Incorporating Value Judgments in Data Envelopment Analysis

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Synopsis

Data Envelopment Analysis (DEA) is a linear programming technique for measuring the relative efficiencies of a set of Decision Making Units (DMUs). Each DMU uses the same set of inputs in differing amounts to produce the same set of outputs in differing quantities. Weights are freely allocated in order to allow these multiple incommensurate inputs and outputs to be reduced to a single measure of input and a single measure of output. A relative efficiency score of a DMU under Constant Returns to Scale is given by maximising the sum of its weighted outputs to the sum of its weighted inputs, such that this ratio can not exceed 1 for any DMU; with the weights derived from the model being taken to represent the value attributed to the inputs and outputs of the assessment.

It is well known in DEA that this free allocation of weights can lead to several problems in the analysis. Firstly inputs and outputs can be virtually ignored in the assessment; secondly any relative relationships between the inputs or outputs can be ignored, and thirdly any relationships between the inputs and outputs can be violated. To avoid/overcome these problems, the Decision Maker's (DM) value judgments are incorporated into the assessment. At present there is one main avenue for the inclusion of values, that of weights restrictions, whereby the size of the weights are explicitly restricted. Thus to include the relative value of the inputs or outputs, the relative value of the weights for these related inputs or outputs are restricted. The popularity of this approach is mainly due to its simplicity and ease of use.

The aim of this thesis is, therefore, firstly, to demonstrate that, although the weights restrictions approach is appropriate for many DMs, for a variety of reasons some DMs, may prefer an alternative form for the expression of their values, e.g. so that they can include local values in the assessment. With this in mind, the second aim of this thesis is to present a possible alternative approach for the DMs to incorporate their values in a DEA assessment and, thirdly, it aims to utilise this alternative approach to improve envelopment.

This alternative approach was derived by considering the basic concept of DEA, which is that it relies solely on observed data to form the Production Possibility Set (PPS), and then uses the frontier of this PPS to derive a relative efficiency score for each DMU. It could be perceived, therefore, that the reason for DMUs receiving inappropriate relative efficiency scores is due to the lack of suitable DEA-efficient comparator DMUs. Thus, the proposed approach attempts to estimate suitable input output levels for these missing DEA-efficient comparator DMUs, i.e. Unobserved DMUs. These Unobserved DMUs are based on the manipulation of observed input output levels of specific DEA-efficient DMUs.

The aim of the use of these Unobserved DMUs is to improve envelopment, and the specific DEA-efficient DMUs that are selected as a basis for the Unobserved DMUs are those that delineate the DEA-efficient frontier from the DEA-inefficient frontier. So, the proposed approach attempts to extend the observed PPS, while assuming that the values of the observed DEA-efficient DMUs are in line with the perceived views of the DM.

The approach was successfully applied to a set of UK bank branches. To illustrate that no approach is all-purpose, and that each has its strengths and weaknesses and, therefore, its own areas of application, a brief comparison is made between the approach of weights restrictions and the approach proposed in this thesis.

This thesis is divided into three sections: A - Overview of the research area; B - An alternative perspective for incorporating values in DEA; C - The use of UDMUs to express the DM's values to improve envelopment

Papers Submitted for Publication from Thesis

1. Thanassoulis, E. and Allen, R., *Simulating Weights Restrictions in Data Envelopment Analysis by Means of Unobserved DMUs. Forthcoming in* **Management Science**.

2. Allen, R., Athanassoupolos, A., Dyson, R.D. and Thanassoulis, E., *Weights Restrictions and Value Judgments in Data Envelopment Analysis: The Evolution, Development and Future Directions. Forthcoming in* **Annals of Operational Research**.

3. Allen R. and Thanassoulis E., *Improving Envelopment in DEA. Under review for publication in* **Journal of The Operational Research Society**.

Papers Presented at Conference

1. Productivity Workshop - Athens, Georgia, USA - October 1994

2. Young O.R. Forum - Warwick Business School, Warwick University - March 1995

3. Young O.R. Conference - York University - March 1996

4. IFORS - Vancouver, Canada - July 1996

5. National O.R. Conference - Warwick University - September 1996

6. Young O.R. Forum - Southampton University - April 1997

Abbreviations

ABC:	Activity Based Costing
ADMU:	Anchor Decision Making Unit
AWR:	Absolute Weights Restriction
CEA:	Controlled Envelopment Analysis
CFA:	Constrained Facet Analysis
CRS:	Constant Returns to Scale
DEA:	Data Envelopment Analysis
DM:	Decision Maker
DMU:	Decision Making Unit [Observed only]
DRS:	Decreasing Returns to Scale
EPPS:	Extended Production Possibility Set
FSRD:	Full Set of Radial DMUs
IM:	Input Minimisation
IRDMU:	Input Radial DMU
IRS:	Increasing Returns to Scale
LWR:	Linked-Dependent Weights Restriction
MPSS:	Most Productive Scale Size
MRS:	Marginal Rate of Substitution
MRT:	Marginal Rate of Transformation
OM:	Output Maximisation
ORDMU:	Output Radial DMU
PPS:	Production Possibility Set
RDMU:	Radial DMU
RSRD:	Reduced Set of Radial DMUs
RWR:	Relative Weights Restriction
SDMU:	Scaled DMU
SE:	Super Efficiency
UDMU:	Unobserved Decision Making Unit
VRS:	Variable Returns to Scale
WR:	Weights Restriction

Sets of DMUs

AJP_{j_0}	Set of Referent DMUs to ADMU j_0 under SE
JE	Set of DEA-efficient DMUs
JE_{j_0}	Set of DEA-efficient DMUs excluding j_0
JF	Set of Class F DMUs under CRS
JFI	Set of Class F DMUs under IM VRS
JFIN	Set of Class NF DMUs under IM VRS
JFO	Set of Class F DMUs under OM VRS
JFON	Set of Class NF DMUs under OM VRS
JIN	Set of Class NF DMUs under CRS

Definitions

This section will define, in very simple terms, some general terminology that will be used in this thesis. The terminology is defined for use in this thesis only.

Absolute Weights Restrictions (AWRs): Restrictions on the actual numerical value of the DEA weights.

Constant Returns to Scale (CRS): Under efficient input to output transformations, scaling the input levels by α leads to a scaling of the output levels by β, such that $\alpha = \beta$.

Data Envelopment Analysis (DEA): A linear programming technique for determining the relative efficiency score of a set of DMUs

DEA-Efficient (Pareto-Koopmans): A DMU is DEA-efficient if no other DMU or linear combination of DMUs provide evidence that some of the inputs or outputs of assessed DMU could have been improved without deterioration to some of its other inputs or outputs.

Decision Maker (DM): The person responsible for the efficiency assessment.

Decision Making Units (DMUs): Organisational units that perform the same function and use the same set of inputs to produce the same set of outputs. e.g. Banks, Schools.

Decreasing Returns to Scale: Under efficient input to output transformations, scaling the input levels by α leads to a scaling of the output levels by β, such that $\alpha > \beta$.

Extended Production Possibility Set (EPPS): The extension and possible modification of the observed PPS, through the use of UDMUs.

Increasing Returns to Scale: Under efficient input to output transformations, scaling the input levels by α leads to a scaling of the output levels by β, such that $\alpha < \beta$.

Input Minimisation (IM): Is a DMU consuming the minimum amount of input to produce its output relative to the other DMUs in the assessment?

Linked-Dependent Weights Restrictions (LWRs): Restrictions on the size of input weights relative to the size of output weights, reflecting the relationship between the inputs and outputs.

Output Maximisation (OM): Is a DMU producing the maximum amount of output from its input levels, relative to the other DMUs in the assessment?

Peers: The DEA-efficient DMUs that are used as a basis for the DEA-efficient input output levels a DMU could attain.

Production Possibility Set (PPS): This set contains all the obtainable input output mixes.

Radial Efficiency: The radial distance of a DMU from the frontier of the PPS.

Relative Efficiency Score: A summary of the measure of the distance between the actual and efficient input output levels of a DMU.

Relative Weights Restrictions (RWRs): Restrictions imposed on the relative size that either input or output weights can take relative to other input or output weights respectively.

Scale Efficiency: A measure of how much of a DMU's inefficiency is solely attributable to its scale of operation.

Slack: The additional improvement required for a DMU to become DEA-efficient (increase in outputs, decrease in inputs), after the radial efficiency of a DMU has been assessed.

Super Efficiency (SE): The relative efficiency score of a DMU relative to the other DMUs in the assessment, excluding itself.

Targets: The input output levels that would render a DMU DEA-efficient.

Technical Efficiency: A measure of efficiency that ignores the effect of scale size of a DMU. That is, a DMU's efficiency is only compared relative to DMUs of a similar scale.

Variable Returns to Scale (VRS): Efficient input to output transformations that do not necessarily follow CRS, i.e. can be IRS, CRS or DRS.

Virtual Weights Restrictions (VWRs): Restrictions on the percentage that an input or output can contribute to the sum of the weighted inputs or outputs respectively.

Classes of DMUs

E: DEA-efficient DMUs that cannot be expressed as linear combinations of other DEA-efficient DMUs.

E': DEA-efficient DMUs that can be expressed as linear combinations of other DEA-efficient DMUs.

F: Radially efficient DMUs but DEA-inefficient, due to the presence of slack values. That is, they can be expressed as a linear combination of other DEA-efficient DMUs plus or minus a slack value.

NE, **NE'** and **NF** are as above only defined for DEA-inefficient DMUs which when projected onto the PPS frontier, are of class **E**, **E'** and **F** respectively.

Section A
Overview of the Research Area of this Thesis

This section covers chapters one to three and is a general introduction to the research area of this thesis. It discusses the general concepts of Data Envelopment Analysis, explains how a relative efficiency score is obtained, and outlines other information provided by the procedure.

More specifically, the need for the inclusion of the DM's value in a DEA assessment, and the current procedures for incorporating values in a DEA assessment are discussed, and their interconnections explored.

The limitations of current procedures to satisfy the possible requirements of certain DMs motivated the need for alternative approaches to incorporate the DM's values, and hence this thesis. These motivating needs, aims and resultant procedure of this thesis are outlined in this section.

1. Chapter One Introduction

1.1 Introduction

In the modern day world it is becoming more and more important for organisations to know how efficiently and effectively they are operating compared to similar organisations (competitors). For example, a department from one university may want to compare its performance with the same department from other universities, or a bank may want to compare the performance of its different branches throughout the country - the latter will in fact be the application of this thesis. What is meant by the word efficient? Efficient means that something is working well, quickly and without waste; whereas the word effective is to produce the desired result. This thesis is concerned only with the assessment of the relative efficiency of an organisation, with relative efficiency being how well, how quickly and without waste, an organisation performs, compared to similar organisations. The concept of relative efficiency will be defined more mathematically later. The question of whether the organisations are producing the desired effect will not be addressed in this thesis, it only addresses the issue of whether an organisation is achieving its goals relatively efficiently.

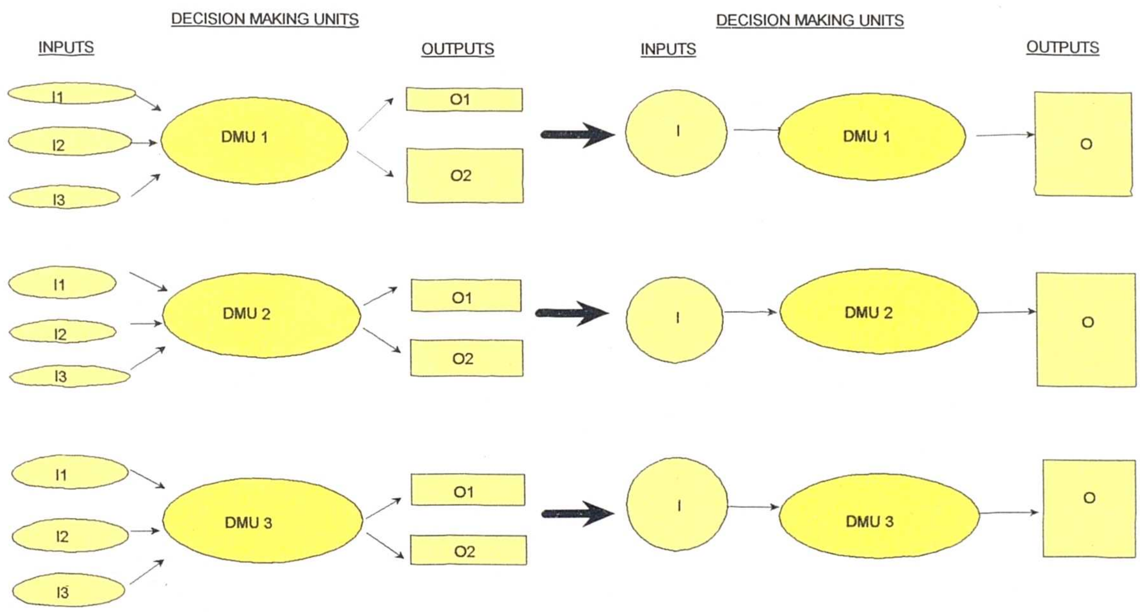
Two fundamental approaches exist for obtaining measures of efficiency, with each approach having numerous methods. These two main approaches are; parametric and non-parametric. The parametric approaches require *a priori* assumptions to be made with regard to the production function (see Färe and Primont [29] p.8 for a formal definition) - these types of approaches will not be considered in this thesis. In the non-parametric approaches no assumptions are necessary with respect to the production function. One such approach is **Data Envelopment Analysis** (DEA) which uses observed data to estimate an efficiency frontier. This is the broad subject area of this thesis. Thus, throughout this thesis it will be assumed that the **Decision Maker** (DM) wants a measure of relative efficiency, as defined by DEA.

This chapter is structured as follows: The next section outlines the general concepts of DEA in non mathematical terms; section three mathematically details DEA; section four graphically illustrates the approach and section five details the other information provided by DEA.

1.2 Data Envelopment Analysis: The Approach

DEA is a mathematical programming technique that is applied to a group of **Decision Making Units** (DMUs) which are organisational units that perform the same task, i.e. bank branches or sales people, each having the same multiple incommensurate inputs and outputs. The initial step in the assessment is to determine a set of relevant inputs and outputs (factors). These factors may be qualitative (weather or location), provided a value can be given to them, or quantitative (number of employees or amount of produce) and should be such that it is desired to minimise input levels and maximise output levels. If large numbers of factors are used in the analysis, the method's ability to distinguish between the relatively efficient and inefficient DMUs decreases. Therefore, only the most important factors should be included, see Golany and Roll [33]. DEA can now be applied to the set of DMUs to determine a relative efficiency measure based on the selected inputs and outputs. These multiple inputs and outputs are reduced to a single input value and a single output value by the allocation of a weight to each input and output, with the only restriction on the weights being that they must be strictly positive. The DMUs' weights are calculated by comparing their observed input output levels to the observed input output levels of all the other DMUs in the assessment, in order to show the DMU in the "*best possible light*" compared to all the other DMUs in the assessment. Finally, a measure of relative efficiency is produced as a ratio of the sum of its weighted outputs to the sum of its weighted inputs. For example, in Figure 1.1 a set of 3 DMUs are to be assessed, each consuming varying amounts of three different inputs to produce varying amounts of two different outputs. Through the free allocation of weights to these three inputs and two outputs, they can be reduced to a single value to represent the amount of input used to produce its output relative to the other DMUs in the data set.

Figure 1.1 - Data Envelopment Analysis: The Approach



Thus DEA is a relative measure, and the addition or the subtraction of DMUs may or may not alter the relative efficiency of a DMU. The efficiency can be viewed from two orientations, and the choice of orientation will depend on the context, i.e. the DM and the organisational nature.

- **Output Maximisation** (OM) - Is the DMU producing the maximum amount of output from its input levels, relative to the other DMUs in the assessment?

- **Input Minimisation** (IM) - Is the DMU consuming the minimum amount of input to produce its output, relative to the other DMUs in the assessment?

Once the orientation of the efficiency measured has been decided, the relatively efficient DMUs which form the "production possibility set frontier" can be identified. This frontier is formed on the assumption that there exists continual linear substitution between any pair of inputs or outputs over the relevant range. Further, this production possibility set frontier is the boundary for the "**Production Possibility Set**" (PPS) which contains all obtainable input and output mixes. For a formal definition of the PPS in DEA see Banker *et al.* [7] p.1081.

Further, DEA not only provides a measure of efficiency, it also provides other useful information, such as targets and peers. Thus the information provided by DEA is:

Efficiency Score:	A summary measure of the distance between the actual and efficient input output levels of a DMU.
Targets:	The input output levels that would render a DMU DEA-efficient.
Peer DMUs:	The DEA-efficient DMUs that are used as a basis for the stated targets of a DEA-inefficient DMU.

Having outlined the approach of DEA, the next section will detail the actual DEA model.

1.3 The Derivation of the DEA Model

This idea of an efficiency measure based on observed data which accounted for multiple inputs and outputs, was first introduced by Farrell [30]. However, his idea remained undeveloped until Charnes *et al.* [16] derived a linear programming problem to measure this efficiency, which assumed **Constant Returns to Scale** (CRS). Consider assessing a set of N DMUs, $j=1,\ldots,N$, each consuming m inputs, x_{ij}, $i=1,\ldots,m$ to produce s outputs, y_{rj}, $r=1,\ldots,s$. The relative efficiency of DMU j_0 is given by the DEA weights ratio models (M1.1) or (M1.2). These are labelled Input Minimisation (IM) and Output Maximisation (OM) respectively, justification for this labelling will be given later. Due to the CRS assumption the relative efficiency scores provided by the two models are the same, see Charnes *et al.* [16].

(M1.1) Input Minimisation		(M1.2) Output Maximisation	
$h^*_{j_0} = Max \dfrac{\sum_{r=1}^{s} \mu_r y_{rj_0}}{\sum_{i=1}^{m} \upsilon_i x_{ij_0}}$		$e^*_{j_0} = Min \dfrac{\sum_{i=1}^{m} \upsilon_i x_{ij_0}}{\sum_{r=1}^{s} \mu_r y_{rj_0}}$	
s.t. $\dfrac{\sum_{r=1}^{s} \mu_r y_{rj}}{\sum_{i=1}^{m} \upsilon_i x_{ij}} \leq 1$	$j=1,\ldots,N$	s.t. $\dfrac{\sum_{i=1}^{m} \upsilon_i x_{ij}}{\sum_{r=1}^{s} \mu_r y_{rj}} \geq 1$	$j=1,\ldots,N$
$\upsilon_i,\ \mu_r \geq \varepsilon$	$\forall i,\ r$	$\upsilon_i,\ \mu_r \geq \varepsilon$	$\forall i,\ r$

υ and μ are the variable weights attached to the inputs and outputs respectively. The relative efficiency score of DMU j_0 is given by $h^*_{j_0}$ in (M1.1), with $h^*_{j_0} = 1/e^*_{j_0}$ in (M1.2).

Thus, the models in the above form can be thought of as a value-based measure of relative efficiency, (see Thanassoulis [45]). These models can be easily converted to ordinary linear programming problems through a simple transformation, ($u_r = t\mu_r$, $v_i = t\upsilon_i$; $t^{-1} = \sum \upsilon_i x_{ij_0}$; with $t > 0$ in (M1.1), see Charnes *et al.* [16]). Thus the relative efficiency score of DMU j_0 is given by the DEA weights model (M1.3) or (M1.4), which are linearisations of (M1.1) and (M1.2) respectively.

(M1.3) Input Minimisation	(M1.4) Output Maximisation
$h^*_{j_0} = Max \sum_{r=1}^{s} u_r y_{rj_0}$	$e^*_{j_0} = Min \sum_{i=1}^{m} v_i x_{ij_0}$
s.t. $\sum_{i=1}^{m} v_i x_{ij_0} = 1$	s.t. $\sum_{r=1}^{s} u_r y_{rj_0} = 1$
$\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \leq 0$ $j=1,\ldots,N$	$\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \leq 0$ $j=1,\ldots,N$
$v_i,\ u_r \geq \varepsilon$ $\forall i,\ r$	$v_i,\ u_r \geq \varepsilon$ $\forall i,\ r$

v_i and u_r are the weights attached to the inputs and outputs respectively and these are the variables of the model. Finally, ε is a non-Archimedian infinitesimal, see Charnes *et al.* [17]. In practical terms this restriction on the weights to be greater than ε, still leads to the virtual zero weighting of an input or output.

From these value-based models, the importance of each input or output to the DMU's relative efficiency score can be determined. This is represented by the value of $v_i x_{ij_0}$ or $u_r y_{rj_0}$, and is given the term **virtual**.

By duality the models (M1.3) and (M1.4), can be expressed in an envelopment form, (M1.5) and (M1.6) respectively. These dual models represent the relative efficiency of a DMU in a production space (see Thanassoulis [45]). From these envelopment models it is clear to see that the models define the relative efficiency of a DMU in terms of Input Minimisation (IM) and Output Maximisation (OM). Hence by duality (M1.3) and (M1.4) are labelled as IM and OM models respectively. Further, the peer DMUs for inefficient DMUs can be readily obtained from these models. The relative efficiency score of DMU j_0 is given by $h^*_{j_0}$ in (M1.5), with $h^*_{j_0} = 1 / e^*_{j_0}$ in (M1.6).

(M1.5) Input Minimisation	(M1.6) Output Maximisation
$h^*_{j_0} = Min\ \theta_0 - \varepsilon\left(\sum_{i=1}^{m} S_i + \sum_{r=1}^{s} S_{m+r}\right)$	$e^*_{j_0} = Max\ z_0 + \varepsilon\left(\sum_{i=1}^{m} S_i + \sum_{r=1}^{s} S_{m+r}\right)$
s.t. $\theta_0 x_{ij_0} - \sum_{j=1}^{N} \lambda_j x_{ij} - S_i = 0 \quad i=1,\dots,m$	s.t. $\sum_{j=1}^{N} \lambda_j x_{ij} + S_i = x_{ij_0} \quad i=1,\dots,m$
$\sum_{j=1}^{N} \lambda_j y_{rj} - S_{m+r} = y_{rj_0} \quad r=1,\dots,s$	$z_0 y_{rj_0} - \sum_{j=1}^{N} \lambda_j y_{rj} + S_{m+r} = 0 \quad r=1,\dots,s$
$\lambda_j,\ S_i,\ S_{m+r} \geq 0 \quad \forall j, i, r$	$\lambda_j,\ S_i,\ S_{m+r} \geq 0 \quad \forall j, i, r$

If $\lambda^*_j > 0$, then the corresponding DMU is a peer to DMU j_0, and * will be used to denote the value of a variable at the optimal solution to the model in which it appears.

S_i and S_{m+r} represent slack variables and if $S^*_i > 0$ or $S^*_{m+r} > 0$ then DMU j_0 has a slack value. So, if $S^*_i > 0$ or $S^*_{m+r} > 0$, for some *i or r* then the DMU either lies on or is projected on a DEA-inefficient frontier segment. A slack in an input, $S^*_i > 0$, represents, in that input only, an additional inefficient use of the input. A slack in an output, $S^*_{m+r} > 0$, represents, in that output only, an additional inefficiency in the production of that output. One way of looking at why slack values are obtained is that there does not exist a relatively efficient DMU or a linear combination of efficient DMUs that have a similar operating mix to these inefficient DMUs. That is, there is a lack of similar comparator DMUs, which could be viewed as **missing data**, see Burgess [13]. This concept of slack values, will be illustrated graphically in the next section.

At this point it is useful to distinguish between the 'radial efficiency' and the 'DEA-efficiency' of a DMU.

- The DEA-efficiency score of DMU j_0 is $h^*_{j_0}$, determined using model (M1.3) or (M1.4) or $1/e^*_{j_0}$ in (M1.5) or (M1.6). A DEA-efficient DMU is considered to be technically efficient and it must therefore satisfy the following conditions:

  - A relative efficiency score of 1.

   - No positive slack values e.g. $S^*_i = S^*_{m+r} = 0; \quad \forall i,r.$

- The radial efficiency of DMU j_0

   - is the inverse of the maximum factor by which its output levels can be raised simultaneously within the PPS, whilst its inputs are held constant. (That is, with reference to (M1.6) the radial efficiency is $1/z^*_0$.)

   - is the factor by which its input levels can be lowered simultaneously within the PPS, whilst its outputs are held constant. (That is, with reference to (M1.5) the radial efficiency is θ^*_0.)

Further, efficiency can be broken down into:

- Technical efficiency: A measure of efficiency given the scale size of a DMU.

- Scale efficiency: A measure of how much of a DMU's inefficiency is soley attributable to its scale of operation.

Since its original formulation, considerable research has been conducted and as a consequence, DEA has expanded. For a brief synopsis of the evolution and the current state-of-the-art in DEA and an up-to-date bibliography see Seiford [43].

Having discussed the CRS DEA model used to determine the relative efficiency scores of DMUs, and to aid in its explanation, the next section will illustrate the method through the use of a simple numerical example.

1.4 Data Envelopment Analysis: A Graphical Illustration

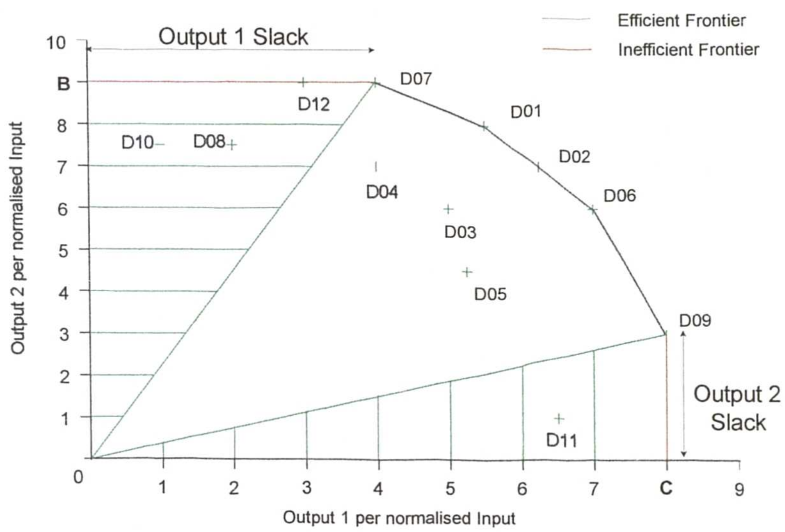
To demonstrate graphically the PPS that is generated by the DEA formulations of (M1.3) and (M1.4), consider assessing a set of 12 DMUs, each consuming a single unit of input to produce two outputs in varying quantities which are shown in Table 1.1. As the DMUs are each consuming a normalised level of input, this allows the two-dimensional representation of the PPS in DEA.

Table 1.1 - Example Data Set 1

	D01	D02	D03	D04	D05	D06	D07	D08	D09	D10	D11	D12
Output 1	5.5	6.25	5	4	5.25	7	4	2	8	1	6.5	3
Output 2	8	7	6	7	4.5	6	9	7.5	3	8	1	9

The PPS for DMUs D01-D12 is plotted in Figure 1.2.

Figure 1.2 - The Production Possibility Set



The DEA-efficient frontier is defined by DMUs D07, D01, D06 and D09 which are Pareto-Koopmans efficient, i.e. scale and technically efficient, see Cooper *et al.* [26]. The DEA-inefficient frontier segments are defined by BD07 and CD09 and are not Pareto-Koopmans efficient. One such DMU is D12 in Figure 1.2, which is clearly relatively inefficient, as it is dominated by D07 and it has a positive output 1 slack value. Similarly for DEA-inefficient DMUs that are projected onto these inefficient frontier segments, slack values are obtained. For example, in Figure 1.2, when D10 and D08 are projected onto the frontier they both have an output 1 slack value, and similarly when D11 is projected onto the frontier it has a slack value for output 2.

At this point, it is useful to clarify the classifications of DMUs under CRS as they will be used throughout this thesis. [It should be noted that at present no formal classification of the DMUs under VRS is given in the literature.] Following the classification of DMUs by Charnes *et al.* [19], class E are those DEA-efficient DMUs that are linearly independent of other DEA-efficient DMUs. Class E' are those DEA-efficient DMUs that can be expressed as a linear combination of other DEA-efficient DMUs. Class F are those DMUs that have a radial efficiency score of 1 but have slack values, e.g. at least one $S^*_i > 0$ or $S^*_{m+r} > 0$ in (M1.5) and (M1.6). Classes NE, NE' and NF represent the classes for the DEA-inefficient DMUs and are based on their radial projections onto the DEA-frontier, i.e. if their projections on the DEA frontier are class E, E' or F respectively. For example, if a DEA-inefficient DMU is radially projected directly onto a class E DEA-efficient DMU, it will be class NE.

The DMUs in Figure 1.2 would be classed as follows: D07, D01, D06, D09 are class E; D02 is class E'; whereas D12 is class F. The remaining DMUs are DEA-inefficient and are classed as follows: D04 and D03 are NE'; D05 is class NE and D08, D10, D11 are class NF. It is this class of DMUs that the approach in this thesis concentrates on.

Further, following the definitions in Bessent *et al.* [12], those of class NE and NE' are termed as **Properly Enveloped** DMUs and have $S^*_i = S^*_{m+r} = 0; \quad \forall i,r$ in (M1.5) and (M1.6) as required. Those DMUs of class F or NF are termed **non-enveloped** DMUs and have at least one $S^*_i > 0$ or $S^*_{m+r} > 0$ in (M1.5) and (M1.6). This implies they do not use all of their inputs and outputs to determine their relative efficiency score, i.e. assign at least one ε weight in (M1.3) and (M1.4). This implies that there are no DEA-efficient DMUs with similar operating mixes to the DEA-inefficient DMUs. Thus the observed data set has no efficient comparator levels for these DMUs to be measured relative to. So it is these DMUs that have relative efficiency scores that may not reflect their true efficiency and are thus the focus of the proposed procedure of section C.

As stated earlier, in addition to providing the DM with a relative efficiency score of a DMU, DEA also provides the DM with information on how the DEA-inefficient DMU can improve its efficiency performance, and which DMUs it might learn from in terms of performance, i.e. use as benchmarks. This additional information will be considered in the next section.

1.5 By-Product of a DEA Assessment: Peers & Targets

The **targets** provided for DEA-inefficient DMUs are based on the performance of DEA-efficient DMUs, their **peers**. Throughout this thesis, it is assumed that the targets are set based on the pre-emptive priority to radially project the DEA-inefficient DMUs onto the DEA-efficient frontier. However, other forms of target setting exist which are based on non-radial measures, see Thanassoulis and Dyson [48]. Target setting relies on a basic PPS assumption, namely - if there are two points that are possible, then a linear combination of these two points is also possible. That is, it is possible to substitute one input/output for another input/output continually between the pair of inputs/outputs. DEA-inefficient DMUs are radially projected onto the frontier, and provided the inefficient DMU is properly enveloped, it is these frontier values that are given as the DMU's targets. Thus their targets are based on the same operating processes as those currently being used. However, if the DMU is non-enveloped, its radial projections are located on an inefficient frontier segment, and the radial targets need a displacement onto the efficient frontier. Thus their targets suggest a change in the DMUs operating practice in order to improve performance and thus the targets are non-radial. This is demonstrated in Figure 1.3.

Figure 1.3 - Radial Target Setting for Inefficient DMUs

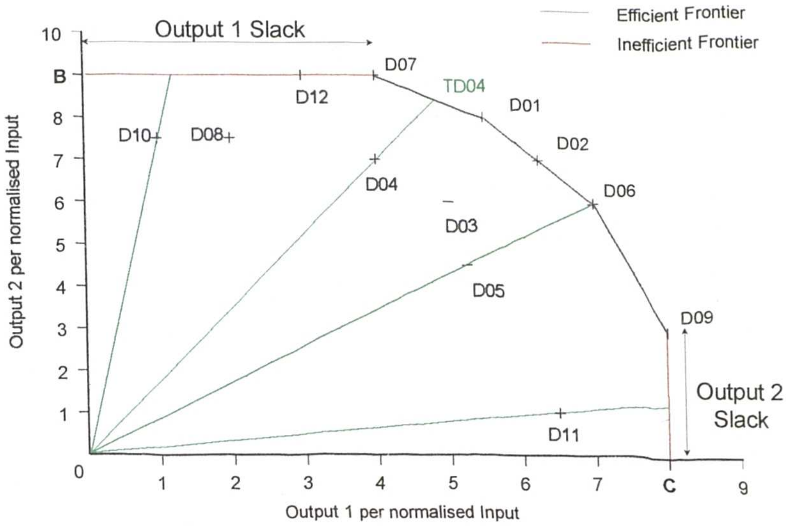


Figure 1.3, shows that properly enveloped DMUs i.e. class NE and NE' are radially projected onto a DEA-efficient frontier segment. For example, DMU D04, a properly enveloped DMU, is projected onto TD04, which is a linear combination of DMUs D07 and D01. Therefore, to reach its radial target, DMU D04 has to increase both its output levels in equal proportions, and thus maintains its current operating mix. Unfortunately, this is not true if the DMU is non-enveloped, i.e. class NF or F, their targets are not based solely on radial increases/decreases to their inputs/outputs, e.g. DMU D10. For these DMUs there is also a suggestion for the DMUs to change their operating process, in order to attain efficiency. This applies to DMUs D12 in Figure 1.3, as DEA suggests the input output levels of D07 as its targets. So for DMU D12 to obtain its target it needs only to increase output 1. Thus, it will have to put more emphasis on producing output 1, while maintaining its present level of output 2. This will require D12 to alter its present operating mix.

Targets cannot be set for DEA-efficient DMUs, as there are no DMUs in the assessment that perform better, so it is not known if it is possible to increase the efficiency of a DMU. Golany and Roll [34] suggest an approach for setting targets and improving efficiency for the DEA-efficient DMUs by the introduction of additional Unobserved DMUs (UDMUs).

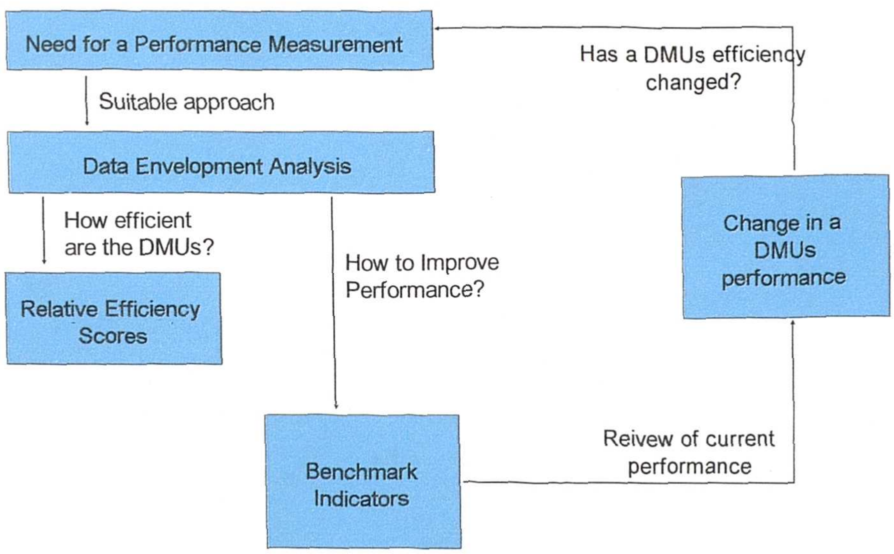
Having outlined the basic principles of DEA and the information provided by the analysis, a summary of the approach is now offered.

1.6 Conclusion

DEA is a linear programming technique for assessing the relative efficiency of a set of DMUs, such as schools and bank branches. Each DMU operates its production process differently but consumes the same set of inputs to produce the same set of outputs. The relative efficiency score is obtained by the free allocation of weights to a set of inputs and outputs, with a DEA-efficient frontier being defined by the observed input output levels of the DMUs. As a by-product of the analysis, inefficient DMUs are offered radial targets and peers, which they might use as benchmarks to improve their performance.

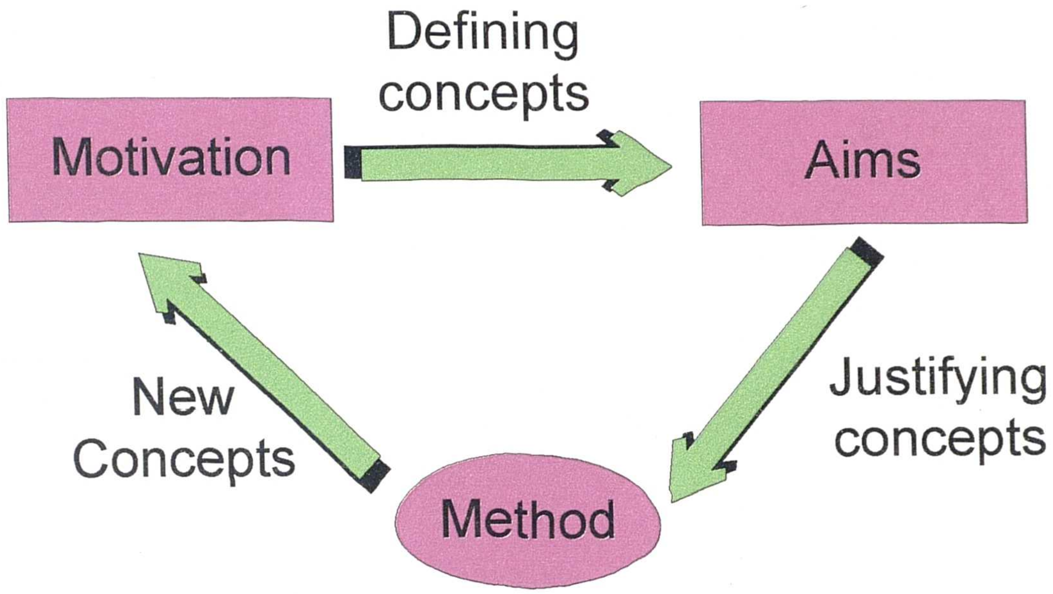
This introductory chapter has outlined the DEA approach for measuring the relative efficiency of DMUs which has grown in popularity since its initial formulation by Charnes *et al.* [16], with the original concept for the need for some form of measure for relative efficiency by Farrell [30]. Figure 1.4 summaries the general process.

Figure 1.4 - Overview of the General Process



The next chapter will discuss the motivating reasons for this thesis, what it will aim to achieve and how it will attempt to achieve these aims.

2. Chapter Two
Why Read This Thesis?



2.1 Introduction

The previous chapter outlined the basic concepts of DEA. This chapter will set out the motivation behind this thesis, what it hopes to achieve and how these aims will be arrived at, with chapter three outlining in greater detail the existing literature in the area of incorporating value judgments in DEA. Hence, some of the concepts introduced here will not be formally defined until chapter three.

This introduction will detail the general motivating reasons for the inclusion of value judgments in DEA assessments. These reasons will be discussed in greater detail in the next chapter see also Allen *et al.* [4].

As detailed in chapter one, DEA treats the observed input and output levels as fixed. Thus the DEA model assigns variable weights to these factors, which are then interpreted as the value attributed to the inputs and outputs. So, an assessed DMU is freely allowed to allocate the weights, in order to show the DMU '*in the best possible light*'.

It should be noted that the DEA model only considers the quantity of the inputs and outputs and does not take into account the value of the inputs and outputs; thus, inappropriate estimates of efficiency may be obtained. There are several reasons for this:

- Non weighting of the inputs and/or outputs

DMUs can attribute low weights (ε, which in practical terms is virtually zero) to their relatively low levels of output and their relatively high levels of input, so that they are effectively ignored in the assessment. In extreme cases it is possible to obtain relative efficiency scores based on the ratio of a single weighted output to a single weighted input.

- Non reflection of the relative relationships between inputs or outputs

DMUs can assign weights to their inputs or outputs in a counter-intuitive manner, that is, ignoring accepted views about the value of the different inputs or outputs. For example, in the assessment of a set of police forces, the solution of a burglary crime appears to be valued more than that of a violent crime by some forces, see Thanassoulis [44].

Further, the weights can be used to estimate marginal rates of substitution/transformations (see Charnes *et al.* [16]). However, with the virtual zero weighting of an input or output this means that these marginal rates cannot be defined.

- Non reflection of dependent relationships between the inputs and outputs

In many assessments of relative efficiency, specific outputs are directly dependent on specific inputs. Hence in this case, it is reasonable to expect the DEA-weights to be linked in some manner. For example, see Thanassoulis *et al.* [47], where in their assessment of perinatal care units, one of the outputs, 'survivals' was directly dependent on an input, 'babies at risk'. Thus it was felt that the weights assigned to these factors should be linked in some way.

In general, to overcome the problem of inappropriate efficiency estimates, the DM's value judgments on the inputs and outputs are incorporated into the analysis. As the term value judgments will be used throughout the thesis, there is a need to define the term:

> "logical constructs expressed as the DM's opinions on the production process under analysis"

Thus, value judgments are DM specific and essentially they begin with the selection of the input output variables. For example, variables that are omitted from the assessment are implicitly given zero weight, see Golany and Roll [33]. However, this is not the type of value judgments that are to be considered in this thesis. The value judgments that are to be considered in this thesis affect the selection of the optimal set of weights for the inputs and outputs when assessing a DMU's relative efficiency. Hence, it could be said that they are the type of values that are incorporated into the analysis in order to obtain a better picture of the DMU's overall relative efficiency. Further, this thesis is concerned with the inclusion of qualitative information on quantitative factors, rather that the actual inclusion of qualitative factors, see Cook *et al.* [23].

Thus, the need for the inclusion of value judgments in a DEA assessment has been established. The next section outlines why there is a need for an alternative approach to those methods available at present.

2.2 Motivation

A number of extensions to the original DEA model have been proposed to overcome the problem of inappropriate efficiency scores, and these are reviewed in chapter three, see also Allen *et al.* [4]. However, there is one primary approach for incorporating value judgments into a DEA assessment, and that is, **Weights Restrictions** (WRs), which explicitly restrict the DEA weights, u_r or v_i in models (M1.3) or (M1.4) of chapter one. The following are observations from this approach that motivated the need for an alternative approach for the incorporation of value judgments in a DEA assessment, which prompted the production of this thesis.

2.2.1 Specification of the Value Judgments

In order to implement weights restrictions, the DM is required to specify information on their values. The format of this information depends on the type of imposed restriction, but the three main forms of expression are:

- Numerical bound values e.g. lower or upper bounds

These can be either the direct or indirect restriction of the contribution of the inputs and/or outputs to the relative efficiency score. In general, there is a lack of objectivity in the setting of numerical bound values, see Roll *et al.* [41]. This is mainly due to the fact that in general the actual numerical weight value holds no real meaning, except in specific applications, see Dyson and Thanassoulis [27]. Thus there is a need for an objective approach to ensure that all inputs and outputs contribute to the relative efficiency score, i.e. no input or output is assigned an ε weight.

- Relative restrictions e.g. marginal rates of substitution

In order to incorporate the relative relationships between inputs or outputs, an explicit definition of the relationship has to be made. Thus, the DM is required to specify global relative relationships between the related inputs or outputs.

It should be noted that this form of restriction is usually used to reflect marginal rates of substitution. Hence in certain applications the DM may have difficulties defining these relationships explicitly or may feel global relationships are inappropriate. So, it follows that the use of relative restrictions may prove problematic to the DM when their production process involves services and the interpretation of the weights as marginal rates holds little meaning. For example, in the assessment of a set of bank branches the outputs may include the number of mortgage applications and the number of counter transactions. In this case, the first output (the number of mortgage applications) is clearly of higher value to the branch. The question then is how to determine the value of this output relative to that of the second output (the number of counter transactions)? Hence an approach that allows the DM an alternative expression for these relationships and for the inclusion of local values would be desirable.

- Linked-dependent restrictions

These are restrictions that link the size of the input weight to the size of the output weight. Once again, this type of restriction requires the explicit definition of the relationship between the inputs and outputs at a global level, which in certain applications may not be appropriate or easy for the DM to define. This is particularly true in a Variable Returns to Scale (VRS) assessment, where the relationship between the inputs and outputs by definition is allowed to vary. Thus there is a need for an approach that allows for the inclusion of local values that apply only to specific operating mixes within the PPS.

This section has briefly highlighted, that for some DMs or certain applications, the specification of their value judgments in the form of weights restrictions may prove to be difficult or inappropriate. Therefore, there is a need to offer the DM an alternative form of expression for their value judgments and an approach that allows for the inclusion of local values for applications where the limitation of weights restrictions to the inclusion of global values may be restrictive, i.e. in Variable Returns to Scale (VRS) applications.

2.2.2 Implementation

The implementation of certain weights restrictions does not guarantee feasible results which may be due to the lack of objectivity in the setting of the restrictions. Further, as will be demonstrated in chapter seven, negative efficiency scores can be obtained through the use of weights restrictions in a VRS DEA assessment. Hence the need for an alternative approach that avoids this problem.

2.2.3 Interpreting the Results

It is known that the incorporation of value judgments in a DEA assessment alters the PPS, see Roll and Golany [42]. However, under weights restrictions, this alteration is only implicit and the explicit input output levels of the altered PPS are not known. Therefore, is it reasonable to interpret the obtained results as relative efficiency scores? Further, at present, targets are based on observed standards and do not truly reflect the input output levels that the DMUs are measured relative to. Thus, to aid the DM in their interpretation of the results and for the setting of objective targets, there is a need for an explicit expression of the value judgments in terms of the inputs and outputs of the assessment.

Having identified the main areas of motivation, the next section will outline the main objectives that this thesis aims to achieve.

2.3 Objectives

The principal objective of this thesis is to offer an alternative approach to weights restrictions for the incorporation of value judgments in DEA assessments, when the use of weights restrictions may be problematic in terms of their specification, their implementation, or the results obtained. Section B of this thesis will establish that a viable alternative to weights restrictions exists in the form of the introduction of DEA-efficient Unobserved DMUs (UDMUs) into the observed data set. Their introduction will attempt to incorporate all three forms of values, as specified above, into the assessment. It should be noted that a more general term of VALUES will be used to define the information incorporated by the approach to be developed in this thesis, as the information ascertained from the DM may reflect a variety of forms of values. That is, it is used to represent one of several values, either a marginal rate of substitution, a marginal rate of transformation or a minimal/maximal weight value.

Finally, section C uses this alternative approach to concentrate on the following main objectives.

2.3.1 Increase Factor Contribution

As noted earlier, in a DEA assessment under free weights it is frequently found that many of the incorporated inputs and outputs do not contribute towards the relative efficiency scores of DEA-inefficient DMUs. That is, although they are thought sufficiently important to be included in the analysis, in the actual assessment of relative efficiency they are not given any value by some DEA-inefficient DMUs. This can be viewed as a lack of DEA-efficient comparator DMUs for the inefficient DMU, i.e. the DEA-efficient DMUs are of a dissimilar operating mix to the DEA-inefficient DMUs. Thus, the approach for including value judgments in this thesis is to provide estimates of DEA-efficient UDMUs with similar input output operating mixes to the DEA-inefficient DMUs, which at present have no observed comparator DEA-efficient DMUs.

2.3.2 Feasible Production Levels

It has been noted that the introduction of values in a DEA assessment leads to the PPS being extended, see Roll and Golany [42] and Dyson *et al.* [28]. The approach proposed in this thesis will allow the DMs to express their value judgments in terms of the input output levels of the assessment. Hence an explicit modification of the PPS is made. Therefore, this thesis aims to ensure that the extensions to the PPS are feasible and consequently, the obtained relative efficiency scores are feasible, and hopefully, further aid the DM in their interpretation of their results.

2.3.3 Varying Local Value Judgments

As mentioned earlier, incorporating value judgments via weights restrictions only allows for the introduction of **global** values. Thus it is assumed that the values hold universally for all the DMUs in that data set, regardless of their operating mix and their priorities in relation to the involved inputs/outputs. The inclusion of global values may be appropriate in some applications, such as Dyson and Thanassoulis [27], where the weights have a meaning universally. However, this may not always be the case, and DMUs with different operating processes may place different values on the different inputs/outputs and the relationships between them. This is particularly important in the case where the DMUs operate under Variable Returns to Scale (VRS). Thus the approach of UDMUs provides the DM with a means for incorporating varying local value judgments into the assessment.

These are the three main objectives of this thesis. The assumptions that are made for this approach are now stated.

2.4 Assumptions

The following assumptions have been made in the proposed approach of section C.

2.4.1 DEA-Efficient DMUs

The values expressed by the DEA-efficient DMUs are acceptable to the DM and thus, they are not being directly asked to express their opinions on which of the observed DEA-efficient DMUs are preferable, i.e. the DM has no model DEA-efficient DMUs. That is, the approach concentrates on expressing the DMs values that will extend the DEA efficient frontier, and although the DEA-efficient DMUs may be discriminated between as a result of the approach, it is not the main aim of the procedure to discriminate between the DEA-efficient DMUs.

2.4.2 Returns to Scale of VRS Frontier

The proposed approach does not set out to explicitly alter the observed returns to scale of the observed DEA frontier. It simply attempts to extend it in an appropriate manner.

2.4.3 Controllable Inputs and Outputs

It is assumed that all the inputs and outputs in the DEA assessment are controllable. That is the DM is able to manipulate the input output levels of the DEA-efficient DMUs in order to derive UDMUs.

2.4.4 Encouragement of Individual Inputs and Outputs

The approach is aimed at encouraging individual inputs and outputs to contribute to the relative efficiency score of DEA-inefficient DMUs, rather than attempting to simultaneously encourage multiple inputs and outputs to contribute to a DEA-inefficient DMU's relative efficiency score. Therefore, if a DEA-inefficient DMUs virtually ignores several inputs and outputs in its initial assessment, full envelopment for these DMUs may not be attained.

One outstanding issue now remains - how is the approach of UDMUs determined suitable for expressing the DM's value judgments in a DEA assessment?

2.5 Methodology

Turning to the issue of how the approach was derived, it was noted earlier that Roll and Golany[42] demonstrate that the inclusion of an absolute weights restrictions in a DEA assessment leads to an implicit modification of the PPS. Similarly, Dyson *et al.* [28] note the correspondence between a single relative weights restriction and an alteration of the PPS. Thus the observation that:

"the inclusion of value judgments in the form of weights restrictions implicitly alter the PPS"

led to an alternative perspective of the problem for the inclusion of values in a DEA assessment, i.e. that of missing suitable comparator DEA-efficient DMUs.

Hence the development of the approach presented in this thesis, which constructs suitable estimates for these missing comparator DEA-efficient DMUs. These suitable estimates are constructed from the observed DEA-efficient DMUs and the DM's values judgments.

2.6 Validity

In order to demonstrate that the proposed approach is a valid one, the theoretical differences between the proposed approach and weights restrictions are highlighted in the concluding chapter, demonstrating that neither approach is all-purpose, and that each has its strengths and weaknesses and, thus, its own individual areas of application. In terms of the usage of this approach, it is difficult to appraise, however, the proposed approach was successfully applied to a set of bank branches operating in the United Kingdom. Although no direct illustrative comparison of the application of the proposed approach and weights restrictions is offered in this thesis, a comparison of the approaches can be found in Allen and Thanassoulis [5].

2.7 By-Product

This thesis highlights that there is a need for alternative means to express values within a DEA assessment. As a result of developing a methodology that enables the defined aims to be achieved, new ideas/thoughts for further areas of research are produced. Those areas of future research generated by this research are detailed in chapter ten.

