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TITLE SOME BOUNDARY HITTING PROBLEMS FOR DIFFUSION PROCESSES

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SOME BOUNDARY HITTING PROBLEMS FOR DIFFUSION PROCESSES

by

Gareth Owen Roberts B.A.Hons.

Thesis submitted for the degree of Doctor of Philosophy at the Department of Statistics, University of Warwick, September 1988.

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CONTENTS

Introduction

This introduction is, in the main, unnecessary, as each chapter contains its own comprehensive introduction, which include explanations about connections between chapters. So, this section contains a brief summary of related research areas, and then moves on to consider some of the intuitive ideas which motivated this work.

The thesis is split into two distinct parts. Part I deals with a stochastic control problem and its application to weak comparison results for solutions to stochastic differential equations, and in part II we look at properties of boundary hitting times for diffusion processes, and in particular, asymptotic approximations for their distribution functions. Each part has a number of work chapters which are, as far as possible, self-contained and each chapter contains its own introduction and reference section. Thus in this form, the thesis is designed to portray the development of the thesis with time, and it is hoped that this will make understanding easier.

Part I is split into two chapters which carry different emphasis. The main theme is to obtain stochastic inequalities for diffusion processes where the drift coefficient of one process dominates that of the other. Here, we concentrate on the case where at least one of the diffusions is symmetric about the origin and the stochastic inequalities obtained are therefore on the modulus of the two processes. In chapter 1, the main result is proved in simple cases where interesting explicit results can be obtained, and in chapter 2 the result is generalized using similar techniques, which connect the result to weak uniqueness theorems such as that of Stroock and Varadhan (1969).

The area of calculating boundary hitting times for diffusion processes, and in particular, Brownian motion, has a long history. Early papers by Darling and Siegert (1953), Breiman (1967), and Shepp (1971) focused on the simplest case of a square-root boundary, where the stationary behaviour exhibited by an appropriate transformation of the Brownian motion, under which the square root boundary is transformed to a constant boundary, enables explicit solutions to be obtained.

The other well known explicit solution is that for a linear boundary. Here, it is easy to use an elementary reflection principle idea to obtain the density of the hitting time of the boundary $f(t) = at + b$ starting at 0:

$$p(t) = \frac{b}{t^{\frac{3}{2}}}\phi\left(\frac{f(t)}{\sqrt{t}}\right),$$

where $\phi(y) = (2\pi)^{-\frac{1}{2}} e^{-\frac{y^2}{2}}$. This is the Bachelier-Levy formula (see for example Levy (1965)).

However, apart from these special cases, we have to be content with approximating solutions. It is not that exact solutions for the distribution of hitting times are unobtainable, but that they are generally of little use in their own right. For example, Durbin (1985) obtains an explicit expression for the density of hitting times. However the formula derived involves evaluation of a complicated limit, which is usually analytically intractable. However, Durbin's solution also leads to a valuable approximation for the density, which is asymptotically correct.

Recent literature has been dominated by applications in sequential analysis. Siegmund (1986) and Lerche (1986) provide good surveys of this area. For this reason, some authors (for example Siegmund (1985), Lai and Siegmund (1977) and Woodroofe (1982)) have prefered to work initially in dicrete time, thus avoiding the need to make invariance principle-type approximations which can be notoriously slow to converge. The main problem with such an approach is that it is analytically less tractable, especially when dealing with the problem of calculating the overshoot at the boundary. Woodroofe (1982) proves a renewal theorem for estimating the overshoot. The discrete time approach is also used in the area of optimal Bayesian sequential analysis (for example Wald and Wolfowitz (1948) and Chernoff (1972)).

Most of the time however, we are able to obtain better approximations in continuous time. Standard analytic techniques for calculating hitting time distributions include the so called method of images (see for example Lerche (1986)), and the method of weighted likelihood functions (Robbins and Siegmund (1973)) where exact solutions for the density are found for an implicitly defined boundary. These methods are very similar, and in fact Lerche (1986) shows that they are equivalent up to time inversion, and that they lead to the tangent approximation which was first proved by Strassen.

The tangent approximation comes from the intuitive idea that the density of the hitting time at time t, should be approximately that for the straight line boundary which is tangent to the curve at that point, and in fact this approximation is asymptotically correct along certain sequences of boundaries that tend to infinity, and moreover this approximation is uniform for t on compact intervals, thus allowing similar statements for the distribution function to hold. This approximation has been extensively used and refined (for example, Jennen and Lerche (1981), Dinges (1982), Jennen (1985) and Klein (1986)).

However, tangent approximation techniques cannot, in general, give good approximations which are valid over the whole real line. Unlike most of the works mentioned above, we intend to look at fixed boundaries, and study the tail behaviour of the distribution of their corresponding hitting times. Also the emphasis has been on obtaining distribution function results as opposed to density estimates, since for any application the local properties of the density function are irrelevant. Whereas, the tangent approximation is locally more accurate than the methods used here, picking up most of the fine structure in the boundary, and is therefore preferable if we are merely interested in the density of the hitting time, the 'global' behaviour of the hitting time is best approximated using our results, which take into account the long term effects on the distribution of the hitting time, of a particular stretch of the boundary function, rather than its immediate effects on the density.

Our approach will follow on more naturally from that of the old school of Breiman and Shepp than those of the more modern approximation methodologists, in that we will focus on the square-root boundary and its stationary behaviour and examine what goes wrong when boundaries are not square-root. We consider boundaries in three main classes depending on whether $f(t)t^{-\frac{1}{2}}$ tends to 0, a finite non-zero limit, or ∞. The behaviour of the hitting time in the finite non-zero limit case is, as you would expect, similar to that of the exact square-root boundary, but in the other two cases, as we shall see, the stationary behaviour breaks down, in different ways. However the stationary behaviour of the square-root boundary still provides the key to the approximation. The strength of the techniques used comes from the fact that the initial approximations are estimates on the distribution of the process at time t conditioned not to have hit the boundary until t, and not approximations on the boundary function itself.

A note about the numbering system is in order. The chapters contain their own internal numbering system for both equations and proved results. So for example, the third proved result of section 2 might be called lemma 2.3. References to results proved in other chapters are prefixed by the number of that chapter, so suppose the above reference above appeared in chapter 2, then it would be refered to as lemma 2.2.3.

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PART I

I.1 Some stochastic control problems and their application to inequalities for diffusions

1.1 Introduction

Benes, Shepp, and Witsenhausen (1980) considered control problems of the type:-

Let $\{X_t^u\}$ be a stochastic integral given by:-

$$dX_t = dB_t + u\, dt, \tag{1.1}$$

where B_t is a 1-dimensional Brownian motion, and $u \in {}^*U = \{\mathcal{F}_t - \text{adapted controls}\}$ $(\mathcal{F}_t = \sigma\{X_s,\ s \leq t\})$.

Let

$$C = \{u;\ -b_1 \leq u_t \leq b_2\ \forall t\}. \tag{1.2}$$

Choose $u \in C \cap {}^*U$ so as to minimise $\mathsf{E}[\int_0^\infty e^{-\alpha t} X_t^2\, dt]$.

Stochastic control problems similar to this are notoriously difficult to solve, and only the simplest can be solved explicitly. In this chapter we generalize some of the results of Benes et al. (1980), in the symmetric case (i.e. $b_1 = b_2$), to even functions f, in other words control problems such as the minimization of $\mathsf{E}[\int_0^\infty f(X_t^u)e^{-\alpha t}dt]$.

The main result is the following.

THEOREM 1.1.

Let C be defined as above with $b_1 = b_2 = b$.

Define

$$S_t = \{|X_t^u|;\ X_t^u \text{ is a solution of (1.1) with } u \in C\}. \tag{1.3}$$

Consider the control $\hat{u}$ given by:-

$$\hat{u}(x) = \begin{cases} -b, & x > 0 \\ b, & x < 0 \\ 0, & x = 0 \end{cases}$$

Then $|X_t^{\hat{u}}|$ attains the stochastic infimum of S_t $\forall t \in [0, \infty)$. In other words:-

$$\mathsf{P}[|X_t^{\hat{u}}| < c] \geq \mathsf{P}[|X_t^u| < c],\ \forall u \in C, c \in \mathbb{R}^+,\ t \in \mathbb{R}^+. \tag{1.5}$$

Many results follow directly from this theorem including the solution of the above exponentially weighted control problem which follows by Fubini's theorem.

The proof of theorem 1.1 involves looking at the associated control problem:-

Choose u so as to minimise $\mathsf{E}[f(X_T^u)|X(0) = x]$ for even and well behaved functions f such that f is non-decreasing for $x \geq 0$, and non-negative.

It turns out that 'f bounded and in C^2' together with the above conditions is sufficient for our purposes.

Define:-

$$\phi(t,x) = \mathsf{E}[f(X_t^a)|X_0^a = x] \tag{1.6}$$

Now $\phi(T-t, X_t)$ is an $\mathcal{F}_t$-martingale for $t \leq T$, and so by a simple application of Ito's formula, for well-behaved f:-

$$\frac{\partial \phi}{\partial t} = \mathcal{L}^a \phi, \tag{1.7}$$

where

$$\mathcal{L}^a = \theta(x)\frac{\partial}{\partial x} + \frac{1}{2}\frac{\partial^2}{\partial x^2}, \tag{1.8}$$

is the infinitesimal generator of X_t^a.

To solve the above control problem, it turns out that we need to prove $\frac{\partial \phi}{\partial x} \geq 0$ for $x \geq 0$, and $\frac{\partial \phi}{\partial x} \leq 0$ for $x \leq 0$.

Section 2 is mainly devoted to a proof of this result. We choose to work directly with $\frac{\partial \phi}{\partial x}$ since this avoids technical problems with the P.D.E., and since $\frac{\partial \phi}{\partial x}$ also satisfies (1.7) for well behaved f.

In order to solve a parabolic P.D.E. such as (1.7) uniquely, it is necessary to consider the problem on a bounded domain with sufficient boundary and initial conditions. With this in mind, we consider a modified diffusion, which lives on the bounded domain $[0,T] \times [0,a]$, and which resembles $|X_t^a|$ for large a, $t \leq T$. The symmetry of f allows us to consider X_t^a and $|X_t^a|$ interchangeably, and this is useful in avoiding technical problems due to the discontinuity of $\theta(x)$ at $x = 0$.

Chapter 3 deals with the control problem and proves theorem 1.1 and various connected results. In chapter 4, explicit solutions for $\phi(t,x)$ and $\mathsf{E}[\int_0^\infty e^{-\alpha t} f(X_t^a)dt]$ are given using the results of Karatzas and Shreve (1984). A simpler expression for $\mathsf{E}[\int_0^\infty e^{-\alpha t} f(X_t^a)dt]$ is then found by solving Bellman's equation for the associated control problem.

1.2 The Partial Differential Equation

The idea is to construct ϕ from its derivative with respect to x, $\frac{\partial \phi}{\partial x}$, which, under certain conditions on f, satisfies the same P.D.E. We need to consider a domain D on which (1.7) has a unique solution under appropriate boundary conditions which are chosen to ensure that we are indeed producing $\frac{\partial \phi}{\partial x}$.

Take:-

$$D = (0,T) \times (0,a). \tag{2.1}$$

It would be natural to consider a solution θ of (1.7) subject to the initial conditions:-

$$\theta(0,x) = f'(x), \ x \in [0,a], \tag{2.2}$$

and the boundary condition:-

$$\theta(t,0) = 0, \ t \in (0,T). \tag{2.3}$$

For a unique solution to (1.7) on D, we also need $\theta(t,a)$ to be specified $\forall t \in (0,T)$. Its value is not obvious for such generalised f. To avoid this problem, we consider a set of modified diffusions $\{_aX_t\}$ which behave like $\{|X_t|\}$ for large a, but which all have boundary conditions at $x = a$ which are immediately apparent.

Define $_aX_t$ by:-

$$d_aX_t = dB_t + G(_aX_t)dt - dl_t^a(_aX_t) \tag{2.4}$$

where $\{l_t^x\}$ is the local time process at x of the diffusion $|_aX_t|$ (see Azéma and Yor (1978)).

Note

(1) This amounts to placing an reflecting boundary at $x = 0$ and $x = a$, (for a discussion of local times and reflecting boundaries, see El Karoui and Chaleyat - Maurel (1978)). The two boundaries are different in the sense that, at $x = 0$, the effect of changing the sign of X^a is that the driving Brownian motion is now $-B_t$, whereas at $x = a$, no such switch occurs. However, the two boundaries act identically in law. The reason for defining $_aX_t$ in this way is so that the following holds:-

$$\text{as } a \to \infty, \ _aX_t \xrightarrow{a.s.} |X_t^a|. \tag{2.5}$$

(2) X_t has the same infinitesimal generator as $|X_t^a|$ for $(t,x) \in D$.

(3) The notation $_aX_t$ will be abbreviated to X_t most of the time where only fixed a is being considered.

Define

$$_a\tilde{\phi}(t,x) = \tilde{\phi}(t,x) = \mathsf{E}^{0,x}[f(_aX_t)] = \mathsf{E}[f(_aX_t)|_aX_0 = x] \tag{2.6}$$

Note: the notation with the a omitted will be used where no confusion is possible, as with X_t.

So we will look for a solution to the equations:-

$$\begin{aligned} \frac{\partial \theta}{\partial t} &= \mathcal{L}^a\theta \text{ on } D, \\ \text{subject to } \theta(0,x) &= f'(x) \\ \text{and } \theta(t,0) &= \theta(t,a) = 0; \end{aligned} \tag{2.7}$$

with the idea of integrating the resulting solution with respect to x and showing this to be $\tilde{\phi}(t,x)$. We will need the following results from the theory of P.D.E.'s.

THEOREM 2.1. *Let D be a domain (as defined above), on which a uniformly parabolic operator $\mathcal{M}$ is defined such that its coefficients are all uniformly Holder continuous. Then the equations:-*

$$\begin{aligned} \mathcal{M}u &= 0 \quad \text{on } D \\ u &= g(t,x) \quad \text{on } \partial D \cap \{t < T\} \end{aligned} \tag{2.8}$$

where g is continuous and bounded, have a unique solution.

This is a simplified version of Theorem 3.6, Chapter 4, Friedman (1975).

THEOREM 2.2. *Under these conditions, assuming also that $g(0,x)$ is continuously differentiable with respect to x for all $x \in (0,a)$, and that this derivative is bounded, then $\frac{\partial u}{\partial x}$ exists and is continuous on D.*

This result follows directly from the corollary to Theorem 10.2, Friedman (1969).

THEOREM 2.3. *Under the conditions of theorem (2.1), the following result holds:-*

Let S be the region $\partial D \cap \{t < T\}$, then,

$$\min_{(t,x)\in S} g(t,x) \le u(t,x) \le \max_{(t,x)\in S} g(t,x) \tag{2.9}$$

This follows immediately from the weak maximum principle for parabolic operators: theorem 3.1, Chapter 6, Friedman (1975).

From these results, we know that θ is well-defined by (2.7), and has a well behaved derivative with respect to x, so long as we stipulate $f'(a) = 0$ and $f \in C^1(\mathbb{R})$.

Now set:-

$$\mu(t,x) = \int_{x_0}^{x} \theta(t,y)dy + Y(t), \tag{2.10}$$

for some arbitrary but fixed $x_0 \in [0,a]$, where $Y(t)$ is given by:-

$$\begin{aligned} Y'(t) &= -b\theta(t,x_0) + \frac{1}{2}\frac{\partial \theta}{\partial x} \\ Y(0) &= f(x_0) \end{aligned}$$

Note: since θ and $\frac{\partial \theta}{\partial x}$ are continuous on D, which is compact, the integral in (2.10) is well defined.

LEMMA 2.4.

$$\frac{\partial \mu}{\partial t} = \mathcal{L}^a \mu \text{ on } \bar{D} \tag{2.12}$$

PROOF:

By theorem (2.2), $\frac{\partial\theta}{\partial x}$ is bounded and hence $\frac{\partial^2\theta}{\partial x^2}$ has a bounded integral for fixed $t \in [0, t]$.

$$\text{Now, } \frac{\partial\theta}{\partial t} = \frac{1}{2}\frac{\partial^2\theta}{\partial x^2} - b\frac{\partial\theta}{\partial x} \text{ on } D, \tag{2.13}$$

so θ has a bounded derivative with respect to t, and by the bounded convergence theorem, $\frac{\partial\mu}{\partial t}$ exists and

$$\begin{aligned}\frac{\partial\mu}{\partial t} &= Y'(t) + \int_{x_0}^{x} \frac{\partial\theta}{\partial t}(t, y)dy \\ &= -b\theta(t, x_0) + \frac{1}{2}\frac{\partial\theta}{\partial x} + \int_{x_0}^{x} \left(\frac{1}{2}\frac{\partial^2\theta}{\partial x^2} - b\frac{\partial\theta}{\partial x}\right) dx \\ &= \mathcal{L}^* \mu, \quad \text{as required.}\end{aligned}$$

LEMMA 2.5.

$M_s = \theta(t - s, X_s)$ is an $\mathcal{F}_s$-martingale, where

$$\mathcal{F}_s = \sigma\{X_u, \ u \leq s\}, \ \ 0 \leq t - s \leq T. \tag{2.14}$$

PROOF: By the generalized Ito formula:-

$$dM_t = -\frac{\partial\mu}{\partial t}dt + \frac{\partial\mu}{\partial x}dX_t + \frac{1}{2}\frac{\partial^2\mu}{\partial x^2}d\langle X\rangle_t \tag{2.15}$$

But,

$$dX_t = d\ell_t^0 - d\ell_t^a + d\tilde{B}_t - bdt, \tag{2.16}$$

where $\tilde{B}_t$ is the Brownian motion given by:-

$$d\tilde{B}_t = \operatorname{sgn}(X_t^*)dB_t. \tag{2.17}$$

Now local times have zero quadratic variation (being non-decreasing), and so:-

$$dM_s = ds\left(\mathcal{L}^* \ \mu - \frac{\partial\mu}{\partial t}\right) + (d\ell_s^0 - d\ell_s^a)\frac{\partial\mu}{\partial x} + d\tilde{B}_s\frac{\partial\mu}{\partial x} \tag{2.18}$$

But by Lemma 2.4, the dt term in (2.18) is zero in $\bar{D}$. Also,

$$d\ell_t^0(X) = 0, \ \ X_t \neq 0,$$
$$d\ell_t^a(X) = 0, \ \ X_t \neq a,$$

and $\frac{\partial\mu}{\partial x}(t, 0) = \frac{\partial\mu}{\partial x}(t, a) = 0$, since $\frac{\partial\mu}{\partial x} = \theta$.

$$\text{So, } dM_s = \frac{\partial\mu}{\partial x} \ d\tilde{B}_s = \theta(s, |X_s|)d\tilde{B}_s. \tag{2.19}$$

But θ is bounded on D, and so $\mu(t - s, X_s)$ is an $\mathcal{F}_s$-martingale.

COROLLARY 2.6.

$$\tilde{\phi}(t,x) = \mu(t,x) \tag{2.20}$$

PROOF: Since M is an $\mathcal{F}_t$-martingale,

$$\mu(t,x) = \mathsf{E}[\phi(0,X_t)] = \mathsf{E}[f(X_t)] = \tilde{\phi}(t,x), \text{as required.}$$

LEMMA 2.7.

$$\frac{\partial \tilde{\phi}}{\partial x} \geq 0, \forall (t,x) \in D. \tag{2.21}$$

PROOF: $f'(x) \geq 0$, so by theorem 2.3,

$$\inf_{(t,x) \in \partial D \cap \{t < T\}} \theta(t,x) \geq 0. \tag{2.22}$$

Thus by theorem 2.3, $\theta(t,x) \geq 0 \ \forall (t,x) \in D$ i.e.

$$\frac{\partial \tilde{\phi}}{\partial x}(t,x) \geq 0 \ \text{ in D.}$$

THEOREM 2.8. *Let f be a function satisfying the above conditions (i.e. f is even, $f \in C^2(\mathbb{R})$, and $f'(x) \geq 0, \forall x \geq 0$). Suppose also that f is bounded and has arbitrarily large points of zero derivative.*

Then

$$\frac{\partial \phi}{\partial x} \geq 0, \forall x \geq 0. \tag{2.23}$$

PROOF: Suppose $|f(x)| < K$.

$$X_t^a = \int_0^t a(X_s^a)ds + B_t$$

so

$$B_t - bt \leq X_t^a \leq B_t + bt,$$

thus

$$\mathsf{P}[\sup_{s \leq t} |X_s^a| > c] \leq \mathsf{P}[\sup_{s \leq t} |B_s| > c - bt]$$

whilst by Doob's inequality,

$$\mathsf{E}[\sup_{s \leq t} B_s^2] \leq 4\mathsf{E}[B_t^2]$$

so, by Chebyshev's inequality,

$$\mathsf{P}[\sup_{s \leq t} |B_s| > c - bt] \leq \frac{1}{(c - bt)^2} \mathsf{E}[\sup_{s \leq t} B_s^2]$$

So that,

$$\mathsf{P}[\sup_{s\le t}|X^a_s| > c] \le \frac{1}{(c-bt)^2}\mathsf{E}[\sup_{s\le t}B_s^2]$$
$$\le \frac{4t}{(c-bt)^2} \to 0 \text{ as } c \to \infty.$$

Now ${}_aX_s = |X^a_s|$, $s \le t$ for all paths not exceeding a, so

$$|\phi(t,x) - {}_a\tilde{\phi}(t,x)| = |\mathsf{E}^{0,x}[f(X^a_t) - f({}_aX_t)]|$$
$$\le |\mathsf{E}^{0,x}[f(X^a_t) - f({}_aX_t)| \sup_{s\le t}|X^a_s| > a]|\mathsf{P}[\sup_{s\le t}[|X^a_s| > a]$$
$$\le 2K\mathsf{P}[\sup_{s\le t}|X^a_s| > a] \to 0 \text{ as } a \to \infty.$$

So, ${}_a\tilde{\phi}(t,x) \to \phi(t,x)$ as $a \to \infty$. Moreover, this convergence is clearly uniform for $x \in [0,d]$ for any constant d. This is sufficient to show that,

$$\frac{\partial_a\tilde{\phi}}{\partial x} \to \frac{\partial\phi}{\partial x} \text{ as } a \to \infty,$$

and so,

$$\frac{\partial\phi}{\partial x} \ge 0 \text{ as required.}$$

1.3 The Control Problem

Armed with the main result of Chapter 2, it is now time to consider the control problem:-

Choose $\mathbf{u}^0$ so as to minimize $\mathsf{E}[f(X_t^{\mathbf{u}^0})]$, where $\mathbf{u}^0$ is an $\mathcal{F}_t$-adapted control, and $-b \le \mathbf{u}^0(x) \le b, \forall x \in \mathbf{R}$.

The following results from Øksendal (1985) stated in the context of the problem are required.

THEOREM 3.1. *Let,*

$$J^{\mathbf{u}}(t,x) = \mathsf{E}[f(X_t^{\mathbf{u}})|X_0 = x]. \tag{3.1}$$

Then suppose there exists $h : [0,T] \times \mathbf{R} \to \mathbf{R}$ *such that:-*

$$\left(\mathcal{L}^{\mathbf{u}} - \frac{\partial}{\partial t}\right) h(t,x) \ge 0, \forall (t,x) \in [0,T] \times [-b,b], \tag{3.2}$$

and $h(0,x) = f(x)$, $h \in C^{1,2}([0,T])$.

Also suppose that there exists a Markov control $\mathbf{u}^0$ *such that:-*

$$\left(\mathcal{L}^{\mathbf{u}^0} - \frac{\partial}{\partial t}\right) h(t,x) = 0, \text{ then} \tag{3.3}$$

$$h(t,x) = J^{\mathbf{u}^0}(t,x), \tag{3.4}$$

and u^0 is our optimal Markov control, i.e. it minimises $J^u(t,x)$.

This result is theorem 10.2 of Øksendal applied to $\phi(T-t,x)$ for some fixed but arbitrarily large T.

THEOREM 3.2. *The optimal Markov control is optimal among the larger set of $\mathcal{F}_t$-adapted controls, where in this case $\mathcal{F}_t = \sigma\{X_s;\ t \geq s\}$.*

This result is theorem 10.3 of Øksendal.

LEMMA 3.3. *$\hat{u}$ minimises $\mathsf{E}[f(X_t^u)]$ for all $\mathcal{F}_t$-adapted admissible controls u.*

PROOF:

By theorems 3.1 and 3.2, all we need to show is that (3.2) holds. But,

$$\left(\mathcal{L}^u - \frac{\partial}{\partial t}\right)\phi(t,x) = (\mathcal{L}^u - \mathcal{L}^{\hat{u}})\phi(t,x)$$
$$= (u-\hat{u})\frac{\partial \phi}{\partial x},$$

where u is a Markov control. But, $-b \leq u(x) \leq b$, so

$$(u-\hat{u})(x) \leq 0,\ x \geq 0,$$
$$(u-\hat{u})(x) \geq 0,\ x \leq 0,$$

i.e. $(u-\hat{u})\frac{\partial\phi}{\partial x} \leq 0, \forall (t,x) \in [0,T] \times \mathbb{R}$

Now by theorem 3.1, $\hat{u}$ is optimal among Markov controls and by theorem 3.2, $\hat{u}$ is optimal among $\mathcal{F}_t$-adapted controls.

THEOREM 3.4.

$|X_t^{\hat{u}}|$ achieves the stochastic infimum of $\{|X_t^u|;\ u \in C\}$.

PROOF: Let,

$$I = I[|X| > c]. \tag{3.5}$$

We can approximate I by a sequence of increasing functions $\{f_i\}$, such that the f_is satisfy the conditions of lemma 3.3 and $\int |f_i - I| dx \to 0$ as $i \to \infty$. By Karatzas and Shreve (1984), $X_t^{\hat{u}}$ has a continuous bounded density, and so since $\mathsf{E}[f_i^u)] \geq \mathsf{E}[f_i(X_t^{\hat{u}})]$ $\forall$ controls u,

$$\mathsf{P}[|X_t^{\hat{u}}| > c] = \mathsf{E}[\lim_{i\to\infty} f_i(X_t^{\hat{u}})] \leq \liminf \mathsf{E}[f_i(X_t^u)] \leq \mathsf{P}[|X_t^u| > c] \tag{3.6}$$

The final inequality follows from:-

$$f_i(X) \leq I(X), \forall \in \mathbb{R}.$$

$\mathsf{P}[|X_t^{\hat{u}}| \geq c] \leq \mathsf{P}[|X_t^u| \geq c]$, $\forall$ controls u.

