On the distribution of periodic orbits and linking numbers for hyperbolic flows

by

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Contents

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Declarations

The main results in Chapter 6 and Section 5.4 are the content of the paper [CS21], and were obtained in collaboration with Richard Sharp. This work has been submitted for publication in a journal. The results of Chapter 7 were also obtained in collaboration with Richard Sharp, and is in the process of being written as a paper. The remaining chapters contain classical background material for which appropriate references will be given where necessary.

I declare the work in this thesis to be my own, except for collaborations and background material as described. This thesis has not been submitted for a degree at any other university.

Abstract

This thesis concerns certain knot theoretic properties of the periodic orbits of hyperbolic flows, with a focus on Anosov flows.

Anosov flows are a class of hyperbolic dynamical system which were introduced in [Ano67] to generalise the behaviour of geodesic flows over negatively curved spaces. They have a countable infinity of periodic orbits which, when the flow is on a 3-dimensional manifold, can be studied as knots. We will study the linking numbers of these orbits with one another.

Our first result relates the helicity of certain Anosov flows to a weighted average of linking numbers of their periodic orbits. This complements a classical result of Arnol'd and Vogel ([Arn86], [Vog02]) which is that, when the manifold is a real homology 3-sphere, the helicity of a volume-preserving flow may be obtained as the limit of normalised linking numbers of long trajectories. Our result is also inspired by work of Contreras [Con95], who studied average linking numbers of hyperbolic flows in S^3.

We then study the number and distribution of periodic orbits with prescribed linking properties relative to a fixed set of orbits. This is inspired by work of McMullen [McM13], which we use to present a new application of the methods of Babillot-Ledrappier [BL98], for counting orbits subject to certain constraints.

The methods used to prove these results come largely from thermodynamic formalism, along with the symbolic coding procedure for hyperbolic flows, developed independently by Bowen [Bow73] and Ratner [Rat73].

Chapter 1

Introduction

The work in this thesis concerns 3-dimensional Anosov flows and some knot-theoretic properties of their periodic orbits. We will present two main results. The first, in Chapters 5 and 6, establishes a connection between the helicity of the flow and the linking numbers of its periodic orbits. The second, in Chapter 7, is a result on the number and distribution of periodic orbits with prescribed linking properties.

Anosov flows are a class of continuous-time dynamical systems which were introduced in [Ano67] to generalise the chaotic behaviour of geodesic flows over negatively curved spaces. These flows have a countable infinity of periodic orbits, growing exponentially with the period. Precisely, the number of periodic orbits N_T of period at most T, satisfies

$$\lim_{T\to\infty} \frac{1}{T} \log N_T = h,$$

where h is the topological entropy of the flow. In 3 dimensions, these orbits can be thought of as knots, allowing one to study quantities from knot theory. The main such quantity appearing throughout this work is the linking number. In S^3 the linking number of knots γ, γ' is defined as the integer $\mathrm{lk}(\gamma, \gamma')$ given by the homology

$$[\gamma'] \in H_1(S^3 \setminus \gamma, \mathbb{Z}) \cong \mathbb{Z}.$$

This definition can be extended to general compact 3-manifolds, provided one of γ and γ' is rationally null-homologous (see Section 2.5). In S^3 the linking number can be evaluated with the Gauss linking integral, which has the form

$$\mathrm{lk}(\gamma, \gamma') = \frac{3}{4\pi} \int_{S^1} \int_{S^1} \frac{\dot{\gamma}(s) \times \dot{\gamma}'(t)}{\|\gamma(s) - \gamma'(t)\|^3} \cdot (\gamma(s) - \gamma'(t))\, ds\, dt.$$

This can also be extended to the general setting, through linking forms, introduced in Section 2.5.

The results in this thesis rely on the use of thermodynamic formalism, a framework rooted in statistical mechanics, which we use for selecting and studying equilibrium states – the optimal measures with which to recognise certain properties of dynamical systems. Equilibrium states are defined as those invariant measures which realise the supremum

$$P(f) = \sup \left\{ h_\mu + \int f \, d\mu : \mu \text{ is a flow-invariant Borel probability} \right\},$$

where h_μ is the measure-theoretic entropy, and f is a continuous real-valued observable. The functional $P(f)$ is called the pressure of f, and we will denote an equilibrium state for f by μ_f.

The thermodynamic approach often allows one to study global quantities associated to a system by analysing averages of local quantities on smaller components of the system. We will mainly use this for the equidistribution theory of periodic orbits in Chapter 5.

Equidistribution theory refers to the convergence to equilibrium states of measures defined on sets of periodic orbits. Let X^t denote our flow. Given a periodic point x with orbit γ of period $\ell(\gamma)$, we define a probability measure μ_γ by

$$\int f \, d\mu_\gamma = \frac{1}{\ell(\gamma)} \int_0^{\ell(\gamma)} f(X^t(x)) \, dt.$$

Given $t > 0$, let $\mathcal{P}_t$ be the collection of prime periodic orbits γ with $\ell(\gamma) \in (t-1, t]$. Define averaging measures μ_t by

$$\mu_t := \frac{\sum_{\gamma \in \mathcal{P}_t} e^{\int f \, d\mu_\gamma} \mu_\gamma}{\sum_{\gamma \in \mathcal{P}_t} e^{\int f \, d\mu_\gamma}}.$$

The following is an example of an equidistribution theorem.

Theorem 1.0.1. *Suppose X^t is a weak-mixing transitive Anosov flow, and f is Hölder continuous. Then, as $t \to \infty$, the measures μ_t converge in the weak* topology to the equilibrium state μ_f.*

This generalises a result of Bowen [Bow72a], in the case $f = 0$. In [Fra77], Bowen's method is adapted to show that for general f, any weak* convergent subsequence of $(\mu_t)_{t \in \mathbb{R}}$, converges to μ_f. The proof of Theorem 1.0.1 using thermodynamic formalism and dynamical zeta functions is due to Parry [Par88].

In Chapter 5, we extend this theory to study convergence of measures

$$\frac{\sum_{\gamma\in\mathcal{P}_t\cap A} e^{\int f\, d\mu_\gamma}\mu_\gamma}{\sum_{\gamma\in\mathcal{P}_t\cap A} e^{\int f\, d\mu_\gamma}},$$

where A is a set of periodic orbits satisfying some homological constraints. We show that when A is the set of orbits which are null in rational homology,

$$\frac{\sum_{\gamma\in\mathcal{P}_t\cap A} e^{\int f\, d\mu_\gamma}\mu_\gamma}{\sum_{\gamma\in\mathcal{P}_t\cap A} e^{\int f\, d\mu_\gamma}} \xrightarrow[t\to\infty]{} \mu_{f^*},$$

where f^* is a function, generally distinct from f, described through analysis of the pressure function. Our methods also apply to the case where A is the set of integrally null homologous orbits.

Using linking forms, we are able to recast topological questions in terms of ergodic averages subject to homological constraints, allowing us to apply the thermodynamic formalism to studying the linking number for orbits of Anosov flows.

One example of an ergodic approach to studying knots arising from dynamical systems is that of Arnol'd [Arn86], centred around the helicity of a vector field. Helicity is a quantity associated to a null-homologous (defined below) 3-dimensional vector field (or differentiable flow), which is invariant under volume preserving diffeomorphism. It is one of the basic quantities conserved by inviscid fluid flow, and has applications in magnetohydrodynamics. We give the definition due to Moffatt [Mof69].

Let M be a smooth closed connected oriented 3-manifold with volume form Ω, and X a divergence-free vector field on M. Assuming X is null-homologous, which is that the interior product $i_X\Omega$ is the derivative of a 1-form α, the helicity of X is defined by

$$\mathcal{H}(X) = \int_M \alpha \wedge i_X\Omega,$$

and is independent of the choice of α. Arnol'd studied vector fields on compact domains in $\mathbb{R}^3$, and proposed that the helicity can be characterised using linking numbers of knots created from trajectories of the flow of X. Gaps in Arnol'ds proof were filled by Vogel [Vog02], in the case of closed 3-manifolds M satisfying $H_1(M,\mathbb{Q}) = \{0\}$.

Precisely, for $x \in M$ and $t > 0$ define $K_t(x)$ to be the closed curve formed by concatenating the flow of x to time t with a minimal geodesic path returning to x. Let Vol denote the volume measure induced by Ω. The result of Arnol'd and Vogel is that the limit

$$\mathcal{A}(x,y) := \lim_{s,t\to\infty} \frac{1}{st}\mathrm{lk}(K_t(x), K_s(y))$$

is in $L^1(\mathrm{Vol}\times\mathrm{Vol})$, and furthermore satisfies

$$\mathcal{H}(X) = \int \mathcal{A}\, d(\mathrm{Vol}\times\mathrm{Vol}).$$

Indeed, the proof of this result uses the linking form to translate to the setting of ergodic theory, making this approach a key inspiration for the results in Chapter 6.

Another motivating work is that of Contreras [Con95], who studied average linking numbers of periodic orbits for hyperbolic flows on basic sets $\Lambda \subset S^3$. The averages are given by

$$L(s,t) = \frac{\sum_{\gamma\in\mathcal{P}_t, \gamma'\in\mathcal{P}_s} \mathrm{lk}(\gamma,\gamma')}{\#\mathcal{P}_t\#\mathcal{P}_s},$$

for $s, t > 0$. Also define $I : S^3 \times S^3 \to \mathbb{R}$ (away from the diagonal) by

$$I(x,y) = \frac{3}{4\pi}\frac{X(x)\times X(y)}{\|x-y\|^3}\cdot(x-y),$$

where X is the vector field of the flow. Contreras showed that

$$\lim_{s,t\to\infty}\frac{1}{st}L(s,t) = \int I\, d(\mu_\Lambda\times\mu_\Lambda),$$

where μ_Λ is the measure of maximal entropy on Λ. Contreras' proof relies on the equidistribution theory of Bowen [Bow72a].

In Chapter 6, we discuss a problem which connects the works of Arnol'd, Vogel and Contreras, in the setting of an Anosov flow on a closed 3-manifold. We show, using the more general equidistribution theory, that in certain cases one can recover helicity as the limit of a weighted average of linking numbers which resembles those considered in Contreras' work.

A different problem, which is the subject of Chapter 7, is to study the distribution of periodic orbits according to their integral homology classes. Let X^t be a topologically weak-mixing transitive Anosov flow on a closed 3-manifold M. Then

$$H_1(M,\mathbb{Z}) \cong \mathbb{Z}^b \oplus \mathrm{Tor},$$

where b is the first Betti number of M, and Tor is a finite abelian group (the torsion). In [BL98], Babillot–Ledrappier found asymptotic formulae which can be applied to study the distribution of periodic orbits according to their torsion free homology vectors $[\gamma] \in \mathbb{Z}^b$. In particular, their results show the following.

Theorem 1.0.2. *There is a closed convex set $\mathcal{C} \subset \mathbb{R}^b$ and an entropy function $H : \mathbb{R}^b \to \mathbb{R}$ such that for each $\alpha \in \mathbb{Z}^b$ there is a positive continuous function $C_\alpha(z,t)$ which is both bounded above and bounded away from zero below, satisfying*

$$\sum_{\gamma \in \mathcal{P}_t} \mathbb{1}_\alpha([\gamma] - \lfloor tz \rfloor) \text{ is } \begin{cases} \text{asymptotic to } C_\alpha(z,t)\frac{e^{H(z)t}}{t^{1+b/2}} & \text{if } z \in \mathcal{C}^\circ \\ \text{eventually } 0 & \text{if } z \notin \mathcal{C}, \end{cases}$$

where $\lfloor \cdot \rfloor$ denotes the component-wise integer part of a vector.

The set $\mathcal{C}$ and the functions H and $C_\alpha(z,t)$ all have explicit descriptions which we will see later. This can be seen to generalise a result of Sharp [Sha93], where it is assumed that $0 \in \mathcal{C}$. In that case we have that

$$\#\{\gamma \text{ periodic } : \ell(\gamma) \leq t \text{ and } [\gamma] = \alpha\} \sim C_\alpha \frac{e^{H(0)t}}{t^{1+b/2}},$$

where C_α is independent of t. In proving the above theorem, it is necessary that the homology classes of periodic orbits generate $H_1(M,\mathbb{Z})$. This is seen to follow from a Chebotarev density theorem for hyperbolic flows proved by Parry–Pollicott in [PP86].

In Chapter 7, we instead consider homology classes in the complement of a link. Let $\gamma_1, \dots, \gamma_n$ be periodic orbits with $[\gamma_i] = 0 \in H_1(M,\mathbb{Z})$ for all i. For each i, replace $\gamma_i \subset M$ with a tubular neighbourhood T_i (a solid torus with meridian γ_i). Defining $M_n := M \setminus (T_1 \cup \ldots \cup T_n)$, we have

$$H_1(M_n, \mathbb{Z}) \cong \mathbb{Z}^{b+n} \oplus \mathrm{Tor}.$$

We study the torsion-free homology vectors $[\gamma]^{(n)} \in \mathbb{Z}^{b+n}$, for the remaining periodic orbits γ. In particular, n of the components of $[\gamma]^{(n)}$ correspond to the linking numbers $\mathrm{lk}(\gamma, \gamma_i)$ for $i = 1, \dots, n$.

In [McM13], McMullen proves a Chebotarev density theorem for this setting, leading to the fact that the $[\gamma]^{(n)}$ generate $\mathbb{Z}^{b+n}$. In Chapter 7 we show how McMullen's work can be applied to obtain the analogous result to Theorem 1.0.2.

A key tool used throughout this thesis is the symbolic coding for Anosov flows, developed independently by Bowen [Bow73] and Ratner [Rat73]. Let $k \in \mathbb{N}$ and Γ be a finite directed graph on k vertices $\{1, \dots k\}$. The shift space $\Sigma(\Gamma) \subset \{1, \dots, k\}^{\mathbb{Z}}$ is the set of bi-infinite walks on Γ. On this space we have the left shift map $\sigma((x_n)_{n\in\mathbb{Z}}) = (x_{n+1})_{n\in\mathbb{Z}}$. Given a continuous and strictly positive function

$r : \Sigma(\Gamma) \to \mathbb{R}$, the suspension space is defined by

$$\Sigma(\Gamma, r) = \{(x,t) \in \Sigma(\Gamma) \times \mathbb{R} : 0 \le t \le r(x)\} \big/ \sim,$$

where $\sim$ is the relation $(x, r(x)) \sim (\sigma(x), 0)$. A suspension flow is defined naturally on $\Sigma(\Gamma, r)$ by $\sigma_t^r[x,s] = [x, s+t]$, where $[x,s]$ is the equivalence class of (x,s) under $\sim$.

The symbolic coding is a semi-conjugacy between the suspension flow σ_t^r and the Anosov flow X^t. Whilst a full conjugacy will in general not exist, the semi-conjugacy has properties which still allow one to translate effectively between suspension and Anosov flows. The result is as follows.

Theorem 1.0.3. *Let X^t be a transitive Anosov flow on a closed manifold M. Then, there exists an aperiodic graph Γ, a Hölder continuous roof function r, and a Hölder continuous surjection $\pi : \Sigma(\Gamma, r) \to M$ satisfying the following*

1. *For all $t \in \mathbb{R}$, $\pi \circ \sigma_t^r = X^t \circ \pi$.*
2. *σ_t^r is transitive.*
3. *σ_t^r is topologically weak-mixing if and only if X^t is topologically weak-mixing.*
4. *There exists $N \in \mathbb{N}$ such that $\#\pi^{-1}(y) \le N$ for all $y \in M$. Furthermore, the set of points with multiple preimages is meagre, and null with respect to any ergodic fully-supported measure.*

1.1 Thesis outline

Let us finish the introduction with an outline of the structure of the thesis.

- In Chapter 2 we give a brief overview of the necessary preliminaries from differential geometry. For the reader who is familiar with differential geometry this is standard material. The chapter also contains a discussion of linking forms, which are crucial in our approach to proving the results of Chapter 6.
- Chapter 3 includes preliminary material from dynamical systems and ergodic theory, again standard for the familiar reader. We also define entropy and pressure, whose finer properties are detailed in Chapter 4.
- In Chapter 4 we define hyperbolic flows and discuss their basic properties. We then formally introduce symbolic dynamics and the coding for hyperbolic

flows. With this coding in place, we review the basic techniques of thermodynamic formalism for symbolic systems, and show how they can be applied to hyperbolic (particularly, Anosov) flows.

- Chapter 5 is the first containing new results. We first discuss classical equidistribution theorems and their thermodynamic proofs, before proving analogous results regarding periodic orbits under homological constraints.

- In Chapter 6, we perform an analysis of the linking form and combine this with the equidistribution theory of Chapter 5 to characterise the helicity in terms of periodic orbit linking.

- In Chapter 7 we show how McMullen's modified symbolic coding can be used to give a new application of the results of Babillot–Ledrappier on the number and distribution of orbits in the first homology group of a link complement.

Chapter 2

Differential geometry preliminaries

Here we discuss some concepts from differential geometry which will be useful for understanding dynamical systems on manifolds. It will be particularly important in the later chapters that differential forms are well understood, since we will use them to study homology and linking numbers.

2.1 Manifolds

Details for Sections 2.1-2.3 can be found in [Lee13], for example.

Definition 2.1.1. A Hausdorff second-countable topological space M is an n-dimensional *manifold* (or n-manifold), if each $p \in M$ has a neighbourhood homeomorphic to an open subset of $\mathbb{R}^n$, i.e. there exist open sets $U \subset M$ and $V \subset \mathbb{R}^n$ such that $p \in U$ and there is a homeomorphism $\varphi : U \to V$. A *submanifold* is a subset of M which is a manifold with respect to the subspace topology.

In this thesis, we always assume our manifolds are connected. The pair (U, φ) is called a *co-ordinate chart* and provides local co-ordinates $x^1, \ldots, x^n$ around p, defined for $q \in U$ by $\varphi(q) = (x^1(q), \ldots, x^n(q))$. An *atlas of charts* is a collection of co-ordinate charts whose domains cover M. For $r \in \mathbb{N}$, a manifold M will be called *smooth* if there is an atlas of charts such that for any two charts (U_1, φ) and (U_2, ψ) with intersecting domains, the transition map

$$\varphi \circ \psi^{-1} : \psi(U_1 \cap U_2) \to \varphi(U_1 \cap U_2)$$

is smooth. For such a manifold, a *smooth structure* is an atlas with smooth transition maps which is maximal (not contained in a larger such atlas). The existence of a smooth structure is Proposition 1.17 in [Lee13]. In this setting, we can discuss regularity of functions between manifolds. Let M, N be smooth manifolds and $f : M \to N$ a function. We say f is smooth (resp. C^r for $r \in \mathbb{N}$) if it is smooth (resp. C^r) under the co-ordinate charts. This means that for each chart (φ, U_1) for M, if (ψ, U_2) is a chart for N satisfying $U_2 \cap f(U_1) \neq \varnothing$, then $\psi \circ f \circ \varphi^{-1}$ is smooth (resp. C^r). The set of smooth (resp. C^r) functions $M \to \mathbb{R}$ will be denoted by $C^\infty(M)$ (resp. $C^r(M)$).

2.2 Tangent bundles

We now introduce *tangent spaces*, which allow us to perform local linear approximation on smooth manifolds. At a point $p \in M$, *tangent vectors* are constructed via directional derivatives of smooth functions. Precisely, a function $v : C^\infty(M) \to \mathbb{R}$ is a *derivation* at p if it is linear and satisfies a product rule at p, meaning

$$v(fg) = f(p)v(g) + g(p)v(f) \text{ for all } f, g \in C^\infty(M).$$

The set of derivations at p is a vector space, which we call the tangent space at p, denoted T_pM. Each $v \in T_pM$ is called a tangent vector, and p is called the *foot point* of v. In local co-ordinates $x^1, \ldots, x^n$ given by a chart (U, φ), a basis of T_pM is given by the partial derivatives $\frac{\partial}{\partial x^i}|_p$, defined by

$$\left.\frac{\partial}{\partial x^i}\right|_p (f) := \frac{\partial (f \circ \varphi^{-1})}{\partial x^i}(\varphi(p)).$$

We will usually write $\frac{\partial f}{\partial x^i}(p)$ for notational ease. This basis will be the one we use for co-ordinate computations.

Definition 2.2.1. The *tangent bundle* is the disjoint union

$$TM = \bigsqcup_{p \in M} T_pM$$

of tangent spaces on M. Define $\pi : TM \to M$ by $\pi(v) = p$ whenever $v \in T_pM$. We may write elements of TM as pairs $(\pi(v), v)$, when we wish to make the foot point explicit.

The following is Proposition 3.18 in [Lee13].

Proposition 2.2.2. *For a smooth n-manifold M, TM has a natural topology which* *supports a smooth structure which makes it a 2n-manifold. With respect to this structure, the foot point map π is smooth.*

The definition of the tangent space leads to the following notion. Given a C^1 map of manifolds $f : M \to N$, and $p \in M$, the *differential* of f at p is a map $Df_p : T_pM \to T_{f(p)}N$ defined by

$$(Df_p(v))(g) = v(g \circ f) \text{ for all } v \in T_pM, g \in C^\infty(N).$$

One can prove the following.

Proposition 2.2.3. *Let M, N, P be smooth manifolds with $f : M \to N$, $g : N \to P$ differentiable maps. Then for all $p \in M$*

1. *Df_p is a linear map $T_pM \to T_{f(p)}N$,*
2. *D satisfies a chain rule, meaning $D(g \circ f)_p = Dg_{f(p)} \circ Df_p : T_pM \to T_{g(f(p))}P$,*
3. *$D(Id_M)_p = Id_{T_pM}$, where Id denotes the identity map,*
4. *If f is a diffeomorphism, then Df_p is a bijection and $(Df_p)^{-1} = D(f^{-1})_{f(p)}$.*

Again this can be interpreted in local co-ordinates. Suppose $x^1, \dots, x^n$ are local co-ordinates in M at p, and $y^1, \dots, y^m$ are local co-ordinates in N at $f(p)$. Denoting the maps $y^i \circ f$ by f^i, Df_p is represented in the usual basis by the $m \times n$ matrix

$$\begin{pmatrix} \frac{\partial f^1}{\partial x^1}(p) & \cdots & \frac{\partial f^1}{\partial x^n}(p) \\ \vdots & \ddots & \vdots \\ \frac{\partial f^m}{\partial x^1}(p) & \cdots & \frac{\partial f^1}{\partial x^n}(p) \end{pmatrix},$$

called the *Jacobian* matrix of f at p.

As well as differentiating functions, tangent spaces allow us to take tangents to curves. Indeed, let $\gamma : I \to M$ be a differentiable map, where $I \subset \mathbb{R}$ is an interval. The *velocity vector* of γ at $t_0 \in I$ is defined by

$$\dot{\gamma}(t_0) = D\gamma_{t_0}\left(\left.\frac{d}{dt}\right|_{t_0}\right) \in T_{\gamma(t_0)},$$

where $\frac{d}{dt}|_{t_0}$ is the usual time derivative in $\mathbb{R}$ (identified with $T_{t_0}I$). In co-ordinates $x^1, \dots, x^n$ in M at $\gamma(t_0)$, we make the identification

$$\gamma(t) = (\gamma^1(t), \dots, \gamma^n(t)),$$

with $\gamma^i = x^i \circ \gamma$. Under these co-ordinates,

$$\dot{\gamma}(t_0) = \sum_{i=1}^{n} \frac{d\gamma^i}{dt}(t_0) \frac{\partial}{\partial x^i}\bigg|_{\gamma(t_0)},$$

resembling the usual velocity of a curve in $\mathbb{R}^n$. For this reason, we may also use the notation $\frac{d\gamma}{dt}(t_0)$ for $\dot{\gamma}(t_0)$. With this identification it is easy to see that every tangent vector is the velocity of a curve in M. The action of $\dot{\gamma}$ on $C^\infty(M)$ can be interpreted as differentiation along γ. For $f \in C^\infty(M)$, $(\dot{\gamma}(t_0))(f) = (f \circ \gamma)'(t_0)$, where $f \circ \gamma$ is differentiated as usual in $\mathbb{R}$. Furthermore, the differential of a smooth function $f : M \to N$ is given by

$$Df_p(v) = \frac{d(f \circ \gamma)}{dt}(0),$$

where γ is a curve in M satisfying $\gamma(0) = p$ and $\dot{\gamma}(0) = v$.

Vector fields

A *vector field* is a map $X : M \to TM$ which satisfies $X(p) \in T_pM$ for all $p \in M$. An *integral curve* of a X is a curve $\gamma : I \to M$ satisfying that $\dot{\gamma}(t) = X(\gamma(t))$ for all $t \in I$. We say that a X is *complete* if for each point $p \in M$, there is a unique infinite integral curve $\gamma_p : \mathbb{R} \to M$ of X, with $\gamma_p(0) = p$. In this case we define the *flow* of X by $X^t(p) = \gamma_p(t)$. In this thesis, flows will be an important class of dynamical system, which we discuss in detail in Section 3.2. In some situations, all regular vector fields will be complete.

Theorem 2.2.4. *If M is compact and $X : M \to TM$ is a C^1 vector field, then X is complete.*

For a proof of the above, note that the proof of Theorem 9.16 in [Lee13] holds in this setting.

The following definition will be useful later.

Definition 2.2.5. Let $X : M \to TM$ be a vector field with local integral curves γ_p at each $p \in M$ satisfying $\gamma(0) = p$. The *Lie derivative* is a map $L_X : C^1(M) \to C^1(M)$ defined by

$$(L_X f)(p) = Df_p(X(p)) := \frac{d(f \circ \gamma_p)}{dt}(0) = \lim_{t \to 0} \frac{f(X^t(p)) - f(p)}{t}.$$

In fact, this definition only requires that f is differentiable along integral curves, not everywhere.

Orientability

Let us define the notion of orientability, which will be a basic assumption for the manifolds in this thesis.

We will continue to denote by M a smooth n-manifold. At $p \in M$, we say two bases $(e_1, \dots, e_n)$, $(e'_1, \dots, e'_n)$ for T_pM are *consistently oriented* if the transition matrix B defined by $e_i = B_{ij}e'_j$ has positive determinant. Consistent orientation forms an equivalence relation of ordered bases, and a *pointwise orientation* of M is a choice of equivalence class at each $p \in M$. Fix a pointwise orientation on M. An *oriented local frame* is an ordered set of vector fields $(X_1, \dots, X_n)$ defined on an open subset $U \subset M$, such that for each $p \in U$, $(X_1(p), \dots, X_n(p))$ is an ordered basis for T_pM which is consistently oriented with the fixed orientation at p. A pointwise orientation is *continuous* if each $p \in M$ is in the domain of some oriented local frame.

Definition 2.2.6. M is *orientable* if there exists a continuous pointwise orientation on M. An *orientation* on M is a choice of continuous pointwise orientation, and once this choice is made, M is *oriented*.

By Proposition 15.6 in [Lee13], we may always assume our usual co-ordinate bases for TM are consistent with the orientation.

2.3 Differential forms

Let us recall some background on multilinear algebra.

Let $k \in \mathbb{N}$ and V a finite-dimensional vector space. A map

$$\alpha : \underbrace{V \times \cdots \times V}_{k \text{ times}} \to \mathbb{R}$$

is *multilinear* if it is linear in each variable (i.e. with the other variables fixed). Let $L_k(V)$ be the set of all such maps. This is itself a vector space, elements of which are called *covariant k-tensors* on V, in the case $k = 1$ they are also called *covectors*. For $\alpha \in L_k(V)$, k is called the *rank* of α. For multilinear maps $\alpha \in L_k(V)$, $\beta \in L_j(V)$, the *tensor product* $\alpha \otimes \beta \in L_{k+j}(V)$ is defined by

$$(\alpha \otimes \beta)(v_1, \dots, v_k, v_{k+1}, \dots v_{k+j}) = \alpha(v_1, \dots, v_k)\beta(v_{k+1}, \dots v_{k+j}).$$

Suppose V has dimension n, and $\{e_1, \dots, e_n\}$ is a basis for V. The dual space V^* of linear maps $V \to \mathbb{R}$ is also n-dimensional. Let $\{E^1, \dots, E^n\}$ denote the dual basis to $\{e_1, \ldots, e_n\}$,, defined by $E^i(e_i) = 1$ for each i. With this notation, $L_k(V)$ has a basis given by

$$\{E^{i_1} \otimes \cdots \otimes E^{i_k} : 1 \leq i_j \leq n \text{ for all } j\},$$

so $L_k(V)$ has dimension n^k. To ease notation, we will write $E^{i_1 \cdots i_k}$ for $E^{i_1} \otimes \cdots \otimes E^{i_k}$.

We say $\alpha \in L_k(V)$ is *alternating* or *anti-symmetric* if

$$\alpha(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_k) = -\alpha(v_1, \ldots, v_j, \ldots, v_i, \ldots, v_k)$$

for all $v_1, \ldots v_k \in V$ and $1 \leq i, j \leq k$. Any $\alpha \in L_k(V)$ has a corresponding alternating tensor $\operatorname{Alt} \alpha \in L_k(V)$ defined by

$$(\operatorname{Alt} \alpha)(v_1, \ldots, v_k) = \frac{1}{k!} \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) \alpha(v_{\sigma(1)}, \ldots, v_{\sigma(k)}),$$

satisfying $\operatorname{Alt} \alpha = \alpha$ if and only if α is alternating. It will be useful to have notation for the subspace of alternating covariant tensors, since they are key in defining *differential forms*. Indeed, denote by $\Lambda_k(V)$ the set of alternating covariant k-tensors. A basis of $\Lambda_k(V)$ is given by the tensors

$$\{E^I : I = i_1 \ldots i_k \text{ with } 1 \leq i_1 < \ldots < i_k \leq n\},$$

so $\Lambda_k(V)$ has dimension $\frac{n!}{k!(n-k)!}$.

For $\alpha \in \Lambda_k(V)$, $\beta \in \Lambda_j(V)$, the *wedge product* $\alpha \wedge \beta \in \Lambda_{k+j}(V)$ is defined by

$$\alpha \wedge \beta = \frac{(k+j)!}{k!j!} \operatorname{Alt}(\alpha \otimes \beta).$$

Given multi-indices I, J such that $E^I \in \Lambda_k(V)$, $E^J \in \Lambda_j(V)$, we have $E^I \wedge E^J = E^{IJ}$. Further the wedge product satisfies:

- bilinearity,
- associativity,
- anti-commutativity i.e. $\alpha \wedge \beta = -(\beta \wedge \alpha)$ for all covariant tensors α and β,
- for any $\alpha^1, \ldots, \alpha^k \in \Lambda_1(V) = L_1(V) = V^*$,

$$(\alpha^1 \wedge \ldots \alpha^k)(v_1, \ldots, v_k) = \det(\alpha^j(v_i))_{ij}.$$

Returning to the case of a smooth n-manifold M, we call

$$\Lambda_k(TM) = \bigsqcup_{p \in M} \Lambda_k(T_pM)$$

the bundle of covariant k-tensors on M.

Definition 2.3.1. A differential k-form (or k-form) is a smooth section of the foot point projection $\Lambda_k(TM) \to M$. This is a smooth map

$$\alpha : M \to \Lambda_k(TM)$$

such that $\alpha(p) \in \Lambda_k(T_pM)$ for all $p \in M$. We will often use the notation α_p for $\alpha(p)$. The number k is the *degree* of α.

The set of k-forms on M will be denoted by $\Omega^k(M)$, and we define $\Omega^*(M) = \bigcup_{k=1}^n \Omega^k(M)$. We use the convention $\Omega^0(M) = C^\infty(M)$, which is natural once we look at local representations of forms. The wedge product of forms $\alpha, \beta \in \Omega^*(M)$ is defined in the obvious way, $(\alpha \wedge \beta)_p = \alpha_p \wedge \beta_p$.

Usually, we will work with forms in local co-ordinates. Recall that in co-ordinates $x^1, \dots, x^n$ on M, the derivations $\frac{\partial}{\partial x^i}$ form the bases of tangent spaces. We will use the notation dx^i for elements of the dual basis. Precisely, in $\Lambda_k(T_pM)$, dx^i_p is the covariant tensor defined by

$$dx^i_p \left(\left. \frac{\partial}{\partial x^i} \right|_p \right) = 1 \text{ and } dx^i_p \left(\left. \frac{\partial}{\partial x^j} \right|_p \right) = 0 \text{ for all } j \neq i.$$

We will use dx^i to denote the 1-form on the whole of M, since x^i will usually denote the local co-ordinates at every point. Again, a multi-index $I = i_1 \dots i_k$ will be used for the shorthand

$$dx^I = dx^{i_1} \wedge \dots \wedge dx^{i_k} \in \Omega^k(M),$$

with $\{dx^I : I \text{ is increasing}\}$ forming a pointwise basis of $\Omega^k(M)$. Thus each $\alpha \in \Omega^k(M)$ has a representation as

$$\alpha_p = \sum_{I \text{ increasing}} \alpha_I(p)\, dx^I,$$

where $\alpha_I \in C^\infty(M)$. Hence it is natural to think of smooth functions as 0-forms.

This notation gives a convenient way of defining the *exterior derivative*

$d\colon \Omega^k(M) \to \Omega^{k+1}(M)$. With α as above, define

$$(d\alpha)_p = \sum_{i=1}^{n} \sum_{I \text{ increasing}} \frac{\partial \alpha_I}{\partial x^i}(p)\, dx^{iI}.$$

In particular, given $f \in C^\infty(M)$,

$$(df)_p = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}(p)\, dx^i,$$

which agrees with the Jacobian Df_p defined earlier. The following definitions will be useful later.

Definition 2.3.2. A k-form β is called *exact* if there is a $(k-1)$-form β such that $d\beta = \alpha$. The form β is closed if $d\beta = 0$.

Remark. All exact forms are closed by anti-commutativity of the wedge product.

Integration

Differential forms allow us to perform integration on manifolds. Here we assume that M is a smooth compact orientable n-manifold. We will begin by differentiating n-forms. Indeed, let $\omega \in \Omega^n(M)$ be supported in a single smooth chart (U, φ) of M. We have that

$$\omega = f\, dx^1 \wedge \ldots \wedge dx^n,$$

where $f \in C^\infty(M)$ is zero outside of U. Define the integral of ω as the integral in $\mathbb{R}^n$ of $f \circ \varphi^{-1}$, i.e.

$$\int_M \omega := \int_{\varphi(U)} f(\varphi^{-1}(x^1, \ldots, x^n))\, dx^1 \ldots dx^n.$$

When the support of ω is not contained in a single chart, we integrate as follows. Let $U_1, \ldots, U_k$ be a cover of the support of ω by chart domains, and $\psi_1, \ldots, \psi_k \in C^\infty(M)$ be a partition of unity subordinate to $U_1, \ldots, U_k$. Then each $\psi_i\omega$ is supported in the domain U_i, and we define the integral by

$$\int_M \omega := \sum_{i=1}^{k} \int_M \psi_i \omega.$$

In Proposition 16.5 of [Lee13] it is shown that this definition is independent of the choice of partition of unity and open cover. Note that the definition does not require compactness of M, just that ω has compact support.

Lower degree forms are integrated over submanifolds of corresponding dimension. Indeed, suppose $\alpha \in \Omega^k(M)$ for $k < n$ and S is a submanifold of dimension k. Then we can view α as a form on S of maximal dimension, and integrate accordingly. Precisely, let $\iota_S : S \to M$ be the inclusion map. Then $\iota_S^*\alpha \in \Omega^k(S)$ and we define

$$\int_S \alpha := \int_S \iota_S^* \alpha.$$

Below we state a corollary of Stokes' Theorem, a classical theorem in differential forms which generalises the fundamental theorem of calculus.

Corollary 2.3.3. *Let M be an oriented smooth n-manifold, and ω a compactly supported $(n-1)$-form. Then*

$$\int_M d\omega = 0.$$

Later, we will use the following charaterisation of cohomology groups, due to de Rham. We denote by $H^k(M,\mathbb{R})$ the kth real cohomology group, dual to the kth real homology grooup $H_k(M,\mathbb{R})$.

Theorem 2.3.4 (de Rham cohomology)**.** *Given $1 \leq k \leq n$, $H^k(M,\mathbb{R})$ can be identified with the collection of closed k-forms, up to addition of exact k-forms. Precisely,*

$$\{\alpha \in \Omega^k(M) : \alpha \text{ is closed}\} \big/ \{\alpha \in \Omega^k(M) : \alpha \text{ is exact}\} \cong H^k(M,\mathbb{R}),$$

with an isomorphism given by

$$[\alpha] \mapsto \left[[c] \mapsto \int_c \alpha \right],$$

where $[c]$ denotes an element of $H_k(M,\mathbb{R})$.

2.4 Riemannian metrics

For detail on this section, see [Lee18]. On a manifold M, a Riemannian metric is an inner product on tangent spaces, which allows one to measure tangent vectors, and thus measure the length of curves and distance of points on M. With the notation for covariant tensors from the last section, a Riemannian metric is a smooth section of the foot point map $\Lambda_2(TM) \to M$, which is symmetric and positive-definite at each point.

Definition 2.4.1. A Riemannian metric is a smooth map $\rho : M \to \Lambda_2(TM)$, satisfying, for all $p \in M$ and $u, v \in T_pM$, that

1. $\rho(p) \in \Lambda_2(T_pM)$,
2. $(\rho(p))(u,v) = (\rho(p))(v,u)$,
3. $(\rho(p))(v,v) \geq 0$.

We will usually write ρ_p for $\rho(p)$. When equipped with a Riemannian metric ρ, we will call M, or (M,ρ), a *Riemannian manifold*.

