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SAM FINCH

THE RANDOM CLUSTER MODEL ON THE TREE

MARKOV CHAINS, RANDOM CONNECTIONS AND BOUNDARY
CONDITIONS
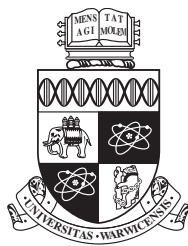
THESIS

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ABSTRACT

The Fortuin-Kasteleyn random cluster model was adapted to the tree by Häggström [38] who considered a Gibbs specification corresponding to "wired boundary conditions" on the tree. Grimmett and Janson [36] generalized this idea by considering boundary conditions defined by equivalence relations on the set of rays of the tree.

In this thesis we continue this project by defining a new object, a "random connection;" a type of random equivalence relation that allows us to redefine what is meant by a cluster of edges on the tree. Our definition is general enough to include Grimmett and Janson's boundary conditions. The random connection approach allows us to reconnect the random cluster model on the tree with Bernoulli bond percolation and we define two critical probabilities for bond percolation on a tree associated with each random connection that allow us to identify three behavioral phases of the associated random cluster model.

We consider some examples of random connections defined by equivalence relations, including the "open" boundary conditions described in [36] where we are able to describe the behaviour of the random cluster model exactly and "Mandelbrot" bondary conditions described by a map from the boundary of a tree to the unit square that defines fractal percolation. In addition we adapt work of Zachary [66] to the wired random cluster model on a tree so as to prove a conjecture of Häggström concerning uniqueness of the Gibbs measure for large bond strengths.
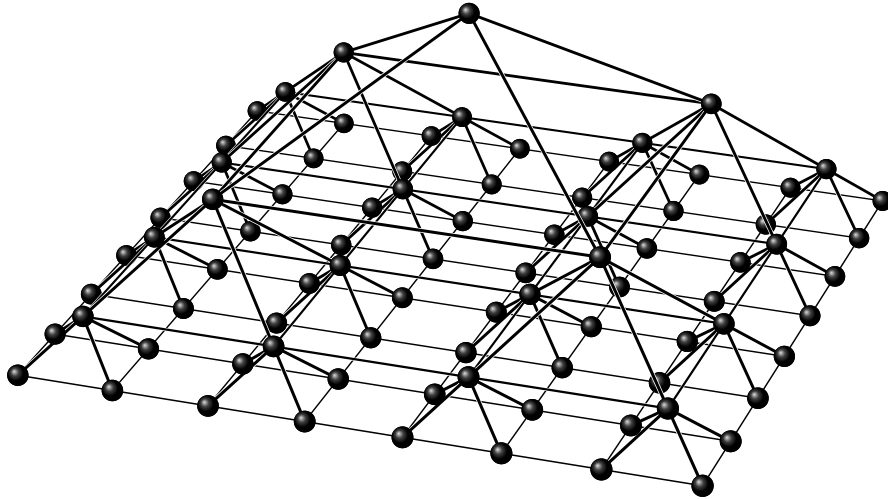
DECALARATION

This thesis is a presentation of my original research work. Wherever contributions of others are involved, every effort is made to indicate this clearly, with due reference to the literature, and acknowledgement of collaborative research and discussions. The work was done under the guidance of Professor Wilfrid Kendall, at the The University of Warwick, Coventry.

CONTENTS

THE RANDOM CLUSTER MODEL ON THE TREE

MARKOV CHAINS, RANDOM CONNECTIONS AND BOUNDARY CONDITIONS

1
MOTIVATION AND CONTEXT

Kendall and Wilson [46], motivated by questions of applied image analysis, considered the Fortuin-Kasteleyn random cluster model on the QuadTree; a graphical structure pictured on the previous page. This thesis concerns questions about the random cluster model on the tree which arose after study of [46]. In this chapter we use the language of image analysis to define the QuadTree and to motivate our study.

Although we have departed quite radically from image analysis problems in this text we pose a specific motivating problem concerning the random cluster model on the QuadTree. We give an overview of some of the existing literature relevant to the subject and describe briefly the material contained in the forthcoming chapters. We will treat the material in this chapter with no great rigour, preferring a historical account. We will provide a more rigorous account of the background material in the next chapter.

1.1 THE QUADTREE

The main focus of this work is to define and study a set of random cluster models on the tree. In order to motivate our study we begin by considering the random cluster model on the QuadTree pictured on the title page.

The QuadTree is a graphical structure used in image analysis. The applications of the QuadTree in image analysis are beyond the scope of this document. We direct the reader to [46] and references therein for some background on the subject.

It will however be helpful for motivation to describe the QuadTree in the language of images.

Before we start let us define an unusual piece of notation which resolves an ambiguity specific to the QuadTree.

From an image analysis point of view, the QuadTree is a tool for understanding $\mathbb{R}^d$. As a graph there is a strong link between the QuadTree and the lattice $\mathbb{Z}^d$. Rectangles, both in $\mathbb{Z}^d$ and in $\mathbb{R}^d$ will play an important role in many of our arguments and we must distinguish between integer intervals and real intervals. The notation $[a,b] = \{x \in \mathbb{R} : a \leq x \leq b\}$ is common throughout mathematics, but in percolation $[a,b]$ is often used to indicate the set $\{a, \ldots, b\}$. We introduce a new bracket

$$\llbracket a,b \rrbracket = [a,b] \cap \mathbb{Z} \tag{1.1}$$

to indicate an interval of integers.

1.1.1 Potts Models on the Lattice

Given a non-negative $n \in \mathbb{N}$ we may partition $\mathbb{R}^d$ into a set of *pixels* by setting

$$\mathcal{P}_{\mathbb{R}^d}^n = \left\{ \left[2^{-n} a_1, 2^{-n} (a_1 + 1) \right] \times \cdots \times \left[2^{-n} a_d, 2^{-n} (a_d + 1) \right] : a_1, \ldots a_d \in \mathbb{Z} \right\} \tag{1.2}$$

to be the *resolution n pixellation of* $\mathbb{R}^d$. Equivalently for a domain $D \subset \mathbb{R}^d$ we may set

$$\mathcal{P}_D^n = \left\{ \mathcal{P} \in \mathcal{P}_{\mathbb{R}^d}^n : \mathcal{P} \subset D \right\}. \tag{1.3}$$

For a general domain D the pixel set $\mathcal{P}_D^n$ does not cover D. We are interested only in those domains which may be partitioned exactly by some subset of $\mathcal{P}_{\mathbb{R}^d}^n$. We say a rectangle $S \subset \mathbb{R}^d$ is a *screen of resolution n* if $\bigcup \left\{ \mathcal{P} \in \mathcal{P}_{\mathbb{R}^d}^n : \mathcal{P} \subset S \right\} = S$. We say S is a *screen* if it is a screen of resolution n for some n. We say the *minimum resolution* of a screen is the smallest n with this property.

Notice that the minimum resolution of any screen must be non-negative as we have not defined $\mathcal{P}_D^n$ for $n < 0$.

For any $n \geq 0$ the set $\mathcal{P}_{\mathbb{R}^d}^n$ is invariant under the symmetries of $\mathbb{Z}^d$. If we were to allow $\mathcal{P}_{\mathbb{R}^d}^{-n}$ to contain cubes in the form $[2^n a_1, 2^n (a_1 + 1)] \times \cdots \times [2^n a_d, 2^n (a_d + 1)]$ this symmetry would be broken as the pixellation would no longer be invariant under a shift of length 1.

We will return briefly to the idea of negative pixelation in Remark 1.6. Unfortunately there has been insufficient time to investigate the idea fully and this topic must be left as a possible direction for future research.

The simplest representation of an image is a bitmap, each pixel $\mathcal{P} \in \mathcal{P}_S^n$ is assigned a colour, this extends in an obvious way to a map (up to a null set) of S to the set of colours. In modern computers the set of colours is large enough to be modelled as a continuum of colours and in practice the "colour" of a pixel will usually represent a wavelet rather than a solid block of colour.

For our purposes we assume that we are dealing with only a finite set of colours and that no two colours are specially related in any way.

A problem of interest in image analysis to assign a probability distribution to the set of maps from $\mathcal{P}_S^n$ to the set of colours. This distribution may than be used either as a prior for Bayesian

image segmentation or reconstruction or as a model for data compression.

A simple model for a bitmap is the Potts model.

Endow the set $\mathcal{P}_S^n$ with a graph structure by naming a set of "lattice edges"

$$\mathbf{L}(\mathcal{P}_S^n) = \left\{ \langle \mathcal{P}, \mathcal{P}' \rangle : \mathcal{P} \cup \mathcal{P}' \text{ is a rectangle} \right\}. \tag{1.4}$$

Let ρ be the discrete metric on the set $[\![1, q]\!]$ (where $[\![1, q]\!]$ is interpreted as a set of q discrete colours). Now let Σ_S be the set of functions $\{\sigma : \mathcal{P}_S^n \to [\![1, k]\!]\}$. Then for $\beta \in \mathbb{R}$ we may assign a measure

$$\mu_S^\beta(\sigma) \propto \exp\left[-\beta \sum_{\langle \mathcal{P}, \mathcal{P}' \rangle \in \mathbf{L}(\mathcal{P}_S^n)} \rho\big(\sigma(\mathcal{P}), \sigma(\mathcal{P}')\big) \right] \tag{1.5}$$

To simplify notation fix $\Sigma = \Sigma_{\mathbb{R}^d}$ to be the space of functions $\{\sigma : \mathcal{P}_{\mathbb{R}^d}^n \to [\![1, k]\!]\}$, and define measures μ_S^β by restricting to the appropriate σ-algebra. We are using Σ here to represent the state of vertex-configurations, as opposed to the more usual Ω which we reserve for the use of edge-configurations introduced below which will be the main focus of this work.

The main concern of this work is the case where $\beta > 0$. Under such conditions a configuration is penalised if the colours of two adjacent pixels $\sigma(\mathcal{P})$ and $\sigma(\mathcal{P}')$ disagree for some edge of the graph $\langle \mathcal{P}, \mathcal{P}' \rangle \in \mathbf{L}(\mathcal{P}_S^n)$. Thus the random configuration favours images with connected "clusters" of pixels with the same colour.

Historically the case where $q = 2$ was studied first. It was famously proposed by Lenz to his student Ising who studied the model on the two dimensional lattice [43]. The model was extended to the four colour case by Ashkin and Teller [2] and to the general case by Potts [55]. The history of the Ising model is well known and we direct the reader to Brush [14] for a historical account.

The accounts described above are all concerned with the infinite lattice $\mathbb{Z}^d$. As $\mathcal{P}_{\mathbb{R}^d}^n$ is isomorphic to $\mathbb{Z}^d$ we shall keep our current notation and not formally introduce the model on $\mathbb{Z}^d$.

However the measure defined in (1.5) makes sense only when S is a finite domain. Next we consider how to extend the Potts model to $\mathcal{P}_{\mathbb{R}^d}^n$.

The Potts model forms an example of a *Gibbs measure*, to each edge we have assigned an "energy function" $\rho(\sigma(\mathcal{P}), \sigma(\mathcal{P}'))$. If we set $H_s(\sigma)$ to be the total energy of a configuration then we have $\mu_S^\beta(\sigma) \propto e^{-\beta H_s(\sigma)}$. For a graph such as $\mathbb{Z}^d$, where there are no triangles a Gibbs measure is a probability measure in this form where energy assigned to each edge $\langle \mathcal{P}, \mathcal{P}' \rangle$ is an arbitrary (and not necessarily symmetric) function of $\sigma(\mathcal{P})$ and $\sigma(\mathcal{P}')$. For a general graph we assign an energy to each "clique" of the graph.

A theorem of Hammersly and Clifford [41], popularized by Preston [56] states that the Gibbs measures are exactly those measures that have the *Markov random field property*

For a set $A \subset \mathcal{P}_S^n$ set N_A to be the set of *neighbours* of A, that is the set of vertices $v \notin A$ such that there is some $u \in A$ with $\langle u, v \rangle \in \mathbf{L}$. Let $\mathscr{F}_A$ be the σ-algebra generated by the random variables $\{\sigma(v) : v \in A\}$ and $\mathscr{T}_A$ the σ-algebra generated by $\{\sigma(v) : v \notin A\}$.

A measure μ on σ is a Markov random field if for any finite set A we have

$$(\mu \,|\, \mathscr{T}_A) = (\mu \,|\, \mathscr{F}_{N_A}). \tag{1.6}$$

That is the distribution of the colouring of A is conditionally independent of the colouring of those pixels outside of A given the neighbourhood of A.

For finite screens the measure μ_S^β is a Markov random field and it may be seen that in the case of the Potts model the left hand side of (1.6) does not depend on the choice of screen S other than in the requirement that $S \ni \mathcal{P}$ for every $\mathcal{P} \in (A \cup N_A)$.

Dobrushin [17], Lanford and Ruelle [47] proposed a definition of general Markov random fields on infinite graphs. For simplicity we consider only the special case of the Potts model on $\mathcal{P}_{\mathbb{R}^d}^n$.

Say a measure μ on Σ satisfies *DLR conditions* (for the (β, q)-Potts model) if for every finite subset $A \subset \mathcal{P}_{\mathbb{R}^d}^n$ we have

$$(\mu \mid \mathscr{T}_A)(\sigma) = \mu_A^\beta(\sigma) \tag{1.7}$$

$$\stackrel{\text{def}}{=} \left(\mu_S^\beta \mid \mathscr{F}_{N_A}\right)(\sigma) \tag{1.8}$$

for an appropriate choice of finite screen S.

We may construct measures that satisfy (1.7) by considering weak limits of models on finite graphs. In particular we introduce sequences of measures with either *free* or *fixed* boundary conditions.

Fix a *boundary condition* $\zeta \in \Sigma$ and let S_i be some sequence of finite screens that exhaust $\mathbb{R}^d$. Consider the two sequences of measures

$$\mu_{\text{free},i}^\beta = \mu_{S_i}^\beta, \tag{1.9}$$

$$\mu_{\zeta,i}^\beta = \mu_{\mathcal{P}_{S_i}^n}^\beta(\zeta). \tag{1.10}$$

The space Σ is compact and a theorem of Prohorov [57] states that any sequence of measures on a compact space contains a weakly convergent subsequence. Thus we assume without loss of generality that the two sequences above converge to weak limits $\mu_{\text{free}}^\beta = \text{wlim}_{i\to\infty} \mu_{\text{free},i}^\beta$ and $\mu_\zeta^\beta = \text{wlim}_{i\to\infty} \mu_{\zeta,i}^\beta$.

In the case of the Potts model it is known that for any sequence of finite screens S_i the free measures above converge to a limit. Moreover there exists some critical value β_c such that if $\beta < \beta_c$ then $\mu_{\zeta,i}^\beta \to \mu_{\text{free},i}^\beta$ weakly as $i \to \infty$ for every configuration ζ. Conversely if $\beta > \beta_c$ then constant limits $\mu_{\zeta^k}^\beta$ (where $\zeta^k(\mathcal{P}) \equiv k$ are the constant configurations for $k \in [\![1, q]\!]$) all differ and the free model may be expressed as the uniform mixture

$$\mu_{\text{free}}^\beta = \frac{1}{q} \sum_{i=1}^q \mu_{\zeta^k}^\beta. \tag{1.11}$$

We will consider this phenomenon in more depth in Section 1.2 where we introduce the Fortuin-Kasteleyn random cluster model. Before that we continue to build up the structure of the QuadTree.

1.1.2 Markov Random Fields on the Tree

In the previous subsection we considered the Potts model on the infinite lattice $\left(\mathcal{P}^n_{\mathbb{R}^d}, \mathbf{L}\left(\mathcal{P}^n_{\mathbb{R}^d}\right)\right)$. On the infinite lattice our choice of the resolution level n does not matter.

However, if we choose a finite screen, say $[0,1]^d$, then the image chosen by a Potts model will have dramatically different properties depending on the choice of resolution.

If n is small the possible images will all be "blocky" and corresponding models of actual images will be unable to represent fine detail. If n is very large the random fluctuations of the Potts model will average out to leave, to a human observer, a single coherent shade rather than a detailed image. To model a "real" image we require aspects of both the high and low resolutions to produce something realistic.

For nonnegative integers $m < n$ set

$$\mathcal{P}^{[\![m,n]\!]}_D = \bigcup_{i=m}^{n} \mathcal{P}^i_D. \tag{1.12}$$

When several resolutions are taken together the pixels exhibit a tree like structure. For every $\mathcal{P} \in \mathcal{P}^n_{\mathbb{R}^d}$ there is a unique smallest $\mathcal{M}(\mathcal{P}) \in \mathcal{P}^{n-1}_{\mathbb{R}^d}$ such that $\mathcal{P} \subset \mathcal{M}(\mathcal{P})$. We name this pixel the *mother* of $\mathcal{P}$. We say $\mathcal{P}_1$ is a *child* of $\mathcal{P}_2$ if $\mathcal{P}_2 = \mathcal{M}(\mathcal{P}_1)$ is the mother of $\mathcal{P}_1$.

Define a set of *directed*, *tree-like* edges,

$$\mathbf{T}(\mathcal{P}) = \left\{ \left| \mathcal{P}_1, \mathcal{P}_2 \right\rangle : \mathcal{P}_1, \mathcal{P}_2 \in \mathcal{P}; \mathcal{P}_2 = \mathcal{M}(\mathcal{P}_2) \right\}. \tag{1.13}$$

We are directing the edges of the tree for convenience only. We will allow undirected edges in the form $\langle \mathcal{P}_1, \mathcal{P}_2 \rangle$ to represent $|\mathcal{P}_1, \mathcal{P}_2 \rangle$ or $|\mathcal{P}_2, \mathcal{P}_1 \rangle$ as appropriate.

The study of Markov random fields on trees was initiated by Spitzer [60] who considered the two state case. Zachary [66] continued this work extending Spitzer's results to the case of countable state space Markov random fields (in our terms a countably infinite set of colours) which we recount briefly here.

We concern ourselves only with the case of finite colour sets. Let T be a finite tree with all edges directed away from some nominated root vertex x and let μ be a Markov random field on T with state space $\Sigma = [\![1,q]\!]^{V(T)}$. We may sample from μ in following way.

First select $\sigma(x)$ according to $\mu(\sigma(x))$. Then μ is a Markov random field and removing each edge $\langle x, u \rangle$ for $u \in N_x$ splits the tree into $|N_x|$ subtrees. Hence the random variables $\{\sigma(v) : v \in N_x\}$ are conditionally independent given $\sigma(x)$.

Arguing inductively we see that for each edge $e = |u, v \rangle$ we may choose some stochastic matrix $M_e : [\![1,q]\!]^2 \to [0,1]$ such that if A is some connected set of vertices containing x with $u \in A$ and $v \in N_A$ we have

$$\mu(\sigma(v) = i \mid \mathscr{F}_A) = M(\sigma(u), i). \tag{1.14}$$

As μ is a Markov random field we may assign some energy function $\rho_e : [\![1,q]\!]^2 \to \mathbb{R}$ such that

$$\mu(\sigma) \propto \exp\left[-\beta \cdot \sum_{e=|u,v\rangle \in E(T)} \rho_e(\sigma(u), \sigma(v)) \right]. \tag{1.15}$$

Now let $\Lambda(T) \subset V(T)$ be the set of *leaves* of T, that is those vertices v with exactly one neighbour (assume the root $x \notin \Lambda(T)$). For an arbitrary function $\Psi : (\Lambda(T) \times [\![1,q]\!]) \to [0,1]$ we may

modify the measure μ by applying an external field to the leaves of T by setting

$$\mu_\Psi(\sigma) \propto \exp\left[-\beta \cdot \sum_{e=|u,v\rangle\in E(T)} \rho_e(\sigma(u),\sigma(v)) + \sum_{v\in\Lambda(T)} \Psi(v)\right]. \quad (1.16)$$

Then μ_Ψ is also a Markov random field and may assign stochastic matrices M_e^Ψ to describe μ_Ψ as a Markov chain.

The notion of a Markov chain extends naturally to the case of an infinite tree. It may be seen that, for an infinite tree $\mathcal{T}$; a measure μ on $\Sigma_\mathcal{T}$ is a Markov chain if and only if, for every finite subtree $T \subset \mathcal{T}$, the restriction of μ to $\mathscr{F}_T$ is a Markov random field.

If $\rho = \{\rho_e : e \in \mathbf{E}(\mathcal{T})\}$ is an interaction then the set of measures $\mathcal{M}_\beta(\rho)$ satisfying DLR conditions for ρ forms a simplex in the sense of Dynkin [21].

Zachary was able to show that

- Every *extremal* element of $\mathcal{M}_\beta(\rho)$ is a Markov chain.

- A measure $\mu \in \mathcal{M}_\beta(\rho)$ is a Markov chain if and only if there exists some function $\Psi : \Lambda(\mathcal{T}) \times [\![1,q]\!]$ such that the restriction of μ to Σ_T is given by (1.16).

In addition Zachary gave necessary and sufficient conditions for a function Ψ to define a Markov chain, and named the such a function an *entrance* law.

Thus the study of Markov random fields on trees reduces to the study of entrance laws. In particular as the extremal elements are specified uniquely by entrance laws then the set of Gibbs measures is unique if and only if there is a unique entrance law for that specification.

The ideas in [66] will form the basis for our investigation of the random cluster model on a tree in Chapter 3.

FIGURE 1
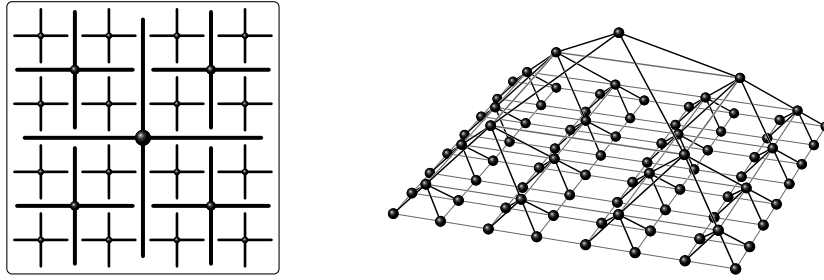
Two graphical representations of the QuadTree. On the left the unit square is partitioned by a sequence of *pixelations*. On the right the partition is represented as a graph, with each vertex representing one pixel. The structure of the underlying pixel set is encoded by adding *tree-like* and *lattice-like* edges.

1.1.3   Constructing the QuadTree

We have described two ways of structuring sets of pixels to form Markov specifications that might describe an image. We have dismissed the lattice structure as insufficient as it is restricted to a single resolution level, where a genuine image may be better described by a "multiresolution" representation. However the tree does not reflect the structure of the cube. In particular pixels than are close in the image may be far from each other on the tree. Any measure on images produced by the tree structure alone will be invariant under the symmetries of the tree, a restriction that does not apply to a genuine image.

To form the QuadTree we combine both structures, the aim being to maintain the multiresolution properties of the tree structure while effectively breaking the symmetry group of the tree better to reflect the finite symmetries of the cube.

Recall we have defined pixel sets

$$\mathcal{P}^n_{\mathbb{R}^d} = \{[2^{-n}a_1, 2^{-n}(a_1+1)] \times \cdots \times [2^{-n}a_d, 2^{-n}(a_d+1)] : a_1, \ldots a_d \in \mathbb{Z}\}$$

(1.17)

$$\mathcal{P}^n_D = \left\{ \mathcal{P} \in \mathcal{P}^n_{\mathbb{R}^d} : \mathcal{P} \subset D \right\}$$

(1.18)

$$\mathcal{P}^{[\![m,n]\!]}_D = \bigcup_{i=m}^{n} \mathcal{P}^i_D.$$

(1.19)

We say two pixels $\mathcal{P}_1, \mathcal{P}_2$ are *adjacent* and write $\mathcal{P}_1 \sim \mathcal{P}_2$ if their union is a (non-square) rectangle.

Let $\mathcal{P} \subset \mathcal{P}_{\mathbb{R}^d}^{[\![1,\infty]\!]}$ be some subset of pixels then we define sets of *lattice like edges* and *tree like* edges

$$\mathbf{L}(\mathcal{P}) = \left\{ \langle \mathcal{P}_1, \mathcal{P}_2 \rangle : \mathcal{P}_1, \mathcal{P}_2 \in \mathcal{P}; \mathcal{P}_1 \sim \mathcal{P}_2 \right\}, \tag{1.20}$$

$$\mathbf{T}(\mathcal{P}) = \left\{ |\mathcal{P}_1, \mathcal{P}_2 \rangle : \mathcal{P}_1, \mathcal{P}_2 \in \mathcal{P}; \mathcal{P}_2 = \mathcal{M}(\mathcal{P}_2) \right\}. \tag{1.21}$$

Hence, associated with the pixel set, there is an implicit partially directed graph $(\mathcal{P}, \mathbf{L}(\mathcal{P}) \cup \mathbf{T}(\mathcal{P}))$. Figure 1 shows a finite pixelation of the unit square and the associated graph structure.

To define the QuadTree, we extend the pixelation of the unit square to infinity by setting

$$\mathbf{V}(\mathcal{Q}) = \mathcal{P}_{[0,1]^d}^{[\![0,\infty]\!]}, \tag{1.22}$$

$$\mathcal{Q} = (\mathbf{V}(\mathcal{Q}), \mathbf{T}(\mathbf{V}(\mathcal{Q})) \cup \mathbf{T}(\mathbf{V}(\mathcal{Q}))). \tag{1.23}$$

The QuadTree may be pictured, in two dimensions at least, by extending the tree in Figure 1 down to infinity and adding in extra lattice-like edges as required. From the point of view of the random cluster model the QuadTree presents two challenges not encountered in the conventional setting of the random cluster model on the integer lattice. Firstly it is not *amenable*, see definition 1.4 below. Similar to the tree, the boundary of a large ball is of comparable size to the ball itself. Secondly the QuadTree has a finite symmetry group, that of the cube.

These two characteristics mean that many of the standard techniques used for the study of Markov random fields are unavailable on the QuadTree.

1.2 PERCOLATION AND THE RANDOM CLUS–
TER MODEL

The random cluster model on a graph $G = (V, E)$ was introduced in 1972 in a series of papers by Fortuin and Kasteleyn [26, 24, 25] to explain connections between results concerning percolation, Ising and Potts models and electrical networks. Rather than considering the space Σ above directly we let Ω_G be the set of functions $E \to \{0, 1\}$.

We say an edge $e \in E$ is *open* if $\omega(e) = 1$ and *closed* if $\omega(e) = 0$. By removing all closed edges from the graph we partition the vertices V into $\kappa(\omega)$ connected components which we refer to as clusters.

Then we may consider measures in the form

$$Q_{p,q}(\omega) \propto q^{\kappa(\omega)} \prod_{e \in E} \left(\frac{p}{1-p} \right)^{\omega(e)}. \tag{1.24}$$

We refer to the measure $Q_{p,q}$ as the *random cluster model* with *bond strength p* and *cluster factor q*.

It is well known that if q is an integer then we may recover the Potts model for $\beta \geq 0$ by setting $\lambda = 1 - e^{-\beta}$ then assigning independently to each cluster a colour chosen uniformly from $[\![1, q]\!]$. The definition of the model above does not require q to be an integer. We will be interested in the random cluster model for all $q > 0$.

The relationship between the random cluster and the Potts models is well, but not perfectly understood, see for example [52, 37, 10, 9]. As we are using the Potts model only as motivation we will not dwell on the details of the relationship.

1.2.1   Domination and Percolation

The space Ω_G comes with a natural partial order which we may exploit by considering *increasing* random variables. We say a random variable $X : \Omega \to \mathbb{R}$ is increasing if $X(\omega_1) \geq X(\omega_2)$ whenever $\omega_1(e) \geq \omega_2(e)$ for every $e \in E$ and an event A is increasing if its indicator $\mathbb{1}_A$ is increasing.

In particular a famous theorem of Fortuin, Kasteleyn and Ginibre [27] gives a sufficient condition for increasing events to be correlated.

THEOREM 1.1: FKG INEQUALITY

Let Ω be a distributive lattice, and let μ be a probability measure on Ω with the property that for any $\omega_1, \omega_2 \in \Omega$

$$\mu(\omega_1 \vee \omega_2) \cdot \mu(\omega_1 \wedge \omega_2) \geq \mu(\omega_1) \cdot \mu(\omega_2) \tag{1.25}$$

then for any pair of increasing random variables X and Y we have

$$\mu(XY) \geq \mu(X) \cdot \mu(Y). \tag{1.26}$$

Holley [42] extended the ideas of Fortuin et al. to compare the probabilities of increasing events over two different measures.