THEOREM 3.5. *Define the policy $\hat{u}$ by:-*

$$\hat{u}(x) = \begin{cases} b, & x > 0 \\ 0, & x = 0 \\ -b, & x < 0 \end{cases} \tag{3.7}$$

Then $|X_t^{\hat{u}}|$ achieves the stochastic supremum of S_t.

PROOF:

As for $|X_t^{\hat{u}}|$.

COROLLARY 3.6. *The control problem:-*

Choose $u \in C$ so as to minimise

$$H^u(t, x) = \mathsf{E}[\int_t^\infty e^{-\alpha s} f(X_s^u) ds | X_t = x], \tag{3.8}$$

has $\hat{u}$ as its solution.

PROOF:

$$\mathsf{E}[f(X_s^u)] \geq \mathsf{E}[f(X_s^{\hat{u}})].$$

So:-

$$\int_t^\infty e^{-\alpha s} \mathsf{E}[f(X_s^u)] ds \geq \int_t^\infty \mathsf{E}[f(X_s^{\hat{u}})] e^{-\alpha s} ds$$

i.e.

$$H^u(t, x) \geq H^{\hat{u}}(t, x), \ \forall u \in C \tag{3.9}$$

THEOREM 3.7. *If $B_t(n)$ is n-dimensional B.M. and ${}_uX_t$ is given by,*

$$d_u X_t(n) = dB_t(n) + u \ dt, \tag{3.10}$$

and

$$\phi_u(t, x) = \mathsf{E}[f(|_u X_t(n)|)], \tag{3.11}$$

where f is a function satisfying the conditions of Lemma 3.3. Then

$$\phi_{\hat{u}}(t, x) \leq \phi_u(t, x) \ \forall \text{ controls } u \text{ s.t. } |u| \leq b. \tag{3.12}$$

where

$$\hat{u}(x) = \begin{cases} \frac{-bx}{|x|}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

PROOF: The proof is essentially the same as that of theorem 3.4 with a few modifications.

Define:-

$$h(t, x) = \inf\{\phi_u(t, x); \ |u| \leq b\} \tag{3.13}$$

Now clearly by symmetry, $h(t,x)$ depends only on t and $|x|$, and so ∇h is parallel to x, where

$$\nabla h^T = \left(\frac{\partial h}{\partial x_1}\frac{\partial h}{\partial x_2}\cdots\frac{\partial h}{\partial x_n}\right).$$

The Bellman equation for the system is:-

$$\min_{\mathbf{u}}\{\mathbf{u}\cdot\nabla h\} + \frac{1}{2}\nabla^2 h + f(\mathbf{x}) = 0. \tag{3.14}$$

Thus our candidate optimal policy is also parallel to $\mathbf{x}$. Now if ${}_uY_t = |{}_u\mathbf{X}_t|$, our problem reduces to the 1-dimensional problem:-

minimise $\mathsf{E}[f({}_uY_t)]$ subject to

$$d_uY_t = dB'_t + \left(\frac{n-1}{{}_uY_t} + u({}_uY)\right)dt, \quad n > 1, \tag{3.15}$$

where B'_t is the 1-dimensional Wiener process given by:-

$$dB'_t = \frac{{}_u\mathbf{X}_t(n)\,.\,d_u\mathbf{B}_t(n)}{{}_uY_t}. \tag{3.16}$$

Proceeding as in chapter 2, $\theta = \frac{\partial h}{\partial|x|}$, satisfies:-

$$\left(\mathcal{L}^a - \frac{\partial}{\partial t}\right)\theta - \frac{n-1}{x^2}\theta = 0 \tag{3.17}$$

where

$$\mathcal{L}^c = \frac{1}{2}\frac{\partial}{\partial x^2} + \left(\frac{n-1}{x} - \dot{o}\,\mathrm{sgn}\,x\right)\frac{\partial}{\partial x}, \tag{3.18}$$

with the convention sgn $0 = 0$.

This is now in a form where we can apply the weak maximum principle, (theorem 3.1, chapter 6, Friedman (1975)), as in lemma 2.7, to a suitably modified diffusion, since $\frac{n-1}{x^2} \geq 0$.

The proof now proceeds in an identical fashion to that of theorem 3.4.

THEOREM 3.8. *Let*

$$T^u(t,x) = \int_0^t I[|X_s^u| > c]ds|X_0^u = x. \tag{3.19}$$

Then $T^a(t,x)$ achieves the stochastic infimum of $\{T^u(t,x);\ u \in C\}$.

PROOF: Let

$$\mu(t,x,y) = \mathsf{E}[g(\int_0^t f(X_s^a)ds - y)] \tag{3.20}$$

where f is a function satisfying the conditions of lemma 3.3, and g is a non-decreasing, positive, C^2 function such that $g(x) = 1$, $\forall x \geq 0$. It is apparent that:-

$$\mu(T-t, X_t^a,\, Y - \int_0^t f(X_s^a)ds) = \mathsf{E}[g(\int_0^T f(X_s^a)ds - Y)|\mathcal{F}_t] \tag{3.21}$$

and so is a martingale. An Ito expansion therefore gives:-

$$\mathcal{L}^{*}\mu - \frac{\partial \mu}{\partial t}t - f(x)\frac{\partial \mu}{\partial y} = 0. \tag{3.22}$$

Assuming the relevant derivatives exist, $\theta = \frac{\partial \mu}{\partial x}$, satisfies:-

$$\mathcal{L}^{*}\theta - \frac{\partial \theta}{\partial t} - f'(x)\frac{\partial \mu}{\partial y} - f(x)\frac{\partial \theta}{\partial y} = 0, \tag{3.23}$$

(the justification of this statement follows almost completely the arguments used in Chapter 2 and therefore will not be given here).

i.e. $\mathcal{P}\theta = f'(x)\frac{\partial \mu}{\partial y}$, where $\mathcal{P}$ is a weakly parabolic operator. Now $f'(x) \geq 0$ on $x \geq 0$, and $\frac{\partial \mu}{\partial y} \leq 0$ because if $y_1 < y_2$, then:-

$$g(\int_0^t f(X_s^*)ds - y_1) \geq g(\int_0^t f(X_s^*)ds - y_2) \tag{3.24}$$

for all $\omega \in \mathcal{F}_t$, and so

$$\mathsf{E}[g(\int_0^t f(X_s^*)ds - y_1)] \geq \mathsf{E}[g(\int_0^t f(X_s^*)ds - y_2)] \tag{3.25}$$

Therefore,

$$\mathcal{P}\theta \leq 0, \ x \geq 0. \tag{3.26}$$

Following the proof of theorem 2.8, we now truncate the diffusion by imposing a reflecting boundary at $x = a$. If we also impose the condition $Y > T$ then:-

$$\begin{aligned} &\theta(t,x,y) = 0, \text{ on } \partial_0 Q \\ &\text{where } Q = (0,T) \times (0,a) \times (0,Y), \\ &\text{and } \partial_0 Q = \partial Q - \{(T,x,y): \ (x,y) \in (0,a) \times (0,Y)\} \end{aligned} \tag{3.27}$$

So using the weak maximum principle, $\theta \geq 0$ on Q. Now all that remains is to use the same limiting arguments as those used in theorem 2.8 and then an appeal to theorems 3.1 and 3.2 to show that u does minimize $\{\mathsf{E}[g(\int_0^t f(X_s^u)ds - y)]\}$. The result now follows by a C^2 approximation scheme such as that used in the proof of theorem 3.4.

1.4 Explicit solutions for $\phi(t,x)$ and $H(t,x)$

Karatzas and Shreve (1984) calculated the transition density of X_t^*, to be:-

$$f_{[X_t^*|X_0^*=x]}(z) = P_t(x,z)$$

$$
= \begin{cases} \frac{1}{(2\pi t)^{\frac{1}{2}}} \left[\exp\left[\frac{-(s-x-bt)^2}{2t}\right] + be^{-bs} \int_{s+x}^{\infty} \exp\left[\frac{-(v-bt)^2}{2t}\right] dv\right], & x \geq 0, s \geq 0 \\ \frac{1}{(2\pi t)^{\frac{1}{2}}} \left[\exp\left[2bx - \frac{(s-x+bt)^2}{2t}\right] + be^{2bx} \int_{x-s}^{\infty} \exp\left[\frac{-(v-bt)^2}{2t}\right] dv\right], & x \geq 0, s \leq 0, \end{cases}
$$

$$
\text{with } P_t(-x, s) = P_t(x, -s). \tag{4.1}
$$

We can use this result to find an explicit expression for $\phi(t, x)$, namely:-

$$
\phi(t, x) = \int_{-\infty}^{\infty} P_t(x, s)\, f(s)\, ds \tag{4.2}
$$

$H^u(t, x)$ can also be expressed in this way. However, a simpler expression can be found by appealing directly to the Bellman equation.

PROPOSITION 4.1.

$$
H^u(t, x) = e^{-\alpha t} H^u(0, x) = e^{-\alpha t} h(x) \tag{4.3}
$$

for some function h.

PROOF: Since u is independent of t, $[X_s^u | X_t^u = x]$ and $[X_{s-t}^u | X_0^u = x]$ are identically distributed. So:-

$$
\begin{aligned}
H^u(t, x) &= e^{-\alpha t} \mathrm{E}\left[\int_t^{\infty} e^{-\alpha(s-t)} f(X_s^u) ds \,|\, X_t^u = x\right] \\
&= e^{-\alpha t} \int_t^{\infty} e^{-\alpha(s-t)} \mathrm{E}[f(X_s^u) | X_0^u = x] ds \\
&= e^{-\alpha t} \int_t^{\infty} e^{-\alpha(s-t)} \mathrm{E}[f(X_{s-t}^u) | X_0^u = x] ds \\
&= e^{-\alpha t} H^u(0, x) = e^{-\alpha t} h(x).
\end{aligned}
$$

The Bellman equation for $H^u(t, x)$, thus reduces to an O.D.E. in $h(x)$:-

$$
\frac{1}{2} h''(x) + u h'(x) + f(x) - \alpha h(x) = 0 \tag{4.4}
$$

Due to the simplicity of u, this equation can be easily solved:-

$$
\begin{aligned}
h(x) = &-2\left[\int_0^x \left(\int_0^s f(\eta) e^{-(b+\bar{b})\eta} d\eta\right) e^{2\bar{b}s} ds\right] e^{-(\bar{b}-b)x} \\
&+ Ae^{(b+\bar{b})x} + Be^{-(\bar{b}-b)x}, \quad x \geq 0
\end{aligned}
$$

$$
\text{and } h(-x) = h(x) \tag{4.5}
$$

$$
\begin{aligned}
= \frac{e^{(\bar{b}-b)x}}{\bar{b}} \int_0^{|x|} e^{(\bar{b}-b)\eta} f(\eta) d\eta - \frac{e^{(b+\bar{b})|x|}}{\bar{b}} \int_0^{|x|} e^{-(b+\bar{b})\eta} f(\eta) d\eta \\
+ Ae^{(b+\bar{b})|x|} + Be^{-(\bar{b}-b)|x|}.
\end{aligned} \tag{4.6}
$$

$$
\text{In the above, } \bar{b} = \sqrt{b^2 + 2\alpha}. \tag{4.7}
$$

Now the heuristic principles of smooth fit, and growth no faster than $f(x)$ lead to an evaluation of the constants. The smooth fit condition was used by Benes et. al. (1980), and Lehoczky and Shreve (1986) provide an insight into this. We get:-

$$h(x) = \frac{e^{(b+\bar{b})|x|}}{\bar{b}} \int_{|x|}^{\infty} f(\eta)e^{-(\bar{b}+b)\eta}\,d\eta + \frac{e^{-(\bar{b}-b)|x|}}{\bar{b}} \int_{0}^{|x|} e^{(\bar{b}-b)\eta} f(\eta)d\eta$$

$$+Be^{-(\bar{b}-b)|x|}, \tag{4.8}$$

where:-

$$B = \frac{b+\bar{b}}{\bar{b}(\bar{b}-b)} \int_{0}^{\infty} f(\eta)e^{-(\bar{b}+b)\eta}\,d\eta. \tag{4.9}$$

To prove (4.8), it is necessary to show that R.H.S. of (4.8) satisfies the conditions of theorem 3.1.

Denote R.H.S. of (4.8) by $l(x)$, i.e.:-

$$l(x) = \frac{e^{(\bar{b}+b)|x|}}{\bar{b}} \int_{|x|}^{\infty} f(\eta)e^{-(\bar{b}+b)\eta}\,d\eta$$

$$+\frac{e^{-(\bar{b}-b)|x|}}{\bar{b}} \int_{0}^{|x|} e^{(\bar{b}-b)\eta} f(\eta)d\eta + Be^{-(\bar{b}-b)|x|} \tag{4.10}$$

Lemma 4.2.

$$l'(x) \geq 0 \text{ for } x \geq 0 \tag{4.11}$$

Proof:

For $x \geq 0$, denote the first two terms of $\phi(x)$ by $g_1(x)$, $g_2(x)$. So,

$$l(x) = g_1(x) + g_2(x) + Be^{-(\bar{b}-b)x} \tag{4.12}$$

$$l'(x) = g_1'(x) + g_2'(x) - (\bar{b}-b)Be^{-(\bar{b}-b)x} \tag{4.13}$$

but $f(x)\uparrow$, and is positive, so

$$l'(x) \geq \frac{\bar{b}+b}{\bar{b}} \left[\int_{x}^{\infty} f(\eta) \left[e^{-(\bar{b}+b)(\eta-x)} - e^{-(\bar{b}+b)\eta}\right] d\eta\right]$$

$$-\frac{e^{(\bar{b}-b)x}}{\bar{b}}(\bar{b}-b) \int_{0}^{x} e^{(\bar{b}-b)\eta} f(\eta)d\eta \tag{4.14}$$

$$\geq \frac{f(x)}{\bar{b}} \left[(b+\bar{b}) \int_{x}^{\infty} e^{-(\bar{b}+b)(\eta-x)} - e^{-(\bar{b}+b)\eta}\right] d\eta$$

$$-(\bar{b}-b) \int_{0}^{x} e^{(\bar{b}-b)\eta} d\eta \tag{4.15}$$

$$= 0.$$

So $l'(x) \geq 0$ for $x \geq 0$.

Lemma 4.3. *Let*

$$M_t^a = e^{-\alpha t} l(X_t^a) + \int_0^t e^{-\alpha s} f(X_s^a)\, ds \tag{4.16}$$

Then M_t^a is an $\mathcal{F}_t$-martingale.

Proof:

$$\alpha l(x) = f(x) + \frac{1}{2} l''(x) + u l'(x), \tag{4.17}$$

and by Ito's formula:-

$$M_t^a = M_s^a + \int_s^t e^{-\alpha r}[-\alpha l(X_r)dr + l'(X_r)(dB_r + u\, dr)$$
$$+ \frac{1}{2} l''(X_r)dr + f(X_r)dr] \tag{4.18}$$

Taking conditional expectations with respect to $\mathcal{F}_s$ and using (4.17), shows that M_t^a is an $\mathcal{F}_t$-martingale, since $l'(x)$ is bounded on compact intervals.

Theorem 4.4. *$h(x)$ is given by (4.8.)*

Proof:

$$h(x) = \mathrm{E}_x[M_\infty^a] = \mathrm{E}_x[M_0^a] = l(x)$$

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I.2 A Weak Comparison Theorem for S.D.E.s

2.1 Introduction

In this chapter we shall develop the ideas of chapter 1 to obtain more general stochastic inequalities. Whereas in chapter 1 the emphasis was on deriving explicit solutions for certain special cases, here we concern ourselves with obtaining inequalities in as general a framework as possible.

Let $\{\Omega,\ \mathcal{F},\ \mathsf{P}\}$ be a probability space on which are defined random variables X, and Y. By a strong comparison result for X and Y, we mean an inequality such as,

$$X \leq Y,\ \mathsf{P}\ a.s.$$

By a weak comparison result to hold we mean:

$$\mathsf{P}[X < x] \geq \mathsf{P}[Y < x],\ \forall x \in \mathbb{R}.$$

Here we concern ourselves with comparison results for processes where the inequalities apply for all time.

The connection between the existence of a unique strong solution for an S.D.E. and strong comparison results where at least one of the S.D.E.s has a strong solution, is well understood (see for example Ikeda and Watanabe (1980) and Le Gall (1982)). Similar techniques can be used in the proofs of theses strong results.

However in the more general case where only the weak uniqueness of solution is guaranteed, such techniques cannot be used. This chapter develops the ideas used in chapter 1, where a stochastic control theory framework is used to give simple proofs of some weak comparison results. The main result that we shall prove is the following:-

Let X be the unique solution to the S.D.E.

$$dX_t = \sigma(t, X_t)\ dB_t + \hat{u}(t, X_t)\ dt,$$

where $\sigma(t, x)$ is a symmetric function of x for each t, and $\hat{u}(t, x) = -\hat{u}(t, -x)\ \forall t, x$. Also, suppose Y is a solution of:

$$dY_t = \sigma(t, Y_t)\ dB_t + u_t\ dt,$$

where u_t is a progressively measurable process with respect to $\mathcal{F}_t$, such that,

$$u \leq \hat{u}(t, X_t),\ X_t \leq 0,$$
$$\text{and } u \geq u(t, X_t),\ X_t \geq 0.$$

Then, $|Y_t|$ dominates $|X_t|$ for all $t \geq 0$.

The main extension to the methods of chapter 1 comes in the proof of the main lemma (Lemma 2.2), where here we have used probabilistic methods: the analytic approach of the earlier chapter being inapplicable in the more general setting. We also make extensive use of the celebrated work of Stroock and Varadhan (1969), and in particular the weak uniqueness conditions this paper gives, together with the equivalence of weak uniqueness with the existence of a unique solution to a related P.D.E..

The use of control theory methods to prove inequalities for processes is not new, for example, Barlow and Jacka (1986) used control theory techniques with application to weak solutions for S.D.E.s. Here, these methods are no more powerful than those in a probabilistic or analytic approach (the Bellman equation being merely a restatement of standard martingale arguments in an Itô calculus context), but it is hoped that the different light they cast on the problem will make the proofs clearer.

2.2 The Main Result

Let $\{\mathcal{F}, \mathcal{F}_t, \ t \geq 0\}$ be a filtration carrying a Brownian motion $\{B_t, \ t \geq 0\}$. Let $\{X_t^u, \ t \geq 0\}$ be a controlled diffusion process given by,

$$dX_t^u = \sigma(t, X_t^u)dB_t + u_t \ dt,$$

where σ is a continuous function of x and t which is bounded away from zero on compact sets, and $u \in C$ where,

$$C = \left\{\mathcal{F}_t - \text{measurable controls } u \text{ such that } u_1(t, X_t^u) \leq u \leq u_2(t, X_t^u)\right\}.$$

Here, u_1 and u_2 are measurable functions of x and t, and are (uniformly in x) Hölder-continuous functions of t, such that,

$$u_1(t, x) = -u_2(t, -x).$$

We will also need to stipulate growth conditions on σ, u_1 and u_2:-

$\exists A > 0$ such that $\forall t, x$,

$$\sigma^2(t, x) + u_1^2(t, x) + u_2^2(t, x) \leq A(1 + x^2).$$

Define the control $\hat{u}$ as follows:-

$$\hat{u} = \begin{cases} u_1(t, X_t^u), & X_t^u \geq 0, \\ u_2(t, X_t^u), & X_t^u \leq 0. \end{cases}$$

We will show that $X_t^{\hat{u}}$ stochastically minimizes $\{|X_t^u|, \ u \in C\}$.

As in chapter 1, we must truncate the diffusion $\{|X_t^u|, \ t \geq 0\}$. However, here we find it more suitable to kill the process at $\pm a$. We prove that $\hat{u}$ is optimal for this problem, and then consider the limit as $a \to \infty$.

So we define ${}_a\tilde{X}_t$ by,

$$d_a\tilde{X}_t = \begin{cases} \sigma(t, {}_a\tilde{X}_t)\, dB_t + \bar{u}({}_a\tilde{X}_t)\, dt, & |{}_a\tilde{X}_t| < a, \\ 0, & |{}_a\tilde{X}_t| = a, \end{cases} \tag{2.1}$$

We will need the following results.

LEMMA 2.1. *Define*

$$S = \left\{ \begin{array}{c} \textit{symmetric functions } f \in C^2(\mathbb{R}) \textit{ such that } f'(x) \geq 0,\ x \geq 0, \\ \textit{and } \exists \textit{ constants } B, k \textit{ such that } f(x) \leq Bx^k, x \geq 0 \end{array} \right\}.$$

The growth condition, together with those imposed on the parameters of X_t^a, merely ensure that $\mathbb{E}[f(X_t^a)]$ is finite $\forall u \in C$. Let

$$\tilde{\phi}(t, x) = \mathbb{E}[f({}_a\tilde{X}_t) | {}_a\tilde{X}_0 = x]$$

for some $f \in S$. Then for each $a \geq 0$, $\tilde{\phi}$ is the unique solution of the parabolic P.D.E.:

$$\mathcal{L}_t \tilde{\phi} = \frac{\partial \tilde{\phi}}{\partial t}, \tag{2.2}$$

with the initial condition,

$$\tilde{\phi}(0, x) = f(x),$$

and boundary conditions,

$$\tilde{\phi}(t, a) = \tilde{\phi}(t, -a) = f(a),$$

where,

$$\mathcal{L}_t = \frac{1}{2}\sigma^2(t, x)\frac{\partial^2}{\partial x^2} + \bar{u}(x)\frac{\partial}{\partial x}$$

is the infinitesimal generator of ${}_a\tilde{X}_t$.

PROOF: For each $r \in S$, equation (2.2) has a unique solution, θ under the given boundary conditions. This is equivalent to the existence of a unique weak solution of the S.D.E. (2.1), and this is guaranteed (see for example Stroock and Varadhan (1969)). It remains to show that $\tilde{\phi} = \theta$. However it is clear by a application of Ito's formula that $\theta(T - t, {}_aX_t)$ is also a martingale by similar argument to those of chapter 1. So,

$$\theta(t, x) = \mathbb{E}[\theta(0, {}_a\tilde{X}_t) | {}_a\tilde{X}_0 = x] = \mathbb{E}[f({}_a\tilde{X}_t)] = \tilde{\phi}(t, x)$$

LEMMA 2.2.

$$\frac{\partial \tilde{\phi}}{\partial x} \geq 0, \quad \textit{for } x \geq 0,$$
$$\frac{\partial \tilde{\phi}}{\partial x} \leq 0, \quad \textit{for } x \leq 0.$$

PROOF: By the Stroock-Varadhan characterization of a diffusion, ${}_a\tilde{X}_t$ is a diffusion since it satisfies (2.2) for all $f \in S$. We therefore have the following standard coupling argument.

Let $\{\mathcal{G}, \mathcal{G}_t, t \geq 0\}$ be a filtration generated by two independent copies of ${}_a\tilde{X}_t$: X^1 and X^2 started at x_1 and x_2 respectively, with $0 \leq x_1 \leq x_2$. Define,

$$\tau = \inf\{s \geq 0;\ |X_s^1| = |X_s^2|\}.$$

Then, clearly τ is a $\{\mathcal{G}_t\}$-stopping time, and

$$\begin{aligned}\tilde{\phi}(t,x_1) - \tilde{\phi}(t,x_2) = &P[\tau < t]\{E[f(X_t^1)|\tau < t] - E[f(X_t^2)|\tau < t]\} \\ &+ P[\tau \geq t]\{E[f(X_t^1)|\tau \geq t] - E[f(X_t^2)|\tau \geq t]\}.\end{aligned}$$

But, conditional on $\tau \geq t$, $|X_t^2| \geq |X_t^1|$ *a.s.* Also, since ${}_a\tilde{X}_t$ is a diffusion satisfying the strong Markov property,

$$E[f(X_t^1)|\tau < t] - E[f(X_t^2)|\tau < t] = 0,$$

and so,

$$\tilde{\phi}(t,x_1) - \tilde{\phi}(t,x_2) \leq 0.$$

THEOREM 2.3. *Suppose $f \in S$, then*

$$E[f({}_a\tilde{X}_t^{\tilde{u}})] = \inf_{u \in C}\{E[f({}_a\tilde{X}_t^{u})]\}.$$

PROOF: The Bellman equation for the problem implies that $\tilde{u}$ is the optimal control if and only if:

$$\mathcal{L}_t^u\tilde{\phi} - \frac{\partial\tilde{\phi}}{\partial t} \geq 0,\ \forall u \in C,\ \forall (t,x) \in [0,t] \times [-a,a], \tag{2.3}$$

since we know we have equality in (2.3) for $u = \tilde{u}$.

However this follows easily from lemma 2.2, because

$$\mathcal{L}_t^u\tilde{\phi} - \frac{\partial\tilde{\phi}}{\partial t} \geq \mathcal{L}_t^{\tilde{u}}\tilde{\phi} - \frac{\partial\tilde{\phi}}{\partial t} = 0$$

by the definition of $\tilde{u}$.

COROLLARY 2.4.

$$E[f(X_t^{\tilde{u}})] = \inf_{u \in C}\{E[f(X_t^u)],\ u \in C\}.$$

PROOF: We look at the limit as $a \to \infty$ of theorem 3. It remains to prove that,

$$\lim_{a \to \infty}[f({}_a\tilde{X}_t^{\tilde{u}})] = f(X_t^{\tilde{u}}),$$

at least for $u = a$. However in this case we have a unique solution in law for both ${}_a\tilde{X}_t$ and X_t, which are identical on the event,

$$E_a = \{|X_s| < a,\ 0 \le s \le t\}.$$

So, denoting the distribution of ${}_a\tilde{X}_t$ by $\tilde{\mu}$ and that of X_t^a by μ:

$$\begin{aligned} \tilde{\phi}(t,x) &= \int_{E_a} f(x)\, d\tilde{\mu}(x) + \int_{E_a^c} f(x)\, d\tilde{\mu}(x), \\ &= \int_{E_a} f(x) d\mu(x) + \int_{E_a^c} f(x)\, d\tilde{\mu}(x), \end{aligned}$$

and,

$$\phi(t,x) - \tilde{\phi}(t,x) \le \int_{E_a^c} f(x)\, d\mu.$$

However, we know by the growth conditions imposed on f and the parameters of X^a, that $\int f(x)\, d\mu$ is finite and so since $\mathrm{P}[E_a^c] \downarrow 0$,

$$\int_{E_a^c} f(x)\, d\mu \downarrow 0,$$

thus completing the proof.