Since ρ_p is an inner product, it defines a norm $\|\cdot\|_\rho : TM \to \mathbb{R}$ given by $\|v\| = (\rho_{\pi(v)}(v,v))^{1/2}$. When the metric is obvious we will write $\|\cdot\|$. With this, define the *Riemannian length* of a piecewise differentiable curve $\gamma : (a,b) \to M$ by

$$\text{length}(\gamma) = \int_a^b \|\dot{\gamma}(t)\|\, dt.$$

Moreover, define the *Riemannian distance* between $x, y \in M$ by

$$d_\rho(x,y) = \inf\{\text{length}(\gamma) : \gamma \text{ piecewise differentiable from } x \text{ to } y\}.$$

Again we will just write $d(x,y)$ when the metric is obvious. The following is Theorem 2.55 in [Lee18].

Proposition 2.4.2. *(M, d_ρ) is a metric space, and the metric topology is the same as that of the original manifold.*

The metric ρ is also used to define the *musical isomorphisms*, which will be useful later.

Definition 2.4.3. Given $p \in M$, each $v \in T_pM$ has a corresponding covector $v^\flat \in T_p^*M$ determined uniquely by $v^\flat(w) = \rho_p(v,w)$ for all $w \in T_pM$. For a vector field, we also write $X^\flat$ to be the 1-form $X_p^\flat = (X(p))^\flat$. In fact, $\flat$ is an isomorphism of vector spaces, and its inverse will be denoted by $\#$, i.e. for $F \in T_p^*M$, $F^\#$ is the unique $v \in T_pM$ such that $v^\flat = F$. We also have the analogous definition for vector fields and 1-forms.

In local co-ordinates, the metric can be evaluated as

$$\rho_p(u,v) = \sum_{i.j=1}^{n} \rho_{ij}(p) u^i v^j,$$

where u^i, v^j are the components of u, v with respect to the usual basis of T_pM, and

$\rho_{ij} : M \to \mathbb{R}$ is the smooth function

$$\rho_{ij}(p) := \rho_p \left(\left.\frac{\partial}{\partial x^i}\right|_p, \left.\frac{\partial}{\partial x^j}\right|_p \right).$$

We use raised indices for the components of a tangent vector v and lowered indices for the components of its corresponding covector $v^\flat$, meaning v^i is the ith component of v with respect to the $\frac{\partial}{\partial x^i}$ basis of T_pM, and v_i is the ith component of $v^\flat$ with respect to the dx^i basis of T_p^*M. Note that for a vector field X, the same notation X^i is used for the ith component map $M \to \mathbb{R}$, as well as the time-i map $M \to M$ of the flow corresponding to X. Any ambiguity on this matter will be settled with context.

Since ρ_p is an inner product, the matrix $(\rho_{ij})_{i,j=1}^n$ is invertible, and we let $(\rho^{ij})_{i,j=1}^n$ be the inverse. Then for $v \in T_pM$, $i \in \{1, \ldots, n\}$,

$$v_i = \sum_{j=1}^n \rho_{ij} v^j \text{ and } v^i = \sum_{j=1}^n \rho^{ij} v_j.$$

The musical isomorphisms also allow us to define an inner product on cotangent spaces T_p^*M. For $F, G \in T_p^*M$, define, by an abuse of notation,

$$\rho_p(F, G) := \rho_p(F^\#, G^\#).$$

For $\alpha, \beta \in \Omega^1(M)$, $\rho(\alpha, \beta)$ will denote the map $p \mapsto \rho_p(\alpha_p, \beta_p)$. Further, we can extend this to general k-forms by defining

$$\rho(\alpha^{(1)} \wedge \ldots \wedge \alpha^{(k)}, \beta^{(1)} \wedge \ldots \wedge \beta^{(k)}) := \det((\rho(\alpha^{(i)}, \beta^{(j)}))_{i,j=1}^n),$$

where each $\alpha^{(i)}, \beta^{(i)} \in \Omega^1(M)$, and extending by linearity.

Volume

The Riemannian metric gives a natural notion of volume on M. It does so by inducing a *volume form*, which is an n-form that is nowhere vanishing. Precisely, the Riemannian volume form is given locally by

$$\Omega_\rho = \sqrt{|\det(\rho_{ij})_{i,j=1}^n|}\, dx^1 \wedge \ldots \wedge dx^n.$$

This induces a volume measure on M by integration, defining

$$\int \psi \, d\mathrm{Vol}_\rho = \int_M \psi \Omega_\rho$$

for all continuous functions $\psi : M \to \mathbb{R}$. We call Vol_ρ the volume or *Liouville measure* associated to ρ. In this thesis, we usually assume Vol_ρ to be normalised.

The Levi-Civita connection

In differential geometry, *connections* give a way of taking directional derivatives of vector fields, naturally extending the theory in Euclidean space. In Riemannian geometry, there is a unique choice of connection which is particularly compatible with the metric.

Continue with a smooth Riemannian n-manifold (M, ρ). Let $\mathfrak{X}(M)$ be the set of vector fields on M. A connection is a map $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$, written $\nabla(X, Y) = \nabla_X Y$, satisfying:

1. For all $X_1, X_2, Y \in \mathfrak{X}(M)$ and $f_1, f_2 \in C^\infty(M)$,

$$\nabla_{f_1 X_1 + f_2 X_2} Y = f_1 \nabla_{X_1} Y + f_2 \nabla_{X_2} Y.$$

2. For all $X, Y_1, Y_2 \in \mathfrak{X}(M)$ and $a_1, a_2 \in \mathbb{R}$,

$$\nabla_X (a_1 Y_1 + a_2 Y_2) = a_1 \nabla_X Y_1 + a_2 \nabla_X Y_2.$$

3. For all $X, Y \in \mathfrak{X}(M)$, and $f \in C^\infty(M)$,

$$\nabla_X (fY) = f \nabla_X Y + (L_X f) Y.$$

The following theorem is sometimes referred to as the fundamental theorem of Riemannian geometry (see Theorem 5.10 in [Lee18]). First define the *Lie bracket* $[\cdot, \cdot] : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ by

$$[X, Y] = L_X L_Y - L_Y L_X.$$

This is a vector field in the sense that at $p \in M$,

$$L_X(L_Y(\cdot)) - L_Y(L_X(\cdot))$$

is a derivation $C^\infty(M) \to \mathbb{R}$.

Theorem 2.4.4. *Let (M, ρ) be a Riemannian manifold. There exists a connection ∇ on M which satisfies that for all $X, Y, Z \in \mathfrak{X}(M)$:*

1. $L_X(\rho(Y, Z)) = \rho(\nabla_X Y, Z) + \rho(Y, \nabla_X Z)$.
2. $\nabla_X Y - \nabla_Y X = [X, Y]$

Furthermore, ∇ is the unique such connection.

The connection ∇ in Theorem 2.4.4 is called the *Levi-Civita* connection, and we will implicitly assume it to be the fixed connection on any Riemannian manifold. In local-coordinates, write

$$\nabla_{\frac{\partial}{\partial x^i}}\left(\frac{\partial}{\partial x^j}\right) = \sum_{k=1}^{n} \Gamma_{ij}^k \frac{\partial}{\partial x^k}.$$

Then the Γ_{ij}^k are smooth functions called the *connection coefficients*, and one can prove (Proposition 4.6 in [Lee18]) that for $X, Y \in \mathfrak{X}(M)$,

$$\nabla_X Y = \sum_{i,j,k=1}^{n} (L_X(Y^k) + X^i Y^j \Gamma_{ij}^k)\frac{\partial}{\partial x^k}.$$

This gives a convenient way to define the derivative of a vector fields along a curve. Indeed, for a curve $\gamma : I \to M$, a *vector field along* γ is a continuous map $V : I \to TM$ such that $V(t) \in T_{\gamma(t)}M$ for all $t \in I$. Let $\mathfrak{X}(\gamma)$ denote the set of such maps. For $V \in X(\gamma)$, define $\nabla_{\dot\gamma} V$ locally by

$$(\nabla_{\dot\gamma} V)(t) = \sum_{i,j,k=1}^{n} \left[(Y^k \circ \gamma)'(t) + (\dot\gamma(t))^i (V(t))^j \Gamma_{ij}^k(\gamma(t))\right] \frac{\partial}{\partial x^k}\bigg|_{\gamma(t)}.$$

Geodesics

The derivative of vector fields along curves allows us to make the following definition.

Definition 2.4.5. A curve γ is a *geodesic* if $\nabla_{\dot\gamma}\dot\gamma = 0$. A geodesic $\gamma : I \to M$ is called *unit-speed* if $\|\dot\gamma(t)\| = 1$ for all $t \in I$. It is called *maximal* if it cannot be extended to a geodesic on a larger interval $J \supset I$.

In Chapter 6 of [Lee18] it is shown that if $\gamma : [a, b] \to M$ is a unit-speed *minimising curve* i.e. it is such that $d(\gamma(a), \gamma(b)) = \text{length}(\gamma)$, then γ is a geodesic. Conversely any geodesic is locally distance minimising. Existence of geodesics is given by the following (Corollary 4.28 in [Lee18]).

Theorem 2.4.6. *For every $p \in M$ and $v \in T_pM$ there is a unique maximal geodesic $\gamma : I \to M$, satisfying $0 \in I$, $\gamma(0) = p$ and $\dot{\gamma}(t) = v$.*

We say that M is *geodesically complete* if each maximal geodesic is defined on the whole of $\mathbb{R}$. The following theorem says that this coincides with the usual notion of completeness in metric spaces.

Theorem 2.4.7 (Hopf-Rinow)**.** *A Riemannian manifold (M, ρ) is complete with respect to the Riemannain distance if and only if it is geodesically complete.*

Remark. In particular, compact Riemannian manifolds are geodesically complete.

Lemma 6.18 in [Lee18] further shows that in a geodesically complete manifold there is a minimising geodesic between any two points.

2.5 Linking numbers and forms

Let M be a closed oriented 3-manifold. A knot is an embedding $\gamma : S^1 \to M$. It will be convenient to use the symbol γ for the embedding itself, but also its image. In the classical situation, where $M = S^3$, we may define the *linking number* of any two disjoint knots as follows.

Definition 2.5.1. Let γ, γ' be disjoint knots in S^3, and S_γ an oriented surface in S^3 whose boundary is γ. We define the linking number $\operatorname{lk}(\gamma, \gamma') \in \mathbb{Z}$ as the algebraic intersection number of γ' with this S_γ.

Remark. The intersection number above is not the number of intersection points $x \in \gamma' \cap S_\gamma$. In fact, each $x \in \gamma' \cap S_\gamma$ which is a transversal intersection point is assigned $+1$ or -1 depending on the relative orientation at x. The algebraic intersection number is the sum of these values, across all such x.

In this setting, with S^3 thought of as the compactification of $\mathbb{R}^3$, the linking number is also given by the *Gauss linking integral*,

$$\operatorname{lk}(\gamma, \gamma') = \frac{3}{4\pi} \int_{S^1} \int_{S^1} \frac{\dot{\gamma}(s) \times \dot{\gamma}'(t)}{\|\gamma(s) - \gamma'(t)\|^3} \cdot (\gamma(s) - \gamma'(t))\, ds\, dt.$$

For details and basic properties concerning these definitions, see Chapter 5, part D of [Rol76].

In the general setting of closed oriented 3-manifolds M, a similar definition of linking number can be given provided at least one of γ, γ' is null-homologous.

Definition 2.5.2. Let γ, γ' be disjoint knots in M with γ rationally null-homologous, i.e. $[\gamma] = 0 \in H_1(M, \mathbb{Q})$. There exists an integer $k \geq 1$ such that γ^k is integrally null-homologous, so there is an oriented surface $S_{\gamma^k} \subset M$ whose boundary is γ^k. We define $\mathrm{lk}(\gamma^k, \gamma') \in \mathbb{Z}$ as above, to be the algebraic intersection number of γ' and S_{γ^k}. With the minimal such k, define

$$\mathrm{lk}(\gamma, \gamma') = \frac{1}{k}\mathrm{lk}(\gamma^k, \gamma') \in \mathbb{Q}.$$

Remark. The condition $[\gamma] = 0 \in H_1(M, \mathbb{Q})$ is equivalent to $[\gamma] = 0 \in H_1(M, \mathbb{R})$. If we make the stronger assumption that $[\gamma] = 0 \in H_1(M, \mathbb{Z})$, then γ itself bounds a surface in M and so the linking number is an integer.

Summarising, knots γ, γ' in M have a well-defined rational linking number if they are disjoint and one of them is rationally null-homologous. In particular, if M is a rational homology 3-sphere, meaning it has the same rational homology groups as S^3, then this holds for each pair of disjoint knots.

It is possible to generalise the Gauss linking integral from S^3 to other 3-manifolds. To do this we introduce *linking forms.*

Let M be a closed oriented 3-manifold. We define a *double form* on M (see Section 7 of [Rha84]). This is essentially a differential form whose coefficients are other differential forms, rather than smooth functions.

Definition 2.5.3. Let x^1, x^2, x^3 be local co-ordinates in $U \subset M$, and y^1, y^2, y^3 in $U' \subset M$. A differential form α of degree p is represented for $x \in U$ by

$$\alpha(x) = \sum_{i_1 < \ldots < i_p} \alpha_{i_1 \ldots i_p}(x)\, dx^{i_1} \wedge \cdots \wedge dx^{i_p},$$

where each $\alpha_{i_1 \ldots i_p}$ is a smooth function. If, instead, $\alpha_{i_1 \ldots i_p}(x)$ is a differential form of degree q, it can be represented for $y \in U'$ by

$$\alpha_{i_1, \ldots i_p}(x)(y) = \sum_{j_1 < \ldots < j_q} a_{i_1 \ldots i_p j_1 \ldots j_q}(x, y)\, dy^{j_1} \wedge \cdots \wedge\, dy^{j_q},$$

where the $a_{i_1 \ldots i_p j_1 \ldots j_q}$ are smooth functions $U \times U' \to \mathbb{R}$. When defined in this way, we call α a (p, q)-form, and write

$$\alpha(x, y) = \sum_{\substack{i_1 < \ldots < i_p \\ j_1 < \ldots < j_q}} a_{i_1 \ldots i_p j_1 \ldots j_q}(x, y)(dx^{i_1} \wedge \cdots \wedge dx^{i_p})(dy^{j_1} \wedge \cdots \wedge dy^{j_q}).$$

We will mainly work with $(1,1)$-forms α, written as

$$\alpha(x,y) = \sum_i \alpha_i(x)(y)\,dx^i = \sum_{ij} a_{ij}(x,y)\,dx^i\,dy^j.$$

Here, for each x, $\alpha_i(x)(\cdot)$ is a 1-form. Letting $\Omega^k(M)$ denote the vector space of k-forms on M, any operator $F : \Omega^1(M) \to \Omega^k(M)$ has a partial action on the second co-ordinate of α, by fixing x and acting on $\alpha_i(x)(\cdot)$. We will use the notation $F_y\alpha$ to denote the resulting $(1,k)$-form. Likewise, writing

$$\alpha(x,y) = \sum_j \beta_j(x)(y)\,dy^j,$$

we obtain a partial action F_x, acting on the $\beta_j(\cdot)(y)$ and giving a $(k,1)$-form.

Similarly, $(1,1)$-forms may be integrated by integrating as single 1-forms in x and then in y. By this, we mean that for fixed y, and curves

$$c_1 : [a_1, b_1] \to M$$

$$c_2 : [a_2, b_2] \to M,$$

$\int_{x\in c_1} \alpha(x,y)$ is the representation of a 1-form at y, which can itself be integrated as usual. The integral of α over $c_1 \times c_2$ is therefore defined by

$$\begin{aligned}\int_{c_1\times c_2} \alpha &= \int_{y\in c_2}\int_{x\in c_1} \alpha(x,y)\\ &= \int_{a_2}^{b_2}\int_{a_1}^{b_1} \alpha(c_1(s), c_2(t))(\dot{c_1}(s), \dot{c_2}(t))\,ds\,dt.\end{aligned}$$

This allows us to define a linking form.

Definition 2.5.4. A linking form is a $(1,1)$-form L on M satisfying that whenever γ, γ' are disjoint knots, one of which is null-homologous,

$$\int_{\gamma\times\gamma'} L = \mathrm{lk}(\gamma,\gamma').$$

Such forms exist for M, and one class of examples is given in [KV03] (generalising [Vog02] for rational homology 3-spheres). To outline their construction, we give a description of the *Green operator* on M, and an associated $(1,1)$-form, known as the *Green kernel.* For this we assume M is equipped with a Riemannian metric ρ.

Recall that the Laplacian operator $\Delta : \Omega^1(M) \to \Omega^1(M)$ is defined by

$$\Delta = d^*d + dd^*,$$

where d is the exterior derivative and d^* is the adjoint of d. Let H denote the orthogonal projection of $\Omega^1(M)$ onto the space of harmonic 1-forms, $\Omega^1_{\text{harm}}(M) = \ker(\Delta)$. (Strictly speaking, one should complete $\Omega^1(M)$ with respect to the inner product to get a Hilbert space, in which case d and Δ become densely defined.)

Definition 2.5.5. The Green operator $G : \Omega^1(M) \to (\Omega^1_{\text{harm}}(M))^{\perp}$ is the unique operator satisfying $G \circ \Delta = \Delta \circ G = \mathrm{Id} - H$ and $G \circ H = 0$.

The existence of G is shown on page 134 of [Rha84]. On $(\Omega^1_{\text{harm}}(M))^{\perp}$, the Green operator acts as an inverse to Δ.

Given an operator $F : \Omega^1(M) \to \Omega^1(M)$, the *kernel* of F (in the sense of [Rha84], Section 17) is a $(1,1)$-form $\alpha(x,y)$ satisfying

$$F(\omega)(x) = \int_{x \in M} \omega(y) \wedge \alpha(x,y)$$

for all $\omega \in \Omega^1(M)$. The Green operator G has such a kernel, which we will denote with $g(x,y)$.

Before we may describe the linking form, we define the *Hodge dual*. This is a map $* : \Omega^k(M) \to \Omega^{3-k}(M)$ for each $k \in \{0,1,2,3\}$, defined uniquely by

$$\alpha \wedge *\beta = \rho(\alpha,\beta)\Omega_\rho,$$

where Ω_ρ is the volume form for ρ, and the inner product of k-forms $\rho(\alpha,\beta)$ is defined in Section 2.4.

The linking form, which we shall henceforth refer to as the *Kotschick–Vogel linking form*, is the $(1,1)$-form

$$L(x,y) := *_y d_y g(x,y),$$

The following appears as Proposition 1 in [KV03].

Proposition 2.5.6 (Kotschick and Vogel, [KV03])**.** *The double form $L(x,y)$ is a linking form. Furthermore, for every 1-form α, there exists a function $h : M \to \mathbb{R}$ such that*

$$\int_{y \in M} L(x,y) \wedge d\alpha(y) = \alpha(x) - H(\alpha)(x) + dh(x).$$

Chapter 3

Dynamical systems preliminaries

In this chapter we will outline some basic dynamical systems and ergodic theory. We will give a general discussion of both discrete-time and continuous-time systems, as well as the definitions of and properties of entropy and pressure, used regularly throughout this thesis.

3.1 Discrete-time systems

In this work a discrete dynamical system will consist of a pair (Y, T) where Y is a compact metric space with distance function d, and $T : Y \to Y$ is a continuous transformation. We will consider iterates of T, denoted for $n \in \mathbb{N}$ by

$$T^n = \underbrace{T \circ \cdots \circ T}_{n \text{ times}}.$$

For a point $x \in Y$, define the set $O^+(x) := \{T^n(x) : n \in \mathbb{N}\}$, the *(forward) orbit* of x. If T is invertible, define the *(full) orbit* $O(x) := \{T^n(x) : n \in \mathbb{Z}\}$, where for $n > 0$, $T^{-n} = (T^{-1})^n$.

Topological dynamics

We introduce some topological notions for dynamical systems which will later be seen to have analogues in the ergodic theory setting.

Definition 3.1.1. $T : Y \to Y$ is:

- *transitive* if for all pairs of non-empty open sets $U, V \subset Y$, there exists $n \in \mathbb{N}$ such that $T^{-n}(U) \cap V \neq \varnothing$.
- *topologically mixing* if for all pairs of non-empty open sets $U, V \subset Y$, there exists $N \in \mathbb{N}$ such that $T^{-n}(U) \cap V \neq \varnothing$ for all $n \geq N$.
- *expansive* if there exists $\delta > 0$ such that if $x, y \in Y$ satisfy $d(T^n(x), T^n(y)) < \delta$ for all $n \in \mathbb{N}$, then x and y are on the same orbit.

One can show that (Y, T) is transitive if and only if there is a point with a dense forward orbit. In fact, it is equivalent that $\{x \in Y : \overline{O^+(x)} = Y\}$ is a countable intersection of open dense sets. In this case, Baire Category Theorem gives that $\{x \in Y : \overline{O^+(x)} = Y\}$ is dense.

We now define an important quantity associated to (Y, T) that will be used throughout this work; the topological entropy.

For $n \in \mathbb{N}$ and $\varepsilon > 0$, we say that $E \subset Y$ is (n, ε)-*separated* if for any distinct $x, y \in E$ there exists $0 \leq k \leq n$ such that $d(T^k(x), T^k(y)) \geq \varepsilon$. Let $N(n, \varepsilon)$ be the maximal cardinality of such a set. In the proof of Lemma 3.1.3 below, we will see that $N(n, \varepsilon)$ always exists, by compactness.

Definition 3.1.2. The *topological entropy* of T is defined by

$$h(T) := \lim_{\varepsilon \to 0} \limsup_{t \to \infty} \frac{\log N(n, \varepsilon)}{n}.$$

An alternative definition is via *spanning sets*. First, define a *Bowen ball* around $x \in Y$ by

$$B(x, \varepsilon, n) := \{y \in Y : d(T^k(x), T^k(y)) < \delta \text{ for all } 0 \leq k < n\}.$$

We say that $E \subset Y$ is (n, ε)-*spanning* if $Y = \bigcup_{x \in E} B(x, \varepsilon, n)$. Let $S(n, \varepsilon)$ be the minimal cardinality of such a set.

Lemma 3.1.3. *The topological entropy $h(T)$ satisfies*

$$h(T) = \lim_{\varepsilon \to 0} \limsup_{t \to \infty} \frac{\log S(n, \varepsilon)}{n}.$$

Proof. It is enough to show that for all $\varepsilon > 0$, $S(n, \varepsilon)$ is finite and

$$S(n, \varepsilon) \leq N(n, \varepsilon) \leq S(n, \tfrac{\varepsilon}{2}).$$

The Bowen balls $\{B(x, \varepsilon, n) : x \in Y\}$ form an open cover of Y. Hence, by compactness, there is a finite subcover, and $S(n, \varepsilon)$ is finite.

For any points $y, y' \in B(x, \frac{\varepsilon}{2}, n)$, we have $d(T^k(y), T^k(y')) < \varepsilon$ for all $0 \leq k \leq n$. This means any (n, ε)-separated set can have at most $S(n, \frac{\varepsilon}{2})$ elements. On the other hand, if E is maximally (n, ε)-separated, it is (n, ε)-spanning, so has at least $S(n, \varepsilon)$ elements. $\square$

Ergodic theory

Ergodic theory is the study of invariant measures for dynamical systems. Here we define mixing properties and entropy with respect to a measure.

Let $(Y, \mathscr{A}, \mu)$ be a probability space, and $T : Y \to Y$ a measurable transformation. We say μ is T-*invariant* (or T preserves μ) if $T_*\mu = \mu$, i.e. $\mu(T^{-1}A) = \mu(A)$ for all $A \in \mathscr{A}$.

Definition 3.1.4. Let μ be T-invariant. We say T (or μ) is:

- *ergodic* if whenever $A \in \mathscr{A}$ is such that $T^{-1}A = A$, we have $\mu(A) \in \{0, 1\}$.
- *weak-mixing* if for all $A, B \in \mathscr{A}$,

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} |\mu(T^{-k}A \cap B) - \mu(A)\mu(B)| = 0.$$

- *mixing* if for all $A, B \in \mathscr{A}$,

$$\lim_{n \to \infty} \mu(T^{-n}A \cap B) = \mu(A)\mu(B).$$

In general, the notions of mixing and topological mixing are not related. From the definitions, we can deduce that if T is mixing it is weak-mixing, and if T is weak-mixing it is ergodic.

Let us now state a fundamental result in ergodic theory, Birkhoff's Ergodic Theorem.

Theorem 3.1.5. *Suppose μ is T-invariant and $f \in L^1(m)$. There exists $\hat{f} \in L^1(m)$ such that for μ-a.e. $x \in Y$,*

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} f(T^i(x)) = \hat{f}(x).$$

Moreover, $\hat{f} \circ T = \hat{f}$ μ-a.e. and $\int \hat{f}\, d\mu = \int f\, d\mu$. If μ is ergodic then $\hat{f}$ is constant μ-a.e., so that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} f(T^i(x)) = \int f\, d\mu.$$

Now we will define the measure-theoretic entropy of the map T, with respect to an invariant measure μ.

Let $\mathscr{B}$ be a sub-σ-algebra of $\mathscr{A}$. For any $\varphi \in L^1(Y, \mathscr{A}, \mu)$, define the *conditional expectation* of φ with respect to $\mathscr{B}$ as the function $E_\mu(\varphi|\mathscr{B}) \in L^1(Y, \mathscr{B}, \mu)$ satisfying

$$\int_B E_\mu(\varphi|\mathscr{B})\, d\mu = \int_B \varphi\, d\mu,$$

for all $B \in \mathscr{B}$.

Proposition 3.1.6. *$E_\mu(\varphi|\mathscr{B})$ exists and is unique μ-a.e.*

Proof. Define measures on $\mathscr{B}$ by $\nu^+(B) = \int_B \varphi^+\, d\mu$ and $\nu^-(B) = \int_B \varphi^-\, d\mu$ for each $B \in \mathscr{B}$. Here $\varphi^+(x) = \max\{\varphi(x), 0\}$ and $\varphi^-(x) = -\min\{\varphi(x), 0\}$, so that $\varphi = \varphi^+ - \varphi^-$. Then ν^+, ν^- are absolutely continuous with respect to μ, and thus there exists Radon-Nikodým derivatives $\frac{d\nu^+}{d\mu}, \frac{d\nu^+}{d\mu} \in L^1(Y, \mathscr{B}, \mu)$, which by definition satisfy

$$\int_B \frac{d\nu^+}{d\mu} - \frac{d\nu^-}{d\mu}\, d\mu = \nu(B) = \int_B \varphi\, d\mu$$

for all $B \in \mathscr{B}$. This justifies existence of conditional expectation.

For a.e. uniqueness, suppose both f and g satisfy the desired properties. Then for each $\varepsilon > 0$, $B_\varepsilon = (f-g)^{-1}(\varepsilon, \infty) \in \mathscr{B}$, and

$$0 = \int_{B_\varepsilon} f - g\, d\mu \geq \varepsilon \mu(B_\varepsilon).$$

Thus $\mu(B_\varepsilon) = 0$. Now,

$$\mu(\{x \in Y : f(x) > g(x)\}) = \mu\left(\bigcup_{n=1}^{\infty} B_{\frac{1}{n}}\right) \leq \sum_{n=1}^{\infty} \mu\left(B_{\frac{1}{n}}\right) = 0.$$

Similarly $\mu(\{x \in Y : f(x) < g(x)\}) = 0$, as required. □

Entropy will be defined through countable partitions of Y. Let $\xi \subset \mathscr{A}$ be such a partition. We define the *conditional information* of ξ with respect to $\mathscr{B}$ by

$$I_\mu(\xi|\mathscr{B}) := -\sum_{V\in\xi} \chi_V \log E_\mu(\chi_V|\mathscr{B}),$$

where we use the convention that $x \log x = 0$ at $x = 0$. Further, we define the *conditional entropy* of ξ with respect to $\mathscr{B}$ by

$$H_\mu(\xi|\mathscr{B}) := \int I_\mu(\xi|\mathscr{B})\, d\mu = -\int \sum_{V\in\xi} E_\mu(\chi_V|\mathscr{B}) \log E_\mu(\chi_V|\mathscr{B})\, d\mu.$$

The partition $\bigvee_{i=0}^{\infty} T^{-i}\xi$ generates a sub-σ-algebra, $\mathscr{C}$. We define the *entropy with respect to* ξ of T by

$$h_\mu(T,\xi) := H_\mu(\xi|T^{-1}\mathscr{C}).$$

Definition 3.1.7. The *measure-theoretic entropy* of T with respect to μ is defined by $h_\mu(T) := \sup\{h_\mu(T,\xi) : \xi \text{ is an } \mathscr{A}\text{-measurable partition}\}$.

In the case where Y is a compact metric space, $\mathscr{A}$ is the Borel σ-algebra, and T is continuous, we can relate the measure-theoretic and topological entropy. First, let $\mathcal{M}(T)$ be the collection of T-invariant Borel probability measures on Y. With the weak* topology, $\mathcal{M}(T)$ is compact, convex, and the ergodic measures are exactly the extremal points of $\mathcal{M}(T)$ (see Chapter 6 in [Wal81]). The following relation is proved as Theorem 8.6 of [Wal81].

Theorem 3.1.8 (Variational Principle)**.**

$$h(T) = \sup\{h_\mu(T) : \mu \in \mathcal{M}(T)\}.$$

A measure for which the supremum in Theorem 3.1.8 is realised is called a *measure of maximal entropy.*

We now define a generalisation of entropy which takes into account a continuous weight function. For $n \in \mathbb{N}$, $\varepsilon > 0$, and $f \in C(Y,\mathbb{R})$, define

$$N(f,n,\varepsilon) := \sup\left\{\sum_{x\in E} \exp \sum_{k=0}^{n-1} f(T^k(x)) : E \text{ is } (n,\varepsilon)\text{-separated}\right\}.$$

Definition 3.1.9. The *(topological) pressure* of f is defined by

$$P(f) := \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{\log N(f,n,\varepsilon)}{n}.$$

The function f is often referred to as a *potential.* Note that setting $f = 0$ gives the entropy. When f is sufficiently regular, we also have a characterisation of pressure via spanning sets. Precisely, we require that f satisfies the *Bowen property*, which is that there exist $C, \delta > 0$ such that whenever $y \in B(x, \delta, n)$,

$$\left| \sum_{k=0}^{n-1} f(T^k(x)) - \sum_{k=0}^{n-1} f(T^k(y)) \right| < C.$$

Later we will see that for the main dynamical systems we consider, any Hölder continuous f satisfies the Bowen property.

To define pressure by spanning sets, let

$$S(f, n, \varepsilon) := \inf \left\{ \sum_{x \in E} \exp \sum_{k=0}^{n-1} f(T^k(x)) : E \text{ is } (n, \varepsilon)\text{-spanning} \right\}.$$

Lemma 3.1.10. *Suppose f satisfies the Bowen property. Then*

$$P(f) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{\log S(f, n, \varepsilon)}{n}.$$

Proof. This proof is similar to that for Lemma 3.1.3 above, with modifications according to f. First, $S(f, n, \varepsilon)$ is finite since f is bounded and $S(0, n, \varepsilon)$ is finite.

Let δ be sufficient for the Bowen property for f and suppose $\varepsilon < \delta$. If E is (n, ε)-separated and U is $(n, \frac{\varepsilon}{2})$-spanning, then each Bowen ball $\{B(y, \frac{\varepsilon}{2}, n) : y \in U\}$ contains at most one element of E. Thus for $x \in E$, letting y_x denote a point in U such that $x \in B(y_x, \frac{\varepsilon}{2}, n)$, we have $y_x = y_{x'}$ only when $x = x'$. This means

$$e^{-C} \sum_{x \in E} \exp \sum_{k=0}^{n-1} f(T^k(x)) < \sum_{x \in E} \exp \sum_{k=0}^{n-1} f(T^k(y_x)) < \sum_{y \in U} \exp \sum_{k=0}^{n-1} f(T^k(y)),$$

allowing us to conclude that $N(\varphi, n, \varepsilon) \leq e^C S(\varphi, n, \frac{\varepsilon}{2})$. Since an (n, ε)-separated set of maximal cardinality is also spanning, we also obtain $S(\varphi, n, \varepsilon) \leq N(\varphi, n, \varepsilon)$. □

We also have a variational principle for pressure (see Theorem 9.10 in [Wal81]). This involves the quantity $h_\mu(T) + \int f \, d\mu$, for $\mu \in \mathcal{M}(T)$, which is sometimes referred to as the *free energy.*

Theorem 3.1.11. *Let $f \in C(Y, \mathbb{R})$. Then*

$$P(f) = \sup \left\{ h_\mu(T) + \int f \, d\mu \, : \, \mu \in \mathcal{M}(T) \right\}.$$

With this general version of the variational principle, we may define measures in $\mathcal{M}(T)$ which will be used throughout this work.

Definition 3.1.12. A measure $\mu \in \mathcal{M}(T)$ is an *equilibrium state* for f if $P(f) = h_\mu(T) + \int f \, d\mu$.

3.2 Continuous-time systems

We now introduce *flows*, dynamical systems parameterised by continuous time. We continue in the setting of a compact metric space Y. A flow is a continuous map $\varphi : \mathbb{R} \times Y \to Y$ which satisfies that for all $x \in Y$ and $s, t \in \mathbb{R}$,

$$\varphi(0, x) = x \text{ and } \varphi(s + t, x) = \varphi(s, \varphi(t, x)).$$

We will often think of φ as a family of transformations $\{\varphi_t : Y \to Y : t \in \mathbb{R}\}$, where φ_t denotes the map $\varphi(t, \cdot)$. We may abuse notation and use the symbol φ_t to refer to the entire family $\{\varphi_t\}_{t\in\mathbb{R}}$. Throughout this work, Y will often have the structure of a smooth manifold, in which case we say the flow is C^r if the map φ is C^r. In this case, we may change notation to indicate the flow is generated by a vector field, as in Section 2.2. Precisely, suppose Y has a smooth structure and let $X : Y \to TY$ be a C^1 vector field, where TY is the tangent bundle to Y. In this case, for each $x_0 \in Y$, the differential equation

$$\begin{aligned} \dot{x}(t) &= X(x(t)) \\ x(0) &= x_0 \end{aligned}$$

has a solution. These solutions define a flow $\varphi_t(x_0) = x(t)$, which we will denote by $X^t = \varphi_t$. In this case, either X or X^t may be used to denote the entire family of maps $\{X^t\}_{t\in\mathbb{R}}$.

We return to the general setting to discuss mixing properties. For a point $x \in Y$, define the set $O^+(x) := \{\varphi_t(x) : t \geq 0\}$, the *(forward) orbit* of x. Also define the *(full) orbit* $O(x) := \{\varphi_t(x) : t \in \mathbb{R}\}$.

Topological dynamics

Definition 3.2.1. We say that (Y, φ) is:

- *transitive* if for all pairs of non-empty open sets $U, V \subset Y$, there exists $t \in \mathbb{R}$ such that $\varphi_t(U) \cap V \neq \varnothing$.
- *topologically weak-mixing* if the only $\psi \in C(Y, S^1)$, $a \in \mathbb{R}$ which solve the equation $\psi \circ \varphi_t = e^{iat}\psi$ for all $t \in \mathbb{R}$, are φ constant and $a = 0$.

- *topologically mixing* if for all pairs of non-empty open sets $U, V \subset Y$, there exists $T \in \mathbb{N}$ such that $\varphi_t(U) \cap V \neq \varnothing$ for all $t \geq T$.
- *expansive* if there exists $\delta > 0$ which satisfies: if there is a continuous function $s : \mathbb{R} \to \mathbb{R}$ with $s(0) = 0$ such that $x, y \in Y$ satisfy $d(\varphi_t(x), \varphi_{s(t)}(y)) < \delta$ for all $t \in \mathbb{R}$, then x and y are on the same orbit.

As in the discrete case, it follows that (Y, φ) is transitive if and only if $\{x \in Y : \overline{O^+(x)} = Y\}$ is a countable intersection of open dense sets. It is also the case that topological mixing implies topological weak-mixing.

We now define pressure for flows, with entropy as a special case. This is essentially the same as for discrete maps, but with time $t \in \mathbb{R}$ replacing $n \in \mathbb{N}$ in the definition of spanning and separated sets.

For $t, \varepsilon > 0$, we say that $E \subset Y$ is (t, ε)-separated if for any distinct $x, y \in E$ there exists $0 \leq s \leq t$ such that $d(\varphi_s(x), \varphi_s(y)) \geq \varepsilon$. Defining a *Bowen ball* around $x \in Y$ by

$$B(x, \varepsilon, t) := \{y \in M : d(\varphi_s(x), \varphi_t(y)) < \delta \text{ for all } 0 \leq s < t\},$$

we say that $E \subset Y$ is (t, ε)-spanning if $Y = \bigcup_{x \in E} B(x, \varepsilon, t)$. For a continuous potential $f \in C(Y, \mathbb{R})$, define

$$N(f, t, \varepsilon) := \sup \left\{ \sum_{x \in E} \exp \int_0^t f(\varphi_s(x))\, ds : E \text{ is } (t, \varepsilon)\text{-separated} \right\}.$$

Definition 3.2.2. The pressure of f is defined by

$$P(f) := \lim_{\varepsilon \to 0} \limsup_{t \to \infty} \frac{\log N(f, t, \varepsilon)}{t}.$$

The topological entropy of φ is defined by $h(\varphi) = P(0)$.

As in the discrete case, we have a characterisation via spanning sets, so long as f satisfies the Bowen property, which for flows means there exist $C, \delta > 0$ such that whenever $y \in B(x, \delta, t)$,

$$\left| \int_0^t f(\varphi_s(x))\, ds - \int_0^t f(\varphi_s(y))\, ds \right| < C.$$

So we may define

$$S(f,t,\varepsilon) := \inf\left\{\sum_{x\in E}\exp\int_0^t f(\varphi_s(x))\,ds : E \text{ is } (t,\varepsilon)\text{-spanning}\right\},$$

and conclude the following.

