 0$. Hence, the degree of $C^\vee$ is the number of points in $\mathrm{V}(f, g)$; the vanishing set of f and g. Since $\deg(g) = d - 1$, this finite set contains $d(d-1)$ points.

If C is singular, then the degree of $C^\vee$ is the number of lines that are tangent to C at a smooth point and pass through the general point y. Those smooth points are contained in the set $\mathrm{V}(f, g)$, but all of the singular points also lie in $\mathrm{V}(f, g)$. The curve $\mathrm{V}(g)$ passes through each node of C with intersection multiplicity two and through each cusp of C with intersection multiplicity 3. Therefore, we conclude that $\deg(C^\vee) = d(d-1) - 3\kappa - 2\delta$. □

Using Lemma 4.4.3, we can compute the degree of the Hurwitz hypersurface for a smooth surface; this formula also follows from Theorem 1.1 in [41].

Proposition 4.4.4. *For an irreducible smooth surface $S \subset \mathbb{P}^3$ of degree d with $d \geq 2$, the degree of the Hurwitz hypersurface $\mathrm{CH}_1(S)$ is $d(d-1)$.*

Proof. Let $H \subset \mathbb{P}^3$ be a general plane and $v \in H$ be a general point. The degree of $\mathrm{CH}_1(S)$ is the number of tangent lines L to S such that $v \in L \subset H$. Bertini's Theorem [18, Theorem 17.16] implies that the intersection $S \cap H$ is a smooth plane curve of degree d. The degree of $\mathrm{CH}_1(S)$ is the number of tangent lines to $S \cap H$ passing through the general point v; see Fig. 4.4. By definition, this is equal to

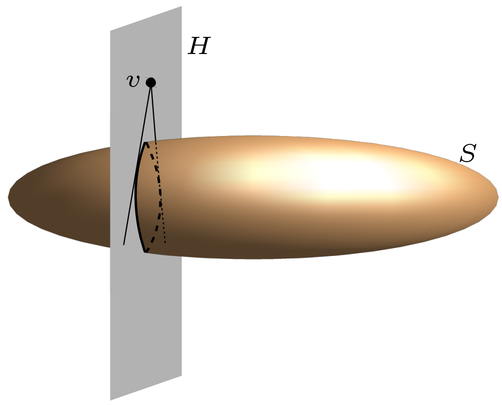

Figure 4.4: The degree of the Hurwitz hypersurface

the degree of the dual plane curve $(S \cap H)^\vee$, so Lemma 4.4.3 shows $\deg\big(\mathrm{CH}_1(S)\big) = d(d-1)$. □

Using Lemma 4.4.3, we can also count the number of bitangents and inflectional tangents to a general smooth plane curve.

Proposition 4.4.5. *A general smooth irreducible curve in $\mathbb{P}^2$ of degree d has exactly $\frac{1}{2}d(d-2)(d-3)(d+3)$ bitangents and $3d(d-2)$ inflectional tangents.*

Proof. Let $C \subset \mathbb{P}^2$ be a general smooth irreducible curve of degree d. A bitangent to C corresponds to a node of $C^\vee$, and an inflectional tangent to C corresponds to a cusp of $C^\vee$; see Fig. 4.3 and [17, pp. 277–278]. Lemma 4.4.3 shows that $C^\vee$ has degree $d(d-1)$. Let κ and δ be the number of cusps and nodes of $C^\vee$, respectively. Applying Lemma 4.4.3 to the plane curve $C^\vee$ yields

$$d = \deg(C) = \deg\big((C^\vee)^\vee\big) = d(d-1)\big(d(d-1)-1\big) - 3\kappa - 2\delta\,.$$

The dual curves C and $C^\vee$ have the same geometric genus; see [43, Proposition 1.5]. Hence, the genus-degree formula [36, p. 54, Eq. (7)] gives

$$\tfrac{1}{2}(d-1)(d-2) = \mathrm{genus}(C) = \mathrm{genus}(C^\vee) = \tfrac{1}{2}\big(d(d-1)-1\big)\big(d(d-1)-2\big) - \kappa - \delta\,.$$

Solving this system of two linear equations in κ and δ, we obtain $\kappa = 3d(d-2)$ and $\delta = \frac{1}{2}d(d-2)(d-3)(d+3)$. $\square$

The next result is the main theorem in this section and solves Problem 4 on Surfaces in [40]. The bidegrees of the bitangent and the inflectional congruence for a general smooth surface appear in [2, Proposition 3.3], and the bidegree of the inflectional congruence also appears in [31, Proposition 4.1].

Theorem 4.4.6. *Let $S \subset \mathbb{P}^3$ be a general smooth irreducible surface of degree d with $d \geq 4$. The bidegree of* $\mathrm{Bit}(S)$ *is* $\big(\frac{1}{2}d(d-1)(d-2)(d-3), \frac{1}{2}d(d-2)(d-3)(d+3)\big)$, *and the bidegree of* $\mathrm{Infl}(S)$ *is* $\big(d(d-1)(d-2), 3d(d-2)\big)$.

Proof. For a general plane $H \subset \mathbb{P}^3$, Bertini's Theorem [18, Theorem 17.16] implies that the intersection $S \cap H$ is a smooth plane curve of degree d. By Proposition 4.4.5, the number of bitangents to S contained in H is $\frac{1}{2}d(d-2)(d-3)(d+3)$, which is the class of $\mathrm{Bit}(S)$. Similarly, the number of inflectional tangents to S contained in H is $3d(d-2)$, which is the class of $\mathrm{Infl}(S)$.

It remains to calculate the number of bitangents and inflectional lines of the surface S that pass through a general point $y \in \mathbb{P}^3$. Following the ideas in [32, p. 230], let $f \in \mathbb{C}[x_0, x_1, x_2, x_3]$ be the defining equation for S in $\mathbb{P}^3$, and consider the polar curve $C \subset S$ with respect to the point y; the set C consists of all points $x \in S$ such that the line through y and x is tangent to S at the point x; see Fig. 4.5. The condition that the point x lies on the curve C is equivalent to saying that the

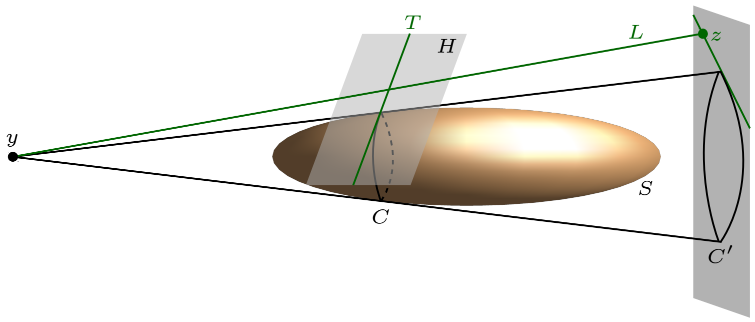

Figure 4.5: Polar curve

point y belongs to $T_x(S)$. As in the proof for Lemma 4.4.3, we have $C = \mathrm{V}(f, g)$ where $g := y_0\frac{\partial f}{\partial x_0} + y_1\frac{\partial f}{\partial x_1} + \cdots + y_3\frac{\partial f}{\partial x_3}$. Thus, the curve C has degree $d(d-1)$.

Projecting away from the point y gives the rational map $\pi_y : \mathbb{P}^3 \dashrightarrow \mathbb{P}^2$. Restricted to the surface S, this map is generically finite, with fibres of cardinality d, and is ramified over the curve C. If C' is the image of C under π_y, then a bitangent to the surface S that passes through y contains two points of C and these points are mapped to a simple node in C'; see Fig. 4.6. All of these nodes in C' have two

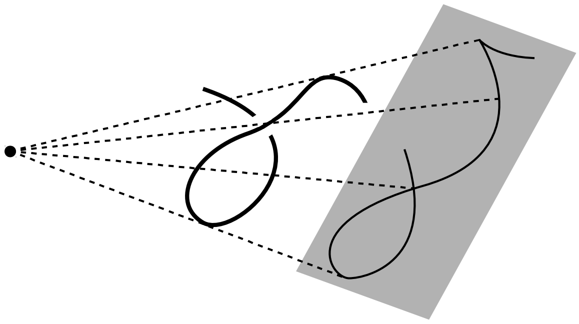

Figure 4.6: A secant projecting onto a node and a tangent projecting to a cusp

distinct tangent lines because no bitangent line passing through y is contained in a bitangent plane that is tangent at the same two points as the line; the bitangent planes to S form a 1-dimensional family, so the union of bitangent lines they contain is a surface in $\mathbb{P}^3$ that does not contain the general point y.

We claim that the inflectional lines to S passing through the point y are exactly the tangent lines of C passing through y. The line between a point $x \in S$ and the point y is parametrized by the map $\ell : \mathbb{P}^1 \to \mathbb{P}^3$ which sends the point $(s:t) \in \mathbb{P}^1$ to the point $(sx_0 + ty_0 : sx_1 + ty_1 : sx_2 + ty_2 : sx_3 + ty_3) \in \mathbb{P}^3$. It follows that this line is an inflectional tangent to S if and only if $f\big(\ell(s,t)\big)$ is divisible by t^3. This is equivalent to the conditions that $\frac{\partial}{\partial t}\big[f\big(\ell(s,t)\big)\big]\big|_{(1,0)} = 0$ and $\frac{\partial^2}{\partial t^2}\big[f\big(\ell(s,t)\big)\big]\big|_{(1,0)} = 0$, which means that $x \in C$ and $y_0\frac{\partial g}{\partial x_0} + y_1\frac{\partial g}{\partial x_1} + \cdots + y_3\frac{\partial g}{\partial x_3} = 0$, or in other words $y \in T_x(C)$. Therefore, the inflectional lines to S passing through y are the tangents to C passing through y, and are mapped to the cusps of C'; again see Fig. 4.6.

Since the bitangent and inflectional lines to S passing through y correspond to nodes and cusps of C', it suffices to count the number κ' of cusps and the number δ' of simple nodes in the plane curve C'. We subdivide these calculations as follows.

$\kappa' = d(d-1)(d-2)$: From our parametrization of the line through points $x \in S$ and y, we see that this line is an inflectional tangent to S if and only if $x \in \mathrm{V}(f,g,h)$ where $h \coloneqq y_0\frac{\partial g}{\partial x_0} + y_1\frac{\partial g}{\partial x_1} + \cdots + y_3\frac{\partial g}{\partial x_3}$. Since $\deg(h) = d-2$ and S is general, the set $\mathrm{V}(f,g,h)$ consists of $d(d-1)(d-2)$ points.

$\deg\big((C')^\vee\big) = \deg(S^\vee)$: By duality, the degree d' of the curve $(C')^\vee$ is the number of tangent lines to $C' \subset \mathbb{P}^2$ passing through a general point $z \in \mathbb{P}^2$. The preimage of z under the projection π_y is a line $L \subset \mathbb{P}^3$ containing y; see Fig. 4.5. Hence, d' is the number of tangent lines to C intersecting L in a point different from

y. For every line T that is tangent to C at a point x and intersects the line L, it follows that the pair L and T spans the tangent plane of S at the point x. On the other hand, given any plane H which is tangent to S at the point x and contains L, we deduce that x must lie on the polar curve C and H is spanned by L and the tangent line to C at x, so this tangent line intersects L. Therefore, d' is the number of tangent planes to S containing L, which is the degree of the dual surface $S^\vee$.

$\deg(S^\vee) = d(d-1)^2$**:** By duality, the degree of $S^\vee$ is the number of tangent planes to the surface S containing a general line, or the number of tangent planes to S containing two general points $y, z \in \mathbb{P}^3$. Thus, this is the number of intersection points of the two polar curves of S determined by y and z, which is the cardinality of the set $\mathrm{V}(f, g, \tilde{g})$ where $\tilde{g} := z_0 \frac{\partial f}{\partial x_0} + z_1 \frac{\partial f}{\partial x_1} + \cdots + z_3 \frac{\partial f}{\partial x_3}$. Since $\deg(\tilde{g}) = d-1$, we conclude that $\deg(S^\vee) = d(d-1)^2$.

Finally, both the surface S and the point y are general, so Lemma 4.4.3 implies that $d(d-1)^2 = \deg\big((C')^\vee\big) = \deg(C')\big(\deg(C')-1\big) - 3d(d-1)(d-2) - 2\delta'$. Since $\deg(C') = \deg(C) = d(d-1)$, we have $\delta' = \frac{1}{2}d(d-1)(d-2)(d-3)$. □

We end this section by proving that the secant locus for an irreducible smooth curve is isomorphic to the bitangent congruence of its dual surface via the natural isomorphism between $\mathrm{Gr}(1,\mathbb{P}^3)$ and $\mathrm{Gr}\big(1,(\mathbb{P}^3)^*\big)$. A subvariety $\Sigma \subset \mathrm{Gr}(1,\mathbb{P}^3)$ is sent under this isomorphism to the variety $\Sigma^\perp \subset \mathrm{Gr}\big(1,(\mathbb{P}^3)^*\big)$ consisting of the dual lines $L^\vee$ for all $L \in \Sigma$. For every congruence $\Sigma \subset \mathrm{Gr}(1,\mathbb{P}^3)$ with bidegree (α,β), the bidegree of $\Sigma^\perp$ is (β,α).

Theorem 4.4.7. *If $C \subset \mathbb{P}^3$ is a nondegenerate irreducible smooth curve, then we have $\mathrm{Sec}(C)^\perp = \mathrm{Bit}(C^\vee)$, the inflectional lines of $C^\vee$ are dual to the tangent lines of C, and $\mathrm{Infl}(C^\vee) \subset \mathrm{Bit}(C^\vee)$.*

Proof. We first show that $\mathrm{Sec}(C)^\perp = \mathrm{Bit}(C^\vee)$. Consider a line L that intersects C at two distinct points x and y, but is equal to neither $T_x(C)$ nor $T_y(C)$. Together the line L and $T_x(C)$ span a plane corresponding to a point $a \in C^\vee$. Similarly, the span of the lines L and $T_y(C)$ corresponds to a point $b \in C^\vee$. Without loss of generality, we may assume that both a and b are smooth points in $C^\vee$. By the Biduality Theorem, the points $a, b \in C^\vee$ must be distinct with tangent planes corresponding to x and y. Thus, the line $L^\vee$ is tangent to $C^\vee$ at the points a, b, and $\mathrm{Sec}(C)^\perp \subset \mathrm{Bit}(C^\vee)$. To establish the other inclusion, let L' be a line that is tangent to $C^\vee$ at two distinct smooth points $a, b \in C^\vee$. The tangent planes at the points a, b correspond to two points $x, y \in C$. If $x \neq y$, then $(L')^\vee$ is the secant to C through these two points. If $x = y$, then the Biduality Theorem establishes that $(L')^\vee$ is the tangent line of C at x. In either case, we see that $\mathrm{Bit}(C^\vee) \subset \mathrm{Sec}(C)^\perp$, so $\mathrm{Sec}(C)^\perp = \mathrm{Bit}(C^\vee)$.

