HEIGHTS ON ELLIPTIC CURVES OVER NUMBER FIELDS, PERIOD LATTICES, AND COMPLEX ELLIPTIC LOGARITHMS

By

Thotsaphon Thongjunthug

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Coventry, England Thotsaphon Thongjunthug

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Declaration

I hereby declare that this thesis is my own work and to the best of my knowledge it contains no materials previously published or written by another person, except where due acknowledgement is made in the thesis. Any contribution made to the research by others, with whom I have worked at the University of Warwick or elsewhere, is explicitly acknowledged in the thesis.

In this thesis, an alternative version of Chapter 2 and 3 has been published in [Tho08] and [Tho10] respectively. Chapter 4 is based on my joint work with Professor John E. Cremona at the University of Warwick; another version of this chapter has been submitted for publication as a joint paper [CT].

Each algorithm based on this thesis has been implemented in MAGMA, a computer package produced and distributed by the School of Mathematics and Statistics of the University of Sydney, Australia.

Finally, I declare that this thesis has not been submitted for a degree at any other institution.

Thotsaphon Thongjunthug

Summary

This thesis presents some major improvements in the following computations: a lower bound for the canonical height, period lattices, and elliptic logarithms.

On computing a lower bound for the canonical height, we have successfully generalised the existing algorithm of Cremona and Siksek [CS06] to elliptic curves over totally real number fields, and then to elliptic curves over number fields in general. Both results, which are also published in [Tho08] and [Tho10] respectively, will be fully explained in Chapter 2 and 3.

In Chapter 4, we give a complete method on computing period lattices of elliptic curves over $\mathbb{C}$, whereas this was only possible for elliptic curves over $\mathbb{R}$ in the past. Our method is based on the concept of *arithmetic-geometric mean* (AGM). In addition, we extend our method further to find elliptic logarithms of complex points. This work is done in collaboration with Professor John E. Cremona; another version of this chapter has been submitted for publication [CT].

In Chapter 5, we finally illustrate the applications of our main results towards certain computations which did not work well in the past due to lack of some information on elliptic curves. This includes determining a Mordell–Weil basis, finding integral points on elliptic curves over number fields [SS97], and finding elliptic curves with everywhere good reduction [CL07].

A number of computer programs have been implemented for the purpose of illustration and verification. Their source code (written in MAGMA) can be found in Appendix A.

Dedicated to my beloved aunt Pissawong Thongjunthug

“You always live on in my heart”

Chapter 1

Introduction

We will first introduce all underlying concepts which are necessary for later chapters, together with an overview of this thesis. In this chapter, we will start by a brief definition of elliptic curves, before move on to describe more specific concepts related to elliptic curves over number fields, and finally, elliptic curves over $\mathbb{C}$. A synopsis of each chapter will be also mentioned where appropriate.

1.1 An Overview of Elliptic Curves

In this section, we will briefly describe the definition an elliptic curve over a general field, and how to construct an operation defining the group law on it.

Definition. Let K be a field. An *elliptic curve* E defined over K (denoted by E/K) is a non-singular projective plane curve of degree 3 over K, with a specified point of inflection O which is also defined over K.

We can assume (see [Mil06, Proposition 1.2]) that E is given by a homogeneous *Weierstrass equation*

$$Y^2Z + a_1XYZ + a_3YZ^2 = X^3 + a_2X^2Z + a_4XZ^2 + a_6Z^3,$$

where all $a_j \in K$ are constants. If $Z \neq 0$, then we can divide every term above by

Z^3 to obtain an affine Weierstrass equation

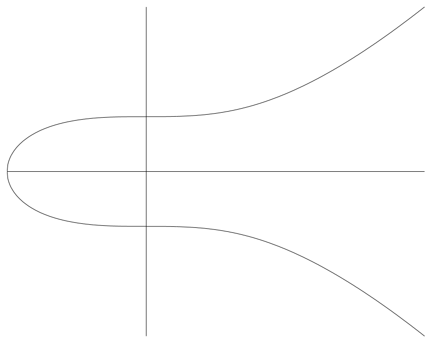
$$E: \quad y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6, \tag{1.1}$$

via $x = X/Z$, $y = Y/Z$. The line $Z = 0$ intersects E at $(0 : 1 : 0)$ with multiplicity 3, so we may take $O = (0 : 1 : 0)$; this is called the *point at infinity* of E. It is also easy to prove (see [Was03, p. 20]) that every vertical line intersects E at O.

From now on, we shall always assume that an elliptic curve E is given by an affine Weierstrass equation (1.1), unless otherwise stated. As in [Sil86, p. 46], we define the following quantities associated to a Weierstrass equation:

$$b_2 = a_1^2 + 4a_2, \quad b_4 = 2a_4 + a_1a_3, \quad b_6 = a_3^2 + 4a_6,$$
$$b_8 = a_1^2a_6 + 4a_2a_6 - a_1a_3a_4 + a_2a_3^2 - a_4^2,$$
$$\Delta = -b_2^2b_8 - 8b_4^3 - 27b_6^2 + 9b_2b_4b_6.$$

The quantity Δ is known as the *discriminant* of E. An example of elliptic curves defined over $K = \mathbb{R}$ is illustrated in Figure 1.1.


(a) $\Delta < 0$

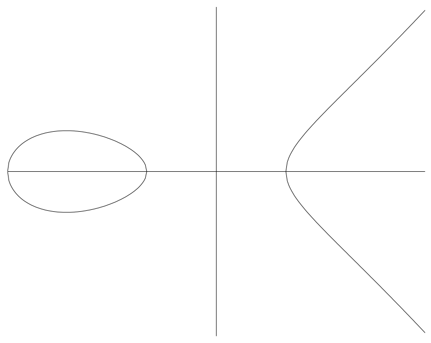

(b) $\Delta > 0$

Figure 1.1: Elliptic curves over $\mathbb{R}$

In this thesis, our field K will be either $\mathbb{R}$, $\mathbb{C}$, or a number field, hence $\text{char}(K) = 0$. For now we note that, since $\text{char}(K) \neq 2$, we can rewrite the Weierstrass equation of E as

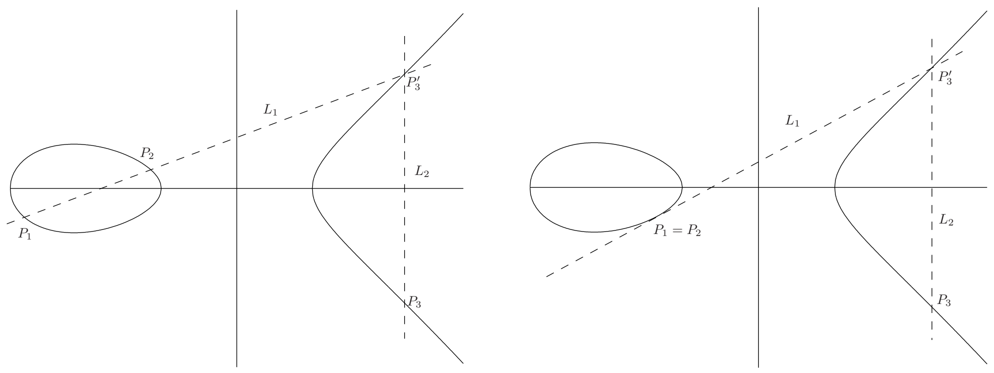
$$E: \quad (2y + a_1x + a_3)^2 = f(x) = 4x^3 + b_2x^2 + 2b_4x + b_6.$$

It can be shown (see [Sil86, Proposition III.1.4]) that $\Delta \neq 0$ if and only if $f(x)$ has three distinct roots, which is equivalent to the non-singularity of E.

Definition. Let E be an elliptic curve defined over a field K, and let $L \supseteq K$ be a field. The set of all *L-points* of E, denoted by $E(L)$, is given by

$$E(L) = \{(x, y) \in L^2 : y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6\} \cup \{O\}.$$

For any two points $P_1, P_2 \in E(L)$, we can construct an operation so-called *addition* (denoted by $+$) geometrically as follows. First, let L_1 be the straight line through P_1, P_2 (or if $P_1 = P_2$, take L_1 to be the tangent line to E at that point). Then L_1 will intersect E at another point, say, P_3'. Let L_2 be the vertical line through P_3'. Then L_2 will intersect E at another point, say P_3. Finally, we define $P_1 + P_2 = P_3$. An example of this process for $L = K = \mathbb{R}$ is shown in Figure 1.2.



(a) $P_1 \neq P_2$ (b) $P_1 = P_2$

Figure 1.2: Addition on elliptic curves

It is readily shown (see, e.g., [Was03, Section 2.2]) that $E(L)$ becomes an abelian group with O as the identity once being equipped with this addition. We say that a point $P \in E(L)$ is a *torsion point* if P has finite order in $E(L)$; otherwise, P is said to be *non-torsion.*

1.2 Elliptic Curves over Number Fields

In this section, we shall first explain the definition of heights on elliptic curves over number fields, and then briefly describe the importance of a lower bound for the canonical height towards computing a Mordell–Weil basis. Throughout this section, our elliptic curve E will be defined over a number field K.

1.2.1 Heights

Roughly speaking, the height function is a way to measure how "complicated" the x-coordinate of a point $P \in E(K)$ is. In this thesis, we will be using the *canonical height*, which can be expressed as a sum of all contributions from *local heights*. It should be noted that normalisation of heights varies in literature. In our case, the local and canonical heights are defined with respect to the divisor $2(O)$. This leads to the same normalisation as the one used in the computer package MAGMA, and gives double the values compared with Silverman's paper [Sil88] where heights are defined with respect to (O).

The Canonical Height

Denote the sets of real and complex archimedean places of K by M_K^{r} and M_K^{c} respectively, and let M_K be the set of all places of K. For $v \in M_K$, let $n_v = [K_v : \mathbb{Q}_v]$, and let σ_v be the associated embedding of K into the completion K_v.

Definition. For $x \in K$, the *absolute value* of x at a place $v \in M_K$ is given by

$$|x|_v = \begin{cases} |\sigma_v(x)| & \text{if } v \in M_K^{\mathrm{r}} \cup M_K^{\mathrm{c}}, \\ \mathcal{N}(\mathfrak{p})^{-\mathrm{ord}_{\mathfrak{p}}(x)/n_{\mathfrak{p}}} & \text{if } v = \mathfrak{p}, \end{cases} \tag{1.2}$$

where $\mathfrak{p}$ is the prime ideal associated to a non-archimedean place v, and $\mathcal{N}$ denotes the norm of an integral ideal of K.

It is a standard fact (see, e.g., [Coh07, Proposition 4.1.14]) that this definition satisfies all axioms of valuation theory and the product formula $\prod_{v\in M_K} |x|_v^{n_v} = 1$.

Definition. For $P \in E(K)$, the *naive height* of P (relative to K) is defined by

$$H_K(P) = \begin{cases} 1 & \text{if } P = O, \\ \displaystyle\prod_{v\in M_K} \max\{1, |x(P)|_v\}^{n_v} & \text{if } P \neq O. \end{cases} \tag{1.3}$$

Definition. For $P \in E(K)$, the *logarithmic height* of P is defined by

$$h(P) = \frac{1}{[K:\mathbb{Q}]} \log H_K(P). \tag{1.4}$$

From this, the *canonical height* of P is given by

$$\hat{h}(P) = \lim_{j\to\infty} \frac{h(2^j P)}{4^j}. \tag{1.5}$$

Observe that $h(P) \geq 0$ for all $P \in E(K)$, thus we also have $\hat{h}(P) \geq 0$ for all $P \in E(K)$. Moreover, we have $\hat{h}(mP) = m^2\hat{h}(P)$ for all $P \in E(K)$ and $m \in \mathbb{Z}$ (see [Sil86, p. 230] for the proof). In particular, if P is a torsion point of order m, then we have

$$0 = \hat{h}(O) = \hat{h}(mP) = m^2\hat{h}(P)$$

(the fact that $\hat{h}(O) = 0$ follows easily from (1.6)), i.e., $\hat{h}(P) = 0$. In fact, the canonical height $\hat{h} : E(K) \to [0,\infty)$ is a positive definite quadratic form on $E(K)/E_{\text{tors}}(K)$, which gives it the structure of a lattice. Hence there exists a positive lower bound for $\hat{h}(P)$ among all non-torsion $P \in E(K)$.

Computing such a lower bound has a number of applications in the arithmetic of elliptic curves. In particular, it is a crucial step in determining a Mordell–Weil basis for $E(K)$; see Section 1.2.2 for more details. In the past, a number of explicit lower bounds for the canonical height on $E(K)$ have been proposed. Some of them, including [HS88, Theorem 0.3], aim to prove *Lang's conjecture* (see [Sil86, Conjecture VIII.9.9]), which states that there exists a constant c_K, depending only on K, such that

$$\hat{h}(P) \geq c_K \log \mathcal{N}(\mathcal{D}_{E/K})$$

for all non-torsion $P \in E(K)$, where $\mathcal{D}_{E/K}$ is the minimal discriminant of E/K. As we will see later on, however, the lower bound obtained by that result is too small for practical use.

In this thesis, we will develop an alternative method for determining a larger positive lower bound for the canonical height on elliptic curves over number fields. The underlying methodology is mainly inspired by the algorithm of Cremona and Siksek [CS06], which allows one to compute such a lower bound for elliptic curves defined over $\mathbb{Q}$ only. Our work on this is divided into two parts, namely, determining certain contributions from all real embeddings, and then from all complex embeddings. Both parts will be described in Chapter 2 and 3 respectively.

Local Height Functions

Finally, we give the definition of local heights. Suppose $P \in E(K)$ with $2P \neq O$. Then one can observe that $x(2P) = g(P)/f(P)$, where

$$f(P) = 4x(P)^3 + b_2 x(P)^2 + 2b_4 x(P) + b_6, \quad g(P) = x(P)^4 - b_4 x(P)^2 - 2b_6 x(P) - b_8.$$

Hence by (1.3), we have

$$\begin{aligned} H_K(2P) &= \prod_{v \in M_K} \max\{1, |x(2P)|_v\}^{n_v} \\ &= \prod_{v \in M_K} \max\{1, |g(P)|_v / |f(P)|_v\}^{n_v} \\ &= \prod_{v \in M_K} |f(P)|_v^{n_v} \cdot \prod_{v \in M_K} \max\{1, |g(P)|_v / |f(P)|_v\}^{n_v} \\ &= \prod_{v \in M_K} \max\{|f(P)|_v, |g(P)|_v\}^{n_v} \end{aligned}$$

(note that $\prod_{v\in M_K} |f(P)|_v^{n_v} = 1$ by the product formula), and so

$$h(2P) = \frac{1}{[K:\mathbb{Q}]} \sum_{v\in M_K} n_v \log\max\{|f(P)|_v, |g(P)|_v\}$$

by (1.4). Together with (1.4) again, this easily yields

$$h(2P) - 4h(P) = \frac{1}{[K:\mathbb{Q}]} \sum_{v\in M_K} n_v \log\Phi_v(P), \tag{1.6}$$

where

$$\Phi_v(P) = \begin{cases} 1 & \text{if } P = O, \\ \dfrac{\max\{|f(P)|_v, |g(P)|_v\}}{\max\{1, |x(P)|_v\}^4} & \text{if } P \neq O. \end{cases} \tag{1.7}$$

Definition. For $v \in M_K$, let K_v be the completion of K at v. The function $\lambda_v : E(K_v) \to \mathbb{R}$ defined by

$$\lambda_v(P) = \log\max\{1, |x(P)|_v\} + \sum_{j=0}^{\infty} \frac{\log\Phi_v(2^jP)}{4^{j+1}}. \tag{1.8}$$

is called the *local height function* at v.

To see the relationship between the canonical height and local heights, we use (1.5) and the telescoping sum to obtain

$$\begin{aligned} \hat{h}(P) &= h(P) + \left[\frac{h(2P)}{4} - h(P)\right] + \left[\frac{h(2^2P)}{4^2} - \frac{h(2P)}{4}\right] + \cdots \\ &= \frac{1}{[K:\mathbb{Q}]} \sum_{v\in M_K} n_v \left(\log\max\{1, |x(P)|_v\} + \frac{\log\Phi_v(P)}{4} + \frac{\log\Phi_v(2P)}{4^2} + \cdots\right) \\ &= \frac{1}{[K:\mathbb{Q}]} \sum_{v\in M_K} n_v\lambda_v(P) \end{aligned} \tag{1.9}$$

(the second equality follows directly from (1.4) and (1.8)). This therefore allows us to obtain $\hat{h}(P)$ by combining the contribution of λ_v on each local model $E(K_v)$, noting that $\lambda_v(P) = 0$ for almost all v.

1.2.2 Mordell–Weil Bases

Recall that $E(K)$ is an abelian group under addition. By the Mordell–Weil theorem (see, e.g., [Sil86, Chapter VIII] for more details), it is also well known that $E(K)$ is finitely generated. It then follows that

$$E(K) \cong E_{\text{tors}}(K) \times \mathbb{Z}^r,$$

where $E_{\text{tors}}(K)$ is the *torsion subgroup* of $E(K)$ (i.e., the set of all torsion points in $E(K)$), and the *rank* $r \geq 0$ of $E(K)$ is the cardinality of a *Mordell–Weil basis* for $E(K)$ (i.e., the set of all non-torsion points in $E(K)$ whose images in $E(K)/E_{\text{tors}}(K)$ form a $\mathbb{Z}$-basis for it).

In general, it turns out that the torsion subgroup of $E(K)$ can be determined more easily than a Mordell–Weil basis for $E(K)$. According to [Sik95], the task of explicit computation of such a basis consists of the following steps:

1. Determine $P_1, \ldots, P_r$ whose images in $E(K)/E_{\text{tors}}(K)$ generate a subgroup of finite index of $E(K)/E_{\text{tors}}(K)$. Usually, these are obtained by performing an m-descent for some $m \geq 2$.

2. A lower bound $\lambda > 0$ for the canonical height $\hat{h}(P)$ is determined, which in turn yields an upper bound on the index $n = [E(K)/E_{\text{tors}}(K) : \langle P_1, \ldots, P_r \rangle]$.

3. A sieving procedure [Sik95, Section 4] is then used to deduce a Mordell–Weil basis for $E(K)$.

In step (2), we certainly wish to have an upper bound for n as small as possible. In particular, $P_1, \ldots, P_r$ will certainly be a Mordell–Weil basis of $E(K)$ if $n = 1$. It follows from the following theorem that, in order to have a *smaller* upper bound for n, one must obtain a *larger* lower bound for the canonical height.

Theorem 1.2.1 (The Geometry of Numbers). *If $E(K)$ contains no points P of infinite order with $\hat{h}(P) \leq \lambda$ for some $\lambda > 0$, then the index n satisfies*

$$n \leq R(P_1, \ldots, P_r)^{1/2}(\gamma_r/\lambda)^{r/2},$$

where $R(P_1, \ldots, P_r) = \det(\langle P_i, P_j \rangle)_{1 \leq i,j \leq r}$ and

$$\langle P_i, P_j \rangle = \frac{1}{2}\left(\hat{h}(P_i + P_j) - \hat{h}(P_i) - \hat{h}(P_j)\right).$$

Moreover, the values γ_r may be taken to be

$$\begin{array}{llll} \gamma_1^1 = 1, & \gamma_2^2 = 4/3, & \gamma_3^3 = 2, & \gamma_4^4 = 4, \\ \gamma_5^5 = 8, & \gamma_6^6 = 64/3, & \gamma_7^7 = 64, & \gamma_8^8 = 2^8, \end{array}$$

and $\gamma_r = (4/\pi)\Gamma(r/2 + 1)^{2/r}$ for $r \geq 9$.

Proof. See [Sik95, Theorem 3.1]. □

As mentioned earlier, we will fully explain a new method for computing λ in Chapter 2 and 3. Some examples on how to determine a Mordell–Weil basis using λ and the process above will be also shown in Chapter 5.

1.3 Elliptic Curves over $\mathbb{C}$

We now move on to elliptic curves defined over $\mathbb{C}$, where we will give a brief introduction on period lattices of elliptic curves and elliptic logarithms of points, which will be the subject of Chapter 4.

Definition. A *lattice* Λ is a free $\mathbb{Z}$-module of rank 2 embedded as a discrete subgroup of $\mathbb{C}$, i.e.,

$$\Lambda = \{n_1 w_1 + n_2 w_2 : n_1, n_2 \in \mathbb{Z}\}$$

for some $w_1, w_2 \in \mathbb{C}$ with $w_1/w_2 \notin \mathbb{R}$.

For a lattice Λ, we can also identify $\mathbb{C}/\Lambda$ with the set

$$F_{w_1,w_2} = \{\lambda_1 w_1 + \lambda_2 w_2 : 0 \leq \lambda_1, \lambda_2 < 1\}$$

called the *(open) fundamental parallelogram* for Λ (or if we allow both $\lambda_j = 1$, we say that it is *closed*). In the topological point of view, this is a *torus*. Clearly, choosing a different $\mathbb{Z}$-basis for Λ yields a different fundamental parallelogram.

Let E be an elliptic curve defined over $\mathbb{C}$. With some change of variables, we can assume that the Weierstrass equation of E is of the form

$$E: \quad Y^2 = 4(X - e_1)(X - e_2)(X - e_3),$$

where all e_j are distinct and $\sum_{j=1}^{3} e_j = 0$. It is well known (see, e.g., [Was03, Chapter 9]) that $E(\mathbb{C}) \cong \mathbb{C}/\Lambda$ for some lattice Λ via the map

$$\begin{aligned} P = (\wp_\Lambda(z), \wp'_\Lambda(z)) \quad &\longleftrightarrow \quad z \pmod{\Lambda}, \\ O \quad &\longleftrightarrow \quad 0 \pmod{\Lambda}. \end{aligned}$$

We say that Λ is the *period lattice* of E, and z is an *elliptic logarithm* of P. The values of $\wp_\Lambda(z)$ and $\wp'_\Lambda(z)$ can be computed using the power series expansion as shown in the following proposition.

Proposition 1.3.1 ([Coh93, Proposition 7.4.4]). *Let $\{w_1, w_2\}$ be a $\mathbb{Z}$-basis for Λ chosen so that $\Im(w_2/w_1) > 0$. Set*

$$\tau = w_2/w_1, \quad q = \exp(2i\pi\tau), \quad u = \exp(2\pi i z/w_1)$$

(here, $i = \sqrt{-1}$). Then

$$\wp_\Lambda(z) = \left(\frac{2i\pi}{w_1}\right)^2 \left(\frac{1}{12} + \frac{u}{(1-u)^2} + \sum_{j=1}^{\infty}\left[\frac{q^j u}{(1-q^j u)^2} + \frac{q^j u^{-1}}{(1-q^j u^{-1})^2} - \frac{2q^j}{(1-q^j)^2}\right]\right)$$

and

$$\wp'_\Lambda(z) = \left(\frac{2i\pi}{w_1}\right)^3 u \left(\frac{1+u}{(1-u)^3} + \sum_{j=1}^{\infty} q^j \left[\frac{1+q^j u}{(1-q^j u)^3} + \frac{q^j+u}{(q^j-u)^3}\right]\right).$$

To be precise, a $\mathbb{Z}$-basis for the period lattice of E is given by any two of the generators w_1, w_2, w_3, where $\wp_\Lambda(w_j/2) = e_j$ and $\wp'_\Lambda(w_j/2) = 0$ for all j. Suppose ℓ_j is the straight line on the complex plane starting from 0 to $w_j/2$. Then we have

$$\frac{w_j}{2} = \int_{\ell_j} dz = \int_{\ell_j} \frac{d\wp_\Lambda(z)}{\wp'_\Lambda(z)} = \int_{\mathcal{C}_j} \frac{dX}{Y}, \tag{1.10}$$

where $\mathcal{C}_j$ is the image of ℓ_j on E under $(\wp_\Lambda, \wp'_\Lambda)$, i.e.,

$$\mathcal{C}_j = \{(\wp_\Lambda(tw_j/2), \wp'_\Lambda(tw_j/2)) : 0 \le t \le 1\}.$$

More generally, if z_P is an elliptic logarithm of $P \in E(\mathbb{C})$, then

$$z_P = \int_{\mathcal{C}_P} \frac{dX}{Y} \pmod{\Lambda},$$

where $\mathcal{C}_P = \{(\wp_\Lambda(tz_P), \wp'_\Lambda(tz_P)) : 0 \le t \le 1\}$.

If E is defined over $\mathbb{R}$, then we obtain one of two special cases for the lattice Λ of E. It can be shown (see, e.g., [Was03, pp. 274–275]) that if E has positive discriminant (see Figure 1.1b), then Λ is *rectangular*, i.e., there exists a $\mathbb{Z}$-basis $\{w_1, w_2\}$ for Λ where $w_1 \in \mathbb{R}$ and $w_2 \in i\mathbb{R}$. In this case, the connected component of the identity (i.e., the one containing O) is parameterised by the line $\{tw_1 : 0 \le t < 1\}$, while the "loop" component is parameterised by the line $\{tw_1 + w_2/2 : 0 \le$

$t < 1\}$. For $E/\mathbb{R}$ with negative discriminant (see Figure 1.1a), we obtain a *skewed* lattice, i.e., there exists a $\mathbb{Z}$-basis for Λ with $w_1 \in \mathbb{R}$ and $\Re(w_2/w_1) = 1/2$. In this case, the whole $E(\mathbb{R})$ is connected and parameterised by the line $\{tw_1 : 0 \leq t < 1\}$.

Finding period lattices and elliptic logarithms is an important computation in its own right, and also has a number of applications towards certain algorithms, including one for determining a lower bound for the canonical height on elliptic curves over number fields, which will be fully explained in Chapter 2 and 3. Although there are some algorithms including [Coh93, Algorithm 7.4.7 and 7.4.8] readily available for computing both period lattices and elliptic logarithms, those algorithms only work for elliptic curves over $\mathbb{R}$. In Chapter 4, we will show how to develop a complete method for computing period lattices and elliptic logarithms for elliptic curves over $\mathbb{C}$ in general, based on the method of *arithmetic-geometric mean* (AGM). As we will see later on, our algorithm will allow one to compute both values with high degree of precision very rapidly.

In conclusion, we have introduced all necessary concepts to be used later on in this thesis, including an overview of each chapter. The next two chapters will focus on development of our first main result, namely, an algorithm for computing a lower bound for the canonical height on elliptic curves over number fields.

Chapter 2

Height Bound I

We will now focus on our first main result, where we develop an algorithm for computing a lower bound for the canonical height on elliptic curves over number fields. Our algorithm, which is inspired by the one of Cremona and Siksek [CS06], involves estimating local heights and solving a system of certain inequalities on both real and complex embeddings.

In this chapter, we will first show how to derive an estimate for local heights, and then show how to solve the system of inequalities mentioned above on *real embeddings*. This in turn will suffice for computing a lower bound for the canonical height on elliptic curves over number fields with at least one real embedding. A more sophisticated method for solving such inequalities on complex embeddings will be explained in Chapter 3.

Another version of this chapter, which is more specific to elliptic curves over *totally real number fields*, has been published in [Tho08].

2.1 Points of Good Reduction

Let E be an elliptic curve defined over a number field K, given by a Weierstrass equation

$$E: \quad y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$$

with all $a_j \in \mathcal{O}_K$, where $\mathcal{O}_K$ is the ring of integers of K. Let Δ be the discriminant of E. As in Chapter 1, we denote the sets of real and complex archimedean places by M_K^{r} and M_K^{c} respectively, and let M_K be the set of all places of K.

For all non-archimedean places v, let $E^{(v)}$ be a local minimal model for E over the completion K_v, while we simply set $E^{(v)} = E$ for all archimedean places v. We define the map

$$\phi: \quad E(K) \to \prod_{v \in S} E^{(v)}(K_v),$$

where $S = M_K^{\mathrm{r}} \cup M_K^{\mathrm{c}} \cup \{\mathfrak{p} : \mathfrak{p} \mid \Delta\}$, in such a way that P is mapped into its corresponding point on:

- $E^{(v)}(\mathbb{R})$, for each $v \in M_K^{\mathrm{r}}$, and
- $E^{(v)}(\mathbb{C})$, for each $v \in M_K^{\mathrm{c}}$, and
- $E^{(v)}(K_v)$, for each non-archimedean place $v \mid \Delta$.

Note that if K has class number greater than 1, then $E^{(v)}$ may differ for different non-archimedean places v, i.e., E may not have a *globally minimal* model. As we will see, our formulae can be simplified if E is given by a globally minimal model.

We wish to compute a positive lower bound λ for the canonical height $\hat{h}$ on $E(K)$. Instead of working on $E(K)$ directly, we determine a positive lower bound μ for the canonical height on the subgroup

$$E_{\mathrm{gr}}(K) = \phi^{-1}\left(\prod_{v \in S} E_0^{(v)}(K_v)\right),$$

where $E_0^{(v)}(K_v)$ is the connected component of the identity (for non-archimedean v, this is the set of points of good reduction). The next lemma shows that we can obtain λ very easily once μ is known.

Lemma 2.1.1. *Let μ be a positive lower bound for the canonical height on $E_{\mathrm{gr}}(K)$. Set*

$$\lambda = \mu / c^2,$$

where c is the least common multiple of the Tamagawa indices

$$c_v = [E^{(v)}(K_v) : E_0^{(v)}(K_v)]$$

for all $v \in M_K$. Then λ is a positive lower bound for the canonical height on $E(K)$.

Proof. Note that c is well-defined since $c_v = 1$ for all $v \notin S$. For all non-torsion point $P \in E(K)$, it is clear that $cP \in E_{\mathrm{gr}}(K)$. Then by quadraticity of $\hat{h}$, we have

$$\mu \leq \hat{h}(cP) = c^2 \hat{h}(P),$$

and so $\hat{h}(P) \geq \mu/c^2$. Hence we can take $\lambda = \mu/c^2$. □

In this chapter, we will first derive an explicit formula for computing μ. The value of μ obtained by this formula, in practice, will not be as good as the one obtained by the algorithm to be derived later on in Chapter 3. Using a number of criteria, our algorithm will check whether a given $\mu > 0$ is a lower bound on $E_{\mathrm{gr}}(K)$. The value of μ then can be refined further by repeating the algorithm.

2.2 Estimation of Local Heights

Recall the definition of local and canonical heights in Section 1.2.1. From (1.9), we have seen that the canonical height can be written as a sum of local heights given by (1.8). This therefore allows us to estimate $\hat{h}(P)$ by approximating each local height λ_v for $v \in M_K$.

2.2.1 Non-Archimedean Cases

For $P \in E(K)$, let $P^{(\mathfrak{p})}$ be its corresponding point of P (via the map ϕ) on the minimal model $E^{(\mathfrak{p})}$. Let $\lambda_{\mathfrak{p}}$ and $\lambda_{\mathfrak{p}}^{(\mathfrak{p})}$ be the local heights associated to E and $E^{(\mathfrak{p})}$ respectively. Assume that E is integral and $E^{(\mathfrak{p})}$ has all coefficients in $\mathcal{O}_{\mathfrak{p}} =$

$\{x \in K : \operatorname{ord}_{\mathfrak{p}}(x) \geq 0\}$, we denote Δ and $\Delta^{(\mathfrak{p})}$ the discriminants of E and $E^{(\mathfrak{p})}$ respectively. These values are related by $\Delta = \left(u^{(\mathfrak{p})}\right)^{12} \Delta^{(\mathfrak{p})}$, for some $u^{(\mathfrak{p})} \in \mathcal{O}_{\mathfrak{p}}$. If E is given by a globally minimal model, then we may take $E^{(\mathfrak{p})} = E$ for all $\mathfrak{p}$.

The following lemma illustrates the relation between $\lambda_{\mathfrak{p}}$ and $\lambda_{\mathfrak{p}}^{(\mathfrak{p})}$.

Lemma 2.2.1.

$$\lambda_{\mathfrak{p}}(P) = \lambda_{\mathfrak{p}}^{(\mathfrak{p})}(P^{(\mathfrak{p})}) + \frac{1}{6}\log|\Delta/\Delta^{(\mathfrak{p})}|_{\mathfrak{p}}.$$

Proof. This follows from the use of two different normalisations of local heights which differ by $\log|\cdot|_{\mathfrak{p}}/6$, and the fact that one of them is independent of the choice of Weierstrass model. For full details, see [CPS06, Section 4]. □

Now for $P \in E_{\mathrm{gr}}(K)$, it follows that $P^{(\mathfrak{p})} \in E_0^{(\mathfrak{p})}(K_{\mathfrak{p}})$ at every prime ideal $\mathfrak{p}$. In this case, we can easily compute $\lambda_{\mathfrak{p}}^{(\mathfrak{p})}(P^{(\mathfrak{p})})$ with the following lemma.

Lemma 2.2.2. *Let $\mathfrak{p}$ be a prime ideal and $P^{(\mathfrak{p})} \in E_0^{(\mathfrak{p})}(K_{\mathfrak{p}}) \setminus \{O\}$ (i.e., P is a point of good reduction). Then*

$$\lambda_{\mathfrak{p}}^{(\mathfrak{p})}(P^{(\mathfrak{p})}) = \log\max\{1, |x(P^{(\mathfrak{p})})|_{\mathfrak{p}}\}.$$

Proof. This is a standard result; see, e.g., [Sil88, Section 5]. Note that the definition that we use of local height of a point with good reduction does not include a multiple of $-\log|\Delta^{(\mathfrak{p})}|_{\mathfrak{p}}$ (cf. [Sil88, p. 351]). □

Definition. Let $x \in K$. The *denominator ideal* of x, denoted by $\operatorname{denom}(x)$, is the integral ideal B such that $\langle x\rangle = AB^{-1}$, where A, B are coprime integral ideals.

The next lemma yields a simplified formula for computing the sum of all non-archimedean local heights on $E_{\mathrm{gr}}(K)$.

Lemma 2.2.3. *Suppose $P \in E_{\mathrm{gr}}(K) \setminus \{O\}$. Then*

$$\sum_{\mathfrak{p}} n_{\mathfrak{p}}\lambda_{\mathfrak{p}}(P) = L(P) - \frac{1}{6}\log\mathcal{N}(M_E),$$

where

$$L(P) = \log \mathcal{N}\left(\prod_{\mathfrak{p} \mid \operatorname{denom}(x(P^{(\mathfrak{p})}))} \mathfrak{p}^{-\operatorname{ord}_{\mathfrak{p}}(x(P^{(\mathfrak{p})}))} \right), \quad M_E = \prod_{\mathfrak{p}} \mathfrak{p}^{\operatorname{ord}_{\mathfrak{p}}(\Delta/\Delta^{(\mathfrak{p})})}.$$

Note that $\mathcal{N}(M_E) = 1$ if E is given by a globally minimal model.

Proof. Since $P \in E_{\mathrm{gr}}(K)$ by assumption, we have $P^{(\mathfrak{p})} \in E_0^{(\mathfrak{p})}(K_{\mathfrak{p}})$ for all $\mathfrak{p}$. It then follows from Lemma 2.2.1 and Lemma 2.2.2 that

$$\begin{aligned} \sum_{\mathfrak{p}} n_{\mathfrak{p}} \lambda_{\mathfrak{p}}(P) &= \sum_{\mathfrak{p}} n_{\mathfrak{p}} \lambda_{\mathfrak{p}}^{(\mathfrak{p})}(P^{(\mathfrak{p})}) + \frac{1}{6} \sum_{\mathfrak{p}} n_{\mathfrak{p}} \log |\Delta/\Delta^{(\mathfrak{p})}|_{\mathfrak{p}} \\ &= \sum_{\mathfrak{p}} n_{\mathfrak{p}} \log \max\{1, |x(P^{(\mathfrak{p})})|_{\mathfrak{p}}\} + \frac{1}{6} \sum_{\mathfrak{p}} n_{\mathfrak{p}} \log |\Delta/\Delta^{(\mathfrak{p})}|_{\mathfrak{p}}. \qquad (2.1) \end{aligned}$$

Clearly, the term $\log\{1, |x(P^{(\mathfrak{p})})|_{\mathfrak{p}}\}$ will vanish if $|x(P^{(\mathfrak{p})})|_{\mathfrak{p}} \leq 1$. Hence the first sum in (2.1) is obtained by all those $\mathfrak{p}$ satisfying $|x(P^{(\mathfrak{p})})|_{\mathfrak{p}} > 1$. Recall from (1.2) that

$$|x(P^{(\mathfrak{p})})|_{\mathfrak{p}} = \mathcal{N}(\mathfrak{p})^{-\operatorname{ord}_{\mathfrak{p}}(x(P^{(\mathfrak{p})}))/n_{\mathfrak{p}}}.$$

Observe that $|x(P^{(\mathfrak{p})})|_{\mathfrak{p}} > 1$ if and only if $\mathfrak{p} \mid \operatorname{denom}(x(P^{(\mathfrak{p})}))$. Therefore, the first sum in (2.1) becomes

$$\sum_{\mathfrak{p}} n_{\mathfrak{p}} \log \max\{1, |x(P^{(\mathfrak{p})})|_{\mathfrak{p}}) = \log \mathcal{N}\left(\prod_{\mathfrak{p} \mid \operatorname{denom}(x(P^{(\mathfrak{p})}))} \mathfrak{p}^{-\operatorname{ord}_{\mathfrak{p}}(x(P^{(\mathfrak{p})}))} \right) = L(P).$$

Secondly, it follows from (1.2) that the second sum in (2.1) is

$$\frac{1}{6} \sum_{\mathfrak{p}} n_{\mathfrak{p}} \log |\Delta/\Delta^{(\mathfrak{p})}|_{\mathfrak{p}} = -\frac{1}{6} \log \mathcal{N}\left(\prod_{\mathfrak{p}} \mathfrak{p}^{\operatorname{ord}_{\mathfrak{p}}(\Delta/\Delta^{(\mathfrak{p})})} \right) = -\frac{1}{6} \log \mathcal{N}(M_E).$$

Finally, if E is given by a globally minimal model, then $\Delta^{(\mathfrak{p})} = \Delta$ for all $\mathfrak{p}$, so $\mathcal{N}(M_E) = 1$. $\square$

2.2.2 Archimedean Cases

For $v \in M_K^{\mathrm{r}} \cup M_K^{\mathrm{c}}$, we define α_v by

$$\alpha_v^{-3} = \inf_{P \in E_0^{(v)}(K_v)} \Phi_v(P)$$

(see (1.7) for the definition of Φ_v). The exponent -3 is introduced in order to simplify expressions appearing later on. These α_v can be computed very rapidly using the method in [CPS06, Section 7 and 9], according as $v \in M_K^{\mathrm{r}}$ and $v \in M_K^{\mathrm{c}}$.

The following lemma gives us an estimate for the archimedean local heights.

Lemma 2.2.4. *If $P \in E_0^{(v)}(K_v) \setminus \{O\}$, then*

$$\log \max\{1, |x(P)|_v\} - \lambda_v(P) \leq \log \alpha_v.$$

Proof. Rearrange (1.8) and use the fact that

$$\sum_{j=0}^{\infty} \frac{\log \Phi_v(2^j P)}{4^{j+1}} \geq \sum_{j=0}^{\infty} \frac{\log(\alpha_v^{-3})}{4^{j+1}} = -\log \alpha_v.$$

□

2.3 Multiplication by n

In this section, we will derive a lower estimate for the contribution that multiplication by n makes towards $L(nP)$, where L is defined as in Lemma 2.2.3.

Let $k_{\mathfrak{p}}$ be the residue class field of $\mathfrak{p}$, and let $e_{\mathfrak{p}}$ be the exponent of the group $E_{\mathrm{ns}}^{(\mathfrak{p})}(k_{\mathfrak{p}}) \cong E_0^{(\mathfrak{p})}(K_{\mathfrak{p}})/E_1^{(\mathfrak{p})}(K_{\mathfrak{p}})$. Define

$$D_E(n) = \sum_{\substack{\mathfrak{p} \text{ prime} \\ e_{\mathfrak{p}} | n}} 2(1 + \mathrm{ord}_{c(\mathfrak{p})}(n/e_{\mathfrak{p}})) \log \mathcal{N}(\mathfrak{p}), \tag{2.2}$$

where $c(\mathfrak{p})$ is the characteristic of $k_{\mathfrak{p}}$. Note that $k_{\mathfrak{p}}$ is a finite field, so $c(\mathfrak{p})$ is always a prime number. In particular, $\mathcal{N}(\mathfrak{p}) = |k_{\mathfrak{p}}| \le c(\mathfrak{p})^{[K:\mathbb{Q}]}$.

Proposition 2.3.1. *If $e_{\mathfrak{p}} \mid n$, then we have the following:*

1. *$\mathcal{N}(\mathfrak{p}) \le (n+1)^{\max\{2,[K:\mathbb{Q}]\}}$.*

2. *$D_E(n)$ is finite. Moreover, if $P \in E_{\mathrm{gr}}(K)$ is non-torsion and $n \ge 1$, then*

$$L(nP) \ge D_E(n).$$

Proof. Suppose $e_{\mathfrak{p}} \mid n$. If $E^{(\mathfrak{p})}$ has bad reduction at $\mathfrak{p}$, then $e_{\mathfrak{p}}$ is $c(\mathfrak{p})$, $\mathcal{N}(\mathfrak{p})+1$, or $\mathcal{N}(\mathfrak{p})-1$ depending on whether $E^{(\mathfrak{p})}$ has additive, non-split multiplicative, or split multiplicative reduction at $\mathfrak{p}$. In any case, this implies

$$n \ge e_{\mathfrak{p}} \ge \mathcal{N}(\mathfrak{p})^{1/[K:\mathbb{Q}]} - 1,$$

and thus $\mathcal{N}(\mathfrak{p}) \le (n+1)^{[K:\mathbb{Q}]}$. Now for $\mathfrak{p}$ at which $E^{(\mathfrak{p})}$ has good reduction, we have

$$E^{(\mathfrak{p})}_{\mathrm{ns}}(k_{\mathfrak{p}}) = E^{(\mathfrak{p})}(k_{\mathfrak{p}}) \cong \mathbb{Z}/d_1\mathbb{Z} \times \mathbb{Z}/d_2\mathbb{Z},$$

where $d_1 \mid d_2$ and $d_2 = e_{\mathfrak{p}}$. It then follows from Hasse's theorem (see, e.g., [Sil86, Theorem V.1.1]) that

$$(\sqrt{\mathcal{N}(\mathfrak{p})} - 1)^2 \le |E^{(\mathfrak{p})}_{\mathrm{ns}}(k_{\mathfrak{p}})| = d_1 d_2 \le e_{\mathfrak{p}}^2 \le n^2.$$

Thus $\mathcal{N}(\mathfrak{p}) \le (n+1)^2$. Combining this with above result, this yields $\mathcal{N}(\mathfrak{p}) \le (n+1)^{\max\{2,[K:\mathbb{Q}]\}}$, which proves (1). It is then immediate that $D_E(n)$ is finite.

To prove the rest of (2), first note that $P \in E_{\mathrm{gr}}(K)$ implies $P^{(\mathfrak{p})} \in E^{(\mathfrak{p})}_0(K_{\mathfrak{p}})$ for every $\mathfrak{p}$. Define

$$E^{(\mathfrak{p})}_n(K_{\mathfrak{p}}) = \{P \in E^{(\mathfrak{p})}_0(K_{\mathfrak{p}}) : \mathrm{ord}_{\mathfrak{p}}(x(P)) \le -2n\}.$$

Then it is known (see [Coh07, Lemma 7.3.28]) that for all $n \geq 1$,

$$E_n^{(\mathfrak{p})}(K_\mathfrak{p})/E_{n+1}^{(\mathfrak{p})}(K_\mathfrak{p}) \cong k_\mathfrak{p}^+ \cong (\mathbb{Z}/c(\mathfrak{p})\mathbb{Z})^t,$$

for some integer $t > 0$. Let $e(\mathfrak{p}) = \operatorname{ord}_{c(\mathfrak{p})}(n/e_\mathfrak{p})$. Then $nP^{(\mathfrak{p})} \in E_{e(\mathfrak{p})+1}^{(\mathfrak{p})}(K_\mathfrak{p})$, i.e.,

$$\operatorname{ord}_\mathfrak{p}(\operatorname{denom}(x(nP^{(\mathfrak{p})}))) \geq 2(e(\mathfrak{p})+1).$$

This implies that $e_\mathfrak{p} \mid n$ is equivalent to $\mathfrak{p} \mid \operatorname{denom}(x(nP^{(\mathfrak{p})}))$. Hence

$$\prod_{\mathfrak{p} \mid \operatorname{denom}(x(nP^{(\mathfrak{p})}))} \mathcal{N}(\mathfrak{p})^{-\operatorname{ord}_\mathfrak{p}(x(nP^{(\mathfrak{p})}))} \geq \prod_{\substack{\mathfrak{p} \text{ prime} \\ e_\mathfrak{p} \mid n}} \mathcal{N}(\mathfrak{p})^{2(e(\mathfrak{p})+1)}.$$

The result then follows after taking logarithms on both sides. □

2.4 A Bound for Multiples of Points of Good Reduction

In this section, we will first derive a bound for the x-coordinates of nP, where $P \in E_{\mathrm{gr}}(K)$ is non-torsion. This in turn yields an explicit lower bound for the canonical height on $E_{\mathrm{gr}}(K)$.

For $\mu > 0$ and $n \in \mathbb{Z}_{>0}$, define $B_n(\mu)$ by

$$\log B_n(\mu) = [K:\mathbb{Q}]n^2\mu - D_E(n) + \frac{1}{6}\log \mathcal{N}(M_E) + \sum_{v \in M_K^{\mathrm{r}}} \log \alpha_v + 2\sum_{v \in M_K^{\mathrm{c}}} \log \alpha_v.$$

Proposition 2.4.1. *If $B_n(\mu) < 1$ then $\hat{h}(P) > \mu$ for all non-torsion $P \in E_{\mathrm{gr}}(K)$. If $B_n(\mu) \geq 1$, then for all non-torsion $P \in E_{\mathrm{gr}}(K)$ with $\hat{h}(P) \leq \mu$, we have*

$$|x(nP)|_v \leq \begin{cases} B_n(\mu) & \text{if } v \in M_K^{\mathrm{r}}, \\ \sqrt{B_n(\mu)} & \text{if } v \in M_K^{\mathrm{c}}. \end{cases}$$

Proof. Suppose $P \in E_{\mathrm{gr}}(K)$ is a non-torsion point with $\hat{h}(P) \leq \mu$. By Lemma 2.2.4, we have

$$\log \max\{1, |x(nP)|_v\} - \lambda_v(nP) \leq \log \alpha_v$$

for all $v \in M_K^{\mathrm{r}} \cup M_K^{\mathrm{c}}$. This implies that

$$\begin{aligned}\sum_{v\in M_K^{\mathrm{r}}} \log\max\{1, |x(nP)|_v\} &+ 2\sum_{v\in M_K^{\mathrm{c}}} \log\max\{1, |x(nP)|_v\} \\ &\leq \sum_{v\in M_K^{\mathrm{r}}} \lambda_v(nP) + 2\sum_{v\in M_K^{\mathrm{c}}} \lambda_v(nP) + \sum_{v\in M_K^{\mathrm{r}}} \log\alpha_v + 2\sum_{v\in M_K^{\mathrm{c}}} \log\alpha_v. \end{aligned} \tag{2.3}$$

Note that $n_v = 1$ for all $v \in M_K^{\mathrm{r}}$ and $n_v = 2$ for all $v \in M_K^{\mathrm{c}}$. By writing $\hat{h}(nP)$ as a sum of local heights (see (1.9)), we have

$$\begin{aligned}\sum_{v\in M_K^{\mathrm{r}}} \lambda_v(nP) + 2\sum_{v\in M_K^{\mathrm{c}}} \lambda_v(nP) &= [K:\mathbb{Q}]\hat{h}(nP) - \sum_{\mathfrak{p}} n_{\mathfrak{p}}\lambda_{\mathfrak{p}}(nP) \\ &= [K:\mathbb{Q}]\hat{h}(nP) - L(nP) + \frac{1}{6}\log\mathcal{N}(M_E) \qquad \text{by Lemma 2.2.3} \\ &\leq [K:\mathbb{Q}]\hat{h}(nP) - D_E(n) + \frac{1}{6}\log\mathcal{N}(M_E) \qquad \text{by Proposition 2.3.1(2)} \\ &\leq [K:\mathbb{Q}]n^2\mu - D_E(n) + \frac{1}{6}\log\mathcal{N}(M_E) \qquad \text{since } \hat{h}(P) \leq \mu.\end{aligned}$$

Combining this with (2.3) and taking the exponential, we obtain

$$\left(\prod_{v\in M_K^{\mathrm{r}}} \max\{1, |x(nP)|_v\}\right)\left(\prod_{v\in M_K^{\mathrm{c}}} \max\{1, |x(nP)|_v\}^2\right) \leq B_n(\mu).$$

But the left-hand side is at least 1. Thus, if $B_n(\mu) < 1$, then we have a contradiction, i.e., $\hat{h}(P) > \mu$ for all non-torsion $P \in E_{\mathrm{gr}}(K)$. On the other hand, it can be seen that $|x(nP)|_v \leq B_n(\mu)$ for all $v \in M_K^{\mathrm{r}}$, and $|x(nP)|_v^2 \leq B_n(\mu)$ for all $v \in M_K^{\mathrm{c}}$. □

We are now ready to state an explicit formula for a lower bound on $E_{\mathrm{gr}}(K)$.

Theorem 2.4.2. *Let $\mathfrak{p}$ be a prime ideal such that*

$$\mathcal{N}(\mathfrak{p}) > \left(\prod_{v \in M_K^{\mathrm{r}}} \sqrt{\alpha_v}\right)\left(\prod_{v \in M_K^{\mathrm{c}}} \alpha_v\right)\mathcal{N}(M_E)^{1/12}. \tag{2.4}$$

Set $n = e_{\mathfrak{p}}$ and

$$\mu_0 = \frac{1}{[K:\mathbb{Q}]n^2}\left(D_E(n) - \sum_{v \in M_K^{\mathrm{r}}} \log \alpha_v - 2\sum_{v \in M_K^{\mathrm{c}}} \log \alpha_v - \frac{1}{6}\log \mathcal{N}(M_E)\right).$$

Then $\mu_0 > 0$, and $\hat{h}(P) \geq \mu_0$ for all non-torsion $P \in E_{\mathrm{gr}}(K)$.

Proof. Suppose $\mathfrak{p}$ is a prime ideal satisfying (2.4). By definition of $D_E(n)$ (see (2.2)), we have

$$D_E(n) \geq 2\log \mathcal{N}(\mathfrak{p}) > \sum_{v \in M_K^{\mathrm{r}}} \log \alpha_v + 2\sum_{v \in M_K^{\mathrm{c}}} \log \alpha_v + \frac{1}{6}\log \mathcal{N}(M_E),$$

which implies that $\mu_0 > 0$. Then for any $\mu < \mu_0$, we have

$$\begin{aligned}[K:\mathbb{Q}]n^2\mu - D_E(n) + \sum_{v \in M_K^{\mathrm{r}}} \log \alpha_v + 2\sum_{v \in M_K^{\mathrm{c}}} \log \alpha_v + \frac{1}{6}\log \mathcal{N}(M_E) \\ < [K:\mathbb{Q}]n^2\mu_0 - D_E(n) + \sum_{v \in M_K^{\mathrm{r}}} \log \alpha_v + 2\sum_{v \in M_K^{\mathrm{c}}} \log \alpha_v + \frac{1}{6}\log \mathcal{N}(M_E) = 0.\end{aligned}$$

Thus $B_n(\mu) < 1$, and so $\hat{h}(P) > \mu$ for all non-torsion $P \in E_{\mathrm{gr}}(K)$ by Proposition 2.4.1. Since this is true for all $\mu < \mu_0$, then $\hat{h}(P) \geq \mu_0$ as claimed. □

Although it is possible to obtain a lower bound for the canonical height on $E_{\mathrm{gr}}(K)$ simply from this theorem, our practical experience shows that this bound is not as good as the one obtained by collecting more information on $x(nP)$. This claim will be illustrated in Example 5.1.1.

2.5 Solving Inequalities I: Real Embeddings

In order to obtain a larger positive lower bound on $E_{\mathrm{gr}}(K)$ than the one obtained by Theorem 2.4.2, we finally concentrate on how to derive an alternative criterion for deciding whether a given $\mu > 0$ is a lower bound. This new criterion, which requires more information on $x(nP)$, will involve solving a system of certain inequalities on each embedding $E^{(v)}$, for every $v \in M_K^{\mathrm{r}} \cup M_K^{\mathrm{c}}$.

Given $\mu > 0$, we wish to check whether $\hat{h}(P) > \mu$ for all non-torsion $P \in E_{\mathrm{gr}}(K)$. If $B_n(\mu) < 1$ for some $n > 0$, then it follows easily from Proposition 2.4.1 that μ is a lower bound. On the other hand, if no such n exists, then Proposition 2.4.1 states that all non-torsion $P \in E_{\mathrm{gr}}(K)$ with $\hat{h}(P) \leq \mu$ must satisfy

$$|x(nP)|_v \leq \begin{cases} B_n(\mu) & \text{if } v \in M_K^{\mathrm{r}}, \\ \sqrt{B_n(\mu)} & \text{if } v \in M_K^{\mathrm{c}}, \end{cases}$$

for all $n > 0$. This can be regarded as a system of inequalities on each embedding $E^{(v)}$. In particular, if such a system has no solution, then this contradicts our assumption that $\hat{h}(P) \leq \mu$ for some non-torsion $P \in E_{\mathrm{gr}}(K)$, so μ must be a lower bound on $E_{\mathrm{gr}}(K)$.

In this section, we will explain how to solve this system of inequalities on each *real embedding* $E^{(v)}$ (i.e., where $v \in M_K^{\mathrm{r}}$). A similar computation on each complex embedding, however, is more sophisticated, and hence will be explained later in Chapter 3. To prove that $\hat{h}(P) > \mu$ for all non-torsion $P \in E_{\mathrm{gr}}(K)$, we attempt to derive a contradiction from these inequalities using an application of *period lattices and elliptic logarithms*, which will be fully described in Chapter 4.

2.5.1 Periods and Elliptic Logarithms

We will now introduce a simplified definition of periods and elliptic logarithms on elliptic curves over $\mathbb{R}$, and use it to obtain a contradiction from the system of inequalities mentioned earlier on each real embedding $E^{(v)}$.

For $v \in M_K^{\mathrm{r}}$, recall that $E^{(v)}$ is of the form

$$E^{(v)}: \quad y^2 + \sigma_v(a_1)xy + \sigma_v(a_3)y = x^3 + \sigma_v(a_2)x^2 + \sigma_v(a_4)x + \sigma_v(a_6),$$

where σ_v is the associated embedding from K to $\mathbb{R}$. With the change of variables

$$x = X - \frac{\sigma_v(b_2)}{12}, \quad y = \frac{Y - \sigma_v(a_1)x - \sigma_v(a_3)}{2},$$

we can rewrite $E^{(v)}$ as

$$Y^2 = 4(X - e_1)(X - e_2)(X - e_3)$$

for some e_1, e_2, e_3 with $\sum_{j=1}^{3} e_j = 0$. Since $E^{(v)}$ is defined over $\mathbb{R}$, then either all $e_j \in \mathbb{R}$, or there is only one $e_j \in \mathbb{R}$. Without loss of generality, we can assume that e_3 is the largest real root.

Recall from Section 1.3 that the connected component $E_0^{(v)}$ can be parameterised by the real line $\{t\Omega_v : 0 \le t < 1\}$, where $\Omega_v \in \mathbb{R}$ is one of the *periods* generating the period lattice of $E^{(v)}$. We will see in Chapter 4 that Ω_v is uniquely determined up to sign, but for now we shall take $\Omega_v > 0$. It then follows from (1.10) that

$$\Omega_v = 2\int_{(e_3,0)}^{O} \frac{dX}{Y} = 2\int_{\beta_v}^{\infty} \frac{dx}{\sqrt{f_v(x)}}$$

(we rearrange O and $(e_3, 0)$ so that $\Omega_v > 0$), where

$$f_v(x) = 4x^3 + \sigma_v(b_2)x^2 + 2\sigma_v(b_4)x + \sigma_v(b_6),$$

and $\beta_v = e_3 - \frac{\sigma_v(b_2)}{12}$ is the largest real root of f_v. If ξ is a real number satisfying $\xi \ge \beta_v$, then there exists η such that $2\eta + \sigma_v(a_1)\xi + \sigma_v(a_3) \ge 0$ and $P = (\xi, \eta) \in$

$E_0^{(v)}(\mathbb{R})$. An *elliptic logarithm* of P is then obtained in a similar way, i.e.,

$$z_{P,v} = \int_O^P \frac{dX}{Y} = \int_\infty^\xi \frac{dx}{\sqrt{f_v(x)}} \pmod{\Omega_v}.$$

Note that $z_{P,v} \in [\Omega_v/2, \Omega_v]$. Moreover, we may take $z_{-P,v} = -z_{P,v} \pmod{\Omega_v}$ (so that $z_{-P,v} \in [0, \Omega_v/2]$). At this point, one may use [Coh93, Algorithm 7.4.7 and 7.4.8] to compute Ω_v and $z_{P,v}$ respectively. We will explain a complete method for computing period lattices and elliptic logarithms on elliptic curves over $\mathbb{C}$ in Chapter 4.

For convenience, we shall define $\varphi_v : E_0^{(v)}(\mathbb{R}) \to [0,1)$, the *normalised elliptic logarithm*, by

$$\varphi_v(P) = \varphi_v((\xi,\eta)) = \begin{cases} \dfrac{z_{P,v}}{\Omega_v} & \text{if } 2\eta + \sigma_v(a_1)\xi + \sigma_v(a_3) \geq 0, \\ 1 - \varphi_v(-P) & \text{otherwise.} \end{cases}$$

For $\xi \geq \beta_v$, we also define

$$\psi_v(\xi) = \varphi_v((\xi,\eta)) \in [1/2, 1),$$

where $(\xi,\eta) \in E_0^{(v)}(\mathbb{R})$ with $2\eta + \sigma_v(a_1)\xi + \sigma_v(a_3) \geq 0$. In other words, $\psi_v(\xi)$ is the normalised elliptic logarithm of the "higher" of the two points on $E_0^{(v)}$ with x-coordinate ξ.

For real ξ_1, ξ_2 with $\xi_1 < \xi_2$, we define the subset $\mathcal{S}^{(v)} \subset [0,1)$ as follows:

$$\mathcal{S}^{(v)}(\xi_1,\xi_2) = \begin{cases} \emptyset & \text{if } \xi_2 < \beta_v, \\ [1-\psi_v(\xi_2), \psi_v(\xi_2)] & \text{if } \xi_1 < \beta_v \leq \xi_2, \\ [1-\psi_v(\xi_2), 1-\psi_v(\xi_1)] \cup [\psi_v(\xi_1), \psi_v(\xi_2)] & \text{if } \xi_1 \geq \beta_v. \end{cases}$$

The following lemma is clear.

Lemma 2.5.1. *Suppose $\xi_1 < \xi_2$ are real numbers. Then $P \in E_0^{(v)}(\mathbb{R})$ satisfies $\xi_1 \le x(P) \le \xi_2$ if and only if $\varphi_v(P) \in \mathcal{S}^{(v)}(\xi_1, \xi_2)$.*

If $\bigcup_j [a_j, b_j]$ is a disjoint union of intervals and $\alpha \in \mathbb{R}$, we define

$$\begin{aligned} \alpha + \bigcup_j [a_j, b_j] &= \bigcup_j [a_j + \alpha, b_j + \alpha], \\ \alpha \bigcup_j [a_j, b_j] &= \bigcup_j [\alpha a_j, \alpha b_j] \quad (\text{for } \alpha > 0). \end{aligned}$$

Lemma 2.5.2. *Suppose $\xi_1 < \xi_2$, and $n \in \mathbb{Z}_{>0}$. Let*

$$\mathcal{S}_n^{(v)}(\xi_1, \xi_2) = \bigcup_{\alpha=0}^{n-1} \left(\frac{\alpha}{n} + \frac{1}{n} \mathcal{S}^{(v)}(\xi_1, \xi_2) \right).$$

Then $P \in E_0^{(v)}(\mathbb{R})$ satisfies $\xi_1 \le x(nP) \le \xi_2$ if and only $\varphi_v(P) \in \mathcal{S}_n^{(v)}(\xi_1, \xi_2)$.

Proof. By Lemma 2.5.1, $P \in E_0^{(v)}(\mathbb{R})$ satisfies $\xi_1 \le x(P) \le \xi_2$ if and only if $\varphi_v(P) \in \mathcal{S}^{(v)}(\xi_1, \xi_2)$. Let ν_n be the multiplication-by-n map on $\mathbb{R}/\mathbb{Z}$. If $\delta \in [0, 1)$, then

$$\nu_n^{-1}(\delta) = \left\{ \frac{\alpha}{n} + \frac{\delta}{n} : \alpha = 0, 1, 2, \ldots, n-1 \right\}.$$

But since φ_v is an isomorphism, we have $\varphi_v(nP) = n\varphi_v(P) \pmod 1$. Hence

$$\varphi_v(nP) \in \mathcal{S}^{(v)}(\xi_1, \xi_2) \iff \varphi_v(P) \in \nu_n^{-1}(\mathcal{S}^{(v)}(\xi_1, \xi_2)) = \mathcal{S}_n^{(v)}(\xi_1, \xi_2).$$

□

This together with Proposition 2.4.1 leads to the following proposition.