Lemma 3.2.3. *Suppose f satisfies the Bowen property. Then*

$$P(f) = \lim_{\varepsilon\to 0}\limsup_{t\to\infty}\frac{\log S(f,t,\varepsilon)}{t}.$$

Proof. This proof is that of Lemma 3.1.10, with obvious modifications. □

Ergodic theory

We now discuss ergodic properties of flows. Let $(Y,\mathscr{A},\mu)$ be a probability space, and $\varphi_t : Y \to Y$ a measurable flow, meaning $\varphi_t A \in \mathscr{A}$ for all $A \in \mathscr{A}$, $t \in \mathbb{R}$. We say μ is φ-invariant (or φ preserves μ) if $\mu(\varphi_t A) = \mu(A)$ for all $A \in \mathscr{A}$, $t \in \mathbb{R}$.

Definition 3.2.4. Let μ be φ-invariant. We say φ (or μ) is:

- *ergodic* if whenever $A \in \mathscr{A}$ is such that $\varphi_t A = A$ for all $t \in \mathbb{R}$, $\mu(A) \in \{0,1\}$.
- *weak-mixing* if for all $A, B \in \mathscr{A}$,

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t |\mu(\varphi_{-s}A\cap B) - \mu(A)\mu(B)|\,ds = 0.$$

- *mixing* if for all $A, B \in \mathscr{A}$,

$$\lim_{t\to\infty}\mu(\varphi_{-t}A\cap B) = \mu(A)\mu(B).$$

Again, ergodicity is implied by weak-mixing, which is implied by mixing. The ergodic theorem for flows is as follows.

Theorem 3.2.5. *Suppose μ is φ-invariant and $f \in L^1(m)$. There exists $\hat{f} \in L^1(m)$ such that for μ-a.e. $x \in Y$,*

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t f(\varphi^s(x))\,ds = \hat{f}(x).$$

Moreover, $\hat{f}\circ\varphi_t = \hat{f}$ μ-a.e. for each $t \in \mathbb{R}$, and $\int \hat{f}\,d\mu = \int f\,d\mu$. If μ is ergodic *then $\hat{f}$ is constant μ-a.e., so that*

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t f(\varphi^s(x))\,ds = \int f\,d\mu.$$

The measure-theoretic entropy of φ with respect to μ is denoted by $h_\mu(\varphi)$ and defined by $h_\mu(\varphi) = h_\mu(\varphi_1)$, where φ is the time 1 map, whose entropy is defined in Definition 3.1.7. We also have a variational principle for flows when Y is a compact metric space. First we discuss the space of invariant measures, using the same notation as in the discrete case. Let $\mathcal{M}(\varphi)$ be the collection of φ-invariant Borel probability measures on Y. Then $\mathcal{M}(\varphi)$ is convex, compact in the weak* topology, and ergodic measures are extremal points (see Theorem 3.1.16 and Proposition 3.3.26 in [FH19]).

Theorem 3.2.6. *Let $f \in C(Y, \mathbb{R})$. Then*

$$P(f) = \sup\left\{h_\mu(\varphi) + \int f\,d\mu \,:\, \mu \in \mathcal{M}(\varphi)\right\}.$$

For a proof of this see Theorem 4.3.8 in [FH19]. As before, a measure in $\mathcal{M}(\varphi)$ which realises the supremum above will be called an equilibrium state for f.

Suspensions

Given a discrete system (Y, T), we can obtain a flow through the process of *suspension.* Precisely, let $r : Y \to \mathbb{R}$ be a strictly positive function, and define

$$Y^r = {}^{(Y\times\mathbb{R})}\!\big/\!{}_{\sim},$$

where $\sim$ is the equivalence relation generated by asserting that $(x, r(x)) \sim (T(x), 0)$ for all $x \in Y$. Precisely, $(x, t) \sim (y, s)$ if there exists some $n \in \mathbb{N}$ such that

$$(y, s) = \left(T^n(x), t - \sum_{i=0}^{n-1} r(T^i(x))\right) \text{ or } (x, t) = \left(T^n(y), s - \sum_{i=0}^{n-1} r(T^i(x))\right).$$

A map $\varphi_t : Y^r \to Y^r$ can be defined on Y^r by $\varphi_t[x, s] = [x, s + t]$, where $[x, s]$ denotes the equivalence class of (x, s) in Y^r. If T is invertible, φ_t is a well defined flow, but otherwise φ_t can only be defined for non-negative t, in which case φ is called a *semi-flow.* We will later discuss suspensions in more detail, in the context of symbolic dynamics (Section 4.2).

Chapter 4

Thermodynamic formalism for hyperbolic flows

4.1 Hyperbolic flows

Hyperbolic flows are a type of continuous-time dynamical system defined to generalise geodesic flows over negatively curved Riemannian manifolds. As such, they display chaotic dynamical behaviour, leading to interesting topological, geometric and ergodic properties. The class of hyperbolic flows is also structurally stable, meaning hyperbolicity persists under small perturbations.

In the early 1970s, Bowen published two influential papers, [Bow72b] and [Bow73], on hyperbolic flows. The former is a study of periodic orbits and their relation to invariant measures, while the latter is a construction of a symbolic model for a hyperbolic flow, similar to that independently found by Ratner [Rat73]. In this section, we define hyperbolic flows and give some basic properties.

4.1.1 Geodesic flows

As the prototypical example in the study of hyperbolic flows, we begin by discussing *geodesic flows* in negative curvature. They are defined as follows.

Let N be a connected Riemannian manifold, $M = T^1N$, and $\pi : M \to N$ the map projecting a tangent vector to its foot point. For any $v \in M$, there is a unique unit-speed geodesic path $c_v : \mathbb{R} \to N$ satisfying $c_v(0) = \pi(v)$ and $\dot{c_v}(0) = v$. Defining $\varphi : \mathbb{R} \times M \to M$ by $\varphi_t(v) = \dot{c_v}(t)$, we see that φ is a flow, called the geodesic flow (over N). There are two examples of 3-dimensional geodesic flows which will be of particular interest in this thesis.

Example 4.1.1. A natural starting point for studying geometry of negatively curved manifolds is compact hyperbolic surfaces. If N is such a surface, it can be realised as the quotient of hyperbolic space $\mathbb{H}^2$ by a Fuchsian group $\Gamma < \mathrm{PSL}_2(\mathbb{R})$. In this case, T^1N can be identified with $\mathrm{PSL}_2(\mathbb{R})/\Gamma$, and the geodesic flow over N (see Figure 4.1) provides a tangible example of an Anosov flow (defined below) on a compact 3-manifold.

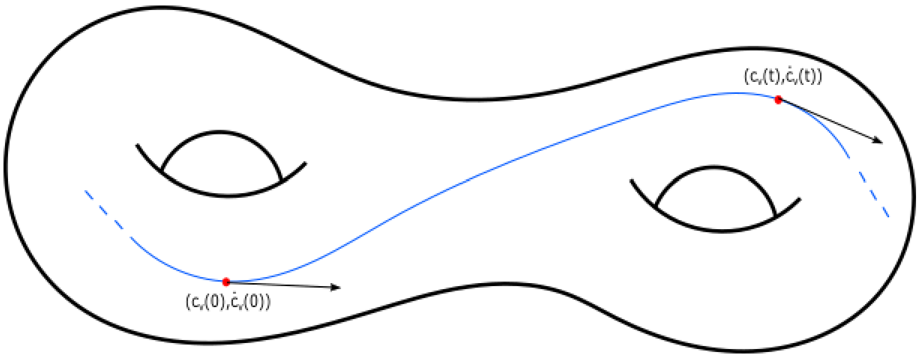

Figure 4.1: The geodesic flow over a surface of genus 2.

Example 4.1.2. This example is not strictly a geodesic flow, but is constructed from a geodesic flow and satisfies the Anosov property, which is defined in the next section.

Let $\Gamma < \mathrm{PSL}_2(\mathbb{R})$ be a Fuchsian group acting freely and properly discontinuously on the hyperbolic plane $\mathbb{H}^2$, such that $S = \mathbb{H}^2/\Gamma$ is a hyperbolic 2-orbifold of genus zero. For example, we may take Γ to be the group generated by the maps which identify the sides of a geodesic quadrilateral in the Poincaré disc, as in Figure 4.2. By Theorem 13.3.6 in [Thu80], the number of cone points p of S satisfies $p \geq 5$, or $p = 4$ and the orders are not all 2, or $p = 3$ and the orders satisfy that the sum of their reciprocals is smaller than 1 (the latter case is that in Figure 4.2). Let $M = \mathrm{PSL}_2(\mathbb{R})/\Gamma$ and $\varphi_t : M \to M$ be the flow given by the quotient of the geodesic flow over $\mathbb{H}^2$. The flow φ will be our main example of an Anosov flow on a rational homology 3-sphere.

Historically, geodesic flows in negative curvature have been among the first cases where chaotic properties, both topological and ergodic, are observed. The study of these systems began with Hadamard [Had98], who considered non-compact negatively curved surfaces in $\mathbb{R}^3$. Hadamard showed that these flows have complicated orbits which resemble the chaotic phenomena earlier studied by Poincaré in celestial mechanics. Later, Artin [Art24] studied the geodesic flow over the modular surface $\mathbb{H}^2/\mathrm{PSL}_2(\mathbb{Z})$, which led to many results for the more general case of

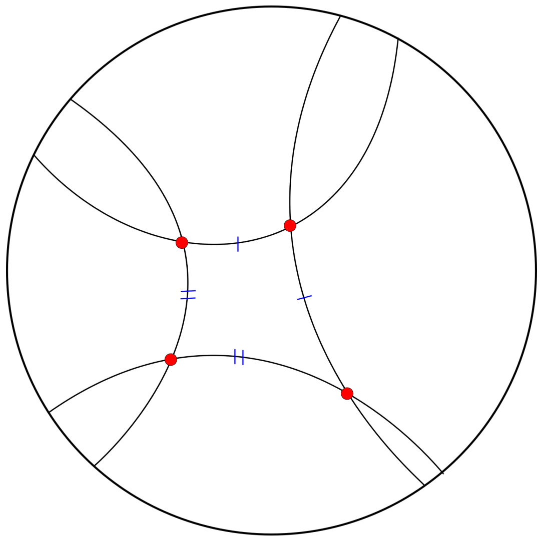

Figure 4.2: A quadrilateral in hyperbolic space.

constant negative curvature. For example, Hedlund showed that constant negative curvature geodesic flows are mixing with respect to the Liouville measure [Hed39], and topologically mixing [Hed36]. They were also shown to be topologically transitive [Löb29]. Shortly after, Hopf [Hop39] showed that in the case of compact surfaces of variable negative curvature, the Liouville measure is ergodic. The method used by Hopf (called the Hopf argument) is still a common approach to proving ergodicity for systems with hyperbolicity.

We will now consider two generalisations of geodesic flows on negatively curved spaces, both formulated in the 1960s. We first discuss Anosov flows, defined in [Ano67], which turn out to be a special case of Smale's Axiom A flows [Sma67], which we define afterwards.

4.1.2 Anosov flows

For this section and the next, let (M, ρ) be a smooth closed connected Riemannian manifold of dimension at least 3. A C^1 flow $X^t : M \to M$ is *Anosov* if M is a *hyperbolic set*, defined as follows.

Definition 4.1.3. An X^t-invariant set $\Lambda \subset M$ is *hyperbolic* if there is a splitting of the tangent sub-bundle $T_\Lambda M$ into DX^t-invariant sub-bundles $T_\Lambda M = E^s \oplus E \oplus E^u$, where E is the one dimensional bundle generated by X, and there exist positive constants C and λ such that

1. $\|DX^t(v)\| \leq Ce^{-\lambda t}\|v\|$ for all $v \in E^s$, $t \geq 0$.

2. $\|DX^{-t}(v)\| \leq Ce^{-\lambda t}\|v\|$ for all $v \in E^u$, $t \geq 0$.

E^s will be called the *stable bundle*, and E^u the *unstable bundle.*

One implication of hyperbolicity is expansivity, as follows (see Corollary 5.3.5 in [FH19]).

Lemma 4.1.4. *Let X^t be a flow restricted to a hyperbolic set. Then X^t is expansive.*

As mentioned above, Anosov flows generalise geodesic flows in negative curvature. For a proof of the following, see Theorem 5.2.4 in [FH19].

Theorem 4.1.5. *Let N be a closed Riemannian manifold with negative sectional curvatures. Then the geodesic flow over N is Anosov.*

For other examples of Anosov flows, see Remark 5.1.3 in [FH19].

4.1.3 Axiom A flows

We continue with a C^1 flow $X^t : M \to M$. We will ask for hyperbolicity on a subset of M with recurrence properties.

Definition 4.1.6. A point $x \in M$ is called *wandering* if there exists a neighbourhood U of x such that for all sufficiently large t, $X^t(U) \cap U = \varnothing$. The *non-wandering set* $\Omega(X)$ is the complement of the set of wandering points.

Definition 4.1.7. X^t is an *Axiom A flow* if $\Omega(X)$ is hyperbolic, and the periodic orbits of $X|_{\Omega(X)}$ are dense in $\Omega(X)$. We may also say that X^t satisfies Axiom A.

These generalise Anosov flows in the following sense.

Lemma 4.1.8. *Suppose $X^t : M \to M$ is a transitive Anosov flow. Then X^t satisfies Axiom A.*

Proof. The transitivity assumption ensures $\Omega(X) = M$, so $\Omega(X)$ is hyperbolic. Further, the Anosov Closing Lemma ([Ano67], details in Section 5.3 of [FH19]) implies that segments of a dense orbit are closely shadowed by periodic orbits, meaning periodic orbits are dense, so X^t satisfies Axiom A. □

Typically, Axiom A flows display different behaviour on disjoint pieces of their non-wandering sets, each invariant under the flow. This is formulated precisely in a decomposition theorem due to Smale [Sma67], which can be stated after the following definitions.

Definition 4.1.9. A set $\Lambda \subset M$ is *basic* if it satisfies the following:

1. Λ is closed and X^t-invariant,
2. Λ is hyperbolic,
3. $X|_\Lambda$ is transitive,
4. There is an open set U with $\Lambda \subset U$ such that $\Lambda = \bigcap_{t\in\mathbb{R}} X^t(U)$,
5. The set of periodic orbits of $X|_\Lambda$ is dense in Λ, and Λ consists of more than a single periodic orbit.

Definition 4.1.10. A periodic orbit of period T is *hyperbolic* if at each point x in the orbit, DX^T_x has no eigenvalues of modulus 1.

Theorem 4.1.11 (Smale [Sma67], Theorem 6.2)**.** *Suppose that X satisfies Axiom A. Then $\Omega(X)$ is a finite disjoint union of basic sets and hyperbolic periodic orbits.*

Remark. For transitive Anosov flows, this decomposition consists solely of the basic set M.

With Theorem 4.1.11 and the invariance of basic sets, we can view X^t as several distinct systems; one for each basic set. This leads us to the following definition.

Definition 4.1.12. For an Axiom A flow X^t and a basic set Λ in the decomposition of $\Omega(X)$, the restricted flow $X^t|_\Lambda$ is called a *hyperbolic flow*. We will usually omit the explicit restriction, and use the notation X^t for both the Axiom A flow on M and the hyperbolic flow on Λ.

4.1.4 Invariant manifolds

For a hyperbolic flow X^t, the hyperbolicity condition on $T_\Lambda M$ has strong geometric implications for the orbits in Λ. This can be understood through the following submanifolds of M, which will be important later.

Definition 4.1.13. The *(strong) stable and unstable manifolds* of a point $x \in \Lambda$ are written as $W^s(x), W^u(x)$ respectively, and are defined by

$$W^s(x) := \{y \in \Lambda : d(X^t(y), X^t(x)) \to 0 \text{ as } t \to \infty\}$$
$$W^u(x) := \{y \in \Lambda : d(X^{-t}(y), X^{-t}(x)) \to 0 \text{ as } t \to \infty\}$$

For $\delta > 0$, the *local stable and unstable manifolds* are defined by

$$W^s_\delta(x) := \{y \in W^s(x) : d(X^t(y), X^t(x)) \leq \delta \text{ for all } t \geq 0\}$$
$$W^u_\delta(x) := \{y \in W^u(x) : d(X^{-t}(y), X^{-t}(x)) \leq \delta \text{ for all } t \geq 0\}$$

The following (see [HPS77]) measures the speed of convergence of orbits of points on shared stable/unstable manifolds. The exponential rate is a result of the hyperbolicity of the flow.

Lemma 4.1.14. *There exists $k, l > 0$ such that for sufficiently small δ*

1. $d(X^t(x), X^t(y)) \leq ke^{-lt}d(x, y)$ *for all* $y \in W^s_\delta(x)$, $t \geq 0$.
2. $d(X^{-t}(x), X^{-t}(y)) \leq ke^{-lt}d(x, y)$ *for all* $y \in W^u_\delta(x)$, $t \geq 0$.

4.2 Symbolic dynamics

For detail on this section, see Chapters 1 and 6 of [PP90].

4.2.1 Shift spaces

We first describe shifts of finite type, a family of discrete dynamical systems. These systems will be particularly useful in this thesis, since their suspensions act as a model for hyperbolic flows.

We consider finite directed graphs on $k \geq 2$ vertices, denoted $\{1, \dots, k\}$. For distinct vertices $i, j \in \{1, \dots, k\}$, ij will denote the directed edge from i to j. A directed graph Γ is then given by $E(\Gamma)$, the set of directed edges present in Γ. We do not consider graphs with multiple directed edges, meaning that any vertices i, j have at most two edges between them, and if there are exactly two, they have opposing directions.

Definition 4.2.1. Let Γ be a directed graph on $\{1, \dots, k\}$. The *shift space* over Γ is the collection of possible bi-infinite walks on Γ, defined by

$$\Sigma(\Gamma) := \{x = (x_n)_{n\in\mathbb{Z}} \in \{1, \dots, k\}^{\mathbb{Z}} : x_n x_{n+1} \in E(\Gamma) \text{ for all } n\}.$$

The *(left) shift map* is $\sigma : \Sigma(\Gamma) \to \Sigma(\Gamma)$ defined by $\sigma((x_n)_{n\in\mathbb{Z}}) = (x_{n+1})_{n\in\mathbb{Z}}$. Similarly, the *one-sided shift space* is defined by

$$\Sigma^+(\Gamma) := \{x = (x_n)_{n\in\mathbb{Z}} \in \{1, \dots, k\}^{\mathbb{N}} : x_n x_{n+1} \in E(\Gamma) \text{ for all } n\},$$

over which we abuse notation and define the shift map $\sigma((x_n)_{n\in\mathbb{N}}) = (x_{n+1})_{n\in\mathbb{N}}$.

These spaces can also be described via matrices. For a $k \times k$ matrix A, and $i, j \in \{1, \dots, k\}$ let A_{ij} denote the entry in row i and column j of A. To Γ, we can associate such an A, by setting

$$A_{ij} = \begin{cases} 1 & \text{if } ij \in E(\Gamma), \\ 0 & \text{otherwise.} \end{cases}$$

A is called the *transition matrix* of Γ, and satisfies

$$\Sigma(\Gamma) := \{(x_n)_{n\in\mathbb{Z}} \in \{1,\ldots,k\}^{\mathbb{Z}} : A_{x_n x_{n+1}} = 1 \text{ for all } n\}.$$

It may be convenient to switch between these descriptions of $\Sigma(\Gamma)$ (and of $\Sigma^+(\Gamma)$), and we will do so freely. We now introduce some useful notation for working in the shift space.

We will call $\{1,\ldots,k\}$ our set of *symbols*. A *word* (of length n) is a string of n symbols $x_0x_1\ldots x_{n-1}$ which is called *admissible* if $A_{x_m x_{m+1}} = 1$ for each $m \in \{0,\ldots,n-2\}$. In $\Sigma^+(\Gamma)$ we will use the notation $\overline{x_0x_1\ldots x_n}$ to denote the periodic sequence $(x_0, x_1, \ldots, x_n, x_0, x_1, \ldots, x_n, x_0, x_1, \ldots, x_n, \ldots)$. We will use the same notation for the analogous bi-infinite sequence in $\Sigma(\Gamma)$. Points of this form are exactly the periodic points of the shift map σ.

Both $\Sigma(\Gamma)$ and $\Sigma^+(\Gamma)$ can be given a metric structure in the following way. Fix $0 < \theta < 1$. For distinct $x, y \in \Sigma(\Gamma)$ set $d(x,y) = \theta^n$, where n is the maximal non-negative integer such that $x_i = y_i$ for all $|i| < n$. The metric on $\Sigma^+(\Gamma)$ is defined analogously, replacing the condition $|i| < n$ with $0 \le i < n$. We will use the notation $d(x,y)$ for both metrics. We may also use $d_\theta(x,y)$ for this metric, when the dependence on θ is important. To describe the open sets arising from d, we introduce some convenient notation.

Definition 4.2.2. A *cylinder set* in $\Sigma(\Gamma)$ is a set of sequences with common first entries, which will be defined by

$$[x_{-(n-1)}\ldots x_{-1}x_0x_1\ldots x_{n-1}] := \{y \in \Sigma(\Gamma) : y_i = x_i \text{ for all } |i| < n\}.$$

Similarly, in $\Sigma^+(\Gamma)$ a cylinder set is defined by

$$[x_0x_1\ldots x_{n-1}] := \{y \in \Sigma^+(\Gamma) : y_i = x_i \text{ for all } 0 \le i < n\}.$$

To be specific, we may call these cylinders of length n, even though in $\Sigma(\Gamma)$ they are described by $2n-1$ symbols.

One then sees that in a shift space, the open ball $B(x,\varepsilon)$ is the cylinder of length n containing x, for n satisfying $\theta^n < \varepsilon \le \theta^{n+1}$. As such, all cylinder sets are both open and closed in the shift space.

Lemma 4.2.3. $\Sigma(\Gamma)$ *and* $\Sigma^+(\Gamma)$ *are compact and totally disconnected.*

Proof. We prove the statement for $\Sigma^+(\Gamma)$; the necessary adaptations for $\Sigma(\Gamma)$ will be clear.

Due to our observation on open balls, to prove compactness it suffices to show that any cover of $\Sigma^+(\Gamma)$ by cylinders has a finite subcover. Indeed, let $\mathcal{C}$ be a collection of cylinders satisfying $\Sigma^+(\Gamma) \subset \bigcup_{C\in\mathcal{C}} C$. Choose a symbol $x_0 \in \{1,\dots,k\}$ such that infinitely many cylinders in $\mathcal{C}$ are needed to cover $[x_0]$. If there is no such symbol, we are done. Similarly, we may choose x_1 such that $[x_0x_1]$ cannot be finitely covered in $\mathcal{C}$. Continuing, we construct cylinders $C_n = [x_0x_1\dots x_n]$ which are not finitely coverable in $\mathcal{C}$. In particular $C_n \notin \mathcal{C}$ for any $n \in \mathbb{N}$. Let $x = (x_n)_{n=0}^{\infty}$. By construction $x \in \Sigma^+(\Gamma)$, so there exists $C \in \mathcal{C}$ with $x \in C$. Since C is a cylinder set, $C = C_n$ for some n, a contradiction.

For total disconnectedness, assume we have a non-empty connected set $A \subset \Sigma^+(\Gamma)$. For each $n \in \mathbb{N}$, A can be written as the disjoint union

$$A = \bigcup_{x_0\dots x_n=1}^{k} A \cap [x_0 \dots x_n].$$

Connectedness of A then implies $A \cap [x_0,\dots,x_n]$ is empty for all but one word, $x_0^A \dots x_n^A$. This is only possible if $A = \{(x_n^A)_{n=0}^{\infty}\}$, a singleton. □

We now describe how certain properties of Γ and A correspond to the topological dynamics of the shift map σ.

Definition 4.2.4. The matrix A is *irreducible* if for each pair $i, j \in \{1,\dots,k\}$, there exists $n(i,j) > 0$ such that $(A^{n(i,j)})_{ij} > 0$. It is *aperiodic* if there exists $n \in \mathbb{N}$ such that all entries of A^n are positive.

These conditions have an interpretation in terms of paths on Γ.

Proposition 4.2.5. *The transition matrix A of Γ is irreducible if and only if there is a directed path in Γ from i to j, for all $i, j \in \{1,\dots,k\}$. Further, A is aperiodic if and only if there are directed paths of uniform length from i to j, for all $i, j \in \{1,\dots,k\}$.*

Proof. We shall prove the stronger statement that for all $i, j \in \{1,\dots,k\}$, $(A^n)_{ij} = P(n,i,j)$, where $P(n,i,j)$ is the number of paths of length n from i to j in Γ. We proceed by induction on n. When $n = 1$, A_{ij} gives precisely the number of directed edges (directed paths of length 1) from i to j, so $P(1,i,j) = A_{ij}$ for all $i,j \in \{1,\ldots,k\}$. Now assume that $P(n-1,i,j) = (A^{n-1})_{ij}$. Then

$$P(n,i,j) = \sum_{l=1}^{k} P(n-1,i,l)P(1,l,j) = \sum_{l=1}^{k} (A^{n-1})_{il}A_{lj} = (A^n)_{ij},$$

completing the proof. □

When A is irreducible (resp. aperiodic), we may instead say Γ is irreducible (resp. aperiodic), or the associated shift space is irreducible (resp. aperiodic).

Proposition 4.2.6. *On $\Sigma(\Gamma)$ or $\Sigma^+(\Gamma)$, the shift map σ is continuous, and is a homeomorphism on the former. If Γ is irreducible, σ is topologically transitive, expansive, and periodic points are dense. If Γ is aperiodic, σ is topologically mixing.*

Proof. Continuity is clear from the definitions. As before, we prove the rest of the statement for $\Sigma^+(\Gamma)$, highlighting modifications for $\Sigma(\Gamma)$ if necessary.

Assume Γ is irreducible. We first show there is a dense orbit of σ (topological transitivity). In the case that Γ is a complete graph, it is easy to construct a point with dense orbit: let x be the sequence given by listing all words of length 1, followed by all words of length 2, followed by all words of length 3, and so on. This gives

$$x = (\underbrace{1,2\ldots,k}_{\text{length 1}},\underbrace{1,1,1,2,\ldots,1,k,2,1,\ldots,2,k,\ldots,k,1,\ldots k,k}_{\text{length 2}},1,1,1,\ldots).$$

Any word w appears in x, so there exists $n \in \mathbb{N}$ such that $\sigma^n(x) \in [w]$. Thus x has a dense orbit. If Γ is not complete, we list additional words to ensure x is admissible. Indeed, suppose we attempt to list all admissible words (of length 1, then length 2 and so on, as before).We may reach consecutive admissible words w_1 and w_2 such that w_1w_2 is not admissible. By irreducibility, there exists a word w such that w_1ww_2 is admissible, so in forming x we list in w_1ww_2 instead of w_1w_2. This ensures $x \in \Sigma^+(\Gamma)$ and each admissible word appears in x. So x has a dense orbit.

For expansivity, notice that if $x,y \in \Sigma^+(\Gamma)$ are distinct, and $d(x,y) \leq \theta$, then

$$d(\sigma(x),\sigma(y)) = \frac{d(x,y)}{\theta}.$$

Thus the orbits under σ of any two distinct points must at some point have distance strictly greater than θ^2. In $\Sigma(\Gamma)$, it is instead only true that

$$d(x,y) \leq \theta \implies d(\sigma(x),\sigma(y)) = \frac{d(x,y)}{\theta} \text{ or } d(\sigma^{-1}(x),\sigma^{-1}(y)) = \frac{d(x,y)}{\theta},$$

meaning the orbits under either σ or σ^{-1} of distinct points are at some point further than θ^2.

To prove periodic points are dense, we show each cylinder contains a periodic point. Suppose $[x_0 \dots x_n]$ is non-empty. By irreducibility, there exists a word w such that $x_n w x_0$ is admissible. This means $\overline{x_0 \dots x_n w}$ is admissible, periodic for σ, and in $[x_0 \dots x_n]$.

Assuming the transition matrix A is aperiodic, we prove topological mixing. Since each open set contains a cylinder, it suffices to prove that for any two words w_1, w_2 producing non-empty cylinders $[w_1], [w_2]$, there exists $M \in \mathbb{N}$ such that for all $m \geq M$, $\sigma^m[w_1] \cap [w_2] \neq \emptyset$. Indeed, aperiodicity gives $N \in \mathbb{N}$ for which A^N has only strictly positive entries. Since Γ is connected, A has no row or column consiting only of 0. Thus A^n has only positive entries for all $n \geq N$. Thus there are words w^n such that $[w_1 w^n w_2] \subset [w_1]$ is non-empty for all $n \geq N$. Thus, letting q be the length of w_1,

$$\sigma^m[w_1] \cap [w_2] \neq \emptyset \text{ for all } m \geq N + q,$$

which completes the proof. □

4.2.2 Function spaces

We will assume from here that Γ is irreducible. With a metric in place, we define the main function spaces over the shift space which will be used throughout this work. Let $C(\Sigma(\Gamma), \mathbb{C})$ denote the set of complex-valued continuous functions on $\Sigma(\Gamma)$. For $n \in \mathbb{N}$ and $f \in C(\Sigma(\Gamma), \mathbb{C})$, define the variation

$$\operatorname{var}_n f := \sup\{|f(x) - f(y)| : x_i = y_i \text{ for all } |i| < n\}.$$

We use variation to describe the space of complex-valued d_θ-Lipschitz functions on $\Sigma(\Gamma)$, denoted by

$$F_\theta := \{f \in C(\Sigma(\Gamma), \mathbb{C}) : \exists L > 0 \text{ with } \operatorname{var}_n f < L\theta^n \text{ for all } n\}.$$

One can analogously define a function space over $\Sigma^+(\Gamma)$, denoted by F_θ^+. This can be thought of as a subset of F_θ, with functions acting only on future co-ordinates of the bi-infinite sequences in $\Sigma(\Gamma)$. Precisely, for $f \in F_\theta^+$, view f as an element of F_θ by setting $f((x_n)_{n=-\infty}^{\infty}) = f((x_n)_{n=0}^{\infty})$.

Remark. Since θ will be allowed to vary in $(0, 1)$, our study of the Lipschitz functions F_θ will also cover the more general class of Hölder continuous functions, since if f is α-Hölder for d_θ, it is Lipschitz for d_{θ^α}, i.e. $f \in F_{\theta^\alpha}$.

Define a norm on F_θ or F_θ^+ by $\|f\|_\theta = \|f\|_\infty + |f|_\theta$, where $\|f\|_\infty$ is the usual supremum norm of f, and $|f|_\theta = \sup\{\frac{\mathrm{var}_n f}{\theta^n} : n \in \mathbb{N}\}$.

Lemma 4.2.7. *F_θ and F_θ^+ are Banach spaces with respect to $||\cdot||_\theta$.*

We now introduce an equivalence relation for F_θ.

Definition 4.2.8. Functions $f, g \in F_\theta$ are *cohomologous*, written $f \sim g$, if there exists a continuous function h such that $f = g + h \circ \sigma - h$. If $f \sim 0$ then f is called a *coboundary.* For a function f, and $n \in \mathbb{N}$, the nth *Birkhoff sum* of f at $x \in \Sigma(\Gamma)$ is defined by $f^n(x) := \sum_{i=0}^{n-1} f(\sigma^i(x))$ for $n > 0$, with the convention $f^0(x) = 0$. Should we ever refer to the multiplicative powers of a function in $\mathbb{C}$, it will be made explicitly clear.

Cohomologous functions have equal Birkhoff sums over periodic orbits of the shift, meaning that whenever $f \sim g$ and $\sigma^n(x) = x$, $f^n(x) = g^n(x)$. A much deeper result is that the converse is true.

Theorem 4.2.9 (Livsic [Liv72])**.** *If $f, g \in F_\theta$ satisfy that $f^n(x) = g^n(x)$ whenever $\sigma^n(x) = x$, then $f \sim g$.*

It can also be shown that, up to cohomology and varying θ, functions in F_θ can always be considered only to depend on future co-ordinates.

Lemma 4.2.10. *If $f \in F_\theta$, there exists $g \in F_{\theta^{1/2}}$ depending only on future co-ordinates, such that $f \sim g$.*

Proof. For each $i \in \{1, \ldots, k\}$, fix a sequence $\alpha^i \in \Sigma(\Gamma)$ such that $\alpha^i_0 = i$. Define $a : \Sigma(\Gamma) \to \Sigma(\Gamma)$ by

$$a(x)_n = \begin{cases} x_n & n \geq 0, \\ \alpha^{x_0}_n & n < 0, \end{cases}$$

i.e. $a(x)$ replaces the past co-ordinates of x with those of α^{x_0}. Observe that, for each $x \in \Sigma(\Gamma)$ and $n \in \mathbb{N}$, $\sigma^n(x)_i = \sigma^n(a(x))_i$ for all $|i| < n+1$. Since $f \in F_\theta$, this means $|f(\sigma^n(a(x))) - f(\sigma^n(x))| \leq |f|_\theta \theta^{n+1}$. So the series

$$h(x) = \sum_{n=0}^{\infty} f(\sigma^n(a(x))) - f(\sigma^n(x))$$

converges and defines a function on $\Sigma(\Gamma)$ whose regularity we will discuss shortly. By the same observation, we obtain $f = g + h \circ \sigma - h$, where

$$g = f \circ a + \sum_{n=0}^{\infty} (f \circ \sigma^{n+1} \circ a - f \circ \sigma^n \circ a \circ \sigma).$$

Due to composition with a in each term, g depends only on future co-ordinates.

We complete the proof by showing that $h \in F_{\theta^{1/2}}$, implying the same for g. Let $N \in \mathbb{N}$, and $x, y \in \Sigma(\Gamma)$ be points such that $x_i = y_i$ for all $|i| < 2N$. Then for all $n \leq N$,

$$\max\{|f(\sigma^n(x)) - f(\sigma^n(y))|, |f(\sigma^n(a(x))) - f(\sigma^n(a(y)))|\} \leq |f|_\theta \theta^{2N-n},$$

as $f \in F_\theta$. Using this and our previous observation

$$\begin{aligned}|h(x) - h(y)| &\leq 2|f|_\theta \sum_{n=0}^{N} \theta^{2N-n} + 2|f|_\theta \sum_{n=N+1}^{\infty} \theta^{n+1} = 2|f|_\theta \left(\theta^{2N}\frac{\theta^{-N}-1}{\theta^{-1}-1} + \frac{\theta^{N+2}}{1-\theta}\right) \\ &= 2|f|_\theta \left(\frac{\theta^{N+1} + \theta^{N+2} - \theta^{2N}}{1-\theta}\right) \leq \frac{4|f|_\theta}{1-\theta}\theta^N.\end{aligned}$$

We conclude that $\frac{\mathrm{var}_{2N} h}{(\theta^{1/2})^{2N}}$ is bounded as N varies. Since $\mathrm{var}_{2N+1} h \leq \mathrm{var}_{2N} h$, we also have boundedness of $\frac{\mathrm{var}_{2N+1} h}{(\theta^{1/2})^{2N+1}}$. Thus $h \in F_{\theta^{1/2}}$. $\square$

When $g \in F_\theta$ depends only on future co-ordinates, we will identify it with the element $g' \in F_\theta^+$, defined by $g'(i_0, i_1, \ldots) = g(x)$, Where $x \in \Sigma(\Gamma)$ is such that $x_l = i_l$ for all $l \in \mathbb{N}$. Such an x exists by irreducibility.

The following cohomology result will be used later.

Proposition 4.2.11 ([PP90], Proposition 5.2). *Let $f \in F_\theta$ and suppose there is $a \in \mathbb{R}$ such that $f^n(x) \in a\mathbb{Z}$ whenever $\sigma^n(x) = x$. Then there is an $a\mathbb{Z}$ valued function $g \in F_\theta$ such that $f \sim g$.*

4.2.3 Suspension flows

We will now discuss the suspensions of shifts of finite type, which we call *suspension flows*. We will also state Theorem 4.2.15 which allows us to model hyperbolic flows by suspension flows. This will be a key tool throughout the thesis. Recall the definitions in Section 3.2.

Definition 4.2.12. Given a real-valued strictly positive function $r \in F_\theta$, the *suspension space* of $\Sigma(\Gamma)$ with *roof* r will be denoted by

$$\Sigma(\Gamma, r) = {}^{\textstyle(\Sigma(\Gamma) \times \mathbb{R})}\big/_{\textstyle\sim},$$

where $\sim$ is the equivalence relation in in Section 3.2, generated by

$$(x, r(x)) \sim (\sigma(x), 0).$$

The suspension flow will be denoted by $\sigma_t^r : \Sigma(\Gamma, r) \to \Sigma(\Gamma, r)$, recalling that

$$\sigma_t^r[x, s] = [x, s + t].$$

One can repeat the same process with $\Sigma^+(\Gamma)$ replacing $\Sigma(\Gamma)$ to obtain a semi-flow, for which we use the same notation σ_t^r. A visual representation is given in Figure 4.3.

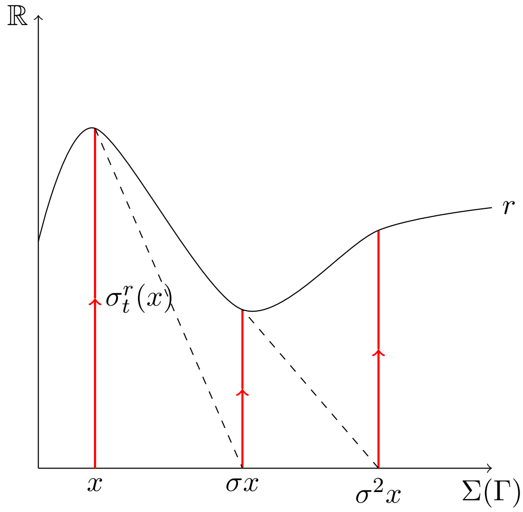

Figure 4.3: Suspension flow.