THEOREM 1.2: HOLLEY

Let Ω be a distributive lattice, and let and let μ_1 and μ_2 be two probability measures on Ω with the property that for any $\omega_1, \omega_2 \in \Omega$

$$\mu_1(\omega_1 \vee \omega_2) \cdot \mu_2(\omega_1 \wedge \omega_2) \geq \mu_1(\omega_1) \cdot \mu_2(\omega_2) \tag{1.27}$$

Then for any increasing random variable X we have

$$\mu_1(X) \geq \mu_2(X). \tag{1.28}$$

We will return to these ideas in more detail in Chapter 2, for now we note that for any finite graph G and $q \geq 0$ and $p \in (0,1)$, then if we set $\pi = \frac{p}{p+(1-p)q}$ and let $\mathbb{Q}_{p,q}$ be the random cluster model as in (1.24) and set $\mathbb{P}_p$ and $\mathbb{P}_\pi$ to be Bernoulli percolation on G with bond probabilities p and π respectively then all measures satisfy (1.25) and the pairs $(\mathbb{P}_\pi, \mathbb{Q}_{p,q})$ and $(\mathbb{Q}_{p,q}, \mathbb{P}_p)$ satisfy (1.27) so that $\mathbb{P}_\pi(X) \leq \mathbb{Q}_{p,q}(X) \leq \mathbb{P}_p(X)$ for any increasing random variable X.

Therefore if we wish to understand the random cluster model, particularly the behaviour of increasing events then it is useful to study Bernoulli bond percolation.

1.2.2 Percolation on the QuadTree

Kendall and Wilson [46] studied the Ising model on the QuadTree by comparison with independent bond percolation.

Let $\Omega_{\mathcal{Q}} = \{0,1\}^{\mathbf{E}(\mathcal{Q})}$ and let $\mathbb{P}$ be the product measure on $\Omega_{\mathcal{Q}}$ such that $\omega(e) = 1$ with probability τ if $e \in \mathbf{T}(\mathcal{Q})$ and with probability λ if $e \in \mathbf{L}(\mathcal{Q})$.

Kendall and Wilson were particularly interested in the number of infinite clusters of the percolation process. The QuadTree combines elements of the lattice $\mathbb{Z}^d$ with the tree. Supercritical percolation on the tree and the lattice behave very differently to each other.

On any graph there exists some critical probability p_c such that if $p < p_c$ then independent percolation with all bonds open with probability p contains no infinitely large clusters whereas if $p > p_c$ at least one infinite cluster exists. We may of course construct graphs where $p_c = 0$ or 1.

On $\mathbb{Z}^d$ if $p > p_c(\mathbb{Z}^d)$ that there can exist at most one infinite cluster see for example Grimmett [34]. On a tree however if $p > p_c$ then there exist infinitely many infinitely large components.

FIGURE 2
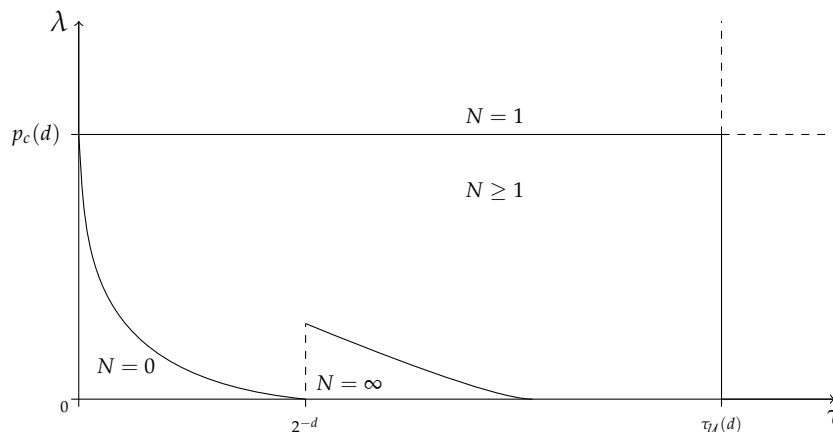
Partial phase diagram for independent bond percolation on the QuadTree from Kendall and Wilson [46]. The number N of infinite clusters may be 0, 1 or ∞.

Kendall and Wilson showed that on the QuadTree, for any $N \in \{0, 1, \infty\}$ there exist pairs pairs of parameters τ, λ for which with probability one there exist N infinite clusters. We summarize the main results of [46] concerning percolation on the QuadTree in Theorem 1.3 and in Figure 2.

THEOREM 1.3: KENDALL AND WILSON

Let $\mathbb{P}$ be independent bond percolation on the QuadTree $\mathcal{Q}$ with parameters (τ, λ) as above.

- If

$$2^d \tau \chi_d(\lambda) \left(1 - \sqrt{1 - \chi_d(\lambda)^{-1}}\right) < 1 \qquad (1.29)$$

  there exist no infinite percolation clusters.
- If $\tau \in \left(2^{-d}, 2^{\frac{1-d}{2}}\right)$ then there is some $\varepsilon = \varepsilon(\tau) > 0$, such that if $\lambda < \varepsilon$ there exist infinitely many infinite clusters.
- When $d = 2$ and $\lambda > p_c(d)$ then for any $\tau > 0$ there exists a unique infinite cluster.
- For $d \geq 2$ there exists some $\tau_u(d)$ such that if $\tau \geq \tau_u$ and $\lambda > 0$ then there exists a unique infinite cluster.

Here $p_c(d)$ is the critical bond probability for Bernoulli percolation on $\mathbb{Z}^d$ and $\chi_d(\lambda)$ is the expected size of the cluster at the origin for Bernoulli percolation on $\mathbb{Z}^d$ with bond probability λ

The QuadTree was not the first example of a graph that exhibits three phases in this way. Grimmett and Newman [31] studied the product graph $T \times \mathbb{Z}^d$ for regular trees T. The authors of that paper were also able to show the existence of both a single infinite cluster phase and a multiple infinite cluster phase.

The existence of both phases is strongly connected with the notion of amenability.

DEFINITION 1.4

Recall for a set of vertices A on a graph $G = (V, E)$ we may define N_A, the *neighbours* of A, as the set of vertices $v \in V \setminus A$ for which $\langle u, v \rangle \in E$ for some $u \in A$.

The *Cheeger constant* of an infinite graph is the infimum over finite sets A

$$h(G) = \inf_A \frac{|N_A|}{|A|}. \tag{1.30}$$

We say a graph is *amenable* if $h(G) = 0$.

A celebrated theorem of Burton and Keane [15] shows that for Bernoulli bond percolation any transitive amenable graph with probability one there exists at most one infinite cluster. The picture is less clear for a nonameanable graph.

A major question in this area was posed by Benjamini and Schramm [5] concerning the number of infinite clusters on a nonamenable graph.

CONJECTURE 1.5: BENJAMINI AND SCHRAMM

If G is a nonameanable transitive graph with one end then there exist $0 < p_c < p_u < 1$ such that Bernoulli p percolation has

- No infinite clusters if $p < p_c$,

- Infinitely many infinite clusters if $p_c < p < p_u$,

- A unique infinite cluster if $p > p_u$.

The result is known under the additional assumption that G is planar, see Benjamini and Schramm [6].

Schonmann [58] has also proved known that for any transitive graph that if $p_1 < p_2$ and Bernoulli-p_1 percolation exhibits a unique infinite cluster, so Bernoulli-p_2 percolation exhibits a unique infinite cluster also. We refer the reader to surveys on the subject by Benjamini and Schramm [5] and Häggström and Jonasson [39].

The QuadTree however is far from transitive, it has only a finite symmetry group. Thus results such as the above may not be applied directly.

REMARK 1.6

We may recover some form of transitivity by a considering a random environment derived from the QuadTree.

We have not defined pixels at resolution $-n$ as there is no obvious way to chose the set $\mathcal{P}_{\mathbb{R}^d}^{-n}$ from the $2^{(nd)}$ possible choices. In addition by fixing $\mathcal{P}_{\mathbb{R}^d}^{-n}$ we reduce the symmetry of $\mathcal{P}_{\mathbb{R}^d}^0$. Our aim then is to choose sets $\mathcal{P}_{\mathbb{R}^d}^{-n}$ uniformly at random in such a way that the random environment is invariant under the group $\mathbb{Z}^d$ of translations.

For some pixel $\mathcal{P} \in \mathbf{V}(\mathcal{Q})$ let $T_{\mathcal{P}}$ be the (homeothetic) map $\mathbb{R}^d \to \mathbb{R}^d$ such that $T_{\mathcal{P}}\left([0,1]^d\right) = \mathcal{P}$. So, choosing $\mathcal{P}$ uniformly from $\mathcal{P}_{[0,1]^d}^n$ we may set $\mathcal{P}_{\mathbb{R}^d}^{k-n} = T_{\mathcal{P}}^{-1}\left(\mathcal{P}_{\mathbb{R}^d}^k\right)$. It is easy to see that the random graph defined by this procedure is invariant under translations in $\mathbb{Z}^d$.

Next we extend this random environment as $n \to -\infty$ by choosing a random sequence of pixels as follows. Let $\Xi = \left(\mathcal{P}_{[0,1]^d}^1\right)^{\mathbb{N}}$

and choose a uniformly distributed i.i.d. sequence of resolution one pixels $\xi_i \in \Xi$. Now set

$$\mathcal{P}_0 = [0,1]^d, \qquad\qquad \mathcal{P}_{i+1} = T_{\xi_i}(\mathcal{P}_i), \qquad (1.31)$$

$$\mathcal{P}_{\mathbb{R}^d}^{-n} = T_{\mathcal{P}_n}^{-1}(\mathcal{P}_{\mathbb{R}^d}^0). \qquad\qquad\qquad\qquad (1.32)$$

We leave it to the reader to check that for any sequence ξ the set $\mathcal{P}_{\mathbb{R}^d}^{-n}$ exactly partitions $\mathcal{P}_{\mathbb{R}^d}^{-(n+1)}$ and that the resulting random pixellation is invariant under both translations by $\mathbb{Z}^d$ and the transformations $T_{\mathcal{P}}$ for any $\mathcal{P} \in \mathbf{V}(\mathcal{Q})$.

If we add the relevant tree and lattice edges to the pixel set $\mathcal{P}_{\mathbb{R}}^{[\![-\infty,0]\!]}$ the resulting random graph may be interpreted as a QuadTree viewed from the bottom layer. Bernoulli percolation and the random cluster model on this "worm's-eye" view of the QuadTree offer many interesting questions for future research. We will return to this structure in Chapter 6 where we consider how the behaviour of the random cluster model on the worm's-eye QuadTree might impact on the behaviour of the random cluster model on the QuadTree.

1.2.3 The Random Cluster Model on the QuadTree

We have not yet discussed how to define the random cluster model on an infinite graph. The DLR approach we have used above was formalized by Grimmett [32], we will deal with this formally in the next chapter. For now we make an observation about the random cluster model on a finite graph.

For a graph G let $\mathbf{Q}_G$ be the (p,q) random cluster model on G. For an edge $e \in \mathbf{E}(G)$ let $\mathcal{T}_e$ be the sigma algebra generated by the random variables $\omega(f)$ for $f \neq e$.

Now suppose we alter a configuration by opening or closing e. This may have one of two effects on the number of clusters; either opening e joins two distinct clusters, hence reduces the

number of clusters by 1 or opening and closing e has no effect on the number of clusters at all.

Let $\mathcal{L}_e$ be the event that there is a "loop" around e, that is the two end vertices of e are connected whether e is open or not. Then $\mathcal{L}_e$ is the event that opening and closing e has no effect on the number of clusters, and we may calculate that

$$\mathbb{Q}_G(\omega(e) \mid \mathscr{T}_e)(\omega) = \begin{cases} p & : \text{if } \omega \in \mathcal{L}_e, \\ \pi & : \text{if } \omega \notin \mathcal{L}_e. \end{cases} \tag{1.33}$$

Here $\pi = \frac{p}{p+(1-p)q}$ as before.

If fact a simple Markov chain argument shows that $\mathbb{Q}_G$ is the only measure on Ω_G that satisfies (1.33). We can use (1.33) as a working definition of the random cluster model on an infinite graph. Although we will not use (1.33) as a formal definition we will show that it is in fact a necessary and sufficient condition for a measure to satisfy DLR conditions for the random cluster model. We will use this observation in many of our arguments concerning the random cluster model.

We may describe the topology on Ω_G by associating an equivalence relation with each subgraph $H \subset G$. Set $\omega \overset{H}{\sim} \omega'$ if $\omega(e) = \omega'(e)$ for every $e \in H$. The open balls of Ω_G are the equivalence classes of $\overset{H}{\sim}$ for finite subgraphs H. Notice that for an infinite graph G the indicator $\mathbb{1}_{\mathcal{L}_e}$ is not a continuous function on Ω.

To see this suppose that when the edge e is closed the end vertices of e are in two separate infinite clusters. Then for any finite subgraph H we may connect the two end vertices of e by opening every vertex outside H.

Hence is it not immediate that any measure satisfies DLR conditions for the random cluster model on an infinite graph. However the indicator $\mathbb{1}_{\mathcal{L}_e}$ while not continuous is *right* continuous with respect to the partial order on Ω_G. That is if $\omega \in \mathcal{L}_e$ then there is some open set of configurations A such that $\omega' \in \mathcal{L}_e$ for any $\omega' \in A$ with $\omega' > \omega$. We will see in the next chapter that if $q > 1$ and H_n is a sequence of finite subgraphs that

exhaust G then the sequence of finite random cluster models $\mathbb{Q}_{H_n}$ are *stochastically increasing* in the sense that for an increasing random variable X the sequence $\mathbb{Q}_{H_n}(X)$ is increasing hence convergent. This will be enough to show that the weak limit of such a sequence satisfies DLR conditions above.

On the QuadTree we have a two parameter percolation process, although we have considered the random cluster model with homogeneous bond weights it is easy to define a random cluster model on the QuadTree with three parameters, tree bond strength τ, lattice bond strength λ and cluster factor q. We denote such a random cluster model by $\mathbb{Q}^q_{\mathcal{Q}(\tau,\lambda)}$.

The event that there exist infinite clusters is clearly increasing, and so if percolation with probabilities λ, τ exhibits only finite clusters then for $q > 1$ the random cluster model with strengths τ, λ exhibits only finite clusters. If percolation with bond probabilities $\frac{\tau}{\tau+(1-\tau)q}, \frac{\lambda}{\lambda+(1-\lambda)q}$ exhibits at least one infinite cluster then the free random cluster model with strengths τ, λ exhibits at least one infinite cluster also.

The event that there exists exactly one infinite cluster is neither increasing nor decreasing and so the above argument fails for the single cluster phase. However a slightly more sophisticated argument, Theorem 2.20 in the next chapter, allows us to bound the single cluster phase on the QuadTree in a similar way.

However, the bounds provided above are limited in their accuracy; even if we could plot the phase diagram exactly for percolation on the QuadTree the margins of error given by the comparison inequalities becomes very large for high q.

So, if we wish to go beyond comparison with independent percolation, to what extent can we can we attack the problem using direct arguments concerning the random cluster model?

The techniques of Kendall and Wilson [46] focus on two parts of the phase diagram (see figure 2), the left axis part where λ is small and the bottom axis there τ is close to zero.

Consider the case of the random cluster model where $q > 1$. We may either fix $\tau \in (0,1)$ and consider a sequence $\lambda_n \to 0$ or fix $\lambda \in (0,1)$ and consider a sequence $\tau_n \to 0$. In both cases this gives us a sequence of random cluster models, $\mathbb{Q}^q_{\mathcal{Q}(\tau,\lambda_n)}$ and $\mathbb{Q}^q_{\mathcal{Q}(\tau_n,\lambda)}$ respectively.

For any increasing random variable X the sequences of expectations $\mathbb{Q}^q_{\mathcal{Q}(\tau,\lambda_n)}(X)$ and $\mathbb{Q}^q_{\mathcal{Q}(\tau_n,\lambda)}(X)$ are decreasing. Again this will be enough to show that the sequences of random cluster models $\mathbb{Q}^q_{\mathcal{Q}(\tau,\lambda_n)}$ and $\mathbb{Q}^q_{\mathcal{Q}(\tau_n,\lambda)}$ converge weakly to some pair of limiting measures.

In fact we may go further. A theorem of Skorohod [59] (Theorem 2.11 in the next chapter) states that if $\mu_n \to \mu$ is a weakly convergent sequence of measures on $\Omega_{\mathcal{Q}}$ (in fact any Polish space) then there is some measure $\overline{\mu}$ on $\Omega_{\mathcal{Q}}^{\mathbb{N}}$ such that $\omega_n \sim \mu_n$ and the sequence $\omega_n \to \omega$ as $n \to \infty$ where $\omega \sim \mu$. We may extend this theorem so that if μ_n is decreasing sequence of measures (in the sense that $\mu_n(X)$ is decreasing for every increasing X) then we may choose ω_n as above to be a decreasing sequence.

Now what can we say about the two limiting measures we have constructed? Consider first the case where $\tau \to 0$. Removing all tree bonds from the QuadTree splits the graph into countably many finite lattices. Let $\omega_n \sim \mathbb{Q}^q_{\mathcal{Q}(\tau_n,\lambda)}$ be a decreasing sequence as above, then every tree edge is closed in ω. Therefore for an edge $e \in \mathbf{L}\left(\mathcal{P}^m_{[0,1]^d}\right)$ we have $\omega \in \mathcal{L}_e$ if and only if $\omega_n \in \mathcal{L}_e$ for every n. It is an easy exercise to deduce from (1.33) that the limit $\text{wlim}_{n\to\infty} \mathbb{Q}^q_{\mathcal{Q}(\tau_n,\lambda)}$ is the measure given by independent random cluster models on the finite graphs $\left(\mathcal{P}^m_{[0,1]^d}, \mathbf{L}\left(\mathcal{P}^m_{[0,1]^d}\right)\right)$.

This does not of course mean that the random cluster model contains no infinite clusters for sufficiently small τ.

Consider the case of independent percolation on the QuadTree, in this case the configurations on the finite layers $\left(\mathcal{P}^m_{[0,1]^d}, \mathbf{L}\left(\mathcal{P}^m_{[0,1]^d}\right)\right)$ are independent whatever the value of τ and it is known (for $d = 2$ at least) that if λ is subcritical then we may choose some τ so small that the finite clusters on the layers are not joined by the few open tree bonds to form an infinite cluster. Conversely if

λ is super critical, even though the layers are finite, the clusters are large enough that no matter how small we make τ we may find some infinite cluster, furthermore it is unique.

For the random cluster model we might suspect that a similar phenomenon occurs.

CONJECTURE 1.7

Let $p_c(q, d)$ be the critical probability for the Fortuin-Kasteleyn random cluster model on $\mathbb{Z}^d$ with cluster factor q.

- If $\lambda < p_c(q, d)$ then there exists some $\tau > 0$ such that the random cluster model $\mathbb{Q}^q_{\mathcal{Q}(\tau,\lambda)}$ exhibits no infinite clusters.

- If $\lambda > p_c(q, d)$ then for every $\tau > 0$ the random cluster model $\mathbb{Q}^q_{\mathcal{Q}(\tau,\lambda)}$ exhibits a unique infinite cluster.

There are technical difficulties in the proof of this statement which we have not yet overcome. We leave this as an open problem for future research.

We move on to the motivating problem for this work. Fix $\tau \in (0, 1)$ and $q > 0$. Let $\lambda_n \to 0$ and let $\widetilde{\mathbb{Q}}_{\tau,q}$ be the weak limit as $n \to \infty$ of the sequence of free random cluster measures $\mathbb{Q}^q_{\mathcal{Q}(\tau,\lambda_n)}$.

If we remove all the lattice edges from the QuadTree we are left with a spanning tree. Now consider some $e = \langle u, v \rangle \in \mathbf{T}(\mathcal{Q})$. The event $\mathcal{L}_e$ cannot occur on a tree and the only measure that satisfies (1.33) is independent bond percolation with probability $\frac{\tau}{\tau + (1-\tau)q}$.

Now consider a decreasing sequence $\omega_n \sim \mathbb{Q}^q_{\mathcal{Q}(\tau,\lambda_n)}$ with $\omega_n \to \omega$ as $n \to \infty$.

If $\frac{\tau}{\tau + (1-\tau)q} > 2^{-d}$ then there is a positive probability that, even if we close e both u and v are contained in infinite clusters in every ω_n. From Theorem 1.3 above if $\frac{\tau}{\tau + (1-\tau)q} > \tau_u$ as well then ω_n contains a single infinite cluster for every n. In particular we may have $\omega_n \in \mathcal{L}_e$ for every $n \in \mathbb{N}$ event though we cannot have $\omega \in \mathcal{L}_e$.

What can we say then about the probabilities $\widetilde{\mathbb{Q}}_{\tau,q}(\omega(e) \mid \mathscr{T}_e)$? Recall that $\mathcal{L}_e$ is the event that both ends of e are in the "same cluster." In this case however we have seen that two vertices may be in different in ω but still be in the same cluster for every ω_n.

Suppose two ω-clusters (say C_1 and C_2) are "close" in the sense that there are infinitely many edges $e_i = \langle u_i, v_i \rangle$ with $u_i \in C_1$ and $v_i \in C_2$. Then for every $n \in \mathbb{N}$ we may argue from (1.33) that with probability one at least one edge e_i is open in ω_n, we should treat C_1 and C_2 as the same cluster from the point of view of our limiting model.

Now consider the sequence of edges e_i. By compactness the sets $(u_i \cup v_i) \subset [0,1]^d$ has a convergent subsequence and as the diameter of the sets u_i and v_i converges to zero the limit must be a point in $\mathbb{R}_d$. We may find some pair of half infinite open paths $\Pi_1 = (\tilde{u}_1, \tilde{u}_2, \dots) \subset C_1$ and $\Pi_2 = (\tilde{v}_1, \tilde{v}_2, \dots)$ such that the sequences $\tilde{u}_i$ and $\tilde{v}_i$ (as subsets of $[0,1]^d$) converge to some point $x \in [0,1]^d$

Alternatively if we can find paths Π_1 and Π_2 as above then (assuming without loss of generality that $\tilde{u}_1, \tilde{v}_i \in \mathcal{P}^i_{[0,1]^d}$) there is a chain of at most d lattice edges between $\tilde{u}_i$ and $\tilde{v}_i$ and so with probability one the clusters C_1 and C_2 are connected in every ω_n.

So given a configuration $\omega \in \Omega_{\mathbf{T}(\mathcal{Q})}$ let $\overset{\omega}{\longleftrightarrow}$ be the smallest equivalence relation on $\mathbf{V}(\mathcal{Q})$ such that $u \overset{\omega}{\longleftrightarrow} v$ if the unique path on the tree between u and v is open or if there is some pair of open half infinite paths $(u = \tilde{u}_1, \tilde{u}_2, \dots)$ and $(v = \tilde{v}_1, \tilde{v}_2, \dots)$ with $\liminf \tilde{u}_i = \liminf \tilde{v}_i$.

Set $\mathcal{L}_e^{\longleftrightarrow}$ for $u = \langle u, v \rangle \in \mathbf{T}(\mathcal{Q})$ to be the event that $u \overset{\omega_e}{\longleftrightarrow} v$, where ω_e is the configuration obtained from ω by closing e.

Set $\pi(\tau) = \frac{\tau}{\tau + (1-\tau)q}$ and consider the question as to whether

$$\widetilde{\mathbb{Q}}_{\tau,q}(\omega(e) \mid \mathscr{T}_e)(\omega) = \begin{cases} \tau & : \text{ if } \omega \in \mathcal{L}_e^{\longleftrightarrow}, \\ \pi(\tau) & : \text{ if } \omega \notin \mathcal{L}_e^{\longleftrightarrow}. \end{cases} \tag{1.34}$$

We are not able to answer this question for every τ. In fact for the Quad tree we will ultimately have to fall back on the bounds given by the comparison with independent bond percolation.

Our interest however is in the specification given by (1.34). Are there any probability measures on $\Omega_{T(Q)}$ that satisfy (1.34) and if so how do they behave?

The aim of this thesis then is to investigate the random cluster model on a tree under general boundary conditions that include those described informally above. In particular we wish to define Gibbs specifications for the random cluster model on a tree in a general context, allowing connections through the boundary.

If we can make the specification (1.34) rigorous then what may we say about the set of measures that satisfy it? In particular does there exist such a measure for every pair τ, q and is this measure unique?

1.3 THE RANDOM CLUSTER MODEL ON THE TREE

The random cluster model was adapted to the tree by Häggström [38]. The standard construction of the random cluster model on a tree will always produce an independent percolation. Häggström's approach was to count only finite clusters, that is to consider all infinite clusters as part of the same cluster. We will return to this idea in more detail in the next chapter.

To adapt our informal definition we may define an event $\mathcal{L}_e^\star$ as the event that the two end vertices of e are both contained either in the same cluster, or in separate infinite clusters in the configuration ω_e.

Substituting $\mathcal{L}_e^\star$ for $\mathcal{L}_e$ in the specification will be equivalent to the formal definition of the *wired* random cluster model we will meet in the next chapter.

1.3.1   Wired Boundary Conditions

Häggström's focus was on the case of regular trees with homogeneous bond weights. Let $\mathcal{T}$ be a regular k-tree, and fix some $\tau \in (0,1)$ and $q > 0$ Now if $\pi(\tau) \leq \frac{1}{k}$ then Bernoulli percolation with bond probability $\pi(\tau)$ forms no infinite clusters on the tree and thus satisfies our informal definition of a wired random cluster model. If $\pi(\tau) > \frac{1}{k}$ then $\mathcal{L}_e^{\star}$ occurs with positive probability and Bernoulli percolation cannot be a wired random cluster model.

In addition to constructing the wired random cluster model formally, and providing a rigorous proof of the claim above Häggström was able to construct processes that satisfied the definition of the wired random cluster model on the tree.

THEOREM 1.8: HÄGGSTRÖM

If $\mathcal{T}$ is a regular tree such that each vertex had degree $k+1$ and each edge has weight τ, then for each $x \in (0,1)$ such that

$$(q-1)x^{k+1} + \left(1 - \tfrac{\tau}{1-\tau} - q\right)x^k + \left(\tfrac{\tau}{1-\tau} + 1\right)x - 1 = 0 \quad (1.35)$$

there exists a measure $\mu_{\tau,k,q}^x$ that satisfies DLR conditions for the wired random cluster model.

_____

If $q \leq 2$ the equation (1.35) has no root in $(0,1)$ for $\pi(\tau) \leq \frac{1}{k}$ and exactly one if $\pi(\tau) > \frac{1}{k}$. If $q > 2$ there is some $\tau_c < \pi^{-1}(\tau) > \frac{1}{k}$ such that (1.35) has N solutions in $(0,1)$ where

$$N = \begin{cases} 0 & : \text{if } \tau < \tau_c, \\ 1 & : \text{if } \tau = \tau_c, \\ 2 & : \text{if } \tau_c < \tau \leq \pi^{-1}\left(\tfrac{1}{k}\right), \\ 1 & : \text{if } \pi(\tau) > \tfrac{1}{k}. \end{cases} \quad (1.36)$$

In addition Häggström was able to show that if $\tau < \tau_c$ then Bernoulli bond percolation as above is the only measure satisfying DLR conditions he conjectured that if $\pi(\tau) > \frac{1}{k}$ then there can exist only one measure that satisfied DLR conditions.

We will recover equation (1.35) in Chapter 3 and prove Häggström's conjecture.

Jonasson [44] extended the wired random cluster model to a general graph and proved uniqueness of the random cluster model for sufficiently high τ. In addition it was shown that on any *non-amenable* graph of bounded degree there is some $q > 0$ such that the wired random cluster model is not unique on some interval of values $\tau \in (\tau_c, \tau_u)$.

1.3.2 Boundary Conditions From Equivalence Relations

We are interested in more general boundary conditions than those considered by Häggström [38] and Jonasson [44]. A more general approach was taken by Grimmett and Janson [36] by considering equivalence relations on the set of *rays* of the tree. A ray on a tree is a half infinite self avoiding path.

Grimmett and Janson considered equivalence relations on the set of rays (under the assumption that two rays are always equivalent if they differ on only a finite number of edges). They then attempted to define the random cluster model on a tree where two clusters are connected if they contain equivalent open rays.