Proposition 2.5.3. *If $B_n(\mu) < 1$ for some integer $n > 0$, then $\hat{h}(P) > \mu$ for all non-torsion $P \in E_{\mathrm{gr}}(K)$. If $B_n(\mu) \ge 1$ for all $n = 1, \ldots, n_{\max}$, then every non-torsion point $P \in E_{\mathrm{gr}}(K)$ with $h(P) \le \mu$ satisfies*

$$\varphi_v(\sigma_v(P)) \in \bigcap_{n=1}^{n_{\max}} \mathcal{S}_n^{(v)}(-B_n(\mu), B_n(\mu))$$

for every $v \in M_K^{\mathrm{r}}$. Here, $\sigma_v : K \to \mathbb{R}$ is the real embedding of K associated to v.

In particular, if the intersection is empty for some $v \in M_K^{\mathrm{r}}$, then $\hat{h}(P) > \mu$ for all non-torsion $P \in E_{\mathrm{gr}}(K)$.

Finally, we remark that if K is also a *totally real number field* (i.e., $M_K^{\mathrm{c}} = \emptyset$), then Proposition 2.5.3 alone will suffice for computing a lower bound for the canonical height on E/K. We shall not discuss such computation in detail here, since this will be a special case of our algorithm to be developed in Section 3.4. Some examples illustrating the applications of this algorithm will be shown in Chapter 5.

To summarise, we have developed all necessary formulas for estimating local heights, which leads to a criterion for deciding if a given $\mu > 0$ is a lower bound for the canonical height. Such criterion requires solving a system of certain inequalities on each embedding of E. In this chapter, we have managed to do this for real embeddings, which turns out to be sufficient for computing a lower bound for the canonical height on elliptic curves over number fields with at least one real embedding. The next chapter will focus on our remaining task, i.e., solving inequalities on complex embeddings.

Chapter 3

Height Bound II: Complex Embeddings

In this chapter, we will continue our work on computing a lower bound for the canonical height from Chapter 2 by introducing a new method for solving a system of certain inequalities on complex embeddings. This together with our work we have done so far will allow us to compute such a lower bound on elliptic curves over *number fields* in general, which will complete our work on height bound.

Let E be an elliptic curve defined over a number field K. Recall the definition of $B_n(\mu)$ in Section 2.4. If $B_n(\mu) \geq 1$, then Proposition 2.4.1 implies that all non-torsion $P \in E_{\mathrm{gr}}(K)$ with $\hat{h}(P) \leq \mu$ satisfy $|x(nP)|_v \leq \sqrt{B_n(\mu)}$ for every $v \in M_K^{\mathrm{c}}$. By computing $B_n(\mu)$ for several $n \in \mathbb{Z}_{>0}$, this yields a system of certain inequalities on each complex embedding $E^{(v)}$. We will see later that each of these inequalities corresponds to a *region* in the fundamental parallelogram for the period lattice of $E^{(v)}$, and solving the system of these inequalities is equivalent to finding the intersection of all such regions.

A combined version of Chapter 2 and this chapter, which explains a complete algorithm for computing a lower bound for the canonical height on elliptic curves over number fields, has been published in [Tho10].

3.1 Corresponding Regions I: An Overview

In this section, we will describe how to visualise an inequality on the x-coordinate of points in $E^{(v)}(\mathbb{C})$ obtained by Proposition 2.4.1 as a *corresponding region* on the fundamental parallelogram for the period lattice of $E^{(v)}$.

3.1.1 Fundamental Parallelograms

For $v \in M_K^{\mathrm{c}}$, let $E^{(v)}$ be the complex embedding of E associated to v. As mentioned in Section 1.3, it is well known that there exists a complex analytic group isomorphism $\varphi_v : E^{(v)}(\mathbb{C}) \to \mathbb{C}/\Lambda$, for some lattice Λ (for more details on computing this isomorphism, see Chapter 4).

Definition. Let Λ be a lattice with $\mathbb{Z}$-basis $\{w_1, w_2\}$. The *(closed) fundamental parallelogram* for Λ is the set

$$F_{w_1,w_2} = \{\lambda_1 w_1 + \lambda_2 w_2 : 0 \leq \lambda_1, \lambda_2 \leq 1\}.$$

Note that every element of $\mathbb{C}/\Lambda$ has a representative in F_{w_1,w_2} which is unique except for points on the boundary of F_{w_1,w_2}. After choosing a lift in F_{w_1,w_2} for each $P \in E^{(v)}(\mathbb{C})$, we may view φ_v as a map $E^{(v)}(\mathbb{C}) \to F_{w_1,w_2} \subset \mathbb{C}$.

Without loss of generality, we can choose a $\mathbb{Z}$-basis for Λ so that the quantity $\tau = w_2/w_1$ satisfies the following:

$$|\tau| \geq 1, \quad |\Re(\tau)| \leq 1/2, \quad \Im(\tau) \geq \sqrt{3}/2. \tag{3.1}$$

Let Λ_τ be the lattice generated by $1, \tau$. Then it is clear that the map $\delta : \mathbb{C} \to \mathbb{C}$ given by $z \mapsto z/w_1$ induces a bijection $\Lambda \to \Lambda_\tau$. To ease notation, we shall denote $F_{1,\tau}$ by F_τ, and let $\mathcal{H}_\tau$ be the "lower half" of F_τ, i.e.,

$$\mathcal{H}_\tau = \{\lambda_1 + \lambda_2 \tau : 0 \leq \lambda_1 \leq 1,\ 0 \leq \lambda_2 \leq 1/2\}.$$

Let $\psi' = \delta \circ \varphi_v$ (viewed as a map $E^{(v)}(\mathbb{C}) \to F_\tau$). Clearly, ψ' maps each $P \in E^{(v)}(\mathbb{C})$ to a point $z \in F_\tau$, and maps either P or $-P$ to a point in $\mathcal{H}_\tau$. Hence we can let

$$\psi_v(P) = \begin{cases} \psi_v'(P) & \text{if } \psi_v'(P) \in \mathcal{H}_\tau, \\ \psi_v'(-P) & \text{if } \psi_v'(P) \notin \mathcal{H}_\tau, \end{cases} \tag{3.2}$$

so that $\psi_v(P) \in \mathcal{H}_\tau$ in all cases.

3.1.2 Visualising the Region

From now on, we shall always assume that our $\mathbb{Z}$-basis $\{w_1, w_2\}$ for Λ is chosen so that τ satisfies (3.1). To see what the region corresponding to an inequality given by Proposition 2.4.1 looks like, we first recall that the Weierstrass parameterisation $\mathbb{C}/\Lambda_\tau \to E_W(\mathbb{C})$, where E_W is the elliptic curve of the form $Y^2 = 4X^3 - g_2(\Lambda_\tau)X - g_3(\Lambda_\tau)$ (for the definition of g_j, see, e.g., [Was03, Section 9.2]), is given by

$$z \mapsto (\wp_{\Lambda_\tau}(z), \wp'_{\Lambda_\tau}(z)). \tag{3.3}$$

Suppose $E^{(v)}$ is given by a Weierstrass equation

$$E^{(v)} : \quad y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$$

for some $a_j \in \mathbb{C}$. Then we have an isomorphism $E_W(\mathbb{C}) \to E^{(v)}(\mathbb{C})$, given by

$$(X, Y) \mapsto (x, y) = \left(w_1^{-2}X - \frac{b_2}{12}, \frac{w_1^{-3}Y - a_1x - a_3}{2} \right).$$

Hence for any $\xi \geq 0$, it is immediate from the triangle inequality and (3.3) that $|x| \leq \xi$ if and only if $|\wp_{\Lambda_\tau}(z)| \leq U_\xi$, where

$$U_\xi = |w_1|^2 \left(\xi + \frac{|b_2|}{12} \right).$$

We can now consider the set $\ell = \{z \in \mathcal{H}_\tau : |\wp_{\Lambda_\tau}(z)| = U_\xi\}$ as a curve[1] on $\mathcal{H}_\tau$ (see Figure 3.1). This is the boundary of the region

$$\mathcal{R}^{(v)}(\xi) = \{z \in \mathcal{H}_\tau : |\wp_{\Lambda_\tau}(z)| \leq U_\xi\}.$$

Since the Weierstrass $\wp$-function becomes a one-to-one continuous map once its domain is restricted to $\mathcal{H}_\tau$, the equation $|\wp_{\Lambda_\tau}(z)| = U_\xi$ yields only one curve on $\mathcal{H}_\tau$. By symmetry (about the mid-point of F_τ), we also have another identical boundary on the upper half of F_τ. Depending on U_ξ, the boundaries on both halves topologically form either one or two identical loops on the torus $\mathbb{C}/\Lambda_\tau$, as shown in Figure 3.2.

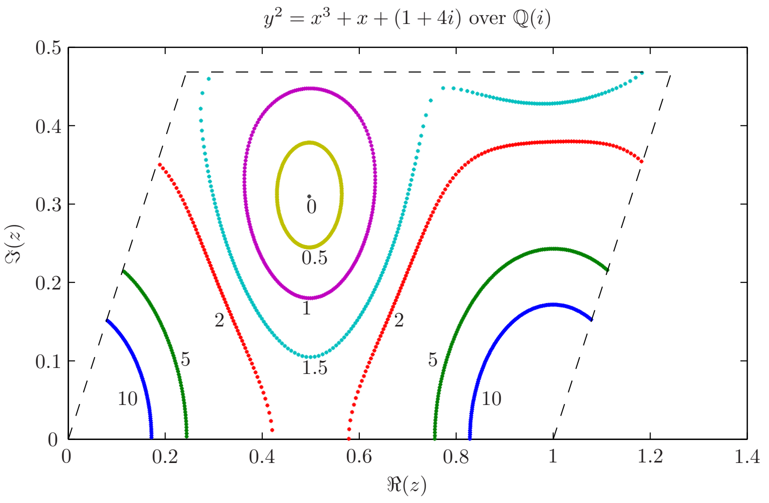


Figure 3.1: The boundary on $\mathcal{H}_\tau$ associated to different U_ξ. Each curve is labelled by the relevant value of ξ.

3.2 Corresponding Regions II: Estimation

In practice, however, it is very difficult to determine the region $\mathcal{R}^{(v)}$ exactly. For example, it is impossible to store an infinitesimal amount of its information on a computer. To circumvent this problem, we approximate $\mathcal{R}^{(v)}$ by a finite number of parallelograms whose union covers $\mathcal{R}^{(v)}$. Denote by $\mathcal{S}^{(v)}$ the finite set of these

[1]This may have either one or two connected components on $\mathcal{H}_\tau$.

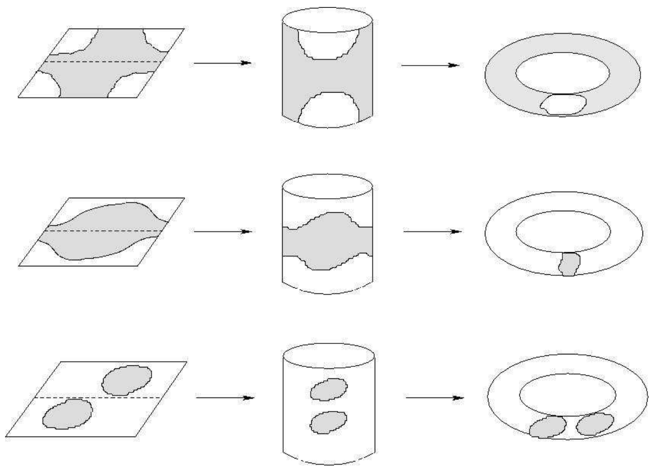


Figure 3.2: Loops on the torus $\mathbb{C}/\Lambda_\tau$ when the boundary varies

parallelograms. A finer approximation to $\mathcal{R}^{(v)}$ then can be obtained by decreasing the size of parallelograms in $\mathcal{S}^{(v)}$.

This section, to which most of our work on height bound is devoted, will focus on a number of approximation techniques which eventually allow us to construct $\mathcal{S}^{(v)}$. For now we mention that $\mathcal{S}^{(v)}(\xi)$ has the following properties:

1. $\bigcup_{C\in\mathcal{S}^{(v)}(\xi)} C \supseteq \mathcal{R}^{(v)}(\xi)$, i.e., the union of all parallelograms in $\mathcal{S}^{(v)}(\xi)$ contains the actual region $\mathcal{R}^{(v)}(\xi)$.

2. Every $C \in \mathcal{S}^{(v)}(\xi)$ contains z such that $|\wp_{\Lambda_\tau}(z)| \le U_\xi$, hence $C \cap \mathcal{R}^{(v)}(\xi) \neq \emptyset$ for all $C \in \mathcal{S}^{(v)}(\xi)$.

3.2.1 The Weierstrass $\wp$-function

Let $q = \exp(2\pi i\tau)$ and let $u = \exp(2\pi i z)$ (where $i = \sqrt{-1}$). For $k \in \mathbb{Z}$, we define

$$f_k(z,\tau) = (2\pi i)^2 \left[\frac{u}{(1-u)^2} + \frac{1}{12} + \sum_{j=1}^{k-1}\left[\frac{q^j u}{(1-q^j u)^2} + \frac{q^j u^{-1}}{(1-q^j u^{-1})^2} - \frac{2q^j}{(1-q^j)^2}\right]\right].$$

It can be seen from Proposition 1.3.1 that $\wp_{\Lambda_\tau}(z) = \lim_{k\to\infty} f_k(z,\tau)$ for all non-lattice points z. By choosing a suitable k, we can bound the error which occurs when $|f_k(z,\tau)|$ is used as the approximation to $|\wp_{\Lambda_\tau}(z)|$, as shown in the next lemma.

Lemma 3.2.1. *For $z \in \mathcal{H}_\tau$ with $z \neq 0, 1$, let $\alpha = \Im(z)/\Im(\tau)$. Define*

$$\epsilon(k) = \frac{4\pi^2}{1-|q|}\left(\frac{|q|^{k+\alpha}}{(1-|q|^{k+\alpha})^2} + \frac{|q|^{k-\alpha}}{(1-|q|^{k-\alpha})^2} + \frac{2|q|^k}{(1-|q|^k)^2}\right).$$

Then $\big||\wp_{\Lambda_\tau}(z)| - |f_k(z,\tau)|\big| \leq \epsilon(k)$.

Proof. By Proposition 1.3.1, we have

$$\wp_{\Lambda_\tau}(z) - f_k(z,\tau) = (2\pi i)^2 \sum_{j=k}^{\infty}\left[\frac{q^j u}{(1-q^j u)^2} + \frac{q^j u^{-1}}{(1-q^j u^{-1})^2} - \frac{2q^j}{(1-q^j)^2}\right].$$

Observe that $|u| = |q|^\alpha$. By the triangle inequality, we obtain

$$|\wp_{\Lambda_\tau}(z) - f_k(z,\tau)| \leq 4\pi^2 \sum_{j=k}^{\infty}\left[\frac{|q|^{j+\alpha}}{(1-|q|^{j+\alpha})^2} + \frac{|q|^{j-\alpha}}{(1-|q|^{j-\alpha})^2} + \frac{2|q^j|}{(1-|q^j|)^2}\right]. \tag{3.4}$$

Since we work on $\mathcal{H}_\tau$, we have $|q| < 1$ and $0 \leq \alpha \leq 1/2$, which implies that $|q|^{j\pm\alpha} < 1$ for all $j \geq 1$. Thus we have the estimate

$$\sum_{j=k}^{\infty}\frac{|q|^{j\pm\alpha}}{(1-|q|^{j\pm\alpha})^2} \leq \frac{1}{(1-|q|^{k\pm\alpha})^2}\sum_{j=k}^{\infty}|q|^{j\pm\alpha} \leq \frac{|q|^{k\pm\alpha}}{(1-|q|^{k\pm\alpha})^2(1-|q|)},$$

and similarly,

$$\sum_{j=k}^{\infty}\frac{2|q|^j}{(1-|q|^j)^2} \leq \frac{2|q|^k}{(1-|q|^k)^2(1-|q|)}.$$

This together with (3.4) and the triangle inequality yields the result. □

One can easily verify that, in the range $0 \leq \alpha \leq 1/2$, the absolute error $\epsilon(k)$ given by Lemma 3.2.1 attains its maximum at $\alpha = 1/2$, and becomes smaller as k increases. Moreover, it can be seen that $\epsilon(k)$ decreases as $\Im(\tau)$ increases. Some examples of maximum values for $\epsilon(k)$ are listed in Table 3.1 (based on $\alpha = 1/2$ and $\Im(\tau) = \sqrt{3}/2$).

Recall that every parallelogram C in $\mathcal{S}^{(v)}(\xi)$ satisfies $|\wp_{\Lambda_\tau}(z)| \leq U_\xi$ for some $z \in C$. In practice, we can compute $|f_k(z,\tau)|$ and add it with the error given by

Table 3.1: Maximum values for $\epsilon(k)$

k	Maximum error
1	3.349
2	0.013
3	5.568×10^{-5}
4	2.413×10^{-7}
5	1.046×10^{-9}
10	1.598×10^{-21}
20	3.731×10^{-45}
23	3.036×10^{-52}

Lemma 3.2.1 to obtain a (small) interval which contains $|\wp_{\Lambda_\tau}(z)|$. On each of the four line segments comprising the boundary of C, we can parameterise $|f_k(z,\tau)|$ by a real-valued function $f_k(x,\tau)$ or $f_k(y,\tau)$, where $x = \Re(z)$ and $y = \Im(z)$. We wish to find the range of f_k when x or y varies along the line. For this computation, we find some techniques from *interval arithmetic* (see [Moo66]) to be very useful.

3.2.2 Interval Arithmetic

Before we proceed to its application, we shall first explain briefly what interval arithmetic is.

Definition. Let $I = [a,b]$ and $J = [c,d]$ (with $a \le b$ and $c \le d$) be two intervals of real numbers. An *arithmetic operation* on intervals I, J is defined by

$$I * J = \{x * y : a \le x \le b,\, c \le y \le d\},$$

where $*$ is an operation on real numbers.

A number of usual arithmetic operations on real numbers can be extended to the ones on intervals. For example,

$$\begin{aligned} &I + J = [a+c, b+d], \quad I - J = [a-d, b-c], \\ &I \cdot J = [\min\{ac, ad, bc, bd\}, \max\{ac, ad, bc, bd\}], \\ &I/J = [a,b] \cdot [1/d, 1/c] \quad \text{(provided that } 0 \notin J). \end{aligned}$$

It can be seen easily that interval addition and interval multiplication are both associative and commutative. Distributivity, however, does not always hold for interval arithmetic. For example,

$$\begin{aligned}[1,3]\cdot([1,3]-[1,3]) = [1,3]\cdot[-2,2] &= [-6,6], \text{ whereas}\\ [1,3]\cdot[1,3]-[1,3]\cdot[1,3] = [1,9]-[1,9] &= [-8,8].\end{aligned}$$

Instead, we always have *subdistributivity*, i.e., $I\cdot(J+K)\subset I\cdot J+I\cdot K$ for all intervals I,J,K.

One important property of interval arithmetic is that it is *inclusion monotonic*, i.e., if $I\subset K$ and $J\subset L$ are intervals, then

$$I+J\subset K+L,\quad I-J\subset K-L,\quad I\cdot J\subset K\cdot L,$$
$$I/J\subset K/L\quad (\text{provided that } 0\notin L).$$

This leads to the following theorem.

Theorem 3.2.2 ([Moo66, Theorem 3.1]). *Let $f(X_1,\dots,X_n)$ be a rational expression with real coefficients in the interval variables $X_1,\dots,X_n$, i.e., a finite combination of $X_1,\dots,X_n$ and a finite set of constant intervals with interval arithmetic operations. Then*

$$X_1'\subset X_1,\ \dots,\ X_n'\subset X_n \quad\Longrightarrow\quad f(X_1',\dots,X_n')\subset f(X_1,\dots,X_n)$$

for every set of intervals $X_1,\dots,X_n$ for which the interval arithmetic operations in f are defined.

Suppose $f(x_1,\dots,x_n)$ is a real rational expression, i.e., f is a quotient of real polynomials in terms of $x_1,\dots,x_n$. Then by Theorem 3.2.2, the resulting interval $F=f(X_1,\dots,X_n)$ will always contain the actual range of $f(x_1,\dots,x_n)$ for $x_j\in X_j$. In particular, F will be the actual range of $f(x_1,\dots,x_n)$ for $x_j\in X_j$ if each variable

x_j occurs only once in f (note that $x_j^2 = x_j \cdot x_j$ is taken as two occurrences). With some techniques, for example, using subdistributivity to group common terms in f, the resulting interval F can be made smaller. For more information on this subject, see [Moo66, Chapter 3 and 6].

Recall the function $f_k(z, \tau)$ in Section 3.2.1. Suppose $z = x + iy \in \mathbb{C}$ is on a fixed line segment L. Depending on L, we can regard z as a function of either x or y (for example, if z is on a vertical line, then x is fixed but y varies). Thus, provided that L is fixed and $z \in L$, we can consider the function $g(z) = |f_k(z, \tau)|^2$ as a real function of one real variable, i.e., either $g(z) = g(z(x))$ or $g(z) = g(z(y))$, depending on L. To ease notation, we shall write

$$f(*) = g(z(*)),$$

where $*$ is either x or y, depending on how z is parameterised along L.

The next proposition shows that we can apply interval arithmetic to $f(*)$.

Proposition 3.2.3. *Define $f(*)$ as above. Then f can be extended to a real rational expression of at most three interval variables, depending on the line segment L.*

Proof. First, we note that

$$f(*) = |f_k(z, \tau)|^2 = \Re(f_k(z, \tau))^2 + \Im(f_k(z, \tau))^2.$$

We will show how to obtain the real part of $f_k(z, \tau)$; the imaginary part of $f_k(z, \tau)$ can be deduced in a similar way.

The real part of $f_k(z, \tau)$ consists of the real parts of the terms

$$\frac{u}{(1-u)^2}, \quad \frac{1}{12}, \quad \frac{q^j u}{(1-q^j u)^2}, \quad \frac{q^j u^{-1}}{(1-q^j u^{-1})^2}, \quad \frac{q^j}{(1-q^j)^2}, \tag{3.5}$$

where $u = \exp(2\pi i z)$ and $q = \exp(2\pi i \tau)$. Write $z = x + iy$. Let

$$x_1 = \exp(-2\pi y), \quad x_2 = \cos(2\pi x), \quad x_3 = \sin(2\pi x).$$

Consider the following two cases:

1. **If L is a non-vertical line** (i.e., $y = \alpha x + \beta$ for some finite α and β), then

$$\Re\left(\frac{u}{(1-u)^2}\right) = \frac{x_1 x_2 (1 + x_1^2) - 2x_1^2}{(1 - 2x_1 x_2 + x_1^2)^2}.$$

   Similarly, it can be shown that the real parts of the other terms in (3.5) can be written as rational expressions in terms of x_1, x_2, x_3.

2. **If L is a vertical line** (i.e., x is fixed), then we have $\Re(u/(1-u)^2)$ as above. Since x_2 and x_3 are now constant, we have $\Re(u/(1-u)^2)$ as a rational expression in terms of x_1 only. This is also the case for the real parts of the other terms in (3.5).

Thus we have $f(*)$ as a real rational expression in terms of x_1, x_2, x_3. Suppose that $a \le x \le b$ and $c \le y \le d$ on L (note that $c, d \ge 0$ since we work on $\mathcal{H}_\tau$). Let

$$\begin{aligned} X_1 &= \exp(-2\pi[c,d]) = [\exp(-2\pi d), \exp(-2\pi c)], \\ X_2 &= \cos(2\pi[a,b]) = [\min_{a\le x\le b} \cos(2\pi x), \max_{a\le x\le b} \cos(2\pi x)], \\ X_3 &= \sin(2\pi[a,b]) = [\min_{a\le x\le b} \sin(2\pi x), \max_{a\le x\le b} \sin(2\pi x)]. \end{aligned} \tag{3.6}$$

After replacing x_1, x_2, x_3 in f with X_1, X_2, X_3 respectively, we finally obtain the interval version of f. $\square$

Since $f(X_1, X_2, X_3)$ is a real rational expression of interval variables, then Theorem 3.2.2 applies. Together with the error term in Lemma 3.2.1, the following proposition is immediate.

Proposition 3.2.4. *Define X_1, X_2, X_3 to be the intervals depending on a line segment L as in* (3.6). *For a fixed $k \in \mathbb{Z}_{>0}$, let $\epsilon = \epsilon(k)$ be the maximum absolute error given by Lemma* 3.2.1. *Then for all $z \in L$, we have*

$$\sqrt{u_1} - \epsilon \leq |\wp_{\Lambda_\tau}(z)| \leq \sqrt{u_2} + \epsilon,$$

where $[u_1, u_2] = f(X_1, X_2, X_3)$ (with $u_2 \geq u_1 \geq 0$).

3.2.3 Approximate Corresponding Regions

We are now ready to construct $\mathcal{S}^{(v)}$, which in turn yields an approximation to the corresponding region $\mathcal{R}^{(v)}$.

Let L be a line segment in the complex plane. By Proposition 3.2.4, the interval

$$I(L) = [\sqrt{u_1} - \epsilon, \sqrt{u_2} + \epsilon]$$

contains the actual range of $|\wp_{\Lambda_\tau}(z)|$ for $z \in L$. We can then extend this notion to any parallelogram C by letting

$$I(C) = \bigcup_{L \in \partial C} I(L),$$

where ∂C is the boundary of C. Note that the four intervals $I(L)$ for $L \in \partial C$ will overlap, so $I(C)$ is an interval.

For $v \in M_K^{\mathrm{c}}$ and $\xi \geq 0$, we define $\mathcal{S}^{(v)}(\xi)$ recursively as follows. First we let

$$\mathcal{S}^{(v,0)}(\xi) = \{\mathcal{H}_\tau\}.$$

Next, for $r \geq 0$, suppose $\mathcal{S}^{(v,r)}(\xi) = \{C_1, \ldots, C_m\}$, where $m = 4^r$. Let

$$\mathcal{S}'^{(v,r+1)} = \{C_{11}, \ldots, C_{14}, \ldots, C_{m1}, \ldots, C_{m4} : C_j = \bigcup_{k=1}^{4} C_{jk}\},$$

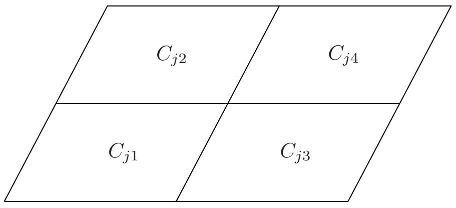


Figure 3.3: Four quarters of C_j

i.e., $C_{j1}, \dots, C_{j4}$ are the four quarters of C_j, as shown in Figure 3.3.

Suppose $E^{(v)}$ is of the form $Y^2 = 4X^3 + AX + B$ for some $A, B \in \mathbb{C}$. Let $P \in E^{(v)}(\mathbb{C})$ be a point with $X(P) = 0$. Let $C_0 \in \mathcal{S}'^{(v,r+1)}$ be the parallelogram containing $\psi_v(P)$ (see (3.2) for its definition). Note that we may have $I(C_0) \cap [0, U_\xi] = \emptyset$. Then we define

$$\mathcal{S}^{(v,r+1)}(\xi) = \{C_0\} \cup \{C \in \mathcal{S}'^{(v,r+1)} : I(C) \cap [0, U_\xi] \neq \emptyset\}.$$

Finally, we let $\mathcal{S}^{(v)}(\xi) = \mathcal{S}^{(v,r)}(\xi)$ for some $r > 0$.

For a set $\mathcal{S}$ of parallelograms in $\mathbb{C}$, we denote $\bigcup_{C \in \mathcal{S}} C$ simply by $\bigcup \mathcal{S}$. It is then obvious from the construction above that

$$\bigcup \mathcal{S}^{(v,0)}(\xi) \supset \bigcup \mathcal{S}^{(v,1)}(\xi) \supset \cdots \supset \bigcup \mathcal{S}^{(v,r)}(\xi) \supset \cdots \supset \mathcal{R}^{(v)}(\xi).$$

In other words, our approximation to $\mathcal{R}^{(v)}$ becomes finer as r increases.

In general, computing $\mathcal{S}^{(v,r+1)}$ with above definition can be very time-consuming. Fortunately, we can usually speed up this process using a combination of the following techniques.

Lemma 3.2.5 (Four-Corner Test). *Suppose $C \in \mathcal{S}'^{(v,r+1)}(\xi)$. Let $z_1, \dots, z_4$ be the corners of C, and let $\epsilon = \epsilon(k)$ be the maximum absolute error given by Lemma* 3.2.1 *for some fixed $k \in \mathbb{Z}_{>0}$. Define*

$$I(z) = [\,|f_k(z,\tau)| - \epsilon, |f_k(z,\tau)| + \epsilon\,].$$

If $I(z_j) \subset [0, U_\xi]$ *for some* $j = 1, \ldots, 4$, *then* $C \in \mathcal{S}^{(v,r+1)}(\xi)$.

Proof. If such condition holds, then we simply have $|\wp_{\Lambda_\tau}(z)| \leq U_\xi$ for some $z \in C$, namely, $z = z_j$. Hence $C \in \mathcal{S}^{(v,r+1)}(\xi)$. □

In practice, checking whether C is in $\mathcal{S}^{(v,r+1)}(\xi)$ by this test is considerably faster than the usual criterion $I(C) \cap [0, U_\xi]$. The next lemma provides a quick way to exclude all parallelograms which are not in $\mathcal{S}^{(v,r+1)}(\xi)$.

Lemma 3.2.6. *For* $r \geq 0$, *let* S_{r+1} *be the set of all parallelograms in* $\mathcal{S}'^{(v,r+1)}(\xi)$ *which satisfy the condition in Lemma* 3.2.5. *Let*

$$\partial S_{r+1} = \{C \in \mathcal{S}'^{(v,r+1)}(\xi) \setminus S_{r+1} : C \text{ is adjacent to } \bigcup S_{r+1}\}.$$

If $I(C) \cap [0, U_\xi] = \emptyset$ *for all* $C \in \partial S_{r+1}$, *then* $\mathcal{S}^{(v,r+1)}(\xi) = S_{r+1}$.

Proof. If all parallelograms in ∂S_{r+1} are excluded from $\mathcal{S}^{(v,r+1)}(\xi)$, then this means that there is no part of the boundary ℓ of the actual region $\mathcal{R}^{(v)}(\xi)$ passing through $\bigcup \partial S_{r+1}$. Thus the one-to-one and continuity properties of the Weierstrass $\wp$-function on $\mathcal{H}_\tau$ imply that the boundary ℓ of $\mathcal{R}^{(v)}(\xi)$ lies entirely in $\bigcup S_{r+1}$, and so all parallelograms in $\mathcal{S}'^{(v,r+1)}(\xi) \setminus S_{r+1}$ can be discarded. □

An illustration of using these lemmas to construct $\mathcal{S}^{(v)}$ is shown[2] in Figure 3.4. In this figure, the process of determining $\mathcal{S}^{(v)}$ consists of the following steps:

1. Starting with $\mathcal{S}'^{(v,r+1)}(\xi)$ for some r, we use Lemma 3.2.5 to identify a number of parallelograms $C \in \mathcal{S}'^{(v,r+1)}(\xi)$ which are also in $\mathcal{S}^{(v,r+1)}(\xi)$ (these are marked by "*"). Let S_{r+1} be the set of all such parallelograms C.

2. Identify all parallelograms in ∂S_{r+1} (these are marked by "?").

3. For each $C \in \partial S_{r+1}$, check if $I(C) \cap [0, U_\xi] = \emptyset$. If so, then $C \notin \mathcal{S}^{(v,r+1)}(\xi)$ and thus can be discarded (this is marked by ".").

[2]Here $\mathcal{S}^{(v)} = \mathcal{S}^{(v,4)}(0.4)$ for the elliptic curve $y^2 = x^3 + x + (1 + 4i)$ defined over $\mathbb{Q}(i)$.

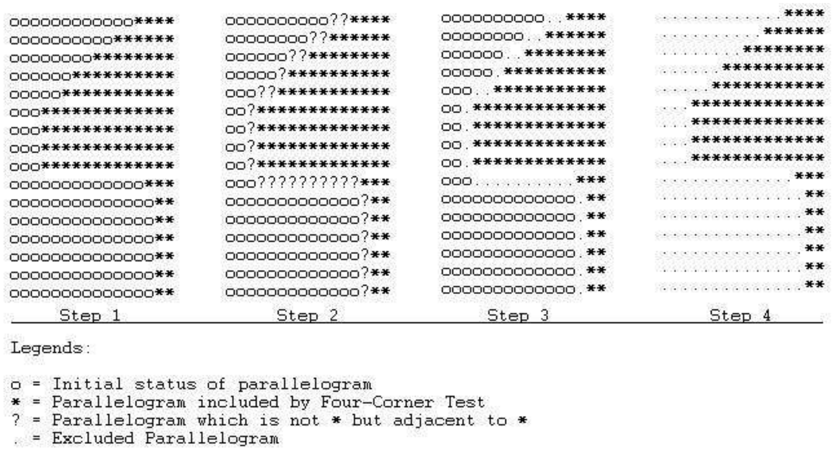

Step 1
Step 2
Step 3
Step 4
Legends:
o = Initial status of parallelogram
* = Parallelogram included by Four-Corner Test
? = Parallelogram which is not * but adjacent to *
. = Excluded Parallelogram

Figure 3.4: An illustration of how to obtain $\mathcal{S}^{(v)}$. The top-left entry represents the parallelogram containing $z = 0$.

4. If it turns out that the set ∂S_{r+1} is entirely discarded, then by Lemma 3.2.6, we have $\mathcal{S}^{(v,r+1)}(\xi) = S_{r+1}$. In other words, every parallelogram in $\mathcal{S}'^{(v,r+1)}(\xi)\setminus S_{r+1}$ is discarded. Finally, we let $\mathcal{S}^{(v)}(\xi) = \mathcal{S}^{(v,r+1)}(\xi)$.

3.3 Solving Inequalities II: Complex Embeddings

In this section, we finally explain how to solve a system of inequalities given by Proposition 2.4.1 on complex embeddings, which is analogous to our previous result in Section 2.5.

As we have already seen, the inequality $|x(P)|_v \leq \xi$ yields the cover $\bigcup \mathcal{S}^{(v)}(\xi)$ which approximates the corresponding region $\mathcal{R}^{(v)}$ in $\mathcal{H}_\tau$. Since the Weierstrass $\wp$-function is even, we also have another identical region in the upper half of F_τ. Let $\mathcal{T}^{(v)}(\xi)$ be the union of both regions. Then clearly $\mathcal{T}^{(v)}(\xi)$ contains the set $\{z \in F_\tau : |\wp_{\Lambda_\tau}(z)| \leq U_\xi\}$.

Recall the isomorphism $\psi'_v : E^{(v)}(\mathbb{C}) \to F_\tau$ from Section 3.1.1. Given a point $P \in E^{(v)}(\mathbb{C})$, we wish to consider all points $Q \in E^{(v)}(\mathbb{C})$ such that $P = nQ$. Let $z = \psi'_v(P)$ and $z' = \psi'_v(Q)$. Then we have

$$z = nz' \pmod{\Lambda_\tau}.$$

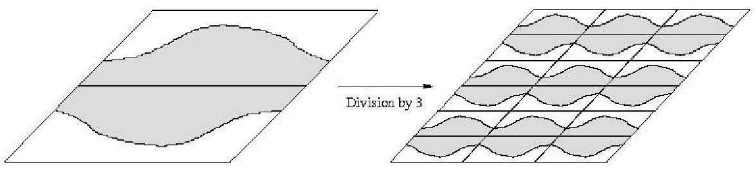


Figure 3.5: Division on F_τ by 3

In fact, if $z = \alpha + \beta\tau$ for some $0 \leq \alpha, \beta \leq 1$, then

$$z' \in \left\{ \frac{\alpha + s}{n} + \frac{(\beta + t)\tau}{n} : 0 \leq s, t \leq n-1 \right\}.$$

This therefore allows us to "divide" $\mathcal{T}^{(v)}(\xi)$ by n (see Figure 3.5 for an illustration) to obtain a new region

$$\mathcal{T}_n'^{(v)}(\xi) = \{z' \in F_\tau : nz' \pmod{\Lambda_\tau} \in C \text{ for some } C \in \mathcal{T}^{(v)}(\xi)\}.$$

Due to the symmetry of $\mathcal{T}_n'^{(v)}$, we can let

$$\mathcal{T}_n^{(v)}(\xi) = \mathcal{T}_n'^{(v)}(\xi) \cap \mathcal{H}_\tau.$$

The following lemma is analogous to Lemma 2.5.2.

Lemma 3.3.1. *If $P \in E^{(v)}(\mathbb{C})$ satisfies $|x(nP)| \leq \xi$, then $\psi_v(P) \in \mathcal{T}_n^{(v)}(\xi)$.*

Proof. If $|x(nP)| \leq \xi$, then we have $\psi_v(nP) \in C$ for some $C \in \mathcal{S}^{(v)}(\xi) \subset \mathcal{T}^{(v)}(\xi)$. Since $n\psi_v(P)$ is either $\psi_v(nP)$ or $-\psi_v(nP) \pmod{\Lambda_\tau}$, in any case we have $\psi_v(P) \in \mathcal{T}_n'^{(v)}(\xi) \cap \mathcal{H}_\tau = \mathcal{T}_n^{(v)}(\xi)$. □

The next proposition, which is analogous to Proposition 2.5.3, follows easily from the previous lemma together with Proposition 2.4.1.

Proposition 3.3.2. *If $B_n(\mu) \geq 1$ for all $n = 1, \ldots, n_{\max}$, then every non-torsion point $P \in E_{\mathrm{gr}}(K)$ with $\hat{h}(P) \leq \mu$ satisfies*

$$\psi_v(\sigma_v(P)) \in \bigcap_{n=1}^{n_{\max}} \mathcal{T}_n^{(v)}(\sqrt{B_n(\mu)})$$

for all $v \in M_K^{\mathrm{c}}$. Here, $\sigma_v : K \to \mathbb{C}$ is the complex embedding of K associated to v.

In particular, if the intersection is empty for some $v \in M_K^{\mathrm{c}}$, then $\hat{h}(P) > \mu$ for all non-torsion $P \in E_{\mathrm{gr}}(K)$.

3.4 An Algorithm for Height Bound

Combining Proposition 2.5.3 and Proposition 3.3.2, we are now ready to state our main theorem.

Theorem 3.4.1. *Let $\mu > 0$. If $B_n(\mu) < 1$ for some $n \in \mathbb{Z}_{>0}$, then $\hat{h}(P) > \mu$ for all non-torsion $P \in E_{\mathrm{gr}}(K)$. Otherwise, if $B_n(\mu) \geq 1$ for all $n = 1, \ldots, n_{\max}$, then every non-torsion point $P \in E_{\mathrm{gr}}(K)$ with $\hat{h}(P) \leq \mu$ satisfies*

$$\varphi_v(\sigma_v(P)) \in \bigcap_{n=1}^{n_{\max}} \mathcal{S}_n^{(v)}(-B_n(\mu), B_n(\mu))$$

for every $v \in M_K^{\mathrm{r}}$, and moreover,

$$\psi_v(\sigma_v(P)) \in \bigcap_{n=1}^{n_{\max}} \mathcal{T}_n^{(v)}(\sqrt{B_n(\mu)})$$

for every $v \in M_K^{\mathrm{c}}$.

In particular, if one of the intersections is empty for some $v \in M_K^{\mathrm{r}} \cup M_K^{\mathrm{c}}$, then $\hat{h}(P) > \mu$ for all non-torsion $P \in E_{\mathrm{gr}}(K)$.

Theorem 3.4.1 in turn yields an algorithm for computing a lower bound for the canonical height on $E_{\mathrm{gr}}(K)$, which consists of the following steps:

1. Given an initial value $\mu > 0$ and the number of steps $n_{\max}$, we start by computing $B_n(\mu)$ for $n = 1, \ldots, n_{\max}$. If $B_n(\mu) < 1$ for some n, then we can conclude immediately that $\hat{h}(P) > \mu$ for all non-torsion $P \in E_{\mathrm{gr}}(K)$.

2. Otherwise, we proceed to compute $\bigcap_{n=1}^{n_{\max}} \mathcal{S}_n^{(v)}(-B_n(\mu), B_n(\mu))$ for every $v \in M_K^{\mathrm{r}}$. If the intersection is empty for some v, then again $\hat{h}(P) > \mu$ for all

non-torsion $P \in E_{\mathrm{gr}}(K)$.

3. If not, then we compute $\bigcap_{n=1}^{n_{\max}} \mathcal{T}_n^{(v)}(\sqrt{B_n(\mu)})$ for every $v \in M_K^{\mathrm{c}}$. Again, if the intersection is empty for some v, then $\hat{h}(P) > \mu$ for all non-torsion $P \in E_{\mathrm{gr}}(K)$. Otherwise, we fail to show that μ is a lower bound on $E_{\mathrm{gr}}(K)$.

4. We can refine μ further in the following way: if μ is shown to be a lower bound, then we increase μ and repeat the process to see if it is still a lower bound. However, if the algorithm fails to show that μ is a lower bound, then we decrease μ (or increase $n_{\max}$) and repeat the process.

5. Return the largest value of μ which is known to be a lower bound for $E_{\mathrm{gr}}(K)$.

Once μ is determined, we can simply use Lemma 2.1.1 to obtain a positive lower bound for the canonical height on $E(K)$. Some examples on how to compute such a lower bound using this algorithm (see Appendix A.3 for its MAGMA code) will be shown in Chapter 5.

3.5 Remarks

Finally, it should be noted that the lower bound we obtain is not model-independent, unlike the one of Hindry and Silverman [HS88, Theorem 0.3]. For example, the values α_v in Section 2.2.2 depend on the coefficients of the Weierstrass equation of E. At present, we have not systematically investigated how the bound obtained by our algorithm is affected by a change of model. As mentioned in Chapter 2, however, our formulas can be simplified if E is given by a globally minimal model.

Regarding the computational complexity, it can be seen that computing $B_n(\mu)$ is less time-consuming than computing $\mathcal{S}_n^{(v)}$, which in turn is less time-consuming than computing $\mathcal{T}_n^{(v)}$. Therefore it is plausible to use $B_n(\mu)$ as the first criterion, followed by the intersection of $\mathcal{S}_n^{(v)}$ and $\mathcal{T}_n^{(v)}$ respectively, as we do in our algorithm.

Let c be the least common multiple of all Tamagawa indices as in Section 2.1. As pointed out by an anonymous referee of [Tho10], it may be possible to obtain a larger lower bound by making use of the explicit formulas for the local heights at non-archimedean places of bad reduction (see, e.g., [Sil88, Theorem 5.2]), provided that c is large. This approach, however, is different to ours which uses the subgroup of points of good reduction. In particular, our lower bound on $E(K)$ will be small if c is large. Nevertheless, it might be an interesting area for further study.

In conclusion, we have completed our work on height bound by introducing a method for solving a system of certain inequalities on complex embeddings. Our method involves a number of approximation techniques which eventually yield an approximate region corresponding to each inequality, where finding a solution to the system of these inequalities is equivalent to finding the intersection of all such regions. Together with our results from Chapter 2, we finally obtain an algorithm for computing a lower bound for the canonical height on elliptic curves over *number fields* in general.

Finally, in order to solve a system of inequalities using the methods in Section 2.5 and 3.3, we need to compute *period lattices* of certain real and complex embeddings, as well as *elliptic logarithms* of certain real and complex points. Nevertheless, algorithms for determining both quantities are currently available only for elliptic curves over $\mathbb{R}$ (see Section 1.3 for more discussion). Motivated by this problem, the next chapter will aim to develop a complete method for computing period lattices of elliptic curves over $\mathbb{C}$, and elliptic logarithms of complex points.

Chapter 4

Period Lattices and Complex Elliptic Logarithms

We will now move on to the second main result of this thesis, where we present a complete method for computing *period lattices* of elliptic curves over $\mathbb{C}$, and then generalise it to compute *elliptic logarithms* of complex points. Based on the complex *arithmetic-geometric mean* (AGM) first studied by Gauss, our method will allow one to compute both quantities to a high degree of precision very quickly. For more background on this chapter, see Section 1.3.

The work in this chapter is done in collaboration with Professor John E. Cremona at the University of Warwick. Another version of this chapter has been submitted for publication as a joint paper [CT].

4.1 Introduction

In this chapter, we will assume that an elliptic curve E is defined over $\mathbb{C}$, and is given by a Weierstrass equation of the form

$$E : Y^2 = 4(X - e_1)(X - e_2)(X - e_3),$$

where all the roots $e_j \in \mathbb{C}$ are distinct and $\sum_j e_j = 0$. As mentioned in Section 1.3, it is well known that there exists an isomorphism (of complex analytic Lie groups) $\mathbb{C}/\Lambda \to E(\mathbb{C})$ for some lattice Λ, given by the map

$$\begin{aligned} z \pmod{\Lambda} &\mapsto P = (\wp_\Lambda(z), \wp'_\Lambda(z)) \\ 0 \pmod{\Lambda} &\mapsto O. \end{aligned} \tag{4.1}$$

Definition. Let E be an elliptic curve defined over $\mathbb{C}$, the *period lattice* of E is the lattice Λ for which $E(\mathbb{C}) \cong \mathbb{C}/\Lambda$ via (4.1).

To be precise, we take Λ to be the lattice of periods of the invariant differential dX/Y on E. It is a discrete subgroup of $\mathbb{C}$ spanned by a $\mathbb{Z}$-basis $\{w_1, w_2\}$ with $w_2/w_1 \notin \mathbb{R}$.

Definition. The inverse map of (4.1) is called the *elliptic logarithm*. For $P \in E(\mathbb{C})$, we say that a value z such that

$$P \mapsto z \pmod{\Lambda}$$

via this inverse is an *elliptic logarithm* of P (note that z is determined modulo Λ).

From this, two natural questions are:

1. Given a Weierstrass equation of E, how can we compute a $\mathbb{Z}$-basis for its period lattice Λ?
2. Given a point $P \in E(\mathbb{C})$, how can we compute its elliptic logarithm z?

For elliptic curves over $\mathbb{R}$, these questions have been answered satisfactorily, since algorithms for computing period lattices of elliptic curves over $\mathbb{R}$ and elliptic logarithms of real points are well-known and available in the literature (see, e.g., [Coh93, Algorithm 7.4.7 and 7.4.8] or [Cre97, §3.7]). The theory behind these algorithms, which heavily relies on the AGM of positive real numbers, is explained succinctly by Bost and Mestre [BM88]. The situation for elliptic curves over $\mathbb{C}$, however, is less satisfactory.

In this chapter, we therefore aim to develop a complete method for computing period lattices and elliptic logarithms for elliptic curves over $\mathbb{C}$. Our approach will closely follow that of [BM88] in the real case, and will also illustrate the connection between the following three classes of objects:

- AGM sequences over $\mathbb{C}$, which were first studied by Gauss and have been explored in depth by Cox [Cox84];
- Chains of lattices in $\mathbb{C}$;
- Chains of 2-isogenies between elliptic curves defined over $\mathbb{C}$.

This connection will allow us to derive an explicit formula for computing the period lattice of E, which yields the first algorithm of this chapter. We then continue further by generalising it to an algorithm for computing elliptic logarithms of points in $E(\mathbb{C})$. Finally, we illustrate the efficiency of both algorithms via some examples.

For computational purposes, we have implemented both algorithms in MAGMA (see Appendix A.1 for the source code); these have been also implemented independently in `Sage` (available from version 4.4) by Professor John E. Cremona.

4.2 AGM Sequences

In this section, we will give a brief overview of arithmetic-geometric mean of complex numbers. For more in-depth survey on this subject, see [Cox84].

Definition. Let $(a, b) \in \mathbb{C}^2$ be a pair of complex numbers satisfying

$$a \neq 0, \quad b \neq 0, \quad a \neq \pm b. \tag{4.2}$$

We say that (a, b) is *good* if $\Re(a/b) \geq 0$, or equivalently

$$|a - b| \leq |a + b|; \tag{4.3}$$

otherwise the pair is said to be *bad*.

Clearly, only one of the pair (a, b), $(a, -b)$ is good, unless $\Re(a/b) = \Re(b/a) = 0$ (or equivalently, $|a - b| = |a + b|$), in which case both pairs are good.

Definition. An *AGM sequence* is a sequence $((a_n, b_n))_{n=0}^{\infty}$ whose pairs satisfy the relation

$$2a_{n+1} = a_n + b_n, \quad b_{n+1}^2 = a_n b_n \tag{4.4}$$

for all $n \geq 0$.

It is easy to see that if any one pair (a_n, b_n) in the sequence satisfies (4.2) then all do, and we will make this restriction henceforth.

Given a starting pair (a_0, b_0), one can obtain uncountably many AGM sequences by iterating the procedure of replacing (a_n, b_n) by their *arithmetic mean* $a_{n+1} = (a_n + b_n)/2$ and their *geometric mean* $b_{n+1} = \sqrt{a_n b_n}$, with a choice of the square root for b_{n+1} at each step. However, we usually prefer to consider the entire sequence as a whole.

Definition. We say that an AGM sequence is *good* if the pairs (a_n, b_n) are good for all but finitely many n. A good AGM sequence in which (a_n, b_n) are good for all $n > 0$ is said to be *optimal*; and *strongly optimal* if in addition (a_0, b_0) is good.

If an AGM sequence is not good, then we say that it is *bad*.

For an optimal AGM sequence $((a_n, b_n))_{n=0}^{\infty}$ with a given starting pair (a_0, b_0), at first it might seem that there could be many such sequences, since there could be several $n \geq 0$ for which both pairs $(a_n, \pm b_n)$ are good. Fortunately, the following lemma shows that this is not the case.

Lemma 4.2.1. *For every starting pair (a_0, b_0), there is exactly one optimal AGM sequence $((a_n, b_n))_{n=0}^{\infty}$, unless a_0/b_0 is real and negative, in which case there are two optimal AGM sequences with different signs of b_1.*

Proof. For $n \geq 0$, let $r_n = a_n/b_n$. Using (4.4), we can rewrite r_{n+1} as

$$r_{n+1} = \pm\frac{1}{2}\left(\sqrt{r_n} + \frac{1}{\sqrt{r_n}}\right).$$

One can then verify the following very easily:

- r_n is real and positive if and only if r_{n+1} is real;
- r_n is real and negative if and only if r_{n+1} is purely imaginary.

If both pairs $(a_{n+1}, \pm b_{n+1})$ are good, then (4.3) implies that r_{n+1} is purely imaginary, and so all preceding ratios r_n are real. Thus equality can hold in (4.3) at most once in any AGM sequence; and only for $n = 0$ or $n = 1$ in an optimal sequence (since $\Re(r_n) \geq 0$ for all $n \geq 1$ in an optimal sequence). In particular, this only holds for $n = 1$ (i.e., both $(a_1, \pm b_1)$ are good) if and only if r_0 is real and negative. □

The following proposition is due to Cox; see [Cox84] for its proof. Note that Cox defines the notion of "good" more strictly than above (when $\Re(a/b) = 0$ he requires $\Im(a/b) > 0$, so that exactly one of $(a, \pm b)$ is good in every case), but in view of the preceding remarks this does not affect the following result.

Proposition 4.2.2. *Given a pair $(a_0, b_0) \in \mathbb{C}^2$ satisfying (4.2), every AGM sequence $((a_n, b_n))_{n=0}^{\infty}$ starting at (a_0, b_0) satisfies the following:*

1. *$\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$ exist and are equal;*
2. *The common limit, say M, is non-zero if and only if the sequence is good;*
3. *$|M|$ attains its maximum (among all AGM sequences starting at (a_0, b_0)) if and only if the sequence is optimal.*

For an AGM sequence $((a_n, b_n))_{n=0}^{\infty}$ starting at (a_0, b_0), we will denote the common limit $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$ by $M_S(a_0, b_0)$, where $S \subseteq \mathbb{Z}_{>0}$ is the set of all indices n for which the pair (a_n, b_n) is bad. For example, $M_\emptyset(a_0, b_0)$ denotes the common limit for the optimal AGM sequence. Note that the sequence is good if and only if S is a finite set. To ease notation, we shall write $M_\emptyset(a_0, b_0)$ as $M(a_0, b_0)$.

4.3 Chains of Lattices

We now move on to consider the second class of objects, namely, *chains of lattices of index* 2. In this section, we will give the definition of a chain and describe its properties, which later will be seen to be analogous to those of an AGM sequence.

Throughout this chapter, a *lattice* will always be a free $\mathbb{Z}$-module of rank 2, embedded as a discrete subgroup of $\mathbb{C}$. Elements of lattices will often be called *periods*, since in our application lattices will arise as period lattices of elliptic curves defined over $\mathbb{C}$.

Definition. A *chain of lattices (of index* 2*)* is a sequence of lattices $(\Lambda_n)_{n=0}^{\infty}$ which satisfies the following conditions:

1. $\Lambda_n \supset \Lambda_{n+1}$ for all $n \geq 0$;
2. $[\Lambda_n : \Lambda_{n+1}] = 2$ for all $n \geq 0$;
3. Λ_0/Λ_n is cyclic for all $n \geq 1$; equivalently, $\Lambda_{n+1} \neq 2\Lambda_{n-1}$ for all $n \geq 1$.

Thus for each $n \geq 1$, we have

$$\Lambda_{n+1} = \langle w \rangle + 2\Lambda_n \tag{4.5}$$

for some $w \in \Lambda_n \setminus 2\Lambda_{n-1}$.

Given an initial lattice Λ_0, there are three choices for Λ_1. When $n \geq 1$, one of the three choices for Λ_{n+1} is excluded since it is contained in $2\Lambda_{n-1}$ (which contradicts the last condition in the definition of a chain), and so there are only two choices for Λ_{n+1}. The number of such chains starting with Λ_0 is therefore uncountable; we will distinguish a countable subset of these as follows. Let

$$\Lambda_\infty = \bigcap_{n=0}^{\infty} \Lambda_n.$$

Then Λ_∞ is free of rank at most 1; the rank cannot be 2 since for all n,

$$[\Lambda_0 : \Lambda_\infty] \geq [\Lambda_0 : \Lambda_n] = 2^n,$$

so $[\Lambda_0 : \Lambda_\infty]$ is infinite.

Definition. A chain of lattices $(\Lambda_n)_{n=0}^{\infty}$ is said to be *good* if Λ_∞ has rank 1; in this case a generator for Λ_∞ will be called a *limiting period* of the chain. If a chain is not good, then we say that it is *bad*.

We will first show that the limiting period is *primitive*, i.e., not in $m\Lambda_0$ for any $m \geq 2$.

Lemma 4.3.1. *Let $(\Lambda_n)_{n=0}^{\infty}$ be a good chain with $\Lambda_\infty = \langle w_\infty \rangle$. Then we have the following:*

1. *w_∞ is primitive; equivalently, Λ_0/Λ_∞ is free of rank 1;*
2. *$\Lambda_n = \langle w_\infty \rangle + 2^n \Lambda_0$ for all $n \geq 0$.*

Proof. Suppose $w_\infty = mw$ for some $m \geq 1$, and $w \in \Lambda_n \subseteq \Lambda_0$ for some $n \geq 0$. If m is odd, then $(m-1)w \in \Lambda_{n+1}$. Since $mw = w_\infty \in \Lambda_{n+1}$, we have $w \in \Lambda_{n+1}$. Hence $w \in \Lambda_\infty$ by induction. Thus $w_\infty = mw = m(m'w_\infty)$ for some $m' \in \mathbb{Z}$, and so $m = 1$.

Next, suppose that $w_\infty = 2w$ for some $w \in \Lambda_0$. By definition of w_∞, we then have $w \notin \Lambda_\infty$, and hence there exists $n > 0$ such that $w \notin \Lambda_n$. This implies that

$w_\infty \in \Lambda_n \setminus 2\Lambda_{n-1}$ (recall that $\Lambda_n \supset 2\Lambda_{n-1}$). But since $w_\infty \in \Lambda_{n+1}$, we have

$$\Lambda_{n+1} = \langle w_\infty \rangle + 2\Lambda_n = \langle 2w \rangle + 2\Lambda_n \subseteq 2\Lambda_0,$$

which contradicts the definition of a chain. Thus w_∞ is primitive, which proves (1).

Since w_∞ is primitive, then $\Lambda_n/2^n\Lambda_0$ is cyclic of order 2^n, and is generated by w_∞ modulo $2^n\Lambda_0$. Hence (2) follows. □

So far, our notion of a good chain has been defined as a property of the chain as a whole, and only used the abstract structure of lattices as free $\mathbb{Z}$-modules. Using the next definition, we will see that this property can be also seen in terms of individual steps $\Lambda_n \supset \Lambda_{n+1}$, when all lattices Λ_n are embedded in $\mathbb{C}$. In view of (4.5), the choice of Λ_{n+1} is determined by the class of w modulo $2\Lambda_n$.

Definition. For $n \geq 1$, we say that $\Lambda_{n+1} \subset \Lambda_n$ is a *right choice* of sublattice of Λ_n if $\Lambda_{n+1} = \langle w \rangle + 2\Lambda_n$, where w is a *minimal* element in $\Lambda_n \setminus 2\Lambda_{n-1}$ (with respect to the usual complex absolute value).

In general, there will be only one right choice at each step; for more details on the exceptional case, see Section 4.3.1.

Lemma 4.3.2. *Let $(\Lambda_n)_{n=0}^\infty$ be a good chain with $\Lambda_\infty = \langle w_\infty \rangle$. Then w_∞ is minimal in Λ_n for all but finitely many $n \geq 0$.*

Proof. Let $w_1 = w_\infty$. By Lemma 4.3.1, w_1 is primitive and there exists $w_2 \in \Lambda_0$ such that $\Lambda_n = \langle w_1, 2^n w_2 \rangle$ for all $n \geq 0$. For a non-zero $w \in \Lambda_n$, we write $w = mw_1 + k2^n w_2$ with $m, k \in \mathbb{Z}$. If $k = 0$, then clearly $|w| = |m||w_1| \geq |w_1|$. On the other hand, if $|k| \geq 1$, then

$$|w/w_1| = |m + k2^n w_2/w_1| \geq 2^n |\Im(w_2/w_1)| \geq 1,$$

for all $n > -\log_2 |\Im(w_2/w_1)|$. This proves the lemma. □

The following proposition yields an alternative definition of a good chain. For now we remark that this is analogous to the definition of a good AGM sequence in Section 4.2; more of its analogues will be seen in later sections.

Proposition 4.3.3. *A chain of lattices $(\Lambda_n)_{n=0}^{\infty}$ is good if and only if $\Lambda_{n+1} \subset \Lambda_n$ is a right choice for all but finitely many $n \geq 1$.*

Proof. Let $(\Lambda_n)_{n=0}^{\infty}$ be a good chain with $\Lambda_\infty = \langle w_\infty \rangle$. Then by Lemma 4.3.2, there exists an integer n_0 such that w_∞ is minimal in Λ_n for all $n \geq n_0$. Since $\Lambda_{n+1} = \langle w_\infty \rangle + 2\Lambda_n$ for all n, then by definition, $\Lambda_{n+1} \subset \Lambda_n$ is a right choice for all $n \geq n_0$.

Conversely, suppose that $\Lambda_{n+1} \subset \Lambda_n$ is a right choice for all $n \geq n_0$ (where $n_0 \geq 1$). Without loss of generality, we may suppose that $n_0 = 1$. Let w_1 be a minimal element of Λ_1. Then w_1 is certainly primitive (as an element of Λ_1, though not necessarily in Λ_0). We claim that $w_1 \in \Lambda_n$ for all $n \geq 1$, so that the chain is good with limiting period w_1.

To prove the claim, suppose that $w_1 \in \Lambda_j$ for all $j \leq n$. Then $\Lambda_n = \langle w_1 \rangle + 2^{n-1}\Lambda_1$, since the latter is contained in the former and both have index 2^{n-1} in Λ_1. Hence $\Lambda_n = \langle w_1, 2^{n-1} w_2 \rangle$, where $w_2 \in \Lambda_1$ is such that $\Lambda_1 = \langle w_1, w_2 \rangle$. By minimality of w_1, the right sublattice Λ_{n+1} of Λ_n is clearly $\langle w_1 \rangle + \Lambda_n$ (note that w_1 is a candidate since $w_1 \in \Lambda_n \setminus 2\Lambda_{n-1}$); in particular, $w_1 \in \Lambda_{n+1}$ as required. □

In the next subsection, we will introduce a special type of a lattice chain, whose properties will be analogous to those of an *optimal* AGM sequence. This type of lattice chain will play an important role in Section 4.5, where we develop an algorithm for computing period lattices of elliptic curves over $\mathbb{C}$.

4.3.1 Optimal Chains and Rectangular Lattices

Definition. A lattice chain $(\Lambda_n)_{n=0}^{\infty}$ is said to be *optimal* if $\Lambda_{n+1} \subset \Lambda_n$ is a right choice for *all* $n \geq 1$.

In general, there will be only one optimal chain for each of the three choices of $\Lambda_1 \subset \Lambda_0$. In order to describe this statement more precisely, however, some preparation is necessary.