Fix one of the metrics d_θ on $\Sigma(\Gamma)$. Equipping $\Sigma(\Gamma)$ with the metric topology, we obtain a topology on $\Sigma(\Gamma, r)$ given by the quotient by $\sim$ of the product topology on $\Sigma(\Gamma) \times \mathbb{R}$. This topology (and the analogue on $\Sigma^+(\Gamma, r)$) is induced by a metric, which we also call d_θ, it is defined as follows, adapted from that seen in Section 4 of [BW72].

We call $[x, t], [y, s] \in \Sigma(\Gamma, r)$ a *horizontal pair* if there exist $u \in [0, 1)$ and $x', y' \in \Sigma(\Gamma)$ such that

$$[x, t] = [x', ur(x')] \text{ and } [y, s] = [y', ur(y')].$$

By the definition of $\sim$, such u, x', y' are unique, and we define

$$d_\theta([x, t], [y, s]) = (1 - u)d_\theta(x', y') + ud_\theta(\sigma(x'), \sigma(y')).$$

We say that $[x, t], [y, s]$ are a *vertical pair* if there exist $t', s' \in [0, 1)$ and $z \in \Sigma(\Gamma)$

such that

$$[x,t] = [z, t'r(z)] \text{ and } [y,s] = [z, s'r(z)].$$

Again such t', s', z are unique, so we define

$$d_\theta([x,t],[y,s]) = |t' - s'|.$$

Outside of these cases we can define the metric via *paths*. Precisely, for $\omega, \omega' \in \Sigma(\Gamma, r)$, a path between ω and ω' is a finite string of points $\omega_0, \ldots, \omega_m \in \Sigma(\Gamma, r)$ satisfying:

- either $\omega_0 = \omega$ and $\omega_m = \omega'$, or $\omega_0 = \omega'$ and $\omega_m = \omega$,
- for each $i \in \{0, \ldots, m-1\}$, ω_i, ω_{i+1} is either a horizontal pair or a vertical pair.

For such a path, define its length as $\sum_{i=0}^{m-1} d_\theta(\omega_i, \omega_{i+1})$. We can then define

$$d_\theta(\omega, \omega') := \inf \left\{ \sum_{i=0}^{m-1} d_\theta(\omega_i, \omega_{i+1}) : \omega_0, \ldots, \omega_m \text{ forms a path between } \omega \text{ and } \omega' \right\}.$$

This is easily seen to define a metric, which will be used when discussing regularity of functions on $\Sigma(\Gamma, r)$.

The following two observations relate the roof function r and the dynamics of the suspension flow σ^r.

Lemma 4.2.13. *If the suspension flow σ^r of an irreducible shift is topologically weak-mixing then r is non-constant.*

Proof. Suppose that $r(x) = t_0$ for all $x \in \Sigma(\Gamma)$. The function $\varphi : \Sigma(\Gamma, r) \to S^1$ defined by $\varphi[x,t] = e^{2\pi i t/t_0}$ is well-defined and continuous. Further, it satisfies $\varphi \circ \sigma_t^r = e^{2\pi i t/t_0}\varphi$, and thus σ^r cannot be topologically weak-mixing. □

We also have that cohomologous roof functions yield topologically conjugate suspension flows.

Proposition 4.2.14. *Suppose $r, r' \in F_\theta$ are real-valued and strictly positive, and satisfy $r = r' + f \circ \sigma - f$ for some $f \in C(\Sigma(\Gamma, \mathbb{R}))$. Then the suspension flows σ^r and $\sigma^{r'}$ are topologically conjugate.*

Proof. One can easily check that the map $\psi : \Sigma(\Gamma, r) \to \Sigma(\Gamma, r')$ defined by

$$\psi[x,t] = [x, t + f(x)]$$

is a conjugacy between σ^r and $\sigma^{r'}$. $\square$

Independently, Bowen [Bow73] and Ratner [Rat73] proved that hyperbolic flows are effectively modelled by suspension flows ([Bow73] is for Axiom A and [Rat73] for Anosov flows). Below we give a summary of their results which is suitable for this work.

Theorem 4.2.15. *Let $X^t : \Lambda \to \Lambda$ be a hyperbolic flow. Then, there exists an aperiodic graph Γ, a roof function r, and a Hölder continuous surjection $\pi : \Sigma(\Gamma, r) \to \Lambda$ satisfying*

1. *For all $t \in \mathbb{R}$, $\pi \circ \sigma^r_t = X^t \circ \pi$.*
2. *σ^r is transitive.*
3. *σ^r is topologically weak-mixing if and only if X^t is topologically weak-mixing.*
4. *There exists $N \in \mathbb{N}$ such that $\#\pi^{-1}(y) \leq N$ for all $y \in M$. Furthermore, the set of points with multiple preimages is meagre, and null with respect to any ergodic fully-supported measure.*

Remark. The map π above is a semi-conjugacy. In general, a full conjugacy does not exist, since this is only possible if Λ is homeomorphic to $\Sigma(\Gamma, r)$. The topology of these two spaces is often incompatible, for example, $\Sigma(\Gamma, r)$ is always 1-dimensional, and Λ need not be.

It is often more convenient to work with the suspension flow than directly with a hyperbolic flow, and Theorem 4.2.15 allows us to translate between the two. Let us briefly describe how the semi-conjugacy is constructed. Further detail can be found in [Bow73] or [Rat73]. Using the structure of the stable and unstable manifolds of X, one can construct a family of Markov sections. This consists of a finite set $\{R_1, \dots R_k\}$ of codimension-one cross sections to the flow, known as *rectangles*, satisfying that every orbit of X^t intersects $R = \bigcup_{i=1}^k R_i$ infinitely often in the past and future, and does so transversally. Letting $\tau : R \to R$ be the first return map to R (or Poincaré map), define a directed graph Γ on k vertices $\{1, 2, \dots, k\}$ as follows. First denote by ij the directed edge from vertex i to vertex j. Include ij in the directed edge set $E(\Gamma)$ whenever $R_i \cap \tau^{-1}(R_j) \neq \emptyset$. This yields a connected directed graph, with at most two edges (of opposite direction) between two vertices. The rectangles are chosen such that there exists $T > 0$ with $M \subset X^{[0,T]}R$, and for each $x \in \Sigma(\Gamma)$, $\bigcap_{n \in \mathbb{Z}} \tau^n(R_{x_n})$ contains exactly one point. With this, we can define $\pi' : \Sigma(\Gamma) \to R$ by $\pi'(x) \in \bigcap_{n \in \mathbb{Z}} \tau^n(R_{x_n})$. Setting $r : R \to \mathbb{R}^+$ to be the first return time to R, we can extend π' to a map $\pi : \Sigma(\Gamma, r) \to M$, by $\pi[x, s] = X^s(\pi'(x))$.

4.3 Thermodynamic formalism

Here we define transfer operators, which can be used to study pressure and to identify equilibrium states for dynamical systems. We do this in the context of symbolic dynamics, before translating to hyperbolic flows through the coding in Theorem 4.2.15. This material can be found in Chapters 2 and 4 of [PP90].

4.3.1 Transfer operators

In this section, Γ will denote any finite directed graph, as in the definition of shift spaces at the start of Section 4.2.1. We consider operators on the Banach space F_θ^+. The shift map induces an operator σ^* on F_θ^+ by setting $\sigma^* g = g \circ \sigma$ for each $g \in F_\theta^+$. The *transfer operator* is in some sense dual to σ^*.

Definition 4.3.1. For each $f \in F_\theta^+$, define the Ruelle transfer operator $\mathcal{L}_f : F_\theta^+ \to F_\theta^+$ (or more generally on $C(\Sigma^+(\Gamma), \mathbb{C})$) by

$$(\mathcal{L}_f g)(x) = \sum_{\sigma(y)=x} e^{f(y)} g(y),$$

for all $x \in \Sigma^+(\Gamma)$. We say that f or $\mathcal{L}_f$ is *normalised* if $\mathcal{L}_f 1 = 1$, where 1 is the constant function $1(x) = 1$ for all x.

Key properties of the transfer operator are summarised in the following theorem.

Theorem 4.3.2 (Ruelle-Perron-Frobenius)**.** *Let $f \in F_\theta^+$ be real-valued, and Γ aperiodic. The following hold.*

1. *$\mathcal{L}_f : C(\Sigma^+(\Gamma), \mathbb{C}) \to C(\Sigma^+(\Gamma), \mathbb{C})$ has a simple maximal real eigenvalue $\lambda > 0$, with eigenfunction $h \in F_\theta^+$ which is strictly positive.*

2. *The remainder of the spectrum of $\mathcal{L}_f : F_\theta^+ \to F_\theta^+$ is contained in a disc of radius stricly smaller than λ.*

3. *There is a unique probability measure m such that $\mathcal{L}_f^* m = \lambda m$, i.e.*

$$\int \mathcal{L}_f g \, dm = \lambda \int g \, dm$$

for all $g \in C(\Sigma^+(\Gamma), \mathbb{C})$.

4. *For each* $g \in C(\Sigma^+(\Gamma), \mathbb{C})$, *we have the uniform convergence*

$$\frac{1}{\lambda^n}\mathcal{L}_f^n g \xrightarrow[n\to\infty]{} h \int g\, dm,$$

once we assume (without loss of generality) that h *is the eigenfunction with* $\int h\, dm = 1$.

A proof of Theorem 4.3.2 may be found in Chapter 2 of [PP90]. We omit it here but note one useful element, a normalisation procedure for any real-valued $f \in F_\theta^+$.

Corollary 4.3.3. *Let* $f \in F_\theta^+$ *be real-valued and assume* $\mathcal{L}_f$ *has a simple maximal real eigenvalue* $\lambda > 0$*, with eigenfunction* h *which is strictly positive. Then* $g = f - \log\lambda + \log h \circ \sigma + \log h$ *is normalised.*

We also have that normalised functions are strictly negative up to cohomology.

Proposition 4.3.4. *Let* $f \in F_\theta^+$ *be normalised. Then there exists* $g \in F_\theta^+$ *such that* $f \sim g$ *and* g *is strictly negative.*

Proof. By aperiodicity, we can choose $N \geq 1$ such that each $x \in \Sigma^+(\Gamma)$ has at least k (the number of symbols) pre-images under σ^N. Then for each $y \in \Sigma^+(\Gamma)$

$$1 = (\mathcal{L}_f^N 1)(\sigma^N(y)) = \sum_{\sigma^N(z)=\sigma^N(y)} e^{f^N(z)},$$

and thus $f^N(y) < 0$. So f^N is a strictly negative function. By Theorem 4.2.9 $f^N \sim Nf$, and so $f \sim \frac{1}{N} f^N$ which is strictly negative. □

The following result gathers important properties of the eigenfunction and eigenmeasure in Theorem 4.3.2.

Proposition 4.3.5. *For* f, λ, h, m *as in Theorem 4.3.2, the measure* hm *is* σ*-invariant, mixing, and fully supported.*

Proof. Given continuous functions $u, v : \Sigma^+(\Gamma) \to \mathbb{R}$, and $n \in \mathbb{N}$ we have $(\mathcal{L}_f^n u) \cdot v = \mathcal{L}_f^n(u \cdot (v \circ \sigma^n))$, so

$$\int h \cdot u\, dm = \frac{1}{\lambda}\int (\mathcal{L}_f h) \cdot u\, dm = \int h \cdot (u \circ \sigma)\, dm,$$

i.e. $hm(u) = hm(u \circ \sigma)$ for each continuous u, meaning hm is σ-invariant.

Let $l, m \in \mathbb{N}$ and fix cylinders $U = [a_0, \ldots, a_l]$ and $V = [b_0, \ldots, b_m]$. Then for the characteristic functions χ_U, χ_V, and $n \in \mathbb{N}$

$$\begin{aligned}\int h\chi_U \cdot (\chi_V \circ \sigma^n)\, dm &= \int \frac{1}{\lambda^n} \mathcal{L}_f^n(h\chi_U \cdot (\chi_V \circ \sigma^n))\, dm \\ &= \int \frac{1}{\lambda^n} (\mathcal{L}_f^n(h\chi_U))\chi_V)\, dm.\end{aligned}$$

It follows that

$$\begin{aligned}\Big| hm(U \cap \sigma^{-n}V) - hm(U)hm(V) \Big| &= \left| \int \left(\frac{1}{\lambda^n} (\mathcal{L}_f^n(h\chi_U)) - hm(U)h \right) \chi_V\, dm \right| \\ &\leq \left\| \frac{1}{\lambda^n} (\mathcal{L}_f^n(h\chi_U)) - hm(U)h \right\|_\infty m(V),\end{aligned}$$

which, by Theorem 4.3.2, converges to zero as $n \to \infty$. Thus hm is mixing with respect to σ.

That hm is fully supported follows from the fact that it is a Gibbs measure (see Corollary 3.2.1 in [PP90]), so gives positive measure to each cylinder set. □

4.3.2 Invariant measures

Return to the setting of an aperiodic shift map on $\Sigma(\Gamma)$ or $\Sigma^+(\Gamma)$. Existence and uniqueness of equilibrium states can be seen for these systems as follows. Recall the notation of Theorem 4.3.2.

Theorem 4.3.6. *Let $f \in F_\theta$ be real-valued, and $g \sim f$ with $g \in F^+_{\theta^{1/2}}$. The normalised Ruelle-Perron-Frobenius eigenfunction h and eigenmeasure m for $\mathcal{L}_g$ are such that hm is an equilibrium state for f. Furthermore hm is unique and $P(f) = \log \lambda$, where λ is the Ruelle-Perron-Frobenius eigenvalue for $\mathcal{L}_g$.*

This is proved in Chapter 3 of [PP90]. From the attainment of the supremum in the variational principle, it immediately follows that

1. $P : C(\Sigma(\Gamma), \mathbb{R}) \to \mathbb{R}$ is monotone increasing with respect to the partial order $f \leq g$ if $f(x) \leq g(x)$ for all x.

2. The function P is convex and Lipschitz continuous with respect to the norm $|\cdot|_\infty$.

3. If $f \sim g + c$ for a constant c, then $P(f) = P(g) + c$.

We will fix the notation m_f for the equilibrium state corresponding to a function f.

The spectral results in Theorem 4.3.2 can now be stated in the generality of complex-valued functions.

Theorem 4.3.7. *Let $f \in F_\theta^+$. Then the spectral radius of $\mathcal{L}_f$ is at most $e^{P(\mathscr{R}f)}$. If $\mathcal{L}_f$ has an eigenvalue of modulus $e^{P(\mathscr{R}f)}$ it is simple and unique with the rest of the spectrum contained in a disc of radius strictly less than $e^{P(\mathscr{R}f)}$. If not, then the spectral radius is strictly less than $e^{P(\mathscr{R}f)}$.*

Further, one can characterise when $\mathcal{L}_f$ attains its maximal spectral radius.

Proposition 4.3.8. *The spectral radius of $\mathcal{L}_f$ is equal to $e^{P(\mathscr{R}f)}$ if and only if there exists $a \in \mathbb{R}$ and $M \in C(\Sigma^+(\Gamma), \mathbb{Z})$ such that $f \sim \mathscr{R}f + ia + 2\pi i M$. Furthermore, if these properties hold, then the maximal eigenvalue is given by $e^{P(\mathscr{R}f)+ia}$.*

Proof. Suppose first that there is $h \in F_\theta^+$ such that $f = \mathscr{R}f + ia + 2\pi i M + h \circ \sigma - h$. Let g be the eigenfunction of $\mathcal{L}_{\mathscr{R}f}$ realising the eigenvalue $e^{P(\mathscr{R}f)}$. Then

$$(\mathcal{L}_f(e^h g))(x) = e^{ia} e^{h(x)} \mathcal{L}_{\mathscr{R}f} g = e^{P(\mathscr{R}f)+ia} e^{h(x)} g(x),$$

so $\mathcal{L}_f$ has an eigenvalue of modulus $e^{P(\mathscr{R}f)}$.

Now we suppose that $\mathcal{L}_f$ has an eigenvalue of modulus $e^{P(\mathscr{R}f)}$. By Corollary 4.3.3, there is no loss in assuming that $u = \mathscr{R}f$ is normalised. Let $h \in F_\theta^+$ be the $\mathcal{L}_f$ eigenfunction for the maximal eigenvalue, so there exists $a \in \mathbb{R}$ such that $\mathcal{L}_f h = e^{ia} h$. By the triangle inequality, $\mathcal{L}_u |h| \geq |h|$. Integrating against m_u we see $\mathcal{L}_u |h| = |h|$ m_u-a.e. Since $\mathcal{L}_u 1 = 1$, we can conclude $|h|$ is constant m_u-a.e, which further implies h is non-zero m_u-a.e. As m_u is fully supported, $|h|$ is constant and h is non-zero everywhere. Since $\mathcal{L}_u(e^{i\Im f} h) = e^{ia} h$, we use a convexity argument to deduce that for $x \in \Sigma^+(\Gamma)$ and any $y \in \sigma^{-1}(x)$, $e^{i\Im f(y)} h(y) = e^{ia} h(x)$. So $e^{i(\Im f - a)} h = h \circ \sigma$. Therefore whenever $\sigma^n(x) = x$, $e^{i(\Im f - a)^n(x)} h(x) = h(x)$ so $(\Im f - a)^n(x) \in 2\pi\mathbb{Z}$. Proposition 5.2 in [PP90] completes the proof. □

The realisation of pressure in terms of the transfer operator also gives a convenient way to extend the pressure function to complex-valued functions. Precisely, for functions $f \in F_\theta^+$ for which $\mathcal{L}_f$ has a unique maximal (in modulus) eigenvalue λ, we set $P(f) = \log |\lambda|$. By perturbation theory of the transfer operator (see [Kat76]), the extended domain $\mathrm{dom}(P)$ is open. Furthermore, P is analytic on $\mathrm{dom}(P)$.

Perturbation theory can also be used to evaluate the derivative of pressure in the following way.

Proposition 4.3.9. *Let $f, g \in F_\theta^+$ be real-valued. Then*

$$\left.\frac{dP(f + sg)}{ds}\right|_{s=0} = \int g \, dm_f$$

Proof. By perturbation theory, any sufficiently small $s \in \mathbb{C}$ is such that $\mathcal{L}_{f+sg}$ has $e^{P(f+sg)}$ as a simple maximal eigenvalue. Let $w(s)$ be the corresponding eigenfunction, which varies analytically with s. It suffices to prove the statement for f normalised, meaning in particular that $w(0) = 1$ and $P(f) = 0$. Let us differentiate the eigenvalue equation pointwise. For $x \in \Sigma^+(\Gamma)$, we obtain

$$\left.\frac{dP(f+sg)}{ds}\right|_{s=0} + w'(0,x) = \sum_{\sigma(y)=x} e^{f(y)}(g(y) + w'(0,y)) = \mathcal{L}_f(g + w'(0))(x).$$

Integrating with respect to m and recalling that $\mathcal{L}_f^* m = m$ gives the result. □

Let us now discuss how invariant measures for σ 'lift' to those in the suspension space. For $m \in \mathcal{M}(\sigma)$, define m^r on $\Sigma(\Gamma, r)$ by

$$\int F \, dm^r = \frac{\int_{x \in \Sigma(\Gamma)} \left(\int_0^{r(x)} F[x,t] \, dt \right) dm(x)}{\int r \, dm}.$$

Under this definition, m^r is invariant, and is ergodic if and only if m is. Furthermore, each measure in $\mathcal{M}(\sigma^r)$ can be constructed in this way. Also, we will see that equilibrium states for σ^r are the lifts of those for σ.

First note that if $F : \Sigma(\Gamma, r) \to \mathbb{R}$ is Hölder continuous, then the function $f : \Sigma(\Gamma) \to \mathbb{R}$ defined by

$$f(x) = \int_0^{r(x)} F[x,t] \, dt \tag{4.1}$$

is itself Hölder continuous. We will use P for both the pressure over the shift and over the suspension flow, with the function on which P is evaluated resolving any ambiguity.

Proposition 4.3.10. *The lift $m^r_{-P(F)r+f}$ of $m_{-P(F)r+f}$ is an equilibrium state for F, on $\Sigma(\Gamma, r)$. Furthermore, $m^r_{-P(F)r+f}$ is unique.*

Proof. This can be seen to follow from the fact that $h_{m^r}(\sigma^r) = \frac{h_m(\sigma)}{\int r \, dm}$ for any $m \in \mathcal{M}(\sigma)$, which was shown by Abramov [Abr59].

Indeed, since $c \mapsto P(-cr + f)$ is strictly decreasing from ∞ to $-\infty$, there is a unique $c \in \mathbb{R}$ such that $P(-cr + f) = 0$. Letting m be the equilibrium state for $-cr + f$, we have that for any other $\nu \in \mathcal{M}(\sigma)$,

$$c = \frac{h_m(\sigma) + \int f \, dm}{\int r \, dm} > \frac{h_\nu(\sigma) + \int f \, d\nu}{\int r \, d\nu}.$$

This is $h_{m^r}(\sigma^r) + \int F \, dm^r > h_{\nu^r}(\sigma^r) + \int F \, d\nu^r$, which gives the result. □

The proof of Proposition 4.3.10 gives a relation between pressure for σ and pressure for σ^r.

Corollary 4.3.11. *With F and f as above, the pressure $P(F)$ is the unique real number satisfying $P(-P(F)r + f) = 0$.*

We are now able to differentiate pressure on the suspension space.

Proposition 4.3.12. *Let F, G be Hölder continuous on $\Sigma(\Gamma, r)$, with corresponding $f, g : \Sigma(\Gamma) \to \mathbb{R}$ defined as in (4.1). Then*

$$\left.\frac{dP(F + sG)}{ds}\right|_{s=0} = \frac{\int g \, dm_{-P(F)r+f}}{\int r \, dm_{-P(F)r+f}}$$

Proof. Let $\beta(s) = P(F + sG)$. By Corollary 4.3.11, $P(-\beta(s)r + f + sg) = 0$. Differentiating, we obtain

$$\frac{d\beta}{ds}(0) \left.\frac{dP(-\beta r + f)}{d\beta}\right|_{\beta=\beta(0)} + \left.\frac{dP(-\beta(0)r + f + sg)}{ds}\right|_{s=0} = 0.$$

The conclusion follows from Proposition 4.3.9. $\square$

The existence of unique equilibrium states for suspension flows also yields unique equilibrium states for hyperbolic flows, through the symbolic coding in Theorem 4.2.15. See [BR75] for the full details.

Proposition 4.3.13. *Let $X^t : \Lambda \to \Lambda$ be a hyperbolic flow, and $\pi : \Sigma(\Gamma, r) \to \Lambda$ the symbolic coding. For $\varphi : \Lambda \to \mathbb{R}$ Hölder continuous, let $m_{\varphi\circ\pi}$ be the equilibrium state for $\varphi \circ \pi$ over σ^r. Then $\pi_* m_{\varphi\circ\pi}$ is an equilibrium state for φ, and is unique.*

Proof. Since π is a semi-conjugacy, Theorem 9.8 in [Wal81] gives that $P(\varphi) \leq P(\varphi \circ \pi)$. Thus, since

$$h_{\pi_* m_{\varphi\circ\pi}}(X^1) + \int \varphi \, d\pi_* m_{\varphi\circ\pi} = h_{m_{\varphi\circ\pi}}(\sigma_1^r) + \int \varphi \circ \pi \, dm_{\varphi\circ\pi} = P(\varphi \circ \pi),$$

$\pi_* m_{\varphi\circ\pi}$ is an equilibrium state for φ and $P(\varphi) = P(\varphi \circ \pi)$. Suppose μ is a second equilibrium state. Then $\mu = \pi_* m$ for some $m \in \mathcal{M}(\sigma^r)$, and Theorem 9.8 of [Wal81] gives

$$h_m(\sigma_1^r) + \int \varphi \circ \pi \, dm \geq h_\mu(X^1) + \int \varphi \circ \pi \, d\mu = P(\varphi) = P(\varphi \circ \pi).$$

Thus $m = m_{\varphi\circ\pi}$ and so $\mu = \pi_* m_{\varphi\circ\pi}$. $\square$

Remark. Existence and uniqueness of equilibrium states for hyperbolic flows can be proved without using the symbolic coding. Another approach is to use a specification property and equidistribution theory, the latter of which is the subject of Chapter 5. The argument is given by Franco [Fra77], following the approach of Bowen [Bow74] for discrete systems.

We will denote the equilibrium state $\pi_* m_{\varphi \circ \pi}$ by μ_φ.

Besides the measure of maximal entropy μ_0, there is another significant equilibrium state, the *Sinai-Ruelle-Bowen (SRB) measure*, which has been studied for its physical relevance.

SRB measure

See Section 7.4 of [FH19] for details of this section. First we define a potential $\varphi^u : \Lambda \to \mathbb{R}$. Recall that in the definition of a hyperbolic set (Definition 4.1.3) we have the unstable subbundle E^u of $T_\Lambda M$. For each $t \geq 0$, define $J_t : \Lambda \to \mathbb{R}$ by

$$J_t(x) := \det DX^t_x|_{E^u_x},$$

a measure of the expansion rate of the flow at x. We then define φ^u by

$$\varphi^u(x) = -\lim_{t\to 0} \frac{1}{t} \log J_t(x) = -\frac{d \log J_t}{dt}\bigg|_{t=0} = -\frac{dJ_t}{dt}\bigg|_{t=0}.$$

That φ^u is Hölder continuous follows from the fact that E^u_x varies Hölder continuously with x, i.e. E^u is spanned by a collection of Hölder continuous vector fields.

Definition 4.3.14. The SRB (or physical) measure for X is the equilibrium state μ_{φ^u} for the potential φ^u.

Let Vol_ρ denote the normalised Riemannian volume measure on M. The following properties of μ_{φ^u} show how it relates to Vol_ρ, and are the justification for describing μ_{φ^u} as physically relevant above.

Theorem 4.3.15. *For $X^t : \Lambda \to \Lambda$ a hyperbolic flow, and $\psi : M \to \mathbb{R}$ continuous, we have*

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \psi(X^t(x))\, dt = \int \psi \, d\mu_{\varphi^u}$$

for Vol_ρ*-a.e.* $x \in W^s(\Lambda)$.

From this one can also prove the following result, which we will use later. First recall that when a measure μ is absolutely continuous to Vol_ρ, it has a Radon-Nikodým derivative $\frac{d\mu}{d\mathrm{Vol}_\rho}$. We say μ is *smooth* if $\frac{d\mu}{d\mathrm{Vol}_\rho}$ is.

Theorem 4.3.16. *Let $X^t : M \to M$ be a weak-mixing transitive Anosov flow. Then $\mathcal{M}(X)$ contains at most one smooth measure, and in the case that $\mu \in \mathcal{M}(X)$ is smooth, $\mu = \mu_{\varphi^u}$. In particular, if Vol_ρ is invariant under X^t, then $\mathrm{Vol}_\rho = \mu_{\varphi^u}$.*

Pressure for Anosov flows

Here we summarise some further results for pressure for Anosov flows $X^t : M \to M$. The results are true in the general setting of hyperbolic flows (replacing M with a basic set Λ). The symbolic coding of hyperbolic flows will be relevant, and we will again use the same symbol P to denote the pressure function for either the shift or the flow.

The following lemma will be used when we discuss large deviations in Section 5.4. The proof is completely analogous to those for Theorem 8.2 and Theorem 9.12 of [Wal81], which deal with a discrete system. In particular, the first statement follows from the fact that the flow is expansive. Once we have established upper semi-continuity, rearranging the variational principle above into this form follows the same argument as the proof of Theorem 9.12 of [Wal81].

Lemma 4.3.17. *The map $\mathcal{M}(X) \to \mathbb{R} : \nu \mapsto h(\nu)$ is upper semi-continuous and*

$$h(\nu) = \inf \left\{ P(\varphi) - \int \varphi \, d\nu \, : \, \varphi \in C(M, \mathbb{R}) \right\}.$$

We will need to use the notion of functions being cohomologous with respect to X or σ^r.

Definition 4.3.18. Functions $\varphi, \psi \in C(M, \mathbb{R})$ are X-cohomologous if there is a continuous function $u : M \to \mathbb{R}$ that is differentiable along flow lines satisfying

$$\varphi - \psi = L_X u,$$

where L_X is the Lie derivative. Similarly, functions $F, G \in C(\Sigma(\Gamma, r), \mathbb{R})$ are σ^r-cohomologous if there is some $I \in C(\Sigma(\Gamma, r), \mathbb{R})$ such that the limit

$$\lim_{\varepsilon \to 0} \frac{I(\sigma^r_\varepsilon[x,t]) - I[x,t]}{\varepsilon}$$

exists for all $[x,t]$, and coincides with $(F - G)[x,t]$.

As for the shift map, X-cohomologous functions have the same integral with respect to every measure in $\mathcal{M}(X)$, and for each constant $c \in \mathbb{R}$, we have

$$P(\varphi + L_X u + c) = P(\varphi) + c.$$

For a periodic orbit γ of X, let

$$\int_\gamma \varphi := \int_0^{\ell(\gamma)} \varphi(X^t x_\gamma)\, dt$$

where $x_\gamma \in \gamma$, and $\ell(\gamma)$ is the period. Then it is clear that for φ and ψ X-cohomologous, $\int_\gamma \varphi = \int_\gamma \psi$. As for the shift map, the converse holds.

Lemma 4.3.19 (Livsic [Liv71])**.** *Suppose that $\varphi, \psi : M \to \mathbb{R}$ are Hölder continuous. If for all periodic orbits γ,*

$$\int_\gamma \varphi = \int_\gamma \psi,$$

then φ and ψ are X-cohomologous.

We will use the following result later.

Lemma 4.3.20. *Suppose that $\varphi : M \to \mathbb{R}$ is Hölder continuous. Then there exists $\varepsilon > 0$ and a Hölder continuous function $v : M \to \mathbb{R}$ such that, for all $x \in M$ and $T \geq 0$, we have*

$$\int_0^T \varphi(X^t x)\, dt \leq (P(\varphi) - \varepsilon)T + v(X^T x) - v(x).$$

Proof. Since $P(\varphi - P(\varphi)) = P(\varphi) - P(\varphi) = 0$, without loss of generality, we may assume that $P(\varphi) = 0$. Let $\pi : \Sigma(\Gamma, r) \to M$ be the symbolic coding map from Theorem 4.2.15. This map is such that the pressure of a function with respect to X^t and of its pull-back by π are equal. Define $q_\varphi : \Sigma(\Gamma) \to \mathbb{R}$ by

$$q_\varphi(x) = \int_0^{r(x)} \varphi(\pi(x, \tau))\, d\tau.$$

By Corollary 4.3.11 we have

$$P(q_\varphi) = P(-P(\varphi)r + q_\varphi) = 0.$$

It then follows from Corollary 4.3.3 and Proposition 4.3.4 that q_φ is σ-cohomologous to a strictly negative function, i.e. that there exists a continuous function $u : \Sigma(\Gamma) \to \mathbb{R}$ such that $q_\varphi + u \circ \sigma - u$ is strictly negative. By compactness, this function is bounded above by $-\varepsilon \|r\|_\infty$ for some $\varepsilon > 0$. This gives

$$\int_\gamma (\varphi + \varepsilon) \leq 0,$$

for all $\gamma \in \mathcal{P}$. Theorem 1 of [PS04] implies that there exists a Hölder continuous function $v : M \to \mathbb{R}$ such that, for all $x \in M$ and $T \geq 0$,

$$\int_0^T \varphi(X^t x)\, dt + \varepsilon T \leq v(X^T x) - v(x),$$

completing the proof. □

As in the symbolic setting, we will need to consider derivatives of pressure functions.

Lemma 4.3.21 ([Lal87],[Sha92]). *Let $\varphi, \psi_1, \ldots, \psi_b : M \to \mathbb{R}$ be Hölder continuous functions. Then the function*

$$\mathbb{R}^b \to \mathbb{R} : (t_1, \ldots, t_b) \mapsto P(\varphi + t_1\psi_1 + \cdots t_b\psi_b).$$

is real-analytic, convex, and satisfies

$$\left.\frac{\partial P(\varphi + t_1\psi_1 + \cdots + t_b\psi_b)}{\partial t_i}\right|_{(t_1,\ldots,t_b)=0} = \int \psi_i \, d\mu_\varphi.$$

The function is strictly convex unless $a_1\psi_1 + \cdots a_b\psi_b$ is X-cohomologous to a constant for some $(a_1, \ldots, a_b) \neq 0$.

Proof. By Proposition 4.3.13, the coding $\pi : \Sigma(\Gamma, r) \to M$ is such that $P(\psi) = P(\psi \circ \pi)$ whenever $\psi \in C(M, \mathbb{R})$. Thus the statement on first derivatives follows from Proposition 4.3.12. The calculations in the proof of Lemma 1 in [Lal87] can be modified to show that the Hessian

$$\nabla^2 P(\varphi \circ \pi + t_1\psi_1 \circ \pi + \cdots + t_b\psi_b \circ \pi)$$

is positive semi-definite, and positive definite if there is no non-zero $(a_1, \ldots, a_b)$ such that $a_1\psi_1 \circ \pi + \cdots a_b\psi_b \circ \pi$ is σ^r-cohomologous to a constant. This completes the proof. □

Chapter 5

Equidistribution theorems

This chapter concerns the equidistribution theory of periodic orbits of Anosov flows. The first result of this type is due to Bowen [Bow72a]. Using the specification property, Bowen showed that for a topologically weak-mixing hyperbolic flow, the periodic orbits equidistribute with respect to the measure of maximal entropy. In fact, this method can be used to prove a more general result regarding equilibrium states for Hölder functions (see [Fra77]). Using thermodynamic formalism, Parry [Par88] obtained the same result for potentials with non-negative pressure. We will discuss the latter approach and show how it can be extended to include negative pressure potentials. Then we will describe a recent result on equidistribution of periodic orbits subject to homological constraints.

5.1 Periodic orbits

Until now we have mentioned periodic orbits without formally defining them, since the notion is clear. For convenience, we will fix some definitions and notation now. Assume X^t is a hyperbolic flow on a basic set $\Lambda \subset M$.

Definition 5.1.1. A point $x \in M$ is periodic if there exists $T > 0$ such that $X^T(x) = x$. For such a point, the closed curve $\gamma : [0, T] \to M$ defined by $\gamma(t) = X^t(x)$ is called a periodic orbit, and $\ell(\gamma) = T$ is the *period* or *length*. Note that length will never refer to the geometric length of the curve unless explicitly stated. A periodic orbit γ is called *prime* if T is minimal.

Let $\mathcal{P}$ denote the set of prime periodic orbits for X^t, and $\tilde{\mathcal{P}}$ the set of all periodic orbits. Similarly, for a suspension flow σ^r, let $\mathcal{P}(\sigma^r)$ denote the set of prime periodic orbits, $\tilde{\mathcal{P}}(\sigma^r)$ the set of all periodic orbits, and $\ell(\eta)$ the period of an orbit $\eta \in \tilde{\mathcal{P}}(\sigma^r)$.

One can find asymptotic growth rates for the number of elements of $\mathcal{P}_{\leq T} := \{\gamma \in \mathcal{P} : \ell(\gamma) \leq T\}$ using thermodynamic formalism.

Theorem 5.1.2 ([PP83])**.** *For any topologically weak-mixing hyperbolic flow X^t,*

$$\#\mathcal{P}_{\leq T} \sim \frac{e^{hT}}{hT},$$

where h denotes the topological entropy of the flow.

Corollary 5.1.3. *For any $a, b \in \mathbb{R}$ with $a < b$, we have*

$$\#\{\gamma \in \mathcal{P} : \ell(\gamma) \in (T+a, T+b]\} \sim (e^{hb} - e^{ha})\frac{e^{hT}}{hT}.$$

This is proved using the symbolic coding from Theorem 4.2.15, as well as *dynamical zeta functions*, which we now introduce.

5.2 Dynamical zeta functions

A dynamical zeta function is a complex function whose analytic properties encode the behaviour of periodic orbits. These properties are closely linked with the spectral theory of the transfer operator, and we will describe this connection precisely. Again we introduce the theory in the context of symbolic dynamics, before using the symbolic coding to translate to hyperbolic dynamics. The main results here can be found in Chapters 5 and 6 of [PP90].

Let $\Sigma(\Gamma)$ be an aperiodic shift space, with shift map σ. Dynamical zeta functions for this system are defined as follows.

Definition 5.2.1. The Artin–Mazur zeta function is the complex function defined by

$$\zeta(z) = \exp \sum_{n=1}^{\infty} \frac{z^n}{n} \#\{x : \sigma^n(x) = x\}.$$

More generally, define a weighted zeta function, for $f \in F_\theta$ by

$$\zeta(z, f) = \exp \sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{\sigma^n(x)=x} e^{f^n(x)}.$$

Clearly $\zeta(\cdot, 0) = \zeta$, and by Theorem 4.2.9, $\zeta(\cdot, f) = \zeta(\cdot, g)$ whenever $f \sim g$.

Let us consider analytic properties of these functions. The next result follows from Proposition 5.1 in [PP90].

Proposition 5.2.2. *For $f \in F_\theta^+$, the radius of convergence of $\zeta(\cdot, f)$ is $e^{-P(\mathscr{R}(f))}$.*

To study periodic orbits of a suspended flow, it is convenient to consider the complex functions

$$\zeta_{-r}(s) := \zeta(1, -sr) = \exp \sum_{n=1}^{\infty} \frac{1}{n} \sum_{\sigma^n(x)=x} e^{-sr^n(x)},$$

where $r \in F_\theta^+$ is real valued and strictly positive. By Proposition 5.2.2, $\zeta_{-r}(s)$ converges when $1 < e^{-P(-\mathscr{R}(s)r)}$. Since P is strictly increasing, this is the case exactly when $\mathscr{R}(s) > c$, where c is such that $P(-cr) = 0$. By Corollary 4.3.11, $c = h(\sigma^r)$, the topological entropy of the suspension flow on $\Sigma(\Gamma, r)$.