For the second part, let L be an inflectional line at a smooth point $a \in C^\vee$. A point $y \in L^\vee \setminus C$ corresponds to a plane H such that $L = T_a(C^\vee) \cap H$, so the line L is also an inflectional line to the plane curve $C^\vee \cap H \subset H$. Regarding L as a subvariety of the projective plane H, its dual variety is a cusp on the plane curve $(C^\vee \cap H)^\vee \subset H^*$; see Fig. 4.3. If $\pi_y : \mathbb{P}^3 \dashrightarrow \mathbb{P}^2 \cong H^*$ denotes the projection away from the point y, then we claim that $(C^\vee \cap H)^\vee$ equals $\pi_y(C)$; for a more general version see [20, Proposition 6.1]. Indeed, a smooth point $z \in \pi_y(C)$ is the projection of a point of C whose tangent line does not contain y. Together this tangent line and the point y span a plane such that its dual point w is contained in the curve $C^\vee \cap H$. Thus, the tangent line $T_z\big(\pi_y(C)\big)$ equals $\pi_y(w^\vee)$; the latter is the line in H^* dual to the point $w \in H$. In other words, we have $\big(\pi_y(C)\big)^\vee \subset C^\vee \cap H$. Since both curves are irreducible, this inclusion must be an equality. Hence, when considering L in the projective plane H, its dual point is a cusp of $\pi_y(C)$. It follows that $L^\vee$ is the tangent line $T_x(C)$, where $x \in C$ is the point corresponding to the tangent plane $T_a(C^\vee)$; see Fig. 4.6. Reversing these arguments shows that the dual of a tangent line to C is an inflectional line to $C^\vee$. Since every tangent line to C is contained in $\mathrm{Sec}(C)$, we conclude that $\mathrm{Infl}(C^\vee) \subset \mathrm{Bit}(C^\vee)$. □

Remark 4.4.8. Theorem 4.4.7 shows that $\mathrm{Infl}(C^\vee)$ is a curve, as $\mathrm{Infl}(C^\vee)^\perp$ is the set of tangent lines to C, so the inflectional locus of a surface in $\mathbb{P}^3$ is not always a congruence.

Remark 4.4.9. For a curve $C \subset \mathbb{P}^3$ with dual surface $C^\vee \subset (\mathbb{P}^3)^*$, Theorem 20 in [26] establishes that $Z_C^\perp = \mathrm{CH}_1(C^\vee)$. Combined with Theorem 4.4.7, we see that the singular locus of the Hurwitz hypersurface $\mathrm{CH}_1(C^\vee)$, for smooth C, has just one component, namely the bitangent congruence.

Remark 4.4.10. For a surface $S \subset \mathbb{P}^3$ with dual surface $S^\vee \subset (\mathbb{P}^3)^*$, Theorem 20 in [26] also establishes that $\mathrm{CH}_1(S)^\perp = \mathrm{CH}_1(S^\vee)$. If both S and $S^\vee$ have mild singularities, then the proof of Lemma 5.1 in [2] shows that $\mathrm{Bit}(S)^\perp = \mathrm{Bit}(S^\vee)$.

4.5 Intersection Theory on $\mathrm{Gr}(1, \mathbb{P}^3)$

In this section, we recast the degree of a subvariety in $\mathrm{Gr}(1, \mathbb{P}^3)$ in terms of certain products in the Chow ring.

Consider a smooth irreducible variety X of dimension n. For each $j \in \mathbb{N}$, the group $Z^j(X)$ of codimension-j cycles is the free abelian group generated by the closed irreducible subvarieties of X having codimension j. Given a variety W of codimension $j-1$ and a nonzero rational function f on W, one can define the cycle $\operatorname{div}(f)$; see [14, Section 1.2]. The group of cycles rationally equivalent to zero is the subgroup generated by the cycles $\operatorname{div}(f)$ for all codimension-$(j-1)$ subvarieties W of X and all nonzero rational functions f on W. The Chow group $A^j(X)$ is the quotient of $Z^j(X)$ by the subgroup of cycles rationally equivalent to zero. We typically write $[Z]$ for the class of a subvariety Z in the appropriate Chow group. Since X is the unique subvariety of codimension 0, we see that $A^0(X) \cong \mathbb{Z}$. We also have $A^1(X) \cong \operatorname{Pic}(X)$. Crucially, the direct sum $A^*(X) \coloneqq \bigoplus_{j=0}^{n} A^j(X)$ forms a commutative $\mathbb{Z}$-graded ring called the *Chow ring* of X. The product arises from intersecting cycles: for subvarieties V and W of X having codimension j and k and intersecting transversely, the product $[V][W] \in A^{j+k}(X)$ is the sum of the irreducible components of $V \cap W$. More generally, intersection theory aims to construct an explicit cycle to represent the product $[V][W]$.

Example 4.5.1. The Chow ring of $\mathbb{P}^n$ is isomorphic to $\mathbb{Z}[H]/(H^{n+1})$ where H is the class of a hyperplane. In particular, any subvariety of codimension d is rationally equivalent to a multiple of the intersection of d hyperplanes.

To a given a vector bundle $\mathcal{E}$ of rank r on X, we associate its *Chern classes* $c_i(\mathcal{E}) \in A^i(X)$ for $0 \le i \le r$; see [42]. When $\mathcal{E}$ is globally generated, these classes are represented by degeneracy loci; the class $c_{r+1-j}(\mathcal{E})$ is associated to the locus of points $x \in X$ where j general global sections of $\mathcal{E}$ fail to be linearly independent. In particular, $c_r(\mathcal{E})$ is represented by the vanishing locus of a single general global section. Given a short exact sequence $0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$ of vector bundles, the Whitney Sum Formula asserts that $c_k(\mathcal{E}) = \sum_{i+j=k} c_i(\mathcal{E}')c_j(\mathcal{E}'')$; see [14, Theorem 3.2]. Moreover, if $\mathcal{E}^* \coloneqq \mathcal{H}om(\mathcal{E}, \mathcal{O}_X)$ denotes the dual vector bundle, then we have $c_i(\mathcal{E}^*) = (-1)^i c_i(\mathcal{E})$ for $0 \le i \le r$; see [14, Remark 3.2.3].

Example 4.5.2. Given nonnegative integers $a_1, a_2, \dots, a_n$, consider the vector bundle $\mathcal{E} \coloneqq \mathcal{O}_{\mathbb{P}^n}(a_1) \oplus \mathcal{O}_{\mathbb{P}^n}(a_2) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^n}(a_n)$. Since each $\mathcal{O}_{\mathbb{P}^n}(a_i)$ is globally generated, the Chern class $c_1\big(\mathcal{O}_{\mathbb{P}^n}(a_i)\big)$ is the vanishing locus of a general homogeneous polynomial $\mathbb{C}[x_0, x_1, \dots, x_n]$ of degree a_i, so $c_1\big(\mathcal{O}_{\mathbb{P}^n}(a_i)\big) = a_i H$ in $A^*(\mathbb{P}^n)$. Hence, the Whitney Sum Formula implies that $c_n(\mathcal{E}) = \prod_{i=1}^{n} c_1\big(\mathcal{O}(a_i)\big) = \prod_{i=1}^{n}(a_i H)$.

Example 4.5.3. If $\mathcal{T}_{\mathbb{P}^n}$ is the tangent bundle on $\mathbb{P}^n$, then we have the short exact sequence $0 \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus(n+1)} \to \mathcal{T}_{\mathbb{P}^n} \to 0$; see [19, Example 8.20.1]. The

Whitney Sum Formula implies that $c_1(\mathcal{T}_{\mathbb{P}^n}) = (n+1)c_1\big(\mathcal{O}_{\mathbb{P}^n}(1)\big) - c_1(\mathcal{O}_{\mathbb{P}^n}) = (n+1)H$ and $c_2(\mathcal{T}_{\mathbb{P}^n}) = c_2\big(\mathcal{O}_{\mathbb{P}^n}(1)^{\oplus(n+1)}\big) = \binom{n+1}{2}H^2$.

Example 4.5.4. Let $Y \subset \mathbb{P}^n$ be a smooth hypersurface of degree d. If $\mathcal{T}_Y$ is the tangent bundle of Y, then we have the exact sequence $0 \to \mathcal{T}_Y \to \mathcal{T}_{\mathbb{P}^n}|_Y \to \mathcal{O}_{\mathbb{P}^n}(d)|_Y \to 0$; see [19, Proposition 8.20]. Setting $h := H|_Y$ in $A^*(Y)$, the Whitney Sum Formula implies that $c_1(\mathcal{T}_Y) = c_1(\mathcal{T}_{\mathbb{P}^n}|_Y) - c_1\big(\mathcal{O}_{\mathbb{P}^n}(d)|_Y\big) = (n+1)h - dh = (n+1-d)h$ and $c_2(\mathcal{T}_Y) = c_2(\mathcal{T}_{\mathbb{P}^n}|_Y) - c_1(\mathcal{T}_Y)c_1\big(\mathcal{O}_{\mathbb{P}^n}(d)|_Y\big) = \big(\binom{n+1}{2} - (n+1-d)d\big)h^2$.

We next focus on the Chow ring of $\mathrm{Gr}(1,\mathbb{P}^3)$; see [1, 42]. Fix a complete flag $v_0 \in L_0 \subset H_0 \subset \mathbb{P}^3$ where the point v_0 lies in the line L_0, and the line L_0 is contained in the plane H_0. The Schubert varieties in $\mathrm{Gr}(1,\mathbb{P}^3)$ are the following subvarieties:

$$\begin{aligned}
\Sigma_0 &:= \mathrm{Gr}(1,\mathbb{P}^3)\,, & \Sigma_1 &:= \{L : L \cap L_0 \neq \varnothing\} \subset \mathrm{Gr}(1,\mathbb{P}^3)\,,\\
\Sigma_{1,1} &:= \{L : L \subset H_0\} \subset \mathrm{Gr}(1,\mathbb{P}^3)\,, & \Sigma_2 &:= \{L : v_0 \in L\} \subset \mathrm{Gr}(1,\mathbb{P}^3)\,,\\
\Sigma_{2,1} &:= \{L : v_0 \in L \subset H_0\} \subset \mathrm{Gr}(1,\mathbb{P}^3)\,, & \Sigma_{2,2} &:= \{L_0\} \subset \mathrm{Gr}(1,\mathbb{P}^3)\,.
\end{aligned}$$

The corresponding classes $\sigma_I := [\Sigma_I]$, called the *Schubert cycles*, form a basis for the Chow ring $A^*(\mathrm{Gr}(1,\mathbb{P}^3))$; see [10, Theorem 5.26]. Since the sum of the subscripts gives the codimension, we have

$$\begin{aligned}
&A^0\big(\mathrm{Gr}(1,\mathbb{P}^3)\big) \cong \mathbb{Z}\sigma_0\,, \quad A^1\big(\mathrm{Gr}(1,\mathbb{P}^3)\big) \cong \mathbb{Z}\sigma_1\,, \quad A^2\big(\mathrm{Gr}(1,\mathbb{P}^3)\big) \cong \mathbb{Z}\sigma_{1,1} \oplus \mathbb{Z}\sigma_2\,,\\
&A^3\big(\mathrm{Gr}(1,\mathbb{P}^3)\big) \cong \mathbb{Z}\sigma_{2,1}\,, \quad A^4\big(\mathrm{Gr}(1,\mathbb{P}^3)\big) \cong \mathbb{Z}\sigma_{2,2}\,.
\end{aligned}$$

To understand the product structure, we use the transitive action of $\mathrm{GL}(4,\mathbb{C})$ on $\mathrm{Gr}(1,\mathbb{P}^3)$. Specifically, Kleiman's Transversality Theorem [25] shows that, for two subvarieties V and W in $\mathrm{Gr}(1,\mathbb{P}^3)$, a general translate U of V under the $\mathrm{GL}(4,\mathbb{C})$-action is rationally equivalent to V and the intersection of U and W is transversal at the generic point of any component of $U \cap W$. Hence, we have $[V][W] = [U \cap W]$. To determine the product $\sigma_{1,1}\sigma_2$, we intersect general varieties representing these classes: $\sigma_{1,1}$ consists of all lines L contained in a fixed plane H_0, and σ_2 is all lines L containing a fixed point v_0. Since a general point does not lie in a general plane, we see that $\sigma_{1,1}\sigma_2 = 0$. Similar arguments yield all products:

$$\begin{aligned}
&\sigma_{1,1}^2 = \sigma_{2,2}\,, \quad \sigma_2^2 = \sigma_{2,2}\,, \quad \sigma_{1,1}\sigma_2 = 0\,, \quad \sigma_1\sigma_{2,1} = \sigma_{2,2}\,,\\
&\sigma_1\sigma_{1,1} = \sigma_{2,1}\,, \quad \sigma_1\sigma_2 = \sigma_{2,1}\,, \quad \sigma_1^2 = \sigma_2 + \sigma_{1,1}\,.
\end{aligned}$$

The degree of a subvariety in $\mathrm{Gr}(1,\mathbb{P}^3)$, introduced in Section 4.1, can be interpreted as certain coefficients of its class in the Chow ring. Geometrically, the order α of a surface $X \subset \mathrm{Gr}(1,\mathbb{P}^3)$ is the number of lines in X passing through the general point v_0. Since we may intersect X with a general variety representing σ_2, it follows that α equals the coefficient of σ_2 in $[X]$. Similarly, the class β of X is the coefficient of $\sigma_{1,1}$ in $[X]$, the degree of a threefold $\Sigma \subset \mathrm{Gr}(1,\mathbb{P}^3)$ is the coefficient of σ_1 in $[\Sigma]$, and the degree of a curve $C \subset \mathrm{Gr}(1,\mathbb{P}^3)$ is the coefficient of $\sigma_{2,1}$ in $[C]$.

The degree of a subvariety in $\mathrm{Gr}(1,\mathbb{P}^3)$ also has a useful reinterpretation via Chern classes of tautological vector bundles. Let $\mathcal{S}$ denote the tautological subbundle, the vector bundle whose fibre over the point $W \in \mathrm{Gr}(1,\mathbb{P}^3)$ is the 2-dimensional vector space $W \subseteq \mathbb{C}^4$. Similarly, let $\mathcal{Q}$ be the tautological quotient bundle whose fibre over W is $\mathbb{C}^4/W$. Both $\mathcal{S}^*$ and $\mathcal{Q}$ are globally generated; $H^0\big(\mathrm{Gr}(1,\mathbb{P}^3),\mathcal{S}^*\big) \cong (\mathbb{C}^4)^*$ and $H^0\big(\mathrm{Gr}(1,\mathbb{P}^3),\mathcal{Q}\big) \cong \mathbb{C}^4$; see [1, Proposition 0.5]. A global section of $\mathcal{S}^*$ corresponds to a nonzero map $\varphi : \mathbb{C}^4 \to \mathbb{C}$, where its value at the point W is $\varphi|_W : W \to \mathbb{C}$. The Chern class $c_2(\mathcal{S}^*)$ is represented by the vanishing locus of φ, so we have $c_2(\mathcal{S}^*) = \sigma_{1,1} = c_2(\mathcal{S})$. For two general sections $\varphi, \psi : \mathbb{C}^4 \to \mathbb{C}$ of $\mathcal{S}^*$, the Chern class $c_1(\mathcal{S}^*)$ is represented by the locus of points W where $\varphi|_W$ and $\psi|_W$ fail to be linearly independent or $W \cap \ker(\varphi) \cap \ker(\psi) \neq \{0\}$. Generality ensures that $\ker(\varphi) \cap \ker(\psi)$ is a 2-dimensional subspace of $\mathbb{C}^4$, so $c_1(\mathcal{S}^*) = -c_1(\mathcal{S}) = \sigma_1$. Similarly, a global section of $\mathcal{Q}$ corresponds to a point $v \in \mathbb{C}^4$; its value at W is simply the image of the point in $\mathbb{C}^4/W$. Thus, $c_2(\mathcal{Q})$ is represented by the locus of those W containing v, which is σ_2. Two global sections of $\mathcal{Q}$ are linearly dependent at W when the 2-dimensional subspace of $\mathbb{C}^4$ spanned by the points intersects W nontrivially, so $c_1(\mathcal{Q}) = \sigma_1$. Finally, for a surface $X \subset \mathrm{Gr}(1,\mathbb{P}^3)$ with $[X] = \alpha\sigma_2 + \beta\sigma_{1,1}$, we obtain

$$\begin{aligned} c_2(\mathcal{Q})\,[X] &= \sigma_2(\alpha\sigma_2 + \beta\sigma_{1,1}) = \alpha\sigma_{2,2}\,, \\ c_2(\mathcal{S})\,[X] &= \sigma_{1,1}(\alpha\sigma_2 + \beta\sigma_{1,1}) = \beta\sigma_{2,2}\,, \end{aligned}$$

so computing the bidegree is equivalent to calculating the products $c_2(\mathcal{Q})\,[X]$ and $c_2(\mathcal{S})\,[X]$ in the Chow ring.