2.8 Conclusion

The principal motivation and aim of this thesis is to provide an alternative to weights restrictions for the incorporation of value judgments in a DEA assessment. Value judgments are incorporated into an assessment for three main reasons:

a) to avoid the non weighting of inputs and outputs;

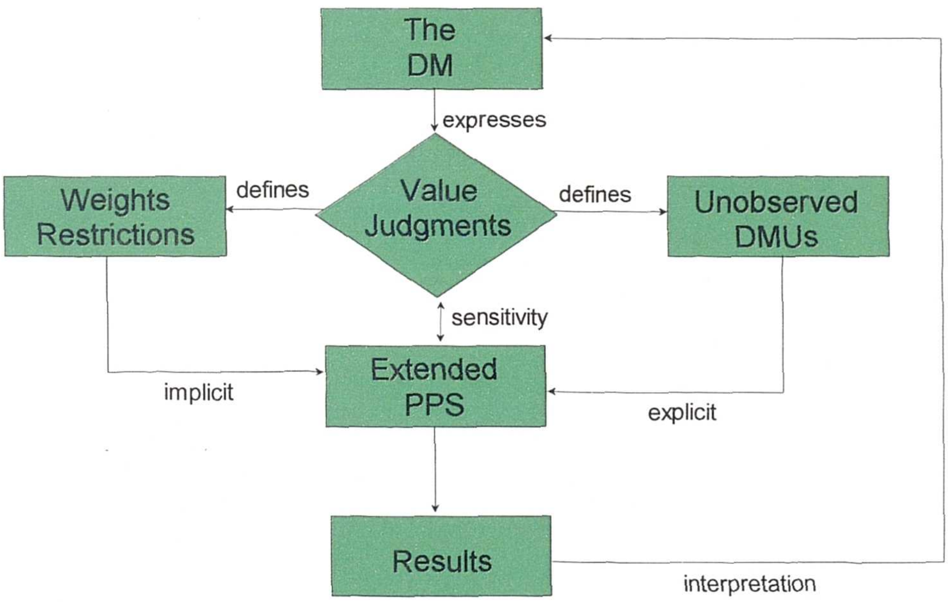
b) to allow the inclusion of relative relationships between the inputs or outputs;

c) to allow the inclusion of linked-dependent relationships between the inputs and outputs.

The approach suggested in this thesis attempts to address these issues by introducing UDMUs into the observed data set.

Figure 2.1 attempts to demonstrate the inter-relationship between some of the problem areas to be investigated.

Figure 2.1 - DEA and Value Judgments



Chapter three will discuss the literature and also highlight some of the points noted in this chapter as the motivating reasons for this thesis; although chapter five will expand these points further.

3. Chapter Three
The Evolution of Incorporating Value Judgments in Data Envelopment Analysis[1]

3.1 Introduction

Farrell [30] originally proposed the concept of a comparative efficiency measure determined from observations rather than by theoretical specification of a production function as followed by economists. This development was operationalised by Charnes *et al.* [16] who established DEA as a prominent methodological tool for assessing the relative efficiency of DMUs. The phenomenal expansion of the method, see Seiford [43], covered a very wide area of applications and theoretical extensions including computations, Ali and Seiford [3] and target setting, Thanassoulis and Dyson [48].

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[1] An earlier version of this chapter is *forthcoming in* **Annals of Operational Research**

The need to incorporate value judgments in DEA has been derived as a consequence of the growing use of DEA in the solution of real life problems. The desire to incorporate value judgments into the assessment has resulted in the development of explicit restrictions of the DEA-weights, with reference to the DEA weights model (M1.3) or (M1.4) in chapter one. Currently, value judgments, generally in the form of weights restrictions, cover a considerable part of the DEA literature without however showing any signs of saturation. The aim of incorporating value judgments is to introduce prior views or information regarding the assessment of the efficiency of DMUs. The incorporation of these opinions will have implications for the relative efficiency score of the assessed DMUs and possibly the peers and radial targets provided.

This chapter reviews the evolution of approaches for capturing value judgments in a DEA assessment and is organised as follows: The next section highlights the application driven motivation for the incorporation of value judgments into a DEA assessment. The third section details the alternative approaches for the inclusion of value judgments and are presented as they arose from the application of DEA to real problems. The fourth section discusses how the incorporation of these value judgments into the assessment affects the interpretation of the relative efficiency score, estimation of targets and the selection of peer comparators for individual DMUs.

3.2 Value Judgments in DEA: Motivation and Purpose

The definition of efficiency in DEA under CRS is specified as the ratio of the weighted sum of outputs to the weighted sum of inputs of a DMU. A linear programming model is solved for each assessed DMU that seeks to derive weights for the inputs and outputs which would maximise its efficiency. The weights represent a *relative value system* for each assessed DMU that provides the highest possible score for the DMU concerned. This is consistent with the notion that the resulting value system is feasible for all other DMUs in the sense that none achieves an efficiency score above a DM specified upper bound.

DEA in its purest form, Charnes *et al.* [16], allows total flexibility in the selection of weights, such that each DMU will achieve the maximum efficiency score feasible for its input and output levels.

This complete flexibility in the selection of weights is important in the identification of inefficient DMUs, which are under-performing even with their own set of weights. As a consequence, the management of an inefficient DMU cannot argue that they were not informed of the importance attached by top management to certain inputs/outputs, as no priorities over the inputs or outputs are included in the analysis.

However, the weights estimated by DEA can prove to be inconsistent with prior knowledge or accepted views on the relative values of the inputs and outputs. For example, in the first application of DEA, by Charnes *et al.* [18] evaluating the performance of "Program Follow Through" (a system of support for under privileged children) in the USA, an analysis of the data shows that many DMUs were rated efficient by placing their output weight solely on "self esteem" and ignoring performance on mathematics and verbal reasoning.

The initial development of DEA by Charnes *et al.* [16] was followed by a rapid expansion of theory and applications without, however, challenging the fundamental basis of DEA insofar as the flexibility in the selection of weights is concerned. The evolution of value judgments (see chapter two for a definition) in the assessment of efficiency followed as a natural by-product of real life applications, some of which are discussed later. A number of reasons motivating the use of value judgments in DEA are discussed next.

- To increase the knowledge of the production process

When assessing the relative efficiency of U.S.A. Air Forces, Clark [21] found that due to the lack of comparability of efficient DMUs with inefficient DMUs, a DMU's efficiency score obtained from the Charnes *et al.* [16] model may not represent a DMU's true efficiency. That is, the DMUs that were found to be relatively efficient under DEA were of different operating mixes to the inefficient DMUs. Consequently, these inefficient

DMUs are projected onto rather arbitrarily generated, artificial frontier facets of the DEA-frontier (created by the inclusion of the ε, see (M1.3) or (M1.4) of chapter one). In addition, these artificially generated frontier facets do not exhibit meaningful efficient trade-offs for the inputs and outputs. Thus to provide further insight into the organisation's operations and the production processes of all the DMUs under analysis, the DEA-frontier is extended using observed Marginal Rates of Substitution (MRS). Consequently, these inefficient DMUs will use all their inputs and outputs to determine their relative efficiency score.

- To enable discrimination between efficient units

The use of DEA by Thompson *et al.* [55] to support the siting of nuclear physics facilities in Texas, highlighted a problem of lack of discrimination when a small set of DMUs is being assessed, as five out of six alternative facilities were found relatively efficient. The discrimination of DEA was improved by defining ranges of acceptable weights, namely assurance regions, which were then introduced to determine the preferred DEA-efficient site.

- To incorporate prior views on the value of individual inputs and outputs

Thanassoulis *et al.* [49] assessing the performance of rates departments, found that the Audit Commission was concerned that some local authorities were deemed efficient due to excessively high weights being placed on the numbers of rebates of taxes and court summonses of recalcitrant tax payers (outputs), while more 'normal' outputs, such as tax accounts administered, were effectively disregarded. Restrictions on the flexibility of weights were imposed by Dyson and Thanassoulis [27] in an attempt to incorporate top management perspectives on the relative importance of the inputs and outputs used in the assessment.

- To reflect the values of certain inputs and/or outputs

Thanassoulis *et al.* [47] assessing the efficiency of perinatal care units in the U.K. required the weight on "babies at risk" (input) to be the same as the weight on "number of survivals" (output). The ratio of the number of survivals to babies at risk was the actual variable to be included in the assessment, and the approach adopted allows the importance of the survival rate ratio to be varied, but not the individual components of the ratio.

In the assessment of the performance of University departments by Beasley [10] and Ahn and Seiford [1] in the U.K. and U.S.A. respectively, it was argued that the Universities with emphasis on postgraduate students should be rewarded in the assessment. Further, Ahn and Seiford [1] sought to guarantee that State funded students should be prioritised in the assessment, as the Universities rely on these students for higher grant support from the State Government.

- To incorporate prior views on efficient and inefficient DMUs

In assessments of efficiency, management often have prior perceptions as to which of the DMUs under assessment they consider to be "good" and "poor" performers. For example, Charnes *et al.* [15], in assessing the performance of banks in the U.S.A., found that "the Charnes *et al.* [15] model recognised some notoriously inefficient banks as DEA-efficient". Managerial perception had to be incorporated within the assessment of efficiency in order to bring the results closer to the prior perceptions of management. This brought forward the "cone ratio" family of models, where the efficiency of banks was assessed on the basis of the input/output values of three preselected banks which were recognised as very good performers. It will be shown later that the preselection of DEA-efficient DMUs for assessing efficiency is a particular type of the cone ratio approach. Nevertheless, the preselection of DMUs for assessing efficiency is in contrast to the rates department study (Thanassoulis *et al.* [49]), which was carried out to challenge perceived wisdom on efficient departments.

- To respect the economic notion of input/output substitution

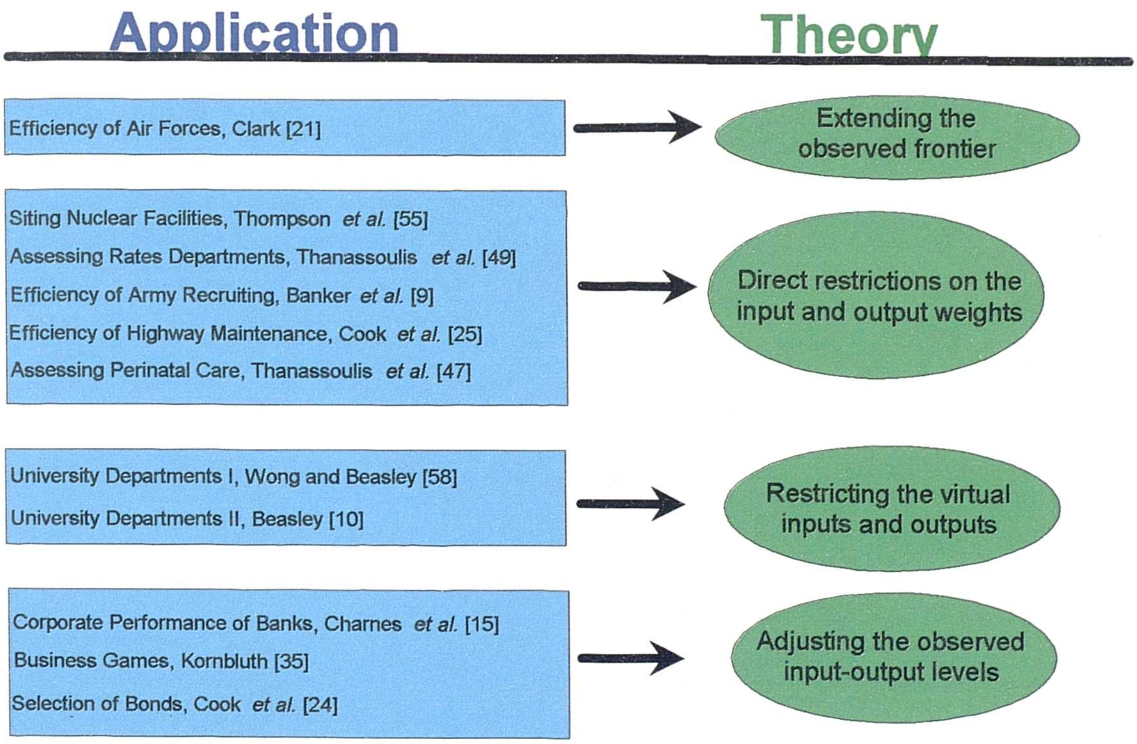
As previously noted, the weights can be used to estimate MRS/MRT, see Charnes *et al.* [16]. Unfortunately, the virtual zero weighting of inputs and outputs means that these marginal rates can not be defined. This creates a problem in relating the DEA-weights with the economic literature.

The aforementioned applications have led to a number of extensions to the original DEA model of Charnes *et al.* [16] for incorporating value judgments in DEA assessments. The next section outlines the main theoretical developments in this area.

3.3 Incorporating Value Judgments in DEA

Figure 3.1, classifies the existing methods for incorporating value judgments in DEA into four approaches and identifies a variety of applications using each approach.

Figure 3.1 - Current Approaches for Incorporating Value Judgments in DEA



The four broad approaches for incorporating value judgments in DEA outlined are:

- Extending the DEA-frontier
- Direct restrictions on the weights
- Restricting the virtual inputs and outputs

- Adjusting the observed input-output levels

These approaches will now be detailed in turn. The discussion will be restricted to incorporating value judgments in the basic DEA model, Charnes *et al.* [16], which implicitly assumes that the DMUs being assessed operate a CRS transformation of the inputs into outputs. This is mainly due to the fact that no real discussion exists in the current literature as to how to meaningfully incorporate value judgments into a VRS DEA assessment. However, this issue will be addressed in chapters seven and eight.

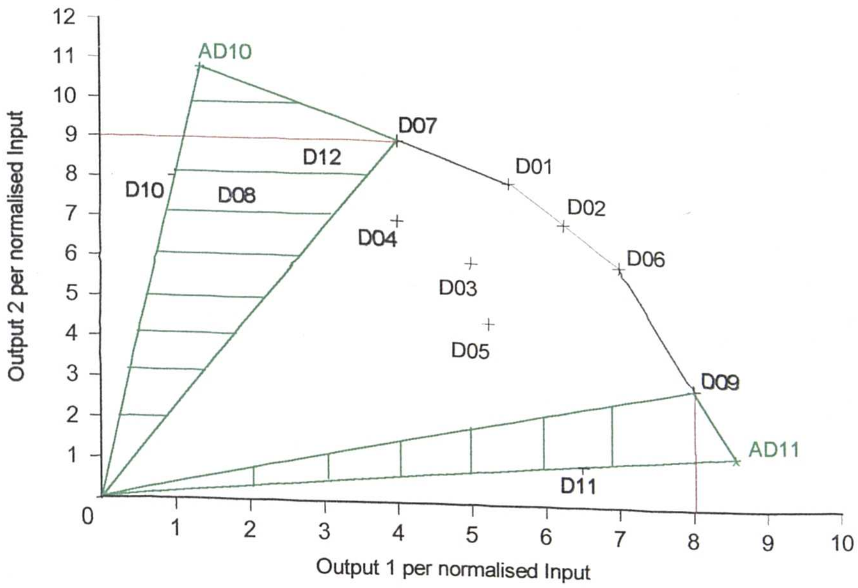
3.3.1 Extending the DEA-Frontier

This approach is motivated by the desire to increase the power of DEA as a management decision making tool. It attempts to extend the DEA-frontier to provide the DM with a greater insight into the production processes of all the DMUs in the assessment by generating artificial efficient frontier facets based on observed trade-offs for inputs and outputs. Consequently, all inefficient DMUs use all the selected inputs and outputs to determine their relative efficiency score, and the artificial efficient facets exhibit some form of meaningful efficient trade-offs for the inputs and outputs. In generating an artificial efficient facet, inefficient DMUs become **quasi-enveloped** DMUs as defined by Bessent *et al.* [12]. This term is used to define those DMUs that are partially enveloped by observed frontier segments and partially by unobserved frontier segments. This methodology was initially discussed by Clark [21] and later by Bessent *et al.* [1], Olesen and Petersen [39] and Lang *et al.* [38]. In addition Chang and Guh [14] and Green *et al.* [31] discuss a similar procedure that differs only in the selection of which frontier segment to extrapolate in order to improve envelopment. To demonstrate the basic concepts of this approach a simple graphical illustration will be used.

<u>A simple example</u>

Consider assessing the set of 12 DMUs shown in Table 1.1 of chapter one. The PPS, of this assessment set is plotted in Figure 3.2. As detailed in chapter one, DMUs D10, D08, D12 and D11 are non-enveloped and following the procedure known as Constrained Facet Analysis (CFA), artificial frontier segments are introduced that extend the observed frontier facets to quasi-envelop the DMUs of class NF and F using observed MRS.

Figure 3.2 - Extending the Production Possibility Set



In order to quasi-envelop, DMUs, D08, D12 and D10, the observed frontier segment of D01D07 is implicitly extended to AD10, based on the assumption that the existing MRS between D01 and D07 can be extended as far as AD10. Similarly, the observed frontier segment of D06D09 is implicitly extended to quasi-envelop D11.

When the methodology is applied to the multi input output case, there may exist a variety of possible frontier facets that can be extended, with different algorithms being presented by Bessent *et al.* [12], Olesen and Petersen [39] and Lang *et al.* [38] for the selection of the most appropriate frontier facet to extend and envelop the inefficient DMUs.

As demonstrated by this example, this approach simply projects existing MRS in the PPS into unknown PPS areas. This assumes that these MRS can be extended as far as required and then hold at these input output levels. It further assumes that the input and output levels of the EPPS are feasible, and thus the obtained lower bound relative efficiency score is feasible. However, as these input output levels are not explicitly stated, how can the

DM be certain that they are feasible? This approach does not involve any direct interaction with the DM for values on the inputs and outputs, and the value judgments are derived directly from the observed data and MRS.

This section has shown that the envelopment of the DMUs can be improved by extrapolating the observed DEA-frontier into unknown production areas to offer a wider range of efficient levels of operating mixes, similar to those of the inefficient DMUs. No explicit specification of the DM's values are incorporated into the analysis, only existing values are extrapolated. Hence, efficient levels are determined assuming that present MRS between DEA-efficient DMUs can be extended, and they do not change as the frontier is extended. Thus, provided this assumption holds, the obtained relative efficiency score should be acceptable. The next section presents the main approach for incorporating value judgments in a DEA assessment - weights restrictions. The alternative types of WRs are presented as motivated by the application of DEA to real life problems.

3.3.2 Direct Restrictions on the Input Output Weights

It is assumed that there are N DMUs, $j=1,...,N$ to be evaluated, each consuming varying amounts, x_{ij} of m different inputs, $i=1,...,m$ to produce varying quantities, y_{rj}, of s different outputs, $r=1,...,s$. In general, these quantities are assumed to be strictly positive, i.e. $x_{ij}>0$ and $y_{rj}>0$, $\forall i,r,j$.

The linear programming model, (M3.1) illustrates some of the direct restrictions on DEA weights typically found. Without r1-r5, (M3.1) reduces to the basic DEA model, Charnes *et al.* [16] for assessing the relative efficiency of DMU j_0.

$$h^*_{j_0} = Max \sum_{r=1}^{s} u_r y_{rj_0}$$

$$s.t. \quad \sum_{i=1}^{m} v_i x_{ij_0} = C$$

$$\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \le 0 \qquad j=1,\dots,N$$

$$\kappa_i v_i + \kappa_{i+1} v_{i+1} \le v_{i+2} \qquad :r1 \qquad \text{(M3.1)}$$

$$\alpha_i \le \frac{v_i}{v_{i+1}} \le \beta_i \qquad :r2$$

$$\gamma_i v_i \ge u_r \qquad :r3$$

$$\delta_i \le v_i \le \tau_i \qquad :r4$$

$$\rho_r \le u_r \le \eta_r \qquad :r5$$

$$-v_i \le -\varepsilon \qquad i=1,\dots,m$$

$$-u_r \le -\varepsilon \qquad r=1,\dots,s$$

(ε is a non-Archimedean infinitesimal)

Where u_r and v_i are the weights attached to the *r*th output and the *i*th input respectively, and are the variables of the model. The Greek letters (κ_i, α_i, β_i, γ_i, δ_i, τ_i, ρ_r, η_r, ϕ_r, ψ_r), are DM-specified constants to reflect their value judgments regarding the relative importance of the input or output factors. The 'normalisation' constant, *C* is arbitrarily set by the DM as an upper limit on efficiency scores. (*C* is typically set to the value of 1 or 100). Constraints of type, r1 and r2 can involve output rather than input weights or a varied number of weights.

The five types of weights restrictions, r1 to r5, listed in (M3.1), can essentially be divided into three categories:

a) Relative Restrictions - Assurance Regions of Type I (ARI)

These types of restrictions are illustrated by r1 and r2 in (M3.1), and are introduced to incorporate into the analysis the relative ordering or values of the inputs/outputs.

Thompson *et al.* [53] termed restrictions such as, r1 and r2 'Assurance Regions type I', (ARI). Form, r1 is similar to the type used in Thompson *et al.* [55] and Kornbluth [35]. The use of form r2 is more prevalent reflecting MRS, although the upper bound, α_i or alternatively the lower bound, β_i is often omitted. Clearly, the bound values for ARI are dependent on the scaling of the inputs and outputs, that is, they are sensitive to the units of measure of the related factors. Throughout this thesis restrictions of this type will be referred to as **Relative Weights Restrictions (RWRs)**.

Charnes *et al.* [15] and Thompson *et al.* [53] note that when imposing relative restrictions there will always exist at least one efficient DMU. Further, the relative efficiency scores obtained from a DEA assessment with the inclusion of RWRs are the same irrespective of the model orientation.

b) Linked-Dependent Restrictions - Assurance Regions of Type II (ARII)

This type of restriction is depicted by r3 in (M3.1). Thompson *et al.* [53] termed relationships between the input and output weights 'Assurance Regions type II' (ARII). The linking of input and output weights is required in many DEA applications, as it is the combination rather than the individual values of the variables that the efficiency measure should reflect. This is, clearly, the case for using ARII in Thanassoulis *et al.* [47] and Thompson *et al.* [51] and [52]. It can be shown that ARII may render (M3.1) infeasible and a DEA model incorporating ARII produces the same relative efficiency scores when switching from an IM to an OM orientation or vice versa, with the ARII being dependent on the scaling of the inputs and outputs. Throughout this thesis restrictions of this type will be referred to as **Linked-Dependent Weights Restrictions (LWRs).**

c) Absolute Restrictions

These restrictions are illustrated by r4 and r5 in (M3.1) and are mainly introduced to prevent the inputs or outputs from being over emphasised or ignored in the analysis. The value of the restriction is context dependent. For example, it may represent either the maximum or minimum cost of the associated factor. Throughout this thesis restrictions of this type will be referred to as **Absolute Weights Restrictions (AWRs).**

The bounds used in the restrictions are dependent on the normalisation constant *C,* in (M3.1), as *C* reflects the scaling of the DEA weights. There is a strong interdependence between the bounds on different weights. For example, setting an upper bound on one input weight imposes a lower bound on the total virtual input of the remaining variables. This, in turn, has implications for the values that the remaining input weights can take, see Roll and Golany [42]. It should be noted that when AWRs are used in a DEA model, switching from an IM to an OM orientation produces different relative efficiency scores, and hence the bounds need to be set in light of the model orientation used. Finally, AWRs may render model, (M3.1) infeasible.

A key difficulty in using any one of the four types of weight restrictions outlined in (a), (b) and (c), is the estimation of the appropriate values for the constants in the restrictions, compatible with the value judgments to be reflected in the efficiency assessments. A number of methods have been developed to aid the estimation of such constants and are now outlined. No method is all-purpose and different approaches may be appropriate in different contexts. This issue is central to the thesis, as it is offering a different approach that may be appropriate in certain applications or to certain DMs.

As the relative efficiency score is dependent on the selected parameter values of the restrictions themselves, the next subsection will discuss how the selection of parameters is made in many practical applications for the different forms of WRs.

3.3.2.1 Estimating the Parameters

a) Parameters in Relative Restrictions

This type of WR is mainly based on the implementation of the economic notion of MRS in the context of the Charnes *et al.* [16] definition. The setting of bounds for relative restrictions in practical applications has been based either solely on expert opinion (Beasley

[10] and Kornbluth [35]), or expert opinion in conjunction with price/cost information (Thompson *et al.* [51], [52]).

b) Parameters in Linked-Dependent Restrictions

Methods for developing suitable LWRs have received little attention in the literature other than Thompson *et al.* [51] and Thanassoulis *et al.* [47] in assessing World Wide Major Oil Companies, and Perinatal Care Units respectively. Thompson *et al.* [51] rely on market prices obtained by corporate/industry reports.

In the assessment of Perinatal Care Units in the U.K., environmental impacts on mortality recognised and they used a standardised survival rate, namely-survival rate of babies at risk, to reflect the quality of perinatal care medical outcomes. This variable was incorporated in the DEA model as two variables: 'Babies at risk' an input and 'survivals' an output.

Evidently the weight of survivals should be linked to that of babies at risk, otherwise a Perinatal Care Units could exploit its high number of survivals or low number of babies at risk to improve its efficiency score irrespective of its actual survival rate. To ensure that the relative efficiencies reflect the actual survival rate, when either survivals or babies at risk are given any weight, the weights for the two variables are linked.

c) Bounds for Absolute Restrictions

Greater attention has been given in the literature to approaches for estimating absolute bound values, due to the absence of a real natural basis for their estimation other than price. Roll *et al.* [41], Roll and Golany [42] and Dyson and Thanassoulis [27], have suggested alternative approaches which rely on relative information obtained from the DMUs included in the analysis. These methods are outlined below:

i. Based on running unbounded DEA models, Chilingerian and Sherman [20], Roll *et al.* [41] and Roll and Golany [42].

A variety of approaches are suggested in these three papers based on a two-phase process. In the first phase, an unbounded DEA model is run and a weights matrix is compiled eliminating either the outlier weights or a certain percentage of the 'extreme' weights.

In the second phase, a number of alternatives are offered. For example, the average weight for each factor is calculated, and a certain amount of allowable variation about each mean is decided upon subjectively, giving an upper and lower bound for each factor weight. It should be noted that alternative optimal solutions may exist for the unbounded DEA model, especially in respect of relatively efficient DMUs. The authors do not clarify how such alternative optimal solutions are to be treated in the context of their method.

ii. Based on average input levels per unit of output, Dyson and Thanassoulis [27]

This method has only been developed for single input multi-output or single output multi-input cases. Considering the single input multi-output case, the weight on the rth output can be interpreted as the marginal resource level that the DMU would attribute to the output r in order to appear at maximum efficiency. Methods outside DEA, notably Ordinary Least Squares (OLS) regression, exist for estimating the average input level per unit of output r. Such estimates can be used to set lower bounds on the DEA output weights. For example, let the input x be regressed on the r outputs yielding equation E2.1.

$$\bar{x} = \sum_{r=1}^{s} \phi_r y_r + \varsigma \qquad :E2.1$$

Where, ϕ_r is the partial regression coefficient of output r and ς is the regression constant. If $\zeta \neq 0$ and is significant, the use of a constant returns to scale DEA model is not appropriate in the assessment. If $\zeta = 0$ or is not significantly different from zero, then ϕ_r can be directly interpreted as the amount of resource used on average per unit of output r.

Once the value of ϕ_r is known it can be used as a reference point to seek a consensus as to how much less than ϕ_r a DMU can sensibly claim to be using per unit of output r by operating very efficiently. This leads to a lower bound on the DEA weight of output r. For example, in assessing tax offices let OLS regression suggest that on average it costs £10 to administer an account. A consensus may now be reached such that the most efficient rates department could not possibly claim it incurs say 10% or less of the average cost to administer an account. This would give a lower bound of £1 for the DEA weight on accounts administered. This approach for setting absolute bounds on the value of the weights has been applied by Cook *et al.* [22] in assessing highway maintenance patrols.

The next section will discuss imposing restrictions on the percentage contribution of the inputs and outputs to the relative efficiency score e.g. virtual restrictions.

3.3.3 Restricting the Virtual Inputs and Outputs

These restrictions are depicted by r6. Wong and Beasley [58] explored the use of such

$$\phi_r \le \frac{u_r y_{rj}}{\sum_{r=1}^{s} u_r y_{rj}} \le \psi_r \qquad :r6$$

restrictions in DEA where $\sum_{r=1}^{s} u_r y_{rj}$ represents the total virtual output of DMU j. The total virtual input or output is included as the denominator in the constraint r6 as a standardisation mechanism that would facilitate the assignment of values to ϕ_r, ψ_r. Rather than restricting the actual DEA weights, the proportion of the total virtual output of DMU j devoted to output r, i.e. the '*importance*' attached to output r by DMU j, can be restricted to range between [ϕ_r, ψ_r], with, ϕ_r and ψ_r determined by expert opinion, see Beasley [10].

Implementing this type of restriction is not straightforward, due to the fact that the implied restrictions on the DEA weights are DMU-specific. Hence, several alternative means of implementation have been suggested by Wong and Beasley [58]. However, as ultimately these restrictions can be reduced to an absolute restriction, a simpler means of implementation would be to determine the possible binding absolute restriction from all the imposed virtual restrictions and then impose this restriction. Clearly, the relative efficiency scores obtained with restrictions applied on the virtual inputs/outputs are sensitive to the orientation of the model (input/output).

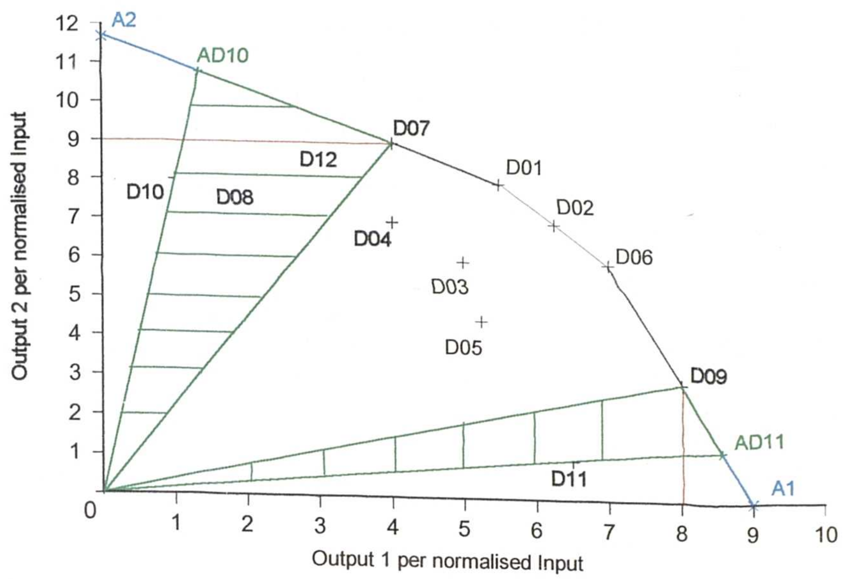
Restrictions on the virtual input/output weights have received relatively little attention in the DEA literature. More research is necessary to explore the *pros* and *cons* of setting restrictions on the virtual inputs and outputs. There has been no attempt to date to compare methods for setting restrictions on the actual DEA weights with those restricting virtual inputs and/or outputs.

Having covered the types of WRs that exist, the next subsection will link the inclusion of WRs in a DEA assessment with extending the PPS.

3.3.4 Linking Weights Restrictions and Extending the Observed Frontier

As outlined in chapter two, the central theme of this thesis is providing an alternative approach to WRs for capturing value judgments in DEA. The alternative approach offered in this thesis, is that of introducing UDMUs into the PPS which extend and possibly modify the PPS. Now, this section attempts to link the two approaches in terms of the current literature. It has been shown by Roll and Golany [42] that in the two input single output case, introducing AWRs on the input weights is equivalent to extending the PPS by the introduction of additional DMUs. For example, consider the data set of, Table 1.1 of chapter one. Introducing bounds on the output weights of $u_1 \geq 0.0571$ and $u_2 \geq 0.0857$ respectively, implicitly leads to the introduction of two UDMUs A1 (9,0) and A2 (0, 11.667) respectively. These two UDMUs extend the DEA-frontier to envelop DMUs of class NF and F. Similarly, Thanassoulis and Allen [50] demonstrated (this is also demonstrated in chapter four) that imposing RWRs of $-1.5u_1 + u_2 \leq 0$ and $u_1 - 3u_2 \leq 0$ (which are coincidental with the MRS between [D06 and D09] and [D07 and D01] respectively) is equivalent to extending the frontier by the introduction of the UDMUs AD11 (8.56, 1.32) and AD10 (1.35, 10.79). In addition these UDMUs are coincidental with the additional DMUs implicitly introduced by CFA, see Figure 3.3.

Figure 3.3 - The Extended Production Possibility Set



Clearly, in terms of the radial efficiency score, the introduction of WRs assumes that the input output levels of the Extended Production Possibility Set (EPPS) are feasible and obtainable. That is, in the simple example above, in order to obtain the relative efficiency score for D10 under the imposed WRs, it is assumed that AD10 is obtainable, although the issue of feasibility may not have been explicitly considered. This consideration of the feasibility of the EPPS will be discussed further in chapter five.

Evidently, the incorporation of value judgments into the assessment via weights restrictions can lead to an implicit extension of the observed PPS. However, it may also lead to an implicit modification of the PPS, with observed efficient DMUs being rendered inefficient. Thus, in the case where the observed efficient DMUs remain so, the inclusion of weights restrictions can be viewed as the removal of slack values, similar to CFA. This only makes assumptions about the feasibility of the introduced efficient input output levels outside the extremes of the observed PPS; whereas weights restrictions that modify the observed PPS make assumptions about the feasibility/under-performance of the observed efficient input output levels.

The incorporation of value judgments into a DEA assessment via directly restricting the input output weight values, has been discussed, and has been shown to ultimately lead to an implicit extension/modification of the observed PPS. The next section will discuss how, rather than use weights restrictions to implicitly extend the PPS, we might explicitly change the PPS to capture value judgments. Thus the PPS itself is transformed to generate an artificial data set that will capture the value judgments in the DEA assessment.

3.3.5 Adjusting the Observed Input Output Levels

Both Charnes *et al.* [15] and Golany [32], derive transformations of the observed input output data, with reference to the envelopment model, in order to simulate relative weights restrictions.

a. Charnes *et al.* [15]

In this method an artificial data set is generated which produces the same relative efficiency scores as imposing RWRs of form r2, in (M3.2). The cone ratio weights DEA model is as follows:

(M3.2) <u>Cone Ratio Weights Model</u>

$$h'_{j_0} = Max\ u^T Y_0$$

$$s.t. \qquad v^T X_0 = 1$$

$$-v^T X + u^T Y \leq 0$$

$$v \in V,\ u \in U$$

Where X is an mxn matrix of input levels, Y is an sxn matrix of output levels, u is an $sx1$ vector of output weights and v is an $mx1$ vector of input weights. X_0 and Y_0 are the $mx1$

vector of input and $sx1$ vector of output levels respectively of the assessed DMU j_0. The WR information is contained in the closed convex cones, $V \subseteq E_+^m$ and $U \subseteq E_+^s$, defining $\overline{V}$ and $\overline{U}$ as their negative polar cones, and information on how to transform the data is contained in $-\overline{V}$ and $-\overline{U}$. Imposed RWRs of form r2 in (M3.2), can be expressed in matrix form for inputs as $V = \{v: Dv \geq 0,\ v \geq 0\}$, and for outputs as $U = \{u: Fu \geq 0,\ u \geq 0\}$. It is worth noting that the weights v and u are allowed to equal zero. Charnes *et al.* [15] derive that the following weights model (M3.3) which provides equivalent relative efficiency scores to those of (M3.2).

(M3.3) <u>Cone Ratio Weights Model</u>

$$h_{j_0}^{l} = Max\ g^T(BY_0)$$

$$s.t. \qquad w^T(AX_0) = 1$$

$$-w^T(AX) + g^T(BY) \leq 0$$

$$w \geq 0,\ g \geq 0$$

where the matrices A and B are defined in relation to matrices D and F above with, $A^T = (D^TD)^{-1}D^T$ and $B^T = (F^TF)^{-1}F^T$, see Charnes *et al.* [15].

Charnes *et al.* [15] also suggest approaches for defining the cones used in (M3.2) such that they favour either specific inputs/outputs or individual DMUs. In their application of the cone-ratio approach to a set of bank branches, the cones favour individual model banks, with these model banks being defined by the DM. For example, suppose that DMUs a and b, are considered as model banks and that the optimal unrestricted DEA weights of DMU a, are $v_1 = a_1$; $v_2 = a_2$ and of DMU b, $v_1 = b_1$, $v_2 = b_2$. It can be deduced that these cones imply that the banks are being assessed under the MRS, as determined by the sets of optimal DEA weights for the *model* DMUs, a and b. That is, $\frac{b_1}{b_2} \leq \frac{v_1}{v_2} \leq \frac{a_1}{a_2}$.

This gives the following matrix $D = \begin{pmatrix} b_2 & -b_1 \\ -a_2 & a_1 \end{pmatrix}$, and from the stated matrix transformations, we obtain $A = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}$, which can then be applied to the observed data to generate an artificial data set. Tone [56] discusses three possible objectives for the selection of suitable cone-ratios and offers three algorithm for the determination of the bound values for these three different objective cone-ratios.

b. Golany [32]

Golany [32] sought to incorporate ordinal relationships of the form $v_1 \geq v_2 \geq v_3 \geq \varepsilon$ among the DEA weights. Without allowing the weights to take a zero value, the relative efficiency scores obtained are the same as those obtained by transforming the input-output data to generate an artificial data set, by accumulating the related factors. Golany's [32] transformations are effectively a special case of the cone ratio transformation. For example, the restrictions $v_1 \geq v_2 \geq v_3 \geq \varepsilon$ can be omitted from a DEA model by replacing x_{2j} with $x_{2j} + x_{1j}$ and x_{3j}, with $x_{3j} + x_{2j} + x_{1j}$, $\forall j$, where x_{ij} is the level of the ith input of the jth DMU.

However, Ali *et al.* [2] pointed out that the data transformations proposed by Golany [32] only provide suitable solutions for strict, not weak, ordinal relationships between DEA weights due to the weights being strictly positive. In addition, they note that the weights themselves can be accumulated, rather than the data, to obtain the same relative efficiency scores as under the original WRs. For example, v_2 is replaced by $v_1 + v_2$ and v_3 is replaced by $v_1 + v_2 + v_3$.

Unfortunately, to interpret the results, the data from this approach requires to be transformed back to the original form. This can prove more cumbersome than the direct application of weights restrictions to the original data.

This section, has illustrated the rich variety of approaches to the use of WRs in DEA. It is clear, however, that no overall approach to setting WRs in DEA has been identified with the different approaches proving to be more appropriate in different contexts. For example, in a single input multi-output case the approach by Dyson and Thanassoulis [27] may prove suitable, while in the case of strong expert identification of good DMUs, the approach by Charnes *et al.* [15] may prove more appropriate. The next section discusses how the results obtained from a DEA analysis are interpreted when value judgments have been incorporated into the analysis.

3.4 Interpreting Results under Value Judgments

In DEA assessments of performance, the results obtained in respect of each DMU reflect its position within the PPS relative to the efficient part of the boundary of the PPS. As discussed earlier the inclusion of value judgments in a DEA model may render parts of the efficient boundary of the PPS inefficient, so that previously defined DEA-efficient DMUs are no longer deemed so. In addition, the efficient boundary may be implicitly extended to include previously undefined efficient input output levels. Clearly, the alterations to the PPS have implications for the interpretation of the relative efficiency score, the radial targets and the peer DMUs. However, there is little discussion on the interpretation of these results in the current literature. Thus the following sections will attempt to give some interpretations of the impact of the inclusion of value judgments in DEA on the efficiency score, the targets and the peers.

3.4.1 The Efficiency Score

Clearly, the introduction of value judgments in the assessment will either reduce or have no impact on the DMUs' relative efficiency. If there is no impact on the relative efficiency score, then the relative efficiency score can be interpreted as in the absence of weights restrictions. However, if the inclusion of the value judgments impacts on the relative efficiency scores, then how should the DM interpret the scores? Bessent *et al.* [12] interpret the results as follows. The standard DEA relative efficiency score is taken to represent an upper bound on the relative efficiency, and the CFA relative efficiency score is treated as a lower bound score. As the inclusion of the value judgments impacts on the

DMU, it implies that the DMU is being measured relative to unobserved input output levels. Therefore the results should be interpreted with caution, as the feasibility of these input output levels is uncertain. Evidently, the interpretation of the efficiency score as a measure of the radial contraction of inputs or radial expansion of outputs, which are feasible under efficient operations, break down under the inclusion of value judgments.

3.4.2 The Radial Targets

Currently, as approaches for including value judgments implicitly modify the PPS, targets are set based on the DEA-efficient DMUs that remain DEA-efficient with the inclusion of the value judgments. However, this may mean that the targets suggested for the DEA-inefficient DMUs are of a very dissimilar operating mix to their present one, when DEA-efficient DMUs have been rendered inefficient by the introduction of the additional information. If the value judgments only extend the present DEA-frontier, the targets will be actually the same as those offered under the standard DEA model. Thus with the inclusion of values, are the targets offered by DEA of any practical use to the DM, if they require substantial changes in their present operating mix? Essentially this will depend on their objectives in improving their efficiency, i.e. if they want to maintain their present operating mix or not. Thus, the setting of objective targets when values have been included in a DEA assessment, is an area in need of further research.

3.4.3 The Peers

It would appear that if the inclusion of weights restrictions in a DEA assessment substantially reduces the number of DEA-efficient DMUs, then the usefulness of the peer information is debatable. They only highlight those DMUs that have favourable operating mixes under weights restrictions, which may be very different from the mixes of the inefficient DMUs.

3.5 Conclusion

The introductory section argued that the incorporation of value judgments in DEA is an area motivated by real life applications. The growing expansion of the weights restrictions methodology since its original development by Thompson *et al.* [53] and Dyson and Thanassoulis [27], heralds encouraging signs regarding the contribution of the method in assessing performance. Taking stock of the evolutionary stages of the weights restriction method it can be said that:

- Weights restrictions are based on mathematical modifications of the Charnes *et al.* [16] model that seek to encapsulate value judgments in the assessment of performance;
- Weights restrictions do not seek to eliminate the fundamental principle of the original DEA model, but rather they seek to ensure that appropriate values are attached to the input/output variables;
- There is no all-purpose method for translating value judgments into restrictions on DEA weights;
- Not fully explored at present are:
    - The mathematical and managerial implications of the introduction of value judgments in DEA models. e.g. target setting.
    - Alternative approaches for including value judgments.
    - The inclusion of value judgments into the VRS model and further into many of the extension models.

The development of the WRs field has led, in turn, to new areas of applications of DEA. One of these areas of concern is the use of DEA as an aid to decision analysis. Cook *et al.* [24] were among the first to propose DEA based models that sought to obtain absolute ranks of efficient DMUs by means of weights restrictions. From the technical point of view, the interest lies in how DEA-efficient DMUs can be ranked on the basis of their ability to assign a balanced magnitude of weights to their inputs and outputs.

To date, the means of capturing value judgments in DEA assessments has been almost exclusively by weights restrictions. Although alternative methods such as those used by Charnes *et al.* [15], Bessent *et al.* [12] and Lang *et al.* [38] exist, and act directly on the PPS, it is possible to think of developing systematic methods to capture progressively the DM's value judgements in DEA assessments by specifying UDMUs, suitably constructed from DMUs. This offers the advantage that the value judgments need only have local rather than global validity. This is in contrast to weights restrictions which reflect value judgments with global validity over the entire PPS. This will be the avenue explored in this thesis.

Section B
An Alternative Perspective for Incorporating Values in DEA

The previous section highlighted current procedures for the inclusion of values in a DEA assessment. This section, which covers chapters four and five, lays the foundations for an alternative means to current approaches for the expression and incorporation of the DM's values in a DEA assessment.

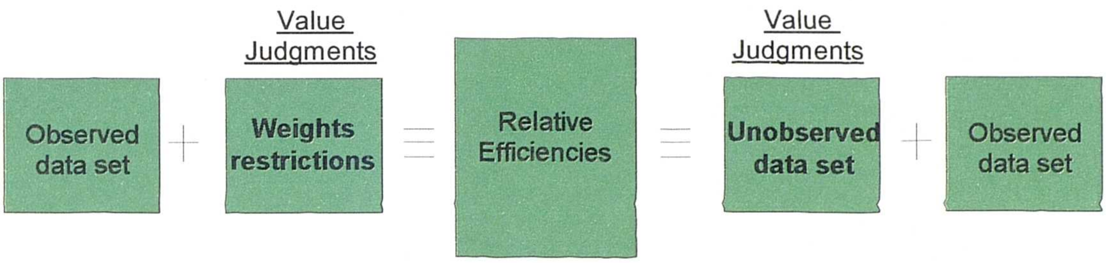
This alternative approach is established by recognising that there is a direct link between the explicit restriction of the weights and an implicit modification of the PPS, and explores the possibility of how a DM could focus on including values in a DEA assessment via an explicit modification of the PPS. Thus, an implicit restriction of the DEA weights will be made.

The section concludes by illustrating why DMs would decide to express their values in the form of UDMUs. It also explains how a combined use of weights restrictions and UDMUs may aid DMs in their expression of values, and the setting of targets. Hence DMs gain a greater understanding of the general implications of the inclusion of values in a DEA assessment.

This section only establishes the principles for an alternative approach. Thus, it lays the foundation for an alternative means for DMs to express and incorporate their values in a DEA assessment, which may be appropriate for DMs who have difficulties with the use or application of weights restrictions.

Section C will use this foundation to build a procedure for incorporating values in DEA using UDMUs. Alternative procedures may be built using the foundations laid in this section.

4. Chapter Four
Simulating Weights Restrictions by Means of Radial DMUs: CRS Case[2]



4.1 Introduction

As illustrated in chapter three, it has long been recognised that complete weights flexibility in DEA often leads to inappropriate estimates of efficiency. At one extreme, a DMU operating under CRS can ignore all but one output and one input variable, and possibly appear DEA-efficient by virtue of offering the best ratio on those two variables from all the DMUs, irrespective of poor performance on the rest of the input and output variables. Alternatively, the weights estimated can be counter intuitive. For example, in an assessment of perinatal care units without Weights Restrictions (WRs), Thanassoulis *et al.* [47] found some DMUs weighted a 'satisfied mother' more heavily than a 'very satisfied mother' with the service received. A number of extensions to the original DEA model have been put forward to overcome the problems created by complete weights flexibility in

[2] An earlier version of this chapter is *forthcoming in* **Management Science**

DEA. (e.g. Dyson and Thanassoulis [27], Wong and Beasley [58], Roll and Golany [42], Thompson *et al.* [53], Charnes *et al.* [15] and Thanassoulis and Dyson [48]).

Attempts at specifying WRs to date have generally been based on estimates of 'sensible' ranges of permissible values for the output or input weights. Methods have differed only on approaches to identify permissible ranges of weights values. See, for example, the alternative approaches offered by Dyson and Thanassoulis [27], Wong and Beasley [58] and Charnes *et al.* [15], among others. It should be noted, however, that at issue is not so much which ranges of input and output weights are permissible, but rather how prior judgments on the relative values of the input and output variables can be clarified and incorporated in DEA assessments. Weights restrictions specifying permissible weights ranges are only a means of specifying and incorporating prior judgments on the input and output variables in DEA assessments. This chapter highlights that there is an alternative approach to capturing and reflecting prior judgments in DEA based on using Unobserved DMUs (UDMUs). This new approach offers an alternative to WRs and requires the DMs to think in terms of comparing DMUs rather than specifying rates of substitution between output or input variables. One key advantage of using UDMUs rather than weights restrictions, is that local as opposed to global information is sought from the DM, which is likely to be more accurate, as well as easier for the certain DMs to provide.

In DEA, efficiency can be defined either in 'weights space' or 'production space', see Thanassoulis [45]. In weights space the efficiency of a DMU is defined as above, in terms of the maximum ratio of the sum of its weighted outputs to the sum of its weighted inputs. In production space, an equivalent input oriented definition of efficiency is the lowest proportion to which all input levels of the DMU can be reduced, providing this is not detrimental to any one of its output levels. This lowest proportion is estimated by using a PPS constructed using the DMU's input output levels, and contains all feasible input output correspondences in the production process operated by the DMUs. In the context of capturing and using value judgments in DEA assessments, UDMUs represent the production space equivalent to WRs in weights space, as this chapter will demonstrate.

That is, weights restrictions use the weights model to explicitly restrict the weights, whereas UDMUs use the envelopment model to implicitly restrict the weights.

As discussed in chapter three, both Charnes *et al.* [15] and Ali *et al.* [2] point out, that WRs imply changes to the PPS and these changes can be simulated by suitable transformations to the observed data. The approach in this chapter maintains the original DMUs, and focuses on the introduction of UDMUs, to simulate the WR relative efficiency scores. The notion that an UDMU is implicitly introduced by an absolute WR was initially highlighted by Roll and Golany [42]. This correspondence between all forms of weights restrictions (relative, linked-dependent, absolute and virtual) and a modification of the PPS is explored further in this chapter, and will be limited to DMUs operating under CRS.

This chapter will initially show how relative and linked-dependent weights restrictions can be simulated by UDMUs. These concepts are then generalised to absolute and virtual weights restrictions.

4.2 Simulating Relative Output Weights Restrictions by Means of Radial DMUs

As detailed in chapter three, the relative efficiency scores under RWRs are independent of the model orientation under CRS. That is, switching from an Input Minimisation (IM) to Output Maximisation (OM) orientation provides the same results. Thus for simplicity, this section will only consider simulating relative efficiency scores obtained under an IM model, although the same approach will hold for the OM model. To illustrate a simple numerical example will be used.

A Simple Example

Relative output weights restrictions in DEA can be simulated by augmenting the set of DMUs with UDMUs. The UDMUs necessary are specific to the set of DMUs and to the WRs being simulated. There is no unique set of UDMUs. The UDMUs simulating a set of WRs are such that the DMUs take the same relative efficiency scores, whether they are assessed under the WRs or within the augmented set of DMUs, without the WRs.