Where $\zeta_{-r}(s)$ converges, it can easily be shown that

$$\begin{aligned}\zeta_{-r}(s) &= \exp \sum_{n=1}^{\infty} \frac{1}{n} \sum_{\eta \in \mathcal{P}(\sigma^r)} e^{-sn\ell(\eta)} \\ &= \prod_{\eta \in \mathcal{P}(\sigma^r)} \left(1 - e^{-s\ell(\eta)}\right)^{-1}.\end{aligned}$$

The latter expression is known as an *Euler product* representation for ζ_{-r}.

We now consider the behaviour of $\zeta_{-r}(s)$ on the critical line $\mathscr{R}(s) = h(\sigma^r)$. Recall that the spectral radius of the transfer operator $\mathcal{L}_{-sr}$ is at most $e^{P(-\mathscr{R}(s)r)}$. The following is discussed after Theorem 5.6 in [PP90].

Theorem 5.2.3. *Suppose r,s are as above with $\mathscr{R}(s) = h(\sigma^r)$. Then*

1. *If $\mathcal{L}_{-sr}$ does not have 1 as an eigenvalue, then there exists $\varepsilon > 0$ such that ζ_{-r} has a nowhere-zero analytic extension to $B(s, \varepsilon)$.*

2. *If $\mathcal{L}_{-sr}$ has 1 as an eigenvalue, there exists $\varepsilon > 0$ such that ζ_{-r} has a nowhere-zero analytic extension to $B(s, \varepsilon) \setminus \{z \in \mathbb{C} : P(-zr) = 0\}$.*

Remark. By the argument at the end of Chapter 5 in [PP90], when 1 is an eigenvalue of $\mathcal{L}_{-sr}$ we may shrink ε to ensure

$$B(s, \varepsilon) \setminus \{z \in \mathbb{C} : P(-zr) = 0\} = B(s, \varepsilon) \setminus \{s\}.$$

This can be further improved if we assume σ^r to be topologically weak-mixing. We will view this as the nondegenerate case, since we work with σ^r which are encoding hyperbolic flows, for which topological weak-mixing is a typical property (see [FMT07]).

Proposition 5.2.4. *Suppose σ^r is topologically weak-mixing. Then ζ_{-r} has a nowhere-zero analytic extension to the line $\mathscr{R}(s) = h(\sigma^r)$, except for the single point $s = h(\sigma^r)$. Furthermore, $h(\sigma^r)$ is a simple pole for ζ_{-r}.*

Proof. By Theorem 5.2.3, ζ_{-r} extends to all s with $\mathscr{R}(s) = h(\sigma^r)$, except those for which the transfer operator $\mathcal{L}_{-sr}$ has 1 as an eigenvalue. Suppose $s_0 = h(\sigma^r) + it_0$ is such that there exists $w \in F_\theta^+$ with $\mathcal{L}_{-sr}w = w$. Since zeta functions are invariant under cohomology of functions, Corollary 4.3.3 allows us to assume $\mathcal{L}_{-h(\sigma^r)r}$ is normalised. Thus by the argument in the proof of Proposition 4.3.8 we have $e^{-it_0 r}w = w \circ \sigma$ with $|w|$ constant and non-zero. It follows that the map $W : \Sigma(\Gamma, r) \to \mathbb{C}$ defined by $W[x,t] = e^{it_0 t}w(x)$ is a well-defined continuous map to the circle of radius $|w|$. Furthermore, it satisfies $W \circ \sigma_u^r = e^{-it_0 u}W$, contradicting the weak-mixing assumption unless $t_0 = 0$.

In Theorem 5.6 of [PP90] it is shown that $\zeta_{-r}(s)(1-e^{P(-sr)})$ is nowhere-zero and analytic in a neighbourhood of $s = h(\sigma^r)$. We can thus deduce the singularity type of $h(\sigma^r)$ as follows.

$$\begin{aligned}\lim_{s\to h(\sigma^r)} \frac{1-e^{P(-sr)}}{s-h(\sigma^r)} &= \lim_{s\to 0} \frac{1-e^{P(-(s+h(\sigma^r))r)}}{s} \\ &= -\frac{\partial e^{P(-(s+h(\sigma^r))r)}}{\partial s}\Big|_{s=0} = \int r\, dm_{-h(\sigma^r)r} \neq 0,\end{aligned}$$

by Proposition 4.3.9. Thus $s = h(\sigma^r)$ is a simple pole. $\square$

In fact, this theory extends to zeta functions which account for weighting by Hölder potentials $F : \Sigma(\Gamma, r) \to \mathbb{R}$. Indeed, setting $f(x) = \int_0^{r(x)} F[x,t]\, dt$, consider the function

$$\zeta_{-r,F}(s) := \zeta(1, -sr + f) = \exp \sum_{n=1}^{\infty} \frac{1}{n} \sum_{\sigma^n(x)=x} e^{-sr^n(x)+f^n(x)}.$$

For a periodic point $x = \sigma^n(x)$, $f^n(x)$ is simply $\int_\eta F$, where $\eta \in \tilde{P}(\sigma^r)$ is the periodic orbit corresponding to x. This leads to an Euler product representation

$$\zeta_{-r,F}(s) = \prod_{\eta\in\mathcal{P}(\sigma^r)} \left(1 - e^{-s\ell(\eta)+\int_\eta F}\right)^{-1},$$

wherever $\zeta_{-r,F}$ converges. Analytic properties of $\zeta_{-r,F}$ are obtained exactly as for ζ_{-r} (see Chapter 6 of [PP90]). We summarise with the following.

Proposition 5.2.5. *Suppose σ^r is topologically weak-mixing. Then $\zeta_{-r,F}(s)$ is nowhere-zero and analytic for $\mathscr{R}(s) > P(F)$, and it has a nowhere-zero analytic extension to the line $\mathscr{R}(s) = P(F)$, except for a simple pole at $s = P(F)$.*

Some simple observations allow us to extend this further, introducing another weight function $G : \Sigma(\Gamma, r) \to \mathbb{R}$ and variable $z \in \mathbb{C}$. We consider the zeta function

$$\begin{aligned}\zeta_{-r,F,G}(s, z) &:= \prod_{\eta \in \mathcal{P}(\sigma^r)} \left(1 - e^{-s\ell(\eta) + \int_\eta F + z \int_\eta G}\right)^{-1} \\ &= \exp \sum_{n=1}^{\infty} \frac{1}{n} \sum_{\sigma^n(x)=x} e^{-sr^n(x)+f^n(x)+zg^n(x)},\end{aligned}$$

where $g(x) = \int_0^{r(x)} G[x, t]\, dt$. This function will be used in Section 5.4, along with the following result.

Proposition 5.2.6. *Suppose σ^r is topologically weak-mixing. Then $\zeta_{-r,F,G}(s, z)$ is nowhere-zero and analytic for $\mathscr{R}(s) > P(F)$ and $|z|$ sufficiently small (depending on s). Furthermore, it has a nowhere-zero analytic extension to the line $\mathscr{R}(s) = P(F)$ for $|z|$ sufficiently small (depending on s), except for a simple pole at $s = P(F)$.*

Proof. Since the pressure function P is continuous, when we have the strict inequality $\mathscr{R}(s) > P(F)$, sufficiently small $|z|$ will ensure $P(F + zG) < \mathscr{R}(s)$, so Proposition 5.2.2 and Corollary 4.3.11 give that $\zeta_{-r,F,G}$ is non-zero and analytic. For the critical line $\mathscr{R}(s) = P(F)$, we consider eigenvalues of the transfer operator as we did in Proposition 5.2.4. Fix $s = P(F) + it_0$. If there are arbitrarily small z for which $\mathcal{L}_{-sr+f+zg}$ has 1 as an eigenvalue, then $\mathcal{L}_{-sr+f}$ has 1 as an eigenvalue, which contradicts the weak-mixing assumption. $\square$

The Euler product representation is a convenient way to define a zeta function for hyperbolic flows $X^t : \Lambda \to \Lambda$.

Let $\varphi : \Lambda \to \mathbb{R}$ be Hölder continuous, and recall the notation for periodic orbits from Section 5.1. We define the zeta function for X, weighted with φ, by

$$\zeta_X(s, \varphi) := \prod_{\gamma \in \mathcal{P}} \left(1 - e^{-s\ell(\gamma) + \int_\gamma \varphi}\right)^{-1}.$$

Recall the semi-conjugacy $\pi : \Sigma(\Gamma, r) \to \Lambda$ from Theorem 4.2.15, from the suspension flow σ^r to X. Since π is in general not a bijection, it is not the case that $\zeta_X(\cdot, \varphi) = \zeta_{-r,\varphi\circ\pi}(\cdot)$. However, the analytic properties are still related through the following construction of Manning [Man71], adapted to the case of flows by Bowen [Bow73]. Details are also given in Appendix III in [PP90].

Theorem 5.2.7. *There is a finite family of suspension spaces $\Sigma(\Gamma_i, r_i)$, $0 \le i \le n$, satisfying the following.*

1. $\Gamma_0 = \Gamma$ *and* $r_0 = r$
2. *For each* $0 \le i \le n$*, the suspension flow* $\sigma_t^{r_i}$ *is semi-conjugate to* X^t *via a map* $\pi_i : \Sigma(\Gamma_i, r_i) \to \Lambda$*, where* $\pi_0 = \pi$.
3. π_i *is not surjective for any* $i \neq 0$.
4. *There exists* $\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_n \in \{0, 1\}$ *such that* $\varepsilon_0 = 0$*, and for all* $\gamma \in \mathcal{P}$,
$$\sum_{i=0}^{n} (-1)^{\varepsilon_i} \#\{\eta \in \mathcal{P}(\sigma^{r_i}) : \pi_i(\eta) = \gamma, \ell(\eta) = \ell(\gamma)\} = 1.$$
5. *For all Hölder continuous* $\varphi : \Lambda \to \mathbb{R}$ *and* $1 \le i \le n$, $P(\varphi \circ \pi_i) < P(\varphi)$.

An example of the usefulness of this result is the following corollary.

Corollary 5.2.8. *With the notation above, we have that*

$$\zeta_X(s, \varphi) = \zeta_{-r, \varphi\circ\pi}(s) \frac{\prod_{\varepsilon_i = 0 \text{ and } i > 0} \zeta_{-r_i, \varphi\circ\pi_i}}{\prod_{\varepsilon_i = 1} \zeta_{-r_i, \varphi\circ\pi_i}}.$$

Using Theorem 5.2.7 we may compare zeta functions for symbolic and hyperbolic flows, in particular obtaining analogues of Propositions 5.2.5 and 5.2.6.

Proposition 5.2.9. *Suppose X is topologically weak-mixing. Then $\zeta_X(s, \varphi)$ is nowhere-zero and analytic for $\mathscr{R}(s) > P(\varphi)$, and it has a nowhere-zero analytic extension to the line $\mathscr{R}(s) = P(\varphi)$, except for a simple pole at $s = P(\varphi)$.*

Proof. With the notation of Theorem 5.2.7, Proposition 5.2.2 says that each $\zeta_{-r_i, \varphi\circ\pi_i}$ is nowhere-zero and analytic on the half plane $\mathscr{R}(s) > P(\varphi \circ \pi_i)$, and $\zeta_{-r, \varphi\circ\pi}$ is nowhere-zero and analytic on the half plane $\mathscr{R}(s) > P(\varphi \circ \pi)$. Since

$$\max_i \{P(\varphi \circ \pi_i)\} < P(\varphi) = P(\varphi \circ \pi),$$

Corollary 5.2.8 says that the analyticity of ζ_X on $\mathscr{R}(s) \ge P(\varphi)$ is completely determined by that of $\zeta_{-r,\varphi\circ\pi}$. By Theorem 4.2.15, σ^r is topologically weak-mixing, so Proposition 5.2.5 completes the proof. □

The following is proved analogously.

Proposition 5.2.10. *Suppose X is topologically weak-mixing and ψ is Hölder continuous. Then $\zeta_X(s, \varphi + z\psi)$ is nowhere-zero and analytic for $\mathscr{R}(s) > P(\varphi)$ and $|z|$ sufficiently small (depending on s). Furthermore, it has a nowhere-zero analytic extension to the line $\mathscr{R}(s) = P(\varphi + z\psi)$ for $|z|$ sufficiently small (depending on s), except for a simple pole at $s = P(\varphi)$.*

5.3 Classical equidistribution theorems

For the remainder of this chapter we assume $X^t : M \to M$ is a topologically weak-mixing transitive Anosov flow. Fix a continuous function $\varphi : M \to \mathbb{R}$. For real numbers $a < b$, write

$$\pi_\varphi(T, \mathbb{1}_{[a,b]}) = \sum_{\gamma \in \mathcal{P}} \mathbb{1}_{[a,b]}(\ell(\gamma) - T) e^{\int_\gamma \varphi}.$$

For $\gamma \in \mathcal{P}$, define a probability measure μ_γ by

$$\int \psi \, d\mu_\gamma = \frac{1}{\ell(\gamma)} \int_\gamma \psi,$$

for each continuous $\psi : M \to \mathbb{R}$. Recall that μ_φ denotes an equilibrium state for φ, which is unique when φ is Hölder continuous. We will discuss the proof of the following result.

Theorem 5.3.1. *Suppose φ is Hölder continuous. Then, for $a < b$, the measures*

$$\frac{1}{\pi_\varphi(T, \mathbb{1}_{[a,b]})} \sum_{\gamma \in \mathcal{P}} \mathbb{1}_{[a,b]}(\ell(\gamma) - T) e^{\int_\gamma \varphi} \mu_\gamma$$

converge weak to μ_φ, as $T \to \infty$, and the same holds if we replace $[a, b]$ by (a, b), $(a, b]$ or $[a, b)$.*

As mentioned at the start of the chapter, the case $\varphi = 0$ is a classical theorem of Bowen [Bow72a] and was reproved using zeta function techniques by Parry [Par84]. For $\varphi = \varphi^u$ (defined in Section 4.3.2), the result is proved in [Par86] and the same arguments cover the cases where $P(\varphi) \geq 0$ (see [Par88], [PP90]). Below, we will show that to prove the theorem for $P(\varphi) < 0$, it is sufficient to be able to estimate the growth of $\pi_\varphi(T, \mathbb{1}_{[a,b]})$. An appropriate estimate is claimed as Proposition 3 of [Pol95], where it is is attributed to Parry [Par88], but the quoted result was only proved by Parry when $P(\varphi) > 0$. We fill this gap by showing the following.

Lemma 5.3.2. *For any continuous function $\varphi : M \to \mathbb{R}$ and $a < b$, we have*

$$\lim_{T\to\infty} \frac{1}{T} \log \pi_\varphi(T, \mathbb{1}_{[a,b]}) = P(\varphi).$$

Proof. We start by assuming that φ is Hölder continuous and $P(\varphi) > 0$. We follow the argument in [Par88]. By Proposition 5.2.9, the zeta function

$$\zeta_X(s, \varphi) = \prod_{\gamma\in\mathcal{P}} \left(1 - e^{-s\ell(\gamma)+\int_\gamma \varphi}\right)^{-1},$$

converges for $\mathscr{R}(s) > P(\varphi)$ and has a nowhere-zero analytic extension to $\mathscr{R}(s) \geq P(\varphi)$, apart from a simple pole at $s = P(\varphi)$ [PP90]. Since $P(\varphi)$ is simple, the logarithmic derivative $\frac{d}{ds} \log \zeta_X(s, \varphi)$ satisfies

$$\frac{d}{ds} \log \zeta_X(s, \varphi) = \frac{1}{s - P(\varphi)} + \alpha(s),$$

where α is analytic. Applying a Tauberian theorem as in [Par88], we deduce that

$$\sum_{\gamma\in\mathcal{P}_{\leq T}} e^{\int_\gamma \varphi} \sim \frac{e^{P(\varphi)T}}{P(\varphi)T}, \tag{5.1}$$

as $T \to \infty$, and hence that

$$\lim_{T\to\infty} \frac{1}{T} \log \pi_\varphi(T, \mathbb{1}_{[a,b]}) = P(\varphi).$$

To extend this growth rate estimate to that for an arbitrary Hölder continuous φ, choose $c > 0$ such that $P(\varphi) + c > 0$ and note that

$$e^{-c(T+b)} \pi_{\varphi+c}(T, \mathbb{1}_{[a,b]}) \leq \pi_\varphi(T, \mathbb{1}_{[a,b]}) \leq e^{-c(T+a)} \pi_{\varphi+c}(T, \mathbb{1}_{[a,b]}).$$

This gives us

$$\lim_{T\to\infty} \frac{1}{T} \log \pi_\varphi(T, \mathbb{1}_{[a,b]}) = -c + P(\varphi + c) = P(\varphi).$$

Note that such an argument does not apply to

$$\sum_{\gamma\in\mathcal{P}_{\leq T}} e^{\int_\gamma \varphi},$$

so our method cannot be used to obtain similar growth rates as in (5.1) when φ has non-positive pressure.

To complete the proof, if φ is only continuous, given $\varepsilon > 0$, we can find a Hölder continuous function φ' with $\|\varphi - \varphi'\|_\infty < \varepsilon$, so that

$$\begin{aligned} P(\varphi) - 2\varepsilon \le P(\varphi') - \varepsilon &\le \liminf_{T\to\infty} \frac{1}{T} \log \pi_\varphi(T, \mathbb{1}_{[a,b]}) \\ &\le \limsup_{T\to\infty} \frac{1}{T} \log \pi_\varphi(T, \mathbb{1}_{[a,b]}) \le P(\varphi') + \varepsilon \le P(\varphi) + 2\varepsilon, \end{aligned}$$

which gives the required limit. □

Remark. One can improve the growth rate estimate to

$$\pi_\varphi(T, \mathbb{1}_{[a,b]}) \sim \left(\int_a^b e^{P(\varphi)x}\, dx \right) \frac{e^{P(\varphi)T}}{T},$$

as $T \to \infty$, using a simplified version of the proof of Theorem 5.4.7 below.

With the growth rate estimate in Lemma 5.3.2, we can prove Theorem 5.3.1 in its full generality using a large deviations result of Pollicott [Pol95], following Kifer [Kif94]. First, recall that $\mathcal{M}(X)$ denotes the set of X^t invariant Borel probability measures on M. For $\mathcal{K} \subset \mathcal{M}(X)$ write

$$\Xi_\varphi(T, \mathbb{1}_{[a,b]}, \mathcal{K}) = \sum_{\substack{\gamma \in \mathcal{P} \\ \mu_\gamma \in \mathcal{K}}} \mathbb{1}_{[a,b]}(\ell(\gamma) - T) \exp\left(\int_\gamma \varphi \right).$$

Theorem 5.3.3 (Pollicott [Pol95])**.** *Suppose $\varphi : M \to \mathbb{R}$ is Hölder continuous. Then, for every weak* compact set $\mathcal{K} \subset \mathcal{M}(X) \setminus \{\mu_\varphi\}$ and $a > b$, we have*

$$\limsup_{T\to\infty} \frac{1}{T} \log \left(\frac{\Xi_\varphi(T, \mathbb{1}_{[a,b]}, \mathcal{K})}{\pi_\varphi(T, \mathbb{1}_{[a,b]})} \right) < 0.$$

Theorem 5.3.1 then follows from this. We omit the proof as it is almost identical to the proof that Theorem 5.4.12 implies Theorem 5.4.13 below.

Remark. If X^t is not weak-mixing then the periods $\ell(\gamma)$ are all integer multiples of some $c > 0$ and the result of Theorem 5.3.1 holds in this case provided $b - a \ge c$.

5.4 Equidistribution of null-homologous orbits

5.4.1 Anosov flows and homology

We wish to consider the homology of periodic orbits in the real and integral homology groups $H_1(M, \mathbb{R}) \cong \mathbb{R}^b$ and $H_1(M, \mathbb{Z}) \cong \mathbb{Z}^b \oplus \mathrm{Tor}$, where $b \geq 0$ is the first Betti number of M, and Tor is a finite abelian group.

The distribution of integral homology classes is studied following the main result in [PP86], which is as follows.

Theorem 5.4.1 (Parry–Pollicott [PP86])**.** *Let G be a finite group for which there is a surjective homomorphism $p : \pi_1(M) \to G$, and fix a conjugacy class $C \subset G$. Then*

$$\lim_{T \to \infty} \frac{\#\{\gamma \in \mathcal{P} : \ell(\gamma) \leq T \text{ and } C(p(\gamma)) = C\}}{\#\{\gamma \in \mathcal{P} : \ell(\gamma) \leq T\}} = \frac{\#C}{\#G},$$

where $C(p(\gamma))$ is the conjugacy class of $p(\gamma)$.

Recall that we have a quotient homomorphism of the abelianisation $\pi_1(M) \to H_1(M, \mathbb{Z})$. This can be used to prove the following.

Proposition 5.4.2 (Parry–Pollicott [PP86])**.** *The set of integral homology classes of periodic orbits of X^t generates $H_1(M, \mathbb{Z})$.*

Proof. Suppose otherwise. Then there is a finite index subgroup $H < H_1(M, \mathbb{Z})$ which contains $\{[\gamma] : \gamma \in \mathcal{P}\}$, where $[\gamma]$ is the integral homology class of γ in $H_1(M, \mathbb{Z})$. Then $G = H_1(M, \mathbb{Z})/H$ is a finite group with a surjective homomorphism (the composition of quotient maps) $\pi_1(M) \to G$. Under this map, each $\gamma \in \mathcal{P}$ is in the same coset $0 + H$, violating Theorem 5.4.1. $\square$

A stronger condition on X is the following.

Definition 5.4.3. We say that X is *homologically full* if every integral homology class in $H_1(M, \mathbb{Z})$ is represented by a periodic orbit.

Homological fullness can be characterised in terms of invariant measures as follows. For each $\nu \in \mathcal{M}(X)$, there is an associated homology class $\Phi_\nu \in H_1(M, \mathbb{R})$, called the *winding cycle* (or asymptotic cycle) for the measure. These cycles were introduced by Schwartzman [Sch57]. We may define Φ_ν using the duality $H_1(M, \mathbb{R}) = H^1(M, \mathbb{R})^*$ and the formula

$$\langle \Phi_\nu, [\omega] \rangle = \int \omega(X)\, d\nu,$$

where ω is a closed 1-form on M, $[\omega] \in H^1(M,\mathbb{R})$ is its cohomology class, and

$$\langle \cdot, \cdot \rangle : H_1(M,\mathbb{R}) \times H^1(M,\mathbb{R}) \to \mathbb{R}$$

is the duality pairing. This is well defined since if $[\omega'] = [\omega]$ then ω and ω' differ by an exact form $d\theta$, say, and we have

$$\int d\theta(X)\, d\nu = \int L_X \theta \, d\nu = 0,$$

That the final integral vanishes follows from the invariance of ν and the dominated convergence theorem.

The following proposition is a consequence of the results in [Sha93].

Proposition 5.4.4. *The following are equivalent:*

(i) X *is homologically full;*

(ii) the map $[\,\cdot\,] : \mathcal{P} \to H_1(M,\mathbb{Z})/\mathrm{Tor}$ *is a surjection;*

(iii) $0 \in \mathrm{int}(\{\Phi_\nu : \nu \in \mathcal{M}(X)\})$.

If X^t is the constant suspension of a diffeomorphism then it cannot be homologically full [Fri82]. In particular, if a transitive Anosov flow is homologically full then it is automatically weak-mixing.

The characterisation of homologically full transitive Anosov flows in part (iii) of Proposition 5.4.4 may be modified to give a statement in terms of the equilibrium states of Hölder continuous functions, as follows.

Proposition 5.4.5 (Sharp [Sha93])**.** *X is homologically full if and only if there exists a Hölder continuous function* $\varphi : M \to \mathbb{R}$ *with* $\Phi_{\mu_\varphi} = 0$.

5.4.2 Pressure and cohomology

Let $X^t : M \to M$ be a homologically full transitive Anosov flow on a manifold M whose first Betti number $b = \dim H_1(M,\mathbb{R})$ is at least 1. Let $\varphi : M \to \mathbb{R}$ be a Hölder continuous function. In this section, we define a pressure function on the cohomology group $H^1(M,\mathbb{R})$ that will allow us to identify the growth rate of periodic orbits in a fixed homology class, weighted by φ. (This generalises results in [Sha93] which were restricted to the case $\varphi = 0$.)

For a closed 1-form ω on M, let $\psi_\omega : M \to \mathbb{R}$ denote the function $\psi_\omega = \omega(X)$, i.e. for $x \in M$, $\psi_\omega(x) = \omega_x(X(x))$. If ω' is in the same cohomology class then

$\omega - \omega' = du$ for some $u \in C^\infty(M)$ and, for any periodic orbit γ,

$$\int_\gamma \psi_\omega - \int_\gamma \psi_{\omega'} = \int_\gamma du(X) = \int_\gamma L_X u = 0,$$

so, by Lemma 4.3.19, ψ_ω and $\psi_{\omega'}$ are X-cohomologous. Thus we can define $\psi_{[\omega]}$, where $[\omega]$ is the cohomology class of ω, as a function on M up to X-cohomology. In particular, this is sufficient for us to use $\psi_{[\omega]}$ to define a pressure function.

Define $\beta_\varphi : H^1(M, \mathbb{R}) \to \mathbb{R}$ by

$$\beta_\varphi([\omega]) = P(\varphi + \psi_{[\omega]}).$$

The next result identifies properties of β_φ and its minimum.

Proposition 5.4.6. *Let $\varphi : M \to \mathbb{R}$ be Hölder continuous. Then β_φ is strictly convex and there exists a unique $\xi(\varphi) \in H^1(M, \mathbb{R})$ such that*

$$\beta_\varphi(\xi(\varphi)) = \inf_{[\omega] \in H^1(M,\mathbb{R})} \beta_\varphi([\omega]).$$

Furthermore, $\mu_{\varphi+\psi_{\xi(\varphi)}}$ is the unique probability measure satisfying

$$h(\mu_{\varphi+\psi_{\xi(\varphi)}}) + \int \varphi \, d\mu_{\varphi+\psi_{\xi(\varphi)}} = \sup\left\{ h(\nu) + \int \varphi \, d\nu \, : \, \nu \in \mathcal{M}(X), \ \Phi_\nu = 0 \right\}.$$

Proof. Fix a basis $c_1, \ldots, c_b$ for the free $\mathbb{Z}$-module $H_1(M, \mathbb{Z})/\mathrm{Tor}$ (regarded as a lattice in $H_1(M, \mathbb{R})$) and take the dual basis $w_1, \ldots, w_b$ for $H^1(M, \mathbb{R})$, i.e.

$$\langle c_i, w_j \rangle = \delta_{ij},$$

where δ_{ij} is the Kronecker symbol. Write $\psi_i = \psi_{w_i}$, $i = 1, \ldots, b$. Then we can view β_φ as a function $\beta_\varphi : \mathbb{R}^b \to \mathbb{R}$ given by

$$\beta_\varphi(t) = P(\varphi + t_1\psi_1 + \cdots t_b\psi_b).$$

By Proposition 4.3.21, this function is strictly convex unless there is a non-zero $a = (a_1, \ldots, a_b) \in \mathbb{R}^b$ such that $a_1\psi_1 + \cdots a_b\psi_b$ is X-cohomologous to a constant. Since the flow is homologically full, we can find $\nu \in \mathcal{M}(X)$ with $\Phi_\nu = 0$, which is equivalent to $\int \psi_i \, d\nu = 0$, $i = 1, \ldots, b$, and therefore if $a_1\psi_1 + \cdots a_b\psi_b$ is X-cohomologous to a constant then the constant is zero. However, this implies that the real homology classes of all periodic orbits lie in the hyperplane $a_1x_1 + \cdots + a_bx_b = 0$ in $H_1(M, \mathbb{R}) \cong \mathbb{R}^b$, which contradicts Proposition 5.4.2, that the homology classes of periodic orbits generate $H_1(M,\mathbb{Z})$ as a group. So β_φ is strictly convex.

We now show that β_φ has a finite minimum. Since β_φ is strictly convex, this minimum will automatically be unique. Since the flow is homologically full, it follows from [Sha93] that β_0 is strictly convex and has a finite minimum. Noting that

$$|\beta_\varphi(t) - \beta_0(t)| \leq \|\varphi\|_\infty,$$

we see that β_φ also has a finite minimum. We call the point where the minimum occurs $\xi(\varphi)$. Clearly, $\nabla\beta_\varphi(\xi(\varphi)) = 0$.

Writing $\psi = (\psi_1, \ldots, \psi_b)$, Proposition 4.3.21 gives that

$$\nabla\beta_\varphi(t) = \int \psi \, d\mu_{\varphi + t_1\psi_1 + \cdots + t_b\psi_b}.$$

In particular, this means

$$\int \psi \, d\mu_{\varphi+\psi_{\xi(\varphi)}} = \nabla\beta_\varphi(\xi(\varphi)) = 0.$$

In terms of the winding cycle, this is $\Phi_{\mu_{\varphi+\psi_{\xi(\varphi)}}} = 0$.

For the second part of the statement, suppose $\Phi_\nu = 0$. Then $\int \psi_{\xi(\varphi)} \, d\nu = 0$ and, using the definition of equilibrium state,

$$\begin{aligned} h(\nu) + \int \varphi \, d\nu &= h(\nu) + \int (\varphi + \psi_{\xi(\varphi)}) \, d\nu \\ &\leq h(\mu_{\varphi+\psi_{\xi(\varphi)}}) + \int (\varphi + \psi_{\xi(\varphi)}) \, d\mu_{\varphi+\psi_{\xi(\varphi)}} \\ &= h(\mu_{\varphi+\psi_{\xi(\varphi)}}) + \int \varphi \, d\mu_{\varphi+\psi_{\xi(\varphi)}}. \end{aligned}$$

Thus we must have that

$$h(\mu_{\varphi+\psi_{\xi(\varphi)}}) + \int \varphi \, d\mu_{\varphi+\psi_{\xi(\varphi)}} = \sup\left\{ h(\nu) + \int \varphi \, d\nu \,:\, \nu \in \mathcal{M}(X),\ \Phi_\nu = 0 \right\},$$

as required. Uniqueness follows from the fact that the inequality above is strict if $\nu \neq \mu_{\varphi+\psi_{\xi(\varphi)}}$. □

5.4.3 Weighted asymptotics for orbits in a fixed homology class

Let $\varphi : M \to \mathbb{R}$ be Hölder continuous and let β_φ and $\xi(\varphi)$ be defined as in Section 5.4.2. To lighten the notation, we shall write

$$\xi = \xi(\varphi) \quad \text{and} \quad \beta = \beta_\varphi(\xi(\varphi)).$$

For $\alpha \in H_1(M,\mathbb{Z})/\mathrm{Tor}$, a compactly supported function $g : \mathbb{R} \to \mathbb{R}$ and $T > 0$, write

$$\pi_\varphi(T,\alpha,g) = \sum_{\gamma \in \mathcal{P}(\alpha)} g(\ell(\gamma) - T) \exp\left(\int_\gamma \varphi\right),$$

where $\mathcal{P}(\alpha) = \{\gamma \in \mathcal{P} : [\gamma] = \alpha\}$. We will prove that the following asymptotic formula holds.

Theorem 5.4.7. *Let $X^t : M \to M$ be a homologically full transitive Anosov flow and let $\varphi : M \to \mathbb{R}$ be a Hölder continuous function. Then, for every compactly supported continuous function $g : \mathbb{R} \to \mathbb{R}$, we have*

$$\pi_\varphi(T,\alpha,g) \sim \frac{1}{(2\pi)^{b/2}\sqrt{\det \nabla^2 \beta_\varphi(\xi)}} \left(\int_{-\infty}^{\infty} e^{\beta x} g(x)\, dx\right) e^{-\langle \alpha, \xi\rangle} \frac{e^{\beta T}}{T^{1+b/2}},$$

as $T \to \infty$. In particular,

$$\lim_{T\to\infty} \frac{1}{T} \log \pi_\varphi(T,\alpha,g) = \beta.$$

Assuming this, an approximation argument immediately gives the following corollary. We shall see later in the section that this implies the equidistribution result we seek.

Corollary 5.4.8. *For real numbers $a < b$,*

$$\lim_{T\to\infty} \frac{1}{T} \log \pi_\varphi(T,\alpha,\mathbb{1}_{[a,b]}) = \beta$$

and the same holds if we replace $[a,b]$ by (a,b), $(a,b]$ or $[a,b)$.

We proceed with the proof of Theorem 5.4.7, following the analysis of [BL98]. For $p \in \mathbb{R}$, δ_p denotes the Dirac measure giving mass 1 to p. For $\varsigma \in \mathbb{R}$, define measures $\mathfrak{M}_{T,\alpha,\varphi,\varsigma}$ on $\mathbb{R}$ by

$$\mathfrak{M}_{T,\alpha,\varphi,\varsigma} = \sum_{\gamma \in \mathcal{P}(\alpha)} e^{-\varsigma \ell(\gamma) + \int_\gamma \varphi + \langle \alpha, \xi\rangle} \delta_{\ell(\gamma) - T}.$$

Write $g_\varsigma(x) = e^{-\varsigma x} g(x)$. We then have the following.

Lemma 5.4.9. *For all $\varsigma \in \mathbb{R}$, we have*

$$\pi_\varphi(T,\alpha,g_\varsigma) = e^{\varsigma T - \langle \alpha,\xi\rangle} \int g \, d\mathfrak{M}_{T,\alpha,\varphi,\varsigma}.$$

Proof. The result follows from the direct calculation

$$\begin{aligned}
\int g \, d\mathfrak{M}_{T,\alpha,\varphi,\varsigma} &= \sum_{\gamma \in \mathcal{P}(\alpha)} g(\ell(\gamma) - T) e^{-\varsigma \ell(\gamma) + \int_\gamma \varphi + \langle \alpha, \xi \rangle} \\
&= e^{-\varsigma T} \sum_{\gamma \in \mathcal{P}(\alpha)} g(\ell(\gamma) - T) e^{-\varsigma(\ell(\gamma) - T) + \int_\gamma \varphi + \langle \alpha, \xi \rangle} \\
&= e^{-\varsigma T} g_\varsigma(\ell(\gamma) - T) e^{\int_\gamma \varphi + \langle \alpha, \xi \rangle} \\
&= e^{-\varsigma T + \langle \alpha, \xi \rangle} \pi_\varphi(T, \alpha, g_\varsigma).
\end{aligned}$$

□

To analyse the measure $\mathfrak{M}_{T,\alpha,\varphi,\varsigma}$, we introduce a complex function

$$Z(s,z) = \sum_{\gamma \in \mathcal{P}} e^{-s\ell(\gamma) + \int_\gamma \varphi + \langle [\gamma], z \rangle},$$

defined, where the series converges, for $(s,z) \in \mathbb{C} \times \mathbb{C}^b / i\mathbb{Z}^b$. We are only interested in z of the form $z = \xi + iv$, with $v \in \mathbb{R}^b / \mathbb{Z}^b$. We will relate $Z(s,z)$ to the logarithm of the zeta function

$$\zeta_X(s,z) := \zeta_X(s, \varphi + \langle \psi, z \rangle) = \prod_{\gamma \in \mathcal{P}} \left(1 - e^{-s\ell(\gamma) + \int_\gamma \varphi + \langle [\gamma], z \rangle}\right)^{-1}.$$

We see that we have

$$\log \zeta_X(s,z) = \sum_{n=1}^{\infty} \frac{1}{n} Z(ns, nz).$$

Theorem 5.2.10 tells us that $\zeta_X(s, \xi + iv)$ converges absolutely for $\mathscr{R}(s) > \beta$ and $|\xi + iv|$ sufficiently small, and that the analytic extension is well understood. We need to show that $Z(s, \xi + iv)$ behaves like $\log \zeta_X(s, \xi + iv)$ and we do this by showing that their difference converges absolutely in a larger half-plane.

Lemma 5.4.10. *There exists $\varepsilon > 0$ such that $Z(s, \xi + iv) - \log \zeta_X(s, \xi + iv)$ converges absolutely for $\mathscr{R}(s) > \beta - \varepsilon$ and $|\xi + iv|$ sufficiently small.*

Proof. By Theorem 5.2.7, it suffices to prove that the analogous statement holds for the suspension flow σ^r from Theorem 4.2.15. Precisely, define the function

$$Z_r(s, \xi + iv) = \sum_{\eta \in \mathcal{P}(\sigma^r)} e^{-s\ell(\eta) + \int_\eta \varphi \circ \pi + \langle \int_\eta \psi \circ \pi, z \rangle},$$

where $\psi = (\psi_1, \ldots, \psi_b)$ is as in the proof of Proposition 5.4.6 and π is the symbolic coding. For convenience, set $F = \varphi \circ \pi$ and $G = \psi \circ \pi$. If $Z_r(s,z) - \log \zeta_{-r,F,G}(s,z)$

converges absolutely for $\mathscr{R}(s) > \beta - \varepsilon$, then Z_r has the convergence properties of $\zeta_{-r,F,G}$. Then Theorem 5.2.7 compares Z with Z_r and ζ_X with $\zeta_{-r,F,G}$ to give the result.