We close this section with three examples demonstrating this approach.

Example 4.5.5. Given a smooth surface S in $\mathbb{P}^3$, we recompute the degree of $\mathrm{CH}_1(S)$; compare with Proposition 4.4.4. Theorem 9 in [26] implies that this degree equals the degree $\delta_1(S)$ of the first polar locus $M_1(S) = \{x \in S : y \in T_x S\}$, where y is a general point of $\mathbb{P}^3$ (this locus is the polar curve in the proof of Theorem 4.4.6). Letting T_S be the tangent bundle of S, Example 14.4.15 in [14] shows that $\delta_1(S) = \deg\big(3h - c_1(T_S)\big)$. Hence, Example 4.5.4 gives $\delta_1(S) = \deg(3h - h(3+1-d)) = (d-1)\deg(h)$. Since S is a degree d surface, the degree of the hyperplane h equals d, so $\delta_1(S) = d(d-1)$.

Example 4.5.6 (Problem 3 on Grassmannians in [40]). Let $S_1, S_2 \subset \mathbb{P}^3$ be general surfaces of degree d_1 and d_2, respectively, with $d_1, d_2 \geq 4$. To find the number of lines bitangent to both surfaces, it suffices to compute the cardinality of $\mathrm{Bit}(S_1) \cap \mathrm{Bit}(S_2)$. Theorem 4.4.6 establishes that, for all $1 \leq i \leq 2$, we have $[\mathrm{Bit}(S_i)] = \alpha_i \sigma_2 + \beta_i \sigma_{1,1}$ where $\alpha_i := \frac{1}{2} d_i(d_i - 1)(d_i - 2)(d_i - 3)$ and $\beta_i := \frac{1}{2} d_i(d_i - 2)(d_i - 3)(d_i + 3)$. It follows that $[\mathrm{Bit}(S_1) \cap \mathrm{Bit}(S_2)] = [\mathrm{Bit}(S_1)][\mathrm{Bit}(S_2)] = (\alpha_1\alpha_2 + \beta_1\beta_2)\sigma_{2,2}$, so the number of lines bitangent to S_1 and S_2 is

$$\tfrac{1}{4} d_1(d_1 - 1)(d_1 - 2)(d_1 - 3) d_2(d_2 - 1)(d_2 - 2)(d_2 - 3) \\ + \tfrac{1}{4} d_1(d_1 - 2)(d_1 - 3)(d_1 + 3) d_2(d_2 - 2)(d_2 - 3)(d_2 + 3)\,.$$

Example 4.5.7. Let $S \subset \mathbb{P}^3$ be a general surface of degree d_1 with $d_1 \geq 4$, and let $C \subset \mathbb{P}^3$ be a general curve of degree d_2 and geometric genus g with $d_2 \geq 2$. To find the number of lines bitangent to S and secant to C, it suffices to compute the cardinality of $\mathrm{Bit}(S) \cap \mathrm{Sec}(C)$. Theorem 4.4.6 and Theorem 4.2.5 imply that

$$\begin{aligned} [\mathrm{Bit}(S)] &= \tfrac{1}{2} d_1(d_1 - 1)(d_1 - 2)(d_1 - 3)\, \sigma_2 + \tfrac{1}{2} d_1(d_1 - 2)(d_1 - 3)(d_1 + 3)\, \sigma_{1,1}\,, \\ [\mathrm{Sec}(C)] &= \big(\tfrac{1}{2}(d_2 - 1)(d_2 - 2) - g\big)\, \sigma_2 + \tfrac{1}{2} d_2(d_2 - 1)\, \sigma_{1,1}\,. \end{aligned}$$

It follows that $[\mathrm{Bit}(S) \cap \mathrm{Sec}(C)] = [\mathrm{Bit}(S)][\mathrm{Sec}(C)] = \gamma\sigma_{2,2}$ where

$$\gamma := \tfrac{1}{4} d_1(d_1 - 1)(d_1 - 2)(d_1 - 3)\big((d_2 - 1)(d_2 - 2) - 2g\big) \\ + \tfrac{1}{4} d_1(d_1 - 2)(d_1 - 3)(d_1 + 3) d_2(d_2 - 1)\,,$$

so the number of lines bitangent to S and secant to C is γ.

4.6 Singular Loci of Congruences

This section investigates the singular points of the secant, bitangent, and inflectional congruences. We begin with the singularities of the secant locus for a smooth irreducible curve.

Proposition 4.6.1. *Let C be a nondegenerate smooth irreducible curve in $\mathbb{P}^3$. If L is a line that intersects the curve C in three or more distinct points, then the line L corresponds to a singular point in* $\mathrm{Sec}(C)$.

Proof. The symmetric square $C^{(2)}$ is the quotient of $C \times C$ by the action of the symmetric group $\mathfrak{S}_2$, so points in this projective variety are unordered pairs of points on C; see [18, pp. 126–127]. The map $\varpi : C^{(2)} \to \mathrm{Sec}(C)$, defined by sending

$\{x, y\}$ to the line spanned by the points x and y if $x \neq y$ or to the tangent line $T_x(C)$ if $x = y$, is a birational morphism. Since $|L \cap C| \geq 3$, the fibre $\varpi^{-1}(L)$ is a finite set containing more than one element. Hence, $\varpi^{-1}(L)$ is not connected and the Zariski Connectedness Theorem [30, Section III.9.V] proves that $\mathrm{Sec}(C)$ is singular at L. □

Lemma 4.6.2. *If $f \in \mathbb{C}[[z, w]]$ satisfies $f(z, w) = -f(w, z)$, then the linear form $z - w$ divides the power series f.*

Proof. We write the formal power series f as a sum of homogeneous polynomials $f = \sum_{i \in \mathbb{N}} f_i$. Since we have $f(z, w) + f(w, z) = 0$, it follows that, in each degree i, we have $f_i(z, w) + f_i(w, z) = 0$. In particular, we see that $f_i(w, w) = 0$. If we consider $f_i(w, z)$ as a polynomial in the variable z with coefficients in $\mathbb{C}[w]$, it follows that w is a root of f_i. Thus, we conclude that $z - w$ divides f_i for all $i \in \mathbb{N}$. □

Theorem 4.6.3. *Let C be a nondegenerate smooth irreducible curve in $\mathbb{P}^3$. If a point in* $\mathrm{Sec}(C)$ *corresponds to a line L that intersects C in a single point x, then the intersection multiplicity of L and C at x is at least 2. Moreover, the line L corresponds to a smooth point of* $\mathrm{Sec}(C)$ *if and only if the intersection multiplicity is exactly 2.*

We thank Jenia Tevelev for help with the following proof.

Proof. Suppose the line L intersects the curve C at the point x with multiplicity 2. Without loss of generality, we may work in the affine open subset with $x_3 \neq 0$, and we assume that $x = (0:0:0:1)$ and $L = \mathrm{V}(x_1, x_2)$. Since C is smooth, there is a local analytic isomorphism φ from a neighbourhood of the origin in $\mathbb{A}^1$ to a neighbourhood of the point x in C. The map φ will have the form $\varphi(z) = \big(\varphi_0(z), \varphi_1(z), \varphi_2(z)\big)$ for some $\varphi_0, \varphi_1, \varphi_2 \in \mathbb{C}[[z]]$. We have $\varphi_0'(0) \neq 0$ and $\varphi_1'(0) = \varphi_2'(0) = 0$ because L is the tangent to the curve C at x. After making an analytic change of coordinates, we may assume that $\varphi(z) = \big(z, \varphi_1(z), \varphi_2(z)\big)$. As L is a simple tangent, at least one of φ_1 and φ_2 must vanish at 0 with order exactly 2. Hence, we may assume that $\varphi_1(z) = z^2 + z^3 f(z)$ and $\varphi_2(z) = z^2 g(z)$ for some $f, g \in \mathbb{C}[[z]]$. The line spanned by the distinct points $\varphi(z)$ and $\varphi(w)$ on the curve C is given by the row space of the matrix

$$\begin{bmatrix} z & z^2 + z^3 f(z) & z^2 g(z) & 1 \\ w & w^2 + w^3 f(w) & w^2 g(w) & 1 \end{bmatrix}.$$

The Plücker coordinates are skew-symmetric power series, so Lemma 4.6.2 implies that they are divisible by $z - w$. In particular, if $f(z) = \sum_i a_i z^i$, then we have

$p_{0,3} = z - w,$

$$p_{0,1} = z\big(w^2 + w^3 f(w)\big) - w\big(z^2 + z^3 f(z)\big) = -zw(z-w)\Big(1 + \sum_i a_i \sum_{j=0}^{i+1} w^j z^{i+1-j}\Big),$$

$$p_{1,3} = z^2 + z^3 f(z) - w^2 - w^3 f(w) = (z-w)\Big(z + w + \sum_i a_i \sum_{j=0}^{i+2} z^j w^{i+2-j}\Big).$$

The symmetric square $(\mathbb{A}^1)^{(2)}$ of the affine line $\mathbb{A}^1$ is a smooth surface isomorphic to the affine plane $\mathbb{A}^2$; see [18, Example 10.23]. Consider the map $\varpi : (\mathbb{A}^1)^{(2)} \to \mathrm{Sec}(C)$ defined by sending the pair $\{z, w\}$ of points in $\mathbb{A}^1$ to the line spanned by the points $\varphi(z)$ and $\varphi(w)$ if $z \neq w$ or to the tangent line of C at $\varphi(z)$ if $z = w$. In other words, the map ϖ sends $\{z, w\}$ to $\big(-zw + h_1(z,w) : \frac{p_{0,2}}{z-w} : 1 : \frac{p_{1,2}}{z-w} : z + w + h_2(z,w) : \frac{p_{2,3}}{z-w}\big)$ where

$$h_1(z,w) := -zw \sum_i a_i \sum_{j=0}^{i+1} w^j z^{i-j+1} \qquad \text{and} \qquad h_2(z,w) := \sum_i a_i \sum_{j=0}^{i+2} z^j w^{i+2-j}.$$

Since the forms zw and $z+w$ are local coordinates of $(\mathbb{A}^1)^{(2)}$ in a neighbourhood of the origin, we conclude that ϖ is a local isomorphism and $\mathrm{Sec}(C)$ is smooth at the point corresponding to L.

Suppose the line L intersects the curve C at the point x with multiplicity at least 3. It follows that the line L is contained in the Zariski closure of the set of lines that intersect C in at least three points or that intersect C in two points, one with multiplicity at least 2. By Proposition 4.6.1 and Lemma 2.3 in [2], we conclude that the line is singular in $\mathrm{Sec}(C)$. $\square$

Corollary 4.6.4. *Let C be a nondegenerate smooth irreducible curve in $\mathbb{P}^3$. If the line L corresponds to a point in $\mathrm{Sec}(C)$, then L corresponds to a singular point of $\mathrm{Sec}(C)$ if and only if one of the following three conditions is satisfied:*

- *the line L intersects the curve C in 3 or more distinct points,*
- *the line L intersects the curve C in exactly 2 points and L is the tangent line to one of these two points,*
- *the line L intersects the curve C at a single point with multiplicity at least 3.*

Proof. Combine Proposition 4.6.1, Lemma 2.3 in [2], and Theorem 4.6.3. $\square$

Analogously, we want to describe the singularities of the inflectional locus $\mathrm{Infl}(S)$ and the bitangent locus $\mathrm{Bit}(S)$ of a surface $S \subset \mathbb{P}^3$.

Theorem 4.6.5. *If $S \subset \mathbb{P}^3$ is an irreducible smooth surface of degree at least 4 which does not contain any lines, then the singular locus of* $\mathrm{Infl}(S)$ *corresponds to lines which either intersect S with multiplicity at least 3 at two or more distinct points, or intersect S with multiplicity at least 4 at some point.*

Proof. We consider the incidence variety

$$\Psi_S := \overline{\{(x,L) : L \text{ intersects } S \text{ at } x \text{ with multiplicity } 3\}} \subset S \times \mathrm{Gr}(1,\mathbb{P}^3)\,.$$

The projection $\pi : \Psi_S \to \mathrm{Infl}(S)$, defined by $(x,L) \mapsto L$, is a surjective morphism. Since S does not contain any lines, all fibres of π are finite and Lemma 14.8 in [18] implies that the map π is finite. Moreover, the general fibre of π has cardinality one, so π is birational. To apply Lemma 4.2.2, we need to examine the singularities of Ψ_S and the differential of π.

Let $f \in \mathbb{C}[x_0,x_1,x_2,x_3]$ be the defining equation for S in $\mathbb{P}^3$. Consider the affine chart in $\mathbb{P}^3 \times \mathrm{Gr}(1,\mathbb{P}^3)$ where $x_0 \neq 0$ and $p_{0,1} \neq 0$. We may assume $x = (1 : \alpha : \beta : \gamma)$ and the line L is spanned by the points $(1 : 0 : a : b)$ and $(0 : 1 : c : d)$. In this affine chart, S is defined by $g_0(x_1,x_2,x_3) := f(1,x_1,x_2,x_3)$. As in the proof of Theorem 4.2.3, we have $x \in L$ if and only if $a = \beta - \alpha c$ and $b = \gamma - \alpha d$. Parametrizing the line L by $\ell(s,t) := (s : s\alpha + t : s\beta + tc : s\gamma + td)$ for $(s:t) \in \mathbb{P}^1$ shows that L intersects S with multiplicity at least m at x if and only if $f\big(\ell(s,t)\big)$ is divisible by t^m. This is equivalent to

$$\frac{\partial}{\partial t}\Big[f\big(\ell(s,t)\big)\Big]\Big|_{(1,0)} = \frac{\partial^2}{\partial t^2}\Big[f\big(\ell(s,t)\big)\Big]\Big|_{(1,0)} = \cdots = \frac{\partial^{m-1}}{\partial t^{m-1}}\Big[f\big(\ell(s,t)\big)\Big]\Big|_{(1,0)} = 0\,.$$

Setting $g_k := \big[\frac{\partial}{\partial x_1} + c\frac{\partial}{\partial x_2} + d\frac{\partial}{\partial x_3}\big]^k g_0$ for $k \geq 1$, the incidence variety Ψ_S can be written on the chosen affine chart as

$$\big\{(\alpha,\beta,\gamma,a,b,c,d) : g_k(\alpha,\beta,\gamma) = 0 \text{ for } 0 \leq k \leq 2,\ a = \beta - \alpha c,\ b = \gamma - \alpha d\big\}\,.$$

As $\dim \Psi_S = 2$, it is smooth at the point (x,L) if and only if its tangent space has dimension 2 or, equivalently, its Jacobian matrix

$$\begin{bmatrix}
\frac{\partial g_0}{\partial x_1}(\alpha,\beta,\gamma) & \frac{\partial g_0}{\partial x_2}(\alpha,\beta,\gamma) & \frac{\partial g_0}{\partial x_3}(\alpha,\beta,\gamma) & 0 & 0 & 0 & 0 \\
\frac{\partial g_1}{\partial x_1}(\alpha,\beta,\gamma) & \frac{\partial g_1}{\partial x_2}(\alpha,\beta,\gamma) & \frac{\partial g_1}{\partial x_3}(\alpha,\beta,\gamma) & 0 & 0 & \frac{\partial g_0}{\partial x_2}(\alpha,\beta,\gamma) & \frac{\partial g_0}{\partial x_3}(\alpha,\beta,\gamma) \\
\frac{\partial g_2}{\partial x_1}(\alpha,\beta,\gamma) & \frac{\partial g_2}{\partial x_2}(\alpha,\beta,\gamma) & \frac{\partial g_2}{\partial x_3}(\alpha,\beta,\gamma) & 0 & 0 & 2\frac{\partial g_1}{\partial x_2}(\alpha,\beta,\gamma) & 2\frac{\partial g_1}{\partial x_3}(\alpha,\beta,\gamma) \\
-c & 1 & 0 & -1 & 0 & -\alpha & 0 \\
-d & 0 & 1 & 0 & -1 & 0 & -\alpha
\end{bmatrix}$$

has rank five. Since S is smooth, the first 2 and the last 2 rows of the Jacobian matrix are linearly independent. If Ψ_S is singular at (x, L), then the third row is a linear combination of the others; specifically, there exist scalars $\lambda, \mu \in \mathbb{C}$ such that $\frac{\partial g_2}{\partial x_j}(\alpha, \beta, \gamma) = \lambda \frac{\partial g_1}{\partial x_j}(\alpha, \beta, \gamma) + \mu \frac{\partial g_0}{\partial x_j}(\alpha, \beta, \gamma)$ for $1 \le j \le 3$. It follows that $g_3(\alpha, \beta, \gamma) = \lambda g_2(\alpha, \beta, \gamma) + \mu g_1(\alpha, \beta, \gamma) = 0$. Thus, the line L intersects the surface S at the point x with multiplicity at least 4 if Ψ_S is singular at (x, L).