Let $\mathcal{R}$ be the set of rays of a tree $\mathcal{T}$. The authors classify equivalence relations $\sim$ according to the properties of the set of equivalent pairs $\left\{ (\Pi_1, \Pi_2) : \Pi_1 \sim \Pi_2 \right\} \subset \mathcal{R}^2$.

We have described above an equivalence relation on the set of rays induced by the map $\mathcal{R} \to [0, 1]^d$. We shall see that as $[0, 1]^d$ is Hausdorff we may immediately conclude that the induced equivalence relation is closed.

Unfortunately the construction of the random cluster model in [36] contains an error. In particular the Gibbs specification defined for the random cluster model is inconsistent. The authors' definition may be recovered for the special class of "open" boundary conditions where the specification is consistent if restricted to some exhaustive set of subgraphs.

There is a lot of work to be done to correct this error. In Chapter 4 we will construct the random cluster model in very general circumstances, This will be enough to redefine the random cluster model with Grimmett-Janson boundary conditions in a consistent way.

1.4 THESIS OVERVIEW

In Chapter 2 we introduce some preliminary material which we shall rely on in later chapters. Although most of this material is not new. We introduce "generalized series and parallel laws" which do not appear to be covered in the literature.

The series and parallel laws of Fortuin [25], well known from electrical network theory concern the replacement of a pair of edges, either in series or in parallel by a single edge. Our generalization allows an *edge like subgraph* to be replaced with a single, appropriately weighted edge. More importantly for our purposes we introduce a decomposition of the random cluster model into a random cluster model on the edge like subgraph and the random cluster model on the new graph formed by replacing the edge like subgraph with a new edge.

In addition we prove Theorem 2.20 concerning the monotonicity of the single cluster phase on the QuadTree mentioned above.

In Chapter 3 We introduce the wired random cluster model of Häggström [38]. We adapt the work of Zachary [66] to the random cluster model by introducing a class of processes that we call "Markov chains." As the random cluster model is not a Markov random fields our Markov chains will not have the Markov property in the conventional sense.

We introduce a weaker conditional independence requirement that describes our notion of a Markov chain and define a set of entrance laws that correspond to the Markov chains in the same way as Zachary's entrance laws correspond to Markov chains

on vertex indexed Markov random fields. Following [66] we are able to show that every extremal random cluster model is a Markov chain and so describe the set of random cluster models in terms of entrance laws. This allows us to prove Häggström's conjecture concerning the uniqueness of the random cluster model.

In Chapter 4 we generalize the notion of a random cluster model by introducing a new object, a *random connection*. Informally a random connection ⇠⇢ is a "different kind of arrow" that may replace the usual connection operator ⟷ in the construction of the random cluster model on an infinite graph. We introduce axioms that a random connection must satisfy to allow a random cluster model to be constructed.

We then consider the behaviour of the random cluster model defined by a general random connection ⇠⇢ and show a connection with the behaviour of Bernoulli bond percolation on the tree.

In Chapter 5 we consider the special case of random cluster models defined by Grimmett-Janson boundary conditions. We pay particular attention to the "open" equivalence relations defined in [36]. Grimmett and Janson were able to show that for sufficiently high bond strengths the random cluster model with open boundary conditions is unique. We go further by demonstrating that any open equivalence relation defines a set of random cluster models that corresponds 1–1 with the set of wired random cluster models on the same underlying tree. In addition we define rigorously the random connection ⇠⇢ and by considering the behaviour of the random connection under Bernoulli bond percolation we are able to give a complete description of the phase behaviour for the random cluster model on a homogeneous tree under open boundary conditions.

In Chapter 6 we conclude by describing the behaviour of the random cluster model on the QuadTree when the strength of the lattice edges is small and discuss topics raised by this work which would constitute interesting and significant avenues for future research.

2 | FUNDAMENTAL CONCEPTS

In this chapter we prepare the ground for our work by introducing some preliminary material.

In addition to an overview of existing material we introduce new "generalized series and parallel laws" which we shall rely on in later chapters and we prove a claim concerning monotonicity of the single cluster phase for the random cluster model on the QuadTree which we made in Chapter 1.

CONTENTS

2.1 PRELIMINARY MATERIAL

2.1.1 Graphs and networks

A *graph* $G = (V, E)$ is a set of *vertices* V, together with a set of *edges* E. An edge is an unordered pair in the form $\langle u, v \rangle$ where $u, v \in V$. A *path* Π on G is a sequence of vertices $(\dots, \Pi_n, \dots)$ with $\langle \Pi_n, \Pi_{n+1} \rangle \in E$ for every n. As with a sequence a path may be *finite*, *infinite* or *doubly-infinite*. We may describe a finite path $(u = \Pi_0, \dots, \Pi_n = v)$ as a path *from u to v*. We say a graph is *connected* if there exists a path from u to v for every $u, v \in V$.

A vertex u of a graph has *degree k* if there exist exactly k edges in the form $\langle u, v \rangle$. We say a graph has degree k if every vertex has degree k. We say a graph has *maximum degree k* if the degree of each vertex is at most k.

We say a path Π is *self-avoiding* if for every $m \neq n$ the two *edges* $\langle \Pi_m, \Pi_{m+1} \rangle$ and $\langle \Pi_n, \Pi_{n+1} \rangle$ are distinct, a *cycle* in G is a finite self avoiding path $(\Pi_0, \dots, \Pi_n)$ with $\Pi_0 = \Pi_n$. Notice that the vertices of a self avoiding path need not be distinct. We say a path Π is *cycle-free* if $\Pi_m \neq \Pi_n$ whenever $m \neq n$.

There are several common generalizations of the basic graph object. We mention a few here. A *weighted graph* is a graph (V, E) and a function $\gamma : E \to \mathbb{R}$ that assigns a weight to each edge. A *multigraph* is a set of vertices together with a *multiset* of edges. That is we allow more than one edge between a given pair of vertices. A *directed graph* is a set of vertices together with a set of *directed* edges. A directed edge is an *ordered* pair in the form $|u, v \rangle$ where $u, v \in V$. A *directed path* is a sequence of vertices Π_n such that $|\Pi_n, \Pi_{n+1} \rangle \in E$ for every n. The notions of a self-avoiding path and a cycle generalize naturally to the directed case.

We will pay particular attention in this work to *trees*. A *tree* is a connected graph that contains no cycles, that is for any two vertices u, v there is a unique self avoiding path from u to v. A

FIGURE 3
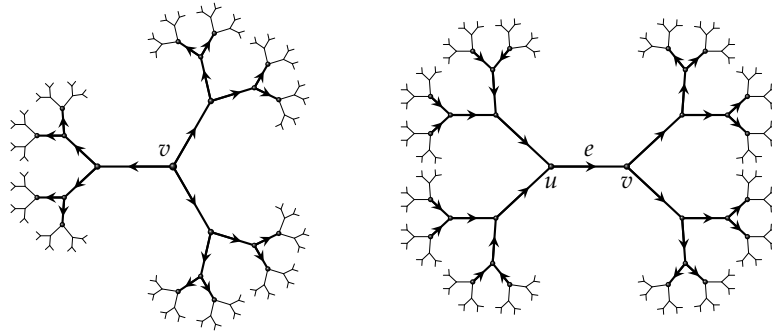
Two ways of rooting a tree. On the left the regular binary tree is rooted at a vertex v by directing all edges away from v. On the right the tree is rooted at a directed edge $e = |u, v\rangle$ by directing all edges either towards u or away from v in such a way that every directed doubly infinite path runs through e.

regular-tree of *degree k* is a tree such that every vertex has degree k. Notice that a regular tree of degree $k > 1$ is necessarily infinite.

A *directed tree* is a directed graph that becomes a tree if we replace every directed edge $|u, v\rangle$ with the undirected edge $\langle u, v\rangle$. We place no restrictions on how edges are directed, however there are two principal configurations that will be important. We say a directed tree is *rooted* at a vertex v if there is a directed path from v to u for every vertex $u \neq v$. We say a directed tree is rooted at a *directed edge $e = |u, v\rangle$* if for every vertex w there is either a directed path from w to v, or from u to w. It is easy to see (refer to figure 3) that rooting a tree at an edge or a vertex specifies the direction of each edge uniquely.

The natural setting for a random cluster model is a weighted graph, as we will be dealing with the series and parallel laws it will also be helpful to allow multiple edges. Although there has been work on percolation in the setting of directed graphs (see for example Durrett [20]) the random cluster model makes no sense on a directed graph as there is no concept of a cluster. However, directed graphs, and in particular directed trees, will be useful to us as a method of navigation around the QuadTree.

Next we define a *network* as a suitable object on which to define a random cluster model. Although we include both multiple and directed edges we stress that most structures we deal with will not have multiple edges (as they may be removed using the series and parallel laws) and that edges are directed only for the convenience of specifying particular events. The underlying percolation and random cluster models will ignore edge directions.

DEFINITION 2.1

A *network* is a *partially directed, weighted multigraph*. That is a set of vertices **V** and a *multiset* **E** containing both directed and undirected edges, together with an associated weight function $\gamma : \mathbf{E} \rightarrow (0,1)$. The weight function is interpreted as a *multifunction* in such a way that multiple instances of an edge $\langle u, v \rangle$ may be assigned different weights.

Formally a network $\mathscr{N}$ is some set of vertices $\mathbf{V}(\mathscr{N})$ together with some arbitrary indexing set $\mathscr{J}_{\mathscr{N}}$ and an associated weight function $\gamma : \mathscr{J}_{\mathscr{N}} \rightarrow (0,1)$. Associate with each $j \in \mathscr{J}_{\mathscr{N}}$ an undirected edge $\langle j \rangle = \langle u, v \rangle$ for $u, v \in \mathbf{V}(\mathscr{N})$.

Recall we wish to downplay the importance of the directions of edges in a network. For this reason the "default" set of edges $\{\langle j \rangle : j \in \mathscr{J}\}$ is undirected. A *labelling* ℓ of a network is a map from $\mathscr{J}_{\mathscr{N}}$ to the set of possible directed and undirected edges in such a way that if $\langle j \rangle = \langle u, v \rangle$ then $\ell(j) \in \{\langle u, v \rangle, |u, v\rangle, |v, u\rangle\}$.

For a labelling ℓ we name $\mathbf{E}_{\ell}$ to be the image of $\mathscr{J}_{\mathscr{N}}$ under ℓ. If necessary we allow $\mathbf{E}_{\ell}$ to be a multiset. We will not normally specify an indexing set and a labelling. Instead, as is conventional with graphs and networks, we will specify the contents of the edge set $\mathbf{E}(\mathscr{N})$, either as a proper set or a multiset. This implicitly defines an indexing set $\mathscr{J}_{\mathscr{N}}$ and a nameless labelling: $\mathscr{J}_{\mathscr{N}} \rightarrow \mathbf{E}(\mathscr{N})$.

As we will not usually be dealing with multigraphs we allow $\mathbf{E}(\mathscr{N}) = \mathscr{J}_{\mathscr{N}}$ to be a proper set of edges with $\langle e \rangle = e$ for every

$e \in \mathscr{J}_{\mathscr{N}}$. We will always assume that a network has finitely or countably many edges and vertices.

REMARK 2.2

We will use letters live u, v and w for *vertices* of a network and letters e and f for edges. As shorthand we will often write $u, v, w \in \mathcal{N}$ or $e, f \in \mathcal{N}$ for $u, v, w \in \mathbf{V}(\mathcal{N})$ and $e, f \in \mathbf{E}(\mathcal{N})$ respectively when it is clear from the context whether we are referring to an edge or a vertex.

2.1.2 Probability and measure

For clarity and definiteness we define briefly some elementary terms from probability theory. This is not intended to be an introduction to the subject and we refer the reader to Williams [63] or Feller [22, 23] for further details.

A *measurable space* $(\Omega, \mathscr{F})$ is a *sample space* Ω, together with a σ-algebra of *events* $\mathscr{F}$. We say a measure μ on $(\Omega, \mathscr{F})$ is a *probability measure* if $\mu(\Omega) = 1$. We will sometimes refer to a probability measure as a *distribution*. We say a probability measure or distribution μ is *supported* on an event A if $\mu(A) = 1$

In this thesis we will deal only with sample spaces equipped with a well defined topology. Thus we may interpret a topological space Ω as the measurable space $(\Omega, \mathscr{B}(\Omega))$ where $\mathscr{B}(\Omega)$ is the *Borel σ-algebra* on Ω.

We say a *random element* is a measurable function $\varepsilon : \Omega \to \Omega'$ from Ω to a second measurable space Ω' If μ is a probability measure on Ω we name the *distribution* of ε under μ to be the *push-forwards* measure $\varepsilon(\mu)$ defined by

$$(\varepsilon(\mu))(A) = \mu\left(\varepsilon^{-1}(A)\right).$$

As a special case of a random element we say a *random variable* X on Ω is a *Borel-measurable* function $\Omega \to \mathbb{R}$.

If μ is a measure on a space Ω and X a random variable we denote by $\mu(X)$ the *expectation of X under μ*. Defined by the Lebesgue integral

$$\mu(X) = \int_\Omega X(\omega) \, d\mu(\omega). \tag{2.1}$$

For some arguments it will be convenient to treat the topic of conditional expectation in a slightly unusual way. For a measure μ on a space Ω, and a σ-algebra $\mathscr{F} \subset \mathscr{B}(\Omega)$ it is well known that for every integrable random variable X there exists a unique (up to μ) $\mathscr{F}$-measurable random variable $\mu(X \mid \mathscr{F})$ such that

$$\int_A \mu(X \mid \mathscr{F})(\omega) \, d\mu(\omega) = \mu(X \cdot \mathbb{1}_A) \tag{2.2}$$

for every $A \in \mathscr{F}$. We refer the reader to Williams [63] for a list of the properties of conditional expectation.

For fixed $\mu, \mathscr{F}$, the operator $X \mapsto \mu(X \mid \mathscr{F})$ is linear and preserves expectation. Thus we may define a *Markov kernel*

$$(\mu \mid \mathscr{F})(\omega, A) = \mu(\mathbb{1}_A \mid \mathscr{F})(\omega). \tag{2.3}$$

We denote the *Markov operator* associated with the above kernel by $(\mu \mid \mathscr{F})$.

We have chosen to consider conditional expectation in this way as a notational convenience rather than for a specific mathematical purpose. In particular, given a σ-algebra $\mathscr{F} \subset \mathscr{B}(\Omega)$ we may define a random, conditioned measure

$$((\mu \mid \mathscr{F})(\omega))(A) = (\mu \mid \mathscr{F})(\omega, A) \tag{2.4}$$

and of course for an event $A \in \mathscr{B}(\Omega)$ we may define a conditional measure

$$(\mu \mid A)(E) = \frac{\mu(A \cap E)}{\mu(E)}. \tag{2.5}$$

2.1.3   Coupling

The technique of coupling is central to many arguments in the field of statistical mechanics, and particularly in interacting particle systems. Suppose we wish to compare two probability distributions (say μ_1 and μ_2). One technique is to specify a *coupling* of μ_1 and μ_2, that is a construction of μ_1 and μ_2 on the same probability space.

Specifically we say a *coupling* of μ_1 and μ_2 is a probability measure ν on a second measurable space Ξ, together with two random elements $\mathcal{E}_1, \mathcal{E}_2 : \Xi \to \Omega$ such that the push forwards distributions satisfy $\mathcal{E}_1(\nu) = \mu_1$ and $\mathcal{E}_2(\nu) = \mu_2$. We may of course extend the notion of a coupling to several measures, not necessarily all referring the same state space.

The simplest example of a coupling is two independent random elements, using the symbols above we set $\Xi = \Omega^2$ and $\nu = \mu_1 \times \mu_2$. Then (Ξ, ν) together with the maps $\mathcal{E}_1 : (\omega_1, \omega_2) \mapsto \omega_1$ and $\mathcal{E}_2 : (\omega_1, \omega_2) \mapsto \omega_2$ is one possible coupling of μ_1 and μ_2.

We will not always wish to describe the mechanism of a coupling explicitly and so we introduce some special notation. For a coupling of say $(\Omega_1, \mu_1), \dots, (\Omega_n, \mu_n)$ we will specify properties of a measure on the product space $\Omega_1 \times \cdots \times \Omega_n$ and the $\mathcal{E}_i$ from above are assumed to be the projections onto the subspaces Ω_i. In this case we will usually denote the product space by Ω and specify a typical element ω as a vector of symbols. For example to couple two measures μ_1 and μ_2 on Ω_1 and Ω_2 with a measure μ_3 on $\mathbb{R}$ we might specify $\omega = (\omega_1, \omega_2, x)$. Then we may specify events in terms of the random elements $\omega_i \in \Omega_i$ and $x \in \mathbb{R}$.

For percolation and the random cluster model the natural probability space $\Omega_{\mathcal{N}} = \{0,1\}^E$ comes equipped with a partial order. Suppose Ω_1 and Ω_2 are partially ordered spaces. We say a random element $\mathcal{E} : \Omega_1 \to \Omega_2$ is *increasing* (respectively decreasing) if for every pair $\omega \le \omega' \in \Omega_1$ we have $\mathcal{E}(\omega) \le \mathcal{E}(\omega')$ (respectively $\mathcal{E}(\omega) \le \mathcal{E}(\omega')$). For $\Omega_2 = \mathbb{R}$ this extends naturally to a partial order on the set of measures.

DEFINITION 2.3

For two probability measures μ_1, μ_2 on some partially ordered space Ω we say μ_1 *stochastically dominates* μ_2 and write $\mu_1 \succ \mu_2$ if for every increasing random variable $X : \Omega \to \mathbb{R}$ we have

$$\mu_1(X) \geq \mu_2(X). \tag{2.6}$$

There is a deep connection between stochastic domination and coupling. Suppose μ_1 and μ_2 are probability measures on Ω and we may find a coupling $\omega \sim \nu$ with $\omega = (\omega_1, \omega_2) \in \Omega^2$ such that $\omega_1 \sim \mu_1$ and $\omega_2 \sim \mu_2$ and $\omega_1 \geq \omega_2$ ν-almost surely. Then it is clear that $\mu_1(X) - \mu_2(X) = \nu(X(\omega_1) - X(\omega_2)) > 0$ for any increasing X.

The converse of this observation also holds and is due to Strassen [61]. The original paper is not concerned with coupling and "Strassen's Theorem" appears only as an special case of a more general result. We refer the reader to Lindvall [49] or the monograph [50] for a proof and a full discussion.

THEOREM 2.4: STRASSEN'S THEOREM

Let μ_1 and μ_2 be probability measures on some partially ordered space Ω. Then if μ_1 and μ_2 are such that $\mu_1(X) \geq \mu_2(X)$ for every *increasing* random variable $X : \Omega \to \mathbb{R}$, that is if $\mu_1 \succ \mu_2$; there exists a measure ν on Ω^2 with first and second marginals μ_1 and μ_2 respectively, such that

$$\omega_1 \geq \omega_2 \text{ for } \nu\text{-almost every } (\omega_1, \omega_2) \in \Omega^2. \tag{2.7}$$

Lastly we introduce some special notation to describe events defined by statements. We may have recourse to define an event using an informal statement. For example if Ω is a space of functions we may define an event A by stating that $\omega \in A$ if ω is continuous, or for a random variable $X : \Omega \to \mathbb{N}$ we may wish to define B as the event that X is an even number.

We use square brackets to define such informal events, that is we may express A and B above as

$$A = [\omega \text{ is continuous}] \qquad\qquad B = [X \text{ is even}]$$

As this notation is by nature informal and its use is largely self explanatory we will not attempt a rigorous definition.

2.2 THE RANDOM CLUSTER MODEL ON A FI-NITE NETWORK

2.2.1 Measurable networks

Recall from Definition 2.1 that a *network* is a partially-directed weighted multigraph. Recall also from the discussion below Definition 2.1 that associated with a network $\mathcal{N}$ there is an arbitrary indexing set $\mathscr{J}_{\mathcal{N}}$.

As $\mathscr{J}_{\mathcal{N}}$ is arbitrary we may assume that it is a set of events in some named measurable space. We will refer to an edge of such a network as a *bond*.

DEFINITION 2.5

A *measurable network* on a measurable space $(\Omega, \mathscr{F})$ is a network $\mathcal{N}$ such that

$$\mathscr{J}_{\mathcal{N}} = \{J_e \in \mathscr{F} : e \in \mathbf{E}(\mathcal{N})\} \tag{2.8}$$

with $\langle J_e \rangle = e$.

We refer to an element of $\omega \in \Omega$ as a *configuration* of $\mathcal{N}$ and name J_e the *bond* at e.

We say a bond J_e is *open* in a configuration ω if $\omega \in J_e$ and *closed* otherwise. We set $\mathcal{H}(\omega) = \{e : \omega \in J_e\}$ to be the set of open bonds.

There is a natural probability space $\Omega_{\mathcal{N}} = \{0,1\}^{\mathbf{E}(\mathcal{N})}$ associated with a network $\mathcal{N}$ which we interpret as the space of functions $\{\omega : \mathbf{E}(\mathcal{N}) \to \{0,1\}\}$. We say a *random network* is a network $\mathcal{N}$ indexed arbitrarily, together with a measure μ on $\Omega_{\mathcal{N}}$.

An *implementation* of a random network $(\mathcal{N}, \mu)$ is a *measurable network* $\mathcal{G}$, isomorphic to $\mathcal{N}$, defined on a Borel space Ω; together with a probability measure ν on Ω such that the joint distribution of the random variables $\{\mathbb{1}_{J_e} : e \in \mathbf{E}(\mathcal{G})\}$ agrees with the distribution of the random variables $\{\omega(e) : e \in \mathbf{E}(\mathcal{N})\}$ under μ.

The *natural* implementation of a random network $(\mathcal{N}, \mu)$ is the measurable network defined on $\Omega_{\mathcal{N}}$ such that

$$J_e = \{\omega \in \Omega_{\mathcal{N}} : \omega(e) = 1\} \tag{2.9}$$

Unless otherwise specified we will always assume that we are working with the natural implementation of a random network.

As $\Omega_{\mathcal{N}}$ is a space of functions it is automatically endowed with a partial order $\leq$ and join and meet operators $\vee$ and $\wedge$. Recall that a random variable X is *increasing* if $X(\omega_1) \leq X(\omega_2)$ whenever $\omega_1 \leq \omega_2$ and *decreasing* if $-X$ is increasing. We say an event $A \in \mathscr{B}(\Omega_{\mathcal{N}})$ is increasing if the indicator $\mathbb{1}_A$ is increasing and decreasing if $A^{\complement}$ is increasing.

If $\mathcal{N}$ is an infinite network the space $\mathcal{N}$ under the product topology is not discrete. We will examine this space in more detail in Section 2.3. We say a random variable $X : \Omega_{\mathcal{N}} \to \mathbb{R}$ is *left continuous* (respectively *right continuous*) if for every $\omega \in \Omega_{\mathcal{N}}$ and $\varepsilon > 0$ there exists some open set $\mathcal{O} \subset \Omega_{\mathcal{N}}$ with $\omega \in \mathcal{O}$ such that $|X(\omega) - X(\omega')| < \varepsilon$ whenever $\omega' \in \mathcal{O}$ and $\omega' \leq \omega$. (Respectively $\omega' \geq \omega$.)

For $\omega \in \Omega_{\mathcal{N}}$ let ω^e and ω_e be the configurations obtained from ω by opening and closing e respectively; that is $\mathcal{H}(\omega^e) = \mathcal{H}(\omega) \cup \{e\}$ and $\mathcal{H}(\omega_e) = \mathcal{H}(\omega) \setminus \{e\}$. For an event $A \in \mathscr{B}(\Omega_{\mathcal{N}})$ and $e \in E$ we define new events A^e and A_e by setting $A^e = [\omega^e \in A]$ and $A_e = [\omega_e \in A]$.

For a set of edges $E \subset \mathbf{E}(\mathcal{N})$ we define two σ-algebras, $\mathscr{F}_E$ and $\mathscr{T}_E$ generated respectively by the set of bonds in E, and those in $\mathbf{E}(\mathcal{N}) \setminus E$ respectively. That is

$$\mathscr{F}_E = \sigma \left\{ J_e : e \in E \right\}, \tag{2.10}$$

$$\mathscr{T}_E = \sigma \left\{ J_e : e \in (\mathbf{E}(\mathcal{N}) \setminus E) \right\}. \tag{2.11}$$

If $\mathcal{G}$ is a subnetwork of $\mathcal{N}$, that is $\mathcal{G}$ is a network with $\mathbf{V}(\mathcal{G}) \subset \mathbf{V}(\mathcal{N})$ and $\mathbf{E}(\mathcal{G}) \subset \mathbf{E}(\mathcal{N})$, or if $e \in \mathbf{E}(\mathcal{N})$ is a single edge, define analogously

$$\mathscr{F}_{\mathcal{G}} = \mathscr{F}_{\mathbf{E}(\mathcal{G})}, \qquad\qquad \mathscr{F}_e = \mathscr{F}_{\{e\}}, \tag{2.12}$$

$$\mathscr{T}_{\mathcal{G}} = \mathscr{T}_{\mathbf{E}(\mathcal{G})}, \qquad\qquad \mathscr{T}_e = \mathscr{T}_{\{e\}}. \tag{2.13}$$

We will be interested in the properties of the random network $(\mathbf{V}(\mathcal{N}), \mathcal{H})$ and in particular the properties of its connected components.

For a network $\mathcal{N} = (V, E)$ with a labelling ℓ and $\omega \in \Omega_{\mathcal{N}}$ we write

$\quad u \xleftarrow{\omega} v \quad$ if there exists a path Π from u to v and a sequence of bonds J_i with $\langle J_i \rangle = \langle \Pi_i, \Pi_{i+1} \rangle$ and $\omega \in J_i$.

$\quad u \xmapsto{\omega, \ell} v \quad$ if there exists an ℓ-*directed* path Π from u to v, and a sequence of bonds J_i with $\ell(J_i) \in \left\{ \langle \Pi_i, \Pi_{i+1} \rangle, |\Pi_i, \Pi_{i+1} \rangle \right\}$ and $\omega \in J_i$.

Thus we may define events

$$[u \longleftrightarrow v] = \{ \omega \in \Omega_{\mathcal{N}} : u \xleftarrow{\omega} v \}, \tag{2.14}$$

$$[u \xmapsto{\ell} v] = \{ \omega \in \Omega_{\mathcal{N}} : u \xmapsto{\omega, \ell} v \}. \tag{2.15}$$

If $\mathbf{E}(\mathcal{N})$ is a set of directed edges we define $u \xmapsto{\omega} v$ and $[u \longmapsto v]$ using the default labelling of $\mathcal{N}$.

If $\mathbf{V}(\mathcal{N})$ is infinite we write $[u \longleftrightarrow \infty]$, $[u \xmapsto{\ell} \infty]$ and $[u \longmapsto \infty]$ for the events that $u \longleftrightarrow v$ (respectively $u \xmapsto{\ell} v$, $u \longmapsto v$) for infinitely many $v \in \mathbf{V}(\mathcal{N})$.

Notice that the event $[u \longleftrightarrow v]$ ignores the directions of edges, and so $\xleftrightarrow{\omega}$ is an equivalence relation for every $\omega \in \Omega_\mathcal{N}$. Thus for any vertex $v \in \mathcal{N}$ we may define the *cluster* at v to be the equivalence class of vertices

$$\mathcal{C}_v(\omega) = \left\{ u \in \mathbf{V}(\mathcal{N}) : u \xleftrightarrow{\omega} v \right\}. \tag{2.16}$$

If $\mathcal{N}$ is a finite network we may set

$$\kappa_\mathcal{N}(\omega) = \left| \{ \mathcal{C}_v : v \in \mathcal{N} \} \right|. \tag{2.17}$$

We will usually omit the subscript on κ when the network is clear from the context.

2.2.2 The random cluster model.

We introduced independent bond percolation informally in Chapter 1. Here we will reintroduce the two models more rigorously. We will use the same notation as in Chapter 1.

Recall that a network $\mathcal{N}$ comes equipped with a weight function $\gamma : \mathbf{E}(\mathcal{N}) \to (0, 1)$. Name the measure $\mathbb{P}_\mathcal{N}$ to be *Bernoulli bond percolation* on $\mathcal{N}$. That is the is the product measure on $\Omega_\mathcal{N}$ such that every bond J_e is open independently with probability $\gamma(e)$.