We say that a lattice $\Lambda \subset \mathbb{C}$ is *rectangular* if it has an *orthogonal* $\mathbb{Z}$-basis $\{w_1, w_2\}$, i.e., one which satisfies $\Re(w_2/w_1) = 0$. For example, the period lattice of an elliptic curve defined over $\mathbb{R}$ with positive discriminant is rectangular, where an orthogonal basis is given by the least real period and the least imaginary period (see Section 1.3 for more details). In fact, rectangular lattices are homothetic to the period lattices of this family of elliptic curves.

If $\{w_1, w_2\}$ is any $\mathbb{Z}$-basis for a lattice Λ, the three non-trivial cosets of 2Λ in Λ are given by $C_j = w_j + 2\Lambda$ for $j = 1, 2, 3$, where $w_3 = w_1 + w_2$. By a *minimal coset representative*, we mean a minimal element of one of these cosets. The next three lemmas explain some of its properties.

Lemma 4.3.4. *Minimal coset representatives are primitive.*

Proof. Let w be a minimal coset representative. Then $w \notin 2\Lambda$, since by definition w does not represent the trivial coset 2Λ. Moreover, if $w = mw'$ for some odd $m \geq 3$, then $|w| = m|w'| > |w'|$. But since both w, w' belong to the same coset modulo 2Λ, this contradicts the minimality of w. Hence $m = 1$, i.e., w is primitive. $\square$

Lemma 4.3.5. *In each coset C_j, the minimal coset representative is unique up to sign, unless Λ is a rectangular lattice with orthogonal $\mathbb{Z}$-basis $\{w_1, w_2\}$, in which case the coset C_3 has four minimal vectors $\pm(w_1 \pm w_2)$.*

Proof. For a rectangular lattice Λ with orthogonal $\mathbb{Z}$-basis $\{w_1, w_2\}$, it is easy to see that the minimal coset representatives are as stated. Conversely, if a lattice Λ has a coset C with at least two pairs of minimal elements $\pm w, \pm w'$, then $w_1, w_2 = (w \pm w')/2 \in \Lambda$ are easily seen to be orthogonal.

Next, we will show that w_1, w_2, and $w = w_1 + w_2$ are non-trivial coset representatives modulo 2Λ. If $w_1 \equiv 0 \pmod{2\Lambda}$, then $w_2 \equiv w \pmod{2\Lambda}$. But then

$|w_2| < |w_1 + w_2| = |w|$, which contradicts the minimality of w in its coset. Hence $w_1 \not\equiv 0 \pmod{2\Lambda}$. Similarly, $w_2 \not\equiv 0 \pmod{2\Lambda}$. Moreover, $w_1 \not\equiv w_2 \pmod{2\Lambda}$ since $w = w_1 + w_2 \not\equiv 0 \pmod{2\Lambda}$. Therefore, w_1, w_2, w do represent the three non-trivial cosets modulo 2Λ.

Finally, it remains to show that $\{w_1, w_2\}$ is a $\mathbb{Z}$-basis for Λ. Suppose the contrary that this is not the case. Then there would exist a non-zero period $w_0 = \alpha w_1 + \beta w_2$ with $0 \le \alpha, \beta < 1$. Clearly, one of $w_0, w_0 - w_1, w_0 - w_2, w_0 - w$ is in the same coset as w, but since all these periods are smaller than w, this yields a contradiction. □

As we will see later on, our algorithm for computing period lattices of elliptic curves will actually compute these minimal coset representatives. To ensure that we thereby obtain a $\mathbb{Z}$-basis for the lattice, the following lemma is required.

Lemma 4.3.6. *For $j = 1, 2, 3$, let w_j be minimal coset representatives for a non-rectangular lattice Λ. Then any two of these w_j form a $\mathbb{Z}$-basis for Λ, and $w_3 = \pm(w_1 \pm w_2)$.*

Proof. We may assume that $|w_1| \le |w_2| \le |w_3|$. Then w_1 is minimal in Λ and w_2 is minimal in $\Lambda \setminus \langle w_1 \rangle$. Hence (by negating w_2 if necessary), $\tau = w_2/w_1$ is in the standard fundamental region for $\mathrm{SL}_2(\mathbb{Z})$ acting on the upper half-plane, $\{w_1, w_2\}$ is a $\mathbb{Z}$-basis for Λ, and $w_3 = w_1 \pm w_2$; the sign depends on that of $\Re(\tau)$. □

The following proposition shows that the limiting period of an optimal chain is closely related to minimal coset representatives.

Proposition 4.3.7. *A good chain of lattices $(\Lambda_n)_{n=0}^{\infty}$ with $\Lambda_\infty = \langle w_\infty \rangle$ is optimal if and only if w_∞ is a minimal coset representative of $2\Lambda_0$ in Λ_0.*

Proof. Suppose that w_∞ is a minimal coset representative. Then it is clear that $\Lambda_{n+1} = \langle w_\infty \rangle + 2\Lambda_n \subset \Lambda_n$ is a right sublattice for all $n \ge 1$, since w_∞ is certainly minimal in $\Lambda_n \setminus 2\Lambda_{n-1}$. Thus the chain $(\Lambda_n)_{n=0}^{\infty}$ is optimal with limiting period w_∞.

Conversely, suppose that a chain $(\Lambda_n)_{n=0}^{\infty}$ is optimal with limiting period w_∞. Let $w \in \Lambda_1$ be a minimal element of $\Lambda_1 \setminus 2\Lambda_0$, so that w is a minimal coset representative for the unique non-trivial coset modulo $2\Lambda_0$ contained in Λ_1. Note that w is unique up to sign, unless Λ_0 is rectangular, in which case (for one of the cosets) there will be two possibilities for w up to sign. By optimality, the sublattice $\Lambda_2 \subset \Lambda_1$ is the right choice. In particular, if Λ_0 is not rectangular, then we must therefore have $\Lambda_2 = \langle w \rangle + 2\Lambda_1$. This, however, may not hold in the rectangular case, but it will hold if we replace w by the other choice of minimal coset representative.

Now we claim that $\Lambda_n = \langle w \rangle + 2\Lambda_{n-1}$ for all $n \geq 2$. We already know this for $n = 2$. If the claim is true for n, then certainly w is also minimal in $\Lambda_n \setminus 2\Lambda_{n-1}$, so the (unique) right choice of sublattice of Λ_n is $\langle w \rangle + 2\Lambda_n$. By optimality, this is Λ_{n+1}, and so the claim holds for $n+1$. Thus $w \in \bigcap_{n=0}^{\infty} \Lambda_n = \langle w_\infty \rangle$, and indeed, $w = \pm w_\infty$, since w is primitive by Lemma 4.3.4. □

This together with Lemma 4.3.5 gives the following conclusion.

Corollary 4.3.8. *Every non-rectangular lattice Λ has precisely three optimal lattice chains $(\Lambda_n)_{n=0}^{\infty}$ (with $\Lambda_0 = \Lambda$), whose limiting periods are the minimal coset representatives in each of the three non-zero cosets of 2Λ in Λ. Every rectangular lattice Λ has precisely four optimal lattice chains associated to it.*

4.4 Chains of 2-Isogenies

We finally consider the last class of objects, where we construct a chain of elliptic curves defined over $\mathbb{C}$ using 2-*isogenies*. Since each elliptic curve uniquely has an associated period lattice, we will see that this chain will be analogous to a chain of lattices defined in Section 4.3. Most of the formulas we use in this section are due to Bost and Mestre [BM88].

Let E_0 be an elliptic curve over $\mathbb{C}$ given by a Weierstrass equation

$$E_0: \quad Y_0^2 = 4(X_0 - e_1^{(0)})(X_0 - e_2^{(0)})(X_0 - e_3^{(0)}) \tag{4.6}$$

where all roots $e_j^{(0)}$ are distinct and $\sum_{j=1}^3 e_j^{(0)} = 0$. Assume that the ordering of $e_j^{(0)}$ is fixed. Similar to [BM88], we define a_n, b_n for $n \geq 0$ by

$$a_0 = \sqrt{e_1^{(0)} - e_3^{(0)}}, \quad b_0 = \sqrt{e_1^{(0)} - e_2^{(0)}},$$
$$a_{n+1} = \frac{a_n + b_n}{2}, \quad b_{n+1}^2 = a_n b_n \quad \text{for } n \geq 0.$$

Note that a_0, b_0 are so far defined only up to sign, and also satisfy (4.2) since all $e_j^{(0)}$ are distinct. Given a_n, b_n, it is easy to see that we can compute a_{n+1} unambiguously, whereas b_{n+1} is obtained up to sign. Starting from a given pair (a_0, b_0), this then determines an *AGM sequence* $((a_n, b_n))_{n=0}^\infty$, where we obtain a different sequence by choosing the sign of b_{n+1} differently at each step. Associated to this AGM sequence, for $n \geq 1$ we let

$$e_1^{(n)} = \frac{a_n^2 + b_n^2}{3}, \quad e_2^{(n)} = \frac{a_n^2 - 2b_n^2}{3}, \quad e_3^{(n)} = \frac{b_n^2 - 2a_n^2}{3}. \tag{4.7}$$

Observe that this identity also holds for $n = 0$, and all $e_j^{(n)}$ are distinct and satisfy $\sum_{j=1}^3 e_j^{(n)} = 0$ for all $n \geq 0$. Hence we can construct a sequence $(E_n)_{n=0}^\infty$ of elliptic curves over $\mathbb{C}$, where E_n is given by the Weierstrass equation

$$E_n : \quad Y_n^2 = 4(X_n - e_1^{(n)})(X_n - e_2^{(n)})(X_n - e_3^{(n)}).$$

For $n \geq 1$, we define $\varphi_n : E_n \to E_{n-1}$ to be the morphism which sends $(X_n, Y_n) \mapsto (X_{n-1}, Y_{n-1})$, where

$$\begin{aligned} X_{n-1} &= X_n + \frac{(e_3^{(n)} - e_1^{(n)})(e_3^{(n)} - e_2^{(n)})}{X_n - e_3^{(n)}} \\ Y_{n-1} &= Y_n \left(1 - \frac{(e_3^{(n)} - e_1^{(n)})(e_3^{(n)} - e_2^{(n)})}{(X_n - e_3^{(n)})^2}\right). \end{aligned} \tag{4.8}$$

Observe that $\ker(\varphi_n) = \langle (e_3^{(n)}, 0) \rangle$. Thus φ_n is a 2-*isogeny* (note that it is a 2-to-1 map). It is well known (see, e.g., [Sil86, Theorem III.6.1]) that there exists a *dual* *isogeny* $\hat{\varphi}_n : E_{n-1} \to E_n$ such that $\hat{\varphi}_n \circ \varphi_n$ is the multiplication-by-2 map on E_n. Note that $\ker(\hat{\varphi}_n) = \langle (e_1^{(n-1)}, 0) \rangle$. However, $\varphi_n \circ \varphi_{n+1}$ is not the multiplication-by-2 map on E_{n+1}, since, for example,

$$\varphi_n \left(\varphi_{n+1} \left((e_1^{(n+1)}, 0) \right) \right) = \varphi_n((e_1^{(n)}, 0)) = (e_1^{(n-1)}, 0) \neq O.$$

This therefore allows us to construct a *chain of* 2-*isogenies*, as depicted below.

$$\begin{tikzcd}
\cdots & E_n \arrow[l, "1"{pos=0, below}] \arrow[r, shift left, "\varphi_n", "3"{pos=0, above}] \arrow[d, "2"{pos=0}] & E_{n-1} \arrow[l, shift left, "\hat{\varphi}_n", "1"{pos=0, below}] \arrow[r, "3"{pos=0, above}] \arrow[d, "2"{pos=0}] & \cdots & E_1 \arrow[l, "1"{pos=0, below}] \arrow[r, shift left, "3"{pos=0, above}] \arrow[d, "2"{pos=0}] & E_0 \arrow[l, shift left, "1"{pos=0, below}] \\
 & {} & {} & & {} &
\end{tikzcd}$$

The number j next to each arrow originating from E_n denotes the point $(e_j^{(n)}, 0)$.

To see the effect of choosing a different sign of b_n, first note that we can rewrite $e_j^{(n+1)}$ given by (4.7) as

$$e_1^{(n+1)} = \frac{e_1^{(n)} + 2a_n b_n}{4}, \quad e_2^{(n+1)} = \frac{e_1^{(n)} - 2a_n b_n}{4}, \quad e_3^{(n+1)} = \frac{-e_1^{(n)}}{2}. \tag{4.9}$$

For $n \geq 1$, if we replace (a_n, b_n) by $(a_n, -b_n)$, then this interchanges $e_1^{(n+1)}$ and $e_2^{(n+1)}$ but leaves $e_3^{(n+1)}$ unchanged. This relabelling of the roots $e_j^{(n+1)}$ therefore has no effect on the curve E_{n+1}, but in turn yields a different curve E_{n+2}. For $n = 0$, recall that a_0, b_0 are determined up to sign. It is easy to see that if only one of a_0, b_0 changes the sign, then this interchanges $e_1^{(1)}$ and $e_2^{(1)}$ but fixes $e_3^{(1)}$; all $e_j^{(1)}$ remain unchanged if both signs of a_0, b_0 are changed.

Given a pair $(a_0, b_0) \in \mathbb{C}^2$ satisfying (4.2), knowing (a_0, b_0) not only tells us which curve E_0 we started with and which 2-torsion point $(e_1^{(0)}, 0)$ we used to construct E_1, it also determines the labelling of all the roots of E_1 (see (4.9)), and hence determines the curve E_2. Thus we have a bijection between:

- The set of all AGM sequences starting at (a_0, b_0), and

- The set of all chains of 2-isogenies starting at E_0 with the subsequence

$$E_2 \to E_1 \to E_0.$$

We now consider what happens to a chain of 2-isogenies $(E_n)_{n=0}^{\infty}$ as $n \to \infty$. Given an elliptic curve $E_0 = E$ over $\mathbb{C}$, we can construct all elliptic curves E_n recursively as above. This construction in turn yields an AGM sequence $((a_n, b_n))_{n=0}^{\infty}$ associated to the isogeny chain. Let $S \subseteq \mathbb{Z}_{>0}$ be the set of all indices for which the pair (a_n, b_n) is bad. It then follows from (4.7) that

$$\lim_{n\to\infty} e_1^{(n)} = \frac{2M_S(a_0,b_0)^2}{3}, \quad \lim_{n\to\infty} e_2^{(n)} = \lim_{n\to\infty} e_3^{(n)} = \frac{-M_S(a_0,b_0)^2}{3}, \tag{4.10}$$

where $M_S(a_0, b_0)$ is the common limit of the AGM sequence. The *limiting curve* E_∞ for the isogeny chain is thus given by

$$E_\infty : \quad Y_\infty^2 = 4\left(X_\infty - \frac{2M_S(a_0,b_0)^2}{3}\right)\left(X_\infty + \frac{M_S(a_0,b_0)^2}{3}\right)^2. \tag{4.11}$$

Observe that E_∞ is a singular curve. We say that the isogeny chain is *good* if the singular point of E_∞ is a node; otherwise it is said to be *bad.*

In Section 4.3, we have seen that the notion of a good chain of lattices, which has been defined as a property of the chain as a whole, can be also considered in terms of individual steps. We will finally show that this is also the case for our notion of a good isogeny chain defined above.

Definition. Let $(E_n)_{n=0}^{\infty}$ be a chain of 2-isogenies. For $n \geq 2$, we say that E_n is the *right choice* for the isogeny chain if its roots satisfy

$$|e_3^{(n)} - e_2^{(n)}| \leq |e_3^{(n)} - e_1^{(n)}|;$$

otherwise we say that E_n is the *bad choice.*

In other words, E_n is the right choice if $e_3^{(n)}$ is closer to $e_2^{(n)}$ than it is to $e_1^{(n)}$. The following lemma is immediate.

Lemma 4.4.1. *Let $(E_n)_{n=0}^{\infty}$ be an isogeny chain, and let $((a_n, b_n))_{n=0}^{\infty}$ be its associated AGM sequence. Then for all $n \geq 1$, the pair (a_n, b_n) is good if and only if E_{n+1} is the right choice for the isogeny chain.*

Proof. From (4.7), we have

$$4|e_3^{(n+1)} - e_2^{(n+1)}| = 4|b_{n+1}^2 - a_{n+1}^2| = |b_n - a_n|^2,$$

and

$$4|e_3^{(n+1)} - e_1^{(n+1)}| = 4|a_{n+1}|^2 = |b_n + a_n|^2.$$

The lemma now follows directly from the definition of a right choice given above. □

The following proposition, which is analogous to Proposition 4.3.3, follows easily from the properties of complex AGM.

Proposition 4.4.2. *The following are equivalent:*

1. *A chain of 2-isogenies $(E_n)_{n=0}^{\infty}$ is good;*
2. *Its associated AGM sequence is good;*
3. *E_n is chosen to be the right choice for all but finitely many $n \geq 2$.*

Proof. Equivalence of (1) and (2) follows easily from Proposition 4.2.2, while equivalence of (2) and (3) is immediate from Lemma 4.4.1 and the definition of a good AGM sequence. □

4.5 Period Lattices of Elliptic Curves

In this section, we will combine all three classes of objects we have introduced so far into an algorithm for computing period lattices of elliptic curves over $\mathbb{C}$.

Our approach will follow closely with the classical definition of periods and elliptic logarithms given in Section 1.3. For an alternative approach, see [CT, Section 6].

4.5.1 General Case

Let E_0 be an elliptic curve over $\mathbb{C}$ of the form (4.6) as before, where all roots $e_j^{(0)}$ are given in some fixed order. As mentioned in Section 1.3, it is well known that $E_0 \cong \mathbb{C}/\Lambda_0$ for some lattice Λ_0 via the map

$$\begin{aligned} P = \big(\wp_{\Lambda_0}(z), \wp'_{\Lambda_0}(z)\big) &\mapsto z \pmod{\Lambda_0} \\ O &\mapsto 0 \pmod{\Lambda_0}. \end{aligned}$$

Let $w_1 \in \Lambda_0$ and let $z_1 = w_1/2$. Then by above isomorphism, we can see that $(\wp_{\Lambda_0}(z_1), \wp'_{\Lambda_0}(z_1))$ is a 2-torsion point in $E_0(\mathbb{C})$. Hence we can assume that

$$\wp_{\Lambda_0}(z_1) = e_1^{(0)}, \quad \wp'_{\Lambda_0}(z_1) = 0$$

(note that there are three ways to choose $e_1^{(0)}$). Define ℓ_1 to be the straight line on the complex plane starting from 0 to z_1. Then we have

$$\frac{w_1}{2} = z_1 = \int_{\ell_1} dz = \int_{\mathcal{C}_1^{(0)}} \frac{dX_0}{Y_0},$$

where $\mathcal{C}_1^{(0)}$ is the path on the elliptic curve E_0 defined by

$$\mathcal{C}_1^{(0)} = \{\big(\wp_{\Lambda_0}(tz_1), \wp'_{\Lambda_0}(tz_1)\big) : 0 \le t \le 1\}.$$

Given E_0, we can construct a chain of 2-isogenies $(E_n)_{n=0}^{\infty}$ and its associated AGM sequence $((a_n, b_n))_{n=0}^{\infty}$ using the method described in Section 4.4. As we will see below, this will also yield the corresponding chain of lattices $(\Lambda_n)_{n=0}^{\infty}$. Note that one can obtain a different isogeny chain with the same starting curve E_0 depending on how the sign of b_n is chosen at each step. We will now combine what we have so far in order to determine z_1 (and hence w_1).

For each $n \geq 1$, since $E_n \cong \mathbb{C}/\Lambda_n$ for some lattice Λ_n, we have the connection between an isogeny chain and a chain of lattices as shown in Figure 4.1.

$$\begin{tikzcd}
\cdots \arrow[r] & \mathbb{C} \arrow[r, "\mathrm{id}"] \arrow[d] & \mathbb{C} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \mathbb{C}/\Lambda_n \arrow[r] \arrow[d, "(\wp_{\Lambda_n}, \wp'_{\Lambda_n})"] & \mathbb{C}/\Lambda_{n-1} \arrow[r] \arrow[d, "(\wp_{\Lambda_{n-1}}, \wp'_{\Lambda_{n-1}})"] & \cdots \\
\cdots \arrow[r] & E_n \arrow[r, "\varphi_n"] & E_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

Figure 4.1: A chain of 2-isogenies linked with a chain of lattices

By definition of φ_n (see (4.8)), it can be verified that

$$\varphi_n^* \left(\frac{dX_{n-1}}{Y_{n-1}} \right) = \frac{dX_n}{Y_n} \tag{4.12}$$

for all $n \geq 1$, where φ_n^* is the pullback of the differential on E_{n-1} by φ_n. This therefore induces the identity map id : $\mathbb{C} \to \mathbb{C}$, which in turn induces the map $\mathbb{C}/\Lambda_n \to \mathbb{C}/\Lambda_{n-1}$ via $z \pmod{\Lambda_n} \mapsto z \pmod{\Lambda_{n-1}}$. Since φ_n is a 2-to-1 map, it then follows that $\Lambda_{n-1} \supset \Lambda_n$ with $[\Lambda_{n-1} : \Lambda_n] = 2$. Moreover, since $\varphi_n \circ \varphi_{n+1}$ is not the multiplication-by-2 map on E_{n+1} for all $n \geq 1$, we have $\Lambda_{n+1} \neq 2\Lambda_{n-1}$. Hence this gives us the relationship between a chain of lattices and a chain of 2-isogenies.

Next, let $\mathcal{C}_1^{(n)}$ be the path on the elliptic curve E_n defined by

$$\mathcal{C}_1^{(n)} = \{ \left(\wp_{\Lambda_n}(tz_1), \wp'_{\Lambda_n}(tz_1) \right) : 0 \leq t \leq 1 \}.$$

Then we can regard $\mathcal{C}_1^{(n)}$ as a map $[0,1] \to E_n$, as shown in the diagram below.

$$\begin{tikzcd}
{[0,1]} \arrow[r, "\mathcal{C}_1^{(n+1)}"] \arrow[d, "\mathrm{id}"] & E_{n+1} \arrow[d, "\varphi_{n+1}"] \\
{[0,1]} \arrow[r, "\mathcal{C}_1^{(n)}"] & E_n
\end{tikzcd}$$

By commutativity of this diagram, we have $\varphi_{n+1} \circ \mathcal{C}_1^{(n+1)} = \mathcal{C}_1^{(n)}$. Together with (4.12), this implies that

$$\int_{\mathcal{C}_1^{(n+1)}} \frac{dX_{n+1}}{Y_{n+1}} = \int_{\mathcal{C}_1^{(n)}} \frac{dX_n}{Y_n},$$

and so

$$\frac{w_1}{2} = z_1 = \int_{\ell_1} dz = \int_{\mathcal{C}_1^{(0)}} \frac{dX_0}{Y_0} = \int_{\mathcal{C}_1^{(1)}} \frac{dX_1}{Y_1} = \cdots = \int_{\mathcal{C}_1^{(n)}} \frac{dX_n}{Y_n} = \cdots$$

Recall the limiting curve E_∞ of the isogeny chain $(E_n)_{n=0}^\infty$ from (4.11). As $n \to \infty$, the path $\mathcal{C}_1^{(n)}$ on E_n approaches to some path on E_∞, say, $\mathcal{C}_1^{(\infty)}$, whose starting point is O. We now describe another end-point of $\mathcal{C}_1^{(\infty)}$ as follows. For all $n \geq 1$, one can easily check from (4.8) that

$$\begin{aligned} \varphi_n(O) = \varphi_n\left((e_3^{(n)}, 0)\right) &= O \\ \varphi_n\left((e_1^{(n)}, 0)\right) = \varphi_n\left((e_2^{(n)}, 0)\right) &= (e_1^{(n-1)}, 0). \end{aligned}$$

Moreover, we can rewrite (4.8) as

$$X_n = \frac{(X_{n-1} + e_3^{(n)}) + s_n}{2}, \quad Y_n = \frac{Y_{n-1}}{1 - \dfrac{(e_3^{(n)} - e_1^{(n)})(e_3^{(n)} - e_2^{(n)})}{(X_n - e_3^{(n)})^2}}, \tag{4.13}$$

where

$$s_n = \sqrt{(X_{n-1} - e_3^{(n)})^2 - 4(e_3^{(n)} - e_1^{(n)})(e_3^{(n)} - e_2^{(n)})}$$

is defined up to sign. By definition of $\mathcal{C}_1^{(n)}$, it is clear that its starting point is always $O \in E_n(\mathbb{C})$. In addition, we already know that $\varphi_n \circ \mathcal{C}_1^{(n)} = \mathcal{C}_1^{(n-1)}$. Hence for all $n \geq 1$, the sign of s_n must be chosen so that $O \in E_{n-1} \mapsto O \in E_n$. This can be achieved provided that s_n satisfies

$$|(X_{n-1} - e_3^{(n)}) - s_n| \leq |(X_{n-1} - e_3^{(n)}) + s_n|. \tag{4.14}$$

By continuity, this criterion for s_n then holds along the path $\mathcal{C}_1^{(n-1)}$. But since another end-point of $\mathcal{C}_1^{(0)}$ is $(e_1^{(0)}, 0)$ by definition, (4.14) eventually implies that $(e_1^{(0)}, 0) \mapsto (e_1^{(1)}, 0)$. Hence by induction, the two end-points of $\mathcal{C}_1^{(n)}$ are O and $(e_1^{(n)}, 0)$ for all $n \geq 0$, and so the two end-points of $\mathcal{C}_1^{(\infty)}$ are

$$P_1 = O, \quad P_2 = \left(\lim_{n\to\infty} e_1^{(n)}, 0\right) = \left(\frac{2M_S(a_0, b_0)^2}{3}, 0\right)$$

(the last equality is from (4.10)), where $((a_n, b_n))_{n=0}^{\infty}$ is the AGM sequence associated to the isogeny chain, and S is the set of all indices for which (a_n, b_n) is bad.

Similar to [BM88], by writing

$$X_\infty = t^2 + \frac{2M_S(a_0, b_0)^2}{3}, \quad Y_\infty = 2t(t^2 + M_S(a_0, b_0)^2),$$

and letting $\tan\theta = t/M_S(a_0, b_0)$, we have

$$\frac{dX_\infty}{Y_\infty} = \frac{d\theta}{M_S(a_0, b_0)}$$

and also

$$t = \frac{Y_\infty}{2\left(X_\infty + \frac{M_S(a_0,b_0)^2}{3}\right)}.$$

Then it is easy to see that

$$P = P_1 \iff t = \infty \iff \cos\theta = 0 \iff \theta = \frac{(2k_1 - 1)\pi}{2}$$
$$P = P_2 \iff t = 0 \iff \sin\theta = 0 \iff \theta = k_2\pi$$

for some $k_1, k_2 \in \mathbb{Z}$. If we choose $k_1 = k_2 = k$ for some $k \in \mathbb{Z}$ (i.e., we choose the values for arctan from the same branch), we finally have

$$w_1 = \cdots = 2\int_{C_1^{(\infty)}} \frac{dX_\infty}{Y_\infty} = \frac{2}{M_S(a_0,b_0)}\int_{k\pi-\frac{\pi}{2}}^{k\pi} d\theta = \frac{\pi}{M_S(a_0,b_0)}.$$

Note that w_1 we just obtained is up to sign. If we had chosen k_1, k_2 differently then we would have obtained some odd multiple of w_1, which would not change w_1 modulo $2\Lambda_0$.

Given (a_0, b_0), we already know that the value of $M_S(a_0,b_0)$ depends on the set S. If we choose $S = \emptyset$, then $|M_S(a_0,b_0)| = |M(a_0,b_0)|$ will attain its maximum among all AGM sequences starting at (a_0,b_0) by Proposition 4.2.2. Thus the period $w_1 \in \Lambda_0$ obtained by making the optimal choice for the AGM sequence with a fixed starting pair (a_0,b_0) will be the minimal one (hence primitive), and *may* also be a *minimal coset representative* modulo $2\Lambda_0$. If we can determine the other two minimal coset representatives w_2, w_3 (by choosing $e_1^{(0)}$ differently and computing w_j in a similar way), then by Lemma 4.3.6, any two of these w_j form a $\mathbb{Z}$-basis for Λ_0.

Recall that

$$a_0 = \sqrt{e_1^{(0)} - e_3^{(0)}}, \quad b_0 = \sqrt{e_1^{(0)} - e_2^{(0)}},$$

i.e., both a_0, b_0 are determined up to sign. Then we have the problem of deciding which one of $\pi/M(a_0, \pm b_0)$ is actually a minimal coset representative of $2\Lambda_0$ in Λ_0, since both are periods of Λ_0 and belong to the same coset modulo $2\Lambda_0$. To avoid this ambiguity, we will regard the pairs $(a_0, \pm b_0)$ as the two results of computing AGM "one step backwards". To be precise, we wish to find a_{-1}, b_{-1} such that

$$2a_0 = a_{-1} + b_{-1}, \quad b_0^2 = a_{-1}b_{-1}.$$

It is then easy to show that

$$a_{-1} = a_0 \pm c_0, \quad b_{-1} = a_0 \mp c_0,$$

where $c_0^2 = a_0^2 - b_0^2$, so both pairs $(a_0, \pm b_0)$ come from $(a_0 + c_0, a_0 - c_0)$.

Let $w = \pi/M(a_0 + c_0, a_0 - c_0)$. One can easily observe that changing the sign of c_0 has no effect on w, whereas changing the sign of a_0 simply negates w. Thus w is uniquely determined up to sign regardless of the signs of a_0, c_0. Moreover, the optimality of $M(a_0 + c_0, a_0 - c_0)$ implies that

$$w = \frac{\pi}{M(a_0 + c_0, a_0 - c_0)} = \frac{\pi}{M(a_0, b_0)},$$

provided that (a_0, b_0) is *good*, i.e., $|a_0 - b_0| \leq |a_0 + b_0|$. Hence if (a_0, b_0) is chosen to be good, then $w_1 = \pi/M(a_0, b_0) = w$ is smaller than $\pi/M(a_0, -b_0)$, and is thus a minimal coset representative modulo $2\Lambda_0$. Finally, it follows from Proposition 4.3.7 that the corresponding lattice chain $(\Lambda_n)_{n=0}^{\infty}$ is optimal with limiting period w_1, and there exists $w_2 \in \Lambda_0$ such that $\Lambda_n = \langle w_1, 2^n w_2 \rangle$ for all $n \geq 0$.

Thus we have proved our first result on period lattices of elliptic curves over $\mathbb{C}$.

Theorem 4.5.1 (Period Lattices of Elliptic Curves over $\mathbb{C}$, first version)**.** *Let E_0 be an elliptic curve over $\mathbb{C}$ of the form* (4.6)*, with period lattice Λ_0. Set*

$$a_0 = \sqrt{e_1^{(0)} - e_3^{(0)}}, \quad b_0 = \sqrt{e_1^{(0)} - e_2^{(0)}},$$

where the sign of b_0 is chosen so that (a_0, b_0) is good (i.e., $|a_0 - b_0| \leq |a_0 + b_0|$). Then

$$w_1 = \frac{\pi}{M(a_0, b_0)}$$

is a primitive period of Λ_0, and is a minimal coset representative modulo $2\Lambda_0$.

Define the other two minimal coset representatives w_2, w_3 in a similar way (by permuting $e_j^{(0)}$). Then any two of these w_j form a $\mathbb{Z}$-basis for Λ_0.

As we can see, this theorem computes each minimal coset representative w_j by choosing $e_1^{(0)}$ differently at a time. Nevertheless, it turns out that we may obtain all w_j by using only one fixed ordering of the roots $e_j^{(0)}$ and three different AGM computations. This alternative method, which also exhibits a certain relationship among all w_j, will be explained in Section 4.5.4.

4.5.2 Special Case I: Rectangular Lattices

For the rest of this chapter, we set $i = \sqrt{-1}$. Recall that if $|a_0 - b_0| = |a_0 + b_0|$, then both $(a_0, \pm b_0)$ are good and $\Re(b_0/a_0) = 0$. This implies that $b_0 = ika_0$ for some $k \in \mathbb{R} \setminus \{0\}$. Let $r = k^2$. Then we have

$$b_0 = ika_0 \iff \frac{e_1^{(0)} - e_2^{(0)}}{e_1^{(0)} - e_3^{(0)}} = -r < 0.$$

Using the fact that $\sum_{j=1}^{3} e_j^{(0)} = 0$, we can rewrite $e_3^{(0)}$ in terms of $e_1^{(0)}, e_2^{(0)}$ and obtain

$$e_2^{(0)} = \left(\frac{1+2r}{1-r}\right) e_1^{(0)}, \quad e_3^{(0)} = \left(\frac{r+2}{r-1}\right) e_1^{(0)},$$

provided that $r \neq 1$. Clearly, the sign of $(1+2r)/(1-r)$ is always opposite to the sign of $(r+2)/(r-1)$ for all $r > 0$ (apart from 1). Geometrically, this means that all $e_j^{(0)}$ are collinear on the complex plane with $e_1^{(0)}$ in the middle. If $r = 1$, then one can easily check that this is still the case, with $e_1^{(0)} = 0$ and $e_2^{(0)} = -e_3^{(0)}$.

To see what the associated period lattice looks like, first we let $w = \pi/M(a_0, b_0)$ and $w' = \pi/M(a_0, -b_0)$. Then both w, w' (up to sign) are the minimal elements in the same coset modulo $2\Lambda_0$. Hence by Lemma 4.3.5, the periods $w_1, w_2 = (w \pm w')/2$ form an orthogonal $\mathbb{Z}$-basis for Λ_0, so the period lattice is *rectangular*. Alternatively, we could obtain a $\mathbb{Z}$-basis for Λ_0 by computing two minimal coset representatives (see Theorem 4.5.1) using the two other roots of E_0 which are not "in the middle" in the role of $e_1^{(0)}$.

Finally, we note that if all $e_j^{(0)}$ are collinear, then we can always "rotate" them by a suitable constant in $\mathbb{C} \setminus \mathbb{R}$ so that the scaled roots ${e'}_j$ are all real. Then one could use an algorithm for computing period lattices of elliptic curves over $\mathbb{R}$ (e.g., [Coh93, Algorithm 7.4.7]) to compute the period lattice of the elliptic curve

$E' : (Y')^2 = 4(X' - e_1')(X' - e_2')(X' - e_3')$. The period lattice of our original elliptic curve E_0 is then obtained after some suitable scaling.

In particular, one can arrange all roots e_j' so that $e_1' > e_2' > e_3'$ and obtain an orthogonal $\mathbb{Z}$-basis for the period lattice of E' by setting

$$w_1 = \frac{\pi}{M\left(\sqrt{e_1' - e_3'}, \sqrt{e_1' - e_2'}\right)}, \quad w_2 = \frac{i\pi}{M\left(\sqrt{e_1' - e_3'}, \sqrt{e_2' - e_3'}\right)}$$

with all positive square roots. In fact, these formulas are familiar from the literature; see, e.g., [Coh93, Algorithm 7.4.7] or [Cre97, (3.7.1)].

4.5.3 Special Case II

If the roots of E_0 are such that

$$\left|\frac{e_1^{(0)} - e_2^{(0)}}{e_1^{(0)} - e_3^{(0)}}\right| = 1 \quad \text{with } e_1^{(0)} - e_2^{(0)} \neq \pm(e_1^{(0)} - e_3^{(0)}),$$

then geometrically all $e_j^{(0)}$ lie on an isosceles triangle having $e_1^{(0)}$ as the vertex where the sides of equal length intersect. As before, one can rotate this triangle by a suitable constant in $\mathbb{C} \setminus \mathbb{R}$ so that $e_1^{(0)} \in \mathbb{R}$, and $e_2^{(0)}, e_3^{(0)}$ are complex conjugates. This yields a new elliptic curve E' over $\mathbb{R}$, whose Weierstrass equation has only one real root. In other words, $E'/\mathbb{R}$ has negative discriminant.

Again, one can use an algorithm for computing period lattices of elliptic curves over $\mathbb{R}$ (e.g., [Coh93, Algorithm 7.4.7]) to compute the period lattice of E'. It is well known that the period lattice of E' is of the form $\Lambda' = \langle w_1', w_2' \rangle$, for some w_1', w_2' satisfying

$$w_1' \in \mathbb{R}, \quad \Re(w_2') = \frac{w_1'}{2}.$$

The period lattice $\Lambda_0 = \langle w_1, w_2 \rangle$ of E_0 can then be obtained by a suitable scaling on w_1', w_2'. Note that this time we have $\Re(w_2/w_1) = 1/2$. This will be illustrated in Example 4.7.4.

4.5.4 A Relationship amongst the Periods

In this subsection, we will present an alternative method for computing period lattices of elliptic curves over $\mathbb{C}$. Unlike Theorem 4.5.1, this time all three minimal coset representatives w_1, w_2, w_3 will be determined by using only one fixed ordering of the roots and three different *strongly optimal* AGM sequences. Finally, we will also show that all w_j obtained by this new method satisfy a certain linear relation.

Let E_0 be an elliptic curve over $\mathbb{C}$ of the form (4.6) as before. Assume that its roots are arranged in some fixed order, say, $(e_1^{(0)}, e_2^{(0)}, e_3^{(0)})$. Then we can compute a_0, b_0 (uniquely up to sign) as before. Let $c_0 = \sqrt{a_0^2 - b_0^2}$, which is again up to sign. We claim that one can always choose the signs of a_0, b_0, c_0 in such a way that the following conditions hold:

$$|a_0 - b_0| \leq |a_0 + b_0|, \quad |c_0 - ib_0| \leq |c_0 + ib_0|, \quad |a_0 - c_0| \leq |a_0 + c_0|. \tag{4.15}$$

To prove this claim, we first consider the case when equality occurs in one of the conditions in (4.15). It might seem possible at first that there could be at least two equalities occurring in (4.15) simultaneously. However, it is very easy to verify that this is not the case. If there exists exactly one equality in (4.15), then one can always choose the sign of the variables appearing in the equality in such a way that all the conditions in (4.15) are satisfied. For example, if

$$a_0 = 1, \quad b_0 = i, \quad c_0 = \sqrt{2}$$

(note that $c_0^2 = a_0^2 - b_0^2$), then we have

$$|a_0 - b_0| = |a_0 + b_0|, \quad |c_0 - ib_0| > |c_0 + ib_0|, \quad |a_0 - c_0| < |a_0 + c_0|.$$

However, if $b_0 = -i$, then a_0, b_0, c_0 now satisfy all the conditions in (4.15).

Now we consider the case when all the conditions in (4.15) are strictly inequalities. In this case, we start by choosing the sign of a_0 arbitrarily. Then we can always choose the signs of b_0, c_0 so that the first and the third conditions in (4.15) hold. The second condition, however, requires some extra work. First, we note that this condition is equivalent to $\Im(c_0/b_0) \geq 0$. Hence, if our chosen b_0, c_0 are such that $\Im(c_0/b_0) \leq 0$, then at first one might consider interchanging b_0 and c_0; this would make the second condition satisfied, whereas other conditions remained unaffected. Unfortunately, such attempt will affect our curve E_0. To be precise, suppose we interchange b_0 and c_0. Then by (4.7), we have

$$\begin{aligned} e_1^{(0)} &= \frac{a_0^2 + b_0^2}{3} \quad \mapsto \quad \frac{a_0^2 + c_0^2}{3} = -e_3^{(0)} \\ e_2^{(0)} &= \frac{a_0^2 - 2b_0^2}{3} \quad \mapsto \quad \frac{a_0^2 - 2c_0^2}{3} = -e_2^{(0)} \\ e_3^{(0)} &= \frac{b_0^2 - 2a_0^2}{3} \quad \mapsto \quad \frac{c_0^2 - 2a_0^2}{3} = -e_1^{(0)}, \end{aligned}$$

i.e., E_0 is mapped to another elliptic curve isomorphic to it. In this case, we should therefore use a new ordering for the roots of E_0. Let

$$a' = ia_0, \quad b' = ic_0, \quad c' = ib_0.$$

Then one can easily check that $(a')^2 - (b')^2 = (c')^2$. Moreover, we have

$$|a' \pm b'| = |a_0 \pm c_0|, \quad |c' \pm ib'| = |ib_0 \mp c_0|, \quad |a' \pm c'| = |a_0 \pm b_0|.$$

Since a_0, b_0, c_0 satisfy all but the second conditions in (4.15), this leads to

$$|a' - b'| < |a' + b'|, \quad |c' - ib'| < |c' + ib'|, \quad |a' - c'| < |a' + c'|,$$

and so a', b', c' satisfy all conditions in (4.15). Defining e_1', e_2', e_3' in a similar way as in (4.7), we finally obtain

$$\begin{aligned}
e_1' &= \frac{(a')^2+(b')^2}{3} = \frac{b_0^2-2a_0^2}{3} = e_3^{(0)} \\
e_2' &= \frac{(a')^2-2(b')^2}{3} = \frac{a_0^2-2b_0^2}{3} = e_2^{(0)} \\
e_3' &= \frac{(b')^2-2(a')^2}{3} = \frac{a_0^2+b_0^2}{3} = e_1^{(0)}.
\end{aligned}$$

This can be summarised into the following proposition.

Proposition 4.5.2. *Let $(e_1^{(0)}, e_2^{(0)}, e_3^{(0)})$ be an ordering of the roots of E_0, and define a_0, b_0, c_0 as before. Then we have one of the following cases:*

1. *a_0, b_0, c_0 readily satisfy all the conditions in (4.15);*

2. *a_0, b_0, c_0 yield an equality in (4.15). In this case, we can choose the sign of the variables appearing in the equality so that all the conditions in (4.15) are satisfied;*

3. *Otherwise, suppose the signs of a_0, b_0, c_0 are chosen so that*

$$|a_0 - b_0| < |a_0 + b_0| \quad \textit{and} \quad |a_0 - c_0| < |a_0 + c_0|.$$

Then if $|c_0 - ib_0| > |c_0 + ib_0|$, the new ordering $(e_3^{(0)}, e_2^{(0)}, e_1^{(0)})$ will give a new set $\{a_0, b_0, c_0\}$ (where a_0, b_0, c_0 is replaced by ia_0, ic_0, ib_0 respectively), whose signs can be chosen so that all conditions in (4.15) hold.

We are now ready to state an alternative version of Theorem 4.5.1.

Theorem 4.5.3 (Period Lattices of Elliptic Curves over $\mathbb{C}$, second version)**.** *Let E_0 be an elliptic curve over $\mathbb{C}$ of the form (4.6), with period lattice Λ_0 as before. Order the roots $(e_1^{(0)}, e_2^{(0)}, e_3^{(0)})$ of E_0 so that the signs of*

$$a_0 = \sqrt{e_1^{(0)} - e_3^{(0)}}, \quad b_0 = \sqrt{e_1^{(0)} - e_2^{(0)}}, \quad c_0 = \sqrt{e_2^{(0)} - e_3^{(0)}}$$

can be chosen to satisfy all the conditions in (4.15)*. Define*

$$w_1 = \frac{\pi}{M(a_0, b_0)}, \quad w_2 = \frac{\pi}{M(c_0, ib_0)}, \quad w_3 = \frac{i\pi}{M(a_0, c_0)}.$$

Then all w_j are primitive periods of Λ_0, and are minimal coset representatives modulo $2\Lambda_0$; any two of these form a $\mathbb{Z}$-basis for Λ_0.

Proof. By Proposition 4.5.2, it is always possible to order $(e_1^{(0)}, e_2^{(0)}, e_3^{(0)})$ so that (a_0, b_0, c_0) satisfies all the conditions in (4.15). We will show that this new definition of w_j still agrees with Theorem 4.5.1 where each root $e_j^{(0)}$ plays the role of $e_1^{(0)}$.

To show this, first note that $w_1 = \pm\pi/M(a_0, b_0)$ as before, since (a_0, b_0) is good. Letting $(e_1', e_2', e_3') = (e_2^{(0)}, e_1^{(0)}, e_3^{(0)})$, we find that $(a', b') = (c_0, ib_0)$ is good, so

$$\frac{\pi}{M(a', b')} = \pm\frac{\pi}{M(c_0, ib_0)} = w_2.$$

Similarly, by letting $(e_1', e_2', e_3') = (e_3^{(0)}, e_1^{(0)}, e_2^{(0)})$, we find that $(a', b') = (ic_0, ia_0)$ is good, and so

$$\frac{\pi}{M(a', b')} = \pm\frac{i\pi}{M(a_0, c_0)} = w_3.$$

Note that these w_j are minimal coset representatives of $2\Lambda_0$ in Λ_0 by Theorem 4.5.1, hence any two of them form a $\mathbb{Z}$-basis for Λ_0 by Lemma 4.3.6. □

Since any two of w_1, w_2, w_3 given by Theorem 4.5.3 form a $\mathbb{Z}$-basis for Λ_0, we have $\pm w_1 \pm w_2 \pm w_3 = 0$ for some suitable signs. We now aim to determine these signs unambiguously.

Suppose that $(e_1^{(0)}, e_2^{(0)}, e_3^{(0)})$ is an ordering of the roots of E_0 which yields (a_0, b_0, c_0) satisfying all conditions in (4.15). Let $(e'^{(0)}_1, e'^{(0)}_2, e'^{(0)}_3)$ be another ordering of the roots of E_0 which also yields (a_0', b_0', c_0') satisfying all conditions in (4.15). If $\{w_1, w_2, w_3\}$ and $\{w_1', w_2', w_3'\}$ are the periods obtained by Theorem 4.5.3

using (a_0, b_0, c_0) and (a'_0, b'_0, c'_0) respectively, then for some suitable signs, we have

$$\pm w_1 \pm w_2 \pm w_3 = 0 \quad \text{and} \quad \pm w'_1 \pm w'_2 \pm w'_3 = 0.$$

But since all w_j, w'_j are of minimal length, we must have $w'_j = \pm w_k$ for some j, k. This leads to a system of two linear equations in terms of w_j. It turns out that all signs in the first equation must be either identical or opposite to those in the second equation; since otherwise we will have either $w_j = 0$ for some j, or $w_j = \pm w_k$ for some $j \neq k$, which contradicts the fact that any two of w_1, w_2, w_3 form a $\mathbb{Z}$-basis.

Finally, to obtain the right signs of w_1, w_2, w_3, we explore all possible cases of rearranging the roots of E_0. For each ordering $(e'^{(0)}_1, e'^{(0)}_2, e'^{(0)}_3)$, we compute (a'_0, b'_0, c'_0) which satisfies all conditions in (4.15) as before (using Proposition 4.5.2 to rearrange the roots if necessary), and rewrite it in terms of a_0, b_0, c_0. Next, note that there are eight possible triples $(\pm w_1, \pm w_2, \pm w_3)$. By applying (a'_0, b'_0, c'_0) to Theorem 4.5.3 and rewriting w'_1, w'_2, w'_3 in terms of w_1, w_2, w_3, we "map" each triple $(\pm w_1, \pm w_2, \pm w_3)$ (which can be regarded as a linear equation) to another triple. By above argument, the right signs of w_1, w_2, w_3 are therefore the ones which remain fixed for any ordering of the roots of E_0. By exhaustive trials and errors, we will eventually see that

$$w_1 - w_2 - w_3 = 0.$$

From this, we can state a more general result in view of complex AGM.

Proposition 4.5.4. *Let $a, b, c \in \mathbb{C} \setminus \{0\}$ satisfying $c^2 = a^2 - b^2$ and the following:*

$$|a - b| \leq |a + b|, \quad |a - c| \leq |a + c|, \quad |c - ib| \leq |c + ib|.$$

Then

$$\frac{1}{M(a,b)} - \frac{1}{M(c,ib)} + \frac{1}{M(ia,ic)} = 0.$$

A more symmetric version of this identity may be obtained by replacing a by

ai and imposing the relation $a^2+b^2+c^2=0$. We have not done so above, since it seemed more natural to state the theorem above as giving the set of all values of $M(a,b)$ rather than the set of all values of $M(ia,b)$.

4.6 Complex Elliptic Logarithms

In this section, we will finally generalise our method for computing period lattices of elliptic curves over $\mathbb{C}$ to compute elliptic logarithms of complex points.

As before, we let E_0 be an elliptic curve over $\mathbb{C}$ given by a Weierstrass equation (4.6). Recall that an *elliptic logarithm* of $P \in E_0(\mathbb{C})$ is a value $z_P \in \mathbb{C}$ such that $P = (\wp_{\Lambda_0}(z_P), \wp'_{\Lambda_0}(z_P))$. Note that z_P is unique modulo Λ_0, where Λ_0 is the lattice of periods of the differential dX_0/Y_0, so that $E_0(\mathbb{C}) \cong \mathbb{C}/\Lambda_0$. We will show that z_P can be determined by a similar method to that shown in Section 4.5.

Let ℓ_P be the straight line from 0 to z_P (which is to be found) on the complex plane. Then

$$z_P = \int_{\ell_P} dz = \int_{\mathcal{C}_P^{(0)}} \frac{dX_0}{Y_0},$$

where $\mathcal{C}_P^{(0)}$ is the path on the elliptic curve E_0 defined by

$$\mathcal{C}_P^{(0)} = \{\big(\wp_{\Lambda_0}(tz_P), \wp'_{\Lambda_0}(tz_P)\big) : 0 \le t \le 1\}.$$

Given E_0, we construct a chain of 2-isogenies $(E_n)_{n=0}^{\infty}$ using the method described in Section 4.4, with a *good* AGM sequence (and preferably, a *strongly optimal* one, as we will see later). Consider the diagram shown in Figure 4.1. For all $n \ge 1$, it follows that

$$\varphi_n \circ \mathcal{C}_P^{(n)} = \mathcal{C}_P^{(n-1)},$$

where $\mathcal{C}_P^{(n)}$ is the path on the elliptic curve E_n defined by

$$\mathcal{C}_P^{(n)} = \{\big(\wp_{\Lambda_n}(tz_P), \wp'_{\Lambda_n}(tz_P)\big) : 0 \le t \le 1\}.$$

Together with (4.12), we again have

$$\int_{\mathcal{C}_P^{(n)}} \frac{dX_n}{Y_n} = \int_{\mathcal{C}_P^{(n-1)}} \frac{dX_{n-1}}{Y_{n-1}},$$

and so

$$z_P = \int_{\mathcal{C}_P^{(0)}} \frac{dX_0}{Y_0} = \int_{\mathcal{C}_P^{(1)}} \frac{dX_1}{Y_1} = \cdots = \int_{\mathcal{C}_P^{(n)}} \frac{dX_n}{Y_n} = \cdots$$

Recall the definition of $\varphi_n : E_n \to E_{n-1}$ from (4.8). In Section 4.5.1, we have seen that (4.8) can be rewritten as (4.13), i.e.,

$$X_n = \frac{(X_{n-1} + e_3^{(n)}) + s_n}{2}, \quad Y_n = \frac{Y_{n-1}}{1 - \dfrac{(e_3^{(n)} - e_1^{(n)})(e_3^{(n)} - e_2^{(n)})}{(X_n - e_3^{(n)})^2}},$$

where

$$s_n = \sqrt{(X_{n-1} - e_3^{(n)})^2 - 4(e_3^{(n)} - e_1^{(n)})(e_3^{(n)} - e_2^{(n)})}.$$

Since the starting point of $\mathcal{C}_P^{(n)}$ is O for all $n \geq 0$, we must again choose the sign of s_n (for $n \geq 1$) so that (4.14) holds, i.e.,

$$|(X_{n-1} - e_3^{(n+1)}) - s_n| \leq |(X_{n-1} - e_3^{(n+1)}) + s_n|,$$

and by continuity, this will be also satisfied along the path $\mathcal{C}_P^{(n)}$.

Note that if P is the 2-torsion point $(e_1^{(0)}, 0) \in E_0(\mathbb{C})$, then the above process is simply what we have seen in Section 4.5.1. In particular, if we construct our isogeny chain using a *strongly optimal* AGM sequence, then by Theorem 4.5.1, we will have $2z_P$ as a minimal coset representative modulo $2\Lambda_0$, where two of which also form a $\mathbb{Z}$-basis for Λ_0. Hence from now on, we will always construct our isogeny chain using a strongly minimal AGM sequence.

For $P = (x_0, y_0) \in E_0(\mathbb{C})$, we can therefore construct the subsequent points $(x_n, y_n) \in E_n$ for all $n \geq 1$ using (4.13) and (4.14). This then gives us a *limiting point* (x_∞, y_∞) on the limiting curve E_∞ (see (4.11) for the equation of E_∞). As discussed earlier, in this case we compute $M_S(a_0, b_0)$ with a strongly minimal sequence. It then follows that

$$z_P = \int_{\mathcal{C}_P^{(0)}} \frac{dX_0}{Y_0} = \cdots = \int_{\mathcal{C}_P^{(\infty)}} \frac{dX_\infty}{Y_\infty},$$

where $\mathcal{C}_P^{(\infty)}$ is the path on E_∞ given by

$$\mathcal{C}_P^{(\infty)} = \left\{ \left(\lim_{n\to\infty} \wp_{\Lambda_n}(tz_P), \lim_{n\to\infty} \wp'_{\Lambda_n}(tz_P) \right) : 0 \le t \le 1 \right\}$$

(for the formulas of $\lim_{n\to\infty} \wp_{\Lambda_n}(z)$ and $\lim_{n\to\infty} \wp'_{\Lambda_n}(z)$, see Proposition 4.6.1), starting from O to (x_∞, y_∞). Note that if we choose $e_1^{(0)}$ differently, then this will result in a different sequence $((x_n, y_n))_{n=0}^\infty$, and hence a different limiting point (x_∞, y_∞).

The next proposition confirms that (x_∞, y_∞) does exist.

Proposition 4.6.1. *Suppose $\{w_1, w_2\}$ is a $\mathbb{Z}$-basis for Λ_0 with $\Im(w_2/w_1) > 0$. Let $\Lambda_n = \langle w_1, 2^n w_2 \rangle$ and let $u = \exp(2\pi i z/w_1)$. Define*

$$\begin{aligned} X_\infty(z) &= \left(\frac{2\pi i}{w_1}\right)^2 \left(\frac{u}{(1-u)^2} + \frac{1}{12} \right) \\ Y_\infty(z) &= \left(\frac{2\pi i}{w_1}\right)^3 \frac{u(1+u)}{(1-u)^3}. \end{aligned}$$

Then as $n \to \infty$, $\wp_{\Lambda_n}(z)$ converges uniformly to $X_\infty(z)$, and $\wp'_{\Lambda_n}(z)$ converges uniformly to $Y_\infty(z)$. In consequence, (x_∞, y_∞) exists and is not equal to O.

Proof. We will first prove the uniform convergence for $\wp_{\Lambda_n}(z)$; such convergence for $\wp'_{\Lambda_n}(z)$ can be proved in a similar way.

Let $X_n(z) = \wp_{\Lambda_n}(z)$, and let

$$\tau_n = \frac{2^n w_2}{w_1}, \quad u = \exp\left(\frac{2\pi i z}{w_1}\right), \quad q_n = \exp(2\pi i \tau_n).$$

Then by Proposition 1.3.1, we have

$$X_n(z) = \left(\frac{2\pi i}{w_1}\right)^2 \left(\frac{u}{(1-u)^2} + \frac{1}{12} + \sum_{j=1}^{\infty}\left[\frac{q_n^j u}{(1-q_n^j u)^2} + \frac{q_n^j u^{-1}}{(1-q_n^j u^{-1})^2} - \frac{2q_n^j}{(1-q_n^j)^2}\right]\right).$$

Since $\Im(w_2/w_1) > 0$, we have $|q_0| < 1$, and so $|q_n| = |q_0|^{2^n} < 1$. By writing $z = \alpha w_1 + \beta w_2$ with $0 \leq \alpha, \beta < 1$, we also have $|u| = \exp(-2\pi\beta\Im(\tau_0)) = |q_0|^\beta < 1$. Hence for all $n > m$, we have

$$\begin{aligned}\frac{|w_1|^2|X_n(z) - X_\infty(z)|}{4\pi^2} &= \left|\sum_{j=1}^{\infty}\left(\frac{q_n^j u}{(1-q_n^j u)^2} + \frac{q_n^j u^{-1}}{(1-q_n^j u^{-1})^2} - \frac{2q_n^j}{(1-q_n^j)^2}\right)\right| \\ &\leq \sum_{j=1}^{\infty}\left[\frac{|q_n^j u|}{(1-|q_n^j u|)^2} + \frac{|q_n^j u^{-1}|}{(1-|q_n^j u^{-1}|)^2} + \frac{2|q_n^j|}{(1-|q_n^j|)^2}\right] \\ &\leq \sum_{j=1}^{\infty}\left[\frac{|q_m^j u|}{(1-|q_m^j u|)^2} + \frac{|q_m^j u^{-1}|}{(1-|q_m^j u^{-1}|)^2} + \frac{2|q_m^j|}{(1-|q_m^j|)^2}\right].\end{aligned}$$

It can be seen that

$$\sum_{j=1}^{\infty}\frac{|q_m^j u|}{(1-|q_m^j u|)^2} \leq \frac{1}{(1-|q_m u|)^2}\sum_{j=1}^{\infty}|q_m|^j \leq \frac{|q_m|}{(1-|q_m|)^3} = \frac{|q_0|^{2^m}}{(1-|q_0|^{2^m})^3}.$$

Similarly,

$$\sum_{j=1}^{\infty}\frac{2|q_m^j|}{(1-|q_m^j|)^2} \leq \frac{2|q_m|}{(1-|q_m|)^3} = \frac{2|q_0|^{2^m}}{(1-|q_0|^{2^m})^3},$$

and

$$\sum_{j=1}^{\infty}\frac{|q_m^j u^{-1}|}{(1-|q_m^j u^{-1}|)^2} \leq \sum_{j=1}^{\infty}\frac{|q_0|^{j\cdot 2^m - 1}}{(1-|q_0|^{j\cdot 2^m - 1})^2} \leq \frac{|q_0|^{2^m}}{|q_0|(1-|q_0|^{2^m-1})^2(1-|q_0|^{2^m})}.$$

Putting everything together, we finally have

$$|X_n(z) - X_\infty(z)| \leq \frac{4\pi^2}{|w_1|^2}\left(\frac{3|q_0|^{2^m}}{(1-|q_0|^{2^m})^3} + \frac{|q_0|^{2^m}}{|q_0|(1-|q_0|^{2^m-1})^2(1-|q_0|^{2^m})}\right)$$

for all $n > m$. Observe that the right-hand side is strictly decreasing as m increases. Hence for any given $\epsilon > 0$, we can always find $m = m(\epsilon)$ (not depending on z) such that $|X_n(z) - X_\infty(z)| < \epsilon$ for all $n > m$. Thus $X_n(z)$ converges uniformly to $X_\infty(z)$, which proves the first part.

To prove the second part, we first note that our isogeny chain $(E_n)_{n=0}^\infty$, where each elliptic curve E_n is of the form

$$E_n : \quad Y_n^2 = 4(X_n - e_1^{(n)})(X_n - e_2^{(n)})(X_n - e_3^{(n)}),$$

is now constructed by a strongly minimal AGM sequence. This in turn yields a corresponding chain of lattices $(\Lambda_n)_{n=0}^\infty$. Let $w_1 \in \mathbb{C}$ be the one obtained by Theorem 4.5.1. Then we have already seen in Section 4.5.1 that w_1 is a minimal coset representative modulo $2\Lambda_0$, and the lattice chain is indeed optimal with limiting period w_1. Hence there exists $w_2 \in \Lambda_0$ such that $\Lambda_n = \langle w_1, 2^n w_2 \rangle$ for all $n \geq 0$. Without loss of generality, we can also assume that $\Im(w_2/w_1) > 0$.

By definition of elliptic logarithm, we have

$$X_n = \wp_{\Lambda_n}(z), \quad Y_n = \wp'_{\Lambda_n}(z),$$

so we can regard X_n, Y_n as functions of z. The first part then implies that $X_n(z)$ converges uniformly to $X_\infty(z)$, and $Y_n(z)$ converges uniformly to $Y_\infty(z)$, for any z. By letting $z = z_P$, we finally have $x_\infty = X_\infty(z_P)$ and $y_\infty = Y_\infty(z_P)$. In addition, if z_P is not a lattice point in Λ_0, then $u(z_P) \neq 1$, and so $(x_\infty, y_\infty) \neq O$. $\square$

Although one can compute the limiting point (x_∞, y_∞) as above, we find that it is more convenient to obtain (x_∞, y_∞) by making some change of variables. Below we will present one possible way to do this; an alternative version can be seen in [CT, Section 8.3].