It remains to show that the differential $d_{(x,L)}\pi : T_{(x,L)}(\Psi_S) \to T_L\big(\operatorname{Infl}(S)\big)$ is not injective if and only if the line L intersects the surface S at the point x with multiplicity at least 4. The differential $d_{(x,L)}\pi$ sends every element in the kernel of the Jacobian matrix to its last four coordinates. This map is not injective if and only if the kernel contains an element of the form $\begin{bmatrix} * & * & * & 0 & 0 & 0 & 0 \end{bmatrix}^\top \neq 0$. Such an element belongs to the kernel if and only if it equals $\begin{bmatrix} \lambda & c\lambda & d\lambda & 0 & 0 & 0 & 0 \end{bmatrix}^\top$ for some $\lambda \in \mathbb{C} \setminus \{0\}$ and $g_1(\alpha, \beta, \gamma) = g_2(\alpha, \beta, \gamma) = g_3(\alpha, \beta, \gamma) = 0$. This shows that the line L intersects the surface S at the point x with multiplicity at least 4 if and only if $d_{(x,L)}$ is not injective.

Finally, the fibre $\pi^{-1}(L)$ consists of more than one point if and only if L intersects S with multiplicity at least 3 at two or more distinct points, so Lemma 4.2.2 completes the proof. □

Proposition 4.6.6. *Let $S \subset \mathbb{P}^3$ be a general irreducible surface of degree at least 4. If L is a line that is tangent to S at three or more distinct points, then the line L corresponds to a singular point of* $\operatorname{Bit}(S)$*.*

Proof. As in the proof of Proposition 4.6.1, the symmetric square $S^{(2)}$ is the quotient of $S \times S$ by the action of the symmetric group $\mathfrak{S}_2$. The projection ϖ from

$$\overline{\big\{(\{x,y\}, L) : x \neq y,\ x, y \in L \subset T_x(S) \cap T_y(S)\big\}} \subset S^{(2)} \times \operatorname{Gr}(1, \mathbb{P}^3)$$

onto $\operatorname{Bit}(S)$, defined by sending the pair $(\{x,y\}, L) \mapsto L$ is a birational morphism. The fibre $\varpi^{-1}(L)$ is a finite set containing more than one element if L is tangent to S in at least three distinct points. Hence, $\varpi^{-1}(L)$ is not connected and the Zariski Connectedness Theorem [30, Section III.9.V] proves that $\operatorname{Bit}(S)$ is singular at L. □

We do not yet have a full understanding of points in $\operatorname{Bit}(S)$ for which the corresponding lines have an intersection multiplicity greater than 4 at a point of S. We know that a line L that is tangent to the surface S at exactly two points corresponds to a smooth point in $\operatorname{Bit}(S)$ if and only if the intersection multiplicity of L and S at both points is exactly 2. Moreover, given a line L that is tangent to S at a single point, the intersection multiplicity of L and S at this point is at least

4, and the line L corresponds to a smooth point of $\mathrm{Bit}(S)$ when the multiplicity is exactly four; see [2, Lemma 4.3]. To complete this picture, we make the following prediction.

Conjecture 4.6.7. *Let $S \subset \mathbb{P}^3$ be a general irreducible surface of degree at least 4. If a point in the bitangent congruence $\mathrm{Bit}(S)$ corresponds to a line L that is tangent to S at a single point x such that the intersection multiplicity of L and S at x is at least 5, then L corresponds to a singular point of $\mathrm{Bit}(S)$.*

Chapter 5

Relative Realizability

An ideal $I \subset \mathbb{C}[c_1, \dots, c_k, x_1, \dots, x_n]$ defines a (not necessarily irreducible) variety $\mathcal{X} = \mathrm{V}(I) \subset \mathbb{A}^k \times \mathbb{A}^n$. We consider the projection

$$\begin{array}{ccc} \mathcal{X} = \mathrm{V}(I) & \subset & \mathbb{A}^k \times \mathbb{A}^n \\ \downarrow & & \\ \mathbb{A}^k. & & \end{array}$$

For fixed $a \in \mathbb{A}^k$ we denote by X_a the fiber of $\mathcal{X}$ over a. We will refer to $\mathcal{X}$ as a family of algebraic varieties over $\mathbb{A}^k$, however no hypothesis of flatness is made on $\mathcal{X}$. The ideal I_a of $X_a \subset \mathbb{A}^n$ is the image of I under the ring homomorphism

$$\begin{array}{rccc} i_a : & \mathbb{C}[c_1, \dots, c_k, x_1, \dots, x_n] & \to & \mathbb{C}[x_1, \dots x_n] \\ & c_i & \mapsto & a_i \\ & x_i & \mapsto & x_i. \end{array}$$

We denote by $\mathrm{T}^n \subset \mathbb{A}^n$ the torus $\mathrm{T}^n = \mathbb{A}^n \smallsetminus \mathrm{V}(x_1 \cdots x_n)$ and, for an ideal $J \subset \mathbb{C}[x_1, \dots, x_n]$, we denote by $\mathrm{V}^{\mathrm{va}}(J)$ the vanishing locus $\mathrm{V}(J) \cap \mathrm{T}^n$ of J in the torus. We study the tropicalization of the intersection $\mathrm{V}^{\mathrm{va}}(I_a)$ of the fibers of $\mathrm{V}(I)$ with the torus T^n. We will simply denote this tropicalization as $\mathrm{Trop}(I_a)$ or $\mathrm{Trop}(X_a)$. Both $\mathrm{Trop}(I_a)$ and $\mathrm{Trop}(X_a)$ strictly mean $\mathrm{Trop}(\mathrm{V}^{\mathrm{va}}(I_a))$.

Let Σ be a tropical curve, i.e., a list of rays $\{\rho_1, \dots, \rho_l\}$ with multiplicities $\{m_1, \dots, m_l\}$. In this chapter we introduce algorithms to describe the set

$$\mathrm{Real}_\Sigma := \{a \in \mathbb{A}^k \mid \dim(\mathrm{Trop}(I_a)) = 1 \text{ and } \mathrm{mult}(\rho_i, \mathrm{Trop}(I_a)) \geq m_i \text{ for } i = 1, \dots, l\}.$$

When Σ is a tropical curve of degree d, and the intersection $\mathrm{V}^{\mathrm{va}}(I_a)$ of the fibers with the torus are known to be degree d curves, then the tropicalization $\mathrm{Trop}(I_a)$ of the fibers over Real_Σ equal Σ. The realizability locus Real_Σ is in general a constructible set and it is not necessarily closed. In this Chapter we will mainly be interested in its Zariski closure $\overline{\mathrm{Real}_\Sigma}$.

The main result of this chapter is Algorithm 5.4.1, which takes as input the ideal I and the tropical curve Σ and produces as output the ideal of the Zariski closure of Real_Σ in $\mathbb{A}^k$. The Algorithm requires some resource-demanding operations, such as repeated computations of comprehensive Gröbner bases.

The structure of the Chapter is the following. In Sections 5.1 and 5.2 we restrict our attention to a single ray. In Sections 5.1 we describe the set of fibers whose fiber contain the ray $\mathrm{pos}(e_1)$. In Sections 5.2 we describe the set of fibers whose fiber contain the ray $\mathrm{pos}(e_1)$ with at least a fixed multiplicity m. Section 5.3 shows how to adapt these algorithms to describe, for any given ray ρ and weight m, the set of fibers whose tropicalization contains the ray ρ with multiplicity at least m. Finally in Section 5.4 we combine all the information coming from the rays of Σ in Algorithm 5.4.1.

5.1 Local Set-Theoretical Computation

Let $I \subset \mathbb{C}[c_1, \dots, c_k, x_1, \dots, x_n]$ be an ideal. In this section we study the set

$$\mathrm{Real}_1(I) \coloneqq \{a \in \mathbb{A}^k \mid \mathrm{pos}(e_1) \subset \mathrm{Trop}(I_a)\}$$

of parameters a such that the tropical variety $\mathrm{Trop}(I_a)$ contains $\mathrm{pos}(e_1)$.

By the Tevelev Lemma (see [44, Lemma 2.2]) the tropical variety $\mathrm{Trop}(I_a)$ intersects the relative interior of the ray $\mathrm{pos}(e_1)$ if and only if $\overline{\mathrm{V^{va}}(I_a)}$ intersects the open torus orbit

$$\mathcal{O}_1 \coloneqq \{(x_1, \dots, x_n) \in \mathbb{A}^n \mid x_1 = 0,\ x_i \neq 0 \text{ for } i = 2, \dots n\}.$$

We note that, as $\mathrm{pos}(e_1)$ is a ray, $\mathrm{Trop}(I_a)$ intersects $\mathrm{pos}(e_1)$ in its relative interior if and only if $\mathrm{pos}(e_1)$ is completely contained in $\mathrm{Trop}(I_a)$. This gives the following equality of sets:

$$\mathrm{Real}_1(I) = \{a \in \mathbb{A}^k \mid \overline{\mathrm{V^{va}}(I_a)} \cap \mathcal{O}_1 \neq \varnothing\}. \tag{5.1}$$

We also consider the following subset of $\mathbb{A}^k \times \mathbb{A}^n$:

$$\mathbf{Real}_1(I) \coloneqq \{(a, x) \in \mathbb{A}^k \times \mathbb{A}^n \mid x \in \overline{\mathrm{V^{va}}(I_a)} \cap \mathcal{O}_1\}. \tag{5.2}$$

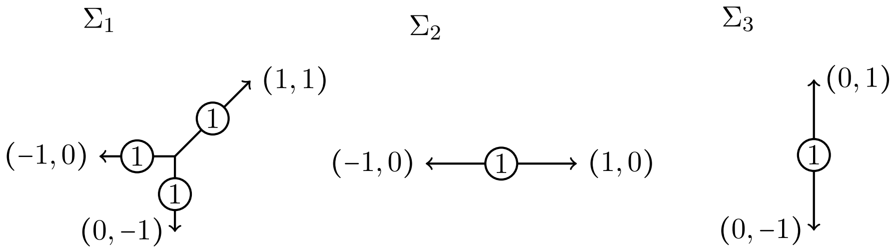

Figure 5.1: The tropical curves of Example 5.1.1.

Whenever the ideal I is clear from the context we will simply denote these sets as Real_1 and $\mathbf{Real}_1$. The sets Real_1 and $\mathbf{Real}_1$ are constructible sets and we denote by $\pi : \mathbf{Real}_1 \to \mathrm{Real}_1$ the restriction of the projection $\mathbb{A}^{k+n} \to \mathbb{A}^k$.

Example 5.1.1. Consider the family $\mathrm{V}(I) \subset \mathbb{A}^2 \times \mathbb{A}^2$ where $I = (c_1x + c_2y + xy) \subset \mathbb{C}[c_1, c_2, x, y]$. For $a = (a_1, a_2) \in \mathbb{C}^2 \setminus \{(0,0)\}$, the tropical variety $\mathrm{Trop}(I_a)$ is depicted in Figure 5.1: for $a_1, a_2 \neq 0$ it is the curve Σ_1, for $a_1 = 0, a_2 \neq 0$ it is the curve Σ_2, for $a_1 \neq 0, a_2 = 0$ it is the curve Σ_3. For $a = (0,0)$ the tropical variety $\mathrm{Trop}(I_a)$ is empty. In particular $e_1 \in \mathrm{Trop}(I_a)$ if and only if $a_1 \neq 0$ and $a_2 = 0$. Equivalently $\mathrm{Real}_1 = \{(c_1, c_2) \mid c_1 \neq 0, c_2 = 0\}$.

The basic algebraic tools we will use are saturation and elimination theory; for a complete introduction see, for example, [11, Section 14.1] and [7, Chapter 3, Section 15.10.6]. Given an ideal I in a ring R, and an element $f \in R$, the saturation ideal $(I : f^\infty)$ is by definition

$$(I : f^\infty) := \{g \in R \mid f^n g \in I \text{ for some } n > 0\}.$$

The ideal I is said to be *saturated* with respect to f if $I = (I : f^\infty)$.

When R is the polynomial ring $R = \mathbb{C}[x_1, \dots, x_n]$ we have $\mathrm{V}(I : f^\infty) = \overline{\mathrm{V}(I) \setminus \mathrm{V}(f)}$. In particular $\mathrm{V}(I : (x_1 \cdots x_n)^\infty) = \overline{\mathrm{V}^{\mathrm{va}}(I)}$.

Given an ideal $I \subset \mathbb{C}[c_1, \dots, c_k, x_1, \dots, x_n]$, the ideal $I \cap \mathbb{C}[c_1, \dots c_k]$ can be computed via elimination theory. The variety $\mathrm{V}(I \cap \mathbb{C}[c_1, \dots c_k])$ is the Zariski closure of the projection of $\mathrm{V}(I) \subset \mathbb{A}^{k+n}$ to $\mathbb{A}^k$.