Let us now show that $Z_r(s,z) - \log \zeta_{-r,F,G}(s,z)$ indeed converges absolutely for $\mathscr{R}(s) > \beta - \varepsilon$. Defining $q_{\varphi+\psi_\xi} : \Sigma(\Gamma) \to \mathbb{R}$ by

$$q_{\varphi+\psi_\xi}(x) = \int_0^{r(x)} (\varphi + \psi_\xi)(\pi[x,t])\, dt,$$

Corollary 4.3.11 gives that $P(-\beta r + q_{\varphi+\psi_\xi}) = 0$, and Proposition 4.3.4 says that there exists a continuous function $u : \Sigma(\Gamma) \to \mathbb{R}$ such that $-\beta r + q_{\varphi+\psi_\xi} + u \circ \sigma - u$ is strictly negative. By compactness, this function is bounded above by $-3\varepsilon\|r\|_\infty$, for some $\varepsilon > 0$. We then have that

$$-\beta\ell(\eta) + \int_\eta F + \left\langle \int_\eta G, \xi \right\rangle \leq -3\varepsilon\ell(\eta),$$

for all $\eta \in \mathcal{P}(\sigma^r)$. For $\varsigma > \beta - \varepsilon$, we have

$$-\varsigma\ell(\eta) + \int_\eta F + \left\langle \int_\eta G, \xi \right\rangle \leq -2\varepsilon < 0$$

and hence

$$\begin{aligned}
&|\log \zeta_{-r,F,G}(\varsigma,\xi) - Z_r(\varsigma,\xi)| = \sum_{n=2}^{\infty} \frac{1}{n} Z_r(n\varsigma, n\xi) \\
&\leq \sum_{n=2}^{\infty} Z_r(n\varsigma, n\xi) = \sum_{n=2}^{\infty} \sum_{\eta\in\mathcal{P}(\sigma^r)} e^{n(-\varsigma\ell(\eta)+\int_\eta F + \langle \int_\eta G,\xi\rangle)} \\
&= \sum_{\eta\in\mathcal{P}(\sigma^r)} \frac{e^{2(-\varsigma\ell(\eta)+\int_\eta F+\langle\int_\eta G,\xi\rangle)}}{1-e^{-\varsigma\ell(\eta)+\int_\eta F+\langle\int_\eta G,\xi\rangle}} \leq C \sum_{\eta\in\mathcal{P}(\sigma^r)} e^{-\varsigma\ell(\eta)+\int_\eta \varphi\circ\pi+\langle\int_\eta G,\xi\rangle} e^{-2\varepsilon\ell(\eta)} \\
&= C \sum_{\eta\in\mathcal{P}(\sigma^r)} e^{-(\varsigma+2\varepsilon)\ell(\eta)+\int_\eta F+\langle\int_\eta G,\xi\rangle} = C \sum_{\eta\in\mathcal{P}(\sigma^r)} e^{-(\beta+\varepsilon)\ell(\eta)+\int_\eta F+\langle\int_\eta G,\xi\rangle} \\
&= C Z_r(\beta+\varepsilon,\xi) < \infty,
\end{aligned}$$

where C is some positive constant (depending on ε). $\square$

For $k \in \mathbb{N}$, let $C^k(\mathbb{R}\times\mathbb{R}^b/\mathbb{Z}^b, \mathbb{R})$ denote the set of C^k functions from $\mathbb{R}\times\mathbb{R}^b/\mathbb{Z}^b$ to $\mathbb{R}$, equipped with the topology of uniform convergence of the jth derivatives, for $0 \leq j \leq k$, on compact sets. The following result is, apart from the weighting by φ, a simplified version of Proposition 2.1 in [BL98].

Proposition 5.4.11. *For each $k \in \mathbb{N}$, there exists*

(i) an open neighbourhood $U = U_1 \times U_2$ of $(0,0) \in \mathbb{R} \times \mathbb{R}^b/\mathbb{Z}^b$;

(ii) a function $p \in C^k(\mathbb{R} \times \mathbb{R}^b/\mathbb{Z}^b, \mathbb{R})$ which satisfies $p(0,0) = 1$ and vanishes outside of U;

(iii) a function $A \in C^k(\mathbb{R} \times \mathbb{R}^b/\mathbb{Z}^b, \mathbb{R})$ such that

$$\lim_{\varsigma \searrow \beta_\varphi(\xi(\varphi))} Z(\varsigma + it, 1, \xi + iv) = -p(t,v)\log(\beta + it - \beta_\varphi(\xi + iv)) + A(t,v),$$

where $\beta_\varphi(u + iv)$ is an analytic extension of $\beta_\varphi(u)$ to

$$\{u \in \mathbb{R}^b : \|u - \xi(\varphi)\| < \delta\} \times U_2,$$

for some small $\delta > 0$.

In particular, the function

$$(t,v) \mapsto \lim_{\varsigma \searrow \beta_\varphi(\xi(\varphi))} Z(\varsigma + it, 1, \xi + iv)$$

is locally integrable on $\mathbb{R} \times \mathbb{R}^b/\mathbb{Z}^b$. Furthermore, for any compact $K \subset \mathbb{R} \times \mathbb{R}^b/\mathbb{Z}^b$, there exist constants $C_1, C_2 > 0$ such that, for any $\varsigma > \beta$, we have

$$|Z(\varsigma + it, \xi + iv)| \leq \begin{cases} -C_1 \log|\beta + it - \beta_\varphi(\xi(\varphi) + iv)| \text{ if } (t,v) \in U, \\ C_2 \text{ if } (t,v) \in K \setminus U. \end{cases}$$

One can then proceed as in section 2 of [BL98] to show that for all continuous compactly supported $g : \mathbb{R} \to \mathbb{R}$,

$$\lim_{T \to \infty} \int_{\mathbb{R}} g \, d\mathfrak{m}_T = \int_{\mathbb{R}} g \, d\mathrm{Leb},$$

where we have defined

$$\mathfrak{m}_T := (2\pi)^{b/2}\sqrt{\det \nabla^2 \beta_\varphi(\xi(\varphi))}T^{1+b/2}e^{\langle \alpha, \xi \rangle}\mathfrak{M}_{T,\alpha,\varphi,\beta}.$$

Finally, applying this to g_β, we can use Lemma 5.4.9 to obtain Theorem 5.4.7.

5.4.4 Large deviations and weighted equidistribution

For $\delta > 0$, write

$$\Xi_\varphi(T, \alpha, \mathbb{1}_{[a,b]}, \mathcal{K}) = \sum_{\substack{\gamma \in \mathcal{P}(\alpha) \\ \mu_\gamma \in \mathcal{K}}} \mathbb{1}_{[a,b]}(\ell(\gamma) - T) \exp\left(\int_\gamma \varphi\right).$$

The growth rate result in Corollary 5.4.8 implies the following large deviations estimate.

Theorem 5.4.12. *Let $X^t : M \to M$ be a homologically full transitive Anosov flow and let $\varphi : M \to \mathbb{R}$ be a Hölder continuous function. Then, for every compact set $\mathcal{K} \subset \mathcal{M}(X)$ such that $\mu_{\varphi+\psi_\xi} \notin \mathcal{K}$, and real numbers $a < b$, we have*

$$\limsup_{T\to\infty} \frac{1}{T} \log\left(\frac{\Xi_\varphi(T, \alpha, \mathbb{1}_{[a,b]}, \mathcal{K})}{\pi_\varphi(T, \alpha, \mathbb{1}_{[a,b]})}\right) < 0.$$

The same holds if we replace $[a,b]$ by (a,b), $(a,b]$ or $[a,b)$.

Proof. This is a standard type of argument which originates from the work of Kifer (for example [Kif90]). Define a function $Q : C(M,\mathbb{R}) \to \mathbb{R}$ by

$$Q(\chi) = P(\varphi + \psi_\xi + \chi).$$

From Corollary 5.4.8, we have

$$\lim_{T\to\infty} \frac{1}{T} \log \pi_\varphi(T, \alpha, \mathbb{1}_{[a,b]}) = \beta = P(\varphi + \psi_\xi) = Q(0). \tag{5.2}$$

Also, for every $\chi \in C(M,\mathbb{R})$, we have

$$\begin{aligned}\sum_{\gamma \in \mathcal{P}(\alpha)} \mathbb{1}_{[a,b]}(\ell(\gamma) - T) e^{\int_\gamma (\varphi+\chi)} &= e^{-\langle \alpha, \xi\rangle} \sum_{\gamma \in \mathcal{P}(\alpha)} \mathbb{1}_{[a,b]}(\ell(\gamma) - T) e^{\int_\gamma (\varphi+\psi_\xi+\chi)} \\ &\le e^{-\langle \alpha, \xi\rangle} \sum_{\gamma \in \mathcal{P}} \mathbb{1}_{[a,b]}(\ell(\gamma) - T) e^{\int_\gamma (\varphi+\psi_\xi+\chi)},\end{aligned}$$

since $\psi_\xi = \langle \psi, \xi\rangle$, and $\int_\gamma \psi$ is the homology of γ. By Lemma 5.3.2, this gives

$$\limsup_{T\to\infty} \frac{1}{T} \log \sum_{\gamma \in \mathcal{P}(\alpha)} \mathbb{1}_{[a,b]}(\ell(\gamma) - T) e^{\int_\gamma (\varphi+\chi)} \le Q(\chi). \tag{5.3}$$

Now define

$$\delta := \inf_{\nu \in \mathcal{K}} \sup_{\chi \in C(M,\mathbb{R})} \left(\int \chi \, d\nu - Q(\chi)\right).$$

Given $\varepsilon > 0$, it follows from the definition of δ that for every $\nu \in \mathcal{K}$, there exists $\chi \in C(M, \mathbb{R})$ such that

$$\int \chi \, d\nu - Q(\chi) > \delta - \varepsilon.$$

Hence

$$\mathcal{K} \subset \bigcup_{\chi \in C(M,\mathbb{R})} \left\{ \nu \in \mathcal{M}(X) : \int \chi \, d\nu - Q(\chi) > \delta - \varepsilon \right\}.$$

Since $\mathcal{K}$ is compact, we can find a finite set of functions $\chi_1, \ldots, \chi_k \in C(M, \mathbb{R})$ with

$$\mathcal{K} \subset \bigcup_{i=1}^{k} \left\{ \nu \in \mathcal{M}(X) : \int \chi_i \, d\nu - Q(\chi_i) > \delta - \varepsilon \right\}.$$

We then have

$$\begin{aligned} \Xi_\varphi(T, \alpha, \mathbb{1}_{[a,b]}, \mathcal{K}) &\le \sum_{i=1}^{k} \sum_{\substack{\gamma \in \mathcal{P}(\alpha) \\ \int \chi_i \, d\mu_\gamma - Q(\chi_i) > \delta - \varepsilon}} \mathbb{1}_{[a,b]}(\ell(\gamma) - T) e^{\int_\gamma \varphi} \\ &\le \sum_{i=1}^{k} \sum_{\gamma \in \mathcal{P}(\alpha)} \mathbb{1}_{[a,b]}(\ell(\gamma) - T) e^{-\ell(\gamma)(Q(\chi_i) + \delta - \varepsilon) + \int_\gamma (\varphi + \chi_i)}. \end{aligned}$$

Recalling the bound (5.3), we have

$$\limsup_{T \to \infty} \frac{1}{T} \log \Xi_\varphi(T, \alpha, \mathbb{1}_{[a,b]}, \mathcal{K}) \le -\delta + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we can combine this with (5.2) to obtain

$$\limsup_{T \to \infty} \frac{1}{T} \log \left(\frac{\Xi_\varphi(T, \alpha, \mathbb{1}_{[a,b]}, \mathcal{K})}{\pi_\varphi(T, \alpha, \mathbb{1}_{[a,b]})} \right) \le -\delta - Q(0).$$

To complete the proof, we show that $\delta + Q(0) > 0$. For any measure $\nu \in$

$\mathcal{M}(X)$, we have

$$
\begin{aligned}
&\sup_{\chi\in C(M,\mathbb{R})}\left(\int \chi\,d\nu - Q(\chi)\right) + Q(0)\\
&= \sup_{\chi\in C(M,\mathbb{R})}\left(\int \chi\,d\nu - P(\varphi+\psi_\xi+\chi)\right) + P(\varphi+\psi_\xi)\\
&= \sup_{\chi\in C(M,\mathbb{R})}\left(\int (\varphi+\psi_\xi+\chi)\,d\nu - P(\varphi+\psi_\xi+\chi)\right) + P(\varphi+\psi_\xi) - \int \varphi+\psi_\xi\,d\nu\\
&= \sup_{\chi\in C(M,\mathbb{R})}\left(\int \chi\,d\nu - P(\chi)\right) + P(\varphi+\psi_\xi) - \int (\varphi+\psi_\xi)\,d\nu\\
&= -\inf_{\chi\in C(M,\mathbb{R})}\left(P(\chi) - \int \chi\,d\nu\right) + P(\varphi+\psi_\xi) - \int (\varphi+\psi_\xi)\,d\nu\\
&= -h(\nu) + P(\varphi+\psi_\xi) - \int (\varphi+\psi_\xi)\,d\nu,
\end{aligned}
$$

where the last equality comes from Lemma 4.3.17. If $\nu \in \mathcal{K}$ then $\nu \neq \mu_{\varphi+\psi_\xi}$ and the uniqueness of equilibrium states gives that

$$-h(\nu) + P(\varphi+\psi_\xi) - \int (\varphi+\psi_\xi)\,d\nu > 0.$$

Since, by Lemma 4.3.17, the map

$$\nu \mapsto -h(\nu) + P(\varphi+\psi_\xi) - \int (\varphi+\psi_\xi)\,d\nu$$

is lower semi-continuous on $\mathcal{M}(X)$ and $\mathcal{K}$ is compact, we see that $\delta + Q(0) > 0$, as required. $\square$

We can now obtain the weighted equidistribution theorem for periodic orbits in a homology class.

Theorem 5.4.13. *Let $X^t : M \to M$ be a homologically full Anosov flow. Let $\varphi : M \to \mathbb{R}$ be Hölder continuous. Then the measures*

$$\frac{1}{\pi_\varphi(T,\alpha,\mathbb{1}_{[a,b]})}\sum_{\gamma\in\mathcal{P}(\alpha)} \mathbb{1}_{[a,b]}(\ell(\gamma)-T)e^{\int_\gamma \varphi}\mu_\gamma$$

converge weak to $\mu_{\varphi+\psi_\xi}$, as $T\to\infty$, and the same holds if we replace $[a,b]$ by (a,b), $(a,b]$ or $[a,b)$.*

Proof. Let $\chi \in C(M, \mathbb{R})$. Given $\varepsilon > 0$, let $\mathcal{K} \subset \mathcal{M}(X)$ be the compact set

$$\mathcal{K} = \left\{ \nu \in \mathcal{M}(X) : \left| \int \chi \, d\nu - \int \chi \, d\mu_{\varphi+\psi_\xi} \right| \geq \varepsilon \right\}.$$

Using Theorem 5.3.3, we have

$$\begin{aligned} &\frac{1}{\pi_\varphi(T, \alpha, \mathbb{1}_{[a,b]})} \sum_{\gamma \in \mathcal{P}(\alpha)} \mathbb{1}_{[a,b]}(\ell(\gamma) - T) e^{\int_\gamma \varphi} \int \chi \, d\mu_\gamma \\ &= \frac{1}{\pi_\varphi(T, \alpha, \mathbb{1}_{[a,b]})} \sum_{\substack{\gamma \in \mathcal{P}(\alpha) \\ \mu_\gamma \notin \mathcal{K}}} \mathbb{1}_{[a,b]}(\ell(\gamma) - T) e^{\int_\gamma \varphi} \int \chi \, d\mu_\gamma + O(e^{-cT}), \end{aligned}$$

for some $c > 0$. Since

$$\begin{aligned} \frac{1}{\pi_\varphi(T, \alpha, \mathbb{1}_{[a,b]})} \sum_{\substack{\gamma \in \mathcal{P}(\alpha) \\ \mu_\gamma \notin \mathcal{K}}} \mathbb{1}_{[a,b]}(\ell(\gamma) - T) e^{\int_\gamma \varphi} \int \chi \, d\mu_\gamma &= (1 - O(e^{-cT})) \int \chi \, d\mu_{\varphi+\psi_\xi} \\ + \frac{1}{\pi_\varphi(T, \alpha, \mathbb{1}_{[a,b]})} \sum_{\substack{\gamma \in \mathcal{P}(\alpha) \\ \mu_\gamma \notin \mathcal{K}}} \mathbb{1}_{[a,b]}(\ell(\gamma) - T) e^{\int_\gamma \varphi} &\left(\int \chi \, d\mu_\gamma - \int \chi \, d\mu_{\varphi+\psi_\xi} \right), \end{aligned}$$

we see that

$$\begin{aligned} \int \chi \, d\mu_{\varphi+\psi_\xi} - \varepsilon &\leq \liminf_{T \to \infty} \frac{1}{\pi_\varphi(T, \alpha, \mathbb{1}_{[a,b]})} \sum_{\gamma \in \mathcal{P}(\alpha)} \mathbb{1}_{[a,b]}(\ell(\gamma) - T) e^{\int_\gamma \varphi} \int \chi \, d\mu_\gamma \\ &\leq \limsup_{T \to \infty} \frac{1}{\pi_\varphi(T, \alpha, \mathbb{1}_{[a,b]})} \sum_{\gamma \in \mathcal{P}(\alpha)} \mathbb{1}_{[a,b]}(\ell(\gamma) - T) e^{\int_\gamma \varphi} \int \chi \, d\mu_\gamma \\ &\leq \int \chi \, d\mu_{\varphi+\psi_\xi} + \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, this completes the proof. $\square$

Chapter 6

Helicity and linking numbers for Anosov flows

This chapter concerns connections between dynamical systems, knots and helicity of vector fields. More specifically, for a divergence-free vector field on a closed 3-manifold that generates an Anosov flow, we show that the helicity of the vector field may be recovered as the limit of appropriately weighted averages of linking numbers of periodic orbits, regarded as knots. This is very much inspired by results of Contreras [Con95] about the linking of periodic orbits of hyperbolic flows on S^3, and it complements a classical result of Arnold (whose proof was completed by Vogel) that, when the manifold is a real homology 3-sphere, the helicity may be obtained as the limit of the normalised linking numbers of typical pairs of long trajectories.

The equidistribution theory introduced in Chapter 5 is essential to prove our main results, and has been submitted for publication together with the contents of this chapter.

6.1 Helicity

Throughout this chapter, M will be a smooth closed connected oriented 3-manifold.

6.1.1 Vector fields and forms

We briefly recall some background on vector fields and forms. Suppose M has a volume form Ω, and let X be a vector field on M generating a flow $X^t : M \to M$.

The divergence of X, denoted by $\mathrm{div}X$, satisfies

$$L_X\Omega = (\mathrm{div}X)\Omega.$$

We say that X is *divergence-free* if $\mathrm{div}X$ is identically zero, which is equivalent to the flow X^t preserving the volume measure which arises from Ω. For the remainder of the section, we will assume that this holds.

For our next assumption we define, for each $k \in \{1, 2, 3\}$, the *interior product* $i_X : \Omega^k(M) \to \Omega^{k-1}(M)$ by

$$i_X\omega(v_1, \ldots, v_{k-1}) = \omega(X, v_1, \ldots, v_{k-1}).$$

We will assume X^t is *null-homologous*, meaning that $i_X\Omega$ is exact. We will characterise this property in terms of asymptotic cycles, after stating some properties of i_X.

If $\omega \in \Omega^k(M)$, $\eta \in \Omega^l(M)$ then

$$i_X(\omega \wedge \eta) = i_X\omega \wedge \eta + (-1)^k(\omega \wedge i_X\eta).$$

In particular, if ω is a 1-form then, since $\omega \wedge \Omega = 0$,

$$i_X\omega \wedge \Omega - \omega \wedge i_X\Omega = i_X(\omega \wedge \Omega) = 0,$$

and so

$$i_X\omega \wedge \Omega = \omega \wedge i_X\Omega \tag{6.1}$$

The Lie derivative, exterior derivative and interior product are related by *Cartan's magic formula*

$$L_X\omega = i_X d\omega + d(i_X\omega). \tag{6.2}$$

Let m denote the volume measure arising from Ω (normalised to be a probability measure). We then have the following key result.

Lemma 6.1.1. *$i_X\Omega$ is exact if and only if $\Phi_m = 0$.*

Proof. Since X is divergence-free, we have $L_X\Omega = 0$. Since $d\Omega = 0$, Cartan's magic formula (6.2) gives $d(i_X\Omega) = 0$, i.e., $i_X\Omega$ is closed. Let $[i_X\Omega] \in H^2(M, \mathbb{R})$ be its cohomology class; we claim that $[i_X\Omega]$ and Φ_m are Poincaré duals. To see this, let

ω be a closed 1-form, then, by (6.1),

$$\begin{aligned}\langle [i_X\Omega], [\omega]\rangle &= \int_M \omega \wedge i_X\Omega = \int_M i_X\omega \wedge \Omega \\ &= \int_M \omega(X)\,\Omega = \int \omega(X)\,dm = \langle \Phi_m, [\omega]\rangle\end{aligned}$$

where the first term is the pairing of $H^2(M,\mathbb{R})$ and $H^1(M,\mathbb{R})$. Therefore, $[i_X\Omega] = 0$ if and only if $\Phi_m = 0$. $\square$

Now suppose that M is equipped with a Riemannian metric ρ (consistent with the volume form Ω). Recalling Section 4.3.2, when X^t is Anosov we have that m is equal to the equilibrium state for the potential φ^u. Thus Lemma 6.1.1 and Proposition 5.4.5 tell us that if X is null-homologous it is homologically full (noting that volume-preserving flows are automatically transitive). Furthermore, with the notation from Section 5.4.2

$$\nabla \beta_{\varphi^u}(0) = \int \psi \, d\mu_{\varphi^u} = \int \psi \, dm = 0,$$

so $\xi(\varphi^u) = 0$. Thus, for null-homologous flows, a special case of Theorem 5.4.13 is the following.

Theorem 6.1.2. *Let $X^t : M \to M$ be a null-homologous volume-preserving transitive Anosov flow on a closed oriented 3-manifold. Then the measures*

$$\frac{1}{\pi_{\varphi^u}(T,\alpha,\mathbb{1}_{[a,b]})} \sum_{\gamma\in\mathcal{P}(\alpha)} \mathbb{1}_{[a,b]}(\ell(\gamma)-T)e^{\int_\gamma \varphi^u}\mu_\gamma$$

converge weak to m, as $T\to\infty$, and the same holds if we replace $[a,b]$ by (a,b), $(a,b]$ or $[a,b)$.*

Theorem 6.1.2 applies in the case of Example 4.1.2, since there M is a real (and rational) homology 3-sphere, i.e. $H_1(M,\mathbb{R}) = \{0\}$ (see [Deh17]). This means any volume-preserving flow is null-homologous, so, since the geodesic flow is volume-preserving and transitive, all assumptions are satisfied. However, in that case all periodic orbits are null-homologous, so the result is identical to in the classical equidistribution theory. In Example 4.1.1, where Theorem 6.1.2 also applies, the situation is different, as there are a countable infinity of periodic orbits in each homology class (see [Sha93]).

6.1.2 Defining helicity

We now define helicity. Let M be a closed oriented 3-manifold with (normalised) volume form Ω. Let X be a divergence-free vector field on M with associated volume-preserving flow $X^t : M \to M$. We assume that X is null-homologous.

Definition 6.1.3. Any 1-form α such that $i_X\Omega = d\alpha$ is called a *form potential* of X. The helicity $\mathcal{H}(X)$ of X is defined by

$$\mathcal{H}(X) = \int_M \alpha \wedge i_X\Omega,$$

where α is a form potential.

Remark. Since X is null-homologous, there exists a form potential α. Any two form potentials must differ by a closed 1-form ω, but by Lemma 6.1.1,

$$\int_M \omega \wedge i_X\Omega = \int_M \omega(X)\,\Omega = 0,$$

so the helicity is independent of this choice.

Helicity was introduced by Woltjer [Wol58], Moreau [Mor61] and Moffat [Mof69] and is an invariant (of volume-preserving diffeomorphisms) which is indicated in [Mof69] to measure the amount of knottedness of flow orbits.

A convenient way of evaluating helicity is given by the musical isomorphisms (see Section 2.4). Define the curl of a vector field Z to be the unique vector field with $Z^\flat$ as a form potential. That is,

$$i_{\operatorname{curl} Z}\Omega = d(Z^\flat).$$

Thus the components of the curl are given by

$$(\operatorname{curl} Z(y))^l = (-1)^{l+1}\left(\frac{\partial Z_{j(l)}}{\partial y^{k(l)}}(y) - \frac{\partial Z_{k(l)}}{\partial y^{j(l)}}(y)\right),$$

where $k(l) < j(l)$ and $\{k(l), j(l)\} = \{1,2,3\} \setminus \{l\}$.

When α is a form potential for X, $\operatorname{curl}(\alpha^\sharp) = X$, and we call $\alpha^\sharp$ a *vector potential* for X. We see that

$$\mathcal{H}(X) = \int_M \alpha \wedge d\alpha = \int_M \alpha(\operatorname{curl}(\alpha^\sharp))\,\Omega = \int_M \rho(X, \alpha^\sharp)\,\Omega.$$

Thus the helicity can be thought of as a scalar product $\langle X, \operatorname{curl}^{-1} X\rangle$ of X and its potential field. Here the curl is not invertible, but we abuse notation due to independence of the preimage choice.

Example 6.1.4. For geodesic flows, helicity is calculated in Example 2.2.1 of [VF94]. There it is shown that for the flows in Example 4.1.1, $\mathcal{H}(X) = -2/c$, where c is the normalising factor for the volume measure, i.e. the volume of M under the canonical Riemannian volume form. If instead of the geodesic flow, one takes the flow generated just by its stable (or unstable) direction, the helicity is seen to be 0.

6.1.3 Arnol'd's asymptotic Hopf invariant

In [Arn86], Arnol'd considered flows on compact domains in $\mathbb{R}^3$, and proposed a characterisation of $\mathcal{H}$ by linking numbers of knots formed from long orbit segments closed up by geodesics. There were some gaps in Arnold's work, which were filled in by Vogel [Vog02] and transferred to the setting of real homology 3-spheres. The result is as follows.

Indeed, suppose M is a real homology 3-sphere and X is a smooth divergence-free vector field on M. Fix Σ a set of minimal geodesic arcs, containing one such arc between x and y for each pair $(x, y) \in M \times M$. For $x \in M$ and $t > 0$, let $c_t(x) \in \Sigma$ be the arc from $X^t(x)$ to x. Define $K_t(x)$ to be the knot given by concatenating $X^{[0,t]}(x)$ and $c_t(x)$, unless $X^t(x) = x$ in which case define $K_t(x) := X^{[0,t]}(x)$. It is in fact not certain $K_t(x)$ is always a knot, for example $c_t(x)$ may intersect $X^{[0,t]}(x)$, however this is not a problem for the main result, as follows.

Theorem 6.1.5 (Arnol'd, Vogel [Vog02])**.** *The limit*

$$\mathcal{A}(x, y) := \lim_{s,t\to\infty} \frac{1}{st} \mathrm{lk}(K_t(x), K_s(y))$$

exists for $(m \times m)$-almost every $(x, y) \in M \times M$, and $\mathcal{A} \in L^1(m \times m)$. We have that

$$\mathcal{H}(X) = \int \mathcal{A}(x, y)\, d(m \times m).$$

Furthermore, if X^t is ergodic with respect to m then we have

$$\mathcal{H}(X) = \mathcal{A}(x, y)$$

for $(m \times m)$-almost every $(x, y) \in M \times M$.

The results in the next section characterise helicity of Anosov flows by just periodic orbits, without the assumption that M is a real homology 3-sphere.

6.2 Statement of results

We restrict now to the case where $X^t : M \to M$ is a homologically full transitive Anosov flow on a closed oriented 3-manifold and consider its periodic orbits as knots. To define the linking number of distinct periodic orbits, at least one of the knots needs to be null-homologous in $H_1(M, \mathbb{R})$. Thus we will consider the sets of periodic orbits

$$\mathcal{P}_T := \{\gamma \in \mathcal{P} : T - 1 < \ell(\gamma) \le T\},$$

$$\mathcal{P}_T(0) := \mathcal{P}(0) \cap \mathcal{P}_T.$$

Since $\mathcal{P}_T(0)$ and $\mathcal{P}_{T+1}$ are disjoint, and $\mathcal{P}_T(0)$ consists of null-homologous orbits, the linking number of a pair of periodic orbits from these two collections are well-defined. The choice of intervals $(T-1, T]$ and $(T, T+1]$ for the period of orbits is somewhat arbitrary. The results below will still hold if we replace them with $[T+a, T+b]$ and $[T+a', T+b']$, for any $a < b$ and $a' < b'$, provided $[a, b]$ and $[a', b']$ are disjoint, and we can replace any [with (and any] with).

We define average linking numbers over the sets of orbits $\mathcal{P}_T(0)$ and $\mathcal{P}_{T+1}$, weighted by a Hölder continuous function $\varphi : M \to \mathbb{R}$, by

$$\mathscr{L}_\varphi(T) := \frac{\displaystyle\sum_{\gamma \in \mathcal{P}_T(0), \gamma' \in \mathcal{P}_{T+1}} \frac{\mathrm{lk}(\gamma, \gamma')}{\ell(\gamma)\ell(\gamma')} \exp\left(\int_\gamma \varphi + \int_{\gamma'} \varphi\right)}{\displaystyle\sum_{\gamma \in \mathcal{P}_T(0), \gamma' \in \mathcal{P}_{T+1}} \exp\left(\int_\gamma \varphi + \int_{\gamma'} \varphi\right)}.$$

It will become clear from our results that dividing by the periods of the orbits gives the correct normalisation. By Proposition 2.5.6, the linking number of two periodic orbits γ, γ' as above is given by

$$\frac{\mathrm{lk}(\gamma, \gamma')}{\ell(\gamma)\ell(\gamma')} = \int L(x, y)(X(x), X(y))\, d(\mu_\gamma \times \mu_{\gamma'})(x, y).$$

So that we can consider integrals of functions rather than forms, we define $\Lambda : (M \times M) \setminus \Delta(M) \to \mathbb{R}$ by

$$\Lambda(x, y) := L(x, y)(X(x), X(y)),$$

where $\Delta(M)$ is the diagonal $\{(x, x) \in M \times M : x \in M\}$ in $M \times M$.

Recall the functions $\xi(\varphi) \in H^1(M, \mathbb{R})$ and $\psi_{\xi(\varphi)}$ defined in Section 5.4.2, when $b = \dim H_1(M, \mathbb{R}) \ge 1$. If $b = 0$, we set $\xi(\varphi) = 0$ and $\psi_{\xi(\varphi)} = 0$. Our main result is the following.

Theorem 6.2.1. *Let $X^t : M \to M$ be a transitive Anosov flow on a closed oriented 3-manifold and let $\varphi : M \to \mathbb{R}$ be a Hölder continuous function. Then*

$$\lim_{T\to\infty} \mathscr{L}_\varphi(T) = \int \Lambda \, d(\mu_{\varphi+\psi_{\xi(\varphi)}} \times \mu_\varphi).$$

As a consequence, for null-homologous volume-preserving flows, we can obtain the helicity $\mathcal{H}(X)$ as the limit of appropriately weighted averages of linking numbers.

Theorem 6.2.2. *Let $X^t : M \to M$ be a null-homologous volume-preserving Anosov flow on a closed oriented 3-manifold. Then*

$$\mathcal{H}(X) = \lim_{T\to\infty} \mathscr{L}_{\varphi^u}(T).$$

We will prove Theorem 6.2.2 assuming we have proved Theorem 6.2.1. The proof of Theorem 6.2.1 appears in the next section.

Proof of Theorem 6.2.2. Since X^t is volume-preserving, $\mu_{\varphi^u} = m$ and, since X is null-homologous, $\Phi_m = 0$. We then have

$$\nabla\beta_\varphi(0) = \int \psi \, d\mu_{\varphi^u} = \int \psi \, dm = 0,$$

so $\xi(\varphi^u) = 0$. Hence we can apply Theorem 6.2.1 to conclude that $\mathscr{L}_{\varphi^u}(T)$ converges to $\int \Lambda \, d(m \times m)$. To complete the proof, we need to show that this integral is equal to the helicity $\mathcal{H}(X)$.

Let α be a 1-form such that $d\alpha = i_X\Omega$. Then, using Proposition 2.5.6,

$$\begin{aligned}
\mathcal{H}(X) &= \int_M \alpha \wedge i_X\Omega \\
&= \int_{x\in M} \left(\left(\int_{y\in M} L(x,y) \wedge i_X\Omega(y) \right) + H(\alpha)(x) - dh(x) \right) \wedge i_X\Omega(x).
\end{aligned}$$

Now, we have that

$$\int_M H(\alpha) \wedge i_X\Omega = \int H(\alpha)(X) \, dm = \langle \Phi_m, H(\alpha) \rangle,$$

$$\int_M dh \wedge i_X\Omega = \int dh(X) \, dm = \langle \Phi_m, dh \rangle.$$

Since $\Phi_m = 0$, we deduce

$$\begin{aligned}
\mathcal{H}(X) &= \int_{x\in M} \left(\int_{y \in M} L(x,y) \wedge i_X \Omega(y) \right) \wedge i_X \Omega(x) \\
&= \int_{(x,y)\in M\times M} L(x,y)(X(x),X(y)) \wedge \Omega(x) \wedge \Omega(y) \\
&= \int_{(x,y)\in M\times M} L(x,y)(X(x),X(y)) \, d(m\times m) \\
&= \int \Lambda \, d(m\times m),
\end{aligned}$$

as required. □

Another consequence of Theorem 6.2.1 is the following result for the geodesic flows discussed in Example 4.1.2.

Theorem 6.2.3. *Let $X^t : M \to M$ be the geodesic flow over a genus zero hyperbolic orbifold. Then*

$$\mathcal{H}(X) = \lim_{T\to\infty} \frac{1}{\#\mathcal{P}_T \ \#\mathcal{P}_{T+1}} \sum_{\gamma\in\mathcal{P}_T, \gamma'\in\mathcal{P}_{T+1}} \frac{\mathrm{lk}(\gamma,\gamma')}{\ell(\gamma)\ell(\gamma')}.$$

Proof of Theorem 6.2.3. Applying Theorem 6.2.3 with $\varphi = 0$, we get that the required limit is $\int \Lambda \, d(\mu_0 \times \mu_0)$. However, for the geodesic flows considered here, the measure of maximal entropy is equal to the volume m. Hence the limit is

$$\int \Lambda \, d(m\times m) = \mathcal{H}(X),$$

as shown in the proof of Theorem 6.2.2. □

Remark. For comparison we state a result of Contreras [Con95] which motivated this work. Contreras studied the asymptotic linking of periodic orbits (without weightings) for hyperbolic flows on basic sets of Axiom A flows on S^3 (we note that S^3 does not support Anosov flows since it is simply connected, [PT72]). The result of Contreras is that the average linking number of periodic orbits for X^t restricted to a non-trivial basic set satisfies

$$\lim_{T\to\infty} \mathscr{L}_0(T) = \int \Lambda \, d(\mu_0 \times \mu_0).$$

In this setting, there is an explicit formula for $\Lambda(x, y)$,

$$\Lambda(x, y) = \frac{3}{4\pi} \frac{X(x) \times X(y)}{\|x - y\|^3} \cdot (x - y),$$

which resembles the integrand of Gauss' linking integral. The above result is proved by comparing this integrand to the distance $\|x - y\|$, and using this comparison to show it is integrable with respect to the orbital measures. The equidistribution of these orbits can then be exploited to complete the proof. We will follow a similar approach to prove Theorem 6.2.1.

6.3 Proof of Theorem 6.2.1

6.3.1 Bounds on the Kotschick–Vogel linking form

For the background to this section, see Sections 27 and 28 of [Rha84]. Denote the Riemannian distance of $x, y \in M$ by $d(x, y)$. We will be interested in estimates on $L(x, y)$ as this distance tends to zero. A $(1, 1)$-form is said to be $O(d^k)$ if its coefficients are. Since the Green kernel $g(x, y)$ (and thus $L(x, y)$) is smooth away from $\Delta(M)$, we will mainly be concerned with the behaviour of the linking form near $\Delta(M)$.

Let $c : \mathbb{R} \to M$ be the unique geodesic with $c(0) = x$, $c(1) = y$, and $\text{length}(c|_{[0,1]})$ minimal. Let $v(x, y) = \dot{c}(0) \in T_x M$ and $w(x, y) = -\dot{c}(1) \in T_y M$. From [Rha84] (page 133),

$$g(x, y) = \omega(x, y) + O(d), \tag{6.3}$$

where $\omega(x, y)$ is the *parametrix*, which after setting $A(x, y) = -\frac{1}{2}d(x, y)^2$, is defined locally by

$$\begin{aligned} \omega(x, y) &= \frac{1}{s_3} \sum_{ij} \frac{1}{d(x, y)} \frac{\partial^2 A}{\partial x^i \partial y^j} dx^i dy^j. \\ &= \frac{1}{s_3} \sum_{ij} \frac{1}{d(x, y)} \frac{\partial v_i}{\partial y^j} dx^i dy^j, \end{aligned}$$

where s_3 is the volume of the 3-dimensional unit sphere, and $(x^i)_{i=1}^3$ (resp. $(y^i)_{i=1}^3$) denote local co-ordinates around x (resp. y). Note that in the above we have omitted evaluating functions at (x, y). We will continue in this way to avoid cumbersome notation where possible, but when we refer to the distance $d(x, y)$ in calculations we will always evaluate at (x, y) to avoid confusion with the exterior derivative.

We now state some properties of the geodesic distance that are useful when working with the parametrix. The functions $v(x, y)$, $w(x, y)$, $A(x, y)$, as well as the local co-ordinate functions x^i, y^i, and their relations are studied in [Rha84] (page 115). We summarise the results in the following lemma.

Lemma 6.3.1. *The functions above satisfy the following, for all pairs* $i, j \in \{1, 2, 3\}$.

(i) $v^i = O(d)$, $w^i = O(d)$

(ii) $y^i - x^i = O(d)$, $y^i - x^i - v^i = O(d^2)$, *and* $x^i - y^i - w^i = O(d^2)$.