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PART II

II.1 Limit Laws For Conditioned Diffusions With Application To Boundary Hitting Times

1.1 Introduction

The existence of power moments for Brownian motion hitting times of exact square root boundaries was first considered by Breiman (1965), and Shepp (1971). They established separately the following result. Let

$$\tau = \inf\{t \geq 1; |B_t| \geq c(1+t)^{\frac{1}{2}}\}, \tag{1.1}$$

then

$$\mathsf{E}[\tau^p] < \infty <=> c < c(p), \tag{1.2}$$

where $c(p)$ is the smallest positive root of the pth confluent hypergeometric function

$$F_p(x) = \sum_{m=0}^{\infty} \frac{(-2x^2)^m p(p-1)...(p-m+1)}{(2m)!}. \tag{1.3}$$

In this chapter we aim to solve the approximate square root boundary problem near $c(p)$. A partial solution was given by Taksar (1982) who looked at boundaries of the type

$$\tau = \inf\{t \geq 1; |B_t| \geq a(t)(1+t)^{\frac{1}{2}}\}, \tag{1.4}$$

where $a(t) \uparrow c(p)$ as $t \to \infty$. He showed that $\mathsf{E}[\tau^p] < \infty$ if

$$\int_1^{\infty} t^{p-1-m(a(t))} dt < \infty, \tag{1.5}$$

where $m(.)$ is the inverse of $c(.)$.

Here, we will prove the following partial converse of Taksar's result.

THEOREM 1.1. *Let τ be a stopping time of the form given in (1.4), then if $\mathsf{E}[\tau^p] < \infty$, we have $\int_1^{\infty} t^{p-1} h(t) dt < \infty$ where*

$$h(t) = \exp - \left\{ \int_0^t \frac{m(a(s))}{s} ds \right\} \tag{1.6}$$

The methods used in this chapter are mainly probabilistic. This has the advantage that it provides intuition into the problem which cannot be provided by the analytic methods used on such problems before.

In the second section, we obtain some general results on constant boundary hitting times for symmetric diffusions. In particular we establish the existence of a limit distribution δ_∞ given by

$$\delta_\infty = \lim \text{law}[X_t | \tau > t] \tag{1.7}$$

As a result of this we can express the tail behaviour of τ', the boundary hitting time as

$$P[\tau' > t] = e^{-\alpha t}(k + o(1)) \tag{1.8}$$

for some constant k depending on X_0. The constant α can be expressed as a moment of δ_∞. Now since square root boundaries for Brownian Motion are equivalent, under scale and time changes, to constant boundaries for an Ornstein-Uhlenbeck process, we can therefore deduce the tail behaviour of τ (the square root stopping time for the Brownian Motion).

$$P[\tau > t] = t^{-\beta}(k' + o(1)) \tag{1.9}$$

where k' is a function of B_0, and β is a function of the corresponding δ_∞ of the Ornstein-Uhlenbeck process. The fundamental result for the approximations in chapter 3 is lemma 2.2 where inequalities are obtained for processes conditioned not to hit two different boundaries. This result is proved by coupling arguments which can be applied to a much more general setting, i.e. any process X which preserves order in the sense that for $s < t$, if X^1 and X^2 are independent copies of X:

$$X_s^1 \overset{st}{\leq} X_s^2 \implies X_t^1 \overset{st}{\leq} X_t^2.$$

In Section 3, this leads us to identify β as $m(c)$. Now the approximate square root boundary problem has a corresponding approximate constant boundary hitting problem for an Ornstein-Uhlenbeck process. It is this correspondence along with some stochastic order relations which allow us to prove theorem 1.1.

In chapter 3, we shall complete this work on approximate square-root boundaries by showing that the integrability conditions given here are in fact necessary and sufficient under certain extra conditions imposed on the function $a(.)$.

1.2 Notation and Preliminaries

Let B_t be a standard Brownian Motion, and let X_t be the associated Ornstein-Uhlenbeck process defined by

$$X_t = \frac{B(e^t)}{e^{t/2}}, \tag{2.1}$$

then X_t satisfies

$$dX_t = dB_t' - \frac{1}{2}X_t dt \tag{2.2}$$

where B' is a Brownian motion. Define,

$$_c\tau = \inf\{t \geq 1; |B_t| \geq ct^{\frac{1}{2}}\} \tag{2.3}$$

$$\text{and } \tau_c = \inf\{t \geq 0; |X_t| \geq c\}, \tag{2.4}$$

then $\tau_c = \log_c \tau$.

More generally, suppose $f(.)$ is a positive function, then

$$_f\tau = \inf\{t \geq 1; |B_t| \geq f(t)\} \tag{2.5}$$

$$\tau_f = \inf\{t \geq 0; |X_t| \geq f(e^t)e^{-t/2}\}, \tag{2.6}$$

and we will write $f(t) = a(t)t^{\frac{1}{2}}$ for the approximate square root case. So in particular, $\tau_{a(t_1)}$ will denote the hitting time of the exact square-root boundary, $f(t) = a(t_1)t^{\frac{1}{2}}$.

Suppose μ is a distribution with support contained in $(-c, c)$, then we define

$$\mu_t(x) = \mathbb{P}[X_t \leq x | \tau_c > t, \quad X_0 \text{ has law } \mu] \tag{2.7}$$

$$\text{and } \mu_t(x) = \int_{-x}^{x} d\mu_t(y). \tag{2.8}$$

Also denote by $\bar{\mu}$ the modulus law of μ,

$$\bar{\mu}(x) = \int_{-x}^{x} d\mu(y). \tag{2.9}$$

We will make extensive use of stochastic order relations, and write,

$$\mu_1 \overset{st}{\leq} \mu_2 \text{ if and only if } \mu_1(x) \geq \mu_2(x), \quad \forall x \in \mathbb{R}.$$

The correspondence between Brownian Motion and the Ornstein-Uhlenbeck process is fundamental to the main results of this paper proved in section three. However, the preliminary results in this section are true in greater generality than just for the Ornstein-Uhlenbeck process. So for the rest of this section, we will assume X is a time-homogeneous symmetric diffusion process,

$$dX_t = \sigma(X_t)dB_t + b(X_t)dt,$$

where $\sigma(x) = \sigma(-x)$, and $b(x) = -b(-x)$. Here b and σ are such that the scale function of X is bounded on bounded intervals, and σ is bounded above and below by positive constants (at least on bounded intervals). The symmetry of the problem allows us to look at $|X|$ and X interchangeably and most of the following results are stochastic inequalities for $|X|$.

The results that follow have been rigorously proved, as the results proved are fundamental to the main results of later chapters. The proofs are long, and (by my own admission) difficult to follow. This is partly due to the fact that the author has attempted to prove the results in a fairly general setting to make the results interesting in their own right. However a further justification for them lies in the fact that not only are the results proved here intuitively clear results, but also it is hoped that the corresponding proofs will give further intuition to the reader, as they did to the author. So although it might be tempting for the reader to skip through the proofs, he is urged to devote some time to them, at least on a second reading.

Lemma 2.1. *Let μ_1, μ_2 be distributions on $(-f(0), f(0))$ for some positive function $f : \mathbb{R}^+ \to \mathbb{R}^+ \cup \{\infty\}$. Suppose $\bar{\mu}_1 \overset{st}{\leq} \bar{\mu}_2$, then $\bar{\mu}_{1_t} \overset{st}{\leq} \bar{\mu}_{2_t}$*

Proof:

$$\text{Let } p_{y,t}(x) = \mathrm{P}[|X_t| < y | \tau_f > t, \quad X_0 = x]. \tag{2.10}$$

Then since $\bar{\mu}_i$ has support $[0, f(0))$,

$$\begin{aligned} \bar{\mu}_{i_t} &= \lim_{c \uparrow f(0)} \int_0^c p_{y,t}(x) d\bar{\mu}_i(x) \\ &= \lim_{c \uparrow f(0)} [p_{y,t}(c)\bar{\mu}_i(c) - \int_0^c \bar{\mu}_i(x) dp_{y,t}(x)], \end{aligned} \tag{2.11}$$

for $i = 1, 2$. Now $\bar{\mu}_1 \overset{st}{\leq} \bar{\mu}_2$, so

$$\begin{aligned} &\bar{\mu}_1(x) - \bar{\mu}_2(x) \geq 0 \\ &\bar{\mu}_{1_t}(y) - \bar{\mu}_{2_t}(y) = \lim_{c \uparrow f(0)} [p_{y,t}(x)(\bar{\mu}_1(c) - \bar{\mu}_2(c)) + \int_0^c (\bar{\mu}_2(x) - \bar{\mu}_1(x)) dp_{y,t}(x)], \end{aligned} \tag{2.12}$$

and the first term on the right hand side is clearly zero.

So it remains to show that $p_{y,t}(x)$ is a decreasing function of x. To prove this we need to show that the conditioned process Y_t, defined by

$$Y_s = [X_s | \tau_f > T],$$

satisfies the strong Markov property, and has almost surely continuous sample paths. Then we can use a pathwise argument on the process started at two different points to give the result.

The strong Markov property follows easily from that of the parent process as follows:

Let $\tau_1, \ldots, \tau_n$ be stopping times for Y such that, $\tau_1 \leq \tau_2 \leq \ldots \leq \tau_n$ *a.s.*

$$\begin{aligned} &\mathrm{P}[Y_{\tau_n+s} < y | Y_{\tau_1}, \ldots, Y_{\tau_n}, \tau_1, \ldots, \tau_n] \\ &\quad = \mathrm{P}[X_{\tau_n+s} < y | \tau > T, X_{\tau_1}, \ldots, X_{\tau_n}, \tau_1, \ldots, \tau_n] \\ &\quad = \mathrm{P}[X_{\tau_n+s} < y | X_{\tau_n}, \ \tau > T, \ \tau_n] \\ &\quad = \mathrm{P}[Y_{\tau_n+s} < y | Y_{\tau_n}]. \end{aligned}$$

Similarly, the almost sure continuity of the sample paths of Y follows from that of X, since

$$\begin{aligned}&\mathsf{P}[Y \text{ is discontinuous on } [0,t]]\\ &\quad = \mathsf{P}[X \text{ is discontinuous on } [0,t] | \tau_f > t]\\ &\quad \le \frac{\mathsf{P}[X \text{ is discontinuous on } [0,t]]}{\mathsf{P}[\tau_f > T]}.\end{aligned} \tag{2.14}$$

Now $\mathsf{P}[\tau_f > T] > 0$ for all t since f is strictly non-zero. So Y_t is an *a.s.* continuous function of time.

Now consider two processes Y^{x_1}, Y^{x_2} started at x_1, x_2 respectively, with $0 \le x_1 \le x_2 < f(0)$, and let

$$\tau^* = \inf\{t; |Y_t^{x_1}| = |Y_t^{x_2}|\}. \tag{2.15}$$

Then

$$\mathsf{P}[Y_t^{x_1}| > |Y_t^{x_2}||\tau^* > t] = 0 \tag{2.16}$$

(by *a.s.* continuity of the sample paths), and so,

$$\mathsf{P}[|Y_t^{x_1}| < x|\tau^* \ge t] \ge \mathsf{P}[|Y_t^{x_2}| < x|\tau^* \ge t]. \tag{2.17}$$

Also

$$\begin{aligned}p_{y,t}(x_i) = {} & \mathsf{P}[|Y_t^{x_i}| < y|\tau^* < t]\mathsf{P}[\tau^* < t]\\ & + \mathsf{P}[|Y_t^{x_i}| < t|\tau^* \ge t]\mathsf{P}[\tau^* \ge t], \quad i = 1,2.\end{aligned} \tag{2.18}$$

Now

$$\mathsf{P}[|Y_t^{x_1}| < y|\tau^* < t,\ Y_{\tau^*},\ \tau^*] = \mathsf{P}[|Y_t^{x_2}| < y|\tau^* < t,\ Y_{\tau^*},\ \tau^*] \tag{2.19}$$

by the strong Markov property, since $\tau > t^*$. So conditioning on the values of τ^*, Y_{τ^*},

$$\mathsf{P}[|Y_t^{x_1}| < y|\tau^* < t] = \mathsf{P}[|Y_t^{x_2}| < y|\tau^* < t]. \tag{2.20}$$

Therefore, $p_{y,t}(x_1) \ge p_{y,t}(x_2)$ and hence

$$\bar{\mu}_{1,\cdot} \overset{st}{\le} \bar{\mu}_{2,\cdot}. \tag{2.21}$$

LEMMA 2.2. *Suppose f, g are positive functions: $\mathsf{R}^+ \to \mathsf{R} \cup \{\infty\}$ such that $f(t) \le g(t) \quad \forall t$. If*

$$\bar{\mu}(t,f) = \text{ distribution of } [|X_t||\tau_f > t,\ X_0 = x], \tag{2.22}$$

then

$$\bar{\mu}(t,f) \overset{st}{\le} \bar{\mu}(t,g). \tag{2.23}$$

PROOF: The idea of the proof is as follows:

We will consider two processes run 'on the events' $[\tau_g > t,\ \tau_f > t]$ and $[\tau_g > t,\ \tau_f \leq t]$ simultaneously, and will prove a coupling inequality for sample paths where the two processes coincide at some time after the latter hits f for the last time before t. This will follow from the Markov property for X. The remaining sample paths satisfy an *a.s.* inequality due to the *a.s.* continuity of the two processes.

Denoting by $\bar{\mu}(t,f)(.)$ the distribution function corresponding to $\bar{\mu}(t,f)$,

$$\bar{\mu}(t,g)(y) = \alpha\bar{\mu}(t,f)(y) + (1-\alpha)\mathsf{P}[|X_t| < y|\tau_g > t,\ \tau_f \leq t] \tag{2.24}$$

where $\alpha = \mathsf{P}[\tau_f > t]/\mathsf{P}[\tau_g > t]$ because $\tau_f \leq \tau_g$. Define a function h by

$$\begin{aligned} h(t) &= f(t) \text{ on } (u,t] \\ &= g(t) \text{ on } [0,u], \end{aligned} \tag{2.25}$$

and let $\{\mathcal{F}_s,\ s \geq 0\}$ be a filtration rich enough to carry mutually independent processes, Z^1, Z^2, Z^3 and X^1, where $Z^1 \stackrel{\mathcal{L}}{=} [X|\tau_g > t,\ \tau_f \leq t]$, $Z^2 \stackrel{\mathcal{L}}{=} [X|\tau_f > t]$, $Z^3 \stackrel{\mathcal{L}}{=} [X|\tau_h > t]$,[1] and X^1 is an independent copy of X. We also denote by $\{\mathcal{G}_s, s \geq 0\}$, the filtration generated by Z^3, and let $\mathcal{H}$ be the $\sigma-$algebra generated by $\{Z_s^1,\ Z_s^2,\ X_s^1;\ 0 \leq s \leq t\}$. Also, the stopping times, $\tau_f(X^1)$, $\tau_g(X^1)$ and $\tau_h(X^1)$ will denote the hitting times: τ_f, τ_g, and τ_h as defined earlier, for the specific process, X^1.

We want to prove $|Z_t^1| \stackrel{st}{\geq} |Z_t^2|$. Z^3 and X^1 are only used for comparison. Define for any $\mathcal{G} \times \mathcal{H}$ adapted Y, Y_1, and Y_2:

$$\tau'(Y) = \sup\{s \leq t;\ |Y_s| \geq f(s)\}, \tag{2.26}$$

and

$$\tau''(Y^1,Y^2) = \inf\{r \geq \tau'(Y^1),\ |Y_r^1| = |Y_r^2|\} \wedge t. \tag{2.27}$$

Now τ' and τ'' are not stopping times and so we must take care about the preservation of the strong Markov property.

Clearly for $\tau''(Z^1,Z^2) = t$, $|Z_t^1| \geq |Z_t^2|$ a. s. due to the almost sure continuity of Z_1 and Z_2, so we concentrate on the case $\tau''(Z^1,Z^2) < t$.

---

[1]Let Z be a process generating a filtration $\{\mathcal{B}_s,\ s \geq 0\}$. Formally we define a probability space $\{\Omega,\ \mathcal{B}_s, s \geq 0,\ \mathsf{P}'\}$ for the conditioned process $\{[Z_s|A],\ 0 \leq s \leq t\}$, where $A \in \mathcal{B}_\infty$ and $\mathsf{P}(A) > 0$, as follows: Suppose $B \in \mathcal{B}$, then,

$$\mathsf{P}'[B] = \frac{\mathsf{P}[B \cap A]}{\mathsf{P}[A]}$$

Note that,

$$\begin{aligned} \mathsf{P}[Z^3_w < y | Z^3_v = x,\ A], \quad v \le w \\ = \mathsf{P}[Z^3_w < y | Z^3_v = x], \quad \forall A \in \mathcal{G}_v \times \mathcal{H}. \end{aligned} \tag{2.28}$$

Consider the events, defined for any $\mathcal{G} \times \mathcal{H}$ adapted Y:

$$\begin{aligned} A_1(Y) = [|Y_s| < f(s),\ s \le v;\ |Y_s| < |Z^1_s|,\ u \le s < v;\ |Z^1_u| = f(u); \\ |Z^1_v| = |Y_v|;\ |Z^1_s| < f(s),\ u < s < t],\ u \le v, \end{aligned} \tag{2.29}$$

$$\begin{aligned} A_2(Y) = [|Y_s| < f(s),\ u < s \le v;\ |Y_u| = f(u);\ |Z^2_v| = |Y_v|; \\ |Y_s| > |Z^2_s|,\ u \le s \le v,\ |Y_v| = |Z^2_v|]\ u \le v. \end{aligned} \tag{2.30}$$

Clearly $A_1(Z^3),\ A_2(Z^3) \in \mathcal{G}_v \times \mathcal{H}$, moreover, we can rewrite A_1 and A_2 as:

$$\begin{aligned} A_1(Y) = [r'(Z^1) = u,\ r''(Z^1, Y) = v], \\ A_2(Y) = [r'(Y) = u,\ r''(Y, Z^2) = v]. \end{aligned}$$

Also by definition,

$$\mathsf{P}[Z^3_w < y | Z^3_v = x,\ A_1(Z^3)] = \mathsf{P}[X^1{}_w < y | X^1{}_v = x,\ r_h(X^1) > t,\ A_1(X^1)],$$

and clearly,

$$\begin{aligned} [r_h(X^1) > t] \supset A_1(X^1) \\ [r_f(X^1) \notin (v, t)] \supset A_1(X^1). \end{aligned}$$

So, by the Markov property for X^1,

$$\begin{aligned} \mathsf{P}[X^1{}_s < y | X^1{}_v = x,\ r_h(X^1) > t,\ A_1(X^1)] &= \mathsf{P}[X^1{}_s < y | X^1{}_v = x,\ r_f(X^1) > v,\ A_1(X^1)] \\ &= \mathsf{P}[X^1{}_s < y | X^1{}_v = x,\ r_f(X^1) > t,\ A_1(X^1)] \end{aligned}$$

$$= \mathsf{P}[Z^2_w < y | r'(Z^1) = u,\ r''(Z^1, Z^2) = v,\ Z^2_v = x], \quad t \ge w \ge v \ge u, \tag{2.31}$$

by the definition of Z^2.

Similarly we can show that,

$$\begin{aligned} \mathsf{P}[Z^3_w < y | Z^3_v = x,\ A_2(Z^3)] &= \mathsf{P}[X^1{}_s < y | X^1{}_v = x,\ r_h > t,\ A_2(X^1)], \\ &= \mathsf{P}[Z^1_w < y | r'(Z^1) = u,\ r''(Z^1, Z^2) = v,\ Z^1_v = x], \quad t \ge s \ge v. \end{aligned} \tag{2.32}$$

So (2.28) gives us,

$$P[Z^2_s < y | Z^2_v = x,\ r'(Z^1) = u,\ r''(Z^1, Z^2) = v < t]$$
$$= P[Z^1_s < y | Z^1_v = x,\ r'(Z^1) = u,\ r''(Z^1, Z^2) = v < t] \quad v \le s \le t, \tag{2.33}$$

and by the continuity of X and hence Z^1, Z^2,

$$r''(Z^1, Z^2) = t \Rightarrow |Z^1_t| \ge |Z^2_t| \quad a.s.$$

So,

$$P[Z^2_s < y | Z^2_v = x,\ r''(Z^1, Z^2) = t] \ge P[Z^1_s < y | Z^1_v = x,\ r''(Z^1, Z^2) = t] \tag{2.34}$$

$$\begin{aligned} P[|Z^1_t| < y] - P[|Z^2_t| < y] &= P[r''(Z^1, Z^2) = t](P[|Z^1_t| < y | r''(Z^1, Z^2) = t] - P[|Z^2_t| < y | r''(Z^1, Z^2) = t]) \\ &+ P[r''(Z^1, Z^2) < t](P[|Z^1_t| < y | r''(Z^1, Z^2) < t] - P[|Z^2_t| < y | r''(Z^1, Z^2) < t]) \end{aligned} \tag{2.35}$$

$$= -P[r''(Z^1, Z^2) = t] P[|Z^2_t| < y | r''(Z^1, Z^2) = t], \text{ for } y \le f(t) \tag{2.36}$$
$$\le 0.$$

So, for some $\alpha \in [0, 1]$:

$$\begin{aligned} \bar{\mu}(t, g)(y) &\le \alpha \bar{\mu}(t, f)(y) + (1 - \alpha)\bar{\mu}(t, f)(y) \\ &= \bar{\mu}(t, f)(y), \end{aligned} \tag{2.37}$$

that is

$$\bar{\mu}(t, f) \overset{st}{\le} \bar{\mu}(t, g). \tag{2.38}$$

Lemma 2.3. *Suppose Z is the process obtained by placing a reflecting boundary at $\pm a$ for the process X, then if ξ_t is the law of Z_t, and the function f is identically a constant a:*

$$\bar{\mu}_t \overset{st}{\le} \bar{\xi}_t. \tag{2.39}$$

Proof: Let $Y_t = [Z_t | \tau_a > T]$, then $Y_t = [X_t | \tau_a > T]$ a.s. since $P[Y_t \text{ hits } \pm a] = 0$.
So we can apply lemma 2.2 to $[Z_t | \tau_\infty > T]$ and $[Z_t | \tau_a > T]$ to give the result.

Theorem 2.4.

(i) *The distribution of $[X_t | \tau_a > t]$ has a limit δ_∞, independent of μ_0, and this convergence is uniform for all μ_0.*

(ii) *δ_∞ satisfies the quasi-stationary relation:*

$$\int p_{y,t}(x) d\bar{\delta}_\infty(x) = \delta_\infty(y), \quad \forall t, y.$$

Furthermore $\bar{\delta}_\infty$ satisfies:

$$\text{If } \bar{\mu} \overset{st}{\leq} \bar{\delta}_\infty\,,\ \text{then } \bar{\mu}_t \overset{st}{\leq} \bar{\delta}_\infty \tag{2.40}$$

$$\text{and if } \bar{\mu} \overset{st}{\geq} \bar{\delta}_\infty\,,\ \text{then } \bar{\mu}_t \overset{st}{\geq} \bar{\delta}_\infty\,. \tag{2.41}$$

PROOF:

(i) Let δ_t^x = distribution of X_t at time t given $|X_0| = x$ and $\tau_a > t$. Then for $0 \leq x_1 \leq x_2$

$$\bar{\delta}_t^{x_1}(y) \geq \bar{\delta}_t^{x_2}(y) \text{ by lemma 2.1 }. \tag{2.42}$$

We need to show that,

$$\bar{\delta}_t^{x_1}(y) - \bar{\delta}_t^{x_2}(y) \to 0 \text{ as } t \to \infty\ \forall y. \tag{2.43}$$

If we consider two processes X^{x_1}, X^{x_2} started at x_1, x_2 respectively, and define

$$\tau' = \inf\{t \geq 0;\ X_t^{x_2} = 0\}, \tag{2.44}$$

we can apply lemma 2.3 to the process W_t defined by

$$\begin{aligned} W_t &= X_t\,,\ t < \tau' \\ &= 0,\ t \geq \tau'. \end{aligned} \tag{2.45}$$

Now,

$$\begin{aligned} \mathsf{P}[X_s \text{ hits 0 before } t] &= \mathsf{P}[|W_t^{x_2}| \leq 0] \\ &\geq \mathsf{P}[|Y_t^{x_2}| \leq 0] \text{ by lemma 2.3 }, \end{aligned} \tag{2.46}$$

where Y_t is W_t with reflecting boundaries at $\pm a$, and defining Z as in lemma 2.3,

$$\begin{aligned} \mathsf{P}[|Y_t^{x_2}| \leq 0] &= \mathsf{P}[Z_s^{x_2} \text{ hits 0 before } t] \\ &\leq \mathsf{P}[Z_s^a \text{ hits 0 before } t], \end{aligned} \tag{2.47}$$

and $\mathsf{P}[Z_s^a$ hits 0 before $t] \to 1$ as $t \to \infty$ since Z_s has a bounded scale function and is confined to a compact interval.

$$\begin{aligned} \bar{\delta}_t^{x_1}(y) - \bar{\delta}_t^{x_2}(y) = {} & \mathsf{P}[\tau' < t|\tau_a > t](\mathsf{P}[|X_t^{x_1}| < y|\tau' > t,\ \tau_a > t] - \mathsf{P}[|X_t^{x_2}| < y|\tau' < t,\ \tau_a > t]) \\ & + \mathsf{P}[\tau' \geq t|\tau_a > t](\mathsf{P}[|X_t^{x_1}| < y|\tau_a > t,\ \tau' > t] - \mathsf{P}[X_t^{x_2} < y|\tau' \geq t,\ \tau_a > t]), \end{aligned} \tag{2.48}$$

and,

$$\mathsf{P}[|X_t^{x_1}| < y | r' = s < t,\ r_a > t,\ |X_s^{x_1}| = k]$$

$$= \mathsf{P}[|X_{t-s}^{k}| < y | r' = s < t,\ r_a > t] \text{ by time homogeneity,} \tag{2.49}$$

$$\leq \mathsf{P}[X_{t-s}^{0} < y | r' = s < t,\ r_a > t] \text{ by lemma 2.1,}$$

$$= \mathsf{P}[|X_t^{x_2} < y | r' > t,\ r' = s < t], \tag{2.50}$$

and so conditioning on the values of r' and $X_{r'}$,

$$\mathsf{P}[|X_t^{x_1}| < y | r' < t,\ r_a > t] - \mathsf{P}[|X_t^{x_2}| < y | r' < t,\ r_a > t] \leq 0. \tag{2.51}$$

So,

$$\bar{\delta}_t^{x_1}(y) - \bar{\delta}_t^{x_2}(y) \leq \mathsf{P}[r' \geq t]$$

$$\to 0 \text{ uniformly for } x_1,\ x_2 \in [-a, a] \text{ as } t \to \infty. \tag{2.52}$$

Now all that remains is to prove that the distributional limit exists for some initial distribution $\bar{\mu}$. We look at $\{\bar{\delta}_t^0\}$ and show that it is a stochastically increasing function of t, i.e. that $\bar{\delta}_{t_1}^0 \overset{st.}{\leq} \bar{\delta}_{t_2}^0$ for $t_1 \leq t_2$. $\bar{\delta}_t^0(y)$ is a continuous function of t (for $t \neq 0$), because,

$$|\bar{\delta}_{t+\delta t}^0(y) - \bar{\delta}_t^0(y)| \leq \mathsf{P}[|X_t| \geq y,\ |X_{t+\delta t}| < y | r_a > t + \delta t,\ X_t = 0]$$

$$= \int_{\eta \geq 0} \mathsf{P}[|X_{t+\delta t}| < y || X_t| = y + \eta,\ r_a > t + \delta t] d\mu_t(y + \eta). \tag{2.53}$$

But

$$\mathsf{P}[|X_{t+\delta t}| < y || X_t| = y + \eta] \to 0 \text{ as } \delta t \to 0\ \forall \eta > 0, \tag{2.54}$$

so,

$$\lim_{\delta_t \downarrow 0} \int_{\eta \geq 0} \mathsf{P}[|X_{t+\delta t}| < y |\ |X_t| = y + \eta,\ r_a > t + \delta t] d\eta_t(y + \eta)$$

$$\leq \mathsf{P}[|X_t| = y | r_a > t] \tag{2.55}$$

(since the integrand is bounded above by 1)

$$\leq \frac{\mathsf{P}[|X_t| = y]}{\mathsf{P}[r_a > t]} = 0. \tag{2.56}$$

This proves right continuity, left continuity follows similarly.