Equations (5.1) and (5.2) suggest a naive approach to compute the closure of $\mathbf{Real}_1$ and Real_1. As $\overline{\mathrm{V}^{\mathrm{va}}(I_a)} = V(I_a : (x_1 \cdots x_n)^\infty)$ and $\mathcal{O}_1 = (\mathrm{V}(x_1) \setminus \mathrm{V}(x_2 \cdots x_n))$, the space $\mathbf{Real}_1$ is given by points (a, x) satisfying

$$x \in \overline{\mathrm{V}^{\mathrm{va}}(I_a)} \cap \mathcal{O}_1 = V(I_a : (x_1 \cdots x_n)^\infty) \cap (\mathrm{V}(x_1) \setminus \mathrm{V}(x_2 \cdots x_n)). \tag{5.3}$$

We now describe a naive algorithm to compute the ideals of $\mathbf{Real}_1$ and Real_1. The naive algorithm approximates the ideal of $\mathbf{Real}_1$ by replacing in Equation (5.3) the saturation of the ideal of the fiber I_a with a saturation of the entire ideal I

$$(I_a : (x_1 \cdots x_n)^\infty) \rightsquigarrow (I : (x_1 \cdots x_n)^\infty)_a, \tag{5.4}$$

and, with the notation $A_2 = (I_a : (x_1 \cdots x_n)^\infty) + (x_1)$, the set theoretic difference by $\mathrm{V}(x_2 \cdots x_n)$ is replaced by a saturation

$$\mathrm{V}(A_2) \smallsetminus \mathrm{V}(x_2 \cdots x_n) \rightsquigarrow \mathrm{V}(A_2 : (x_2 \cdots x_n)^\infty).$$

Finally from the approximation $J = (A_2 : (x_2 \cdots x_n)^\infty)$ of $\mathrm{Ideal}(\mathbf{Real}_1)$, elimination theory is used to approximate the ideal $\mathrm{Ideal}(\mathrm{Real}_1)$

$$\mathrm{Ideal}(\mathrm{Real}_1) \rightsquigarrow J \cap \mathbb{C}[c_1, \ldots, c_k].$$

To sum up, the naive algorithm performs the following operations:

1. $A_1 = (I : x_1^\infty)$
2. $A_2 = A_1 + (x_1)$
3. $J = (A_2 : (x_2 \cdots x_n)^\infty)$
4. $C = J \cap \mathbb{C}[c_1, \ldots, c_k]$
5. `return` (C, J).

The naive approach does not work well as the following example shows.

Example 5.1.2. Let us consider the ideal $I = (c_2 x^2 + c_2 xy + c_1) \subset \mathbb{C}[c_1, c_2, x, y]$. The tropical variety $\mathrm{Trop}(I_{(a_1,a_2)})$ for $a_2 \neq 0$ is depicted in Figure 5.2: it is the tropical curve Σ_1 for $a_1, a_2 \neq 0$ and the tropical curve Σ_2 for $a_1 = 0, a_2 \neq 0$. The tropical variety $\mathrm{Trop}(I_{(a_1,a_2)})$ is empty for $a_1 \neq 0$ and $a_2 = 0$, and it is the whole $\mathbb{R}^2$ when $a_1 = a_2 = 0$. In particular we have $\mathrm{Real}_1 = \{(0,0)\}$. We would wish the naive Algorithm to produce as output the ideal (c_1, c_2). However, running it we get

- $A_1 = (I : x^\infty) = I$,
- $A_2 = A_1 + (x) = I + (x) = (c_1, x)$,
- $J = (A_2 : y^\infty) = (c_1, x)$,
- $C = J \cap \mathbb{C}[c_1, c_2] = (c_1)$.

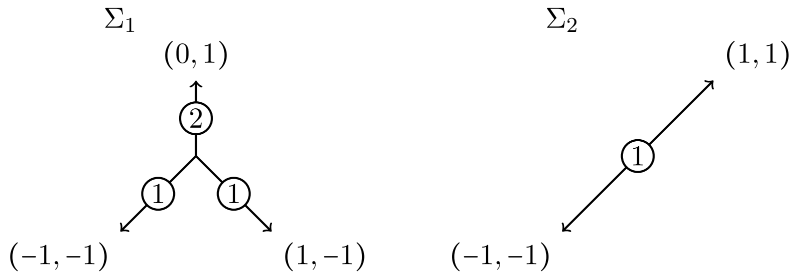

Figure 5.2: The tropical curves of Example 5.1.2.

This happens because performing the saturation on the entire family as we do in Equation (5.4) is a weaker operation than performing it on the fiber. We will give a precise statement of this fact in Corollary 5.1.9.

We note that, in Example 5.1.2, while the ideal $I = (c_2x^2 + c_2xy + c_1)$ is saturated with respect to (x), the ideal $I + (c_1) = (c_2x^2 + c_2xy, c_1)$ is not. Actually, we can observe that running again the naive Algorithm on $I' = I + (c_1)$, we get as output $C' = (1)$ as desired. This is a general fact: the ideal of Real_1 can be computed by a repeated application of the naive Algorithm. More precisely, we introduce the following algorithm.

Algorithm 5.1.3.

Input:

I, ideal in $\mathbb{C}[c_1, \dots, c_k, x_1, \dots, x_n]$.

Output:

C, ideal in $\mathbb{C}[c_1, \dots, c_k]$ satisfying $\mathrm{V}(C) = \overline{\mathrm{Real}_1}(I)$,

J, ideal in $\mathbb{C}[c_1, \dots, c_k, x_1, \dots x_n]$ satisfying $\mathrm{V}(J_a) = \overline{(\mathbf{Real}_1(I))_a}$ for a general in $\mathrm{V}(C)$.

$\{$
$C = (0) \subset \mathbb{C}[c_1, \dots, c_k]$;
$J_{\mathrm{old}} = (1)$;
$J = (0) \subset \mathbb{C}[c_1, \dots, c_k, x_1, \dots, x_n]$;
`while` $(J_{\mathrm{old}} \neq J)$ `do` $\{$
$\quad J_{\mathrm{old}} = J$;
$\quad A_1 = ((I + C) : x_1^\infty)$;
$\quad A_2 = A_1 + (x_1)$;
$\quad J = (A_2 : (x_2 \cdot \ldots \cdot x_n)^\infty)$;

$C = \mathrm{Radical}(J \cap \mathbb{C}[c_1, \dots, c_k]);$
}
`return` $(C, J);$
}

Proof of correctness. This follows from Lemma 5.1.4, Proposition 5.1.11 and Theorem 5.1.12. □

For the proofs of the following results we denote by $A_1^{(i)}$, $A_2^{(i)}$, $J^{(i)}$, $C^{(i)}$ the ideals A_1, A_2, J, C as computed at the i^{th} iteration of the `while` loop of Algorithm 5.1.3.

Lemma 5.1.4. *For every $i > 0$ we have $C^{(i)} \subset C^{(i+1)}$ and $J^{(i)} \subset J^{(i+1)}$. In particular, by Noetherianity, Algorithm 5.1.3 terminates after a finite number of steps.*

Proof. Fix $i > 0$ and consider the ideal $C^{(i)} \subset \mathbb{C}[c_1, \dots, c_k]$. We have $C^{(i)} \subset A_1^{(i+1)} \subset A_2^{(i+1)} \subset J^{(i+1)}$, and hence $C^{(i)} \subset C^{(i+1)} = \mathrm{Radical}(J^{(i+1)} \cap \mathbb{C}[c_1, \dots, c_k])$.

From $C^{(i)} \subset C_1^{(i+1)}$ it follows that $A_1^{(i)} \subset A_1^{(i+1)}$, $A_2^{(i)} \subset A_2^{(i+1)}$ and, finally, $J^{(i)} \subset J^{(i+1)}$. □

Remark 5.1.5. We have $I \subset A_1^{(0)} \subset A_2^{(0)} \subset J_1^{(0)}$. In particular, using Lemma 5.1.4 we have that $\mathrm{V}(J) \subset \mathrm{V}(I)$.

We recall the following fact about saturations.

Lemma 5.1.6. *Let $J, K \subset \mathbb{C}[y_1, \dots, y_m]$ be ideals and $f \in \mathbb{C}[y_1, \dots, y_m]$. Then $(J : f^\infty) + K \subset (J + K : f^\infty)$ and $((J : f^\infty) + K : f^\infty) = (J + K : f^\infty)$.*

We will mostly use Lemma 5.1.6 in the form of the following corollary.

Corollary 5.1.7. *Let $J \subset \mathbb{C}[c_1, \dots, c_k, x_1, \dots, x_n]$ be an ideal and $f \in \mathbb{C}[x_1, \dots, x_n]$, then, for every $a \in \mathbb{C}^k$, $(J : f^\infty)_a \subset (J_a : f^\infty)$ and $(J_a : f^\infty) = ((J : f^\infty)_a : f^\infty)$.*

Proof. This follows from Lemma 5.1.6, after identifying the ideal $J_a \subset \mathbb{C}[x_1, \dots, x_n]$ with the ideal $J + (c_1 - a_1, \dots, c_k - a_k) \subset \mathbb{C}[c_1, \dots, c_k, x_1, \dots, x_n]$. □

We are now ready to prove the first relation between the ideals C and J and the spaces Real_1 and $\mathbf{Real}_1$, namely that $\mathrm{Real}_1 \subset \mathrm{V}(C)$ and $\mathbf{Real}_1 \subset \mathrm{V}(J)$.

Proposition 5.1.8. *Let (C, J) be the output of Algorithm 5.1.3 with input I. Then we have $\mathrm{Real}_1 \subset \mathrm{V}(C)$ and $\mathbf{Real}_1 \subset \mathrm{V}(J)$.*

Proof. As $\mathrm{Real}_1 = \pi(\mathbf{Real}_1)$ and $\pi(\mathrm{V}(J)) \subset \mathrm{V}(C)$, it is enough to prove that $\mathbf{Real}_1 \subset \mathrm{V}(J)$. We prove by induction that $\mathbf{Real}_1 \subset \mathrm{V}(J^{(i)})$ for every $i \geq 0$, the base case $i = 0$ being trivial. Fix $(a, x) \in \mathbf{Real}_1$ and assume $(a, x) \in \mathrm{V}(J^{(i)})$. From $(a, x) \in \mathrm{V}(J^{(i)})$ it follows that $a \in \mathrm{V}(C^{(i)})$. Fix a polynomial $f \in C^{(i)}$. If we regard f as a polynomial in $\mathbb{C}[c_1, \dots, c_k]$ we have $f(a) = 0$ as $a \in \mathrm{V}(C^{(i)})$. Similarly, if we regard f as a polynomial in $\mathbb{C}[c_1, \dots, c_k, x_1, \dots, x_n]$, we have that f_a is the zero polynomial in $\mathbb{C}[x_1, \dots, x_n]$. As a result, for every $f \in C^{(i)}$, we have $f_a = 0$, which means that $C_a^{(i)} = (0)$. It follows that $I_a = I_a + C_a^{(i)} = (I + C^{(i)})_a$. As we assumed that $(a, x) \in \mathbf{Real}_1$ we get, by the definition of $\mathbf{Real}_1$, that x is in $\overline{\mathrm{V}^{\mathrm{va}}(I_a)}$. We have $\overline{\mathrm{V}^{\mathrm{va}}(I_a)} = \overline{\mathrm{V}^{\mathrm{va}}(I_a : x_1^\infty)} = \overline{\mathrm{V}^{\mathrm{va}}((I + C^{(i)})_a : x_1^\infty)}$. In particular, $x \in \mathrm{V}((I + C^{(i)})_a : x_1^\infty)$. Corollary 5.1.7 implies that $x \in \mathrm{V}(((I + C^{(i)}) : x_1^\infty)_a) = \mathrm{V}((A_1^{(i+1)})_a)$. By the definition of $\mathbf{Real}_1$ we have $x \in \mathcal{O}_1 = \mathrm{V}(x_1) \setminus \mathrm{V}(x_2 \cdots x_n)$. From $x \in \mathrm{V}(x_1)$ it follows that $x \in \mathrm{V}((A_1^{(i+1)})_a + (x_1)) = \mathrm{V}((A_2^{(i+1)})_a)$. From $x \notin \mathrm{V}(x_2 \cdots x_n)$ it follows that $x \in \mathrm{V}((A_2^{(i+1)})_a) \setminus \mathrm{V}(x_2 \cdots x_n) \subset \mathrm{V}((A_2^{(i+1)})_a : (x_2 \cdots x_n)^\infty)$. Again by Corollary 5.1.7, we have that $x \in \mathrm{V}((A_2^{(i+1)} : (x_2 \cdots x_n)^\infty)_a) = \mathrm{V}(J_a^{(i+1)})$. This last condition is equivalent to $(a, x) \in \mathrm{V}(J^{(i+1)})$, which concludes the proof. □

Corollary 5.1.7 shows that the saturation ideal of a fiber is a contained in the fiber of the saturation of the entire family. The following lemma shows that this containment is generally an equality.

Lemma 5.1.9. *Let $J \subset R := \mathbb{C}[c_1, \dots, c_k, x_1, \dots, x_n]$ be an ideal and suppose that $C = J \cap \mathbb{C}[c_1, \dots, c_k]$ is radical. Suppose that J is saturated with respect to x_1. Then there exists an open dense subset $A \subset \mathrm{V}(C)$ such that, for every $a \in A$, J_a is saturated with respect to x_1.*

The proof of Lemma 5.1.9 is postponed to Section 5.2.1, where we describe an algorithm to compute the complement of the set A.

Corollary 5.1.10. *Let $J \subset \mathbb{C}[c_1, \dots, c_k, x_1, \dots, x_n]$ be an ideal and suppose that $C = J \cap \mathbb{C}[c_1, \dots, c_k]$ is radical. Then there exists an open dense subset $A \subset \mathrm{V}(C)$ such that, for every $a \in A$, $(J : x_1^\infty)_a = (J_a : x_1^\infty)$.*

Proof. Applying Lemma 5.1.9 to $(J : x_1^\infty)$ we get that there exists an open dense subset $A \subset \mathrm{V}(C)$ such that, for every $a \in A$, $(J : x_1^\infty)_a$ is saturated with respect to x_1. Equivalently $(J : x_1^\infty)_a = ((J : x_1^\infty)_a : x_1^\infty)$ and the result now follows from Corollary 5.1.7. □

Lemma 5.1.9 allows us to show that the fibers of $\mathrm{V}(J)$ agree with the fibers of $\mathbf{Real}_1$. This is the content of the following proposition.

Proposition 5.1.11. *Let (C, J) be the output of Algorithm 5.1.3 with input I. Then for $a \in \mathrm{V}(C)$ general we have $\mathrm{V}(J_a) = \overline{(\mathbf{Real}_1)_a}$.*

Proof. We have

$$\begin{aligned}
(\mathbf{Real}_1)_a &= \overline{\mathrm{V}^{\mathrm{va}}(I_a)} \cap \mathcal{O}_1 \\
&= \mathrm{V}\left(I_a : (x_1 \cdots x_n)^\infty\right) \cap (\mathrm{V}(x_1) \smallsetminus \mathrm{V}(x_2 \cdots x_n)) \\
&= \mathrm{V}\left((I_a : (x_1 \cdots x_n)^\infty) + (x_1)\right) \smallsetminus \mathrm{V}(x_2 \cdots x_n) \\
&= \mathrm{V}\left(((I_a : (x_1 \cdots x_n)^\infty) + (x_1)) : (x_2 \cdots x_n)^\infty\right) \smallsetminus \mathrm{V}(x_2 \cdots x_n) \\
&= \mathrm{V}\left(((I_a : x_1^\infty) + (x_1)) : (x_2 \cdots x_n)^\infty\right) \smallsetminus \mathrm{V}(x_2 \cdots x_n),
\end{aligned}$$

where the last equality comes from Lemma 5.1.6. It follows that

$$\overline{(\mathbf{Real}_1)_a} = \mathrm{V}\left(((I_a : x_1^\infty) + (x_1)) : (x_2 \cdots x_n)^\infty\right).$$

By construction we have

$$J = (((I + C : x_1^\infty) + (x_1)) : (x_2 \cdots x_n)^\infty) \tag{5.5}$$

and, therefore,

$$J_a = (((I + C : x_1^\infty) + (x_1)) : (x_2 \cdots x_n)^\infty)_a .$$

We now show that the ideal J satisfies the assumptions of Corollary 5.1.10, namely that $C' = J \cap \mathbb{C}[c_1, \ldots, c_k]$ is radical. By construction we have $C = \mathrm{Radical}(C')$. Moreover, by Equation (5.5), we have $C \subset J$ and therefore $C \subset C'$. This shows that $C' = C$ and in particular $C' = J \cap \mathbb{C}[c_1, \ldots, c_k]$ is radical. Similarly we show that $(I + C) \cap \mathbb{C}[c_1, \ldots c_k] = C$, so that $I + C$ satisfies the assumptions of Corollary 5.1.10. We $C \subset (I + C) \cap \mathbb{C}[c_1, \ldots c_k]$ because $C \subset I + C$ and, since $I \subset J$, we also have $I \cap \mathbb{C}[c_1, \ldots, c_k] \subset J \cap \mathbb{C}[c_1, \ldots, c_k] = C$. This shows that $C = I \cap \mathbb{C}[c_1, \ldots, c_k]$.