Given an elliptic curve E_0 of the form (4.6) as before, let

$$a_0 = \sqrt{e_1^{(0)} - e_3^{(0)}}, \quad b_0 = \sqrt{e_1^{(0)} - e_2^{(0)}}.$$

Recall that we wish to construct an isogeny chain $(E_n)_{n=0}^{\infty}$ using a strongly minimal AGM sequence. Hence we compute the AGM sequence $((a_n, b_n))_{n=0}^{\infty}$ using the method in Section 4.2, in such a way that $|a_n - b_n| \leq |a_n + b_n|$ for all $n \geq 0$. For $P = (x_0, y_0) \in E_0(\mathbb{C}) \setminus \{O\}$, we define

$$u_0 = \sqrt{x_0 - e_3^{(0)}}, \quad v_0 = \sqrt{x_0 - e_2^{(0)}}.$$

The sign of u_0 can be chosen arbitrarily. To choose the sign of v_0, we first recall that for all $n \geq 1$ we map $(x_{n-1}, y_{n-1}) \in E_{n-1} \mapsto (x_n, y_n) \in E_n$ via (4.13) in such a way that s_n satisfies the criterion (4.14). It can be verified that such criterion is equivalent to

$$|u_{n-1} - v_{n-1}| \leq |u_{n-1} + v_{n-1}| \tag{4.16}$$

(the situation when this becomes an equality will be explained later). Hence we choose the sign of v_0 so that $|u_0 - v_0| \leq |u_0 + v_0|$. Next, we define

$$t_0 = \begin{cases} \sqrt{x_0 - e_1^{(0)}} & \text{if } x_0 = e_j^{(0)} \text{ for some } j = 1, 2, 3, \\ \dfrac{y_0}{2u_0v_0} & \text{otherwise.} \end{cases}$$

Note that if P is a 2-torsion point in $E_0(\mathbb{C})$, then t_0 is determined up to sign. This will have no effect on our result, since we will obtain *half* a primitive period of Λ_0 in this case. For a non-2-torsion point P, one can easily observe that if we had chosen the other sign for u_0, then v_0 would also have been negated, but t_0 remains unchanged. In fact, t_0 is where we embed the information on the sign of z_P.

For $n \geq 1$, we define u_n, v_n, t_n in a similar way, i.e.,

$$u_n = \sqrt{x_n - e_3^{(n)}}, \quad v_n = \sqrt{x_n - e_2^{(n)}}, \quad t_n = \sqrt{x_n - e_1^{(n)}} = \frac{y_n}{2u_n v_n}.$$

We will show that these quantities can be determined by $u_{n-1}, v_{n-1}, t_{n-1}$ obtained earlier. In (4.13), one can rewrite x_n in terms of u_n to obtain (after some algebra)

$$u_n = \frac{u_{n-1} + v_{n-1}}{2}.$$

Note that the sign of u_n is determined unambiguously. For v_n, it is also easy to check that

$$v_n = \sqrt{x_n - e_2^{(n)}} = \sqrt{u_n^2 - c_n^2},$$

where $c_n^2 = a_n^2 - b_n^2$. Again, we choose the sign of v_n so that (4.16) holds. Similarly, one can show using (4.8) that

$$t_n = \frac{u_n t_{n-1}}{v_n}.$$

Recall that if we had chosen the other sign for u_0, then by (4.16), v_0 would be also negated, while t_0 remains unchanged. From these new formulas, it then follows that this will also negate both u_n, v_n, while again t_n will remain unchanged.

By definition, we have $u_n^2 = x_n - e_3^{(n)}$, $v_n^2 = x_n - e_2^{(n)}$, and $t_n^2 = x_n - e_1^{(n)}$. Since $x_\infty = \lim_{n\to\infty} x_n$ exists by Proposition 4.6.1, and $\lim_{n\to\infty} e_j^{(n)}$ exists for all $j = 1, 2, 3$ (see (4.10)), then each u_n, v_n, t_n has a limit as $n \to \infty$. Let

$$U = \lim_{n\to\infty} u_n = \lim_{n\to\infty} v_n, \quad T = \lim_{n\to\infty} t_n,$$
$$M = M(a_0, b_0) \quad (\text{where } |a_0 - b_0| \leq |a_0 + b_0|).$$

We finally have

$$\begin{aligned} x_\infty &= (\lim_{n\to\infty} u_n)^2 + \lim_{n\to\infty} e_3^{(n)} = U^2 - \frac{M^2}{3} \\ y_\infty &= 2(\lim_{n\to\infty} t_n)(\lim_{n\to\infty} u_n)(\lim_{n\to\infty} v_n) = 2TU^2. \end{aligned} \tag{4.17}$$

It is easy to see that if $P = (e_1^{(0)}, 0)$, then $t_n = 0$ for all $n \geq 0$, so $T = 0$.

Consider the limiting curve E_∞. Similar to [BM88], we again define t, θ by

$$X_\infty = t^2 + \frac{2M^2}{3}, \quad Y_\infty = 2t(t^2 + M^2), \quad \tan\theta = \frac{t}{M}.$$

As before, this gives us

$$\frac{dX_\infty}{Y_\infty} = \frac{d\theta}{M}$$

and

$$t = \frac{Y_\infty}{2\left(X_\infty + \frac{M^2}{3}\right)}.$$

Hence we have

$$P = O \iff t = \infty \iff \cos\theta = 0 \iff \theta = \frac{(2k+1)\pi}{2} \text{ for some } k \in \mathbb{Z}.$$

Let $\tan\theta^* = T/M$. Then we have

$$z_P = \int_{\mathcal{C}_P^{(\infty)}} \frac{dX_\infty}{Y_\infty} = \frac{1}{M}\int_{\frac{(2k+1)\pi}{2}}^{\theta^*} d\theta = \frac{\theta^* - \frac{(2k+1)\pi}{2}}{M}.$$

If we choose k differently, then z_P will be changed by adding a multiple of $w_1 = \pi/M$, which is a primitive period of Λ_0 by Theorem 4.5.1. Thus z_P we just obtained is unique modulo Λ_0. By letting $k = 0$ and using the fact that $\tan(\theta - \pi/2) = -1/\tan(\theta)$, we finally have

$$z_P = \frac{\theta^* - \frac{\pi}{2}}{M} = \frac{-1}{M}\arctan\left(\frac{M}{T}\right). \tag{4.18}$$

If $T = 0$ (i.e., $x_0 = e_1^{(0)}$), then one can use the fact that $\arctan(x) \to \pi/2$ as $x \to \infty$ to obtain $z_P = -\pi/(2M)$, which (up to sign) is half of the primitive period w_1 obtain by Theorem 4.5.1.

To summarise, we have the following algorithm.

Algorithm 4.6.2 (Elliptic Logarithms of Complex Points)**.** Given an elliptic curve E_0 defined over $\mathbb{C}$ and a point $P \in E_0(\mathbb{C})$, return an elliptic logarithm of P.

Input: An elliptic curve E_0 of the form (4.6), and $P = (x_0, y_0) \in E_0(\mathbb{C})$.

1. If $P = O$, **return** $z_P = 0$.
2. Let $a_0 = \sqrt{e_1^{(0)} - e_3^{(0)}}$ and $b_0 = \sqrt{e_1^{(0)} - e_2^{(0)}}$. Choose the sign of b_0 so that $|a_0 - b_0| \le |a_0 + b_0|$.
3. Let $u_0 = \sqrt{x_0 - e_3^{(0)}}$ and $v_0 = \sqrt{x_0 - e_2^{(0)}}$. Choose the sign of v_0 so that $|u_0 - v_0| \le |u_0 + v_0|$.
4. Let
$$t_0 = \begin{cases} \sqrt{x_0 - e_1^{(0)}} & \text{if } x_0 = e_j^{(0)} \text{ for some } j = 1, 2, 3, \\ \dfrac{y_0}{2u_0v_0} & \text{otherwise.} \end{cases}$$
If $t_0 = 0$, **return** $z_P = \pi/(2M(a_0, b_0))$.
5. Set $n = 1$. Repeat the following:
   (a) Let
   $$a_n = \frac{a_{n-1} + b_{n-1}}{2}, \quad b_n = \sqrt{a_{n-1}b_{n-1}}, \quad c_n^2 = a_n^2 - b_n^2.$$
   Choose the sign of b_n so that $|a_n - b_n| \le |a_n + b_n|$.

   (b) Let $u_n = (u_{n-1} + v_{n-1})/2$ and $v_n = \sqrt{u_n^2 - c_n^2}$. Choose the sign of v_n so that $|u_n - v_n| \le |u_n + v_n|$.

   (c) Let $t_n = u_n t_{n-1}/v_n$.

   (d) $n \leftarrow n + 1$.

until $|a_n - b_n|$ and $|u_n - v_n|$ are sufficiently small. Let M and T be the limiting values of a_n and t_n respectively.

Output:

$$z_P = \frac{-1}{M} \arctan\left(\frac{M}{T}\right).$$

Remark. If the criterion (4.16) becomes an equality, then we have $\Re(u_{n-1}/v_{n-1}) = 0$, or equivalently, x_{n-1} lies on the straight line joining $e_2^{(n-1)}$ and $e_3^{(n-1)}$. To avoid the ambiguity of the sign of v_{n-1}, one can recover all $e_j^{(n-1)}, x_{n-1}$, and rearrange the roots of E_{n-1} so that the new u'_{n-1}, v'_{n-1} satisfy a strict inequality. Nevertheless, we will see in Example 4.7.5 that both $\pm v_{n-1}$ are equally good for computing z_P.

Moreover, the requirement for sufficiently small $|u_n - v_n|$ as another stopping criterion in Algorithm 4.6.2 may be omitted in practice, since our experience shows that both AGM sequences $((a_n, b_n))_{n=0}^{\infty}$ and $((u_n, v_n))_{n=0}^{\infty}$ seem to converge roughly at the same rate.

Finally, note that our formulas shown in Algorithm 4.6.2 are somewhat similar to the ones in Cohen's algorithm [Coh93, Algorithm 7.4.8] for computing elliptic logarithms of real points (where our u_n is called c_n in his algorithm). Using the fact that $U^2 = T^2 + M^2$, we can rewrite z_P as

$$z_P = \frac{-1}{M} \arcsin\left(\frac{M}{U}\right),$$

which is similar (up to sign) to the output of Cohen's algorithm. By writing z_P this way, however, we have an ambiguity of the sign of z_P, since this information is embedded in T. Our formulas remove this ambiguity.

4.7 Examples

Finally, we will illustrate our method for computing period lattices of elliptic curves over $\mathbb{C}$ and elliptic logarithms of complex points via some examples.

For computational purposes, we have implemented our algorithms in MAGMA (see Appendix A.1 for the code), including our own function for computing an optimal AGM sequence (since the existing function in MAGMA does not always return an optimal one). Note that all complex numbers in our examples are computed correctly up to 100 decimal places, but only the first 20 decimal places are shown.

Example 4.7.1. Let E be the elliptic curve over $\mathbb{C}$ given by the Weierstrass equation $Y^2 = 4(X - e_1)(X - e_2)(X - e_3)$, where

$$e_1 = 3 - 2i, \quad e_2 = 1 + i, \quad e_3 = -4 + i.$$

Observe that $\sum_{j=1}^{3} e_j = 0$. We will compute the period lattice of E using the method described in Section 4.5.4. To do this, first we let $E_0 = E$ and calculate

$$a_0 = \sqrt{e_1 - e_3}, \quad b_0 = \sqrt{e_1 - e_2}, \quad c_0 = \sqrt{a_0^2 - b_0^2},$$

in such a way that the signs of a_0, b_0, c_0 satisfy all conditions in (4.15), i.e.,

$$|a_0 - b_0| \leq |a_0 + b_0|, \quad |a_0 - c_0| \leq |a_0 + c_0|, \quad |c_0 - ib_0| \leq |c_0 + ib_0|.$$

In this example, one can verify that such a_0, b_0, c_0 are

$$\begin{aligned} a_0 &= 2.70331029534753078867\ldots - i0.55487525889334275023\ldots \\ b_0 &= 1.67414922803554004044\ldots - i0.89597747612983812471\ldots \\ c_0 &= 2.23606797749978969640\ldots. \end{aligned}$$

In fact, all conditions in (4.15) are strictly inequalities in this case, so the period lattice of E is non-rectangular. By Theorem 4.5.3, we have

$$\begin{aligned} w_1 &= 1.29215151748713051904\ldots + i0.44759218107818896608\ldots \\ w_2 &= 1.42661373451784507587\ldots - i0.80963848056301882107\ldots \\ w_3 &= -0.13446221703071455682\ldots + i1.25723066164120778715\ldots \end{aligned}$$

and any two of w_j form a $\mathbb{Z}$-basis for the period lattice Λ of E. In our computation, we also have $|w_1 - w_2 - w_3| \approx 10^{-100}$, which agrees with the result given by Proposition 4.5.4. Note that these w_j are minimal coset representatives of 2Λ in Λ.

Next, we wish to compute an elliptic logarithm of the point $P = (2-i, 8+4i) \in E(\mathbb{C})$ (which has infinite order), i.e., a value z_P such that $P = (\wp_\Lambda(z_P), \wp'_\Lambda(z_P))$. Using a_0, b_0 as above, Algorithm 4.6.2 shows that

$$\begin{aligned} x_\infty &= 1.67097624471645689380\ldots - i1.23329436157704253331\ldots \\ y_\infty &= 7.78679958972849436041\ldots + i4.93520281519385276354\ldots \end{aligned}$$

and then

$$z_P = -0.72212997914002299126\ldots + i0.01717122412650902249\ldots.$$

Let $u_P = \exp(2\pi i z_P / w_1)$. One can verify that

$$\begin{aligned} \left| x_\infty - \left(\frac{2\pi i}{w_1}\right)^2 \left(\frac{u_P}{(1-u_P)^2} + \frac{1}{12}\right) \right| &\approx 10^{-100} \\ \left| y_\infty - \left(\frac{2\pi i}{w_1}\right)^3 \frac{u(1+u)}{(1-u)^3} \right| &\approx 10^{-100}. \end{aligned}$$

which agrees with our result in Proposition 4.6.1.

Note that z_P is unique modulo Λ. Depending on a $\mathbb{Z}$-basis $\{w_1, w_2\}$ for Λ, it can be seen that z_P we just obtained may not necessarily lie in the fundamental parallelogram

$$F_{w_1,w_2} = \{\lambda_1 w_1 + \lambda_2 w_2 : 0 \le \lambda_1, \lambda_2 < 1\}.$$

In this example, one can check that

$$\begin{aligned} z_P &= (-0.33249952362000772434\ldots)w_1 - (0.20502411273191295799\ldots)w_2 \\ &\equiv (0.66750047637999227565\ldots)w_1 + (0.79497588726808704200\ldots)w_2, \end{aligned}$$

and so z_P is not in the fundamental parallelogram spanned by $\{w_1, w_2\}$. Finally, we verify that

$$|\wp_\Lambda(z_P) - x(P)| \approx 10^{-99}, \quad |\wp'_\Lambda(z_P) - y(P)| \approx 10^{-100}.$$

Moreover, we have

$$\left|\wp_\Lambda\left(\frac{w_1}{2}\right) - e_1\right| \approx 10^{-99}, \quad \left|\wp_\Lambda\left(\frac{w_2}{2}\right) - e_2\right| \approx 10^{-100}, \quad \left|\wp_\Lambda\left(\frac{w_3}{2}\right) - e_3\right| \approx 10^{-100},$$

and $\wp'_\Lambda(w_j/2) \approx 0$ for all $j = 1, 2, 3$.

Finally, one can verify the above results by the following MAGMA instructions together with our code given in Appendix A.1 (`elog.m`) and Appendix A.6.3 (`wp.m`):

```
> Attach("elog.m"); // main program for computing AGM and periods
> Attach("wp.m");   // for computing Weierstrass \wp-function and its derivative
> C<i> := ComplexField(100);
> e1 := 3 - 2*i;
> e2 := 1 + i;
> e3 := -e1-e2;
> // SetVerbose("Elog", 1); // enable this line to see more details
> // Find the periods of E
> w1, w2, w3 := Explode(PeriodLattice([e1,e2] : Prec := 95));
> // Verify the linear relation given by Proposition 4.5.4
> // x-coordinates
> Abs(w1-w2-w3);
1.178164435751500540627255249934484036365875621234243886016913231010440340300022\
6132528675450933671261E-100
> // Verify if w1, w2, w3 are correct
> Abs(WeierstrassP([w2, w1], w1/2, 50) - e1);
2.207475313300438969965429438127258471881267306170402792154448031338244171032286\
6144311648981370662044E-99
> Abs(WeierstrassP([w2, w1], w2/2, 50) - e2);
5.391460741767790455448686628279229528753749205619468454035459216926777491280000\
5892434123803155519330E-100
> Abs(WeierstrassP([w2, w1], w3/2, 50) - e3);
7.318690268199678038113754829536449141684434896706657626149989699528984134911169\
4598918797783156020261E-100
```

```
> // y-coordinates: these should be approximately zero
> Abs(WeierstrassPDash([w2, w1], w1/2, 50));
2.828767677547349380629893987641710495095290298688032884567349272309132610926654\
42834638866832639717 37E-100
> Abs(WeierstrassPDash([w2, w1], w2/2, 50));
4.385616334586878455250971239232603561266412821159883821009511029706809733696494\
185198376944649816 2005E-99
> Abs(WeierstrassPDash([w2, w1], w3/2, 50));
5.99971141235607783993827798548907710252418335884779083090787357542560402559964\
805192383869256391 0701E-99
> // Compute elliptic logarithm
> P := [2-i, 8+4*i];
> z := EllipticLog([e1,e2], P : Prec := 95);
> // Verify if z is correct
> Abs(WeierstrassP([w2, w1], z, 50) - P[1]);     // x-coordinate of P
1.091837172464425937884949679637405102772833151683440639386936502505803179811 61\
31177681241180456324 60E-99
> Abs(WeierstrassPDash([w2, w1], z, 50) - P[2]); // y-coordinate of P
5.075687725141168939929731359015049047410116531098947371267710764692997910937 17\
36088802877998175472 63E-99
```

Example 4.7.2 (Rectangular Lattice)**.** Let E be the elliptic curve over $\mathbb{C}$ given by the Weierstrass equation $Y^2 = 4(X - e_1)(X - e_2)(X - e_3)$, where

$$e_1 = 1 + 3i, \quad e_2 = -4 - 12i, \quad e_3 = 3 + 9i.$$

Observe that $\sum_{j=1}^{3} e_j = 0$ and all e_j are collinear. By letting $E_0 = E$ and computing a_0, b_0, c_0 as before, we have

$$\begin{aligned} a_0 &= 1.47046851723128684330\ldots - i2.04016608641756892919\ldots \\ b_0 &= -3.22578581905571472955\ldots - i2.32501487101070997214\ldots \\ c_0 &= 2.75099469475848456460\ldots - i3.81680125374499001591\ldots . \end{aligned}$$

This time, however, we have $|a_0 - b_0| = |a_0 + b_0|$, while the other two conditions in (4.15) are strictly inequalities. Let Λ be the period lattice of E. Then we have two minimal elements (up to sign) in one coset of 2Λ in Λ, so Λ is rectangular.

To obtain an orthogonal $\mathbb{Z}$-basis for Λ, first we let $w, w' = \pi/M(a_0, \pm b_0)$. In this example, we have

$$\begin{aligned} w &= -0.29920293143872535713\ldots + i1.10940038117892953702\ldots \\ w' &= 1.14708588706988127437\ldots + i0.06697438037476960963\ldots . \end{aligned}$$

One can check that $|w| = |w'|$. Let $w_1, w_2 = (w \pm w')/2$. Then w_1, w_2 form an orthogonal $\mathbb{Z}$-basis for Λ, as proved in Lemma 4.3.5. Here, we have

$$\begin{aligned} w_1 &= 0.42394147781557795862\ldots + \imath 0.58818738077684957333\ldots \\ w_2 &= -0.72314440925430331575\ldots + \imath 0.52121300040207996369\ldots. \end{aligned}$$

Note that $\Re(w_2/w_1) = 0$.

Let z_P be an elliptic logarithm of the point $P = (3+2i, 28-14i) \in E(\mathbb{C})$ (which has infinite order). Algorithm 4.6.2 shows that

$$\begin{aligned} z_P &= -0.42599662534207481578\ldots - \imath 0.02491254923738153924\ldots \\ &\equiv (0.62858224538977667533\ldots)w_1 + (0.37134662195976180031\ldots)w_2. \end{aligned}$$

Finally, we verify that

$$|\wp_\Lambda(z_P) - x(P)| \approx 10^{-98}, \quad |\wp'_\Lambda(z_P) - y(P)| \approx 10^{-97},$$

and moreover,

$$|\wp_\Lambda(w_1/2) - e_2| \approx 10^{-99}, \quad |\wp_\Lambda(w_2/2) - e_3| \approx 0, \quad |\wp_\Lambda(w/2) - e_1| \approx 10^{-99},$$
$$|\wp'_\Lambda(w_1/2)| \approx 10^{-99}, \quad |\wp'_\Lambda(w_2/2)| \approx 10^{-99}, \quad |\wp'_\Lambda(w/2)| \approx 10^{-100}.$$

The following MAGMA instructions show how to obtain an orthogonal $\mathbb{Z}$-basis for Λ, again using the files `elog.m` and `wp.m`. An elliptic logarithm of P can be verified in a similar way as shown in Example 4.7.1.

```
> Attach("elog.m"); // main program for computing AGM and periods
> Attach("wp.m");   // for computing Weierstrass \wp-function and its derivative
> C<i> := ComplexField(100);
> // SetVerbose("Elog", 1); // enable this line to see more details
> e1 := 1+3*i;
> e2 := -4-12*i;
> e3 := -e1-e2;
> a := Sqrt(e1-e3);
> a;
1.470468517231286843302541764159328827579346329250632025852570540491786842264OO\
1831020031879802582806 - 2.040166086417568929195632588758543678573450706458213l\
372909432911028539995823185100167267554486347 58*i
> b;
-3.225785819055714729551628940618265080655744847105054607424327456289107263187l\
6200153841119742819l589 - 2.325014871010709972142178001058151104070502895062003\
693010129666408387578241476849515074193271660734*i
> Abs(Abs(a-b)- Abs(a+b)); // verify that |a - b| = |a + b|
4.571949565129099928863134193221363837807639979291194644860503372910530543626 95\
2782701792087440688119E-100
> pi := Pi(C);
> w := pi/AGM(a, b : Prec := 95);
> ww := pi/AGM(a, -b : Prec := 95);
> w1 := (w + ww)/2;
> w2 := (w - ww)/2;
> w1;
0.423941477815577958621041084515833252185402964678086454162415533317456830086 06\
6212712828355935397572l + 0.588187380776849573332658158493542874116418229902161\
032616351774522890245692514604597024020198772552 5*i
> w2;
-0.723144409254303315752332386891048872474471228199333225462465432140491455480 3\
3996131372107069265601 97 + 0.521213000402079963694722832114395581558966797057 97\
729701168525359961118233764861018527690427741753 03*i
> // Verify if {w1, w2} is an orthogonal basis
> Re(w2/w1);
-7.635260342301538195695485577916344946955478625099457683581219845132855310712 5\
0703326559859341122137 9E-101
> // Verify if w1, w2, w are correct
> // x-coordinates
> Abs(WeierstrassP([w1, w2], w1/2, 50) - e2);
1.648439858590825743501586494255081471967042800445054699634228184072313914725 47\
421825725315481631191 2E-99
> Abs(WeierstrassP([w1, w2], w2/2, 50) - e3);
0.000000000000000000000000000000000000000000000000000000000000000000000000000 0\
0000000000000000000000
> Abs(WeierstrassP([w1, w2], w/2, 50) - e1);
1.680813384038495230854190222632050379213389432395222242358419076313951655575 88\
77120681837852611103 06E-99
> // y-coordinates
> Abs(WeierstrassPDash([w1, w2], w1/2, 50));
6.692724015514553816405544279770415011482854826668085765557724361561660086006 328\
1676381838174025125113E-99
> Abs(WeierstrassPDash([w1, w2], w2/2, 50));
1.370873209236313498506941124173271409145236185788992096873002760788026391162 00\
9130596388077486470522E-99
> Abs(WeierstrassPDash([w1, w2], w/2, 50));
3.122595141716013610384293298845699563580815141409717877955675289079274207750 11\
1710974566400775635803E-99
```

Example 4.7.3. Let $K = \mathbb{Q}(\theta)$ where θ is a root of the polynomial $x^3 - 2$. Let E be the elliptic curve defined over K given by the Weierstrass equation

$$E: \quad Y^2 = 4(X-\theta)(X-1)(X+1+\theta).$$

Note that K has one real embedding and one conjugate pair of complex embeddings.

Let E_1, E_2 be the real and complex embedding of E respectively, i.e.,

$$\begin{aligned} E_1: &\quad Y^2 = 4(X - \sqrt[3]{2})(X-1)(X+1+\sqrt[3]{2}) \\ E_2: &\quad Y^2 = 4(X - \omega\sqrt[3]{2})(X-1)(X+1+\omega\sqrt[3]{2}) \end{aligned}$$

where $\omega = \exp(2\pi i/3)$ is a cube root of unity. Since E_1 has three real roots, then the period lattice of E_1 is rectangular. In fact, by letting $e_1^{(0)} = \sqrt[3]{2}, e_2^{(0)} = 1, e_3^{(0)} = -1 - \sqrt[3]{2}$, we can compute a_0, b_0, c_0 satisfying all the conditions in (4.15) as follows:

$$\begin{aligned} a_0 &= 1.87612422291002530767\ldots \\ b_0 &= 0.50982452853395859808\ldots \\ c_0 &= 1.80552514518487755254\ldots. \end{aligned}$$

One can then verify that $|c_0 - ib_0| = |c_0 + ib_0|$. As before, we compute

$$w = \frac{\pi}{M(c_0, ib_0)} = 2.90130425944817643666\ldots - i1.70677932803214980295\ldots$$
$$w' = \frac{\pi}{M(c_0, -ib_0)} = \bar{w},$$

and let $w_1, w_2 = (w \pm w')/2$. Then w_1, w_2 form an orthogonal $\mathbb{Z}$-basis for the period lattice of E_1. In this example, we have $w_1 = \Re(w)$ and $w_2 = i\Im(w)$.

Nevertheless, the period lattice of E_2 is non-rectangular, since all roots of E_2 are not collinear. In fact, by letting $e_1^{(0)} = -1 - \omega\sqrt[3]{2}, e_2^{(0)} = 1, e_3^{(0)} = \omega\sqrt[3]{2}$ (note that we use Proposition 4.5.2 here to ensure that a_0, b_0, c_0 satisfy all the conditions in (4.15)), we have

$$\begin{aligned} a_0 &= 1.10851094368231305521\ldots - i0.98431471713501219051\ldots \\ b_0 &= 0.43669517024285334726\ldots - i1.24929660083200513980\ldots \\ c_0 &= 1.34004098848655674756\ldots - i0.40712323180652750769\ldots. \end{aligned}$$

In fact, one can check that all the conditions in (4.15) are strictly inequalities, hence this also confirms that the period lattice of E_2 is non-rectangular. By Theorem 4.5.3, we finally obtain

$$\begin{aligned} w_1 &= 1.28194824894788708942\ldots + i1.88277404359595361782\ldots \\ w_2 &= 2.36557653380849535471\ldots - i0.03808700290170419307\ldots \\ w_3 &= -1.08362828486060826529\ldots + i1.92086104649765781090\ldots \end{aligned}$$

with $|w_1 - w_2 - w_3| \approx 10^{-100}$, as claimed by Proposition 4.5.4.

Example 4.7.4. Let E be the elliptic curve over $\mathbb{C}$ given by the Weierstrass equation $Y^2 = 4(X - e_1)(X - e_2)(X - e_3)$, where

$$e_1 = -1 - 3i, \quad e_2 = 3 + i, \quad e_3 = -2 + 2i.$$

Observe that $\sum_{j=1}^{3} e_j = 0$ and $|e_1 - e_3| = |e_2 - e_3|$. Thus e_1, e_2, e_3 form an isosceles triangle, as explained in Section 4.5.3. By letting $E_0 = E$ and computing a_0, b_0, c_0 satisfying all the conditions in (4.15) as before, we have

$$\begin{aligned} a_0 &= 1.74628455779589152702\ldots - i1.43161089573822132705\ldots \\ b_0 &= 0.91017972112445468260\ldots - i2.19736822693561993207\ldots \\ c_0 &= 2.24711142509587014360\ldots - i0.22250788030178260411\ldots. \end{aligned}$$

Hence by Theorem 4.5.3, we obtain

$$\begin{aligned} w_1 &= 0.81646689790312614904\ldots + i1.10773333340066743861\ldots \\ w_2 &= 1.36061503191563570645\ldots - i0.20595647167234558716\ldots \\ w_3 &= -0.54414813401250955741\ldots + i1.31368980507301302578\ldots \end{aligned}$$

with $|w_1 - w_2 - w_3| \approx 10^{-100}$, as claimed by Proposition 4.5.4. In addition, one can check that $\Re(w_1/w_3) = 1/2$ as claimed in Section 4.5.3. Let Λ be the period lattice of E. We finally verify that $|\wp_\Lambda(w_j/2) - e_j| \approx 10^{-100}$ for all $j = 1, 2, 3$, and

$$|\wp'_\Lambda(w_1/2)| \approx 10^{-99}, \quad |\wp'_\Lambda(w_2/2)| \approx 10^{-100}, \quad |\wp'_\Lambda(w_3/2)| \approx 10^{-99}.$$

Example 4.7.5. Let E be the elliptic curve over $\mathbb{C}$ given by the Weierstrass equation $Y^2 = 4(X - e_1)(X - e_2)(X - e_3)$, where

$$e_1 = -2 - 2i, \quad e_2 = -1 + 6i, \quad e_3 = 3 - 4i.$$

By Theorem 4.5.3, the period lattice Λ of E has the following minimal coset representatives; two of which form a $\mathbb{Z}$-basis for Λ:

$$\begin{aligned} w_1 &= 1.04665075729832942736\ldots + i0.45525281255263173893\ldots \\ w_2 &= 0.67791651620742852409\ldots - i0.77797238161544820221\ldots \\ w_3 &= 0.36873424109090090326\ldots + i1.23322519416807994115\ldots \end{aligned}$$

with $|w_1 - w_2 - w_3| \approx 10^{-101}$, which again agrees with Proposition 4.5.4.

Now we wish to find an elliptic logarithm of the point

$$P = (1 + i, \sqrt{12 + 492i}) = (1 + i, 15.8768\ldots + i15.4942\ldots) \in E(\mathbb{C}).$$

Letting $E_0 = E$ and computing u_0, v_0, we have

$$u_0^2 = x(P) - e_3^{(0)} = -2 + 5i, \quad v_0^2 = x(P) - e_2^{(0)} = 2 - 5i = -u_0^2.$$

Thus $|u_0 - v_0| = |u_0 + v_0|$. If we choose

$$v_0 = \sqrt{2 - 5i} = 1.9216\ldots - i1.3009\ldots,$$

then Algorithm 4.6.2 gives

$$z_P^{(1)} = -0.52013573443395982317\ldots + i0.13628275717388366013\ldots.$$

However, if we choose $-v_0$, then we obtain

$$z_P^{(2)} = 0.15778078177346870092\ldots - i0.64168962444156454207\ldots.$$

But then $z_P^{(2)} - z_P^{(1)} = w_2$. Hence both choices for v_0 are equally good for computing elliptic logarithms. Finally, we verify that

$$\left|x(P) - \wp_{\Lambda_0}(z_P^{(1)})\right| \approx 10^{-98}, \quad \left|y(P) - \wp'_{\Lambda_0}(z_P^{(1)})\right| \approx 10^{-98}.$$

In conclusion, we have presented a complete method, based on complex AGM, for computing period lattices of elliptic curves defined over $\mathbb{C}$, and generalised it into an algorithm for computing elliptic logarithms of complex points. As we can see from the above illustrative examples, this work, which is done in collaboration with Professor John E. Cremona, finally allows one to compute both quantities on any elliptic curves over $\mathbb{C}$, while such computations in the past were possible only for elliptic curves over $\mathbb{R}$. For more information on precise running time of complex AGM, see Dupont's thesis [Dup06] or his paper [Dup].

In the next chapter, we will bring all the main results we have obtained so far to illustrate their applications in assisting some essential computations in the arithmetic of elliptic curves over number fields.

Chapter 5

Applications

In this final chapter, we will illustrate the applications of all the main results we have obtained so far towards some computations in the arithmetic of elliptic curves over number fields, whose existing methods experienced some difficulties in the past due to lack of certain information on elliptic curves.

In this chapter, we will start by showing how to compute a lower bound for the canonical height (see Chapter 2 and 3) and use it to determine Mordell–Weil bases for elliptic curves over number fields. Then we will move on to demonstrate an algorithm of Smart and Stephens [SS97] for computing integral points on elliptic curves over number fields, which involves determining complex elliptic logarithms (see Chapter 4) of all generators of Mordell–Weil bases. Finally, we will conclude this chapter by illustrating some examples of finding all elliptic curves with everywhere good reduction based on the method of Cremona and Lingham [CL07], which requires integral points on elliptic curves of a certain type over number fields.

5.1 Height Bound III: Examples

In this section, we will show several illustrative examples on how to use Theorem 3.4.1 to determine a positive lower bound for the canonical height on elliptic curves over number fields. Note that our computation, which also involves real and complex elliptic logarithms (see Chapter 4), will be more sophisticated if the base number fields are not totally real. We have implemented our algorithm for computing the following examples in MAGMA; its source code can be found in Appendix A.3. For a brief demonstration of how to use our program, see Example 5.1.4.

5.1.1 Case I: Totally Real Number Fields

We will first concentrate on the case when our elliptic curves are defined over totally real number fields. As we will see, this will require periods of elliptic curves over $\mathbb{R}$ and elliptic logarithms of real points, which can be obtained by Algorithm 4.6.2 or Cohen's algorithms [Coh93, Algorithm 7.4.7 and 7.4.8]. For the relevant notations, the reader should refer to Chapter 2.

Example 5.1.1. Let $E = E_1$, where E_1 is the elliptic curve defined over $K = \mathbb{Q}(\sqrt{2})$ given by the Weierstrass equation

$$E_1 : \quad y^2 = x^3 + x + (1 + 2\sqrt{2}).$$

The discriminant Δ of E is $-3952 - 1728\sqrt{2}$. Moreover, we have $\langle \Delta \rangle = \mathfrak{p}_1^8 \mathfrak{p}_2^2 \mathfrak{p}_3$, where

$$\mathfrak{p}_1 = \langle \sqrt{2} \rangle, \quad \mathfrak{p}_2 = \langle 7, 3 + \sqrt{2} \rangle, \quad \mathfrak{p}_3 = \langle 769, 636 + \sqrt{2} \rangle,$$

are prime ideals. Since $\mathrm{ord}_{\mathfrak{p}_j}(\Delta) < 12$ for all j, then E is given by a globally minimal model, and so $M_E = 1$.

As explained in Section 3.4, our algorithm, based on Theorem 3.4.1, will start by checking whether a given $\mu > 0$ is a lower bound for the canonical height on $E_{\mathrm{gr}}(K)$ by computing $B_n(\mu)$ for $n = 1, \ldots, n_{\max}$. If $B_n(\mu) < 1$ for some n, then μ is indeed a lower bound. Otherwise, we proceed to compute $\bigcap_{n=1}^{n_{\max}} \mathcal{S}_n^{(v)}(-B_n(\mu), B_n(\mu))$ for every real archimedean place $v \in M_K^{\mathrm{r}}$ (here, we do not have to compute any $\mathcal{T}_n^{(v)}$, since K is totally real). If the intersection is empty for some v, then μ is a lower bound. Note that we obtain no conclusion if none of the intersections is empty.

In this example, we define v_+, v_- to be the real archimedean place of K whose associated real embedding sends $\sqrt{2} \mapsto \pm 1.414\ldots$ respectively. By letting $\mu = 1$ and $n_{\max} = 5$, we have

$$B_1(\mu) = 8.117389, \quad B_2(\mu) = 8.186971 \times 10^2, \quad B_3(\mu) = 7.213201 \times 10^7,$$
$$B_4(\mu) = 5.421641 \times 10^{12}, \quad B_5(\mu) = 5.685757 \times 10^{21}.$$

Since none of these is less than 1, we have to compute $\bigcap_{n=1}^{n_{\max}} \mathcal{S}_n^{(v)}(-B_n(\mu), B_n(\mu))$ for every $v \in M_K^{\mathrm{r}}$. Recall from Section 2.5 that $\mathcal{S}_n^{(v)}(\xi_1, \xi_2)$ is defined in terms of $\psi_v(\xi_1), \psi_v(\xi_2)$, where $\psi_v : E_0^{(v)}(\mathbb{R}) \to [1/2, 1)$ is the *normalised elliptic logarithm* of the "higher" of the two points on $E_0^{(v)}$ with the same x-coordinate. For $v = v_+$, one can check that the corresponding real embedding $E^{(v)}$ has only one real root at $\beta_v = -1.352786$. Using Algorithm 4.6.2, we have (after normalisation)

$$\psi_v(B_1(\mu)) = 0.891227,$$

which yields[1] $\mathcal{S}_1^{(v)}(-B_1(\mu), B_1(\mu)) = [0.108773, 0.891227]$.

Computing $\mathcal{S}_n^{(v)}(-B_n(\mu), B_n(\mu))$ for all $n = 2, \ldots, n_{\max}$ in a similar way, we will eventually see that $\bigcap_{n=1}^{n_{\max}} \mathcal{S}_n^{(v)}(-B_n(\mu), B_n(\mu)) \neq \emptyset$. A similar procedure also shows that another intersection associated to $v = v_-$ is non-empty. Hence we fail to show that $\mu = 1$ is a lower bound on $E_{\mathrm{gr}}(K)$, in which case we shall repeat the above computation with a smaller μ (and/or a larger $n_{\max}$). On the other hand, if μ is known to be a lower bound, then we can repeat such process with a larger μ to see if it is still a lower bound. This refinement can be done repeatedly as required.

After a number of refinements as shown in Table 5.1, our algorithm finally shows that

$$\hat{h}(P) > \mu = 0.2415$$

[1]Only $\psi_v(B_1(\mu))$ is required in this case, since $-B_1(\mu) < \beta_v < B_1(\mu)$.

Table 5.1: Illustration of the algorithm for Example 5.1.1

Initial μ	Initial $n_{\max}$	Is any $B_n(\mu) < 1$?	Is any intersection empty?	Is μ a lower bound?	Next μ	Next $n_{\max}$
1.0000	5	No	No	Fail	0.5000	6
0.5000	6	No	No	Fail	0.2500	7
0.2500	7	No	No	Fail	0.1250	8
0.1250	8	Yes	Skipped	Yes	0.1875	8
0.1875	8	No	Yes	Yes	0.2187	8
0.2187	8	No	Yes	Yes	0.2343	8
0.2343	8	No	Yes	Yes	0.2421	8
0.2421	8	No	No	Fail	0.2382	9
0.2382	9	No	Yes	Yes	0.2402	9
0.2402	9	No	Yes	Yes	0.2412	9
0.2412	9	No	Yes	Yes	0.2416	9
0.2416	9	No	No	Fail	0.2414	10
0.2414	10	No	Yes	Yes	0.2415	10
0.2415	10	No	No	Fail	0.2415	11
0.2415	11	No	Yes	Yes		

for all non-torsion points $P \in E_{\mathrm{gr}}(K)$. Nevertheless, the lower bound for $E_{\mathrm{gr}}(K)$ derived from Theorem 2.4.2 is not as good as this one. In this example, we have

$$\alpha_{v_+} = 1.096562, \quad \alpha_{v_-} = 1.001830,$$

and so $\alpha_{v_+}\alpha_{v_-} = 1.098569$. We now choose a prime ideal $\mathfrak{p}$ whose norm is greater than $\sqrt{\alpha_{v_+}\alpha_{v_-}}$, and set $n = e_{\mathfrak{p}}$. To minimise n, we choose $\mathfrak{p} = \langle\sqrt{2}\rangle$ to obtain $n = e_{\mathfrak{p}} = 2$. Then we have $D_E(2) = 1.386294$, which finally yields the lower bound

$$\mu_0 = \frac{D_E(n) - \log(\alpha_{v_+}\alpha_{v_-})}{[K:\mathbb{Q}]n^2} = \frac{1.386294 - \log(1.098569)}{8} = 0.1615.$$

In order to obtain a lower bound for the canonical height on $E(K)$, we first note that the Tamagawa indices c_v at $v = \mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3$ are 4, 2, and 1 respectively. Moreover, one can easily see that both real embeddings of E have only real root, so $c_{v_+} = c_{v_-} = 1$. Hence $c = \mathrm{lcm}\{4, 2, 1\} = 4$. By Lemma 2.1.1, we finally have

$$\hat{h}(P) > \mu/c^2 = 0.2415/4^2 = 0.0150$$

for all non-torsion points $P \in E(K)$.

Table 5.2: Illustration of the algorithm for Example 5.1.2

Initial μ	Initial $n_{\max}$	Is any $B_n(\mu) < 1$?	Is any intersection empty?	Is μ a lower bound?	Next μ	Next $n_{\max}$
1.0000	5	No	No	Fail	0.5000	6
0.5000	6	No	No	Fail	0.2500	7
0.2500	7	No	No	Fail	0.1250	8
0.1250	8	No	Yes	Yes	0.1875	8
0.1875	8	No	No	Fail	0.1562	9
0.1562	9	No	No	Fail	0.1406	10
0.1406	10	No	Yes	Yes	0.1484	10
0.1484	10	No	No	Fail	0.1445	11
0.1445	11	No	No	Fail	0.1425	12
0.1425	12	No	No	Fail	0.1416	13
0.1416	13	No	No	Fail	0.1411	14
0.1411	14	No	Yes	Yes	0.1413	14
0.1413	14	No	Yes	Yes	0.1414	14
0.1414	14	No	Yes	Yes	0.1415	14
0.1415	14	No	Yes	Yes		

Example 5.1.2. Let $E = E_2$, where E_2 is the elliptic curve defined over $K = \mathbb{Q}(\sqrt{7})$ given by the Weierstrass equation

$$E_2 : \quad y^2 + (3 + 3\sqrt{7})xy + y = x^3 + (26 + 4\sqrt{7})x^2 + x.$$

The discriminant Δ of E is $-937513 - 299394\sqrt{7}$. Moreover, $\langle\Delta\rangle$ can be factorised into a product of prime ideals as $\mathfrak{p}_1\mathfrak{p}_2\mathfrak{p}_3$, where

$$\mathfrak{p}_1 = \langle 4219, 1083 + \sqrt{7}\rangle, \quad \mathfrak{p}_2 = \langle 4657, 3544 + \sqrt{7}\rangle, \quad \mathfrak{p}_3 = \langle 12799, 5358 + \sqrt{7}\rangle.$$

Again, since $\operatorname{ord}_{\mathfrak{p}_j}(\Delta) < 12$ for all j, E is already given by a globally minimal model, and thus $M_E = 1$. Our algorithm shows that

$$\hat{h}(P) > 0.1415$$

for all non-torsion points $P \in E_{\mathrm{gr}}(K)$. This is obtained after a number of refinements as shown in Table 5.2.

Finally, we note that the Tamagawa indices c_v at $v = \mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3$ are all 1. In addition, since both real embeddings of E have three real roots, we have $c_v = 2$ for both $v \in M_K^{\mathrm{r}}$, and so $c = 2$. Hence by Lemma 2.1.1, we have

$$\hat{h}(P) > 0.1415/2^2 = 0.0353$$

for all non-torsion points $P \in E(K)$.

Example 5.1.3. Let $E = E_3$, where E_3 is the elliptic curve defined over $K = \mathbb{Q}(\sqrt{10})$ given by a Weierstrass equation

$$E_3 : \quad y^2 = x^3 + 125.$$

Note that K has class number 2. Decomposing the discriminant Δ of E into prime ideals, it can be seen that $\langle \Delta \rangle = \langle -2^4 3^3 5^6 \rangle = \mathfrak{p}_1^{12}\mathfrak{p}_2^3\mathfrak{p}_3^3\mathfrak{p}_4^8$, where

$$\mathfrak{p}_1 = \langle 5, \sqrt{10} \rangle, \quad \mathfrak{p}_2 = \langle 3, 4 + \sqrt{10} \rangle, \quad \mathfrak{p}_3 = \langle 3, 2 + \sqrt{10} \rangle, \quad \mathfrak{p}_4 = \langle 2, \sqrt{10} \rangle.$$

Observe that the model of E is now minimal everywhere except at $\mathfrak{p}_1$. With the substitutions

$$x = (\sqrt{10})^2 x', \quad y = (\sqrt{10})^3 y',$$

we have a new elliptic curve $E' : y'^2 = x'^3 + 1/8$. This time, however, the model of E' is minimal everywhere except at all prime ideals dividing 2. Thus we let $E^{(\mathfrak{p}_1)} = E'$ and $E^{(\mathfrak{p})} = E$ for any $\mathfrak{p} \neq \mathfrak{p}_1$ in our computation. Our algorithm then shows that

$$\hat{h}(P) > 0.2859$$

for all non-torsion points $P \in E_{\mathrm{gr}}(K)$. This is based on a number of refinements as shown in Table 5.3.

To derive a lower bound on $E(K)$, we first note that the Tamagawa indices c_v at $v = \mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3, \mathfrak{p}_4$ are 1, 2, 2, and 1 respectively. Moreover, we have $c_v = 1$ for both $v \in M_K^{\mathrm{r}}$, since both real embeddings of E have only one real root. Hence $c = 2$,

Table 5.3: Illustration of the algorithm for Example 5.1.3

Initial μ	Initial $n_{\max}$	Is any $B_n(\mu) < 1$?	Is any intersection empty?	Is μ a lower bound?	Next μ	Next $n_{\max}$
1.0000	5	No	No	Fail	0.5000	6
0.5000	6	No	No	Fail	0.2500	7
0.2500	7	Yes	Skipped	Yes	0.3750	7
0.3750	7	No	No	Fail	0.3125	8
0.3125	8	No	No	Fail	0.2812	9
0.2812	9	Yes	Skipped	Yes	0.2968	9
0.2968	9	No	No	Fail	0.2890	10
0.2890	10	No	No	Fail	0.2851	11
0.2851	11	Yes	Skipped	Yes	0.2871	11
0.2871	11	No	No	Fail	0.2861	12
0.2861	12	No	No	Fail	0.2856	13
0.2856	13	Yes	Skipped	Yes	0.2858	13
0.2858	13	Yes	Skipped	Yes	0.2860	13
0.2860	13	No	No	Fail	0.2859	14
0.2859	14	Yes	Skipped	Yes		

and thus by Lemma 2.1.1,

$$\hat{h}(P) > 0.2859/2^2 = 0.0714$$

for all non-torsion points $P \in E(K)$.

5.1.2 Case II: Number Fields with Complex Embeddings

Next, we will consider the case when our elliptic curves are defined over non-totally real number fields. The reader should refer to Chapter 3 for the relevant notations.

Let K be a non-totally real number field (i.e., one having non-real complex embeddings), and let E be an elliptic curve defined over K. It can be seen from Theorem 3.4.1 that, in order to determine a positive lower bound for the canonical height on $E_{\mathrm{gr}}(K)$, we may have to compute $\bigcap_{n=1}^{n_{\max}} \mathcal{T}_n^{(v)}(\sqrt{B_n(\mu)})$ for every complex archimedean place $v \in M_K^{\mathrm{c}}$, in addition to $\bigcap_{n=1}^{n_{\max}} \mathcal{S}_n^{(v)}(-B_n(\mu), B_n(\mu))$ for every $v \in M_K^{\mathrm{r}}$. To obtain $\mathcal{T}_n^{(v)}(\xi)$, we first need to construct the approximate corresponding region $\mathcal{S}^{(v)}(\xi)$. Assume that for each $v \in M_K^{\mathrm{c}}$, the associated complex embedding

$E^{(v)}$ of E is of the form

$$E^{(v)} : Y^2 = 4X^3 + AX + B$$

for some $A, B \in \mathbb{C}$. Then it can be seen from Section 3.1.2 that the definition of $\mathcal{S}^{(v)}(\xi)$ requires the quantity

$$U_\xi = |w_1|^2 \left(\xi + \frac{b_2}{12} \right),$$

where b_2 is an invariant as defined in Chapter 1 for $E^{(v)}$, and $w_1 \in \mathbb{C}$ is one of the two vectors forming a $\mathbb{Z}$-basis for the period lattice Λ of $E^{(v)}$, in such a way that $\Lambda = \langle w_1, w_2 \rangle$ and $\tau = w_2/w_1$ satisfies (3.1), i.e.,

$$|\tau| \geq 1, \quad |\Re(\tau)| \leq 1/2, \quad \Im(\tau) \geq \sqrt{3}/2.$$

One can then use Theorem 4.5.3 (together with some linear transformation if necessary) to obtain $\Lambda = \langle w_1, w_2 \rangle$ whose τ satisfies (3.1).

Furthermore, one can see from Section 3.2.3 that construction of $\mathcal{S}^{(v)}(\xi)$ requires a parallelogram C_0 containing an elliptic logarithm of a point $P \in E^{(v)}(\mathbb{C})$ with $X(P) = 0$. Although one can use Algorithm 4.6.2 to compute an elliptic logarithm of P, it should be noted that this is rarely required in practice, since C_0 is normally obtained as one of the parallelograms C satisfying $I(C) \cap [0, U_\xi] \neq \emptyset$.

We will now illustrate our algorithm for elliptic curves defined over non-totally real number fields with the following examples. For the rest of this chapter, we shall let $i = \sqrt{-1}$.

Example 5.1.4. Let $E = E_4$, where E_4 is the elliptic curve defined over $K = \mathbb{Q}(i)$ given by the Weierstrass equation

$$E_4 : \quad y^2 = x^3 + (91 - 26i)x - (144 + 323i).$$

Table 5.4: Illustration of the algorithm for Example 5.1.4

μ	$n_{\max}$	Is any $B_n(\mu) < 1$?	Is any $\bigcap \mathcal{T}_n^{(v)}$ empty?	Is μ a lower bound?
0.20	4	No	No	Fail
0.10	4	No	Yes	Yes
0.15	4	No	Yes	Yes
0.18	4	No	Yes	Yes

The discriminant of E can be factorised into a product of prime ideals as $\mathfrak{p}_1\mathfrak{p}_2\mathfrak{p}_3^8$, where

$$\mathfrak{p}_1 = \langle 799 + 1124i\rangle, \quad \mathfrak{p}_2 = \langle 7 - 12i\rangle, \quad \mathfrak{p}_3 = \langle 1 + i\rangle.$$

Hence the model of E is globally minimal, and so $M_E = 1$. Based on a number of refinements as shown in Table 5.4, our algorithm shows that

$$\hat{h}(P) > 0.18$$

for all non-torsion $P \in E_{\mathrm{gr}}(K)$. Note that in this example we only have to compute $\mathcal{T}_n^{(v)}(\sqrt{B_n(\mu)})$ but not $\mathcal{S}_n^{(v)}(-B_n(\mu), B_n(\mu))$, since K has no real embedding. In addition, we choose $\mathcal{S}^{(v)} = \mathcal{S}^{(v,4)}$ for $v \in M_K^{\mathrm{c}}$.

It can be checked that the Tamagawa indices c_v of E at $v = \mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3$ are all 1. Moreover, we have $c_v = 1$ where v is the only complex archimedean place of K. Hence $c = 1$, and by Lemma 2.1.1,

$$\hat{h}(P) > 0.18$$

for all non-torsion $P \in E(K)$. One can verify the above results using our MAGMA code (note that we require `elog.m` and every file mentioned in Appendix A.3 onwards) together with the following instructions:

```
> // Note that all of these files are required
> Attach("elog.m");
> Attach("alphas.m");
> Attach("heightbound.m");
> Attach("intersect_real.m");
> Attach("intersect_complex.m");
> Attach("interval_arith.m");
> Attach("interval_wp.m");
> Attach("wp.m");
> // SetVerbose("Bound", 1); // enable this line to see more details
> // Define elliptic curve E
> _<x> := PolynomialRing(Integers());
> K<i> := NumberField(x^2+1);
> E := EllipticCurve([91-26*i, -144-323*i]);
> // Check if 0.2 is a lower bound on E(K)
> // (to get a lower bound on E_{gr}(K), multiply by a square of lcm of
> // all Tamagawa indices)
> IsLowerBound(E, 0.2 : n_max := 4);
false
> // Fail to show that 0.2 is a lower bound, so try something smaller
> IsLowerBound(E, 0.1 : n_max := 4);
true
> // So 0.15 is a lower bound, try to check a bigger lower bound
> IsLowerBound(E, 0.15 : n_max := 4);
true
> IsLowerBound(E, 0.18 : n_max := 4);
true
```

On the other hand, the lower bound on $E_{\mathrm{gr}}(K)$ (and also $E(K)$ in this case) obtained by Theorem 2.4.2 is not as good as this one. In this example, we have

$$\alpha_v = 4.715889.$$

Choose a prime ideal $\mathfrak{p}$ with $\mathcal{N}(\mathfrak{p}) > \alpha_v$, say, $\mathfrak{p} = \langle 5, 2+i \rangle$, and set $n = e_{\mathfrak{p}} = 5$. Then we have $D_E(5) = 3.218876$, which yields the lower bound

$$\mu_0 = \frac{3.218876 - 2\log(4.715889)}{2 \cdot 5^2} = 2.34 \times 10^{-3}.$$

Finally, one can verify that the lower bound obtained by Hindry and Silverman [HS88, Theorem 0.3] is

$$\hat{h}(P) \geq 3.0624 \times 10^{-25}$$

for all non-torsion $P \in E(K)$. We leave it to the reader to compare the results.

Example 5.1.5. The following elliptic curve is from Cremona's paper [Cre94, Example 2]. Let $E = E_5$, where E_5 is the elliptic curve defined over $K = \mathbb{Q}(i)$ given by the Weierstrass equation

$$E_5 : \quad y^2 + iy = x^3 + (1-i)x^2 - ix.$$

One can easily observe that $P_0 = (0,0) \in E(K)$. Let Δ be the discriminant of E. Then we have $\langle \Delta \rangle = \mathfrak{p}$, where $\mathfrak{p} = \langle 13 + 8i \rangle$ is prime. Moreover, we have the Tamagawa index $c_{\mathfrak{p}} = 1$, and also $c_v = 1$ where v is the only complex archimedean place of K. Hence $c = 1$. Using the fact that $\hat{h}(P_0) = 0.0230$, we set our initial guess μ to be smaller than 0.0230, say, $\mu = 0.01$. Our algorithm shows that

$$B_5(\mu) = 0.7772 < 1.$$

Thus by Proposition 2.4.1, $\hat{h}(P) > 0.01$ for all non-torsion $P \in E_{\mathrm{gr}}(K)$. Since $c = 1$, we also have $\hat{h}(P) > 0.01$ for all non-torsion $P \in E(K)$ by Lemma 2.1.1.

Example 5.1.6. Let $K = \mathbb{Q}(\theta)$ where θ is a root of the polynomial $x^3 - 2$. Let $E = E_6$, where E_6 is the elliptic curve defined over K given by the Weierstrass equation

$$E_6 : \quad y^2 = x^3 - (\theta^2 + 3\theta)x + \theta^2.$$

Let Δ be the discriminant of E. The prime ideal factorisation of $\langle \Delta \rangle$ is $\mathfrak{p}_1^{16}\mathfrak{p}_2$, where

$$\mathfrak{p}_1 = \langle 2, \theta \rangle, \quad \mathfrak{p}_2 = \langle 390433, 218056 + \theta \rangle.$$

It can be verified that the model of E is globally minimal, and so $M_E = 1$. Our algorithm shows that

$$\hat{h}(P) > 0.25$$

for all non-torsion $P \in E_{\mathrm{gr}}(K)$, which is obtained after a number of refinements as shown in Table 5.5. Recall that if $\bigcap \mathcal{S}_n^{(v)} = \emptyset$ for some $v \in M_K^{\mathrm{r}}$, then μ is a lower bound and so there is no need to compute $\bigcap \mathcal{T}_n^{(v)}$ for any $v \in M_K^{\mathrm{c}}$.

Finally, we note that the Tamagawa indices c_v at $v = \mathfrak{p}_1, \mathfrak{p}_2$ are 2 and 1 respectively. Moreover, since E has only one real embedding, say, $E^{(v_1)}$ with three real roots, we have $c_{v_1} = 2$. Denote the only complex archimedean place of K by v_2.

Table 5.5: Illustration of the algorithm for Example 5.1.6

μ	$n_{\max}$	Is any $B_n(\mu) < 1$?	Is any $\bigcap \mathcal{S}_n^{(v)}$ empty?	Is any $\bigcap \mathcal{T}_n^{(v)}$ empty?	Is μ a lower bound?
0.50	3	No	No	No	Fail
0.20	3	No	Yes	Skipped	Yes
0.30	3	No	No	No	Fail
0.25	3	No	Yes	Skipped	Yes

Then again $c_{v_2} = 1$, and so $c = \operatorname{lcm}\{1, 2\} = 2$. Thus by Lemma 2.1.1, we have

$$\hat{h}(P) > 0.25/2^2 = 0.0625$$

for all non-torsion $P \in E(K)$. Note that we have obtained no additional information from the complex place in this specific example; however, there is no reason to suppose that this would be the case in general.

In the next section, we will explain how to use a lower bound for the canonical height to derive Mordell–Weil bases for elliptic curves defined over number fields. This method will be illustrated when we revisit all the examples recently shown.

5.2 Mordell–Weil Bases

Computing Mordell–Weil bases for elliptic curves over number fields is one of the most difficult computations in the arithmetic of elliptic curves, and so far there is no known procedure which can determine such a basis in general. In this section, we will illustrate an application of a lower bound for the canonical height in assisting such computation. For more background on this section, see Section 1.2.2 or [Cre97, Section 3.5].

Let E be an elliptic curve defined over a number field K. Recall from Section 1.2.2 that, given some non-torsion points $P_1, \ldots, P_r \in E(K)$ whose images in $E(K)/E_{\mathrm{tors}}(K)$ generate a subgroup of finite index of $E(K)/E_{\mathrm{tors}}(K)$, it is possible to "saturate" these points (which are normally obtained by performing an

m-descent for some $m \geq 2$) to obtain a full Mordell–Weil basis for $E(K)$. The saturation process consists of the following steps:

1. Determine an upper bound ℓ for the index $n = [E(K)/E_{\text{tors}}(K) : \langle P_1, \ldots, P_r \rangle]$ using the geometry of numbers (Theorem 1.2.1), which then requires a positive lower bound for the canonical height on $E(K)$ obtained by Theorem 3.4.1.

2. For each prime $p \leq \ell$, determine whether $p \mid n$, or equivalently, whether there exists $a_1, \ldots, a_r \in \mathbb{Z}$, not all divisible by p, such that

$$\sum_{j=1}^{r} a_j P_j = pQ \tag{5.1}$$

   for some $Q \in E(K)$. Without loss of generality, we can assume that $|a_j| \leq p/2$.

3. If there exists a solution to (5.1), let a_i be the minimal non-zero coefficient (in absolute value). If $a_i = \pm 1$, then we can simply replace P_i by Q; otherwise, we find a coefficient a_j not divisible by a_i. Write $a_j = a_i q + b$ with $0 < b < |a_i|$. Observe that

$$a_i P_i + a_j P_j = a_i(P_i + qP_j) + bP_j.$$

   This then allows us to replace the generator P_i by $P_i + qP_j$, replace a_j by b, and replace i by j. This time, the index of the sublattice generated by the new $P_1, \ldots, P_r$ in $E(K)$ will be at most $\ell |a_i|/p$.

4. Repeat the above steps until the index n is not divisible by any primes. The final set $\{P_1, \ldots, P_r\}$ will be a Mordell–Weil basis for $E(K)$ modulo torsion.

Nevertheless, the upper bound ℓ obtained by Theorem 1.2.1 can be very large even though the points $P_1, \ldots, P_r$ may already form a Mordell–Weil basis, and so there can be too many primes p to consider. Fortunately, it is possible to quickly eliminate some of p from our consideration before we actually have to solve (5.1).

5.2.1 Sieving Procedure

This procedure was initially explained in [Sik95, Section 4], and has been described in full details later by Prickett in his thesis [Pri04]. For convenience, we shall give a summary here.

For a given prime $p \leq \ell$, let $P_{r+1}, \dots, P_{r+s}$ be a basis for $E_{\text{tors}}(K)/pE_{\text{tors}}(K)$. Our aim now is to determine the set

$$V_p = \{\mathbf{a} \in \mathbb{F}_p^{r+s} : \sum_{j=1}^{r+s} a_j P_j \in pE(K)\}.$$

It can be seen easily that the index n is divisible by p if and only if $V_p \neq \{\mathbf{0}\}$. We choose a prime ideal $\mathfrak{p}$ such that E is minimal at $\mathfrak{p}$, and satisfies the following:

1. E has good reduction at $\mathfrak{p}$;
2. $\#E(k_\mathfrak{p})$ is divisible by p, but not p^2 (Here, $k_\mathfrak{p}$ is the residue class field).

Write $\#E(k_\mathfrak{p}) = lp$. Clearly, $p \nmid l$ due to the choice of $\mathfrak{p}$.

Let π be a uniformiser at $\mathfrak{p}$. Now for each P_j, we compute $P_j' \equiv lP_j \pmod{\pi}$ for all $j = 1, \dots, r+s$. If $P_j' \equiv O \pmod{\pi}$ for every j, then this yields no information, and so we should start with a new prime $\mathfrak{p}$ satisfying the above conditions. Otherwise, there exists a point, say, $P_1' \not\equiv O \pmod{\pi}$. The condition (2) then ensures that $lE(k_\mathfrak{p})$ is a cyclic group of order p. Thus for all $j = 1, \dots, r+s$, we have

$$P_j' \equiv m_j P_1' \pmod{\pi}$$

for some $m_j \in \mathbb{Z}$. It then follows that if $\mathbf{a} \in V_p$, then $\mathbf{a}$ satisfies the relation

$$\sum_{j=1}^{r+s} m_j a_j \equiv 0 \pmod{p}.$$

By solving all of these $r+s$ relations over $\mathbb{F}_p$, we eventually reduce V_p into a smaller set. In particular, if such $r+s$ relations are independent, then $V_p = \{\mathbf{0}\}$, and so the index n is not divisible by p.