Also, $\bar{\delta}_{n s}^0(y) \leq \bar{\delta}_{(n-1)s}^0(y)$, $n \in \mathbb{N}$, $t \in \mathbb{R}^+$ by induction, since suppose the inequality holds up to $n = m - 1$ then,

$$\bar{\delta}_{m s}^0(y) = \int p_{y,s}(z) d\bar{\delta}_{(m-1)s}^0(z)$$

$$\leq \int p_{y,s}(z) d\bar{\delta}_{(m-2)s}^0(z)$$

$$= \bar{\delta}_{(m-1)s}^0(y). \tag{2.57}$$

Clearly, the inductive hypothesis is true for $n = 1$, so $\{\bar{\delta}^0_{ns}(y),\ n \in \mathbb{N}\}$ is a stochastically increasing sequence. However by letting $s \to 0$, and using the t- continuity of $\bar{\delta}^0_t(y)$, it follows that $\{\bar{\delta}^0_t,\ t \geq 0\}$ is a stochastically ordered set such that,

$$\bar{\delta}^0_{t_1} \overset{st.}{\leq} \bar{\delta}^0_{t_2} \text{ for } 0 \leq t_1 \leq t_2. \tag{2.58}$$

Since $\bar{\delta}^0_t$ is stochastically bounded above by a point mass at a, $\bar{\delta}^0_t$ therefore has a limit $\bar{\delta}^0_\infty$ which is also the limit for all initial distributions. Moreover this convergence is uniform by theorem 2.4.

(ii)

$$\int p_{y,t}(x)d\bar{\delta}^0_s(x) = \bar{\delta}_{s+t}(y)\ \forall t, y \tag{2.59}$$

So, taking the limit as $s \to \infty$ (formally the left hand side is integrated by parts, and then we apply dominated convergence before integrating by parts back again), we get,

$$\int p_{y,t}(x)d\bar{\delta}_\infty(x) = \bar{\delta}_\infty(y),\ \forall t, y. \tag{2.60}$$

thus proving the first part. Furthermore,

$$\begin{aligned}\bar{\mu}_t(y) &= \int p_{y,t}(x)d\bar{\mu}(x)\\ &\geq \int p_{y,t}(x)d\bar{\delta}_\infty(x) = \bar{\delta}_\infty(y).\end{aligned} \tag{2.61}$$

The inequality above follows from the first part of the proof of lemma 2.1.

The second stochastic inequality follows similarly.

COROLLARY 2.5.

$$\bar{\delta}^a_\infty \overset{st.}{\leq} \bar{\delta}^{a+\epsilon}_\infty. \tag{2.62}$$

PROOF: This result follows from lemma 2.2 by taking the limit on both sides of the inequality,

$$\bar{\mu}^a_t \overset{st.}{\leq} \bar{\mu}^{a+\epsilon}_t \tag{2.63}$$

(where $\bar{\mu}^a_t(y) = \mathsf{P}[|X_t| < y | \tau_a > t]$).

LEMMA 2.6.

$$\mathsf{P}[\tau < t|\mu_1] \leq \mathsf{P}[\tau < t|\mu_2] \text{ if } \bar{\mu}_1 \overset{st.}{\leq} \bar{\mu}_2,$$

where τ is a symmetric boundary hitting stopping time.

PROOF: Suppose $|x_1| < |x_2|$ then $\mathsf{P}[\tau < t|X_0 \sim x_1] \leq \mathsf{P}[\tau < t|X_0 \sim x_2]$ by a similar argument to that in lemma 2.1.

Define $p_t(x) = \mathsf{P}[\tau > t|X_0 = x]$ Now $p_t(x)$ is an increasing for $x \geq 0$ function and $\mathsf{P}[\tau < t|\mu_1] = \int p_t(x)d\mu_1(x) \leq \int p_t(x)d\mu_2(x) = \mathsf{P}[\tau < t|\mu_2]$. The inequality follows from the stochastic ordering of μ_1, μ_2.

1.3 The Approximate Square-root Boundary

In order to prove the main result it is necessary to look at a geometric partition for the Brownian motion time (corresponding to a uniform partition for the Ornstein-Uhlenbeck process). This allows us to make approximations for the approximate square-root case in terms of the exact square-root results obtained by Breiman (1967). So in this section X_t is the Ornstein-Uhlenbeck process corresponding to B_t, and we shall use the two processes interchangeably.

Firstly we need the following lemma.

LEMMA 3.1. *Suppose $m(.)$ is the inverse function of $c(.)$. Then*

$$\mathsf{P}[\tau > t \mid B_1 \sim \delta^c_\infty] = t^{-m(c)}, \tag{3.1}$$

where $\tau = \inf\{t \geq 1; |B_t| = c\sqrt{t}\}$.

PROOF: Fix $r < 1$, and let $t = r^{-n}$ for some integer n. Now we will assume we are conditioning on the event $[\tau > 1,\ B_1 \sim \delta^c_\infty]$ in all that follows:

$$\begin{aligned}\mathsf{P}[\tau > t] &= \prod_{i=1}^{n} \mathsf{P}[\tau > r^{-i} \mid \tau > r^{-(i-1)}] \\ &= \prod_{i=1}^{n} \mathsf{P}[\tau > r^{-i} \mid \tau > r^{-(i-1)},\ B_{r^{-(i-1)}} \sim \delta^c_\infty] \\ &= [k(r)]^n \\ &= t^{g(r)}, \text{ say.}\end{aligned} \tag{3.2}$$

We will show that g is a constant function and that $\mathsf{P}[\tau > t] = t^{g(r)}$ for all $t \geq 1$. By a similar argument we can show that,

$$\mathsf{P}[\tau > t] = t^{g(r^{\frac{1}{m}})},$$

so for $m \in \mathbb{N}$, $g(r) = g(r^{\frac{1}{m}})$, over all $r < 1$, $m \in \mathbb{N}$. So we have,

$$g(r) = g(r^{\frac{m}{n}}), \text{ for integers } m, n.$$

Now fix arbitrary t. We can pick sequences $\{a_i\}$, $\{t_i\}$, such that for each $i \geq 1$, a_i and t_i can be written in the form $r^{\frac{m}{n}}$ for some integers, m, n, and $a_i \uparrow t$ and $t_i \downarrow t$.

$$\mathsf{P}[\tau > t_i] \leq \mathsf{P}[\tau > t] \leq \mathsf{P}[\tau > a_i], \tag{3.3}$$

and

$$t_i^{g(r)} \leq \mathsf{P}[\tau > t] \leq a_i^{g(r)},\ \forall i = 1, 2, \ldots.$$

So taking limits as $i \uparrow \infty$,

$$P[r > t] = t^{g(r)}, \ \forall t \geq 1.$$

But by Breiman (1967), $g(r) = m(c)$.

Proof of Theorem 1.1: Assume firstly that $B_1 = X_0 \sim \delta_\infty^{a(0)}$, then

$$\bar{\delta}_\infty^{a(0)} \overset{st.}{\leq} \bar{\delta}_\infty^{a(t)}. \tag{3.4}$$

Suppose

$${}_c r = \inf\{t \geq 1, \ |B_t| \geq ct^{\frac{1}{2}}\}, \tag{3.5}$$

and

$${}_f r = \inf\{t \geq 1, \ |B_t| \geq f(t)\}, \tag{3.6}$$

and denote by ${}_c\mu_t$, the law of $[X_t|_c r > t]$ and ${}_f\mu_t$ the law of $[X_t|_f r > t]$, in the usual way. Then it is clear from lemmas 2.2 and theorem 2.4(ii) that,

$${}_f\bar{\mu}_t \overset{st.}{\leq} {}_{a(t)}\bar{\mu}_t \overset{st.}{\leq} \bar{\delta}_\infty^{a(t)}, \tag{3.7}$$

for all $t \geq 1$. It is this powerful distributional inequality that allows us to prove the theorem.

Now let $t = r^{-n}$ for some $r < 1$. Then,

$$P[{}_f r > t] = \prod_{i=1}^{n} P[{}_f r > r^{-i}|{}_f r > r^{-(i-1)}]. \tag{3.8}$$

But,

$$\begin{aligned} &P[{}_f r > r^{-i}|{}_f r > r^{-(i-1)}] \\ =&P[{}_f r > r^{-i}|{}_f r > r^{-(i-1)}, \ X_{(i-1)\log\frac{1}{r}} \sim {}_f\mu_{(i-1)\log\frac{1}{r}}], \\ \geq&P[{}_f r > r^{-i}|{}_f r > r^{-(i-1)}, \ X_{(i-1)\log\frac{1}{r}} \sim \delta_\infty^{a((i-1)\log\frac{1}{r})}], \end{aligned} \tag{3.9}$$

from 3.7, and by lemma 2.6. Now define the function $\bar{a}(t)$ as follows:

$$\bar{a}(t) = \begin{cases} a(t), & 1 \leq t \leq r^{-(i-1)} \\ a(r^{-(i-1)}), & r^{-(i-1)} \leq t \leq r^{-i} \end{cases} \tag{3.10}$$

So $\bar{a}(t) \leq a(t), \ 1 \leq t \leq r^{-i}$, and if $\bar{f}(t) = \bar{a}(t)t^{\frac{1}{2}}$:

$$\begin{aligned} &P[{}_f r > r^{-i}|{}_f r > r^{-(i-1)}, \ X_{(i-1)\log\frac{1}{r}} \sim \delta_\infty^{a((i-1)\log\frac{1}{r})}] \\ \geq&P[{}_{\bar{f}} r > r^{-i}|{}_{\bar{f}} r > r^{-(i-1)}, \ X_{(i-1)\log\frac{1}{r}} \sim \delta_\infty^{a((i-1)\log\frac{1}{r})}] \\ =&r^{m(a(r^{-(i-1)}))}. \end{aligned} \tag{3.11}$$

So,

$$\begin{aligned} \mathsf{P}[{}_f r > t] &\geq \prod_{i=1}^{n} r^{m(a(r^{-(i-1)}))} \\ &= \prod_{i=1}^{n} t^{\frac{-m(a(r^{-(i-1)}))}{n}} = M_r(t), \text{ say.} \end{aligned} \tag{3.12}$$

for $t = r^{-n}$ for some $n \in \mathbb{N}$. So,

$$\frac{M_r(t)}{M_r(r^n t)} = \prod_{i=0}^{n-1} r^{m(a(r^i))},$$

and,

$$\frac{M_r(t) - M_r(r^n t)}{t(1-r^n)M_r(tr^n)} = \frac{-1 + \prod_{i=0}^{n-1} r^{m(a(r^i))}}{t(1-r^n)} = -b(r), \text{ say.} \tag{3.13}$$

Now choose $n = \frac{\log(1-\delta/t)}{\log r}$, for suitable r,

$$\frac{M_r(t) - M_r(t-\delta)}{\delta M_r(t-\delta)} = \frac{-1 + \prod_{i=0}^{n-1} r^{m(a(r^i))}}{\delta}.$$

Now $m(a(t)) \downarrow$ with t since $a(t) \uparrow$ so,

$$\frac{1 - \left(1 - \frac{\delta}{t}\right)^{m(a(t))}}{\delta} \leq b(r) \leq \frac{1 - \left(1 - \frac{\delta}{t}\right)^{m(a(t-\delta))}}{\delta}.$$

Now we define,

$$M(t) = \liminf_{r \uparrow 1} M_r(t), \tag{3.14}$$

and so taking limits as $r \uparrow 1$,

$$\frac{M(t) - M(t-\delta)}{\delta M(t-\delta)} \geq \frac{-1 + \left(1 - \frac{\delta}{t}\right)^{m(a(t-\delta))}}{\delta}, \tag{3.15}$$

and now letting $\delta \downarrow 0$, we see that $M'(t)$ exists and,

$$\begin{aligned} \frac{M'(t)}{M(t)} &\geq \frac{-m(a(t))}{t} \\ M(t) &\geq \exp\left\{-\int_1^t \frac{m(a(s))}{s} ds\right\} \end{aligned} \tag{3.16}$$

that is

$$\mathsf{P}[{}_f r > t] \geq \exp\left\{-\int_1^t \frac{m(a(s))}{s} ds\right\}. \tag{3.17}$$

It remains to show that the result holds for all initial distributions, μ_0 of X. In fact this proof idea though simple will be used extensively in chapters 3 and 4, and to avoid repetition the proof is only given in full here. We will show that

$$\mathsf{E}[r^p|\mu_0] < \infty \Rightarrow \mathsf{E}[r^p|\delta_\infty^{a(0)}] < \infty$$

for arbitrary μ_0.

$$E_{\mu_0}[\tau^p] = \int E_x[\tau_\tau^p]\, d\delta_\infty^{a(0)}(x)$$
$$= \int \eta(x)\, d\delta_\infty^{a(0)}(x), \text{ say.} \tag{3.18}$$

Now, $\eta(x)$ decreases with x for $x \geq 0$. Also if $E_{\mu_0}[\tau^p] < \infty$, then $\exists x < c$ such that

$$E_x[\tau^p] < \infty,$$

However,

$$E_x[\tau^p] \geq E_x[\tau^p \mid X \text{ hits 0 before } f(t)\ ]\ P[X \text{ hits 0 before } f(t)\ ],$$

and,

$$P[X \text{ hits 0 before } f(t)] > 0.$$

Moreover it is clear that, since X is strongly Markov,

$$E_x[\tau^p | X \text{ hits 0 before } f(t)] \geq E_0[\tau^p],$$

so $E_x[\tau^p]$ must be finite implying that the result must also follow for μ_0.

COROLLARY 3.2. *Suppose*

$$\left| \int^\infty \log t\ dm(a(t))\ dt \right| < \infty \tag{3.19}$$

then the necessary conditions in Theorem 1 and the sufficient condition given by Taksar are equivalent and we have a necessary and sufficient conditions for the finiteness of $E[\tau^p]$.

PROOF:

$$\int_1^t \frac{-m(a(s))}{s} ds = -\log t\ m(a(t)) + \int^t \log s\ dm(a(s)) ds.$$

So,

$$\exp\left\{-\int_1^t \frac{m(a(s))}{s} ds\right\} = t^{-m(a(t))} \exp\left\{\int^t \log s\ dm(a(s))\right\}. \tag{3.20}$$

and clearly if the integral $\int^t \log s\, dm(a(s))$ is finite, then this expression is bounded by multiples of $t^{-m(a(t))}$ which is Taksar's approximation.

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II.2 Conditional Diffusions: Their Infinitesimal Generators And Limit Laws

2.1 Introduction

In this chapter we consider the effect on Ito diffusions of the form

$$X_t = x + \int_0^t \sigma(X_s)dB_s + \int_0^t \xi(x_s)ds, \tag{1.1}$$

of conditioning on the event $E = (\tau = \infty)$, where

$$\tau = \inf\{t \geq 0; X_t \notin (a,b)\}.$$

More precisely, since under the conditions imposed on σ and τ in this chapter, E is a null event, we consider the weak limit $\bar{X}$ of the process $(\bar{X}_t^T)$ where $\bar{X}^T$ is the solution to (1.1) conditioned on $E_T = (\tau > T)$.

In order to identify the $(\bar{X}^T)$ (and hence $\bar{X}$), we make use of two techniques: the first is some simple functional analysis—the relevant results are covered in section 2.2, and the appendix, and the second is that of enlargement of filtrations—an area covered in great depth in Jeulin and Yor (1985)—the relevant result is introduced in section 2.3.

Some of the results of this chapter were first proved in Pinsky (1985), although the techniques used in Pinsky's paper are quite different, using results from large deviation theory. Our approach here has been to try to preserve probabilistic intuition, whilst still providing a rigorous treatment.

The main results of the chapter are as follows:

(i) $\bar{X}$ is a diffusion with generator $\bar{G} = G + \sigma^2 \frac{e_1'}{e_1}\frac{d}{dx}$, where G is the infinitesimal generator of X, σ is as in (1.1), and e_1 is the eigenfunction corresponding to λ_1, the largest eigenvalue of G, on the interval $[a,b]$.

(ii) $\lim_{t\to\infty} \mathsf{P}_x[X_t \leq y | E_t] = \delta_\infty(y)$ exists, is independent of x, and is given by, $\delta_\infty(dy) \propto e_1(y)m'(y)dy$, where m is the speed measure of X;

(iii) $\lim_{t\to\infty} \mathsf{P}_x[\bar{X}_t \leq y] = v_\infty(y)$ exists, is independent of x, and is given by $v_\infty(dy) \propto e_1^2(y)m'(y)dy$.1

To prove (i) we first establish that

$$\lim_{t\to\infty} e^{-\lambda_1 t}\mathsf{P}_x[E_t] = \frac{e_1(x)}{(\int_a^b e_1^2(x)m'(x)dx)^{1/2}} \tag{1.2}$$

(a result implicit in Itô and McKean,1974) and then use results on enlargement of filtrations and on convergence of solutions of SDE's.

To prove (ii) we establish a generalisation of (1.2) and (iii) then follows by a simple conditioning argument.

In section 5, we focus on two important examples. The first is that of the Ornstein-Uhlenbeck process, an example which will be used extensively in subsequent chapters. The other example is that of Brownian motion. Here we are able to use a limiting argument to prove the existence of a Brownian taboo process first proved by Knight (1969).

2.2 Preliminaries

2.2.1

We consider diffusions of the form

$$dX_t = \sigma(X_t)dB_t + \xi(X_t)dt \tag{2.1}$$

where σ and ξ are locally Lipshitz functions and σ is bounded away from 0, at least on any compact interval.

We wish to study the asymptotic behaviour of the stopping time

$$\tau = \inf\{t \geq 0; X_t = a \text{ or } X_t = b\}, \tag{2.2}$$

$a < b$; we fix a and b for the rest of the paper.

We denote by G the infinitesimal generator of X:

$$G = \frac{1}{2}\sigma^2(x)\frac{d^2}{dx^2} + \xi(x)\frac{d}{dx} \equiv \frac{1}{2}\frac{d}{dm}\frac{d}{ds}; \tag{2.3}$$

here s and m are respectively the scale and speed measures of X.

Most of the elementary function analytic results given below can be found in Curtain and Pritchard (1977).

Denote by V the vector space,

$$V = \{\text{functions } f \in C^2[a,b]; f(a) = f(b) = 0\} \tag{2.4}$$

then V admits a natural inner-product which makes G a self-adjoint linear operator:

$$\langle f, g\rangle = \int_a^b f(x)g(x)\rho(x)dx \tag{2.5}$$

where

$$\rho(x) \equiv \frac{dm}{dx} = \frac{1}{\sigma^2(x)}\exp\left(2\int_a^x \frac{\xi(u)}{\sigma^2(u)}du\right)$$

since then,

$$\begin{aligned}\langle f, Gg\rangle &= \frac{1}{2}\int_a^b f\frac{d^2g}{ds^2}ds(x)\\ &= \frac{1}{2}\int_a^b \frac{df}{ds}\frac{dg}{ds}ds(x)\end{aligned}$$

so that G is clearly self-adjoint.

The spectral theorem now implies that the eigenvalues of G are all real, and the (suitably scaled) eigenfunctions of G form an orthonormal basis for the pre-Hilbert space $H = (V, \langle .,.\rangle)$, the finiteness of the domain of G implying that the spectrum is purely discrete, and so consists of only the eigenvalues of G. Moreover, the eigenvalues of G are all simple; this is implied by the uniqueness of solution of the elliptic equation $Gy = \lambda y$ for given $y(a)$ and $y'(a)$.

Now let $\gamma(x, t)$ be defined by

$$\begin{aligned}\gamma(x,t) &= \mathrm{P}_x[X_s \in (a,b) \quad \forall s \leq t]\\ &= \mathrm{P}_x[\tau > t] \qquad (2.6)\end{aligned}$$

Standard arguments will establish that γ is the unique solution of the PDE,

$$\begin{aligned} G\gamma = \frac{\partial\gamma}{\partial t} \qquad &\text{on} \qquad (a,b)\times(0,\infty),\\ \gamma(0,x) = 1 \qquad &x \in (a,b), \qquad (2.7)\\ \gamma(a,t) = \gamma(b,t) = 0 \qquad &t\in[0,\infty).\end{aligned}$$

More generally, letting $f : [a, b] \to \mathbb{R}$ be a bounded, piecewise continuous function then under the conditions imposed on σ and ξ,

$$\gamma_f(x,t) = \mathrm{E}_x[f(X_t)I(\tau > t)]$$

is the unique solution of the PDE

$$\begin{aligned} G\gamma_f = \frac{\partial\gamma_f}{\partial t} \qquad &\text{on} \qquad (a,b)\times(0,\infty),\\ \gamma(x,0) = f(x) \qquad &\text{on} \qquad (a,b), \qquad (2.8)\\ \gamma(a,t) = \gamma(b,t) = 0 \qquad &: \qquad t\in[0,\infty).\end{aligned}$$

Now $\gamma_f(.,t) \in V \quad \forall t > 0$ and so we may write:

$$\gamma_f(x,t) = \sum_{i=1}^{\infty} a_i(t)e_i(x),$$

where the $\{e_i(t) : i \geq 1\}$ are the orthonormal eigenfunctions of G with corresponding eigenvalues $\{\lambda_i : i \geq 1\}$, and the $\{a_i(.), i \geq 1\}$ are C^1 –functions.

On applying (2.8), it is clear that $a_i(t) = b_i e^{\lambda_i t}$ whilst, since $\gamma(x,0) \equiv f(x)$ is in the closure of V, we see that $b_i = \langle e_i, f\rangle$ (see for example Curtain and Pritchard(1977), for more details).

To complete our preliminary remarks, notice that all the eigenvalues of G must be strictly negative, since if λ was a non-negative eigenvalue of G with corresponding normalised eigenfunction e, then we would have

$$\frac{1}{2}\sigma^2\frac{d^2e}{dx^2} + \xi\frac{de}{dx} - \lambda e = 0 \tag{2.9}$$

and applying the strong maximum principle to e and $-e$ (see Friedman, 1975), we would deduce that e was identically zero, contradicting $||e|| = 1$. Note that the supremum of the spectrum is certainly attained, see for example Krein and Rutman (1948). Recall also that without loss of generality, we may choose $e_1(x)$, the eigenfunction corresponding to λ_1 (the largest eigenvalue) to be strictly positive on (a,b). A simple proof of these results in enough generality for use in this chapter is also given in the appendix.

2.2.2

Our plan of campaign is as follows: we propose to show first that defining

$$_f\phi(x,t) = e^{-\lambda_1 t}\gamma_f(x,t);$$

(i)

$$_f\phi(x,t) \to k_f e_1(x) \text{ uniformly in } x, \tag{2.10}$$

(ii)

$$_f\phi'_x \to k_f e'_1(x) \text{ uniformly in } x,$$

for a suitable constant k_f; and then using arguments based on the theory of enlargement of filtrations, we can show that,

$$((X_s : s \le t | \tau > T) \overset{T\to\infty}{\Rightarrow} (\bar{X}_s ; s \le t),$$

where $\bar{X}$ is a diffusion on $[a,b]$ with generator:

$$G = \frac{1}{2}\sigma^2\frac{d^2}{dx^2} + \left(\frac{\sigma^2 e'_1}{e_1} + \xi\right)\frac{d}{dx} \tag{2.11}$$

2.3 The Conditioned Diffusion

We are now in a position to establish the main results of the chapter. We first establish the following results about ϕ, given by (2.8) and (2.10).

THEOREM 3.1. *${}_f\phi(.,t) \in V$ for each $t > 0$, and is given by:*

$$ {}_f\phi(x,t) = \sum_{i=1}^{\infty} e^{(\lambda_i - \lambda_1)t} \langle f, e_i \rangle e_i(x). \tag{3.1} $$

Moreover (i) $\quad {}_f\phi(x,t) \xrightarrow{t\to\infty} \langle f, e_1\rangle e_1(x)$ *uniformly in x,*

and $\quad$ (ii) ${}_f\phi_x(x,t) \xrightarrow{t\to\infty} \langle f, e_1\rangle e_1'(x)$ *uniformly in x.*

PROOF: ${}_f\phi(.,t) = e^{-\lambda_1 t}\gamma_f(.,t)$ so is clearly in $V \quad \forall t > 0$.
$\langle {}_f\phi, e_i\rangle = e^{-\lambda_1 t}\langle \gamma_f, e_i\rangle e^{\lambda_i t}$, and substituting in (2.8) we see that,

$$ \begin{aligned} \langle \gamma_f(.,t), e_i\rangle &= \langle \gamma_f(.,0), e_i\rangle e^{\lambda_i t} \\ &\equiv \langle f, e_i\rangle e^{\lambda_i t}, \end{aligned} $$

establishing (3.1).