By applying Corollary 5.1.10 to J and to $I + C$ we get that, for $a \in \mathrm{V}(C)$ general,

$$\begin{aligned}
J_a &= ((I + C : x_1^\infty) + (x_1))_a : (x_1 \cdots x_n)^\infty \\
&= ((I + C : x_1^\infty)_a + (x_1)) : (x_1 \cdots x_n)^\infty \\
&= (((I + C)_a : x_1^\infty) + (x_1)) : (x_1 \cdots x_n)^\infty .
\end{aligned}$$

where the first equality is Corollary 5.1.10 applied to J, and the third equality is

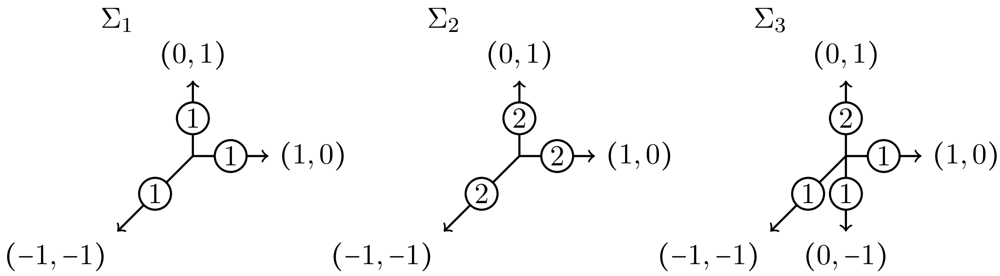

Figure 5.3: The tropical varieties of Example 5.2.1.

Corollary 5.1.10 applied to $I + C$. The result follows from $I_a = (I + C)_a$. $\square$

We are now ready to prove the main results of this section.

Theorem 5.1.12. *Let (C, J) be the output of Algorithm 5.1.3 with input I. Then we have $\overline{\mathrm{Real}_1} = \mathrm{V}(C)$.*

Proof. Let $A \subset \mathrm{V}(C)$ an open dense subset in which the condition of Proposition 5.1.11 holds. We have $\mathrm{V}(J) \cap \pi^{-1}(A) = \mathbf{Real}_1 \cap \pi^{-1}(A)$. In particular $\mathrm{Real}_1 = \pi(\mathbf{Real}_1) \supset \pi(\mathrm{V}(J) \cap A)$ which is dense in $\mathrm{V}(C)$. This shows $\overline{\mathrm{Real}_1} = \mathrm{V}(C)$. $\square$

5.2 Local Weighted Computation

The aim of this section is to describe the Zariski closure of the set

$$\mathrm{Real}_{1,m}(I) := \{a \in \mathbb{A}^k \mid \dim(\mathrm{Trop}(I_a)) = 1 \text{ and } \mathrm{mult}(\mathrm{pos}(e_1), \mathrm{Trop}(I_a)) \geq m\}.$$

We will often omit the ideal I and simply write $\mathrm{Real}_{1,m}$.

Example 5.2.1. Consider in $\mathbb{C}[c_1, c_2, x, y]$ the following ideals: $I_1 = (c_1, x + y + 1)$, $I_2 = (c_2, x+7y+4)$, $I_3 = (c_1-c_2-1, x+c_1y+1)$. Let $I \subset \mathbb{C}[c_1, c_2, x, y]$ be the intersection $I = I_1 \cap I_2 \cap I_3$. The tropicalization of the fiber over a of the projection of $\mathrm{V}(I)$ to $\mathbb{C}^2$ is empty outside $\mathrm{V}(c_1c_2(c_1 - c_2 - 1)) \subset \mathbb{C}^2$ and it is depicted in Figure 5.3 for $a \in \mathrm{V}(c_1c_2(c_1 - c_2 - 1))$: it is the tropical curve Σ_1 for a in $\mathrm{V}(c_1c_2(c_1 - c_2 - 1)) \smallsetminus \{(0,0), (0,-1), (1,0)\}$, it is the tropical curve Σ_2 for a in $\{(0,0), (1,0)\}$ and it is the tropical curve Σ_2 for $a = (0,-1)$. In particular the realization space $\mathrm{Real}_{1,2}$ consists of the points $(0,0)$ and $(1,0)$.

The computation of the multiplicity of the ray $\mathrm{pos}(e_1)$ in the tropicalization of a very affine curve $X \subset \mathrm{T}^n$ can be based on the following generalization of Tevelev Lemma (see, for example, [38, Section 2.5]).

Theorem 5.2.2. *Let $X \subset \mathrm{T}^n$ be an irreducible curve. Then* $\mathrm{mult}(\mathrm{pos}(e_1), \mathrm{Trop}(I))$ *equals the cardinality, counted with multiplicity, of $\overline{X} \cap \mathcal{O}_1$.*

Since we do not assume any flatness condition on the ideal I, the first step in order to describe $\mathrm{Real}_{1,m}$ is to compute the set of parameters a for which $\mathrm{Trop}(I_a)$ has dimension 1. This is carried out in Section 5.2.1. In Section 5.2.2 we use Theorem 5.2.2 to study the multiplicity of $\mathrm{pos}(e_1)$ in the tropicalization of the fibers $\mathrm{V}(I_a)$. Finally, in Section 5.2.3, we combine all this information to describe $\mathrm{Real}_{1,m}$.

5.2.1 Higher dimensional fibers

In this section we describe the set of parameters a such that the tropicalization of the fiber $\mathrm{Trop}(I_a)$ has dimension at most 1. The dimension of $\mathrm{Trop}(I_a) = \mathrm{Trop}(\mathrm{V}^{\mathrm{va}}(I_a))$ equals the dimension of $\mathrm{V}^{\mathrm{va}}(I_a)$. Consider an irreducible component Y of $\mathrm{V}(I)$ and let Z be the closure of the projection of Y to $\mathbb{A}^k$. Let d be the dimension of Z and let $d+l$ be the dimension of Y. By [30, Theorem 2 of Chapter 1 Section 8], for every $a \in Z$, every component of the fiber Y_a has dimension at least l. Now suppose that $l \geq 2$ and fix a point $a \in Z$. If the fiber Y_a is not empty its dimension, and a fortiori the dimension of $\mathrm{V}(I_a)$, is at least 2. Denote by $\overline{Y}$ the closure of Y in $\mathbb{P}^n \times \mathbb{A}^k$. By the Main Theorem of Elimination Theory (see [11, Theorem 14.1]) the fiber $(\overline{Y})_a$ is not empty. As a result, a necessary condition for $\dim(\mathrm{V}^{\mathrm{va}}(I_a)) \leq 1$ is that the fiber $(\overline{Y})_a \subset \mathbb{P}^n$ does not intersect the torus. Finally, a necessary condition for the fiber $(\overline{Y})_a$ not to intersect the torus is that the ideal of $(\overline{Y})_a$ is not saturated.

Example 5.2.3. Consider the ideals $I_1 = (x + cy + cz + c) \subset \mathbb{C}[c, x, y, z]$, $I_2 = (x + 2y - z + 3, y + 2z + 2) \subset \mathbb{C}[c, x, y, z]$, and let I be the intersection ideal $I_1 \cap I_2$. The tropicalization of the fiber $\mathrm{V}(I_a)$, for $a \neq 0$, is the tropical standard plane. However, for $a = 0$, the tropicalization of $\mathrm{Trop}(\mathrm{V}(I_0))$ is the tropical standard line, because $\mathrm{V}(x)$ does not intersect the torus.

Let $J \subset \mathbb{C}[c_1, \ldots, c_k, x_1, \ldots, x_n]$ be an ideal saturated with respect to x_n and suppose that $C = J \cap \mathbb{C}[c_1, \ldots, c_k]$ is prime. A *comprehensive* Gröbner basis is a Gröbner basis $\mathcal{L}$ of J with the extra property that $\mathcal{L}_a = \{f_a \mid a \in \mathcal{L}\}$ is a Gröbner basis of J_a for every $a \in \mathbb{A}^k$. Comprehensive Gröbner bases were first introduced, together with an algorithm to compute them, in [47]. A more efficient algorithm to compute comprehensive Gröbner bases is described in [24]. Let $\mathcal{L}$ be a comprehensive Gröbner basis of J with respect to the graded reverse lexicographical order (see Example 2.2.2). For each $f \in \mathcal{L}$ write $f = \sum_{i \geq 0} f_i x_n^i$, where $f_i \in \mathbb{C}[c_1, \ldots, c_k, x_1, \ldots, x_{n-1}]$

for every i. We can assume that $f_0 \notin J$ or $f = f_0$. Indeed, if $f = f_0 + x_n f'$ with $f_0 \in J$, then $x_n f' \in J$ and hence, by saturation, $f' \in J$. Therefore we can obtain a new comprehensive Gröbner basis of J by removing f from $\mathcal{L}$ and adding to it f_0 and f'. Let $\mathcal{L}_1$ be the subset of $\mathcal{L}$ that consists of polynomials f that satisfy $f \neq f_0$. For every $f \in \mathcal{L}_1$ we denote by $M_f \subset \mathbb{C}[c_1, \ldots, c_k]$ the ideal generated by the coefficients of f_0, where f_0 is now regarded as a polynomial in $(\mathbb{C}[c_1, \ldots, c_k])[x_1, \ldots, x_{n-1}]$. Let $L \subset \mathbb{C}[c_1, \ldots, c_k]$ be the ideal $\bigcap_{f \in \mathcal{L}_1} M_f$.

Proposition 5.2.4. *Let $J \subset \mathbb{C}[c_1, \ldots, c_k, x_1, \ldots, x_n]$ be an ideal saturated with respect to x_n and suppose that $C = J \cap \mathbb{C}[c_1, \ldots, c_k]$ is prime. Then, with the notation introduced above, for every $a \in \mathrm{V}(C) \setminus \mathrm{V}(L+C)$, the ideal J_a is saturated with respect to x_n, moreover the set $\mathrm{V}(C) \setminus \mathrm{V}(L+C)$ is dense in $\mathrm{V}(C)$.*

Proof. Let $\mathcal{L}$ be the comprehensive Gröbner basis of J introduced above and fix $a \in \mathrm{V}(C)$. We have that $\mathcal{L}_a := \{f_a \mid f \in \mathcal{L}\}$ is a Gröbner basis of J_a. Let $\mathcal{L}'_a$ be the set obtained by dividing every element $f_a \in \mathcal{L}_a$ by the highest power of x_n that divides f_a. By [39, Lemma 12.1] $\mathcal{L}'_a$ is a Gröbner basis of $(J_a : x_n^\infty)$. Fix $f \in \mathcal{L}$. If $f \notin \mathcal{L}_1$ then $f \in \mathbb{C}[c_1, \ldots, c_k, x_1, \ldots, x_{n-1}]$ and x_n does not divide f_a for every a. If $f \in \mathcal{L}_1$ and $a \notin \mathrm{V}(M_f)$, by construction, f_a is not divisible by x_n. To sum up, for $a \in \mathrm{V}(C) \setminus \mathrm{V}(L+C)$ none of the polynomials in $\mathcal{L}_a$ is divisible by x_n. As a result we have $\mathcal{L}_a = \mathcal{L}'_a$ and therefore $J_a = (J_a : x_n^\infty)$, since they are generated by the same set of polynomials. To conclude we show that the complement of $\mathrm{V}(C + M_f)$ is dense in $\mathrm{V}(C)$ for every $f \in \mathcal{L}_1$. Let $f \in \mathcal{L}_1$ and write $f_0 = \sum g_u x^u$ with $g_u \in \mathbb{C}[c_1, \ldots, c_k]$ for every u. By the definition of $\mathcal{L}_1$, we have $f \neq f_0$. By our assumption on $\mathcal{L}$ this implies that $f_0 \notin J$ and therefore not all the g_u can be in C. As C is prime, it follows that $\mathrm{V}(C + M_f)$ is a proper closed set in $\mathrm{V}(C)$. ☐

Proof of Lemma 5.1.9. This is a weak formulation of Proposition 5.2.4. ☐

Proposition 5.2.4 allows us to write the following algorithm.

Algorithm 5.2.5.

Input:

J, ideal in $\mathbb{C}[c_1, \ldots, c_k, x_1, \ldots, x_n]$ saturated with respect to x_n.

Output:

$C + L$, ideal in $\mathbb{C}[c_1, \ldots, c_k]$ satisfying $\mathrm{V}(C + L) = \{a \mid J_a \neq (J_a : x_n)\}$.

{

TerminationCondition = False;

```
while (TerminationCondition = False) do
    {
    C = J ∩ ℂ[c_1, ..., c_k];
    C = Radical(C);
    J = J + C;
    compute a comprehensive Gröbener basis 𝓑 of J with respect to the graded
reverse lexicographical order;
    𝓛 = {};
    while(𝓑 ≠ ∅) do
        {
        let f be the first element of 𝓑;
        write f = f_0 + x_n f', where f_0 ∈ ℂ[c_1, ..., c_k, x_1, ..., x_{n-1}];
        if (f_0 ∈ J AND f' ≠ 0) then
            {add f_0 to 𝓛; remove f from 𝓑; add f' to 𝓑};
        else
            {add f to 𝓛; remove f from 𝓑};
        }
    L = C;
    𝓛_1 = {f ∈ 𝓛 | f ∉ ℂ[c_1, ..., c_k, x_1, ..., x_{n-1}]};
    for f in 𝓛_1 do
        {
        write f as ∑ a_i x_n^i with a_i ∈ ℂ[c_1, ..., c_k][x_1, ..., x_{n-1}];
        let M_f ⊂ ℂ[c_1, ..., c_k] be the ideal generated by the coefficients of a_0;
        L = L ∩ M_f;
        };
    if (J + L ≠ (J + L : x_n^∞) OR L = (1)) then TerminationCondition = true;
    J = J + L;
    }
return C + L;
}
```

Proof of correctness. By Proposition 5.2.4 at each step of the `while` loop, the ideal L is such that $\mathrm{V}(L)$ contains the set of a such that $J_a \neq (J_a : x_n)$. Moreover, by Noetherianity and by Proposition 5.2.4 the `while` loop terminates after finitely many steps. When the algorithm terminates we have $J + L \neq (J + L : x_n^\infty)$, in which case $\mathrm{V}(L) = \{a \mid J_a \neq (J_a : x_n)\}$, or $L = (1)$, in which case $\mathrm{V}(L) = \{a \mid J_a \neq (J_a : x_n)\} = \varnothing$. □

Algorithm 5.2.5 can be adapted to describe the set of parameters a such that J_a is not saturated with respect to $x_1 \cdots x_n$.

Algorithm 5.2.6.

Input:

J, ideal in $\mathbb{C}[c_1, \ldots, c_k, x_1, \ldots, x_n]$ saturated with respect to $x_1 \cdots x_n$.

Output:

L, ideal in $\mathbb{C}[c_1, \ldots, c_k]$ satisfying $\mathrm{V}(L) = \{a \mid J_a \neq (J_a : (x_1 \cdots x_n))\}$.