If $\mathcal{N}$ is a finite measurable network the number of clusters κ is bounded above by the number of vertices. Thus for a finite network $\mathcal{N}$ we may define the random cluster model in closed form as follows.

$$\mathbb{Q}_{\mathcal{N},q}(\omega) = Z_{\mathcal{N},q}^{-1} \left(\prod_{e \in \mathbf{E}(\mathcal{N})} \left(\frac{\gamma(e)}{1 - \gamma(e)} \right)^{\omega(e)} \right) q^{\kappa(\omega)} \tag{2.18}$$

$$\text{where} \quad Z_{\mathcal{N},q} = \sum_{\omega \in \Omega_\mathcal{N}} \left(\prod_{e \in \mathbf{E}(\mathcal{N})} \left(\frac{\gamma(e)}{1 - \gamma(e)} \right)^{\omega(e)} \right) q^{\kappa(\omega)} \tag{2.19}$$

is a normalizing constant.

FIGURE 4
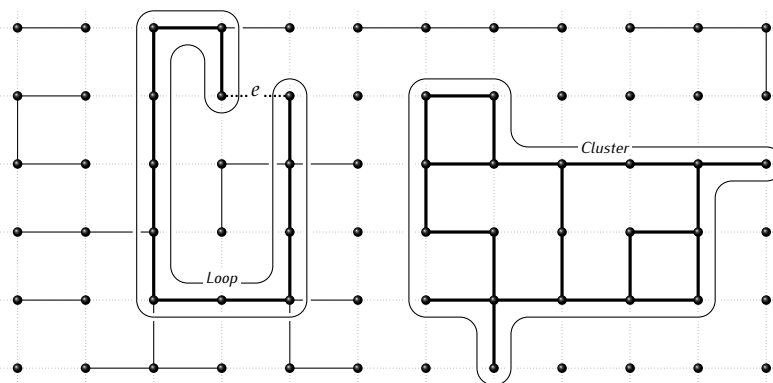
A random configuration of $\mathbb{Z}^2$. Marked are a *loop* at a bond e and a *cluster*. Loops are of crucial importance to the random cluster model as the existence of a loop dictates whether the inclusion or exclusion of a bond affects the number of clusters.

2.2.3 Basic properties

In this section we outline several key properties of the random cluster model. These properties stem from a characterization of the random cluster model in terms of the conditional probabilities $\mathbb{Q}_{\mathcal{N},q}(J_e \mid \mathscr{T}_e)$. In order to express this characterization we introduce a *loop event* $\mathcal{L}_e$.

The term "loop" is already used in graph theory to denote an edge from a vertex to itself. As such edges do not effect any event in percolation theory, and we do not allow them in our definition of a network we shall give the term a special meaning in the context of random networks. Let $\mathcal{N}$ be a measurable network and $\omega \in \Omega_{\mathcal{N}}$ a configuration of $\mathcal{N}$. Then for an edge $e = \langle u, v \rangle$ a *loop at* e is an undirected open path from u to v that does not include e.

Figure 4 shows a loop at an edge e. Notice the existence of a loop at e is not affected by the state of the bond J_e.

DEFINITION 2.6

We say a configuration contains a *loop* at an edge $e = \langle u, v \rangle$ if it contains an *open* self avoiding path $\Pi = (u = \Pi_0, \ldots, \Pi_n = v)$ from u to v such that for every $i \in [\![1, n]\!]$ we have $\langle \Pi_{i-1}, \Pi_i \rangle \neq e$.

For an edge $e \in \mathbf{E}(\mathcal{N})$; if there exists a loop at e then opening and closing e has no effect on the event $[u \longleftrightarrow v]$ for any $u, v \in \mathbf{V}(\mathcal{N})$. Conversely if there is no loop at e then opening e connects the two distinct clusters containing the end vertices of e.

So we may define the event that there is a loop at e as

$$\mathcal{L}_e = [\tilde{u} \longleftrightarrow \tilde{v}]_e \qquad\qquad \text{for } e = \langle \tilde{u}, \tilde{v} \rangle. \qquad (2.20)$$

and we have

$$\kappa(\omega_e) = \kappa(\omega^e) + 1 - \mathbb{1}_{\mathcal{L}_e}(\omega) \qquad\qquad (2.21)$$

From the definition of the random cluster model (2.18) we see that

$$\frac{\mathbb{Q}_{\mathcal{N},q}(\omega^e)}{\mathbb{Q}_{\mathcal{N},q}(\omega_e)} = \begin{cases} \frac{\gamma(e)}{1-\gamma(e)} & : \text{if } \omega \in \mathcal{L}_e \\ \frac{\gamma(e)}{q(1-\gamma(e))} & : \text{if } \omega \notin \mathcal{L}_e \end{cases} \qquad (2.22)$$

Therefore as $\mathcal{L}_e$ is $\mathscr{T}_e$-measurable we have

$$\mathbb{Q}_{\mathcal{N},q}(J_e \mid \mathscr{T}_e) = \begin{cases} \gamma(e) & : \text{if } \omega \in \mathcal{L}_e \\ \frac{\gamma(e)}{\gamma(e)+q(1-\gamma(e))} & : \text{if } \omega \notin \mathcal{L}_e. \end{cases} \qquad (2.23)$$

This fact is crucial to the study of the random cluster model and will be central to many of our arguments. The quotient in the right hand side of equation (2.23) will play an important role in this work. Thus we name functions

$$\pi_q(p) = \frac{p}{p + (1-p)q} \qquad \pi_q^{-1}(p) = \frac{pq}{pq + (1-p)}. \qquad (2.24)$$

In addition we allow the functions above to apply to entire networks, that is if $\mathcal{N}$ is a network with weight function γ, we set $\pi(\mathcal{N})$ to be the network with the same edges and vertices as $\mathcal{N}$ but with weight function γ', where $\gamma'(e) = \pi(\gamma(e))$

THEOREM 2.7

Let G be a finite multigraph and $\mathcal{N}, \mathcal{N}'$ be two networks with underlying graphs G and weight functions γ, γ'. Fix $q \geq 1$ and $q' \leq q$.

- $\mathbb{Q}_{\mathcal{N},q'}$ is the unique probability measure that satisfies (2.23).

- $\mathbb{Q}_{\mathcal{N},q}$ satisfies the strong FKG condition (1.25).

- If either; for every $e \in \mathbf{E}(G)$, $\gamma(e) \geq \gamma'(e)$; or for every $e \in \mathbf{E}(G)$, $\pi(\gamma(e)) \geq \pi(\gamma'(e))$ then $\mathbb{Q}_{\mathcal{N},q} \succ \mathbb{Q}_{\mathcal{N}',q'}$.

The proof of this theorem is elementary and may be found in almost any book on the random cluster model. We refer the reader to Grimmett [35] or Georgii et al. [29] for the full proof.

Rather than a full proof, we sketch the main ideas using a Markov chain based coupling argument. We will rely on this and similar couplings in later chapters.

For every edge $e \in \mathbf{E}(\mathcal{N})$ define functions $\theta^e_{\mathcal{N},q} : \Omega_{\mathcal{N}} \times [0,1] \to \{0,1\}$ and $\Theta^e_{\mathcal{N},q} : \Omega_{\mathcal{N}} \times [0,1] \to \Omega_{\mathcal{N}}$ by

$$\theta^e_{\mathcal{N},q}(\omega, \lambda) = \begin{cases} 1 & : \text{if } \lambda < \pi_q(\gamma(e)), \\ 0 & : \text{if } \lambda > \gamma(e), \\ \mathbb{1}_{\mathcal{L}_e}(\omega) & : \text{if } \lambda \in [\pi_q(\gamma(e)), \gamma(e)]. \end{cases} \tag{2.25}$$

$$\left(\Theta^e_{\mathcal{N},q}(\omega, \lambda) \right)(f) = \begin{cases} \theta^e_{\mathcal{N},q}(\omega, \lambda_i) & : \text{if } f = e, \\ \omega(f) & : \text{if } f \neq e. \end{cases} \tag{2.26}$$

We have chosen $\theta^e_{\mathcal{N},q}$ in such a way that if λ is a uniform $[0,1]$ random variable then $\theta^e_{\mathcal{N},q}(\omega, \lambda)$ is a Bernoulli random variable with expectation $\mathbb{Q}_{\mathcal{N},q}(J_e \mid \mathscr{T}_e)(\omega)$. Hence we have $\int_0^1 \Theta^e_{\mathcal{N},q}(\omega, \lambda)d\lambda = \left(\mathbb{Q}_{\mathcal{N},q} \mid \mathscr{T}_e \right)(\omega)$ and in particular $\mathbb{Q}_{\mathcal{N},q}$ is invariant under the Markov kernel $(f, \omega) \mapsto \int_0^1 f\left(\Theta^e_{\mathcal{N},q}(\omega, \lambda) \right)d\lambda$.

Notice also that as $\mathcal{L}_e$ is an increasing event then, if $q \geq 1$, the random variable $\omega \mapsto \theta^e_{\mathcal{N},q}(\omega, \lambda)$ is increasing for every $\lambda \in (0,1)$.

Next set ω_0 to be the constant configuration $\omega_0(e) \equiv 0$ choose independent sequences $\Lambda = (\lambda_1, \lambda_2, \ldots)$ and $E = (e_1, e_2, \ldots)$ uniformly from $[0,1]^{\mathbb{N}}$ and $\mathbf{E}(\mathcal{N})^{\mathbb{N}}$ respectively.

Now define a sequence $\omega_n = \omega_n(\mathcal{N}, q)$ inductively by setting $\omega_{i+1} = \Theta^{e_i}_{\mathcal{N},q}(\omega_i, \lambda_i)$.

It is easy to see that the sequence $\omega_n(\mathcal{N}, q)$ is an irreducible Markov chain with invariant measure $\mathbb{Q}_{\mathcal{N},q}$. Therefore $\mathbb{Q}_{\mathcal{N},q}$ is specified uniquely by the functions $\theta^e_{\mathcal{N},q}$ and, as any measure satisfying (2.23) is an invariant measure for the chain ω_n; $\mathbb{Q}_{\mathcal{N},q}$ is the unique measure with conditional probabilities (2.23).

Now for any two pairs $(\mathcal{N}, q)$ and $(\mathcal{N}', q')$ we may define a sequence of couplings $\omega_n = (\omega_n(\mathcal{N}, q), \omega_n(\mathcal{N}', q'))$. Furthermore if $(\mathcal{N}, q)$ and $(\mathcal{N}', q')$ satisfy the conditions of the theorem then for every $\omega \geq \omega' \in \Omega_{\mathcal{N}}$, $\lambda \in [0,1]$ and $e \in \mathbf{E}(\mathcal{N})$ we have $\theta^e_{\mathcal{N},q}(\omega, \lambda) \geq \theta^e_{\mathcal{N}',q'}(\omega', \lambda)$ and we must have $\omega_n(\mathcal{N}, q) \geq \omega_n(\mathcal{N}', q')$ for every n. By considering the stationary distribution of the chain ω_n we must have $\mathbb{Q}_{\mathcal{N},q} \succ \mathbb{Q}_{\mathcal{N}',q'}$.

We have proved two parts of Theorem 2.7. We leave it to the reader to check the strong FKG condition (equation (1.25)). This is simply a matter of checking that for any pair of configurations ω_1, ω_2 we have $\kappa(\omega_1 \vee \omega_2) + \kappa(\omega_1 \wedge \omega_2) \geq \kappa(\omega_1) + \kappa(\omega_2)$, which follows inductively by checking the inequality for configurations that differ on a single edge. We refer the reader to Grimmett [35] for details.

Instead we will check the conclusions of Theorem 1.1 directly, that is we show that for any network $\mathcal{N}$ and $q \geq 1$ we have

$$(\mathbb{Q}_{\mathcal{N},q} \,|\, A) \succ \mathbb{Q}_{\mathcal{N},q} \tag{2.27}$$

for all increasing events A.

The proof is identical to the argument above, but this time set

$$(\Theta \,|\, A)^e_{\mathcal{N},q}(\omega, \lambda) = \begin{cases} \Theta^e_{\mathcal{N},q}(\omega, \lambda) & : \text{if } \omega_e \in A \\ 1 & : \text{if } \omega_e \notin A \end{cases} \tag{2.28}$$

Now define a second chain by $\omega_i^A = (\theta \mid A)_{\mathcal{N},q}^{e_i}(\omega_{i-1}^A, \lambda_i)$. This chain is identical to the previous Markov Chain except that a bond will never close if this would cause the chain to leave A. Furthermore the event A forms the only recurrence class of the chain ω_n^A.

Therefore $(\mathbb{Q}_{\mathcal{N},q} \mid A)$ is the unique invariant measure and, arguing as above, $(\Theta \mid A)_{\mathcal{N},q}^e(\omega, \lambda) \geq \Theta_{\mathcal{N},q}^e(\omega, \lambda)$ for every $\omega \in \Omega_{\mathcal{N}}$ and we have $(\mathbb{Q}_{\mathcal{N},q} \mid A) \succ \mathbb{Q}_{\mathcal{N},q}$.

2.3 PROBABILITY ON INFINITE PRODUCT SPACES

The random cluster model is a probability measure on the space of *edge configurations* of a graph G, in which edges can be either "open" or "closed." We have already seen that the natural probability space for a random cluster model, or for Bernoulli percolation on a finite network is the product space $\Omega_{\mathcal{N}} = \{0,1\}^{\mathbf{E}(\mathcal{N})}$. If $\mathcal{N}$ is an infinite network than the natural probability space is isomorphic to $\{0,1\}^{\mathbb{N}}$ and it is natural to consider the *Tychonoff product topology*.

For an infinite network $\mathcal{N}$ let $\mathbb{G}_{\mathcal{N}}$ be the set of *finite* subnetworks of $\mathcal{N}$.

We say a set $\mathbb{G} \subset \mathbb{G}_{\mathcal{N}}$ is *exhaustive* if for every finite network $\mathcal{G} \in \mathbb{G}_{\mathcal{N}}$ there exists some $\mathcal{G}' \in \mathbb{G}$ with $\mathcal{G} \subset \mathcal{G}'$.

Alternatively we say a *sequence* of finite subnetworks $\mathcal{G}_n \in \mathbb{G}_{\mathcal{N}}$ is exhaustive if for every $\mathcal{G} \in \mathbb{G}_{\mathcal{N}}$ there exists some $N \in \mathbb{N}$ such that $\mathcal{G} \subset \mathcal{G}_n$ for every $n > N$.

We say a set $\{X_{\mathcal{G}} : \mathcal{G} \in \mathbb{G}\}$ indexed by some exhaustive set $\mathbb{G}$ has property P as $\mathcal{G} \uparrow \mathcal{N}$ if for every exhaustive sequence $\mathcal{G}_n \in \mathbb{G}$ the sequence $X_{\mathcal{G}_n}$ has property P as $n \to \infty$. We may write "as $\mathcal{G}$ *exhausts* $\mathcal{N}$" for as $\mathcal{G} \uparrow \mathcal{N}$.

Recall the definitions of $\mathscr{F}_{\mathcal{G}}$ and $\mathscr{T}_{\mathcal{G}}$ for subnetworks $\mathcal{G}$. For an infinite network we may define the *tail σ-algebra*

$$\mathscr{T} = \bigcap_{\mathcal{G} \in \mathbf{G}_{\mathcal{N}}} \mathscr{T}_{\mathcal{G}}. \tag{2.29}$$

Chose some $\xi \in \Omega_{\mathcal{N}}$ arbitrarily. For $\mathcal{G} \subset \mathcal{N}$ we say a configuration *ω agrees with ξ on $\mathcal{G}$* if $\omega(e) = \xi(e)$ for every $e \in \mathbf{E}(\mathcal{G})$ and *ω agrees with ξ off $\mathcal{G}$* if $\omega(e) = \xi(e)$ for every $e \in \mathbf{E}(\mathcal{N}) \setminus \mathbf{E}(\mathcal{G})$. We name two *cylinder sets* associated with $\mathcal{G}$ and ξ.

$$\mathcal{O}_{\mathcal{G}}^{\xi} = \bigcap_{e \in \mathcal{G}} \left\{ \omega \in \Omega_{\mathcal{N}} : \omega(e) = \xi(e) \right\}, \tag{2.30}$$

$$\Omega_{\mathcal{G}}^{\xi} = \bigcap_{e \notin \mathcal{G}} \left\{ \omega \in \Omega_{\mathcal{N}} : \omega(e) = \xi(e) \right\}. \tag{2.31}$$

Notice that there is a natural bijection between the three sets $\Omega_{\mathcal{G}}$, $\Omega_{\mathcal{G}}^{\xi}$ and $\{ \mathcal{O}_{\mathcal{G}}^{\xi} : \xi \in \Omega_{\mathcal{N}} \}$.

DEFINITION 2.8

The *Tychonoff product topology* on $\Omega_{\mathcal{N}}$ is the topology generated by the countable set of open cylinders $\{ \mathcal{O}_{\mathcal{G}}^{\xi} : \xi \in \Omega_{\mathcal{N}}, \mathcal{G} \in \mathbf{G}_{\mathcal{N}} \}$.

It is a fundamental fact of topology, see for example Willard [62] that this topology is compact, countably generated and Hausdorff.

Notice also that for finite subgraphs $\mathcal{G}_1$ and $\mathcal{G}_2$ and configurations ξ_1 and ξ_2 the event $\mathcal{O}_{\mathcal{G}_1}^{\xi_1} \cap \mathcal{O}_{\mathcal{G}_2}^{\xi_2}$ is $\mathscr{F}_{\mathcal{G}_1 \cup \mathcal{G}_2}$-measurable, hence may be expressed as a union of cylinder sets in the form $\mathcal{O}_{\mathcal{G}_1 \cup \mathcal{G}_2}^{\zeta}$. In particular the set of open cylinders forms a basis for the product topology.

As for every finite $\mathcal{G} \in \mathbf{G}_{\mathcal{N}}$ the space $\Omega_{\mathcal{G}}$ is finite each set $\mathcal{O}_{\mathcal{G}}^{\xi}$ is both open and closed. In particular all $\mathscr{F}_{\mathcal{G}}$-measurable functions are continuous and take only finitely many values. These continuous simple functions play a key role in our understanding of the space of probability measures on $\Omega_{\mathcal{N}}$.

DEFINITION 2.9

We say a random variable $X : \Omega_{\mathcal{N}} \to \mathbb{R}$ is a *continuous simple function* if X is $\mathscr{F}_{\mathcal{G}}$-measurable for some finite $\mathcal{G} \in \mathbf{G}_{\mathcal{N}}$.

2.3.1 Topology of probability measures

We may extend any topology on a sample space to a topology on the space of probability measures.

DEFINITION 2.10

- We say a sequence of measures μ_n converges *weakly* to a measure μ and write $\mu_n \xrightarrow{w} \mu$ if for every continuous simple function X we have $\mu_n(X) \to \mu(X)$.

- For a network $\mathcal{N}$ let $\mathscr{P}_{\mathcal{N}} = \mathscr{P}(\Omega_{\mathcal{N}})$ be the set of probability measures on $\Omega_{\mathcal{N}}$ with the associated topology of weak convergence.

Notice we have defined weak convergence in terms of continuous simple functions rather than the usual definition involving bounded continuous functions. As any continuous function may be expressed as a uniform limit of continuous simple functions is is easy to see that there is no loss of generality in weakening the definition.

It is known (see for example Billingsley [7]) that as the space $\Omega_{\mathcal{N}}$ is compact any sequence of measures contains a weakly convergent subsequence. The space $\Omega_{\mathcal{N}}$ is metrizable as well as compact. By Prohorov's Theorem [57] the topology of weak convergence is metrizable also. Both the Levy-Prohorov and the Vasserstein metrics may be shown to generate the weak topology.

Say a sequence of measures μ_n is *Cauchy* if $\mu_n(X)$ is a Cauchy sequence for every continuous simple function X. If μ_n is Cauchy we write $\text{wlim}_{n \to \infty} \mu_n$ to represent the weak limit of the sequence μ_n.

Recall the event $\mathcal{L}_e$. The definition extends naturally to an infinite network, however the indicator function $\mathbb{1}_{\mathcal{L}_e}$ is not continuous. In Chapter 1 we gave an informal definition of the random

cluster model on an infinite graph as a measure that satisfies the conditional specification (2.23).

As this specification is not continuous the weak limit of a sequence random cluster measures might not satisfy (2.23) and so might not be a random cluster model.

The key tool we will use to overcome this difficulty is that of stochastic domination. We will give a brief overview of some aspects of the weak topology from the perspective of stochastic domination and coupling. This will enable us to construct random cluster measures as weak limits in certain situations, in particular as weak limits of *monotonic* sequences of measures.

A famous theorem of Skorohod [59] states that for any weakly convergent sequence of measures we may find a coupling that converges almost surely. This theorem is valid not just for Ω_N, but for any Polish space, see for example Billingsley [8]. We prove only the special case where we may write down the coupling explicitly.

THEOREM 2.11: SKOROHOD

Let μ_n be a sequence of measures on Ω_N. Then $\mu_n \overset{w}{\Rightarrow} \mu$ if and only if there exists a coupling with typical element $\bar{\omega} = (\omega, \omega_1, \omega_2, ...)$ with marginals $\omega \sim \mu$, $\omega_i \sim \mu_i$ such that $\omega_n \to \omega$ as $n \to \infty$ for almost every $\bar{\omega}$.

_____

PROOF _____

First if $\bar{\omega}$ is distributed as in the statement of the theorem then for any continuous random variable $X : \Omega_N \to \mathbb{R}$ the sequence $X(\omega_n) \to X(\omega)$ almost surely as $n \to \infty$. Ω_N is compact and so X is bounded, therefore $\mu_n(X) \to \mu(X)$ as $n \to \infty$ by the dominated convergence theorem.

Next suppose $\mu_n \overset{w}{\Rightarrow} \mu$. Let $\Lambda = (\lambda_1, \lambda_2, \ldots) \in (0,1)^{\mathbb{N}}$ be an i.i.d. sequence of uniform $(0,1)$ random variables.

Order $\mathbf{E}(\mathcal{N})$ arbitrarily and set $\omega_i(\Lambda)$ by fixing the values of bonds e_i inductively, setting

$$\mathbb{1}_{J_{e_i}}(\omega_n(\Lambda)) = \begin{cases} 1 & : \text{if } \lambda_i \geq 1 - \mu_n\big(J_{e_i} \,\big|\, J_{e_1}, \ldots, J_{e_{i-1}}\big), \\ 0 & : \text{otherwise} \end{cases} \quad (2.32)$$

and choosing $\omega(\Lambda) \sim \mu$ in the same way.

For a given $n \in \mathbb{N}$ we say that ω_n fails at stage i if i is smallest number such that $\omega_n(\Lambda)$ and $\omega(\Lambda)$ disagree on e_i. We may write the probability that ω_n fails at stage i as a bounded rational function of a finite number of probabilities $\mu_n\big(\mathcal{O}_g^{\xi}\big)$, $\mu\big(\mathcal{O}_g^{\xi}\big)$ and so if $\mu_n \xrightarrow{w} \mu$ the probability that ω_n fails at stage i converges to the probability that ω fails at stage i, which, trivially, is zero.

Now fix ω and suppose ω_n has not failed before stage i. Set $\tilde{\lambda}_i = 1 - \mu\big(J_{e_i} \,\big|\, J_{e_1}, \ldots, J_{e_{i-1}}\big)(\omega)$ then ω_n fails at stage i only if $\big|\lambda_i - \tilde{\lambda}_i\big|$ is sufficiently small. Moreover the size of this interval must shrink to zero for μ-almost very ω, for if not then the probability that ω_n fails at or before stage i would not disappear as $n \to \infty$.

Therefore for almost every Λ, ω_n fails at stage i for only finitely many n, hence $\omega_n \to \omega$ as $n \to \infty$ almost surely as required. $\qquad \Box$

Skorohod's Theorem demonstrates that there is a close link between coupling and weak convergence. Strassen's Theorem (Theorem 2.4 above) gives us a link between coupling and stochastic domination. Our next aim is to exploit these links to express the notion of weak convergence meaningfully in terms of stochastic domination.

From the definition of stochastic domination(Definition 2.3) if we wish to show $\mu \succ \nu$ then we must check that $\mu(X) \geq \nu(X)$ for every increasing function X. However, weak convergence gives us control only over continuous functions, and not all increasing functions are continuous. (Consider for example the indicator $\mathbb{1}_{\mathcal{L}_e}$.)

Before we proceed we prove a technical lemma to show that for the purpose of stochastic domination it is enough to consider only the continuous simple functions.

LEMMA 2.12

If μ_1 and μ_2 be measures such that $\mu_1(X) \leq \mu_2(X)$ for any increasing *continuous simple function* X then $\mu_1 \prec \mu_2$.

PROOF

Suppose μ_1 and μ_2 are as in the statement of the theorem. We use the sequential compactness the set of probability measures to construct an ordered coupling of μ_1 and μ_2 as the limit of a sequence of couplings.

For any $\mathcal{G} \in \mathbb{G}_N$ we may define restricted measures $\mu_1^{\mathcal{G}}, \mu_2^{\mathcal{G}}$ on $\Omega_{\mathcal{G}}$ in the obvious way. As every $\mathscr{F}_{\mathcal{G}}$-measurable random variable is a continuous simple function we must have $\mu_1^{\mathcal{G}} \prec \mu_2^{\mathcal{G}}$. By Strassen's theorem, (Theorem 2.4), we may chose a coupling $(\tilde{\omega}_1^{\mathcal{G}}, \tilde{\omega}_2^{\mathcal{G}}) \sim \tilde{\nu}_{\mathcal{G}}$ on $\Omega_{\mathcal{G}}^2$ such that $\tilde{\omega}_1^{\mathcal{G}} \sim \mu_1^{\mathcal{G}}$, $\tilde{\omega}_2^{\mathcal{G}} \sim \mu_2^{\mathcal{G}}$ and $\tilde{\omega}_1^{\mathcal{G}} \leq \tilde{\omega}_2^{\mathcal{G}}$ $\tilde{\nu}_{\mathcal{G}}$-almost surely.

Extend this to a coupling of μ_1 and μ_2 by setting

$$\nu_{\mathcal{G}}(A \times B) = \int_{\Omega_{\mathcal{G}}^2} \mu_1(A \mid \mathscr{F}_{\mathcal{G}})(\tilde{\omega}_1^{\mathcal{G}}) \cdot \mu_2(B \mid \mathscr{F}_{\mathcal{G}})(\tilde{\omega}_2^{\mathcal{G}}) \; d\tilde{\nu}_{\mathcal{G}}(\tilde{\omega}_1^{\mathcal{G}}, \tilde{\omega}_2^{\mathcal{G}}).$$

Informally, we sample from $\nu_{\mathcal{G}}$ by choosing $(\tilde{\omega}_1^{\mathcal{G}}, \tilde{\omega}_2^{\mathcal{G}}) \in \Omega_{\mathcal{G}}^2$ according to the ordered coupling $\tilde{\nu}_{\mathcal{G}}$, then choosing $(\omega_1^{\mathcal{G}}, \omega_2^{\mathcal{G}}) \in \Omega_N^2$ according to the product measure $\mu_1 \times \mu_2$ conditioned to agree with $(\tilde{\omega}_1^{\mathcal{G}}, \tilde{\omega}_2^{\mathcal{G}})$ on $\mathcal{G}$.

We have chosen $\nu_{\mathcal{G}}$ in such a way that if $(\omega_1^{\mathcal{G}}, \omega_2^{\mathcal{G}}) \sim \nu_{\mathcal{G}}$ then $\omega_1^{\mathcal{G}} \sim \mu_1$, $\omega_2^{\mathcal{G}} \sim \mu_2$ and $\omega_1(e) \leq \omega_2(e)$ for every $e \in \mathbf{E}(\mathcal{G})$.

The space Ω_N^2 is compact and metrizable, hence by Prohorohov's Theorem the set of measures $\{\nu_{\mathcal{G}} : \mathcal{G} \in \mathbb{G}_N\}$ is tight and we may choose some sequence $\mathcal{G}_n \uparrow N$ such that $\nu_{\mathcal{G}_n} \xrightarrow{w} \nu$ for some $\nu \in \mathscr{P}(\Omega_N^2)$.