(iii) $\dfrac{\partial v^i}{\partial x^j} = -\delta^i_j + O(d)$, *and* $\dfrac{\partial v^i}{\partial y^j} = \delta^i_j + O(d)$,

(iv) $\dfrac{\partial A}{\partial x^i} = v_i$, *and* $\dfrac{\partial A}{\partial y^i} = w_i$.

Here δ^i_j *is the Kronecker delta symbol.*

The remainder of this section is dedicated to proving the following.

Lemma 6.3.2. *There exists* $K > 0$ *such that for all* $x, y \in M$,

$$|\Lambda(x, y)| < \frac{K}{d(x, y)}.$$

By (6.3) it suffices to show that $*_y d_y \omega(x, y)(X(x), X(y)) = O(d^{-1})$. To understand the coefficients of the (1,1)-form $*_y d_y \omega(x, y)$, we will use the following geometric identity. Given a vector field Z on M,

$$*d(Z^\flat) = (\operatorname{curl} Z)^\flat. \tag{6.4}$$

First, taking x, y close enough, we can assume they are in the same co-ordinate chart, meaning that they also have the same metric components. From now on, we will write simply ρ_{ij}, instead of $\rho_{ij}(x) = \rho_{ij}(y)$, and $\hat{\rho}$ will denote the matrix of metric components. All sums in what follows are taken over indices ranging from 1 to 3, unless otherwise specified.

We wish to apply (6.4) to the y-part of the (1,1)-form $\omega(x, y)$. Given $x \in M$, let

$$\alpha^{i,x}_j(y) = \frac{1}{d(x, y)} \frac{\partial v_i}{\partial y^j}(y), \text{ and } \alpha^{i,x}(y) = \sum_j \alpha^{i,x}_j(y)\, dy^j.$$

Then $\omega(x,y) = \frac{1}{s_3}\sum_i \alpha^{i,x}(y)\,dx^i$, and

$$\begin{aligned}
s_3 *_y d_y\omega(x,y) &= \sum_i (*d\alpha^{i,x})(y)dx^i = \sum_i (\mathrm{curl}(\alpha^{i,x})^\sharp)^\flat dx^i \\
&= \sum_i \left(\sum_l (-1)^{l+1} \left(\frac{\partial \alpha_j^{i,x}}{\partial y^k} - \frac{\partial \alpha_k^{i,x}}{\partial y^j} \right) \frac{\partial}{\partial y^l} \right)^\flat dx^i \\
&= \sum_i \left(\sum_l (-1)^{l+1} \left(\frac{\partial v_i}{\partial y^j}\frac{\partial d(x,y)^{-1}}{\partial y^k} - \frac{\partial v_i}{\partial y^j}\frac{\partial d(x,y)^{-1}}{\partial y^j} \right) \frac{\partial}{\partial y^l} \right)^\flat dx^i \\
&= \sum_{il\lambda} \rho_{l\lambda}(-1)^{l+1} \left(\frac{\partial v_i}{\partial y^j}\frac{\partial d(x,y)^{-1}}{\partial y^k} - \frac{\partial v_i}{\partial y^j}\frac{\partial d(x,y)^{-1}}{\partial y^j} \right) dx^i dy^\lambda \\
&= \sum_{il\lambda} \rho_{l\lambda}(\nabla_y d(x,y)^{-1} \times \nabla_y v_i)^l dx^i dy^\lambda,
\end{aligned}$$

where $k = k(l) < j(l) = j$ are such that $\{k(l), j(l)\} = \{1,2,3\} \setminus \{l\}$. Using the component formulae for the musical isomorphism $\flat$ and standard rules for the cross and dot product in $\mathbb{R}^3$, we obtain

$$\begin{aligned}
s_3 *_y d_y\omega(x,y)(X(x),X(y)) &= \sum_{il\lambda} \rho_{l\lambda}(\nabla_y d(x,y)^{-1} \times \nabla_y v_i)^l X(x)^i X(y)^\lambda \\
&= \sum_{il} X(y)_l(\nabla_y d(x,y)^{-1} \times \nabla_y v_i)^l X(x)^i \\
&= \sum_{ilp} X(y)_l \rho_{ip}(\nabla_y d(x,y)^{-1} \times \nabla_y v^p)^l X(x)^i \\
&= \sum_{lp} X(y)_l(\nabla_y d(x,y)^{-1} \times X(x)_p \nabla_y v^p)^l \\
&= (X(y)^\flat)^T \cdot \left(\nabla_y d(x,y)^{-1} \times \sum_p X(x)_p \nabla_y v^p \right) \\
&= -\nabla_y d(x,y)^{-1} \cdot \left(\hat{\rho} X(y) \times \sum_p X(x)_p \nabla_y v^p \right).
\end{aligned}$$

Applying Lemma 6.3.1, we have

$$\begin{aligned}
\nabla_y d(x,y)^{-1} &= -\frac{1}{d(x,y)^2}\nabla_y d(x,y) = -\frac{d(x,y)}{d(x,y)^3}\nabla_y d(x,y) = -\frac{1}{2d(x,y)^3}\nabla_y d(x,y)^2 \\
&= \frac{1}{d(x,y)^3}\nabla_y A = \frac{1}{d(x,y)^3}(w^\flat)^T = \frac{1}{d(x,y)^3}\hat{\rho} w.
\end{aligned}$$

Recalling that $\frac{\partial v^p}{\partial y^j} = \delta^p_j + O(d)$, we have

$$\sum_p X(x)_p \nabla_y v^p = (X(x)^\flat)^T + O(d) = \hat{\rho} X(x) + O(d).$$

By Taylor's theorem applied to X, for x and y sufficiently close, this gives

$$\sum_p X(x)_p \nabla_y v^p = \hat{\rho} X(y) + O(d).$$

Collecting terms from above, we obtain

$$*_y d_y \omega(x,y)(X(x), X(y)) = -\frac{1}{d(x,y)^3} \hat{\rho} w \cdot (\hat{\rho} X(y) \times O(d)).$$

Since $w = O(d)$ and $\hat{\rho}$ and X are bounded, Lemma 6.3.2 is proved.

6.3.2 Completing the proof of Theorem 6.2.1

We are now ready to prove Theorem 6.2.1. We will follow the method in [Con95], aided by Theorems 5.3.1 and 5.4.13, and Lemma 6.3.2.

We define measures

$$\mu_{\varphi,T} = \frac{\sum\limits_{\gamma \in \mathcal{P}_T} e^{\int_\gamma \varphi} \mu_\gamma}{\sum\limits_{\gamma \in \mathcal{P}_T} e^{\int_\gamma \varphi}}, \qquad \mu^0_{\varphi,T} = \frac{\sum\limits_{\gamma \in \mathcal{P}_T(0)} e^{\int_\gamma \varphi} \mu_\gamma}{\sum\limits_{\gamma \in \mathcal{P}_T(0)} e^{\int_\gamma \varphi}}.$$

By Theorem 5.3.1, $\mu_{\varphi,T+1}$ converges to the equilibrium state μ_φ. If M is a real homology 3-sphere then $\mu^0_{\varphi,T}$ also converges to μ_φ. On the other hand, if $H_1(M,\mathbb{R})$ has dimension at least one then, by Theorem 5.4.13, $\mu^0_{\varphi,T}$ converges to the equilibrium state $\mu_{\varphi+\psi_{\xi(\varphi)}}$. To simplify notation, we shall write

$$\varphi^* = \begin{cases} \varphi & \text{if } \dim H_1(M,\mathbb{R}) = 0 \\ \varphi + \psi_{\xi(\varphi)} & \text{if } \dim H_1(M,\mathbb{R}) \geq 1, \end{cases}$$

so that the limit of $\mu^0_{\varphi,T}$ is denoted μ_{φ^*} in all cases. By definition,

$$\mathscr{L}_\varphi(T) = \int \Lambda \, d(\mu^0_{\varphi,T} \times \mu_{\varphi,T+1}).$$

Therefore, if either of the following limits exist, then we have the equality

$$\lim_{T\to\infty} \mathscr{L}_\varphi(T) = \lim_{T\to\infty} \int \Lambda \, d(\mu^0_{\varphi,T} \times \mu_{\varphi,T+1}). \tag{6.5}$$

We have that $\mu^0_{\varphi,T}$ converges to μ_{φ^*} and $\mu_{\varphi,T+1}$ converges to μ_φ. We will use this to prove that integral in (6.5) converges to the integral over $\mu_{\varphi^*} \times \mu_\varphi$. First we must prove $\int \Lambda \, d(\mu_{\varphi^*} \times \mu_\varphi)$ exists. We do this with the following lemma from [Con95].

Lemma 6.3.3 ([Con95], Lemma 2.4)**.** *Let (Y, d) be a separable metric space with Borel probability measures μ and ν.*

(i) If $x \in Y$ is such that $\liminf_{\rho\to 0} \frac{\log \mu(B(x,\rho))}{\log \rho} > 1$, then $\int \frac{1}{d(x,a)} \, d\mu(a)$ exists.

(ii) If this limit is uniformly greater than 1 ν-a.e, then $\int \frac{1}{d(x,a)} \, d(\mu(x) \times \nu(a))$ exists.

By Lemmas 6.3.2 and 6.3.3, to prove $\int \Lambda \, d(\mu_{\varphi^*} \times \mu_\varphi)$ exists we need only show

$$\liminf_{\rho\to 0} \frac{\log \mu_{\varphi^*}(B(x,\rho))}{\log \rho} > 1$$

uniformly μ_φ-almost everywhere. We will use a bound on the measure μ_{φ^*} which resembles a Gibbs property. This relies on φ^* satisfying the Bowen property, defined in Section 3.2. For now we will assume φ^* satisfies the Bowen property, proving later that it holds for all Hölder potentials.

Lemma 6.3.4 (Franco [Fra77])**.** *Suppose $\chi : M \to \mathbb{R}$ satisfies the Bowen property, and $\delta > 0$ is small. Then, there exists $C_\delta > 0$ such that for any $L > 0$, $x \in M$,*

$$\mu_\chi(B(x, \delta, L)) \leq C_\delta \exp\left(\int_0^L \chi(X^t(x)) \, dt - P(\varphi)L \right).$$

Proof. The method we follow is based on that of Franco ([Fra77], Proposition 2.11). We first consider bounding the orbital measures $\mu_{\chi,T}(B(x, \delta, L))$, for $T > L$. To obtain bounds involving pressure, we first construct a large separated set of periodic points contained in $B(x, \delta, L)$.

Let $|\mathcal{P}_T|$ denote the set of points on the orbits in $\mathcal{P}_T$. By expansivity, there exists a constant $q > 0$ such that for $y, y' \in |\mathcal{P}_T|$, $y' \notin X^{[-q,q]}y$ implies y and y' are $(T, 2\delta)$-separated. Let $\delta' = \min\{q, \delta\}$ and choose an integer $S > 2q/\delta'$. Consider an orbit $\gamma \in \mathcal{P}_T$. We can divide γ into consecutive closed segments $I_1, \ldots, I_m$, such that each segment has the same orbit length l, for some $l \in (\delta'/2, \delta')$. By the definition of q, if $|i - j| > S \pmod m$, we have $I_i \cap X^{[-q,q]} I_j = \varnothing$. We will now distribute the segments into collections $E_1, \dots, E_{2(S+1)}$ such that if $I_i, I_j \in E_k$ are distinct, then $|i - j| > S \pmod{m}$. We do this with the following process:

1. Put $I_1 \in E_1$, then add the $\lfloor \frac{m}{S+1} \rfloor - 1$ other segments $I_{S+2}, I_{2S+3}, I_{3S+4}, \dots$
2. Put $I_2 \in E_2$, and the $\lfloor \frac{m}{S+1} \rfloor - 1$ other segments $I_{S+3}, I_{2S+4}, I_{3S+5}, \dots$
3. Repeat this process until collections $E_1, \dots, E_{S+1}$ are full, at which point at most $S + 1$ segments remain.
4. Put each of the remaining segments (if any) into a collection on its own, and leave the remaining collections (if any) empty.

Now, we have that

$$\mu_{\chi,T}(B(x, \delta, L) \cap \gamma) = \sum_{k=1}^{2(S+1)} \mu_{\chi,T} \left(B(x, \delta, L) \cap \bigcup_{I_i \in E_k} I_i \right),$$

so there exists k^* such that E_{k^*} satisfies

$$\mu_{\chi,T} \left(B(x, \delta, L) \cap \bigcup_{I_i \in E_{k^*}} I_i \right) \geq \frac{1}{2(S+1)} \mu_{\chi,T}(B(x, \delta, L) \cap \gamma).$$

Form a set A_γ by picking one point (wherever possible) from $B(x, \delta, L) \cap I_i$, for each $I_i \in E_{k^*}$. For $y \in A_\gamma \cap I_i$, set

$$R_y = \{t \in (-\tfrac{\ell(\gamma)}{2}, \tfrac{\ell(\gamma)}{2}) : X^t y \in B(x, \delta, L) \cap I_i\} \subset [-\delta, \delta].$$

If we let $A = \cup_{\gamma \in \mathcal{P}_{T+1}} A_\gamma$, then A is $(T, 2\delta)$-separated and we have

$$\mu_{\chi,T}(B(x, \delta, L)) \leq 2(S+1)\mu_{\chi,T} \left(\bigcup_{y \in A} X^{R_y} y \right) = 2(S+1) \frac{\sum_{y \in A} \lambda(R_y) e^{\int_{\gamma_y} \chi}}{\sum_{\gamma \in \mathcal{P}_{T+1}} e^{\int_\gamma \chi}},$$

where λ is Lebesgue measure on the real line, and γ_y refers to the periodic orbit containing y. Now, since $A \subset B(x, \delta, L)$, $X^L A$ is $(T - L, 2\delta)$-separated and by the Bowen property there is $C > 0$ such that for each $y \in A$,

$$\left| \int_0^L \chi(X^t(x))\, dt - \int_0^L \chi(X^t(y))\, dt \right| < C.$$

Thus

$$\frac{\sum_{y\in A}\lambda(R_y)e^{\int_{\gamma_y}\chi}}{\sum_{\gamma\in\mathcal{P}_{T+1}}e^{\int_\gamma \chi}} \leq 2\delta e^{C+\int_0^L \chi(X^t x)\,dt}\frac{\sum_{y\in X^L A}e^{\int_0^{T-L}\chi(X^t y)\,dt}}{\sum_{\gamma\in\mathcal{P}_{T+1}}e^{\int_\gamma \chi}}.$$

Combining the above with Lemmas 2.6 and 2.8 in [Fra77], we have $C_\delta > 0$ such that

$$\mu_{\chi,T}(B(x,\delta,L)) \leq C_\delta \exp\left(\int_0^L \chi(X^t(x))\,dt - P(\varphi)L\right).$$

By Theorem 5.3.1,

$$\mu_\chi(B(x,\delta,L)) \leq \liminf_{T\to\infty}\mu_{\chi,T}(B(x,\delta,L)),$$

which completes the proof. □

Remark. The results from [Fra77] that are used in the proof above are only proved for periodic orbits whose least period lies within a small range $(T-\varepsilon, T+\varepsilon)$, as opposed to our range of $(T, T+1]$. Strictly speaking, one should obtain the result of Theorem 6.2.1 for these smaller ranges and then apply an additive argument to the limit in order to obtain it for the larger range.

Lemma 6.3.5. *Suppose χ satisfies the Bowen property. Then*

$$\liminf_{\rho\to 0}\frac{\log \mu_\chi(B(x,\rho))}{\log \rho} > 1$$

uniformly in x.

Proof. We follow the proof of Lemma 2.6 in [Con95]. By Lemma 4.3.20, there exists $\varepsilon > 0$ and a Hölder continuous function $v : M \to \mathbb{R}$ such that

$$\int_0^L \chi(X^t x)\,dt - P(\chi)L \leq -\varepsilon L + v(X^L x) - v(x),$$

for all $x \in M$ and $L \geq 0$. Thus, the proof of Lemma 6.3.4 tells us that for $\delta > 0$ sufficiently small and $T > L$,

$$\mu_{\chi,T}(B(x,\delta,L)) \leq C'_\delta e^{-\varepsilon L},$$

where $C'_\delta = C_\delta e^{2\|v\|_\infty}$, and C_δ is that from Lemma 6.3.4.

By compactness of M, there exist constants $\lambda, k_1, k_2 > 0$ such that $\|DX^t_x\| \leq \lambda^t$ for $t \geq 0$, and $k_1 < \|X\| < k_2$. So whenever $\rho\lambda^L \leq \delta/2$, $B(x,\rho) \subset B(x,\delta/2,L)$ and for any $a > 0$,

$$X^{[-a,a]}B(x,\rho) \subset B(x,\delta/2 + ak_2, L).$$

Now, any orbit intersecting $X^{[-a,a]}B(x,\rho)$ does so for time at least $2a$ and any orbit intersecting $B(x,\rho)$ does so for time at most $2\rho/k_1$. Setting $a = \delta/2k_2$,

$$\mu_{\chi,T}(B(x,\rho)) \leq \frac{2\rho}{2ak_1}\mu_{\chi,T}(X^{[-a,a]}B(x,\rho)) \leq \frac{\rho}{ak_1}\mu_{\chi,T}(B(x,\delta,L)).$$

Since $B(x,\rho)$ is open and $\mu_{\chi,T} \to \mu_\chi$, we have

$$\begin{aligned}\mu_\chi(B(x,\rho)) &\leq \liminf_{T\to\infty} \mu_{\chi,T}(B(x,\rho)) \\ &\leq \frac{\rho}{2ak_1} \liminf_{T\to\infty} \mu_{\chi,T}(B(x,\delta,L)) \leq \frac{C'_\delta \rho}{2ak_1} e^{-\varepsilon L}.\end{aligned}$$

Now, if we set $\rho > 0$ sufficiently small and consider $L = L(\rho) := \frac{\log\delta/2 - \log\rho}{\log\lambda}$, then $\rho \leq \delta/2\lambda^L$, and so

$$\frac{\log \mu_\chi(B(x,\rho))}{\log\rho} \geq 1 + \frac{\varepsilon}{\log\lambda} + O\left(\frac{1}{\log(1/\rho)}\right).$$

Thus we have that

$$\liminf_{\rho\to 0} \frac{\log \mu_\chi(B(x,\rho))}{\log\rho} \geq 1 + \frac{\varepsilon}{\log\lambda} > 1,$$

uniformly in x. $\square$

Having shown that our integral exists, we are left to show it is the value of the limit on the right-hand side of (6.5). We again examine the behaviour of Λ near the diagonal. First we show that the diagonal has zero measure with respect to the product of two equilibrium states of Hölder continuous functions.

Lemma 6.3.6. *Let $\chi : M \to \mathbb{R}$, $\chi' : M \to \mathbb{R}$ be Hölder continuous. Then*

$$(\mu_{\chi'} \times \mu_\chi)(\Delta(M)) = 0.$$

Proof. We can cover $\Delta(M)$ by products $B(x,\delta,T) \times B(x,\delta,T)$, where x runs over a (T,δ)-spanning set $E_{\delta,T}$. Applying Lemma 6.3.4 followed by Lemma 4.3.20, we have some $\varepsilon, k_\delta, k'_\delta > 0$ such that

$$
\begin{aligned}
(\mu_{\chi'} \times \mu_\chi)(\Delta(M)) &\leq \sum_{x \in E_{\delta,T}} \mu_{\chi'}(B(x,\delta,T))\mu_\chi(B(x,\delta,T)) \\
&\leq k_\delta \sum_{x \in E_{\delta,T}} \exp\left(\int_0^T \chi'(X^t x) + \chi(X^t x)\, dt - T(P(\chi') + P(\chi))\right) \\
&\leq k'_\delta \sum_{x \in E_{\delta,T}} \exp\left(\int_0^T \chi'(X^t x)\, dt - P(\chi')T - \varepsilon T\right),
\end{aligned}
$$

Since $E_{\delta,T}$ was an arbitrary (T,δ)-spanning set,

$$
(\mu_{\chi'} \times \mu_\chi)(\Delta(M)) \leq k'_\delta e^{-(P(\chi')+\varepsilon)T} \inf\left\{\sum_{x \in E} e^{\int_0^T \chi'(X^t x)\, dt} : E \text{ is } (T,\delta)\text{-spanning}\right\}.
$$

Provided $\delta > 0$ is chosen sufficiently small, we can take $T \to \infty$ and use the topological definition of pressure to conclude that $(\mu_{\chi'} \times \mu_\chi)(\Delta(M)) = 0$. □

Lemma 6.3.7. *If there exists a nested collection $\{B_R\}_{0<R\leq R_0}$ of open neighbourhoods of $\Delta(M)$ with $\bigcap_{0<R\leq R_0} B_R = \Delta(M)$, such that*

$$
\lim_{R\to 0} \lim_{T\to\infty} \int_{B_R} \Lambda\, d(\mu^0_{\varphi,T} \times \mu_{\varphi,T+1}) = 0,
$$

then the following limit exists, and equality holds:

$$
\lim_{T\to\infty} \int \Lambda\, d(\mu^0_{\varphi,T} \times \mu_{\varphi,T+1}) = \int \Lambda\, d(\mu_{\varphi^*} \times \mu_\varphi).
$$

Proof. Suppose the hypothesis is satisfied. We have that $\mu^0_{\varphi,T} \times \mu_{\varphi,T+1} \to \mu_{\varphi^*} \times \mu_\varphi$ in the weak* topology. Let $A_R = (M \times M) \setminus B_R$, then as Λ is continuous away from the diagonal,

$$
\lim_{T\to\infty} \int_{A_R} \Lambda\, d(\mu^0_{\varphi,T} \times \mu_{\varphi,T+1}) = \int_{A_R} \Lambda\, d(\mu_{\varphi^*} \times \mu_\varphi).
$$

Now, as Λ is $(\mu_{\varphi^*} \times \mu_\varphi)$–integrable and, by Lemma 6.3.6, $(\mu_{\varphi^*} \times \mu_\varphi)(\Delta(M)) = 0$, we have

$$
\lim_{R\to 0} \int_{B_R} \Lambda\, d(\mu_{\varphi^*} \times \mu_\varphi) = 0.
$$

These facts, along with the hypothesis and the triangle inequality, yield that, for

$\delta > 0$, there exist $T_0 > 0$ such that $T > T_0$ implies

$$\left| \int \Lambda \, d(\mu^0_{\varphi,T} \times \mu_{\varphi,T+1}) - \int \Lambda \, d(\mu_{\varphi^*} \times \mu_\varphi) \right| < \delta.$$

$\square$

We must now exhibit the sets B_R and show they satisfy the required property. We consider bounds for the integral of Λ with respect to $\mu_{\varphi,T+1}$.

Lemma 6.3.8. *There exists $\delta, R, \alpha, Q > 0$, independent of T, such that for all $x \in M$ and $\delta/2\lambda^T \leq R$,*

$$\int_{B(x,R)\setminus B(x,\delta/2\lambda^T)} |\Lambda(x,y)| \, d\mu_{\varphi,T+1}(y) \leq QR^\alpha.$$

Proof. Choose δ as in the proof of Lemma 6.3.5 to be smaller than the expansivity constant for X. Our calculations there also show that there exists $R > 0$ such that whenever $0 < \rho < R$

$$\mu_{\varphi,T+1}(B(x,\rho)) \leq \frac{\rho}{ak_1} \mu_{\varphi,T+1}(B(x,\delta,L(\rho))).$$

Now, by the proof of Lemma 6.3.4, there exists some $C_\delta > 0$ such that for $T > L(\rho)$,

$$\mu_{\varphi,T+1}(B(x,\delta,L(\rho))) \leq C_\delta \exp\left(\int_0^{L(\rho)} \varphi(X^t(x))dt - P(\varphi)L(\rho) \right). \tag{6.6}$$

By Lemma 4.3.20 we have $\varepsilon, K_\delta, K'_\delta > 0$ such that

$$\mu_{\varphi,T+1}(B(x,\rho)) \leq \frac{K_\delta \rho}{2ak_1} e^{-\varepsilon L(\rho)} = K'_\delta \rho^{1+\frac{\varepsilon}{\log\lambda}}.$$

Set $\alpha = \varepsilon/\log\lambda$, and define $N_T = \min\{n \in \mathbb{N} : R/2^n \leq \delta/2\lambda^T\}$. Let

$$A_n(x) = B(x, R/2^{n-1}) \setminus B(x, R/2^n); \text{ for } x \in M \text{ and } 1 \leq n \leq N_T - 1,$$

$$A_{N_T}(x) = B(x, R/2^{N_T-1}) \setminus B(x, \delta/2\lambda^T).$$

Now, splitting our integral over these annuli, and using Lemma 6.3.2, we have

$$\begin{aligned}\int_{B(x,R)\setminus B(x,\delta/2\lambda^T)} |\Lambda(x,y)|\, d\mu_{\varphi,T+1}(y) &= \sum_{n=1}^{N_T} \int_{A_n(x)} |\Lambda(x,y)|\, d\mu_{\varphi,T+1}(y) \\ &\le \sum_{n=1}^{N_T} \frac{2^n A}{R} \mu_{\varphi,T+1}(B(x,R/2^{n-1})) \\ &\le 2AK'_\delta R^\alpha \sum_{n=0}^{N_T-1} \frac{1}{2^{\alpha n}} \le \frac{2AK'_\delta}{1-2^{-\alpha}} R^\alpha.\end{aligned}$$

Setting $Q = 2AK'_\delta/(1-2^{-\alpha})$, we are done. □

Now, let

$$B_R = \bigcup_{x\in M} (\{x\} \times B(x,R)) \text{ and } D = \bigcup_{x\in M} (\{x\} \times B(x,\delta/2\lambda^T)).$$

It is clear that the B_R limit to the diagonal in the required way, so we are done if we show the integral limit property in Lemma 6.3.7. By Fubini's Theorem,

$$\int_{B_R\setminus D} |\Lambda|\, d(\mu^0_{\varphi,T} \times \mu_{\varphi,T+1}) \le QR^\alpha.$$

It remains to describe the integral on D. As our measures are supported on periodic orbits, we have

$$\int_D |\Lambda|\, d(\mu^0_{\varphi,T} \times \mu_{\varphi,T+1}) = \int_{x\in|\mathcal{P}_T(0)|} \int_{y\in B(x,\delta/2\lambda^T)\cap|\mathcal{P}'_T|} |\Lambda(x,y)|\, d\mu^0_{\varphi,T}(x) d\mu_{\varphi,T+1}(y),$$

where $|\mathcal{P}_T(0)|, |\mathcal{P}_{T+1}|$ denote the set of points on orbits in $\mathcal{P}_T(0), \mathcal{P}_{T+1}$ respectively. Now, as we chose δ smaller than the expansivity constant, we ensure that for any $x \in |\mathcal{P}_T(0)|$,

$$B(x,\delta/2\lambda^T) \cap |\mathcal{P}_{T+1}| = \varnothing.$$

If this were not the case then there would be a distinct periodic orbit from that of x which intersects $B(x,\delta/2\lambda^T)$, whilst also having comparable period to x. This violates expansivity, so the above integral is zero and we can conclude

$$\lim_{T\to\infty} \left| \int_{B_R} \Lambda\, d(\mu^0_{\varphi,T} \times \mu_{\varphi,T+1}) \right| \le QR^\alpha.$$

Thus

$$\lim_{R\to 0} \lim_{T\to\infty} \left| \int_{B_R} \Lambda\, d(\mu^0_{\varphi,T} \times \mu_{\varphi,T+1}) \right| = 0$$

as required.

We have now proved Theorem 6.2.1 under the assumption that Hölder continuous functions satisfy the Bowen property. We prove this holds below.

Lemma 6.3.9. *Suppose $\chi : M \to \mathbb{R}$ is Hölder continuous. Then χ satisfies the Bowen property.*

Proof. Suppose χ is α-Hölder, with constant H. Let $t > 0$ be large and define

$$D_t(x,y) := \left| \int_0^t \chi(X^s(x))ds - \int_0^t \chi(X^s(y))ds \right|,$$

for $x, y \in M$. Clearly D_t is a pseudo-metric. We will use the structure of stable and unstable manifolds in M to prove the result. Let $0 < \delta < 1$, and consider the following cases for $y \in B(x, \delta, t)$.

1. Suppose y is on the same orbit as x, writing $y = X^{t_0}(x)$. By compactness, there exists $a, b, c > 0$ such that $a \le \|X\| \le b$, and $|\chi(z)| \le c$ for all $z \in M$. Thus $|t_0| \le \delta/a$. This gives

$$D_t(x,y) = \left| \int_0^{t_0} \chi(X^s(x))ds + \int_t^{t+t_0} \chi(X^s(x))ds \right| \le 2c/a.$$

2. Suppose $y \in W_r^s(x)$ for some small $r > 0$. As in Lemma 4.1.14, there are $k, l > 0$ such that $d(X^s(x), X^s(y)) \le ke^{-ls}d(x,y) \le ke^{-ls}r$ for all $s \ge 0$. Thus

$$\begin{aligned} D_t(x,y) &= \left| \int_0^t \chi(X^s(x))ds - \int_0^t \chi(X^s(y))ds \right| \\ &\le H \int_0^t d(X^s(x), X^s(y))^\alpha ds \\ &\le H(kr)^\alpha \int_0^t e^{-\alpha ls} ds = \frac{H(kr)^\alpha}{\alpha l}(1 - e^{-\alpha lt}) < \frac{H(kr)^\alpha}{\alpha l}. \end{aligned}$$

3. Now suppose $y \in W_r^u(x)$. Then, with k, l as above and $s \le t$,

$$\begin{aligned} d(X^s(x), X^s(y)) &= d(X^{-(t-s)}X^t(x), X^{-(t-s)}X^t(y)) \\ &\le ke^{-l(t-s)}d(X^t(x), X^t(y)) \le kre^{-l(t-s)} \end{aligned}$$

Thus we again obtain

$$\begin{aligned} D_t(x,y) &= \left| \int_0^t \chi(X^s(x))ds - \int_0^t \chi(X^s(y))ds \right| \\ &\leq H \int_0^t d(X^s(x), X^s(y))^\alpha ds \\ &\leq H(kr)^\alpha \int_0^t e^{-\alpha l(t-s)} ds = \frac{H(kr)^\alpha}{\alpha l}(1 - e^{-\alpha l t}) < \frac{H(kr)^\alpha}{\alpha l}. \end{aligned}$$

To complete the proof, we use a construction similar to that used in [Bow73] to find Markov partitions. Let $0 < r < 1$ be sufficient for Lemma 4.1.14. Given $x \in M$, consider the splitting $T_x M = E_x^s \oplus E_x \oplus E_x^u$, associated to φ. Take a small (compared to r) 2-dimensional disc $D \subset M$ centred at x tangent to the $E_x^s \oplus E_x^u$ plane and transverse to the flow. D is foliated by pieces of local stable sets, i.e. there exists $U \subset D$ such that $\mathscr{F} = \{W_r^s(p) \cap D : p \in U\}$ foliates D. Also, D contains a piece of $W_r^u(x)$ which is transverse to this foliation, meaning $D \cap W_r^u(x)$ intersects each leaf of the foliation exactly once. Given δ sufficiently smaller (independently of x) than the radius of D, $y \in B(x, \delta, t)$ implies y passes through D under the flow. Thus we have $w \in D$, and $t_0 \in \mathbb{R}$ such that $y = X^{t_0}(w)$. Now, w belongs to a leaf $W_r^s(p) \cap D \in \mathscr{F}$. Let z be the intersection point of $W_r^s(p) \cap D$ and $W_r^u(x)$. By the triangle inequality

$$D_t(y,x) \leq D_t(y,w) + D_t(w,z) + D_t(z,x),$$

but by the calculations above, $D_t(y,w)$, $D_t(w,z)$, and $D_t(z,x)$ are all bounded in t, so we are done. $\square$

Remark. It is interesting to ask whether one can obtain versions of Theorem 6.2.1 and Theorem 6.2.2 with $\mathcal{P}_{T+1}$ replaced by $\mathcal{P}_{T+1}(0)$. One suspects this is the case; however, we were unable to prove the estimate (6.6) for the orbital measures corresponding to the null-homologous orbits $\mathcal{P}_{T+1}(0)$.

Remark. As we remarked after its definition, the helicity is an invariant of volume-preserving diffeomorphisms. Arnold [Arn86] conjectured that it is invariant under volume-preserving homeomorphisms. Unfortunately, our results do not shed any light on this conjecture since a homeomorphism need not preserve the quantities $\int_\gamma \varphi^u$. Indeed, if they are preserved then the flows are already smoothly conjugate [lLM88]. For further discussion of this problem, see [MS13].

Chapter 7

Distribution of periodic orbits in the homology group of a knot complement

In this chapter we consider counting periodic orbits of Anosov flows which satisfy certain homological constraints. We have already seen some results of this type implicitly in Chapter 5. A result of Sharp [Sha93] is that, when the flow is homologically full, for each homology class $\alpha \in H_1(M, \mathbb{Z})/\mathrm{Tor}$

$$\#(\mathcal{P}_{\leq T} \cap \mathcal{P}(\alpha)) \sim Ce^{-\langle \alpha, \xi \rangle} \frac{e^{\beta(\xi)T}}{T^{1+b/2}},$$

where $\beta = \beta_0, \xi = \xi(0)$ are as in Section 5.4.2, b is the first Betti number of M, and C is a constant with an explicit formula. This result can be recovered from those of Babillot–Ledrappier [BL98], who considered a larger class of Anosov flows. Without assuming homological fullness, it is possible that the prime period of orbits in any fixed homology class is bounded, and thus one must vary the homology class in line with the period to obtain meaningful results. Recall the function ψ from Section 5.4.2, and assume that $\{(\ell(\gamma), \int_\gamma \psi) : \gamma \in \mathcal{P}\}$ generates $\mathbb{R} \times \mathbb{Z}^b$. Babillot–Ledrappier proved that for compactly supported real-valued integrable functions g_0, g,

$$\sum_{\gamma \in \mathcal{P}} g_0(\ell(\gamma) - T) g([\gamma] - \lfloor Tz \rfloor) \sim C(z, T) \frac{e^{H(z)T}}{T^{1+b/2}}$$

where

$$z \in \left\{ \int \psi \, d\mu : \mu \in \mathcal{M}(X) \right\}^{\circ},$$

$C(z,T)$ is an oscillating function bounded below away from zero, and H is a natural entropy function. As was discussed in Section 5.4.3, the approach of Babillot–Ledrappier can be used to prove the non-weighted version of Theorem 5.4.7, with the weighted version requiring a minor modification.

The results of this chapter are similar to those above, where instead of homology classes in $H_1(M,\mathbb{Z})$, we consider those in $H_1(M \setminus L,\mathbb{Z})$, where L is a link made up of finitely many periodic orbits. Our results will rely on the work of McMullen [McM13], who showed how to modify the symbolic coding in Theorem 4.2.15 to obtain meaningful information about homology in $M \setminus L$.

For the remainder of this chapter, M will be a smooth connected closed oriented Riemannian 3-manifold, and X^t will be a transitive Anosov flow on M.

7.1 Homology and symbolic dynamics

Let us comment on how periodic orbits are encoded by the map $\pi : \Sigma(\Gamma, r) \to M$ in Theorem 4.2.15. Given a periodic orbit γ of X, there is a periodic orbit η of σ^r, with $\ell(\gamma) = \ell(\eta)$ such that $\gamma = \pi(\eta)$. Furthermore, η is unique as long as γ does not pass through the boundary of some rectangle in the Markov family. In 3-dimensions, the rectangles are such that their boundary consists of finitely many pieces of unstable and stable manifolds. In each piece there can be at most one periodic orbit, since orbits sharing a stable or unstable manifold must diverge in either the past or future. Thus there are only finitely many periodic orbits with multiple preimages under π.

We wish to consider the homology of periodic orbits M, after removing finitely many orbits. Fix $n \geq 1$ and a set of n distinct orbits $L_n := \{\gamma_1, \dots, \gamma_n\} \subset \mathcal{P}(0)$. Let $\mathcal{P}^* = \mathcal{P} \setminus L_n$. For each $1 \leq i \leq n$, replace $\gamma_i \subset M$ with a tubular neighbourhood T_i. Then, defining $M_n := M \setminus \bigcup_{i=1}^n T_i$, we have that

$$H_1(M_n, \mathbb{R}) \cong H_1(M, \mathbb{R}) \oplus \mathbb{R}^n.$$

In particular, M_n has first Betti number $b + n$, where b is the first Betti number of M. In this chapter, for $\gamma \in \mathcal{P}^*$, we write $[\gamma] \in \mathbb{Z}^{b+n}$ for the torsion free part of the integral homology class of γ in M_n.

We wish to adjust the approach in [BL98] to find analogous results for the orbits $\mathcal{P}^*$ and their homology classes in M_n. We require the following result, analogous to Proposition 5.4.2.