It should be noted, however, that this method may sometimes fail to prove that a point is p-saturated even though it actually is. For more details on modification of this method, see [Pri04].

5.2.2 Examples Revisited

We will now revisit all the elliptic curves shown in Section 5.1 and illustrate how to obtain their Mordell–Weil bases using a lower bound for the canonical height together with the sieving procedure.

Example 5.2.1. Let $K = \mathbb{Q}(\sqrt{2})$ and let $E = E_1$ as defined earlier. In Example 5.1.1, we have obtained from our algorithm that $\hat{h}(P) > 0.0150$ for all non-torsion $P \in E(K)$. In fact, one can check that the torsion subgroup of $E(K)$ is trivial. We now wish to determine whether $E(K) = \langle P_1 \rangle$, where

$$P_1 = (1, 1+\sqrt{2}) \in E(K).$$

Using MAGMA, we know that $\hat{h}(P_1) = 0.5033$, and the rank of $E(K)$ is at most 1. Hence $E(K)$ has rank 1. By Theorem 1.2.1, we have

$$n = [E(K) : \langle P_1 \rangle] \leq \sqrt{0.5033/0.0150} = 5.7927.$$

It therefore remains to check whether the index n is divisible by any primes $p \leq 5$. Note that this upper bound can be computed using our MAGMA function `UpperBound4Index()` in the file `heightbound.m` (see Appendix A.3) as follows:

```
> Attach("heightbound.m");
> // Define elliptic curve E
> _<x> := PolynomialRing(Integers());
> K<a> := NumberField(x^2-2);
> E := EllipticCurve([1, 1+2*a]);
> P1 := E![1,1+a];
> // Use 0.0150 as a lower bound for the canonical height on E(K)
> UpperBound4Index([P1], 0.0150);
5.79270969603967816405459250
```

In this example, one can easily check using division polynomial that $P_1 \notin pE(K)$ for all $p = 3, 5$, so n must be 1, 2, or 4. A simple observation also shows that $P_1 = 2P$ where

$$P = (1 - \sqrt{2}, 1 - 2\sqrt{2}) \in E(K),$$

and $P \neq 2Q$ for any $Q \in E(K)$. Hence we have $[E(K) : \langle P \rangle] = 1$, i.e., P generates $E(K)$.

Example 5.2.2. Let $K = \mathbb{Q}(\sqrt{7})$ and let $E = E_2$ as defined earlier. In Example 5.1.2, we have shown that $\hat{h}(P) > 0.0353$ for all non-torsion $P \in E(K)$. Again, one can check that the torsion subgroup of $E(K)$ is trivial, and the points

$$P_1 = (0, 0), \quad P_2 = (1, \sqrt{7})$$

are in $E(K)$. We wish to show that $E(K) = \langle P_1, P_2 \rangle$.

Using MAGMA, one can see that

$$\hat{h}(P_1) = 0.8051, \quad \hat{h}(P_2) = 1.4957.$$

By computing the height pairing matrix, we have

$$R(P_1, P_2) = \det \begin{pmatrix} \langle P_1, P_1 \rangle & \langle P_1, P_2 \rangle \\ \langle P_2, P_1 \rangle & \langle P_2, P_2 \rangle \end{pmatrix} = \begin{vmatrix} 0.8051 & -0.1941 \\ -0.1941 & 1.4957 \end{vmatrix} = 1.1665 \neq 0.$$

Hence P_1 and P_2 are independent. From MAGMA, we also know that the rank of $E(K)$ is at most 2. Hence $E(K)$ has rank 2. By Theorem 1.2.1, we finally have

$$n = [E(K) : \langle P_1, P_2 \rangle] \leq \frac{2\sqrt{1.1665}}{0.0353 \cdot \sqrt{3}} = 35.2450.$$

Thus we shall apply the sieving procedure for all primes $p \leq 31$.

Using a similar argument as in the previous example, we deduce from sieving

Table 5.6: Sieving procedure for the elliptic curve E_2

p	$\mathfrak{p}$	$\#E(k_{\mathfrak{p}})$	l	(m_1, m_2)	(a_1, a_2)
2	$\langle 17 \rangle$	318	159	$(1,2)$	$(0,0),(0,1)$
3	$\langle 3, 1+\sqrt{7} \rangle$	6	2	$(1,2)$	$(0,0)$
	$\langle 5 \rangle$	24	8	$(1,3)$	
5	$\langle 2, 1+\sqrt{7} \rangle$	5	1	$(1,3)$	$(0,0)$
	$\langle 11 \rangle$	115	23	$(1,4)$	
7	$\langle 19, 8+\sqrt{7} \rangle$	21	3	$(1,7)$	$(0,0)$
	$\langle 29, 6+\sqrt{7} \rangle$	35	5	$(1,1)$	
11	$\langle 47, 30+\sqrt{7} \rangle$	44	4	$(1,5)$	$(0,0)$
	$\langle 113, 32+\sqrt{7} \rangle$	99	9	$(1,8)$	
13	$\langle \sqrt{7} \rangle$	13	1	$(1,6)$	$(0,0)$
	$\langle 103, 78+\sqrt{7} \rangle$	91	7	$(1,8)$	
17	$\langle 29, 23+\sqrt{7} \rangle$	34	2	$(1,2)$	$(0,0),(1,8),(2,-1),$ $(3,7),(4,-2),(5,6),$ $(6,-3),(7,5),(8,-4)$
19	$\langle 31, 21+\sqrt{7} \rangle$	38	2	$(1,11)$	$(0,0)$
	$\langle 37, 9+\sqrt{7} \rangle$	38	2	$(1,7)$	
23	$\langle 11 \rangle$	115	5	$(1,4)$	$(0,0)$
	$\langle 337, 119+\sqrt{7} \rangle$	322	14	$(1,17)$	
29	$\langle 103, 25+\sqrt{7} \rangle$	116	4	$(1,8)$	$(0,0)$
	$\langle 149, 56+\sqrt{7} \rangle$	145	5	$(1,25)$	$(0,0)$
31	$\langle 137, 12+\sqrt{7} \rangle$	155	5	$(1,2)$	$(0,0)$
	$\langle 139, 110+\sqrt{7} \rangle$	155	5	$(1,5)$	$(0,0)$

that $V_p = \{\mathbf{0}\}$ for every $p \leq 31$ except for $p = 2, 17$. For each $p \leq 31$, the choice of $\mathfrak{p}$ and their corresponding (m_1, m_2) is shown in Table 5.6. For $p = 2$, the sieving method yields a possible set for V_2. To be precise, we have to check whether $(0,1) \in V_2$, i.e. if there exists a point $Q \in E(K)$ such that

$$P_2 = (1, \sqrt{7}) = 2Q.$$

Using 2-division polynomial, it turns out that there is no such $x(Q) \in K$ which satisfies the polynomial. Hence Q does not exist, and so $V_2 = \{\mathbf{0}\}$.

It still remains to find V_{17}. In this case, it suffices to check only one pair of (a_1, a_2), say, $(a_1, a_2) = (1, 8)$. Again, by division polynomial, one can eventually show that $P_1 + 8P_2 \neq 17Q$ for any $Q \in E(K)$. Thus $n = 1$, i.e., $E(K) = \langle P_1, P_2 \rangle$.

For large p, note that the p-division polynomial technique may become very inefficient due to the difficulty in finding all roots of a polynomial of degree p^2. As suggested in [Sik95, Section 4.2], it is perhaps more practical to solve (5.1) by computing elliptic logarithms of all $P^{(v)} \in E^{(v)}$, where $E^{(v)}$ is the embedding of E associated to an archimedean place $v \in M_K$, and $P^{(v)}$ is the image of P on $E^{(v)}$.

Example 5.2.3. Let $K = \mathbb{Q}(\sqrt{10})$ and let $E = E_3$ as defined earlier. Note that $E_{\text{tors}}(K)$ is a cyclic group of order 2 generated by the point $T = (-5, 0) \in E(K)$. We have shown in Example 5.1.3 that $\hat{h}(P) > 0.0714$ for all non-torsion $P \in E(K)$. Let $P_1 = (5, 5\sqrt{10}) \in E(K)$. From MAGMA, we know that $\hat{h}(P_1) = 0.6532$, and the rank of $E(K)$ is at most 1. Hence $E(K)$ has rank 1. By Theorem 1.2.1, we have

$$n = [E(K)/E_{\text{tors}}(K) : \langle P_1 \rangle] \leq \sqrt{0.6532/0.0714} = 3.0229.$$

In fact, we verify that $P_1 \notin pE(K)$ for $p = 2, 3$. Hence $n = 1$, and so

$$E(K) = \langle T \rangle \times \langle P_1 \rangle \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}.$$

Example 5.2.4. Let $K = \mathbb{Q}(i)$ and let $E = E_4$ as defined earlier. One can check that the torsion subgroup of $E(K)$ is trivial. Let

$$P_1 = (1 + 5i, 2 - i), \quad P_2 = \left(\frac{-32 - 53i}{2}, \frac{-663 + 49i}{4} \right).$$

Then we have $P_1, P_2 \in E(K)$. We will show that $E(K) = \langle P_1, P_2 \rangle$.

In Example 5.1.4, we have obtained from our algorithm that $\hat{h}(P) > 0.18$ for all non-torsion $P \in E(K)$. Moreover, one can check using MAGMA that

$$\hat{h}(P_1) = 1.2326, \quad \hat{h}(P_2) = 4.2894,$$

and the rank of $E(K)$ is at most 2. Computing the height pairing matrix, one can see that

$$R(P_1, P_2) = \det(\langle P_j, P_k \rangle)_{1 \leq j,k \leq 2} = 3.6050 \neq 0,$$

i.e., P_1 and P_2 are independent. Thus $E(K)$ has rank 2. It then follows from

Table 5.7: Sieving procedure for the elliptic curve E_4

p	$\mathfrak{p}$	$\#E(k_{\mathfrak{p}})$	l	(m_1, m_2)	(a_1, a_2)
2	$\langle 3 \rangle$	10	5	$(1, 0)$	$(0, 0)$
	$\langle 53, 23 + i \rangle$	54	27	$(1, 1)$	
3	$\langle 13, 8 + i \rangle$	21	7	$(1, 2)$	$(0, 0)$
	$\langle 17, 4 + i \rangle$	24	8	$(1, 1)$	
5	$\langle 3 \rangle$	10	2	$(1, 2)$	$(0, 0)$
	$\langle 5, 2 + i \rangle$	5	1	$(1, 3)$	
7	$\langle 13, 8 + i \rangle$	21	3	$(0, 1)$	$(0, 0)$
	$\langle 73, 46 + i \rangle$	63	9	$(1, 4)$	
11	$\langle 7 \rangle$	44	4	$(1, 2)$	$(0, 0)$
	$\langle 109, 33 + i \rangle$	110	10	$(1, 7)$	

Theorem 1.2.1 that

$$n = [E(K) : \langle P_1, P_2 \rangle] \leq \frac{2\sqrt{3.6050}}{\sqrt{3} \cdot 0.18} = 12.1801.$$

The sieving procedure then shows that n is not divisible by any primes $p \leq 11$; see Table 5.7. Therefore $n = 1$, and so $E(K) = \langle P_1, P_2 \rangle$. In fact, it can be verified that P_1 has the smallest canonical height among non-torsion $P \in E(K)$, with $\hat{h}(P_1) = 1.2326$. Compare this with our lower bound $\hat{h}(P) > 0.18$.

On the other hand, if we had used the lower bound obtained by Theorem 2.4.2 (i.e., $\hat{h}(P) > 2.34 \times 10^{-3}$ for all non-torsion $P \in E(K)$), then it would follow from Theorem 1.2.1 that $n \leq 936$. Finally, we note that the lower bound obtained by Hindry and Silverman [HS88, Theorem 0.3] (i.e., $\hat{h}(P) > 3.0624 \times 10^{-25}$ for all non-torsion $P \in E(K)$) would lead to $n \leq 7.1591 \times 10^{24}$, which would make it considerably harder to verify that $n = 1$.

Example 5.2.5. Let $K = \mathbb{Q}(i)$ and let $E = E_5$ as defined earlier. We have already shown in Example 5.1.5 that $\hat{h}(P) > 0.01$ for all non-torsion $P \in E(K)$. One can check that E has trivial torsion subgroup and the point $P_0 = (0, 0) \in E(K)$. In Cremona's paper [Cre94, Example 2], it has been asked whether $E(K) = \langle P_0 \rangle$. We will show that this is the case.

Using MAGMA, one can check that the rank of $E(K)$ is at most 1. Since P_0 is non-torsion, the rank of $E(K)$ is also at least 1. Hence $E(K)$ has rank 1. Theorem

1.2.1 then implies that

$$n = [E(K) : \langle P_0 \rangle] \leq \sqrt{0.0230/0.01} = 1.5173 < 2,$$

i.e., $n = 1$. Hence $E(K) = \langle P_0 \rangle$.

Example 5.2.6. Let $K = \mathbb{Q}(\theta)$ where θ is a root of the polynomial $x^3 - 2$, and let $E = E_6$ as defined earlier. One can verify that the torsion subgroup of $E(K)$ is trivial, and

$$P_1 = (0, \theta), \quad P_2 = (1 + \theta, 1), \quad P_3 = (3 - 9\theta + 7\theta^2, 31 + 23\theta - 36\theta^2),$$

are in $E(K)$. We wish to confirm that $E(K) = \langle P_1, P_2, P_3 \rangle$.

In Example 5.1.6, we have shown that $\hat{h}(P) > 0.0625$ for all non-torsion $P \in E(K)$. In addition, one can check using MAGMA that

$$\hat{h}(P_1) = 0.6303, \quad \hat{h}(P_2) = 0.8045, \quad \hat{h}(P_3) = 2.4430,$$

and the rank of $E(K)$ is at most 3. Computing the height pairing matrix, we have

$$R(P_1, P_2, P_3) = \det(\langle P_j, P_k \rangle)_{1 \leq j,k \leq 3} = 0.6263 \neq 0,$$

i.e., P_1, P_2, P_3 are independent. Thus $E(K)$ does have rank 3. Then by the geometry of numbers (Theorem 1.2.1), we obtain

$$n = [E(K) : \langle P_1, P_2, P_3 \rangle] \leq \sqrt{2(0.6263)}/(\sqrt{0.0625})^3 = 71.6300.$$

Using the sieving procedure, we can eventually show (details omitted) that n is not divisible by any primes $p \leq 71$. Therefore $n = 1$, and so

$$E(K) = \langle P_1, P_2, P_3 \rangle.$$

It can be verified that P_1 has the smallest canonical height among non-torsion $P \in E(K)$, with $\hat{h}(P_1) = 0.6303$. Compare this with our lower bound $\hat{h}(P) > 0.0625$.

5.2.3 Comparison with a Searching Points Method

Let E be an elliptic curve over a number field K. As suggested by a referee of [Tho10], we shall briefly describe an alternative way, as illustrated in [Sil90], to derive a Mordell–Weil basis for $E(K)$, and finally compare it with our method.

Suppose we can find a set of points $\{P_1, \ldots, P_r\} \subset E(K)$ which bijects to a basis for the group $E(K)/mE(K)$ for some $m \geq 2$. Let

$$C_1 = \max\{\hat{h}(Q) : Q = n_1P_1 + \cdots + n_rP_r, \text{ with } 0 \leq n_1, \ldots, n_r < m\}.$$

Then [Sil90, Proposition 7.2] says that the set $S = \{R \in E(K) : \hat{h}(R) \leq C_1\}$ generates $E(K)$. Using a result of [CPS06] or [Sil90], one can compute a constant C_2 satisfying $h(P) - \hat{h}(P) \leq C_2$ for all $P \in E(K)$, where $h(P)$ denotes the Weil height of the x-coordinate of P. It then follows that

$$h(R) \leq C_1 + C_2$$

for all $R \in S$. This, in principle, will allow one to search for R. If there exists $R \in S$ which is not a linear combination of $P_1, \ldots, P_r$, then we can replace some P_j by the linear combination of R. Repeating this process until no such R exists, the final set of $P_1, \ldots, P_r$ will eventually be a Mordell–Weil basis for $E(K)$.

The difficulty of this method lies in searching for points. Even though the x-coordinates have bounded height, this can be a non-trivial task especially if $[K : \mathbb{Q}]$ is large. In contrast, our method completely circumvents this problem. If $P_1, \ldots, P_r$ do not yet form a Mordell–Weil basis, we can use the sieving procedure to derive a new set of candidates. This process, which can be done more quickly than searching for points, however requires an upper bound for the index

$[E(K)/E_{\mathrm{tors}}(K) : \langle P_1, \ldots, P_r \rangle]$, which in turn requires a lower bound for the canonical height on $E(K)$.

5.3 Integral Points on Elliptic Curves

In this section, we will explain how Mordell–Weil bases, periods of elliptic curves, and complex elliptic logarithms can assist in finding integral points on elliptic curves over number fields. The method to be described here is a summary of a paper by Smart and Stephens [SS97] with some modifications. Some illustrative examples, which are computed by the algorithm based on this method (see Appendix A.2 for its MAGMA source code), will be also given at the end of this section.

5.3.1 Introduction

Definition. Let E be an elliptic curve defined over a number field K. We say that a point $P = (x, y) \in E(K)$ is an *integral point* if both $x, y \in \mathcal{O}_K$, where $\mathcal{O}_K$ is the ring of integers of K.

If the rank of $E(K)$ is non-zero, then we have already seen that there are infinitely many points in $E(K)$. However, this does not imply that the set of all integral points is also infinite. In fact, it has been proved by Siegel [Sie26] that there are only finitely many integral points in $E(K)$.

Suppose $\{P_1, \ldots, P_r\}$ is a Mordell–Weil basis for $E(K)$. Then every point $P \in E(K)$ can be written as

$$P = q_1 P_1 + \cdots + q_r P_r + T, \tag{5.2}$$

for some $T \in E_{\mathrm{tors}}(K)$ and $q_1, \ldots, q_r \in \mathbb{Z}$. If P is an integral point, then Siegel's Theorem implies that there exists an upper bound on each coefficient $|q_j|$. Let

$$Q = \max_{1 \le j \le r} \{|q_j|\}.$$

Provided that an upper bound for Q is known, then we can, in principle, obtain all integral points in $E(K)$ simply by brute-force search.

Given a Mordell–Weil basis for $E(K)$, Smart and Stephens [SS97] have proposed a method for computing an upper bound for Q. This method can be roughly described as follows. For each archimedean place $v \in M_K^{\mathrm{r}} \cup M_K^{\mathrm{c}}$, we let $E^{(v)}$ be the associated (real or complex) embedding of E. Then on each $E^{(v)}$, the method will initially compute a rather large bound Q_v, and then repeatedly apply LLL basis reduction [LLL82] to reduce Q_v as much as possible. Finally, we take the maximum among all Q_v to be an upper bound for Q.

5.3.2 Initial Bounds

For each $v \in M_K^{\mathrm{r}} \cup M_K^{\mathrm{c}}$, let $E^{(v)}$ be the associated (real or complex) embedding of E. Without loss of generality, we can assume that $E^{(v)}$ is of the form

$$E^{(v)}: \quad Y_v^2 = 4X_v^3 + A_v X_v + B_v$$

for some $A_v, B_v \in \mathbb{C}$, depending on E and v. Recall from Section 1.3 that there exists an isomorphism (of complex analytic Lie groups) $\mathbb{C}/\Lambda_v \to E^{(v)}(\mathbb{C})$ for some lattice Λ_v, given by the map

$$\begin{array}{rcl} z \pmod{\Lambda_v} & \mapsto & (\wp_{\Lambda_v}(z), \wp'_{\Lambda_v}(z)) \\ 0 \pmod{\Lambda_v} & \mapsto & O. \end{array}$$

We will denote the inverse of this map, *the elliptic logarithm* on $E^{(v)}$, by φ_v. For an integral point $P \in E(K)$, let $P^{(v)}$ be its associated image on $E^{(v)}$. Our aim is to estimate both lower and upper bound for $|\varphi_v(P^{(v)})|$. Combining both bounds then yields an initial upper bound for Q.

The following lemma gives an upper bound for $|\varphi_v(P^{(v)})|$; see [SS97] for the detailed proof.

Lemma 5.3.1. *If $Q \geq Q_0$, then $|\varphi_v(P_v)| \leq c_9 \exp(-c_{10}Q^2)$, for some explicitly computable constants*[2] *Q_0, c_9 and c_{10} depending only on E and v.*

Note that the constants Q_0, c_9, c_{10} are defined by a number of intermediate constants, which are well explained in [SS97] and are defined accordingly in our algorithm (see Appendix A.2 for the code). For now we mention that the following information is required in order to define these constants.

1. **A Mordell–Weil basis for $E(K)$.** This is essential for computing height pairing matrix, whose least eigenvalue is required for defining Q_0 and c_{10}.

2. **The period lattice Λ_v of $E^{(v)}$.** This can be obtained by Theorem 4.5.3, and is required for the constant c_9. Without loss of generality, we can assume that $\Lambda_v = \langle w_1, w_2 \rangle$ with $\tau = w_2/w_1$ satisfying (3.1), i.e.,

$$|\tau| \geq 1, \quad |\Re(\tau)| \leq 1/2, \quad \Im(\tau) \geq \sqrt{3}/2.$$

3. **Difference between the Weil (logarithmic) height and the canonical height on $E(K)$.** This result can be found in [Sil90] and more recently in [CPS06], which is required for computing Q_0 and c_9.

Since $\varphi_v : E^{(v)}(\mathbb{C}) \to \Lambda_v$ is a group isomorphism, it then follows from (5.2) that

$$\begin{aligned} \varphi_v(P^{(v)}) &\equiv \varphi_v(T^{(v)}) + \sum_{j=1}^{r} q_j \varphi_v(P_j^{(v)}) \pmod{\Lambda_v} \\ &= \varphi_v(T^{(v)}) + \sum_{j=1}^{r} q_j \varphi_v(P_j^{(v)}) + m_1 w_1 + m_2 w_2 \end{aligned} \tag{5.3}$$

for some $m_1, m_2 \in \mathbb{Z}$. Let $t = \operatorname{ord}(T)$. Then we have $\varphi_v(T^{(v)}) = (n_1 w_1 + n_2 w_2)/t$, for some integers $0 \leq n_1, n_2 < t$. Together with (5.3), this leads to

$$t\varphi_v(P^{(v)}) = \sum_{j=1}^{r} q_j t \varphi_v(P_j^{(v)}) + (m_1 t + n_1) w_1 + (m_2 t + n_2) w_2.$$

[2]All constants c_j are indexed so that they match the ones defined in [SS97].

In our notations, the following theorem yields a lower bound for this linear form in elliptic logarithms.

Theorem 5.3.2 ([Dav95, Théorème 2.1])**.** *Let* $Q' = \max\{Qt, |m_1t+n_1|, |m_2t+n_2|\}$*. There exist explicitly computable constants*[3] d_8, d_9, d_{10} *and* h_E*, such that, if* $Q' > \exp(d_8)$*, then*

$$\log|t\varphi_v(P^{(v)})| > -d_{10}(\log Q' + \log([K:\mathbb{Q}]d_9))(\log\log Q' + h_E + \log([K:\mathbb{Q}]d_9))^{r+3}.$$

For now we note that determining d_8, d_9, d_{10} and h_E requires elliptic logarithms $\varphi_v(P_j^{(v)})$ for all $1 \le j \le r$, which can be computed using Algorithm 4.6.2. For more details on how to compute these constants, see [Sma98, Appendix A]; these are also defined accordingly in our algorithm shown in Appendix A.2.

In order to make $t\varphi_v(P^{(v)})$ lie in the fundamental parallelogram spanned by w_1, w_2, we require that

$$|m_jt + n_j| < t\sum_{k=1}^{r}|q_k| = rQt, \quad \text{for } j = 1, 2.$$

Thus $Q' < rQt \le rQe_{\text{tors}}$, where e_{tors} is the exponent of $E_{\text{tors}}(K)$. Observe that $t \mid e_{\text{tors}}$. Combining Lemma 5.3.1 and Theorem 5.3.2, the following proposition is immediate.

Proposition 5.3.3 (Principal Inequality)**.** *If* $Q > \max\{Q_0, \exp(d_8)\}$*, then*

$$c_{10}Q^2 < d_{10}(\log(rQe_{\text{tors}}) + \log([K:\mathbb{Q}]d_9))(\log\log(rQe_{\text{tors}}) + h_E + \log([K:\mathbb{Q}]d_9))^{r+3} + \log(e_{\text{tors}}c_9).$$

This proposition therefore gives us an initial upper bound for Q, that is, either the one obtained from the above inequality or $\max\{Q_0, \exp(d_8)\}$, whichever is greater. Denote this initial upper bound by Q_v.

[3] The notations are as defined in [Sma98, Appendix A], with c_j being replaced by d_j.

5.3.3 Bound Reduction

In general, the initial upper bound Q_v we just obtained is considerably too large for a practical use. The next step in Smart and Stephens' method is therefore to reduce Q_v as much as possible. This can be achieved using an application of LLL basis reduction [LLL82].

To use an LLL basis reduction, we first choose a constant $C \approx Q_v^{\frac{r+2}{2}}$. Consider the $r+2$-dimensional lattice generated by the columns of the matrix

$$\mathcal{L} = \begin{pmatrix} 1 & \dots & 0 & 0 & 0 \\ 0 & \dots & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \dots & 1 & 0 & 0 \\ [C\Re(\varphi_v(P_1{}^{(v)}))] & \dots & [C\Re(\varphi_v(P_r{}^{(v)}))] & [C\Re(w_1)] & [C\Re(w_2)] \\ [C\Im(\varphi_v(P_1{}^{(v)}))] & \dots & [C\Im(\varphi_v(P_r{}^{(v)}))] & [C\Im(w_1)] & [C\Im(w_2)] \end{pmatrix}$$

(see [Sma98, p. 84]), where $[\cdot]$ is the rounding towards 0, i.e.,

$$[x] = \begin{cases} \lfloor x \rfloor & \text{if } x \geq 0, \\ \lceil x \rceil & \text{if } x < 0. \end{cases}$$

In general, C can be very large, hence one needs to compute the periods w_1, w_2 and complex elliptic logarithms of the points $P_1^{(v)}, \dots, P_r^{(v)}$ to a very high degree of precision in order to ensure that their integer parts are correct.

Next, we let

$$\boldsymbol{\ell} = \mathcal{L} \begin{pmatrix} tq_1 \\ \vdots \\ tq_r \\ m_1 t + n_1 \\ m_2 t + n_2 \end{pmatrix} = \begin{pmatrix} tq_1 \\ \vdots \\ tq_r \\ \lambda_1 \\ \lambda_2 \end{pmatrix},$$

where

$$\begin{aligned}
\lambda_1 &= (m_1t+n_1)[C\Re(w_1)]+(m_2t+n_2)[C\Re(w_2)]+\sum_{j=1}^{r} tq_j[C\Re(\varphi_v(P_j{}^{(v)}))],\\
\lambda_2 &= (m_1t+n_1)[C\Im(w_1)]+(m_2t+n_2)[C\Im(w_2)]+\sum_{j=1}^{r} tq_j[C\Im(\varphi_v(P_j{}^{(v)}))].
\end{aligned}$$

After applying LLL algorithm [LLL82, Proposition (1.11)], one shall obtain another basis $\{\boldsymbol{b}_1,\dots,\boldsymbol{b}_{r+2}\}$ which spans the same lattice as $\mathcal{L}$ does. Since $\boldsymbol{\ell}\neq\mathbf{0}$, the LLL algorithm assures that

$$\|\boldsymbol{\ell}\|^2 \geq 2^{-r-1}\|\boldsymbol{b}_1\|^2.$$

Recall that $|q_j|\leq Q_v$ for all $1\leq j\leq r$, and $|m_jt+n_j|<rQ_vt\leq rQ_ve_{\text{tors}}$ for $j=1,2$. This leads to the following proposition.

Proposition 5.3.4. *If* $\sqrt{2^{-r-1}\|\boldsymbol{b}_1\|^2-rQ_v^2e_{\text{tors}}^2}>3rQ_ve_{\text{tors}}/\sqrt{2}$, *then*

$$Q_v^2\leq\frac{1}{c_{10}}\left(\log(Cc_9e_{\text{tors}})-\log\left(\sqrt{2^{-r-1}\|\boldsymbol{b}_1\|^2-rQ_v^2e_{\text{tors}}^2}-\frac{3}{\sqrt{2}}rQ_ve_{\text{tors}}\right)\right). \tag{5.4}$$

Proof. By definition of $[\cdot]$, we have

$$\begin{aligned}
|\lambda_1+i\lambda_2-Ct\varphi_v(P^{(v)})| &\leq \sqrt{2}\left(\sum_{j=1}^{r}\frac{t|q_j|}{2}+\frac{|m_1t+n_1|}{2}+\frac{|m_2t+n_2|}{2}\right)\\
&\leq \frac{3}{\sqrt{2}}rQ_ve_{\text{tors}}.
\end{aligned}$$

Thus

$$|\lambda_1+i\lambda_2|\leq\frac{3}{\sqrt{2}}rQ_ve_{\text{tors}}+C|t\varphi_v(P^{(v)})|\leq\frac{3}{\sqrt{2}}rQ_ve_{\text{tors}}+Cc_9\exp(-c_{10}Q_v^2).$$

Now we have

$$\begin{aligned} 2^{-r-1}\|\boldsymbol{b}_1\|^2 \leq \|\boldsymbol{\ell}\|^2 &= t^2\sum_{j=1}^{r} q_j^2 + |\lambda_1 + i\lambda_2|^2 \\ &\leq rQ_v^2 e_{\text{tors}}^2 + \left(\frac{3}{\sqrt{2}} rQ_v e_{\text{tors}} + Cc_9 \exp(-c_{10}Q_v^2)\right)^2 . \end{aligned}$$

Hence

$$\sqrt{2^{-r-1}\|\boldsymbol{b}_1\|^2 - rQ_v^2 e_{\text{tors}}^2} - \frac{3}{\sqrt{2}} rQ_v e_{\text{tors}} \leq Cc_9 \exp(-c_{10}Q_v^2),$$

and (5.4) then follows after taking logarithms on both sides, which will be well-defined provided that

$$\sqrt{2^{-r-1}\|\boldsymbol{b}_1\|^2 - rQ_v^2 e_{\text{tors}}^2} > 3rQ_v e_{\text{tors}}/\sqrt{2}.$$

□

Note that if $\mathcal{L}$ does not satisfy the condition in Proposition 5.3.4, we can redefine $\mathcal{L}$ with a larger C until the condition is satisfied. Moreover, once we obtain a smaller Q_v, we can repeat the above process with this new Q_v until no further reduction is possible. Finally, we take the maximum among all reduced Q_v to be an upper bound for Q. Searching for all integral points then becomes an easy task if this bound is feasible.

5.3.4 Examples

We have implemented Smart and Stephens' method into an algorithm and use it to compute the following examples; its MAGMA code can be found in Appendix A.2. A demonstration of how to use this code will be shown in Example 5.3.6.

Example 5.3.5. We will first verify the result from [SS97, Example 2]. Let E be the elliptic curve defined over $K = \mathbb{Q}(\sqrt{-2})$ given by the Weierstrass equation

$$E: \quad y^2 = x^3 - 16x + 16.$$

It can be checked that the torsion subgroup of $E(K)$ is trivial. In [SS97], it is claimed that $E(K) = \langle P_1, P_2 \rangle$, where

$$P_1 = (0, 4), \quad P_2 = (2, -2\sqrt{-2}).$$

We will first confirm that this is indeed the case. Using our algorithm for computing a lower bound for the canonical height, we obtain

$$\hat{h}(P) > 0.012$$

for all $P \in E(K)$. Then one can see from the height pairing matrix that

$$R(P_1, P_2) = \det(\langle P_j, P_k \rangle)_{1 \leq j,k \leq 2} = 0.0330 \neq 0,$$

i.e. P_1 and P_2 are independent. In addition, one can check using MAGMA that the rank of $E(K)$ is at most 2. Hence $E(K)$ has rank 2. By the geometry of numbers (Theorem 1.2.1), we have

$$n = [E(K) : \langle P_1, P_2 \rangle] \leq 17.4808.$$

In fact, one can verify that $n = 1$ after applying the sieving procedure for all primes $p \leq 17$. Hence $E(K) = \langle P_1, P_2 \rangle$.

Next, we wish to compute all integral points $P \in E(K)$. As discussed earlier, this is equivalent to finding an upper bound for $Q = \max\{|q_1|, |q_2|\}$, where $P = q_1 P_1 + q_2 P_2$. In this example, E has only one complex embedding $E^{(v)}$. Using

Table 5.8: LLL reduction used in Example 5.3.5

Previous Q_v	C being chosen	New Q_v
10^{61}	1.0000×10^{244}	106
106	1.2625×10^{8}	27
27	5.3144×10^{5}	25
25	3.9062×10^{5}	24

Theorem 4.5.3 to compute the period lattice Λ_v of $E^{(v)}$, we have $\Lambda_v = \langle w_1, w_2\rangle$ with

$$w_1 = -i1.225694\ldots, \quad w_2 = 1.496729\ldots.$$

Note that w_1, w_2 are chosen so that $\tau = w_2/w_1$ satisfies (3.1) as required. Moreover, one can compute both lower and upper bounds for $h(P) - \hat{h}(P)$ for all $P \in E(K)$ using, for example, [Sil90, Theorem 1.1][4], and obtain

$$-5.461894 \leq h(P) - \hat{h}(P) \leq 6.211695.$$

Using the above quantities, our algorithm shows that

$$Q_0 = 12.2286, \quad c_9 = 2106.0087, \quad c_{10} = 0.0256.$$

In addition, we obtain the following quantities for David's constants:

$$d_8 = 31.5690, \quad d_9 = 4.7156, \quad d_{10} = 1.9249 \times 10^{110}, \quad h_E = 11.6136.$$

This finally yields $Q_v \leq 10^{61}$ as an initial upper bound for Q. After applying LLL basis reduction repeatedly until no further reduction is possible (see Table 5.8), we are finally able to reduce an upper bound for Q to 24. A quick search within this range then reveals all integral points in $E(K)$, as listed (up to inverse) in Table 5.9.

Note that the quantities Q_0, c_9, c_{10} we obtained from our algorithm are slightly

[4]We use Silverman's bounds in this example so that our constants can be compared with the ones shown in [SS97, Example 2] directly. In our algorithm (see Appendix A.2), we will use [CPS06, Theorem 1] to compute these bounds.

Table 5.9: Integral points on $y^2 = x^3 - 16x + 16$ over $\mathbb{Q}(\sqrt{-2})$

(q_1, q_2)	$P = q_1P_1 + q_2P_2$
$(1, 0)$	$(0, 4)$
$(2, 0)$	$(4, 4)$
$(3, 0)$	$(-4, -4)$
$(4, 0)$	$(8, -20)$
$(5, 0)$	$(1, -1)$
$(6, 0)$	$(24, 116)$
$(0, 1)$	$(2, -2\sqrt{-2})$
$(1, 1)$	$(4\sqrt{-2}, -12 + 8\sqrt{-2})$
$(1, -1)$	$(-4\sqrt{-2}, -12 - 8\sqrt{-2})$
$(2, 1)$	$(-4 + 4\sqrt{-2}, 20)$
$(2, -1)$	$(-4 - 4\sqrt{-2}, 20)$
$(5, 1)$	$(-10 - 4\sqrt{-2}, 28 - 18\sqrt{-2})$
$(5, -1)$	$(-10 + 4\sqrt{-2}, 28 + 18\sqrt{-2})$
$(3, 2)$	$(60 - 40\sqrt{-2}, 316 - 480\sqrt{-2})$
$(3, -2)$	$(60 + 40\sqrt{-2}, 316 + 480\sqrt{-2})$

different from the ones shown in [SS97, Example 2] due to some modifications in the formulas; this, however, has no effect on the final result.

Example 5.3.6. Let $K = \mathbb{Q}(\theta)$ where θ is a root of the polynomial $x^3 - 2$. In Example 5.2.6, we have readily verified that the elliptic curve E_6/K given by

$$E_6 : \quad y^2 = x^3 - (\theta^2 + 3\theta)x + \theta^2$$

has $\{P_1, P_2, P_3\}$ as a $\mathbb{Z}$-basis for $E(K)$, where

$$P_1 = (0, \theta), \quad P_2 = (1 + \theta, 1), \quad P_3 = (3 - 9\theta + 7\theta^2, 31 + 23\theta - 36\theta^2).$$

Thus any integral point P can be expressed as $P = q_1P_1 + q_2P_2 + q_3P_3$ for some $q_1, q_2, q_3 \in \mathbb{Z}$. We now wish to find all integral points in $E_6(K)$, that is, to find an upper bound for $Q = \max\{|q_1|, |q_2|, |q_3|\}$. To ease notation, we shall write $E = E_6$ and let v_r, v_c be the real and complex archimedean place of K respectively.

In order to determine such an upper bound, we will start by computing some certain constants associated to E/K. By [CPS06, Theorem 1], we first obtain

$$-1.196864 \leq h(P) - \hat{h}(P) \leq 0.174492$$

for all $P \in E(K)$. Moreover, our algorithm shows that $h_E = 11.9773$. The next step is to compute other constants associated to each real and complex embedding of E respectively.

Consider the case when $v = v_r$. Using Theorem 4.5.3 and some linear transformation, one can see that a $\mathbb{Z}$-basis $\{w_1, w_2\}$ for the period lattice of the real embedding $E^{(v)}$ is given by

$$w_1 = i1.658105\ldots, \quad w_2 = -1.815187\ldots.$$

Observe that $\tau = w_2/w_1$ satisfies (3.1). Combining all information on E/K and $E^{(v)}$ we have obtained so far, our algorithm shows that

$$c_9 = 19.6306, \quad c_{10} = 0.2017, \quad Q_0 = 2.8967,$$

and also

$$d_8 = 32.5576, \quad d_9 = 5.5533, \quad d_{10} = 7.9894 \times 10^{161}.$$

Hence by Proposition 5.3.3, we obtain $Q_v = 10^{87}$ as an initial upper bound for Q. After applying LLL basis reduction repeatedly as shown in Table 5.10, one can finally reduce Q_v to 11.

For $v = v_c$, we also obtain from Theorem 4.5.3 that the period lattice of the complex embedding $E^{(v)}$ is given by $\Lambda_v = \langle w_1, w_2 \rangle$, where

$$w_1 = 1.106543\ldots + i1.444101\ldots, \quad w_2 = -1.838531\ldots + i1.133717\ldots.$$

Again, w_1, w_2 are chosen so that $\tau = w_2/w_1$ satisfies (3.1). A similar computation as before also shows that

$$c_9 = 214.9545, \quad c_{10} = 0.1009, \quad Q_0 = 3.3679,$$

$$d_8 = 32.5576, \quad d_9 = 5.7684, d_{10} = 6.2775 \times 10^{161}.$$

Table 5.10: LLL reduction used in Example 5.3.6

$v = v_r$			$v = v_c$		
Previous Q_v	C being chosen	New Q_v	Previous Q_v	C being chosen	New Q_v
10^{87}	1.0000×10^{522}	67	10^{87}	1.0000×10^{261}	60
67	2.7207×10^{16}	12	60	4.6656×10^{10}	14
12	5.1598×10^{9}	11	14	7.5295×10^{6}	13

Table 5.11: Integral points on the elliptic curve E_6

(q_1, q_2, q_3)	$P = q_1P_1 + q_2P_2 + q_3P_3$
$(1,0,0)$	$(0,\theta)$
$(0,1,0)$	$(1+\theta,1)$
$(1,-1,0)$	$(-\theta,-2\theta)$
$(2,-1,0)$	$(9+\theta,-27-4\theta)$
$(0,0,1)$	$(3-9\theta+7\theta^2, 31+23\theta-36\theta^2)$
$(2,-2,-1)$	$(139+111\theta+87\theta^2, 2837+2253\theta+1788\theta^2)$

In consequence, Proposition 5.3.3 yields $Q_v = 10^{87}$ as an initial bound, which is eventually reduced to 13 after successive LLL reductions as shown in Table 5.10. Hence we have $Q \leq \max\{11, 13\} = 13$. The complete list of all integral points (up to inverse) in $E(K)$ is shown in Table 5.11; this is computed using our MAGMA code in Appendix A.2 (`intpts.m`) together with the following instructions:

```
> Attach("nfhtbound.m");   // from Cremona - for computing CPS bound
> Attach("intpts.m");      // main program for computing integral points
> Attach("elog.m");        // for computing periods and elliptic logarithms
> SetVerbose("Intpts", 1); // minimal printing
> // Define elliptic curve E
> _<x> := PolynomialRing(Integers());
> K<a> := NumberField(x^3-2);
> E := EllipticCurve([-a^2-3*a, a^2]);
> // Generators for E(K)
> P1 := E![0,a];
> P2 := E![1+a,1];
> P3 := E![3-9*a+7*a^2, 31+23*a-36*a^2];
> L, _ := IntegralPoints(E, [P1,P2,P3]);
Maximum absolute bound on coefficients = 13
Exact arithmetic
[ 1, 0, 0 ] ---> (0 : a : 1)
[ 0, 1, 0 ] ---> (a + 1 : 1 : 1)
[ 1, -1, 0 ] ---> (-a : -2*a : 1)
[ 2, -1, 0 ] ---> (a + 9 : -4*a - 27 : 1)
[ 0, 0, 1 ] ---> (7*a^2 - 9*a + 3 : -36*a^2 + 23*a + 31 : 1)
[ 2, -2, -1 ] ---> (87*a^2 + 111*a + 139 : 1788*a^2 + 2253*a + 2837 : 1)
********************************************
> L;
[ (0 : a : 1), (a + 1 : 1 : 1), (-a : -2*a : 1), (a + 9 : -4*a - 27 : 1), (7*a^2
    - 9*a + 3 : -36*a^2 + 23*a + 31 : 1), (87*a^2 + 111*a + 139 : 1788*a^2 +
    2253*a + 2837 : 1) ]
```

In conclusion, it should be noted that although Smart and Stephens' method [SS97], in principle, allows one to find all integral points on any elliptic curves over number fields, it requires a number of certain results on elliptic curves which may not be obtained easily, especially in the past. For example, lack of an algorithm for computing period lattices of arbitrary elliptic curves over $\mathbb{C}$ would prevent one to apply the method to most elliptic curves other than the ones having real coefficients. Our main results on height bound (see Chapter 3), period lattices and complex elliptic logarithms (see Chapter 4) therefore enhance Smart and Stephens' method by minimising its limitations.

5.4 Elliptic Curves with Everywhere Good Reduction

We finally come to the last section of this thesis, where we will illustrate an application of integral points, whose computation requires all the main results of this thesis, on finding *elliptic curves with everywhere good reduction* over some quadratic number fields. The method for finding this family of elliptic curves is due to Cremona and Lingham [CL07], which will be explained very briefly in this section.

5.4.1 Cremona–Lingham's Method: An Overview

Definition. Let K be a number field with ring of integers $\mathcal{O}_K$, and let $\mathcal{S}$ be a finite set of prime ideals of $\mathcal{O}_K$. We say that $x \in K$ is an *$\mathcal{S}$-integer* if $\operatorname{ord}_{\mathfrak{p}}(x) \geq 0$ for all prime ideals $\mathfrak{p} \notin \mathcal{S}$.

It is easy to verify that that the set of all $\mathcal{S}$-integers is a ring, which will be denoted by $\mathcal{O}_{K,\mathcal{S}}$ from now on.

For a finite set $\mathcal{S}$ of prime ideals of $\mathcal{O}_K$ and $m \in \mathbb{Z}_{>0}$, we define

$$K(\mathcal{S}, m) = \{x \in K^*/K^{*m} : \operatorname{ord}_{\mathfrak{p}}(x) \equiv 0 \pmod{m} \text{ for all } \mathfrak{p} \notin \mathcal{S}\}.$$

Here, $K^* = K \setminus \{0\}$. For convenience, we will also abuse the notation and say that an element $x \in K^*$ is in $K(\mathcal{S}, m)$ if $xK^{*m} \in K(\mathcal{S}, m)$. The following proposition can be proved very easily.

Proposition 5.4.1 ([CL07, Proposition 2.1]). *Let m, n be coprime. Then*

$$K(\mathcal{S}, mn) \cong K(\mathcal{S}, m) \times K(\mathcal{S}, n)$$

via the map $w \mapsto (w, w)$, with inverse map $(u, v) \mapsto v^{am}u^{bn}$, where $am + bn = 1$.

For this application, we will see later that we will need to consider $K(\mathcal{S}, m)$ for $m = 4, 6, 12$, and also the set $K(\mathcal{S}, 6)_{12}$ which is the image of the natural map $K(\mathcal{S}, 12) \to K(\mathcal{S}, 6)$. By Proposition 5.4.1, it then suffices to compute $K(\mathcal{S}, m)$ only for $m = 2, 3, 4$. For $m = 2, 3$, this can be computed easily using the MAGMA function `pSelmerGroup()`. A set of MAGMA functions for determining $K(\mathcal{S}, 4)$ has been implemented by Professor John Cremona who kindly supplied me with them.

Let E be an elliptic curve defined over K, given by a Weierstrass equation (1.1) as before, i.e.,

$$E: \quad y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6.$$

Definition. We say that E has *good reduction at* $\mathfrak{p}$ if it has a $\mathfrak{p}$-integral model (i.e., $\mathrm{ord}_{\mathfrak{p}}(a_j) \geq 0$ for all j) whose discriminant Δ is a $\mathfrak{p}$-unit (i.e., $\mathrm{ord}_{\mathfrak{p}}(\Delta) = 0$).

By Shafarevich's Theorem (see [Sil86, pp. 263–264] for the proof), it is well known that there are finitely many isomorphism classes of E/K having good reduction outside a finite set of primes $\mathcal{S}$. The following proposition shows the connection between $K(\mathcal{S}, 6)_{12}$ and the set of all elliptic curves over K with good reduction outside $\mathcal{S}$.

Proposition 5.4.2. *Let E be an elliptic curve defined over K with j-invariant $j(E) = j \neq 0, 1728$ and good reduction at all primes $\mathfrak{p} \notin \mathcal{S}$. Set $w = j^2(j - 1728)^3$.*

Then

$$\Delta \in K(\mathcal{S}, 12), \quad j \in \mathcal{O}_{K,\mathcal{S}}, \quad w \in K(\mathcal{S}, 6)_{12}.$$

Conversely, if $j \in \mathcal{O}_{K,\mathcal{S}}$ *with* $w = j^2(j-1728)^3 \in K(\mathcal{S}, 6)_{12}$*, then the elliptic curve*

$$E : \quad y^2 = x^3 - 3u^2 j(j-1728)x - 2u^3 j(j-1728)^2$$

with $u \in K^*$ *satisfying* $(3u)^6 w \in K(\mathcal{S}, 12)$*, has* $j(E) = j$ *and good reduction at all primes outside the extended set* $\mathcal{S}^{(6)} = \mathcal{S} \cup \{\mathfrak{p} : \operatorname{ord}_{\mathfrak{p}}(6) > 0\}$*.*

Proof. See [CL07, Proposition 3.2]. □

To obtain elliptic curves E with $j(E) \neq 0, 1728$ and good reduction outside $\mathcal{S}$, the strategy of Cremona and Lingham's method is to consider each class $w \in K(\mathcal{S}, 6)_{12}$ in turn, and determine all possible $j \in \mathcal{O}_{K,\mathcal{S}}$ satisfying $w \equiv j^2(j-1728)^3 \pmod{K^{*6}}$. For each of such j, one then obtain an elliptic curve E with good reduction outside $\mathcal{S}^{(6)}$ using the converse of Proposition 5.4.2. If such E also has good reduction at all primes $\mathfrak{p} \mid 6$, then E has good reduction outside $\mathcal{S}$, and we discard E otherwise. By [CL07, Proposition 3.4], the complete set of all curves with j-invariant j and good reduction outside $\mathcal{S}$ is obtained by twisting E with $u \in K(\mathcal{S}, 2)$.

Definition. Let E be an elliptic curve defined over a number field K, and let $\mathcal{S}$ be a finite set of prime ideals of $\mathcal{O}_K$. A point $P = (x, y) \in E(K)$ is said to be an *$\mathcal{S}$-integral point* if both $x, y \in \mathcal{O}_{K,\mathcal{S}}$. In addition, if $\mathcal{S} = \emptyset$ (i.e., $\mathcal{O}_{K,\mathcal{S}} = \mathcal{O}_K$), then we simply say that P is an *integral point.*

The next proposition shows that all possible j can arise from $\mathcal{S}$-integral points on certain elliptic curves over K.

Proposition 5.4.3. *Let* $w \in K(\mathcal{S}, 6)$*. Each* $j \in \mathcal{O}_{K,\mathcal{S}} \setminus \{0, 1728\}$ *with* $j^2(j-1728)^3 \equiv w \pmod{K^{*6}}$ *has the form* $j = x^3/w = 1728 + y^2/w$*, where* $P = (x, y)$

(with $xy \neq 0$) is an $\mathcal{S}$-integral point on the elliptic curve

$$E_w : \quad y^2 = x^3 - 1728w.$$

Proof. See [CL07, Proposition 3.3]. $\square$

It should be noted, however, that not all the values j obtained by Proposition 5.4.3 are $\mathcal{S}$-integral. Furthermore, not every $\mathcal{S}$-integral j arising from an $\mathcal{S}$-integral points on some E_w will necessarily be the j-invariant of a suitable elliptic curve, unless j is derived from $w \in K(\mathcal{S}, 6)_{12}$.

To summarise, in order to find elliptic curves with *everywhere* good reduction, we set $\mathcal{S} = \emptyset$ and apply Cremona and Lingham's method [CL07]. For those curves with j-invariant neither 0 nor 1728, the computation proceeds as follows:

1. Compute $K(\emptyset, 6)$ from $K(\emptyset, 2)$ and $K(\emptyset, 3)$, and determine a (finite) representative set W of $w \in K(\emptyset, 6)_{12}$.
2. For each $w \in W$, find all *integral points* on the elliptic curve E_w/K such that $j = x^3/w \in \mathcal{O}_K$.
3. If such j satisfies $j^2(j-1728)^3 \in K(\emptyset, 6)_{12}$, then we determine $u_0 \in K^*$ such that $(3u_0)^6 j^2 (j-1728)^3 \in K(\emptyset, 12)$. Let E be the elliptic curve

$$E : \quad y^2 = x^3 - 3u_0^2 j(j-1728)x - 2u_0^3 j(j-1728)^2.$$

   Check if E has good reduction at all primes $\mathfrak{p}$ dividing 6; discard E if not.
4. Repeat step (3) for each quadratic twist $E^{(u)}$, where $u \in K(\emptyset, 2)$.

Since $\mathcal{S} = \emptyset$, it is immediate from [CL07, Proposition 4.1] that there is no elliptic curve E/K with $j(E) = 0$ and everywhere good reduction. For $j = 1728$, finding elliptic curves with j-invariant j and everywhere good reduction does not involve searching for integral points at all; see [CL07, Proposition 4.2] for more details.

5.4.2 Examples I: Real Quadratic Fields

As we have seen in Section 5.3, the applications of all the main results of this thesis finally allow one to use Smart and Stephens' method [SS97] to find all integral points on elliptic curves over number fields with less restriction. This in turn benefits to the determination of elliptic curves with everywhere good reduction using Cremona and Lingham's algorithm [CL07]. In particular, we are able to settle some inconclusive cases appearing in Cremona's compiled list[5] on elliptic curves over $K = \mathbb{Q}(\sqrt{d})$ with everywhere good reduction for $2 \leq d \leq 100$. For more information on imaginary quadratic fields $\mathbb{Q}(\sqrt{-d})$ (with $2 \leq d \leq 100$), see Section 5.4.3.

In this subsection, we will illustrate these new results in full details. Note, however, that on some real quadratic fields K we may not fully confirm non-existence of elliptic curves over K with everywhere good reduction, nor that the list of such elliptic curves is complete, owing to the difficulty in searching for non-torsion points on certain elliptic curves. Furthermore, we carry out our computation for all $d \leq 100$ which are inconclusive from Cremona's table, apart from $d = 71, 79, 91$ in which there are too many elliptic curves E_w/K whose Mordell–Weil bases are unknown.

Based on the tables shown later in this subsection (see next page for the description), we obtain the following conclusion in addition to what we already know from Cremona's table.

Proposition 5.4.4. *Let $2 \leq d \leq 100$. Then we have the following:*

1. *For $d = 55, 78, 95$, there is no elliptic curve over $\mathbb{Q}(\sqrt{d})$ with everywhere good reduction.*

2. *For $d = 38, 41, 65$, we have the complete list of all elliptic curves over $\mathbb{Q}(\sqrt{d})$ with everywhere good reduction.*

Proof. For (1), see Table 5.17, 5.22, and 5.24. For (2), see Table 5.12, 5.13, and 5.20. □

[5] Available at `http://www.warwick.ac.uk/~masgaj/ecegr/ecegrqf.html` (last checked on November 22, 2010).

Description of Tables

In the following pages, we will illustrate in detail how to find all elliptic curves over a quadratic number field K with everywhere good reduction using Cremona and Lingham's method (see Section 5.4.1). For each K, we give a table whose columns represent the following information:

# Index of each w.

$\boldsymbol{w}$ Each $w \in W$, where W is the set of representatives for $K(\emptyset, 6)_{12}$; note that this is unique modulo K^{*6}. If a fundamental unit ε of $\mathcal{O}_K$ exists, then w will be expressed in terms of ε.

Torsion All generators of the torsion subgroup of $E_w(K)$, where E_w is the elliptic curve $y^2 = x^3 - 1728w$. Each generator is denoted by $\langle T, t\rangle$, where $T \in E_w(K)$ is a generator, and $t = \mathrm{ord}(T)$. If the torsion subgroup is trivial, we simply write "O".

Rank The rank of $E_w(K)$.

Mordell–Weil basis A Mordell–Weil basis for $E_w(K)$. If the rank is 0, we simply write "–".

Integral Points The list of all integral points in $E_w(K)$. If no such point exists, we simply write "–".

$\boldsymbol{j \neq 0, 1728}$ The list of all j associated to each integral point in $E_w(K)$. If no j exists, or the corresponding j is 0 or 1728, we simply write "–" in that entry.

If some information is currently unknown, then we put "?" in that entry.

Recall that not all j shown in the table may yield an elliptic curve over K with everywhere good reduction. However, if there exists j which gives rise to such curves, then that j and its associated integral point will be shown in **bold**.

Moreover, the details of all curves arising from that j will be shown in the second table, whose columns are as follows:

j The j-invariant of elliptic curves

Index of each elliptic curve E/K with j-invariant j, having everywhere good reduction.

$\boldsymbol{a_1, a_2, a_3, a_4, a_6}$ The a-invariants of the Weierstrass equation of E.

$\boldsymbol{\Delta}$ The discriminant of E. In case Δ cannot be expressed exactly (for example, when there is no globally minimal model for the curve), then the ideal $\langle\Delta\rangle$ will be shown instead.

Torsion The torsion subgroup of $E(K)$, represented by the same notation as above.

Rank The rank of $E(K)$.

At the end, a summary line will be given. This can be either a **conclusion** (that our list of elliptic curves over K with everywhere good reduction is complete, or that there is no such curve), or a **conjecture** (especially when there exists w which currently cannot be completely settled).

Table 5.12: Finding elliptic curves over $\mathbb{Q}(\sqrt{38})$ with everywhere good reduction ($a = \sqrt{38}$, $\varepsilon = 37 + 6a$, $\mathcal{N}(\varepsilon) = 1$)

#	w	Torsion	Rank	Mordell–Weil basis	Integral points	$j \neq 0, 1728$
1	1	$\langle(12,0),2\rangle$	1	$\left(\frac{119180}{31^2}, \frac{6671392a}{31^3}\right)$	$(12,0)$	–
2	-1	$\langle(-12,0),2\rangle$	2	$\left(\frac{-10}{9}, \frac{182a}{27}\right), (20, 16a)$	$(20,16a)$, $(-12,0)$, $(1014, 5238a)$	-8000, –, -1042590744
3	ε^3	$\langle(444+72a,0),2\rangle$	2	$\left(\frac{9142108+1482504a}{2025}, \frac{39062491552+6336741600a}{91125}\right)$, $(740+120a, -25200-4088a)$	$\boldsymbol{(740+120a, -25200-4088a)}$, $(444+72a, 0)$, $(2442+396a, 170100+27594a)$	**8000**, –, 287496
4	$-\varepsilon^3$	$\langle(-444-72a,0),2\rangle$	1	$(-296-48a, 11096+1800a)$	$(-296-48a, 11096+1800a)$, $(-444-72a, 0)$, $(3552+576a, -299592-48600a)$	512, –, -884736
5	$\bar{\varepsilon}^2$	O	0	–	–	–
6	$-\bar{\varepsilon}^2$	O	1	$(-24+4a, 1528-248a)$	$(-24+4a, 1528-248a)$	$-768-128a$
7	ε	O	1	$\left(\frac{12176+444a}{289}, \frac{218232-113312a}{4913}\right)$	–	–
8	$-\varepsilon$	O	0	–	–	–
9	Conjugate to #5					
10	Conjugate to #6					
11	Conjugate to #7					
12	Conjugate to #8					

List of curves[a]

j	#	a_1	a_2	a_3	a_4	a_6	Δ	Torsion	Rank
8000	1	a	$1+a$	$1+a$	$15+4a$	$21+4a$	ε^3	$\langle\left(-\frac{3}{2}, \frac{-2+a}{4}\right),2\rangle$	0
	2	a	$1-a$	$1+a$	$15-5a$	$21-5a$	$\bar{\varepsilon}^3$	$\langle\left(-\frac{3}{2}, \frac{-2+a}{4}\right),2\rangle$	0

Conclusion: All elliptic curves over $\mathbb{Q}(\sqrt{38})$ with everywhere good reduction have been found.

[a] These curves were initially found by Cremona; we confirm that no other curves can arise from the remaining cases. Note also that, although their j-invariant is rational, these curves are not isomorphic to any elliptic curves over $\mathbb{Q}$.

Table 5.13: Finding elliptic curves over $\mathbb{Q}(\sqrt{41})$ with everywhere good reduction
($a = \frac{1+\sqrt{41}}{2}$, $\varepsilon = 37 - 10a$, $\mathcal{N}(\varepsilon) = -1$)

#	w	Torsion	Rank	Mordell–Weil basis	Integral points	$j \neq 0, 1728$
1	1	$\langle (12,0), 2 \rangle$	0	–	$(12,0)$	–
2	-1	$\langle (-12,0), 2 \rangle$	0	–	$(-12,0)$	–
3	$\bar{\varepsilon}^3$	$\langle (324+120a,0), 2 \rangle$	0	–	$(324+120a,0)$	–
4	$-\bar{\varepsilon}^3$	$\langle (-324-120a,0), 2 \rangle$	0	–	$(-324-120a,0)$	–
5	ε^2	O	1	$(593-160a, 18915-5110a)$	$\mathbf{(593-160a, 18915-5110a)}$	$\mathbf{152753-40800a}$
6	$-\varepsilon^2$	O	1	$(177-48a, -3465+936a)$	$(177-48a, -3465+936a)$	$-3537+1296a$
7	$\bar{\varepsilon}$	O	2	$(20+8a, -48-16a)$, $(6913+2560a, 885221+327670a)$	$(20+8a, -48-16a)$, $\mathbf{(6913+2560a, 885221+327670a)}$	$960+256a$, $\mathbf{29013115611+10739384330a}$
8	$-\bar{\varepsilon}$	O	2	$(700-120a, 17216-5760a)$, $(17, -103-64a)$	$(700-120a, 17216-5760a)$, $\mathbf{(17, -103-64a)}$, $(44+8a, 240+128a)$	$39757240000-10740736000a$, $\mathbf{181781-49130a}$, $412608-113152a$
9	Conjugate to #5					
10	Conjugate to #6					
11	Conjugate to #8					
12	Conjugate to #7					

List of curves[a]

j	#	a_1	a_2	a_3	a_4	a_6	Δ	Torsion	Rank
$152753-40800a$	1	1	0	0	$-27-10a$	0	$\bar{\varepsilon}^4$	$\langle (0,0), 2 \rangle$, $\langle \left(\frac{-14-5a}{4}, \frac{14+5a}{8}\right), 2 \rangle$,	0
$111953+40800a$	1	1	0	0	$-37+10a$	0	ε^4	$\langle (0,0), 2 \rangle$, $\langle \left(\frac{18-5a}{4}, \frac{-18+5a}{8}\right), 2 \rangle$,	0
$29013115611+10739384330a$	1	1	0	0	$148-40a$	$37-10a$	$-\varepsilon^5$	$\langle \left(-\frac{1}{4}, \frac{1}{8}\right), 2 \rangle$	0
$39752499941-10739384330a$	1	1	0	0	$108+40a$	$27+10a$	$-\bar{\varepsilon}^5$	$\langle \left(-\frac{1}{4}, \frac{1}{8}\right), 2 \rangle$	0
$181781-49130a$	1	1	a	a	3	-2	$-\bar{\varepsilon}$	$\langle (1,1), 4 \rangle$	0
$132651+49130a$	1	1	$1-a$	$1+a$	$3-a$	$-2-a$	$-\varepsilon$	$\langle (1,-3), 4 \rangle$	0

Conclusion: All elliptic curves over $\mathbb{Q}(\sqrt{41})$ with everywhere good reduction have been found.