Now $G_f\phi$ is continuous and zero at a and b (for all $t > 0$), so is clearly in V. Thus,

$$ G_f\phi = e^{-\lambda_1 t}G\gamma_f = \sum_{0}^{\infty} e^{-\lambda_1 t}\langle G\gamma_f, e_i\rangle e_i, $$

but $\langle G\gamma_f, e_i\rangle = \langle \gamma_f, Ge_i\rangle$ (since G is self-adjoint) and so,

$$ \begin{aligned} \langle G\gamma_f, e_i\rangle &= \lambda_i\langle \gamma_f, e_i\rangle, \text{ and} \\ e^{-\lambda_1 t}G\gamma_f &= \sum_{i=1}^{\infty} e^{(\lambda_i-\lambda_1)t}\langle f, e_i\rangle \lambda_i e_i. \end{aligned} $$

Thus,

$$ e^{-\lambda_1 t}G\gamma = G\phi \xrightarrow{L^2} G\beta, $$

where $\beta = \langle f, e_1\rangle$. Rewriting G as $\frac{1}{2}\frac{ds}{dm}\frac{d}{ds} \equiv \frac{1}{\rho}\frac{d}{dx}\frac{d}{dx}$, ρ is continuous on $[a,b]$ and bounded away from 0; also let $\|f\|_2 = (\langle f, f\rangle)^{\frac{1}{2}}$,

$$ \begin{aligned} \text{then } &|\phi_x(x,t) - \phi_x(y,t) - (\beta_x(x) - \beta_x(y))| \\ &= 2\Big|\int_y^x \rho G(\phi - \beta)dx\Big| \\ &\le 2\|G\phi(.,t) - G\beta(.)\|_2\Big(\int_y^x \rho(x)dx\Big)^{\frac{1}{2}}, \end{aligned} $$

since ρ and $(s')^{-1}$ are bounded on $[a,b]$ under the conditions on σ and ξ. Thus, $\phi_x(x,t) - \phi_x(a,t) \xrightarrow{t\to\infty} \beta_x(x) - \beta_x(a)$ uniformly in x, and ϕ_x is uniformly (for $t > 0$)

continuous in x. Now it is clear that $\phi \xrightarrow{L^2} \beta$, and

$$\|\phi_\epsilon - \beta_\epsilon\|_2^2 = \int_a^b (\phi_\epsilon - \beta_\epsilon)^2 \frac{dm}{dx} dx$$
$$= \int_a^b (\phi_\epsilon - \beta_\epsilon)^2 \frac{\rho}{s'} ds(x)$$
$$\leq 2\sup_{[a,b]} \left(\frac{\rho(x)}{s'(x)}\right) \langle(\phi - \beta), G(\phi - \beta)\rangle.$$

Therefore $\phi_x \xrightarrow{L^2} \beta_x$, and we may conclude that $\phi_x \to \beta_x$ uniformly in x. So $\phi_x \to \beta_x$ uniformly (in x) and $\phi \to \beta$ uniformly (in x).

LEMMA 3.2.

$$\frac{\phi_x(x,t)}{\phi(x,t)} \xrightarrow{t \to \infty} \frac{\beta'(x)}{\beta(x)}$$

uniformly in x on any interval $I_\epsilon = [a + \epsilon, b - \epsilon]$ (for $\epsilon > 0$).

PROOF: The result is an immediate corollary of Theorem 3.1, recalling that $\beta, \phi > 0$ on (a, b).

We now require a result from the theory of enlargement of filtrations:

THEOREM 3.3. *(Jeulin and Meyer, 1985). Suppose $(\mathcal{F}_t; t \geq 0)$ is a given function, and $A \in \mathcal{F}_\infty$; denote by $(\mathcal{F}_t^A; t \geq 0)$ the filtration obtained by an initial enlargement of $(\mathcal{F}_t)$ with the event A; that is,*

$$\mathcal{F}_t^A = \sigma(\mathcal{F}_t, A).$$

Then denoting by M^A, the $\mathcal{F}_t$- martingale:

$$M_t^A = \mathrm{P}[A|\mathcal{F}_t],$$

if M is an $\mathcal{F}_t$- martingale:

$$N_t \equiv M_t - 1_A \int_0^t \frac{d\langle M, M^A\rangle_s}{M_s^A} + 1_{A^c} \int \frac{d\langle M, M^A\rangle_s}{1 - M_s^A}$$

is an $\mathcal{F}_t^A$- martingale.

COROLLARY 3.4. *Conditional on $[\tau > t]$, the diffusion X_s satisfies the SDE*

$$dX_s = \sigma(X_s)d\tilde{B}_s + \left(\xi(X_s) + \frac{\sigma^2(X_s)\phi_x(X_s, t-s)}{\phi(X_s, t-s)}\right) ds \qquad (0 \leq s \leq t)$$

where $\tilde{B}$ is a Brownian motion (conditional on $\tau > t$).

PROOF: If we apply Theorem 3.3 with $A = E_t = (\tau > t)$, and $M = B$, we see that

$$M_s^A = \gamma(X_s, t-s)I_{E_s}, \text{ and}$$
$$dM_s^A = \sigma(X_s)\gamma_x(X_s, t-s)dB_s, \text{ at least on } A,$$

so that,

$$\widetilde{B}_s \equiv B_s - \int_0^s \frac{\sigma(X_u)\gamma_x(X_u,t-u)}{\gamma(X_u,t-u)}du \equiv B_s - \int_0^s \frac{\sigma(X_u)\phi_x(X_u,t-u)}{\phi(X_u,t-u)}du$$

is a martingale conditional on $(\tau > t)$. Moreover, it is clear that $\langle\widetilde{B}\rangle_s = s$, since quadratic variation is independent of the filtration being considered. So that $\widetilde{B}$ is a Brownian motion (at least on $0 \le s \le t$). The result now follows by substituting

$$dB_s = d\widetilde{B}_s + \sigma(X_s)\frac{\phi_x(X_s,t-s)}{\phi(X_s,t-s)}ds$$

into the original SDE for X.

In order to establish our result about the weak convergence of $(X_s \ : 0 \le s \le t|\tau > T)$ as $T \to \infty$ we need a standard result on convergence of SDEs:

THEOREM 3.5. *Suppose $\sigma_n(.,t), \xi_n(.,t)$ are uniformly (in t) Lipshitz functions which converge uniformly (in (x,t)) to $\sigma_\infty(.,t)$ and $\xi_\infty(.,t)$ respectively, where σ_∞ and ξ_∞ are uniformly Lipshitz in x, and $(\sigma_n \ 1 \le n \le \infty)$ are uniformly bounded away from zero. Let $X_t^n(x)$ $(1 \le n \le \infty)$ be the unique solution to the SDEs:*

$$X_t^n(x) = \int_0^t \sigma_n(X_s^n,s)dB_s + \int_0^t \xi_n(X_s^n,s)ds,$$

then, $X^n(x) \to X^\infty(x)$ uniformly in L^2 and a.s. on any interval $[0,t]$; that is,

$$\sup_{0\le s\le t} |X_s^n(x) - X_s^\infty| \underset{a.s.}{\overset{L^2}{\to}} 0 \ \ \forall t \ge 0.$$

We may now state the main result.

THEOREM 3.6. *Let X^T be the conditioned process, $\{X_t : (0 \le t \le T)|\tau > T\}$, then, $X^T(x) \overset{w}{\to} \widetilde{X}(x)$ (X^T converges weakly to $\widetilde{X}$)where $\widetilde{X}$ is the solution to the SDE,*

$$\widetilde{X}(x) = x + \int_0^t \sigma(\widetilde{X}_s)dB_s + \int_0^t \left(\xi(\widetilde{X}_s) + \frac{\sigma^2(\widetilde{X}_s)\beta'(\widetilde{X}_s)}{\beta(\widetilde{X}_s)}\right) ds,$$

and $\mathrm{P}[a < \widetilde{X}_s < b \ \forall s \in [0,t]] = 1$, $\forall t \ge 0$, provided $x \in (a,b)$.

To prove Theorem 3.6 we first establish the following lemmas.

LEMMA 3.7. *Define $\tau_\epsilon(Y) = \inf\{t \ge 0 : \ Y_t \notin I_\epsilon\}$ for any process Y, with I_ϵ defined as in lemma 3.2. Then,*

$$(X^T_{t\wedge\tau_\epsilon(X^T)}) \overset{w}{\to} (\widetilde{X}_{t\wedge\tau_\epsilon(\widetilde{X})}). \tag{3.3}$$

PROOF: Define $\tilde{\xi}^\epsilon_T$ and $\tilde{\xi}^\epsilon_\infty$ as follows:

$$\tilde{\xi}^\epsilon_T(x,s) = \begin{cases} \xi(x) + \frac{\sigma^2\phi_x(x,T-s)}{\phi(x,T-s)}, & x \in I_\epsilon \\ \xi(x) + \frac{\sigma^2\phi_x(a+\epsilon,T-s)}{\phi(a+\epsilon,T-s)}, & x \leq a+\epsilon \\ \xi(x) + \frac{\sigma^2\phi_x(b-\epsilon,T-s)}{\phi(b-\epsilon,T-s)}, & x \geq b-\epsilon. \end{cases}$$

$$\tilde{\xi}^\epsilon_\infty(x,s) = \begin{cases} \xi(x) + \frac{\sigma^2\beta'(x)}{\beta(x)}, & x \in I_\epsilon \\ \xi(x) + \frac{\sigma^2\beta'(a+\epsilon)}{\beta(a+\epsilon)}, & x \leq a+\epsilon \\ \xi(x) + \frac{\sigma^2\beta'(b-\epsilon)}{\beta(b-\epsilon)}, & x \geq b-\epsilon. \end{cases}$$

Fix $\epsilon > 0$, and let,

$$\tilde{X}^T_t = x + \int_0^t \sigma(\tilde{X}^T_s)dB_s + \int_0^t \tilde{\xi}^\epsilon_T(\tilde{X}^T_s, T-s)ds \tag{3.4}$$

$$\tilde{X}^\infty_t = x + \int_0^t \sigma(\tilde{X}^\infty_s)dB_s + \int_0^t \xi^\epsilon_\infty(\tilde{X}^\infty_s)ds, \tag{3.5}$$

for some fixed Brownian motion B_t. Then it is clear that $\tilde{\xi}^\epsilon_T$, $\tilde{\xi}^\epsilon_\infty$ and σ are uniformly Lipshitz, and $\tilde{\xi}^\epsilon_T \to \tilde{\xi}^\epsilon_\infty$ uniformly on $[0,t]$, and applying Theorem 3.5, $\sup_{0<s<t}|\tilde{X}^T - \tilde{X}^\infty| \xrightarrow{a.s.} 0$ and so, since it is clear that, in this case $\tau_\epsilon(\tilde{X}^T) \xrightarrow{a.s.} \tau_\epsilon(\tilde{X}^\infty)$, (see Barlow and Jacka, 1986 for a proof of this statement) we see that, $X^T_{t\wedge\tau_\epsilon} \xrightarrow{a.s.} \tilde{X}_{t\wedge\tau_\epsilon}$.

LEMMA 3.8. *The stopping times $\tau_\epsilon \equiv \tau_\epsilon(\tilde{X}) \uparrow \infty$ a.s. as $\epsilon \downarrow 0$.*

PROOF: Clearly the (τ_ϵ) increase as $\epsilon \downarrow 0$. To prove that $\tau_\epsilon \xrightarrow[\epsilon\downarrow 0]{a.s.} \infty$ we show that $\tau_\epsilon \xrightarrow{P} \infty$ and use the fact that they increase.

From lemma 3.7 we know that (in the notation of Corollary 3.4),

$$\tau^T_\epsilon \equiv \tau_\epsilon(\tilde{X}^T) \xrightarrow[T\to\infty]{a.s.} \tau_\epsilon(\tilde{X}^\infty) \equiv \tau_\epsilon(\tilde{X}),$$

so to show that $\tau_\epsilon \xrightarrow{P} \infty$, it is sufficient to show that,

$$\lim_{\epsilon\downarrow 0}\left(\lim_{T\to\infty} \mathsf{P}[\tau^T_\epsilon < t]\right) = 0, \ \forall t \geq 0. \tag{3.6}$$

Now, $\mathsf{P}[\tau^T_\epsilon(\tilde{X}^T) < t] = \mathsf{P}[\tau_\epsilon(X) < t|\tau < T]$ where X is the unconditioned process given in 2.1, and denoting by R_ϵ and S_ϵ, the stopping times;

$$R_\epsilon = \inf\{s \geq 0;\ X_s(x) \leq a+\epsilon\}$$

$$S_\epsilon = \inf\{s \geq 0;\ X_s(x) \geq b-\epsilon\},$$

we see that,

$$P[\tau_\epsilon(X) < t | \tau > T] \le \frac{P_x[R_\epsilon < t \wedge S_\epsilon] P_{a+\epsilon}[\tau > T - t] + P_x[S_\epsilon < t \wedge R_\epsilon] P_{b-\epsilon}[\tau > T - t]}{P_x[\tau > T]}$$
$$\le \frac{e^{\lambda_1 t}\{\phi(a+\epsilon, T-t) + \phi(b-\epsilon, T-t)\}}{\phi(x,T)},$$

and letting $T \to \infty$, we see that

$$\lim_{T\to\infty} P[\tau_\epsilon(\tilde{X}^T) < t] \le \frac{e^{\lambda_1 t}\{\beta(a+\epsilon) + \beta(b-\epsilon)\}}{\beta(x)},$$

so that, $\lim_{\epsilon\downarrow 0}(\lim_{T\to\infty} P[\tau_\epsilon(\tilde{X}^T) < t]) = 0$ for each t as required, since $\lambda_1 < 0$ and $\beta(a) = \beta(b) = 0$.

PROOF OF THEOREM 3.6: If we apply lemma 3.8 to $\tilde{X}^\infty$ we see that,

$$\tilde{X}^\infty_{\cdot\wedge\tau_\epsilon} \xrightarrow[\epsilon\downarrow 0]{a.s.} \tilde{X}^\infty_\cdot$$

whilst,

$$\tilde{X}^T_{t\wedge\tau_\epsilon} \xrightarrow[\epsilon\downarrow 0]{a.s.} \tilde{X}^T_t, \quad \text{on } [0,T]$$

(since $\tau_\epsilon(\tilde{X}^T) \to \tau(\tilde{X}^T)$) and thus the result follows by taking the relevant limits in (3.3).

2.4 Identifying The Exit And Entrance Laws Of The Conditional Diffusion

Given the results of Cox and Rösler (1984) and others, it is tempting to directly identify the limit law of $\tilde{X}$ via the dual diffusion $\tilde{X}^*$, however, the conditions in Cox and Rösler do not apply so we instead first identify δ_∞, where,

$$\delta_\infty(y) = \lim_{t\to\infty} P_x[X_t \le y | \tau > t] \tag{4.1}$$

THEOREM 4.1. *Let $\delta_\infty(y)$ be as given in (4.1), then δ_∞ is given by,*

$$\delta_\infty(dy) = \frac{e_1(y) m'(y) dy}{(1, e_1)}$$
$$= e_1(y) \exp\left\{2\int_a^y \xi(t)dt\right\} dy \tag{4.2}$$

PROOF: Applying theorem 3.1 to the functions $f_1 \equiv 1$ and $f_2 = I[x \le y]$, we see that

$$P[X_t \le y | \tau > t] = \frac{f_2\phi(x,t)}{f_1\phi(x,t)}$$

and taking the limit as $t \to \infty$ we see that,

$$\delta_\infty(y) = \frac{(f_2, e_1)}{(f_1, e_1)}$$
$$= \frac{\int_a^y e_1(x)m'(x)dx}{(1, e_1)}$$

We now define ν_t^{x} and ν_∞ as follows:

$$\nu_t^{x_0}(dx) = \mathsf{P}[\tilde{X}_t \in dx | \tilde{X}_0 = x_0],$$
$$\nu_\infty(dx) = \lim_{t \to \infty} \mathsf{P}_{x_0}[\tilde{X}_t \in dx],$$

(we shall see that ν_∞ is independent of x_0).

THEOREM 4.2. *Define $\beta(x)$, δ_t^x and $\delta_{t,s}$ as follows:*

$$\beta(x) = (1, e_1)e_1(x),$$
$$\delta_t^{x_0}(dx) = \mathsf{P}[X_t \in dx | \tau > t, X_0 = x_0],$$
$$\delta_{t,s}^{x_0}(dx) = \mathsf{P}[X_t \in dx | \tau > t + s, X_0 = x_0].$$

Then $\delta_{t,s}^{x_0}$ satisfies,

$$\delta_{t,s}^{x_0}(dx) = \nu_t^{x_0}(dx)\frac{\beta(x_0)}{\phi(x_0, t+s)}\frac{\phi(x, s)}{\beta(x)}, \tag{4.3}$$
$$\delta_t^{x_0}(dx) = \lim_{s \downarrow 0}\delta_{t,s}^{x_0}(dx) = \frac{\nu_t^{x_0}(dx)}{\beta(x)}\frac{\beta(x_0)}{\phi(x_0, t)}, \tag{4.4}$$

and

$$\nu_\infty(dx) = \lim_{t \to \infty}\nu_t^{x_0}(dx) = \beta(x)\delta_\infty(dx), \tag{4.5}$$
$$= e_1^2(x)m'(x)dx.$$

PROOF: Consider first,

$$l_t = \mathsf{P}[X_t \in dx \cap \tau > t + s + T | X_0 = x_0]e^{-\lambda_1(t+s+T)}.$$

Now

$$l_t = \mathsf{P}[\tau > t + s | X_0 = x_0]e^{-\lambda_1(t+s)}\mathsf{P}[X_t \in dx | \tau > t + s, X_0 = x_0]$$
$$\times \mathsf{P}[\tau > t + s + T | X_t \in dx, \tau > t + s]e^{-\lambda_1 T}$$
$$= \phi(x_0, t+s)\delta_{t,s}^{x_0}(dx)m_T, \tag{4.6}$$

where

$$m_T = \int \mathrm{P}[\tau > t + s + T \cap X_{t+s} \in dy | X_t = x, \tau > t + s] e^{-\lambda_1 T}, \tag{4.7}$$

by conditioning on the value of X_{t+s}. Now we see that

$$\begin{aligned} m_T &= \int \mathrm{P}[\tau > t + s + T | X_{t+s} \in dy, X_t = x, \tau > t + s] \mathrm{P}[X_{t+s} \in dy | \tau > t + s, X_t = x] e^{-\lambda_1 T} \\ &= \int \phi(T, y) \delta_s^x(dy). \end{aligned}$$

Conversely,

$$l_T = \mathrm{P}[X_t \in dx | \tau > t + s + T, X_0 = x_0] \mathrm{P}[\tau > t + s + T | X_0 = x_0] e^{-\lambda_1 (t+s+T)} \tag{4.8}$$

Now letting $T \to \infty$ we obtain from (4.6) and (4.7):

$$\begin{aligned} \lim_{T \to \infty} l_T &= \phi(x_0, t + s) \delta_{t,s}^{x_0}(dx) \lim_{T \to \infty} \int \phi(T, y) \delta_s^x(dy) \\ &= \phi(x_0, t + s) \delta_{t,s}^{x_0}(dx) \int \beta(y) \delta_s^x(dy), \end{aligned} \tag{4.9}$$

whilst from (4.8) we obtain

$$\lim_{T \to \infty} l_T = \nu_t^{x_0}(dx) \beta(x_0), \tag{4.10}$$

so that equating (4.9) and (4.10)

$$\delta_{t,s}^{x_0}(dx) = \frac{\nu_t^{x_0}(dx) \beta(x_0)}{\phi(x_0, t + s) \int \beta(y) \delta_s^x(dy)}. \tag{4.11}$$

Similarly, we see that setting

$$\begin{aligned} k_T &= \mathrm{P}[X_t \in dx,\ \tau > t + T |\ X_0 = x_0] e^{-\lambda_1 (T+t)} \\ k_T &= \mathrm{P}[\tau > t | X_0 = x_0] e^{-\lambda_1 t} \mathrm{P}[X_t \in dx | \tau > t,\ X_0 = x_0] \\ &\quad \times \mathrm{P}[\tau > t + T | X_t = x,\ \tau > t] e^{-\lambda_1 T}, \end{aligned}$$

and

$$k_T = \mathrm{P}[X_t \in dx | \tau > t + T,\ X_0 = x_0] \mathrm{P}[\tau > t + T] e^{-\lambda_1 (t+T)}.$$

Letting $T \to \infty$ we obtain,

$$\phi(x_0, t) \delta_t^{x_0}(dx) \beta(x) = \nu_t^{x_0}(dx) \beta(x_0),$$

establishing (4.4). Substituting (4.4) in (4.11), we see that

$$\int \beta(y) \delta_s^{x_0}(dy) = \int \frac{\nu_t^{x_0}(dx) \beta(x_0)}{\phi(x_0, t)} = \frac{\beta(x_0)}{\phi(x_0, t)}$$

establishing (4.3).

Finally letting $t \to \infty$ in (4.4) we establish (4.5).

We see therefore that $\bar{X}$ converges in distribution to $\bar{X}_\infty \sim \nu_\infty$, and that the transition density $\bar{p}$ for $\bar{X}$ is given by,

$$\begin{aligned}\bar{p}(x, y; s, t) &= \frac{\partial}{\partial y} P[\bar{X}_t \leq y | \bar{X}_s = x] \\ &= \frac{\partial}{\partial y} \nu^x_{t-s}(y) \\ &= \beta(y) \frac{\phi(x, t-s)}{\beta(x)} \frac{\partial \delta^x_{t-s}(y)}{\partial y} \\ &= \frac{e_1(y)}{e_1(x)} m'(y) \sum_{i=1}^{\infty} e_i(y) e_i(x) e^{(\lambda_i - \lambda_1)(t-s)} \end{aligned} \qquad (4.12)$$

(for $x, y \in (a, b)$, $0 \leq s < t < \infty$) from (4.3) and theorem 4.1.

2.5 Some Applications

2.5.1 The Ornstein-Uhlenbeck Process

We call a diffusion process an Ornstein-Unlenbeck (θ, α) process if

$$\sigma(x) \equiv \theta \text{ and } \xi(x) = -\alpha x.$$

We will look at the Ornstein-Uhlenbeck $(1, \frac{1}{2})$ process and its conditioning with respect to τ. Applying Theorems (4.1) and (4.2) we get

$$\delta_\infty(dx) = \frac{e_1(x) e^{-x^2/2} dx}{\int_a^b e_1(y) e^{-y^2/2} dy} \qquad (5.1)$$

and

$$\nu_\infty(dx) = e_1^2(x) e^{-x^2/2} dx. \qquad (5.2)$$

Furthermore, we see that $\bar{G}$ is given by,

$$\bar{G} = \frac{1}{2} \frac{d^2}{dx^2} + \left[\frac{e_1'(x)}{e_1(x)} - \frac{x}{2} \right] \frac{d}{dx}. \qquad (5.3)$$

It remains to identify $e_1(.)$. Solutions of

$$\left(\frac{1}{2} \frac{d^2}{dx^2} - \frac{x}{2} \frac{d}{dx} \right) \beta = \lambda \beta, \qquad (5.4)$$

are parabolic cylinder functions. We look for an appropriate λ which makes β equal to 0 at a and b and positive on (a, b). In the case $-a = b$, we can exhibit β as a Kummer function (see, for example, Abramowitz and Stegun, 1972), as follows. Clearly the symmetry of the problem now dictates that we search for an even solution and the general even solution of (5.4) is

$$\beta = kM(\lambda, \frac{1}{2}, x^2/2)$$

where $M(\lambda, \frac{1}{2}, .)$ is a Kummer function. So if we define a function $C(.)$ by $C(p)$ = smallest solution of $M(-p, \frac{1}{2}, x^2/2) = 0$, and let $m(.)$ be the inverse function of $C(.)$ (as in Breiman (1967) and Taksar (1982)), then it is clear that λ_1 must be $-m(b)$, and

$$e_1(x) = kM(-m(b), \frac{1}{2}, x^2/2), \tag{5.5}$$

where k is a normalising constant.

This evaluation of λ_1 as $-m(b)$ agrees with the work of Breiman (1967) and Shepp (1971).

2.5.2 Brownian Motion With Constant Drift

Here $\sigma = 1$ and $\xi = \alpha$, and we find that,

$$e_1(x) = ke^{-\alpha x} \cos\left[\frac{\pi}{b-a}\left(x - \frac{a+b}{2}\right)\right] \tag{5.6}$$

and

$$\lambda_1 = -\left(\frac{\pi^2}{2(b-a)^2} + \frac{\alpha^2}{2}\right). \tag{5.7}$$

From these, we can deduce from Theorems 4.1 and 4.2,

$$\delta_\infty(dx) = ke^{\alpha x} \cos\left[\frac{\pi}{b-a}\left(x - \frac{a+b}{2}\right)\right] \tag{5.8}$$

and

$$\nu_\infty(dx) = k\cos^2\left[\frac{\pi}{b-a}\left(x - \frac{a+b}{2}\right)\right]. \tag{5.9}$$

We can thus exhibit $\tilde{G}$: by Theorem 3.6,

$$\tilde{G} = \frac{1}{2}\frac{d^2}{dx^2} - \frac{\pi}{b-a}\tan\left[\frac{\pi}{b-a}\left(x - \frac{a+b}{2}\right)\right]. \tag{5.10}$$

It is interesting to look at the limiting case in (5.10) where $a = 0$ and $b \to +\infty$. Suppose $({}^{\alpha}X_t)$ is a Brownian motion with drift α, and let ${}^{\alpha}(\tilde{X}_t^k)$ be the conditional process where $a = 0$, $b = k$.

We can apply Theorem 3.5 on the interval $[n^{-1}, n]$ and note that,

$$\lim_{k\to\infty} \frac{-\pi}{k}\tan\left(\frac{\pi x}{k} - \frac{\pi}{2}\right) = \lim_{k\to\infty} \frac{\pi}{k}\cot\left(\frac{\pi x}{k}\right)$$
$$= \frac{1}{x}.$$

It is clear that $({}^{\alpha}\tilde{X}_t^k) \xrightarrow{k\to\infty} BES_3(t)$ where $(BES_3(t))$ is a 3-dimensional Bessel process (with infinitesimal generator $\frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{x}\frac{d}{dx}$.