{
`for` i `from` 1 `to` n `do`
{
let J_i be the ideal obtained from J by swapping x_n and x_i;
let L_i be the output of Algorithm 5.2.5 on input J_i;
}
$L = \sum L_i$;
`return` L;
}

Proof of correctness. As $(J_a : x_1 \cdots x_n) = (\ldots((J_a : x_1) : x_2) \ldots : x_n)$, we have that $J_a \neq (J_a : (x_1 \cdots x_n))$ if and only if $J_a \neq (J_a : x_i)$ for some i. The correctness now follows from the correctness of Algorithm 5.2.5.

We are now ready to describe the main algorithm of this section.

Algorithm 5.2.7.

Input:

I, ideal in $\mathbb{C}[c_1, \ldots, c_k, x_1, \ldots, x_n]$ saturated with respect to $x_1 \cdots x_n$.

Output:

L, ideal in $\mathbb{C}[c_1, \ldots, c_k]$ satisfying $\mathrm{V}(L) = \overline{\{a \mid 0 \leq \dim(\mathrm{V}^{\mathrm{va}}(I_a)) \leq 1\}}$.

{
$C = I \cap \mathbb{C}[c_1, \ldots, c_k]$;
$\{C_1, \ldots, C_m\}$ = `PrimaryDecomposition`(C);
let $\bar{I}$ be the homogenization of I in $(\mathbb{C}[c_1, \ldots, c_k])[x_0, \ldots, x_n]$;
`for` i `from` 1 `to` m `do`
{
$I_i = \bar{I} + C_i$;

```
        C' = C_i;
        I' = I_i;
        L_i = (0) ⊂ ℂ[c_1, ..., c_k];
        while (dim(C') + 1 ≤ dim(I')) do
            {
            {Q_i1, ..., Q_il} = PrimaryDecomposition(I');
            for j from 1 to l do
                {
                D_j = Q_ij ∩ ℂ[c_1, ..., c_k];
                if (Radical(D_j) = Radical(C_i) AND dim(Q_ij) > dim(C_i) + 1) then
                    {compute the output L_ij of Algorithm 5.2.6 on input Q_ij +
Radical(C_i)}
                else L_ij = (0) ⊂ ℂ[c_1, ... c_k];
                }
            L_i = ∑_j L_ij;
            I' = (I' + L_i : (x_0···x_n)^∞);
            C' = I' ∩ ℂ[c_1, ..., c_k];
            L_i = C' + L_i;
            }
        }
L = C + ⋂_i L_i;
return L;
}
```

Proof of correctness. Let $\{C_1, \dots, C_m\}$ be the primary ideals of a primary decomposition of C. The output of the Algorithm is the ideal $L = C + \bigcap_i L_i$, so it will suffices to show that $\mathrm{V}(C_i + L_i)$ is the closure of the set $\{a \in \mathrm{V}(C_i) \mid 0 \leq \dim(\mathrm{V}^{\mathrm{va}}(I_a)) \leq 1\}$ for $i = 1, \dots, m$.

Let C_i be a primary component of C such that $\dim(C_i) + 1 \geq \dim(I + C_i)$. The Algorithm computes $L_i = (0)$. For $a \in \mathrm{V}(C_i)$ general we have $0 \leq \dim(I_a) \leq 1$. Therefore we have, as claimed, $\mathrm{V}(C_i + L_i) = \overline{\{a \in \mathrm{V}(C_i) \mid 0 \leq \dim(\mathrm{V}^{\mathrm{va}}(I_a)) \leq 1\}}$.

Let now C_i be a primary component of C such that $\dim(C_i)+1 < \dim(I+C_i)$. We first show that the `while` loop terminates after finitely many steps. To do so, it suffices to show that the ideal L_i as computed at the j^{th} iteration of the loop strictly contains the deal L_i as computed at the $(j-1)^{\mathrm{th}}$ iteration. At the end of each iteration of the `while` loop the ideal I' is redefined to contain L_i. Therefore it suffices to show that, at each iteration, the ideal L_i is not contained in I'. Since $\dim(C_i) + 1 < \dim(I')$ there exists some primary component Q_{ij} of I' such that

$\mathrm{V}(Q_{ij})$ projects dominantly to $\mathrm{V}(C_i)$ and $\dim(C_i) + 1 < \dim(Q_{ij})$. This matches the condition of the inner `if` operator, therefore the Algorithm computes the output L_{ij} of Algorithm 5.2.6. By the correctness of Algorithm 5.2.6 and Lemma 5.1.9 we have that L_{ij}, and hence L_i, is not contained in I'. This shows that the `while` loop terminates. We now show that the set of parameters $a \in \mathrm{V}(C_i)$ such that $0 \le \dim(I_a) \le 1$ is contained in $V(C_i + L_i)$ at each iteration of the `while` loop. As before, let Q_{ij} be a primary component of I' such that $\mathrm{V}(Q_{ij})$ projects dominantly to $\mathrm{V}(C_i)$ and $\dim(C_i)+1 < \dim(Q_{ij})$. By [30, Theorem 2 of Chapter 1 Section 8] the fibers of the projective closure of $\mathrm{V}(Q_{ij})$ all have dimension at least 2. In particular the fiber $\mathrm{V}^{\mathrm{va}}((Q_{ij})_a)$ has dimension less than 2 if and only if $\mathrm{V}^{\mathrm{va}}(Q_{ij})$ is empty. As a result the set of parameters a such that $\mathrm{V}^{\mathrm{va}}((Q_{ij})_a)$ is empty is contained in the set of parameters a such that $(Q_{ij})_a$ is not saturated with respect to $x_0 \cdots x_n$. Since $\mathrm{V}^{\mathrm{va}}(I_a)$ contains $\mathrm{V}^{\mathrm{va}}((Q_{ij})_a)$, we have that the set of parameters $a \in \mathrm{V}(C_i)$ such that $0 \le \dim(I_a) \le 1$ is contained in $\mathrm{V}(C_i + L_i)$. Finally, since the `while` loop terminates, we have at the last iteration $\dim(C_i) + 1 \ge \dim(I + C_i)$. Therefore the proof that $\mathrm{V}(C_i + L_i) = \overline{\{a \in \mathrm{V}(C_i) \mid 0 \le \dim(\mathrm{V}^{\mathrm{va}}(I_a)) \le 1\}}$ can be reduced to the previous case. □

5.2.2 Hilbert functions

Theorem 5.2.2 is a useful tool to compute the multiplicity of $\operatorname{pos} e_1$ on the fibers of $\mathrm{V}(I)$. We now discuss how to use this result to compute the locus in $\mathrm{V}(C)$ where the multiplicity is at least m.

When the projection $\pi : \mathrm{V}(J) \to \mathrm{V}(C)$ is quasi finite, the degree of π equals the degree of the general fiber $\mathrm{V}(J_a)$ and hence, by Proposition 5.1.11 and Theorem 5.2.2, the multiplicity of $\operatorname{pos}(e_1)$ in $\operatorname{Trop}(I_a)$. This gives, however, no control on the special fibers, as the following example shows.

Example 5.2.8. Let $I = (c_1^3 + c_1^2 - c_2^2, 2xc_2 + 2yc_2 - c_1, 2xc_1^2 + 2yc_1^2 + 2xc_1 + 2yc_1 - c_2, 4x^2c_1 + 8xyc_1 + 4y^2c_1 + 4x^2 + 8xy + 4y^2 - 1) \subset \mathbb{C}[c_1, c_2][x, y]$. The tropical variety $\operatorname{Trop}(I_a)$ is empty for $a \notin \mathrm{V}(c_1^3 + c_1^2 - c_2^2)$, it is the tropical curve Σ_1 depicted in Figure 5.4 for $(0,0) \ne a \in \mathrm{V}(c_1^3 + c_1^2 - c_2^2)$, and it is the tropical curve Σ_2 depicted in Figure 5.4 for $a = (0,0)$.

Proposition 5.2.2 relates the multiplicity of e_1 in $\operatorname{Trop}(I_a)$ with the cardinality of $\overline{\mathrm{V}(I_a)} \cap \mathcal{O}_1$. The following proposition relates this cardinality with the Hilbert function of J_a.

Proposition 5.2.9. *Suppose that for $a \in \mathrm{V}(C)$ general the dimension of $\mathrm{V}(J_a)$ is 0. Denote by $\bar{J}$ the homogenization of J in $(\mathbb{C}[c_1, \ldots, c_k])[x_0, \ldots, x_n]$, and by $\bar{J}_a$ the*

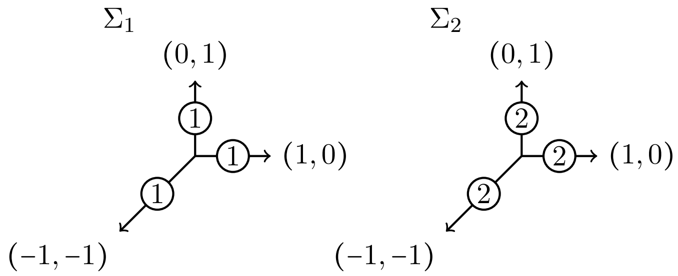

Figure 5.4: The tropical varieties of Example 5.2.8

homogenization of J_a *in* $\mathbb{C}[x_0,\ldots,x_n]$*. Let* r *be the Castelnuovo-Mumford regularity* $\operatorname{reg}(\bar{J})$ *and denote by* $h_{\bar{J}_a}$ *the Hilbert polynomial of* $\bar{J}_a$*. Then for every* $a \in \mathrm{V}(C)$ *we have* $\operatorname{mult}(\operatorname{pos}(e_1), I_a) \le h_{\bar{J}_a}(r)$*, and the equality holds on an open dense subset of* $\mathrm{V}(C)$*.*

Proof. We recall that $\mathbf{Real}_1$ is defined as

$$\mathbf{Real}_1 := \{(a,x) \in \mathbb{A}^{k+n} \mid x \in \overline{\mathrm{V}^{\mathrm{va}}(I_a)} \cap \mathcal{O}_1\}.$$

Fix $a \in \mathrm{V}(C)$ such that $\dim(\mathrm{V}^{\mathrm{va}}(I_a)) \le 1$. The multiplicity $\operatorname{mult}(\operatorname{pos}(e_1), I_a) \le h_{\bar{J}_a}(r)$ equals the cardinality of $(\mathbf{Real}_1)_a$. By Proposition 5.1.11, $(\mathbf{Real}_1)_a \subset \mathrm{V}(J_a)$ for every $a \in \mathrm{V}(C)$ and, by Theorem 5.1.12, $(\mathbf{Real}_1)_a = \mathrm{V}(J_a)$ for $a \in \mathrm{V}(C)$ general. Moreover, since $\dim(J_a)$ is 0, its cardinality equals the cardinality of its closure $\mathrm{V}(\bar{J}_a)$ in the projective space. To conclude the proof it suffices to show that the Hilbert function of $\bar{J}_a$ computed at r coincides with the Hilbert polynomial of $\bar{J}_a$ computed at r. This follows from [11, Corollary 20.19] once we identify the ideal $\bar{J}_a \subset \mathbb{C}[x_0,\ldots,x_n]$ with the ideal $(\bar{J}, c_1-a_1,\ldots,c_k-a_k) \subset \mathbb{C}[c_1,\ldots,c_k,x_0,\ldots,x_n]$. □

We now describe how to compute the set of parameters $a \in \mathrm{V}(C)$ such that $h_{\bar{J}_a}(r) \ge m$. We denote by $\bar{J}(r)$ the vector space $\bar{J} \cap (\mathbb{C}[c_1,\ldots,c_k])[x_0,\ldots,x_n]_r$. Let $\mathcal{L} = \{f_1,\ldots,f_l\}$ be a homogeneous basis of $\bar{J}$. A set of generators of $\bar{J}(r)$ as a vector space is given by all the degree r polynomials that can be obtained by multiplying an element of $\mathcal{L}$ by a monomial. Let M be the matrix of coefficients of this basis. It is a matrix with entries in $\mathbb{C}[c_1,\ldots,c_k]$. Let D the ideal generated by $\left(\binom{n+r}{r} - m + 1\right)$-minors of M. For $a \in \mathrm{V}(C)$ the degree r component of $\bar{J}_a$ is a vector space $(\bar{J}_a)(r) \subset \mathbb{C}[x_0,\ldots,x_n]_r$. Denote by M_a the matrix obtained by evaluating the entries of M in a. By construction M_a is the coefficients matrix of a set of generators of $\bar{J}_a(r)$. In particular $\bar{J}_a(r)$ has dimension at most $\binom{n+r}{r} - m$ if and only if $a \in \mathrm{V}(D)$ and, equivalently, we have $h_{\bar{J}_a}(r) \ge m$ if and only if $a \in \mathrm{V}(D)$. This construction is the content of the following algorithm.

Algorithm 5.2.10.

Input:

I, ideal in $\mathbb{C}[c_1,\ldots,c_k,x_1,\ldots,x_n]$ such that for $a\in \mathrm{V}(I)\cap\mathbb{A}^k$ general, $\dim(\mathrm{V}^{\mathrm{va}}(I_a))\leq 1$,

m, positive integer.

Output:

D, ideal in $\mathbb{C}[c_1,\ldots,c_k]$ such that $h_{\bar{J}_a}(r)\geq m$ if and only if $a\in\mathrm{V}(D)$.

{
let E be the homogenization of J in $(\mathbb{C}[c_1,\ldots,c_k])[x_0,x_1,\ldots,x_n]$ and L a list of generators of E;
$B=\{\}$;
`for` f `in` L `do`
{ add every monomial x^u of degree $m-\deg(f)$ to B } ;
let M be the matrix of coefficients of B;
compute the ideal D of the $(\binom{n+r}{r}-m+1)$- minors of M;
`return` D;
}

Proposition 5.2.11. *Suppose that, for $a\in\mathrm{V}(C)$ general, we have $\dim(\mathrm{V}^{\mathrm{va}}(I_a))\leq 1$. Let $a\in\mathrm{V}(C)$ be a parameter such that $\mathrm{mult}(\mathrm{pos}(e_1),\mathrm{Trop}(I_a))\geq m$, then $a\in\mathrm{V}(D)$. In other words $\mathrm{Real}_{1,m}\subset\mathrm{V}(C,D)$.*

Proof. As $\mathrm{mult}(\mathrm{pos}(e_1),\mathrm{Trop}(I_a))\geq m$ then, by Proposition 5.2.9, $h_{J_a}(r)\geq m$. As a result the matrix M_a has rank at most $\binom{n+r}{r}-m$ and therefore $a\in\mathrm{V}(D)$. □

When $D\subset C$ the ideal of $\mathrm{Real}_{1,m}$ is described by the following proposition.

Proposition 5.2.12. *Suppose that, for $a\in\mathrm{V}(C)$ general, we have $\dim(\mathrm{V}^{\mathrm{va}}(I_a))\leq 1$ and suppose that $D\subset C$. Then we have $\mathrm{Ideal}(\mathrm{Real}_{1,m})=C$.*

Proof. By Proposition 5.2.11 we have $\mathrm{Ideal}(\mathrm{Real}_{1,m})\supset C+D=C$. Moreover by the definition of D we have $h_{J_a}(r)\geq m$ for every $a\in\mathrm{V}(C,D)=\mathrm{V}(C)$ and by Proposition 5.2.9 we have that $h_{J_a}(r)=\mathrm{mult}(\mathrm{pos}(e_1),\mathrm{Trop}(I_a))$ for a general in $\mathrm{V}(C)$. This shows that $\mathrm{Real}_{1,m}$ is dense in $\mathrm{V}(C)$. □

5.2.3 The main algorithm

We are now ready to describe an algorithm to compute the Zariski closure of $\mathrm{Real}_{1,m}$. The algorithm proceeds as follows. It first compute the ideals C and J using Algorithm 5.1.3 on input I. It then computes the ideals J, C and D. If $D \not\subset C$ it starts again running Algorithm 5.1.3 on the restricted family $I + C + D$.