Proposition 7.1.1. *Suppose X^t is topologically weak-mixing. The set $\{[\gamma] : \gamma \in \mathcal{P}^*\}$ generates $\mathbb{Z}^{b+n}$.*

Proof. This is a corollary of the main result in [McM13], where an analogue of Theorem 5.4.1 is proved for Anosov flows. This yields that for a finite abelian group G, and a surjective homomorphism $a : \pi_1(M_n) \to G$, given any $g \in G$,

$$\lim_{t\to\infty} \frac{\#\{\gamma \in \mathcal{P}^* : \ell(\gamma) \leq t,\, a(\gamma) = g\}}{\#\{\gamma \in \mathcal{P}^* : \ell(\gamma) \leq t\}} = \frac{1}{\#G}. \tag{7.1}$$

If $\mathbb{Z}^{b+n}$ is not generated by the classes of orbits in $\mathcal{P}^*$, then there is some proper cofinite subgroup $H \leq \mathbb{Z}^{b+n}$, such that

$$\left\{[\gamma] : \gamma \in \mathcal{P}^*\right\} \subset H.$$

Set $G = \mathbb{Z}^{b+n}/H$. Then we have a homomorphism $a : \pi_1(M_n) \to G$, given by quotient maps, for which the result of (7.1) is contradicted, since all orbits satisfy $a(\gamma) = 0$. $\square$

Let us now describe how the homology of periodic orbits can be encoded using the symbolic dynamics. Recall the Markov family of rectangles $\{R_1, \dots, R_k\}$ discussed after the statement of Theorem 4.2.15. Arguing as in [McM13], we will show that there is no loss of generality in assuming L_n only intersects $R = \bigcup_{i=1}^k R_i$ at the boundary of rectangles i.e.

$$L_n \cap R \subset \bigcup_{i=1}^k \partial R_i.$$

Indeed, if this were not the case and $x \in L_n \cap R_i^\circ$ (there can only be finitely many such points by transversality), then we could split R_i into two pieces $R_{i,1}, R_{i,2}$ at the point x. The family $R_1, \dots, R_{i-1}, R_{i,1}, R_{i,2}, R_{i+1}, \dots, R_k$ would still yield a semi-conjugacy with the properties in Theorem 4.2.15, but with x not in the interior of any rectangle. With this in mind, we will henceforth assume L_n only intersects the boundary of rectangles.

Given $1 \leq i, j \leq k$, let E_{ij} be the set of points on flow lines going from R_i° to R_j°, and let

$$U = \left(\bigcup_{i=1}^k R_i^\circ\right) \cup \left(\bigcup_{1\leq i,j\leq k} E_{ij}\right).$$

By the above assumption, $U \subset M \setminus L_n$. Let $\bar{\Gamma}$ be the graph obtained by removing the direction from edges of Γ, but retaining any multiple edges between vertices. We can choose an embedding $\iota : \bar{\Gamma} \hookrightarrow U$ which in turn induces a surjective homomorphism $\iota_* : \pi_1(\bar{\Gamma}) \to \pi_1(M \setminus L_n)$.

For each $i \in \{1, \ldots, k\}$, fix a path $p(1,i)$ from 1 to i in $\bar{\Gamma}$, noting that paths in $\bar{\Gamma}$ need not follow the directions of edges of Γ. Such paths always exist since Γ is aperiodic. Let $p(i,1)$ be the path from i to 1 obtained by following $p(1,i)$ backwards. For any vertex j which satisfies that $ij \in E(\Gamma)$, let $e(i,j)$ be the corresponding edge between i, j in $\bar{\Gamma}$. For such i, j, form a loop $K(i,j)$ by concatenating as follows

$$K(i,j) = 1 \xrightarrow{p(1,i)} i \xrightarrow{e(i,j)} j \xrightarrow{p(j,1)} 1.$$

This has a homotopy class $[K(i,j)] \in \pi_1(\bar{\Gamma})$. Let $q : \pi_1(M_n) \to \mathbb{Z}^{b+n}$ be the projection onto the torsion-free part of homology, and define $f : \Sigma(\Gamma) \to \mathbb{Z}^{b+n}$ by $f(x) = (q \circ \iota_*)([K(x_0, x_1)])$. Clearly, f is locally constant. Further, as $q \circ \iota^*$ is a homomorphism, we have that for a periodic point $x = \overline{x_0 x_1 \ldots x_{n-1}}$, the Birkhoff sum

$$\begin{aligned} f^n(x) &= f(x) + f(\sigma x) + \ldots + f(\sigma^{n-1} x) \\ &= (q \circ \iota^*)([K(x_0, x_1)][K(x_1, x_2)] \cdots [K(x_{n-2}, x_{n-1})][K(x_{n-1}, x_0)]) \\ &= (q \circ \iota^*)([c(x_0, x_1, \ldots, x_{n-1}, x_0)]), \end{aligned}$$

where $c(x_0, x_1, \ldots, x_{n-1}, x_0)$ is the cycle

$$x_0 \xrightarrow{e(x_0,x_1)} x_1 \xrightarrow{e(x_1,x_2)} x_2 \to \ldots \to x_{n-2} \xrightarrow{e(x_{n-2},x_{n-1})} x_{n-1} \xrightarrow{e(x_{n-1},x_0)} x_0.$$

This construction leads to the following.

Lemma 7.1.2. *Let $\eta \in \mathcal{P}(\sigma^r)$, and $x \in \Sigma(\Gamma)$ the corresponding periodic point for σ, with period $n \in \mathbb{N}$. Then $\ell(\pi(\eta)) = \ell(\eta) = r^n(x)$, and $f^n(x) = [\pi(\eta)]$.*

7.2 Counting orbits by linking

We will study the homology of orbits in the following way. Let $z \in \mathbb{R}^{b+n}$, and $T > 0$. Denote by $\lfloor Tz \rfloor \in \mathbb{Z}^{b+n}$ the integer part (taken component-wise) of Tz. Further, let $g_0 : \mathbb{R} \to \mathbb{R}$, $g : \mathbb{Z}^{b+n} \to \mathbb{R}$ be integrable functions with compact support. Using the method of [BL98], we will study the functional N_T^z defined by

$$N_T^z(g_0 \otimes g) = \sum_{\gamma \in \mathcal{P}^*} g_0(\ell(\gamma) - T) g([\gamma] - \lfloor Tz \rfloor).$$

We will be particularly interested in the case where g_0 is the characteristic function of an interval $[a, b]$, and $g = \delta_\alpha$ for some $\alpha \in \mathbb{Z}^{b+n}$. In this case, we have

$$N_T^z(g_0 \otimes g) = \#\{\gamma \in \mathcal{P}^* : \ell(\gamma) \in [T + a, T + b] \text{ and } [\gamma] = \alpha + \lfloor Tz \rfloor\}$$

In general, since g_0, g have compact support, N_T^z will vanish eventually (with T) unless there are orbits $\gamma^{(1)}, \gamma^{(2)}, \ldots \in \mathcal{P}^*$ such that $\ell(\gamma^{(m)}) \to \infty$ and $\frac{[\gamma^{(m)}]}{\ell(\gamma^{(m)})} \to z$. This occurs whenever there are periodic points $x_m \in \Sigma(\Gamma)$, of prime period n_m, such that

$$\frac{f^{n_m}(x_m)}{r^{n_m}(x_m)} \to z.$$

Since the measures

$$\left\{\mu_x = \frac{1}{n}\sum_{i=0}^{n-1} \delta_{\sigma^i(x)} : x \in \Sigma(\Gamma) \text{ is periodic of period } n\right\} \subset \mathcal{M}(\sigma)$$

are weak*-dense in $\mathcal{M}(\sigma)$, we will only consider z taken from the set

$$\mathcal{C} := \left\{\frac{\int f\, dm}{\int r\, dm} : m \in \mathcal{M}(\sigma)\right\}.$$

Let us now give another description of $\mathcal{C}$, using the suspension flow σ^r.

We 'lift' from $\Sigma(\Gamma)$ to $\Sigma(\Gamma, r)$ in the following way. Let $g : \Sigma(\Gamma) \to \mathbb{R}$ be continuous, and define $\tilde{g} : \Sigma(\Gamma, r) \to \mathbb{R}$ by

$$\tilde{g}[x, t] = \frac{g(x)}{r(x)} \frac{\pi}{2} \sin\left(\frac{\pi}{r(x)} t\right),$$

where (x, t) is the unique representative of its equivalence class satisfying $0 \le t < r(x)$. Under this definition, $\tilde{g}$ inherits the regularity of g, and satisfies

$$g(x) = \int_0^{r(x)} \tilde{g}[x, t]\, dt.$$

Further, for $\eta \in \mathcal{P}(\sigma^r)$ corresponding to a point $x \in \Sigma(\Gamma)$ of minimal period n,

$$\frac{g^n(x)}{r^n(x)} = \frac{1}{\ell(\eta)} \int_\eta \tilde{g},$$

As seen in Section 4.2, there is a one-to-one correspondence between invariant measures $\mathcal{M}(\sigma)$ and $\mathcal{M}(\sigma^r)$. This gives, setting $F = \tilde{f}$, that

$$\mathcal{C} = \left\{ \int F \, d\tilde{m} \, : \, \tilde{m} \in \mathcal{M}(\sigma^r) \right\}.$$

We can characterise the interior of $\mathcal{C}$ using pressure functions. For $z \in \mathcal{C}$, define

$$H(z) = \sup \left\{ h_{\tilde{m}}(\sigma^r) \, : \, \tilde{m} \in \mathcal{M}(\sigma^r), \, \int F \, d\tilde{m} = z \right\}.$$

For $u \in \mathbb{R}^{b+n}$, define $\beta(u) = P(\langle u, F \rangle)$. We have the following lemma.

Lemma 7.2.1. *The map $u \mapsto \nabla\beta(u)$ is a diffeomorphism between $\mathbb{R}^{b+n}$ and $\mathcal{C}^\circ$. Furthermore, H is differentiable on $\mathcal{C}^\circ$ and $(\nabla\beta)^{-1} = -\nabla H$.*

Proof. This follows from Theorem 26.5 in [Roc70]. Since

$$\beta(u) = h_{m_{\langle u,F \rangle}}(\sigma^r) + \left\langle u, \int F \, dm_{\langle u,F \rangle} \right\rangle$$

for a unique measure $m_{\langle u,F \rangle}$ (the equilibrium state of $\langle u, F \rangle$), we have

$$\beta(u) = \sup_{z \in \mathcal{C}} \{ H(z) + \langle u, z \rangle \},$$

with the supremum achieved uniquely at $z = \int F \, dm_{\langle u,F \rangle}$. This shows that $-H$ is the Legendre transform of β, as defined on page 256 of [Roc70]. By Theorem 26.5 in [Roc70], $\nabla\beta$ is a diffeomorphism onto its image, with inverse $-\nabla H$. Thus it suffices to show that $\nabla\beta(\mathbb{R}^{b+n}) = \mathcal{C}^\circ$. To do so, we follow the approach in Lemma 7 of [MT90].

Let $z \in \mathcal{C}^\circ$ and $\varepsilon > 0$ such that $B(z, 2\varepsilon) \subset \mathcal{C}$. Then $z + \frac{u}{\|u\|}\varepsilon \in \mathcal{C}$. Thus

$$\langle u, z \rangle + \|u\|\varepsilon \leq \sup_{z' \in \mathcal{C}} \langle u, z' \rangle = \sup \left\{ \int F \, d\tilde{m} : \tilde{m} \in \mathcal{M}(\sigma^r) \right\} \leq \beta(u).$$

Let $e_z : \mathbb{R}^{b+n} \to \mathbb{R}$ be defined by $e_z(u) = \beta(u) - \langle u, z \rangle$. Then by the above inequality, $e_z(u) \leq r$ only if $\|u\| \leq r/\varepsilon$. In particular, $e_z(u) \geq 0$ for all $u \in \mathbb{R}^{b+n}$. Thus e_z has a finite minimum attained at $u^z \in \mathbb{R}^{b+n}$. This means $\nabla e_z(u^z) = 0$, which is exactly $\nabla\beta(u^z) = z$. $\square$

This allows us to introduce the following notation. Given $z \in \mathcal{C}^\circ$, let u^z be defined by $\nabla\beta(u^z) = z$, and $m^z \in \mathcal{M}(\sigma^r)$ be the equilibrium state for $\langle u^z, F \rangle$. By Proposition 4.3.12, this satisfies $\int F \, dm^z = z$. Furthermore, $H(z)$ is attained at m^z.

Before stating our main theorem, we make precise an assumption we will need. We say that X has property (B) if there is no suspension space $\Sigma(\Gamma, r)$ satisfying the properties in Theorem 4.2.15, with r cohomologous to a locally constant function.

Property (B) is in some sense typical amongst Anosov flows. Precisely, if $A(M)$ denotes the set of Anosov flows on M, we have the following.

Proposition 7.2.2. *For any $1 \leq l \leq \infty$, there is a C^1 open, C^l dense subset of $A(M)$ which only contains flows satisfying (B).*

This proposition follows from Theorem 1.6 in [FMT07], since if an Anosov flow X^t is modelled by a suspension σ^r with locally constant roof function r, there is a perturbation Z of X (which can be chosen arbitrarily close to X), such that Z^t is Anosov and modelled by $\sigma^{r'}$, where r' is rational-valued and locally constant. This means periodic orbits of $\sigma^{r'}$ (and hence those of Z) have periods contained in a discrete subgroup of $\mathbb{R}$. It is a result of Bowen [Bow72b] that this violates the topological weak-mixing property.

The following is the main theorem of this chapter.

Theorem 7.2.3. *Assume that X satisfies property (B). Then for integrable functions g_0, g with compact support, and $z \in \mathcal{C}^\circ$, $N_T^z(g_0 \otimes g)$ is asymptotic , as $T \to \infty$, to*

$$\frac{e^{TH(z)+\langle u^z, Tz - \lfloor Tz \rfloor\rangle}\sqrt{|\det H''(z)|}}{T(2\pi T)^{\frac{b+n}{2}}}\int_{\mathbb{R}\times\mathbb{Z}^{b+n}} e^{\beta(u^z)x - \langle u^z, y\rangle} g_0(x) g(y)\, dx\, dy.$$

In particular, for real numbers $a < b$, $\alpha \in \mathbb{Z}^{b+n}$,

$$N_T^z(\mathbb{1}_{[a,b]} \otimes \delta_\alpha) = \#\{\gamma \in \mathcal{P}^* : \ell(\gamma) \in [T+a, T+b] \text{ and } [\gamma] = \alpha + \lfloor Tz \rfloor\},$$

and we have

$$N_T^z(\mathbb{1}_{[a,b]} \otimes \delta_\alpha) \sim \frac{e^{TH(z)+\langle u^z, Tz - \lfloor Tz \rfloor - \alpha\rangle}\sqrt{|\det H''(z)|}}{T(2\pi T)^{\frac{b+n}{2}}\beta(u^z)} e^{\beta(u^z)b} - e^{\beta(u^z)a}.$$

Proof. We will show that

$$Y := \left\{ \left(\ell(\eta), \int_\eta F\right) : \eta \in \mathcal{P}(\sigma^r) \right\}$$

generates $\mathbb{R}\times\mathbb{Z}^{b+n}$, after which the proof is essentially that of Theorem 1.2 in [BL98]. We give details on this at the end of the chapter.

Let $\langle Y \rangle$ be the group generated by Y. It suffices to show that the only character of $\mathbb{R} \times \mathbb{Z}^{b+n}$ which is trivial on $\langle Y \rangle$ is trivial everywhere. Characters of $\mathbb{R} \times \mathbb{Z}^{b+n}$ have the form

$$\chi_{t,u}(x, y) = e^{2\pi i(tx + \langle u,y \rangle)},$$

where $t \in \mathbb{R}$, $u \in \mathbb{R}^{b+n}/\mathbb{Z}^{b+n}$. Suppose $\chi_{t,u}$ is trivial on $\langle Y \rangle$. Then

$$e^{2\pi i(tr^n(x)) + \langle u, f^n(x) \rangle)} = 1$$

whenever $\sigma^n(x) = x$ in $\Sigma(\Gamma)$. Since $\{f^n(x) : \sigma^n(x) = x\}$ generates $\mathbb{Z}^{b+n}$ we have $e^{2\pi i \langle u,m \rangle} = 1$ for all $m \in \mathbb{Z}^{b+n}$, so $u = 0$. This leaves $tr^n(x) \in \mathbb{Z}$ whenever $\sigma^n(x) = x$. By Proposition 4.2.11, tr is cohomologous to a locally constant function. By our assumption, this is impossible unless $t = 0$. $\square$

7.3 Equidistribution of orbits with prescribed linking

Here we will consider equidistribution of periodic orbits according to their homology in M_n. First, given $z \in \mathcal{C}^\circ$ we let $\mu^z := \pi_* m^z$ be the equilibrium state for X which corresponds to m^z. The main theorem is as follows.

Theorem 7.3.1. *Suppose X satisfies property (B), and $z \in \mathcal{C}^\circ$, $\alpha \in \mathbb{Z}^{b+n}$. Then the measures*

$$\frac{1}{N_T^z(\mathbb{1}_{[a,b]} \otimes \delta_\alpha)} \sum_{\gamma \in \mathcal{P}(\alpha + \lfloor Tz \rfloor)} \mathbb{1}_{[a,b]}(\ell(\gamma) - T)\mu_\gamma$$

converge weak to μ^z, as $T \to \infty$.*

This theorem follows from Theorem 7.2.3, along with the large deviations results discussed in Section 5.4.4 (to which we refer for notation). In particular, we can use the same approach as in Theorem 5.4.12 to prove the following.

Theorem 7.3.2. *Suppose X satisfies property (B). Then, for every compact set $\mathcal{K} \subset \mathcal{M}(X)$ such that $\mu^z \notin \mathcal{K}$, and real numbers $a < b$, we have*

$$\limsup_{T \to \infty} \frac{1}{T} \log \left(\frac{\Xi_0(T, \alpha + \lfloor Tz \rfloor, \mathbb{1}_{[a,b]}, \mathcal{K})}{N_T^z(\mathbb{1}_{[a,b]} \otimes \delta_\alpha)} \right) < 0.$$

7.4 Completing the proof of Theorem 7.2.3

With the setup from Section 7.1, Theorem 7.2.3 is essentially an application of Theorem 1.2 in [BL98]. In this section we detail the method in [BL98], and explain how it is used in our setting. We will also highlight the necessary modification to prove Proposition 5.4.11.

Let $l > 0$ and fix a Hölder continuous function $G : \Sigma(\Gamma, r) \to \mathbb{R}^l$, such that no non-trivial linear combination of $G_1, \ldots, G_l$ is σ^r-cohomologous to 0. Let $\mathcal{C}_G$ be the set

$$\mathcal{C}_G := \left\{ \int G\, dm : m \in \mathcal{M}(\sigma^r) \right\}.$$

Proceeding identically as in the proof of Proposition 7.2.1, for each $z \in \mathcal{C}_G^\circ$, there is a unique $u^z \in \mathbb{R}^l$ such that

$$z = \int G\, dm_{\langle u^z, G\rangle}.$$

We then denote by m^z the equilibrium state $m_{\langle u^z, G\rangle}$, by $H(z)$ the entropy h_{m^z}, and by $\beta(u^z)$ the pressure $H(z) + \int \langle u^z, G\rangle\, dm^z$.

Set J to be the group generated by the integrals

$$\left\{ \int_\eta G : \eta \in \mathcal{P}(\sigma^r) \right\}.$$

By our assumption on the components of G, J is of the form $\mathbb{R}^q \times \mathbb{Z}^{l-q}$ for some $0 \le q \le l$. We say that σ^r satisfies assumption (A) if

$$\left\{ \left(\ell(\eta), \int_\eta G \right) : \eta \in \mathcal{P}(\sigma^r) \right\}$$

generates $\mathbb{R} \times J$.

For $z \in \mathbb{R}^l$, and $g_0 : \mathbb{R} \to \mathbb{R}$, $g : J \to \mathbb{R}$ integrable with compact support, define

$$K_T^z(g_0 \otimes g) = \sum_{\eta \in \mathcal{P}(\sigma^r)} g_0(\ell(\eta) - T) g\left(\int_\eta G - \lfloor Tz \rfloor \right),$$

where $\lfloor Tz \rfloor$ denoted the unique element of J such that $Tz = \lfloor Tz \rfloor + E$, where E is a fixed fundamental domain for J.

Theorem 7.4.1 (Babillot–Ledrappier [BL98])**.** *Suppose σ^r is topologically weak-mixing and satisfies (A). Then for integrable functions g_0, g with compact support, and $z \in \mathcal{C}_G^\circ$, $K_T^z(g_0 \otimes g)$ is asymptotic, as $T \to \infty$, to*

$$\frac{e^{TH(z) + \langle u^z, Tz - \lfloor Tz \rfloor\rangle} \sqrt{|\det H''(z)|}}{T(2\pi T)^{\frac{l}{2}}} \int_{\mathbb{R} \times J} e^{\beta(u^z)x - \langle u^z, y\rangle} g_0(x) g(y)\, dx\, dy.$$

By our comments at the start of Section 7.1, Theorem 7.4.1 completes the proof of Theorem 7.2.3, setting $l = b + n$ and $G = F$.

To prove Theorem 7.4.1, define counting measures through which to analyse $K_T^z(g_0 \otimes g)$. For $z \in \mathcal{C}_G^\circ$, let $\frac{\lfloor Tz \rfloor}{T}$ be denoted by z_T. For $c \in \mathbb{R}$, $u \in \mathbb{R}^l$, we have that

$$\begin{aligned} K_T^z(e^{-c(\cdot)}g_0 \otimes e^{\langle u, \cdot \rangle} g) &= \sum_{\eta \in \mathcal{P}(\sigma^r)} e^{-c(\ell(\eta)-T)+\langle u, \int_\eta G - \lfloor Tz \rfloor \rangle} g_0(\ell(\eta)-T) g\left(\int_\eta G - \lfloor Tz \rfloor\right) \\ &= e^{T(c-\langle u, z_T \rangle)} \sum_{\eta \in \mathcal{P}(\sigma^r)} e^{-c\ell(\eta)+\langle u, \int_\eta G \rangle} g_0(\ell(\eta)-T) g\left(\int_\eta G - \lfloor Tz \rfloor\right) \\ &= e^{T(c-\langle u, z_T \rangle)} M_T^{(c,u,z)}(g_0 \otimes g), \end{aligned}$$

Where $M_T^{(c,u,z)}$ is the measure defined by

$$M_T^{(c,u,z)} = \sum_{\eta \in \mathcal{P}(\sigma^r)} e^{-c\ell(\eta)+\langle u, \int_\eta G \rangle} \delta_{\ell(\eta)-T} \otimes \delta_{\int_\eta G - \lfloor Tz \rfloor}.$$

We will be particularly interested in the behaviour of the measure $M_T^{(\beta(u^{z_T}), u^{z_T}, z_T)}$, henceforth denoted by M_T^z. By the calculations above, one sees that

$$K_T^z(g_0 \otimes g) = e^{tH(z_T)} \int_{\mathbb{R} \times J} e^{\beta(u^{z_T})x - \langle u^{z_T}, y \rangle} g_0(x) g(y) \, dM_T^z(x,y). \tag{7.2}$$

Since we have $e^{tH(z_T)} = e^{tH(z)} e^{tH(z_T)-tH(z)} \sim e^{tH(z)+\langle u^z, Tz - \lfloor Tz \rfloor \rangle}$, to prove Theorem 7.2.3 it suffices to show that for any compactly supported h,

$$M_T^z(h) \underset{T\to\infty}{\longrightarrow} \frac{\sqrt{|\det H''(z)|}}{T(2\pi T)^{\frac{l}{2}}} \int_{\mathbb{R} \times J} h(x,y) \, dx \, dy, \tag{7.3}$$

where dy refers to the Haar measure on J.

Our principal tool for this will be the series

$$Z_r(s,w) := \sum_{\eta \in \mathcal{P}(\sigma^r)} e^{-s\ell(\eta)+\langle w, \int_\eta G \rangle}.$$

This function was defined, more generally, in the proof of Lemma 5.4.10 in Chapter 5. There it is shown, by comparison with a zeta function, that Z_r is well-defined and analytic on a half-plane described by $s = c + it \in \mathbb{C}$ and $w = u + iv \in \mathbb{C}^l$. Precisely, $Z_r(s,w)$ converges whenever $\mathscr{R}(s) > \beta(\mathscr{R}(w))$. Here we will consider w to be such that $\Im(w)$ is an element of J^*, the Pontryagin dual group to J. When J is of the form $\mathbb{R}^q \times \mathbb{Z}^{l-q}$, J^* is of the form $\mathbb{R}^q \times \mathbb{R}^{l-q}/\mathbb{Z}^{l-q}$.

The series $Z_r(s,w)$ will be seen to appear in Fourier inversion formulae for $M_T^z(h)$, but we must first discuss some technical properties. We will be interested in $Z_r(s,w)$ when $\mathscr{R}(s) = \beta(\mathscr{R}(w))$. The necessary properties for $Z_r(s,w)$ are summarised in the following technical lemma. For $N \in \mathbb{N}$, let $\mathbf{C}^N$ denote the function space $C^N(\mathbb{R} \times J^*, \mathbb{C})$, with the topology of uniform convergence of derivatives on compact sets.

Lemma 7.4.2 ([BL98], Proposition 2.1)**.** *For any compact set $K_0 \subset \mathbb{R}^l$, there exist*

(i) *An open neighbourhood $U = U_1 \times U_2$ of $(0,0) \in \mathbb{R} \times J^*$;*

(ii) *A function $p \in \mathbf{C}^N$ such that p vanishes outside of U and $p(0,0) = 1$;*

(iii) *A continuous map $u \to A_u$ from K_0 to $\mathbf{C}^N$, such that for $u \in K_0$,*

$$\lim_{c \searrow \beta(u)} Z_r(c+it, u+iv) = -p(t,v)\log(\beta(u) + it - \beta(u+iv)) + A_u(t,v),$$

where $\beta(u+iv)$ is the analytic extension of $\beta(u)$ to $K_0 \times U_2$. In particular, $(t,v) \to \lim_{c\searrow\beta(u)} Z_r(c+it, u+iv)$ is locally integrable on $\mathbb{R} \times J^*$.

Moreover, for any compact set $K \subset \mathbb{R}\times J^*$, there exists positive constants $C_1, C_2 > 0$ such that for any $c > \beta(u)$,

$$|Z_r(c+it, u+iv)| \leq \begin{cases} -C_1 \log|\beta(u) + it - \beta(u+iv)| & \text{if } (t,v) \in U \\ C_2 & \text{if } (t,v) \in K \setminus U. \end{cases}$$

Proof. First, define $\hat{G} : \Sigma(\Gamma) \to \mathbb{R}^l$, by

$$\hat{G}(x) = \int_0^{r(x)} G[x,t]\, dt.$$

We will use fine spectral properties of the transfer operator $\mathcal{L}_{s,w} := \mathcal{L}_{-sr+\langle w, \hat{G}\rangle}$, defined in Chapter 4. By the results in Chapter 4, we may assume

1. For each s, w, the spectral radius of the transfer operator satisfies $\mathrm{sr}(\mathcal{L}_{s,w}) \leq 1$;
2. If s, w are real then $\mathcal{L}_{s,w}1 = 1$.
3. If $\mathcal{L}_{s,w}$ has an eigenvalue of modulus 1 then it is simple and unique, and the remainder of the spectrum is contained in a disc of radius strictly smaller than 1.

Fix a compact set $K_0 \subset \mathbb{R}^l$, and let

$$Y_0 = \{(\beta(u), u) : u \in K_0\}.$$

For any $(s, w) \in Y_0$, 1 is a maximal simple unique eigenvalue of $\mathcal{L}_{s,w}$, and the remainder of the spectrum is contained in a strictly smaller disc. Thus, using perturbation theory (see Theorem 4.6 of [PP90]), there exists $0 < \theta_0 < 1$ such that there is an open neighbourhood $Y_0 \subset V_0$ such that for all $(s, w) \in V_0$, the spectrum of $\mathcal{L}_{s,w}$ consists of a simple unique maximal eigenvalue $k_0(s, w)$, and other values contained in a disc of radius θ_0. Now, Suppose instead that (s, w) is such that $(t, v) \neq (0, 0)$. Then 1 is not an eigenvalue of $\mathcal{L}_{s,w}$, since if it was, Proposition 4.3.8 would say

$$e^{i(-tr^n(x)+\langle v,\hat{G}^n(x)\rangle)} = 1$$

whenever $\sigma^n(x) = x$. By assumption (A), this gives

$$e^{i(-tx+\langle v,y\rangle)} = 1$$

for all $(x, y) \in \mathbb{R} \times J$. This is only possible if $(t, v) = (0, 0) \in \mathbb{R} \times J^*$.

The spectral radius of $\mathcal{L}_{\beta(u)+it,u+iv}$ may still be maximal for $(t, v) \neq (0, 0)$. We choose closed sets $K_u \subset \mathbb{R} \times J^*$ for each $u \in K_0$ such that $\mathrm{sr}(\mathcal{L}_{\beta(u)+it,u+iv}) = 1$ if $(t, v) \in K_u$, and $\mathrm{sr}(\mathcal{L}_{\beta(u)+it,u+iv}) < 1$ otherwise. Fix a compact set $K \subset \mathbb{R} \times J^*$ and define

$$Y_1 = \{(\beta(u) + it, u + iv) : u \in K_0, (t, v) \in K_u \cap K\}.$$

Again by perturbation theory, there exists $0 < \theta_1 < 1$ and an open subset $Y_1 \subset V_1$, disjoint from V_0, such that for $(s, w) \in V_1$ the spectrum of $\mathcal{L}_{s,w}$ consists of a simple unique maximal eigenvalue $k_1(s, w)$, and other values contained in a disc of radius θ_1. Furthermore, there is $0 < \theta_2 < 1$ and an open neighbourhood V_2 of $Y \setminus (V_0 \cup V_1)$ such that for $(s, w) \in V_2$, all spectral values are contained in a disc of radius θ_2. We also have that the functions k_0, k_1 vary analytically with (s, w).

We wish to use the functions k_0, k_1 to understand the growth of $Z_r(s, w)$. For these purposes, it is more convenient to use the series

$$\tilde{Z}_r(s, w) := \sum_{\eta\in\tilde{\mathcal{P}}(\sigma^r)} e^{-s\ell(\eta)+\langle w,\int_\eta G\rangle} = \sum_{m=1}^{\infty} Z_r(ms, mw).$$

The calculations in the proof of Lemma 5.4.10 show that Z_r and $\tilde{Z}_r$ have the same analytic properties on the critical line $\mathscr{R}(s) = \beta(\mathscr{R}(w))$, so there is no loss in working with $\tilde{Z}_r$.

We write $\tilde{Z}_r(s,w) = \sum_{n=1}^{\infty} \frac{1}{n} \tilde{Z}_{r,n}(s,w)$, where

$$\tilde{Z}_{r,n}(s,w) := \sum_{\sigma^n(x)=x} e^{-sr^n(x)+\langle w, \hat{G}^n(x)\rangle}.$$

By Chapter 10 of [PP90], we have that for $(s,w) \in V_i$,

$$\tilde{Z}_{r,n}(s,w) = k_i(s,w)^n + a_n^i(s,w),$$

where $k_2(s,w) = 0$ and $|a_n^i(s,w)| \leq C_i \theta_i^n$ for a constant C_i. Thus for $(s,w) \in V_i$, $\tilde{Z}_r(s,w) = K_i(s,w) + A^i(s,w)$, where $A^i(s,w) = \sum_{n=1}^{\infty} \frac{a_n^i(s,w)}{n}$ is absolutely convergent and analytic on U_i, and when $\mathscr{R}(s) > \beta(\mathscr{R}(w))$,

$$K_i = \sum_{n=1}^{\infty} k_i(s,w)^n = \log(1 - k_i(s,w)).$$

For the case $\mathscr{R}(s) = \beta(\mathscr{R}(w))$, the function

$$(0,0) \neq (t,v) \mapsto -\log(1 - k_i(\beta(u) + it, u + iv))$$

is well defined as the limit of $\log(1 - k_i(c + it, u + iv))$ as c decreases to $\beta(u)$.

Choose a partition of unity p_0, p_1, p_2 of class $\mathbf{C}^N$ subordinate to V_0, V_1, V_2, such that p_0 is of the form

$$p_0(c + it, u + iv) = p'(c - \beta(u))p(t,v),$$

for functions p', p. Then we can deduce the following.

$$\lim_{c \searrow \beta(u)} \tilde{Z}_r(c + it, u + iv) = -p(t,v) \log(1 - k_0(\beta(u) + it, u + iv) + B_u(t,v), \quad (7.4)$$

where

$$B_u(t,v) = \left(\sum_{i=0}^{2} p_i A^i - p_1 \log(1 - k_1) \right) (\beta(u) + it, u + iv)$$

is $\mathbf{C}^N$ and varies continuously with u.

We now apply Weierstrass' preparation theorem to the function $1 - k_0(s,w)$ in a neighbourhood of a point of the form $(\beta(u), u)$. Since, by Proposition 4.3.9,

$$\left. \frac{\partial k_0(s,w)}{\partial s} \right|_{(s,w)=(\beta(u),u)} = -\int r \, dm_{-\beta(u)r + \langle u, \hat{G} \rangle} \neq 0,$$

the preparation theorem gives

$$1 - k_0(s, w) = a_u(s, w)(s - b_u(w)), \tag{7.5}$$

where a_u, b_u are analytic. Differentiating both sides of (7.5) with respect to s, we obtain that

$$a_u(\beta(u), u) = \int r \, dm_{-\beta(u)r + \langle u, \hat{G} \rangle}$$

which is strictly positive and bounded away from 0 uniformly for $u \in K_0$. Thus there exists a neighbourhood U of $(0, 0)$ in $\mathbb{R} \times J^*$ and $\delta > 0$ such that

$$\mathscr{R}(a_u(c + it, u + iv)) > \delta$$

whenever $(t, v) \in U$ and $c \geq \beta(u)$. We also have that $b_u(u) = \beta(u)$, so setting

$$\beta(u + iv) := b_u(u + iv)$$

gives an analytic extension of β on a complex neighbourhood of K_0. Therefore, for $u \in K_0$ and $(t, v) \in U$, there exists an analytic function $C_u(t, v)$ such that

$$\log(1 - k_0(\beta(u) + it, u + iv)) = \log(\beta(u) + it - \beta(u + iv)) + C_u(t, v).$$

This, with (7.4), completes the main part of the proof. The remainder can be found at the end of the appendix to [BL98]. $\square$

Remark. In Chapter 5, as Proposition 5.4.11, we stated a simplified but weighted version of Lemma 7.4.2. The proof of Lemma 7.4.2 is entirely analogous if one replaces $Z_r(s, w)$ with the weighted version

$$\sum_{\eta \in \mathcal{P}(\sigma^r)} e^{-s\ell(\eta) + \int_\eta I + \langle w, \int_\eta G \rangle},$$

for some Hölder continuous function $I : \Sigma(\Gamma, r) \to \mathbb{R}$. This allows us to prove Proposition 5.4.11.

With Lemma 7.4.2 we are ready to begin Fourier analysis of M_T^z. Let H^+ be the class of non-negative real valued functions $h : \mathbb{R} \times J \to \mathbb{R}$, such that the Fourier transform $\hat{h}$ has compact support and is an element of $\mathbf{C}^N$. For $c > \beta(u)$, we apply the Fourier inversion formula to

$$M_T^{(c,u,z)}(h) = \sum_{\eta \in \mathcal{P}(\sigma^r)} e^{-c\ell(\eta) + \langle u, \int_\eta G \rangle} h\left(\ell(\eta) - T, \int_\eta G - \lfloor Tz \rfloor\right).$$

Interpreting dv as the Haar measure on J^*, we obtain

$$\frac{1}{(2\pi)^{l+1}} \sum_{\eta \in \mathcal{P}(\sigma^r)} e^{-c\ell(\eta)+\langle u, \int_\eta G\rangle} \int_{\mathbb{R}\times J^*} e^{-it(\ell(\eta)-T)+i\langle v, \int_\eta G - \lfloor Tz \rfloor\rangle} \hat{h}(-t,v)\, dtdv,$$

which in turn reduces to

$$\frac{1}{(2\pi)^{l+1}} \int_{\mathbb{R}\times J^*} e^{iT(t-\langle v, z_T\rangle)} \hat{h}(-t,v) Z_r(c+it, u+iv)\, dtdv.$$

The final equality is obtained with Fubini's Theorem for Z_r on the compact support K of $\hat{h}$, where Lemma 7.4.2 justifies convergence. Lemma 7.4.2 also allows us to apply the dominated convergence theorem to show that

$$\lim_{c\searrow\beta(u)} M_T^{(c,u,z)}(h) = \frac{1}{(2\pi)^{l+1}} \int_{\mathbb{R}\times J^*} e^{iT(t-\langle v, z_T\rangle)} \hat{h}(-t,v) \lim_{c\searrow\beta(u)} Z_r(c+it, u+iv)\, dtdv \tag{7.6}$$

which is finite by Lemma 7.4.2. On the other hand, since h is non-negative everywhere, $M_T^{(c,u,z)}(h)$ increases as c decreases to $\beta(u)$, so

$$\lim_{c\searrow\beta(u)} M_T^{(c,u,z)}(h) = M_T^{(\beta(u),u,z)}(h) \tag{7.7}$$

In fact, it follows that equations (7.6) and (7.7) also hold for any linear combination of functions of the form $(x,y) \mapsto e^{i(tx+\langle v,y\rangle)} h(x,y)$, where $h \in H^+$ and $(t,v) \in \mathbb{R}\times J^*$. Let us denote by H the collection of all such linear combinations.

Summarising this for the measure M_T^z, we have that for any function $h \in H$,

$$M_T^z(h) = \frac{1}{(2\pi)^{l+1}} \int_{\mathbb{R}\times J^*} e^{iT(t-\langle v, z_T\rangle)} \hat{h}(-t,v) \lim_{c\searrow\beta(u^z)} Z_r(c+it, u^z+iv)\, dtdv.$$

Following Section 2.2 of [BL98], one can prove that the limit (7.3) holds for $h \in H$, uniformly in z. Rewriting, we have the uniform convergence

$$\frac{(2\pi T)^{\frac{l}{2}} T}{\sqrt{|\det H''(z)|}} M_T^z(h) \to \lambda(h),$$

where λ is the Haar measure on J. To complete the proof of Theorem 7.2.3, this extends to all continuous functions of compact support.

Lemma 7.4.3 ([BL98], Lemma 2.4)**.** *Let ν_T^z be a family of measures on $\mathbb{R}\times J$. If for any $h \in H$,*

$$\nu_T^z(h) \underset{T\to\infty}{\longrightarrow} \lambda(h)$$

uniformly in z, then the same is true for any continuous function of compact support.

This can then be extended to integrable functions h by approximation.

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