In order to see how relative output WRs can be simulated by UDMUs, consider the following set of 4 DMUs, each one using the same normalised level of 12 units of input, to secure the following levels on two outputs:

Table 4.1 - Example Data Set 2

	D1	D2	D3	D4
Output 1	1	3	3.75	1.5
Output 2	3	2	1	1.5

It is desired to assess the DMUs in Table 4.1 under the assumption that output 1 is more valuable than output 2. Under this assumption, the efficiency of DMU j_0 offering output levels (y_{1j_0}, y_{2j_0}) is the optimal value $h^*_{j_0}$ in (M4.1).

$$h^*_{j_0} = Max\ u_1 y_{1j_0} + u_2 y_{2j_0}$$

$$
\begin{aligned}
s.t.\quad & u_1 + 3u_2 \le 12 && :D1 \qquad \text{(M4.1)}\\
& 3u_1 + 2u_2 \le 12 && :D2\\
& 3.75u_1 + u_2 \le 12 && :D3\\
& 1.5u_1 + 1.5u_2 \le 12 && :D4\\
& -u_1 + u_2 \le 0 && :rr1\\
& -u_2 \le -\varepsilon &&
\end{aligned}
$$

(ε is a non-Archimedean infinitesimal)

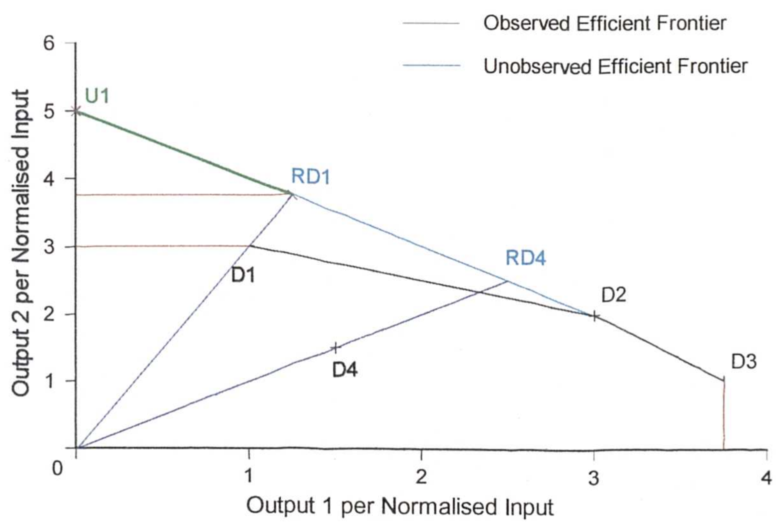
The model (M4.1) is, in essence, the basic DEA model developed by Charnes *et al.* [16], in which u_1 and u_2 are the weights attached to output 1 and output 2 respectively. It differs from the basic DEA model, only in that it includes the relative weights restriction (RWR), rr1, whereby $u_1 \ge u_2$ reflects the perception that output 1 is at least as valuable as output 2.

The WR, rr1, only needs to be simulated for those DMUs of (M4.1), for which rr1 is binding at all optimal solutions to the model. For the remaining DMUs there exists an optimal solution where rr1 is redundant. Consider replacing rr1 in (M4.1) by one or more UDMUs. Obviously, the UDMU must be DEA-efficient, otherwise its introduction into the observed data set would have no impact on the relative efficiency of the DMUs. It is shown in Appendix 4.1 that:

Any UDMU offering output levels such that $y_1 + y_2 = 5$ and $y_1 < 3y_2/2$ can simulate rr1 in (M4.1).

Figure 4.1 shows how the PPS originally defined by the DMUs in Table 4.1 is implicitly altered by the introduction of the weights restriction, rr1.

Figure 4.1 - Extended Production Possibility Set



Thus any UDMU that lies on the line segment, RD1U1, in Figure 4.1, has output levels that meet these two conditions and hence will simulate rr1 in (M4.1), when added to the data set. One such DMU is RD1, (1.25, 3.75) which is essentially the radial expansion of the output levels of D1. Thus adding this DMU to the observed data set and using (M4.2) to determine the relative efficiency scores, the same scores are obtained as when solving (M4.1).

$$h_{j_0}^* = Max\ u_1 y_{1j_0} + u_2 y_{2j_0}$$

s.t.	$u_1 + 3u_2 \leq 12$	:D1	
	$3u_1 + 2u_2 \leq 12$	:D2	(M4.2)
	$3.75u_1 + u_2 \leq 12$	:D3	
	$1.5u_1 + 1.5u_2 \leq 12$	:D4	
	$1.25u_1 + 3.75u_2 \leq 12$	:RD1	
	$-u_1, -u_2 \leq -\varepsilon$		

(ε *is a non-Archimedean infinitesimal)*

Under the model (M4.1), the efficient boundary is D3D2, whereas under the model (M4.2), the efficient frontier is D3D2RD1, with the same relative efficiency scores being obtained.

As there are no DMUs to the left of the radial OD1RD1, the only part of the efficient boundary used in assessing DMUs in Figure 4.1 is D3D2RD1. Thus as long as RD1 is used as an UDMU, the required part of the efficient boundary is specified and all other UDMUs are redundant. The UDMU, RD1 is the radial projection of D1, using its relative efficiency under (M4.1). UDMUs determined by the radial expansion of the DMUs' output levels will be defined as **Output Radial** DMUs (ORDMUs). Due to the CRS assumption, UDMUs based on the radial contraction of the input levels will also lead to the same relative efficiency scores as under ORDMUs. UDMUs determined by the radial contraction of the DMUs' input levels shall be termed **Input Radial** DMUs (IRDMUs). The construction of UDMUs by means of radial expansion or contraction of DMUs using their WRs relative efficiency scores, is developed further in the next section.

4.3 Simulating Relative and Linked-Dependent Weights Restrictions by Means of Radial DMUs

The principle of simulating output WRs by means of ORDMUs demonstrated in the preceding section, can be extended to the general case involving the assessment of any number of DMUs using multiple inputs to produce multiple outputs. However, ORDMUs are not the only set of Radial DMUs (RDMUs) that are capable of simulating the imposed relative weights restrictions. It is now shown that:

- Radial expansions of the output levels of the DMUs can be used to construct Output Radial DMUs (ORDMUs)

OR

- Radial contractions of the input levels of the DMUs can be used to construct Input Radial DMUs (IRDMUs)

Augmenting the observed data set with either the ORDMUs or the IRDMUs will simulate the imposed WRs relative efficiency scores. Only one set is required, and it is for the DM to decide which set to calculate - this may relate to whether the organisation is in a growth or downsizing phase. The term Radial DMUs (RDMUs) will be used to define the set of ORDMUs or IRDMUs that are being implemented in order to simulate the WRs relative efficiency scores.

This section specifies two sets of RDMUs which can simulate relative and linked-dependent weights restrictions in the general case. The first set contains as many UDMUs as there are observed DMUs. This is referred to as the "Full Set of Radial DMUs", (FSRD). The second set of RDMUs is a subset of FSRD and will be referred to as the "Reduced Set of Radial DMUs", (RSRD).

4.3.1 Specifying a Full Set of Radial DMUs

It will be assumed that there are N DMUs, $j=1,..,N$, with DMU j using input levels, x_{ij}, $i=1,...,m$ to produce outputs levels, y_{rj}, $r=1,...,s$. Further it will be assumed that the N DMUs are to be assessed under a set of input and output relative and linked-dependent weights restrictions defining a feasible set of output weights. The relative efficiency, $h^*_{j_0}$ of DMU j_0 under the input and output relative weights restrictions, V and U, and linked-dependent weights restrictions UV is given by (M4.3).

$$
\begin{aligned}
& h^*_{j_0} = Max \sum_{r=1}^{s} u_r y_{rj_0} \\
s.t. \quad & \sum_{i=1}^{m} v_i x_{ij_0} = 1 \qquad\qquad\qquad \text{(M4.3)} \\
& \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \le 0 \qquad j=1,\dots,N \\
& v \in V,\ u \in U \\
& uv \in UV
\end{aligned}
$$

where $u = (u_r,\ r=1,\dots,s)$ and $v = (v_i,\ i=1,\dots,m)$ are output and input weights respectively and are the variables in the model. Two sets of radial DMUs will now be defined that will simulate the relative efficiency scores obtained in (M4.3) excluding the set of weights restriction defined by U, V and UV.

Full Set of Output Radial DMUs (FSORD)

Define a set of ORDMUs, $jt=1,\dots,N$, such that DMU jt has output levels, y_{rjt}, $r=1,\dots,s$ and input levels, x_{ijt}, $i=1,\dots,m$ as follows:

$$
y_{rjt} = \frac{y_{rj}}{h^*_j} \qquad x_{ijt} = x_{ij} \qquad j=1,\dots,N \qquad \text{(4.1)}
$$

h^*_j is determined by means of (M4.3).

The ORDMUs defined in (4.1) simulate the weights restrictions defined by U, V and UV in (M4.2) by virtue of Theorem 4.1.

Theorem 4.1

Let h_j^*, $j=1,\ldots,j_0,\ldots,N$, be as defined in (M4.3). Let $jt=1,\ldots,N$ be RDMUs having the input output levels defined in (4.1), and let $h_{j_0}^{/}$ be as defined in (M4.4), so that

$$
\begin{aligned}
& h_{j_0}^{/} = Max \sum_{r=1}^{s} \delta_r y_{rj_0} \\
s.t. \quad & \sum_{i=1}^{m} \gamma_i x_{ij_0} = 1 && && \text{(M4.4)} \\
& \sum_{r=1}^{s} \delta_r y_{rjt} - \sum_{i=1}^{m} \gamma_i x_{ijt} \leq 0 && jt=1,\ldots,N \\
& \delta_r, \gamma_i \geq \varepsilon && \forall r,i
\end{aligned}
$$

where ε, y_{rj} and x_{ij} are as in (M4.3). Then for DMU j_0 it follows that:

$$h_{j_0}^{/} = h_{j_0}^* \qquad (4.2).$$

The proof of Theorem 4.1 can be found in Appendix 4.2.

The set of RDMUs $jt=1,\ldots,N$, whose input output levels are those defined in (4.1) is referred to as a FSORD.

Alternatively, a set of input radial DMUs can be used to simulate these weights restrictions relative efficiency scores.

Full Set of Input Radial DMUs (FSIRD)

Define a set of *IRDMUs* $jp=1,\ldots,N$, such that DMU jp has output levels, y_{rjp}, $r=1,\ldots,s$ and input levels, x_{ijp}, $i=1,\ldots,m$ as follows:

$$y_{rjp} = y_{rj} \qquad x_{ijp} = h^*_j x_{ij} \qquad j=1,\ldots,N \qquad (4.3)$$

h^*_j is determined from model (M4.3).

Clearly, the RDMUs defined in (4.1) can be scaled to obtain the RDMUs defined in (4.3) and therefore simulate the WRs defined by *U*, *V* and *UV* in (M4.3) by virtue of Theorem 4.1. The set of DMUs *jp=1,...,N*, whose input output levels are those defined in (4.3) is referred to as a FSIRD

Where the relative efficiency score of a DMU is $h^*_{j_0} = 1$, the RDMU is a duplicate of the original DMU as can be deduced from (4.1) or (4.3). More generally these RDMUs can be expressed in terms of the existing inputs and outputs. Evidently the FSORD or the FSIRD can be reduced to obtain a smaller set of RDMUs that is necessary and sufficient to simulate the relative efficiency scores under relative and linked-dependent weights restrictions.

4.3.2 Specifying a Reduced Set of Radial DMUs

Clearly, those RDMUs that duplicate observed DMUs, as well as those RDMUs that can be expressed as a linear combination of other DEA-efficient DMUs and/or RDMUs are redundant. That is, those RDMUs that are of class E' as defined by Charnes *et al.* [19] are redundant. This will hold for both the ORDMUs and the IRDMUs, and so for simplicity, only the case for the ORDMUs will be considered. It can be seen, for example, from Figure 4.1 that the removal of RD4 (class E') will not affect the relative efficiency scores of the DMUs, and so it is not necessary for the simulation of the WR relative efficiency scores.

The aforementioned observations can be readily generalised so that

> the FSORD/FSIRD which consists of *N* DMUs constructed using expression (4.1) or (4.3) can be reduced by eliminating any RDMUs that are linearly dependent on other DMUs and RDMUs in the FSORD/FSIRD. The resulting RSORD/RSIRD is necessary and sufficient to simulate the relative efficiency scores under weights restrictions as simulated by the FSORD/FRIRD.

One way to identify and eliminate the redundant RDMUs (class E') is by reference to their '**Super Efficiency**' (SE) (see Andersen and Petersen [6]). The super efficiency of DMU j_0 in (M4.4) is assessed by dropping the constraint $\sum_{r=1}^{s}\delta_r y_{rj_0} - \sum_{i=1}^{m}\gamma_i x_{ij_0} \leq 0$ from the model.

In respect of DMU j_0 the model can now yield an efficiency score of over 1, and hence the term 'Super Efficiency'. Super efficiencies can be used to identify redundant RDMUs as follows.

Let DMU jt_0 be the non-duplicate RDMU to be tested for redundancy. As the aim is to eliminate the RDMUs that are linear combinations of other DMUs, the only DMUs that these RDMUs can be linearly dependent on, are those that are of class E and E' as defined by Charnes *et al.* [19] under (M4.4). Let *JE* denote the set of DMUs of class E and E' in (M4.4). Solve model (M4.5) to compute its SE, $h^{//}_{jt_0}$ where

$$h^{//}_{jt_0} = Max \sum_{r=1}^{s}\psi_r y_{rjt_0}$$

$$s.t. \quad \sum_{i=1}^{m}\omega_i x_{ijt_0} = 1 \qquad \text{(M4.5)}$$

$$\sum_{r=1}^{s}\psi_r y_{rj} - \sum_{i=1}^{m}\omega_i x_{ij} \leq 0 \qquad j \in JE$$

$$\sum_{r=1}^{s}\psi_r y_{rjt} - \sum_{i=1}^{m}\omega_i x_{ijt} \leq 0 \qquad jt \in Jrt$$

$$\psi_r,\ \omega_i \geq \varepsilon \qquad \forall r,i$$

where $Jrt = \{jt \mid jt \in FSRD,\ jt \neq jt_0$ and for $j=jt,\ h^*_j < 1\}$, h^*_j being as defined in (M4.3). Thus *Jrt* consists of non-duplicate RDMUs, excluding DMU jt_0. Notation in (M4.5) is otherwise as in (M4.3).

Proposition 4.1

If $h^{//}_{jt_0} > 1$ or if (M4.5) has no feasible solution, RDMU jt_0 is non-redundant RDMU.

Proposition 4.2

If $h^{//}_{jt_0} = 1$ then RDMU jt_0 is a redundant RDMU.

Proposition 4.3

$h_{jt_0}^{//}$ cannot be less than 1.

For the proof of these propositions see Appendix 4.3.

In light of the above Propositions, a RSRD can be constructed from the FSRD by eliminating all RDMUs which:

- Duplicate original DMUs;
- Yield a super efficiency of 1 in (M4.5).

The concept of using SE to identify class E DMUs is easily implemented and SE scores can be readily estimated by commercial software. (e.g. Warwick DEA Software, Thanassoulis and Emrouznejad [46]).

It should be noted that, for explanation purposes all possible RDMUs were determined. Clearly, the number of redundant RDMUs can be reduced by initially only determining RDMUs that correspond to those DMUs that are effected by the weights restrictions.

This section has shown how relative and linked-dependent weights restrictions can be simulated by the use of Radial DMUs (RDMUs). The next section will consider the case for absolute and virtual weights restrictions.

4.4 Simulating Absolute and Virtual Weights Restrictions by Means of Radial DMUs

As discussed in chapter three, virtual weights restrictions restrict the percentage contribution of individual inputs or outputs to the normalised input or output of a DMU. This corresponds to DMU specific restrictions on the respective DEA weights. For their implementation to have any real meaning in terms of relative efficiency, the virtual constraint for each DMU must be added to the constraint set for all the DMUs, as detailed in chapter three. In this case, their implementation reduces to that of the introduction of a single binding absolute weights restriction. Thus this section will only discuss the simulation of absolute weights restrictions by means of RDMUs.

Frequently, absolute weights restrictions lead to infeasible solutions. However, it shall be assumed that the implemented absolute restrictions provide feasible results. In chapter three, it was noted that when imposing absolute and virtual weights restrictions, the relative efficiency scores were dependent upon the model orientation. This implies that the unobserved frontier that is implicitly introduced when assessing a DMU under the absolute weights restrictions, is different for the different model orientations. Clearly, this has implications for the interpretation of the results. As the DMUs are operating under CRS it is therefore expected that the relative efficiency scores will be the same for each model. Unfortunately, this problem is beyond the area of interest of this thesis and will not be investigated further. Evidently, it follows that each orientation requires a different set of RDMUs to simulate the weights restrictions relative efficiency scores. However, the process of determination of these RDMUs will be the same irrespective of the orientation and thus only the IM case will be considered.

Consider assessing N DMUs, $j=1,..,N$, with DMU j using input levels, x_{ij}, $i=1,...,m$ to produce outputs levels, y_{rj}, $r=1,...,s$. The relative efficiency, $e^*_{j_0}$ of DMU j_0 under input and output weights restrictions, ar1 and ar2, respectively is determined for model (M4.6).

$$
\begin{aligned}
& e^*_{j_0} = Max \sum_{r=1}^{s} u_r y_{rj_0} \\
s.t. \quad & \sum_{i=1}^{m} v_i x_{ij_0} = 1 && && \text{(M4.6)} \\
& \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \le 0 && && j=1,...,N \\
& \rho_i \le v_i \le \zeta_i && :ar1 && i=1,...,m \\
& \rho_r \le u_r \le \zeta_r && :ar2 && r=1,...,s
\end{aligned}
$$

where v and u are the weights attached to the inputs and outputs respectively and ρ and ζ are DM specified bound values on the input and output weights.

The DM has the choice of determining either the FSORD or FSIRD as discussed in section 4.3.2. It will be assumed that the DM has decided to define a set of Input Radial DMUs (IRDMUs), $jp=1,\dots,N$ such that RDMU jp has output levels, y_{rjp}, $r=1,\dots,s$ and input levels x_{ijp} $i=1,\dots,m$ as follows:

$$y_{rjp} = y_{rj} \qquad x_{ijp} = e^*_j x_{ij} \qquad j=1,\dots,N \qquad (4.4)$$

e^*_j is determined from (M4.6). Clearly, in practice only the RDMUs that correspond to those DMUs that are effected by the weights restrictions are required to be determined.

The ORDMUs defined in (4.4) simulate the WRs defined by ar1 and ar2 by virtue of Theorem 4.2.

<u>Theorem 4.2</u>

Let e^*_j, $j=1,\dots,N$ be the efficiency scores obtained from (M4.6) and let $jp=1,\dots,N$ be the RDMUs having input output levels defined in (4.4). Let $e^{/}_{j_0}$ be as defined in (M4.7)

$$
\begin{aligned}
& e^{/}_{j_0} = Max \sum_{r=1}^{s} \tau_r y_{rj_0} \\
s.t. \quad & \sum_{i=1}^{m} \delta_i x_{ij_0} = 1 && \text{(M4.7)} \\
& \sum_{r=1}^{s} \tau_r y_{rj} - \sum_{i=1}^{m} \delta_i x_{ij} \le 0 && j=1,\dots,N \\
& \sum_{r=1}^{s} \tau_r y_{rjp_0} - \sum_{i=1}^{m} \delta_i x_{ijp_0} \le 0 \\
& \delta_i,\ \tau_r \ge \varepsilon && \forall i,r
\end{aligned}
$$

Notation in (M4.7) as in (M4.6). Then for DMU j_0 it follows that:

$$e^{/}_{j_0} = e^{*}_{j_0} \qquad (4.5)$$

The proof of Theorem 4.2 can be found in Appendix 4.4.

Thus, when only the specific RDMU, jp_0 that corresponds to the assessed DMU j_0 is added, it is found that the correct relative efficiency scores are obtained. This would imply that when DMUs are assessed under absolute weights restrictions they are being assessed relative to different standards. That is different hypothetical frontiers. As the hypothetical frontier that is generated is dependent on the assessed DMU, it would indicate that the relative efficiency measures are not truly comparable, see Appendix 4.5. The impact of this on the interpretation of the results and the implications for the imposed values, will be discussed in chapter five. Clearly, this has implications for the use of absolute weights restrictions in DEA.

Thus having demonstrated that weights restrictions can be simulated by a modification of the PPS, a summary of the results will now be drawn.

4.5 Conclusion

Value judgments hitherto have been reflected in DEA assessments by means of restrictions on the values that the DEA weights can take. This chapter has introduced the foundations for a new approach to capturing and using value judgments in DEA and establishes that weights restrictions can be simulated by means of UDMUs, thereby providing an alternative avenue to capturing and using value judgments in DEA assessments. Expressing value judgments via UDMUs offers the following advantages;

(i) Value judgments can be expressed locally

(ii) Non linear MRS between variables can be expressed

(iii) The feasibility of the extended PPS can be considered/ascertained

To date much of the effort in capturing value judgments in DEA assessments has been focused on weights restrictions and their construction. This approach virtually ignores the fact that the issue is capturing value judgments rather than the weights restrictions. This chapter switches the focus from weights restrictions back to the real issue of expressing values in a DEA assessment and offers an alternative to weights restrictions for their inclusion in DEA. Hence, one way to view the impact of explicit restrictions on DEA weights, is to say that they implicitly add UDMUs to the observed data set. These UDMUs extend the DEA-efficient frontier of the PPS in such a way that values are incorporated into the assessment. However, as the modifications to the PPS under weights restrictions are only implicit, the use of weights restrictions to express the DM's views can hide important assumptions about feasible input output transformations and/or value judgments. This lays the foundation for a new direction of capturing and using value judgments in DEA assessments, and is the avenue of exploration developed in this thesis.

The next chapter discusses how the insight offered by Radial DMUs (RDMUs) into the impact of value judgments on the PPS, provides motivating reasons for the DMs to express their values directly in the form of UDMUs in DEA.

5. Chapter Five
Why Express Value Judgments Via UDMUs?

5.1 Introduction

The primary aim of this chapter is to demonstrate *in principle* the motivating reasons for why a DM would want to express value judgments in the form of Unobserved DMUs (UDMUs). To achieve this, it is necessary to demonstrate some of the problems that exist with the traditional approach for the incorporation of value judgments, namely that of weights restrictions. This demonstration aims to highlight that there is a need, in certain situations, for an alternative approach to weights restrictions for capturing value judgments in DEA. Subsequently, it will also demonstrate how the Radial DMUs (RDMUs) of chapter four can be used in conjunction with weights restrictions to aid the DM in the determination of appropriate values and how to implement local values.

As demonstrated in earlier chapters, the main avenue for the inclusion of value judgments in a DEA assessment is by means of weights restrictions. Chapters three and four establish a link between the incorporation of value judgments and an Extended Production

Possibility Set (EPPS). This link suggests that Weights Restrictions (WRs) implicitly express value judgments as input output levels i.e. Unobserved DMUs (UDMUs). Thus, if the focus of the inclusion of value judgments is transferred from:

- the explicit restriction of the weights,
  with an implicit expression of input output levels,

to:

- the explicit expression of input output levels,
  with an implicit restriction of the DEA-weights,

an alternative approach for the expression of value judgments in a DEA assessment can be derived. However, having identified the possibility of an alternative approach to weights restrictions, namely that of UDMUs, there is a need to offer motivating reasons for the use of this alternative, when in many cases, weights restrictions are simple, workable and acceptable. There are several reasons why, in principle, this alternative approach may be desirable to a DM. These reasons were outlined in chapter two and will now be detailed further in light of chapter four.

The chapter will be structured as follows: Section two details how Radial DMUs (RDMUs) can aid the DM's interpretation of the results under WRs. Section three suggests how to combine the use of weights restrictions and Radial DMUs to incorporate varying local values into the assessment. Section four concludes.

5.2 Interpreting the Results

As previously mentioned, when value judgments have been incorporated in a DEA assessment, the Production Possibility Set (PPS) may be implicitly extended (this may include a modification of the present PPS). The implicit input output levels of the Extended Production Possibility Set (EPPS) are unknown and therefore may not be attainable. Thus, while the DMUs can be ranked on their relative efficiency under weights restrictions, the obtained relative efficiency scores cannot be readily interpreted in terms of attainable input output levels.. Thus, other than as a clearer overall picture of a DMUs relative efficiency, how should a DM interpret the relative efficiency scores of the DMUs? How can the DM be certain that their DMUs are being measured relative to realistic production processes? This raises the question of how to set objective targets under weights restrictions? This section, therefore, proposes to illustrate how expressing the DM's value judgments via UDMUs can aid the DM in the interpretation of the results as the DM is provided with an explicit account of the modified PPS.

The initial consideration is that of the feasibility of the EPPS which, clearly, has implications for the interpretation of the relative efficiency scores, the targets, peers and the imposed weights restrictions.

5.2.1 Feasibility of the Extended Production Possibility Set

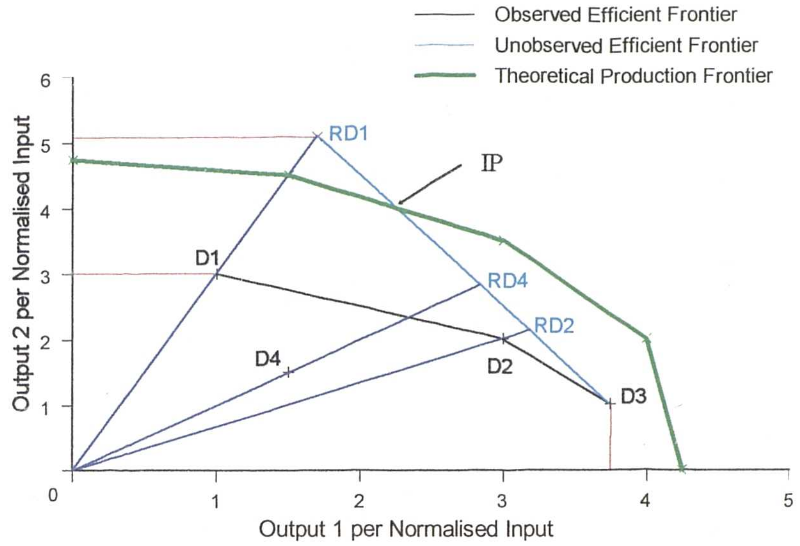
At present, in terms of the obtained relative efficiency score, it is assumed that the input output levels of the implicit EPPS are feasible and, thus, the scores obtained are valid. That is, it is assumed that the input output levels implicitly introduced by the weights restrictions (WRs) lie within an economically defined production frontier, see Färe and Primont [29], which here shall be termed a Theoretical Production Set (TPS). However, as the actual input output levels that the DMUs are measured relative to, are unknown, this assumption may or may not be viable. That is, the input output levels of the implicit EPPS may be outside the theoretical production set or be deemed unachievable by the DM. Thus without an explicit expression of the EPPS, the DM cannot be certain as to the acceptability of their relative efficiency scores, and the results may mislead the DM with respect to the DMUs' true inefficiency. Essentially, if the implicit EPPS under WRs is not achievable, then the relative efficiency score is not a valid measure of performance. Thus, caution should be used in interpreting the results without knowledge of the EPPS. However the real issue at question here is one of whether or not the imposed global values are appropriate.

This problem may be overcome through the use of the Radial DMUs (RDMUs) defined in chapter four. These RDMUs allow the value judgments captured by the Weights Restrictions (WRs) to be expressed in the form of the inputs and outputs of the process under analysis. Thus the RDMUs provide information to the DM on the feasibility of the input output levels that the DMUs are being assessed relative to under Weights Restrictions (WRs). This, in turn, aids the DM in the interpretation of the relative efficiency score and the feasibility of the imposed weights restrictions or, more specifically, the value judgments that they reflect. Section three of this chapter deals with the case where, given the explicit expression of the EPPS, the DM feels that the global values are inappropriate and therefore would like to impose local values. To demonstrate how RDMUs may aid the DM in the interpretation of their results, a simple example shall be used.

A Simple Example

Consider assessing the 4 DMUs shown in Table 3.1 of chapter three with the additional information that output 1 is deemed twice as valuable as output 2, expressed in the form of $u_1 - 2u_2 \geq 0$, and further suppose that the Theoretical Production Set is known. Construct a Full Set of Output Radial DMUs (ORDMUs), {RD1, RD4, RD2} (using (4.1) of chapter four) that will simulate the relative efficiency scores under the weights restriction and thus define an Extended Production Possibility Set for the problem. The three production sets (theoretical, observed and unobserved) are plotted in Figure 5.1.

Figure 5.1 **- *Extended Production Possibility Set and the Theoretical Production Set***



The marginal rates of substitution (MRS) expressed as the Relative Weights Restriction (RWR): $u_1 - 2u_2 \geq 0$ is translated into terms of the input output levels of the assessment by their expression in the form of the RDMUs: RD1, RD4 and RD2. These three ORDMUs modify the observed DEA frontier to form a new partly observed, partly unobserved DEA efficient frontier that is defined by D3RD1. However, it can clearly be seen in Figure 5.1 that RD1 and part of the unobserved frontier segment IPRD1 lie outside the theoretical production set, i.e. it is thought that these input output levels are not achievable in theoretical terms. Hence this would imply that for any inefficient DMU projected onto the IPRD1 segment of the frontier, that their relative efficiency score is not achievable. However, UDMUs RD4 and RD2 lie within the theoretical production set, and their relative efficiency scores should, in theory, be achievable, thus implying that the imposed global values are NOT realistic for all the DMUs. That is, they are appropriate for D2 and D4 but inappropriate for D1. Hence in this specific example, there may be a need for the introduction of local rather than global marginal rates of substitution (MRS), or a modification of the imposed global MRS.

So, without this explicit expression of the EPPS, the DM would not have been aware that some of its DMUs were being measured relative to unrealistic production processes. Clearly, in providing the DM with an insight into the feasibility of the EPPS, the DM is also provided with an insight into the feasibility of the globally imposed value judgments, and whether they are appropriate globally, locally or not at all.

The next section deals with problems that arise with the inclusion of absolute/virtual weights restrictions.

5.2.2 A Meaningful Relative Measure

As demonstrated in chapter four, absolute/virtual restrictions cannot be simulated by a Full Set of Radial DMUs (FSRD). This implies that the inclusion of absolute/virtual restrictions leads to a different, *implicitly introduced, unobserved frontier* for assessed DMUs, i.e. the implicitly introduced frontier is DMU specific. Hence the DMUs are being measured relative to different standards. This has serious implications for the meaning and interpretation of the results. Obviously, the scores cannot be considered as relative efficiency scores, as the *DMUs are not measured relative to the same input output levels.* Thus how should the DM interpret the results? Further, there is a mismatch of the relative efficiency scores in that the different orientations provide different relative efficiency scores, which contradicts the CRS assumption. This merely highlights the need for an alternative approach to weights restrictions for ensuring that all the selected variables contribute to the relative efficiency score.

One possible approach to overcome this problem is through the combined use of weights restrictions and the RDMUs that were defined in chapter four. Suppose that some absolute/virtual restrictions have been imposed in the DEA assessment. The obtained scores are then used to determine a Full Set of Radial DMUs (FSRD) that individually are required to simulate the absolute relative efficiency scores as detailed in chapter four. However, if the DM now re-assesses the DMUs without the WRs but allowing all the members of the Full Set of Radial DMUs (FSRD) to be considered as peer DMUs. Hence the problem is converted back to a normal DEA assessment with one common PPS, partly observed and partly unobserved for all the assessed DMUs. As the assessment is now a standard DEA assessment with an extended data set, the scores can be considered as relative once again. So, the DM has gained a relative efficiency score for all its DMUs relative to the same PPS, where the contribution of specific inputs and outputs has been limited in a similar manner to their initial requirements. (Although the obtained results may not reflect this due to multiple optimal solutions.)

Having considered the actual interpretation of the relative efficiency score, there is also the interpretation of the targets and peers to be considered, given that the DM has information on the input output levels of the EPPS, which will be discussed in the next section.

5.2.3 Targets and Peers

This section proposes to suggest how the use of RDMUs can aid the DM in the setting of alternative targets to those suggested at present under weights restrictions. Currently the targets provided under WRs are based solely on those DMUs that remain DEA-efficient, as it is known that these input output levels are achievable. However, as a consequence of the inclusion of weights restrictions, many of the DEA-efficient DMUs under the standard DEA model, Charnes *et al.* [16] are rendered DEA-inefficient. In general those DMUs that remain DEA-efficient are those with favourable operating mixes, i.e. consume less of the higher valued inputs and produce more of the higher valued outputs. Thus, suggested targets for a DMU which has been rendered inefficient by weights restrictions may involve either a reduction to some of its output levels or an increase in some of its inputs. Such targets could be perceived as counter-intuitive as they require increases in inputs and reductions in outputs. Although, these proposed changes do render the DMU DEA-efficient and are suggested so that higher levels of the higher valued outputs and lower levels of the higher valued inputs are achieved, they may not be objective for the DM.

One possible interpretation of these suggested targets is that all the DMUs should tend towards consuming less of the higher valued inputs and produce more of the higher valued outputs. That is, tend towards a single desirable operating mix. However, this is not in line with the principles of DEA, as it implies that the introduced value judgments are limiting the allowable range of efficient operating mixes, making some operating mixes appear less desirable than others; whereas DEA allows DMUs of different operating mixes to be deemed relatively efficient. RDMUs can be used to provide the DM with an alternative perspective for the impact of the inclusion of values in a DEA assessment. This alternative indicates to the DM targets that would maintain their present operating mix and render them DEA-efficient. The more unfavourable the present operating mix under the values involved the lower the input/higher the output levels needed to render the DMU DEA-efficient. Thus, indicating to the DM that it is the levels of inputs and outputs that are inefficient rather than their operating mix.

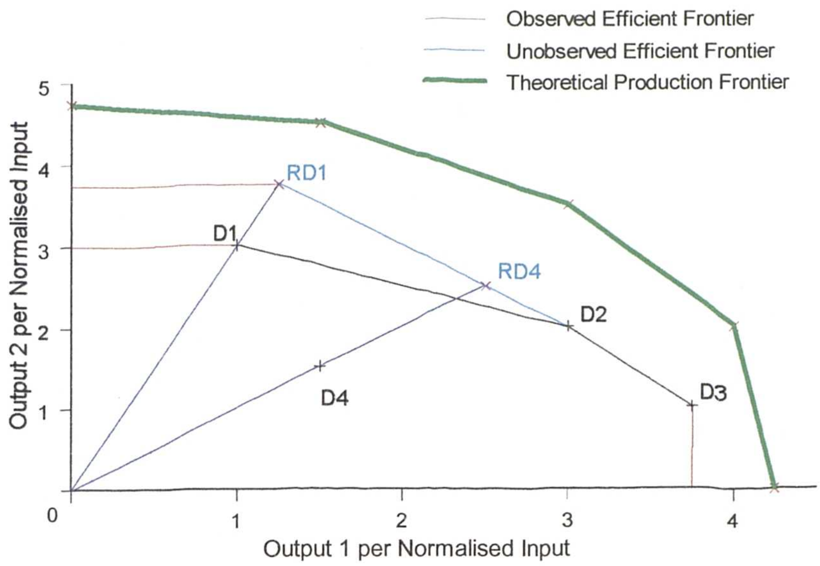
Hence through the use of RDMUs to express values, targets that maintain the DMU's present operating mix can be suggested. These suggested targets will unfortunately be based on unobserved input output levels. However, they could be interpreted as indicating to the DM that it is not their operating mix that is inefficient, but that their inputs or outputs levels are inefficient given the DM's values. Therefore, with the use of UDMUs to express the DM's values the obtained targets imply an extending of the PPS with the inclusion of values. Whereas the obtained targets with the inclusion of values in the form of weights restrictions imply the values are narrowing the PPS.

At the least RDMUs are providing the DM with an alternative perspective for improving their efficiency given the DMs values and their operating mix. To illustrate consider a simple example.

A Simple Example

Consider assessing the DMUs of Table 3.1 of chapter three with the inclusion of the value judgments in the form of relative restriction of $u_1 - u_2 \geq 0$. This section assumes that the imposition of the weights restriction renders no RDMU outside the Theoretical Production Frontier and hence the imposed values are appropriate.

Figure 5.2 **-** ***Interpretation of the Targets and Peers***



If the DM now determines the output radials, RD1 and RD4 that will simulate the relative efficiency scores under the weights restriction. Then assessing the DMUs with these Radial DMUs (RDMUs) as well as the observed DMUs allowed as peers, leads to a new DEA-frontier being defined D3D2RD1. As can be seen from Figure 5.2 the inclusion of the relative restriction renders D1 DEA-inefficient. Thus for D1 to be deemed DEA-efficient, with the imposed WR, DEA suggests targets values of (3,2) with D2 as its peer. These suggestions imply that DMU D1 should change its operating mix and decrease its output 2 level by 1 unit so that it can increase its output 1 level (the higher valued output) by 2 more units. However, the DM may or may not consider *these targets to be objective.*

RDMUs can be used to suggest alternative targets which are based on their present operating mix. Thus, in the simple example, this would require an increase of 0.75 units in output 1 and 0.25 in output 1. Hence to achieve efficiency with the use of RDMUs to suggest targets, a total increase of 1 unit of output is required, rather than the 2 units for the targets suggested under weights restrictions. The targets now convey to the DMs that, given the standards of the other DMUs with different operating mixes and the values of their outputs, to be deemed relatively efficient while maintaining their present mix, they must increase, overall, their output levels by 20%.

Further, both D1 and D4 have the same suggested target output levels under weights restrictions, but this does not seem appropriate, as they do not reflect either the difference in current output, their relative inefficiency scores nor their previous efficiency scores without weights restrictions.

The use of RDMUs to aid the DM in their interpretation of the results and in the setting of alternative targets, has been discussed in the above section. The next section will discuss how the DM may amend the input output levels of the RDMUs in order to incorporate local value judgments.

5.3 The Combined use of Weights Restrictions and RDMUs

In a combined use of weights restrictions and RDMUs, the latter can be used to fine tune the DM's value judgments initially conveyed by means of the weights restrictions. In practical applications, the information on which weights restrictions are based is often subject to uncertainty, as a DM cannot be confident on the precise numerical expressions of their preferences. In these instances RDMUs can be used in an attempt to give more precise expression to the DM's value judgments. The next section will consider the case when the DM finds the RDMUs inappropriate and, therefore wants to adjust their imposed weights restriction values.

5.3.1 Introducing Local Value Judgments

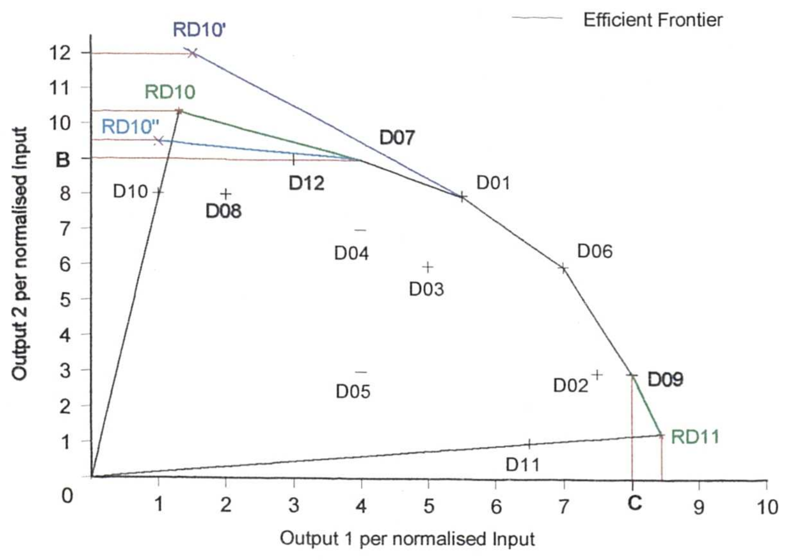
Consider the case, when the DM has calculated the Reduced Set of Radial DMUs (RSRD), following the steps outlined in chapter four, with the RSRD representing the minimum required number of RDMUs in order to simulate the weights restrictions. In examining the input output levels of these RDMUs, the DM considers the imposed values are in certain cases inappropriate, and would therefore like to alter their imposed values. This can be achieved by modifying the imposed weights restrictions. However in this case the DM will just alter their global values as opposed to introducing local values.

Alternatively, the RSRD could be modified and the DM can introduce varying local values. If this latter option is chosen, the question, then becomes how to modify the RSRD? Clearly, different members of the RSRD will be modified differently. This will lead to different value judgments being introduced for the various operating mixes that exist within the PPS, i.e. local values are being introduced. One advantage of this approach is that it is more in line with the principles of DEA, in that DMUs with different operating mixes place different emphasis on the different inputs and outputs. Essentially, this approach leads to local marginal rates of substitution (MRS) being introduced, based on the local modifications of the global rate introduced by weights restrictions. To illustrate consider a simple example.

A Simple Example

Consider assessing the set of DMUs shown in table 1.1 of chapter one, with the inclusion of relative weights restrictions (RWRs): $0.5u_2 \leq u_1 \leq 4u_2$. Using (4.1) to calculate the set of ORDMUs {RD08, RD12 RD10 RD11} which can be reduced to a Reduced Set of Radial DMUs (RSRD) {RD10, RD11}, following the procedure outlined in chapter four, that will simulate the imposed weights restrictions. Figure 5.3 plots the EPPS for this specific example as defined by the RSRDs.

Figure 5.3 - Extended Production Possibility Set



The output levels of RDMU RD10 convey the global extent to which output 1 is preferred over output 2. Thus, if output 1 is less preferred to output 2, as the restriction $u_1 - 0.5u_2 \geq 0$ states, then the output levels of DMU RD10 are (1.30, 10.35). However, had the output weights restriction been $u_1 \geq u_2$, indicating that output 1 is preferred to output 2, then the output levels of the requisite UDMU would be (1.5, 12.0), placing RD10 at RD10'. A much stronger preference of $u_1 - 2u_2 \geq 0$ would lead to a RDMU at (2, 16), which is not plotted in Figure 5.3. The output levels of RDMU RD10 and RD10' were estimated, assuming that the output mix of DMU D10 remains constant.

Alternatively the DM may deem that altering RDMU RD10 from (1.30, 10.35) to (1, 9.5) would be desirable, which provides a UDMU of a different output mix to D10. This leads to a new UDMU at RD10'' (see Figure 5.3). The higher efficiency of DMU D10 under RD10'' rather than RD10, reflects a weaker preference of output 1 relative to output 2.

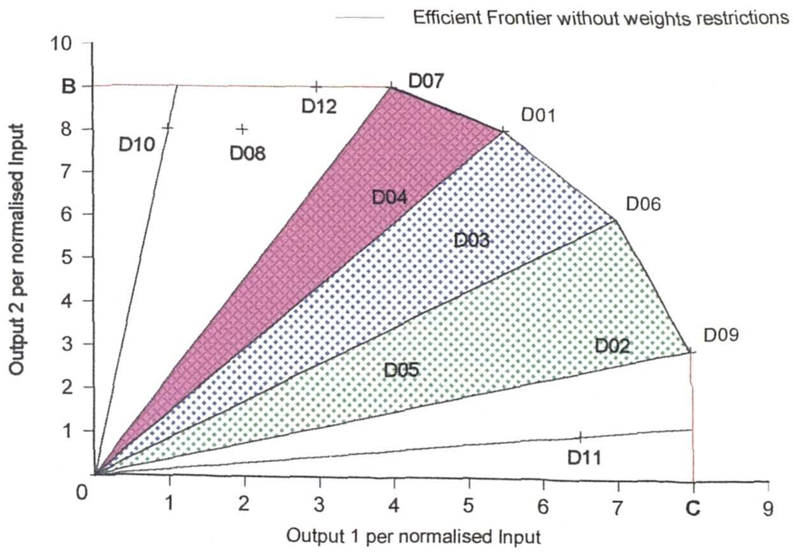
Extending this idea to the multiple input output case, the RSRD will consist of many RDMUs. Thus, it is for the DM to decide how to individually modify these RDMUs as they feel appropriate. In general these modifications will vary from RDMU to RDMU and thus local values are introduced based on the global values imposed by the weights restrictions.

This section has illustrated how to introduce a local single value into the PPS. That is, it has been shown how the DM may take into account the local changes in the values of the inputs and outputs (values specific to the DMUs operating mix) within the PPS in their relative efficiency measure. Suppose that the DM now wants to introduce several local marginal rates of substitution (MRS). That is, take into account the changes in the values of the inputs and outputs as the unobserved frontier is extended. This will be considered in the next section.

5.3.2 Introducing Varying Local Value Judgments

As shown in the previous section, a single local value can be introduced which is acceptable to the DM in terms of the input output levels that are introduced. This, however, raises the question of whether these local values are in line with the observed values? For example, do they apply over similar ranges of input and output levels? If not, then although a single local value may be viable, it may not be truly meaningful in terms of the observed values. To illustrate this consider assessing the DMUs of Table 1.1, the PPS is plotted in Figure 5.4.

Figure 5.4 - Production Possibility Set

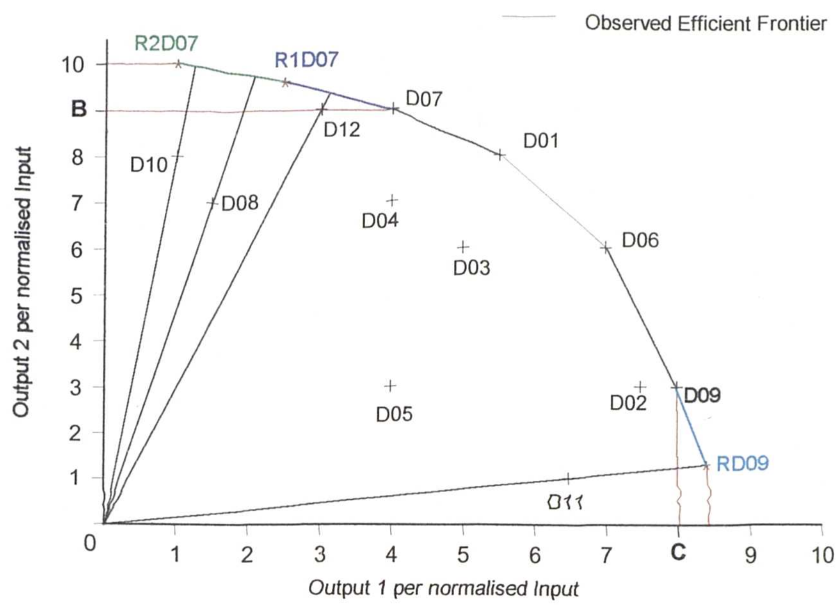


Examining the DEA-efficient frontier and its marginal rates of substitution (MRS), as Figure 5.4 demonstrates, there are three different marginal rates of substitution (MRS), one between D09 and D06 (1 unit of output 1 is worth 3 units of output 2); a second between D06 and D01 (1 unit of output 1 is worth 1.33 units of output 2); and a third between D01 and D07 (1 unit of output 1 are worth 0.67 unit of output 2). Clearly, the value of output 1 relative to output 2 is decreasing as the quantity of its level decreases and appears to change as output 1 level decreases by 1.5 units. However the frontier implicitly introduced by the weights restriction, $u_1 \geq 0.5u_2$ only allows one set of MRS that extends over 3 units of output 1, see Figure 5.3. An extended frontier more in line with the observed frontier would exhibit two MRS over the 3 units of output 1.

The question is, therefore, how to introduce local varying values? Clearly, this will involve the introduction of several additional UDMUs. The question then becomes how to determine input output levels for the additional UDMUs. This can be achieved by modifying the Full Set of Radial DMUs (FSRD) (as defined in chapter four) that simulate the relative efficiency scores under the WRs rather than the RSRD as in the previous section. As detailed in chapter four, the RSRD can be obtained by removing the linearly dependent RDMUs from the FSRD. Instead of removing the linearly dependent RDMUs, the DM is required to modify the input output levels of these RDMUs. In doing this several different local MRS can be introduced, as the RDMUs are no longer linear combinations of other DEA-efficient DMUs.

Consider for example introducing varying local values when assessing the DMUs of Table 1.1, plotted in Figure 5.4. As noted earlier the FSORD consists of {RD12, RD08, RD10, RD11}. However, as detailed above, the observed frontier appears to have a change of MRS approximately every 1.5 units of output 1. Thus, to maintain the continuity of the observed frontier, the extended frontier should have local values that hold over similar ranges of output 1. Imposing the global values in the form of relative restrictions, leads to values that hold over approximately 3 units of output 1, D07RD10. Suppose that the DM considers a single MRS between D07 and RD10 to be unacceptable, and at least two MRS should exist, due to the change in the output 1 level. This can be achieved through the modification of the FSRD. In practice the precise determination of the ranges over which the local values should and do hold is difficult to determine. However, the DM should have an idea of ranges of inputs or outputs over which they would like their local values to hold.

Figure 5.5 - Introducing Varying Values



It shall be assumed that the DM decides to modify RD10 and RD08, (not shown in Figure 5.5) in order to capture two different MRS expressing the preference of the DM for output 1 over output 2. The DM decides that more desirable output levels for the DMUs that express their preferences would be R1D07 (2.3, 9.5) and R2D07 (1, 9.75). Introducing these two UDMUs into the observed data set introduces two MRSs of $u_1 - 0.294u_2 \geq 0$ for output 1 between 2.3 and 4 units of output 1 and $u_1 - 0.192u_2 \geq 0$, for output 1 between 1 and 2.3 units of output 1.

Clearly in the multiple input multiple output case, there will be many more RDMUs that require adjustments to their input output levels in order to express the DM's varying local values. Further, it is not so straightforward to translate the adjustments to the RDMUs back to weights restrictions. However, this graphical example has illustrated that through the combined use of weights restrictions and RDMUs, the DM may include varying local values into the assessment. The values are varying in the sense that as the unobserved frontier created by the RDMUs is extended the marginal rates of substitution that are exhibited between connecting RDMUs are different. The values are local in the sense that the RDMUs from the differing areas of the PPS will have different modifications to their input output levels and thus the marginal rates of substitution between the same inputs and outputs will be different in the different areas of the PPS.

This section has illustrated how the value of the outputs can be varied by DMU comparisons, and how varying local values that are more in line with the DMUs operating processes, can be implemented. The process illustrated above of modifying explicitly the imposed value judgments by means of UDMUs can be applied more generally.

5.4 Conclusion

This chapter has discussed and illustrated that the use of UDMUs as an alternative means to weights restrictions for expressing the DMs values in a DEA assessment is, in principle, a valid one. There are several motivating reasons for their use, as the approach offers the DM a variety of different options for the expression of their values and their inclusion in the assessment to weights restrictions. The key advantages in the use of UDMUs to express values in a DEA assessment are now highlighted:

- Aid the DM in their interpretation of the results

Without the explicit expression of the modified PPS under weights restrictions, the DM cannot be certain as to the meaning of their results. That is, whether they are feasible and, if so, whether the imposed weights restrictions are appropriate. Thus, through the use of UDMUs the DM is given an explicit expression of the input output levels that the DMUs may be measured relative to, in order to obtain their relative efficiency score. Hopefully, this will aid the DM in their interpretation of their results and give them a clearer picture of the impact of the imposed values. At the very least it offers the DM an alternative perspective for the interpretation of the results under weights restrictions.

- Varying Local Values

In general, the observed DMUs have different operating mixes and, therefore, place different emphasis on the inputs and outputs in the assessment. Thus UDMUs allow the

DM to express differences that may exist between the values of the inputs and outputs of the assessment at the various operating mixes.

- Alternative form of expression for value judgments

The approach of weights restrictions requires the precise definition of the relationships between the inputs and/or outputs or maximum/minimum weight values, which may in certain situations be difficult for the DMs to define. Thus UDMUs allow the DM to express their values in an alternative form: DMU comparisons.

This chapter has established, in principle, that there are several advantages to expressing the DM values in the form of the inputs and outputs of the production process. However, throughout this chapter the input output levels of the UDMUs have been determined through the initial introduction of weights restrictions. That is, the UDMUs are dependent on the initial specification and implementation of weights restrictions. Thus, it could be said that the UDMUs have been used to supplement the DM with additional information that may aid them in obtaining more meaningful and useful results.

The next chapter attempts to further develop these concepts of expressing values via UDMU, by proposing an approach for the determination of UDMUs independently of weights restrictions in order to capture value judgments in a DEA assessment. The specific aim of the introduction of these UDMUs will be to improve envelopment, that is they are aimed at simply extending the observed DEA frontier rather than modifying and extending it simultaneously.