Algorithm 5.2.13.

Input:

I, ideal in $\mathbb{C}[c_1, \ldots, c_k, x_1, \ldots, x_n]$,

m, positive integer.

Output:

D, ideal in $\mathbb{C}[c_1, \ldots, c_k]$ satisfying $\mathrm{V}(D) = \overline{\mathrm{Real}_{1,m}} \subset \mathbb{A}^k$.

{
$C = D = (0) \subset \mathbb{C}[c_1, \ldots, c_k]$;
doFirstIteration = `true`;
`while` ($D \not\subset C$ `OR` doFirstIteration = `true`) `do`
    {
    doFirstIteration = `false`;
    computeLocCond = `true`;
    `while`(computeLocCond) `do`
        {
        compute the output (C, J) of Algorithm 5.1.3 with input $I + C + D$;
        compute the output C' of Algorithm 5.2.7 with input $(I{+}C : (x_1 \cdots x_n)^\infty)$;
        `if` $C = C'$ `then` computeLocCond = `false` ;
        $C = C'$;
        };
    compute the output D of Algorithm 5.2.10 on input $(I + C, m)$;
    };
`return` C;
}

Proof of correctness. When the algorithm reaches its termination we are in assumptions of Proposition 5.2.12 and therefore $\overline{\mathrm{Real}_{1,m}} = \mathrm{V}(C)$. □

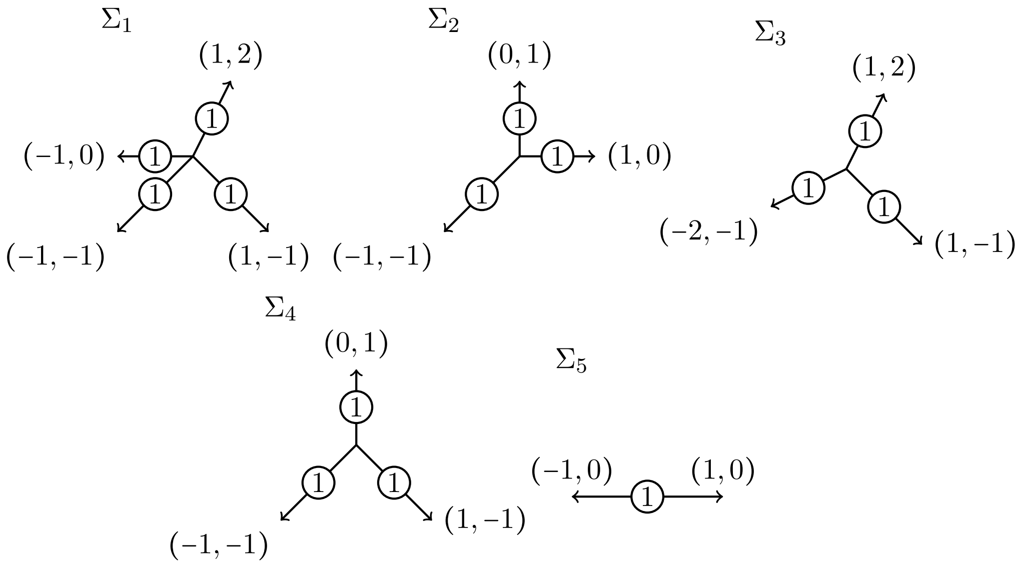

Figure 5.5: The tropical varieties of Example 5.2.14

Example 5.2.14. Let I be the ideal $I = (c_1y + xy + xy^2 + c_1c_2x^2 + c_2x^2y + x^2y) \subset \mathbb{C}[c_1, c_2, x, y]$. The tropical variety $\mathrm{Trop}(I_a)$ is depicted in Figure 5.5: it is the tropical curve Σ_1 for $a_1 \neq 0, a_2 \neq 0, -1$, it is the tropical curve Σ_2 for $a_1 = 0, a_2 \neq -1$, it is the tropical curve Σ_3 for $a_1 \neq 0, a_2 = -1$, it is the tropical curve Σ_4 for $a_1 \neq 0, a_2 = 0$ and it is the tropical curve Σ_5 for $a_1 = 0, a_2 = -1$. Let $m = 2$, we have $\mathrm{Real}_{1,2} = \varnothing$. Algorithm 5.2.13 first computes the ideals $C = \mathrm{Ideal}(\mathrm{Real}_1) = (c_1)$ and $J = \mathrm{Ideal}(\mathbf{Real}_1) = (c_1, x, y + 1)$. Finally it computes that the Hilbert function of I_0 is less than 2.

Example 5.2.15. Let I be the ideal $I = I_0 \cdot I_1 \subset \mathbb{C}[c, x, y, z]$ where $I_0 = (x + y + z + 1, 2x - 3y + 5z - 2, c)$ and $I_1 = (x + 3y - 2z + 6, c(-x + 4y + 2z - 1))$. The tropical variety $\mathrm{Trop}(I_a)$ does not depend on a and it is depicted in Figure 5.6. For $m = 2$ we have $\mathrm{Real}_{1,2} = \varnothing$. The algorithm computes the ideal $\mathrm{Ideal}(\mathrm{Real}_{1,2}) = \mathbb{C}[c]$ in the following way. It first computes $C = \mathrm{Ideal}(\mathrm{Real}_1(I)) = (0) \subset \mathbb{C}[c]$ and $J = \mathrm{Ideal}(\mathbf{Real}_1(I)) = I + (x)$. Then it computes that the Hilbert function $h_{J_a}(2)$ of J_a computed at the regularity 2 is at least 2 in $\mathrm{V}(c)$. Then the algorithm computes $C = \mathrm{Ideal}(\mathrm{Real}_1(I + (c))) = (c) \subset \mathbb{C}[c]$ and $J = \mathrm{Ideal}(\mathbf{Real}_1(I + (c))) = I + (x, c)$. The ideal J has two components, corresponding to the two components I_0 and I_1 of I, and the component corresponding to I_1 is discarded since the fiber of $I_1 + (c)$ has dimension 2. Finally, the computation of the Hilbert function shows that $\mathrm{Ideal}(\mathrm{Real}_{1,2}) = \mathbb{C}[c]$.

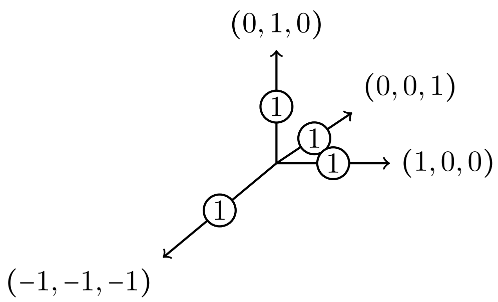

Figure 5.6: The tropical variety $\mathrm{Trop}(\mathrm{V}(I_a))$ of Example 5.2.15.

5.3 Change of Coordinates

In the previous sections we studied the realizability space $\mathrm{Real}_{1,m}$ for the ray $\mathrm{pos}(e_1)$ with multiplicity m. In this section we show how to extend these results to the case of an arbitrary rational ray ρ. We denote by $\mathrm{Real}_{\rho,m}$ the set of parameters $a \in \mathbb{A}^k$ such that the tropicalization of X_a is a curve that contains the ray ρ with multiplicity at least m.

To a $m \times n$ integer matrix $A = (a_{i,j})$ one can associate a monomial map φ, that is the regular morphism

$$\begin{array}{cccc} \varphi_A: & \mathrm{T}^n & \to & \mathrm{T}^m \\ & (x_1,\dots,x_n) & \mapsto & (\prod_{i=1}^n x_i^{a_{1,i}},\dots,\prod_{i=1}^n x_i^{a_{m,i}}). \end{array}$$

The tropicalization $\mathrm{Trop}(\varphi_A)$ of φ is by definition the linear map $\mathrm{Trop}(\varphi_A): \mathbb{R}^n \to \mathbb{R}^m$ associated to the transpose matrix $A^\intercal$. The reason for this definition is the following (see [29, Corollary 3.2.15]): for any subvariety X of T^n the tropicalization of the Zariski closure of its image under φ_A equals the image of its tropicalization under $\mathrm{Trop}(\varphi_A)$, in symbols

$$\mathrm{Trop}(\overline{\varphi_A(X)}) = \mathrm{Trop}(\varphi_A)(\mathrm{Trop}(X)). \tag{5.6}$$

Let $\rho \subset \mathbb{R}^n$ be a rational ray and let v be the first integer point of ρ. By the Lemma in [35] there exists a $\mathrm{GL}_n(\mathbb{Z})$ matrix B such that $B \cdot e_1 = v$. We denote by A the transpose of B. Let $\varphi_A : T^n \to T^n$ be the monomial map associated to A, and denote by $\varphi_A^* : \mathbb{C}[x_1^\pm,\dots,x_n^\pm] \to \mathbb{C}[x_1^\pm,\dots,x_n^\pm]$ the associated ring homomorphism. As $A \in \mathrm{GL}_n(\mathbb{Z})$ is invertible, φ_A^* is invertible with inverse $\varphi_{A^{-1}}^*$. Given a variety $X \subset \mathrm{T}^n$, by Equation (5.6) we have that $\mathrm{mult}(\mathrm{pos}(v),\mathrm{Trop}(X)) = \mathrm{mult}(\mathrm{pos}(e_1),\mathrm{Trop}(\overline{\varphi_A(X)}))$. The variety $\overline{\varphi_A(X)}$ is easily described: if X is the vanishing locus of the Laurent polynomials $f_1, \dots, f_l \in \mathbb{C}[x_1^{\pm}, \dots, x_n^{\pm}]$, then $\overline{\varphi_A(X)}$ is the vanishing locus of the Laurent polynomials $(\varphi_A^*)^{-1}(f_1), \dots, (\varphi_A^*)^{-1}(f_l)$.

Let now $I \subset \mathbb{C}[c_1, \dots, c_k][x_1, \dots, x_n]$. We denote again by φ_A^* the ring homomorphism $\varphi_A^* : \mathbb{C}[c_1, \dots, c_k][x_1^{\pm}, \dots, x_n^{\pm}] \to \mathbb{C}[c_1, \dots, c_k][x_1^{\pm}, \dots, x_n^{\pm}]$. We have the following equality:

$$\mathrm{Real}_{\rho,m}(I) = \mathrm{Real}_{1,m}((\varphi_A^*)^{-1}\, I).$$

This allows to compute $\mathrm{Real}_{\rho,m}(I)$ via Algorithm 5.2.13.

We conclude this section by describing an algorithm to explicitly compute a matrix $B \in \mathrm{GL}_n(\mathbb{Z})$ satisfying the property $Be_1 = v$. The Smith normal form of an integer matrix $M \in \mathbb{Z}^{n\times m}$ is a matrix $D \in \mathbb{Z}^{n\times m}$ that is zero outside the main diagonal and whose elements on the diagonal $(d_1, \dots, d_l)$ satisfy $d_1|d_2 \dots |d_l$, where $l = \min(n, m)$. The Smith normal form D is related to M via the expression $D = PMQ$ where P is an invertible $n \times n$ integer matrix and Q is an invertible $m \times m$ integer matrix.

Algorithm 5.3.1.

Input:

v, a primitive integer vector $v \in \mathbb{Z}^n$.

Output:

B, a $\mathrm{GL}_n(\mathbb{Z})$ matrix such that $Be_1 = v$.

{
Compute the a Smith normal expression $D = Pv^{\top}Q$ for the row vector $v^{\top}$;
$d = D_{1,1}$;
$B = dP(Q^{\top})^{-1}$;
`return` B;
}

Proof of correctness. The Smith normal form D is a $1 \times n$ integer matrix $D = (d, 0, \dots, 0)$. Moreover, P is a 1×1 invertible matrix, so we have $P = (1)$ or $P = (-1)$. We have $Pv^{\top}Q = D = de_1^{\top}$ and hence $PQ^{\top}v = de_1$. It follows that $dP(Q^{\top})^{-1}e_1 = d(PQ^{\top})^{-1}e_1 = v$. □

5.4 The Global Algorithm

In this section we glue the information coming from Algorithm 5.2.13 to describe the closure of the locus of parameters $a \in \mathbb{C}^k$ such that $\mathrm{Trop}(I_a)$ is a curve and contains a given list of rays $\Sigma = \{\rho_1, \dots, \rho_s\}$ with at least multiplicities $m_1, \dots, m_s$.

Algorithm 5.4.1.

Input:

I, ideal in $\mathbb{C}[c_1, \dots, c_k, x_1, \dots, x_n]$,

Σ, a collection of rays in $\mathbb{R}^n$ and multiplicities $\Sigma = \{(\rho_1, m_1), \dots, (\rho_s, m_s)\}$.

Output:

C, ideal in $\mathbb{C}[c_1, \dots, c_k]$ satisfying $\mathrm{V}(C) = \overline{\mathrm{Real}_\Sigma} \subset \mathbb{A}^k$.

{
$C = C_0 = (0) \subset \mathbb{C}[c_1, \dots, c_k]$;
doFirstIteration =`true`;
`while` ($C \neq C_0$ or doFirstIteration) `do`
    {
    doFirstIteration =`false`;
    $C_0 = C$;
    `for` (ρ_i, m_i) `in` Σ `do`
        {
        compute a matrix $A \in \mathrm{GL}_n(\mathbb{Z})$ such that $A^\top \cdot v = e_1$ via Algorithm 5.3.1;
        compute $\mathrm{Ideal}(\mathrm{Real}_{1,m_i}((\varphi_A^*)^{-1}(I) + C))$ via Algorithm 5.2.13;
        $C = \mathrm{Ideal}(\mathrm{Real}_{1,m_i}((\varphi_A^*)^{-1}(I) + C))$;
        }
    }
`return` C;
}

We need to repeat the procedure in Algorithm 5.4.1 as Algorithm 5.2.13 assures that the multiplicity of $\mathrm{pos}(e_1)$ in $\mathrm{Trop}(I_a)$ is at least m only for a dense subset in its output. The following is an example of when a single iteration is not sufficient.

Example 5.4.2. Let I be the ideal $I = (c + y + xy + y^2) \subset \mathbb{C}[c, x, y]$. The tropical variety $\mathrm{Trop}(I_a)$ is depicted in Figure 5.7: it is Σ_1 for $a \neq 0$ and it is Σ_2 for $a = 0$. Let $\Sigma = \{(\mathrm{pos}(e_1), 2), (\mathrm{pos}(e_2), 1)\}$. We have $\mathrm{Real}_\Sigma = \varnothing$. To compute the ideal $\mathrm{Ideal}(\mathrm{Real}_\Sigma)$ Algorithm 5.4.1 needs two iterations of the `while` loop. It first computes $\mathrm{Real}_{\mathrm{pos}\, e_1, 2}(I) = \mathbb{A}^1$ and $\mathrm{Real}_{\mathrm{pos}\, e_2, 1}(I) = \mathrm{V}(c)$. It then computes that $\mathrm{Real}_{\mathrm{pos}\, e_1, 2}(I + (c)) = \varnothing$

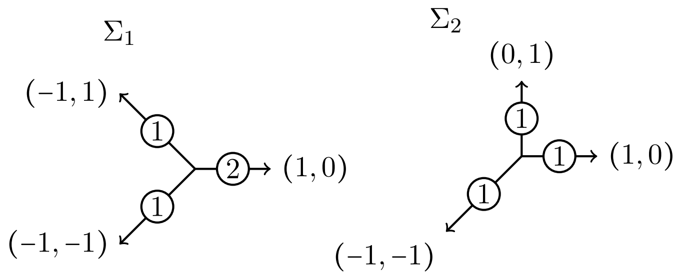

Figure 5.7: The tropical varieties of Example 5.4.2

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