[a]The first two curves were previously found by Comalada [Com90, Theorem 2]; we complete the list by finding other curves.

Table 5.14: Finding elliptic curves over $\mathbb{Q}(\sqrt{43})$ with everywhere good reduction
($a = \sqrt{43}$, $\varepsilon = 3482 + 531a$, $\mathcal{N}(\varepsilon) = 1$)

#	w	Torsion	Rank	Mordell–Weil basis	Integral points	$j \neq 0, 1728$
1	1	$\langle (12,0), 2\rangle$	1	$\left(\frac{60288556}{549081}, \frac{71340219776a}{406869021}\right)$	$(12,0)$	–
2	-1	$\langle (-12,0), 2\rangle$	1	$(960, 4536a)$	$(960, 4536a), (-12, 0)$	-884736000, –
3	$\bar{\varepsilon}^3$	$\langle (41784 - 6372a, 0), 2\rangle$	2	$\left(\frac{4833016-737028a}{49}, \frac{14443490160-2202611600a}{343}\right)$, $(69640 - 10620a, 23012360 - 3509352a)$	$(69640 - 10620a, 23012360 - 3509352a)$, $(41784 - 6372a, 0)$, $(229812 - 35046a, -155333430 + 23688126a)$	8000, –, 287496
4	$-\bar{\varepsilon}^3$	$\langle (-41784 + 6372a, 0), 2\rangle$	0	–	$(-41784 + 6372a, 0)$	–
5[a]	$\bar{\varepsilon}^2$	O	≤ 1	?	?	?
6	$-\bar{\varepsilon}^2$	O	1	$(3200 - 488a, -294088 + 44848a)$	$(3200 - 488a, -294088 + 44848a)$, $(-944 + 144a, -138808 + 21168a)$	$-2048 + 512a$ $-483328 - 73728a$
7	ε	O	0	–	–	–
8	$-\varepsilon$	O	0	–	–	–
9	Conjugate to #5					
10	Conjugate to #6					
11	Conjugate to #7					
12	Conjugate to #8					

Conjecture: No elliptic curve over $\mathbb{Q}(\sqrt{43})$ with everywhere good reduction exists.

[a] We cannot find a Mordell–Weil basis for $E_w(\mathbb{Q}(\sqrt{43}))$ due to the difficulty in searching for a point on the curve. Any elliptic curve with everywhere good reduction, if exists, will arise only from this case.

Table 5.15: Finding elliptic curves over $\mathbb{Q}(\sqrt{46})$ with everywhere good reduction
($a = \sqrt{46}$, $\varepsilon = 24335 - 3588a$, $\mathcal{N}(\varepsilon) = 1$)

#	w	Torsion	Rank	Mordell–Weil basis	Integral points	$j \neq 0, 1728$
1	1	$\langle (12,0), 2 \rangle$	1	$(\frac{5423}{450}, \frac{9373a}{13500})$	$(12, 0)$	–
2	-1	$\langle (-12,0), 2 \rangle$	0	–	$(-12, 0)$	–
3	ε^3	$\langle (292020 - 43056a, 0), 2 \rangle$	2	$(486700 - 71760a, 425172384 - 62688248a)$, $(\frac{128021336-18875716a}{49}, \frac{2047081399240-301825685912a}{343})$	$(486700 - 71760a, 425172384 - 62688248a)$, $(292020 - 43056a, 0)$, $(1606110 - 236808a, 2869913592 - 423145674a)$	8000 –, 287496
4	$-\varepsilon^3$	$\langle (-292020 + 43056a, 0), 2 \rangle$	1	$(\frac{1044823225-154050780a}{6084}, \frac{116177050458217-17129371579866a}{474552})$	$(-292020 + 43056a, 0)$	–
5	$\bar{\varepsilon}^2$	O	0	–	–	–
6	$-\bar{\varepsilon}^2$	O	1	$(8600 + 1268a, -1515064 - 223384a)$	$(8600 + 1268a, -1515064 - 223384a)$	$-1280 - 128a$
7[a]	ε	O	≤ 1	?	?	?
8	$-\varepsilon$	O	0	–	–	–
9	Conjugate to #5					
10	Conjugate to #6					
11	Conjugate to #7					
12	Conjugate to #8					

Conjecture: No elliptic curve over $\mathbb{Q}(\sqrt{46})$ with everywhere good reduction exists.

[a]We cannot find a Mordell–Weil basis for $E_w(\mathbb{Q}(\sqrt{46}))$ due to the difficulty in searching for a point on the curve. Any elliptic curve with everywhere good reduction, if exists, will arise only from this case.

Table 5.16: Finding elliptic curves over $\mathbb{Q}(\sqrt{51})$ with everywhere good reduction ($a = \sqrt{51}$, $\varepsilon = 50 - 7a$, $\mathcal{N}(\varepsilon) = 1$)

#	w	Torsion	Rank	Mordell–Weil basis	Integral points	$j \neq 0, 1728$
1	1	$\langle (12, 0), 2\rangle$	0	–	–	–
2	ε^3	$\langle (600 - 84a, 0), 2\rangle$	1	$(1000 - 140a, 39592 - 5544a)$	$(1000 - 140a, 39592 - 5544a)$, $(600 - 84a, 0)$, $(3300 - 462a, -267246 + 37422a)$	8000 –, 287496
3	-1	$\langle (-12, 0), 2\rangle$	0	–	–	–
4	$-\varepsilon^3$	$\langle (-600 + 84a, 0), 2\rangle$	1	$\left(\frac{135000-18900a}{17}, \frac{289247112-40502616a}{289}\right)$	$(-600 + 84a, 0)$	–
5	ε^2	O	0	–	–	–
6	$\bar{\varepsilon}$	O	1	$(1000 - 84a, 28000 - 5040a)$	$(1000 - 84a, 28000 - 5040a)$	$204733761728 - 28668371200a$
7	$-\varepsilon^2$	O	0	–	–	–
8[a]	$-\bar{\varepsilon}$	O	≤ 1	?	?	?
9	Conjugate to #5					
10	Conjugate to #6					
11	Conjugate to #7					
12	Conjugate to #8					

Conjecture: No elliptic curve over $\mathbb{Q}(\sqrt{51})$ with everywhere good reduction exists.

[a]We cannot find a Mordell–Weil basis for $E_w(\mathbb{Q}(\sqrt{51}))$ due to the difficulty in searching for a point on the curve. Any elliptic curve with everywhere good reduction, if exists, will arise only from this case.

Table 5.17: Finding elliptic curves over $\mathbb{Q}(\sqrt{55})$ with everywhere good reduction ($a = \sqrt{55}$, $\varepsilon = 89 + 12a$, $\mathcal{N}(\varepsilon) = 1$)

#	w	Torsion	Rank	Mordell–Weil basis	Integral points	$j \neq 0, 1728$
1	1	$\langle (12,0), 2\rangle$	1	$\left(\frac{131748211326659}{5124566322000}, \frac{432149202134683869017a}{25940042265331800000}\right)$	$(12,0)$	–
2	ε^3	$\langle (1068+144a,0), 2\rangle$	1	$\left(\frac{1447051+195108a}{100}, \frac{2461120046+331857477a}{1000}\right)$	$(1068+144a,0)$	–
3	-1	$\langle (-12,0), 2\rangle$	1	$\left(\frac{899}{80}, \frac{-12103a}{1600}\right)$	$(-12,0)$	–
4	$-\varepsilon^3$	$\langle (-1068-144a,0), 2\rangle$	1	$(2848+384a, 220528+29736a)$	$(2848+384a, 220528+29736a)$, $(-1068-144a, 0)$	-32768, –
5	ε^2	O	0	–	–	–
6	$\bar{\varepsilon}$	O	0	–	–	–
7	$-\varepsilon^2$	O	0	–	–	–
8	$-\bar{\varepsilon}$	O	0	–	–	–
9	Conjugate to #5					
10	Conjugate to #6					
11	Conjugate to #7					
12	Conjugate to #8					

Conclusion: No elliptic curve over $\mathbb{Q}(\sqrt{55})$ with everywhere good reduction exists.

Table 5.18: Finding elliptic curves over $\mathbb{Q}(\sqrt{59})$ with everywhere good reduction ($a=\sqrt{59}$, $\varepsilon = 530+69a$, $\mathcal{N}(\varepsilon)=1$)

#	w	Torsion	Rank	Mordell–Weil basis	Integral points
1	1	$\langle (12,0),2\rangle$	1	$(\frac{17430989226389123544459}{1452556063128834244400}, \frac{6992270055695438354038717165 9a}{175065010936808294509965447200 0})$	$(12,0)$
2	-1	$\langle (-12,0),2\rangle$	1	$(\frac{-133}{16}, \frac{-283a}{64})$	$(-12,0)$
3	$\bar{\varepsilon}^3$	$\langle (6360-828a,0),2\rangle$	2	$(10600-1380a, 1366568-177912a)$, $(\frac{27766759598762519208150197 50-36149177590841770289855917 5a}{1021323545187387576079 68}, \frac{290749715481802500138851657089478506630287-3785238876299689191321306737490620613892 7a}{461593850041207606005082215321454 08})$	$(10600-1380a, 1366568-177912a)$ $(6360-828a,0)$, $(34980-4554a, -9224334+1200906a)$
4	$-\bar{\varepsilon}^3$	$\langle (-6360+828a,0),2\rangle$	2	$(\frac{18550-2415a}{2}, \frac{5810733-756493a}{4})$, $(\frac{50000200-6509460a}{59}, \frac{3840591605112-500002437752a}{3481})$	$(-6360+828a,0)$
5[a]	$\bar{\varepsilon}^2$	O	≤ 2	?	?
6	$-\bar{\varepsilon}^2$	O	2	$(-552+72a, 12168-1584a)$, $(8304-1080a, 1069560-139248a)$	$(-552+72a, 12168-1584a)$, $(8304-1080a, 1069560-139248a)$ $(-612+80a, -4912+640a)$
7[a]	ε	O	≤ 1	?	?
8	$-\varepsilon$	O	1	$(\frac{13348+1734a}{225}, \frac{3157622+410526a}{3375})$	–
9	Conjugate to #5				
10	Conjugate to #6				
11	Conjugate to #7				
12	Conjugate to #8				

#	$j \neq 0, 1728$
1	–
2	–
3	8000, –, 287496
4	–
5	?
6	$-635904-82944a$, $-330587136-42508800a$, $-8689728-1131520a$
7	?
8	–

Conjecture: No elliptic curve over $\mathbb{Q}(\sqrt{59})$ with everywhere good reduction exists.

[a]We cannot find Mordell–Weil bases for both $E_w(\mathbb{Q}(\sqrt{59}))$ due to the difficulty in searching for a point on both curves. Any elliptic curves with everywhere good reduction, if exist, will arise only from these two cases.

Table 5.19: Finding elliptic curves over $\mathbb{Q}(\sqrt{62})$ with everywhere good reduction ($a = \sqrt{62}$, $\varepsilon = 63 + 8a$, $\mathcal{N}(\varepsilon) = 1$)

#	w	Torsion	Rank	Mordell–Weil basis	Integral points	$j \neq 0, 1728$
1	1	$\langle (12,0), 2\rangle$	1	$\left(\frac{31126430}{1034289}, \frac{21343982842a}{1051871913}\right)$	$(12,0)$	–
2	-1	$\langle (-12,0), 2\rangle$	0	–	$(-12,0)$	–
3	$-\bar{\varepsilon}^3$	$\langle (-756+96a,0), 2\rangle$	1	$\left(\frac{30492-3872a}{25}, \frac{8377936-1064000a}{125}\right)$	$(-756+96a,0)$	–
4	$\bar{\varepsilon}^3$	$\langle (756-96a,0), 2\rangle$	2	$(1260-160a, -56000+7112a)$, $\left(\frac{1892045988-240259808a}{680625}, \frac{115201755674768-14630652232000a}{561515625}\right)$	$(1260-160a, -56000+7112a)$, $(756-96a, 0)$, $(4158-528a, 378000-48006a)$	8000 –, 287496
5[a]	$\bar{\varepsilon}^2$	O	≤ 1	?	?	?
6	$-\bar{\varepsilon}^2$	O	0	–	–	–
7[a]	$-\varepsilon$	O	≤ 1	?	?	?
8	ε	O	0	–	–	–
9	Conjugate to #5					
10	Conjugate to #6					
11	Conjugate to #7					
12	Conjugate to #8					

Conjecture: No elliptic curve over $\mathbb{Q}(\sqrt{62})$ with everywhere good reduction exists.

[a]We cannot find Mordell–Weil bases for both $E_w(\mathbb{Q}(\sqrt{62}))$ due to the difficulty in searching for a point on both curves. Any elliptic curves with everywhere good reduction, if exist, will arise only from these two cases.

Table 5.20: Finding elliptic curves over $\mathbb{Q}(\sqrt{65})$ with everywhere good reduction
($a = \frac{1+\sqrt{65}}{2}$, $\varepsilon = 9 - 2a$, $\mathcal{N}(\varepsilon) = -1$)

#	w	Torsion	Rank	Mordell–Weil basis	Integral points	$j \neq 0, 1728$
1	1	$\langle(12,0),2\rangle$	2	$(257, 511 - 1022a)$, $(17, -7 + 14a)$	$\mathbf{(257, 511 - 1022a)}$, $\mathbf{(17, -7 + 14a)}$, $(12, 0)$	$\mathbf{16974593}$, $\mathbf{4913}$,–
2	ε^3	$\langle(108 - 24a, 0), 2\rangle$	2	$(217 - 48a, -3997 + 882a)$, $\left(\frac{3353-741a}{16}, \frac{240667-53123a}{64}\right)$	$\mathbf{(217 - 48a, -3997 + 882a)}$, $\mathbf{(596225 - 131584a, 614101215 - 135529408a)}$, $(108 - 24a, 0)$	$\mathbf{8049 - 1666a}$ $\mathbf{161923694525417 - 35735839572482a}$, –
3	-1	$\langle(-12, 0), 2\rangle$	0	–	$(-12, 0)$	–
4	$-\varepsilon^3$	$\langle(-108 + 24a, 0), 2\rangle$	2	$(1937 + 480a, 112889 + 33278a)$, $(-71 + 16a, -1267 + 280a)$	$\mathbf{(1937 + 480a, 112889 + 33278a)}$, $\mathbf{(-71 + 16a, -1267 + 280a)}$, $(108 - 24a, 0)$	$\mathbf{126187854952935 + 35735839572482a}$ $\mathbf{6383 + 1666a}$, –
5	$\bar{\varepsilon}^2$	O	1	$\left(\frac{25969+6843a}{441}, \frac{5346127+1493361a}{9261}\right)$	–	–
6	ε	O	0	–	–	–
7	$-\bar{\varepsilon}^2$	O	1	$(-12 - 3a, -288 - 81a)$,	$(-12 - 3a, -288 - 81a)$,	$432 - 81a$
8	$-\varepsilon$	O	2	$(148 + 24a, -1504 - 576a)$, $(20 - 3a, -128 + 31a)$	$(148 + 24a, -1504 - 576a)$, $(20 - 3a, -128 + 31a)$	$119055040 + 33720320a$, $848 + 745a$
9	Conjugate to #5					
10	Conjugate to #8					
11	Conjugate to #7					
12	Conjugate to #6					

List of curves[a] [b]
($\mathfrak{p}_1 = \langle 2, 2 + a\rangle$, $\mathfrak{p}_2 = \langle 2, 1 + a\rangle$)

j	#	a_1	a_2	a_3	a_4	a_6	$\langle\Delta\rangle$	Torsion	Rank
16974593	1	0	a	0	$-9675 - 2741a$	$-568506 - 160998a$	$\mathfrak{p}_1^{12}\mathfrak{p}_2^{12}$	$\langle(-37 - 11a, 0), 2\rangle, \langle(-38 - 11a, 0), 2\rangle$	≤ 2
	2	0	$2a$	0	$-2720 + 601a$	$-66880 + 14740a$	$\mathfrak{p}_1^{24}\mathfrak{p}_2^{12}$	$\langle(-24 + 4a, 0), 2\rangle, \langle(44 - 11a, 0), 2\rangle$	≤ 2
4913	1	0	a	0	$-635 - 181a$	$-8554 - 2422a$	$\mathfrak{p}_1^{12}\mathfrak{p}_2^{12}$	$\langle(-13 - 4a, 0), 2\rangle, \langle(-9 - 3a, -28 - 8a), 4\rangle$	≤ 2
	2	0	$2a$	0	$-160 + 41a$	$-576 + 132a$	$\mathfrak{p}_1^{24}\mathfrak{p}_2^{12}$	$\langle(12 - 3a, 0), 2\rangle, \langle(-4, 0), 2\rangle$	≤ 2
$8049 - 1666a$	1	0	$2 + a$	0	$25 + 7a$	$-28 - 8a$	$\mathfrak{p}_1^{12}\mathfrak{p}_2^{12}$	$\langle(1, 0), 2\rangle$	≤ 2
	2	0	$1 + 2a$	0	$-4085 + 909a$	$134771 - 29735a$	$\mathfrak{p}_1^{24}\mathfrak{p}_2^{12}$	$\langle(65 - 15a, 544 - 120a), 4\rangle$	≤ 2
$6383 + 1666a$	1	0	$1 + 2a$	0	$-14536 - 4120a$	$907280 + 256940a$	$\mathfrak{p}_1^{12}\mathfrak{p}_2^{12}$	$\langle(55 + 15a, 0), 2\rangle$	≤ 2
	2	0	$2 + a$	0	$-4 - 4a$	$-8 + 4a$	$\mathfrak{p}_1^{24}\mathfrak{p}_2^{12}$	$\langle(6, -4a), 4\rangle$	≤ 2

Continue...

[a]Note that $K = \mathbb{Q}(\sqrt{65})$ has class number 2, so we do not always have a globally minimal model for elliptic curves defined over K. In fact, there is no globally minimal model for any elliptic curves with everywhere good reduction given in this list.

[b]The first four curves were also found by Comalada [Com90, Theorem 2].

List of curves (continued)

$$\mathfrak{p}_1 = \langle 2, 2+a\rangle, \quad \mathfrak{p}_2 = \langle 2, 1+a\rangle$$
$$\mathfrak{p}_3 = \langle 1783, 119+a\rangle, \quad \mathfrak{p}_4 = \langle 73, 24+a\rangle$$
$$\mathfrak{p}_5 = \langle 1783, 1663+a\rangle, \quad \mathfrak{p}_6 = \langle 73, 48+a\rangle.$$

$j = 161923694525417 - 35735839572482a$		
	Curve #1	Curve #2
a_1	1	0
a_2	$2+a$	$2a$
a_3	$2+a$	0
a_4	$-21468356714742 - 6079026789064a$	$-333242682087072 + 73355938366985a$
a_6	$-70962492158002003969 - 20096248456507716887a$	$-311818848627775357 5488 + 688087595049844553252a$
$\langle\Delta\rangle$	$\mathfrak{p}_3^{12}\mathfrak{p}_4^{12}$	$\mathfrak{p}_1^{24}\mathfrak{p}_2^{12}\mathfrak{p}_3^{12}\mathfrak{p}_4^{12}$
Torsion	$\langle\left(\frac{14442791+4123028a}{4}, \frac{-14442799-4123032a}{8}\right), 2\rangle$	$\langle(-7417048 + 1878644a, 0), 2\rangle$
Rank	≤ 2	≤ 2

$j = 126187854952935 + 35735839572482a$		
	Curve #1	Curve #2
a_1	1	0
a_2	$2a$	$2a$
a_3	$2a$	0
a_4	$-27547383503792 + 6079026789066a$	$-98182631945888 + 21018349077737a$
a_6	$-91058615802697592007 + 20096220909124213078a$	$-60155770251453159 5840 + 132331628833007719364a$
$\langle\Delta\rangle$	$\mathfrak{p}_5^{12}\mathfrak{p}_6^{12}$	$\mathfrak{p}_1^{24}\mathfrak{p}_2^{12}\mathfrak{p}_5^{12}\mathfrak{p}_6^{12}$
Torsion	$\langle\left(\frac{18565823-4123032a}{4}, \frac{-18565823+4123024a}{8}\right), 2\rangle$	$\langle(8294824 - 2049324a, 0), 2\rangle$
Rank	≤ 2	≤ 2

Conclusion: All elliptic curves over $\mathbb{Q}(\sqrt{65})$ with everywhere good reduction have been found.

Table 5.21: Finding elliptic curves over $\mathbb{Q}(\sqrt{67})$ with everywhere good reduction
($a = \sqrt{67}$, $\varepsilon = 48842 + 5967a$, $\mathcal{N}(\varepsilon) = 1$)

#	w	Torsion	Rank	Mordell–Weil basis	Integral points	$j \neq 0, 1728$
1	1	$\langle (12,0), 2 \rangle$	1	$\left(\frac{13036}{441}, \frac{-175648a}{9261}\right)$	$(12,0)$	–
2	-1	$\langle (-12,0), 2 \rangle$	1	$\left(\frac{-584}{49}, \frac{-248a}{343}\right)$	$(5280, 46872a)$, $(-12,0)$	-147197952000, –
3[a]	ε^3	$\langle (586104 + 71604a, 0), 2 \rangle$	≤ 2	?	?	?
4[a]	$-\varepsilon^3$	$\langle (-586104 - 71604a, 0), 2 \rangle$	≤ 2	?	?	?
5[a]	ε^2	O	≤ 1	?	?	?
6	$-\varepsilon^2$	O	1	$(-5304 - 648a, 1955448 + 238896a)$	$(-5304 - 648a, 1955448 + 238896a)$	$-6110208 + 746496a$
7	$\bar{\varepsilon}$	O	0	–	–	–
8	$-\bar{\varepsilon}$	O	0	–	–	–
9	Conjugate to #5					
10	Conjugate to #6					
11	Conjugate to #7					
12	Conjugate to #8					

Conjecture: No elliptic curve over $\mathbb{Q}(\sqrt{67})$ with everywhere good reduction exists.

[a] We cannot find Mordell–Weil bases for $E_w(\mathbb{Q}(\sqrt{67}))$ due to the difficulty in searching for a point on these curves. Any elliptic curves with everywhere good reduction, if exist, will arise only from these three cases.

Table 5.22: Finding elliptic curves over $\mathbb{Q}(\sqrt{78})$ with everywhere good reduction ($a = \sqrt{78}$, $\varepsilon = 53 + 6a$, $\mathcal{N}(\varepsilon) = 1$)

#	w	Torsion	Rank	Mordell–Weil basis	Integral points	$j \neq 0, 1728$
1	1	$\langle (12,0), 2\rangle$	2	$(36, 24a), \left(\frac{135}{8}, \frac{201a}{32}\right)$	$(36, 24a)$, $(12, 0)$, $(30, -18a)$, $(3756, -26064a)$	46656, $-$, 27000, 52987905216
2	$\bar{\varepsilon}^3$	$\langle (636 - 72a, 0), 2\rangle$	1	$\left(\frac{115169-13038a}{50}, \frac{77331254-8756055a}{500}\right)$	$(636 - 72a, 0)$	$-$
3	-1	$\langle (-12, 0), 2\rangle$	1	$\left(\frac{15}{2}, \frac{21a}{4}\right)$	$(-12, 0)$	$-$
4	$-\bar{\varepsilon}^3$	$\langle (1272 - 144a,$ $-68040 + 7704a), 6\rangle$	2	$(-212 + 24a, 22256 - 2520a)$, $\left(\frac{-583+66a}{2}, \frac{86242-9765a}{4}\right)$	$(-212 + 24a, 22256 - 2520a)$, $(152428 - 17256a, -84149936 + 9528120a)$ $(1272 - 144a, -68040 + 7704a)$, $(0, -22680 + 2568a)$, $(22680 - 2568a, 4830408 - 546936a)$, $(-636 + 72a, 0)$, $(2226 - 252a, -150228 + 17010a)$ $(0, 22680 - 2568a)$, $(216 - 24a, -23112 + 2616a)$, $(1272 - 144a, 68040 - 7704a)$	64, -23788477376, -13824, $-$, $-39191040 + 4437504a$, $-$, -74088, $-$, $-39191040 - 4437504a$, -13824
5	ε^2	O	0	$-$	$-$	$-$
6	$\bar{\varepsilon}$	O	1	$\left(\frac{30304-3356a}{121}, \frac{7324408-831168a}{1331}\right)$	$-$	$-$
7	$-\varepsilon^2$	O	1	$(120 + 12a, 2808 + 312a)$	$(120 + 12a, 2808 + 312a)$	$-15475968 + 1752192a$
8	$-\bar{\varepsilon}$	$\langle (0, 216 - 24a), 3\rangle$	0	$-$	$(0, 216 - 24a), (0, -216 + 24a)$	$-,-$
9	Conjugate to #5					
10	Conjugate to #6					
11	Conjugate to #7					
12	Conjugate to #8					

Conclusion: No elliptic curve over $\mathbb{Q}(\sqrt{78})$ with everywhere good reduction exists.

Table 5.23: Finding elliptic curves over $\mathbb{Q}(\sqrt{87})$ with everywhere good reduction
($a = \sqrt{87}$, $\varepsilon = 28 + 3a$, $\mathcal{N}(\varepsilon) = 1$)

#	w	Torsion	Rank	Mordell–Weil basis	Integral points	$j \neq 0, 1728$
1	1	$\langle (12,0), 2\rangle$	0	–	–	–
2	$-\varepsilon^3$	$\langle (-336-36a,0), 2\rangle$	1	$(168+18a, 9234+990a)$	$(168+18a, 9234+990a)$, $(-336-36a, 0)$, $(336+36a, -12312-1320a)$, $(7728+828a, 960336+102960a)$	-216, –, -1728, -21024576
3	-1	$\langle (-12,0), 2\rangle$	0	–	$(-12, 0)$	–
4	ε^3	$\langle (336+36a,0), 2\rangle$	1	$\left(\frac{7964484080+853337580a}{94221}, \frac{5410595932122920+580070159494056a}{155747313}\right)$	$(336+36a, 0)$	–
5	ε^2	O	0	–	–	–
6	$-\bar{\varepsilon}$	O	1	$\left(\frac{217776-21924a}{6241}, \frac{158740128-16682304a}{493039}\right)$	–	–
7	$-\varepsilon^2$	O	0	–	–	–
8[a]	$\bar{\varepsilon}$	O	≤ 1	?	?	?
9	Conjugate to #5					
10	Conjugate to #6					
11	Conjugate to #7					
12	Conjugate to #8					

Conjecture: No elliptic curve over $\mathbb{Q}(\sqrt{87})$ with everywhere good reduction exists.

[a]We cannot find a Mordell–Weil basis for $E_w(\mathbb{Q}(\sqrt{87}))$ due to the difficulty in searching for a point on the curve. Any elliptic curve with everywhere good reduction, if exists, will arise only from this case.

Table 5.24: Finding elliptic curves over $\mathbb{Q}(\sqrt{95})$ with everywhere good reduction ($a = \sqrt{95}$, $\varepsilon = 39 + 4a$, $\mathcal{N}(\varepsilon) = 1$)

#	w	Torsion	Rank	Mordell–Weil basis	Integral points	$j \neq 0, 1728$
1	1	$\langle (12, 0), 2\rangle$	1	$\left(\frac{823516}{17405}, \frac{170043328a}{5134475}\right)$	$(12, 0)$	–
2	$\bar{\varepsilon}^3$	$\langle (468 - 48a, 0), 2\rangle$	1	$\left(\frac{9636276-988336a}{2025}, \frac{42273107296-4337147584a}{91125}\right)$	$(468 - 48a, 0)$	–
3	-1	$\langle (-12, 0), 2\rangle$	1	$\left(\frac{156964}{3645}, \frac{14420224a}{492075}\right)$	$(-12, 0)$	–
4	$-\bar{\varepsilon}^3$	$\langle (-468 + 48a, 0), 2\rangle$	1	$(-312 + 32a, 12008 - 1232a)$	$(-312 + 32a, 12008 - 1232a)$, $(-468 + 48a, 0)$, $(3744 - 384a, -324216 + 33264a)$	512, –, -884736
5	$\bar{\varepsilon}^2$	O	0	–	–	–
6	ε	O	0	–	–	–
7	$-\bar{\varepsilon}^2$	O	0	–	–	–
8	$-\varepsilon$	O	0	–	–	–
9	Conjugate to #5					
10	Conjugate to #6					
11	Conjugate to #7					
12	Conjugate to #8					

Conclusion: No elliptic curve over $\mathbb{Q}(\sqrt{95})$ with everywhere good reduction exists.

5.4.3 Examples II: Imaginary Quadratic Fields

In this subsection, we will illustrate a similar computation on finding elliptic curves with everywhere good reduction over $K = \mathbb{Q}(\sqrt{-d})$ for $2 \leq d \leq 100$. Most cases, as mentioned in another Cremona's table[6], have been already proved by various mathematicians, including:

- **Kida** [Kid01, Theorem 1] has proved non-existence of such curve for $d = 35$, 37, 51, and 91.
- **Setzer** has proved non-existence for several d up to 161; see [Set78, Theorem 4(a)] for the complete list. He has also showed existence of elliptic curves with everywhere good reduction for $d = 65$ [Set78, Theorem 4(b)].
- **Cremona** has used his joint method [CL07] to confirm non-existence of such curve for $d = 23$.

In general, we also have the following theorem due to Setzer.

Theorem 5.4.5 ([Set78, Theorem 5])**.** *If the class number of $K = \mathbb{Q}(\sqrt{-d})$ ($d > 0$) is prime to 6, then there is no elliptic curve over K with everywhere good reduction.*

Using all the main results we have so far to assist in computing Mordell–Weil bases and integral points, we are eventually able to use Cremona and Lingham's method to show non-existence of elliptic curves with everywhere good reduction over more imaginary quadratic fields in addition to the above results[7]. To be precise, we obtain the following conclusion from our tables to be shown in the following pages.

Proposition 5.4.6. *For $d = 26, 29, 31, 59, 83, 87$, there is no elliptic curve defined over $\mathbb{Q}(\sqrt{-d})$ with everywhere good reduction.*

Proof. See Table 5.25, 5.26, 5.27, 5.30, 5.32, 5.33; note that we cannot use Theorem 5.4.5 since these imaginary quadratic fields have class number not prime to 6. □

[6]Available at `http://www.warwick.ac.uk/~masgaj/ecegr/egr_imag.txt` (last checked on November 30, 2010).

[7]Except at $d = 89$, due to the difficulty in finding Mordell–Weil bases for most $E_w(K)$.

Table 5.25: Finding elliptic curves over $\mathbb{Q}(\sqrt{-26})$ with everywhere good reduction ($a = \sqrt{-26}$)

#	w	Torsion	Rank	Mordell–Weil basis	Integral points	$j \neq 0, 1728$
1	1	$\langle (12,0), 2\rangle$	2	$(4, 8a), \left(\frac{11}{2}, \frac{31a}{4}\right)$	$(4, 8a), (12, 0), (-2876, 30248a), (-42, 54a)$	64, –, -23788477376, -74088
2	-1	$\langle (-12,0), 2\rangle$	1	$\left(\frac{-2173}{50}, \frac{27797a}{500}\right)$	$(-12, 0)$	–
3	$(1+a)^2$	O	1	$\left(\frac{-15220+2888a}{961}, \frac{-6717568+125528a}{29791}\right)$	–	–
4	$-(1+a)^2$	O	0	–	–	–
5	Conjugate to #3					
6	Conjugate to #4					

Conclusion: No elliptic curve over $\mathbb{Q}(\sqrt{-26})$ with everywhere good reduction exists.

Table 5.26: Finding elliptic curves over $\mathbb{Q}(\sqrt{-29})$ with everywhere good reduction ($a = \sqrt{-29}$)

#	w	Torsion	Rank	Mordell–Weil basis	Integral points	$j \neq 0, 1728$
1	1	$\langle (12, 0), 2 \rangle$	0	–	$(12, 0)$	–
2	-1	$\langle (-12, 0), 2 \rangle$	0	–	$(-12, 0)$	–
3	$-721 + 20a$	O	1	$(356 + 80a, 3280 + 2368a)$	$(356 + 80a, 3280 + 2368a)$	$224704 - 15360a$
4	$721 - 20a$	O	1	$(188 - 16a, -2192 + 352a)$	$(188 - 16a, -2192 + 352a)$	$5056 - 2048a$
5	Conjugate to #3					
6	Conjugate to #4					

Conclusion: No elliptic curve over $\mathbb{Q}(\sqrt{-29})$ with everywhere good reduction exists.

Table 5.27: Finding elliptic curves over $\mathbb{Q}(\sqrt{-31})$ with everywhere good reduction $(a = \frac{1+\sqrt{-31}}{2})$

#	w	Torsion	Rank	Mordell–Weil basis	Integral points	$j \neq 0, 1728$
1	1	$\langle (12,0), 2 \rangle$	1	$\left(-\frac{484}{25}, \frac{-2128+4256a}{125}\right)$	$(12,0)$	–
2	-1	$\langle (-12,0), 2 \rangle$	1	$\left(-\frac{30032476}{680625}, \frac{-29261304464+58522608928a}{561515625}\right)$	$(-12,0)$	–
3	$-8+a$	O	0	–	–	–
4	$8-a$	O	0	–	–	–
5	Conjugate to #3					
6	Conjugate to #4					

Conclusion: No elliptic curve over $\mathbb{Q}(\sqrt{-31})$ with everywhere good reduction exists.

Table 5.28: Finding elliptic curves over $\mathbb{Q}(\sqrt{-38})$ with everywhere good reduction ($a = \sqrt{-38}$)

#	w	Torsion	Rank	Mordell–Weil basis	Integral points	$j \neq 0, 1728$
1	1	$\langle (12,0), 2\rangle$	2	$(-20, 16a)$, $\left(\frac{10}{9}, \frac{-182a}{27}\right)$	$(-20, 16a)$, $(12, 0)$, $(-1014, 5238a)$	-8000, –, -1042590744
2	-1	$\langle (-12,0), 2\rangle$	1	$\left(\frac{-119180}{961}, \frac{6671392a}{29791}\right)$	$(-12, 0)$	–
3[a]	$(8-2a)^2$	O	≤ 1	?	?	?
4	$-(8-2a)^2$	O	0	–	–	–
5	Conjugate to #3					
6	Conjugate to #4					

Conjecture: No elliptic curve over $\mathbb{Q}(\sqrt{-38})$ with everywhere good reduction exists.

[a]We cannot find a Mordell–Weil basis for $E_w(\mathbb{Q}(\sqrt{-38}))$ due to the difficulty in searching for a point on the curve. Any elliptic curve with everywhere good reduction, if exists, will arise only from this case.

Table 5.29: Finding elliptic curves over $\mathbb{Q}(\sqrt{-53})$ with everywhere good reduction ($a = \sqrt{-53}$)

#	w	Torsion	Rank	Mordell–Weil basis	Integral points	$j \neq 0, 1728$
1	1	$\langle (12,0), 2 \rangle$	0	–	$(12,0)$	–
2	-1	$\langle (-12,0), 2 \rangle$	0	–	$(-12,0)$	–
3^a	$-208-8a$	O	1, 2	$P_1 = (-64-8a, -896+32a)$, ?	$(-64-8a, -896+32a)$, ?	$-1088+384a$, ?
4^a	$208+8a$	O	1, 2	$P_2 = (-16+16a, -976+112a)$, ?	$(-16+16a, -976+112a)$, ?	$1024-1024a$, ?
5	Conjugate to #3					
6	Conjugate to #4					

Conjecture: No elliptic curve over $\mathbb{Q}(\sqrt{-53})$ with everywhere good reduction exists.

[a] One can check using MAGMA that the rank r of each $E_w(K)$ is at most 3. By applying 2-descent twice (using MAGMA code implemented by Nils Bruin), currently we can show that 3 out of 7 non-trivial elements of the 2-Selmer group have homogeneous spaces with no rational points, hence $r \leq 2$. Since we can find a non-torsion point, then $1 \leq r \leq 2$. Note that if the Parity Conjecture is true, then $r = 1$, and each $E_w(K)$ will be generated by P_1, P_2 respectively.

Table 5.30: Finding elliptic curves over $\mathbb{Q}(\sqrt{-59})$ with everywhere good reduction ($a = \frac{1+\sqrt{-59}}{2}$)

#	w	Torsion	Rank	Mordell–Weil basis	Integral points	$j \neq 0, 1728$
1	1	$\langle (12,0), 2 \rangle$	1	$(24+4a, 96+32a)$	$(24+4a, 96+32a)$, $(12,0)$, $(28-4a, -128+32a)$	$-4416+7168a$ –, $2752-7168a$
2	-1	$\langle (-12,0), 2 \rangle$	1	$\left(\frac{89680564260+92640196a}{10214134225}, \frac{-5062251572865 9840-22095057306784a}{1032291475449625}\right)$	$(-12,0)$	–
3	$-6+7a$	O	2	$(21+13a, -123+112a)$, $(-24-8a, 312-40a)$	$(21+13a, -123+112a)$, $(-24-8a, 312-40a)$, $(48+16a, -120-184a)$, $(12-8a, -96+40a)$	$-671+1785a$, $-1536-512a$, $12288+4096a$, $832+192a$
4[a]	$6-7a$	O	0	–	–	–
5	Conjugate to #3					
6	Conjugate to #4					

Conclusion: No elliptic curve over $\mathbb{Q}(\sqrt{-59})$ with everywhere good reduction exists.

[a]One can check using MAGMA that the rank r of $E_w(K)$ is at most 2. By applying 2-descent twice, we can show that only one non-trivial case exists, hence $r = 0$.

Table 5.31: Finding elliptic curves over $\mathbb{Q}(\sqrt{-61})$ with everywhere good reduction ($a = \sqrt{-61}$)

#	w	Torsion	Rank	Mordell–Weil basis	Integral points	$j \neq 0, 1728$
1	1	$\langle (12,0), 2\rangle$	2	$\left(\frac{7644}{961}, \frac{-133488a}{29791}\right)$, $\left(\frac{131}{25}, \frac{-637a}{125}\right)$	$(12,0)$	–
2^a	-1	$\langle (-12,0), 2\rangle$	≤ 2	?	$(-12,0)$, ?	–, ?
3^a	$3-16a$	O	≤ 1	?	?	?
4	$-3+16a$	O	1	$(-216-48a, -9000)$	$(-216-48a, -9000)$	$-13824-82944a$
5	Conjugate to #3					
6	Conjugate to #4					

Conjecture: No elliptic curve over $\mathbb{Q}(\sqrt{-61})$ with everywhere good reduction exists.

[a]We cannot find Mordell–Weil bases for both $E_w(\mathbb{Q}(\sqrt{-61}))$ due to the difficulty in searching for a point on both curves. Any elliptic curves with everywhere good reduction, if exist, will arise only from these two cases.

Table 5.32: Finding elliptic curves over $\mathbb{Q}(\sqrt{-83})$ with everywhere good reduction $(a = \frac{1+\sqrt{-83}}{2})$

#	w	Torsion	Rank	Mordell–Weil basis	Integral points	$j \neq 0, 1728$
1	1	$\langle (12,0), 2 \rangle$	1	$(4a, -48+16a)$	$(4a, -48+16a), (12,0),$ $(4-4a, 32+16a)$	$-1344-1280a, -$ $-2624+1280a$
2	-1	$\langle (-12,0), 2 \rangle$	1	$\left(\frac{653541657746014240-69426505269939844a}{74038045930647561}, \frac{-989386371008251115684647856+4454586114510582333440912a}{20145702766276473744881691}\right)$	$(-12,0)$	–
3	$-12-5a$	O	0	–	–	–
4	$12+5a$	O	2	$(-1740+76a, 13872+15176a),$ $(96+4a, 912+56a)$	$(-1740+76a, 13872+15176a)$ $(96+4a, 912+56a)$	$-14480448+42574400a$ $34752-3520a$
5	Conjugate to #3					
6	Conjugate to #4					

Conclusion: No elliptic curve over $\mathbb{Q}(\sqrt{-83})$ with everywhere good reduction exists.

Table 5.33: Finding elliptic curves over $\mathbb{Q}(\sqrt{-87})$ with everywhere good reduction ($a = \frac{1+\sqrt{-87}}{2}$)

#	w	Torsion	Rank	Mordell–Weil basis	Integral points	$j \neq 0, 1728$
1	1	$\langle (12,0), 2 \rangle$	0	–	$(12, 0)$	–
2	-1	$\langle (-12,0), 2 \rangle$	0	–	$(-12, 0)$	–
3[a]	$14 + 13a$	O	0	–	–	–
4[b]	$-14 - 13a$	O	2	$P_1 = (-66 - 3a, -138 + 105a)$, $P_2 = \left(\frac{-93-21a}{4}, \frac{2157+213a}{8}\right)$	$(-66 - 3a, -138 + 105a)$	$4455 - 648a$
5	Conjugate to #3					
6	Conjugate to #4					

Conclusion: No elliptic curve over $\mathbb{Q}(\sqrt{-87})$ with everywhere good reduction exists.

[a]One can check using MAGMA that the rank r of $E_w(K)$ is at most 2. By applying 2-descent twice, we can show that only one non-trivial case exists, hence $r = 0$.

[b]Note that the sieving procedure fails at $p = 29$; in which case we prove that $[E_w(K) : \langle P_1, P_2 \rangle] \neq 29$ by solving 29-division polynomial explicitly.

Appendix A

MAGMA Source Code

As mentioned earlier, we have implemented our algorithms based on the main results of this thesis in MAGMA. For convenience, we shall split our source code into several files according to their applications.

A.1 Period Lattices and Complex Elliptic Logarithms

We have implemented all necessary functions for computing complex *arithmetic-geometric mean* (AGM), period lattices of elliptic curves over $\mathbb{C}$, and elliptic logarithms of complex points (see Chapter 4). In the following file, some important functions include:

`AGM()` This function computes an AGM of two complex numbers based on a specified set of all indices for which the pair in the AGM sequence is bad.

`PeriodLattice()` Given an elliptic curve $E/\mathbb{C}$, this function will compute all three minimal coset representatives of Λ modulo 2Λ, where Λ is the period lattice of E. Any two of these minimal coset representatives form a $\mathbb{Z}$-basis for Λ.

`EllipticLog()` Given an elliptic curve $E/\mathbb{C}$ and a point $P \in E(\mathbb{C})$, this function computes an elliptic logarithm of P.

For more details on these functions, see the documentation inside the code.

```
/******************************************************************************
 * elog.m
 * Computing Complex AGM, Period lattices, and Complex Elliptic Logarithms
 * By Thotsaphon Thongjunthug
 * Last updated: 02 December 2010
 * Any errors should be reported to <nookaussie@yahoo.com>
 ******************************************************************************/

declare verbose Elog, 1; // 0 = no printing, 1 = print more details

/******************************************************************************
 * Main intrinsic functions
 ******************************************************************************/

/**
 * Given a, b, compute AGM(a, b) based on the AGM sequence having (finite
 * number of) bad pairs specified by S.
 * Input:
 *   a, b = two complex numbers
 *      S = the set of all indices at which the pair in the AGM sequence is bad
 * Output:
 *   Complex AGM of a, b, subject to |a_n - b_n| < 10^-Prec
 * Parameter:
 *   Prec = precision used as the stopping criterion
 *          (should be less than the precision of a, b to avoid infinite loop
 **/
intrinsic AGM(a::FldComElt, b::FldComElt, S::SeqEnum : Prec := 25) -> FldComElt
{Compute AGM(a, b) based on the AGM sequence having bad indices specified by S.}
    require Precision(a) eq Precision(b): "a, b must have the same precision";
    require (Prec ge 0) and (Prec in Integers()):
        "Precision must be a non-negative integer";
    require Precision(a) gt Prec: "Prec must be less than the precision of a";
    n := 0;
    Sort(~S); // sort S in increasing order
    repeat
        n +:= 1;
        a1 := (a + b)/2;
        b1 := Sqrt(a*b);
        // Find the right choice for b1
        if Abs(a1-b1) gt Abs(a1+b1) then
            b1 := -b1;
        elif (Abs(a1-b1) eq Abs(a1+b1)) and (Im(b1/a1) lt 0) then
            // Re(b1/a1) = 0, i.e. b1 = a1*i (up to sign)
            // Choose b1 so that Im(b1/a1) > 0 (Cox's convention)
            vprintf Elog: "|a_%o-b_%o| = |a_%o+b_%o|: use Cox's convention\n",
               n, n, n, n;
            b1 := -b1;
        end if;
        a := a1;
        if (#S ne 0) and (S[1] eq n) then
            b := -b1; // bad choice
            vprint Elog: "Choose bad choice for AGM at n = ", S[1];
            Remove(~S, 1); // remove the first index from the list
        else
            b := b1;  // good choice
        end if;
        vprintf Elog: "|a_%o-b_%o| = %o\n", n, n, Abs(a-b);
    until Abs(a - b) lt (10^-Prec);
    vprint Elog: "---";
    if #S ne 0 then
        vprint Elog: "Some indices still remain in S, need higher precision";
    end if;
    return a;
end intrinsic;


/**
 * Given a, b, compute an optimal AGM(a, b) (i.e. no bad pair allowed in the
 * AGM sequence for all n>0)
 * Input:
 *   a, b = two complex numbers
 * Output:
 *   An optimal AGM of a, b, subject to |a_n - b_n| < 10^-Prec
 * Parameter:
 *   Prec = precision used as the stopping criterion
 **/
intrinsic AGM(a::FldComElt, b::FldComElt : Prec := 25) -> FldComElt
{Compute an optimal AGM of a, b.}
    return AGM(a, b, []: Prec := Prec);
end intrinsic;


/**
 * Given an elliptic curve E: Y^2 = 4*(X-e1)*(X-e2)*(X-e3) with e_j distinct
 * and e1+e2+e3=0, compute the three minimal coset representatives of
 * \Lambda modulo 2*\Lambda, where \Lambda is the period lattice of E.
 * Any two of them form a \Z-basis for \Lambda.
 *
 * Note: Users may need to do an extra work to obtain an orthogonal basis
 * in case of a rectangular lattice.
 *
 * Input:
 *   E = [e1, e2] = two roots of E (note that e3 = -e1-e2)
 * Output:
 *   [w1, w2, w3] = the three minimal coset representatives
 * Parameter:
 *   Prec = precision used as the stopping criterion when computing AGM
 **/
intrinsic PeriodLattice(E::SeqEnum : Prec := 25) -> SeqEnum
{Given E = [e1,e2], compute all three minimal coset representatives; two of which
form a \Z-basis for the period lattice of the ellipic curve Y^2=4*(X-e1)*(X-e2)*(X-e3),
with e1+e2+e3=0.}
    require #E eq 2: "E must contain exactly two complex numbers";
    require (Prec ge 0) and (Prec in Integers()):
        "Precision must be a non-negative integer";
    e1, e2 := Explode(E);
    require e1 ne e2: "All roots of E must be distinct";
    require Precision(e1) eq Precision(e2): "e1,e2 must have the same precision";
    e3 := -e1-e2;

    C := Parent(e1); i := C!Sqrt(-1);
    a := Sqrt(e1-e3);
    b := Sqrt(e1-e2);
    c := Sqrt(e2-e3);

    // Rearrange e1, e2, e3 (and thus redefine a, b, c) if necessary to make
    // w1, w2, w3 satisfy the 3-term relation |w1-w2-w3| = 0.
    // First, check if there is an equality among a,b,c (can happen at most once)
    equality := false;
    if Abs(a-b) eq Abs(a+b) then
        equality := true;
        if Abs(c-i*b) gt Abs(c+i*b) then
            b := -b;
        end if;
        if Abs(a-c) gt Abs(a+c) then
            a := -a;
        end if;
    elif Abs(c-i*b) eq Abs(c+i*b) then
        equality := true;
        if Abs(a-b) gt Abs(a+b) then
            b := -b;
        end if;
        if Abs(a-c) gt Abs(a+c) then
```

```
            c := -c;
        end if;
    elif Abs(a-c) eq Abs(a+c) then
        equality := true;
        if Abs(a-b) gt Abs(a+b) then
            a := -a;
        end if;
        if Abs(c-i*b) gt Abs(c+i*b) then
            c := -c;
        end if;
    end if;

    // No equality from this point, choose the sign of a arbitrarily
    // and choose b, c to satisfy other conditions
    if not equality then
        if Abs(a-b) gt Abs(a+b) then
            b := -b;
        end if;
        if Abs(a-c) gt Abs(a+c) then
            c := -c;
        end if;
        if Abs(c-i*b) gt Abs(c+i*b) then
            // Change the order to (e3, e2, e1);
            vprint Elog: "Changing the order of e1, e2, e3 ...";
            e1 := e3;
            vprint Elog: "new e1 = ", e1;
            return PeriodLattice([e1, e2] : Prec := Prec);
        end if;
    end if;
    // Now a, b, c are valid, proceed to compute w1, w2, w3
    vprintf Elog: "a = %o\nb = %o\nc = %o\n", a, b, c;
    pi := Pi(C);
    w1 := pi/AGM(a, b : Prec := Prec);
    w2 := pi/AGM(c, i*b : Prec := Prec);
    w3 := i*pi/AGM(a, c : Prec := Prec);
    vprintf Elog: "w1 = %o\nw2 = %o\nw3 = %o\n", w1, w2, w3;

    // Test relationship among w1, w2, w3;
    disc := Abs(w1-w2-w3);
    vprintf Elog: "|w1 - w2 - w3| = %o: ", disc;
    if disc lt 10^(-Prec) then
        vprint Elog: "OK";
    end if;
    return [w1,w2,w3];
end intrinsic;


/**
 * Given an elliptic curve E by a Weierstrass model over complex numbers,
 * compute the three minimal coset representatives of \Lambda modulo
 * 2*\Lambda, where \Lambda is the period lattice of E. Any two of them form
 * a \Z-basis for \Lambda.
 * Input:
 *   E = an elliptic curve over complex numbers
 * Output:
 *   [w1, w2, w3] = the three minimal coset representatives
 * Parameter:
 *   Prec = precision used as the stopping criterion when computing AGM
 **/
intrinsic PeriodLattice(E::CrvEll : Prec := 25) -> SeqEnum
{Given an elliptic curve E by a Weierstrass model defined over complex numbers,
compute all three minimal coset representatives; two of which form a \Z-basis
for the period lattice of E.}
    require Type(BaseRing(E)) eq FldCom: "E must be defined over complex numbers";
    // Transform E into the form Y^2 = 4*(X-e1)*(X-e2)*(X-e3) with e1+e2+e3=0
    e1, e2 := Explode(TransformModel(E));
    return PeriodLattice([e1, e2] : Prec := Prec);
end intrinsic


/**
```

```
 * Compute an elliptic logarithm of complex points on elliptic curves of the
 * form Y^2 = 4*(X-e1)*(X-e3)*(X-e3) with e1+e2+e3=0.
 * Input:
 *   E = [e1, e2]
 *   P = [x,y] = a point on E
 * Output:
 *   an elliptic logarithm of P
 * Parameter:
 *   Prec = precision used as the stopping criterion when computing AGM
 **/
intrinsic EllipticLog(E::SeqEnum, P::SeqEnum : Prec := 25) -> FldComElt
{Compute an elliptic logarithm of a point P = [x,y] on an elliptic curve of the
form Y^2 = 4*(X-e1)*(X-e3)*(X-e3) with e1+e2+e3=0.}
    // Verify if inputs are valid
    require #E eq 2: "E must contain exactly 2 complex numbers";
    require #P eq 2: "P must contain exactly 2 complex numbers";
    require (Prec ge 0) and (Prec in Integers()):
        "Precision must be a non-negative integer";
    e1, e2 := Explode(E);
    require e1 ne e2: "All roots of E must be distinct";
    require Precision(e1) eq Precision(e2): "e1,e2 must have the same precision";
    require Precision(e1) gt Prec: "Prec must be less than the precision of E";
    e3 := -e1-e2;
    x, y := Explode(P);
    require Precision(x) eq Precision(y): "x, y must have the same precision";
    require Precision(x) eq Precision(e1): "Precision of E and its point must be the same";

    if Round(Abs(y^2 / (4*(x-e1)*(x-e2)*(x-e3)))) ne 1 then
        error "P is not on E";
    end if;

    a := Sqrt(e1 - e3); b := Sqrt(e1 - e2);
    u := Sqrt(x - e3); v := Sqrt(x - e2);
    // Use two strongly optimal sequences - choose both (a,b), (u,v) to be good
    // If equality holds, either choice will do
    if Abs(a-b) gt Abs(a+b) then
        b := -b;
    end if;
    if Abs(u-v) gt Abs(u+v) then
        v := -v;
    end if;

    // Define t
    if x in [e1, e2, e3] then
        t := Sqrt(x - e1);
    else
        t := y/(2*u*v);
    end if;
    // Special case: t = 0
    if t eq 0 then
        return Pi(Parent(e1))/(2*AGM(a, b: Prec := Prec));
    end if;

    n := 0;
    repeat
        // Compute new a, b
        new_a := (a + b) / 2; new_b := Sqrt(a*b);
        a := new_a; b := new_b;
        // Optimal sequence: choose right choice at every step
        if Abs(a-b) gt Abs(a+b) then
            b := -b;
        end if;
        c := Sqrt(a^2 - b^2);

        // Compute new u, v, t
        u := (u + v)/2; v := Sqrt(u^2 - c^2);
        // Optimal sequence: choose right choice at every step
        if Abs(u-v) gt Abs(u+v) then
            v := -v;
        end if;
        t := u*t/v;
        n := n + 1;
        vprintf Elog: "|a_%o-b_%o| = %o\n", n, n, Abs(a-b);
        vprintf Elog: "|u_%o-v_%o| = %o\n", n, n, Abs(u-v);
    until Abs(a - b) lt (10^-Prec);
    // xinf := t^2 + 2/3*(a^2); yinf := 2*t*(t^2 + a^2);
    return -Arctan(a/t)/a;
end intrinsic;


/**
 * Compute an elliptic logarithm of complex points on elliptic curves
 * given by a Weierstrass equation [a1,a2,a3,a4,a6]
 * Input:
 *   E = an elliptic curve in standard Weierstrass form
 *   P = a point on E
 * Output:
 *   an elliptic logarithm of P
 * Parameter:
 *   Prec = precision used as the stopping criterion when computing AGM
 **/
intrinsic EllipticLog(E::CrvEll, P::PtEll : Prec := 25) -> FldComElt
{Compute an elliptic logarithm of a point P on E, where E is an elliptic
curve over C given by a Weierstrass equation.}
    if P eq Identity(E) then
        return BaseRing(E)!0;
    end if;
    newE, newP := TransformModel(E, P);
    return EllipticLog(newE, newP : Prec := Prec);
end intrinsic;


/*****************************************************************************
 * Auxiliary intrinsic functions
 *****************************************************************************/

/**
 * Transform an elliptic curve given by a Weierstrass model [a1,a2,a3,a4,a6]
 * (defined over C) into the form E': Y^2 = 4*(X-e1)*(X-e2)*(X-e3) with
 * e1+e2+e3=0.
 * Input:
 *   E = an elliptic curve of the form [a1, a2, a3, a4, a6]
 * Output:
 *   [e1, e2]
 **/
intrinsic TransformModel(E::CrvEll) -> SeqEnum
{Transform an elliptic curve over C given by a Weierstrass model [a1,a2,a3,a4,a6]
into the form E': Y^2 = 4*(X-e1)*(X-e2)*(X-e3) with e1+e2+e3=0.}
    C := BaseRing(E);
    require Type(C) eq FldCom: "E must be defined over complex numbers";
    _<t> := PolynomialRing(C);
    c4, c6 := Explode(cInvariants(E));
    R := Roots(t^3 - (c4/48)*t - c6/864);
    e1 := R[1][1]; e2 := R[2][1];
    return [e1, e2];
end intrinsic;


/**
 * Transform an elliptic curve given by a Weierstrass model [a1,a2,a3,a4,a6]
 * (defined over C) into the form E': Y^2 = 4*(X-e1)*(X-e2)*(X-e3) with
 * e1+e2+e3=0, and map a given point P on E to its image on E'
 * Input:
 *   E = an elliptic curve of the form [a1, a2, a3, a4, a6]
 *   P = a point on E
 * Output:
 *   [e1, e2], [X,Y], where [X,Y] is the image of P on E'
 **/
intrinsic TransformModel(E::CrvEll, P::PtEll) -> SeqEnum, SeqEnum
{Transform an elliptic curve over C given by a Weierstrass model [a1,a2,a3,a4,a6]
into the form E': Y^2 = 4*(X-e1)*(X-e2)*(X-e3) with e1+e2+e3=0, and map a point
P on E to its image on E'.}
```

```
    require P ne Identity(E): "P must not be the point at infinity";
    newE := TransformModel(E);
    a1, _, a3, _, _ := Explode(aInvariants(E));
    b2 := bInvariants(E)[1];
    X := P[1] + b2/12;
    Y := 2*P[2] + a1*P[1] + a3;
    return newE, [X, Y];
end intrinsic;


/**
 * Reduce any given z into the one inside the fundamental parallelogram
 * spanned by L = [w1, w2], and return [a, b] such that
 * reduced z = z' = a*w1 + b*w2, with 0 <= a,b < 1
 * Input:
 *   L = [w1, w2] = fundamental parallelogram
 *   z = the complex number to be reduced
 * Output:
 *   z' = the reduced version of z inside the paralellogram
 *   [a, b] = coordinates of new z on the parallelogram
 **/
intrinsic Reduce2FP(L::SeqEnum, z::FldComElt) -> FldComElt, SeqEnum
{Reduce z modulo the lattice L}
    require #L eq 2: "L must have exactly two numbers";

    w1, w2 := Explode(L); tau := w2/w1;
    denom := w1;
    IsDenomW1 := true;
    if Im(tau) eq 0 then
        error "w2/w1 must not be real";
    elif Im(tau) lt 0 then
        tau := 1/tau;
        IsDenomW1 := false;
        denom := w2;
    end if;

    // Reduce z into the fundamental parallelogram
    z := z/denom;
    if IsDenomW1 then
        beta := Im(z)/Im(tau);
        alpha := Re(z - beta*tau); // already real, just put it to avoid error
        z := z - Floor(beta)*tau - Floor(alpha);
    else
        alpha := Im(z)/Im(tau);
        beta := Re(z - alpha*tau); // already real, just put it to avoid error
        z := z - Floor(alpha)*tau - Floor(beta);
    end if;
    beta := beta - Floor(beta);
    alpha := alpha - Floor(alpha);
    z := z*denom;
    return z, [alpha, beta];
end intrinsic;


/**
 * Given a period lattice L = <w1,w2>, apply linear transformation in SL(2,\Z)
 * so that we obtain a new basis {w1', w2'} with tau = w2'/w1' in the
 * fundamental domain
 * (Based on Algorithm 7.4.2 in Cohen's "A Course in Computational Algebraic
 * Number Theory")
 * Input:
 *   L = [w1, w2]
 * Output:
 *   [w1', w2'] = a new basis with the above property
 *   A = transformation matrix where [w2' w1']^T = A*[w2 w1]^T
 **/
intrinsic TransformLattice(L::SeqEnum) -> SeqEnum, Mtrx
{Apply linear transformation to a basis for a lattice so that the new basis
\{w1',w2'\} has tau = w2'/w1' in the fundamental domain of SL(2,\Z)}
    w1, w2 := Explode(L);
    tau := w2/w1;
    require Im(tau) gt 0: "Im(w2/w1) must be positive";
    A := Matrix(2, [1,0,0,1]);
    while true do
        n := Round(Re(tau));
        tau -:= n;
        A := Matrix(2, [1,-n,0,1]) * A;

        if Abs(tau) ge 1 then
            break;
        end if;

        tau := -1/tau;
        A := Matrix(2, [0,-1,1,0]) * A;
    end while;
    new_w2 := A[1,1]*w2 + A[1,2]*w1;
    new_w1 := A[2,1]*w2 + A[2,2]*w1;
    return [new_w1, new_w2], A;
end intrinsic;
```