Section C
Improved Envelopment Via UDMUs

Having established a viable alternative perspective to current approaches for the inclusion of values in a DEA assessment, this section, which covers chapters six to nine, focuses on developing practical procedures for using this alternative avenue for the expression of value judgments.

This section breaks with the traditional approach of weights restrictions for the inclusion of value judgments in DEA. UDMUs are used to incorporate value judgments in a DEA assessment, with the input output levels of the UDMUs being expressed independently of weights restrictions. Thus the DM directly expresses their values in the form of UDMUs.

The specific aim of this section is to improve envelopment of the DEA-inefficient DMUs, that is, simply extend the frontier rather than modify and extend it simultaneously. Thus, essentially the UDMUs are being used to derive a similar end result to the inclusion of lower bound absolute restrictions, in the assessment, except inputs and outputs are being forced into minimal/maximal contribution levels via the relationships between the inputs and/or outputs of the process under analysis.

This alternative approach provides the DM with the ability to include varying local values and also takes into account the feasibility of the Extended Production Possibility Set (EPPS).

6. Chapter Six
Incorporating Values and Improving Envelopment Via UDMUs: CRS Case[3]

6.1 Introduction

The preceding chapters have demonstrated that the use of Weights Restrictions (WRs) to incorporate value judgments in a DEA assessment implicitly add Unobserved DMUs (UDMUs) to the observed data set, and only allow global values to be introduced into the assessment. Chapter five has demonstrated the advantages of expressing value judgments via UDMUs which have been determined through the initial imposition of weights restrictions. This chapter presents an approach where UDMUs are used directly to incorporate value judgments in DEA assessments, without the use of Weights Restrictions (WRs). That is, rather than use the weights model to introduce value judgments, the envelopment model will be used to introduce UDMUs that explicitly modify the PPS and implicitly restrict the weights.

One approach already exists that works directly on the PPS to ensure that at least a pre-specified number of variables are given more than a minimal weight of ε (see model (M1.3) of chapter one) in computing the efficiency score of a DMU, namely that of

[3] An earlier version of this chapter is under review for publication in **Journal of Operational Research Society**

'Constrained Facet Analysis' (CFA) proposed by Bessent *et al.* [12] and extended as 'Controlled Envelopment Analysis' (CEA) by Lang *et al.* [38]. However, these methods do not take into account the DM's value judgments on the worth of the inputs and outputs in the assessment. They merely ensure inputs or outputs do not receive an ε weight. Consequently these methods assume that the existing observed marginal rates of substitution (MRS) can be extrapolated as far as necessary into unknown production areas. This assumes that the unobserved operating mixes will have the same MRS as certain observed mixes. This may or may not be true as, in general, the observed frontier exhibits a variety of MRS, *thus it is reasonable to assume that unknown production areas will also* exhibit a variety of MRS.

Further, the use of UDMUs in this thesis to capture value judgments resembles, but differs from, that of Golany and Roll [33], who also use the envelopment model to introduce their values. They introduce standard DMUs into the DEA assessment in order to enable the DM to estimate targets for improved performance for all DMUs, including those which would otherwise be DEA-efficient. These standard DMUs are DM specified and are denoted as standard as they are taken to represent ideal standard performances; whereas in this chapter, the approach is attempting to incorporate values that will extend the DEA-efficient frontier, rather than directly modify and narrow it, which is the principal aim of Roll and Golany [42].

The use of UDMUs in this proposed approach has the advantage that an explicit account can be taken of information in respect of any technological or policy limitations on the production process. A further advantage is that an explicit account can also be taken of the value judgments at specific localities of the PPS. Chapter five discussed these advantages in greater detail.

Throughout this chapter the term values is used to represent one of several values, either a marginal rate of substitution, a marginal rate of transformation or a minimal/maximal weight value. DMUs will be used to denote observed DMUs, and UDMUs to denote unobserved DMUs.

The chapter is structured as follows. The second section details the approach to be developed, with two simple illustrations; the third section establishes a need for the procedure; section four details from where to extend the frontier; section five identifies which input output levels to adjust to encourage the non-ε weighting of individual factors; section six provides a basis for determining estimates for DEA-efficient UDMUs; section seven discusses their implementation; section eight summarises the procedure and section nine applies the procedure to a set of bank branches.

6.2 Incorporating Values & Improving Envelopment by Means of UDMUs: An Outline

The aim of this chapter is to construct a set of UDMUs which when introduced into an observed data set will incorporate values and improve envelopment, while placing minimum informational requirements on the DM. It should be noted that the aim is purely to improve envelopment, that is, reduce the number of input output variables that are allocated an ε weight in terms of the weights model. This does not, however, guarantee full envelopment.

In essence the approach aims to obtain a more appropriate measure of efficiency by extending the DEA-efficient frontier by means of UDMUs. That is, suitably defined UDMUs are introduced into the observed data set that will extend the observed DEA frontier in such a manner that it will result in the improved envelopment of DMUs which have unusual input-output mixes. These DMUs could not be enveloped previously due to the lack of suitable DEA-efficient comparator DMUs. Thus this approach considers the problem of the inclusion of values as one of missing data, see Burgess [13]. It attempts therefore to specify input output levels for these missing DMUs, hence providing comparator DEA-efficient DMUs where, at present, none exist.

As minimal information requirements are to be placed on the DM, the estimates for these DEA-efficient UDMUs are to be based on the input output levels of selected DEA-efficient DMUs. Thus, there are two main questions to be addressed in determining the estimates of the DEA-efficient UDMUs:

- Which DMUs to extend the frontier from?
- How to adjust the input output levels of these selected DMUs to derive suitable UDMUs that will improve envelopment?

The approach is aimed at improving envelopment by means of value judgments. Thus it is assumed that the values of the DEA-efficient DMUs are acceptable to the DM, and they are not being asked to directly express their perceived views on the DEA-efficient DMUs. That is, it is concerned with the non-enveloped DMUs, similar to the rates department assessment of Dyson and Thanassoulis [27]. Thus, it was decided that the frontier will be extended from those DMUs that are:

DEA-efficient and delineate the DEA-efficient from the DEA-inefficient parts of the PPS boundary.

These DMUs shall be termed ANCHOR DMUs (ADMUs) and an approach for identifying these ADMUs will be detailed in the next section. Clearly, DMUs can be classed as either ADMUs or non-ADMUs. Having identified these ADMUs, suitable estimates for DEA-efficient DMUs are made by adjusting their input output levels. The exact adjustments to the input output levels are detailed in section 6.5, and are dependent on whether the UDMU to be constructed is attempting to encourage the non-ε weighting of an individual input or output. However, as they are only estimates, their DEA-efficiency cannot be guaranteed and it may be that not all the estimates of DEA-efficient UDMUs will in fact be DEA-efficient.

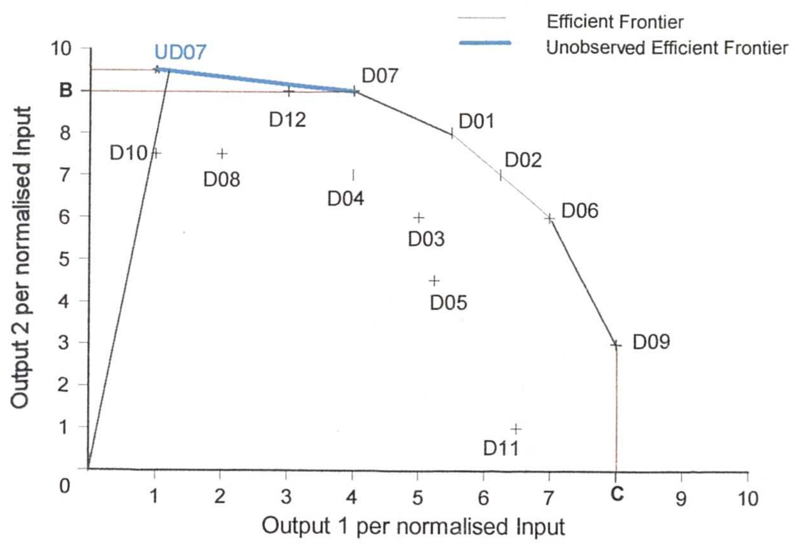
It should be noted that the procedure only considers attempting to prevent individual inputs and outputs from being ignored. That is, how to determine UDMUs that, when added to the data set will, in principle, encourage the non-ε weighting of individual inputs or outputs, one at a time and NOT the simultaneous non-ε weighting of a combination of inputs or outputs. Thus, if a DMU ignores several inputs or outputs, then it may be necessary to simultaneously adjust the input output levels of several of the inputs and outputs of the ADMUs to further decrease the number of their inputs and outputs given an ε weight.

To demonstrate the procedure to be developed for incorporating values and improving envelopment, two graphical examples are considered, one to encourage the non-ε weighting of a single output, and one to encourage the non-ε weighting of a single input.

6.2.1 Encouraging the Non-ε Weighting of an Individual Output

Consider assessing the set of 11 DMUs in Table 1.1 of chapter one, plotted in Figure 6.1.

Figure 6.1 - Extended Production Possibility Set



The DEA-efficient boundary is defined by D07, D01, D06 and D09, the DEA-inefficient boundary segments are CD09 and BD07, with any DMUs that lie on these segments (class F), or which are projected onto these segments (class NF), allocate one of their outputs an ε weight. (Alternatively, it can be said that they have a positive output slack.) It is evident from Figure 6.1 that to ensure proper envelopment, with minimal alteration to the observed frontier, it should be extended from D07 and D09 (both class E), in order to prevent output 1 and output 2 respectively from being ignored in the assessment. Thus D07 and D09 are the ADMUs in Figure 6.1. Now, in order to emphasise the value of output 2 over output 1, consider adjusting the output levels of DMU D07 so that its output 1 is set to the minimum production level deemed to be feasible (because of policy and/or technical reasons). It is assumed that the minimum feasible level of output 1 is 1 unit (per unit of input). Now the DM is requested to determine how to raise the level of output 2 to compensate for this loss of 3 units of output 1. The DM decides that locally no more than 0.5 units of output 2 would be necessary, to compensate for the loss of 3 units of output 1 per unit of input. This means that DMU UD07, which produces 9.5 units of output 2 and 1 unit of output 1 per unit of input, would be deemed by the DM as equally efficient as DMU D07. From Figure 6.1 it can be seen that this is sufficient to ensure the proper envelopment of the previously non-enveloped DMUs D10, D08 and D12.

Hence to encourage an individual output, k to use a non-ε weight, selected ADMUs require the lowering of their output *k* levels.

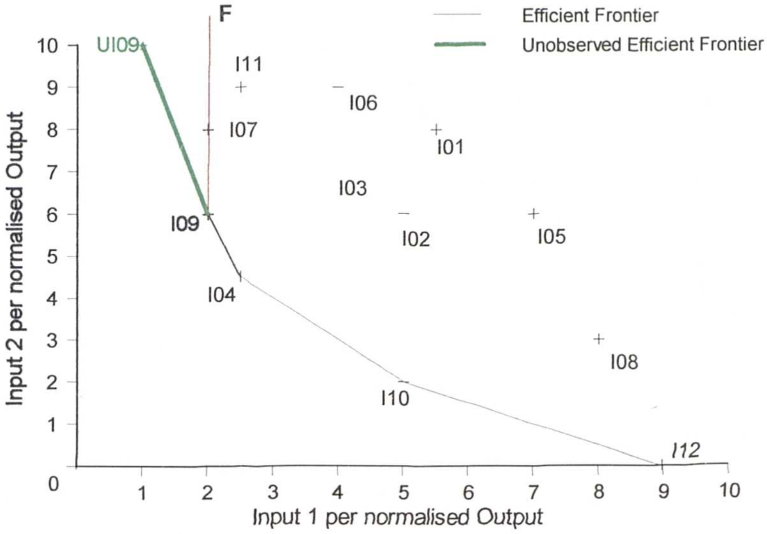
6.2.2 Encouraging the Non-ε Weighting of an Individual Input

Consider assessing a set of 12 DMUs producing a single unit of output and consuming two inputs in the quantities shown in Table 6.1, with Figure 6.2 plotting the PPS generated by these 12 DMUs. It should be noted that in practice I12 is unrealistic but is included in the data set for illustrative purposes only.

Table 6.1 - Example Data Set 3

	I01	I02	I03	I04	I05	I06	I07	I08	I09	I10	I11	I12
Input 1	5.5	5	4	4	7	4	2	8	2	5	2.5	9
Input 2	8	6	7	3	6	9	8	3	6	2	9	0

Figure 6.2 **-** ***Extended Production Possibility Set***



The DEA-efficient boundary is mapped by I12, I10, I04 and I09, with the PPS being bounded by the DEA-efficient boundary and the line FI09. DMUs I07 (class F) and I11 (class NF) attain their *maximum efficiency score by allocating an ε weight to their input 2*, with I09 as their sole peer DMU. Clearly, the frontier needs to be extended from I09 in order to improve envelopment, and so I09 is an ADMU. Suppose that the DM considers adjustments to the input output levels of I09 and deems that UI09 (1,10) would be considered equally as efficient as I09. Thus the DM considers 1 unit of input 1 to have a local value of 4 units of input 2. Thus UDMU, UI09 is sufficient to class I07 and I11 as NE, and ensures that their input 2 receives a non-ε weight.

Thus to encourage the non-ε weighting of an input, a DEA-efficient DMU must be introduced that consumes <u>more</u> of that input than the observed inefficient DMU.

These graphical illustrations demonstrate that DMUs which are non-enveloped have radial projections onto the DEA-inefficient parts of the PPS boundary. Thus, in the general case, the UDMUs to be introduced should at least extend the DEA-efficient part of the PPS boundary to envelop as large a part of the DEA-inefficient boundary of the PPS as possible. A summary of the steps required in the proposed procedure for suitably extending the PPS will now be outlined, with each step explained in a later section of this chapter:

i. Assess the DMUs to determine the DEA-efficient DMUs and the initial envelopment of the DMUs.

ii. Identify the ADMUs.

iii. Identify the individual inputs and outputs of the ADMUs that need to be adjusted in order to improve envelopment.

iv. Construct suitable estimates of DEA-efficient UDMUs.

v. Re-assess the DMUs permitting DMUs and UDMUs to be peers.

vi. If the DM feels the results are unsatisfactory then repeat steps (iv) and (v). Otherwise stop.

Thus, the next section will begin with identifying the DEA-efficient DMUs and the initial envelopment of the inefficient DMUs.

6.3 Assessing Envelopment: Step (i)

The initial step in the procedure is to determine the DEA-efficient DMUs and the envelopment of the DMUs, to establish a need for the procedure. Thus, consider assessing a set of N DMUs, $j=1,\ldots,N$, each using varying amounts of m different inputs, x_{ij}, $i=1,\ldots,m$ to secure varying quantities of s different outputs, y_{rj}, $r=1,\ldots,s$. The DEA weights model yielding the DEA-efficiency score $h^*_{j_0}$ of DMU j_0, under CRS is (see Charnes *et al.* [16])

(M6.1) Weights Model	(M6.2) Envelopment Model
$h^*_{j_0} = Max \sum_{r=1}^{s} u_r y_{rj_0}$	$h^*_{j_0} = Min\ q_0 - \varepsilon\left(\sum_{i=1}^{m} S_i + \sum_{r=1}^{s} S_{m+r}\right)$
$s.t.\quad \sum_{i=1}^{m} v_i x_{ij_0} = 1$	$s.t.\quad q_0 x_{ij_0} - \sum_{j=1}^{N} \lambda_j x_{ij} - S_i = 0 \qquad i=1,\ldots,m$
$\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \le 0 \qquad j=1,\ldots,N$	$\sum_{j=1}^{N} \lambda_j y_{rj} - S_{m+r} = y_{rj_0} \qquad r=1,\ldots,s$
$v_i,\ u_r \ge \varepsilon \qquad \forall i,\ r$	$\lambda_j,\ S_i,\ S_{m+r} \ge 0 \qquad \forall j,\ i,\ r$

In (M6.1) v_i and u_r are the weights attached to the inputs and outputs respectively, ε is a non-Archimedian infinitesimal, see Charnes *et al.* [17]. In (M6.2) S represent slack variables. Let * denote the value of a variable at the optimal solution to the model in which it appears. Let *JE* be the set of DEA-efficient DMUs provided by (M6.1), with a DEA-efficient DMU as defined in chapter one.

Clearly, if all the inefficient DMUs are properly enveloped i.e. $S^*_i = S^*_{m+r} = 0$ for all DMUs in (M6.2) then there is no need for the continuation of the procedure. However, in general, this will not be the case, and so the proposed steps (ii) to (v) provide the DM with a means of including values and improving envelopment.

The next section will outline an approach for identifying which of the DEA-efficient DMUs identified by (M6.1) are ADMUs.

6.4 Identifying Anchor DMUs: Step (ii)

As noted in section 6.2, ADMUs are those DMUs that delineate the DEA-efficient frontier from the DEA-inefficient frontier. Evidently in the graphical illustrations of section 6.2 the ADMUs are easily identified. [It should also be noted that DMUs can be classed either as ADMUs or as non-ADMUs.] Unfortunately, in the multiple input output case, the identification of the ADMUs is not so straightforward. However, ADMUs can be identified using the concept of Super Efficiency (SE), introduced by Andersen and Petersen [6] (see chapter four). Let JE_{j_0} be the set *JE* defined with reference to (M6.1) excluding DMU $j_0 \in JE$. To determine the SE of $j_0 \in JE$ with respect to JE_{j_0} solve the following envelopment model:

$$h_{j_0}^{/} = Min\ z_0 - \varepsilon\left(\sum_{i=1}^{m} H_i + \sum_{r=1}^{s} H_{m+r}\right)$$

$$s.t. \quad z_0 x_{ij_0} - \sum_{j \in JE_{j0}} \tau_j x_{ij} - H_i = 0 \qquad i=1,\dots,m \qquad \text{(M6.3)}$$

$$\sum_{j \in JE_{j0}} \tau_j y_{rj} - H_{m+r} = y_{rj_0} \qquad r=1,\dots,s$$

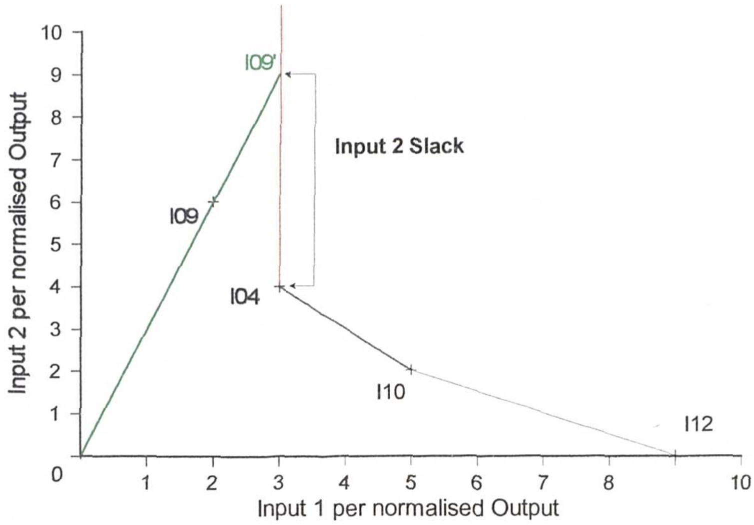
$$\tau_j,\ H_i,\ H_{m+r} \geq 0 \qquad \forall i,\ r,\ j \in JE_{j_0}.$$

H represent slack variables. Let $^{/}$ denote the value of a variable at the optimal solution to model (M6.3).

Model (M6.3) is the Charnes *et al.* [16] basic DEA model but with reference only to the DEA-efficient DMUs identified by (M6.1) excluding DMU j_0. It should be noted that there are two possible outcomes to (M6.3) in terms of the solution, feasible or infeasible.

For those DMUs with feasible solutions to (M6.3), their status as to whether or not they are ADMUs is decided by reference to the classifications of DMUs in DEA introduced by Charnes *et al.* [19]. Firstly, as the ADMUs are to delineate the DEA-efficient from the DEA-inefficient boundary, they clearly must be of class E. Secondly, if they are excluded from the data set, the inefficient facet of the frontier would be altered. This implies ADMUs can be rendered class F with respect to the other DEA-efficient DMUs by radially adjusting their input output levels. That is, when the ADMU is excluded from the constraint set, there exists no suitable DEA-efficient comparators to ensure no positive slack variable exist at the optimal solution to (M6.3). Thus, if a DMU has a feasible solution to (M6.3), then to be considered as an ADMU it must have a SE value in (M6.3) that is greater than 1 and have at least one positive slack variable. Figure 6.3 illustrates the case for I09 of Table 1.1.

Figure 6.3* - *Identifying I09 as an ADMU



Clearly, assessing I09 under (M6.3) will provide an optimal value of greater than one. In addition, there is a positive slack corresponding to its input 2, and hence it meets the conditions to be deemed an Anchor DMU (ADMU). Note that when I09 is assessed under (M6.3) its point of reference for its SE score is I09′, which is a class F DMU. Thus I09 can be rendered a class F DMU with reference to the other DEA-efficient DMUs by radially extrapolating its input levels.

By the nature of an infeasible solution to (M6.3), see Land [37], the DMUs delineate the DEA-efficient from the DEA-inefficient. It implies that the DMU cannot be expressed in terms of the other DMUs. Figure 6.4 demonstrates the case for I12, which consumes 8 units of input 1 and zero units of input 2 per unit of output and yields no feasible solution to (M6.3).

Figure 6.4 - The Super Efficiency of I12

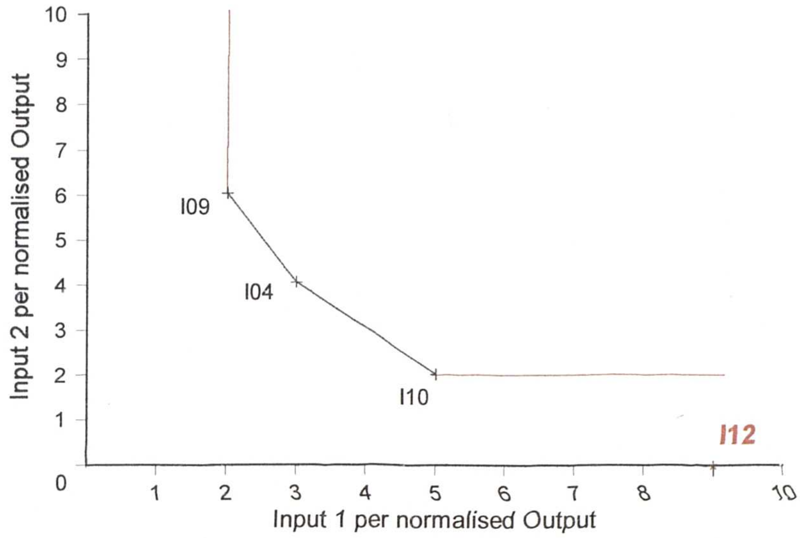


Figure 6.4 illustrates that when assessing the SE of I12 under (M6.3) there is no DEA-efficient frontier against which to measure its SE, and so the model is infeasible. Clearly, DMUs such as I12 in Figure 6.4, with infeasible solutions to (M6.3), delineate the DEA-efficient from the DEA-inefficient frontier *and thus, by definition, they are ADMUs.*

Concluding, in the general case, a set of ADMUs, *JA* can be identified as follows: for all the DEA-efficient DMUs assessed under (M6.3) that satisfy one of the following:

- $h'_{j_0} > 1$, and at least one $H'_i > 0$ or $H'_{m+r} > 0$

or

- (M6.3) has no feasible solution.

For proof of these two statements, see Appendix 6.1

Thus, the process for identifying the ADMUs has been outlined, but there still remains the question of how to determine the input output levels for the UDMUs. The initial consideration in the determination of the input output levels of the UDMUs is how to adjust specific inputs and/or outputs of a ADMU in order to encourage the non-ε weighting of specific inputs and outputs. The next section will address how to identify which of the ADMU's inputs and/or outputs need these specific adjustments.

6.5 Which Input and/or Output Levels of an ADMUs to Adjust? Step (iii)

The graphical illustration of section 6.2 (where there are only two ADMUs and only two possible outputs that can be ignored) demonstrated that in order to encourage the non-ε weighting of input *k*, for a selected ADMU, its input *k* level needs to be raised. Similarly, to encourage the non-ε weighting of output *k* for a selected ADMU, its output *k* level needs to be lowered. However, in the multiple input output case, it is not so straightforward and, in general, the number of ADMUs and number of possible inputs and outputs that can be assigned an ε weight is not coincidental. Hence Proposition 6.1 and Proposition 6.2 state the general cases, for encouraging the non-ε weighting of individual variables.

Proposition 6.1: Encouraging the non-ε weighting of an input

To encourage the non-ε weighting of input *k* for a set of selected ADMUs their input *k* levels need to be raised in order to construct estimates of suitable DEA-efficient DMUs that will, in principle, improve envelopment.

Proposition 6.2: Encouraging the non-ε weighting of an output

To encourage the non-ε weighting of output *k* for a set of selected ADMUs their output *k* levels need to be lowered in order to construct estimates of suitable DEA-efficient DMUs that will, in principle, improve envelopment.

However, the problem of how to identify the set of ADMUs needs to be addressed. That is, how to identify for each ADMU, which of its inputs and/or outputs need to be adjusted as the basis of the construction of UDMUs, that are necessary in order to improve envelopment. It is proposed to identify these inputs and/or outputs by utilising information on the positive slack variables of the class NF and F DMUs, from the initial DEA assessment of (M6.2). The motivation for this is, that although it is known that the ADMUs delineate the DEA-efficient from the DEA-inefficient frontier, it is not known if any DEA-inefficient DMUs are projected onto these DEA-inefficient frontier segments.

Thus combining this information should indicate to the DM whether or not the ADMUs connect to a DEA-inefficient frontier that inefficient DMUs are projected onto.

Let *JIN* be the set of class NF DMUs, with optimal values of q^*_j with reference to (M6.2) corresponding to the assessed DMU j. Construct a set of class F DMUs, with input output levels as defined in (6.1), corresponding to these class NF DMUs.

$$x_{ijf} = q^*_j x_{ij} \qquad y_{ijf} = y_{ij} \qquad j \in JIN \tag{6.1}$$

Thus, the class NF DMUs have their input levels radially reduced in line with their radial DEA-efficiency yielded by (M6.2). Let the set *JF* contain the observed class F DMUs in (M6.2) and the adjusted class F DMUs of (6.1), with $jf=1,\ldots,|JF|$. Let *JA* denote the set of ADMUs, defined with reference to (M6.3). For each $j_0 \in JA$ solve (M6.4).

$$\hat{h}_{j_0} = Min\ z_0 - \varepsilon\left(\sum_{i=1}^{m} H_i + \sum_{r=1}^{s} H_{m+r}\right)$$

$$s.t. \quad z_0 x_{ij_0} - \sum_{j \in JE_{j0}} \tau_j x_{ij} - \sum_{jf \in JF} \tau_{jf} x_{ijf} - H_i = 0 \qquad i=1,\ldots,m \tag{M6.4}$$

$$\sum_{j \in JE_{j0}} \tau_j y_{rj} + \sum_{jf \in JF} \tau_{jf} y_{rjf} - H_{m+r} = y_{rj_0} \qquad r=1,\ldots,s$$

$$\tau_j,\ \tau_{jf},\ H_i,\ H_{m+r} \geq 0 \qquad \forall i,\ r,\ j \in JE_{j_0},\ jf \in JF$$

Notation in (M6.4) as in (M6.3). Let ^ denote the value of a variable at the optimal solution to model (M6.4).

Let AJP_{j_0} denote the set of referent DMUs for ADMU j_0 in (M6.4). If (M6.4) provides a feasible solution, then ADMU j_0 requires adjustments to its input and/or output levels as outlined below.

Stages for identifying which inputs and outputs of the ADMUs to adjust

a) Identify each class F DMU that is a referent DMU to ADMU j_0 in (M6.4), i.e. each $\hat{\tau}_{jf} > 0$ and thus $jf \in AJP_{j_0}$.

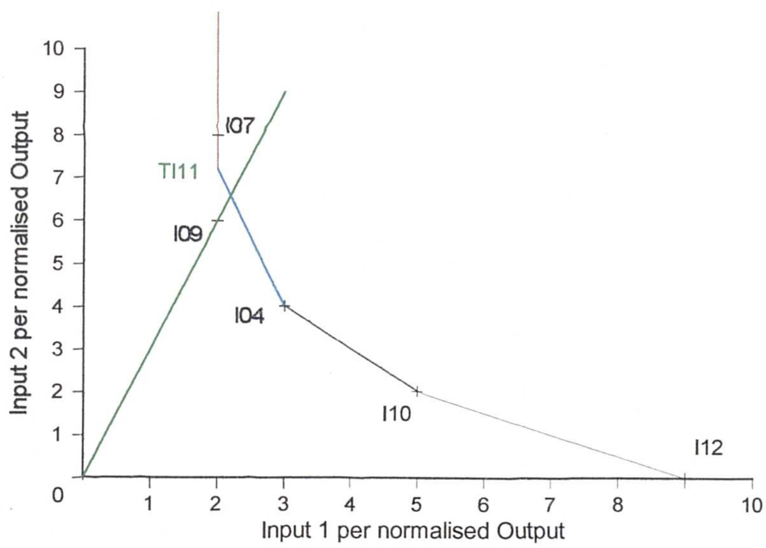
b) For each of these $jf \in AJP_{j_0}$ identify the positive slack variables for their original DMU in (M6.2).

c) For each input or output of the ADMU corresponding to the positive slack variable in (M6.2), at least one estimate of a DEA-efficient UDMU is to be constructed following the initial adjustments outlined in Proposition 6.1 and Proposition 6.2.

For proof that these steps will improve envelopment when solving (M6.2) inclusive of the UDMUs constructed see Appendix 6.2.

To illustrate the procedure, consider the DMUs of Table 6.1, as discussed in Section 6.2.2, where DMU I11 is of class NF. Construct a class F DMU corresponding to this class NF DMU using (6.1), denoted by TI in Figure 6.5. Section 6.4, identified I09 as an ADMU, so DMU I09 is assessed in (M6.4), to determine if any of its inputs and/or outputs need adjusting. Figure 6.5 graphically illustrates this case.

Figure 6.5 - Identifying Which Input Output Levels of I09 to Adjust



Clearly, when I09 is assessed under (M6.4), its referent DMUs are I04 and TI11. Now it is known that TI11 has a positive output 1 slack and hence, ADMU connects to a DEA-inefficient frontier section that has inefficient DMUs either on the frontier or are projected onto the frontier in (M6.2). So, ADMU I09 is required to increase its input 2 level to create an UDMU in order to encourage the non-ε weighting of input 2 in the assessment, as detailed in Section 6.2.2. Further, it should be noted that I09 will be a peer DMU to I11 in (M6.2) as detailed earlier.

It should be noted that this approach may not provide the DM with all possible adjustments to the inputs and outputs of an ADMU in order to improve envelopment, as alternative optimal solutions may exist. Clearly, there may be other possible approaches for identifying the required adjustments to the inputs and/or outputs of an ADMU in order to improve envelopment. Although the procedure presented here will increase the number of enveloped DMUs.

Thus to conclude once a DMU is identified as an ADMU *from model (M6.3) to identify* which the set of specific inputs and/or outputs of each *ADMU* that require adjustments in order to construct suitable estimates for UDMUs that will improve envelopment, model (M6.4) is used. The next section will suggest approaches for compensating for the raising of an input or the lowering of an output in order to construct suitable estimates for the DEA-efficient UDMUs.

6.6 Constructing Suitable Estimates for DEA-Efficient UDMUs: Step (iv)

To enable the DM to construct suitable estimates of DEA-efficient UDMUs that will, in principle, improve envelopment, there are several questions that need to be addressed. What level to raise the inputs to and what levels to reduce the outputs to? How to arrive at these levels? How to maintain the efficiency of the DMU? As these three questions are interlinked, and essentially determine the input output levels of the UDMUs, they will be considered simultaneously. However, the following offers only some general guidelines to the DM for the construction of the UDMUs. The actual adjustments will be for the DM to decide and will be dependent on the relationships between the inputs and outputs of the assessment.

6.6.1 Encouraging the Non-ε Weighting of an Individual Output

To encourage the non-ε weighting of an individual output, DEA-efficient estimates of UDMUs are constructed from an ADMU by initially reducing the level of one of its outputs identified following the procedure outlines in the previous section. Ideally for maximum envelopment output levels will be reduced to zero. However, in many practical contexts, zero output levels are impossible or simply not acceptable. For example, if in an assessment of a school's efficiency, one of the outputs is the mean mathematics score of pupils at exit, a zero level is not very likely even if feasible in principle. Thus each output will be reduced to a minimal production level determined by the DM, given the technical constraints or management policies, and the input levels of the ADMUs.

Further, the DM may feel that to reduce the output level of the ADMU directly to its minimum level would encompass several different values (marginal rates of substitution and/or transformation). That is, the DM feels that to reduce the output directly to a minimum level would only introduce a single value, whereas in reality several values may exist over the output levels that the reduction encompasses. Thus the DM would prefer to reduce the ADMUs output level in stages, with different values introduced at each stage. These different values are introduced by varying the adjustments to the remaining input and/or output levels of the ADMU in order to compensate for the reduced output production. This leads to the question of how to compensate for the reduced output level. Essentially there are two approaches:

- Raising of the other output levels

As demonstrated graphically in the two output case, if one output level is reduced, then to maintain efficiency, the output level of the other output must be raised. Generalising, a decrease in the level of one output will require an increase in production of some or all of the remaining $s-1$ outputs in order to maintain the DMU's efficiency. The rise of level in some or all of the remaining $s-1$ outputs will be dependent on the relationships between the outputs and between the input and the outputs. For example, if the level of output l is dependent on the level of output k, then the reduction in level of output k will automatically lead to the reduction in output l. Furthermore, the DMs may or may not have a preference over the relative changes in the output levels, in that they may want an increase in level of specific outputs in preference to a global increase in the $s-1$ outputs.

- Lowering of input levels

Alternatively the reduction in output k could be compensated for by reducing the level of the input used to produce this output. Obviously, this reduction will depend on the relationship between the reduced output and the input, and will be DM defined. Further, if as mentioned above, any of the remaining $s-1$ outputs are dependent on the reduced output, then their level will also have to be reduced. Consequently, the input level will have to be further reduced to take into account this additional decrease in output.

6.6.2 Encouraging the Non-ε Weighting of an Individual Input

To encourage the non-ε weighting of an individual input the DEA-efficient estimates of the UDMUs are constructed from an ADMU by increasing the level of one of its inputs as outlined in the previous section. The level by which this input is raised will depend on technical constraints and managerial policy. In addition, if the consumption level is increased in steps, a variety of different values can be incorporated into the assessment, provided that the adjustments to the other inputs and outputs of the ADMU vary from step to step in order to compensate and maintain efficiency. Thus the question that remains is how to compensate for the increase in input in order to maintain efficiency? There are two approaches:

- Raising of output levels

Intuitively, it is reasonable to expect an increase in output as the direct consequence of an increase in input. Thus UDMUs can be created by increasing the level of the related outputs to account for the increased amount of input. These increases will be dependent on the existing relationships between the inputs, with the increase in one input possibly implying an increase/decrease in a dependent input. This approach simultaneously incorporates information on the relationship of the inputs and outputs.

- Lowering of the other input levels

In the multiple input output case, the rise in one input level needs to be compensated for by a lowering of some or all of the *m-1* inputs, taking into account the relationship between the inputs and the outputs. Further, the DMs may or may not have preferences over the relative change in the input levels, which will effect the adjustments, as in the output case.

The DM is now in a position to provide the essential information required to construct the necessary UDMUs. Clearly, the determination of the input output levels of the UDMUs is for the DM to decide. This will depend on their values and the existing relationships between the inputs and/or outputs. However, the value judgments that are extracted at this stage are only implicitly felt by the DMs and they will need to be helped to articulate them.

For simplicity, to check if the specified UDMUs are DEA-efficient, the DM may assess the UDMUs relative to the DEA-efficient DMUs only. If it is found that some of the UDMUs are DEA-inefficient then their input output levels may be adjusted. Further, DEA-efficiency of a UDMU is not sufficient to guarantee the UDMU will improve envelopment; this depends on the input output levels of the UDMU.

6.7 Implementation: Step (v)

Once the ADMUs relating to a set of *N* DMUs have been identified and their associated UDMUs created, following the steps of the previous sections, the DMUs can be assessed using model (M6.1), permitting UDMUs as well as DMUs to be peers. The number of properly enveloped DMUs should be larger than in the absence of the UDMUs. This follows from Appendix 6.2. However, full envelopment is not guaranteed. The increase in the envelopment of DMUs will depend on the input output levels of the UDMUs in relation to the DMUs. If the DM feels that the improvement to envelopment is not sufficient, then the UDMUs may be modified and/or new UDMUs added using the same process.

The next section will summarise the procedure.

6.8 Incorporating Values & Improving Envelopment by Means of UDMUs: A Summary

Consider a set of N DMUs using m inputs, x_{ij}, $i=1,\ldots,m$ to produce s different outputs y_{rj}, $r=1,\ldots,s$. The following steps can increase the number of properly enveloped DMUs in assessments of DEA efficiency, but does not guarantee full envelopment.

i. Model (M6.1) is used to identify the set of DEA-efficient DMUs *JE*. The DMUs in *JE* are of class E and E' as defined by Charnes *et al.* [19]. If all DMUs $j \notin JE$ are properly enveloped as defined in Lang *et al.* [38] stop. Otherwise go to (ii).

ii. For each $j \in JE$ solve model (M6.3) to determine $h_j^{/}$ as defined in that model. The set of ADMUs *JA*, can be identified from this model with $JA = \{j|\ h_j^{/} > 1$, and at least one $H_i^{/} > 0$ or $H_{m+r}^{/} > 0$, or DMU j has no feasible solution$\}$.

iii. In respect of each $j \in JA$ solve (M6.4) and identify the inputs and outputs of each ADMU that require necessary adjustments as outlined in Section 6.5. Using Proposition 6.1 and Proposition 6.2 to initiate the construction of at least one UDMU.

iv. In respect of each ADMU, for each output and input identified in step (iii) at least one UDMU is constructed. The construction of each UDMU is DM defined and based on the their local values, the relationship between the inputs and outputs and any technological and policy constraints that may exist.

v. Assess the DMUs using model (M6.1) but permitting both DMUs and the UDMUs created in step (iv) to be peer DMUs. The number of properly enveloped DMUs should be greater than the number initially found in step (i).

vi. If the DM wants to see a further increase in the number of DMUs enveloped, repeat steps (iv) and (v). Otherwise stop.

The next section demonstrates the use of the aforementioned process on a real data set.

6.9 An Application of the Use of UDMUs to Incorporate Values and Improve Envelopment in DEA

In this section the use of UDMUs to improve envelopment in DEA will be illustrated by applying the theory to a real data set, where the DMUs consume multiple inputs in order to produce multiple outputs. For an illustration of a single input multiple output case, and as a means of comparison to the approach detailed in this chapter to the approach of absolute weights restrictions, see Allen and Thanassoulis [5].

The following is an example to illustrate the proposed approach for incorporating value judgments in a DEA assessment under CRS. A set of 668 bank branches shall be assessed using the following set of input and output variables, which were selected by the DM and are deemed to provide appropriate estimates of resource efficiency given the limited available data. That is, the extent to which the resources of a branch can be reduced, while maintaining their current level of sales/services. See Berger and Humphrey [11], for a survey of performance measurement in banks.

Table 6.2 - The Inputs and Output Used to Assess the 668 Bank Branches

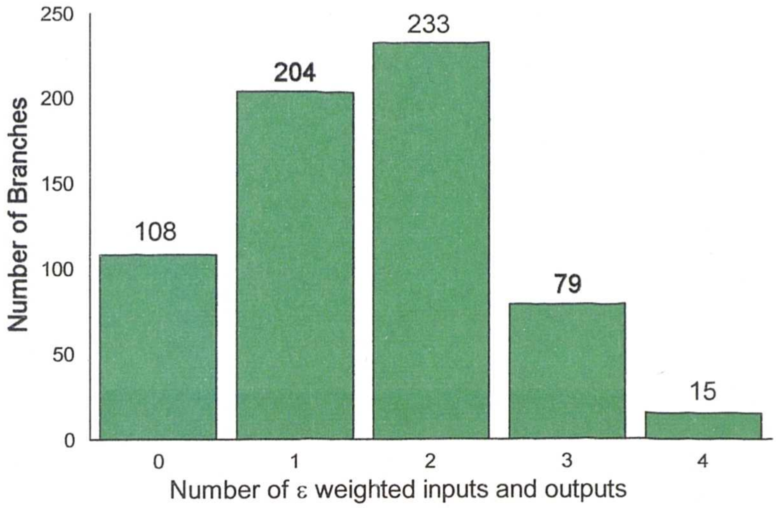
Inputs	Outputs
1. Total Costs (TC)	1. Number of Mortgages Applications (MT)
2. Number of Facilities (FA)	2. Number of Protection Applications (AP)
	3. Number of Insurance Applications (AI)
	4. Number of New Saving Accounts (SV)
	5. Number of Counter Transactions (CT)

The first input is the total staff costs incurred at each branch; the second input is the number of facilities (e.g. computers, desks etc.) at each branch. The first output is the number of mortgage applications made; the second output is the number of applications for protection sales that are made, e.g. life and medical insurance and are sold only by trained staff; the third output is the number of investment applications that are made, which are regulated sales e.g. peps and unit trusts, and are sold only by trained staff; the fourth output is the number of new saving accounts made, and the last output is the number of counter transactions made, e.g. paying in/withdrawal transactions at the tills. The actual input output levels of the branches can be found in Appendix 6.3.

Step (i)

The initial envelopment of the branches was assessed by solving model (M6.1). It was found that 29 branches were DEA-efficient, and so there were 639 inefficient branches. Figure 6.6 summarises the number of inputs and outputs assigned an ε weight in the assessment for each of the 639 inefficient branches. As there are a total of 7 variables, it is possible for a DMU to assign a maximum of 5 ε weights. However, it was found that only a maximum of 4 ε weights were assigned by the inefficient branches. With an ε weighting of a factor in the weights model being equivalent to a positive slack value in the envelopment model. [There may have been alternative optimal solutions which change the envelopment of the DMUs as shown in Figure 6.6, but in general the number of DMUs would not be properly enveloped.]

Figure 6.6 - Number of ε Weighted Input and Output Variables Per DEA-Inefficient Branch in (M6.1)



Although there are a large number of branches that are properly enveloped, the majority of the inefficient branches do not use all their inputs and outputs to determine their relative efficiency score and, in general, at least 1 or 2 of their inputs and outputs are allocated an ε weight, i.e. are essentially ignored in the analysis. Hence, it could be concluded that the attained scores do not reflect the true inefficiency of the majority of branches. That is, for the majority of branches, their true inefficiency is higher than it actually indicated by their DEA-inefficiency because the latter effectively ignores several inputs and outputs. Thus this step has established the need for a procedure to improve the envelopment of the branches. It is worth noting here that the 29 DEA-efficient branches enveloped only 108 inefficient branches, which is a ratio of approximately 1 to 4.

At present, the DM does not use weights restrictions, because it is felt that there is a lack of objectivity in the setting of restrictions, especially as the weights cannot be used to represent marginal rates of substitution, so relative restrictions were inappropriate. However, it was felt by the DM that some means of introducing values into the assessment was required in order to obtain relative efficiency scores that were more representative of a branch's true efficiency.

Independently of their DEA analysis the organisation conduct an Activity Based Costing (ABC) analysis, see Johnson and Kaplan [36] on their branches. ABC is an accounting based assessment aimed at aiding the DM in improving the relative efficiency of their branches. This involves gathering information on the costs assigned to the outputs, and it was found that this existing information could be readily used to determine the input output levels for the unobserved branches. The DM felt there may be problems relating the results to the bank managers, but as these unobserved branches were to be based on information collected for their ABC assessment, it was felt they would have credibility. Thus it has been established that there existed both a need for the procedure and a suitable approach for the DM.

Step (ii)

The identification of the anchor branches of the assessment, was achieved by solving model (M6.3). It was found that 27 of the 29 DEA-efficient branches were anchor branches.

Step (iii)

Having identified the potential branches for the basis of the unobserved branches, there now remains the question of which inputs and outputs of these anchor branches require adjustments in order to improve envelopment. To illustrate this step of the approach, the assessment of branch D150 under (M6.4) shall be considered. Firstly, model (M6.4) was solved in order to determine which of the radially adjusted class F DMUs were D150's referent DMUs. It was found that the observed inefficient branches from (M6.1) which correspond to those radially adjusted class F branches identified in (M6.4) as D150's referent branches were: D046, D246 and D308. Secondly, the slack values for each of these 3 branches in (M6.2) were identified and are shown in Table 6.3. These are used as the basis for the construction of any unobserved branches to be based on the input output levels of D150.

Table 6.3 - Results of Step (iii) for Branch D150

Observed class NF branch corresponding to the referent branch in (M6.4)	Positive slacks in (M6.2)
D046	CT
D246	AI & SV
D308	TC, AI & SV

Hence for branch D150 a minimum of 4 unobserved branches needed to be constructed, as 3 class NF branches have in total 4 different inputs and outputs with positive slack values in (M6.2). Table 6.4 shows the basis for the construction of these 4 unobserved branches.

Table 6.4 - The Basis for the Construction of the Unobserved Branches Based on Branch D150

Constructed unobserved branch	Basis of unobserved branch
A1D150	Raising of D150's TC Level
A2D150	Lowering of D150's AI Level
A3D150	Lowering of D150's CT Level
A4D150	Lowering of D150's SV Level

For the details of which input output levels of each anchor branch are to be raised or lowered, as required, in order to construct at least one unobserved branch as appropriate see Appendix 6.4. However, it should be noted that it was found that only 23 anchor branches required adjustments to their input output levels. The remaining 4 anchor branches only had anchor branches as their referent branches in (M6.4).

It was decided by the DM to initially only introduce one unobserved branch per input raised or output lowered. That is, only one local value is to be introduced in extending the PPS, so the minimum number of unobserved branches (75) were constructed.

Turning to the actual construction of the unobserved branches, the DM felt it was unrealistic for an unobserved branch to have zero levels for any of the outputs. Thus, an interesting point here is to note the maximum observed input and minimum observed output levels, as they may be a useful source of information for the consideration of appropriate levels to raise the inputs and lower the outputs. Table 6.5 displays these maximum observed input and minimum observed output levels.

Table 6.5 - Observed Maximum Input Levels and Minimum Output Levels

Inputs			Outputs				
TC	FA		AI	AP	CT	MT	SV
426178	20		2	5.8	33077	11	459

Step (iv)

As mentioned previously, the information collected by the DM for their Activity Based Costing (ABC) analysis, see Johnson and Kaplan [36], could easily be used as a basis for the determination of the unobserved branches, as the information establishes relationships between the inputs and outputs.

As the number of unobserved branches to be determined is large, their construction shall be demonstrated by the construction of the unobserved branches based on branch D150. As demonstrated earlier 4 unobserved branches A1D150, A2D150, A3D150 and A4D150 are to be constructed based on D150's input output levels. The input output levels for these unobserved branches are shown in Table 6.6 and they were decided by the DM, drawing from their ABC accounting information on the inputs and outputs of the assessment.

Table 6.6 - Input Output Levels of the Unobserved Branch Based on D150

	FA	TC	AI	AP	CT	MT	SV
D150	13	309822	219.8	219.2	305347	384	3902
A1D150	14.0	364145	248	245	331254	421	4125
A2D150	12.0	268966	2	174	29700	332	3816
A3D150	10.8	202908	179	174	3000	324	2654
A4D150	11.8	239765	187	185	245120	345	450

Clearly, for A1D150, the DM felt that a raise in the total costs at branch D150 would be compensated by a general increase in all the outputs and the number of facilities at the branch. Similarly, in lowering the individual outputs of the branch, the DM felt this would impact on all the other outputs and the inputs of the branch, as can be seen from the construction of A2D150, A3D150 and A4D150. The input output levels of the final non-redundant unobserved branches can be found in Appendix 6.5.

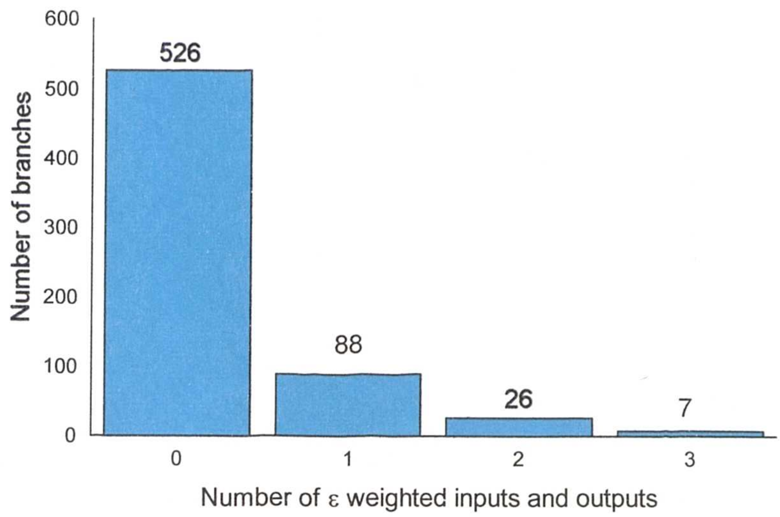
It should be noted here that the input output levels of the unobserved branches shown in Appendix 6.5 were not the initial estimates. They are the result of several modifications to their initial specification following their assessment under model (M6.1). Further, although their DEA-efficiency was secured, it is not sufficient to guarantee that they will contribute to improvement in the envelopment of the branches. In particular, if the operating mixes of the unobserved branches are not similar to those of the non-enveloped inefficient branches, then they may not impact on the envelopment of the non-envelopment branches. This is difficult to ensure in practice.

Step (v)

Finally, the DM is now in the position to re-assess the observed branches permitting both the observed and unobserved branches to be used as peer branches in (M6.2). The inclusion of the unobserved branches has also discriminated between the observed DEA-efficient branches. The inclusion of the 75 unobserved branches has reduced the number of observed DEA-efficient branches from 29 to only 19 and as expected, in general, the relative efficiency scores are lower in the presence of the unobserved branches.

However, the main aim of this procedure was to improve envelopment and incorporate values, thus the effect of the unobserved branches on the envelopment of the observed branches needs to be considered. It should be noted that only 48 of the unobserved branches were actually used to improve the envelopment of the 531 initially non-enveloped inefficient branches. Figure 6.7 summarises the envelopment of the 649 inefficient branches, with the inclusion of the 75 unobserved branches. [There may have been alternative optimal solution which change the envelopment of the DMUs, as shown in Figure 6.7, but in general the number of enveloped DMUs would still be vastly increased.]

Figure 6.7 **- *Number of ε Weighted Input and Output Variables Per DEA-Inefficient Branch with an Extended Data Set in (M6.1)***



Clearly, the number of properly enveloped branches has been vastly improved by the introduction of the unobserved branches from 108 to 526. Thus the scores should reflect more appropriate efficiency estimates.