In the case $\alpha \leq 0$, this implies that

$$ {}^{\alpha}\tilde{X}_t^{\infty} = (BES_3), \tag{5.11}$$

where $({}^{\alpha}\tilde{X}_t^{\infty})$ is the process conditional not to hit zero. This is clear since,

$$\mathsf{P}[{}^{\alpha}X \text{ hits } k \text{ before } 0] \xrightarrow{k\to\infty} 0,$$

and so the 'k conditioning' becomes negligible. However for $\alpha > 0$,

$$\mathsf{P}[{}^{\alpha}X \text{ hits } k \text{ before } 0] \not\xrightarrow{k\to\infty} 0,$$

so we cannot deduce (5.11) in this case. These results can be compared with the similar results obtained by Iglehart (1974a,b) for a random walk.

2. Appendix

We present here a simple proof of the existence of the largest eigenvalue and positivity of the corresponding eigenfunction. We hope that this will be of use to probablists not familiar with functional analysis.

Define a function $z(.) : \mathbb{R}^- \to \mathbb{R} \cup \{\infty\}$ by;

$$z(\lambda) = \inf\{x > a \text{ such that } y(x) = 0\} \tag{A.1}$$

where,

$$Gy = \lambda y, \tag{A.2}$$

and $y(a) = 0, y'(a) = k > 0$, where k is fixed $\forall \lambda \leq 0$.

We will need the following lemmas about z.

LEMMA A.1. *$z(.)$ is non-decreasing.*

PROOF: Let $\lambda_1 < \lambda_2$, and suppose $z(\lambda_2) < z(\lambda_1) \leq \infty$. Also let y_λ be the solution of (A.2). In this case we can write $y_{\lambda_2}(x) = g(x) y_{\lambda_1}(x)$, for some function g such that,

$$G'g = (\lambda_2 - \lambda_1)g \quad \text{on } (a, z(\lambda_2))$$

for some elliptic operator G'. Also $g(z(\lambda_2)) = 0$, and define $g(a) = 1$ to make g C^2 on $[a, z(\lambda_2)]$. Furthermore, by the maximum principle for elliptic operators (see Friedman, 1975) we know that g attains its bounds on the boundary of $[a, z(\lambda_2)]$, and $0 \leq g(z) \leq 1$ in this region.

However,

$$G(y_{\lambda_1} - y_{\lambda_2}) = \lambda_1(y_{\lambda_1} - y_{\lambda_2}) + (\lambda_1 - \lambda_2)y_{\lambda_2}$$

and

$$(y_{\lambda_1} - y_{\lambda_2})(a) = (y_{\lambda_1} - y_{\lambda_2})' = 0.$$

So in a small neighbourhood of a, $(a, a+\epsilon)$, $(y_{\lambda_1} - y_{\lambda_2}) < 0$, since $(\lambda_1 - \lambda_2)y_{\lambda_2} < 0$. But this implies that $g(z) > 1$ for $z \in (a, a+\epsilon)$, giving a contradiction and proving the lemma.

LEMMA A.2. *$z(.)$ is continuous.*

PROOF: $y_\lambda(x)$ is a continuous function of λ $\forall x < \infty$. (This is easily seen by looking at the Green function expansion for $(y_{\lambda+\epsilon} - y_\lambda)$.)

Also $y'_\lambda(z(\lambda)) < 0$, since under the conditions imposed on σ and ξ, if $y'_\lambda(z(\lambda)) = y_\lambda(z(\lambda)) = 0$ then $y \equiv 0$.

Now fix λ such that $z(\lambda) < \infty$. Then for some arbitrarily small $\delta > 0$,

$$y_\lambda(z(\lambda) + \delta) = -c,$$

for some $c > 0$. By the continuity of y, we can choose ϵ small enough so that,

$$|y_{\lambda+\epsilon}(z(\lambda) + \delta) - y_\lambda(z(\lambda) + \delta)| < c, \tag{A.3}$$

and so, $y_{\lambda+\epsilon}(z(\lambda) + \delta) < 0$ and,

$$z(\lambda + \epsilon) < z(\lambda) + \delta$$

by the intermediate value theorem. But δ is arbitrary, so $z(.)$ is right-continuous.

Suppose $z(.)$ is left discontinuous at λ, and let,

$$\lim_{l \to \lambda} z(l) = \alpha.$$

Note this limit certainly exists since z is non-decreasing. Let

$$R = [\lambda - \epsilon, \lambda],$$

and

$$A = \{\beta \in R \text{ such that } z(\beta) \leq \alpha\}.$$

Now since $y_\beta(x)$ is a continuous function of both β and x, A is closed in R. However, $A = [\lambda - \epsilon, \lambda)$ which is not closed in R giving a contradiction, and thus completing the proof.

LEMMA A.3. *The image of z is $(a, \infty]$.*

PROOF: By lemma A.2, all we need to show is that we can find λ_1 such that $z(\lambda_1) = \infty$ and λ_2 such that $z(\lambda_2)$ is arbitrarily small.

Clearly, $z(0) = \infty$ by the maximum principle. Also for arbitrary b, eigenvalues of G on $[a, b]$ span V. Let λ be an eigenvalue on $[a, b]$, then $z(\lambda) \leq b$. But b is arbitrary, completing the proof.

We are now in a position to prove the main result of this appendix.

THEOREM A.4. *Let λ = the supremum of the spectrum of G on $[a, b]$. Then,*

(i) *λ is an eigenvalue of G on $[a, b]$, and*

(ii) *if $e(.)$ is the eigenfunction corresponding to λ, then e has no zero on (a, b), and e is the only eigenfunction of G on $[a, b]$ with this property.*

PROOF: Lemma A.3 implies that an eigenfunction which has no zero on $[a, b]$ exists. It remains to show that any non-maximal eigenvalue cannot be non-zero on (a, b).

Suppose $\gamma < \lambda$ is an eigenvalue with eigenfunction e_γ which is non-zero on (a, b). Choose $\mu (> \gamma)$ to be another eigenvalue with eigenfunction e_μ.

Now, $z(\mu) \geq b$, so e_μ is also non-zero on (a, b) by lemma A.1. So we can choose e_μ and e_γ to be positive on (a, b), contradicting their orthogonality which is implied by Spectral theory, and thus completing the proof.

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II.3 Asymptotic Properties of Boundary Hitting Times for Brownian Motion

3.1 Introduction

The calculation of the distribution of boundary hitting times for Brownian motion has been found to be intractable, the simplest of problems leading to complex P. D. equations, and solutions at best being given in terms of implicit eigenfunctions. However, very often we are merely interested in studying the asymptotic behaviour of such hitting times.

In this chapter, we attempt to describe the asymptotic properties of certain classes of these boundary hitting times. The approach is to consider these different types of boundary: approximate square-root; lower case, (that is boundaries f of the form $f = a(t)t^{\frac{1}{2}}$ where $a(t) \downarrow 0$ as $t \to \infty$) and upper case, (that is boundaries f of the form $f = a(t)t^{\frac{1}{2}}$ where $a(t) \uparrow \infty$) as $t \to \infty$. For each of these cases we prove results which are much stronger than any previously derived.

The main results are summarised as follows.

(1) Under certain regularity conditions on the boundary, the exact asymptotic behaviour of approximate square-root boundaries is exhibited.

(2) For lower case boundaries, bounds are given on the asymptotic behaviour of the hitting time.

(3) In the upper case boundary case, the exact behaviour is given.

(4) Necessary and sufficient conditions for a boundary hitting time to be almost surely finite are given. This is a generalisation of the law of the iterated logarithm.

The first result completes the work of chapter 1, where a lower bound for $P[\tau > t]$ is found. Here we show that a multiple of this expression is also an upper bound. This result improves on the result of Taksar (1982). The methods used here are typical of the whole chapter, and involve mainly intuitive probabilistic arguments. Stochastic inequalities on the distribution at time t conditional on the event $[\tau > t]$ are used to give bounds on the distribution function of τ. This differs greatly from the approach of Taksar (1982) and from Breiman (1967) and Shepp (1971), who were the first to consider the exact square-root hitting time. Breiman calculated the Laplace transform for such hitting times and derived results about the distribution function of the hitting time from this. Taksar used this result for giving a bound on the distribution function of certain types of approximate square-root boundaries. We find here that approximating the conditional distribution of B_t given the event $[\tau > t]$ gives much stronger results as well as providing more insight into the nature of the problem. This problem is considered in section three.

In section four, we look at the lower case boundary case. The main result, theorem 4.1 is an improvement on the results obtained by Lai (1977). Again the methods used depend crucially on the techniques developed in chapter 1.

The fifth section deals with the interesting area of boundaries 'between' $t^{\frac{1}{2}}$ and $\sqrt{2t\log\log t}$. The stationary behaviour of the distribution of B_t conditioned on $[\tau > t]$ breaks down here, but we still find that the same approximation techniques provide results on the exact asymptotic behaviour of the hitting time, although under slightly more restrictive conditions (theorem 5.2). An interesting consequence of this is a simple new proof of the well known Kolmogorov-Petrowski theorem which is a generalisation of the law of the iterated logarithm. This is proved in corollary 5.3. We also give an example to show how theorem 2.2 can be adapted when the regularity conditions are not satisfied.

3.2 Notation and Preliminaries

We recall the following distributions from chapter 2.

Let X_t be a time-homogenous diffusion process, and let τ be the stopping time for X,

$$\tau = \inf\{t \geq 0 : X_t = -b \text{ or } a\},$$

and define the distributions

$$\delta_\infty = \lim_{t\to\infty} \text{law}[X_t|\tau > t],$$
$$\nu_\infty = \lim_{t\to\infty}\lim_{s\to\infty} \text{law}[X_t|\tau > s].$$

Specifically we will write $\delta_\infty(\alpha, a)$ for the limit law δ_∞ when X is an Ornstein-Uhlenbeck process with parameters $(1,\alpha)$ on the interval $[-a, a]$, where the Ornstein-Uhlenbeck (α,β) process satisfies the S. D. E.,

$$dX_t = \alpha dB_t - \beta X_t dt.$$

Also, we write $m(\alpha, a)$ for the exponential decay rate of τ, ie.,

$$\mathsf{P}[\tau > t|X_0 \sim \delta_\infty(\alpha, a)] = e^{-m(\alpha,a)t}.$$

That is, $-m(\alpha, a)$ is the largest eigenvalue of the infinitesimal generator of X, in this case m is related to the confluent hypergeometric function, see section 2.5.2.

We will use the following convention from chapter 1: suppose a random variable Y, has a distribution μ, then we also denote the measure induced by the distribution by μ, and we use $\bar{\mu}$, to denote the modulus law, i.e.

$$\bar{\mu}(x) = \int_0^x d\bar{\mu}(y) = \int_{-x}^x d\mu(y).$$

We must establish the following results about the behaviour of m.

Lemma 2.1.

$$m(\epsilon,\delta) = 2\epsilon m(\frac{1}{2},\delta(2\epsilon)^{\frac{1}{2}})$$

Proof: Suppose X is an O. U. $(1,\epsilon)$ process, and

$$\tau = \inf\{t \geq 0 : |X_t| \geq \delta\}.$$

Let

$$Y_t = (2\epsilon)^{\frac{1}{2}} X_{t/2\epsilon}.$$

Then Y is an O. U. $(1,\frac{1}{2})$ process, and if

$$\tau' = \inf\{t \geq 0; \ |X_t| \geq \delta(2\epsilon)^{\frac{1}{2}}\},$$

then

$$[\tau' = 2\epsilon t] = [\tau = t].$$

Also it is clear that $\delta_m(\frac{1}{2},\delta(2\epsilon)^{\frac{1}{2}}) = (2\epsilon)^{\frac{1}{2}}\delta_m(\epsilon,\delta)$ by this transformation, so

$$\begin{aligned} \mathsf{P}[\tau > t] = \mathsf{P}[\tau' > 2\epsilon t] &= e^{-2\epsilon t m(\frac{1}{2},\delta(2\epsilon)^{\frac{1}{2}})} \\ &= e^{-tm(\epsilon,\delta)}. \end{aligned}$$

So $m(\epsilon,\delta) = 2\epsilon m(\frac{1}{2},\delta(2\epsilon)^{\frac{1}{2}})$.

Recall from chapter 2 the following definitions.

(1) $V(a) = \{C^2$ functions constrained to be 0 at $\pm a\}$

(2) $\langle\cdot,\cdot\rangle_a$ = natural inner product on $V(a)$ chosen to make an operator G self-adjoint.

In the case $G = \frac{1}{2}\frac{d^2}{dx^2} - \frac{x}{2}\frac{d}{dx}$,

$$\langle f,g\rangle_a = \int_{-a}^{a} fge^{-x^2/2}dx.$$

Now fix a and consider points $a+\epsilon$ for small positive ϵ. We will denote by ρ_b the eigenfunction corresponding to the largest negative eigenvalue, $-m(\frac{1}{2},b)$.

$$G\rho_b = -m(\frac{1}{2},b)\rho_b(x) \text{ for } x \in [-b,b].$$

Lemma 2.2.

$$[m(\frac{1}{2},a+\epsilon) - m(\frac{1}{2},a)]\langle\rho_a,\rho_{a+\epsilon}\rangle_{a+\epsilon} = -\rho'_{a+\epsilon}(a+\epsilon)\rho_a(a+\epsilon)$$

Proof: Now $\rho_a \notin V(a+\epsilon)$, and taking its analytic continuation to the interval $[-(a+\epsilon), a+\epsilon]$, and using the inner-product on $[-(a+\epsilon), a+\epsilon]$,

$$
\begin{aligned}
\langle G\rho_a, \rho_{a+\epsilon}\rangle_{a+\epsilon} &= \int_{-a-\epsilon}^{a+\epsilon} \frac{1}{2}\frac{d^2\rho_a}{ds^2}\rho_{a+\epsilon}\,ds \\
&= \int_{-a-\epsilon}^{a+\epsilon} \frac{1}{2}\frac{d^2\rho_{a+\epsilon}}{ds^2}\rho_a\,ds - \frac{1}{2}[\rho'_{a+\epsilon}\rho_a]_{-a-\epsilon}^{a+\epsilon} \\
&= \langle G\rho_{a+\epsilon}, \rho_a\rangle_{a+\epsilon} - \rho'_{a+\epsilon}(a+\epsilon)\rho_a(a+\epsilon)
\end{aligned}
$$

where s is the natural scale for G, $\rho'_{a+\epsilon}(x)$ is the derivative with respect to s. The second line follows from two integrations by parts, and the third uses the even nature of $\rho_a, \rho_{a+\epsilon}$. So,

$$
[m(\tfrac{1}{2}, a+\epsilon) - m(\tfrac{1}{2}, a)]\langle\rho_a, \rho_{a+\epsilon}\rangle_{a+\epsilon} = -\rho'_{a+\epsilon}(a+\epsilon)\rho_a(a+\epsilon) \qquad (2.1)
$$

Now $\rho_b(x) = k(b)M(-m(\frac{1}{2}, b), \frac{1}{2}, \frac{x^2}{2})$ where M is a confluent hypergeometric function and $k(b)$ is an L^2 normalising constant. So we need the following lemma about M.

Lemma 2.3.

(i) *$M(\alpha,\ \delta,\ \frac{x^2}{2})$ is a continuous function of α for non-integer valued $\delta > 0$ and this continuity is uniform for $x \in [-A, A]$ for some fixed but arbitrarily large $A > 0$.*

(ii) *$M(-m(\delta, b),\ \delta,\ \frac{x^2}{2})$ is bounded for $(b, x) \in [a, a+R'] \times [-A, A]$ for some constant $R' > 0$.*

Proof:

(i) A power series expansion for the confluent hypergeometric function (see for example Abramowitz and Stegun, 1972) is Kummer's expansion,

$$
M(\alpha,\ \delta,\ y) = \sum_{i=0}^{\infty} \frac{(\alpha)_i}{(\delta)_i}\frac{y^i}{i!}
$$

where $(\beta)_i = \beta(\beta+1)(\beta+2)\ldots(\beta+i-1)$.

This series is absolutely convergent for all y and α. Also the series is eventually monotone, since all the terms in the expansion for which $i \geq \max\{|\alpha|+1,\ |\delta|+1\}$ have the same sign.

Now fix $R' > 0$. We choose $I \geq m(\delta, b) + 1$ for all $b \in [a, a+R']$. We can certainly do this since $m(\delta, b)$ is non-increasing as a function of b, so it suffices to take $I \geq m(\delta, a) + 1$.

$$
M(\alpha,\ \delta,\ y) = \sum_{i=0}^{I-1} \frac{y^i(\alpha)_i}{(\delta)_i i!} + (\alpha)_I \sum_{i=I}^{\infty} \frac{y^i(\alpha+I)_{i-I}}{(\delta)_i i!}
$$

We would like to show that,

$$
|M(\alpha,\ \delta,\ y) - M(\alpha-\epsilon,\ \delta,\ y)| \to 0 \text{ as } \epsilon \to 0 \qquad (2.2)
$$

uniformly for $y \in [0, R]$, for some positive constant R from which assertion (i) will follow.

The first term in the above expression is a polynomial in y and α, and so satisfies,

$$\sum_{i=0}^{I-1} \frac{y^i((\alpha)_i - (\alpha-\epsilon)_i)}{(\delta)_i i!} = \epsilon P_1(\alpha, y)$$

for some polynomial P_1.

Similarly, $(\alpha)_I - (\alpha-\epsilon)_I = \epsilon\alpha P_2(\alpha)$ for some polynomial P_2. So it remains to consider the term:

$$T(\alpha, y) = \sum_{i=I}^{\infty} \frac{y^i(\alpha+I)_{i-I}}{(\delta)_i i!}.$$

$$T(\alpha,y) - T(\alpha-\epsilon, y) = \sum_{i=I}^{\infty} \frac{y^i[(\alpha+I)_{i-I} - (\alpha+I-\epsilon)_{i-I}]}{(\delta)_i i!}$$
$$\leq T(\alpha, R) - T(\alpha-\epsilon, R), \text{ for } y \in [0, R],$$

since all the terms are positive. Also,

$$T(\alpha, R) - T(\alpha - \epsilon, R) \to 0 \text{ as } \epsilon \to \infty$$

since the individual terms tend to 0, and T is given by an absolutely convergent series. This establishes equation (2) and assertion (i) follows by taking $R \geq \frac{A^2}{2}$.

(ii) M is a continuous function of α, y, and so is bounded on any compact set. Now $m(\frac{1}{2}, b)$ is a decreasing function of b, so

$$m(\delta, a+R') \leq m(\delta, b) \leq m(\delta, a) \text{ for } b \in [a, a+R'].$$

This implies that, $M(-m(\delta, b),\ \delta,\ \frac{x^2}{2})$ is bounded for $(b, \alpha) \in [a, a+R'] \times [-A, A]$.

Lemma 2.4. *$\exists k_1,\ k_2 > 0$ such that for $b \in [a,\ a+R']$,*

$$k_1 \leq k(b) \leq k_2.$$

Proof:

$$k(b) = \left[\int_{-b}^{b} M^2(-m(\tfrac{1}{2}, b),\ \tfrac{1}{2},\ \tfrac{x^2}{2})dm(x)\right]^{-\frac{1}{2}}$$

where $m(x)$ denotes the speed measure for the O.U.$(1,\ \frac{1}{2})$ process. Define,

$$\gamma(b, \lambda) = \int_{-b}^{b} M^2(-\lambda,\ \tfrac{1}{2},\ \tfrac{x^2}{2})dm(x).$$

Now $m(x)$ is absolutely continuous with respect to Lebesgue measure, so lemma 2.3 implies that $\gamma(b, \lambda)$ is continuous for $(b, \lambda) \in [a, a+R] \times [m(\frac{1}{2}, a+R'),\ m(\frac{1}{2}, a)]$. So γ is bounded in this region and also clearly cannot be zero anywhere (this would imply $M(-\lambda,\ \frac{1}{2},\ \frac{x^2}{2}) = 0$), and must therefore be bounded away from 0. Hence the existance of k_1, k_2, since $m(.)$ is non-increasing.

DEFINITION.

We call a function f locally Lipshitz in D if for each point $x \in D$, $\exists$ an open region $N(x) \ni x$ and a constant $c > 0$ such that,

$$|f(x) - f(y)| \leq c|x - y| \qquad \forall y \in N(x).$$

LEMMA 2.5. *$m(\frac{1}{2},\ a)$ is locally Lipshitz for $a \in (0, \infty)$.*

PROOF:

$$\rho_b'(x) = -2x\ k(b)\ m(\tfrac{1}{2},\ b) M(-m(\tfrac{1}{2},\ b) + 1,\ \tfrac{3}{2}, \tfrac{x^2}{2}).$$

So letting $\delta = \frac{a}{2}$ in lemma 2.3, we see that $\rho_b'(x)$ is bounded for $(b, x) \in [a,\ a + R'] \times [a,\ a + R']$. Also, $\rho_a(\cdot)$ is clearly locally Lipshitz since it is continuously differentiable. So $\exists k_3 > 0$ such that

$$\rho_b'(a + \epsilon) \leq k_3,$$

and

$$|\rho_a(a + \epsilon)| \leq k_3 \epsilon$$

for small enough ϵ. Also, as $\epsilon \downarrow 0$,

$$\begin{aligned}
&\lim_{\epsilon \downarrow 0} |\langle \rho_a, \rho_{a+\epsilon} \rangle_{a+\epsilon}| \\
&\geq \frac{k_1^{\frac{1}{2}}}{k_2^{\frac{1}{2}}} \int_{-a}^{a} M(m(\tfrac{1}{2}, a),\ \tfrac{1}{2},\ \tfrac{x^2}{2}) M(\lim_{\epsilon \downarrow 0} m(\tfrac{1}{2}, a + \epsilon),\ \tfrac{1}{2},\ \tfrac{x^2}{2}) dm(x), \\
&\quad > 0.
\end{aligned}$$

So for small enough ϵ,

$$|\langle \rho_a,\ \rho_{a+\epsilon} \rangle_{a+\epsilon}| \geq k_4, \text{ say.}$$

Finally, from lemma 2.2 we see that,

$$|m(\tfrac{1}{2},\ a + \epsilon) - m(\tfrac{1}{2},\ a)| \leq \frac{k_3}{k_4} \epsilon \text{ as required.}$$

This proves the right hand local Lipshitz property. The left hand property follows similarly.

2.3 Approximate Square-Root Boundaries

Armed with lemmas 2.1 and 2.2, we are able to completely describe the asymptotic behaviour of approximate square root boundaries under certain conditions on the boundary.

DEFINITION. *We call f a simple approximate square root boundary if $f(t) = t^{\frac{1}{2}}a(t)$, where,*

(A1) *$a(t)$ is asymptotically non-decreasing to a limit $a(\infty)$.*

(A2) *$a(\cdot)$ is differentiable and $a^2(t) + ta'(t)a(t)$ is asymptotically non-decreasing.*

THEOREM 3.1. *Suppose f is a simple approximate square root boundary, B_t is a Brownian motion, and τ the hitting time,*

$$\tau = \inf\{t \geq 1, \ |B_t| \geq f(t)\}.$$

Then,

$$P[\tau > t] = O\left(\exp\left\{-\int^t \frac{m(\frac{1}{2}, a(s))}{s} ds\right\}\right)$$

as $t \to \infty$ as $t \to \infty$, where $a(t) = f(t)t^{-\frac{1}{2}}$.

PROOF:

$$P[\tau > t] \geq k \exp\left\{-\int^t \frac{m(\frac{1}{2}, a(s))}{s} ds\right\}$$

has already been proved in chapter 1. We concentrate here on the other inequality.

Define the following deterministic space and time change,

$$X_t = \frac{B_{\alpha(t)}}{g(\alpha(t))}$$

where $g(t) = \frac{t^{\frac{1}{2}}a(t)}{a(\infty)}$ and $\alpha(\cdot)$ is the solution of,

$$\alpha'(t) = g^2(\alpha(t)), \tag{3.1}$$

such that $\alpha(1) = 0$. X_t satisfies the SDE

$$dX_t = dB'_t - X_t g(\alpha(t))g'(\alpha(t))dt,$$

where B' is another Brownian motion.

Write $r(t) = g(\alpha(t))g'(\alpha(t))$.

Since f is a simple approximate square root boundary, $r(t)$ is asymptotically non-decreasing. We may assume that r is always non-decreasing without loss of generality since we're only interested in the asymptotic behaviour of it.

As usual, we denote by μ_t the distribution of X_t given $\alpha(\tau) > t$, and let β be the inverse function of α. The idea of the proof is to approximate μ_t by $\delta_m(r(t), a(\infty))$.

Firstly, assume $\mu_0 = \delta_m(r(0), a(\infty))$, then $r(s) \leq r(t)\ \forall s \leq t$. The idea is to approximate with the Ornstein Uhlenbeck $(1, r(t))$ process, so:

$$\bar{\mu}_t \overset{st.}{\geq} \bar{\sigma}_t$$

where $\sigma_t =$ distribution of $[X_t | \alpha(\tau') > t]$,

$$dX'_s = dB'_s - r(t)\, ds\, X'_s,$$

and τ' is the corresponding stopping time for X'.