A.2 Integral Points on Elliptic Curves

The following file is our implementation (with some modifications) of Smart and Stephens' algorithm [SS97] for finding integral points on elliptic curves over number fields. The only main function in this file is `IntegralPoints()`; see the documentation inside the code for more details.

```
/******************************************************************************
 * intpts.m
 * Computing all integral points on elliptic curves over number fields
 * Based on Smart & Stephens' paper (Math. Proc. Camb. Phil. Soc. 122 (1997),
 * pp. 9-16) with some modifications
 *
 * By Thotsaphon Thongjunthug
 * Last updated: 07 December 2010
 * Any errors should be reported to <nookaussie@yahoo.com>
 *
 * Required packages:
 * 1) elog.m - for computing periods and elliptic logarithms
 * 2) John Cremona's "nfhtbound.m" - for Cremona-Prickett-Siksek's bound
 *    (available freely on his webpage at
 *    <http://www.warwick.ac.uk/~masgaj/ftp/progs/magma/index.html>)
 ******************************************************************************/

/**
 * Declare printing verbose
 * 0 = return result only, no detail printed
 * 1 = minimal amount of details shown
 * 2 = all details (e.g. values of all constants) shown (for debugging only)
 **/
declare verbose Intpts, 2;


/******************************************************************************
 * Auxiliary local functions I:
 * computing constants for an upper bound on linear form in logarithm.
 * The indices of c are as defined in Smart & Stephens' paper
 ******************************************************************************/

/**
 * Compute constant c3
 * Input:
```

```
 *   L = a sequence of points on elliptic curves over number field
 * Output:
 *   minimum eigenvalue of the height paring matrix
 **/
function c3(L)
    M := Eigenvalues(HeightPairingMatrix(L));
    M := SetToSequence(M);
    M := [m[1] : m in M]; // ignore multiplicity
    M := Minimum(M); // least eigenvalue
    return M;
end function;


/**
 * Compute constant c6
 * Input:
 *   E = elliptic curve over real/complex numbers in standard Weierstrass form
 * Require:
 *   elog.m
 **/
function c6(E)
    R := TransformModel(E);
    Append(~R, -R[1]-R[2]);
    R := [Abs(r) : r in R];
    R := 2*Maximum(R);
    return R;
end function;


/**
 * Compute constant c8, and the periods w1, w2 of the period lattice of a
 * given elliptic curve E (with w2/w1 in the fundamental domain).
 * Input:
 *   E = elliptic curve over real/complex numbers in standard Weierstrass form
 * Output:
 *   L = the constant c8
 *   [w1, w2] = periods of E
 * Parameter:
 *   Prec = precision for computing periods
 * Require:
 *   elog.m
 **/
function c8(E : Prec := 25)
    w1, w2, _ := Explode(PeriodLattice(E : Prec := Prec));
    if Im(w2/w1) gt 0 then
        L := [w1, w2];
    else
        L := [w2, w1];
    end if;
    // Apply transformation by SL(2,Z) so that tau = w2/w1 is in the
    // fundamental domain
    L, _ := TransformLattice(L);
    wList := [L[1], L[2], L[1]+L[2]];
    wList := [Abs(w) : w in wList];
    gamma := Maximum(wList);
    return gamma, L;
end function;


/**
 * Compute absolute logarithmic height of an element in n-projective space
 * over a number field K
 * Input:
 *   X = a sequence of elements in a number field K
 **/
function AbsLogHeight(X)
    // X cannot be zero vector
    if #X eq #[x : x in X | x eq 0] then
        error "X cannot be zero vector";
    end if;
    // Find all prime ideal that divides some denominators of x_i
    I := {};
    K := Parent(X[1]); // assume each x_i is in the same field
    O := RingOfIntegers(K);
    for x in X do
        den := O ! Denominator(x);
        L := Decomposition(den);
        L := {l[1] : l in L}; // ignore multiplicity
        I := I join L;
    end for;

    h := 0;
    // Non-archimedean contributions
    for p in I do
        M := [Rationals()| ];
        for x in X do
            if x eq 0 then
                Append(~M, 0);
            else
                Append(~M, Norm(p)^(-Valuation(x, p)));
            end if;
        end for;
        M := Maximum(M);
        M := Log(M);
        h +:= M;
    end for;

    // Archimedean contributions
    s1, s2 := Signature(K);
    // M = [log max(|x_1|_v,..., |x_n|_v) : v in M_K]
    M := [];
    for x in X do
        C := Conjugates(x);
        newC := [];
        // Real embedding contributions
        for i := 1 to s1 do
            Append(~newC, Abs(C[i]));
        end for;
        // Complex embedding contributions
        for i := 1 to s2 do
            Append(~newC, Abs(C[s1+(2*i-1)])^2);
        end for;

        if #M eq 0 then
            M := newC;
        else
            for i := 1 to (s1+s2) do
                if newC[i] gt M[i] then
                    M[i] := newC[i];
                end if;
            end for;
        end if;
    end for;
    M := [Log(m) : m in M];

    // Overall absolute logarithmic height
    h +:= (&+M);
    h /:= Degree(K);
    return h;
end function;


/*******************************************************************************
 * Auxiliary local functions II:
 * Computing constants for David's lower bound on linear form in logarithms.
 * Notation used as in Appendix A of Smart's book
 * "The Algorithmic Resolution of Diophantine Equations", with c's being
 * replaced by d's
 ******************************************************************************/

/**
```

```
 * Compute the "height" of elliptic curve
 * Input:
 *   E = elliptic curve over number field in standard Weierstrass form
 **/
function h_E(E)
    j := jInvariant(E);
    C4, C6 := Explode(cInvariants(E));
    g2 := C4/12;
    g3 := C6/216;
    m := Maximum([1, AbsLogHeight([1, g2, g3]), AbsLogHeight([1, j])]);
    return m;
end function;


/**
 * Compute the list of modified height of a point P in E(K)
 * depending on embedding
 * Input:
 *   E = elliptic curve over a number field K in standard Weierstrass form
 *   P = a point in E(K)
 *   ElogEmbedP = elliptic logarithm of the image of P in some embedding
 *   D7 = constant d7 (depending on embedding)
 **/
function h_m(E, P, ElogEmbedP, D7)
    K := BaseRing(E);
    L := [Height(P), h_E(E), D7/Degree(K)*Abs(ElogEmbedP)^2];
    L := Maximum(L);
    return L;
end function;


/**
 * Compute two extra h_m's based on the two periods
 * Similar to h_m, but now ElogEmbedP becomes a period of the fundamental
 * parallelogram of some embedding of E
 * Input:
 *         E = elliptic curve over a number field in standard Weierstrass form
 *   Periods = [w1, w2] = period lattice of some real/complex embedding of E
 *        D7 = constant d7 (depending on embedding)
 **/
function Extra_h_m(E, Periods, D7)
    D := Degree(BaseRing(E));
    h := h_E(E);
    h1 := Maximum([0, h, D7/D*Abs(Periods[1])^2]);
    h2 := Maximum([0, h, D7/D*Abs(Periods[2])^2]);
    return [h1, h2];
end function;


/**
 * Compute constant d8
 * Input:
 *       E = elliptic curve over a number field K in standard Weierstrass form
 *       L = a sequence of points in E(K) (e.g. Mordell-Weil basis)
 *    Elog = a sequence of (pre-computed) elliptic logarithms of points in L
 *           on some fixed embedding
 * Periods = [w1, w2] = period lattice of some embedding of E
 *      D7 = constant d7 (depending on embedding)
 **/
function d8(E, L, Elog, Periods, D7)
    C := [Exp(1)*h_E(E)];
    D := Degree(BaseRing(E));
    for i := 1 to #L do
        Append(~C, h_m(E, L[i], Elog[i], D7)/D);
    end for;
    C := C cat [t/D : t in Extra_h_m(E, Periods, D7)];
    C := Maximum(C);
    return C;
end function;
```

```
/**
 * Compute constant d9
 * Input:
 *   E = elliptic curve over a number field in standard Weierstrass form
 *   L = a sequence of points in E(K) (e.g. Mordell-Weil basis)
 *   Elog = a sequence of (pre-computed) elliptic logarithms of points in L
 *          on some fixed embedding
 *   Periods = [w1, w2] = period lattice of some embedding of E
 *   D7 = constant d7 (depending on embedding)
 **/
function d9(E, L, Elog, Periods, D7)
    D := Degree(BaseRing(E));
    C := [];
    for i := 1 to #L do
        tmp := Exp(1) * Sqrt(D * h_m(E, L[i], Elog[i], D7) / D7);
        tmp /:= Abs(Elog[i]);
        C[i] := tmp;
    end for;

    // Take minimum among extra_h_m
    Ehm := Extra_h_m(E, Periods, D7);
    tmp1 := Exp(1) * Sqrt(D*Ehm[1]/D7) / Abs(Periods[1]);
    tmp2 := Exp(1) * Sqrt(D*Ehm[2]/D7) / Abs(Periods[2]);
    C := C cat [tmp1, tmp2];
    C := Minimum(C);
    return C;
end function;


/**
 * Compute constant d10
 * Input:
 *   E = elliptic curve over a number field in standard Weierstrass form
 *   L = a sequence of points in E(K) (e.g. Mordell-Weil basis)
 *   Elog = a sequence of (pre-computed) elliptic logarithms of points in L
 *          on some fixed embedding
 *   Periods = [w1, w2] = period lattice of some embedding of E
 *   D7 = constant d7 (depending on embedding)
 **/
function d10(E, L, Elog, Periods, D7)
    D := Degree(BaseRing(E));
    n := #L+2;
    scalar := 2 * 10^(8 + 7*n) * (2/Exp(1))^(2*n^2);
    scalar *:= (n+1)^(4*n^2 + 10*n) * D^(2*n + 2);
    scalar *:= (Log(d9(E, L, Elog, Periods, D7)))^(-2*n-1);
    for i := 1 to #L do
        scalar *:= h_m(E, L[i], Elog[i], D7);
    end for;
    scalar *:= &*(Extra_h_m(E, Periods, D7));
    return scalar;
end function;


/**
 * Compute the right-hand side of the Principal Inequality
 * Input:
 *   D = Degree(K), where K = number field
 *   r = rank(E(K))
 *  C9 = constant c9
 * C10 = constant c10
 *  D9 = constant d9
 * D10 = constant d10
 *   h = h_E(E), where E = elliptic curve over K
 *   Q = initial bound for the coefficients of P_i's, where P_i's are points
 *       in a Mordell-Weil basis for E(K)
 * expTors = exponent of the torsion subgroup of E(K)
 **/
function RHS(D, r, C9, C10, D9, D10, h, Q, expTors)
    bound := (Log(Log(Q*r*expTors)) + h + Log(D*D9))^(r+3);
    bound *:= D10*(Log(Q*r*expTors) + Log(D*D9));
    bound +:= Log(C9*expTors);
    bound /:= C10;
    return bound;
end function;


/**
 * Approximate initial bound on Q = max_{1 <= i <= r}{|q_i|}
 * Input:
 *   D = Degree(K), where K = number field
 *   r = rank of E(K)
 *  Q0 = constant Q0 (in S&S paper, this is called K0)
 *  C9 = constant c9
 * C10 = constant c10
 *  D8 = constant d8  (from function d8())
 *  D9 = constant d9  (from function d9())
 * D10 = constant d10 (from function d10())
 *   h = h_E(E)
 * expTors = exponent of the torsion subgroup of E(K)
 *
 * Revised: 05 May 2009
 **/
function InitialQ(D, r, Q0, C9, C10, D8, D9, D10, h, expTors)
    minQ := Maximum(Q0, Exp(D8));

    // Try to approximate Q such that Q^2 = RHS(Q) (i.e. Q makes both sides
    // of the Principal Inequality equal)
    // Firstly, set a guess for Q, say minQ + 1 (so that Q > minQ)
    // For simplicity, let's round Q up to the nearest power of 10
    Q := minQ + 1;
    x := Ceiling(Log(10, Q)); // x = log_10(Q)

    // Check if Q satisfies the Principal Inequality, i.e. if Q^2 < RHS(Q)
    // If so, repeat with the larger Q until we find the first Q that
    // violates the Principal Inequality
    // N.B. This loop will eventually terminate
    exp_OK := 0;   // the exponent that satisfies P.I.
    exp_fail := 0; // the exponent that fails P.I.
    while 10^(2*x) lt RHS(D, r, C9, C10, D9, D10, h, 10^x, expTors) do
        exp_OK := x; // Principal Inequality satisfied
        x *:= 2;     // double x, and retry
    end while;
    exp_fail := x; // x that fails the Principal Inequality

    // So now x = log_10(Q) must lie between exp_OK and exp_fail
    // Refine x further using "binary search"
    repeat
        x := Floor((exp_OK + exp_fail)/2);
        if 10^(2*x) ge RHS(D, r, C9, C10, D9, D10, h, 10^x, expTors) then
            exp_fail := x;
        else
            exp_OK := x;
        end if;
    until (exp_fail - exp_OK) le 1;
    return exp_fail; // over-estimate
end function;


/**
 * Reduce the bound Q by LLL reduction until no further improvement
 * is possible. This function initially requires high precision to
 * proceed, although this should be done automatically by now
 * Input:
 *   Pts = sequence of points in E(K)
 *   j = j-th embedding (based on the index used in Conjugates()
 *   EmbedL = a sequence of points on EmbedE (e.g. points in a Mordell-Weil
 *            basis when embedded into EmbedE)
 *   Elog = a sequence of (pre-computed) elliptic logarithms of points in EmbedL
 *   C9 = constant c9
 *   C10 = constant c10
 *   Periods = [w1, w2] = period lattice of EmbedE
 *   expTors = exponent of the torsion subgroup of E(K), K = number field
 *   initQ = initial guess for Q to be reduced by LLL
 **/
function ReducedQ(Pts, j, EmbedL, Elog, C9, C10, Periods, expTors,
    initQ)
    r := #EmbedL;
    newQ := initQ;
    EmbedE := Curve(EmbedL[1]);

    // Repeat LLL reduction until no further reduction is possible
    repeat
        Q := newQ;
        S := r*(Q^2)*(expTors^2);
        T := 3*r*Q*expTors/Sqrt(2);

        // Create the basis matrix
        C := 1;
        repeat
            C *:= Q^Ceiling((r+2)/2);
            L := ZeroMatrix(Integers(), r+2, r+2);

            // Elliptic logarithm should have precision "suitable to" C
            // e.g. If C = 10^100, then Re(Elog[i]) should be computed
            // correctly to at least 100 decimal places
            pow10_C := Ceiling(Log(10, C));
            if pow10_C gt Precision(Elog[1]) then
        vprint Intpts, 2:
                    "Need higher precision, recompute elliptic logarithm ...";
                // Re-compute periods and elliptic logarithms
                // to the right precision
                // First, re-embed E into higher precision
                E := Curve(Pts[1]); // elliptic curve over number field
                a1, a2, a3, a4, a6 := Explode(aInvariants(E));
                a1 := Conjugate(a1, j : Precision := pow10_C+10);
                a2 := Conjugate(a2, j : Precision := pow10_C+10);
                a3 := Conjugate(a3, j : Precision := pow10_C+10);
                a4 := Conjugate(a4, j : Precision := pow10_C+10);
                a6 := Conjugate(a6, j : Precision := pow10_C+10);
                EmbedE := EllipticCurve([a1, a2, a3, a4, a6]);
                EmbedL := [];
                _, Periods := c8(EmbedE : Prec := pow10_C);
                X := [Conjugates(P[1] : Precision := pow10_C+10) : P in Pts];
                Y := [Conjugates(P[2] : Precision := pow10_C+10) : P in Pts];
                for i := 1 to #Pts do
                    P := Points(EmbedE, X[i][j])[1];
                    if Abs(P[2] - Y[i][j]) lt 10^(pow10_C/2) then
                        Append(~EmbedL, P);
                    else
                        Append(~EmbedL, -P);
                    end if;
                end for;
                Elog := [EllipticLog(EmbedE, P : Prec := pow10_C) :
                    P in EmbedL];
                vprint Intpts, 2: "Elliptic logarithm recomputed";
            end if;
            w1, w2 := Explode(Periods);

            // Assign all non-zero entries
            for i := 1 to r do
                L[i,i] := 1;
                L[r+1, i] := Truncate(C*Re(Elog[i]));
                L[r+2, i] := Truncate(C*Im(Elog[i]));
            end for;
            L[r+1, r+1] := Truncate(C*Re(w1));
            L[r+1, r+2] := Truncate(C*Re(w2));
            L[r+2, r+1] := Truncate(C*Im(w1));
            L[r+2, r+2] := Truncate(C*Im(w2));
            L := Transpose(L); // In Magma, basis is spanned by row vector!

            // LLL reduction and constants
            L := LLL(L);
            b1 := L[1]; // 1st row of reduced basis
            // Norm(b1) = square of Euclidean norm
            lhs := 2^(-r-1)*Norm(b1) - S;
        until (lhs ge 0) and (Sqrt(lhs) gt T);

        newQ := ( Log(C*C9*expTors) - Log(Sqrt(lhs) - T) ) / C10;
        newQ := Floor(Sqrt(newQ));
        pow10 := Floor(Log(10, C));
        vprintf Intpts, 2: "Choose C = %.4o x 10^%o. ", 1.*C/10^pow10, pow10;
        vprintf Intpts, 2: "After reduction, Q <= %o\n", newQ;
    until ((Q - newQ) le 1) or (newQ le 1);
    return newQ;
end function;



/*****************************************************************************
 * Main intrinsic functions
 *****************************************************************************/

/**
 * Search for all integral points on elliptic curves over number fields
 * within a given bound
 * Input:
 *   E = elliptic curve over a number field K in standard Weierstrass form
 *   L = a sequence of points in a Mordell-Weil basis for E(K)
 *   Q = a maximum for the absolute bound on all coefficients
 *       in the linear combination of points in L
 * Output:
 *   S1 = sequence of all integral points on E(K) modulo [-1]
 *   S2 = sequence of tuples representing the points as a linear combination
 *        of points in L
 * Parameter:
 *   Prec = Precision used for checking integrality of points.
 *          (Default = 0 - only exact arithmetic will be performed)
 **/
intrinsic IntegralPoints(E::CrvEll, L::[PtEll], Q::RngIntElt : Prec := 0) ->
  SeqEnum, SeqEnum
{Given an elliptic curve E over a number field, its Mordell-Weil basis L, and
a positive integer Q, return the sequence of all integral points modulo [-1]
of the form P = q_1*L[1] + ... + q_r*L[r] + T with some torsion point T and
|q_i| <= Q, followed by a sequence of tuple sequences representing the points
as a linear combination of points. An optional tolerance may be given to speed
up the computation when checking integrality of points.}

    // Check input validity
    require IsNumberField(BaseRing(E)):
        "Elliptic curve must be defined over a number field";
    require (Prec ge 0) and (Prec in Integers()):
        "Precision must be a non-negative integer";
    require &and[P in E : P in L]: "All points in L must be in E(K)";

    // Find the generators of the torsion subgroup of E(K)
    Tors, map := TorsionSubgroup(E);
    expTors := Exponent(Tors);
    G := Generators(Tors);
    if (#L eq 0) and (#G eq 0) then
        return [], []; // nothing to do
    end if;
    Tors := [map(g) : g in G];   // each generator of E(K)_tors
    OrdG := [Order(g) : g in G]; // order of each generator

    // Create all possible (r+#Tors)-tuples
    r := #L; // r = rank of E(K)
    C := [0 : i in [1..(r + #Tors)]];
    ListC := [];
    for i := 0 to Q do
        C[1] := i;
        Append(~ListC, C);
    end for;

    for i := 2 to r do
```

```
        tmp := [];
        for j := 1 to Q do
            for l in ListC do
                tup := l;
                tup[i] := j;
                Append(~tmp, tup);

                // Avoid having its negative in the list
                // Only use when all previous entries in tuple are zero
                for k := 1 to i-1 do
                    if tup[k] ne 0 then
                        tup[i] := -j;
                        Append(~tmp, tup);
                        break;
                    end if;
                end for;

            end for;
        end for;
        ListC := ListC cat tmp;
    end for;

    // Add torsion point, if any
    if #Tors ne 0 then
        for i := 1 to #Tors do
            tmp := [];
            for j := 1 to (OrdG[i]-1) do
                for l in ListC do
                    tup := l;
                    tup[r+i] := j;
                    Append(~tmp, tup);
                end for;
            end for;
            ListC := ListC cat tmp;
        end for;
    end if;
    Remove(~ListC, 1); // remove point at infinity

    L := L cat Tors;
    vprint Intpts, 2: "Generators = ", L;
    PtsList := [];
    CoeffList := [];

    // Skip the complex arithmetic and only perform exact arithmetic if tol = 0
    if Prec eq 0 then
        vprint Intpts : "Exact arithmetic";
        for l in ListC do
            P := &+[l[i]*L[i] : i in [1..#L]];
            if IsIntegral(P[1]) and IsIntegral(P[2]) then
                vprintf Intpts: "%o ---> %o\n", l, P;
                Append(~PtsList, P);
                TupList := [ <L[i], l[i]> : i in [1..#L] | l[i] ne 0 ];
                Append(~CoeffList, TupList);
            end if;
        end for;
        vprint Intpts: "*"^45;
        return PtsList, CoeffList;
    end if;

    // Suggested by John Cremona
    // Point search. This is done via arithmetic on complex points on each
    // embedding of E. Exact arithmetic will be carried if the resulting
    // complex points are "close" to being integral, subject to some precision

    // Embed each generator of the torsion subgroup
    basePrec := Maximum([30, Prec+10]);
    X := [Conjugates(P[1] : Precision := basePrec) : P in (L cat Tors)];
    Y := [Conjugates(P[2] : Precision := basePrec) : P in (L cat Tors)];
    // Create all embeddings of E
    K := BaseRing(E);
    s1, s2 := Signature(K);
```

```
a1, a2, a3, a4, a6 := Explode(aInvariants(E));
a1 := Conjugates(a1 : Precision := basePrec);
a2 := Conjugates(a2 : Precision := basePrec);
a3 := Conjugates(a3 : Precision := basePrec);
a4 := Conjugates(a4 : Precision := basePrec);
a6 := Conjugates(a6 : Precision := basePrec);
EmbedEList := [EllipticCurve([a1[j], a2[j], a3[j], a4[j], a6[j]]): j in
    [1..(s1+2*s2)]];

// Use precision to decide a possibility of "being integral".
// Note that too large precision may lead to missing some integral points,
// while too small precision may slow the computation.
vprint Intpts: "Precision = ", Prec;

// Create the matrix containing all embeddings of the integral basis of K
// as its columns
IntBasis := IntegralBasis(K);
C := ComplexField(basePrec);
B := Matrix(C, #IntBasis, Degree(K),
    [Conjugates(a : Precision := basePrec) : a in IntBasis]);
// Note that B is always invertible, so we can take its inverse
B := B^(-1);

// Modified on 29-30 May 2009
// For each possible tuple representing the coefficients in a linear
// combination of points in L, compute x(P) on each embedding and store
// them. This is to:
// 1) avoid any possible risks of Magma being unable to recognise complex
//    points when calling them from a nested sequence, as happen in the
//    previous version, and
// 2) maintain the same (or even faster) computational speed.
// Technically, the drawback of this version is that it consumes much more
// memory than the previous one, although this is not really a huge problem
// in practice for most computers these days
x_tuple := [(C!0) : i in [1..Degree(K)]];
x_coord := [x_tuple : i in [1..#ListC]];
for j in ([1..s1] cat [s1+1..s1+2*s2 by 2]) do
    // Create the embedding of each point in L (one embedding at a time)
    EmbedL := [];
    for i := 1 to (r + #Tors) do
        P := Points(EmbedEList[j], X[i][j])[1];
        // Choose the right sign for the y-coordinate
        //if (Y[i][j] ne 0) and (Re(P[2]/Y[i][j]) lt 0) then
        if Abs(P[2] - Y[i][j]) lt 10^-(basePrec/2) then
            Append(~EmbedL, P);
        else
            Append(~EmbedL, -P);
        end if;
    end for;

    for n in [1..#ListC] do
        l := ListC[n];
        P := &+[l[i]*EmbedL[i] : i in [1..#L]];
        x_coord[n][j] := P[1];
        // For each pair of complex embeddings, we only have to
        // compute P on just one embedding in the pair. Another P on
        // another embedding is simply the complex conjugate
        if j gt s1 then
            x_coord[n][j+1] := Conjugate(P[1]);
        end if;
    end for;
end for;

// Point search
for n in [1..#ListC] do
    // Check if the x-coordinate of P is "close to" being integral
    // If so, compute P exactly and check if it is integral; skip P otherwise
    XofP := Matrix([x_coord[n]]);
    // Write x(P) w.r.t. the integral basis of (K)
    // Due to the floating arithmetic, some entries in LX may have very tiny
    // imaginary part, which can be thought as zero
```

```
        LX := XofP * B;
        LX := [Abs( Re(LX[1,i]) - Round(Re(LX[1,i])) ): i in [1..#IntBasis]];
        LX := &and[dx lt 10^-Prec : dx in LX];
        if not LX then
            // x-coordinate of P is not integral, skip P
            continue;
        end if;

        // Now check P by exact arithmetic
        // Add P and the list of tuples representing P into the list
        // if P is integral
        l := ListC[n];
        P := &+[l[i]*L[i] : i in [1..#L]];
        if IsIntegral(P[1]) and IsIntegral(P[2]) then
            vprintf Intpts: "%o ---> %o\n", l, P;
            Append(~PtsList, P);
            TupList := [ <L[i], l[i]> : i in [1..#L] | l[i] ne 0 ];
            Append(~CoeffList, TupList);
        end if;
    end for;
    vprint Intpts: "*"^45;
    return PtsList, CoeffList;
end intrinsic;


/**
 * Compute all integral points on elliptic curve over a number field.
 * This function simply computes a suitable bound Q, and return
 * IntegralPoints(E, L, Q : tol := ...).
 * Input:
 *   E = elliptic curve over a number field K in standard Weierstrass form
 *   L = a sequence of points in the Mordell-Weil basis for E(K)
 * Output:
 *   S1 = sequence of all integral points on E(K) modulo [-1]
 *   S2 = sequence of tuples representing the points as a linear combination
 *        of points in L
 * Parameter:
 *   Prec = precision used for checking integrality of points.
 *         (Default = 0 - only exact arithmetic will be performed)
 * Require:
 *   elog.m - for computing elliptic logarithms
 *   nfhtbound.m - for computing Cremona-Prickett-Siksek height bounds
 **/
intrinsic IntegralPoints(E::CrvEll, L::[PtEll] : Prec := 0) -> SeqEnum, SeqEnum
{Given an elliptic curve over a number field and its Mordell-Weil basis, return
the sequence of all integral points modulo [-1], followed by a sequence of tuple
sequences representing the points as a linear combination of points. An optional
tolerance may be given to speed up the computation when checking integrality of
points. (This function simply computes a suitable Q and call
IntegralPoints(E, L, Q: tol := ...)}

    // Check input validity
    require IsNumberField(BaseRing(E)):
        "Elliptic curve must be defined over a number field";
    require (Prec ge 0) and (Prec in Integers()):
        "Precision must be a non-negative integer";
    require &and[P in E : P in L]: "All points in L must be in E(K)";

    if #L eq 0 then
        return IntegralPoints(E, [], 0 : Prec := Prec);
    end if;

    K := BaseRing(E);
    s1, s2 := Signature(K);
    a1, a2, a3, a4, a6 := Explode(aInvariants(E));

    // Set initial precision for computing embeddings, periods, elliptic logs
    // N.B. We only require high-precision elliptic logs during LLL process,
    // so if our initial precision is already high enough, then we do not need
    // to re-compute them again.
    initPrec := 300; // this can be changed arbitrarily
```

```
// Embed E into various (real/complex) embeddings.
a1 := Conjugates(a1 : Precision := initPrec);
a2 := Conjugates(a2 : Precision := initPrec);
a3 := Conjugates(a3 : Precision := initPrec);
a4 := Conjugates(a4 : Precision := initPrec);
a6 := Conjugates(a6 : Precision := initPrec);
b2 := Conjugates(bInvariants(E)[1] : Precision := initPrec);
pi := Pi(RealField(initPrec));

// Embed generators in the Mordell-Weil basis
X := [Conjugates(P[1] : Precision := initPrec) : P in L];
Y := [Conjugates(P[2] : Precision := initPrec) : P in L];

// Find the generators of the torsion subgroup of E(K)
Tors, map := TorsionSubgroup(E);
expTors := Exponent(Tors);
G := Generators(Tors);
Tors := [map(g) : g in G]; // generators of torsion subgroup
OrdG := [Order(g) : g in G]; // their orders

// Global constants (i.e. do not depend on any embedding of E)
C2 := -CPSLowerHeightBound(E);
C3 := c3(L);
h := h_E(E);
vprint Intpts, 2: "Global constants";
vprintf Intpts, 2: "c2 = %.4o\n", C2;
vprintf Intpts, 2: "c3 = %.4o\n", C3;
vprintf Intpts, 2: "h_E = %.4o\n", h;
vprint Intpts, 2: "-"^45;

Q := [];
// Find the most reduced bound on each embedding of E
// But first let's adjust the index
for i := 1 to (s1+s2) do
    if i le s1 then
        j := i;
        nv := 1;
        vprintf Intpts, 2: "Real embedding #%o\n", j;
    else
        j := s1 + (2*(i-s1)-1);
        nv := 2;
        vprintf Intpts, 2: "Complex embedding #%o\n", i-s1;
    end if;

    // Create complex embedding of E
    ee := EllipticCurve([a1[j], a2[j], a3[j], a4[j], a6[j]]);
    // Local constants (depending on embedding)
    // C9, C10 are used for the upper bound on the linear form in logarithm
    C4 := C3 * Degree(K) / (nv*(s1+s2));
    vprintf Intpts, 2: "c4 =  %.4o\n", C4;
    C5 := C2 * Degree(K) / (nv*(s1+s2));
    vprintf Intpts, 2: "c5 = %.4o\n", C5;
    C6 := c6(ee);
    vprintf Intpts, 2: "c6 = %.4o\n", C6;
    delta := 1 + (nv-1)*pi;
    C8, Periods := c8(ee : Prec := initPrec-10);
    vprintf Intpts, 2: "c8 = %.4o\n", C8;
    // N.B. Periods w1, w2 are such that w2/w1 is in the fundamental domain
    vprintf Intpts, 2: "Periods = %o\n", Periods;
    C7 := 8*(delta^2) + (C8^2)*Abs(b2[j])/12;
    vprintf Intpts, 2: "c7 = %.4o\n", C7;
    C9 := C7*Exp(C5/2);
    vprintf Intpts, 2: "c9 = %.4o\n", C9;
    C10 := C4/2;
    vprintf Intpts, 2: "c10 = %.4o\n", C10;
    Q0 := Sqrt( ( Log(C6+Abs(b2[j])/12) + C5) / C4 );
    vprintf Intpts, 2: "Q0 = %.4o\n", Q0;

    // Constants for David's lower bound on the linear form in logarithm
    w1, w2 := Explode(Periods);
    EmbedL := [];
```

```
        // Find images of all points in a Mordell-Weil basis for E(K) on
        // the embedding ee, and then compute complex elliptic logarithms.
        for k := 1 to #L do
            P := Points(ee, X[k][j])[1];
            if Abs(P[2] - Y[k][j]) lt 10^(initPrec/2) then
                Append(~EmbedL, P);
            else
                Append(~EmbedL, -P);
            end if;
        end for;
        //EmbedL := [[X[i][j], Y[i][j]] : i in [1..#L]];
        vprintf Intpts, 2: "Computing elliptic logarithms...";
        Elog := [EllipticLog(ee, P: Prec := initPrec-10) : P in EmbedL];
        vprint Intpts, 2: " : Done";

        D7 := 3*pi / ((Abs(w2)^2) * Im(w2/w1));
        vprintf Intpts, 2: "d7 = %.4o\n", D7;
        D8 := d8(E, L, Elog, Periods, D7);
        vprintf Intpts, 2: "d8 = %.4o\n", D8;
        D9 := d9(E, L, Elog, Periods, D7);
        vprintf Intpts, 2: "d9 = %.4o\n", D9;
        D10 := d10(E, L, Elog, Periods, D7);
        vprintf Intpts, 2: "d10 = %.4o\n", D10;

        // Find the reduced bound for the coefficients in the linear
        // logarithmic form
        initQ := InitialQ(Degree(K), #L, Q0, C9, C10, D8, D9, D10, h, expTors);
        vprintf Intpts, 2: "Initial Q <= 10^%o\n", initQ;
        initQ := 10^initQ;
        new_Q := ReducedQ(L, j, EmbedL, Elog, C9, C10, Periods, expTors,
            initQ);
        Append(~Q, new_Q);
        vprint Intpts, 2: "-"^45;
    end for;
    Q := Maximum(Q);
    vprintf Intpts: "Maximum absolute bound on coefficients = %o\n", Q;
    return IntegralPoints(E, L, Q : Prec := Prec);
end intrinsic;
```

A.3 Height Bound I: Main Functions

For the rest of this appendix, we will give our own implementation of an algorithm for determining a positive lower bound for the canonical height on elliptic curves over number fields, based on Theorem 3.4.1. Although all of the following files are required for this algorithm, the only function that should be called by users is `IsLowerBound()`; see the file below for more details.

```
/******************************************************************************
 * heightbound.m
 * Computing a positive lower bound for the canonical height on elliptic
 * curves over number fields
 * (Based on Thongjunthug's paper Math. Comp. 79 (2010), pp. 2431-2449)
 *
 * By Thotsaphon Thongjunthug
 * Last updated: 07 December 2010
 * Any errors should be reported to <nookaussie@yahoo.com>
 *
 * Required packages:
 * 1) elog.m - for computing periods and elliptic logarithms
 * 2) alphas.m - for computing alpha_v for all archimedean place v
 ****************************************************************************/

/**
 * Declare printing verbose
 * 0 = minimal printing,
 * 1 = full printing (for debugging purpose only)
 **/
declare verbose Bound, 1;

/****************************************************************************
 * Auxiliary local functions
 ****************************************************************************/

/**
 * Find all Tamagawa indices at every non-archimedean places of K, and
 * also c_v for all real archimedean places v of K (note that c_v = 1 for
 * all complex archimedean places v of K). Return their LCM.
 * Input:
 *   E = elliptic curve over a number field K in standard Weierstrass form
 * Output:
 *   A least common multiplier of all Tamagawa indices
 **/
function LCMTamagawa(E)
    I := LocalInformation(E);
    S := [t[4] : t in I]; // Tamagawa index at each bad place
    // Real archimidean places
    n, _ := Signature(BaseRing(E)); // no. of real embedding
    A := aInvariants(E);
    A := [Conjugates(a) : a in A];
    for i := 1 to n do
        _<x> := PolynomialRing(Parent(A[1][i])); // parent = some real field
        // Set y = 0 in y^2 + a1*x*y + a3*y ==> 0 = x^3 + a2*x^2 + a4*x + a6
        // and check the number of real roots.
        f := x^3 + A[2][i]*x^2 + A[4][i]*x + A[5][i];
        if #(Roots(f)) eq 1 then
            Append(~S, 1);
        else
            Append(~S, 2);
        end if;
    end for;
    vprint Bound: "Tamagawa indices = ", S;
    return LeastCommonMultiple(S);
end function;


/**
 * Extra constant in B_n (take care when E is not globally minimal)
 * Input:
 *   E = elliptic curve over number field K in standard Weierstrass form
 *   P = sequence of prime ideals of O_K
 *   L = sequence of minimal models at each prime p in P
 **/
function ExtraConstant(E, P, L)
    discE := Discriminant(E);
    n := #P; // should be equal to #L
    extraConst := 1;
    for i := 1 to n do
        discEp := Discriminant(L[i]);
        extraConst *:= Norm(P[i])^Valuation(discE/discEp, P[i]);
    end for;
    extraConst := Log(extraConst)/6;
    return extraConst;
end function;


/**
 * Find all prime ideals whose norm <= n
 * last fixed on 8 Oct 2007
 * Input:
 *   R = ring of integers
 *   n = an upper bound for norm
 **/
function AllPrimeIdeals(R, n)
    I := [];
    S := [t : t in [1 .. n] | IsPrime(t)];
    for x in S do
        tmp := Decomposition(R, x);
        // Decomposition always return a list (seq) of tuples
        // The first element in each tuple is the prime ideal
        tmp := [ t[1] : t in tmp ];
        // note that there is always at least 1 elt in tmp
        // (really, up to Degree(K) elements)
        // check norm of, say, first, prime ideal in tmp
        if Norm(tmp[1]) le n then
            I:= I cat tmp;
        end if;
    end for;
    return I;
end function;


/**
 * Fine local minimal model of E at each non-archimedean place
 * N.B. All models must be integral
 * Input:
 *   E = ellliptic curve over number K in standard Weierstrass form
 * Output:
 *   P = sequence of bad primes p
 *   L = sequence of locally minimal model of E at those p
 *       dividing the discriminant of E
 **/
function LocalMinimalModel(E)
    L := []; P := [];
    // Decompose the discriminant of E
    O := RingOfIntegers(BaseRing(E));
    disc := O ! Discriminant(E);
    c4, c6 := Explode(cInvariants(E));
    c4 := O!c4; c6 := O!c6;
    tmp := Decomposition(disc);

    // check minimality of each bad place
    for D in tmp do
        // D[1] = bad place, D[2] = multiplicity in discriminant
        // E is minimal at p if ord_p(disc) < 12
        // otherwise, minimal at p if ord_p(c4) < 4 or ord_p(c6) < 6
        if (D[2] lt 12) or (Valuation(c4, D[1]) lt 4) or
            (Valuation(c6, D[1]) lt 6) then
            Append(~L, E); // E is already minimal at p
        else
            // otherwise, find an integral minimal model at p
            minModelp := MinimalModel(E, D[1]);
            Append(~L, minModelp);
        end if;
        Append(~P, D[1]);
    end for;
    return P, L;
end function;


/**
 * Given a prime ideal p, calculate e_p (the exponent of the group E_ns(k_p)
 * where k_p is residue class field).
 * Input:
 *   E = elliptic curve over number field K in standard Weierstrass form
 *   Plcs = sequence of all bad prime ideals in E
 *   L = sequnece of locally minimal elliptic curves at each place in Plcs
 *   p = a prime ideal of O_K
 **/
function FindEp(E, Plcs, L, p)
    norm_p := Norm(p);
    F, phi := ResidueClassField(p);
```

```
// make sure we're working on a model that is minimal at p
if p in Plcs then
    // p is a bad place (i.e. p | disc(E))
    // Get the corresponding minimal model at p
    E := L[Index(Plcs, p)];
end if;

pl := Place(p);

// Case 1: Good reduction - E_ns is ell. curve over finite field
if (Valuation(Discriminant(E), pl) eq 0) then
    A := phi(Coefficients(E));
    E := EllipticCurve(A);
    // Need group structors - prod. of (up to) 2 cyclic group
    // of finite order
    G := Generators(E);
    G := [Order(g) : g in G];
    // N.B: e_p = exponent = max(d1,d2)
    G, _ := Max(G);
    return G;
end if;

// Otherwise, E has bad reduction at p
// calculate c4, c6
b2, b4, b6, _ := Explode(bInvariants(E));
c4, c6 := Explode(cInvariants(E));

// Case 2.1: Additive reduction
if Valuation(c4, pl) gt 0 then
    // E_ns = k_p+
    // if |k_p| is prime, E_ns is cyclic and so e_p = |E_ns| = N(p)
    // otherwise, e_p = char(k_p)
    if IsPrime(#F) then
        return norm_p;
    else
        return Characteristic(F);
    end if;
end if;

// case 1: char(k_p) = 2
if IsZero(norm_p mod 2) then
    a1, a2, a3, _, _ := Explode(Coefficients(E));
    a1 := phi(a1);
    a2 := phi(a2);
    a3 := phi(a3);

    f := func<x | x^2 + a1*x + (a3/a1 + a2)>;
    for r in F do
        if f(r) eq 0 then
            // a root exist => split mult.
            return (norm_p - 1);
        end if;
    end for;
    // otherwise, non-split
    return (norm_p + 1);
end if;

// case 2: char(k_p) = 3
if IsZero(norm_p mod 3) then
    //print "b2 = ", b2;
    b2 := phi(b2);
    if IsSquare(b2) then
        return (norm_p - 1); // split
    else
        return (norm_p + 1); // non-split
    end if;
end if;

// case 3: char(k_p) != 2,3
//c6 := -b2^3 + 36*b2*b4 - 216*b6;
c4 := phi(c4);
```

```
    c6 := phi(c6);
    if IsSquare(c4 * c6) then
        return (norm_p - 1); // split
    else
        return (norm_p + 1); // non-split
    end if;
end function;


/**
 * Calculate D_E(n)
 * Fixed on 8 Jan 2008
 * Input:
 *   E = elliptic curve over number field in standard Weierstrass form
 *   n = a positive integer
 **/
function D_E(E, n)
    K := BaseRing(E);
    O := RingOfIntegers(K);
    r, _ := Max([2, Degree(K)]);
    P := AllPrimeIdeals(O, (n+1)^r);
    S := 0;
    Plcs, L := LocalMinimalModel(E);

    // Choose p such that e_p divides n
    for p in P do
        e_p := FindEp(E, Plcs, L, p);
        if ((n mod e_p) ne 0) then
            continue;
        end if;
        cp, _ := ResidueClassField(p);
        cp := Characteristic(cp); // this must be prime number
        S +:= 2 * (1 + Valuation(n/e_p, cp)) * Log(Norm(p));
    end for;
    return S;
end function;

//load "intersect_complex.m";

/******************************************************************************
 * Main intrinsic functions
 ******************************************************************************/

/**
 * Decide if a given number \lambda is a positive lower bound for the canonical
 * height on an elliptic curve E defined over a number field K.
 * Input:
 *   E = elliptic curve over K
 *   lambda = initial guess for a lower bound on E(K)
 * Output:
 *   Return true if \lambda is a lower bound.
 *   If the algorithm FAILS to show that \lambda is a lower bound, return false
 * Parameter:
 *   n_max = maximum number for computing B_n(\mu) (i.e. for 1 <= n <= n_max)
 *   initRes = initial resolution for region intersection (required only when E
 *             has complex embeddings.
 * Note that if initRes = n, then the grid has 2^n-by-2^n dimension
 **/
intrinsic IsLowerBound(E::CrvEll, lambda::FldReElt :
    n_max := 5, initRes := 4) -> BoolElt
{Check if \lambda > 0 is a lower bound for the canonical height on an elliptic
curve E defined over number field. If so, return true. If the algorithm fails
to confirm this, false is returned.}
    //Pts := [], initRes := 4, showIntersection := false)
    K := BaseRing(E);
    require Type(K) eq FldNum: "E must be defined over a number field";
    require IsIntegralModel(E): "E must be an integral model";
    require lambda gt 0: "The number to be checked must be positive";
    require (n_max ge 1) and (n_max in Integers()):
        "n_max must be a positive integer";
    vprintf Bound: "Computing alphas...";
    prodAlphas := &*Alphas(E);
    vprint Bound: " : Done";
    P, L := LocalMinimalModel(E);
    extraConst := ExtraConstant(E, P, L);

    // Now we work on E_gr(K)
    c := LCMTamagawa(E); mu := lambda*c^2;
    vprintf Bound: "Check if mu = %o is a lower bound on E_gr(K)\n", mu;
    // Step 1: Compute all B_n(mu) for 1 <= n <= n_max
    // If some of them is less than 1, then mu is a lower bound on E_gr,
    // and thus we return true.
    Bns := [];
    for n := 1 to n_max do
        Bn:= Exp(Degree(K)*(n^2)*mu - D_E(E, n) + extraConst) * prodAlphas;
        vprintf Bound: "B_%o(mu) = %o\n", n, Bn;
        if Bn lt 1 then
            vprintf Bound: "*** B_%o(mu) < 1, we have a lower bound! ***\n", n;
            return true;
        end if;
        Append(~Bns, Bn);
    end for;

    // Create s real embeddings and t complex embeddings of E
    Es := []; Et := [];
    s, t := Signature(K);
    A := aInvariants(E);
    A := [Conjugates(a) : a in A];

    for i := 1 to s do
        RR := RealField(Precision(A[1][i]));
        // to make sure that curve will be defined over R rather than C
        Append(~Es, EllipticCurve([RR| a[i] : a in A ]));
    end for;
    for i := 1 to t do
        Append(~Et, EllipticCurve([ a[s+(2*i-1)] : a in A ]));
    end for;

    // Step 2: Real Embeddings
    // Find intersection of subintervals of [0, 1]
    j := 1;
    for E in Es do
        vprintf Bound: "Real embedding #%o\n", j;
        D := FindSn(E, -Bns[1], Bns[1], 1);
        vprint Bound: "S_1 is ", D;
        if #D eq 0 then
            vprint Bound: "*** Empty intersection of intervals ***";
            return true;
end if;
        for n := 2 to n_max do
            tmp := FindSn(E, -Bns[n], Bns[n], n);
            vprintf Bound: "S_%o is %o\n", n, tmp;
            vprint Bound: "The Intersection now is";
            D := Intersection(D, tmp);
            vprint Bound: D;
            vprint Bound: "-"^40;
            // if the intersection is empty, we again have a lower bound
            if (#D eq 0) then
                vprint Bound: "*** Empty intersection of intervals ***";
                return true;
    end if;
        end for;
        vprint Bound: "="^75;
        j +:= 1;
    end for;

    // Step 3: Complex Embeddings
    // Find intersection of regions on each fundamental paralellogram
    j := 1;
    flag := GetVerbose("Bound");
    flag := flag eq 1; // convert to true/false
    for E in Et do
```

```
        vprintf Bound: "Complex embedding #%o\n", j;
        D := ZRegion(E, Sqrt(Bns[1]), initRes : ShowPlot := flag);
        D := GridEntryTransform(D);
        oldLevel := 1;
        for n := 2 to n_max do
            vprint Bound: "n = ", n;
    tmp := ZRegion(E, Sqrt(Bns[n]), initRes : ShowPlot := flag);
            tmp := GridEntryTransform(tmp);
            // If the region is the whole lowe-half fundamental parallelogram,
            // then we don't have to do intersection and manification
            if not (&and(tmp)) then
                tmp := DivByN(tmp, n); // find T^{(v)}_n(\sqrt{B_n(\mu)})
                vprintf Bound: "After division by %o, the size is %o\n",
                    n, #tmp;
                // Apply region's scaling if necessary before checking
                // the intersection.
                refineLevel := LCM(n, oldLevel);
                D := Magnify(D, Integers() ! (refineLevel/oldLevel));
                tmp := Magnify(tmp, Integers() ! (refineLevel/n));
                D := IntersectTwoGrids(D, tmp);
                oldLevel := refineLevel;
            end if;
            // If the intersection is empty, we have a lower bound
            if not(&or(D)) then
        vprint Bound: "*** Empty intersection of regions ***";
                return true;
            end if;
        end for;
        vprint Bound: "="^75;
        j +:= 1; // move to next complex embedding
    end for;
    // Otherwise, fail to show that mu is a lower bound on E_gr(K)
    return false;
end intrinsic;


/**
 * Compute an upper bound for the index n = [E(K)/E_tors(K) : <Pts>], where
 * Pts is the set of generators in a Mordell-Weil basis of E(K), using the
 * geometry of numbers
 * (see Siksek's "Infinite descent on elliptic curves", Theorem 3.1)
 * Input:
 *   Pts = sequence of points in a Mordell-Weil basis
 *   lambda = a positive lower bound for the canonical height on E(K)
 **/
intrinsic UpperBound4Index(Pts::SeqEnum, lambda::FldReElt) -> FldReElt
{Compute an upper bound for the index [E(K)/E_tors(K) : <P_1,...,P_r>] using
the geometry of numbers.}
    E := Curve(Pts[1]);
    require &and[P in E : P in Pts]: "All points must be on the same curve";
    detR := Determinant(HeightPairingMatrix(Pts));
    r := #Pts;
    // Here, gamma := [gamma_r^r]
    if r lt 9 then
      gamma := [1, 4/3, 2, 4, 8, 64/3, 64, 2^8];
      gamma := gamma[r];
    else
      gamma := ((Gamma(1 + r/2))^2) * (4 * Pi(RealField()))^r;
    end if;
    // calculate the upper bound of n
    n := Sqrt(detR * gamma / (lambda^r));
    return n;
end intrinsic;
```

A.4 Height Bound II: Computing α_v

This file computes the quantity α_v (see Section 2.2.2 for its definition) for every archimedean place v, using the method mentioned in [CPS06, Section 7 and 9].

```
/******************************************************************************
 * alphas.m
 * Computing alpha_v for all archimedean place v of a number field K
 * (Based on Section 7 and 9 of Cremona-Prickett-Siksek's paper
 * J. Number Theory 116 (2006), pp. 42-68).
 *
 * By Thotsaphon Thongjunthug
 * Last updated: 08 December 2010
 * Any errors should be reported to <nookaussie@yahoo.com>
 ******************************************************************************/

forward RefineAlphaBound; // require for this recursive function

/******************************************************************************
 * Generic local function: used by both real and complex cases
 ******************************************************************************/

/**
 * Find max{|P(z)|,|Q(z)|} at z}
 * Input:
 *   P, Q = real/complex polynomial in z
 *   z = a real/complex number
 **/
function MaxAbsPQ(P, Q, z)
    tmp := [Abs(Evaluate(P, z))];
    Append(~tmp, Abs(Evaluate(Q, z)));
    tmp, _ := Max(tmp);
    return tmp;
end function;


/******************************************************************************
 * Auxiliary local functions I:
 * Calculate alpha_v, where v is a real archimedean place
 ******************************************************************************/

/**
 * Only return real roots of f lying between a and b, no multiplicity.
 * Sort in increasing order.
 * Input:
 *   f = real polynomial
 *   a, b = two real numbers (with a <= b)
 * Output:
 *   a sequence of roots between a and b
 **/
function JustRoots(f, a, b)
    if a gt b then
        error "a must be < or = b";
    end if;
    R := Roots(f);
    R := [r[1] : r in R];
    R := [r : r in R | (r ge a) and (r le b)];
    Sort(~R);
    return R;
end function;


/**
 * Check if a real value t is in a given real interval I
 **/
function IsInInterval(I,t)
    a := I[1]; b := I[2];
    return ((t ge a) and (t le b));
end function;


/**
 * Find d = inf{|f(x)|, |g(x)|} where x is real and satisfies
 * (this is called e in CPS' paper!)
 * |x| <= 1 and f(x) >= 0.
 * Input:
 *   E = elliptic curve over reals
 *   f, g = real polynomials in x
 * Output:
 *   A value of d > 0. If such d does not exist (due to the region is empty),
 * then -1 is returned.
 **/
function Find_d(E, f, g)
    local leftPt, rightPt;
    // Find the starting point of x in E_0(R)
    // i.e. beta = max real root of RHS
    _, a2, _, a4, a6 := Explode(aInvariants(E));
    _<x> := PolynomialRing(Parent(a2));
    rhs := x^3 + a2*x^2 + a4*x + a6;
    tmp := Roots(rhs); // real roots
    tmp := [t[1] : t in tmp];
    beta, _ := Max(tmp);

    // Define the domain D = {x:|x|<=1 and f(x)>=0}
    if beta gt 1 then
        return -1; // D empty
    end if;

    a, _ := Max([-1., beta]);
    b := 1.;
    tmp := JustRoots(f, a, b);
    R := {@a@};
    for i in tmp do
        R := R join {@i@};
    end for;
    R := R join {@b@};

    r := #R - 1;
    D := [];
    includeStatus := false;
    for i := 1 to r do
        midX := (R[i] + R[i+1])/2;
        if Evaluate(f, midX) gt 0 then
            // in case the root has multiplicity 2
            if (#D eq 0) then
                Append(~D, [ R[i], R[i+1] ]);
                leftPt := R[i];
            elif (rightPt eq R[i]) then
                Prune(~D);
                Append(~D, [leftPt, R[i+1]]);
            else
                Append(~D, [ R[i], R[i+1] ]);
                leftPt := R[i];
            end if;
            rightPt := R[i+1];
            includeStatus := true; // next pivot point is included
            continue;
        end if;

        if (Evaluate(f, R[i]) eq 0) and not includeStatus then
            Append(~D, [ R[i], R[i] ]);
            leftPt := R[i];
            rightPt := R[i];
        end if;
        includeStatus := false;
    end for;
    if (Evaluate(f, R[r+1]) eq 0) and not includeStatus then
        Append(~D, [ R[r+1], R[r+1] ]);
    end if;


    if (#D eq 0) then
        //print "FindD: Valid region of x is empty";
        return -1;
    end if;

    // list all roots of f+g, f-g, f', g'
    R := JustRoots(f + g, a, b);
    R := R cat JustRoots(f - g, a, b);
    R := R cat JustRoots(Derivative(f), a, b);
    R := R cat JustRoots(Derivative(g), a, b);
    //R := [r : r in R | (r ge -1) and (r le 1)];
    //Sort(~R);

    // here, i is the interval
    Vals := [];
    for i in D do
        Vals := Vals cat [ MaxAbsPQ(f,g,i[1]), MaxAbsPQ(f,g,i[2]) ];
        //print "Vals init = ", Vals;
        for j in R do
            if IsInInterval(i,j) then
                Append(~Vals, MaxAbsPQ(f,g,j));
                //Exclude(~R,j);
            end if;
        end for;
        //print Vals;
    end for;

    //if IsZero(#Vals) then
    //  print "ERROR: Vals is empty";
    //end if;
    d, _ := Min(Vals);
    //print "The infimum is ", d;
    return d;
end function;


/**
 * Find d' = inf{|F(x)|, |G(x)|} where x is real and satisfies
 * |x|<=1 and F(x)>=0 (this is called e' in CPS' paper!)
 * Input:
 *   E = elliptic curve over reals
 *   F, G = real polynomials in x
 * Output:
 *   a value of d'. If such d does not exist (due to the region
 * is empty), then -1 is returned.
 **/
function Find_ddash(E, F, G)
    local leftPt, rightPt;
    // Find the starting point of x in E_0(R)
    // i.e. beta = max real root of RHS
    _, a2, _, a4, a6 := Explode(Coefficients(E));
    P<x> := PolynomialRing(RealField());
    a2 := P ! a2;
    a4 := P ! a4;
    a6 := P ! a6;
    rhs := x^3 + a2*x^2 + a4*x + a6;
    tmp := Roots(rhs);
    tmp := [t[1] : t in tmp];
    beta, _ := Max(tmp);

    // init the domain D'
    if beta le -1 then
        DDashInit := [ [-1., 1./beta], [0., 1.] ];
    elif beta le 1 then
        DDashInit := [ [0., 1.] ];
    else
        DDashInit := [ [0, 1./beta] ];
    end if;
```

```
DDash := [];
for I in DDashInit do
    a := I[1];
    b := I[2];
    tmp := JustRoots(F, a, b);
    R := {@a@};
    for i in tmp do
        R := R join {@i@};
    end for;
    R := R join {@b@};

    r := #R - 1;
    includeStatus := false;
    for i := 1 to r do
        midX := (R[i] + R[i+1])/2;
        if Evaluate(F, midX) gt 0 then
            // In case the root has multiplicity 2
            if (#DDash eq 0) then
                Append(~DDash, [ R[i], R[i+1] ]);
                leftPt := R[i];
            elif (rightPt eq R[i]) then
                Prune(~DDash);
                Append(~DDash, [leftPt, R[i+1]]);
            else
                Append(~DDash, [ R[i], R[i+1] ]);
                leftPt := R[i];
            end if;
            rightPt := R[i+1];
            includeStatus := true; // next pivot point is included
            continue;
        end if;

        if (Evaluate(F, R[i]) eq 0) and not includeStatus then
            Append(~DDash, [ R[i], R[i] ]);
            leftPt := R[i];
            rightPt := R[i];
        end if;
        includeStatus := false;
    end for;
    if (Evaluate(F, R[r+1]) eq 0) and not includeStatus then
        Append(~DDash, [ R[r+1], R[r+1] ]);
    end if;
end for;

if (#DDash eq 0) then
    //print "FindDDash: Valid region of x is empty";
    return -1;
end if;

// list all roots of F+G, F-G, F', G'
R := JustRoots(F + G, -1, 1);
R := R cat JustRoots(F - G, -1, 1);
R := R cat JustRoots(Derivative(F), -1, 1);
R := R cat JustRoots(Derivative(G), -1, 1);
//R := [r : r in R | (r ge -1) and (r le 1)];
//Sort(~R);

// here, I is the interval
Vals := [];
for I in DDash do
    Vals := Vals cat [ MaxAbsPQ(F,G,I[1]), MaxAbsPQ(F,G,I[2]) ]; // end points
    //print "Vals init = ", Vals;
    for j in R do
        if IsInInterval(I,j) then
            Append(~Vals, MaxAbsPQ(F,G,j));
            //Exclude(~R,j);
        end if;
    end for;
    //print Vals;
end for;
```

```
    //if IsZero(#Vals) then
    //  print "ERROR: Vals is empty";
    //end if;
    d, _ := Min(Vals);
    //print "The infimum is ", d;
    return d;
end function;


/**
 * Calculate the value of alpha of a given elliptic curve over real numbers
 * Input:
 *   E = elliptic curve over reals
 **/
function AlphaReal(E)
    b2, b4, b6, b8 := Explode(bInvariants(E));
    _<x> := PolynomialRing(Parent(b2)); // assume all b have the same precision
    f := 4*x^3 + b2*x^2 + 2*b4*x + b6;
    g := x^4 - b4*x^2 - 2*b6*x - b8;
    F := 4*x + b2*x^2 + 2*b4*x^3 + b6*x^4; // = f(x)/(x^4) and let x := 1/x
    G := 1 - b4*x^2 - 2*b6*x^3 - b8*x^4; // similarly

    d := Find_d(E, f, g);
    d_dash := Find_ddash(E, F, G);

    // take care of cases when region of valid x is empty
    if (d lt 0) then
        if (d_dash ge 0) then
            return d_dash^(-1/3);
        else
            error "Both regions of X are empty (should not happen), please report!";
        end if;
    elif (d_dash ge 0) then
        alpha, _ := Min([d, d_dash]);
        return alpha^(-1/3);
    else
        return d^(-1/3);
    end if;
end function;


/******************************************************************************
 * Auxiliary local functions II:
 * Calculate alpha_v for complex archimedean places v in K
 ******************************************************************************/

/**
 * Calculate min H = min{h((m+ni)/10: m^2 + n^2 <= 100}, m, n integer
 * where h = max{|P(z)|, |Q(z)|}.
 * Input:
 *   P, Q = polynomials defined over C
 **/
function AlphaInitialGuess(P, Q)
    i := BaseRing(P)!Sqrt(-1);
    initMin, _ := Max(Abs(Evaluate(P, -1)), Abs(Evaluate(Q, -1)));
    for m := -10 to 10 do
        boundN := Floor(Sqrt(100 - m^2));
        for n := -boundN to boundN do
            h := MaxAbsPQ(P, Q, (m + n*i)/10 );
            if h lt initMin then
                initMin := h;
            end if;
        end for;
    end for;
    return initMin;
end function;


/**
 * Compute BigEpsilon(z, eta)
 * Input:
 *   P, Q = complex polynomials
 *   z, eta = complex numbers
 **/
function BigEpsilon(P, Q, z, eta)
    d1 := Degree(P);
    d2 := Degree(Q);
    sum1 := 0;
    sum2 := 0;
    for n := 1 to d1 do
        P := Derivative(P); // P^(n)
        sum1 +:= (eta^n) * Abs(Evaluate(P, z)) / Factorial(n);
    end for;
    for n := 1 to d2 do
        Q := Derivative(Q); // Q^(n)
        sum2 +:= (eta^n) * Abs(Evaluate(Q, z)) / Factorial(n);
    end for;
    eps, _ := Max([sum1, sum2]);
    return eps;
end function;


/**
 * Approximate alpha_PQ = inf max{|P(z)|, |Q(z)|}, z on unit circle
 * using repeated quadrisection method recursively.
 * Input:
 *   P, Q = complex polynomials
 *   mu = accuracy level (need Exp(-mu) close to 1)
 *   S = nested sequence representing a square
 *   alpha = initial alpha to be refined
 *   level = how many times one wishes to refine alpha
 **/
function RefineAlphaBound(P, Q, mu, S, alpha, level)
    // Step 1: check if the square S intersects unit circle
    // if not, return alpha
    a, b := Explode(S[1]);
    r := S[2];

    // Modified: 25 Mar 2010 (bug found by Robert Bradshaw)
    // If [a, b] = [-1, -1], the below trick won't work.
    // But of course the intersection won't be empty.
    if level eq 0 then
        level := 1;
        S1 := <[a, b], r/2>;
        S2 := <[a, b+r/2], r/2>;
        S3 := <[a+r/2, b], r/2>;
        S4 := <[a+r/2, b+r/2], r/2>;
        alpha := RefineAlphaBound(P, Q, mu, S1, alpha, level);
        alpha := RefineAlphaBound(P, Q, mu, S2, alpha, level);
        alpha := RefineAlphaBound(P, Q, mu, S3, alpha, level);
        alpha := RefineAlphaBound(P, Q, mu, S4, alpha, level);
    end if;

    C := [ S[1], [a, b+r], [a+r, b], [a+r, b+r] ];
    C := [ c[1]^2 + c[2]^2 : c in C]; // square of modulus of each corner
    C := [ (c le 1) : c in C]; // check if each corner is in the circle
    if not (true in C) then
        return alpha;
    end if;

    // Step 2:
    if (a+r/2)^2 + (b+r/2)^2 le 1 then
        u := [a+r/2, b+r/2]; // u = mid-point
        eta := r/Sqrt(2);
    else
        // u = any corner that in unit circle
        ind := Index(C, true); // position of the corner in D
        if ind eq 1 then
            u := [a, b];
        elif ind eq 2 then
            u := [a, b+r];
        elif ind eq 3 then
            u := [a+r, b];
        else
            u := [a+r, b+r];
        end if;
        eta := r * Sqrt(2);
    end if;

    // Step 3 and 4: check condition for minimality
    i := BaseRing(P)!Sqrt(-1);
    u := u[1] + u[2]*i;
    h := MaxAbsPQ(P, Q, u);
    epsilon := BigEpsilon(P, Q, u, eta);
    if (h - epsilon) gt (alpha * Exp(-mu)) then
        return alpha;
    else
        alpha, _ := Min([alpha, h]);
    end if;

    // Step 5 and so on: split S into 4 quadrants and recursively apply
    // this function to each S_i
    level := level + 1;
    S1 := <[a, b], r/2>;
    S2 := <[a, b+r/2], r/2>;
    S3 := <[a+r/2, b], r/2>;
    S4 := <[a+r/2, b+r/2], r/2>;
    alpha := RefineAlphaBound(P, Q, mu, S1, alpha, level);
    alpha := RefineAlphaBound(P, Q, mu, S2, alpha, level);
    alpha := RefineAlphaBound(P, Q, mu, S3, alpha, level);
    alpha := RefineAlphaBound(P, Q, mu, S4, alpha, level);
    return alpha;
end function;


/**
 * Find alphas for a given elliptic curve over complex number
 * Use repeated quadrisection method from Section 9 of CPS' paper.
 * Input:
 *   E = elliptic curve over C
 *   mu = accuracy level (need Exp(-mu) close to 1)
 **/
function AlphaComplex(E : mu := 0.005)
    b2, b4, b6, b8 := Explode(bInvariants(E));
    _<x> := PolynomialRing(Parent(b2));
    f := 4*x^3 + b2*x^2 + 2*b4*x + b6;
    g := x^4 - b4*x^2 - 2*b6*x - b8;
    F := 4*x + b2*x^2 + 2*b4*x^3 + b6*x^4; // = f(x)/(x^4) and let x := 1/x
    G := 1 - b4*x^2 - 2*b6*x^3 - b8*x^4; // similarly

    // S = square [-1,1] X [-1,1], represents as <[a,b], h, v> where
    // [a,b] = lower left corner, r = length (same for all sides)
    S := <[-1.,-1.], 2.>;

    // Step 1: find inf max{|f(z)|, |g(z)|}, where z on closed unit disc
    // Initial guess: alpha_fg = min H = min{h((m+ni)/10: m^2 + n^2 <= 100}
    // where h = max{|f(z)|, |g(z)|}
    alpha_fg := AlphaInitialGuess(f, g);
    alpha_fg := RefineAlphaBound(f, g, mu, S, alpha_fg, 0);

    // Step 2: find inf max{|F(z)|, |G(z)|}, where z on closed unit disc
    // Initial guess: alpha_FG = min H = min{h((m+ni)/10: m^2 + n^2 <= 100}
    // where h = max{|F(z)|, |G(z)|}
    alpha_FG := AlphaInitialGuess(F, G);
    alpha_FG := RefineAlphaBound(F, G, mu, S, alpha_FG, 0);


    // Then alpha^(-3) = Min(alpha_fg, alpha_FG)
    // Note that we need alpha
    alpha, _ := Min([alpha_fg, alpha_FG]);
    alpha := (alpha*Exp(-mu))^(-1/3);
    return alpha;
```

```
end function;


/******************************************************************************
 * Main intrinsic function
 ******************************************************************************/

/**
 * Return the list of alpha_v for all archimedean place v.
 * For those alphas from complex embeddings, the returned values are squared
 * Input:
 *   E = elliptic curve over number field K
 * Output:
 *   A sequence of all values of alpha_v
 **/
intrinsic Alphas(E::CrvEll) -> SeqEnum
{Given an elliptic curve E over a number field K, compute all alpha_v associated to
E for all archimedean places v of K. If v is a complex place, then the returned
value will be alpha_v^2.}
    A:= aInvariants(E);
    tmp := [Conjugates(a) : a in A];
    s, t := Signature(BaseRing(E));
    alphas := [];

    // Real embeddings
    for j := 1 to s do
        Erj := EllipticCurve([RealField() ! a[j] : a in tmp]);
        Append(~alphas, AlphaReal(Erj));
    end for;

    // Complex embeddings
    // N.B.: we pick only one conjugate from its conjugacy pair
    for j := 1 to t do
        Ecj := EllipticCurve([b[s + 2*j - 1] : b in tmp]);
        Append(~alphas, (AlphaComplex(Ecj))^2);
    end for;
    return alphas;
end intrinsic;
```