It should be noted that although the construction of these unobserved branches appears to be a rather large number, it is not unexpected. Consider the number of DEA-efficient branches (observed and unobserved) to the number of properly enveloped branches, approximately, 1 to 8 compared to the initial ratio of 1 to 4. So, the number of unobserved branches constructed in order to improve envelopment is not an unusually large number.

6.10 Conclusion

This chapter has developed a procedure for incorporating values aimed at improving the number of properly enveloped DMUs in DEA assessments where the DMUs operate under CRS. Properly enveloped DMUs have no positive slack values at their optimal solution. Hence, all their inputs and outputs are taken into account in assessing their performance.

A key feature of the procedure developed, and its difference from previous approaches to ensure DMUs assign realistic weights to their inputs and outputs, is that it implicitly restricts rather than explicitly restricts the DEA-weights, by using the envelopment model rather than the weights model to achieve the end result. The UDMUs are created as local variations of DEA-efficient DMUs. These local variations are DM defined, thus, hopefully, they are reasonable extensions to the observed PPS. One way to look at the approach developed here is to say that it attempts to fill in for missing data by asking the DMs about potentially efficient input output levels close to certain DEA-efficient DMUs. The DMs are only required to provide values on a local level, i.e. for specific DMUs only. These judgments can be in the form of comparing a DMU with an UDMU or by offering MRS. Such local information may prove easier for some DMs to provide than the alternative of specifying global values.

The procedure does not guarantee full envelopment of all DMUs, as the judgments provided by the DMs throughout the procedure cannot be guaranteed to always lead to DEA-efficient UDMUs, as required. *However, the procedure does provide a mechanism* whereby the information provided by the DMs can be built upon, in order to modify the PPS, thereby improving the envelopment of the DMUs in the DEA assessment. In this context the use of UDMUs has some important advantages over the more traditional use of weights restrictions to capture value judgments in DEA.

- The trade offs the DMs are asked to make between output and/or input levels are local to the part of the PPS in which the ADMU is located.
- The trade offs between output and/or input levels can be given by the DMs in the form of comparisons of the ADMU with trial UDMUs.
- The modification to the PPS has been made explicit and therefore the feasibility of the EPPS can be considered; whereas under weights restrictions, the modification of the PPS is implicit and therefore its feasibility is not considered.

Some DMs may find the use of comparisons of DMUs easier than the specification of global values (MRS, MRT or maximum/minimum weight values). However, it should be noted that the subjective nature of the information provided by the DMs can mean that some of the UDMUs created may not be DEA-efficient. These UDMUs will be redundant and will not increase the number of properly enveloped DMUs, as originally intended.

The next chapter details the DEA Variable Returns to Scale (VRS) model and how Radial DMUs (RDMUs) can be used to simulate weights restrictions.

7. Chapter Seven
Data Envelopment Analysis Under Variable Returns to Scale With and Without Values

7.1 Introduction

The preceding chapters of this thesis have only considered the incorporation of value judgments in a DEA assessment when the DMUs are operating under CRS. The focus will now turn to the case where the DMUs operate in a Variable Returns to Scale (VRS) environment. Hence it is necessary to first detail the standard VRS DEA assessment, which was introduced by Banker *et al.* [7]. This model differs from the Charnes *et al.* [16] model in that it assesses a DMU given its scale of operation.

This chapter is structured as follows: Section two details the VRS DEA model; section three illustrates the PPS under VRS; section four details how weights restrictions can be simulated under VRS; section five highlights some of the problems encountered when using weights restrictions in a VRS DEA assessment.

7.2 The Variable Returns to Scale DEA Model

Consider assessing a set of N DMUs, $j=1,\dots,N$ each consuming in varying amounts m different inputs, x_{ij}, $i=1,\dots,m$ to produce in varying quantities s different outputs, y_{rj}, $r=1,\dots,s$. The DEA weights models (M7.1) and (M7.2) provide the relative efficiency score of DMU j_0 with an Input Minimisation (IM) and Output Maximisation (OM) orientation respectively. In general, under the VRS assumption, for inefficient DMUs $\hat{h}_{j_0} \neq 1/\bar{e}_{j_0}$. Thus care must be taken by the DM in selecting the appropriate model orientation for the assessment.

(M7.1) Input Minimisation	(M7.2) Output Maximisation
$\hat{h}_{j_0} = Max \sum_{r=1}^{s} u_r y_{rj_0} + \omega$	$\bar{e}_{j_0} = Min \sum_{i=1}^{m} \rho_i x_{ij_0} - \varpi$
$s.t. \quad \sum_{i=1}^{m} v_i x_{ij_0} = 1$	$s.t. \quad \sum_{r=1}^{s} \delta_r y_{rj_0} = 1$
$\sum_{r=1}^{s} u_r y_{rj} + \omega - \sum_{i=1}^{m} v_i x_{ij} \leq 0 \quad j=1,\dots,N$	$\sum_{r=1}^{s} \delta_r y_{rj} - \sum_{i=1}^{m} \rho_i x_{ij} + \varpi \leq 0 \quad j=1,\dots,N$
$v_i, u_r \geq \varepsilon \quad \forall i, r$	$\rho_i, \delta_r \geq \varepsilon \quad \forall i, r$
ω free	ϖ free

In (M7.1) v_i and u_r are the weights attached to the inputs and outputs respectively and in (M7.2) ρ_i and δ_r are the weights attached to the inputs and outputs respectively. In (M7.1) and (M7.2) ω and ϖ respectively can be used to ascertain the nature of returns to scale efficient DMUs. Table 7.1 shows how to identify the nature of returns to scale of a DEA-efficient DMU, following Banker and Thrall [8].

Table 7.1* - *How to Identify Returns to Scale of DEA-Efficient DMUs

Value of ω or ϖ	Nature of Returns to Scale
$\omega > 0$ or $\varpi > 0$ at ALL multiple optimal solutions	Increasing
$\omega = 0$ or $\varpi = 0$ at ANY optimal solution	Constant
$\omega < 0$ or $\varpi < 0$ at ALL multiple optimal solutions	Decreasing

The DEA dual models corresponding to these DEA weights models, (M7.1) and (M7.2), are known as the envelopment models and they are:

(M7.3) Input Minimisation		(M7.4) Output Maximisation	
$\hat{h}_{j_0} = Min\ \phi_0 - \varepsilon\left(\sum_{i=1}^{m} H_i + \sum_{r=1}^{s} H_{m+r}\right)$		$\bar{e}_{j_0} = Max\ \theta_0 + \varepsilon\left(\sum_{i=1}^{m} G_i + \sum_{r=1}^{s} G_{m+r}\right)$	
s.t. $\phi_0 x_{ij_0} - \sum_{j=1}^{N} \lambda_j x_{ij} - H_i = 0$	$i=1,\ldots,m$	s.t. $\sum_{j=1}^{N} \tau_j x_{ij} + G_i = x_{ij_0}$	$i=1,\ldots,m$
$\sum_{j=1}^{N} \lambda_j y_{rj} - H_{m+r} = y_{rj_0}$	$r=1,\ldots,s$	$\theta_0 y_{rj_0} - \sum_{j=1}^{N} \tau_j y_{rj} + G_{m+r} = 0$	$r=1,\ldots,s$
$\sum_{j=1}^{N} \lambda_j = 1$		$\sum_{j=1}^{N} \tau_j = 1$	
$\lambda_j, H_i, H_{m+r} \geq \varepsilon$	$\forall j, i, r$	$\tau_j, G_i, G_{m+r} \geq \varepsilon$	$\forall j, i, r$

The sum of the λ and τ are set to one to prohibit the extrapolations of scales of operation. Let * denote the value of a variable at the optimal solution to the model in which it appears.

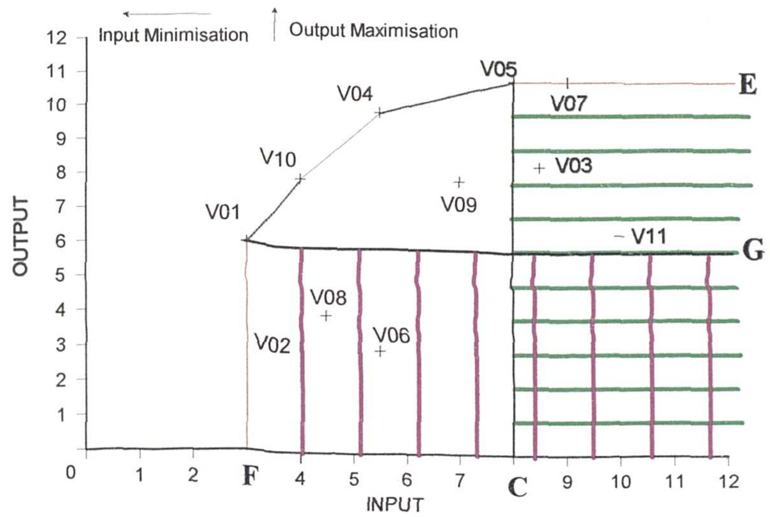
To illustrate the generated DEA PPS when the DMUs operate under VRS, consider assessing a set of 11 DMUs each consuming a single input to produce a single output. The actual input output levels are shown in Table 7.2 along with their DEA efficiency scores, to illustrate that in general for inefficient DMUs, $\hat{h}_{j_0} \neq \frac{1}{\bar{e}_{j_0}}$

Table 7.2* - *Example Data Set 4

DMUs	Input	Output	IM, $\hat{\phi}$	OM, $\overline{\theta}$
V01	3	6	100.00	100.00
V02	3	3.5	100.00	58.33
V03	8.5	8.5	51.47	77.27
V04	5.5	10	100.00	100.00
V05	8	11	100.00	100.00
V06	5.5	3	54.55	30.00
V07	9	11	88.89	100.00
V08	4.5	4	66.67	46.15
V09	7	8	57.14	75.47
V10	4	8	100.00	100.00
V11	10	6.5	32.50	59.09

Figure 7.1 plots the Production Possibility Set for the DMUs.

Figure 7.1* - *Production Possibility Set



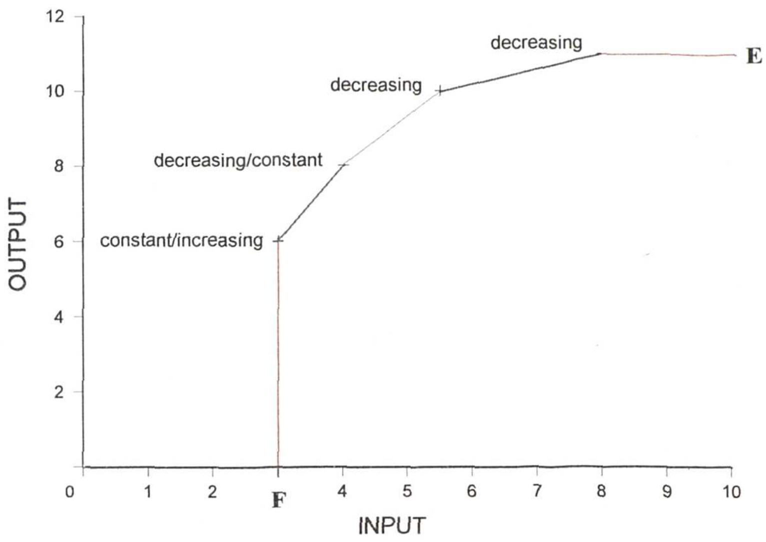
The DEA-efficient boundary is mapped by V01, V10, V04 and V05, with the PPS being defined by the DEA-efficient frontier, the input axis and the lines FV01 and EV05. Under an IM model DMUs V02, V08 and V06 allocate their output level an ε weight. (Alternatively they have a positive output slack.) Thus their relative efficiency score is the ratio of a value representing their scale of operation, i.e. ω in (M7.1), and their weighted input. More generally, any inefficient DMU in this assessment that lies below the line GV01, will allocate its output an ε weight. Under an OM model, DMUs V03, V07 and V11 allocate their input level an ε weight, and their relative efficiency score is the ratio of a value representing their scale of operation i.e. ϖ in (M7.2) and their weighted output. In general, for this assessment, any inefficient DMU that lies to the right of the line CV05 will allocate their input an ε weight.

Generalising to the multiple input multiple output case any inefficient DMUs that lie on or are projected onto an inefficient frontier facet, allocate at least one input or output an ε weight. Alternatively and equivalently these DMUs have at least one positive slack value in the envelopment model. Thus, this simple graphical example illustrates that under VRS the envelopment of the DMUs is dependent on the model orientation. This is due to the fact that the projection of the DMUs onto the DEA-frontier is dependent on the model orientation. However, the defined DEA-efficient frontier is coincidental for the two orientations. Further, the complete weights flexibility offered by the VRS DEA model may lead to inappropriate estimates of efficiency. In extreme cases, relative efficiency scores can be obtained that are based on the DMUs scale of operation and a single weighted input or output. This is not even dependent on any input to output ratio, which is the essence of an efficiency measure. That is, the efficiency score is determined by the DMU's weighted scale to one of its weighted inputs/outputs in the IM/OM case respectively. Thus the score cannot truly be deemed an efficiency score. Evidently to overcome this problem, additional information must be added to the model. This could either be in the form of weights restrictions or the introduction of additional DEA-efficient DMUs.

Although the envelopment of the DMUs is model orientation dependent, the definition of a properly enveloped DMU is independent of the model orientation. A properly enveloped DMU is one that has no positive slack values in their DEA envelopment model (M7.3) or (M7.4) as solved, i.e. $H^*_i = H^*_{m+r} = 0$ or $G^*_i = G^*_{m+r} = 0$ respectively $\forall i$ and r.

Under VRS, as well as exhibiting varying marginal rates of substitution, the DEA-frontier exhibits varying rates of returns to scale. Figure 7.2 illustrates the returns to scale exhibited by the production frontier generated by the DMUs of Table 7.2. Clearly, the production frontier exhibits increasing, constant and decreasing returns to scale. The scale value given for the inefficient DMUs is the scale that they would be operating at if they were efficient.

Figure 7.2 - Variable Returns of the Production Possibility Frontier



Having considered the standard VRS DEA models, the next section will consider how UDMUs can be used to simulate weights restrictions under VRS. The purpose of this is to allow for the comparison of the two approaches for capturing value judgments (see chapter nine).

7.3 Simulating Weights Restrictions for DMUs

As already detailed, under VRS the model orientation becomes significant, with the different models providing different relative efficiency scores. Thus, in all cases, the weights restrictions will have a different impact on the relative efficiency scores. Therefore a different set of Radial DMUs (RDMUs) is required to simulate the relative efficiency scores under each orientation. This in turn implies that the RDMUs have to be constructed in a specific manner in order to simulate the imposed WRs, i.e. Input Radial DMUs (IRDMUs) for an IM model and Output Radial DMUs (ORDMUs) for an OM model. The simulation of WRs under VRS will now be consider, essentially this is the same as the simulation of weights restrictions under CRS, except for the above mentioned specification of the necessary Radial DMUs.

Consider assessing a set of N DMUs the jth using input levels x_{ij}, $i=1,...,m$ to produce output levels, y_{rj}, $r=1,...,s$, with additional constraints on the weights, r1-r5 in various forms of weights restrictions. [r1-r2: relative restrictions, r3: linked-dependent restrictions, r4-r5: absolute restrictions.]

(M7.5) Input Minimisation			(M7.6) Output Maximisation		
$h^*_{j_0} = Max \sum_{r=1}^{s} u_r y_{rj_0} + \omega$			$\tilde{h}_{j_0} = Min \sum_{i=1}^{m} \rho_i x_{ij_0} - \varpi$		
s.t. $\sum_{i=1}^{m} v_i x_{ij_0} = 1$			s.t. $\sum_{r=1}^{s} \delta_r y_{rj_0} = 1$		
$\sum_{r=1}^{s} u_r y_{rj} + \omega - \sum_{i=1}^{m} v_i x_{ij} \leq 0$			$\sum_{r=1}^{s} \delta_r y_{rj} - \sum_{i=1}^{m} \rho_i x_{ij} + \varpi \leq 0$		
$u_r - \pi_r u_{r+1} \leq 0$	:r1	*for some r*	$\delta_r - \pi_r \rho_{r+1} \leq 0$	:r1	*for some r*
$v_i - \sigma_i v_{i+1} \leq 0$	:r2	*for some i*	$\rho_i - \sigma_i \rho_{i+1} \leq 0$	:r2	*for some i*
$u_r - \chi_i v_i \leq 0$	:r3	*for some i,r*	$\delta_r - \chi_i \rho_i \leq 0$	:r3	*for some i,r*
$u_r \geq \gamma_r \geq \varepsilon$	:r4	$r=1,...,s$	$\delta_r \geq \gamma_r \geq \varepsilon$	:r4	$r=1,...,s$
$v_i \geq \kappa_i \geq \varepsilon$	:r5	$i=1,...,m$	$\rho_i \geq \kappa_i \geq \varepsilon$	:r5	$i=1,...,m$

The notation in (M7.5) and (M7.6) as in (M7.1) and (M7.2) respectively. The Greek letters (π, σ, χ, γ, κ) are DM specified constants that reflect their judgments on the values of the inputs and outputs in the context of the assessment being undertaken. The two models require different sets of Radial DMUs (RDMUs) to simulate their weights restrictions, with, the construction of the Radial DMUs being the same irrespective of the type of weights restrictions that are to be simulated. Thus, the construction of the RDMUs for the two orientations will be considered separately:

- Input Minimisation Case

Solve (M7.5) to obtain h^*_j, $j=1,...,N$. Construct a Full Set Input Radial DMUs (FSIRD), which consists of RDMUs, $jt = 1,...,N$, such that RDMU jt has output levels y_{rjt}, $r=1,...,s$ and input levels x_{ijt}, $i=1,...,m$ where

$y_{rjt} = y_{rj}$	$x_{ijt} = h_j^* x_{ij}$	$j=1,...,N$	(7.1)

- Output Maximisation Case

Solve (M7.6) to obtain $\tilde{h}_j$, $j=1,...,N$. Construct a Full Set of Output Radial DMUs (FSORD), which consists of RDMUs, $jp = 1,...,N$, such that RDMU jp has output levels y_{rjp}, $r=1,...,s$ and input levels x_{ijp}, $i=1,...,m$ where

$y_{rjp} = \tilde{h}_j \, y_{rj}$	$x_{ijp} = x_{ij}$	$j=1,...,N$	(7.2)

As shown in chapter four, weights restrictions can be separated into two categories in order to simulate their relative efficiency scores by RDMUs, see Appendix 4.5.

- For the simulation of relative and linked-dependent restrictions r1-r3 follow Theorem 7.1.

- For the simulation of absolute restrictions r4-r5 follow Theorem 7.2.

Theorem 7.1: Relative and Linked-Dependent Restrictions

For the case when only weights restrictions of type r1-r3 have been imposed in (M7.5) and (M7.6), to give h_j^* and $\tilde{h}_j$ respectively. Let $jt=1,...,N$ and $jp=1,...,N$ be RDMUs having the input output levels defined in (7.1) and (7.2) respectively. Solve the models (M7.7) and (M7.8) respectively.

(M7.7) Input Minimisation		(M7.8) Output Maximisation	
$e^*_{j_0} = Max \sum_{r=1}^{s} \beta_r y_{rj_0} + \varphi$		$\tilde{e}_{j_0} = Min \sum_{i=1}^{m} \vartheta_i x_{ij_0} - \psi$	
s.t. $\sum_{i=1}^{m} \alpha_i x_{ij_0} = 1$		s.t. $\sum_{r=1}^{s} \mu_r y_{rj_0} = 1$	
$\sum_{r=1}^{s} \beta_r y_{rjt} + \varphi - \sum_{i=1}^{m} \alpha_i x_{ijt} \leq 0$	$jt=1,\ldots,N$	$\sum_{r=1}^{s} \mu_r y_{rjp} - \sum_{i=1}^{m} \vartheta_i x_{ijp} + \psi \leq 0$	$jp=1,\ldots,N$
$\alpha_i,\ \beta_r \geq \varepsilon$	$\forall i, r$	$\vartheta_i,\ \mu_r \geq \varepsilon$	$\forall i, r$
φ free		ψ free	

In (M7.7) α_i and β_r are the weights attached to the inputs and outputs respectively, in (M7.8) ϑ_i and μ_r are the weights attached to the inputs and outputs respectively. In (M7.7) and (M7.8) φ and ψ respectively reflect the scale of operation that DEA-efficient DMUs operate under. Then for DMU j_0 it follows that:

$h^*_{j_0} = e^*_{j_0}$	$\tilde{h}_{j_0} = \tilde{e}_{j_0}$	(7.3)

The proof of Theorem 7.1 can be found in Appendix 7.1.

Similar to the CRS case, this Full Set of Radial DMUs (FSRD) can be reduced to provide a necessary and sufficient set of RDMUs to simulate relative and linked-dependent weights restrictions. This can be achieved, by following the procedure outlined in chapter four, for reducing the FSRD to the RSRD.

Theorem 7.2: Absolute Restrictions

For the case when weights restrictions of type r4-r5 have been imposed in (M7.5) and (M7.6), to give h^*_j and $\tilde{h}_j$ respectively. Let $jt=1,\ldots,N$ and $jp=1,\ldots,N$ be RDMUs having the input output levels defined in (7.1) and (7.2) respectively. Solve the models (M7.9) and (M7.10) as required.

(M7.9) Input Minimisation	(M7.10) Output Maximisation
$f^*_{j_0} = Max \sum_{r=1}^{s} \beta_r y_{rj_0} + \varphi$	$\tilde{f}_{j_0} = Min \sum_{i=1}^{m} \vartheta_i x_{ij_0} - \psi$
s.t. $\sum_{i=1}^{m} \alpha_i x_{ij_0} = 1$	s.t. $\sum_{r=1}^{s} \mu_r y_{rj_0} = 1$
$\sum_{r=1}^{s} \beta_r y_{rj} + \varphi - \sum_{i=1}^{m} \alpha_i x_{ij} \leq 0$ $\quad j=1,\ldots,N$	$\sum_{r=1}^{s} \mu_r y_{rj} - \sum_{i=1}^{m} \vartheta_i x_{ij} + \psi \leq 0$ $\quad j=1,\ldots,N$
$\sum_{r=1}^{s} \beta_r y_{rjt_0} + \varphi - \sum_{i=1}^{m} \alpha_i x_{ijt_0} \leq 0$	$\sum_{r=1}^{s} \mu_r y_{rjp_0} - \sum_{i=1}^{m} \vartheta_i x_{ijp_0} + \psi \leq 0$
$\alpha_i, \beta_r \geq \varepsilon$ $\quad \forall i, r$	$\vartheta_i\ \mu_r \geq \varepsilon$ $\quad \forall i, r$
φ *free*	ψ *free*

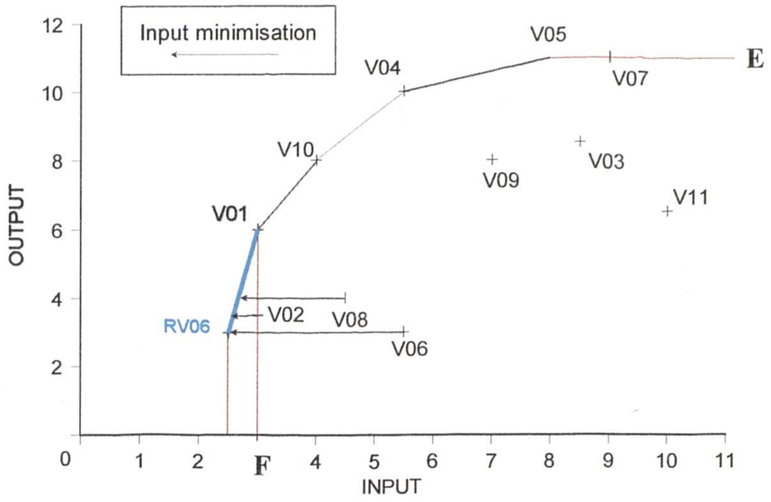
Notation in (M7.9) and (M7.10) as in (M7.7) and (M7.8) respectively. Then for DMU j_0 it follows that:

$h^*_{j_0} = f^*_{j_0}$	$\tilde{h}_{j_0} = \tilde{f}_{j_0}$	(7.4).

The proof of Theorem 7.2 can be found in Appendix 7.2.

To illustrate the simulation of weights restrictions in a VRS assessment, consider assessing the set of DMUs shown in Table 7.2. Figure 7.3 plots the Extended Production Possibility Set (EPPS) for this assessment under Input Minimisation.

Figure 7.3 - Extended Production Possibility Set



As stated in section 7.2, under an IM VRS model DMUs V02, V08 and V06 are non-enveloped and allocate their output an ε weight. To overcome this problem a linked-dependent weights restriction $v - 6u \leq 0$ can be imposed, which can be simulated by a set of RDMUs determined using (7.1). The FSRD consists of {RV06, RV08, RV06}, which can be reduced to a RSRD {RV06} following the procedure of chapter four, and this can be seen in Figure 7.3. Thus re-assessing the observed data set, without the weights restriction but with RDMU RV06 added to the assessment set, the same relative efficiency scores are obtained. So, the DM is provided with an idea of an implicit extension to the observed frontier, RV06V01 that is sufficient to simulate the relative efficiency scores under the linked-dependent restrictions.

This section has shown how to simulate weights restrictions in a VRS environment via the use of RDMUs. The next section will highlight several possible problematic outcomes from the use of weights restrictions in a VRS DEA assessment.

7.4 Possible Problematic Outcomes

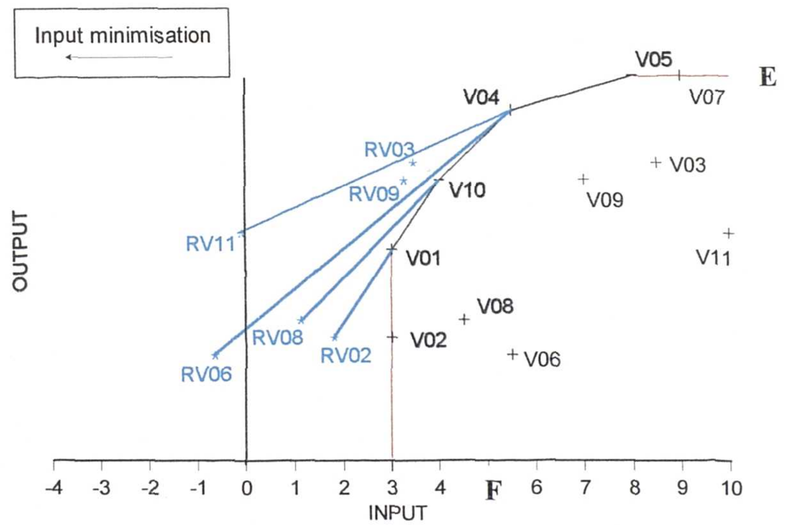
Unfortunately the use of weights restrictions to capture value judgments in a VRS DEA assessment can lead to several problematic results, three of which will now be discussed.

In general, these problems are associated with the fact that weights restrictions implicitly restrict the scale of operation of the DMUs.

7.4.1 DMU Dependent Implicit Extensions of the PPS: Absolute Restrictions

The use of absolute weights restrictions (AWR), e.g. r4 and r5 in (M7.5) in a DEA assessment leads to the implicit extensions of the DEA-frontier being DMU dependent. (Although this is also a problem in the CRS case.) This is now demonstrated graphically. Consider assessing the DMUs of Table 7.2, with an absolute weights restriction (AWR) of $u \geq 0.16$. Using the expression (7.1) to determine the Input Radial DMUs (IRDMUs), corresponding to the implicit extensions to the DEA-frontier under the AWR. These implicit modifications to the PPS are shown in Figure 7.4, with the IRDMUs denoted as RV. For example, RV08 represents the IRDMU corresponding to V08 required to simulate its relative efficiency score under the absolute restriction and RV02 is necessary to simulate the absolute relative efficiency score for V02.

Figure 7.4 - Absolute Restrictions Under VRS



Clearly, by translating the Absolute Weight Restriction (AWR) into terms of the inputs and outputs of the production process under analysis, it can be seen that the DMUs receive their relative efficiency scores relative to different implicit modifications of the observed

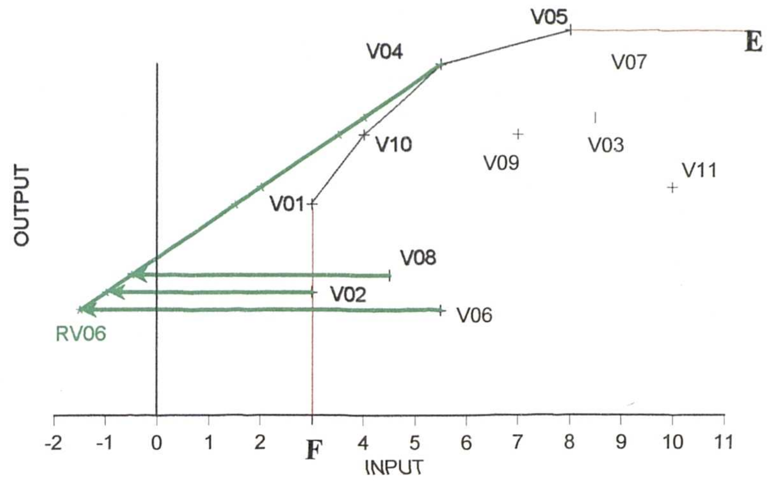
DEA-frontier. For example, RV02 represent input output levels such that when V02 is measured relative to this RDMU, its relative efficiency score under the absolute restriction is simulated. Similarly, RV08 represent input output levels such that when V08 is measured relative to this RDMU, its relative efficiency score under the absolute restriction is simulated. Clearly, the input output levels of RV02 are inefficient relative to RV08, hence implying that under the absolute restriction, the frontier that V02 is measured relative to, is inefficient compared to the frontier that V08 is measured relative to. Thus, this interpretation of the scores as being representative of a relative measure of efficiency is questionable. *This, in turn, implies that the use of AWRs to ensure the* maximum/minimum contribution of variables in a DEA assessment *under* VRS is questionable, as they cannot be interpreted as *relative efficiency scores. Chapter five* suggests two approaches to overcome this problem.

7.4.2 Misspecification of Returns to Scale Leading to Negative Relative Efficiency Scores

Another clear cause for concern is the possibility of obtaining negative relative efficiency scores under a VRS DEA model. This is demonstrated in Figure 7.4 by IRDMUs, RV06 and RV11, which are the input output levels that are the radial reductions of the input output levels of V06 and V11 under absolute restrictions respectively. See Appendix 7.3 for evidence that negative efficiency can be obtained. Evidently, this outcome would suggest that the imposed weights restrictions are inappropriate.

This can be perceived as a problem due to the fact that the imposed restrictions are forcing the DMUs to be assessed under a nature of returns to scale that is inappropriate for the DMUs. This can perhaps be more easily seen, when a linked-dependent weights restrictions is imposed, e.g. $-u + v \leq 0$. Through the use of Radial DMU, constructed from (7.1), the implicit modification of the PPS with the inclusion of the linked-dependent restriction: $-u + v \leq 0$, can be plotted, see Figure 7.5.

Figure 7.5 - Extended Production Possibility Set



The RSRD for this assessment consists of RV06 and is sufficient to simulate the linked-dependent relative efficiency scores. Clearly, the introduced weights restriction forces DMUs V08, V02 and V06 to be assessed under a decreasing returns to scale, which provides meaningless relative efficiency scores for these three DMUs, as the local boundary of V01V10 is exhibiting constant returns to scale. In a sense this restriction is imposing non-increasing returns to scale between the related input and output on the set of DMUs i.e. $v \leq u$, hence it imposes an inappropriate nature of returns to scale on these DMUs. Whereas, the earlier restriction of $v \leq 6u$, see Figure 7.3, could be thought of as imposing an increasing returns to scale on the related input and output. Thus when the latter restriction is imposed the model provides reasonable relative efficiency scores as the DEA frontier is extended in an appropriate manner.

This clearly links to the next problem with the use of weights restrictions in a VRS DEA assessment.

7.4.3 Inappropriate Nature of Returns to Scale Value

Banker and Thrall [8] discuss the Most Productive Scale Size (MPSS) of a DMU and note that in the discussion of the observed input-output levels of a DMU, dividing by $\Sigma\lambda^*$ from model (M7.3) would render a DMU MPSS. Thus, in essence the value of the sum of lambdas is indicating the distance of the DMU from MPSS. As the scale variable (ω in (M7.3)) is the dual variable to the restricted sum of lambdas, it is reasonable to assume that the size of the scale variable provides the DM with some indication of the distance of the DMU from the MPSS, rather than just an indication of whether the assessed DMU is operating at increasing/constant/decreasing returns to scale, when DEA-efficient. This would imply that if the scale variable takes a very large positive or negative value, the scale of operation suggested by DEA would not reflect a reasonable scale of operation in practice, and thus the obtained relative efficiency score would be questionable.

When weights restrictions are imposed, as the variable used to ascertain the returns to scale of efficient DMUs is unrestricted, frequently to satisfy the constraints, this variable may take inappropriate values in order to obtain the assessed DMU's relative efficiency score. In the sense that the scale of operation of a DMU indicated by the DEA results may be practice be unreal, i.e. the absolute value of ω in (M7.3) is extremely large.

Clearly, these observations indicate a cause for concern in the use of weights restrictions to capture value judgments in a VRS DEA assessment. The approach of this thesis endeavours to overcome these difficulties.

7.5 Conclusion

This chapter has briefly considered the use of DEA to assess the relative efficiency of DMUs operating in a VRS environment. It has also demonstrated that the envelopment of a DMU and its relative inefficiency score is dependent on the model orientation.

Further, the simulation of weights restrictions was considered. It was found that unlike the CRS case, only a specific set of RDMUs can be used to simulate the relative efficiency scores. It also highlighted several disturbing outcomes from the use of weights restrictions to capture value judgments under VRS. This motivates the need for alternative approaches to weights restrictions for capturing values in a VRS DEA assessment.

The next chapter will propose an approach similar to that detailed in chapter six for capturing value judgments via UDMUs in a DEA assessment where the DMUs operate under VRS.

8. Chapter Eight
Incorporating Values and Improving Envelopment Via UDMUs: VRS Case

8.1 Introduction

As highlighted in chapter three, to date there has been very little attention in the literature given to approaches for the inclusion of values into a VRS DEA assessment. Chapter seven illustrated that the main use for their inclusion in a CRS DEA assessment, that of weights restrictions, does not lend itself readily to implementation in the VRS assessment. This chapter therefore demonstrates how the procedure proposed in chapter six as an alternative to weights restrictions in the CRS case can be readily implemented in a VRS assessment. This is mainly due to the ability of the approach to allow varying local values and the relationship between the inputs and outputs to be included in the assessment, which are particularly important in a VRS assessment.

When the DMUs are assessed under VRS, a variable to represent the DMU's scale of operation is introduced, (see ω in model (M7.1) of chapter seven). In extreme cases it is possible to obtain relative efficiency scores that are based purely on the ratio of a measure of the DMU's scale to a weighted input in the input minimisation case, or to a weighted output in the output maximisation case. That is, the sum of weighted outputs/inputs in the

IM/OM case can be zero, with all the weight being applied to the DMU's scale to maximise its efficiency score. Clearly, in these extreme cases the relative efficiency scores do not, in fact, represent efficiency scores and it is necessary therefore to introduce additional information into the assessment to overcome this problem.

In a standard CRS DEA assessment this is usually done by the introduction of some weights restrictions. However, the inclusion of the variable representing the scale of operation of a DMU can give rise to difficulties in their use. Chapter seven illustrated for an IM model that WRs can implicitly extend the observed frontier into areas of production that provide negative relative efficiency scores, which is a cause for concern in the implementation of WRs in the VRS case. Hence alternative approaches that avoid these difficulties due to the implicit modification of the Production Possibility Set (PPS) are necessary.

The use of the approach developed in chapter six which utilises the envelopment model to express the DM's values and thus explicitly modifying the PPS, avoids these difficulties. This approach therefore perceives the concept of the inclusion of values in a DEA assessment as a problem of missing data, i.e. the UDMUs are attempts at specifying estimates of efficient levels of inputs and outputs for operating processes which at present are only observed at inefficient levels. The approach is not concerned with directly expressing values on the DEA-efficient DMUs.

The chapter is structured as follows: Section two considers the approach for improving envelopment; the third section establishes a need for the procedure; section four details from where to extend the frontier; section five identifies which input output levels to adjust to encourage the non-ε weighting of individual factors; section six provides a means for constructing suitable DEA-efficient UDMUs; section seven discusses their implementation and section eight summarises the procedure; section nine applies the procedure to a set of bank branches.

8.2 Incorporating Values & Improving Envelopment by Means of UDMUs: An Outline

This chapter focuses on adapting the approach derived in chapter six for the CRS case to the VRS case. In this case estimates of DEA-efficient UDMUs are to be suitably constructed from the observed input output levels of specific DEA-efficient DMUs, their estimated scale of operation and the DM's value judgments, while taking into account technological and managerial constraints. It should be noted that the proposed approach is an attempt to encourage the non-ε weighting of individual inputs and outputs and NOT the simultaneous non-ε weighting of a combination of inputs or outputs. Hence it does not guarantee full envelopment. Further, as noted in chapter six the improvement to the envelopment of the DMUs will depend on the specification of the UDMUs.

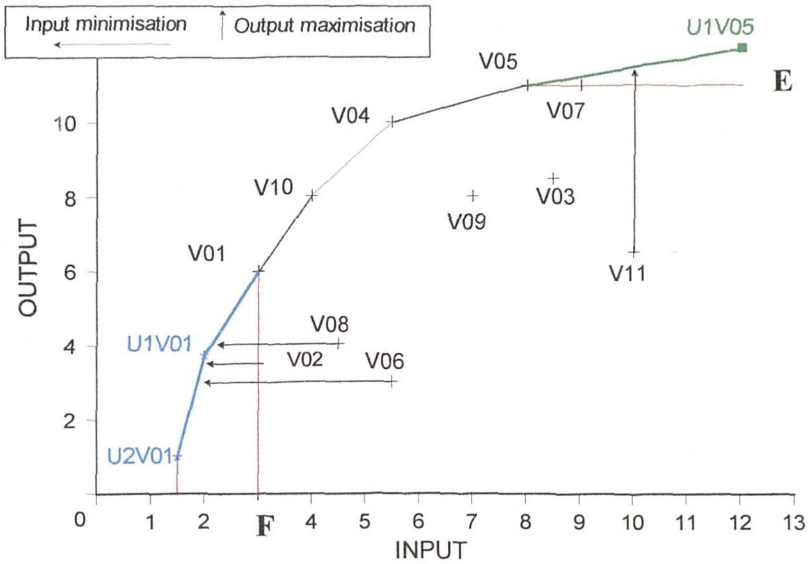
As the procedure is an adaptation of the one presented in chapter six, the main steps involved will now be outlined and then explained in later sections of this chapter.

i. Assess the DMUs to determine the DEA-efficient DMUs and the envelopment of the DMUs.
ii. Identify the ADMUs.
iii. Identify which input and/or output levels of the ADMUs need to be individually raised and/or lowered to, in principle, improve envelopment.
iv. Construct estimates of DEA-efficient UDMUs.
v. Re-assess the DMUs permitting both DMUs and UDMUs to be peer DMUs.
vi. If the DM feels envelopment is unsatisfactory, repeat steps (iv) and (v). Otherwise stop.

To demonstrate how UDMUs can be used independently of weights restrictions to include values in DEA, a simple graphical example will be considered using the 11 DMUs of Table 7.1 of chapter seven, where each DMU consumes a single input to produce a single output. Further, as chapter seven demonstrated, under VRS the envelopment of the

DMUs and the use of UDMUs to incorporate values are model orientation dependent, and so the two orientations will be considered separately.

Figure 8.1* - *Extended Production Possibility Set



8.2.1 Input Minimisation: Encouraging the non-ε weighting of an output

As noted in chapter seven, under IM in Figure 8.1 DMUs, V02, V08 and V06 are non-enveloped and their output receives an ε weight. (These DMUs have a positive output slack.) Evidently, to overcome this problem and provide meaningful relative efficiency scores for these DMUs, some form of values regarding the relationship between the input and output must be incorporated into the analysis, that exhibits increasing returns to scale. It will be assumed that these values are to be expressed via the inclusion of additional UDMUs into the assessment set. Clearly, the DEA-frontier must be extended from V01, which is therefore an ADMU, and from Figure 8.1 it can be seen that it delineates the DEA-efficient frontier from the DEA-inefficient frontier. Thus, if the introduced UDMUs are to maintain the returns to scale of the observed frontier, they must exhibit in conjunction with V01 varying increasing returns to scale.

Consider the introduction of two UDMUs U1V01 (2, 3.75) and U2V01 (1.5, 1). These UDMUs improve envelopment and introduce two marginal rates of transformation. Thus

V08 and V02 are measured against the extended frontier U1V01V01 and V06 against U2V01U1V01. There are three conclusions that can be drawn from the input minimisation orientation VRS DEA model:

- To encourage the non-ε weighting of an individual output, it is necessary to introduce DEA-efficient DMUs into the observed data set that have similar operating mixes to the non-enveloped DMUs but produce less of the ignored output.
- ADMUs will require adjustments to their <u>input</u> levels in order to incorporate values and improve envelopment.
- The scale of operation of the ADMU must be considered in determining the input output levels of the UDMUs.

8.2.2 Output Maximisation: Encouraging the non-ε weighting of an input

Clearly, in the OM case, in Figure 8.1 DMUs V03, V07 and V11 are non-enveloped and their input receives an ε weight. To overcome this problem information on the relationship between the input and output once again needs to be introduced into the assessment. However, in this case the relationship must exhibit decreasing returns to scale in order to improve envelopment while at the same time maintaining the efficiency of the DEA-efficient DMUs. Clearly, in this case the frontier needs to be extended from V05, and thus it is the ADMU of the observed data set. Therefore the introduced UDMU should, in conjunction with V05, exhibit decreasing returns. One such DMU would be U1V05, (12,12). Hence in the VRS DEA output maximisation model:

- To encourage the non-ε weighting an individual input, DEA-efficient DMUs need to be introduced into the observed data set that have similar operating mixes to the non-enveloped DMUs but consume more of the ignored input.
- ADMUs will require adjustments to their output levels, in order to incorporate values and improve envelopment.
- The scale of operation of the ADMU needs to be considered in determining the input output levels of the UDMUs.

Having outlined the procedure graphically, the next section will begin the formal procedure for including values and improving envelopment for the multiple input output case.

8.3 Assessing Envelopment: Step (i)

Initially the DEA-efficient DMUs have to be identified, along with establishing a need for the procedure. Thus, consider assessing a set of N DMUs that consume varying amounts of m different inputs, x_{ij}, $i=1,...,m$ to produce varying quantities of s different outputs, y_{rj}, $r=1,...,s$. The VRS DEA models (M7.3) and (M7.4) provide the relative efficiency score of DMU j_0 under an IM and OM model respectively. Let JE define the set of DEA-efficient DMUs under (M7.3) and (M7.4). Although both the IM and the OM models are discussed, it is assumed that the DM solves only one, either the IM model (M7.3) or the OM model (M7.4), not both. This model will then be used throughout the procedure, due to the envelopment of the DMUs being model orientation dependent, as discussed in chapter seven, hence the constructed UDMUs will be model orientation dependent.

Clearly, if all the inefficient DMUs are properly enveloped i.e. $H^*_i = H^*_{m+r} = 0\ \forall i,r$ in (M7.3) or $G^*_i = G^*_{m+r} = 0\ \forall i,r$ in (M7.4) then there is no need for the proposed procedure. However, in general this will not be the case and this initial assessment should establish the need for the following procedure of steps (ii) to (v). Thus proceeding to step (ii) which identifies the ADMUs.

8.4 Identifying the ADMUs: Step (ii)

As stated in chapter six the ADMUs are: DEA-efficient and delineate the DEA-efficient from the DEA-inefficient frontier. Hence the same procedure will be followed in order to identify them, that is, the use of SE with respect to JE (the set of DEA-efficient DMUs). Let JE_{j_0} be JE excluding DMU j_0. The envelopment models (M8.1) or (M8.2), as required, is solved in respect of each $j_0 \in JE$.

(M8.1) Input Minimisation	(M8.2) Output Maximisation
$\bar{h}_{j_0} = Min\ \bar{f}_0 - \varepsilon\left(\sum_{i=1}^{m} S_i + \sum_{r=1}^{s} S_{m+r}\right)$	$\hat{h}_{j_0} = \hat{f}_0 + \varepsilon\left(\sum_{i=1}^{m} H_i + \sum_{r=1}^{s} H_{m+r}\right)$
s.t. $\bar{f}_0 x_{ij_0} - \sum_{j \in JE_{j0}} \delta_j x_{ij} - S_i = 0 \quad i=1,\ldots,m$	s.t. $\sum_{j \in JE_{j0}} \gamma_j x_{ij} + H_i = x_{ij_0} \quad i=1,\ldots,m$
$\sum_{j \in JE_{j0}} \delta_j y_{rj} - S_{m+r} = y_{rj_0} \quad r=1,\ldots,s$	$\hat{f}_0 y_{rj_0} - \sum_{j \in JE_{j0}} \gamma_j y_{rj} + H_{m+r} = 0 \quad r=1,\ldots,s$
$\sum_{j \in JE_{j0}} \delta_j = 1$	$\sum_{j \in JE_{j0}} \gamma_j = 1$
$\delta_j, S_i, S_{m+r} \geq 0 \quad \forall j \in JE_{j_0}, i, r$	$\gamma_j, H_i, H_{m+r} \geq 0 \quad \forall j \in JE_{j_0}, i, r$

Notation in (M8.1) and (M8.2) as in (M7.3) and (M7.4) respectively. Let ‾ and ^ denote the value for a variable at the optimal solution to (M8.1) and (M8.2) respectively.

Let *JA* denote the set of ADMUs for the assessment, with the criteria for classifying ADMUs under VRS being the same as the CRS criteria for an ADMU in chapter six. Thus a DEA-efficient DMU must meet either of the following conditions corresponding to its assessed model above, to be classed as an ADMU:

(M8.1) Input Minimisation	(M8.2) Output Maximisation
♦ $\bar{h}_{j_0} > 1$ and at least one $\bar{S}_i > 0$ or $\bar{S}_{m+r} > 0$	♦ $\hat{h}_{j_0} < 1$ and at least one $\hat{H}_i > 0$ or $\hat{H}_{m+r} > 0$
or	or
♦ An infeasible solution	♦ An infeasible solution

Proof of these statements can be found in Appendix 8.1.

As illustrated in section 8.2, specific adjustments to selected inputs and/or outputs of the ADMUs are required to be altered to construct UDMUs, that will in principle improve envelopment. The next section outlines one approach for identifying which inputs and/or outputs of an ADMU require adjustments in order to improve envelopment.

8.5 Which Inputs and/or Output Levels of a ADMU to Adjust? Step (iii)

As detailed in chapter six, in the multiple input output case, to prevent individual inputs and outputs from being ignored in the assessment, several ADMUs may require adjustments to the same inputs and outputs in order to encourage the non-ε weighting of an individual input and output. Proposition 6.1 and Proposition 6.2 of chapter six outline the proposed manner for the required adjustments to the selected inputs and/or outputs of an ADMU. These propositions are:

Proposition 8.1: Encouraging the non-ε weighting of an individual input

To encourage the non-ε weighting of input k, raise the levels of input k for a set of selected ADMUs, in order to construct estimates of suitable DEA-efficient UDMUs that will, in principle, improve envelopment.

Proposition 8.2: Encouraging the non-ε weighting of an individual output

To encourage the non-ε weighting of output k, lower the levels of output k for a set of selected ADMUs, in order to construct estimates of suitable DEA-efficient UDMUs that will, in principle, improve envelopment.

Thus, the question now becomes how to identify which of the inputs and/or outputs of an ADMU to raise and/or lower. The approach proposed, follows that of chapter six and attempts to use the information from the initial DEA assessment to determine a basis for their identification.

Let *JFIN* and *JFON* be the set of class NF DMUs, with optimal values of, f_j^* and $f_j^/$ with reference to (M7.3) and (M7.4) respectively for DMU j. Determine a set of class F DMUs, with input output levels as defined in (8.1) and (8.2) respectively, corresponding to these class NF DMUs.

(8.1) Input Minimisation	(8.2) Output Maximisation
$x_{ijfi} = \phi_j^* x_{ij} \quad y_{rjfi} = y_{rj} \quad \forall j \in JFIN$	$x_{ijfo} = x_{ij} \quad y_{rjfo} = \theta_j' y_{rj} \quad \forall j \in JFON$

In (8.1) the class NF DMUs have their input levels radially reduced in line with their radial DEA-efficiency yielded by (M7.3) and in (8.2) class NF DMUs have their output levels radially increased in line with their radial DEA-efficiency yielded by (M7.4) respectively. Let *JFI*, $jfi=1,\ldots,|JFI|$ be the set of observed class F DMUs of (M7.3) and the class F DMUs created by means of (8.1). Similarly, let *JFO*, $jfo=1,\ldots,|JFO|$ be the set of observed class F DMUs of (M7.4) and the class F DMUs created by means of (8.2). Let *JA* denote the set of ADMUs defined with reference to (M8.1) and (M8.2) as required. For each $j_0 \in JA$ solve model (M8.3) or (M8.4) as required.

(M8.3) Input Minimisation	(M8.4) Output Maximisation
$h_{j_0}^{//} = Min\ f_0^{//} - \varepsilon\left(\sum_{i=1}^{m} S_i + \sum_{r=1}^{s} S_{m+r}\right)$	$\tilde{h}_{j_0} = Max\ \tilde{f}_0 + \varepsilon\left(\sum_{i=1}^{m} H_i + \sum_{r=1}^{s} H_{m+r}\right)$
s.t. $f_0^{//} x_{ij_0} - \sum_{j \in JE_{j_0}} \delta_j x_{ij} - \sum_{jfi \in JFI} \delta_{jfi} x_{ijfi} - S_i = 0$	s.t. $\sum_{j \in JE_{j_0}} \gamma_j x_{ij} + \sum_{jfo \in JFO} \gamma_{jfo} y_{rjfo} + H_i = x_{ij_0}$
$\sum_{j \in JE_{j_0}} \delta_j y_{rj} + \sum_{jfi \in JFI} \delta_{jfi} y_{rjfi} - S_{m+r} = y_{rj_0}$	$\tilde{f}_0 y_{rj_0} - \sum_{j \in JE_{j_0}} \gamma_j y_{rj} - \sum_{jfo \in JFO} \gamma_{jfo} y_{rjfo} + H_{m+r} = 0$
$\sum_{j \in JE_{j_0}} \delta_j + \sum_{jfi \in JFI} \delta_{jfi} = 1$	$\sum_{j \in JE_{j_0}} \gamma_j + \sum_{jfo \in JFO} \gamma_{jfo} = 1$
$\delta_{jfi}, \delta_j, S_i, S_{m+r} \geq 0 \quad \forall i, r, jfi, j \in JE_{j_0}$	$\gamma_{jfo}, \gamma_j, H_i, H_{m+r} \geq 0 \quad \forall i, r, jfo, j \in JE_{j_0}$

Notation in (M8.3) and (M8.4) as in (M8.1) and (M8.2) respectively. Let $^{//}$ and $^{\sim}$ denote the value for a variable at the optimal solution to (M8.1) and (M8.2) respectively. Let AP_{j_0} denote the set of referent DMUs to ADMU j_0 from the solved model.