For any continuous simple function X and edge $e \in \mathbf{E}(\mathcal{N})$ the functions $X_1 : (\omega_1, \omega_2) \mapsto X(\omega_1)$, $X_2 : (\omega_1, \omega_2) \mapsto X(\omega_2)$ and $\mathbb{1}_{[\omega_1(e) > \omega_2(e)]} : (\omega_1, \omega_2) \mapsto \omega_1(e)\,(1 - \omega_2(e))$ are continuous.

Therefore we have $\nu(X_1) = \mu_1(X)$ and $\nu(X_2) = \mu_2(X)$ so by Carathéodory's Extension Theorem, ν has marginal distributions μ_1 and μ_2. Furthermore. $\nu[\omega_1(e) > \omega_2(e)] = 0$ therefore $\omega_1(e) \leq \omega_2(e)$-$\nu$ almost surely. Hence ν gives us the required coupling and $\mu_1 \prec \mu_2$ by Theorem 2.11. ❑

Suppose that μ_n is an *increasing* (respectively decreasing) sequence of measures, that is if $\mu_{n+1} \succ \mu_n$ for every n (respectively $\mu_{n+1} \prec \mu_n$) we say μ_n *increases to* μ and write $\mu_n \uparrow \mu$ ($\mu_n \downarrow \mu$, decreases to μ) if in addition $\mu_n \overset{w}{\Rightarrow} \mu$.

We may combine Theorem 2.11 and Lemma 2.12 to prove a useful monotone convergence theorem for the space $\mathscr{P}_{\mathcal{N}}$ associated with an infinite network $\mathcal{N}$.

THEOREM 2.13

Let μ_n be some sequence of probability measures on $\Omega_{\mathcal{N}}$.

- If μ_n is an increasing sequence of probability measures then there exists some measure μ such that $\mu_n \overset{w}{\Rightarrow} \mu$. The measure μ may be characterized as the unique smallest probability measure such that $\mu \succ \mu_n$ for every $n \in \mathbb{N}$. In addition $\mu_n(X) \to \mu(X)$ for every left-continuous random variable X.

- The sequence $\mu_n \overset{w}{\Rightarrow} \mu$ if and only if there exist both an increasing sequence $\nu_n \uparrow \mu$ and a decreasing sequence $\bar{\nu}_n \downarrow \mu$ such that $\nu_n \prec \mu_n \prec \bar{\nu}_n$ for every $n \in \mathbb{N}$.

PROOF

Firstly if μ_n is increasing then for each n we may specify a coupling $(\omega_n, \omega_{n+1}) \sim \nu_n$ of μ_n and μ_{n+1} such that $\omega_n \leq \omega_{n+1}$ almost surely. If we define a Markov transition kernel $\theta_n(A, \omega) =$

$\nu_n(\Omega_{\mathscr{N}} \times A \mid \omega_n = \omega)$ then we may construct a discrete Markov chain $\omega = (\omega_1, \omega_2, \ldots)$ with $\omega_n \sim \mu_n$ and $\omega_n \leq \omega_{n+1}$ almost surely. Now set $\omega(e) = \sup_{n \in \mathbb{N}} \omega_n(e)$ and let μ be the distribution of ω. Arguing as in Theorem 2.11 we have $\mu_n(X) \to \mu(X)$ for every left continuous random variable X.

If μ' is such that $\mu' \succ \mu_n$ for every n then for every finite $\mathcal{G} \in \mathbb{G}_{\mathscr{N}}$ and increasing $\mathscr{F}_{\mathcal{G}}$-measurable X we have $\mu'(X) \geq \mu(X)$ and so $\mu' \succ \mu$ by Lemma 2.12.

For the second statement if $\mu_n \xrightarrow{w} \mu$ then we may find a coupling $\omega = (\omega, \omega_1, \omega_2, \ldots)$ as in Theorem 2.11. Setting ν_n to be the distribution of $\inf_{i>n} \omega_n(e)$ and $\overline{\nu}_n$ to be the distribution of $\sup_{i>n} \omega_n(e)$ we have $\nu_n \uparrow \mu$ and $\overline{\nu}_n \downarrow \mu$ as required. ❑

COROLLARY 2.14

If μ_n and ν_n are two sequences of probability measures on $\Omega_{\mathscr{N}}$ with $\mu_n \prec \nu_n$ for every $n \in \mathbb{N}$ then and $\mu_n \xrightarrow{w} \mu$, $\nu_n \xrightarrow{w} \nu$ as $n \to \infty$ we have $\mu \prec \nu$.

PROOF

From Theorem 2.13 we may choose an increasing sequence μ'_n and a decreasing sequence ν'_n of probability measures such that $\mu'_n \prec \mu_n \prec \nu_n \prec \nu'_n$ for every $n \in \mathbb{N}$ and $\mu'_n \uparrow \mu$, $\nu'_n \downarrow \nu$ as $n \to \infty$.

So for any *increasing* continuous simple function X we have

$$\mu(X) = \lim_{n \to \infty} \mu'_n(X) \leq \lim_{n \to \infty} \nu'(X) = \nu(X). \tag{2.33}$$

Therefore $\mu \prec \nu$ by Lemma 2.12 ❑

2.3.2   The random cluster model on an infinite graph

We turn our attention back to the random cluster model.

If $\mathcal{N}$ is a countably infinite network we may still define Bernoulli bond percolation on $\mathcal{N}$ as an infinite product measure. However, if the weight function $\gamma(e)$ is bounded above then number of clusters $\kappa_{\mathcal{N}}(\omega)$ is infinite $\mathbb{P}_{\mathcal{N}}$-almost surely. Therefore we may not define the random cluster model directly using the closed form definition (2.18).

We may however define measures which obey the rules of the random cluster model (Theorem 2.7) on some finite subnetwork $\mathcal{G} \in \mathbb{G}_{\mathcal{N}}$, but are "fixed" outside $\mathcal{G}$.

Fix an infinite network $\mathcal{N}$ and a cluster factor $q > 0$. Now let $\mathcal{G}$ be a finite subnetwork of $\mathcal{N}$. Then only a finite number of clusters intersect $\mathcal{G}$ and we may set

$$\kappa_{\mathcal{G}}(\omega) = \left|\{\mathcal{C}_v(\omega) : v \in \mathcal{G}\}\right|. \tag{2.34}$$

Recall from (2.31) the *cylinder spaces* $\Omega_{\mathcal{G}}^{\xi}$ of configurations that agree with ξ off $\mathcal{G}$. For finite $\mathcal{G} \in \mathbb{G}_{\mathcal{N}}$ the space $\Omega_{\mathcal{G}}^{\xi}$ is finite and we may define a measure $\mathbb{Q}_{\mathcal{G}}^{\xi} = \mathbb{Q}_{\mathcal{N},\mathcal{G}}^{\xi,q}$ as the *free cylinder measure* with *boundary condition* ξ by setting

$$\mathbb{Q}_{\mathcal{G}}^{\xi}(\omega) = Z_{\mathcal{G},\xi}^{-1} \; \mathbb{1}_{\Omega_{\mathcal{G}}^{\xi}}(\omega) \prod_{e \in \mathbf{E}(\mathcal{G})} \left(\frac{\gamma(e)}{1-\gamma(e)}\right)^{\omega(e)} q^{\kappa_{\mathcal{G}}(\omega)} \tag{2.35}$$

$$\text{where} \quad Z_{\mathcal{G},\xi} = \sum_{\omega \in \Omega_{\mathcal{G}}^{\xi}} \left(\prod_{e \in \mathbf{E}(\mathcal{G})} \left(\frac{\gamma(e)}{1-\gamma(e)}\right)^{\omega(e)}\right) q^{\kappa_{\mathcal{G}}(\omega)}. \tag{2.36}$$

We say that a measure $\mathbb{Q}$ is an *infinity free* random cluster model on $\mathcal{N}$ if for every finite subnetwork $\mathcal{G} \subset \mathcal{N}$ we have

$$(\mathbb{Q} \mid \mathscr{T}_{\mathcal{G}})(\xi) = \mathbb{Q}_{\mathcal{G}}^{\xi}. \tag{2.37}$$

The term infinity free is defined in contrast to the term *infinity wired*. The infinity wired random cluster model was first introduced on the tree by Häggström [38] and extended to a general

graph by Jonasson [44]. The wired model is defined in a similar manner to the free model by changing the way we count clusters. In particular all infinite clusters are considered as a single "*giant*" clusters by setting $\kappa_g^\star(\omega)$ to be the number of *finite* clusters that intersect $\mathcal{G}$. We may then define the *wired cylinder measure* $\overline{\mathbb{Q}}_\mathcal{G}^\xi$ by substituting $\kappa_g^\star$ into the definition (2.35).

We say that a measure $\mathbb{Q}$ is an *infinity wired* random cluster model on $\mathcal{N}$ if for every finite subnetwork $\mathcal{G} \subset \mathcal{N}$ we have

$$(\mathbb{Q} \,|\, \mathscr{T}_\mathcal{G})(\xi) = \overline{\mathbb{Q}}_\mathcal{G}^\xi. \tag{2.38}$$

Arguing as above it is easy to see that the measure $\mathbb{Q}_\mathcal{G}^\xi$ satisfies the conditional specification (2.23) for every edge $e \in \mathcal{G}$. Furthermore the conclusions of Theorem 2.7 hold for all free cylinder measures. The specification (2.23) does not in general hold for wired cylinder measures. For an edge $e = \langle u, v \rangle \in \mathbf{E}(\mathcal{G})$, let ω be a configuration in which u and v are members of separate infinite clusters. Then, although there is no loop at e, including or removing e has no effect on the number of finite clusters.

We may recover a version of (2.23) by defining an analogue of the event $\mathcal{L}_e$ to fit with the wired random cluster model. Recall that informally $[u \longleftrightarrow v]$ is the event that u and v are members of the same cluster. The equivalent for the wired specification therefore is that u and v are not members of distinct clusters which are not both infinite. Define a new set of events

$$[u \Longleftrightarrow v] = [u \longleftrightarrow v] \cup ([u \longleftrightarrow \infty] \cap [v \longleftrightarrow \infty]) \tag{2.39}$$

as the wired equivalent of the events $[u \longleftrightarrow v]$.

Intuitively $u \Longleftrightarrow v$ if there is a path between u and v, where we may consider two half infinite paths as a single path "through infinity". Then we may set

$$\mathcal{L}_{\langle u,v \rangle}^\star = [u \Longleftrightarrow v]_{\langle u,v \rangle} \tag{2.40}$$

analogously to (2.20).

We have defined $\mathcal{L}_e^\star$ so that for $e \in \mathbf{E}(\mathcal{G})$ we have $\kappa_\mathcal{G}^\star(\omega_e) - \kappa_\mathcal{G}^\star(\omega^e) = 1 - \mathbb{1}_{\mathcal{L}_e^\star}(\omega)$. Arguing as for Theorem 2.7 we have

THEOREM 2.15

- The conclusions of Theorem 2.7 hold for both $\mathbb{Q}_\mathcal{G}^\xi$ and $\overline{\mathbb{Q}}_\mathcal{G}^\xi$.

- If $\xi' \leq \xi$ and $q \geq 1$ then both $\mathbb{Q}_\mathcal{G}^{\xi'} \prec \mathbb{Q}_\mathcal{G}^\xi$ and $\overline{\mathbb{Q}}_\mathcal{G}^{\xi'} \prec \overline{\mathbb{Q}}_\mathcal{G}^\xi$.

- $\overline{\mathbb{Q}}_\mathcal{G}^\xi \succ \mathbb{Q}_\mathcal{G}^\xi$ whenever $q \geq 1$.

- If the configuration ξ does not contain two distinct infinite clusters then $\mathbb{Q}_\mathcal{G}^\xi = \overline{\mathbb{Q}}_\mathcal{G}^\xi$.

The proof is again elementary, and follows the same steps as the proof of Theorem 2.7. It is enough to check the domination for the implied transition functions

$$
\theta_{\mathcal{N},q}^e(\omega, \lambda) = \begin{cases} 1 & : \text{if } \lambda < \pi_q(\gamma(e)), \\ 0 & : \text{if } \lambda > \gamma(e), \\ \mathbb{1}_{\mathcal{L}_e}(\omega) & : \text{if } \lambda \in [\pi_q(\gamma(e)), \gamma(e)], \end{cases} \tag{2.41}
$$

$$
\overline{\theta}_{\mathcal{N},q}^e(\omega, \lambda) = \begin{cases} 1 & : \text{if } \lambda < \pi_q(\gamma(e)), \\ 0 & : \text{if } \lambda > \gamma(e), \\ \mathbb{1}_{\mathcal{L}_e^\star}(\omega) & : \text{if } \lambda \in [\pi_q(\gamma(e)), \gamma(e)]. \end{cases} \tag{2.42}
$$

We have mentioned in passing the notion of *infinity free* and *infinity wired random cluster models*. We will use these concepts as our definition of the random cluster model. This approach was pioneered by Dobrushin [17, 18], Lanford and Ruelle [47] and adapted for the random cluster model by Grimmett [32].

DEFINITION 2.16

Name the sets of *infinity free* and *infinity wired random cluster models* respectively as

$$
\mathscr{R}_{\mathcal{N},q} = \bigcap_{\mathcal{G} \in \mathbf{G}} \left\{ \mu \,\middle|\, (\mu \,|\, \mathscr{T}_\mathcal{G})(\xi) = \mathbb{Q}_\mathcal{G}^\xi \right\}, \tag{2.43}
$$

$$
\mathscr{R}_{\mathcal{N},q}^\star = \bigcap_{\mathcal{G} \in \mathbf{G}} \left\{ \mu \,\middle|\, (\mu \,|\, \mathscr{T}_\mathcal{G})(\xi) = \overline{\mathbb{Q}}_\mathcal{G}^\xi \right\}. \tag{2.44}
$$

From Theorem 2.15 we may characterize the random cluster models as follows.

$$\mathscr{R}_{\mathcal{N},q} = \left\{ \mu \left| \mu(J_e \mid \mathscr{T}_e)(\omega) = \left\{ \begin{array}{ll} \gamma(e) & : \text{if } \omega \in \mathcal{L}_e \\ \pi(\gamma(e)) & : \text{if } \omega \notin \mathcal{L}_e \end{array} \right. \right. \right\} \quad (2.45)$$

$$\mathscr{R}_{\mathcal{N},q}^{\star} = \left\{ \mu \left| \mu(J_e \mid \mathscr{T}_e)(\omega) = \left\{ \begin{array}{ll} \gamma(e) & : \text{if } \omega \in \mathcal{L}_e^{\star} \\ \pi(\gamma(e)) & : \text{if } \omega \notin \mathcal{L}_e^{\star} \end{array} \right. \right. \right\} \quad (2.46)$$

We are interested primarily in the structure of the sets $\mathscr{R}_{\mathcal{N},q}$ and $\mathscr{R}_{\mathcal{N},q}^{\star}$. Before proceeding we must check that the sets $\mathscr{R}_{\mathcal{N},q}$ and $\mathscr{R}_{\mathcal{N},q}^{\star}$ are non empty.

DEFINITION 2.17

Define the set

$$\Xi_{\mathcal{N},q} = \left\{ \xi \in \Omega_{\mathcal{N}} \left| Q_{\mathcal{G}}^{\xi} \text{ is Cauchy as } \mathcal{G} \uparrow \mathcal{N}. \right. \right\} \quad (2.47)$$

and for $\xi \in \Xi_{\mathcal{N},q}$ set

$$Q_{\mathcal{N},q}^{\xi} = \text{wlim}_{\mathcal{G} \uparrow \mathcal{N}} Q_{\mathcal{G}}^{\xi} \quad (2.48)$$

and let

$$\mathcal{W}_{\mathcal{N},q} = \left\{ Q_{\mathcal{N},q}^{\xi} \left| \xi \in \Xi_{\mathcal{N},q} \right. \right\}. \quad (2.49)$$

be the set of weak limits.

Define equivalents $\Xi_{\mathcal{N},q}^{\star}$, $\overline{Q}_{\mathcal{N},q}^{\xi}$ and $\mathcal{W}_{\mathcal{N},q}^{\star}$ for the wired random cluster model.

Of special importance are the weak limits obtained by constant configurations, $\xi_1(e) \equiv 1$ and $\xi_0(e) \equiv 0$.

THEOREM 2.18

For every $q \geq 1$ we have $\xi_0, \xi_1 \in \left(\Xi_{\mathcal{N},q} \cap \Xi_{\mathcal{N},q}^{\star} \right)$. Furthermore

$$Q_{\mathcal{N},q} = \mathrm{wlim}_{\mathcal{G} \uparrow \mathcal{N}} \, Q_{\mathcal{G}}^{\xi_0} \in \mathscr{R}_{\mathcal{N},q}, \tag{2.50}$$

$$\overline{Q}_{\mathcal{N},q} = \mathrm{wlim}_{\mathcal{G} \uparrow \mathcal{N}} \, \overline{Q}_{\mathcal{G}}^{\xi_1} \in \mathscr{R}_{\mathcal{N},q}^{\star} \tag{2.51}$$

and for any $\mu \in \mathscr{R}_{\mathcal{N},q} \cup \mathscr{R}_{\mathcal{N},q}^{\star}$ we have

$$Q_{\mathcal{N},q} \prec \mu \prec \overline{Q}_{\mathcal{N},q}. \tag{2.52}$$

This was first proved for the free model by Grimmett [32] and for the wired model by Jonasson [44]. We omit the proof which follows directly from the observation that if $q \geq 1$ then for any increasing, exhaustive sequence $\mathcal{G}_n \in \mathbb{G}_{\mathcal{N}}$ the sequence $Q_{\mathcal{G}_n}^{\xi_0}$ is increasing and $\overline{Q}_{\mathcal{G}_n}^{\xi_1}$ is decreasing.

The sets $\mathscr{R}_{\mathcal{N},q}$ and $\mathscr{R}_{\mathcal{N},q}^{\star}$ are obviously convex as they are defined in terms of conditional probabilities. Thus it is natural to consider the extremal sets. We have defined the random cluster models in terms of a Gibbs Specification. There is a well established theory of the geometry of the sets of measures satisfying such specifications. We state the conclusions of the general theory, details of which may be found in Dynkin [21] or Georgii [28].

THEOREM 2.19

For every $\mu \in \mathscr{R}_{\mathcal{N},q}$

- $\mu\big(\Xi_{\mathcal{N},q}\big) = 1$ and $Q_{\mathcal{N},q}^{\xi} = (\mu \mid \mathscr{T})(\xi) \in \mathscr{R}_{\mathcal{N},q}$ μ-almost surely.

- The set $\mathscr{E}_{\mathcal{N},q} = \big\{ \mu \in \mathscr{R}_{\mathcal{N},q} : (\mu \mid \mathscr{T})(\xi) \equiv \mu \big\}$ of *tail trivial measures* is exactly the set of *extremal elements* of the set $\mathscr{R}_{\mathcal{N},q}$.

Respectively for the wired model the set of tail trivial measures $\mathscr{E}_{\mathcal{N},q}^{\star} = \big\{ \mu \in \mathscr{R}_{\mathcal{N},q}^{\star} : (\mu \mid \mathscr{T})(\xi) \equiv \mu \big\}$ forms the extremal elements of $\mathscr{R}_{\mathcal{N},q}^{\star}$ with $\overline{Q}_{\mathcal{N},q}^{\xi} = (\mu \mid \mathscr{T})(\xi) \in \mathscr{R}_{\mathcal{N},q}^{\star}$ μ-almost surely.

2.3.3   Monotonicity of the single cluster phase on the QuadTree

We conclude this section by returning to the QuadTree.

We saw in Chapter 1 that supercritical Bernoulli percolation on the QuadTree exhibits either infinitely many infinite clusters or only a single infinite cluster. Kendall and Wilson [46] provided bounds for the single cluster phase for Bernoulli percolation.

The exhibition of a single infinite cluster is not an increasing event, for if ω contains a unique infinite cluster there may exist some path Π for which no vertex is contained within the infinite cluster. Then we may open every bond in Π to create a second infinite cluster.

Here we show that the single cluster phase of the random cluster model is still monotonic in the sense that if $\mu_1 \prec \mu_2$ are free or wired random cluster models (with $q > 1$ in both cases) then if μ_1 has the single cluster property so does μ_2.

Recall the graph $\mathcal{Q}(\tau, \lambda)$ — the d-dimensional QuadTree with edge weights τ for *tree-like* edges and λ for *lattice-like* edges – defined in Section 1.1.3 for fixed $d \geq 2$.

THEOREM 2.20

For $q, q' \geq 1$ and $\tau \leq \tau', \lambda' \leq \lambda'$ be such that $\pi_q(\tau) \leq \pi_{q'}(\tau')$ and $\pi_q(\lambda) \leq \pi_{q'}(\lambda')$.

- If $Q_{\mathcal{Q}(\tau, \lambda), q}$ exhibits a unique infinite cluster then $Q_{\mathcal{Q}(\tau', \lambda'), q'}$ exhibits a unique infinite cluster also.

- If $Q_{\mathcal{Q}(\tau, \lambda), q}$ exhibits a unique infinite cluster then $\overline{Q}_{\mathcal{Q}(\tau, \lambda), q}$ exhibits a unique infinite cluster also.

- If $\overline{Q}_{\mathcal{Q}(\tau, \lambda), q}$ exhibits a unique infinite cluster then $\overline{Q}_{\mathcal{Q}(\tau', \lambda'), q'}$ exhibits a unique infinite cluster also.

PROOF

We only consider the case for the free random cluster models, the proof only requires stochastic domination and a certain invariance requirement shared by both the free and wired models.

First for some large integer N, which we will fix below and an arbitrary pixel $\mathcal{P} \in \mathbf{V}(\mathcal{Q})$ let $\mathcal{Q}_N$, $\mathcal{Q}_N^{\complement}$ and $\mathcal{Q}_{\mathcal{P}} \subset \mathcal{Q}$ be the subnetworks of $\mathcal{Q}$ with $\mathbf{V}(\mathcal{Q}_N) = \mathcal{P}_{[0,1]^d}^{[\![0,N]\!]}$, $\mathbf{V}(\mathcal{Q}_N^{\complement}) = \mathcal{P}_{[0,1]^d}^{[\![N,\infty]\!]}$ and $\mathbf{V}(\mathcal{Q}_{\mathcal{P}}) = \mathcal{P}_{\mathcal{P}}^{[\![0,\infty]\!]}$; where each subnetwork contains all appropriate edges of $\mathcal{Q}$.

First each subgraph $\mathcal{Q}_{\mathcal{P}}$ for $\mathcal{P} \in \mathcal{P}_{[0,1]^d}^N$ is isomorphic to $\mathcal{Q}$, let μ_N be the measure obtained by choosing the configuration of each subgraph $\mathcal{Q}_{\mathcal{P}}$, for $\mathcal{P} \in \mathcal{P}_{[0,1]^d}^N$ independently according to $\mathbb{Q}_{\mathcal{Q}(\tau,\lambda),q}$ with the remaining edges closed.

Now we may express μ_N as the limit as $M \to \infty$ of the free random cluster model on the graph $\mathcal{Q}_M \cap \bigcup_{\mathcal{P} \in \mathcal{P}_{[0,1]^d}^N} \mathcal{Q}_{\mathcal{P}}$. Therefore it is easy to see that we must have $\mu_N \prec \mathbb{Q}_{\mathcal{Q}_N^{\complement}(\tau,\lambda),q}$.

Now for $\xi \leq \xi' \in \Omega_{\mathcal{Q}}$ consider the measures $\left(\mathbb{Q}_{\mathcal{Q}(\tau,\lambda),q} \middle| \mathscr{F}_{\mathcal{Q}_N} \right)(\xi)$ and $\left(\mathbb{Q}_{\mathcal{Q}(\tau',\lambda'),q'} \middle| \mathscr{F}_{\mathcal{Q}_N} \right)(\xi')$. by the FKG inequality we have

$$\left(\mathbb{Q}_{\mathcal{Q}(\tau',\lambda'),q'} \middle| \mathscr{F}_{\mathcal{Q}_N} \right)(\xi') \succ \left(\mathbb{Q}_{\mathcal{Q}(\tau,\lambda),q} \middle| \mathscr{F}_{\mathcal{Q}_N} \right)(\xi) \qquad (2.53)$$

$$\succ \left(\mathbb{Q}_{\mathcal{Q}(\tau,\lambda),q} \middle| \mathscr{F}_{\mathcal{Q}_N} \right)(\xi_0) \qquad (2.54)$$

$$= \mathbb{Q}_{\mathcal{Q}_N^{\complement}(\tau,\lambda),q} \succ \mu_N. \qquad (2.55)$$

We may use these observations to create an ordered coupling $\omega = (\omega_1, \omega_2, \omega_3) \sim \nu$ with $\omega_1 \sim \mu_N$, $\omega_2 \sim \mathbb{Q}_{\mathcal{Q}(\tau,\lambda),q}$ and $\omega_3 \sim \mathbb{Q}_{\mathcal{Q}(\tau',\lambda'),q'}$ in such a way that $\omega_1 \leq \omega_2 \leq \omega_3$ but for any $\mathscr{F}_{\mathcal{Q}_N}$ measurable X we have $X(\omega_3)$ independent of ω_1.

To achieve this choose a pair ξ, ξ' such that $\xi \sim \mathbb{Q}_{\mathcal{Q}(\tau,\lambda),q}$ and $\xi' \sim \mathbb{Q}_{\mathcal{Q}(\tau',\lambda'),q'}$ and choose ω_1 according to μ_N independently of ξ and ξ'.

From above we may choose ω_2 according to $\left(\mathbb{Q}_{\mathcal{Q}(\tau,\lambda),q} \middle| \mathscr{F}_{\mathcal{Q}_N} \right)(\xi)$ and ω_3 according to $\left(\mathbb{Q}_{\mathcal{Q}(\tau',\lambda'),q'} \middle| \mathscr{F}_{\mathcal{Q}_N} \right)(\xi')$ to get the required coupling.

Now set $\rho = [0,1]^d$ be the root vertex of $\mathcal{Q}$ and set

$$p = \mathbb{Q}_{\mathcal{Q}(\tau,\lambda),q}[\rho \longleftrightarrow \infty]. \tag{2.56}$$

Choose vertices $u, v \in \mathbf{V}(\mathcal{Q})$ and $\varepsilon > 0$ arbitrarily. By the martingale convergence theorem; for ν almost every ω

$$\mathbb{Q}_{\mathcal{Q}(\tau',\lambda'),q'}(u \longleftrightarrow \infty \mid \mathscr{F}_{\mathcal{Q}_N})(\omega_3) \to \mathbb{1}_{[u \longleftrightarrow \infty]}(\omega_3). \tag{2.57}$$

Therefore we may choose N large enough so that with probability at least $1 - \varepsilon$ either

$$\mathbb{Q}_{\mathcal{Q}(\tau',\lambda'),q'}(u \longleftrightarrow \infty \mid \mathscr{F}_{\mathcal{Q}_N})(\omega_3) < p$$

or

$$\mathbb{Q}_{\mathcal{Q}(\tau',\lambda'),q'}(u \longleftrightarrow \infty \mid \mathscr{F}_{\mathcal{Q}_N})(\omega_3) > 1 - \exp\left[2^d \frac{\log(1-\tau)\log(\varepsilon)}{\log(1-p)}\right].$$

Now colour a pixel $\mathcal{P} \in \mathcal{P}^N_{[0,1]^d}$ green if $\mathcal{P} \xleftrightarrow{\omega_1} \infty$ and blue if there is some ω_3-open path from u to $\mathcal{P}$ within the subgraph $\mathcal{Q}_N$. Then each pixel is coloured green independently with probability p and independently of all blue pixels.

Now if a pixel is both green and blue then as $\omega_3 \geq \omega_1$ we must have $u \xleftrightarrow{\omega_3} \infty$. Say a blue pixel $\mathcal{P}$ dies if every tree bond $|\mathcal{P}, \mathcal{P}'\rangle$ is ω_3-closed.

If all blue pixels die then we cannot have $u \xleftrightarrow{\omega_3} \infty$. Furthermore as the random cluster model is dominated by Bernoulli percolation each green pixel dies with probability at least $(1-\tau)^{2^d}$.