Also,

$$\bar{\delta}_\infty(r(t),\ a(\infty)) \overset{st}{\leq} \bar{\delta}_\infty(r(0),\ a(\infty)),$$

since $r(t)$ increases by a transformation similar to that given in the proof of corollary 1.2.5 , and so by lemma 1.2.2 ,

$$\bar{\sigma}_t \overset{st}{\geq} \bar{\delta}_\infty(r(t),\ a(\infty)),$$

and so

$$\bar{\mu}_t \overset{st}{\geq} \bar{\delta}_\infty(r(t),\ a(\infty)). \tag{3.2}$$

Now,

$$\begin{aligned}
&\mathsf{P}[\alpha(\tau) > t | X_0 \sim \delta_\infty(r(0),\ a(\infty)),\ \alpha(\tau) > t - \epsilon],\\
&\quad \leq \mathsf{P}[\alpha(\tau) > t | \alpha(\tau) > t - \epsilon,\ X_{t-\epsilon} \sim \delta_\infty(r(t-\epsilon),\ a(\infty))], \quad \text{from (3.2)}\\
&\quad \leq \mathsf{P}[\alpha(\tau) > t | \alpha(\tau) > t - \epsilon,\ X_{t-\epsilon} \sim \delta_\infty(r(t),\ a(\infty))]\\
&\quad \leq \exp -\{m(r(t),\ a(\infty))\epsilon\},
\end{aligned}$$

since $r(s) \leq r(t)$ for $s \in [t-\epsilon, t]$.

$$\frac{\mathsf{P}[\alpha(\tau) > t] - \mathsf{P}[\alpha(\tau) > t - \epsilon]}{\mathsf{P}[\alpha(\tau) > t - \epsilon]\epsilon} \leq -\frac{1 - \exp -\{m(r(t),\ a(\infty))\epsilon\}}{\epsilon},$$

so in the limit as $\epsilon \downarrow 0$, $\mathsf{P}[\alpha(\tau) > t] \leq M(t)$, where $M(t)$ satisfies

$$\frac{M'(t)}{M(t)} = \frac{d}{ds} \exp -\{m(r(t),\ a(\infty))s\}|_{s=0} = -m(r(t),\ a(\infty)),$$

with $M(0) = 1$. So

$$\mathsf{P}[\tau > t] \leq \exp - \int^{\beta(t)} m(r(s),\ a(\infty))ds.$$

But $m(r(s),\ a(\infty)) = 2r(s)m(\frac{1}{2},\ a(\infty)(2r(s)^{\frac{1}{2}})$ by lemma 2.1 so,

$$\begin{aligned}
\mathsf{P}[\tau > t] &\leq \left\{\exp - \int^{\beta(t)} 2r(s)m(\tfrac{1}{2},\ a(\infty)(2r(s))^{\frac{1}{2}})ds\right\},\\
&= \exp - \left\{\int^{t} \frac{a^2(\infty)m(\frac{1}{2},\ a(\infty)(2r(s))^{\frac{1}{2}})2r(s)}{ua^2(u)} du\right\}.
\end{aligned}$$

This last step follows from the change of variable, $u = \alpha(s)$. Now $r(s) = \beta(u)\beta'(u)$ and

$$\beta'(t) = \frac{1}{2t^{\frac{1}{2}}} \frac{a(t)}{a(\infty)} + \frac{t^{\frac{1}{2}} a'(t)}{a(\infty)}.$$

So,

$$g(u)g'(u) = \frac{1}{2}\frac{a^2(u)}{a^2(\infty)} + \frac{ua'(u)a(u)}{a(\infty)},$$

and

$$-\log \mathsf{P}[r > t] \geq \int^t \frac{m(\frac{1}{2},\ a(u)[1 + \frac{2ua'(u)a(u)}{a^2(u)}]^{\frac{1}{2}})}{u} h(u)du,$$

where

$$h(u) = 1 + \frac{2ua'(u)}{a(u)},$$

$$-\log \mathsf{P}[r > t] \geq \int^t \frac{m(\frac{1}{2},\ a(u)) - \frac{k'2ua'(u)}{a(u)}}{u} du,$$

and k' is a local Lipshitz constant for the function $m(\frac{1}{2},\cdot)$ around $a(\infty)$.

But $\int^t \frac{2k'ua'(u)}{a(u)u} du$ converges, so

$$-\log \mathsf{P}[r > t] \geq \text{ constant } + \int^t \frac{m(\frac{1}{2},\ a(u))}{u} du.$$

Here we have used that a' is non-negative. Therefore,

$$\mathsf{P}[r > t] \leq k \exp\left\{-\int^t \frac{m(\frac{1}{2},\ a(u))}{u} du\right\},$$

for some constant k.

Now it remains to show that the result holds for all initial distributions, but this is clear from the arguments in the complementary result, theorem 1.1.1, which show that all starting points yield the same asymptotic behaviour for the hitting time, thus completing the proof.

Remark

Condition (A2) is never satisfied for functions a of the form

$$a'(t) < o(\frac{1}{t^2}),$$

and so it might seem rather restrictive. However, the result for the case $\int \log t a'(t)dt < \infty$ is covered by corollary 1.3.4 . In the case $\int a'(t)\ \log t\ dt = \infty$, (A2) is satisfied in almost all cases of interest. For example, in the case where $a(\cdot)$ is twice differentiable it is sufficient that $a''(\cdot)$ is asymptotically increasing. Also, cases where $a''(\cdot)$ exhibits some sort of oscillation can often be tackled by bounding above by a function satisfying (A2).

3.4 Lower Case Boundaries

An obvious way to proceed in this case, since the rate of increase of the boundary function is 'relatively small' is to look at the approximations obtained by approximating the distribution of the conditioned Brownian motion itself. This has the added advantage that no normalizing time change is necessary. However we can obtain marginally better results by the natural scaling and its appropriate normalizing time change, which converts the problem to a constant boundary hitting problem.

DEFINITION. *f is a simple lower case boundary if $f(t)f'(t)$ is asymptotically non-increasing to 0.*

THEOREM 4.1. *Suppose f is a simple lower case boundary and α is the solution of*

$$\alpha'(t) = f^2(\alpha(t)) \tag{4.1}$$

such that $\alpha(1) = 0$. Let,

$$\tau = \inf\{t;\ |B_t| \geq af(t)\}$$

then

$$\mathsf{P}[\tau > t] = \exp\left\{-\frac{\beta(t)\pi^2}{8a^2}\right\} h(t),$$

where, $\beta(\cdot)$ is the inverse function of $\alpha(\cdot)$, and,

$$\exp\left\{k_1 \int_0^{-\beta(t)} r(s)ds\right\} \leq h(t) \leq k_2 t^p,$$

where $r(t) = f(\alpha(t))f'(\alpha(t))$ and k_1, k_2, p are positive constants.

PROOF: Consider the first inequality first. Let

$$X_t = \frac{B_{\alpha(t)}}{f(\alpha(t))}, \tag{4.2}$$

then

$$dX_t = dB'_t - X_t r(t)dt,$$

where B' is another Brownian motion.

The general idea is to approximate μ_t, the distribution of $[X_t|\tau > t]$, by the stationary distribution for the O.U. (1, $r(t)$) process with boundaries $-a$, a: $\delta_m(r(t), a)$.

$$\frac{\mathsf{P}[\beta(\tau) > t]}{\mathsf{P}[\beta(\tau) > t - \epsilon]} = \mathsf{P}[\beta(\tau) > t | \beta(\tau) > t - \epsilon],$$

and assuming $X_0 \sim \delta_\infty(r(0), a)$,

$$\begin{aligned}
&\mathsf{P}[\beta(r) > t | \beta(r) > t - \epsilon,\ X_0 \sim \delta_\infty(r(0), a)] \\
&\quad = \mathsf{P}[\beta(r) > t | \beta(r) > t - \epsilon,\ X_{t-\epsilon} \sim \mu_{t-\epsilon}] \\
&\quad \geq \mathsf{P}[\beta(r) > t | \beta(r) > t - \epsilon,\ X_{t-\epsilon} \sim \delta_\infty(r(t-\epsilon), a)],
\end{aligned}$$

since $\delta_\infty(r(t-\epsilon), a) \overset{st}{\geq} \mu_{t-\epsilon}$, by theorem 1.2.4. Now X'_s is a process satisfying

$$\begin{aligned}
X'_s &= X_s, \qquad s \leq t - \epsilon; \\
dX'_s &= dB_s - r(t) X'_s ds, \qquad t - \epsilon < s \leq t.
\end{aligned}$$

Then $X'_s \overset{st}{\geq} X_s$, $0 \leq s \leq t$, (see lemma 1.2.2) and so,

$$\begin{aligned}
&\mathsf{P}[\beta(r) > t | \beta(r) > t - \epsilon,\ X_{t-\epsilon} \sim \delta_\infty(r(t-\epsilon), a)] \\
&\quad \geq \mathsf{P}[\beta(r') > t | \beta(r') > t - \epsilon,\ X'_{t-\epsilon} \sim \delta_\infty(r(t-\epsilon), a)],
\end{aligned}$$

where r' is the corresponding stopping time for X'.

Now let μ'_t = distribution of $[X'_t | r' > t]$, then $\bar{\mu}_t \overset{st}{\leq} \bar{\delta}_\infty(r(t), a)$ since

$$\bar{\delta}_\infty(r(t-\epsilon), a) \overset{st}{\leq} \bar{\delta}_\infty(r(t), a).$$

So,

$$\begin{aligned}
&\mathsf{P}[\beta(r') > t | \beta(r') > t - \epsilon,\ X'_{t-\epsilon} \sim \delta_\infty(r(t-\epsilon), a)] \\
&\quad \geq \mathsf{P}[\beta(r') > t |\ \beta(r') > t - \epsilon,\ X'_{t-\epsilon} \sim \delta_\infty(r(t), a)] \\
&\quad = \exp\{-tm(r(t), a)\},
\end{aligned}$$

and so,

$$\frac{\mathsf{P}[\beta(r) > t]}{\mathsf{P}[\beta(r) > t - \epsilon]} \geq \exp\{-\epsilon m(r(t), a)\}.$$

So $\mathsf{P}[\beta(r) > t] \geq M(t)$, where

$$\frac{M(t)}{M(t-\epsilon)} = \exp\{-\epsilon m(r(t), a)\},$$

and $M(0) = 1$. Taking limits as $\epsilon \downarrow 0$,

$$\begin{aligned}
\frac{M'(t)}{M(t)} &= -m(r(t), a), \\
M(t) &= \exp\{-\int_0^t m(r(s), a) ds\}.
\end{aligned}$$

However for small $r(s)$,

$$m(r(s), s) = \frac{\pi^2}{8a^2} - O(r(s)),$$

see Abramowitz and Stegun (1971). So

$$M(t) \geq \exp\left\{\frac{-\pi^2 t}{8a^2}\right\} \exp\left\{k_1 \int_0^t r(s)ds\right\}.$$

Then if $\mu_t =$ distribution of $\{X_s | r > e^t\}$,

$$\bar{\mu}_t \overset{st}{\geq} \delta_{m}(\tfrac{1}{2},\ a(t))$$

and so we can use a similar argument to that for the previous inequality to show that,

$$\begin{aligned} &P[r > e^t] \leq M(t), \text{ where} \\ &\frac{M'(t)}{M(t)} = -m(\tfrac{1}{2},\ a(t)) \\ &\text{and } a(t) = at^{1/2} f(t) \\ &M(t) = \exp\{-\int_0^t m(\tfrac{1}{2},\ a(e^s))ds\}, \end{aligned}$$

and for small $a(s)$, from Abramowitz and Stegun (1971),

$$M(t) = \exp\left\{-\int_0^t \left(\frac{\pi^2}{8a^2(e^s)} + O(1)\right) ds\right\}.$$

So

$$\begin{aligned} P[r > t] &\leq \exp\left\{-\int^{\log t} \left(\frac{\pi^2}{8a^2(e^s)} + O(1)\right)\right\}, \\ &\leq kt^p \exp\left\{-\int^t \frac{\pi^2}{8a^2(u)u} du\right\}, \\ &= kt^p \exp\left\{-\int_0^t \frac{\pi^2}{8a^2 f^2(u)} du\right\}, \\ &= kt^p \exp\left\{-\frac{\pi^2}{8a^2}\beta(t)\right\}. \end{aligned}$$

We can now apply identical arguments to those of theorem 1.1.1 to show that these distributional inequalities hold for all initial distributions.

3.5 Upper Case Boundaries

LEMMA 5.1.

$$m(z) = \frac{ze^{-z^2/2}}{2\sqrt{2}\Gamma(\frac{3}{2})}(1 + O(\frac{1}{z^2})).$$

PROOF: $m(z) \downarrow 0$ as $z \to \infty$, so we consider $M(-b, \frac{1}{2}, z)$ for small positive b.

$$M(-b, \frac{1}{2}, z) = 1 - 2bF(b, z),$$
$$\text{where } F'(b, z) = M(1-b, \frac{3}{2}, z),$$
$$\text{and } F(b, 0) = 0.$$

So recalling the definition of m,

$$m(a) = 1/2F(1 - m(a), \tfrac{a^2}{2}).$$

Now we are interested in the behaviour of $m(a)$ for large a, so we use an approximation for M given in Abramowitz and Stegun (1971),

$$M(\alpha, \beta, z) = \frac{e^z z^{\alpha-\beta}\Gamma(\beta)}{\Gamma(\alpha)}[1 + O(z^{-1})],$$
$$\text{and so } F(b, z) = \frac{e^z z^{-b-\frac{1}{2}}\Gamma(\frac{3}{2})}{\Gamma(1-b)}[1 + O(z^{-1})].$$

Furthermore, the $O(z^{-1})$ term is uniformly bounded for b in some neighbourhood of 0 so,

$$m(a) = \frac{1}{2}e^{-a^2/2}\left(\frac{a^2}{2}\right)^{\frac{1}{2}+m(a)}\frac{\Gamma(1-m(a))}{\Gamma(\frac{3}{2})}(1 + O(a^{-2}))$$
$$= \frac{e^{-a^2/2}\Gamma(1-m(a))}{2\sqrt{2}\Gamma(\frac{3}{2})}a^{2m(a)+1}(1 + O(a^{-2})).$$

But $m(a)$ is bounded for $a \in [A, \infty)$ say, so $m(a) = o(e^{-a})$, and $a^{2m(a)} - 1 = o(a^{-2})$. Also, this implies that,

$$m(a) = O(ae^{-a^2/2}),$$

so that, $\Gamma(1 - m(a)) - 1 = o(a^{-2})$, and,

$$m(a) = \frac{ae^{-a^2/2}}{2\sqrt{2}\Gamma(\frac{3}{2})}(1 + O(a^{-2}))$$

as required.

DEFINITION. *A function f is called a simple upper-case boundary if $f \in C^2$, $f(t) = a(t)t^{\frac{1}{2}}$, and the following conditions are satisfied:*

(B1) *a is asymptotically non-decreasing to ∞.*

(B2) *If $\alpha(.)$ is the solution of,*

$$\alpha'(t) = f^2(\alpha(t)),$$

with $\alpha(1) = 0$, and,

$$r(s) = f(\alpha(s))f'(\alpha(s)) = \frac{1}{2}\frac{\alpha''(s)}{\alpha'(s)},$$

then $r(s)$ is asymptotically non-decreasing to ∞.

(B3) *Let*

$$p(u) = a(u)\left[1 + \frac{2ua'(u)}{a(u)}\right]^{1/2},$$

then $p(.)$ is asymptotically increasing.

(B4)

$$\int^{\infty} \frac{a(u)e^{-a^2(u)/2}}{u}\left(\frac{a'(u)}{p'(p^{-1}(a(u)))} - 1\right) du < \infty.$$

THEOREM 5.2. *Suppose f is a simple upper-case boundary, and $\{B_t, t \geq 1\}$ is a Brownian motion with hitting time,*

$$\tau_f(B) = \inf\{t \geq 1;\ |B_t| \geq f(t)\}.$$

Then,

$$\mathsf{P}[\tau_f(B) > t] = O\left(\exp\left\{-\int_0^t \frac{m(a(s))}{s} ds\right\}\right),$$

as $t \to \infty$.

PROOF: The result is identical in form to that for the simple approximate square-root boundary, and we proceed in a similar fashion.

Let $X_t = e^{-\frac{t}{2}} B_{e^t}$. As usual, we assume to begin with that $X_0 \sim \delta_m(a(0))$, and $a(.)$ is non-decreasing $\forall t \geq 0$. Under these assumptions,

$$\bar{\mu}_t \overset{st}{\leq} \text{law}\ [\bar{X}_t | \tau_g(x) > t],$$

where $g(s) = a(t)$, $s \leq t$, and

$$\text{law}\ [\bar{X}_t | \tau_g(X) > t] \overset{st}{\leq} \text{law}[\bar{X}_t | \tau_g(X) > t,\ X_0 \sim \delta_m(a(t))].$$

These results follow from lemma 1.2.2 and corollary 1.2.5 . But,

$$\text{law}[X_t | \tau_g(X) > t,\ X_0 \sim \delta_m(a(t))] = \delta_m(a(t))$$

so, $\bar{\mu}_t \overset{st}{\leq} \bar{\delta}_m(a(t))$, and

$$\mathsf{P}[\tau_a(X) > t] \geq \exp - \left\{\int_0^t \frac{m(a(s))}{s} ds\right\}.$$

For the other inequality, let $Y_t = \frac{B_{\alpha(t)}}{f(\alpha(t))}$ where α is defined as in definition 3.1. We also let $\beta = \alpha^{-1}$. Y satisfies,

$$dY_t = dB_t^* - r(t)dt,$$

where B^* is an associated Brownian motion. We assume in the usual way that r is actually non-decreasing everywhere. Denoting by μ_t the law of $[Y_t | \tau_f(Y) > t]$, we can follow the proof of theorem 3.1 to obtain,

$$\bar{\mu}_t = \bar{\delta}_m(r(t),\ 1)),$$

$$\mathsf{P}[\tau_f(Y) > t] \leq \exp - \left\{ \int_0^t m(r(s),\ 1)ds \right\},$$

$$\mathsf{P}[\tau_f(B) > t] \leq \exp - \left\{ \int_0^{\beta(t)} m(r(s),\ 1)ds \right\},$$

$$-\log \mathsf{P}[\tau_f(B) > t] \geq \int_0^t \frac{m(\frac{1}{u},\ p(u))}{u} du.$$

Using the transformation $a(v) = p(u)$, (note that this is certainly always possible by restriction (B3)), we obtain,

$$-\log \mathsf{P}[\tau_f > t] \geq \int_0^{a^{-1}p(t)} \frac{m(\frac{1}{u},\ a(v))a'(v)}{p^{-1}(a(v))p'(a(v))} dv,$$

$$\geq \int_0^t \frac{m(a(v))}{p^{-1}(a(v))} \frac{a'(v)}{p'(p^{-1}(a(v))} dv,$$

since the integrand is positive, and $(a^{-1}p)(t) \geq t$. Also, $p^{-1}a(v) \leq v\ \forall v$, so,

$$-\log \mathsf{P}[\tau > t] \geq \int_0^t \frac{m(a(v))}{v} \frac{a'(v)}{p'(p^{-1}(a(v)))} dv,$$

$$\geq k + \int_0^t \frac{m(a(v))}{v} dv,$$

for some constant k, by (B4) and lemma 5.1.

The final result for general μ_0 follows in the usual way . Note that the two inequalities have been proved here using the same initial distribution. This simplifies the argument for general μ_0.

Remarks

The result is most interesting in the case when the boundary is attained with probability 1, i.e. boundaries a such that,

$$a(t) = o((\log \log t)^{\frac{1}{2}}).$$

For such boundaries, (B3) will be satisfied unless a exhibits some oscillatory behaviour. Any attempt to give more explicit expressions for (B1)–(B4) would only lead to a weakening of the result. However, almost all cases of interest can be either solved directly or by means of an approximation scheme.

Example

Consider the boundary,

$$a(t) = \sqrt{2l_3(t)}$$

where $l_i(\cdot) = \log l_{i-1}(\cdot)$, $l_1(\cdot) = \log(\cdot)$. It is easy to check that (B1), (B2) and (B3) are satisfied. For (B4), we must consider

$$\int_0^t \frac{\sqrt{2l_3(u)}}{ul_2(u)} \left(\frac{a'(u)}{p'(p^{-1}(a(u))} - 1 \right) du.$$

After much algebra we obtain,

$$\frac{a'(u)}{p'(p^{-1}(a(u))} \approx \frac{w(l(w))(l_2(w))(l_3(w))^{\frac{1}{2}}}{u(l(u))(l_2(u))(l_3(u))^{\frac{1}{2}}},$$

where

$$u = w\left(1 + \frac{1}{2w^2 e^{w^2} e^{e^{w^2}}}\right).$$

And so,

$$\frac{a'(u)}{p'(p^{-1}(a(u)))} - 1 \leq o(e^{-e^{u^2/2}} e^{-u^2/2} u^{-2}).$$

So (B4) is clearly satisfied, and

$$\mathsf{P}[\tau > t] = O\left(\exp\left\{\int_0^t \frac{(2l_3(u))^{\frac{1}{2}}\, du}{ul_2(u)}\right\}\right).$$

Now consider a similar curve,

$$a(t) = \sqrt{2l_3(t)(1 + \sin t)}.$$

In this case (B1), (B2), and (B3) are all contravened while (B4) holds (this follows from the calculation for $\sqrt{2l_3(t)}$. However, if we define the function,

$$a_\theta(t) = \sqrt{2l_3(t)(1 + e^{-\theta t}(1 + \sin t)},$$

then a_θ satisfies (B1), (B2), and (B4), and we can therefore consider $a(\cdot)$ as $\lim_{\theta\downarrow 0} a_\theta(\cdot)$ and it is not too difficult to prove

$$\mathsf{P}[\tau_a > t] = \lim_{\theta\downarrow 0} \mathsf{P}[\tau_{a_\theta} > t].$$

A simple consequence of theorem 5.1 is the following well known result, a generalisation of the law of the iterated logarithm:

COROLLARY 5.3 (KOLMOGOROV-ERDŐS-FELLER-PETROWSKI).

$$\tau_f < \infty \text{ a.s. } \Leftrightarrow \int_0^\infty \frac{a(s)e^{-\frac{a^2(s)}{2}}}{s} ds = \infty$$

PROOF: This follows immediately from lemma 5.1 and theorem 5.2..

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II.4 One Sided Boundary hitting problems

4.1 Introduction

In this chapter, we look at the behaviour of stopping times τ, such as,

$$\tau = \inf\{t \geq 0;\ X_t \geq f(t)\},$$

where X is a diffusion process. We attempt to extend the ideas of previous chapters to 1-sided boundaries. The main results, giving the asymptotic behaviour of one-sided boundaries for Brownian motion are stated in section three. We give first order expansions for the distribution function of the hitting times by considering the four distinct cases determined by the behaviour of $a(t) = t^{-1/2}f(t)$:

(i) Approximate square root boundaries, that is boundaries such that $a(t) \to a(\infty)$, a finite non-zero limit.

(ii) Lower case boundaries, that is $a(t) \to 0$.

(iii) Positive upper case boundaries, $a(t) \to \infty$.

(iv) Negative upper case boundaries, $a(t) \to -\infty$.

In section 2 we consider the time homogeneous problem, that is we look at hitting times of the form:

$$\tau = \inf\{t \geq 0:\ X_t \geq b\},$$

where X is time homogeneous. For the applications in section 3 we are only interested in the case where X is an Ornstein-Uhlenbeck process. In this particular case, we are able to prove analogous results to those of chapter 2. In fact we can carry out a self-adjoint analysis of the the infinitesimal generator $\mathcal{L}$ of X, but in general this is not possible.

In chapter 2, the finiteness of the domain of $\mathcal{L}$ simplifies the problem for three main reasons:

(1) Firstly the stationary behaviour exhibited by X with respect to the hitting time $\tau = \inf\{t \geq 0;\ X_t = a \text{ or } b\}$ in the finite interval case does not necessarily follow in the semi-infinite case. More precisely, if

$$\mu_t = \text{law } [X_t | \tau > t],$$

then in the finite case it is always true that μ_t has a limit which we call δ_{ab}, but this is not always the case when $a = \infty$.

(2) Secondly, in the finite interval case, at least for well-behaved $\mathcal{L}$, the spectrum of $\mathcal{L}$ is purely discrete on the space S, where,

$$S = \{C^2 \text{ functions } f \text{ on } (a,b) \text{ such that } f(a) = f(b) = 0\}.$$

This allows us to take eigenfunction expansions in terms of an infinite sum, and furthermore the eigenfunctions are always in S.

(3) Thirdly, suppose we form the following subspaces of S by imposing the integrability conditions on function in S:

$$L_i = \{f \in S;\ f^i(x)m'(x) \in L(a,b)\},$$

where m' is the speed measure of X. The nature of the problem compels us to work in the space L_1 since the inherent probabilistic restriction on δ_m is that it integrates to unity, whereas in L_2 we are able to define an inner-product to make $\mathcal{L}$ self-adjoint, and the eigenfunction expansions are easier to handle as well as having been more widely studied. Of course in the finite interval case, trivially we have $S = S_1 = S_2$, and so we can work with whatever structure we choose. However in the semi-infinite case, $L_1 \neq L_2$, and so in general it is not possible to adopt the self-adjoint approach.

In section 3, we look at the problem of 1-sided boundary hitting problems for Brownian motion. Some of the proofs are similar to those of the previous chapters and so to avoid repetition, parts of these proofs are merely sketched. Also, the theorems of section 3 are by no means a definitive collection of results that can be proved by the methods which are developed in section 2 and previous chapters. Extensions to higher dimensions and to other diffusions are obvious examples of areas where the methods can be applied, but also the consideration of different classes of boundaries can yield similar inequalities. An example of this is theorem 3.1, where we consider a class of functions of the form $f(t) = a(t)t^{\frac{1}{2}}$ where $a(t)$ is asymptotically increasing to a finite limit $a(\infty)$. An analagous theorem for the case $a(t)$ asymptotically decreasing to $a(\infty)$ is easily derived with all the inequalities running the other way. So in a sense, the results of this section should be viewed as illustrative of the power of the techniques used.

4.2 The Time-homogeneous Problem

4.2.1

Let $X_t(\alpha)$ be an Ornstein-Uhlenbeck $(1,\alpha)$ process, i.e.

$$dX_t(\alpha) = dB_t - \alpha\ X_t\ dt,$$

and define the stopping time:

$$\tau = \inf\{t \geq 0:\ X_t(\alpha) \geq b\},$$

and its distribution function:

$$\phi(t,x) = P[\tau > t |\ X_0(\alpha) = x].$$

Then the backward equation for ϕ is,

$$\mathcal{L}(\alpha)\phi = \frac{\partial \phi}{\partial t},$$

where $\mathcal{L}$ is the infintesimal generator of $X(\alpha)$.

The method of solution which naturally suggests itself, analagous to the approach of chapter 2, is taking an appropriate eigenfunction expansion in a space where $\mathcal{L}$ is a self adjoint operator. Proceeding in this fashion, we define the pre-Hilbert space $H = \{\tilde{L}_2(b,\alpha),\ \langle .,. \rangle\}$ as follows:

$$\tilde{L}_2(b,\alpha) = \left\{ \begin{array}{c} C^2\text{-functions } f \text{ such that} \\ \int_{-\infty}^{b} f^2(x)e^{-\alpha x^2}\,dx < \infty, \text{ and } f(b) = 0 \end{array} \right\},$$

and,

$$\langle f,g \rangle = \int_{-\infty}^{b} f(x)g(x)e^{-\alpha x^2}\,dx.$$

It is clear that $\phi \in \tilde{L}_2$ since $|\phi| \leq 1$ and it is easily checked that $\mathcal{L}(\alpha)$ is self-adjoint on H. We will need the following results about the spectrum of $\mathcal{L}(\alpha)$.