If in (M8.3) and (M8.4) ADMU j_0 provides a feasible solution, then the ADMU j_0 requires adjustments to its input and/or output levels as detailed below.

Stages for identifying which inputs and outputs of the ADMUs to adjust

a) Identify each class F DMU that is a referent DMU to ADMU j_0 in (M8.3) or (M8.4) as required, i.e. each $jfi \in AP_{j_0}$ in (M8.3) or each $jfo \in AP_{j_0}$ in (M8.4).

b) For each of these $jfi \in AP_{j_0}$ in (M8.3) or $jfo \in AP_{j_0}$ in (M8.4) identify the positive slack variables for their original DMU in (M7.3) or (M7.4) respectively.

c) For each input or output of the ADMU corresponding to the positive slack variable in (M7.3) or (M7.4) respectively, at least one estimate of a DEA-efficient UDMU is to be constructed following the initial adjustments as defined by Proposition 8.1 and Proposition 8.2.

The proof of that the above steps will improve envelopment in (M7.3) or (M7.4) respectively inclusive of the UDMUs constructed can be found in Appendix 8.2.

It should be noted here that this approach may not identify all the possible necessary adjustments to inputs and/or outputs of an ADMU due to multiple optimal solutions. Clearly, there may be alternative approaches. Although, the one presented here will increase the number of enveloped DMUs.

Having identified the inputs and/or outputs of an ADMU that are to be raised and/or lowered to improve envelopment, in principle, there now exists/remains the question of how to compensate for these adjustments, i.e. how to determine the actual input output levels of the UDMUs. The next section will deal with this issue.

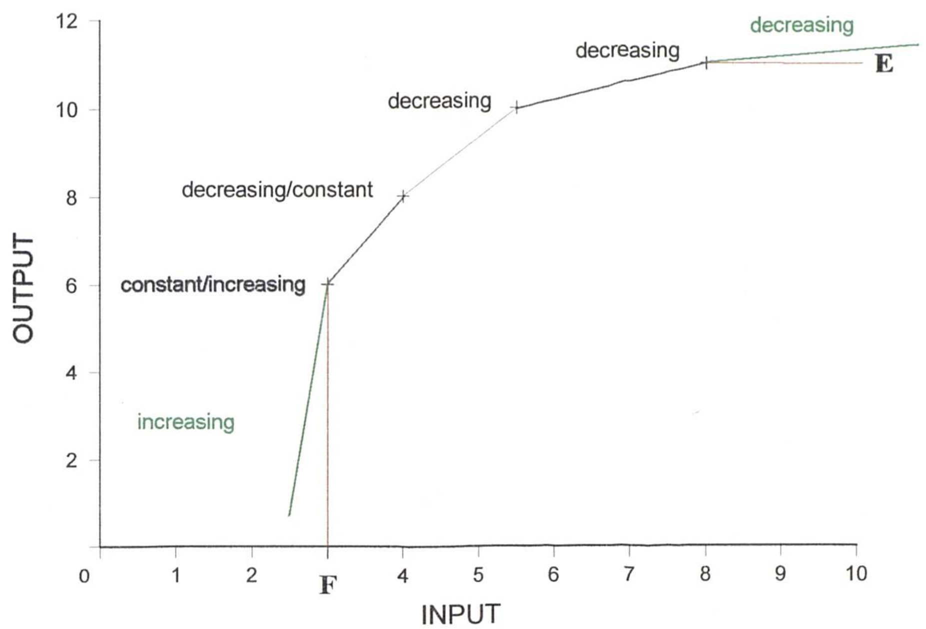
8.6 How to Construct Estimates for DEA-Efficient UDMUs? Step (iv)

As illustrated in chapter seven to simulate weights restrictions in a VRS environment, the necessary adjustments to the input output levels of the DMUs in order to determine a set of Radial DMUs are dependent on the model orientation. However, it has been established in Proposition 8.1 and Proposition 8.2, that in order to use UDMUs independently of weights restrictions to encourage the non-ε weighting of an individual input, it is required to raise this input level for a set of ADMUs regardless of the model orientation. Similarly, to encourage the non-ε weighting of an individual output, it is required to lower this output level for a set of ADMUs irrespective of the model orientation. So, it has already been established that the initial adjustments to the ADMUs for the construction of UDMUs that will attempt to improve envelopment are independent of the model orientation. This, therefore, implies that if the UDMUs are to improve envelopment, the adjustments to the remaining input/output levels of the ADMUs must be dependent on the model orientation. Therefore in the IM case, adjustments to the input levels of the ADMUs are required in the construction of the UDMUs. Similarly in the OM case, adjustments to the output levels of the ADMUs are required in some way. Chapter six offers the DM some general guidelines for the adjustments of the inputs outputs of the ADMUs in order to construct their UDMUs. But, the DM is limited to one option for each orientation for the adjustments:

- IM orientation model requires input adjustments
- OM orientation model requires output adjustments

Further, as the DMUs are now operating in a Variable Returns to Scale environment, there is the additional concept of what nature of returns to scales should the UDMUs exhibit? It has been assumed that the additional UDMUs should attempt to maintain the observed nature of returns to scale of the DEA frontier. Thus the nature of the returns to scale of the ADMU that the UDMU is being constructed from should be considered. Consider the earlier simple example and the nature of returns to scale of the DEA-frontier, see Figure 8.2.

Figure 8.2 - The Nature of the Returns to Scale of the DEA Frontier



Clearly, if it is desired to only extend the DEA-frontier from V01 by means of UDMUs, then the UDMUs, in conjunction with V01, must exhibit IRS, whilst, if it is desired to extend the DEA-frontier from V05, the UDMUs, in conjunction with V05, must exhibit DRS.

Some very general guidelines are now offered to the DM for the construction of their UDMUs considering the returns to scale of the ADMU on which they are based. That is, it would be expected that the UDMUs operate under the same returns to scale as their ADMU. Further, if several UDMUs are introduced per ADMU, then these UDMUs should operate under returns to scale appropriate for those of the ADMU. See Table 8.1, for some general guidelines of what returns to scale of UDMUs are appropriate for the returns to scale of the ADMU. These guidelines are based on the concept that the extended frontier should exhibit VRS and therefore the introduction of any additional UDMUs should attempt to be consistent with this. However, in practice this may be difficult to achieve and it will also depend on the number of UDMUs that the DM wants to introduce.

Table 8.1* - *Basic Guidelines for the Construction of the UDMUs in Terms of Appropriate Returns to Scale

	Nature of Returns to Scale of UDMUs	
Nature of Returns to Scale of ADMU	**Increase input and output levels of ADMU**	**Decrease input and output levels of ADMU**
Increasing	Initial UDMU: Increasing Subsequent UDMUs: Increasing, Constant, Decreasing	Initial UDMU: Increasing Subsequent UDMUs: Increasing
Constant	Initial UDMU: Constant Subsequent UDMUs Constant, Decreasing	Initial UDMU: Constant Subsequent UDMUs Constant, Increasing
Decreasing	Initial UDMU: Decreasing Subsequent UDMUs: Decreasing	Initial UDMU: Decreasing Subsequent UDMUs Decreasing, Constant, Increasing

Evidently, the construction of the input output levels of the UDMUs is for the DM to decide. This will depend on their values and the existing relationships between the inputs and outputs. However, the DM should now be in a position to estimate a set of UDMUs for introduction into the observed data set that will, in principle, improve envelopment. As stated earlier the DEA-efficiency of UDMUs is not guaranteed by their construction, so for simplicity, the DM may feel it appropriate to first check the DEA-efficiency of their UDMUs by assessing them relative only to the DEA-efficient DMUs, particularly, if there are a large number of DMUs in the assessment. If UDMUs are found to be inefficient, their input output levels may be adjusted until their DEA-efficiency is obtained.

8.7 Implementation: Step (v)

Once the ADMUs relating to a set of N DMUs have been identified and their associated UDMUs created, the DMUs can be assessed using model (M7.1) or (M7.2) as required, allowing DMUs and UDMUs to be peer DMUs. The number of properly enveloped DMUs should be greater than in the absence of the UDMUs, see Appendix 8.2. However, the increase in the number of properly enveloped DMUs will depend on the specification of the UDMUs and further adjustments to their input output levels may be required to further increase the envelopment of the DMUs.

An algorithmic summary of the suggested procedure is now given.

8.8 Incorporating Values & Improving Envelopment Via UDMUs: A Summary

Consider a set of N DMUs using m inputs, x_{ij}, $i=1,\ldots,m$ to produce s different outputs y_{rj}, $r=1,\ldots,s$, using an IM model. The following steps can increase the number of properly enveloped DMUs in assessments of DEA efficiency, but does not guarantee full envelopment. (A similar summary can be specified for an output maximisation model.)

i. The model (M7.1) is used to identify the set of DEA-efficient DMUs JE which are of class E and E' as defined by Charnes *et al.* [19]. If all DMUs $j \notin JE$ are properly enveloped stop. Otherwise go to (ii).

ii. In respect of each $j \in JE$ solve model (M8.1) to determine $\bar{h}_{j_0}$ as defined in that model. The set of ADMUs $JA = \{j \mid \bar{h}_j > 1$, and at least one $\bar{S}_i > 0$ or $\bar{S}_{m+r} > 0$, or DMU j has no feasible solution in (M8.1)$\}$.

iii. In respect of each $j \in JA$ solve model (M8.4) and use (M7.3) to identify the inputs and output of each ADMU that require necessary adjustments following Proposition 8.1 and Proposition 8.2 to initiate the construction of at least one UDMU.

iv. In respect of each ADMU, the DM specifies UDMUs based on the results of step (iii). That is, for each input and/or output identified in (M7.3) at least one UDMU is constructed given the DM's local values, returns to scale of the ADMU, the model orientation and any technological and policy constraints.

v. Assess the DMUs using model (M7.1) but permitting both DMUs and the UDMUs created in step (iv) to be peer DMUs. The number of properly enveloped DMUs should be greater than the number initially found in step (i).

vi. If the DMs consider further envelopment of the DMUs is required, repeat steps (iv) and (v). Otherwise stop.

The next section demonstrates the use of the foregoing process on a real data set.

8.9 An Application of the Use of UDMUs to Capture Value Judgments and Improve Envelopment in DEA

In this section the use of UDMUs to incorporate value judgments and improve envelopment will be illustrated by applying the theory to the same data set of chapter six. However, in this application it is assumed that the branches operate under VRS. It was felt that the same data set could be assessed in a VRS environment, as the application is being used to merely illustrate the procedure in a VRS environment. Although, this does highlight the subjective nature of DEA and the need for an objective procedure for the selection of an appropriate model. Clearly, if a VRS DEA model is applied VRS is assumed to hold, whereas if a CRS model is applied CRS is assumed to hold.

Consider assessing the set of 668 bank branches of chapter six, each consuming two inputs to produce five outputs detailed in Table 6.2. An Input Minimisation model is used.

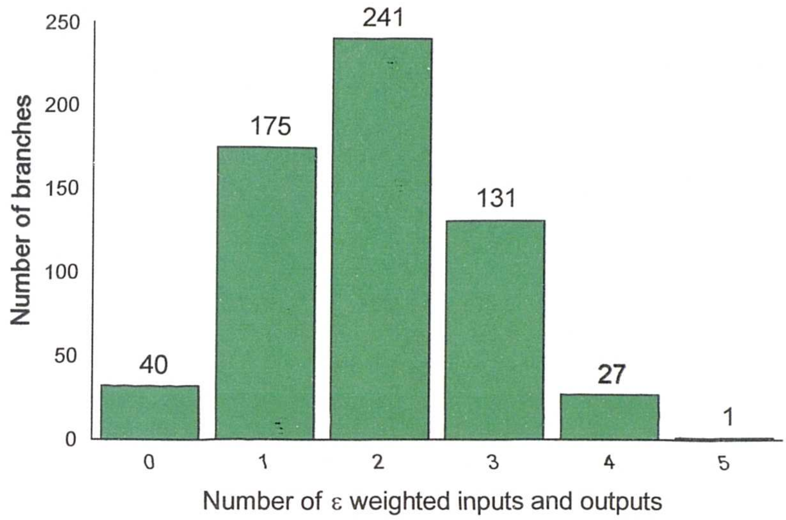
Step (i)

The initial step of the procedure is to assess the branches under model (M7.1) to identify the DEA-efficient branches and establish that there is a need to include values and improve the envelopment of the branches.

As there are 7 factors in the assessment and a VRS IM model is being applied, it is possible for a maximum of 6 ε-weights to be assigned by the branches in their assessment.

However, it was found that all DMUs use at least two factors to determine their relative efficiency scores and that there were 53 efficient branches. The frequencies of ε weights assigned by the 615 inefficient branches in (M7.1) are shown in Figure 8.3. [There may have been alternative optimal solutions which change the envelopment of the DMUs as shown in Figure 8.3, but in general the number of DMUs would not be properly enveloped.]

Figure 8.3 - The Number of ε Weighted Inputs and/or Outputs Per DEA-Inefficient Branch in (M7.1)



Clearly, under the standard VRS model, the majority of branches do not use all their factors to determine their relative efficiency score. Thus, this implies that in general, the observed efficient branches are of dissimilar operating mixes to the inefficient branches. It should be noted that 53 efficient branches envelop only 40 inefficient branches, which is an approximate ratio of 1 to 0.8.

This step clearly establishes a need for a procedure to improve the envelopment of the inefficient branches. As established in chapter six the proposed approach developed in this thesis is suitable for the DM and the implementation of their values, using the information collected from their ABC analysis to determine the unobserved branches.

Step (ii)

To determine which of the DEA-efficient branches are anchor branches, model (M8.1) was solved. It was found that 52 of the 53 DEA-efficient branches were anchor branches.

Step (iii)

Having identified the potential branches for the basis of the unobserved branches, it is now required to identify which of the inputs and/or outputs of these branches require adjustments to their inputs and outputs in order to improve envelopment. To demonstrate, the assessment of branch D586 under (M8.3) will be considered. Solving (M8.3) for D586 to determine its radially adjusted class F referent branches, it was found that D586 had two referent branches. The original class NF branches from (M7.3) corresponding to these two referent branches in (M8.3) are D332 and D485. Referring to the assessment of these branches in (M7.3), their positive slack values are shown in Table 8.2, and are the basis for the construction of the unobserved branches based on D586.

Table 8.2 - Results of Step (iii) for Branch D586

Observed Class NF Branches Corresponding to D586's Referent Branches in (M8.3)	Positive Slack Values in (M7.3)
D332	AP & MT
D485	AP & SV

The 2 class NF branches have in total positive slack values for 3 different inputs and outputs in model (M7.3). Thus, a minimum of 3 unobserved branches are to be determined using the input and output levels of branch D586 as a basis for their construction, as outlined in Table 8.3.

Table 8.3 - The Basis for the Construction of the Unobserved Branches Based on Branch D586

Constructed Unobserved Branch	Basis of Unobserved Branch
A1D586	Lowering of AP
A2D586	Lowering of MT
A3D586	Lowering of SV

For details of which inputs and/or outputs of the ADMUs are to be adjusted in order to improve envelopment see Appendix 8.3. It was also noted that only 29 of the anchor branches actually required adjustments to their input output levels.

Step (iv)

The actual construction of the unobserved branches now needs to be considered.

It was decided that if the anchor branch operated under CRS, then the DM would construct unobserved branches such that they would, hopefully, operate at CRS and estimate only one unobserved branch per lowering of an output or raising of an input. However, if the anchor branch was found to exhibit increasing or decreasing returns to scale, then the DM would attempt to construct unobserved branches that would exhibit variable returns to scale when added to the observed set of branches. In practice this is difficult to achieve, but by the adjustments to the inputs and outputs of the ADMU, varying rates of transformation may be achieved. The DM only wanted to estimate a maximum of two unobserved branches per lowering of an output or raising of an input level. This was done following the guidelines in Table 8.1. In total 97 unobserved branches were constructed.

As in the CRS case, the construction of the unobserved branches was based on the information gathered for the ABC analysis. The only real difference in the construction of the unobserved branches in the VRS to the CRS assessment is for those branches that are found to exhibit IRS or DRS. The construction of two unobserved branches based on the lowering of D586's SV level will be used as an illustration, as it was found that D586 exhibits IRS. Table 8.4 displays the input output levels of the observed branch and the two constructed unobserved branches, A3D586 and B3D586.

Table 8.4 - Unobserved Branches Based on a Reduction in Number of Saving Accounts (SV) Held at D586

	FA	TC	AI	AP	CT	MT	SV
D586	4	75429	84.2	25.8	94140	57	**1136**
A3D586	3.6	63654	71	18	92921	41	**848**
B3D586	3	59280	56	11	90412	31	**450**

As in the construction of the unobserved branches in the CRS assessment, the DM felt that a lowering of an output would lead to a reduction in its inputs and the other outputs. Clearly, SV has been lowered in two stages, with different adjustments to the inputs and outputs at each stage. As the observed branch exhibits IRS, the unobserved branches should exhibit varying rates of IRS. In an attempt to ensure this the ratios of the reductions of the inputs and outputs of D586 were varied at the different stages, in constructing A3D586 and B3D586.

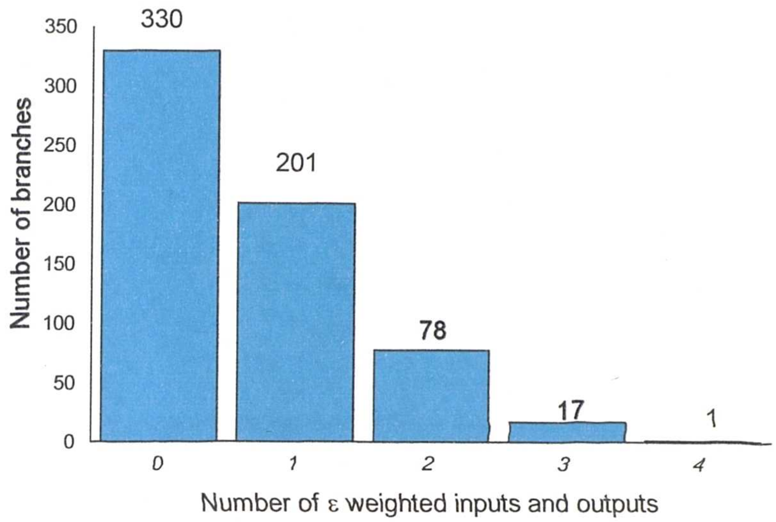
The final input output levels of the 97 unobserved branches can be found in Appendix 8.4. It should be noted that these unobserved branches are the result of several iterations under model (M7.1) with the assessment set containing only the DEA-efficient branches and the unobserved branches.

Step (v)

Finally, the DM is now in the position to re-assess the observed branches with the inclusion of the unobserved branches in the assessment, in (M7.1).

As the unobserved branches attempt to include values while improving envelopment, it would naturally be expected that the DEA-efficiency of the observed DEA-efficient branches would be discriminated between. This was found with the inclusion of the observed branches, as only 41 of the observed DEA-efficient branches remained DEA-efficient with 627 inefficient branches. Figure 8.4 summarises the effect on envelopment of these 97 unobserved branches. [Evidently, there may have been multiple optimal solutions that provide different envelopment results to those shown in Figure 8.4, but the results to envelopment would be vastly improved.]

Figure 8.4* - *The Number of ε Weighted Factors Per DEA-inefficient Branch in (M7.1) with an Extended Data Set



Clearly, the number of properly enveloped branches has been vastly improved by the introduction of the 97 unobserved branches into the assessment set. Thus the scores should reflect more appropriate measures of relative efficiency.

Finally, although at first it may appear that the determination of 97 unobserved branches is rather excessive, it is not unexpected, due to the number of inputs and outputs involved in the assessment and the initial number of non-enveloped branches.

8.10 Conclusion

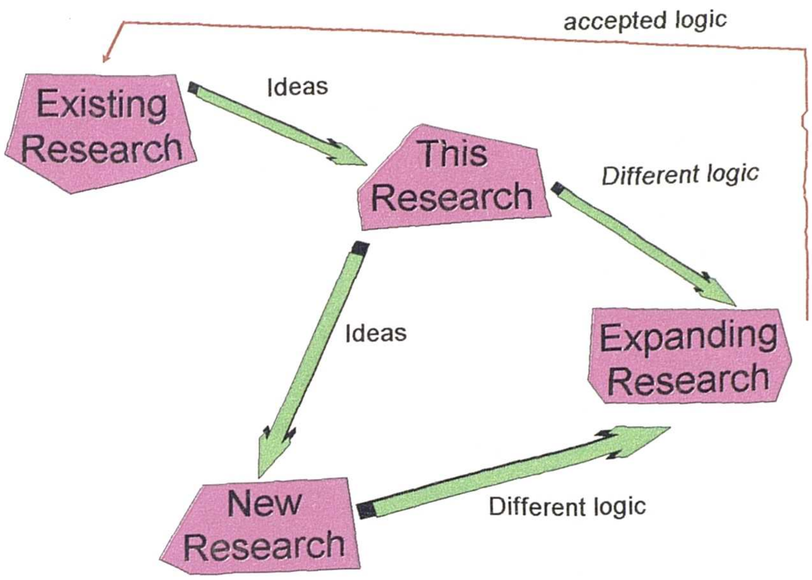
This chapter has adapted the approach developed in chapter six for introducing value judgments and improving envelopment in a CRS DEA assessment to a VRS DEA assessment. The DM's value judgments with regard to unknown production areas have been captured via the inputs and outputs of the production process, i.e. UDMUs. As in the CRS case these UDMUs have been constructed based on the observed DEA-efficient standards and information provided by the DM. However, they have also considered the returns to scale of their base ADMU, thus hopefully extending the frontier with suitable returns to scale being exhibited.

The main advantages of this approach have generally already been discussed in chapters five and six. However, this chapter has highlighted how the ability of the approach to express values at varying local levels and incorporate the relationship between the inputs and outputs, readily allows it to be applied to VRS DEA assessments.

Evidently the difficulty in the approach is the specification of the UDMUs. If inappropriately specified they do not impact on the envelopment of the inefficient DMUs, as it is rather a subjective procedure. Thus this area of the approach is in need of further research. However, the proposed approach is merely a starting point for an alternative perspective for perceiving the problem of how to include value judgments in a DEA assessment.

The next chapter summarises and concludes. It also offers the reader some thoughts on further research into the area of the inclusion of values judgments in a DEA assessment.

9. Chapter Nine
Summary, Conclusions and Further Explorations



9.1 Summary

Initially, this thesis presented an approach for incorporating value judgments in a DEA assessment via the introduction of Unobserved DMUs (UDMUs) into the assessment set, it then focused on the use of Unobserved DMUs to capture the DM's values that will improve the envelopment of the DMUs. These UDMUs are in essence based on observed DMUs and the DM's values. The main motivating reason for this research was the lack of information provided by present approaches on the explicit expression of the impact of the inclusion of values on the Production Possibility Set. Further motivating reasons for the research and the developed approach can be found in chapters two and five, with a review of present approaches for the inclusion of values in chapter three. Chapter four demonstrated, in the general case, that the inclusion of values in the form of weights restrictions implicitly modifies the Production Possibility Set; thus demonstrating that the problem of including values into a DEA assessment could be considered as a problem of missing data i.e. lack of suitable DEA-efficient comparator DMUs. This, switches the perceived view of the inclusion of values in a DEA assessment from the weights model to the envelopment model, and it is this link that laid the foundations for the approach proposed in this thesis. A brief outline of the steps involved in the procedure developed will now be provided:

i. Assess the observed data using an appropriate standard DEA model to identify the DEA-efficient DMUs and the initial envelopment of the DMUs. This step establishes the need for an approach for the inclusion of values that will improve the envelopment of the inefficient DMUs.

ii. Identify the ADMUs, which are those DEA-efficient DMUs that delineate the DEA-efficient frontier from the DEA-inefficient frontier. This is done using the concept of Super Efficiency, see Andersen and Petersen [6].

iii. Identify the specific inputs and/or outputs of the ADMUs that need to be individually raised and/or lowered respectively in order to improve envelopment. The proposed approach for identifying these inputs and/or outputs utilises the information provided from step (i) and the positive slack values of the inefficient DMUs.

iv. The DM constructs suitable estimates of DEA-efficient DMUs based on
    a) The information provided by step (iii).
    b) The input output levels and returns to scales of the ADMUs.
    c) Their perceived local values.
    d) Management policies and technological constraints.

v. Re-assess the DMUs under the appropriate standard DEA model, allowing both the observed and unobserved DMUs to be considered as peer DMUs. The obtained results should reflect the inclusion of the DM's values and an improvement to the envelopment of the DMUs.

vi. If the envelopment of the DMUs is not satisfactory to the DM, repeat steps (iv) & (v), otherwise stop.

Having outlined the general procedure developed in the thesis, which is applicable to DMUs operating under CRS and VRS, the next section will outline the advantages offered by the approach, its limitations and indicates the circumstances under which a DM would opt to select this approach for incorporating their values and improving envelopment in a DEA assessment.

9.2 Conclusions

The developed approach explicitly modifies the observed PPS in order to express the DM's preferences/values on the inputs and outputs used in the assessment in order to improve envelopment. This use of UDMUs to modify the PPS could be thought of as filling in for missing data in the observed data set. That is, estimates of efficient input output levels are being made, based on certain observed efficient input output levels and the DM's preferences.

In essence capturing value judgments via unobserved DMUs offers the following advantages:

- Alternative Expression of the DM's Values

In certain situations, DMs may find it difficult to express their value judgments via the specification of specific global marginal rates of substitution/transformation or maximal/minimal weight values. In these cases the DMs are provided with an alternative means of specifying their preferences/values in terms of the inputs and outputs with regard to specific production processes. Thus value judgments are expressed by the comparison of input output levels.

- Inclusion of the Relationship Between the Inputs and Outputs

UDMUs can be generated by simultaneously manipulating the input output levels of certain observed DEA-efficient DMUs. This directly incorporates any relationships which may exist between the inputs and outputs.

- Inclusion of Varying Local Values

In certain applications, global values may not be appropriate. That is, the relative value of the inputs or outputs may be dependent on the levels in which they are observed, i.e. variable local marginal rates of substitution are appropriate. In general as the observed frontier exhibits several different values between the inputs and outputs, it is only reasonable to assume that any extension to this existing frontier may also exhibit a variation in values. Thus capturing values via UDMUs allows varying local values to be incorporated.

- Consideration of the Feasibility of the Extended PPS

UDMUs explicitly modify the PPS to implicitly restrict the weights, rather than explicitly modify the weights, thereby implicitly modifying the PPS. In acting directly on the PPS the input output levels of the extended PPS are considered and hence controlled. This avoids DM infeasible extensions to the PPS being made and, therefore, unrealistic relative efficiency scores being obtained. For example, in the VRS case it avoids extensions into unknown production areas that provide negative efficiency scores.

- Aid the DM in the Interpretation of the Results

As stated, UDMUs allow the feasibility of the EPPS to be considered. Thus if the EPPS is deemed feasible by a DM, then the obtained results can be deemed feasible. Further, the DMs are provided with targets of similar operating processes to their present ones, which although unobserved, may be more meaningful and objective to the DM in certain situations. At the very least they provide an alternative suggestion to the DM for how efficiency may be improved.

- Limitations

Unfortunately, no approach is all-purpose or without limitations, and the proposed approach has several drawbacks, namely the determination of the UDMUs. This is rather time consuming, with the DEA-efficiency of the UDMUs not guaranteed and their introduction into the observed data set does not guarantee full envelopment.

Having established that the proposed approach offers both a different perspective towards the inclusion of values in a DEA assessment, and an alternative to existing approaches, a brief comparison of the approach of UDMUs and weights restrictions will now be made. Essentially this aims to indicate that each approach has its advantages and disadvantages, and suggests that each approach has a viable use and is appropriate for certain applications or specific DMs. A brief comparison of the two approaches is now made.

The three types of weights restrictions, absolute, relative and linked-dependent weights restrictions are used to incorporate different types of information. Absolute restrictions attempt to restrict the maximum/minimum contribution of specific inputs and/or outputs to the relative efficiency score. Hence in many respects these restrictions attempt to extend the PPS and are concerned with the non-enveloped DMUs of the assessment, similar to the proposed approach of this thesis. Relative restrictions attempt to reflect the DM's relative values on the inputs or outputs in the assessment. So, these restrictions attempt to extend and modify the PPS and, in several applications of these relative restrictions they are used to directly discriminate between the DEA-efficient DMUs, Thompson *et al.* [55]. In these cases the use of relative restrictions, is not concerned with the non-enveloped DMUs of the assessment. Linked-Dependent restrictions are used to reflect the DM's values on dependent relationships between the inputs and outputs of the assessment, so in essence they are not concerned with the non-enveloped DMUs of the assessment. The approach of this thesis is primarily interested in the use of the DM's values to extend the PPS and envelop the previously non-enveloped DMUs, while incorporating the relative and linked-dependent relationships that may exist between the inputs and/or outputs. So, it is assumed that the perceived values of the DEA-efficient DMUs are acceptable to the DM, whereas in weights restrictions this assumption, in general, is made.

Table 9.1 outlines, in general, the differences between the two approaches for the inclusion of value judgments in a DEA assessment.

Table 9.1* - *A General Comparison of Weights Restrictions and UDMUs

	Weights Restrictions	UDMUs
Specification	Explicit definition of global relationships between or on the values of the inputs and/or outputs.	Local values in form DMUs comparisons. Time consuming.
Implementation	Iterative. Infeasible solutions possible. Suitable software required. Time consuming.	Iterative.
Extended Production Possibility Set	Implicit extension therefore feasibility of input output levels not considered.	Explicit therefore the feasibility of the input output levels considered.
Targets and Peers	Observed data only. Drastic changes in input output mix could arise.	Observed and Unobserved data used. Input output mix is stable.
Returns to Scale	No account is given to the returns to scale of DMUs.	Accounts for the variation of the returns to scale of the DMUs.

Essentially the information required for, and the results provided by the use of weights restrictions and UDMUs, is of a different format, hence the approaches may be appropriate for different DMs. As the information is incorporated into the assessment differently, each format may be appropriate for different applications.

However, as discussed in chapter five, a combined use of weights restrictions and Radial DMUs can aid the DM in their interpretation of the results and implementation of their values.

Having identified the main conclusions of this thesis, some further possible explorations will be proposed which may extend the ideas presented in this thesis.

9.3 Further Explorations

Possible areas which may provide fruitful research, beginning with this thesis which raises a number of issues worthy of further exploration including the following:

a) ADMUs - Adjustment of Their Input Output Levels

Alternative approaches for identifying the inputs and outputs of an ADMU that require adjustments, in order to construct UDMUs that will in principle improve envelopment could be formulated.

b) Specification of UDMUs

i. Adaptation of a Current Approach

The adjustments to the data sets suggested by Charnes *et al.* [15] may prove valuable in the determination of suitable input output levels for the UDMUs. In the approach suggested by Charnes *et al.* [15], a new data set is generated through the use of cone-ratio information, thus the adjustments suggested are applied by adding related inputs or outputs. However, as the information provided for the cone-ratio represents substitution rates, if the related inputs or outputs are substituted rather than accumulated under their appropriate rates, individual UDMUs could be generated, instead of an entirely new data set. Further, this is only necessary for the DEA-efficient DMUs - as they are the only DMUs that will possibly remain efficient under the substitutions. So, adding these UDMUs to the observed data set should modify the PPS in the desired manner to express the DM's values in the assessment.

ii. Combining Existing Approaches

As illustrated in this thesis, including values into a DEA assessment modifies the PPS. At present the DEA-efficiency of the UDMUs and their impact on the envelopment of the inefficient DMUs is rather hit and miss. Thus, rather than depending solely on the DMs insight into the production process to determine suitable estimates for UDMUs, it may be possible to integrate knowledge of the theoretical production function into the model, to aid in extending the observed production frontier in an appropriate manner. One such approach that attempts to combine incorporating values in a DEA assessment with stochastic approaches is Olesen and Petersen [40].

c) Suitable DEA-Efficient Frontier?

As stated in chapter two, it has been assumed in the development of the procedure of section C that the values of the DEA-efficient DMUs are acceptable to the DM. However, there may be situations where this assumption is not acceptable to the DM and they may feel that the input output levels offered by the ADMUs relative to the other DMUs may not be truly efficient. That is, it would be preferable for the DM to manipulate the input output levels of other DEA-efficient DMUs to improve envelopment, incorporate values and discriminate between the DEA-efficient DMUs. This would involve a development of the current procedure. Similarly, if the DM feels that the identified returns to scale of the DEA frontier is not appropriate, then a development of the procedure would be necessary to allow the DM to include UDMUs to modify the returns to scale of the frontier.

d) Improving Envelopment

The procedure developed in section C is aimed at encouraging individual inputs and outputs to contribute to DEA-inefficient DMU's relative efficiency scores. However, the DM may want to simultaneously encourage several inputs or outputs to contribute to a DMU's relative efficiency scores. This will require a development of the proposed approach.

The approach is limited here to an application with controllable inputs, and further research of a more general nature is required into the impacts of the inclusion of values into an assessment where the following may exist:

- Exogeneously Fixed Variables/Categorical Variables

The approach detailed in this thesis assumes that all the inputs and outputs can be freely modified as required. However, this may not always be the case, such as with exogeneously fixed and categorical variables, and the approach needs to be extended in order to take these variables into account, which is true of most approaches for the inclusion of values in a DEA assessment.

- Values Over Time

In general very little attention has been given to incorporating values into a DEA assessment over time, with the exception of Thompson *et al.* [54]. Thus, how to adapt this procedure for such an assessment would be an interesting research question.

- The Sensitivity of the Results to the Inclusion of Value Judgments.

This thesis has concentrated on how to actually capture value judgments in a DEA assessment. A possible further avenue of research leading on from this thesis, would be to consider the sensitivity of the results to changes in the values included via both weights restrictions and UDMUs. For example, there are limits to the changes in the modification of specific relative values of the inputs or outputs and the changes in the obtained results. Does modifying the input/output levels at local levels have a large impact on the variation in the obtained results? Do modifications to specific inputs outputs have a greater impact on the relative efficiency score than others?

- Target Setting

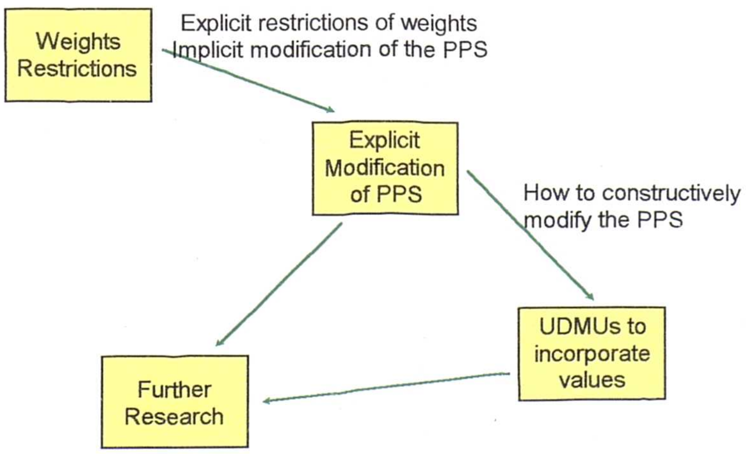
As demonstrated in chapter five, there is a clear need for an approach for setting meaningful targets once value judgments have been included in the assessment. It can be seen that basing targets on observed DMUs only, can provide results that suggest decreases in output levels or increases in input levels for the DMUs to be deemed relatively efficient. Clearly, these targets contradict the objectives of setting targets and provide the DM with no incentive and meaning. However, basing targets on unobserved input output levels can also be problematic in that, as the levels have not actually been observed, it is not known whether they can be achieved (this will always be the case for DEA-efficient DMUs). Therefore, some form of compromise needs to be found.

- Absolute Weights Restrictions

The results reported in chapter four and seven suggest that the use of absolute/virtual restrictions in their present format is questionable, although there are special cases for their implementation where their use is acceptable. See, for example Dyson and Thanassoulis [27]. Further research is required into the use of weights restrictions as, in many applications, their use is extremely simple, and is a desirable approach for capturing value judgments.

Finally, the research process of this thesis is shown in Figure 9.1.

Figure 9.1 - The Research Process of this Thesis



Appendices

Appendix 4

Appendix 4.1

Simulating Relative Output Weights Restrictions: A Specific Example

For ease of explanation, the model (M4.1) is reproduced here as model (A4.M1).

$$h^*_{j_0} = Max\ u_1 y_{1j_0} + u_2 y_{2j_0}$$

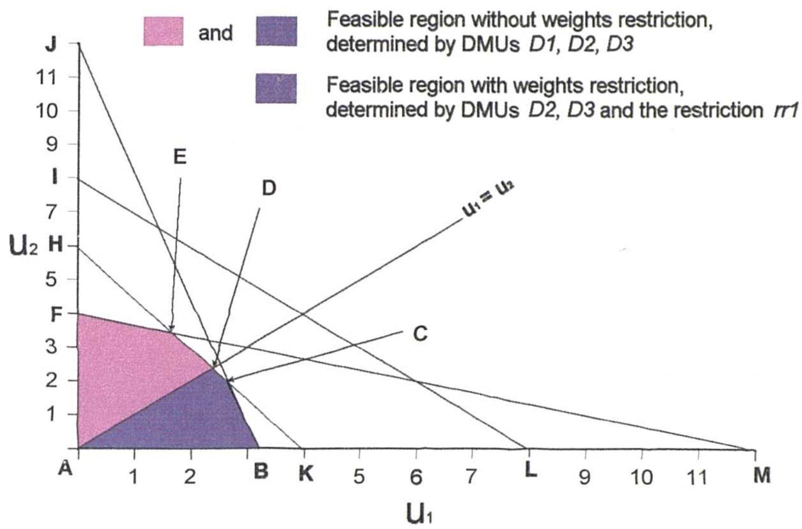
$u_1 + 3u_2 \leq 12$	:D1	(A4.M1)
$3u_1 + 2u_2 \leq 12$	:D2	
$3.75u_1 + u_2 \leq 12$	:D3	
$1.5u_1 + 1.5u_1 \leq 12$	D4	
$-u_1 + u_2 \leq 0$	:rr1	
$-u_2 \leq -\varepsilon$		

(ε is a non-Archimedean infinitesimal)

Notation as in (M4.1).

The conditions under which rr1 is binding for all optimal solutions of (A4.M1), can be readily deduced by examining its feasible region shown in Figure A4.1. The lines FM, HK, JB and IL are defined when the constraints corresponding to D1, D2, D3 and D4 respectively are binding, while $u_1 = u_2$ represents the case when rr1 is binding in (A4.M1).

Figure A4.1 - Output Weight Space of (A4.M1)



The feasible region of (A4.M1) is ABCD and is defined by the DMUs D3, D2 and rr1. Given that $y_{rj} \geq 0 \ \forall j,r$ with at least one of these being strictly positive, it follows that one of the vertices B, C or D will be optimal when assessing a DMU in (A4.M1). The weights restriction rr1 will only be binding for those DMUs of (A4.M1) which have D as their UNIQUE optimal solution. From the graph, it can be deduced that D will be the unique optimal solution for DMUs of (A4.M1) if their objective function slope in (A4.M1), $-y_{1j}/y_{2j}$, (not plotted), is larger than the slope of the line HK (representing D2) which is -3/2. Thus, rr1 is binding for a DMU in (A4.M1) if $-y_{1j}/y_{2j} > -3/2$ or

$$y_{1j} < \frac{3y_{2j}}{2} \qquad \text{(A4.1)}$$

Thus for all the DMUs with output levels that satisfy (A4.1) the optimal solution to (A4.M1) provides a unique optimal solution and rr1 will not be redundant.

Effectively the introduction of the weights restriction rr1 in (A4.M1) introduces a new vertex, D into its feasible region, see Figure A4.1 and when a DMU of (A4.M1) has its unique optimal solution at D, the efficiency score of this DMU is affected by rr1.

Now, consider replacing rr1 in (A4.M1) by one UDMU. Obviously, the UDMU must be DEA-efficient, otherwise its introduction will have no impact on the relative efficiency scores of the DMUs. Without rr1, the feasible region for (A4.M1) is the area labelled ABCEF in Figure A4.1. If rr1 is to be simulated, then the DEA-efficient UDMU must reduce ABCEF to create a subset of it, sub-ABCEF which will need to be such that

(i) it contains D

and

(ii) D is optimal for given values of y_{1j} and y_{2j} in (A4.M1), whether the feasible region is ABCD or sub-ABCEF.

To satisfy condition (i) the UDMU must be DEA-efficient and define a line that introduces the vertex D into the feasible region. At D, $u_1 = u_2 = 2.4$ and for DEA-efficiency it is required that $u_1y_1 + u_2y_2 = 12$. Hence UDMUs must offer output levels (y_1, y_2) such that $y_1+y_2=5$.

Condition (ii) is satisfied by (A4.1).

Hence any UDMU offering output levels such that $y_1 + y_2 = 5$ and $y_1 < 3y_2/2$ can simulate rr1 in (A4.M1).

APPENDIX 4.2

Proof of Theorem 4.1

For ease of explanation models (M4.3) and (M4.4) are reproduced here as (A4.M2) and (A4.M3) respectively.

(A4.M2)	(A4.M3)
$h^*_{j_0} = Max \sum_{r=1}^{s} u_r y_{rj_0}$	$h^/_{j_0} = Max \sum_{r=1}^{s} \delta_r y_{rj_0}$
s.t. $\sum_{i=1}^{m} v_i x_{ij_0} = 1$	s.t. $\sum_{i=1}^{m} \gamma_i x_{ij_0} = 1$
$\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \le 0 \quad j=1,\dots,N$	$\sum_{r=1}^{s} \delta_r y_{rjt} - \sum_{i=1}^{m} \gamma_i x_{ijt} \le 0 \quad jt=1,\dots,N$
$u \in U, \quad v \in V, \quad uv \in UV$	$\delta_r, \gamma_i \ge \varepsilon \quad \forall r, i$

Notation in (A4.M2) and (A4.M3) as in (M4.3) and (M4.4) respectively. The RDMUs, $jt=1,\dots,N$ in (A4.M3) are derived using (4.1). It is necessary to show that $h^/_{j_0} = h^*_{j_0}$.

Proof

Let $u^*_r, v^*_i, \gamma^*_i, \delta^*_r$, be respectively the optimal values of u_r, $r=1,\dots,s$, v_i, $i=1,\dots,m$ and δ_r, $r=1,\dots,s$, γ_i, $i=1,\dots,m$ obtained in (A4.M2) and (A4.M3) respectively.

From the constraints of (A4.M3) for $jt_0=j_0$ it follows that $\sum_{r=1}^{s} \delta^*_r y_{rj_0} \le \sum_{i=1}^{m} \gamma^*_i x_{ij_0} = 1$. Using (4.1) this gives $\dfrac{\sum_{r=1}^{s} \delta^*_r y_{rj_0}}{h^*_{j_0}} = \dfrac{h^/_{j_0}}{h^*_{j_0}} \le 1$, or

$$h^/_{j_0} \le h^*_{j_0} \qquad \text{(A4.2)}$$

The solution, $\delta_r = u^*_r$, $\forall r$ and $\gamma_i = v^*_i$, $\forall i$, is feasible in (A4.M3). To show this it is only necessary to show that, $\delta_r = u^*_r$, $\forall r$ and $\gamma_i = v^*_i$, $\forall i$, satisfies the constraints $jt=1,\dots,N$ in

(A4.M3). A feasible solution u_r^*, $r=1,...,s$, v_i^* $i=1,...,m$ which is feasible for one assessed DMU j_0 of (A4.M2) will also be feasible for another assessed DMU $j \neq j_0$ of (A4.M2) provided the weights restrictions in (A4.M2) all have zero RHS value, see Appendix 4.5. Thus, for any DMU j in (A4.M2), $\sum_{r=1}^{s} u_r^* y_{rj} = h_j \sum_{i=1}^{m} v_i^* x_{ij}$, with $h_j \leq h_j^* \leq 1$, where h_j^* is the efficiency yielded by (A4.M2). So, it follows that, $\frac{\sum_{r=1}^{s} u_r^* y_{rj}}{h_j^*} \leq \sum_{i=1}^{m} v_i^* x_{ij}$ and by recourse to (4.1), for $j=jt$ the following holds:

$$\sum_{r=1}^{s} u_r^* y_{rjt} - \sum_{i=1}^{m} v_i^* x_{ijt} \leq 0 \qquad jt=1,...,N \qquad \text{(A4.3)}$$

Thus $\delta_r = u_r^*$, $\forall r$ and $\gamma_i = v_i^*$, $\forall i$ satisfy the constraints $jt=1,...,N$ of (A4.M3) and provide a feasible solution to this model. This implies $\sum_{r=1}^{s} u_r^* y_{rj_0} = h_{j_0}^*$ is a feasible objective function value to (A4.M3) and so

$$h_{j_0}^* \leq h_{j_0}^{/} \qquad \text{(A4.4)}$$

Clearly (A4.2) and (A4.4) imply

$$h_{j_0}^{/} = h_{j_0}^* \qquad \text{(A4.5)}$$

QED

APPENDIX 4.3

Proof of Propositions 4.1-4.3

Let $h_{jt_0}^{//}$ be as defined in model (M4.5), with the following envelopment model being the dual to (M4.5).

$$h_{jt_0}^{//} = Min \quad \theta_{jt_0} - \varepsilon\left(\sum_{i=1}^{m} S_i + \sum_{r=1}^{s} S_{m+r}\right)$$

$$s.t. \quad \theta_{jt_0} x_{ijt_0} - \sum_{jt \in Jrt} \lambda_{jt} x_{ijt} - \sum_{j \in JE} \lambda_j x_{ij} - S_i = 0 \qquad i=1,\dots,m \qquad \text{(A4.M4)}$$

$$\sum_{jt \in Jrt} \lambda_{jt} y_{rjt} + \sum_{j \in JE} \lambda_j y_{rj} - S_{m+r} = y_{rjt_0} \qquad r=1,\dots,s$$

$$S_i,\ S_{m+r},\ \lambda_j,\ \lambda_{jt} \geq 0 \qquad \forall i,\ r,\ j,\ jt.$$

S represent the slack variables. Let * denote the value of the corresponding variable at the optimal solution to (A4.M4).

Proposition 4.1

a) If $h_{jt_0}^{//} > 1$ then DMU jt_0 is not a redundant RDMU.

Proof

Clearly if $h_{jt_0}^{//} > 1$ then $\theta_{jt_0}^{*} > 1$. Further, at least one S^*_i will be zero as the minimisation of θ_{jt_0} has pre-emptive priority. Hence within (A4.M4) the input output levels of RDMU jt (x_{ijt}, $i=1,\dots,m$; y_{rjt}, $r=1,\dots,s$) cannot be expressed as a linear combination of other DMUs or RDMUs. Hence RDMU jt_0 does not generate a redundant constraint in (A4.M4) and thus it is not a redundant RDMU.

b) If (A4.M4) has no feasible solution then RDMU jt_0 is not redundant.

Proof

If no feasible solution to (A4.M4) exists, then RDMU jt_0 cannot be expressed as a linear combination of the other DMUs or RDMUs. Hence RDMU jt_0 is not a redundant RDMU.

Proposition 4.2

If $h^{//}_{jt_0}$ = 1 in (A4.M4) then RDMU jt_0 is a redundant RDMU.

Proof

Consider the optimal solution to (A4.M4), where $h^{//}_{jt_0} = \theta^*_{jt_0} - \varepsilon\left(\sum_{i=1}^{m} S_i^* + \sum_{r=1}^{s} S_{m+r}^*\right) = 1$, with DMU jt_0 being DEA-efficient, i.e. $\theta^*_{jt_0} = 1$, and S_i^* = S_{m+r}^* $= 0$ $\forall r, i$. Hence it can be seen from the constraints of (A4.M4) that the optimal λ values in (A4.M4) express the input output levels of RDMU jt_0 as a linear combination of the input output levels of other DMUs or RDMUs in (A4.M4). Therefore RDMU jt_0 is a redundant RDMU in (A4.M4).

Proposition 4.3

At the optimal solution to (A4.M4) $h^{//}_{jt_0}$ cannot be less than 1.

Proof

From the construction of RDMU jt_0 we know that (A4.M2) has a feasible solution (u^*, v^*) that renders RDMU jt_0 DEA-efficient in the sense that $\sum_{r=1}^{s} u_r^* y_{rjt_0} = 1$, $\sum_{i=1}^{m} v_i^* x_{ijt_0} = 1$ and therefore $h^{//}_{jt_0} = 1$. Appendix 4.2 showed that any solution (u^*, v^*) to (A4.M2) is feasible in (A4.M3). Further, since (A4.M4), which is the dual to (A4.M3), but contains only a subset of the constraints of (A4.M3) the solution will also be a feasible in (A4.M4). Thus at the optimal solution to (A4.M4) we cannot have $h^{//}_{jt_0} < 1$.

QED.

Appendix 4.4

Proof of Theorem 4.3

For ease of explanation models (M4.6) and (M4.7) are reproduced here as models (A4.M5) and (A4.M6) respectively.

(A4.M5)			(A4.M6)	
$e^*_{j_0} = Max\sum_{r=1}^{s} u_r y_{rj_0}$			$e^/_{j_0} = Max\sum_{r=1}^{s} \tau_r y_{rj_0}$	
s.t. $\sum_{i=1}^{m} v_i x_{ij_0} = 1$			s.t. $\sum_{i=1}^{m} \delta_i x_{ij_0} = 1$	
$\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \le 0$		$j=1,\ldots,N$	$\sum_{r=1}^{s} \tau_r y_{rj} - \sum_{i=1}^{m} \delta_i x_{ij} \le 0$	$j=1,\ldots,N$
$\rho_i \le v_i \le \zeta_i$	:ar1	$i=1,\ldots,m$	$\sum_{r=1}^{s} \tau_r y_{rjp_0} - \sum_{i=1}^{m} \delta_i x_{ijp_0} \le 0$	
$\rho_r \le u_r \le \zeta_r$	:ar2	$r=1,\ldots,s$	$\delta_i, \tau_r \ge \varepsilon$	$\forall i, r$

Notation in (A4.M5) and (A4.M6) as in (M4.5) and (M4.6) respectively. The RDMU jp_0 in (A4.M6) is derived using (4.4). It is necessary to show that $e^/_{j_0} = e^*_{j_0}$.

Proof

Let $u^*_r, v^*_i, \delta^*_i, \tau^*_r$, be respectively the optimal values of u_r, $r=1,\ldots,s$, v_i, $i=1,\ldots,m$, and δ_i $i=1,\ldots,m$; τ_r $r=1,\ldots,s$ obtained in (A4.M5) and (A4.M6) respectively.

From the constraints of (A4.M6) for $jp_0=j_0$ it follows that $\sum_{r=1}^{s} \tau^*_r y_{rjp_0} \le \sum_{i=1}^{m} \delta^*_i x_{ijp_0} = 1$.