Therefore if there are $K > 0$ green pixels we have

$$p > \mathbb{Q}_{\mathcal{Q}(\tau',\lambda'),q'}(u \longleftrightarrow \infty \mid \mathscr{F}_{\mathcal{Q}_N})(\omega_3) > (1-\tau)^K. \tag{2.58}$$

And in particular with probability at least $1 - \varepsilon$ there are either no blue pixels or at least $\frac{\log(\varepsilon)}{\log(1-p)}$ blue pixels. Therefore with

probability $1 - 2\varepsilon$ there are either no blue pixels or at least one pixel that is both blue and green.

So to finish notice that any green pixel is is the root of an infinite cluster of ω_1 and so contained in some infinite cluster of ω_2. As ω_2 contains only a single infinite cluster then all green pixels are connected in ω_2 and so all green pixels are connected in ω_3.

Arguing as above with probability at least $1 - 4\varepsilon$ either one of u, v is connected to only finitely many vertices or $u \xleftrightarrow{\omega_3} v$. As ε is arbitrary and there are only countably many pairs of vertices u, v there can only be one ω_3 infinite cluster. ❏

2.4 GENERALIZED SERIES AND PARALLEL LAWS

Part of the original motivation for defining the random cluster model was the observation that independent percolation and the Ising/Potts models satisfy versions of the series and parallel laws of electrical networks. In an electrical circuit an electrician may replace a single resistor with two resistors either in series or in parallel without affecting the rest of the circuit, provided he chooses the values of the new resistors correctly. There is a similar rule for the random cluster model. An edge e in a network $\mathcal{N}$ may be replaced by two edges, e_1 and e_2, in parallel if $1 - \gamma(e) = (1 - \gamma(e_1))(1 - \gamma(e_2))$; or in series if $\pi(\gamma(e)) = \pi(\gamma(e_1))\pi(\gamma(e_2))$; without affecting the random cluster model on the rest of $\mathcal{N}$. For details see the original papers [26, 24, 25] or for example Grimmett [35]

Of course a clever electrician need not be restricted to the series and parallel laws, he may replace a single resistor with any network of resistors so long as the resistance across the network is the same as the original resistance, whether calculated or simply measured using an ohmmeter. We present here a generalized version of the series and parallel laws for the random cluster

model whereby a single edge on a weighted graph may be replaced by a second weighted graph as long as the probability of a path across the new graph is correct.

The motivation for this theorem is not in fact that we wish to replace a complicated graph with a single edge, rather we wish to make explicit a coupling that is implicit in the original series and parallel laws.

Let $\mathcal{N}$ be a measurable network containing some edge $e = \langle u, v \rangle$. Let $\mathcal{G}$ be a second network containing vertices u' and v' with $\mathbb{Q}_{\mathcal{G},q}[u' \longleftrightarrow v'] = \pi(\gamma(e))$. Now suppose we remove e from $\mathcal{N}$ and replace it with a copy of $\mathcal{G}$ by attaching u' to u and v' to v, how does the random cluster model on this new graph behave?

Theorem 2.23 below may be interpreted as a method of sampling from the random cluster model on the new network. First choose a configuration ω of $\mathcal{N}$ according to $\mathbb{Q}_{\mathcal{N},q}$. Next, choose a configuration ω' of $\mathcal{G}$ as follows. If $\omega(e) = 1$ then choose ω' according to $\left(\mathbb{Q}_{\mathcal{G},q} \mid u' \longleftrightarrow v' \right)$. If $\omega(e) = 0$ then choose ω' according to $\left(\mathbb{Q}_{\mathcal{G},q} \mid u' \nleftrightarrow v' \right)$. Then $\omega \times \omega'$ gives a configuration of the new network distributed according to the q-random cluster model.

2.4.1 Gluing Networks

In order to make rigorous the informal idea of replacing a subnetwork with an edge we define a gluing operation in which we join two separate graphs by identifying two pairs of vertices. This will allow us to define an *edge like subnetwork* (Definition 2.22 below) that may be replaced by a single edge.

Consider two networks $\mathcal{N_1}$ and $\mathcal{N_2}$ containing vertices $u_1, v_1 \in \mathcal{N_1}$ and $u_2, v_2 \in \mathcal{N_2}$. Interpret $\frac{\mathcal{N_1} \sqcup \mathcal{N_2}}{u_1 \sim u_2, v_1 \sim v_2}$ as the network whose vertices are the quotient set $\frac{\mathbf{V}(\mathcal{N_1}) \sqcup \mathbf{V}(\mathcal{N_2})}{u_1 \sim u_2, v_1 \sim v_2}$ with edge set $\mathbf{E}(\mathcal{N_1}) \sqcup \mathbf{E}(\mathcal{N_2})$ interpreted in the obvious way.

Notice in particular that if $\mathcal{N} = \frac{\mathcal{N_1} \sqcup \mathcal{N_2}}{u_1 \sim u_2, v_1 \sim v_2}$ we have $\Omega_{\mathcal{N}} = \Omega_{\mathcal{N_1}} \times \Omega_{\mathcal{N_2}}$. First we examine how the random cluster measure on $\mathcal{N}$ relates to the product measure $\mathbb{Q}_{\mathcal{N_1}} \times \mathbb{Q}_{\mathcal{N_2}}$.

FIGURE 5
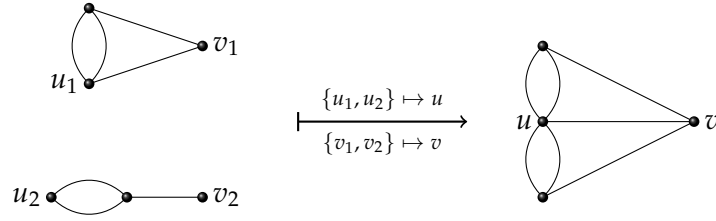
An illustration of the gluing operation $(\mathcal{N}_1, \mathcal{N}_2) \mapsto \frac{\mathcal{N}_1 \sqcup \mathcal{N}_2}{u_1 \sim u_2, v_1 \sim v_2}$ from Lemma 2.21.

Lemma 2.21 describes this relationship. To state the lemma we name a new simple multigraph, the *digon*. A digon is a network with two vertices u, v and two undirected edges e_1 and e_2 from u to v.

LEMMA 2.21

Let $\mathcal{N} = \frac{\mathcal{N}_1 \sqcup \mathcal{N}_2}{u_1 \sim u_2, v_1 \sim v_2}$ be as above and name a digon D with edge weights $\gamma(e_i) = \pi^{-1}(\mathbb{Q}_{\mathcal{N}_i} [u_i \longleftrightarrow v_i])$ for $i \in \{0, 1\}$.

If $\psi : \Omega_{\mathcal{N}_1} \times \Omega_{\mathcal{N}_2} \to \Omega_D$ is the map defined by setting

$$\mathbb{1}_{J_{e_i}}(\psi(\omega_1, \omega_2)) = \mathbb{1}_{[u_i \longleftrightarrow v_i]}(\omega_i)$$

Then the push forwards measure $\psi(\mathbb{Q}_{\mathcal{N}})$ agrees with $\mathbb{Q}_D$ and the conditional measure $(\mathbb{Q}_{\mathcal{N}} | \psi)(\omega) = (\mathbb{Q}_{\mathcal{N}_1} \times \mathbb{Q}_{\mathcal{N}_2} | \psi)(\omega)$ for all $\omega \in \Omega_D$.

PROOF

Recall that $\Omega_{\mathcal{N}} = \Omega_{\mathcal{N}_1} \times \Omega_{\mathcal{N}_2}$. For $\omega = (\omega_1, \omega_2)$. Let $\kappa_{\mathcal{N}}(\omega)$ be the number of clusters of ω when interpreted as a configuration on $\mathcal{N}$ and let $\kappa_{\mathcal{N}_1}(\omega)$ and $\kappa_{\mathcal{N}_2}(\omega)$ be the number of clusters of ω_1 and ω_2 respectively interpreted as configurations of $\mathcal{N}_1$ and $\mathcal{N}_2$.

Let $A = [u_1 \longleftrightarrow v_1] \times [u_2 \longleftrightarrow v_2] \subset \Omega_{\mathcal{N}_1} \times \Omega_{\mathcal{N}_2}$ be the event that both edges of D are open in the configuration $\psi(\omega)$. Notice that

$$\kappa_{\mathcal{N}} = \kappa_{\mathcal{N}_1} + \kappa_{\mathcal{N}_1} + \mathbb{1}_A - 2. \tag{2.59}$$

To see this connect $\mathcal{N}_1$ and $\mathcal{N}_2$ by first identifying u_1 with u_2 and then v_1 with v_2. When we identify u_1 with u_2 we reduce the number of clusters by one. Then connecting v_1 to v_2 we reduce the number of clusters by one again unless v_1 and v_2 are already in the same cluster, that is if $u_1 \longleftrightarrow v_1$ in $\mathcal{N}_1$ and $u_2 \longleftrightarrow v_2$ in $\mathcal{N}_2$.

So we have

$$\frac{\mathbb{Q}_{\mathcal{N}}(\omega)}{\mathbb{Q}_{\mathcal{N}_1}(\omega_1)\mathbb{Q}_{\mathcal{N}_2}(\omega)} \propto q^{\mathbb{1}_A(\omega)}. \tag{2.60}$$

We may interpret (2.60) as a Radom-Nikodym derivative and by comparison with the closed form definition of the random cluster model (2.18) we have

$$\frac{d\mathbb{Q}_{\mathcal{N}}}{d(\mathbb{Q}_{\mathcal{N}_1} \times \mathbb{Q}_{\mathcal{N}_2})}(\omega) = \frac{d\mathbb{Q}_D}{d\mathbb{P}_{\pi(D)}}(\psi(\omega)). \tag{2.61}$$

In particular $\frac{d\mathbb{Q}_{\mathcal{N}}}{d\left(\mathbb{Q}_{\mathcal{N}_1} \times \mathbb{Q}_{\mathcal{N}_2}\right)}(\omega)$ is measurable with respect to the σ-algebra generated by ψ and so $(\mathbb{Q}_{\mathcal{N}} \mid \psi)(\omega) = (\mathbb{Q}_{\mathcal{N}_1} \times \mathbb{Q}_{\mathcal{N}_2} \mid \psi)(\omega)$ for all $\omega \in \Omega_D$. Furthermore we have

$$\frac{d\psi(\mathbb{Q}_{\mathcal{N}})}{d\psi(\mathbb{Q}_{\mathcal{N}_1} \times \mathbb{Q}_{\mathcal{N}_2})}(\omega) = \frac{d\mathbb{Q}_D}{d\mathbb{P}_{\pi(D)}}(\omega). \tag{2.62}$$

Therefore, as $\psi(\mathbb{Q}_{\mathcal{N}_1} \times \mathbb{Q}_{\mathcal{N}_2}) = \mathbb{P}_{\pi(D)}$ we have $\psi(\mathbb{Q}_{\mathcal{N}}) = \mathbb{Q}_D$ as required. ❑

2.4.2 Edge–like subgraphs

Lemma 2.21 provides the engine for the generalized series and parallel laws. It remains to translate the lemma into a more usable form.

The gluing operation above may be used as a formal tool for adding a single edge to a network. If $\mathcal{N}$ is some network containing vertices u, v we may set E to be a network containing a

single edge $e = \langle u', v' \rangle$ with weight γ. The network $\frac{\mathcal{N} \sqcup E}{u \sim u', v \sim v'}$ is simply a copy of $\mathcal{N}$ with an extra edge between u' and v'.

If $\mathcal{G} \ni (u', v')$ is a second network, with $\mathbb{Q}_{\mathcal{G}} [u' \longleftrightarrow v'] = \pi(\gamma)$ then Lemma 2.21 states that the random cluster models on $\frac{\mathcal{N} \sqcup \mathcal{G}}{u \sim u', v \sim v'}$ and $\frac{\mathcal{N} \sqcup E}{u \sim u', v \sim v'}$ behave in similar ways.

We reinterpret Lemma 2.21 as a partial decomposition of the random cluster model on a network in the form $\frac{\mathcal{N} \sqcup \mathcal{G}}{u \sim u', v \sim v'}$ into loosely dependent random cluster models on $\mathcal{G}$ and on $\frac{\mathcal{N} \sqcup E}{u \sim u', v \sim v'}$.

Given a finite network it is easy to spot subgraphs which may be replaced by a single edge without resorting to a rigorous characterization of such subgraphs. Thus we define an *edge like subgraph* simply as a subgraph that may be replaced by an edge in the manner specified above.

DEFINITION 2.22

For a network $\mathcal{N}$ we say $\mathcal{G} \subset \mathcal{N}$ is an *edge-like subgraph* of $\mathcal{N}$ if there exists a graph $\mathcal{N}' \ni u', v'$ such that $\mathcal{N}$ is isomorphic to $\frac{\mathcal{N}' \sqcup \mathcal{G}}{u' \sim u, v' \sim v}$ for some $u, v \in \mathcal{G}$.

Let $(\mathcal{N} /\!\!/ \mathcal{G})$ be the graph obtained from $\mathcal{N}'$ by adding a single edge $e_{\mathcal{G}} = \langle u', v' \rangle$ with weight $\gamma(e_{\mathcal{G}}) = \pi^{-1}(\mathbb{Q}_{\mathcal{G}} [u \longleftrightarrow v])$.

For $\mathcal{G} \subset \mathcal{N}$ and $\mathcal{N}'$ as above recall that $\Omega_{\mathcal{N}} = \Omega_{\mathcal{N}'} \times \Omega_{\mathcal{G}}$. Setting $\omega = (\omega_1 \times \omega_2) \in \Omega_{\mathcal{N}'} \times \Omega_{\mathcal{G}}$ we may define projections $\psi_{\mathcal{G}} : \Omega_{\mathcal{N}} \to \Omega_{\mathcal{G}}$ and $\psi_{(\mathcal{N} /\!\!/ \mathcal{G})} : \Omega_{\mathcal{N}} \to \Omega_{(\mathcal{N} /\!\!/ \mathcal{G})}$ such that

$$\mathbb{1}_{J_e}(\psi_{\mathcal{G}}(\omega)) = \omega_1(e) \tag{2.63}$$

$$\mathbb{1}_{J_e}\left(\psi_{(\mathcal{N} /\!\!/ \mathcal{G})}(\omega)\right) = \begin{cases} \mathbb{1}_{[u' \leftrightarrow v']}(\omega_2) & : \text{if } e = e_{\mathcal{G}} \\ \omega_1(e) & : \text{otherwise} \end{cases} \tag{2.64}$$

2.4.3 Generalized series and parallel laws

THEOREM 2.23: THE GENERALIZED SERIES AND PARALLEL LAW

If $\mathcal{G} \subset \mathcal{N}$ is an *edge-like subgraph* then

- $\psi_{(\mathcal{N}/\!\!/\mathcal{G})}(\mathbb{Q}_{\mathcal{N}}) = \mathbb{Q}_{(\mathcal{N}/\!\!/\mathcal{G})}$,

- for any $\mathscr{F}_{\mathcal{G}}$-measurable random variable X

$$\mathbb{Q}_{\mathcal{N}}\left(X \,\Big|\, \psi_{(\mathcal{N}/\!\!/\mathcal{G})} \right)(\omega) = \mathbb{Q}_{\mathcal{G}}\left(X \,\Big|\, \mathbb{1}_{[u \leftrightarrow v]} \right)(\psi_{\mathcal{G}}(\omega)).$$

PROOF

Let $\mathcal{G} \subset \mathcal{N}$ and $\mathcal{N}'$ be as in Definition 2.22 and let E be the graph containing a single edge $e = \langle u, v \rangle$ with bond weight $\gamma(e) = \pi^{1}(\mathbb{Q}_{\mathcal{G}}[u \leftrightarrow v])$. Then $(\mathcal{N}/\!\!/\mathcal{G}) = \frac{\mathcal{N}' \sqcup E}{u' \sim u, \, v' \sim v}$ up to some trivial isomorphism.

From Lemma 2.21 we see that

$$\frac{d\mathbb{Q}_{\mathcal{N}}}{d(\mathbb{Q}_{\mathcal{N}'} \times \mathbb{Q}_{\mathcal{G}})}(\omega) = \frac{d\mathbb{Q}_{(\mathcal{N}/\!\!/\mathcal{G})}}{d(\mathbb{Q}_{\mathcal{N}'} \times \mathbb{Q}_{E})}\left(\psi_{(\mathcal{N}/\!\!/\mathcal{G})}(\omega) \right) \qquad (2.65)$$

is $\psi_{(\mathcal{N}/\!\!/\mathcal{G})}$-measurable and so

$$\frac{d\mathbb{Q}_{\mathcal{N}}}{d(\mathbb{Q}_{\mathcal{N}'} \times \mathbb{Q}_{\mathcal{G}})}(\omega) = \frac{d\psi_{(\mathcal{N}/\!\!/\mathcal{G})}(\mathbb{Q}_{\mathcal{N}})}{d\psi_{(\mathcal{N}/\!\!/\mathcal{G})}(\mathbb{Q}_{\mathcal{N}'} \times \mathbb{Q}_{\mathcal{G}})}\left(\psi_{(\mathcal{N}/\!\!/\mathcal{G})}(\omega) \right) \quad (2.66)$$

$$= \frac{d\mathbb{Q}_{(\mathcal{N}/\!\!/\mathcal{G})}}{d(\mathbb{Q}_{\mathcal{N}'} \times \mathbb{Q}_{E})}\left(\psi_{(\mathcal{N}/\!\!/\mathcal{G})}(\omega) \right). \qquad (2.67)$$

Hence

$$\frac{d\psi_{(\mathcal{N}/\!\!/\mathcal{G})}(\mathbb{Q}_{\mathcal{N}})}{d\mathbb{Q}_{(\mathcal{N}/\!\!/\mathcal{G})}} = \frac{d\psi_{(\mathcal{N}/\!\!/\mathcal{G})}(\mathbb{Q}_{\mathcal{N}'} \times \mathbb{Q}_{\mathcal{N}})}{d(\mathbb{Q}_{\mathcal{N}'} \times \mathbb{Q}_{E})}. \qquad (2.68)$$

By expressing $\psi_{(\mathcal{N}/\!\!/\mathcal{G})}$ as a product map we see that

$$\psi_{(\mathcal{N}/\!\!/\mathcal{G})}(\mathbb{Q}_{\mathcal{N}'} \times \mathbb{Q}_{\mathcal{G}}) = \mathbb{Q}_{\mathcal{N}'} \times \mathbb{Q}_{E} \qquad (2.69)$$

hence $\psi_{(\mathcal{N} /\!/ \mathcal{G})}(\mathbb{Q}_{\mathcal{N}}) = \mathbb{Q}_{(\mathcal{N} /\!/ \mathcal{G})}$.

To prove the second part of the theorem let X be some $\mathscr{F}_{\mathcal{G}}$-measurable random variable and choose a $\psi_{(\mathcal{N} /\!/ \mathcal{G})}$-measurable test function Y. Then as $\frac{d\mathbb{Q}_{\mathcal{N}}}{d\psi_{(\mathcal{N} /\!/ \mathcal{G})}(\mathbb{Q}_{\mathcal{N}'} \times \mathbb{Q}_{\mathcal{G}})}$ is also $\psi_{(\mathcal{N} /\!/ \mathcal{G})}$-measurable we have

$$\mathbb{Q}_{\mathcal{N}}(XY) = \mathbb{Q}_{\mathcal{N}}\left((\mathbb{Q}_{\mathcal{N}'} \times \mathbb{Q}_{\mathcal{G}})\left(X \,\Big|\, \psi_{(\mathcal{N} /\!/ \mathcal{G})} \right) Y \right) \tag{2.70}$$

Furthermore as $(\mathbb{Q}_{\mathcal{N}'} \times \mathbb{Q}_{\mathcal{G}})$ is a product measure and $\psi_{(\mathcal{N} /\!/ \mathcal{G})}$ depends on bonds in $\mathcal{G}$ only through the indicator $\mathbb{1}_{[u \longleftrightarrow v]}(\psi_{\mathcal{G}}(\omega))$ we have

$$\mathbb{Q}_{\mathcal{N}}(XY) = \mathbb{Q}_{\mathcal{N}}\left(\mathbb{Q}_{\mathcal{G}}\left(X \,\Big|\, \mathbb{1}_{[u \longleftrightarrow v]} \right)(\psi_{\mathcal{G}}(\omega))Y \right). \tag{2.71}$$

❏

2.4.4 An example

We illustrate Theorem 1.4 with an example.

Define the *diamond* $\mathcal{D}_{k,p,\eta}$ to be the weighted graph consisting of $k+2$ vertices, u, v and $w_1, \ldots, w_k$ and $2k$ edges $\langle u, w_i \rangle$, $\langle w_i, v \rangle$ for $i = 1 \ldots k$ with weights $\gamma\langle u, w_i \rangle = p$ and $\gamma\langle w_i, v \rangle = \eta$. Suppose we wish to simulate $\mathbb{Q}_{\mathcal{D}_{k,p,\eta}}$ using a random number generator.

The calculation below plays a key role in Grimmett's [35, §10.10] calculation of the critical point for the wired model on the regular tree and we will refer to it in Chapter 3. The exact simulation will play a part in establishing a non-uniqueness phase for the random cluster model for some of the new boundary conditions on trees we define in Chapter 4.

We may use the series and parallel laws to calculate the probability $\mathbb{Q}_{\mathcal{D}_{k,p,\eta}}[u \longleftrightarrow v] = 1 - (1 - \pi^{-1}(\pi(p)\pi(\eta)))^k$. This calculation is elementary and appears in [35]. See Figure 6 for an illustration.

FIGURE 6
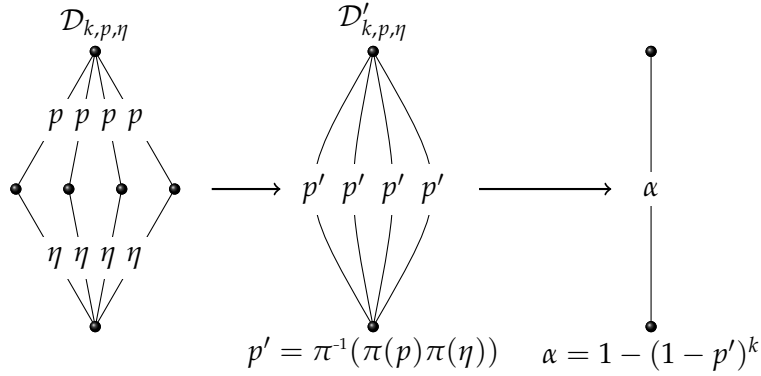
We simplify the "diamond" $\mathcal{D}_{k,p,\eta}$ (here $k = 4$) using the series law and then the parallel law. We may simulate the random cluster model $\mathbb{Q}_{\mathcal{D}_{k,p,\eta}}$ by working backwards and building up the graph from simpler pieces.

First let $\mathcal{G}_i$ be the edge like subgraph of $\mathcal{D}_{k,p,\eta}$ comprised of the two edges $\langle u, w_i \rangle, \langle w_i, v \rangle$. Each graph $\mathcal{G}_i$ is a tree and so the random cluster model $\mathbb{Q}_{\mathcal{G}_i}$ is independent bond percolation with bond probabilities $\pi(p)$ and $\pi(\eta)$. We may apply Theorem 2.23 k times, each time replacing $\mathcal{G}_i$ with a single edge e_i.

So let $\mathcal{D}'_{k,p,\eta}$ be the multigraph consisting of two vertices u, v and k edges e_i between u and v, each with weight $p' = \pi^{-1}(\pi(p)\pi(\eta))$ and define a map $\psi : \Omega_{\mathcal{D}_{k,p,\eta}} \to \mathcal{D}'_{k,p,\eta}$ by

$$\psi \cdot \omega(e_i) = \omega \langle u, w_i \rangle \wedge \omega \langle w_i, v \rangle.$$

By k inductive applications of Theorem 2.23 we may calculate $\mathbb{Q}_{\mathcal{D}'_{k,p,\eta}} = \psi\left(\mathbb{Q}_{\mathcal{D}_{k,p,\eta}}\right)$ and $\left(\mathbb{Q}_{\mathcal{D}_{k,p,\eta}} \mid \psi\right)$ is the measure given by choosing the configuration of each pair of bonds $(\langle u, w_i \rangle, \langle w_i, v \rangle)$ independently as either both open if e_i is open, or according to $\left(\mathbb{Q}_{\mathcal{D}_{1,p,\eta}} \mid u \nleftrightarrow v\right)$ if e_i is closed.

For $\omega \in \Omega_{\mathcal{D}'_{k,p,\eta}}$, the number of clusters $\kappa(\omega) = 2 - \mathbb{1}_{[u \leftrightarrow v]}$ depends only on the event $[u \leftrightarrow v]$ and so

$$\left(\mathbb{Q}_{\mathcal{D}'_{k,p,\eta}} \mid u \leftrightarrow v\right) = \left(\mathbb{P}_{\mathcal{D}'_{k,p,\eta}} \mid u \leftrightarrow v\right)$$

and from the definition of the random cluster model we have

$$\mathbb{Q}_{\mathcal{D}'_{k,p,\eta}}[u \longleftrightarrow v] = \pi\left(\mathbb{P}_{\mathcal{D}'_{k,p,\eta}}[u \longleftrightarrow v]\right)$$
$$= \pi\left(1 - \left(1 - p'\right)^k\right).$$

So far we have done nothing new, we have simply used the ordinary series and parallel laws to calculate the probability $\mathbb{Q}_{\mathcal{D}_{k,p,\eta}}[u \longleftrightarrow v]$. The innovation is that we may reverse the process to build up the distribution from conditioned percolation processes.

To sample from $\mathbb{Q}_{\mathcal{D}_{k,p,\eta}}$ first set

$$\alpha = 1 - \left(1 - p'\right)^k, \qquad \beta = \frac{1 - \pi(p)}{1 - \pi(p)\pi(\eta)},$$
$$\gamma = \frac{(1 - \pi(p))\pi(\eta)}{1 - \pi(p)\pi(\eta)}, \qquad \theta(x) = 1 - \left(\frac{1 - \alpha}{1 - \alpha x}\right)^{1/k}.$$

Now choose $(x_1, \ldots, x_k, y_1, \ldots, y_k, N)$ uniformly from $[0,1]^{2k} \times \{1, \ldots, k\}$ and set

$$\omega_0\langle u, w_i\rangle = \mathbb{1}_{[x_i > \beta]} \qquad \omega_0\langle w_i, v\rangle = \mathbb{1}_{[x_i < \gamma]}$$

and for an edge $e \in \{\langle u, w_i\rangle, \langle w_i, v\rangle\}$

$$\omega_1(e) = \begin{cases} 1 & : \text{if } i = N \text{ or } y_i < \theta(y_N) \\ \omega_0(e) & : \text{otherwise} \end{cases}$$

Lastly we set $\omega = \omega_1$ with probability $1 - (1 - \pi^{-1}(\pi(p)\pi(\eta)))^k$ and $\omega = \omega_0$ with probability $(1 - \pi^{-1}(\pi(p)\pi(\eta)))^k$. We claim ω is distributed as $\mathbb{Q}_{\mathcal{D}_{k,p,\eta}}$.

As the x_i are independent uniform $[0,1]$ random variables it is easy to see that ω_0 is distributed as $\left(\mathbb{P}_{\mathcal{D}_{1,p,\eta}} \mid u \nleftrightarrow v\right)^k$. From above, $\left(\mathbb{Q}_{\mathcal{D}_{k,p,\eta}} \mid \psi\right) = \left(\mathbb{P}_{\mathcal{D}_{k,p,\eta}} \mid \psi\right)$ so we need only check that $\psi(\omega)$ is distributed as $\psi\left(\mathbb{Q}_{\mathcal{D}_{k,p,\eta}}\right)$.

In particular we need to show that the indicators $(\psi(\omega_1))(e_i) = \mathbb{1}_{[i = N \text{ or } y_i < \theta(y_N)]}$ are distributed as independent Bernoulli(p') random variables conditioned on the event that $\sum_{i=1}^k G_i > 0$.

Set

$$z_i^n = \begin{cases} 1 - (1 - \alpha y_n)^{1/k} & : \text{ if } i = n \\ 1 - (1 - y_i)(1 - z_n^n) & : \text{ if } i \neq n \end{cases}$$

We have chosen z^n so that $\min_{i=1...k} z_i^n = z_n^n \leq 1 - (1 - \alpha)^{1/k} = p'$ and we may calculate the determinant $\left\| \frac{\partial z_i^n}{\partial y_j} \right\| = \frac{\alpha}{k}$.