A.5 Height Bound III: Intersection of Intervals

This file consists of all necessary functions for compute $\mathcal{S}_n^{(v)}(\xi_1, \xi_2)$ (see Section 2.5 for its definition) for each real archimedean place v. Note that $\mathcal{S}_n^{(v)}$ is a disjoint union of subintervals of $[0, 1]$. As we have seen in Section 2.5, an algorithm for computing periods and elliptic logarithms of real points is also required for this computation; see Appendix A.1 for its implementation.

```
/******************************************************************************
 * intersect_real.m
 * Computing S^{(v)}_n(\xi_1, \xi_2) for all real archimedean place v
 *
 * By Thotsaphon Thongjunthug
 * Last updated: 08 December 2010
 * Any errors should be reported to <nookaussie@yahoo.com>
 *
 * Required package:
 *   elog.m - for computing normalised elliptic logarithms of real points
 ******************************************************************************/
```

```
/****************************************************************************
 * Auxiliary local functions
 ***************************************************************************/

/**
 * Return the list of real roots of f(x) = 4x^3 + b2*x^2 + 2*b4*x + b6
 * in decreasing order
 * Input:
 *   E = elliptic curve defined over real numbers
 * Output:
 *   a sequence of real roots
 **/
function RootsF(E)
    b2, b4, b6, b8 := Explode(bInvariants(E));
    _<t> := PolynomialRing(Parent(b2));
    p := 4*t^3 + b2*t^2 + 2*b4*t + b6;
    R := Roots(p); // real roots only
    R := [e[1] : e in R]; // get rid of multiplicity
    Sort(~R);
    Reverse(~R); // decreasing order
    return R;
end function;


/**
 * Find the intersection of two real intervals I1, I2
 * Input:
 *   I1 = [a, b], I2 = [c, d] : real intervals
 **/
function IntervalsMeet(I1,I2)
    if (#I1 eq 0) or (#I2 eq 0) then
        // One interval is empty, nothing to do
        return [];
    end if;

    a := I1[1]; b := I1[2];
    c := I2[1]; d := I2[2];

    if (b lt c) or (d lt a) then
        // Both intervals are disjoint
        return [];
    elif (a le c) and (d le b) then
        return I2; // subset of I1
    elif (c le a) and (b le d) then
        return I1; // subset of I2
    elif (a lt c) and (b lt d) then
        return [c,b];
    elif (c lt a) and (d lt b) then
        return [a,d];
    else
        // Should not happen, just for checking
        error "You miss some case";
    end if;
end function;


/**
 * Compute a normalised elliptic logarithm of the "higher" point of the two
 * having the same x-coordinate.
 * Input:
 *   E = elliptic curve over reals in standard Weierstrass form
 *   x = a real number for the x-coordinate
 **/
function NormalisedElog(E, x)
    // First, find a positive period of E
    r := Precision(BaseRing(E));
    // Warning: PeriodLattice() needs curves over C
    prec := Precision(BaseRing(E));
    C := ComplexField(prec);
    EC := ChangeRing(E, C);
    w1, w2, _ := Explode(PeriodLattice(EC : Prec := r-10));
    if Im(w2/w1) gt 0 then
        L := [w1, w2];
    else
        L := [w2, w1];
    end if;
    L, _ := TransformLattice(L);
    if Abs(Im(L[1])) lt 10^(-r/2) then
        w := Re(L[1]);
    else
        w := Re(L[2]);
    end if;
    if w lt 0 then
        w := -w; // need positive real period
    end if;
    //"w = ", w;

    // Find the "higher" point
    a1, a2, a3, a4, a6 := Explode(aInvariants(E));
    _<t> := PolynomialRing(Parent(a1));
    f := t^2 + a1*x*t + a3*t - x^3 - a2*x^2 - a4*x - a6;
    tmp := Roots(f); // real roots
    tmp := [y[1] : y in tmp];
    // Chose y such that 2*y + a1*x + a3 >= 0
    if (2*tmp[1] + a1*x + a3) ge 0 then
      y := tmp[1];
    else
      y := tmp[2];
    end if;

    PC := Points(EC, x)[1]; // embed P to E(C)
    if Abs(PC[2]-y) gt 10^(-r/2) then
        PC := -PC;
    end if;

    // Compute elliptic logaithm and scale it to [0,1]
    elog := EllipticLog(EC, PC : Prec := r-10);
    elog := Re(elog/w); // already real, just to make sure
    return elog - Floor(elog); // mod 1, and force it to be real
end function;


/**
 * Find S^{(v)}(e1, e2)
 * Input:
 *   e1, e2 = two real number with e1 < e2
 * Output:
 *   a nested sequence representing disjoint union of subinterval
 **/
function FindS(E, e1, e2)
    if (e1 ge e2) then
        error "e1 must be less than e2";
    end if;

    // beta  = max real root of f(x) = 4x^3 + ...
    beta := (RootsF(E))[1];

    if (e2 lt beta) then
        return [];
    elif (e1 lt beta) then
        el2 := NormalisedElog(E, e2);
        return [ [1 - el2, el2] ];
    else
        el1 := NormalisedElog(E, e1);
        el2 := NormalisedElog(E, e2);
        if (el1 eq 0.5) then
            if (el2 eq 0.5) then
                return [[0.5, 0.5]];
            else
                return [[1 - el2, el2]];
            end if;
        else
            return [ [1 - el2, 1 - el1], [el1, el2] ];
        end if;
    end if;
end function;


/******************************************************************************
 * Main intrinsic functions
 ******************************************************************************/

/**
 * Compute S^{(v)}_n(e1, e2)
 * Input:
 *   E = elliptic curve over reals in standard Weierstrass form
 *   e1, e2 = two real numbers (with e1 < e2)
 *   n = a positive integer
 * Output:
 *   a nested sequence representing a disjoint union of subintervals of [0,1]
 **/
intrinsic FindSn(E::CrvEll, e1::FldReElt, e2::FldReElt, n::RngIntElt) -> SeqEnum
{Compute S^v_n(e1, e2) for real archimedean place v}
    require e1 lt e2 : "e1 must be less than e2";
    require n gt 0: "n must be a positive integer";
    D := [];
    S := FindS(E, e1, e2);
    //print "n =", n, "S = ", S;
    s := #S;

    if s eq 0 then
        return [];
    end if;

    a := (S[1])[1];
    b := (S[1])[2];
    if s eq 1 then
        if (a eq 0) and (b eq 1) then
            return [[0., 1.]];
        end if;

        for t := 0 to (n-1) do
            Append(~D, [(t+a)/n, (t+b)/n]);
        end for;
        return D;
    end if;

    // s = 2
    c := (S[2])[1];
    d := (S[2])[2];
    if (a eq 0) and (d eq 1) then
        D := [[0, b/n]];
        for t := 0 to (n-2) do
            Append(~D, [(t+c)/n, (t+1+b)/n]);
        end for;
        Append(~D, [(n-1+c)/n, (n-1+d)/n]);
        return D;
    end if;

    // otherwise, ordinary append will do
    for t := 0 to (n-1) do
        Append(~D, [(t+a)/n, (t+b)/n]);
        Append(~D, [(t+c)/n, (t+d)/n]);
    end for;
    return D;
end intrinsic;


/**
 * Find the intersection of two disjoint unions of intervals
 * Input:
 *   I1, I2 = nested sequences representing disjoint unions of intervals
 * Output:
 *   a nested sequences representing disjoint union of intervals
 **/
intrinsic Intersection(I1::SeqEnum , I2::SeqEnum) -> SeqEnum
{Find the intersection of two disjoint unions of intervals}
    D:= [];
    for I in I1 do
        for J in I2 do
            tmp := IntervalsMeet(I,J);
            if (#tmp ne 0) then
                Append(~D, tmp);
            end if;
        end for;
    end for;
    return D;
end intrinsic;
```

A.6 Height Bound IV: Intersection of Regions

The following files provides functions for computing the approximate corresponding region $\mathcal{T}_n^{(v)}$ (see Section 3.3 for the definition) for each complex archimedean place v. Again, we use our implementation in Appendix A.1 to compute the period lattice of each complex embedding.

A.6.1 `intersect_complex.m`

This is the main file which does most of the task of computing $\mathcal{T}_n^{(v)}$.

```
/******************************************************************************
 * intersect_complex.m
 * Computing T^{(v)}(\xi) and functions to be used for intersecting regions
 * on half fundamental parallelograms
 *
 * By Thotsaphon Thongjunthug
 * Last updated: 08 Decemner 2010
 * Any errors should be reported to <nookaussie@yahoo.com>
 *
 * Required packages:
 * 1) elog.m - for computing periods of elliptic curve over C
 * 2) interval_wp.m - for HasBoundary() function
 * 3) wp.m - for computing error term when computing \wp(z) using
 *    only finite number of terms in the power series expansion.
 ******************************************************************************/

/******************************************************************************
 * Auxiliary local functions/procedures
 ******************************************************************************/

/**
 * Return half fundamental paralelogram discretised into n-by-n block
 * Input:
 *   tau = w2/w1
 *   n = dimension of the discritisation
 * Output
 *   <dw1, dw2, [<pivot, status>, ...]>
 **/
function HFPDiscretise(tau, n)
```

```
    dw1 := 1.;
    dw2 := tau/2;
    pivot := 0.;
    U := car<Parent(tau), Integers()>;
    L := [U|];
    dw1 := dw1/n;
    dw2 := dw2/n;
    for i := 1 to n do
        for j := 1 to n do
            // initial status is set to be '2' (uncertain) for all blocks
            Append(~L, <pivot, 2>);
            pivot +:= dw2;
        end for;
        pivot := i*dw1;
    end for;
    return <dw1, dw2, L>;
end function;


/**
 * Plot a given brief half fundamental paralellogram into figure
 * (for debug only) - outout is rotated-right of the real figure
 * Status - 0 = No (cell excluded), else = Yes (included)
 * Input:
 *   L = sequence of digits representing cells in the H.F.P.
 **/
procedure GridPlot(L)
    // Check dimension = must be square (nxn)
    if not IsSquare(#L) then
        error "Dimension of L must be square";
    else
        dim := Integers() ! Sqrt(#L);
    end if;
    i := 1;
    for c in L do
        if c eq 0 then
            printf "x";
        elif c eq 1 then
            printf "y";
        elif c eq 2 then
            printf "f";
        else
            printf "u";
        end if;

        if (i mod dim) eq 0 then
            printf "\n";
            i := 1;
        else
            i +:= 1;
        end if;
    end for;
end procedure;


/******************************************************************************
 * Main intrinsic functions
 ******************************************************************************/

/**
 * Intersect two (brief version of) regions on half fundamental
 * parallelograms (HFP)
 * Input:
 *   L1, L2 = sequence of true/false representing the region (true = in region)
 * Output:
 *   A sequence of true/false representing the intersection
 **/
intrinsic IntersectTwoGrids(L1::SeqEnum[BoolElt], L2::SeqEnum[BoolElt])
    -> SeqEnum[BoolElt]
{Intersect two regions of the same size}
    // Both L1 and L2 must be of identical size.
    // All entries in both L1, L2 are booleans.
    require #L1 eq #L2: "Both HFPs must be of identical dimension";
    n := #L1; // = #L2 as well
    L := [];
    for i := 1 to n do
        Append(~L, L1[i] and L2[i]);
    end for;
    return L;
end intrinsic;


/**
 * Divide a given region L by n. If L has dimention m-by-m, then the new
 * divided region has dimention (m*n)-by-(m*n)
 * Input:
 *   L = a sequence of true/false representing a region
 *   n = a positive integer
 * Output:
 *   a sequence of true/false representing a new region
 **/
intrinsic DivByN(L::SeqEnum, n::RngIntElt) -> SeqEnum
{Division of region by n}
    // Check dimension of L - must be square
    require IsSquare(#L): "Dimension of L must be square";
    require n ge 1: "n must be a positive integer";
    oldDim := Integers()! Sqrt(#L);

    // "shrinked" block: dimension = oldDim x oldDim
    // split shrink block into oldDim columns
    Cols := [];
    for i := 1 to oldDim do
        Col := [];
        for j := oldDim*(i-1)+1 to oldDim*i do
            Append(~Col, L[j]);
        end for;
        Append(~Cols, Col);
    end for;

    // Build up each big column.
    // Swap method - top of 1st column concats to top of oldDim-th column.
    // Count the total entry - stop when # = oldDim * n
    BigCol := [];
    for i := 1 to oldDim do
        BigCol[i] := []; // initialise blank big column
    end for;
    for i := 1 to oldDim do
        ind := 1;
        repeat
            if (ind mod 2) eq 1 then
        // pick entry from i-th column
        BigCol[i] := BigCol[i] cat Cols[i];
            else
        BigCol[i] := BigCol[i] cat Reverse(Cols[oldDim-i+1]);
            end if;
            ind +:= 1;
        until ind gt n;
    end for;

    // Concat oldDim big columns together, and keep doing this n times.
    // Overall dim = (n*oldDim) x (n*oldDim)
    LL := [];
    for i := 1 to n do
        for j := 1 to oldDim do
            LL := LL cat BigCol[j];
        end for;
    end for;
    return LL;
end intrinsic;


/**
```

```
 * Magnify a brief half fundamental parallelogram by n folds
 * Input:
 *   L = sequence of true/false representing a region, say, size m-by-m
 *   n = a positive integer
 * Output:
 *   a new sequence of true/false of size (m*n)-by-(m*n)
 **/
intrinsic Magnify(L::SeqEnum, n::RngIntElt) -> SeqEnum
{Magnify a brief half fundamental parallelogram by n folds}
    // Check dim of L - must be square
    require IsSquare(#L): "Dimension of L must be square";
    oldDim := Integers() ! Sqrt(#L);
    require n ge 1: "n must be a positive integer";

    // initialise BigCol - n of them
    BigCol := [];
    for i := 1 to n do
        BigCol[i] := [];
    end for;

    LL := [];
    // for each cell, discretise it into nxn subcells
    // do this by column
    for i := 1 to (oldDim^2) do // no. of org. cells
        for j := 1 to n do // no. of subcol in each cell
            for k := 1 to n do // no. of copies of that org. cell to subcol
        Append(~BigCol[j], L[i]);
            end for;
        end for;
        if (i mod oldDim) eq 0 then // 1 old col done
            // concat those n subcols and put into LL
            for k := 1 to n do
        LL := LL cat BigCol[k];
            end for;
            // clear all n subcols
            BigCol := [];
            for k := 1 to n do
        BigCol[k] := [];
            end for;
        end if;
    end for;
    return LL;
end intrinsic;


/**
 * Determine a region T^v_n(\xi).
 * This is done by simple methods, but still need more subtle way to confirm
 * that the pre-excluded cells are indeed excluded.
 * Input:
 *   E = elliptic curve defined over C in standard Weierstrass form
 *   xi = an upper bound for |X|, where Y^2 = 4*X^3 + A*X + B
 *   initRes = initial resolution (dimension) for the region
 * Output:
 *   ?
 * Parameter:
 *   ShowPlot = if true, print the region
 *
 **/
intrinsic ZRegion(E::CrvEll, xi::FldReElt, initRes::RngIntElt :
    ShowPlot := false) -> SeqEnum
{Find the region T^v(\xi)}
    prec := Precision(BaseRing(E));
    w1, w2, _ := Explode(PeriodLattice(E: Prec := prec-10));
    if Im(w2/w1) gt 0 then
        L := [w1, w2];
    else
        L := [w2, w1];
    end if;
    L, _ := TransformLattice(L);
    w1, w2 := Explode(L); tau := w2/w1;
```

```
b2 := bInvariants(E)[1];
u_xi := (xi + Abs(b2)/12) * Abs(w1)^2; // U_{\xi}

// Create a discritisation of half fundamental paralellogram,
// both full version L and brief version C.
n := 2^initRes; // dimension needs to be a power of 2
// Format: L = <dw1, dw2, [<pivot, status>, ...]>
dw1, dw2, cells := Explode(HFPDiscretise(tau, n));
// initial C
C := [c[2] : c in cells]; // status of each cell (2 = uncertain)

// Stage 1: Four-Corner Test
for ind := 1 to (n^2) do
    pivot, _ := (cells[ind])[1];
    // Error term:
    // Using 23 terms yields accuracy of computing \wp(z)
    // within 50 decimal places
    err := EstimateWPMaxError(23, 23);
    // Temporary excluded the corner containing 0 (mod \Lambda)
    if (ind eq 1) or (ind eq (n*(n-1)+1)) then
        continue;
    // find p(z,tau) where z are the four corners of the cell c
    elif Abs(WeierstrassP([1, tau], pivot, 23))+err le u_xi then
        C[ind] := 1;
    elif Abs(WeierstrassP([1, tau], pivot + dw1, 23))+err le u_xi then
        C[ind] := 1;
    elif Abs(WeierstrassP([1, tau], pivot + dw2, 23))+err le u_xi then
        C[ind] := 1;
    elif Abs(WeierstrassP([1, tau], pivot + dw1 + dw2, 23))+err le u_xi then
        C[ind] := 1;
    end if;

    if C[ind] eq 1 then
        // Stage 2: Identify "possibly false" boundary
        // set it to '0' first in order to distinguish the region
        // but is still subject to verification
        // left cell
        if ind gt n then
            if C[ind - n] ne 1 then
                C[ind - n] := 0;
            end if;
        end if;

        // right cell
        if ind le n*(n-1) then
            if C[ind + n] ne 1 then
                C[ind + n] := 0;
            end if;
        end if;

        // top cell
        if (ind mod n) ne 0 then
            if C[ind + 1] ne 1 then
                C[ind + 1] := 0;
            end if;
        end if;

        // bottom cell
        if (ind mod n) ne 1 then
            if C[ind - 1] ne 1 then
                C[ind - 1] := 0;
            end if;
        end if;
    end if;
end for;

// Stage 3: Confirm "possibly false" boundary that it is indeed excluded
boundaryConfirmed := true;
for ind := 1 to (n^2) do
    if C[ind] eq 0 then
        // lower horizontal
```

```
            m := 0.; k := Im(pivot);
            X := [Re(pivot), Re(pivot)+dw1];
            if HasBoundary(X, m, k, u_xi, tau) then
                C[ind] := 1;
                //printf "C[%o]: lower boundary not confirmed\n", ind;
                boundaryConfirmed := false;
                //print "==========";
                continue;
            end if;
            // upper horizontal
            m := 0.; k := Im(pivot + dw2);
            X := [Re(pivot+dw2), Re(pivot+dw1+dw2)];
            if HasBoundary(X, m, k, u_xi, tau) then
                C[ind] := 1;
                //printf "C[%o]: upper boundary not confirmed\n", ind;
                boundaryConfirmed := false;
                //print "==========";
                continue;
            end if;

            // ADDED: 18 Nov 2010
            // left/right vertical (i.e. Re(tau)=0 )
            if Re(tau) eq 0 then
               if HasBoundaryVertical(Re(pivot), Im(pivot), Im(pivot+dw2),
                    u_xi, tau) then
            C[ind] := 1;
            //printf "C[%o]: left vertical boundary not confirmed\n", ind;
                    boundaryConfirmed := false;
                    //print "==========";
                    continue;
                end if;
        if HasBoundaryVertical(Re(pivot+dw1), Im(pivot+dw1), Im(pivot+dw1+dw2),
                    u_xi, tau) then
            C[ind] := 1;
            //printf "C[%o]: right vertical boundary not confirmed\n", ind;
                    boundaryConfirmed := false;
                    //print "==========";
                    continue;
                end if;
            else
                // left slant
                m := Im(tau)/Re(tau); k := Im(pivot) - m*Re(pivot);
                if (m lt 0) then
                    X := [Re(pivot+dw2), Re(pivot)];
                elif (m gt 0) then
                    X := [Re(pivot), Re(pivot+dw2)];
                end if;
                if HasBoundary(X, m, k, u_xi, tau) then
                    C[ind] := 1;
                    //printf "C[%o]: left boundary not confirmed\n", ind;
                    boundaryConfirmed := false;
                    //print "==========";
                    continue;
                end if;

                // right slant
                m := Im(tau)/Re(tau); k := Im(pivot) - m*Re(pivot+dw1);
                if (m lt 0) then
                    X := [Re(pivot+dw1+dw2), Re(pivot+dw1)];
                elif (m gt 0) then
                    X := [Re(pivot+dw1), Re(pivot+dw1+dw2)];
                end if;
                if HasBoundary(X, m, k, u_xi, tau) then
                    C[ind] := 1;
                    printf "C[%o]: right boundary not confirmed\n", ind;
                    boundaryConfirmed := false;
                    //print "==========";
                else
                    // cell can be excluded
                    C[ind] := 0;
                    if ShowPlot then
```

```
                        printf "C[%o] = %o\n", ind, C[ind];
                    end if;
                    //print "==========";
                end if;
    end if;
        end if;
    end for;

    // Stage 4: shade region correctly
    if boundaryConfirmed then
        if ShowPlot then
            print "Excluded boundary confirmed! Shading remaining region ...";
        end if;
        for ind := 1 to (n^2) do
            if C[ind] eq 2 then
                C[ind] := 0; // 0 = exclude from the region
            end if;
        end for;
    else
        if ShowPlot then
            print "Excluded boundary not entirely confirmed";
        end if;
    end if;

    if ShowPlot then
        GridPlot(C);
        printf "\n";
        print "Size Z = ", #C;
        print "-----------------------------------------";
    end if;
    return C;
end intrinsic;


/**
 * Convert grid entry from 0 -> false and from 1 -> true.
 * If contains any other number, print error message
 * Input:
 *   C = sequence of binaries
 * Output:
 *   sequence of true/false
 **/
intrinsic GridEntryTransform(C::SeqEnum[RngIntElt]) -> SeqEnum[BoolElt]
{Transform all binaries in a grid into true/false entries}
    L := [];
    for c in C do
        if c eq 0 then
            Append(~L, false);
        elif c eq 1 then
            Append(~L, true);
        else
            error "Grid must only contain 0 and 1";
        end if;
    end for;
    return L;
end intrinsic;
```

A.6.2 wp.m

This file consists of functions for computing the approximate Weierstrass $\wp$-function (i.e., using only finite number of terms in the power series expansion), and the maximum error caused by such approximation. For more details, see Section 3.2.1.

```
/******************************************************************************
 * wp.m
 * Computing Weierstrass \wp-function and the error term of using finite
 * number of terms in the power series expansion as the approximate for \wp(z)
 *
 * By Thotsaphon Thongjunthug
 * Last updated: 08 December 2010
 * Any errors should be reported to <nookaussie@yahoo.com>
 ******************************************************************************/

/******************************************************************************
 * Main intrinsic functions
 ******************************************************************************/

/**
 * Estimate the maximum error of |p(z,tau)-f(u)*(2*pi*i)^2| on the lower-half
 * fundamental parallelogram, where u = Exp(2*pi*i*z) and
 * f(u) = u/(1-u)^2 + 1/12 + some terms from the right-hand side of the
 * expression of p(z,tau) (See Silverman's paper).
 *
 * Input:
 *   n1 = starting index for the infinite sum of terms
 *        (q^n)u/(1-(q^n)u)^2 + (q^n)(1/u)/(1-(q^n)(1/u))^2
 *   n2 = starting index for the infinite sum of terms 2*(q^n)/(1-q^n)^2
 *
 * Output:
 *   the maximum absolute error
 * Parameter:
 *   q = Exp(2*pi*i*tau),   default = Exp(-Sqrt(3)*pi)
 *   alpha = Im(z)/Im(tau), default = 1/2
 * (The default values of q and alpha are ones of the worst case scenario)
 **/
intrinsic EstimateWPMaxError(n1::RngIntElt, n2::RngIntElt :
    q := Exp(-Sqrt(3)*Pi(RealField())), alpha := 1/2) -> FldReElt
{Estimate the error when using finite number of terms in the power series
expansion as the approximate to the Weierstrass p-function}
    require (n1 ge 1) and (n2 ge 1): "n1 and n2 must be positive integers";
    pi := Pi(RealField());
    err := (q^(n1 + alpha))/(1 - q^(n1 + alpha))^2;
    err +:= (q^(n1 - alpha))/(1 - q^(n1 - alpha))^2;
    err +:= 2*(q^n2)/(1 - q^n2)^2;
    err *:= 4*(pi^2);
    err /:= (1 - q);
    return err;
end intrinsic;


/**
 * Calculate Weierstrass \wp-function p(z) for a given z using
 * only finite number of terms in the power series
 * (see Proposition 7.4.4 and Algorithm 7.4.5 in Cohen's book)
 * Input:
 *   L = [w1, w2] with Im(w2/w1) > 0
 *   z = a complex number
 *   k = number of terms to be used
 **/
intrinsic WeierstrassP(L::SeqEnum, z::FldComElt, k::RngIntElt) -> FldComElt
{Compute Weierstrass p-function of z using up to k terms in the power series
expansion}
    require #L eq 2: "L must contain exactly two complex numbers";
    w1, w2 := Explode(L);
    require Im(w2/w1) gt 0: "Im(w2/w1) must be positive";

    L, _ := TransformLattice(L);
    w1, w2 := Explode(L);
    tau := w2/w1;
    // reduce z
    z := z/w1;
    n := Round(Im(z)/Im(tau));
    z := z - n*tau;
    z := z - Round(Re(z));
    if z eq 0 then
        error "z is a lattice point";
    end if;

    // Compute precision needed, based on the error term
    err := EstimateWPMaxError(k, k);
    err := Ceiling(-Log(10, err));
    C := Parent(tau);
    pi := Pi(C); i := C!Sqrt(-1);
    q := Exp(2*pi*i*tau);
    u := Exp(2*pi*i*z); // since already let z <- z/w1
    f := 1/12 + u/(1-u)^2;
    // Reset n up to k-1
    // (22 term -> max abs. err. = 3.4291 x 10^-52)
    // (50 term -> max abs. err. = 2.3251 x 10^-118)
    for n := 1 to (k-1) do
        tmp := u * ( 1/(1 - (q^n)*u)^2 + 1/((q^n) - u)^2 );
        tmp -:= 2/(1 - q^n)^2;
        tmp *:= q^n;
        f +:= tmp;
    end for;
    f *:= (2*pi*i/w1)^2;
    return f;
end intrinsic;


/**
 * Calculate the 1st derivative of the Weierstrass \wp-function p(z)
 * for a given z approximately, using Alg 7.4.5, and a finite of terms
 * in Prop 7.4.4 in Cohen's book
 * Input:
 *   L = [w1, w2], the periods of L that E(C) = C/
 *   z = a complex number that we want to find \wp(z, L)
 *   k = number of terms in the infinite sum
 **/
intrinsic WeierstrassPDash(L::SeqEnum, z::FldComElt, k::RngIntElt)
  -> FldComElt
{Compute the value of the first derivative of Weierstrass \\wp-function at z,
where z is given with respect to the fundamental parallelogram spanned by L.
This function uses the first k-1 terms in the infinite sum formula}
    require #L eq 2: "L must contain exactly two complex numbers";
    w1, w2 := Explode(L);
    require Im(w2/w1) gt 0: "Im(w2/w1) must be positive";

    L, _ := TransformLattice(L);
    w1, w2 := Explode(L);
    tau := w2/w1;
    // reduce z
    z := z/w1;
    n := Round(Im(z)/Im(tau));
    z := z - n*tau;
    z := z - Round(Re(z));

    // now compute p(z,tau)
    if z eq 0 then
      error "z is a lattice point";
    end if;

    C := Parent(w1);
    i := C!Sqrt(-1);
    pi := Pi(C);
    q := Exp(2*pi*i*tau);
    u := Exp(2*pi*i*z); // since already let z <- z/w1
    f := (1 + u)/(1 - u)^3;

    for n := 1 to (k-1) do
      tmp := ( 1 + (q^n)*u )/( 1 - (q^n)*u )^3;
      tmp +:= ((q^n) + u)/((q^n) - u)^3;
      tmp *:= q^n;
      f +:= tmp;
    end for;
    f *:= u * (2*pi*i/w1)^3;
    return f;
end intrinsic;
```

A.6.3 interval_wp.m

This file involves computing the interval version of the function f mentioned in Proposition 3.2.3. Note that this also requires some basic arithmetic operations on real intervals, which are implemented as shown in the next subsection.

```
/******************************************************************************
 * interval_wp.m
 * Computing interval version of the approximate Weierstrass \wp-function
 *
 * By Thotsaphon Thongjunthug
 * Last updated: 08 December 2010
 * Any errors should be reported to <nookaussie@yahoo.com>
 *
 * Required packages:
 * 1) interval_arith.m - for basic interval arithmetic
 ******************************************************************************/

// required for recursive definition
forward HasBoundary;
forward HasBoundaryVertical;

/******************************************************************************
 * Auxiliary local functions
 ******************************************************************************/

/**
 * (verified 13 Jul 08)
 * Compute the real part of a slant boundary y = m*x + c
 * Input:
 *   L = [x1, x2] = range of x
 *   m = gradient of the boundary (m = 0 ==> horizontal boundary)
 *   c = constant
 *   k = number of terms used in the power series expansion
 *   tau = represent the fundamental parallelogram {1, tau}
 * Output:
 *   interval containing the range of the real part
 **/
function realPartSlant(L, m, c, k, tau)
    if #L ne 2 then
        error "L must contain exactly two real numbers";
    end if;
    x1, x2 := Explode(L);
    if x1 gt x2 then
        error "x1 must be < or = x2";
    end if;
    pi := Pi(Parent(x1));
    y1 := -2*pi*(m*x1 + c);
    y2 := -2*pi*(m*x2 + c);
    if y1 le y2 then
        Y := [y1, y2];
    else
        Y := [y2, y1];
    end if;
    orgY := Y;
    Y := invExp(Y);

    // Re( u/(1-u)^2 )
    // denominator
    denom := invMul([2., 2.], invCos([2*pi*x1, 2*pi*x2]));
    denom := invSub(Y, denom);
    denom := invMul(Y, denom);
    denom := invAdd([1., 1.], denom);
    //print "denom 1 = ", denom;
    // In case the interval contains negative numbers, try to adjust it
    if denom[1] le 0 then
        //print "problem here (real part 1)";
        //print "Caution at L = ", L;
        //print "Y = ", Y;
        //print "m = ", m;
        //print "denom 1 = ", denom;
        // test for existence of local extremum in L
        diffden := invPow(Y, 2);
        diffden := invSub(invCos([2*pi*x1, 2*pi*x2]), diffden);
        diffden := invMul(diffden, [m, m]);
        diffden := invAdd(invSin([2*pi*x1, 2*pi*x2]), diffden);
        diffden := invMul(diffden, Y);
        if (diffden[1] le 0) and (0 le diffden[2]) then
            // may have local extremum - should not happen
            error "0 is in denominator interval, please report!";
        else
            // strict behaviour
            denx1 := 1 - 2*Exp(y1)*Cos(2*pi*x1) + Exp(2*y1);
            denx2 := 1 - 2*Exp(y2)*Cos(2*pi*x2) + Exp(2*y2);
            denom := [Min(denx1, denx2), Max(denx1, denx2)];
            //print "Denom 1 modified = ", denom; // successfully modified
            //print "*********************";
        end if;
    end if;

    // numerator
    numer := invAdd([1., 1.], invPow(Y, 2));
    numer := invMul(numer, invCos([2*pi*x1, 2*pi*x2]));
    numer := invSub(numer, invMul([2., 2.], Y));
    numer := invMul(numer, Y);
    rp := invDiv(numer, invPow(denom, 2));
    rp := invAdd(rp, [1./12, 1./12]);

    for n := 1 to (k-1) do
        aqn := Exp(-2*pi*Im(tau)*n); // = |q^n|
        A := Cos(2*pi*n*Re(tau));
        B := Sin(2*pi*n*Re(tau));
        qnu := invAdd(orgY, [-2*pi*Im(tau)*n, -2*pi*Im(tau)*n]);
        qnu := invExp(qnu);
        // Re (q^n*u / (1 - q^n*u)^2)
        // denom
        denom := invMul([A, A], invCos([2*pi*x1, 2*pi*x2]));
        denom := invSub(denom, invMul([B, B], invSin([2*pi*x1, 2*pi*x2])));
        denom := invMul(denom, [2., 2.]);
        denom := invSub(qnu, denom);
        denom := invMul(denom, qnu);
        denom := invSub([1., 1.], denom);

        //print "denom 2 = ", denom;
        // numerator
        numer := invMul([A, A], invCos([2*pi*x1, 2*pi*x2]));
        numer := invSub(numer, invMul([B, B], invSin([2*pi*x1, 2*pi*x2])));
        numer := invMul(numer, invAdd([1., 1.], invPow(qnu, 2)));
        numer := invSub(numer, invMul([2., 2.], qnu));
        numer := invMul(numer, qnu);
        rp := invAdd(rp, invDiv(numer, invPow(denom, 2)));

        // Re( q^n*u / (q^n - u)^2 )
        // denominator
        denom := invMul([A, A], invCos([2*pi*x1, 2*pi*x2]));
        denom := invAdd(denom, invMul([B, B], invSin([2*pi*x1, 2*pi*x2])));
        denom := invMul(denom, qnu);
        denom := invMul(denom, [2., 2.]);
        denom := invSub([aqn^2, aqn^2], denom);
        denom := invAdd(denom, invPow(Y, 2));
        //print "denom 3 = ", denom;
        // numerator
        numer := invMul([A, A], invCos([2*pi*x1, 2*pi*x2]));
        numer := invAdd(numer, invMul([B, B], invSin([2*pi*x1, 2*pi*x2])));
        numer := invMul(numer, invAdd([aqn^2, aqn^2], invPow(Y, 2)));
        numer := invSub(numer, invMul([2., 2.], qnu));
        numer := invMul(numer, qnu);
        //print "real num 3 = ", numer;
        rp := invAdd(rp, invDiv(numer, invPow(denom, 2)));

        // Constant real part
        // denominator
        denom := [1 - 2*A*aqn + aqn^2, 1 - 2*A*aqn + aqn^2];
        //print "denom 4 = ", denom;
        // numerator
        numer := [aqn*A*(1+aqn^2) - 2*aqn^2, aqn*A*(1+aqn^2) - 2*aqn^2];
        numer := invMul(numer, [2., 2.]);
        rp := invSub(rp, invDiv(numer, invPow(denom, 2)));
    end for;
    rp := invMul(rp, [-4*pi, -4*pi]);
    //print "rp = ", rp;
    return rp;
end function;


/**
 * (verified 13 Jul 08)
 * Compute the imaginary part of a slant boundary y = m*x + c
 * Input:
 *   L = [x1, x2] = range of x
 *   m = gradient of the boundary (m = 0 ==> horizontal boundary)
 *   c = constant
 *   k = number of terms used in the power series expansion
 *   tau = represent the fundamental parallelogram {1, tau}
 * Output:
 *   interval containing the range of the imaginary part
 **/
function imPartSlant(L, m, c, k, tau)
    if #L ne 2 then
        error "L must have exactly 2 real numbers";
    end if;
    x1, x2 := Explode(L);
    if x1 gt x2 then
        error "x1 must be <= x2";
    end if;
    pi := Pi(Parent(x1));
    y1 := -2*pi*(m*x1 + c);
    y2 := -2*pi*(m*x2 + c);
    if y1 le y2 then
        Y := [y1, y2];
    else
        Y := [y2, y1];
    end if;
    orgY := Y;
    Y := invExp(Y);

    // Im( u/(1-u)^2 )
    // denominator
    denom := invMul([2., 2.], invCos([2*pi*x1, 2*pi*x2]));
    denom := invSub(Y, denom);
    denom := invMul(Y, denom);
    denom := invAdd([1., 1.], denom);
    // In case the interval contains negative numbers, try to adjust it
    if denom[1] le 0 then
        //print "problem here (Im part 1)";
        //print "Caution at L = ", L;
        //print "Y = ", Y;
        //print "m = ", m;
        //print "denom im 1 = ", denom;
        // test for existence of local extremum in L
        diffden := invPow(Y, 2);
        diffden := invSub(invCos([2*pi*x1, 2*pi*x2]), diffden);
        diffden := invMul(diffden, [m, m]);
        diffden := invAdd(invSin([2*pi*x1, 2*pi*x2]), diffden);
        diffden := invMul(diffden, Y);
        if (diffden[1] le 0) and (0 le diffden[2]) then
            // may have local extremum - should not happen
            error "0 is in denominator interval, please report!";
        else
            // strict behaviour
            denx1 := 1 - 2*Exp(y1)*Cos(2*pi*x1) + Exp(2*y1);
            denx2 := 1 - 2*Exp(y2)*Cos(2*pi*x2) + Exp(2*y2);
            denom := [Min(denx1, denx2), Max(denx1, denx2)];
            //print "Denom 1 modified = ", denom; // successfully modified
            //print "*********************";
        end if;
    end if;

    // numerator
    numer := invSub([1., 1.], invPow(Y, 2));
    numer := invMul(numer, Y);
    numer := invMul(numer, invSin([2*pi*x1, 2*pi*x2]));
    ip := invDiv(numer, invPow(denom, 2));

    for n := 1 to (k-1) do
        aqn := Exp(-2*pi*Im(tau)*n); // = |q^n|
        A := Cos(2*pi*n*Re(tau));
        B := Sin(2*pi*n*Re(tau));
        qnu := invAdd(orgY, [-2*pi*Im(tau)*n, -2*pi*Im(tau)*n]);
        qnu := invExp(qnu);
        // Im (q^n*u / (1 - q^n*u)^2)
        // denom
        denom := invMul([A, A], invCos([2*pi*x1, 2*pi*x2]));
        denom := invSub(denom, invMul([B, B], invSin([2*pi*x1, 2*pi*x2])));
        denom := invMul(denom, [2., 2.]);
        denom := invSub(qnu, denom);
        denom := invMul(denom, qnu);
        denom := invSub([1., 1.], denom);
        //print "denom im 2 = ", denom;
        // numerator
        numer := invMul([B, B], invCos([2*pi*x1, 2*pi*x2]));
        numer := invAdd(numer, invMul([A, A], invSin([2*pi*x1, 2*pi*x2])));
        numer := invMul(numer, qnu);
        numer := invMul(numer, invSub([1., 1.], invPow(qnu, 2)));
        ip := invAdd(ip, invDiv(numer, invPow(denom, 2)));

        // Im( q^n*u / (q^n - u)^2 )
        // denominator
        denom := invMul([A, A], invCos([2*pi*x1, 2*pi*x2]));
        denom := invAdd(denom, invMul([B, B], invSin([2*pi*x1, 2*pi*x2])));
        denom := invMul(denom, qnu);
        denom := invMul(denom, [2., 2.]);
        denom := invSub([aqn^2, aqn^2], denom);
        denom := invAdd(denom, invPow(Y, 2));
        //print "denom im 3 = ", denom;
        // numerator
        numer := invMul([A, A], invSin([2*pi*x1, 2*pi*x2]));
        numer := invSub(numer, invMul([B, B], invCos([2*pi*x1, 2*pi*x2])));
        numer := invMul(numer, qnu);
        numer := invMul(numer, invSub([aqn^2, aqn^2], invPow(Y, 2)));
        //print "num im 3 = ", numer;
        ip := invAdd(ip, invDiv(numer, invPow(denom, 2)));

        // Constant part
        // denominator
        denom := [1 - 2*A*aqn + aqn^2, 1 - 2*A*aqn + aqn^2];
        //print "denom im 4 = ", denom;
        // numerator
        numer := [aqn*B*(1-aqn^2), aqn*B*(1-aqn^2)];
        numer := invMul(numer, [2., 2.]);
        ip := invSub(ip, invDiv(numer, invPow(denom, 2)));
    end for;
    ip := invMul(ip, [-4*pi, -4*pi]);
    return ip;
end function;


/**
 * Added 18 Nov 2010
 * Compute the real part for a vertical boundary (rare case)
 * Input:
 *   x = fixed x-coordinate
 *   c, d = range of y-coordinates (c <= d)
 *   k = number of terms used in the power series expansion
 *   tau = the fundamental parallelogram {1, tau} (here, Re(tau) = 0)
 * Output:
 *   a real interval containing the range of the real part.
 **/
function realPartVertical(x, c, d, k, tau)
    if c gt d then
        error "c must be < or = d";
    end if;
    pi := Pi(Parent(x));
    y1 := -2*pi*c; y2 := -2*pi*d;
    Y := [y2, y1];
    orgY := Y;
    Y := invExp(Y);
    //print "Y = ", Y;

    // Re( u/(1-u)^2 )
    // denominator
    denom := invMul([2., 2.], invCos([2*pi*x, 2*pi*x]));
    denom := invSub(Y, denom);
    denom := invMul(Y, denom);
    denom := invAdd([1., 1.], denom);
    //print "denom 1 = ", denom;

    // numerator
    numer := invAdd([1., 1.], invPow(Y, 2));
    numer := invMul(numer, invCos([2*pi*x, 2*pi*x]));
    numer := invSub(numer, invMul([2, 2], Y));
    numer := invMul(numer, Y);
    rp := invDiv(numer, invPow(denom, 2));
    rp := invAdd(rp, [1./12, 1./12]);

    for n := 1 to (k-1) do
        aqn := Exp(-2*pi*Im(tau)*n); // = |q^n|
        qnu := invAdd(orgY, [-2*pi*Im(tau)*n, -2*pi*Im(tau)*n]);
        qnu := invExp(qnu);
        // Re (q^n*u / (1 - q^n*u)^2)
        // denom
        denom := invCos([2*pi*x, 2*pi*x]);
        denom := invMul(denom, [2., 2.]);
        denom := invSub(qnu, denom);
        denom := invMul(denom, qnu);
        denom := invSub([1., 1.], denom);

        //print "denom 2 = ", denom;
        // numerator
        numer := invCos([2*pi*x, 2*pi*x]);
        numer := invMul(numer, invAdd([1., 1.], invPow(qnu, 2)));
        numer := invSub(numer, invMul([2., 2.], qnu));
        numer := invMul(numer, qnu);
        rp := invAdd(rp, invDiv(numer, invPow(denom, 2)));

        // Re( q^n*u / (q^n - u)^2 )
        // denominator
        denom := invCos([2*pi*x, 2*pi*x]);
        denom := invMul(denom, qnu);
        denom := invMul(denom, [2., 2.]);
        denom := invSub([aqn^2, aqn^2], denom);
        denom := invAdd(denom, invPow(Y, 2));
        //print "denom 3 = ", denom;
        // numerator
        numer := invCos([2*pi*x, 2*pi*x]);
        numer := invMul(numer, invAdd([aqn^2, aqn^2], invPow(Y, 2)));
        numer := invSub(numer, invMul([2., 2.], qnu));
        numer := invMul(numer, qnu);
        //print "real num 3 = ", numer;
        rp := invAdd(rp, invDiv(numer, invPow(denom, 2)));

        // Constant real part
        // denominator
        denom := [1 - 2*aqn + aqn^2, 1 - 2*aqn + aqn^2];
        //print "denom 4 = ", denom;
        // numerator
        numer := [aqn*(1+aqn^2) - 2*aqn^2, aqn*(1+aqn^2) - 2*aqn^2];
        numer := invMul(numer, [2., 2.]);
        rp := invSub(rp, invDiv(numer, invPow(denom, 2)));
    end for;
    rp := invMul(rp, [-4*pi, -4*pi]);
    //print "rp = ", rp;
    return rp;
end function;


/**
 * Added 18 Nov 2010
 * Compute the imaginary part for a vertical boundary (rare case)
 * Input:
 *   x = fixed x-coordinate
 *   c, d = range of y-coordinates (c <= d)
 *   k = number of terms used in the power series expansion
 *   tau = the fundamental parallelogram {1, tau} (here, Re(tau) = 0)
 * Output:
 *   a real interval containing the range of the imaginary part.
 **/
function imPartVertical(x, c, d, k, tau)
    if c gt d then
        error "c must be < or = d";
    end if;
    pi := Pi(Parent(x));
    y1 := -2*pi*c; y2 := -2*pi*d;
    Y := [y2, y1];
    orgY := Y;
    Y := invExp(Y);

    // Im( u/(1-u)^2 )
    // denominator
    denom := invMul([2., 2.], invCos([2*pi*x, 2*pi*x]));
    denom := invSub(Y, denom);
    denom := invMul(Y, denom);
    denom := invAdd([1., 1.], denom);
    // numerator
    numer := invSub([1., 1.], invPow(Y, 2));
    numer := invMul(numer, Y);
    numer := invMul(numer, invSin([2*pi*x, 2*pi*x]));
    ip := invDiv(numer, invPow(denom, 2));

    for n := 1 to (k-1) do
        aqn := Exp(-2*pi*Im(tau)*n); // = |q^n|
        qnu := invAdd(orgY, [-2*pi*Im(tau)*n, -2*pi*Im(tau)*n]);
        qnu := invExp(qnu);
        // Im (q^n*u / (1 - q^n*u)^2)
        // denom
        denom := invCos([2*pi*x, 2*pi*x]);
        denom := invMul(denom, [2., 2.]);
        denom := invSub(qnu, denom);
        denom := invMul(denom, qnu);
        denom := invSub([1., 1.], denom);
        //print "denom im 2 = ", denom;
        // numerator
        numer := invAdd(numer, invSin([2*pi*x, 2*pi*x]));
        numer := invMul(numer, qnu);
        numer := invMul(numer, invSub([1., 1.], invPow(qnu, 2)));
        ip := invAdd(ip, invDiv(numer, invPow(denom, 2)));

        // Im( q^n*u / (q^n - u)^2 )
        // denominator
        denom := invCos([2*pi*x, 2*pi*x]);
        denom := invMul(denom, qnu);
        denom := invMul(denom, [2., 2.]);
        denom := invSub([aqn^2, aqn^2], denom);
        denom := invAdd(denom, invPow(Y, 2));
        //print "denom im 3 = ", denom;
        // numerator
        numer := invSin([2*pi*x, 2*pi*x]);
        numer := invMul(numer, qnu);
        numer := invMul(numer, invSub([aqn^2, aqn^2], invPow(Y, 2)));
        //print "num im 3 = ", numer;
        ip := invAdd(ip, invDiv(numer, invPow(denom, 2)));
    end for;
    ip := invMul(ip, [-4*pi, -4*pi]);
    return ip;
end function;


/*****************************************************************************
 * Main intrinsic functions
 *****************************************************************************/

/**
 * Check if there is part of the boundary of the region R^v(\xi) on a given
 * (slant or horizontal) boundary y = m*x + c of a parallelogram.
 * Input:
 *   L = [x1, x2] = range of x-coordinates
 *   m = the gradient of the parallelogram's boundary
 *   c = a constant
 *   B = bound for the function f(X1, X2, X3) (normally is the U_\xi)
 *   tau = representing the fundamental parallelogram {1, tau}
 * Output:
 *   true if we (suspect) that it may contain part of R^v(\xi); false otherwise
 **/
intrinsic HasBoundary(L::SeqEnum, m::FldReElt, c::FldReElt, B::FldReElt,
    tau::FldComElt) -> BoolElt
{Check if there is part of the boundary of the region R^v(\xi) on a given slant
(or horizontal) boundary of a parallelogram.}
    require #L eq 2: "L must have exactly two real numbers";
    x1, x2 := Explode(L);
    require x1 le x2: "x1 must be < or = x2";
    require B ge 0: "B must be non-negative";
    require Im(tau) ge 0: "Im(tau) must be non-negative";

    r := realPartSlant(L, m, c, 23, tau);
    //if #r eq 1 then
    //    print "*** Null denominator ***";
    //    return true;
    //end if;
    err := EstimateWPMaxError(23, 23);
    F := invPow(r, 2);
    F := invAdd(F, invPow(imPartSlant(L, m, c, 23, tau), 2));
    F := [Sqrt(F[1]), Sqrt(F[2])];
    F := invSub(F, [B, B]);
    F := [F[1]-err, F[2]+err];
    if not ((F[1] le 0) and (0 le F[2])) then
        return false;
    end if;

    // If still not return false, try to bisect the interval and check
    midPt := (x1 + x2)/2;
    L1 := [x1, midPt];
    L2 := [midPt, x2];
    l1 := HasBoundary(L1, m, c, B, tau);
    l2 := HasBoundary(L2, m, c, B, tau);
    if not(l1) and not(l2) then
```

```
        return false;
    else
        return true;
    end if;
end intrinsic;


/**
 * Check if there is part of the boundary of the region R^v(\xi) on a given
 * vertical boundary of a parallelogram.
 * Input:
 *   x = a fixed x-coordinate
 *   c, d = the range of y-coordinates
 *   B = bound for the function f(X1, X2, X3) (normally is the U_\xi)
 *   tau = representing the fundamental parallelogram {1, tau}
 * Output:
 *   true if we (suspect) that it may contain part of R^v(\xi); false otherwise
 **/
intrinsic HasBoundaryVertical(x::FldReElt, c::FldReElt, d::FldReElt,
    B::FldReElt, tau::FldComElt) -> BoolElt
{Check if there is part of the boundary of the region R^v(\xi) on a given
vertical boundary of a parallelogram.}
    require c le d: "c must be < or = d";
    require B ge 0: "B must be non-negative";
    require Im(tau) ge 0: "Im(tau) must be non-negative";

    r := realPartVertical(x, c, d, 23, tau);
    //if #r eq 1 then
    //    print "*** Null denominator ***";
    //    return true;
    //end if;
    err := EstimateWPMaxError(23, 23);
    F := invPow(r, 2);
    F := invAdd(F, invPow(imPartVertical(x, c, d, 23, tau), 2));
    F := [Sqrt(F[1]), Sqrt(F[2])];
    F := invSub(F, [B, B]);
    F := [F[1]-err, F[2]+err];
    if not ((F[1] le 0) and (0 le F[2])) then
        return false;
    end if;

    // If still not return false, try to bisect the interval and check
    l1 := HasBoundaryVertical(x, c, d/2, B, tau);
    l2 := HasBoundaryVertical(x, d/2, d, B, tau);
    if not(l1) and not(l2) then
        return false;
    else
        return true;
    end if;
end intrinsic;
```

A.7 Height Bound V: Interval Arithmetic

This file provides some basic arithmetic operations on real intervals. For more information on the subject of interval arithmetic, see, e.g., [Moo66].

```
/******************************************************************************
 * interval_arith.m
 * Functions for basis arithmetic on real intervals
 *
 * By Thotsaphon Thongjunthug
 * Last updated: 08 December 2010
 * Any errors should be reported to <nookaussie@yahoo.com>
 ******************************************************************************/
```

```
/*****************************************************************************
 * Unary Operations
 ****************************************************************************/

/**
 * Cosine function for the interval [a, b]
 * Input:
 *   L = [a, b] with a <= b
 **/
intrinsic invCos(L::SeqEnum[FldReElt]) -> SeqEnum[FldReElt]
{Cosine function for a real interval L = [a, b].}
    // Check validity of L = [a, b]
    require #L eq 2: "Interval must contain exactly two real numbers";
    a, b := Explode(L);
    require a le b: "a must be < or = b";

    // Check if L contains any multiples of Pi
    pi := Pi(Parent(a));
    lb := Ceiling(a/pi);
    ub := Floor(b/pi);
    width := ub - lb;
    if ((lb*pi) gt b) or ((ub*pi) lt a) then
        // L contains no extremum for cos function
        return [Min(Cos(a), Cos(b)), Max(Cos(a), Cos(b))];
    elif width gt 0 then
        // L contains both even and odd multiples of pi, so return [-1, 1]
        return [-1, 1];
    elif (lb mod 2) eq 0 then
        // lb = ub, and is even, then maximum of cosine = 1
        return [Min(Cos(a), Cos(b)), 1];
    else
        // minimum of cosine = -1
        return [-1, Max(Cos(a), Cos(b))];
    end if;
end intrinsic;


/**
 * Sine function for the interval [a, b]
 * Input:
 *   L = [a, b], with a <= b
 **/
intrinsic invSin(L::SeqEnum[FldReElt]) -> SeqEnum[FldReElt]
{Sine function for a real interval L = [a, b].}
    // Check validity of L = [a, b]
    require #L eq 2: "Interval must contain exactly two real numbers";
    a, b := Explode(L);
    require a le b: "a must be < or = b";

    // Check if L contains any multiples of Pi/2
    pi := Pi(Parent(a));
    lb := Ceiling(2*a/pi);
    ub := Floor(2*b/pi);
    // only care the odd multiple of pi/2 - N . 8/7/08
    if (lb mod 2) eq 0 then
        lb := lb + 1;
    end if;
    if (ub mod 2) eq 0 then
        ub := ub - 1;
    end if;
    width := (ub - lb)/2;

    // Case 1: when L contains no odd multiple of Pi/2
    if ((lb*pi/2) gt b) or ((ub*pi/2) lt a) then
        return [Min(Sin(a), Sin(b)), Max(Sin(a), Sin(b))];
    end if;
    // Case 2: width >= 1, so L contains two odd multiples of Pi/2
    if width gt 1 then
        return [-1, 1];
    end if;
    // Case 2: width = 0 (so lb = ub)
    if (lb mod 4) eq 1 then
        // Max of sin = 1
        return [Min(Sin(a), Sin(b)), 1];
    else
        // Mim of sin = -1
        return [-1, Max(Sin(a), Sin(b))];
    end if;
end intrinsic;


/**
 * Exponential function for the interval
 * Input:
 *   L = [a, b], with a <= b
 **/
intrinsic invExp(L::SeqEnum[FldReElt]) -> SeqEnum[FldReElt]
{Exponential function for a real interval L = [a, b].}
    // Check validity of L = [a, b]
    require #L eq 2: "Interval must contain exactly two real numbers";
    a, b := Explode(L);
    require a le b: "a must be < or = b";
    return [Exp(a), Exp(b)];
end intrinsic;


/**
 * Let L be an interval, compute L^n = L * ... * L, n times
 * Input:
 *   L = [a, b], with a <= b
 *   n = a non-negative integer
 **/
intrinsic invPow(L::SeqEnum[FldReElt], n::RngIntElt) -> SeqEnum[FldReElt]
{Compute n-th power of a real interval L.}
    // Check validity of L = [a, b]
    require #L eq 2: "Interval must contain exactly two real numbers";
    a, b := Explode(L);
    require a le b: "a must be < or = b";
    require n ge 0: "n must be a non-negative integer";

    if (n mod 2) eq 0 then
        // take care when a and b is of different sign
        if (a le 0) and (b ge 0) then
            return [0, Max(a^n, b^n)];
        else
            return [Min(a^n, b^n), Max(a^n, b^n)];
        end if;
    end if;
    return [Min(a^n, b^n), Max(a^n, b^n)];
end intrinsic;


/******************************************************************************
 * Binary Operations
 ******************************************************************************/

/**
 * Add two real intervals
 * Input:
 *   L, K = real intervals
 **/
intrinsic invAdd(L::SeqEnum[FldReElt], K::SeqEnum[FldReElt]) -> SeqEnum[FldReElt]
{Add two real intervals.}
    require (#L eq 2) and (#K eq 2):
        "Both L, K must contain exactly two real numbers";
    a, b := Explode(L);
    c, d := Explode(K);
    require (a le b) and (c le d): "Check if a <= b and c <= d";
    return [a+c, b+d];
end intrinsic;
```

```
/**
 * For real intervals L, K, compute L - K
 * Input:
 *   L, K = real intervals
 **/
intrinsic invSub(L::SeqEnum[FldReElt], K::SeqEnum[FldReElt]) -> SeqEnum[FldReElt]
{Subtract two real intervals.}
    return invAdd(L, [-K[2], -K[1]]);
end intrinsic;


/**
 * For real intervals L, K, compute L*K
 * Input:
 *   L, K = real intervals
 **/
intrinsic invMul(L::SeqEnum[FldReElt], K::SeqEnum[FldReElt]) -> SeqEnum[FldReElt]
{Multiply two real intervals.}
    require (#L eq 2) and (#K eq 2):
        "Both L, K must contain exactly two real numbers";
    a, b := Explode(L);
    c, d := Explode(K);
    require (a le b) and (c le d): "Check if a <= b and c <= d";

    lb, _ := Min([a*c, a*d, b*c, b*d]);
    ub, _ := Max([a*c, a*d, b*c, b*d]);
    return [lb, ub];
end intrinsic;


/**
 * For real intervals L, K = [c, d], compute L/K = L*[1/d, 1/c]
 * provided that 0 is not in K
 * Input:
 *   L, K = real intervals
 **/
intrinsic invDiv(L::SeqEnum[FldReElt], K::SeqEnum[FldReElt]) -> SeqEnum[FldReElt]
{For real intervals L, K (with 0 not in K), compute L/K.}
    require (#L eq 2) and (#K eq 2):
        "Both L, K must contain exactly two real numbers";
    a, b := Explode(L);
    c, d := Explode(K);
    require (a le b) and (c le d): "Check if a <= b and c <= d";
    if (c le 0) and (0 le d) then
        error "K must not contain 0";
    end if;
    return invMul(L, [1/d, 1/c]);
end intrinsic;
```

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