Using (4.4) this gives for $jp_0=j_0$, $\dfrac{\sum_{r=1}^{s} \tau^*_r y_{rj_0}}{e^*_{j_0}} = \dfrac{e^/_{j_0}}{e^*_{j_0}} \le 1$, or

$$e^/_{j_0} \le e^*_{j_0} \qquad \text{(A4.6)}$$

The solution, $\tau_r = u_r^*$, $\forall r$ and $\delta_i = v_i^*$, $\forall i$, is feasible in (A4.M6). To show this it is only necessary to recall that, $\tau_r = u_r^*$, $\forall r$ and $\delta_i = v_i^*$, $\forall i$, satisfies the constraint jp_0 in (A4.M6) which is true by virtue of (4.4), so the following holds:

$$\sum_{r=1}^{s} u_r^* y_{rjp_0} - \sum_{i=1}^{m} v_i^* x_{ijp_0} \le 0 \qquad \text{(A4.7)}$$

Thus the solution $\tau_r = u_r^*$, $\forall r$ and $\delta_i = v_i^*$, $\forall i$ is feasible in (A4.M6), this implies $\sum_{r=1}^{s} u_r^* y_{rj_0} = e_{j_0}^*$ is a feasible objective function value to (A4.M6) and so

$$e_{j_0}^* \le e_{j_0}^{/} \qquad \text{(A4.8)}$$

Clearly (A4.6) and (A4.8) imply

$$e_{j_0}^{/} = e_{j_0}^* \qquad \text{(A4.9)}$$

QED

Appendix 4.5

Linear and Non-Linear Programming Equivalencies

This appendix will show that:

(i) If the model (M1.3) with the inclusion of relative and linked-dependent weights restrictions with a zero RHS value is solved, then a feasible solution for DMU j_0, will be feasible for any other assessed DMU $j \neq j_0$.

(ii) If the model (M1.3) with the inclusion of absolute restrictions or relative and linked-dependent weights restrictions with a non-zero RHS value is solved, then a feasible solution for DMU j_0, may not be feasible for any other assessed DMU $j \neq j_0$.

Proof

Consider assessing a set N DMUs, $j=1,...,N$ with DMU j using input levels x_{ij}, $i=1,...,m$ to produce output levels y_{rj}, $r=1,...,s$. Further, it is assumed that the DMUs are to be assessed with additional constraints on their DEA weights. The relative efficiency $h^*_{j_0}$ of DMU j_0 is given by (A4.M7).

$$
\begin{aligned}
& h^*_{j_0} = Max \sum_{r=1}^{s} u_r y_{rj_0} & & \\
s.t.\quad & \sum_{i=1}^{m} v_i x_{ij_0} = 1 & & \\
& \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \leq 0 & & j=1,...,N \\
& u_r \geq \alpha_r & :r1 \quad & r=1,...,s \\
& v_i \geq \gamma_i & :r2 \quad & i=1,....,m \\
& u_k - u_{k+1} \leq 0 & :r3 \quad & \text{for some } k \\
& v_l - v_{l+1} \leq 0 & :r4 \quad & \text{for some } l \\
& u_p - v_t \leq 0 & :r5 \quad & \text{for some } p, t
\end{aligned}
\qquad \text{(A4.M7)}
$$

v_i and u_r are the weights attached to the inputs and outputs respectively and α and γ are DM specified bound values.

To convert the model into a non-linear model, the following transformations are used:

$$u_r = \mu_r t \qquad v_i = \beta_i t \qquad t^{-1} = \sum_{i=1}^{m} \beta_i x_{ij_0}$$

which, with $t > 0$ gives:

$$h^*_{j_0} = Max \frac{\sum_{r=1}^{s} \mu_r y_{rj_0}}{\sum_{i=1}^{m} \beta_i x_{ij_0}} \qquad \text{(A4.M8)}$$

$$s.t. \quad \frac{\sum_{r=1}^{s} \mu_r y_{rj}}{\sum_{i=1}^{m} \beta_i x_{ij}} \le 1 \qquad j=1,\dots,N$$

$$\mu_r \ge \alpha_r \sum_{i=1}^{m} \beta_i x_{ij_0} \quad :r1 \quad r=1,\dots,s$$

$$\beta_i \ge \gamma_i \sum_{i=1}^{m} \beta_i x_{ij_0} \quad :r2 \quad i=1,\dots,m$$

$$\mu_k - \mu_{k+1} \le 0 \quad :r3 \quad \text{for some } k$$

$$\beta_l - \beta_{l+1} \le 0 \quad :r4 \quad \text{for some } l$$

$$\mu_p - \beta_t \le 0 \quad :r5 \quad \text{for some } p,\ t$$

β_i and μ_r are the weights attached to the inputs and outputs respectively and α and γ are the DM specified bound values on the numerical DEA weight values of (A4.M7).

Let μ^*_r and β^*_i be optimal solution values for μ_r and β_i respectively in (A4.M8).

Proof of (i): Weights Restrictions with a Zero RHS Value

If model (A4.M7) only contains additional constraints on the DEA-weights of a similar form to r3-r5, then a feasible solution for DMU j_0 (μ^*_r and β^*_i) in (A4.M8) will also be feasible when assessing DMU $j \neq j_0$ in (A4.M8) as all this does is change the objective function of the model.

Proof of (ii): Weights Restrictions with a Non-Zero RHS Value

If model (A4.M7) contains additional constraints on the DEA-weights in the form of r1-r2, (or r3-r5 with a non-zero RHS value) then a feasible solution for DMU j_0 (μ^*_r and β^*_i) in (A4.M8) may not be feasible when assessing DMU $j \neq j_0$ in (A4.M8) as the constraints r1-r2, are now DMU dependent, and thus, in addition to changing the objective function of the model, the constraints r1-r2, are also changed.

Appendix 6

Appendix 6.1

Identifying Anchor DMUs

Consider assessing a set of N DMUs $j=1,\ldots,N$, each using varying amounts of m different inputs, x_{ij}, $i=1,\ldots,m$ to produce varying quantities of s different outputs, y_{rj}, $r=1,\ldots,s$. Let the set JE consist of the DEA-efficient DMUs identified using model (M6.2) and let JE_{j_0} be the set JE excluding DMU j_0. In respect of each DEA-efficient DMU j_0 solve the envelopment model (A6.M1).

$$
\begin{aligned}
h^{/}_{j_0} &= Min\ z_0 - \varepsilon\left(\sum_{i=1}^{m} H_i + \sum_{r=1}^{s} H_{m+r}\right) && && \\
s.t.\quad & z_0 x_{ij_0} - \sum_{j \in JE_{j_0}} \tau_j x_{ij} - H_i = 0 && i=1,\ldots,m && \text{(A6.M1)} \\
& \sum_{j \in JE_{j_0}} \tau_j y_{rj} - H_{m+r} = y_{rj_0} && r=1,\ldots,s && \\
& H_i,\ H_{m+r},\ \tau_j \geq 0 && \forall i, r, j \in JE_{j_0} &&
\end{aligned}
$$

H_i and H_{m+r} represent slack variables. Superscripts $^{/}$ will be used to denote the value of a variable at the optimal solution to (A6.M1).

DMU j_0 is classed as an ADMU if:

a) $h^{/}_{j_0} > 1$ and it has at least one $H^{/}_i > 0$ or $H^{/}_{m+r} > 0$.

or

b) (A6.M1) has no feasible solution.

Proof of (a)

Consider assessing DMU j_0 under model (A6.M2), after scaling its inputs to $z^{/}_0 x_{ij_0}$, $i=1,\ldots,m$ to give it a radial efficiency of 1 in (A6.M1).

$$g_0 = Max \sum_{r=1}^{s} S_{m+r} + \sum_{i=1}^{m} S_i$$

$$s.t. \quad z^{/}_0 x_{ij_0} - \sum_{j \in JE_{j_0}} \lambda_j x_{ij} = S_i \qquad i=1,\ldots,m \qquad \text{(A6.M2)}$$

$$-y_{rj_0} + \sum_{j \in JE_{j_0}} \lambda_j y_{rj} = S_{m+r} \qquad r=1,\ldots,s$$

$$\lambda_j,\ S_i,\ S_{m+r} \geq 0 \qquad \forall i,\ r,\ j \in JE_{j_0}$$

$z^{/}_0$ is the optimal value of z_0 in (A6.M1) and S_i and S_{m+r} represent slack variables.

If (a) holds then DMU j_0 will yield $g_0 > 0$ in (A6.M2), and by definition the assessed DMU is deemed to be of class F. This shows that DEA-efficient DMU j_0 can be rendered class F under SE with respect to JE_{j_0}, and therefore it is an ADMU.

Proof of (b)

DEA-efficient DMUs fall into two categories ADMUs and non-ADMUs. A non-ADMU in (A6.M1) meets the following conditions:

- $h^{/}_{j_0} = 1$

or

- $h^{/}_{j_0} \neq 1$ and $H^{/}_i = H^{/}_{m+r} = 0$ for $i=1,\ldots,m$ and $r = 1,\ldots,s$.

DMU j_0 does not meet these conditions when (A6.M1) has no feasible solution and so it must be an ADMU.

Appendix 6.2

Improving Envelopment

Consider assessing a set of N DMUs $j=1,\ldots,N$, each using m different inputs, x_{ij}, $i=1,\ldots,m$ to produce s different outputs, y_{rj}, $r=1,\ldots,s$, under model (M6.2). Let some of the DMUs be non-enveloped in the sense of Lang *et al.* [38]. Then introducing DEA-efficient UDMUs as local variations of ADMUs in the manner outlined in Theorem 6.1 for determining the input output levels of UDMUs will, in principle, increase the number of properly enveloped DMUs. An outline of the steps involved in the proof is now given:

(i) It is feasible that if a class F DMU *jf* is a referent DMU to ADMU *jp* in (M6.4) then DMU *jf* will have ADMU *jp* as a peer DMU in (M6.2).

(ii) Introducing an UDMU *ja* created from ADMU *jp* will in principle improve envelopment of DMU *jf* which had DMU *jp* as a peer.

The above steps will now be detailed.

Proof of (i)

It is feasible that if a class F DMU *jf* is a referent DMU to ADMU *jp* in (M6.4) then DMU *jf* will have ADMU *jp* as a peer DMU in (M6.2).

Consider using model (M6.2), reproduced here as (A6.M3) for convenience, to assess the efficiency of DMU j_0.

$$h^*_{j_0} = Min\ q_0 - \varepsilon\left(\sum_{i=1}^{m} S_i + \sum_{r=1}^{s} S_{m+r}\right)$$

$$s.t. \quad q_0 x_{ij_0} - \sum_{j=1}^{N} \kappa_j x_{ij} - S_i = 0 \qquad i=1,\ldots,m \qquad \text{(A6.M3)}$$

$$\sum_{j=1}^{N} \kappa_j y_{rj} - S_{m+r} = y_{rj_0} \qquad r=1,\ldots,s$$

$$\kappa_j,\ S_i,\ S_{m+r} \geq 0 \qquad \forall j,\ i,\ r$$

S represent slack variables. Let * denote the value of a variable at the optimal solution to (A6.M3).

Suppose that all class NF DMUs have been adjusted using (6.1), so that they are now class F DMUs. Let JF, $jf=1,\ldots,|JF|$ denote the set of observed and radially adjusted class F DMUs. Let JA be the set of ADMUs. Consider assessing each $j_0 \in JA$ under (A6.M4).

$$h'_{j_0} = Min\ z_0 - \varepsilon\left(\sum_{i=1}^{m} H_i + \sum_{r=1}^{s} H_{m+r}\right)$$

$$\text{s.t.} \quad z_0 x_{ij_0} - \sum_{j \in JE_{j0}} \tau_j x_{ij} - \sum_{jf \in JF} \tau_{jf} x_{ijf} - H_i = 0 \qquad i=1,\ldots,m \qquad \text{(A6.M4)}$$

$$\sum_{j \in JE_{j0}} \tau_j y_{rj} + \sum_{jf \in JF} \tau_{jf} y_{rjf} - H_{m+r} = y_{rj_0} \qquad r=1,\ldots,s$$

$$H_i,\ H_{m+r},\ \tau_j,\ \tau_{jf} \geq 0 \qquad \forall i,r,\ j \in JE_{j_0},\ jf \in JF$$

H represent slack variables and (A6.M4) allows only the class F DMUs of JF and the DEA-efficient DMUs excluding DMU j_0 to be referent DMUs to DMU j_0. Let JP_{j_0} be the set of DEA-efficient referent DMUs to DMU j_0 in (A6.M4). Let $'$ denote the value of a variable at the optimal solution to (A6.M4).

Let the assessed ADMU j_0 in (A6.M4) be ADMU jp i.e. $j_0 = jp$. It is found that ADMU jp has one class F DMU, jf_a identified as one of its referent DMUs, i.e. $\tau'_{jf_a} > 0$. Thus, at the optimal solution to (A6.M4) the input output levels of jf_a can be expressed as a linear combination of DMU jp and other DEA-efficient DMUs plus possibly a slack value, i.e.

$$x_{ijf_a} = \frac{z'_{jp} x_{ijp} - \sum_{j \in JP_{jp}} \tau'_j x_{ij} - H'_i}{\tau'_{jf_a}} \qquad i=1,\ldots,m$$

$$y_{rjf_a} = \frac{y_{rjp} - \sum_{j \in JP_{jp}} \tau'_j y_{rj} + H'_{m+r}}{\tau'_{jf_a}} \qquad r=1,\ldots,s \qquad \text{(A6.1)}$$

In (A6.1) DMU jp has coefficients of z'_{jp} / τ'_{jf_a} for x_{ijp}, $i=1,\ldots,m$ and $1 / \tau'_{jf_a}$ for y_{rjp}, $r=1,\ldots,s$. Therefore when assessing jf_a in (A6.M3) it is feasible one of its peers will be DMU jp, i.e. $\kappa^*_{jp} > 0$. This holds, if the original DMU in (A6.M3) corresponding to DMU jf_a is a class F or NF DMU.

Proof of (ii)

Introducing an UDMU *ja* created from ADMU *jp* will in principle improve the envelopment of non-enveloped DMUs that had *jp* as a peer in (A6.M3).

Let DMU ja be an UDMU which is DEA-efficient and created from ADMU jp. Adding DMU ja to the DMUs can, in principle, decrease the number of positive slack values at the optimal solution to (A6.M3) and so increase the number of *properly enveloped DMUs.* To see how the addition of an UDMU ja to the observed data set can increase the number of properly enveloped DMUs consider using model (A6.M3) to assess the efficiency of DMU j_0 which had ADMU jp as one of its peers. Following the addition of a single UDMU ja created from DMU jp as in (a) or (b) below, the model solved to assess DMU j_0 is (A6.M5).

$$\hat{h}_{j_0} = Min\ \theta_0 - \varepsilon\left(\sum_{i=1}^{m} SS_i + \sum_{r=1}^{s} SS_{m+r}\right)$$

$$s.t. \quad \theta_0 x_{ij_0} - \sum_{j=1}^{N} \lambda_j x_{ij} - \lambda_{ja} x_{ija} - SS_i = 0 \qquad i=1,\ldots,m \qquad \text{(A6.M5)}$$

$$\sum_{j=1}^{N} \lambda_j y_{rj} + \lambda_{ja} y_{rja} - SS_{m+r} = y_{rj_0} \qquad r=1,\ldots,s$$

$$\lambda_j,\ SS_i,\ SS_{m+r} \geq 0 \qquad \forall j,\ i,\ r$$

Models (A6.M3) and (A6.M5) differ only in that the latter contains the additional variable λ_{ja} corresponding to the UDMU ja. Superscripts * will be used to denote the value of a variable at the optimal solution to model (A6.M5).

There are two approaches for the basis of the creation of UDMUs to be used in (A6.M5):

- Encourage the non-ε weighting of an individual output: Lower an output level.
- Encourage the non-ε weighting of an individual input: Raise an input level.

These will be considered now:

a) Encouraging the non-ε weighting of an individual output: Let DMU j_0 in (A6.M3) have a $S^*_{m+k} > 0$ for one k with ADMU jp as one of its peer DMUs.

As the introduction of the UDMU is to encourage the non-ε weighting of output k, the output k level of ADMU jp will be set to zero (it is assumed here that the DM specified minimum level for the output is zero, but it could be a minimum output level). One way to construct a DEA-efficient DMU, is to raise the remaining $s-1$ output levels of ADMU jp. Thus, an UDMU ja is created as a local variation of ADMU jp with input output levels of

$$\begin{aligned} &x_{ija} = x_{ijp} && i=1,\ldots,m \\ &\mathbf{y_{kja} = 0} && \\ &y_{rja} = y_{rjp} + B_r && \forall r \neq k \end{aligned} \qquad \text{(A6.2)}$$

where B_r, $r = 1,\ldots,s$ $r \neq k$ are DM specified levels of sufficient size to enable DMU ja to be deemed by the DM to be DEA-efficient. Consider the solution to (A6.M5). Depending on the values of B_r, $\forall r \neq k$ in (A6.2) it will be the case that $\lambda^*_{ja} > 0$.

To see this note that if in (A6.M5) the following holds, $\lambda^*_{jp} + \lambda^*_{ja} < \kappa^*_{jp}$ and $\kappa^*_j = \lambda^*_j$ $\forall j \neq jp$, (A6.M5) will give a lower optimal objective function value than (A6.M3) and would therefore be preferred to the original optimal solution to (A6.M3), provided it is feasible.

To see that the optimal objective value will be lower in (A6.M5) than in (A6.M3), consider some binding constraint i' at the optimal solution to (A6.M3). [Non-binding constraints will not affect the optimal solution.] The constraint reduces to

$\sum_{j=1}^{N} \kappa^*_j x_{i'j} = q^*_0 x_{i'j_0}$ in (A6.M3) and when $\lambda^*_{jp} + \lambda^*_{ja} < \kappa^*_{jp}$ and $\kappa^*_j = \lambda^*_j$ $\forall j \neq jp$ it becomes in (A6.M5),

$$\sum_{\substack{j=1 \\ j \neq jp}}^{N} \kappa^*_j x_{i'j} + (\lambda^*_{jp} + \lambda^*_{ja}) x_{i'jp} = \theta^*_0 x_{i'j_0} \qquad \text{(A6.3).}$$

Since $\sum_{j=1}^{N} \kappa^*_j x_{i'j} = q^*_0 x_{i'j_0}$, from (A6.M3) and $\lambda^*_{jp} + \lambda^*_{ja} < \kappa^*_{jp}$ and $\kappa^*_j = \lambda^*_j$ $\forall j \neq jp$, then provided $\lambda^*_{ja} > 0$, (A6.4) can be balanced with a solution value of $q^*_0 > \theta^*_0$, as required.

To see that the solution in which $\lambda^*_{jp} + \lambda^*_{ja} < \kappa^*_{jp}$ and $\kappa^*_j = \lambda^*_j$ $\forall j \neq jp$ can be feasible in (A6.M5) consider some binding constraint r' at the optimal solution to (A6.M3). [Non-binding constraints will not effect the optimal solution.] The constraint reduces to $\sum_{j=1}^{N} \kappa^*_j y_{r'j} = y_{r'j_0}$ in (A6.M3) and when $\lambda^*_{jp} + \lambda^*_{ja} < \kappa^*_{jp}$ and $\kappa^*_j = \lambda^*_j$ $\forall j \neq jp$ it becomes in (A6.M5),

$$\left[\sum_{\substack{j=1 \\ j \neq jp}}^{N} \kappa^*_j y_{r'j} + (\lambda^*_{jp} + \lambda^*_{ja}) y_{r'jp} \right] + \lambda^*_{ja} B_{r'} = y_{r'j_0} \qquad \text{(A6.4).}$$

Since $\sum_{j=1}^{N} \kappa^*_j y_{r'j} = y_{r'j_0}$, from (A6.M3) $\lambda^*_{jp} + \lambda^*_{ja} < \kappa^*_{jp}$ and $\kappa^*_j = \lambda^*_j$ $\forall j \neq jp$ the sum in the squared brackets is less than the RHS of (A6.4). However, depending on the size of $B_{r'}$, $r=1,\ldots,s$, provided $\lambda^*_{ja} > 0$ (A6.4) can be balanced and the solution $\lambda^*_{jp} + \lambda^*_{ja} < \kappa^*_{jp}$, $\kappa^*_j = \lambda^*_j$ $\forall j \neq jp$ can be feasible in (A6.M5).

Note that when $\lambda^*_{jp} + \lambda^*_{ja} < \kappa^*_{jp}$ and $\kappa^*_j = \lambda^*_j$ $\forall j \neq jp$ at the solution to (A6.M5) it follows that $\sum_{j=1}^{N} \lambda^*_j y_{kj} < \sum_{j=1}^{N} \kappa^*_j y_{kj}$, which implies that for the constraint corresponding to output k, $y_{kj_0} + SS^*_{m+k} < y_{kj_0} + S^*_{m+k}$ which implies that $SS^*_{m+k} < S^*_{m+k}$.

The fact that $\lambda^*_{ja} > 0$ and the slack of output k is reduced for model (A6.M5) to yield an improved objective function value in comparison to that of model (A6.M3) means that (A6.M5) is more likely than (A6.M3) to identify DMU j_0 as a properly enveloped DMU.

QED

b) Encouraging the non-ε weighting of an individual input:
Let DMU j_0 in (A6.M3) have a $S^*_{m+k} > 0$ for one k with ADMU jp as one of its peer DMUs.

As the introduction of the UDMU is to encourage the non-ε weighting of input k, the input k level of ADMU jp will be raised to a DM determined amount and to construct a DEA-efficient DMU, the remaining $m-1$ input levels of ADMU jp will be lowered. Thus, an UDMU ja is created as a local variation of ADMU jp with input output levels of:

$$\begin{aligned} & x_{kja} = B_k \\ & x_{ija} = x_{ijp} - B_i \qquad \forall i \neq k \qquad\qquad \text{(A6.5)} \\ & y_{rja} = y_{rjp} \qquad r=1,\ldots,s \end{aligned}$$

where B_i, $i=1,\ldots,m$ are DM specified levels of sufficient size to enable DMU ja to be deemed by the DM to be DEA-efficient. Consider the solution to (A6.M5). Depending on the values of B_i, $\forall i \neq k$ in (A6.2) it will be the case that $\lambda^*_{ja} > 0$.

To see this note that if in (A6.M5) the following holds $\lambda^*_{jp} + \lambda^*_{ja} = \kappa^*_{jp}$ $\lambda^*_{ja} > 0$ and $\kappa^*_j = \lambda^*_j$ $\forall j \neq jp$, (A6.M5) will give a lower optimal objective function value than (A6.M3) and would therefore be preferred to the original optimal solution to (A6.M3), provided it is feasible.

To see that the solution in which $\lambda^*_{jp} + \lambda^*_{ja} = \kappa^*_{jp}$ and $\kappa^*_j = \lambda^*_j$ $\forall j \neq jp$ can be feasible in (A6.M5) consider some binding constraint i' at the optimal solution to (A6.M3). [Non-binding restrictions will not effect the optimal solution]. The constraint reduces to

$\sum_{j=1}^{N} \kappa^*_j x_{i'j} = q^*_0 x_{i'j}$ in (A6.M3) and when $\lambda^*_{jp} + \lambda^*_{ja} = \kappa^*_{jp}$ and $\kappa^*_j = \lambda^*_j \ \forall j \neq jp$ it becomes in (A6.M5),

$$\left[\sum_{\substack{j=1 \\ j \neq jp}}^{N} \kappa^*_j x_{i'j} + (\lambda^*_{jp} + \lambda^*_{ja}) x_{i'jp}\right] - \lambda^*_{ja} B_{i'} = \theta_0 x_{i'j_0} \qquad \text{(A6.6).}$$

Since $\sum_{j=1}^{N} \kappa^*_j x_{i'j} = q^*_0 x_{i'j_0}$, $\lambda^*_{jp} + \lambda^*_{ja} = \kappa^*_{jp}$, and $\kappa^*_j = \lambda^*_j \ \forall j \neq jp$, provided $\lambda^*_{ja} > 0$, which will depend on the size of $B_{i'}$, $i=1,\ldots,m$, (A6.6) can be balanced and the solution $\lambda^*_{jp} + \lambda^*_{ja} = \kappa^*_{jp}$, $\kappa^*_j = \lambda^*_j \ \forall j \neq jp$ is feasible in (A6.M5), with $q^*_0 > \theta^*_0$, as required.

To see that the slack value for input k in (A6.M3) will be reduced in (A6.M5) consider the constraint for input k, in both (A6.M5) and (A6.M3). Given $q^*_0 > \theta^*_0$, thus $\sum_{j \neq jp} \kappa^*_j x_{kj} + \kappa^*_{jp} x_{kjp} + S^*_k > \sum_{j \neq jp} \lambda^*_j x_{kj} + \lambda^*_{jp} x_{kjp} + \lambda^*_{ja} B_k + SS^*_k$, which given $\lambda^*_{jp} + \lambda^*_{ja} = \kappa^*_{jp}$, and $\kappa^*_j = \lambda^*_j \ \forall j \neq jp$, becomes $S^*_k > \lambda^*_{ja}(B_k - x_{kjp}) + SS^*_k$. Thus as $B_k - x_{kjp} > 0$, it follows that $S^*_k > SS^*_k$, as required.

The fact that $\lambda^*_{ja} > 0$ and the slack of input k is reduced for model (A6.M5) to yield an improved objective function value in comparison to that of model (A6.M3) means that (A6.M5) is more likely than (A6.M3) to identify DMU j_0 as a properly enveloped DMU.

QED

Appendix 6.3

The Input Output Levels of the 668 Bank Branches

	FA	TC	AI	AP	CT	MT	SV
D001	18	348979	90	98	171547	433	2365
D002	5	113681	66.8	59.2	73122	135	908
D003	9	166150	68.8	50.2	71717	145	1039
D004	14	246690	125.2	85.8	208661	138	1819
D005	11	220527	79.8	76.2	87609	214	1397
D006	12	217428	96.8	45.2	157956	131	1890
D007	13	292577	291	87	220520	231	3008
D008	8	141005	109.8	82.2	73449	146	1173
D009	7	133718	66.8	90.2	83332	154	1361
D010	16	426178	425	170	309067	499	3751
D011	13	247055	119	147	141121	233	2241
D012	11	243548	91.2	53.8	235292	154	2931
D013	12	221789	139.2	97.8	122234	214	2050
D014	8	209780	80.2	29.8	153411	85	3698
D015	11	218225	254.4	192.6	97238	445	2235
D016	11	179705	99	31	143335	103	1777
D017	14	236557	167.8	71.2	148008	142	2127
D018	14	346445	279.2	79.8	294343	222	3070
D019	13	220796	178.2	46.8	129931	173	1235
D020	17	255853	156	154	216165	221	3034
D021	16	373041	211.6	115.4	361563	247	4179
D022	14	346074	261.8	64.2	311410	201	4838
D023	13	298947	221.6	151.4	126260	265	2577
D024	12	215757	79.8	104.2	112168	205	2805
D025	11	212659	79.8	117.2	157647	191	2351
D026	16	269273	151.2	60.8	154678	288	1903
D027	18	424608	176.4	113.6	405456	215	5256
D028	13	257118	108	117	190710	199	2475
D029	7	182931	84.8	62.2	98231	177	1844
D030	15	287927	90.2	112.8	183855	221	3072
D031	13	278747	139.6	181.4	228746	236	3077
D032	12	202337	59.2	53.8	121679	150	1875
D033	14	256603	279.2	175.8	184091	135	2807
D034	14	232306	158.2	134.8	145362	172	1932
D035	14	267262	124.2	100.8	252473	287	3323
D036	12	246253	94.4	70.6	252237	146	3055
D037	15	327193	259.8	124.2	299664	246	3426
D038	13	296114	187.8	106.2	208103	204	2744
D039	15	323106	140.2	100.8	238893	240	3125
D040	13	202059	130	44	156817	158	1717
D041	14	201541	43.2	36.8	151411	118	1753
D042	13	327320	158.6	113.4	299782	253	3425
D043	13	283126	292	119	207944	247	3583
D044	9	262724	214	101	225774	181	2895
D045	13	303453	273	155	175349	178	3054

	FA	TC	AI	AP	CT	MT	SV
D046	17	383999	295.6	220.4	243386	416	3699
D047	18	292884	108.8	118.2	196735	143	2781
D048	16	346997	292.8	178.2	299039	318	4092
D049	15	362072	388.6	123.4	261621	370	4114
D050	12	349679	258.2	104.8	263755	283	3462
D051	12	336254	331.6	103.4	257118	198	3111
D052	9	213500	100.2	62.8	208846	210	2796
D053	16	324816	207.8	148.2	281568	272	3867
D054	15	301841	252	144	258578	288	3756
D055	15	307322	230	213	216594	305	3358
D056	11	181358	129	70	135374	111	1904
D057	7	195615	132.4	49.6	147443	87	1960
D058	11	224209	128.8	85.2	143342	149	2157
D059	13	315102	191	118	282725	253	3215
D060	15	271441	151.8	153.2	198352	251	2955
D061	11	291443	186	118	207785	163	2849
D062	17	281264	84.6	95.4	188053	130	3208
D063	20	232424	262.4	195.6	323954	243	4040
D064	12	394359	385.2	53.8	300915	151	4049
D065	10	297971	149.6	49.4	261883	143	2805
D066	17	333761	207.8	72.2	266985	190	3230
D067	12	290597	215	161	229720	224	3143
D068	15	301013	224.4	120.6	201293	215	3525
D069	14	203188	117.8	106.2	166491	215	2915
D070	12	307713	227.8	52.2	268981	137	3829
D071	12	264766	106.6	76.4	190862	95	2181
D072	7	100787	50.4	31.6	93314	67	1168
D073	11	330134	158.2	89.8	228199	207	2867
D074	13	281979	208.8	112.2	212371	173	2847
D075	11	242217	120.8	90.2	198876	132	2144
D076	8	210356	63.6	73.4	137259	128	2163
D077	12	270460	122	106	195208	188	2044
D078	12	232651	196.2	105.8	141878	198	2353
D079	10	176162	180.2	83.8	112616	177	1730
D080	11	260061	214.2	125.8	208730	176	3034
D081	10	252864	140.4	71.6	170127	181	2150
D082	12	222368	195.2	93.8	253055	165	3068
D083	14	265792	175.8	110.2	162078	228	2510
D084	11	251244	74.2	86.8	214424	150	2670
D085	11	250654	193.4	79.6	220631	150	2153
D086	11	295310	183	85	186356	150	2138
D087	9	171616	186	89	93838	157	1786
D088	16	382264	188	100	365142	277	4810
D089	20	325982	154	124	273763	222	4071
D090	20	375028	272	184	282881	217	4746
D091	13	255169	110.6	67.4	170573	130	2505
D092	12	289716	224.6	140.4	240899	165	3630
D093	9	175814	69	33	134835	91	1584
D094	15	279750	108.6	114.4	184479	179	3148
D095	13	274795	173.2	136.8	227396	211	3277
D096	14	266468	194	104	228538	242	3039

	FA	TC	AI	AP	CT	MT	SV
D097	11	247452	252	65	190863	127	2633
D098	13	229810	166.4	90.6	162772	207	2372
D099	16	267628	172.8	138.2	174140	189	3001
D100	12	311026	211.8	108.2	286208	183	3532
D101	7	87478.8	41.2	21.8	77764	61	939
D102	10	265609	200.4	65.6	245214	128	2795
D103	10	143883	64.2	100.8	78004	109	1751
D104	14	357627	249.6	170.4	301109	401	4464
D105	12	242939	112.4	105.6	209342	225	2783
D106	8	179852	94.4	56.6	159862	134	1537
D107	13	257634	133	113	152466	167	2722
D108	12	229236	114.2	52.8	201695	140	2537
D109	15	290117	286.4	151.6	220802	213	2401
D110	9	205792	28.8	20.2	107408	92	932
D111	15	239897	84.2	97.8	177716	217	2794
D112	8	120909	40.6	40.4	87154	81	1336
D113	9	232284	143.4	62.6	204678	138	2418
D114	19	403231	316.4	110.6	296117	322	4459
D115	8	180841	74.4	47.6	151311	137	2305
D116	12	256000	137	93	165308	158	2890
D117	14	293327	127.4	172.6	230984	251	3510
D118	14	253676	114.8	60.2	225245	99	3853
D119	17	341366	290.2	140.8	272316	315	3930
D120	6	121779	41	42	95938	51	1143
D121	10	197194	149	73	152851	148	2245
D122	14	294696	131.8	118.2	234348	271	2826
D123	12	211878	238.4	30.6	150637	101	1875
D124	11	184024	158.6	80.4	182318	151	2419
D125	12	313156	140.4	105.6	274460	207	3381
D126	10	211802	254.2	121.8	123081	221	2519
D127	14	269720	97.4	127.6	179834	320	1908
D128	14	212207	132.4	72.6	217176	128	2631
D129	5	85463	18.6	9.4	83834	18	811
D130	12	229011	178.4	68.6	169093	182	2203
D131	10	157935	66.2	47.8	126843	131	1680
D132	7	119127	89.4	67.6	94358	135	1652
D133	13	240356	156.8	96.2	186431	180	2328
D134	9	262033	58.6	72.4	228161	196	3295
D135	11	211443	74.8	56.2	169751	65	2708
D136	12	268007	102.8	128.2	294837	169	3222
D137	12	260697	167.2	110.8	218654	207	3569
D138	12	273226	144.2	122.8	211595	181	3631
D139	9	158942	66	27	131879	100	1222
D140	12	275861	158.8	85.2	189362	192	2407
D141	10	206883	155	90	183185	261	2396
D142	9	168743	77.6	49.4	137138	91	1656
D143	11	190971	135	76	114416	148	1704
D144	13	264453	115.2	61.8	228291	125	3144
D145	13	223070	84	72	154708	202	1514
D146	13	320652	126.4	86.6	293982	184	3855
D147	5	83071	65.2	37.8	92322	49	1118

	FA	TC	AI	AP	CT	MT	SV
D148	12	216945	195	34	189687	71	2246
D149	12	247993	142	49	213584	164	2874
D150	13	309822	219.8	219.2	305347	384	3902
D151	10	140620	60.6	30.4	124316	52	1361
D152	15	276942	204.8	95.2	205887	246	3221
D153	10	224518	181.2	56.8	221863	117	2310
D154	12	228684	146.2	131.8	180819	169	2692
D155	6	89259	15	19	75413	52	1076
D156	9	142728	70.2	52.8	115238	66	1687
D157	6	177352	96.6	65.4	161430	125	1802
D158	7	108785	32.8	11.2	165833	27	1577
D159	7	82600	33.8	7.2	104881	25	1039
D160	8	123921	50.2	30.8	124851	52	1966
D161	11	150508	90	52	153620	100	2316
D162	10	218513	106.2	79.8	206719	146	2610
D163	10	164247	215.2	70.8	100060	125	1631
D164	12	256178	201.2	121.8	169915	178	2440
D165	14	374038	204.6	124.4	332874	240	3997
D166	6	106870	76.8	37.2	92559	66	1178
D167	7	97005	46.2	31.8	90444	46	1516
D168	12	261053	162.6	113.4	195751	167	2696
D169	11	246265	179.2	66.8	251575	130	2827
D170	10	211316	177.8	95.2	222463	184	3008
D171	12	219253	172	33	178625	102	2339
D172	13	264868	167.8	90.2	232836	184	2474
D173	11	229036	90.4	67.6	240709	218	2931
D174	9	228600	100	119	184489	115	3088
D175	10	124559	71	32	137097	114	1888
D176	12	333821	227.8	52.2	274768	159	2221
D177	8	222396	120.8	61.2	176658	113	2226
D178	13	249710	183.2	49.8	175085	217	2334
D179	13	331268	344.4	67.6	219816	132	2906
D180	13	268598	207.6	59.4	213187	107	1823
D181	7	170762	137.2	103.8	126425	247	2033
D182	8	197739	91.4	69.6	134551	57	2212
D183	15	297941	190	154	251605	351	3211
D184	10	184098	61.4	72.6	131961	94	1751
D185	12	275054	109	117	200350	165	2645
D186	7	150468	51.6	65.4	112436	159	1503
D187	13	321024	255.4	150.6	178128	277	3787
D188	11	198165	82.8	69.2	166895	149	2399
D189	12	312115	109.4	121.6	295765	293	4422
D190	7	212062	71.8	58.2	240528	83	2343
D191	14	296582	137.4	208.6	253888	305	3636
D192	12	197520	82.6	47.4	185890	118	2256
D193	12	204091	172.2	100.8	185530	162	2460
D194	12	311798	204.6	80.4	290263	209	3494
D195	10	153568	26.8	25.2	150723	69	1620
D196	8	155095	82.2	48.8	156631	114	1720
D197	8	180738	92.6	45.4	123963	130	1863
D198	6	81654	31.6	27.4	82234	77	593

	FA	TC	AI	AP	CT	MT	SV
D199	9	126371	68.2	19.8	128091	70	1271
D200	7	48167	29	41	54199	55	745
D201	10	171680	104.4	61.6	173108	100	2022
D202	4	112796	56.6	31.4	98369	43	1171
D203	5	97679.5	35.6	15.4	96460	42	929
D204	9	215029	158	65	119031	144	1781
D205	11	219169	62.2	59.8	225217	145	2892
D206	11	219180	178.4	73.6	207221	143	2343
D207	12	261667	136.6	130.4	193506	145	2999
D208	6	94828	66.2	37.8	78550	95	1179
D209	10	133517	36.6	28.4	163630	89	1735
D210	9	138656	83.4	14.6	155043	48	1869
D211	9	133900	84.4	27.6	141488	68	1548
D212	9	209815	131	67	209385	141	2188
D213	9	137022	66.4	53.6	106290	102	1624
D214	10	235258	119.2	32.8	193841	98	1928
D215	13	233840	101.8	81.2	186180	197	2367
D216	12	175714	52	66	178108	106	2638
D217	7	200271	141.8	73.2	167738	115	2415
D218	12	268135	156	89	228969	241	3023
D219	6	151725	43	21	176284	79	1791
D220	7	148540	82	26	155682	45	1928
D221	12	318500	462.6	111.4	261076	257	3249
D222	11	250804	132.6	113.4	193864	141	3172
D223	10	194382	90.8	51.2	121670	106	1444
D224	11	232738	140.6	63.4	167402	153	1982
D225	8	133473	38.6	24.4	159762	86	1663
D226	14	266619	216.8	50.2	233227	107	2904
D227	15	309587	189.6	69.4	243866	163	3362
D228	10	193734	101.6	60.4	112777	102	2102
D229	10	165837	63	89	112005	129	1285
D230	7	139787	69.8	76.2	120096	149	1658
D231	8	113250	57.2	31.8	111528	61	1302
D232	14	247014	122.2	42.8	210918	124	2383
D233	13	249385	152.4	137.6	170327	213	3026
D234	10	220741	152	68	171533	122	2471
D235	9	241263	238.2	60.8	201392	130	2265
D236	11	245416	295.8	79.2	210179	214	3154
D237	11	208741	87	80	191984	138	2154
D238	7	151764	39	76	111507	170	1854
D239	10	178186	133.2	73.8	122906	165	1741
D240	16	407944	118.2	103.8	357359	213	3780
D241	6	85272	64.4	36.6	76099	80	1083
D242	10	165188	97.8	41.2	130275	65	1828
D243	6	124244	42.8	46.2	111154	85	1873
D244	12	269438	150.2	97.8	228715	222	3116
D245	9	191717	156.2	58.8	194483	91	2374
D246	6	137785	52.8	88.2	113225	192	1426
D247	9	176726	133.4	71.6	112939	98	1604
D248	7	110648	42.8	26.2	79103	49	1216
D249	7	103378	40.2	14.8	100039	45	1164

	FA	TC	AI	AP	CT	MT	SV
D250	7	143533	188	72	80370	74	1607
D251	7	94095.3	32	18	105126	69	1314
D252	8	141341	41.6	23.4	141318	102	1693
D253	7	139853	108	61	107724	84	1550
D254	13	199264	108.2	66.8	152804	137	2426
D255	9	214125	212	82	156127	193	1943
D256	6	127643	68.4	63.6	102313	136	1629
D257	12	170294	50	38	107399	158	1018
D258	10	164548	103	43	127800	100	1462
D259	11	186147	136.6	28.4	188243	65	2037
D260	7	102967	40.8	58.2	125720	62	1649
D261	12	245058	149.2	20.8	228704	77	2463
D262	9	221150	117.4	69.6	190731	157	2180
D263	8	117243	64	31	83107	86	789
D264	6	87461	62.4	10.6	126187	46	1306
D265	10	263041	135	77	231816	140	2297
D266	9	154909	106	51	156737	97	2182
D267	10	230791	177.8	103.2	260150	161	3075
D268	9	186120	58.4	50.6	191286	112	2176
D269	9	144726	101.8	68.2	141830	77	1895
D270	13	296717	241.4	195.6	196523	179	3100
D271	8	199298	152.4	76.6	179886	151	2310
D272	10	207568	171.4	59.6	154077	166	1998
D273	8	127683	107.4	88.6	98225	110	1424
D274	10	243701	138.4	42.6	250138	171	3190
D275	13	304205	160.4	58.6	295183	196	3566
D276	6	80351	36.6	35.4	79013	59	1094
D277	6	122454	49.2	49.8	64841	87	876
D278	7	178567	60	33	160995	99	1434
D279	9	193791	247.2	53.8	145054	94	1878
D280	9	178376	246.4	69.6	156106	111	2238
D281	9	141987	76	77	112966	110	1774
D282	8	104325	51.8	56.2	95618	110	1341
D283	6	96846.3	23.2	5.8	96357	26	865
D284	11	178251	86.4	55.6	185628	128	2151
D285	9	122185	45.8	18.2	123258	55	1536
D286	7	109082	71	23	94714	51	1207
D287	7	161356	87.6	26.4	122090	61	1230
D288	9	223391	89.4	70.6	182467	128	2947
D289	11	231144	127	113	192154	140	2570
D290	11	188266	120	44	171854	126	2112
D291	10	242955	138	75	217335	144	2665
D292	9	165396	112	38	189720	113	1995
D293	8	178188	139.2	40.8	167984	92	2171
D294	11	201587	122.4	31.6	166383	110	2009
D295	7	109796	59	29	110162	76	1468
D296	9	217038	126.8	83.2	157387	210	1999
D297	7	110566	86.6	31.4	77534	45	1279
D298	10	195523	142.6	107.4	184462	140	2347
D299	9	148957	64.6	36.4	155547	63	2070
D300	11	191093	77.6	60.4	181160	117	2323

	FA	TC	AI	AP	CT	MT	SV
D301	11	241254	214.2	92.8	190167	218	2747
D302	8	211983	48.8	22.2	187506	72	1952
D303	6	114105	36	27	107268	36	1094
D304	11	213519	81.6	115.4	151269	187	2090
D305	14	215573	62.4	103.6	168175	166	2687
D306	16	280000	148.8	99.2	153609	182	2846
D307	9	128280	104.2	16.8	130057	69	1285
D308	7	212814	81.8	107.2	152093	160	2183
D309	9	178336	93.4	70.6	125302	138	1611
D310	8	202945	118.4	26.6	205568	82	1959
D311	10	188870	96	74	193281	142	2654
D312	11	281514	110.4	57.6	248472	119	2597
D313	7	127794	85.8	61.2	94351	79	1455
D314	8	164244	133.8	89.2	134394	137	2198
D315	10	274133	205	75	233524	191	3005
D316	7	97159	73.2	39.8	88090	60	1395
D317	10	199783	128.6	72.4	142733	119	1898
D318	7	99039	18.8	15.2	126452	32	1648
D319	10	174945	64.8	58.2	217763	130	2021
D320	9	153426	75.2	56.8	94855	122	1405
D321	7	154685	106.4	32.6	150298	65	2150
D322	8	141575	65.4	67.6	124020	158	1516
D323	9	159040	69.6	85.4	128224	208	1790
D324	11	168599	62.6	51.4	178370	84	2581
D325	8	151234	74.2	50.8	148568	103	1964
D326	9	168292	183.8	83.2	184363	145	2178
D327	14	268305	194.8	82.2	196912	142	2788
D328	8	123006	115	63	72257	97	1019
D329	6	102551	42.2	67.8	64768	124	916
D330	7	96744	42.6	26.4	95499	48	983
D331	8	137605	62.2	39.8	150480	83	1779
D332	6	115480	87	25	120447	54	1430
D333	12	231016	162.8	116.2	170078	143	2817
D334	5	142177	26.6	11.4	154080	24	2125
D335	10	219311	235.2	74.8	184837	159	2429
D336	8	169242	151	42	165787	100	1632
D337	8	138002	108.4	81.6	101167	128	1764
D338	14	348659	190.2	94.8	280026	150	3001
D339	9	140729	61.4	64.6	108385	111	1551
D340	12	241572	140.4	78.6	113804	124	1601
D341	8	172652	139.8	111.2	119663	160	2224
D342	10	169870	172	67	124470	84	2314
D343	7	97839	101	40	76003	109	1398
D344	6	114370	67.4	39.6	91935	90	1435
D345	7	109372	69.8	29.2	129232	46	1384
D346	5	134977	95	21	155918	87	1767
D347	11	288744	143	99	268668	105	3152
D348	5	172052	96.2	38.8	181622	106	2356
D349	5	140441	84	34	93919	58	1292
D350	11	175806	100	38	152469	87	1789
D351	6	88537	21.8	21.2	87899	39	941

	FA	TC	AI	AP	CT	MT	SV
D352	10	209013	110	71	150654	130	1704
D353	5	70640	39.8	43.2	94159	38	1027
D354	9	229348	135.8	58.2	218245	160	2413
D355	6	91637	27.4	26.6	124269	86	1717
D356	6	81661	20.2	31.8	63536	53	998
D357	8	171867	62.8	26.2	136959	64	1517
D358	5	116068	28.8	28.2	118858	61	1055
D359	7	103098	58.4	53.6	100565	45	2202
D360	8	130012	114	34	129807	103	1618
D361	8	159189	128.4	50.6	174014	112	2072
D362	8	112028	75.6	79.4	119722	161	2135
D363	10	156824	64.6	34.4	174971	129	2265
D364	9	173723	136	45	199399	97	1877
D365	6	122112	97	53	73787	113	1111
D366	6	101989	47.6	20.4	158533	57	1662
D367	5	77857	54.4	35.6	98649	74	1071
D368	8	178412	141.6	117.4	164828	124	2504
D369	7	138336	107.8	88.2	118857	173	1562
D370	6	110229	53.4	55.6	72974	58	1022
D371	8	180716	161.2	132.8	128072	202	2071
D372	7	137308	91	47	127514	108	1392
D373	10	238238	47.2	119.8	239744	109	3724
D374	5	96292	26.2	35.8	67687	83	896
D375	12	302561	152.4	52.6	198768	175	2105
D376	9	138515	60	19	124047	69	1232
D377	9	220912	115.6	66.4	165175	122	2292
D378	10	218944	119	47	177620	167	2200
D379	6	120742	61.6	40.4	113527	84	1658
D380	12	200005	99.8	41.2	225903	122	2762
D381	6	125143	97	19	167305	60	1876
D382	14	214909	141	57	182276	94	1801
D383	6	115842	94.6	35.4	105815	70	1474
D384	7	176476	82.4	20.6	221452	54	2031
D385	8	139892	51.6	38.4	135160	104	1861
D386	6	92224	41.4	24.6	108397	39	1427
D387	11	199045	87	94	200593	182	2753
D388	6	95841	42.8	33.2	73062	77	1090
D389	10	201329	165.4	71.6	192719	83	2450
D390	5	129540	25.2	50.8	135062	37	1906
D391	6	139460	99.8	50.2	91115	109	1480
D392	8	219361	160	49	201009	122	2381
D393	11	191449	123.2	78.8	186431	170	3023
D394	8	136602	68.8	58.2	91004	131	1368
D395	11	241028	118.4	71.6	213722	148	2644
D396	6	117602	56	17	145945	27	1655
D397	13	241963	107.4	124.6	145482	215	2291
D398	7	111838	69.2	19.8	113130	54	1291
D399	7	133599	57.4	18.6	130868	68	1207
D400	8	140511	41.2	29.8	109008	55	1383
D401	6	122426	30.6	20.4	134267	45	1356
D402	9	176496	173.4	61.6	135838	119	1573

	FA	TC	AI	AP	CT	MT	SV
D403	11	210037	102.8	52.2	193353	108	2888
D404	6	109319	84	64	101133	97	1536
D405	12	215228	110.4	34.6	176534	154	2559
D406	12	244922	132.6	79.4	239718	161	3251
D407	11	156125	105.2	74.8	114076	160	1829
D408	8	208967	171.6	78.4	174997	74	2227
D409	12	204672	101.8	28.2	160859	139	1548
D410	6	128580	90.4	84.6	103694	115	1484
D411	11	181788	125.8	21.2	186543	90	2683
D412	10	168999	202.6	81.4	147486	133	2281
D413	6	77461	67.2	16.8	72425	30	860
D414	9	255123	91.4	51.6	222161	141	3178
D415	7	78106	56.4	50.6	50343	94	790
D416	7	93086	58.8	41.2	131875	76	1420
D417	8	152893	73.2	44.8	157066	55	1855
D418	6	124496	59.2	50.8	118232	126	1511
D419	7	109771	50.4	30.6	134626	48	1375
D420	6	140307	49.8	62.2	122597	83	1629
D421	6	87437	61.8	52.2	93963	87	937
D422	6	131389	40.8	34.2	150912	67	1602
D423	5	84067	17	8	101911	14	1126
D424	4	60508	34.6	20.4	71906	52	944
D425	5	89133.3	109.4	13.6	90781	46	758
D426	7	133997	129.6	92.4	77177	140	1487
D427	9	204740	159.8	111.2	175686	149	2441
D428	7	157744	76.6	31.4	202240	83	1988
D429	6	81351	78.4	43.6	71089	63	706
D430	9	120911	42.8	20.2	128776	81	1467
D431	7	100905	56.6	23.4	134966	40	1544
D432	5	110823	71.2	12.8	106104	32	1040
D433	15	358819	415.6	77.4	254256	227	3773
D434	5	87480	57.4	25.6	117176	40	1257
D435	6	96421	29.8	30.2	104613	79	1535
D436	8	156071	40.4	45.6	136781	96	1508
D437	9	156786	38.4	17.6	180448	46	2177
D438	7	221211	304.2	95.8	150855	102	1961
D439	7	77232	45	26	84438	41	912
D440	7	131481	70.4	87.6	168825	133	1740
D441	16	229783	113.8	101.2	161799	220	2005
D442	6	125848	94.6	96.4	100916	113	1631
D443	9	159391	118.2	41.8	143698	68	2153
D444	7	124968	68.8	56.2	90631	61	1118
D445	7	123964	122.2	11.8	147494	26	1561
D446	7	107809	99.6	29.4	134417	73	1586
D447	11	289886	84.2	63.8	237484	116	3251
D448	11	220353	168.6	32.4	158698	72	2446
D449	11	282651	141.6	157.4	196081	207	2790
D450	6	107923	67.2	25.8	116731	46	970
D451	4	91877	31.2	27.8	64133	48	770
D452	9	148294	102.4	49.6	124612	126	1769
D453	7	117577	62.8	41.2	116542	96	1712