Therefore as N is chosen uniformly the random variables $z^{N,i}$ are distributed as uniform $[0, 1]$ random variables conditioned on the event that at least one is smaller that p'. Furthermore for $i \neq n$ we have $z_i^n < 1 - (1 - \alpha)^{1/k}$ if and only if $y_i < \theta(y_n)$.

3 | THE RANDOM CLUSTER MODEL ON A TREE

We consider the random cluster model on a tree under the "wired" boundary conditions of Häggström [38]. Our approach combines the method of Zachary [66] with the method of Grimmett [35, Section 10.9]. In particular we adapt the notion of a Markov chain to the random cluster model on a tree and show that the set of Markov chains contains the extremal set of random cluster models. Using the Generalized Series and Parallel Laws of Chapter 2 we may identify each Markov chain belonging to the set of (p,q) random cluster models on a tree with an "entrance law". On a regular tree this approach allows us to reconstruct the branching construction of Häggström [38] and to prove a conjecture from that paper regarding the uniqueness phase of the random cluster model.

CONTENTS

3.1 CONSTRUCTION OF THE WIRED MODEL

Fix $q > 0$ and let $\mathcal{T}$ be a weighted tree. Recall the definition of the infinity free and infinity wired random cluster models from subsection 2.3.2. As $\mathcal{T}$ is a tree and contains no cycles, we have $\mathcal{L}_e = \varnothing$ for every $e \in \mathcal{T}$.

Hence for every finite subtree $T \subset \mathcal{T}$ and boundary condition $\xi \in \Omega_{\mathcal{T}}$; $\mathbb{Q}_T^\xi(\mathcal{L}_e) = 0$ and so $\mathbb{Q}_T^\xi(J_e \,|\, \mathscr{T}_e) = \pi_q(p)\ \mathbb{Q}_T^\xi$-almost surely. Therefore $\mathbb{Q}_T^\xi = \left(\mathbb{P}_{\pi_q(T)} \,\Big|\, \mathscr{T}_T \right)(\xi)$.

In particular we have $\mathbb{Q}_{\mathcal{T},q}^\xi = \mathbb{P}_{\mathcal{T}}$ and the study of the free random cluster model is reduced to the study of Bernoulli percolation on the tree.

In this chapter we explore the wired random cluster model on the tree. This model was introduced by Häggström [38] on the regular isotropic tree, and in more generality by Jonasson [44]. Recall from Theorem 2.15 that we may characterize the set of wired random cluster models

$$\mathscr{R}_{\mathcal{T}}^\star = \left\{ \varphi : \varphi(J_e \,|\, \mathscr{T}_e) = \pi_q(p) + (p - \pi_q(p))\mathbb{1}_{\mathcal{L}_e^\star} \right\}, \qquad (3.1)$$

and that the set $\mathscr{R}_{\mathcal{T}}^\star$ is closed and convex with extremal measures

$$\mathscr{E}_{\mathcal{T}}^\star = \left\{ \varphi \in \mathscr{R}_{\mathcal{T}}^\star : \varphi(A) \in \{0,1\} \forall A \in \mathscr{T} \right\}. \qquad (3.2)$$

Our aim is to adapt the work of Spitzer [60] and Zachary [66] to the random cluster model in combination with the method of Grimmett [35, §10.10]. Both Spitzer and Zachary considered Markov random fields indexed by the vertices of a tree. In both papers it was shown that any extremal Markov random field may be expressed as a Markov chain. In the two state case by Spitzer [60] and for countable state space by Zachary [66]. Furthermore Zachary was able to show that the set of Markov chains for a given state space is in one to one correspondence with a set of *entrance laws* defined for the specification.

REMARK 3.1

Before we begin we comment on an idiosyncrasy of the random cluster model on the general tree. Suppose $\mu \in \mathscr{R}_{\mathcal{T},q}^\star$ is a random cluster model for $q \neq 1$. Suppose further that for some (hence every) $v \in \mathbf{V}(\mathcal{T})$ we have $\mu[v \longleftrightarrow \infty] = 0$. Then for every $e \in \mathbf{E}(\mathcal{T})$ we have $\mu(\mathcal{L}_e^\star) = 0$ and μ must be independent bond percolation with bond probabilities $\pi(\gamma(e))$.

Now suppose $\mathcal{T}$ is a regular k-tree with $\gamma(e) = \tau$ then if $\tau \leq \pi^{-1}\left(\frac{1}{k}\right)$ we have $\mathbb{P}_{\pi(\mathcal{T})}(v \longleftrightarrow \infty) = 0$ and $\mathbb{P}_{\pi(\mathcal{T})}$ is a random cluster model. Conversely if $\tau > \pi^{-1}\left(\frac{1}{k}\right)$ then for every $e \in \mathbf{E}(\mathcal{T})$ we have $\mathbb{P}_{\pi(\mathcal{T})}(\mathcal{L}_e^\star) > 0$ and $\mathbb{P}_{\pi(\mathcal{T})}$ is not a random cluster model.

We may conclude informally that subcritical percolation on a homogeneous tree is a random cluster model, whereas supercritical percolation is not. This observation is not restricted to homogeneous trees, and will play a key role in this chapter, however there are exceptions to this rule that we must discount. Let $\mathcal{T}'$ be a regular k-tree with some nominated infinite path Π from ρ to infinity. Set $\gamma\left(\langle \Pi_n, \Pi_{n+1} \rangle\right) = 1 - 2^n$ and $\gamma(e) = \pi^{-1}\left(\frac{1}{2k}\right)$ for all other edges. Then $\mathbb{P}_{\pi(\mathcal{T}')}$ is "supercritical" in that $\mathbb{P}_{\pi(\mathcal{T}')}(\rho \longleftrightarrow \infty) > 0$ but for every $e \in \mathbf{E}(\mathcal{T}')$ we have $\mathbb{P}_{\pi(\mathcal{T}')}(\mathcal{L}_e^\star) = 0$ and $\mathbb{P}_{\pi(\mathcal{T}')} \in \mathscr{R}_{\mathcal{T}',q}^\star$.

The main purpose of this chapter is prepare the ground for Chapter 4, where we focus on homogeneous trees, and to prove Conjecture 1.9 of Häggström [38], which again concerns homogeneous trees. However we prefer to present arguments in as much generality as possible.

In order to avoid this kind of exception we will always assume that for any weighted tree the weight function $\gamma : \mathbf{E}(T) \to (0,1)$ is bounded below by some $\varepsilon > 0$.

One interpretation of a Markov chain on the vertices of a tree is as follows. A Markov random field φ on a tree is a Markov chain if and only if for any finite subtree T the restriction of φ to the vertices of T is a Markov random field on T. We have seen

FIGURE 7

$$T \subset \mathcal{T} \qquad\qquad T^\star = T \cup \partial T$$
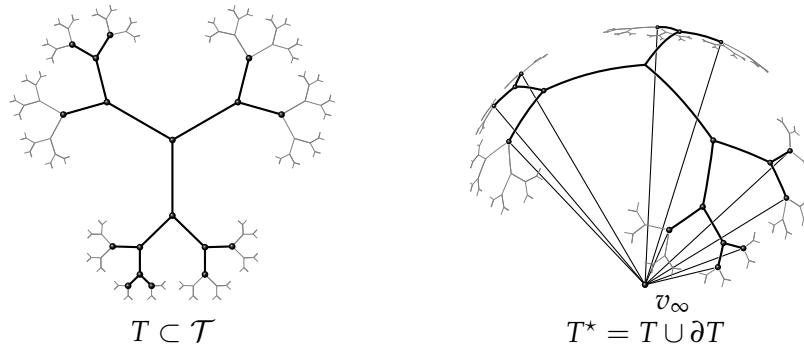
Augmenting a subtree by adding a ghost vertex. We may always describe an extremal wired random cluster model by a consistent set of random cluster models on such finite graphs.

that any random cluster model on a finite tree is independent bond percolation. Hence for the wired random cluster model on a tree we may not use such a definition directly. In order to make sense of this interpretation of a Markov chain for the random cluster measure we augment certain finite subtrees by including a ghost vertex v_∞ to represent the infinite cluster. We add extra edges between v_∞ and the leaf vertices of the tree (see Figure 7.) A random cluster measure φ will be a Markov chain if the push forwards measure of φ under an appropriate projection is a random cluster model on the augmented tree $T^\star$, whenever T is a member of a certain class of finite subtrees $\mathbb{T}$

Recall that for a nominated vertex $v \in \mathbf{V}(\mathcal{T})$ we may root $\mathcal{T}$ as v by directing all edges away from v. Alternatively for a directed edge e we may root the tree at e by directing all bi-infinite paths through e. Let $\mathbf{E}_v(\mathcal{T})$ be the set of directed edges of the tree rooted at v and $\mathbf{E}_e(\mathcal{T})$ be the edges of the tree rooted at e. If $e = |u,v\rangle$ is a directed edge then as $\mathcal{T}$ is a tree we may remove e from $\mathcal{T}$ to split the tree into two components, the *ancestors* of e and the *descendants* of e, $A(e) \ni u$ and $D(e) \ni v$ respectively, by convention we direct both subtrees to agree with $\mathbf{E}_e(\mathcal{T})$. Define the *children* of e to be the set of directed edges

$$\chi(|u,v\rangle) = \{|v,w\rangle \in D(e)\} . \qquad\qquad (3.3)$$

Now let $\mathbb{T}$ be the set of finite subtrees of $\mathcal{T}$ such that for every directed edge $e \in \mathbf{E}(T)$ either $\chi(e) \subset \mathbf{E}(T)$ or $\chi(e) \cap \mathbf{E}(T) = \varnothing$. Introduce a ghost vertex v_∞ and for a directed edge $e = |u, v\rangle \in \mathbf{E}(\mathcal{T})$ let $e^\star = |v, v_\infty\rangle$.

Next for each $T \in \mathbb{T}$ name the *leaves*, the *boundary* and the *augmentation* of T respectively as follows.

$$\Lambda T = \{e \in T : \chi(e) \cap T = \varnothing\}, \tag{3.4}$$

$$\partial T = \{e^\star : e \in \Lambda T\}, \tag{3.5}$$

$$T^\star = (\mathbf{V}(T) \cup \{v_\infty\}, \mathbf{E}(T) \cup \partial T). \tag{3.6}$$

In addition let $\mathbb{T}^\star = \{T^\star : T \in \mathbb{T}\}$ be the set of finite augmented trees.

Our main arguments will revolve around random cluster models on finite augmented trees, together with the generalized series and parallel laws of Theorem 2.23. It will be helpful to name some further subgraphs in order to dissect our augmented trees effectively. Recall from Remark 2.2 that for a subtree $T \in \mathbb{T}$ we may write $e \in T$ or $e \in T^\star$ as shorthand for $e \in \mathbf{E}(T)$ or $e \in \mathbf{E}(T^\star)$

For a directed edge e let $\chi^\star(e)$ be the graph containing edges f and $f^\star$ for every $f \in \chi(e)$. If $e \in T \in \mathbb{T}$ let $D_T(e) = D(e) \cap T$ and $\Lambda D_T(e) = D(e) \cap \Lambda T$. Define $\partial D_T(e) = \{f^\star : f \in \Lambda D(e)\}$ and $D_T^\star(e) = D_T(e) \cup \partial D_T(e)$ as above.

For a tree $T \in \mathbb{T}$ and $e \in \Lambda T$ define the *e-extension* of T to be the undirected tree $(T + e) \in \mathbb{T}$ with $\mathbf{E}_\rho(T + e) = \mathbf{E}_\rho(T) \cup \chi(e)$ for any and every $\rho \in T$.

Next for a directed edge $e = |u, v\rangle$ let $[e \downarrow_\infty] = [v \longleftrightarrow \infty]_e$ be the event that there is an open directed path from e to infinity in $D(e)$. Let $[e \downarrow_{v_\infty}] = [v \longleftrightarrow v_\infty]_e \subset \Omega_{T^\star}$ be the event that there is an open directed path from e to v_∞ in $D_T(e)$. For finite trees $S, T \in \mathbb{T}$

with $S \subset T$ define maps $\psi_{S,T} : \Omega_{T^\star} \to \Omega_{S^\star}$ and $\psi_T : \Omega_{\mathcal{T}} \to \Omega_{T^\star}$ such that

$$\mathbb{1}_{J_e}(\psi_{S,T}(\omega)) = \begin{cases} \omega(e) & : \text{if } e \in S, \\ \mathbb{1}_{[f \downarrow_{v_\infty}]}(\omega) & : \text{if } e = f^\star \in \partial S, \end{cases} \tag{3.7}$$

$$\mathbb{1}_{J_e}(\psi_T(\omega)) = \begin{cases} \omega(e) & : \text{if } e \in T, \\ \mathbb{1}_{[f \downarrow_\infty]}(\omega) & : \text{if } e = f^\star \in \partial T. \end{cases} \tag{3.8}$$

Notice that these maps commute in the sense that $\psi_{S,T} \cdot \psi_{T,W} = \psi_{S,W}$ and $\psi_{S,T} \cdot \psi_T = \psi_S$. Furthermore the map $\psi_{S,T}$ preserves the events $[e \downarrow_{v_\infty}]$ and $\mathcal{L}_e$ interpreted as subsets either of $\Omega_{S^\star}$ or of $\Omega_{T^\star}$. Similarly for the map ψ_T we have $\psi_T^{-1}[e \downarrow_{v_\infty}] = [e \downarrow_\infty]$ and $\psi_T^{-1}(\mathcal{L}_e) = \mathcal{L}_e^\star$.

It is easy to see that if $S, T \in \mathbb{T}$ with $S \subset T$ then for each $e \in \Lambda S \setminus \Lambda T$ the subgraph $D_T^\star(e)$ is an edge like subgraph of $T^\star$. In particular applying Theorem 2.23 once for each $e \in \Lambda S \setminus \Lambda T$ we see that if γ_S and γ_T are weightings of $S^\star$ and $T^\star$ such that

$$\gamma_S(e) = \begin{cases} \gamma_T(e) & : \text{if } e \in S, \\ \pi^1\left(\mathbb{Q}_{D_T^\star(e)}[f \downarrow_{v_\infty}]\right) & : \text{if } e = f^\star \in \partial S. \end{cases} \tag{3.9}$$

then the push forwards measure $\psi_{S,T}(\mathbb{Q}_{T^\star}) = \mathbb{Q}_{S^\star}$.

Now suppose $\varphi \in \mathscr{R}_{\mathcal{T}}^\star$ is a random cluster measure on $\mathcal{T}$, what can we say about the push forwards measures $\psi_T(\varphi)$? The map ψ_T follows the same rules as $\psi_{S,T}$ so it is natural to ask whether $\psi_T(\varphi)$ is a random cluster model on $T^\star$.

Recall that for a Markov specification on the vertices of a tree we may characterize the set of Markov chains as the set of Markov random fields whose restrictions to finite trees are Markov random fields.

For the random cluster model we will see that there is an analogous class of random cluster measures which we shall name the Markov chains. These measures may be characterized as exactly those measures $\varphi \in \mathscr{R}_{\mathcal{T}}^\star$ for which $\psi_T(\varphi)$ is a random cluster measure on $T^\star$ for every $T \in \mathbb{T}$.

The following observation motivates our definition of a Markov chain. Let $\mathbb{Q}_{T^\star}$ be the random cluster measure associated with any weighting of $T^\star$. For an edge $e \in T$, removing e splits $T^\star$ into two parts, $A_T^\star(e)$ and $D_T^\star(e)$ connected by a single vertex v_∞. It is easy to see, either from the characterization (2.23) or an application of Therorem 2.23 that the two σ-algebras $\mathscr{F}_{A_T^\star(e)}$ and $\mathscr{F}_{D_T^\star(e)}$ are independent under the conditioned measure $\left(\mathbb{Q}_{T^\star} \,\middle|\, J_e^{\complement}\right)$.

We may deduce that if $\psi_T(\varphi)$ is a random cluster model for every $T \in \mathbb{T}$ then φ must have the following property.

DEFINITION 3.2

We say a measure φ on $\Omega_{\mathcal{T}}$ is a *Markov chain* if for every directed edge $e \in \mathcal{T}$ the σ-algebras $\mathscr{F}_{A(e)}$ and $\mathscr{F}_{D(e)}$ are independent under the conditioned measure $\left(\varphi \,\middle|\, J_e^{\complement}\right)$.

As an aid to understanding we make the following claim about Markov chains.

PROPOSITION 3.3

A measure $\varphi \in \mathscr{R}_{\mathcal{T}}^\star$ is a Markov chain if and only if for every $T \in \mathbb{T}$ the push forwards measure $\psi_T(\varphi)$ is a random cluster model on $T^\star$ for some weight function $\gamma_{T,\varphi}$ such that $\gamma_{T,\varphi}(e) = \gamma(e)$ for every $e \in T$.

Proposition 3.3 is a strictly weaker statement than the forthcoming Theorem 3.8 which we prove in Section 3.2.

Now we follow the main steps of Zachary [66]. First we connect the set of Markov chains with the extremal elements of $\mathscr{R}_{\mathcal{T}}$.

THEOREM 3.4

Every extremal random cluster model $\varphi \in \mathscr{E}_{\mathcal{T}}^\star$ is a Markov chain.

PROOF

Given $\varphi \in \mathscr{R}_{\mathcal{T}}^{\star}$ let $\varphi_e = (\varphi \,|\, \omega(e) = 0)$ and for $T \in \mathbb{T}$ suppose $T^{\star}$ is weighted in such a way that $\gamma_T(e) = \gamma(e)$ for every $e \in T$. We will only be interested in $(\mathbb{Q}_{T^{\star}} \,|\, \mathscr{T}_{\partial T})$ so bond strengths of ∂T may be chosen arbitrarily.

Now choose $\xi \in \Omega_{\mathcal{T}}$. As $\varphi \in \mathscr{R}_{\mathcal{T}}^{\star}$ we have $(\varphi \,|\, \mathscr{T}_T)(\xi) = \overline{\mathbb{Q}}_T^{\xi}$. Set $\varphi_T = \psi_T\left(\overline{\mathbb{Q}}_T^{\xi}\right)$. Firstly the event $[e \downarrow_{\infty}]$ is $\mathscr{T}_T$-measurable for every $e \in \Lambda T$; therefore $\varphi_T(J_{e^{\star}}) = \mathbb{1}_{J_{e^{\star}}}(\psi_T(\xi))$ for every $e \in \Lambda T$.

Furthermore for $e \in T$ and $\omega \in \Omega_{\mathcal{T}}^{\xi}$

$$\overline{\mathbb{Q}}_{\mathcal{T}}^{\xi}(J_e \,|\, \mathscr{T}_e)(\omega) = \pi(p) + (p - \pi(p))\mathbb{1}_{\mathcal{L}_e^{\star}}(\omega) \tag{3.10}$$

$$= \pi(p) + (p - \pi(p))\mathbb{1}_{\mathcal{L}_e}(\psi_T(\omega)) \tag{3.11}$$

$$= \mathbb{Q}_{T^{\star}}(J_e \,|\, \mathscr{T}_e)(\psi_T(\omega)). \tag{3.12}$$

Hence we have $\varphi_T = (\mathbb{Q}_{T^{\star}} \,|\, \mathscr{T}_T)$ from Theorem 2.7.

Therefore the σ-algebras $\mathscr{F}_{A(e)}$ and $\mathscr{F}_{D(e)}$ are independent under $(\varphi_e \,|\, \mathscr{T}_T)(\xi)$ and so are independent under $(\varphi_e \,|\, \mathscr{T})(\xi)$ for φ almost every ξ by the reverse martingale convergence theorem. If φ is extremal we have $(\varphi \,|\, \mathscr{T})(\xi) = \varphi$ for almost every ξ and so φ is a Markov chain. $\qquad\square$

Our next task is to define an *entrance law*. We have claimed that for every Markov chain φ and subtree $T \in \mathbb{T}$ the push forwards measure $\psi_T(\varphi)$ is a random cluster measure on $T^{\star}$ with cluster factor q. For an edge $e \in T$, the conditional probabilities $\varphi(J_e \,|\, \mathscr{T}_e)$ and $\psi_T(\varphi)(J_e \,|\, \mathscr{T}_e)$ are set by the conditional specifications of equations (2.37) and (2.23) respectively, and so the bond weights $\gamma_T(e)$ must be constant over all $T \ni e$.

Therefore to specify a Markov chain it will be enough to fix the weights of bonds $e^{\star}$ for every directed edge e.

Let $\overline{\mathbf{E}} = \overline{\mathbf{E}}(\mathcal{T})$ be the set of directed edges $\{|u, v\rangle : \langle u, v \rangle \in \mathbf{E}(\mathcal{T})\}$. For a function $\Theta : \overline{\mathbf{E}} \to [0, 1]$ and subtree $T \in \mathbb{T}$, let T^{Θ} be the network with underlying graph $T^{\star}$ such that the bond weights

$\gamma(e)$ agree with those of $\mathcal{T}$ and $\gamma(e^{\star}) = \Theta(e)$ for every edge $e \in \Lambda T$.

An entrance law H will be a special function : $\overline{\mathbf{E}} \to (0,1)$ such that the set of random cluster models $\mathbb{Q}_T^H = \mathbb{Q}_{T^H}$ are coherent under the maps $\psi_{S,T}$.

There is an extra complication for the random cluster model. The constant function $H_0 \equiv 0$ induces a consistent set of random cluster measures $\{\mathbb{Q}_T^0 : T \in \mathbb{T}\}$ with weak limit $\mathbb{Q}_T^0 \to \mathbb{P}_{\pi(\mathcal{T})}$ as $T \uparrow \mathcal{T}$. Independent bond percolation trivially satisfies our definition of a Markov chain, but we have seen that supercritical bond percolation is not a wired random cluster model on the tree. Furthermore if $\mathbb{P}_{\pi(\mathcal{T})}$ is supercritical the push forwards measure $\psi_T\left(\mathbb{P}_{\pi(\mathcal{T})}\right) \neq \mathbb{Q}_T^0$.

We introduce an extra *robustness* condition that will ensure that $\psi_T\left(\lim_{S \uparrow T} \mathbb{Q}_S^H\right) = \mathbb{Q}_T^H$.

DEFINITION 3.5

For a nominated root vertex $\rho \in \mathbf{V}(\mathcal{T})$, we say a function $H_\rho : \mathbf{E}_\rho \to [0,1]$ is a *rooted entrance law* if for every $e \in \mathbf{E}_\rho$

$$1 - H_\rho(e) = \prod_{f \in \chi(e)} \left(1 - \frac{\gamma(f)H_\rho(f)}{1 + (1-\gamma(f))\left(1 - H_\rho(f)\right)(q-1)}\right). \quad (3.13)$$

We say H_ρ is *robust* if in addition the independent percolation $\overline{\mathbb{P}}_{H_\rho}$, given by

$$\overline{\mathbb{P}}_{H_\rho}(J_e) = \frac{\pi(\gamma(e))\left(1 - \pi\left(H_\rho(e)\right)\right)}{1 - \pi(\gamma(e))\pi\left(H_\rho(e)\right)} \quad (3.14)$$

is subcritical.

We say a function $H : \overline{\mathbf{E}} \to [0,1]$ is an *entrance law* if for every $\rho \in \mathbf{V}(\mathcal{T})$ the restriction H_ρ of H to $\mathbf{E}_\rho$ is a rooted entrance law. We say H is robust if H_ρ is robust for every choice of root vertex ρ.

There is only a superficial difference between an entrance law and a rooted entrance law. If ρ and ρ' are adjacent vertices then $\mathbf{E}_\rho$ and $\mathbf{E}_{\rho'}$ differ only by the reversal of a single edge $\langle \rho, \rho' \rangle$. From the definition (3.13), given H_ρ, there is a unique choice of $H_{\rho'}$ that agrees with H_ρ on all directed edges $\mathbf{E}_\rho \cap \mathbf{E}_{\rho'}$. Furthermore from (3.13) the new value $H_{\rho'}(|\rho', \rho\rangle) < 1$ and so the right hand side of (3.14) is nonzero, therefore $H_{\rho'}$ is robust if and only if H_ρ is robust. Extending this by an induction on the graph theoretic distance between ρ and ρ' is is easy to see that for every (robust) rooted entrance law H_ρ there exists a unique (robust) entrance law H that agrees with H_ρ on $\mathbf{E}_\rho$.

Next we use Theorem 2.23 to justify (3.13) in the definition above.

LEMMA 3.6

A function $H : \overline{\mathbf{E}} \to [0,1]$ is an entrance law if and only if for every $S \subset T \in \mathbb{T}$ the push forwards measure

$$\psi_{S,T}(\mathbf{Q}_T^H) = \mathbf{Q}_S^H. \tag{3.15}$$

PROOF

Choose $S, T \in \mathbb{T}$ with $S \subset T$. For every leaf $e \in \Lambda S$, either $e \in \Lambda T$ in which case $\chi(e) \cap \mathbf{E}(T) = \varnothing$ or $e \notin \Lambda T$, in which case $\chi(e) \subset \mathbf{E}(T)$. Hence $T \setminus S$ is a forest and each connected component contains $\chi(e)$ for exactly one $e \in \Lambda S \setminus \Lambda T$. Therefore we may define a sequence of trees S_i with $S_0 = S$ and $S_{i+1} = S_i + e_i$ for some $e_i \in \Lambda S \setminus \Lambda T$. As $T \in \mathbb{T}$ is finite the sequence must terminate with $S_n = T$,

Now $\chi^\star(e_i)$ is an edge like subgraph of S_{i+1} and S_i is isomorphic to $(S_{i+1} /\!\!/ \chi^\star(e_i))$. Setting $e_i = |u_i, v_i\rangle$ and letting $\mathbf{Q}_{\chi(e_i)}^H$ be the random cluster model on $\chi^\star(e_i)$ with bond weights inherited from T^H, we have from Theorem 2.23 that $\mathbf{Q}_{S_i}^H = \psi_{S_i, S_{i+1}}\left(\mathbf{Q}_{S_{i+1}}^H\right)$ if and only if $\pi(H(e_i)) = \mathbf{Q}_{\chi(e_i)}^H[v_i \longleftrightarrow v_\infty]$.

Similar to the example in subsection 2.4.4 we may use the series and parallel laws to calculate

$$Q^{\mathrm{H}}_{\chi(e_i)}[v_i \leftrightarrow v_\infty] = \pi\left(1 - \prod_{f \in \chi(e)} 1 - \pi^{-1}(\pi(\gamma(e))\pi(\eta(f)))\right)$$

(3.16)

$$= \prod_{f \in \chi(e)} \left(1 - \frac{\gamma(f)\mathrm{H}_\rho(f)}{1+(1-\gamma(f))(1-\mathrm{H}_\rho(f))(q-1)}\right).$$

(3.17)

So if H is an entrance law then

$$Q^{\mathrm{H}}_S = \psi_{S,S_1} \ldots \psi_{S_{n-1},T}(Q^{\mathrm{H}}_T)$$

(3.18)

$$= \psi_{S,T}(Q^{\mathrm{H}}_T).$$

(3.19)

Conversely if (3.13) fails for some directed edge $e \in \mathcal{T}$ we may choose $S, T \in \mathbb{T}$ with $T = S + e$ and from above we see that $Q^{\mathrm{H}}_S \neq \psi_{S,T}(Q^{\mathrm{H}}_T)$. ◻

REMARK 3.7

Notice that there is an iterative construction of the random cluster models Q^{H}_T implicit in the proof above. Choose an undirected edge $e \in T$ and let S_0 be the tree containing the single edge e. Next choose a sequence of trees $S_{i+1} = S_i + e_i$. Select $\omega_0 \in \Omega_{S_0^\star}$ according to $Q^{\mathrm{H}}_{S_0}$ and select $\omega_{i+1} \in \Omega_{S_{i+1}^\star}$ inductively by replacing $e_i^\star$ with a configuration of $\chi^\star(e_i)$ chosen according to $\left(Q^{\mathrm{H}}_{\chi(e_i)} \,\middle|\, v_i \leftrightarrow v_\infty\right)$ if $J_{e_i} = 1$, or according to $\left(Q^{\mathrm{H}}_{\chi(e_i)} \,\middle|\, v_i \nleftrightarrow v_\infty\right)$ if $J_{e_i} = 0$.