LEMMA 2.1. *The spectrum of $\mathcal{L}(\alpha)$ in H is purely discrete.*

PROOF: Define the pre-Hilbert space $H^* = \{L_2^*,\ \langle .,. \rangle_*\}$ as follows:

$$L_2^* = \left\{ \begin{array}{c} C^2 - \text{functions } f \text{ on } (-\infty, b) \text{ such that} \\ \int_{-\infty}^{b} f^2(x)\,dx < \infty \text{ and } f(b) = 0 \end{array} \right\},$$

and,

$$\langle f,g \rangle_* = \int_{-\infty}^{b} fg\,dx.$$

Molchanov (1953) showed that the spectrum of an operator $G = \frac{1}{2}\frac{d^2}{dx^2} - a(x)\frac{d}{dx}$ in H^* is purely discrete if and only if:

$$\lim_{A \to \infty} \int_{A}^{A+\epsilon} a(x)\,dx = \infty,$$

for arbitrary positive ϵ, and where $a(x)$ is bounded below. Consider the operator,

$$G = \frac{1}{2}\frac{d^2}{dx^2} + \left(\frac{\alpha}{2} - \frac{\alpha^2 x^2}{2}\right),\ x \leq b.$$

Clearly G satisfies Molchanov's conditions and so has a purely discrete spectrum.

Now f is an eigenfunction of G on H^* with corresponding eigenvalue λ if and only if $f(x)e^{\frac{\alpha x^2}{2}}$ is an eigenfunction of $\mathcal{L}$ in H with eigenvalue λ. That is the spectra of G on H^* and $\mathcal{L}(\alpha)$ on H coincide, and therefore the spectrum of $\mathcal{L}(\alpha)$ on H is discrete.

Lemma 2.2. *The eigenvalues of $\mathcal{L}$ on H are negative.*

Proof: From the proof of lemma 2.1, the spectrum of $\mathcal{L}(\alpha)$ in H coincides with the spectrum of G in H^*. However, suppose λ is a non-negative eigenvalue of G on H^*. By successive applications of the maximum principle on intervals $[-R, b]$ as $R \to \infty$, we see that f is either non-increasing or non-decreasing. But $f(b) = 0$ and $f \in L^2(-\infty, b)$, so $f \equiv 0$ giving a contradiction, and completing the proof.

Now by the Spectral Theorem, the eigenfunctions of $\mathcal{L}(\alpha)$ form an orthonormal basis for H. These results allow us to carry out a self-adjoint analysis in the manner of that of chapter 2. We will denote by $-n(\alpha, b)$ the largest eigenvalue of $\mathcal{L}(\alpha)$ on H, and let $e^{-\frac{\alpha x^2}{2}} e_1(x)$ be its corresponding normalised eigenfunction, i.e. e_1 is the corresponding eigenfunction on H^*. The existence of $-n(\alpha, b)$ is clear from lemmas 2.1 and 2.2. In this context we can apply the results of sections 2 and 3 of chapter 2 to obtain the following analagous results which are summarised below:

Theorem A. *A process $\tilde{X} = \lim_{T\to\infty}[X|\ \tau > T]$ exists as a weak limit and satisfies the S.D.E.*

$$d\tilde{X}_t = dB_t + dt\left(\frac{e_1'(\tilde{X}_t)}{e_1(\tilde{X}_t)} - \alpha\tilde{X}_t\right)$$

for a suitable Brownian motion B.

Theorem B. *The process $\tilde{X}$ has a distributional limit given by:*

$$v_\infty(dx) = \lim_{t\to\infty} \text{law } \tilde{X}_t = e^{-\alpha x^2} e_1^2(x)dx.$$

Theorem C. *Let μ_t denote the distribution of $[X_t|\ \tau > t]$, then the following limit exists:*

$$\delta_\infty(dx) = \lim_{t\to\infty} \mu_t = e^{-\alpha x^2} \frac{e_1(x)}{\langle 1, e_1\rangle} dx,$$

and has the following properties:

(i)

$$\int_{-\infty}^{b} \phi(t,x)\ \delta_\infty(dx) = e^{-n(\alpha,b)t},\ \forall t \geq 0.$$

(ii) *If $\mu_0 \overset{st.}{\leq} \delta_\infty$ then $\mu_t \overset{st.}{\leq} \delta_\infty$, and if $\mu_0 \overset{st.}{\geq} \delta_\infty$, then $\mu_t \overset{st.}{\geq} \delta_\infty$.*

Theorem D. *If δ_∞^b denotes the limit distribution corresponding to the hitting time of b, then:*

$$\delta_\infty^b \leq \delta_\infty^{b+\epsilon} \text{ for } \epsilon > 0.$$

In our particular problem, solutions of

$$\mathcal{L}(\alpha)\phi = \lambda\phi \tag{2.1}$$

are parabolic cylinder functions, and we shall see that the only $\tilde{L}_2$ solution of (2.1) can be expressed in terms of a Whittaker function $D_\lambda(.)$. Firstly though, for notational simplicity, we reduce the problem to the case $\alpha = 1/2$.

LEMMA 2.3.

$$n(\alpha, b) = 2\alpha n(\frac{1}{2}, b(2\alpha)^{\frac{1}{2}}).$$

PROOF: Define the process,

$$Y_t = (2\alpha)^{\frac{1}{2}} X_{t/2\alpha}.$$

Then Y is an O.U. $(1, \frac{1}{2})$ process, and a time change of τ corresponds to the following hitting time for Y:

$$\tau' = \inf\{t \geq 0;\ Y_t \geq b(2\alpha)^{\frac{1}{2}}\},$$

and satisfies the following identity:

$$[\tau' = 2\alpha t] = [\tau = t].$$

Also it is clear that $\delta_\infty(\frac{1}{2}, b(2\alpha)^{\frac{1}{2}}) = (2\alpha)^{\frac{1}{2}}\delta_\infty(\alpha, b)$ by this transformation, so assuming an initial distribution of $\delta_\infty(\alpha, b)$ for X,

$$\begin{aligned} \mathrm{P}[\tau > t] = \mathrm{P}[\tau' > 2\alpha t] &= e^{-2\alpha t n(\frac{1}{2}, b(2\alpha)^{\frac{1}{2}})} \\ &= e^{-t n(\alpha, b)}. \end{aligned}$$

So $n(\alpha, b) = 2\alpha n(\frac{1}{2}, b(2\alpha)^{\frac{1}{2}})$.

4.2.2

For the rest of this section, we shall assume that $\alpha = 1/2$, and we abbreviate $n(\frac{1}{2}, b)$ by $n(b)$.

LEMMA 2.5. *$n(b)$ is a decreasing function of b.*

PROOF: The proof of this result revolves around the intuitive idea that since $n(b)$ is the exponential decay rate of the hitting time of b, then it must be greater for smaller b.

Consider $b_1 < b_2$, and denote by $\phi(t, x; b_i)$, $i = 1, 2$ the distribution function of $\tau(b_i)$, $i = 1, 2$ the respective hitting times of b_1 and b_2. Furthermore, denote by $\delta_\infty^{b_i}$ the limit distribution corresponding to $\tau(b_i)$, $i = 1, 2$. Clearly,

$$\phi(t, x, b_1) \leq \phi(t, x, b_2),$$

and by theorem D,

$$\delta_\infty^{b_1} \overset{st}{\leq} \delta_\infty^{b_2}.$$

Using this, together with the fact that ϕ is a non-increasing function of x, we see that:

$$\begin{aligned} e^{-n(b_1)t} &= \int \phi(t,x;b_1)\, d\delta_{\infty}^{b_1}(x) \\ &\leq \int \phi(t,x;b_2)\, d\delta_{\infty}^{b_1}(x) \\ &\leq \int \phi(t,x;b_2)\, d\delta_{\infty}^{b_2}(x) = e^{-n(b_2)t}. \end{aligned} \tag{2.2}$$

So,

$$n(b_1) \geq n(b_2),\ \forall b_1 \leq b_2.$$

Furthermore, since $d\delta_{\infty}^{b_2}$ is absolutely continuous with respect to Lebesgue measure, equality in 2.2 is only possible when $\phi(t,x;b_1) = \phi(t,x;b_2)$ for almost all x for each $t \geq 0$. But

$$\phi(t,x;b_2) - \phi(t,x;b_1) = \mathrm{P}[\max_{s \leq t} X_s \in (b_1, b_2)] > 0.$$

This implies that $n(b_1) > n(b_2)$ completing the proof.

To continue our investigation of n we need to study the corresponding eigenfunctions and thereby find an implicit characterization of n.

Eigenfunctions of G on H^* are solutions of,

$$\frac{1}{2}\frac{d^2 e_i(x)}{dx^2} + \left(\frac{1}{4} - \frac{x^2}{8} - \lambda_i\right) e_i(x) = 0,$$

such that $e_i(b) = 0$. So,

$$e_i(x) = kD_{2\lambda_i}(-x),$$

where k is a L_2^*-normalising constant, and λ_i is such that $D_{2\lambda_i}(b) = 0$.

Since $e_1(x)$ is positive for $x \in (-\infty, b)$, this leads to the following characterization of n.

Let $z(\lambda)$ be the smallest zero of the equation;

$$D_{2\lambda}(-x) = 0.$$

Then $-n$ is the inverse function of z. See Abramowitz and Stegun (1971) for a summary of the properties of the Whittaker function that we have used.

We now extend our notation for the dominating eigenfunction and write e_b for the eigenfunction corresponding to the eigenvalue $-n(b)$, on the interval $(-\infty, b]$. We shall also denote by $\tilde{L}_2(b)$ the relevant $\tilde{L}_2$ space, and by $\langle .,. \rangle_b$, the corresponding inner-product.

LEMMA 2.6.

$$[n(b+\epsilon) - n(b)]\langle e_b, e_{b+\epsilon}\rangle_{b+\epsilon} = -\frac{1}{2}e'_{b+\epsilon}(b+\epsilon)e_b(b+\epsilon),$$

where e' denotes the derivative with respect to s, the natural scale of $\mathcal{L}$.

PROOF: Now $e_b \notin L_2(b+\epsilon)$, and taking its analytic continuation to the interval $(-\infty, b+\epsilon]$, and using the inner-product on $(-\infty, b+\epsilon]$,

$$\begin{aligned}
(\mathcal{L}e_b, e_{b+\epsilon})_{b+\epsilon} &= \int_{-\infty}^{b+\epsilon} \frac{1}{2}\frac{d^2 e_b}{ds^2} e_{b+\epsilon}\, ds \\
&= \int_{-\infty}^{b+\epsilon} \frac{1}{2}\frac{d^2 e_{b+\epsilon}}{ds^2} e_b\, ds - \frac{1}{2}[e'_{b+\epsilon} e_b]_{-\infty}^{b+\epsilon} \\
&= (Le_{b+\epsilon}, e_b)_{b+\epsilon} - \frac{1}{2}e'_{b+\epsilon}(b+\epsilon)e_a(b+\epsilon).
\end{aligned}$$

The second line follows from two integrations by parts. So,

$$[n(b+\epsilon) - n(b)](e_b, e_{b+\epsilon})_{b+\epsilon} = -\frac{1}{2}e'_{b+\epsilon}(b+\epsilon)e_b(b+\epsilon).$$

We wish to establish continuity results about $D_\mu(x)$ as a function of μ in order to take the limit as $\epsilon \downarrow 0$ in lemma 2.5. Now $D_\mu(x)$ can be written,

$$D_\mu(x) = A(\mu)e^{-\frac{1}{4}x^2} M(-\frac{\mu}{2}, \frac{1}{2}, \frac{1}{2}x^2) + B(\mu)xe^{-\frac{1}{4}x^2} M(-\frac{\mu}{2}+\frac{1}{2}, \frac{3}{2}, \frac{1}{2}x^2),$$

where M is the Kummer function and A and B are given by:

$$\begin{aligned}
A(\mu) &= \frac{1}{\sqrt{\pi}}\frac{\Gamma(\frac{\mu}{2}+\frac{1}{2})}{2^{-\frac{\mu}{2}}}\cos\pi(-\frac{\mu}{2}), \\
B(\mu) &= -\frac{1}{\sqrt{\pi}}\frac{\Gamma(\frac{\mu}{2}+1)}{2^{-\frac{\mu}{2}-\frac{1}{2}}}\sin\pi(-\frac{\mu}{2}).
\end{aligned}$$

Now since $n(.)$ is decreasing by lemma 2.4, and clearly $A(\lambda)$ and $B(\lambda)$ are C^∞ functions of λ, we can directly apply lemma 3.2.3 to give the necessary continuity properties. Furthermore, since the derivative of the Whittaker function is given by,

$$D'_\mu(x) = -\frac{1}{2}xD_\mu(x) - \mu D_{\mu-1}(x),$$

we can derive similar results for $D'_\mu(x)$ also. We summarise these results in the following lemma, which then enables us to state the main result of this subsection.

LEMMA 2.7.

(i) *$D_\mu(x)$ is a continuous function of μ and this continuity is uniform in x for x on compact sets.*

(ii) *$D_{-2n(b)}(x)$ is a bounded function for b and x on compact sets.*

(iii) *$D'_\mu(x)$ is a continuous function of μ and this continuity is uniform in x for x on compact sets.*

(iv) *$D'_{-2n(b)}(x)$ is a bounded function for b and x on compact sets.*

THEOREM 2.8. *$n(b)$ is a continuously differentiable function such that,*

$$n'(b) = -\frac{1}{2}(\epsilon_b'(b))^2 = -\frac{1}{2}\frac{(D'_{-2n(b)}(-x))^2}{||D_{-2n(b)}(-x)||^2}.$$

PROOF: Lemma 2.5 can be rewritten:

$$n(b+\epsilon) - n(b) = -\frac{D'_{-2n(b+\epsilon)}(-b)D_{-2n(b)}(-b-\epsilon)}{2\langle D_{-2n(b)},\ D_{-2n(b+\epsilon)}\rangle_{b+\epsilon}}. \tag{2.3}$$

We wish to consider the limit as $\epsilon \downarrow \infty$ in 2.3. Firstly we look at the denominator.

We know that $n^+(b+\epsilon) = \lim_{\epsilon\downarrow\infty} n(b+\epsilon)$ exists since $n(b)$ is decreasing. It is also clear that $D_{-2n^+(b)}(-x) \geq 0$ for $x \in (-\infty, b)$, and is not identically zero (since no parabolic cylinder function is). So $D_{-2n^+(b)}(-x)$ is a non-negative eigenfunction of $\mathcal{L}$ on $(-\infty, b)$. This implies that $n(b) = n^+(b)$ since if not then $D_{-2n(b)}$ and $D_{-2n^+(b)}$ are orthogonal eigenfunctions on H, and this is contradicted by taking their inner-product since $D_{-2n(b)}$ is a positive eigenfunction on $(-\infty, b)$. Thus we have proved that n is a right continuous function. Now lemma 2.6 ensures that the integrals converge to the expected limits,

$$\langle D_{-2n(b)},\ D_{-2n(b+\epsilon)}\rangle_{b+\epsilon} \to ||D_{-2n(b)}||^2 > 0, \text{ as } \epsilon \downarrow 0.$$

Now we can strengthen lemma 2.6 to say that $D_{-2n(x)}(-x)$ is a continuous function of both b and x, and we can thus take limits in 2.3 to obtain the required result.

4.3 Time-dependent boundaries for Brownian motion

THEOREM 3.1. *Let $\{\Omega,\ \mathcal{F},\ \mathcal{F}_t, t \geq 1,\ \mathsf{P}\}$ be a filtered probability space on which is defined a Brownian motion $\{B_t, t \geq 1\}$. Let $f(t)$ be a 1-sided boundary of the form:*

$$f(t) = a - b(t)t^{\frac{1}{2}},\ t \geq 1,\ a - b(1) > 0,$$

where $b(t) - at^{-\frac{1}{2}}$ is asymptotically non-increasing to a limit $b(\infty)$, and $f(t)f'(t)$ is asymptotically non-increasing to $\frac{1}{2}b(\infty)^2$. Let

$$\tau = \inf\{t \geq 1;\ B_t \geq f(t)\},$$

then,

$$\mathsf{P}[\tau > t] = O\left(\exp\left\{-\int_0^t \frac{n(-b(s))}{s}ds\right\}\right),\ \text{as } t \to \infty.$$

PROOF: Let

$$X_t = e^{-\frac{t}{2}}B_{e^t}.$$

Then, $\{\tau = t\} = \{\tau' = \log t\}$, where,

$$\tau' = \inf\{t \geq 0;\ X_t \geq ae^{-\frac{t}{2}} - b(e^t)\}.$$

Let $g(t) = ae^{-\frac{t}{2}} - b(e^t)$, and fix T, such that $g(t)$ is non-decreasing for $t \geq T$. Adopting the notation introduced in chapter 2, we let μ_t =law$\{X_t |\ \tau' > t\}$, and we assume,

$$\mu_T \sim \delta_\infty^{g(T)}.$$

Then the results of chapter 1 show us that;

$$\mu_t \overset{st}{\leq} \delta_\infty^{g(t)},\ t \geq T,$$

(where, $Z_1 \overset{st}{\leq} Z_2$ implies that $\mathsf{P}[Z_1 < z] \geq \mathsf{P}[Z_2 < z],\ \forall z \in \mathbb{R}$).

Now we define the stopping time τ^* as follows:

$$\tau^* = \inf\{s \geq 0;\ B_s \geq f^*(s)\},$$

where,

$$f^*(s) = \begin{cases} f(s), & s \leq t \\ f(t), & s \geq t, \end{cases}$$

for some fixed $t \geq T$. Then,

$$\begin{aligned} \mathsf{P}[\tau' > t + \epsilon | \tau' > t,\ \mu_T \sim \delta_\infty^{g(T)}] &\geq \mathsf{P}[\tau' > t + \epsilon | \tau' > t,\ \mu_t \sim \delta_\infty^{g(t)}], \\ &\geq \mathsf{P}[\tau^* > t + \epsilon | \tau^* > t,\ \mu_t \sim \delta_\infty^{g(t)}], \\ &= e^{-\epsilon n(g(t))},\ t \geq T. \end{aligned}$$

So,

$$\mathsf{P}[\tau' > t | \tau' > T,\ \mu_t \sim \delta_\infty^{g(T)}] \geq \exp\left[-\epsilon \sum_{i=1}^{\text{int}(\frac{t-T}{\epsilon}+1)} n(g(T + i\epsilon))\right].$$

But this is true for arbitrarily small ϵ, so by taking the limit as $\epsilon \to 0$, as in chapters 1 and 2:

$$\mathsf{P}[\tau' > t | \tau' > T,\ \mu_T \sim \delta_\infty^{g(T)}] \geq \exp\left\{\int_T^t -n(g(s))ds\right\},\ t \geq T.$$

Also using the standard argument using the recurrence of Brownian motion and thus showing that all starting distributions have the same asymptotic behaviour for the hitting time, we can say:

$$\mathsf{P}[\tau' > t |\ \tau' > T] \geq \exp\left\{k_1 - \int_T^t n(g(s))ds\right\},\ t \geq T,$$

for some constant k_1, and so:

$$\mathsf{P}[\tau' > t] \geq k_2 \ \exp\left\{\int_0^t -n(g(s))ds\right\}.$$

Now by the transformation $s = \log u$:

$$\mathsf{P}[\tau' > t] \geq k_2 \exp\left\{\int_1^{e^t} \frac{-n(au^{-\frac{1}{2}} - b(u))}{u} du\right\},$$

and,

$$\mathsf{P}[\tau > t] \geq k_2 \exp\left\{\int_1^t \frac{-n(au^{-\frac{1}{2}} - b(u))}{u} du\right\}.$$

We have now given a lower bound for the distribution function of τ. The proof of the upper bound follows in a similar way to that of Theorem 3.3.1 . Define the following transformation:

$$Y_t = \frac{B_{\alpha(t)}}{f(\alpha(t))},$$

where α is a time change given by:

$$\alpha'(t) = f^2(\alpha(t)).$$

As in chapter 3, we obtain the inequality,

$$\mathsf{P}[\tau > t] \leq k_3 \exp\left\{\int_1^t \frac{-n(a(u^{-\frac{1}{2}} - b(u))}{u} du\right\},$$

for some constant k_3. Note that this uses crucially the existence of a local Lipschitz constant of $n(.)$.

It remains to show that $\int_1^t \frac{n(au^{-\frac{1}{2}} - b(u))}{u} du$ is asymptotically equivalent to $\int_1^t \frac{n(-b(u))}{u} du$. However, again by the local Lipschitz property of $n(.)$ around $b(\infty)$, noting that $n(.)$ is decreasing:

$$n(-b(u)) - k_4 au^{-\frac{1}{2}} \leq n(au^{-\frac{1}{2}} - b(u)) \leq n(-b(u)),$$

for some constant k_4. But, $u^{-\frac{1}{2}} \in L^1(1, \infty)$, and so,

$$\exp\left\{\int_1^t \frac{-n(-b(u))}{u} du\right\} \leq \exp\left\{\int_1^t \frac{-n(au^{-\frac{1}{2}} - b(u))}{u} du\right\} \leq k_5 \exp\left\{\int_1^t \frac{-n(-b(u))}{u} du\right\},$$

thus completing the proof.

For our result on lower case boundaries we need to define a well behaved class of functions.

DEFINITION. *A function $f = t^{1/2}a(t) \in C^+$ if it satisfies:*

(i) $a(t) \to 0$ *as* $t \to \infty$.

(ii) $f(t)f'(t)$ *is asymptotically decreasing to* 0.

THEOREM 3.2. *Suppose $f \in C^+$, then for $a(t) = f(t)t^{-1/2}$,*

$$\mathsf{P}[\tau > t] = O\left(\exp\left\{-\int_1^t \frac{n(a(s))}{s} ds\right\}\right).$$

PROOF: In the usual way we derive the inequalities:

$$\mathsf{P}[\tau > t] \leq k_1 \exp\left\{-\int_1^t \frac{n(a(s))}{s} ds\right\},$$

and

$$\mathsf{P}[\tau > t] \geq k_2 \exp\left\{-\int_{\beta(1)}^{\beta(t)} n(r(s), 1) ds\right\},$$

where $\beta(t) = \int_1^t \frac{dy}{f^2(y)}$, α is the inverse of β, and $r(s) = f(\alpha(s))f'(\alpha(s))$, k_1 and k_2 are positive constants. Also,

$$\begin{aligned}
\int_{\beta(1)}^{\beta(t)} n(r(s), 1) ds &= \int_{\beta(1)}^{\beta(t)} 2r(s) n((2r(s))^{1/2})\, ds, \\
&= \int_1^t n((a^2(s) + 2sa(s)a'(s))^{1/2}) \left(\frac{1}{s} + \frac{2a'(s)}{a(s)}\right)\, ds, \qquad (3.1) \\
&\leq \int_1^t \frac{n((a^2(s) + 2sa(s)a'(s))^{1/2})}{s}\, ds, \\
&\leq \int_1^t \frac{n(a(s)) + k_3 sa(s)a'(s)}{s}\, ds, \\
&\leq k_3(a^2(t) - a^2(1)) + \int_1^t \frac{n(a(s))}{s}\, ds, \\
&\leq k_4 + \int_1^t \frac{n(a(s))}{s}\, ds.
\end{aligned}$$

Here the equality 3.1 follows from lemma 2.3 and the subsequent inequalities follow from the facts that n is a decreasing function and is continuously differentiable, and this leads to the existance of positive constants k_3 and k_4. So,

$$\mathsf{P}[\tau > t] \geq k_5 \exp\left\{-\int_1^t \frac{n(a(s))}{s}\, ds\right\}$$

for some constant k_5, completing the proof.

DEFINITION. *f is a simple positive upper case function if $f(t)f'(t)$ is asymptotically non-decreasing to ∞.*

THEOREM 3.3. *Suppose $f(t) = a(t)t^{1/2}$ is a simple positive upper case function, then*

$$\mathsf{P}[\tau > t] = O\left(\exp\left\{-\int_1^t \frac{n(a(s))}{s} ds\right\}\right).$$

PROOF: Since f is a simple positive upper case function, we derive the usual inequalities:

$$P[\tau > t] \geq k_1 \exp\left\{-\int_1^t \frac{n(a(s))}{s} ds\right\},$$

and

$$P[\tau > t] \leq k_2 \exp\left\{-\int_{\beta(1)}^{\beta(t)} n(r(s), 1) ds\right\},$$

where $r(s) = f(\alpha(s))f'(\alpha(s))$, $\alpha'(s) = f^2(\alpha(s))$, and β is the inverse of α. Again k_1 and k_2 are positive constants. But,

$$\int_{\beta(1)}^{\beta(t)} n(r(s), 1) ds \geq \int_1^t \frac{n\left(a(s)(1 + \frac{2sa'(s)}{a(s)})^{1/2}\right)}{s} ds$$
$$\geq \int_1^t \frac{n(a(s) + sa'(s))}{s} ds,$$

and $n'(b)$ is bounded for large b. This can be seen, for example from theorem 2.8 and the fact that n maps $[b, \infty)$ onto a subset of the compact region $[0, n(b)]$ and $e_b(x) = k_3 D_{-2n(b)}$ for some positive constant k_3. So, let the lower bound on $n'(b)$ be $-k_4$:

$$\int_1^t \frac{n(a(s) + sa'(s))}{s} ds \geq \int_1^t \frac{n(a(s))}{s} ds - k_4 \int_1^t \frac{n'(a(s))sa'(s)}{s} ds$$
$$= \int_1^t \frac{n(a(s))}{s} ds - k_4(n(a(t)) - n(a(1))).$$

So clearly,

$$P[\tau > t] \leq k_5 \exp\left\{-\int_1^t \frac{n(a(s))}{s}\right\}$$

for some constant k_5 as required.

Unfortunately, for the negative upper-case boundary, the hitting time becomes 'too fast' and conditional distributions change too quickly to allow our methods to give such good estimates. We can, however, derive the following theorem which will be stated without proof since the methodology does not involve any new ideas.

THEOREM 3.4. *Suppose f is a simple negative upper case boundary, that is a negative upper case boundary such that $a(t)a'(t)$ is asymptotically increasing to ∞. Then its hitting time τ satisfies:*

$$P[\tau > t] \leq O\left(\exp\left\{\int_1^t \frac{-n(a(s))}{s} ds\right\}\right),$$
$$\text{and } P[\tau > t] \geq O\left(\exp\left\{\int^{\beta(t)} -n(r(s), 1)\, ds\right\}\right).$$

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TITLE SOME BOUNDARY HITTING PROBLEMS FOR DIFFUSION PROCESSES

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