	FA	TC	AI	AP	CT	MT	SV
D454	10	143737	48.2	55.8	157963	130	2296
D455	7	153923	57.6	46.4	131675	102	1733
D456	13	283033	112.8	174.2	258765	256	2872
D457	9	159122	120.8	48.2	158443	64	2178
D458	7	138458	87.4	18.6	157377	90	2104
D459	5	76492	24.6	25.4	75915	45	1055
D460	4	88371	80.4	34.6	82654	54	1463
D461	6	129105	54.6	18.4	133570	50	1511
D462	8	139990	128.6	43.4	124502	115	1826
D463	5	94247.3	24.4	23.6	152279	39	1565
D464	4	77552	32.6	36.4	68490	63	968
D465	5	105872	65.6	19.4	146828	68	1526
D466	9	146107	71	38	129517	111	1595
D467	6	110575	123.4	29.6	120886	61	1290
D468	5	123023	77.8	46.2	124459	64	1366
D469	7	129878	40.8	19.2	100667	75	1256
D470	5	123404	39	32	158321	64	1670
D471	7	125840	46.2	26.8	113607	51	1159
D472	9	145747	74	55	111080	96	1757
D473	14	307290	147.2	105.8	256414	203	3692
D474	9	118704	52.2	22.8	109196	74	1242
D475	8	144874	101.4	40.6	143514	64	1848
D476	14	318881	108.2	97.8	248725	192	3521
D477	10	157042	69.6	30.4	120931	40	1612
D478	10	136273	62.4	36.6	146461	79	2242
D479	6	97062	23.4	26.6	100931	62	1091
D480	7	141114	47.6	47.4	111341	87	1259
D481	5	92598.3	57	14	117305	44	995
D482	9	127771	121.2	63.8	100073	99	1943
D483	9	151012	88	59	118519	101	1700
D484	6	114443	35.4	33.6	86668	85	1037
D485	5	89412	79.8	27.2	88576	63	1035
D486	11	246803	110.2	79.8	218748	184	2951
D487	10	250013	147.8	53.2	272918	112	3296
D488	17	233805	86.8	86.2	169574	95	2304
D489	7	98103	43	33	106850	34	1257
D490	9	262661	262.6	109.4	210548	187	2518
D491	6	90810	60.6	46.4	79781	80	1109
D492	7	116577	62.4	36.6	99491	78	1313
D493	7	79678	46.6	21.4	78294	62	954
D494	7	135004	80.4	27.6	214303	53	1602
D495	9	117658	64.8	32.2	123914	60	1294
D496	9	112299	76.4	46.6	116229	90	2041
D497	6	100139	53.4	49.6	114858	80	1817
D498	6	85913	42	51	83213	81	1119
D499	5	100219	29.8	24.2	115463	55	1340
D500	5	71908	42.8	30.2	80349	54	1118
D501	7	156944	52.8	28.2	166900	86	1879
D502	7	104975	48.4	22.6	120452	56	1722
D503	6	113416	45	18	105092	41	1122
D504	8	222116	568.2	64.8	160318	143	2358

	FA	TC	AI	AP	CT	MT	SV
D505	5	138328	81.8	36.2	156789	62	1567
D506	7	134724	105.2	66.8	130997	68	2194
D507	9	187082	201.8	58.2	173338	161	2284
D508	7	140587	79.2	47.8	142177	99	1599
D509	10	159647	74.8	40.2	176044	107	2131
D510	5	87426	110	10	108234	29	1203
D511	7	136964	49	35	175854	66	1872
D512	6	103282	57.4	10.6	107334	28	1092
D513	10	265252	153.4	103.6	197245	231	2330
D514	12	271627	220.6	142.4	252224	194	2838
D515	3	50634	2	12	33972	21	459
D516	5	98871.3	46.8	19.2	113446	33	1436
D517	5	90981	50.8	20.2	73987	48	904
D518	6	126376	43	24	171025	58	1617
D519	5	134116	103	22	126776	62	1271
D520	7	134229	96.2	62.8	175966	129	2240
D521	3	46365	53.4	26.6	33077	45	492
D522	5	94502.5	51.6	26.4	85412	35	856
D523	10	257417	125.8	50.2	262727	108	2884
D524	6	151439	67.4	55.6	141717	85	1324
D525	6	110960	58.8	14.2	144136	51	1708
D526	6	136151	49.8	24.2	144971	37	1334
D527	10	183764	108.2	68.8	183962	121	2907
D528	8	147852	52.4	17.6	191190	50	2012
D529	5	68604	56.4	18.6	49943	43	670
D530	6	124473	75.6	21.4	122435	58	1584
D531	5	98109	26.6	21.4	127634	35	1252
D532	6	105656	46.2	57.8	119692	98	1256
D533	6	96752	121.6	44.4	86517	85	1403
D534	5	78712.8	76.8	21.2	104708	27	1056
D535	12	237348	70.2	76.8	207601	139	2968
D536	7	101542	76.8	49.2	81879	110	1412
D537	4	71793	64	51	49488	68	718
D538	12	210536	135	82	164879	149	2135
D539	5	125912	153.4	41.6	153254	63	1550
D540	7	108013	33.8	30.2	141834	67	1579
D541	7	122749	71.6	13.4	143319	47	1579
D542	10	182765	233.2	67.8	83302	205	1302
D543	6	82656	32.2	28.8	105510	55	1006
D544	10	163125	63.6	77.4	159815	149	2238
D545	7	81694.5	54.8	29.2	101679	62	997
D546	6	123849	139	70	76282	153	1076
D547	6	110339	17	18	167463	63	1360
D548	6	99016.3	58	18	133229	41	1620
D549	6	104112	78	64	91428	186	1577
D550	6	117713	31.8	23.2	130872	89	1358
D551	6	190273	52.8	53.2	270171	106	2778
D552	14	235917	79.8	95.2	256965	170	2836
D553	15	338595	175.4	89.6	304831	284	3738
D554	5	91732	40.2	13.8	119758	55	1221
D555	6	112465	122.8	97.2	79985	102	1053

	FA	TC	AI	AP	CT	MT	SV
D556	9	134357	88.8	50.2	110007	115	1949
D557	5	91141	47.4	51.6	103131	120	1783
D558	10	184847	200.4	136.6	164833	291	2843
D559	4	100453	82.4	17.6	80682	52	1151
D560	10	148089	84.2	48.8	105496	94	1989
D561	5	108038	29.6	14.4	126453	43	1349
D562	6	101605	52.6	19.4	162497	37	1432
D563	12	232849	62.2	66.8	142954	153	1413
D564	12	197197	155.8	71.2	133706	223	1599
D565	5	90569	42.2	34.8	102254	62	1346
D566	4	71965	64.2	30.8	78273	45	1203
D567	15	239491	353.8	124.2	219499	216	2894
D568	6	111790	45.8	15.2	124049	48	1189
D569	6	90615	32.2	38.8	114042	60	1461
D570	6	112808	64.4	26.6	140414	44	1534
D571	7	120279	136.4	49.6	132924	91	2631
D572	5	90447	69.4	34.6	108762	41	1419
D573	4	80333	18.6	10.4	88372	24	881
D574	6	87810	73.6	64.4	75109	89	1223
D575	5	105218	28	25	103471	51	1184
D576	12	226603	320.6	90.4	161248	157	2711
D577	5	75663	55.6	53.4	83122	91	1254
D578	4	76032	80.8	27.2	69866	61	1184
D579	4	80974	47.2	20.8	114277	21	1144
D580	4	75678	27	11	76266	14	970
D581	5	69031	59.6	16.4	114902	72	1361
D582	5	114711	91.4	18.6	124462	47	1184
D583	8	119294	42.2	24.8	139287	66	1747
D584	6	168720	90.2	43.8	179994	71	1745
D585	8	198203	109.2	35.8	173596	75	1874
D586	4	75429	84.2	25.8	94140	57	1136
D587	11	165512	57.8	30.2	157597	120	2748
D588	4	58621	20	12	59850	17	867
D589	5	103974	55.2	19.8	133576	30	1214
D590	6	126137	42.2	23.8	152647	63	1808
D591	5	94604.5	35.2	21.8	133290	40	1449
D592	4	79619	101.8	30.2	52780	62	684
D593	5	104233	46	14	145918	38	1418
D594	5	78543	65.6	19.4	101214	29	998
D595	7	166582	135.6	34.4	177346	119	2573
D596	7	124345	47	61	90302	91	1594
D597	7	133141	97	39	157664	80	1695
D598	9	168026	82.2	41.8	134379	108	1844
D599	7	175826	147.8	31.2	183238	117	2087
D600	7	176677	111.6	115.4	184482	128	2489
D601	5	106943	29.6	18.4	148236	36	1452
D602	7	88371.8	43	8	95577	23	1029
D603	10	155506	101.8	55.2	109545	125	1515
D604	8	169227	56.6	37.4	163844	58	1466
D605	6	155210	55.2	30.8	184487	61	1465
D606	5	74786.5	57.6	25.4	89815	43	1239

	FA	TC	AI	AP	CT	MT	SV
D607	6	124664	201	85	104674	163	1424
D608	6	72696	59.8	45.2	74634	49	1009
D609	5	159023	55.2	23.8	218233	60	2001
D610	4	72636	48.6	8.4	55669	49	861
D611	12	250467	174.6	120.4	216383	210	3370
D612	5	115575	59.8	40.2	146592	60	1736
D613	6	86935.5	54.4	31.6	127113	66	1481
D614	5	113734	62.4	25.6	127705	44	1335
D615	5	76983	104.2	30.8	78454	72	1050
D616	4	79970	53.8	19.2	98582	36	1212
D617	6	129984	74.4	24.6	123504	83	1032
D618	5	105805	50	10	124235	44	1216
D619	5	87153.3	49.6	18.4	123634	32	961
D620	9	155240	230.6	20.4	181656	45	2330
D621	5	87956.3	56.6	22.4	113319	52	733
D622	6	117442	41.2	9.8	137045	49	1079
D623	7	121313	97	61	96183	81	1567
D624	6	99069.5	74.8	17.2	115073	51	1112
D625	5	109754	54.2	26.8	170832	33	1833
D626	5	110236	41	51	113130	105	1536
D627	6	107268	23.2	26.8	140824	66	1694
D628	6	149704	41.6	17.4	103421	62	854
D629	5	136822	9.2	20.8	170739	28	1949
D630	5	63268	9.6	9.4	119047	11	689
D631	6	85839	36.6	23.4	101921	46	1333
D632	7	138967	60	21	204067	84	2130
D633	5	105565	38.6	12.4	118184	27	1081
D634	10	203684	92.4	54.6	129371	198	1371
D635	12	188442	116.8	44.2	144154	149	1127
D636	7	109962	48	10	139616	47	1572
D637	13	222079	88.2	87.8	136529	196	2093
D638	6	128547	66.2	20.8	145288	75	1337
D639	7	111266	30.6	9.4	94329	45	1221
D640	4	73587	68.4	24.6	60136	56	940
D641	5	73643.5	35.2	14.8	105956	48	1079
D642	8	166758	68.6	84.4	138966	158	2003
D643	8	181778	118.2	88.8	177945	165	2055
D644	8	184804	149.8	80.2	162171	120	2141
D645	7	143193	253.6	88.4	120967	152	1976
D646	11	224308	131.2	59.8	204836	155	2988
D647	9	88386	34.4	12.6	57557	84	624
D648	9	162749	68.8	93.2	120992	140	1978
D649	10	193811	123.2	103.8	122536	217	1956
D650	10	225163	276.8	83.2	194617	191	2665
D651	9	135030	50	33	97209	116	1261
D652	5	188991	39.4	34.6	232743	51	2196
D653	10	200688	110.8	88.2	133678	149	1837
D654	5	65575.3	42.6	11.4	85829	24	856
D655	10	183845	55.4	31.6	185073	88	2317
D656	15	328362	204.4	124.6	251008	234	3154
D657	12	215995	84.6	86.4	170788	132	2390

	FA	TC	AI	AP	CT	MT	SV
D658	11	283324	73.2	58.8	307855	109	3637
D659	13	281175	173	73	240532	180	2421
D660	15	325208	121	59	236546	170	2012
D661	14	203211	115.4	80.6	136444	211	1493
D662	12	213326	159	95	179739	225	1908
D663	16	235844	147.6	105.4	157237	295	2200
D664	12	182404	95.8	67.2	133291	132	2112
D665	12	245645	200.6	155.4	136915	318	2651
D666	7	83135	32.2	18.8	59747	61	997
D667	10	217959	60.6	85.4	197193	127	3016
D668	12	248850	149.8	46.2	154412	150	1996

Appendix 6.4

The Input Output Levels of Each Anchor Branch that Require Adjustments in Order to Construct at Least one Unobserved Branch

	FA	TC	AI	AP	CT	MT	SV	
D014			✓	✓		✓		3
D015			✓	✓	✓		✓	4
D063	✓		✓		✓	✓	✓	5
D150		✓	✓		✓		✓	4
D348		✓		✓	✓	✓		4
D362	✓		✓		✓			3
D440			✓			✓	✓	3
D442			✓			✓		2
D463			✓				✓	2
D494				✓		✓	✓	3
D504				✓			✓	2
D539						✓	✓	2
D549	✓		✓	✓			✓	4
D551				✓		✓	✓	3
D555			✓			✓		2
D557			✓		✓	✓		3
D571	✓		✓	✓		✓		4
D581	✓		✓	✓		✓	✓	5
D600		✓	✓		✓	✓		4
D607				✓			✓	2
D625			✓	✓		✓	✓	4
D630	✓		✓	✓		✓	✓	5
D645					✓	✓		2
								75

Appendix 6.5

The 48 Unobserved Branches Used to Improve Envelopment

	FA	TC	AI	AP	CT	MT	SV
A3D014	7.1	194114	52	26	150008	11	3599
A3D015	9.85	166164	194	146	30000	402	1824
A4D015	10	176851	197	148	73321	408	402
A1D063	19	200133	2	169	319642	182	3971
A2D063	18.9	169192	211	159	30000	198	3605
A3D063	18.9	208352	206	168	318457	11	3971
A5D063	24	263223	304	248	356023	279	4289
A1D150	14	364145	248	245	331254	421	4125
A2D150	12	268966	2	174	297000	332	3816
A4D150	11.8	239765	187	185	245120	345	450
A3D348	3.7	114113	74	24	30000	84	1834
A1D362	9.7	136215	112	110	139854	210	2441
A3D362	7	74798	48	53	30000	138	1652
A1D440	6	99357	2	46	158541	74	1649
A2D440	6	99018	46	52	161214	11	1662
A1D442	5.05	96058	2	76	100001	84	1550
A2D442	5.05	96905	74	78	100004	14	1552
A1D463	3.9	80053	2	5	146543	24	1441
A2D463	4.1	75142	19	18	137124	31	455
A3D494	6	99784	66	17	186457	41	450
A1D504	7	197038	482	4	153854	110	2275
A2D504	6.9	159574	509	46	139541	120	450
A2D539	4.2	95693	132	34	129541	48	450
A2D549	5	88095	2	48	91138	142	1519
A3D549	5	89260	57	5	91122	142	1522
A4D549	4.8	75125	60	53	79912	130	450
A1D551	5.05	174422	35	4	261122	74	2564
A2D551	5.05	169043	31	30	261241	11	2597
A3D551	5	147608	37	36	238254	78	450
A2D557	3.5	58330	27	31	30000	97	1241
A3D557	4.2	81115	31	34	100214	11	1687
A1D571	9.1	168215	192	98	164214	162	3184
A2D571	6.2	112018	2	32	125014	74	2541
A4D571	6.1	109048	86	27	123512	11	2545
A1D581	8.4	88003	68	24	141904	135	1612
A2D581	4.1	58367	2	10	104123	51	1201
A3D581	4	62975	36	5	111000	56	1297
A4D581	4	60138	35	9	107942	11	1251
A5D581	4.2	54464	42	11	99545	61	450
A1D607	5	103065	159	5	100341	131	1378
A2D607	4.9	99745	164	61	92572	138	450
A2D625	4.1	101003	34	5	169574	20	1775
A3D625	4.5	96981	31	14	167954	11	1760
A4D625	4.2	84413	40	18	152213	25	450
A5D630	3.1	44986	8	8	85197	10	371
A1D645	5	91644	203	56	30000	102	1421
A2D645	5.9	110442	162	68	110732	11	1894

Appendix 7

Appendix 7.1

Proof of Theorem 7.1: The Input Minimisation Case

For ease of explanation models (M7.5) excluding r4 and r5 and (M7.7) are reproduced here as (A7.M1) and (A7.M2) respectively.

(A7.M1)	(A7.M2)
$h^*_{j_0} = Max \sum_{r=1}^{s} u_r y_{rj_0} + \omega$	$e^*_{j_0} = Max \sum_{r=1}^{s} \beta_r y_{rj_0} + \varphi$
$s.t. \quad \sum_{i=1}^{m} v_i x_{ij_0} = 1$	$s.t. \quad \sum_{i=1}^{m} \alpha_i x_{ij_0} = 1$
$\sum_{r=1}^{s} u_r y_{rj} + \omega - \sum_{i=1}^{m} v_i x_{ij} \leq 0 \quad j=1,\ldots,N$	$\sum_{r=1}^{s} \beta_r y_{rjt} + \varphi - \sum_{i=1}^{m} \alpha_i x_{ijt} \leq 0 \quad jt=1,\ldots,N$
$u_r - \pi_r u_{r+1} \leq 0$:r1 *for some i,r*	$\alpha_i, \beta_r \geq \varepsilon$ $\forall i,r$
$v_i - \sigma_i v_{i+1} \leq 0$:r2 *for some i,r*	φ *free*
$u_r - \chi_i v_i \leq 0$:r3 *for some i,r*	
$v_i, u_r \geq \varepsilon$ $\forall i,r$	
ω *free*	

Notation in (A7.M1) and (A7.M2) as in (M7.5) and (M7.7) respectively. The RDMUs, *jt=1,...,N* in (A7.M2) are derived using (7.1). It is necessary to show that $h^*_{j_0} = e^*_{j_0}$

Proof

Let * denote the value of a variable at the optimal solution to the model in which it appears.

From the constraints of (A7.M2) it follows that $e^*_{j_0} = \sum_{r=1}^{s} \beta^*_r y_{rj_0} + \varphi^* \leq \left(\sum_{i=1}^{m} \alpha^*_i x_{ij_0} = 1 \right)$.

This, using (7.1) gives for $jt = j_0$, $\sum_{r=1}^{s} \beta^*_r y_{rj_0} + \varphi^* = e^*_{j_0} \leq \sum_{i=1}^{m} \alpha^*_i x_{ij_0} h^*_{j_0}$

$$h^*_{j_0} \geq e^*_{j_0} \qquad \text{(A7.1)}$$

The solution, $\beta_r = u^*_r$, $r=1,...,s$, $\alpha_i = v^*_i$ $i=1,...,m$ and $\varphi = \omega^*$ is feasible in (A7.M2). To show that this is true it is only necessary to show that $\beta_r = u^*_r$ $r=1,...,s$, $\alpha_i = v^*_i$ $i=1,...,m$ and $\varphi = \omega^*$ satisfies the constraints $jt=1,...,N$ in (A7.M2). A feasible solution u^*_r, $r=1,...,s$, v^*_i $i=1,...,m$ which is feasible for one assessed DMU j_0 of (A7.M1) will also be feasible for another assessed DMU $j \neq j_0$ of (A7.M1) provided the weights restrictions in (A7.M1) all have zero RHS value, see Appendix 4.5. Thus, in (A7.M2) for DMU j in (A7.M1) $\sum_{r=1}^{s} u^*_r y_{rj} + \omega^* = h_j \sum_{i=1}^{m} v^*_i x_{ij}$, where $h_j \leq h^*_j \leq 1$ and h^*_j is the efficiency of DMU j yielded by (A7.M1). Hence $\sum_{r=1}^{s} u^*_r y_{rj} + \omega^* \leq \sum_{i=1}^{m} v^*_i x_{ij} h^*_j$ and by reference to (7.1) for $j=jt$ the following holds:

$$\sum_{r=1}^{s} u^*_r y_{rjt} + \omega^* - \sum_{i=1}^{m} v^*_i x_{ijt} \leq 0 \qquad jt=1,...,N \qquad \text{(A7.2)}$$

Thus $\beta_r = u^*_r$, $\forall r$ and $\alpha_i = v^*_i$ $\forall i$ and $\varphi = \omega^*$ satisfies the constraints $jt=1,...,N$ of (A7.M2) and it is a feasible solution to the model. This implies $\sum_{r=1}^{s} u^*_r y_{rj_0} + \omega^* = h^*_{j_0}$ is a feasible objective function value to (A7.M2), hence

$$h^*_{j_0} \leq e^*_{j_0} \qquad \text{(A7.3)}$$

Clearly, (A7.1) and (A7.3) imply

$$h^*_{j_0} = e^*_{j_0} \qquad \text{(A7.4)}$$

QED

A similar proof for the OM model can be constructed.

Appendix 7.2

Proof of Theorem 7.2: The Input Minimisation Case

For ease of explanation models (M7.5) excluding r1-r3 and (M7.9) are reproduced here as (A7.M3) and (A7.M4) respectively.

(A7.M3)	(A7.M4)
$h^*_{j_0} = Max \sum_{r=1}^{s} u_r y_{rj_0} + \omega$ s.t. $\sum_{i=1}^{m} v_i x_{ij_0} = 1$ $\sum_{r=1}^{s} u_r y_{rj} + \omega - \sum_{i=1}^{m} v_i x_{ij} \le 0 \quad j=1,\dots,N$ $u_r \ge \gamma_r \ge \varepsilon$:r4 $r=1,\dots,s$ $v_i \ge \kappa_i \ge \varepsilon$:r5 $i=1,\dots,m$ ω *free*	$f^*_{j_0} = Max \sum_{r=1}^{s} \beta_r y_{rj_0} + \varphi$ s.t. $\sum_{i=1}^{m} \alpha_i x_{ij_0} = 1$ $\sum_{r=1}^{s} \beta_r y_{rj} + \varphi - \sum_{i=1}^{m} \alpha_i x_{ij} \le 0 \quad j=1,\dots,N$ $\sum_{r=1}^{s} \beta_r y_{rjt_0} + \varphi - \sum_{i=1}^{m} \alpha_i x_{ijt_0} \le 0$ $\alpha_i, \beta_r \ge \varepsilon \quad \forall i,r$ φ *free*

Notation in (A7.M3) and (A7.M4) as in (M7.5) and (M7.9) respectively. The RDMU jt_0 in (A7.M4) is derived using (7.1). It is necessary to show that $h^*_{j_0} = f^*_{j_0}$.

Proof

Let * denote the value of a variable at the optimal solution to the model in which it appears.

From the constraints of (A7.M4) it follows that $f^*_{j_0} = \sum_{r=1}^{s} \beta^*_r y_{rj_0} + \varphi^* \le \left(\sum_{i=1}^{m} \alpha^*_i x_{ij_0} = 1\right)$.

This using (7.1) gives for $jt = j_0$, $\sum_{r=1}^{s} \beta^*_r y_{rj_0} + \varphi^* = f^*_{j_0} \le \sum_{i=1}^{m} \alpha_i x_{ij_0} h^*_{j_0}$,

$$h_{j_0}^* \geq f_{j_0}^* \qquad \text{(A7.5)}$$

The solution, $\beta_r = u_r^*$, $r=1,\dots,s$, $\alpha_i = v_i^*$ $i=1,\dots,m$ and $\varphi = \omega^*$ are feasible in (A7.M4). To show that this is true, recall that $\beta_r = u_r^*$ $r=1,\dots,s$, $\alpha_i = v_i^*$ $i=1,\dots,m$ and $\varphi = \omega^*$ satisfies the constraint jt_0 in (A7.M4) which is true by the virtue of (7.1), so the following holds:

$$\sum_{r=1}^{s} u_r^* y_{rjt_0} + \omega^* - \sum_{i=1}^{m} v_i^* x_{ijt_0} \leq 0 \qquad \text{(A7.6)}$$

Thus the solution, $\beta_r = u_r^*$, $r=1,\dots,s$, $\alpha_i = v_i^*$ $i=1,\dots,m$ and $\varphi = \omega^*$ is feasible in (A7.M4) and provide a feasible solution to this model. This implies that $\sum_{r=1}^{s} u_r^* y_{rj_0} + \omega^* = h_{j_0}^*$ is a feasible objective function value to (A7.M4) and so

$$h_{j_0}^* \leq f_{j_0}^* \qquad \text{(A7.7)}$$

Clearly, (A7.5) and (A7.7) imply

$$h_{j_0}^* = f_{j_0}^* \qquad \text{(A7.8)}$$

QED

A similar argument can be constructed for the OM model.

Appendix 7.3

Proof of Negative Relative Efficiency Scores

Consider assessing a set of N DMUs, $j=1,\ldots,N$ each consuming m different inputs, x_{ij}, $i=1,\ldots,m$ to produce s different outputs, y_{rj}, $r=1,\ldots,s$. The VRS DEA weights models (A7.M5) and (A7.M6) provide the relative efficiency scores of DMU j_0 with an IM and OM orientation respectively.

(A7.M5) Input Minimisation	(A7.M6) Output Maximisation
$h^*_{j_0} = Max \sum_{r=1}^{s} u_r y_{rj_0} + w1 - w2$	$\bar{h}_{j_0} = Min \sum_{i=1}^{m} \rho_i x_{ij_0} + w3 - w4$
$s.t. \quad \sum_{i=1}^{m} v_i x_{ij_0} = 1$	$s.t. \quad \sum_{r=1}^{s} \delta_r y_{rj_0} = 1$
$\sum_{r=1}^{s} u_r y_{rj} + w1 - w2 - \sum_{i=1}^{m} v_i x_{ij} + S_j = 0 \quad \forall j$	$\sum_{r=1}^{s} \delta_r y_{rj} - \sum_{i=1}^{m} \rho_i x_{ij} - w3 + w4 + H_j = 0 \quad \forall j$
$v_i,\ u_r \geq \varepsilon \quad \forall i, r$	$\delta_r,\ \rho_i \geq \varepsilon \quad \forall i, r$
$u_k - v_l \leq 0 \quad$ *for some k,l*	$u_k - v_l \leq 0 \quad$ *for some k,l*
$w1,\ w2,\ S_j \geq 0 \quad \forall j$	$w3,\ w4,\ H_j \geq 0 \quad \forall j$

Notation in (A7.M5) and (A7.M6) as in (M7.1) and (M7.2) respectively, except the variable that can be used to ascertain the returns to scale that the DMU is operating under is now expressed as two variables. These two variables must be non-negative and hence only one of these variables can be basic. e.g. $w1>0 \Rightarrow w2=0$ and $w2 > 0 \Rightarrow w1=0$, see Winston [57] p.172.

Essentially, the scale variable acts as an additional slack variable. Thus in order to balance a constraint, with the introduction of a weights restriction, the scale variable may take an inappropriately large value. In the IM case if $w2 > \sum_{r=1}^{s} u_r y_{rj_0}$, then a negative relative efficiency score is obtained. The same occurs in the OM case, if $w4 > \sum_{i=1}^{m} \rho_i x_{ij_0}$. This will only happen when the imposed weights restrictions provide infeasible solutions in the CRS case.

Appendix 8

It should be noted that the proofs presented in these Appendices are similar to those presented in Appendix 6.1 and 6.2.

Appendix 8.1

Identifying Anchor DMUs: The Input Minimisation Case

Consider assessing a set of N DMUs $j=1,\dots,N$, each using varying amounts of m different inputs, x_{ij}, $i=1,\dots,m$ to produce varying quantities of s different outputs, y_{rj}, $r=1,\dots,s$. Let the set JE consist of the DEA-efficient DMUs identified using model (M7.3) and let JE_{j_0} be the set JE without DMU j_0. In respect of each DEA-efficient DMU j_0 solve the envelopment model (A8.M1).

$$
\begin{aligned}
& h^*_{j_0} = Min\ \theta_0 - \varepsilon\left(\sum_{i=1}^{m} G_i + \sum_{r=1}^{s} G_{m+r}\right) \\
s.t.\quad & \theta_0 x_{ij_0} - \sum_{j\in JE_{j0}} \lambda_j x_{ij} - G_i = 0 && i=1,\dots,m && \text{(A8.M1)} \\
& \sum_{j\in JE_{j0}} \lambda_j y_{rj} - G_{m+r} = y_{rj_0} && r=1,\dots,s \\
& \sum_{j\in JE_{j0}} \lambda_j = 1 \\
& G_i,\ G_{m+r},\ \lambda_j \ge 0 && \forall i, r, j \in JE_{j_0}
\end{aligned}
$$

G_i and G_{m+r} represent slack variables. Let * denote the value of a variable at the optimal solution to (A8.M1).

DMU j_0 is classed as an ADMU if:

a) $h^*_{j_0} > 1$ and it has at least one $G^*_i > 0$ or $G^*_{m+r} > 0$.

or

b) (A8.M1) has no feasible solution

<u>Proof of (a)</u>

Consider assessing DMU j_0 under model (A8.M2), after scaling its inputs to $\theta^*_0 x_{ij_0}$, $i=1,\ldots,m$ to give it a radial efficiency of 1 in (A8.M1).

$$g_0 = Max \sum_{r=1}^{s} S_{m+r} + \sum_{i=1}^{m} S_i$$

$$s.t. \quad \theta^*_0 x_{ij_0} - \sum_{j \in JE_{j0}} \lambda_j x_{ij} = S_i \qquad i=1,\ldots,m \qquad \text{(A8.M2)}$$

$$-y_{rj_0} + \sum_{j \in JE_{j0}} \lambda_j y_{rj} = S_{m+r} \qquad r=1,\ldots,s$$

$$\sum_{j \in JE_{j0}} \lambda_j = 1$$

$$\lambda_j,\ S_i,\ S_{m+r} \geq 0 \qquad \forall i,\ r,\ j \in JE_{j_0}$$

θ_0^* is the optimal solution to (A8.M1) and S_i and S_{m+r} are slack variables.

If (a) holds then DMU j_0 will yield $g_0 > 0$, and by definition the assessed DMU is deemed to be of class F. This shows that the DEA-efficient DMU j_0 can be rendered class F under SE with respect to JE_{j_0}. Hence DMU j_0 is an ADMU.

<u>Proof of (b)</u>

DEA-efficient DMUs fall into two categories ADMUs and non-ADMUs. A non-ADMU in (A8.M1) meets the following conditions:

- $h^*_{j_0} = 1$

or

- $h^*_{j_0} \neq 1$ and $G^*_i = G^*_{m+r}, = 0$ for $i=1,\ldots,m$ and $r = 1,\ldots,s$.

DMU j_0 does not meet these conditions when (A8.M1) has no feasible solution and so it must be an ADMU.

Similar arguments for (a) and (b) can be constructed for the Output Maximisation Case.

Appendix 8.2

Improving Envelopment: The Input Minimisation Case

Consider assessing a set of N DMUs $j=1,...,N$, each using m different inputs, x_{ij}, $i=1,...,m$ to produce s different outputs, y_{rj}, $r=1,...,s$, under model (M7.3). Let some of the DMUs be non-enveloped i.e. weight some inputs or outputs with ε. Then introducing DEA-efficient UDMUs as local variations of ADMUs in the manner outlined in section 8.5 for determining the input output levels of UDMUs will, in principle, increase the number of properly enveloped DMUs. An outline of the steps involved in the proof is now given:

(i) It is feasible that if a class F DMU *jf* is a referent DMU to ADMU in (M8.3) then DMU *jf* will have ADMU *jp* as a peer DMU in (M7.3).

(ii) Introducing an UDMU *ja* created from ADMU *jp* will in principle improve envelopment of DMU *jf* which had DMU *jp* as a peer.

The above steps will now be detailed.

Proof of (i)

It is feasible that if a class F DMU *jf* is a referent DMU to ADMU *jp* in (M8.3) then DMU *jf* will have ADMU *jp* as a peer DMU in (M7.3).

Consider using model (M7.3), reproduced here as (A8.M3) for convenience, to assess the relative efficiency of DMU j_0.

$$h^*_{j_0} = Min\ f_0 - \varepsilon\left(\sum_{i=1}^{m} S_i + \sum_{r=1}^{s} S_{m+r}\right)$$

$$\text{s.t.} \quad f_0 x_{ij_0} - \sum_{j=1}^{N} \kappa_j x_{ij} - S_i = 0 \qquad i=1,\dots,m \qquad \text{(A8.M3)}$$

$$\sum_{j=1}^{N} \kappa_j y_{rj} - S_{m+r} = y_{rj_0} \qquad r=1,\dots,s$$

$$\sum_{j=1}^{N} \kappa_j = 1$$

$$\kappa_j,\ S_i,\ S_{m+r} \geq 0 \qquad \forall j,\ i,\ r$$

S represent slack variables. Let * denote the value of a variable at the optimal solution to (A8.M3).

Suppose that all class NF DMUs have been adjusted using (8.1), so that they are now class F DMUs. Let *JFI*, $jf=1,\dots,|JFI|$ denote the set of these observed and radially adjusted class F DMUs and let *JA* be the set of ADMUs for the assessment. Thus consider assessing each $j_0 \in JA$ under (A8.M4).

$$h^/_{j_0} = Min\ z_0 - \varepsilon\left(\sum_{i=1}^{m} H_i + \sum_{r=1}^{s} H_{m+r}\right)$$

$$\text{s.t.} \quad z_0 x_{ij_0} - \sum_{j \in JE_{j_0}} \tau_j x_{ij} - \sum_{jf \in JFI} \tau_{jf} x_{ijf} - H_i = 0 \qquad i=1,\dots,m \qquad \text{(A8.M4)}$$

$$\sum_{j \in JE_{j_0}} \tau_j y_{rj} + \sum_{jf \in JFI} \tau_{jf} y_{rjf} - H_{m+r} = y_{rj_0} \qquad r=1,\dots,s$$

$$\sum_{j \in JE_{j_0}} \tau_j + \sum_{jf \in JFI} \tau_{jf} = 1$$

$$H_i,\ H_{m+r},\ \tau_j,\ \tau_{jf} \geq 0 \qquad \forall i,\ r,\ j \in JE_{j_0},\ jf \in JFI$$

H represent slack variables and (A8.M4) is the normal DEA model, with only the DEA-efficient DMUs, excluding DMU j_0 and the class F DMUs of *JFI* allowed as referent DMUs. Let JP_{j_0} be the set of DEA-efficient referent DMUs to DMU j_0 in (A8.M4). Let $^/$ denote the value of a variable at the optimal solution to (A8.M4)

Let the assessed ADMU j_0 in (A8.M4) be ADMU *jp*, i.e. j_0=*jp*. It is found that it has one class F DMU, jf_a identified as one of its referent DMUs, i.e. $\tau^{l}_{jf_a} > 0$. Thus, at the optimal solution to (A8.M4) the input output levels of jf_a can be expressed as a linear combination of DMU *jp* and other DEA-efficient DMUs plus possibly a slack value, i.e.

$$x_{ijf_a} = \frac{z^{l}_{jp} x_{ijp} - \sum_{j \in JP_{jp}} \tau^{l}_{j} x_{ij} - H^{l}_{i}}{\tau^{l}_{jf_a}} \qquad i=1,\dots,m$$

$$y_{rjf_a} = \frac{y_{rjp} - \sum_{j \in JP_{jp}} \tau^{l}_{j} y_{rj} + H^{l}_{i}}{\tau^{l}_{jf_a}} \qquad r=1,\dots,s \qquad \text{(A8.1)}$$

In (A8.1) DMU *jp* has coefficients of $z^{l}_{jp} / \tau^{l}_{jf_a}$ for x_{ijp}, $i=1,\dots,m$ and $1/\tau^{l}_{jf_a}$ for y_{rjp}, $r=1,\dots,s$. Therefore when assessing jf_a in (A8.M3) it is feasible one of its peers will be DMU *jp*, i.e. $\kappa^{*}_{jp} > 0$. This holds if the original DMU in (A8.M3) corresponding to DMU jf_a is a class F or NF DMU.

<u>*Proof of (ii)*</u>

Introducing an UDMU *ja* created from ADMU *jp* that will in principle improve the envelopment of non-enveloped DMUs that had *jp* as a peer in (A8.M3).

Let DMU *ja* be an UDMU which is DEA-efficient and created from ADMU *jp*. Adding DMU *ja* to the DMUs can, in principle, increase the number of properly enveloped DMUs at the optimal solution to (A8.M3). To see how the addition of an UDMU *ja* to the observed data set can increase the number of properly enveloped DMUs consider using model (A8.M3) to assess the efficiency of DMU j_0 which had ADMU *jp* as one of its peers. Following the addition of a single UDMU *ja* created from DMU *jp* as in (a) or (b) below, the model solved to assess DMU j_0 is (A8.M5).

$$\hat{h}_{j_0} = Min\ q_0 - \varepsilon\left(\sum_{i=1}^{m} SS_i + \sum_{r=1}^{s} SS_{m+r}\right)$$

$$s.t. \quad q_0 x_{ij_0} - \sum_{j=1}^{N} \lambda_j x_{ij} - \lambda_{ja} x_{ija} - SS_i = 0 \qquad i=1,\ldots,m \qquad \text{(A8.M5)}$$

$$\sum_{j=1}^{N} \lambda_j y_{rj} + \lambda_{ja} y_{rja} - SS_{m+r} = y_{rj_0} \qquad r=1,\ldots,s$$

$$\sum_{j=1}^{N} \lambda_j + \lambda_{ja} = 1$$

$$\lambda_{ja}\ \lambda_j,\ SS_i,\ SS_{m+r} \geq 0 \qquad \forall j,\ i,\ r$$

Models (A8.M3) and (A8.M5) differ only in that the latter contains the additional variable λ_{ja} corresponding to an UDMU *ja*. Superscripts * will be used to denote the value of a variable at the optimal solution to the model (A8.M5).

There are two approaches for the creation of the UDMU to be used in (A8.M5):

- Encourage the non-ε weighting of an individual output: Lower an output level.
- Encourage the non-ε weighting of an inidividual input: Raising an input level.

With adjustments to the ADMU's input levels to compensate in each case. These two approaches will be considered now:

a) Encouraging the non-ε weighting of an individual output:
Let DMU j_0 in (A8.M3) have a $S^*_{m+k} > 0$ for one *k* with ADMU *jp* as one of its peer DMUs.

As the UDMU is to encourage the non-ε weighting of output *k*, the output *k* level of ADMU *jp* will be set to zero and to construct a DEA-efficient DMU, the input levels of ADMU *jp* will be lowered. Thus, an UDMU *ja* is created as a local variation of ADMU *jp* with input output levels of

$$x_{ija} = x_{ijp} - B_i \qquad i=1,\ldots,m$$
$$y_{kja} = 0 \qquad \text{(A8.2)}$$
$$y_{rja} = y_{rjp} \qquad \forall r \neq k$$

where B_i; $i = 1,\ldots,m$ are DM specified levels of sufficient size to enable DMU ja to be deemed by the DM to be DEA-efficient. Consider the solution to (A8.M5). Depending on the values of B_i, $i=1,\ldots,m$ in (A8.M5) it will be the case that $\lambda^*_{ja} > 0$.

To see this note that if in (A8.M5) the following holds $\lambda^*_{jp} + \lambda^*_{ja} = \kappa^*_{jp}$ and $k^*_j = \lambda^*_j$ $\forall j \neq jp$, this will give a lower optimal objective function value to (A8.M5) than to (A8.M3) and would therefore be preferred to the original optimal solution to (A8.M3), provided it is feasible.

To see that the solution in which $\lambda^*_{jp} + \lambda^*_{ja} = \kappa^*_{jp}$ and $\kappa^*_j = \lambda^*_j$ $\forall j \neq jp$ can be feasible in (A8.M5) consider some binding constraint i' at the optimal solution to (A8.M3). [Non-binding constraints will not effect the optimal solution.] The constraint reduces to $\sum_{j=1}^{N} \kappa^*_j x_{i'j} = f^*_0 x_{i'j_0}$ in (A8.M3) and when $\lambda^*_{jp} + \lambda^*_{ja} = \kappa^*_{jp}$ and $\kappa^*_j = \lambda^*_j$ $\forall j \neq jp$ it becomes in (A8.M5).

$$\sum_{\substack{j=1 \\ j \neq jp}}^{N} \kappa^*_j x_{i'j} + \left(\lambda^*_{jp} + \lambda^*_{ja}\right) x_{i'jp} - \lambda^*_{ja} B_{i'} = q^*_0 x_{i'j_0} \qquad \text{(A8.3)}$$

Since $\sum_{j=1}^{N} \kappa^*_j x_{i'j} = f^*_0 x_{i'j_0}$, $\lambda^*_{jp} + \lambda^*_{ja} = \kappa^*_{jp}$, and $k^*_j = \lambda^*_j$ $\forall j \neq jp$ depending on the size of B_i, $i=1,\ldots,m$, and <u>provided</u> $\lambda^*_{ja} > 0$ (A8.3) can be balanced with $f^*_0 > q^*_0$ and the solution $\lambda^*_{jp} + \lambda^*_{ja} = \kappa^*_{jp}$, $\kappa^*_j = \lambda^*_j$ $\forall j \neq jp$ is feasible in (A8.M5).

To see that the slack value for output k will be reduced, consider the constraint for output k, in both (A8.M5) and (A8.M3), thus $SS^*_{m+k} - \sum_{j=1}^{N} \lambda^*_j y_{kj} - \lambda^*_{ja} y_{kja} = S^*_{m+k} - \sum_{j=1}^{N} \kappa^*_j y_{kj}$, and given $\lambda^*_{jp} + \lambda^*_{ja} = \kappa^*_{jp}$, and $k^*_j = \lambda^*_j$ $\forall j \neq jp$ becomes $SS^*_{m+k} - \lambda^*_{ja}\left(y_{kja} - y_{kjp}\right) = S^*_{m+k}$, since $y_{kja} = 0$, it follows that $SS^*_{m+k} < S^*_{m+k}$.

The fact that $\lambda^*_{ja} > 0$ and the slack of output k is reduced for model (A8.M3) to yield an improved objective function value in comparison to that of model (A8.M3) means that (A8.M5) is more likely than (A8.M3) to identify DMU j_0 as a properly enveloped DMU.

b) Encouraging the non-ε weighting of an individual input: Let DMU j_0 in (A8.M3) have a $S^*_k > 0$ for one k with ADMU jp as one of its peer DMUs.

The introduced DMU is to encourage the non-ε weighting of input k, so the input k level of ADMU jp will be raised to a DM determined amount, and to construct a DEA-efficient DMU, the remaining m-1 input levels of the ADMU jp will be lowered. Thus, an UDMU, ja is created as a local variation of the ADMU jp with input output levels of:

$x_{kja} = B_k$		
$x_{ija} = x_{ijp} - B_i$	$\forall i \neq k$	(A8.4)
$y_{rja} = y_{rjp}$	$r=1,\ldots,s$	

where B_i, $i=1,\ldots,m$ are DM specified levels of sufficient size to enable DMU ja to be deemed by the DM to be DEA-efficient. Consider the solution to (A8.M5). Depending on the values of B_i, $\forall i \neq k$ in (A8.M5) it will be the case that $\lambda^*_{ja} > 0$.

It should be noted that **(a)** proves that $\lambda^*_{jp} + \lambda^*_{ja} = \kappa^*_{jp}$ and $\kappa^*_j = \lambda^*_j \;\; \forall j \neq jp$ provides a feasible solution in (A8.M3), for a binding constraint i' with $x_{i'ja} = x_{ijp} - B_{i'}$, provided $f^*_0 > q^*_0$.

To see that the slack value for input k will be reduced consider the constraint for input k, in both (A8.M5) and (A8.M3). Since $f^*_0 > q^*_0$, $\sum_{j \neq jp} \kappa^*_j x_{kj} + \kappa^*_{jp} x_{kjp} + S^*_k > \sum_{j \neq jp} \lambda^*_j x_{kj} + \lambda^*_{jp} x_{kjp} + \lambda^*_{ja} B_k + SS^*_k$, which given that $\lambda^*_{jp} + \lambda^*_{ja} = \kappa^*_{jp}$, and $\kappa^*_j = \lambda^*_j \;\; \forall j \neq jp$, becomes $S^*_k > \lambda^*_{ja}(B_k - x_{kjp}) + SS^*_k$. Thus as $B_k - x_{kjp} > 0$, it follows that $S^*_k > SS^*_k$ as required.

The fact that $\lambda^*_{ja} > 0$ and the slack of input k is reduced for model (A8.M3) to yield an improved objective function value in comparison to that of model (A8.M3) means that (A8.M5) is more likely than (A8.M3) to identify DMU j_0 as a properly enveloped DMU.

A similar proof can be constructed for the Output Maximisation Model.

Appendix 8.3

The Input Output Levels that Need to be Adjusted for Each ADMU in Order to Construct at Least One Unobserved Branch

	FA	TC	AI	AP	CT	MT	SV	TOTAL	RTNs
D014				✓		✓		2	CRS
D015	✓		✓	✓			✓	4	CRS
D181		✓	✓	✓	✓		✓	5	CRS
D200			✓		✓	✓	✓	4	IRS
D348				✓		✓		2	CRS
D353			✓			✓	✓	3	IRS
D373			✓					1	DRS
D440			✓			✓		2	CRS
D442			✓		✓	✓		3	CRS
D460						✓		1	IRS
D463			✓					1	CRS
D521	✓	✓		✓		✓	✓	5	IRS
D539						✓	✓	2	CRS
D549			✓	✓			✓	3	CRS
D551			✓	✓		✓	✓	4	CRS
D555							✓	1	CRS
D557			✓	✓	✓	✓		4	CRS
D558			✓	✓	✓	✓	✓	5	CRS
D571	✓		✓	✓	✓	✓		5	CRS
D577			✓		✓	✓	✓	4	IRS
D579			✓	✓		✓	✓	4	IRS
D581	✓		✓	✓		✓	✓	5	CRS
D586				✓		✓	✓	3	IRS
D600		✓	✓		✓	✓		4	CRS
D607				✓		✓	✓	3	CRS
D625			✓	✓		✓	✓	4	CRS
D630				✓		✓		2	CRS
D645					✓	✓		2	CRS
D652		✓	✓	✓			✓	4	CRS
								92	

Appendix 8.4

Input Output Levels of the 97 Unobserved Branches

	FA	TC	AI	AP	CT	MT	SV
A1D014	7.1	194087	59	5	151345	47	3601
A2D014	6.9	192014	58	15	152081	11	3605
A3D015	16	254875	371	254	100245	648	2541
A1D015	9.8	197600	2	146	96947	396	2189
A2D015	9.9	197209	192	5	96824	394	2194
A4D015	9.9	175851	198	148	73521	408	450
A5D181	12	198415	245	197	151044	335	2472
A1D181	5.9	149001	2	75	120654	208	1962
A2D181	5.9	131101	97	5	120975	203	1957
A3D181	5.9	113825	120	84	30000	224	1698
A4D181	5.8	113486	121	82	98045	221	450
A1D200	5.8	42207	2	24	53294	28	651
A2D200	6.35	42375	23	34	42607	45	556
B2D200	5.9	34554	16	24	30000	36	345
A3D200	5.95	42308	2	21	53784	11	652
A4D200	6	38576	11	21	49124	32	450
A3D348	4	126213	42	2	175457	64	2268
A4D348	4.05	153126	54	18	174214	11	2268
A1D353	3.95	53488	2	15	93945	18	964
A2D353	3.9	53069	8	14	93987	10	967
A3D353	4.25	60915	29	35	93007	31	761
B3D353	4	54392	17	26	90124	22	450
A1D373	8.9	172079	2	48	230759	48	3659
A1D440	5.95	103057	2	49	161941	80	1648
A3D440	5.9	98718	46	52	163214	11	1662
A1D442	5	95858	2	72	100041	84	1555
A2D442	5	85035	81	84	30000	102	1306
A3D442	5	96605	72	74	100054	11	1552
A1D460	3	90850	37	10	81991	11	1370
A1D463	4.1	84353	2	8	147043	18	1469
A1D521	6.5	50515	102	72	36085	87	801
B1D521	9.2	57487	154	137	37751	150	998
A3D521	2	40532	19	5	32687	18	406
A4D521	2	39215	20	12	32514	11	402
A5D521	2.1	41893	17	14	29812	24	324
A1D539	4.1	110962	103	24	147974	11	1460
A2D539	4.2	97893	124	27	129041	41	450
A1D549	5	88195	2	43	91138	140	1513
A2D549	5	89260	57	5	91122	142	1522
A4D549	4.8	75125	60	53	79912	157	450
A1D551	4.95	185051	2	16	265874	64	2654
A2D551	5.05	177422	24	4	261122	74	2564
A3D551	5.05	171043	31	30	261241	11	2597
A4D551	5	150608	37	36	238254	78	450
A4D555	5.05	93079	102	76	76001	75	450
A1D557	4.2	84123	2	26	99542	84	1692
A2D557	4	85977	23	5	99687	76	1691

A3D557	3.5	58330	27	21	30000	76	1241
A4D557	4.2	81115	31	34	100214	11	1687
A1D558	8.9	158514	2	83	160210	210	2767
A2D558	8.9	154514	127	5	159874	201	2757
A3D558	9.1	140285	170	114	30000	280	2504
A4D558	9	134097	138	108	159974	11	2752
A5D558	8.9	140285	172	106	139974	276	450
A1D571	10	169215	192	106	164214	162	3214
A2D571	6.1	110018	2	27	121014	84	2541
A3D571	6.2	111018	82	5	124325	57	2547
A4D571	6.2	86104	102	26	30000	71	2264
A5D571	6.2	107048	84	24	124512	11	2545
A1D577	3.8	60309	2	25	82794	62	1170
A3D577	4.4	63403	48	47	62354	87	1009
B3D577	3.95	53324	41	40	30000	78	845
A4D577	3.9	58440	27	26	82648	11	1178
A5D577	4.1	58260	39	39	80012	76	450
A1D579	3	72096	2	9	108421	14	1054
A2D579	2.9	73996	21	5	107845	12	1068
A3D579	2.95	72066	26	8	106548	11	1097
A4D579	3	62049	33	12	97546	17	450
A1D581	8.5	82903	76	32	141024	109	1612
A2D581	4.1	58367	2	10	104123	51	1201
A3D581	4	62975	36	3	111000	56	1297
A5D581	4	60138	35	9	107942	11	1251
A6D581	4.2	54464	42	11	99545	61	450
A2D586	2.9	67131	26	11	93654	11	1049
A1D586	3.45	69054	47	19	93945	37	1105
B1D586	2.9	65494	19	4	93745	14	1056
A3D586	3.6	65654	71	18	92921	41	848
B3D586	3	59280	56	11	90412	31	450
A1D600	8.4	207341	140	146	212234	150	2702
A2D600	6.05	163073	2	84	179528	90	2385
A3D600	5.8	130073	94	92	30000	104	2034
A4D600	5.9	150243	73	75	178912	11	2401
A2D607	5	103065	159	5	100341	131	1378
A3D607	4.95	99914	146	64	99784	11	1378
A4D607	4.9	99745	164	61	92572	138	450
A1D625	4.2	101303	2	14	168547	18	1795
A2D625	4.2	102003	30	5	168174	20	1745
A3D625	4.4	99881	24	10	165654	11	1740
A4D625	4.2	84413	40	18	152213	25	450
A2D630	4.15	57996	6	3	112041	8	570
A3D630	4.1	58068	5	5	110092	2	594
A2D645	5	91644	203	56	30000	102	1421
A3D645	5.9	110442	162	68	110732	11	1894
A2D652	4.1	179871	2	15	225745	26	2001
A4D652	3.9	140523	28	25	200069	32	450
A1D652	9	233249	134	136	269874	195	2564
A3D652	3.95	173871	17	5	227987	24	2001

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