3.2 EQUIVALENCE OF MARKOV CHAINS AND ENTRANCE LAWS

So far we have defined two objects. A Markov chain is a measure on $\Omega_{\mathcal{T}}$ which satisfies a weak conditional independence condi-

tion. An entrance law is a function on the set of directed edges of a tree which gives rise to a set of random cluster models, indexed by $\mathbb{T}$, that are coherent under the projections $\{\psi_{S,T} : S \subset T \in \mathbb{T}\}$.

The iterative construction of Remark 3.7 suggests that there is a large volume limit of the set of measures $\{\mathbb{Q}_T^H : T \in \mathbb{T}\}$ and the conditional independence assumption is consistent with the conditional independence properties of the finite random cluster models $\{\mathbb{Q}_{T^\star} : T \in \mathbb{T}\}$. In this section we explore the relationship between entrance laws and Markov chains. The *robustness* condition (3.14) will play a key role.

THEOREM 3.8

If $\varphi \in \mathscr{R}_T^\star$ is a Markov chain then the function

$$H_\varphi : e \mapsto \pi^{-1}\big(\varphi\big(e \downarrow_\infty \,\big|\, J_e^{\complement}\big)\big) \tag{3.20}$$

is an entrance law and the push forwards measures of φ under the maps $\{\psi_T : T \in \mathbb{T}\}$ satisfy $\psi_T(\varphi) = \mathbb{Q}_T^{H_\varphi}$ for every $T \in \mathbb{T}$.

PROOF

Choose a subtree $T \in \mathbb{T}$ arbitrarily and set $\varphi_T = \psi_T(\varphi)$. First we check that $\varphi_T = \mathbb{Q}_T^{H_\varphi}$.

For any $\omega \in \Omega_{T^\star}$ and edge $e \in T$, the event $\psi_T^{-1}\{\omega^e, \omega_e\} \in \mathscr{T}_e \subset \mathscr{F}_T$. As $\mathcal{L}_e^\star = \psi_T^{-1}(\mathcal{L}_e)$ and $\varphi \in \mathscr{R}_T^\star$ satisfies the characterization (3.1) we have

$$\frac{\varphi_T(\omega^e)}{\varphi_T(\omega_e)} = \frac{p}{1-p} q^{\mathbb{1}_{\mathcal{L}_e}(\omega)-1}. \tag{3.21}$$

Thus we need only check the finite conditional probabilities (2.23) for edges $e^\star \in \partial T$.

Choose an edge $e \in \Lambda T$ and fix $\omega \in \Omega_{T^\star}$ such that $\omega(e) = 0$ and $\omega(e^\star) = 1$. As the edges e and $e^\star$ are in series we have

$$\mathbb{1}_{\mathcal{L}_{e^\star}}(\omega^e) = \mathbb{1}_{\mathcal{L}_e}(\omega) \tag{3.22}$$

$$\mathbb{1}_{\mathcal{L}_{e^\star}}(\omega) = \mathbb{1}_{\mathcal{L}_e}(\omega_{e^\star}) = 0. \tag{3.23}$$

Now as φ is a Markov chain and $\omega(e) = 0$ we have

$$\frac{\varphi_T(\omega)}{\varphi_T(\omega_{e^\star})} = \frac{\pi\big(\mathrm{H}_\varphi(e)\big)}{1 - \pi\big(\mathrm{H}_\varphi(e)\big)} = \frac{\mathrm{H}_\varphi(e)}{\big(1 - \mathrm{H}_\varphi(e)\big) q} \tag{3.24}$$

where $\mathrm{H}_\varphi(e) = \pi^{-1}\big(\varphi\big(e \downarrow_\infty \big| J_e^{\complement}\big)\big)$ as in the statement of the theorem.

Now as $e \in T$ we have from (3.21) that

$$\frac{\varphi_T(\omega^e)}{\varphi_T(\omega)} = \frac{p}{1 - p} q^{\mathbb{1}_{\mathcal{L}_e}(\omega) - 1}, \tag{3.25}$$

$$\frac{\varphi_T(\omega_{e^\star}^e)}{\varphi_T(\omega_{e^\star})} = \frac{p}{(1 - p) q}. \tag{3.26}$$

Taking the product of (3.24), (3.25) and (3.26) we see

$$\frac{\varphi_T(\omega^e)}{\varphi_T(\omega_{e^\star}^e)} = \frac{\varphi_T(\omega^e)}{\varphi_T(\omega)} \frac{\varphi_T(\omega)}{\varphi_T(\omega_{e^\star})} \frac{\varphi_T(\omega_{e^\star})}{\varphi_T(\omega_{e^\star}^e)} \tag{3.27}$$

$$= \frac{\mathrm{H}_\varphi(e)}{1 - \mathrm{H}_\varphi(e)} q^{\mathbb{1}_{\mathcal{L}_e}(\omega) - 1} \tag{3.28}$$

$$= \frac{\mathrm{H}_\varphi(e)}{1 - \mathrm{H}_\varphi(e)} q^{\mathbb{1}_{\mathcal{L}_{e^\star}}(\omega^e) - 1} \tag{3.29}$$

Therefore we have confirmed the finite conditional bond probabilities of (2.23) for $e^\star$ in both ω and ω^e and by Theorem 2.7 we have $\psi_T(\varphi) = \mathbb{Q}_T^{\mathrm{H}_\varphi}$ for arbitrary $T \in \mathbb{T}$. Furthermore we have $\mathbb{Q}_S^{\mathrm{H}_\varphi} = \psi_S(\varphi) = \psi_{S,T}\psi_T(\varphi) = \psi_{S,T}(\mathbb{Q}_T^{\mathrm{H}_\varphi})$ and so H_φ is an entrance law by Lemma 3.6. ❏

Now, if φ_1 and $\varphi_2 \in \mathscr{R}_{\mathcal{T}}^\star$ are two Markov chains with $\mathrm{H}_{\varphi_1} = \mathrm{H}_{\varphi_2}$ then for any finite subtree $T \in \mathbb{T}$ and $\mathscr{F}_T$-measurable event A we must have $\varphi_1(A) = \varphi_2(A)$ by Theorem 3.8 and as $\mathbb{T}$ exhausts $\mathcal{T}$ we must have $\varphi_1 = \varphi_2$ by Carathéodory's extension theorem.

Conversely we have seen that not all entrance laws may be obtained in this way from random cluster models, The constant entrance law $\mathrm{H} \equiv 0$ is a counterexample if $\mathbb{P}_{\pi(T)}$ is supercritical. Next we show that the property of robustness is a necessary and

sufficient property for an entrance law to give rise to a random cluster model.

THEOREM 3.9

If $H : \overline{\mathbf{E}} \to [0,1]$ is an entrance law there exists a unique Markov chain $\mathbb{Q}_{\mathcal{T}}^{\mathrm{H}}$ such that for every $T \in \mathbb{T}$ and $\mathscr{F}_T$-measurable random variable X we have $\mathbb{Q}_{\mathcal{T}}^{\mathrm{H}}(X) = \mathbb{Q}_T^{\mathrm{H}}(X)$

Furthermore $\mathbb{Q}_{\mathcal{T}}^{\mathrm{H}} \in \mathscr{R}_{\mathcal{T}}^{\star}$ if and only if H is robust.

PROOF

First from Lemma 3.6, for any $S \subset T \in \mathbb{T}$ and $\mathscr{F}_S$-measurable X we have $\mathbb{Q}_T^{\mathrm{H}}(X) = \mathbb{Q}_S^{\mathrm{H}}(X)$. As $\Omega_{\mathcal{T}}$ is compact we may set $\mathbb{Q}_{\mathcal{T}}^{\mathrm{H}}$ to be the weak limit as $T \uparrow \mathcal{T}$. For any $e \in T$ the σ-algebras $\mathscr{F}_{A_T(e)}$ and $\mathscr{F}_{D_T(e)}$ are independent under $\left(\mathbb{Q}_T^{\mathrm{H}} \,\big|\, J_e^{\complement}\right)$ and hence under $\left(\mathbb{Q}_{\mathcal{T}}^{\mathrm{H}} \,\big|\, J_e^{\complement}\right)$. By the martingale convergence theorem $\mathscr{F}_{A_T(e)}$ and $\mathscr{F}_{D_T(e)}$ are independent under $\left(\mathbb{Q}_{\mathcal{T}}^{\mathrm{H}} \,\big|\, J_e^{\complement}\right)$ and so $\mathbb{Q}_{\mathcal{T}}^{\mathrm{H}}$ is a Markov chain.

Notice that for general entrance laws we have claimed only that $\mathbb{Q}_{\mathcal{T}}^{\mathrm{H}}(X) = \mathbb{Q}_T^{\mathrm{H}}(X)$ for $\mathscr{F}_T$-measurable X. We have not claimed that $\psi_T(\mathbb{Q}_{\mathcal{T}}^{\mathrm{H}}) = \mathbb{Q}_T^{\mathrm{H}}$. From Theorem 3.8 if $\mathbb{Q}_{\mathcal{T}}^{\mathrm{H}} \in \mathscr{R}_{\mathcal{T}}^{\star}$ then $\psi_T(\mathbb{Q}_{\mathcal{T}}^{\mathrm{H}})$ is a random cluster model on $T^{\star}$ for every $T \in \mathbb{T}$. Furthermore if $\psi_T(\mathbb{Q}_{\mathcal{T}}^{\mathrm{H}})$ is a random cluster model on $T^{\star}$ then $\mathbb{Q}_{\mathcal{T}}^{\mathrm{H}}$ must satisfy (3.1). Hence $\mathbb{Q}_{\mathcal{T}}^{\mathrm{H}} \in \mathscr{R}_{\mathcal{T}}^{\star}$ if and only if $\psi_T(\mathbb{Q}_{\mathcal{T}}^{\mathrm{H}}) = \mathbb{Q}_T^{\mathrm{H}}$.

We show that $\psi_T(\mathbb{Q}_{\mathcal{T}}^{\mathrm{H}}) = \mathbb{Q}_T^{\mathrm{H}}$ if and only if H is robust by constructing a large coupling of the set of measures $\{\mathbb{Q}_T^{\mathrm{H}} : T \in \mathbb{T}\}$.

Let Ω be the product space $\prod_{T \in \mathbb{T}} \Omega_{T^{\star}}$. We consider an element $\omega \in \Omega$ as a set of random elements $\{\omega_T \in \Omega_{T^{\star}} : T \in \mathbb{T}\}$. We say ω is ψ-coherent if $\omega_S = \psi_{S,T}(\omega_T)$ for every $S \subset T \in \mathbb{T}$ and let $\Psi \subset \Omega$ be the event that ω is ψ-coherent. If ω is ψ-coherent then we may set $\omega_{\mathcal{T}} \in \Omega_{\mathcal{T}}$ to be the configuration such that $\omega_{\mathcal{T}}(e) = \omega_T(e)$ whenever $e \in T$. We say ω is *robust* if ω is ψ-coherent and $\omega_T = \psi_T(\omega_{\mathcal{T}})$ for every $T \in \mathbb{T}$.

If $\mathscr{G}$ is a σ-algebra of events in $\Omega_{\mathcal{T}}$, we interpret $\mathscr{G}$ as the σ-algebra of events in Ω generated by events expressible in the form $\Psi \cap [\omega_{\mathcal{T}} \in A]$ for $A \in \mathscr{G}$.

For $T \in \mathbb{T}$ set $\Omega_T = \prod_{\substack{S \in \mathbb{T} \\ S \subseteq T}} \Omega_S$ (with typical element ω_T). For any $\omega \in \Omega_{T^\star}$ there exists exactly one ψ-coherent $\omega_T \in \Omega_T$ with $\omega_T = \omega$. Let μ_T be the distribution on Ω_T such that ω is ψ-coherent almost surely and ω_T is distributed according to $\mathbb{Q}_T^H$.

If $\omega_T \sim \mu_T$ then, as H is an entrance law, for any $S \subset T$, $\omega_S \sim \mathbb{Q}_S^H$. Hence μ_S is the push forwards measure of μ_T under the natural projection $\Omega_T \to \Omega_S$. Therefore as $\mathbb{T}$ is countable and each Ω_T is trivially Polish, by Daniell's theorem on the existence of random sequences, [45, Theorem 6.14], there exists a unique measure μ on Ω such that ω is almost surely ψ-coherent and ω_T is distributed as $\mathbb{Q}_T^H$ for every $T \in \mathbb{T}$. Hence if $\omega \sim \mu$ we have $\omega_T \sim \mathbb{Q}_T^H$.

If ω is robust μ-almost surely then $\psi_T(\omega_T) = \omega_T \sim \mathbb{Q}_T^H$, hence $\mathbb{Q}_T^H$ satisfies (3.1) and $\mathbb{Q}_T^H \in \mathscr{R}_T^\star$.

Conversely as ω is μ-almost surely ψ-coherent, then if $\omega_T \in [e \downarrow_{v_\infty}] \subset \Omega_{T^\star}$ for some, and hence every, $T \in \mathbb{T}$ we must have $\omega_T \in [e \downarrow_\infty]$. Therefore, for μ-almost every ω, $\omega_T \in \mathcal{L}_e$ only if $\omega_T \in \mathcal{L}_e^\star$.

Hence if $\mathbb{Q}_T^H$ is a random cluster model we have

$$\mathbb{Q}_T^H(J_e) = \pi(\gamma(e)) + (\gamma(e) - \pi(\gamma(e))) \, \mu(\omega_T \in \mathcal{L}_e^\star) \qquad (3.30)$$
$$\mathbb{Q}_T^H(J_e) = \pi(\gamma(e)) + (\gamma(e) - \pi(\gamma(e))) \, \mu(\omega_T \in \mathcal{L}_e). \qquad (3.31)$$

Therefore if $q \neq 1$ we must have $\mu[\omega_T \in \mathcal{L}_e^\star] = \mu[\omega_T \in \mathcal{L}_e]$ and so, for μ-almost every ω, $\omega_T \in \mathcal{L}_e$ if and only if $\omega_T \in \mathcal{L}_e^\star$.

That is $\mathbb{Q}_T^H$ is a random cluster model if and only if ω is μ-almost surely robust.

To complete the proof we show that ω is μ-almost surely robust if and only if H is robust.

Consider the distribution $\left(\mathbb{Q}_{\chi(e)}^H \, \big| \, v \nleftrightarrow v_\infty \right)$ for $e = |u, v\rangle$. Each $f \in \chi(e)$ is open independently with probability $\frac{\pi(\gamma(e))(1 - \pi(H(e)))}{1 - \pi(\gamma(e))\pi(H(e))}$, and if f is open $f^\star$ is closed. Using the iterative construction of Remark 3.7 we see that $\mu\left(e \downarrow_\infty \, \big| \, J_{e^\star}^{\complement} \right) = \overline{\mathbb{P}}_H(e \downarrow_\infty)$.

Now, suppose $\overline{\mathbb{P}}_H$ is supercritical. Recall from Remark 3.1 that we are considering only those trees with $\gamma(e)$ bounded strictly below 1, then $\overline{\mathbb{P}}_H(J_e \,|e \downarrow_\infty) \leq \pi(\gamma(e))$ is bounded strictly below 1 also and the $(\overline{\mathbb{P}}_H)$ probability that there is exactly one open path from v to infinity is zero.

Hence if H is not robust then with positive probability we may find some edge e with $\omega_\mathcal{T} \in \mathcal{L}_e^\star$ but $\omega_T \notin \mathcal{L}_e$. Therefore $\mathbb{Q}_\mathcal{T}^H \in \mathscr{R}_\mathcal{T}^\star$ if and only if H is robust. ❏

3.3 ADDITIONAL RESULTS FOR GENERAL TREES

To summarise, there is a one to one correspondence between the set $\mathscr{E}_\mathcal{T}^\star$ of extremal random cluster measures and some subset of the robust entrance laws. Therefore $\mathscr{R}_\mathcal{T}^\star$ is nonempty if and only if there exists a robust entrance law and $\mathscr{R}_\mathcal{T}^\star$ is a singleton if and only if there exists a unique robust entrance law.

We are unable to create a complete picture of the set $\mathscr{R}_{\mathcal{T},q}^\star$ for general trees. On a general tree the question of whether a given entrance law is robust is non trivial. In Section 3.4 we restrict our attention to homogeneously weighted regular trees where the question of robustness is resolved in the upcoming Lemma 3.14.

In Chapter 4 we present results for the random cluster model on a tree under more general boundary conditions, we will concentrate on regular trees, partially for simplicity and partially because we will rely on results in this chapter which we may only prove on the homogeneous tree.

For now we will present a few results that may be proved on the general tree.

We have already seen that all extremal random cluster models may be constructed as weak limits of the cylinder measures $\mathbb{Q}_\mathcal{T}^\xi$

for boundary conditions ξ for some, but not in general every $\xi \in \Omega_{\mathcal{T}}$.

On the tree, we may take an alternative approach, by considering sequences of functions that converge to entrance laws.

For a function $\Theta : \overline{\mathbf{E}} \to [0,1]$ recall the random cluster measures $\mathbb{Q}_T^\Theta$ on $\Omega_{T^\star}$ formed by choosing the edge weights of ∂T according to Θ. We begin by making an observation that the random cluster models $\mathbb{Q}_T^\Theta$ are well behaved with respect to the product topology on the set of functions $\overline{\mathbf{E}} \to [0,1]$.

LEMMA 3.10

If Θ_n is a sequence of functions $\overline{\mathbf{E}} \to [0,1]$ such that for each $e \in \overline{\mathbf{E}}$, $\Theta_n \to H$ as $n \to \infty$ for some entrance law H then for every $T \in \mathbb{T}$ the measure $\mathbb{Q}_T^{\Theta_n} \to \psi_T(\mathbb{Q}_{\mathcal{T}}^H)$.

PROOF

From Theorem 3.9 $\mathbb{Q}_{\mathcal{T}}^H$ is a Markov chain with $\psi_T(\mathbb{Q}_{\mathcal{T}}^H) = \mathbb{Q}_T^H$. As $T^\star$ is a finite graph the probabilities $\mathbb{Q}_T^{\Theta_n}(\omega)$ may be expressed as a continuous function of the edge weights. Hence $\mathbb{Q}_T^{\Theta_n}(\omega) \to \mathbb{Q}_T^H(\omega)$ for every $\omega \in \Omega_{T^\star}$. $\qquad\qquad\square$

Notice that for $\xi \in \Omega_{\mathcal{T}}$ we may set $\Theta^\xi(e) \equiv \mathbb{1}_{[e\downarrow\infty]}(\xi)$ and the measures $\mathbb{Q}_T^\Theta$ and $\mathbb{Q}_T^{\Theta^\xi}$ agree on $\mathscr{F}_T$.

Recall that to specify an entrance law H it is enough to fix the rooted entrance law H_ρ for any choice of root vertex ρ. For a general function Θ there is no such restriction. Therefore it will be convenient to fix a root ρ and restrict our attention to subtrees $\mathbb{T}_\rho = \{T \in \mathbb{T} : \rho \in T\}$ and consider the restriction $\Theta_\rho : \mathbf{E}_\rho \to [0,1]$. As for any $S \in \mathbb{T}$ we may choose $T \in \mathbb{T}_\rho$ with $S \subset T$ we will incur no loss of generality from such a restriction.

Given a function $\Theta : \mathbf{E}_\rho \to [0,1]$ and some subtree $T \in \mathbb{T}_\rho$ set $\Theta_T : \overline{\mathbf{E}}_\rho \to [0,1]$ to be the function that agrees with Θ for every $e \in \mathcal{T}_\rho \setminus T$ and satisfies (3.13) for every $e \in T$. Notice that if

$\Theta_{\mathcal{T}} = \lim_{T\uparrow\infty}\Theta_T$, exists it must be an entrance law by continuity of (3.13).

DEFINITION 3.11

We say $\Theta : \mathbf{E}_\rho \to [0,1]$ is a *sub-entrance law* if $\Theta(e) \geq 1 - \prod_{f\in\chi(e)}(1 - \pi^{-1}(1 - \pi(\gamma(f))\pi(\Theta(f))))$ and a *super-entrance law* if $\Theta(e) \leq 1 - \prod_{f\in\chi(e)}(1 - \pi^{-1}(1 - \pi(\gamma(f))\pi(\Theta(f))))$ (cf. equation (3.13)). We say Θ is robust if it the bond percolation $\overline{\mathbb{P}}_\Theta$ from (3.14) is subcritical.

Notice that if $\mathcal{T}$ is a regular tree then any constant function $\Theta(e) \equiv \theta$ is either a sub-entrance law or a super-entrance law.

If Θ is a sub (respectively super)-entrance law then $\Theta_T(e)$ is increasing (respectively decreasing) in T. Therefore the (edgewise) limit, $\Theta_{\mathcal{T}} = \lim_{T\uparrow\infty}\Theta_T$, exists and must be an entrance law by continuity of (3.13).

The bond probabilities $\overline{\mathbb{P}}_\Theta(e)$ are decreasing in $\Theta(e)$, Therefore if Θ is a robust sub-entrance law then $\Theta_{\mathcal{T}}$ is robust also.

LEMMA 3.12

If $\xi \in \Omega_{\mathcal{T}}$ is such that $\overline{\mathbb{Q}}_T^\xi \xrightarrow{w} \varphi$ as $T \uparrow \mathcal{T}$ for some measure φ on $\Omega_{\mathcal{T}}$ then φ is a Markov chain and for every $S \in \mathbb{T}$ $\psi_S\left(\overline{\mathbb{Q}}_T^\xi\right) \xrightarrow{w} \psi_S(\varphi)$ as $T \uparrow \mathcal{T}$.

PROOF

Given $\xi \in \Omega_{\mathcal{T}}$ set $\Theta^\xi(e) \equiv \mathbb{1}_{[e\downarrow\infty]}(\xi)$ it is easy to see that the measures $\mathbb{Q}_T^{\Theta^\xi}$ and $\overline{\mathbb{Q}}_T^\xi$ agree on $\mathscr{F}_T$.

Now if $\Theta_T^\xi \to \Theta_{\mathcal{T}}^\xi$ as $T \uparrow \mathcal{T}$ then from Lemma 3.10 $\overline{\mathbb{Q}}_T^\xi$ converges weakly to the Markov chain $\mathbb{Q}_{\mathcal{T}}^{\Theta_{\mathcal{T}}^\xi}$.

If Θ_T^ξ does not converge then by compactness we may choose sequences T_n and T_n' such that $\Theta_{T_n}^\xi \to H$ and $\Theta_{T_n'}^\xi \to H'$ as $n \to \infty$ for distinct entrance laws H and H'. Hence $\overline{\mathbb{Q}}_{T_n}^\xi \xrightarrow{w} \mathbb{Q}_{\mathcal{T}}^H$ and $\overline{\mathbb{Q}}_{T_n'}^\xi \xrightarrow{w} \mathbb{Q}_{\mathcal{T}}^{H'}$ with $\mathbb{Q}_{\mathcal{T}}^H \neq \mathbb{Q}_{\mathcal{T}}^{H'}$ from Theorem 3.9. □

Let $\overline{\Theta} : \overline{\mathbf{E}}(\mathcal{T}) \to [0,1]$ be the constant function $\overline{\theta}(e) \equiv 1$, Then $\overline{\Theta}$ is clearly a super entrance-law and we may set

$$\overline{\mathsf{H}}_{\mathcal{T}} = \lim_{T \uparrow \mathcal{T}} \overline{\Theta}_{\mathcal{T}}. \tag{3.32}$$

Then $\overline{\mathsf{H}}_{\mathcal{T}}$ is an entrance law and for any entrance law H we must have $\mathsf{H} \leq \overline{\mathsf{H}}_{\mathcal{T}}$ by monotonicity of (3.13). Furthermore $\mathbb{Q}_T^{\overline{\mathsf{H}}} = \overline{\mathbb{Q}}_T^{\xi_1}$ for every $T \in \mathbb{T}$. Therefore if $q \geq 1$ then $\mathbb{Q}_{\mathcal{T}}^{\overline{\mathsf{H}}} = \overline{\mathbb{Q}}_{\mathcal{T},q}$ is a random cluster model hence $\overline{H}$ is robust.

If $q < 1$ the maximal entrance law $\overline{H}$ still exists. However for the general case we may not show that $\overline{H}$ is robust. For the regular homogeneous tree it is always the case that $\overline{H}$ is robust, we will prove this in Section 3.4. This fact is also proved in Häggström [38] although not explicitly.

We will leave open the question of whether there exists a tree $\mathcal{T}$ and some $q < 1$ such that $\overline{H}$ is not robust and $\mathscr{R}_{\mathcal{T},q}^{\star} = \varnothing$. In Chapter 4, Example 4.13 we will see an example of a boundary condition on a tree that admits no random cluster model for a particular value of $q < 1$.

However when $q < 1$ we may prove that there exists at most one random cluster model on a given tree.

THEOREM 3.13

If $q < 1$ then either $\mathscr{R}_{\mathcal{T}}^{\star} = \{\overline{\mathbb{Q}}_{\mathcal{T}}\}$ or it is empty. If $\mathbb{P}_{\mathcal{T}}$ is supercritical then $\mathscr{R}_{\mathcal{T}}^{\star} = \{\overline{\mathbb{Q}}_{\mathcal{T}}\}$ and $\overline{\mathbb{Q}}_{\mathcal{T}} \neq \mathbb{P}_{\pi(\mathcal{T})}$. If $\mathbb{P}_{\pi(\mathcal{T})}$ is subcritical then $\mathscr{R}_{\mathcal{T}}^{\star} = \{\overline{\mathbb{Q}}_{\mathcal{T}}\}$ and $\overline{\mathbb{Q}}_{\mathcal{T}} = \mathbb{P}_{\pi(\mathcal{T})}$.

PROOF

From above we need only show that there can exist at most one random cluster model. We may assume without loss of generality that $\overline{H}$ is robust. For if not then $\mathscr{R}_{\mathcal{T}}^{\star}$ is empty and there is nothing to prove.

Consider a multigraph with two edges, weighted as p_1 and p_2. For $q < 1$ we may sample from the random cluster model as follows. First choose edges open independently with probabilities

$\pi(p_1)$ and $\pi(p_2)$. If at least one edge is closed accept the configuration, if both edges are open accept the sample with probability q and reject with probability $1 - q$. If the sample is rejected redraw in the same manner rejecting as many times as necessary. It is easy to check that this procedure generates a random cluster model on the two bond multigraph.

For two graphs $\mathcal{G}_1 \ni u_1, v_1$ and $\mathcal{G}_2 \ni u_2, v_2$ set $\mathcal{N} = \frac{\mathcal{G}_1 \sqcup \mathcal{G}_2}{u_1 \sim u_2, v_1 \sim v_2}$ as in Theorem 2.23. Then we may sample from $\mathbb{Q}_{\mathcal{N}}$ in the same way, by choosing configurations independently according to $\mathbb{Q}_{\mathcal{G}_1}$ and $\mathbb{Q}_{\mathcal{G}_2}$ and rejecting the configuration with probability q if both $u_1 \longleftrightarrow v_1$ in $\mathcal{G}_1$ and $u_2 \longleftrightarrow v_2$ in $\mathcal{G}_2$.

Next, consider a directed edge $e = |u, v\rangle \in T \in \mathbb{T}$ and let H be any entrance law. Recall the measures $\mathbb{Q}_T^{\mathrm{H}}$ and $\overline{\mathbb{Q}}_T^1$ on Ω_T and define random cluster measures $\mathbb{Q}_{D_T(e)}^{\mathrm{H}}$ and $\overline{\mathbb{Q}}_{D_T(e)}^1$ on $\Omega_{D_T^\star(e)}$ analogously.

Define $[e \downarrow_{\partial T}] = \bigcup_{f \in \Lambda T} [v \longleftrightarrow v_\infty]_e^f$ as the event that there is an open directed path from v to v_f for some $\langle v_f, v_\infty \rangle \in \partial D_T(e)$. We claim that for any entrance law H we have

$$\mathbb{Q}_{D_T(e)}^{\mathrm{H}}[e \downarrow_{\partial T}] \geq \overline{\mathbb{Q}}_{D_T(e)}^1[e \downarrow_{v_\infty}] \geq \mathbb{Q}_{D_T(e)}^{\mathrm{H}}[e \downarrow_{v_\infty}] = \mathrm{H}(e). \quad (3.33)$$

For the right hand inequality we already have $\overline{\mathbb{Q}}_{D_T(e)}^1[e \downarrow_{v_\infty}] = \overline{\Theta}_T(e) \geq \mathrm{H}(e)$. To prove the left hand inequality we use induction on the depth of $D_T(e)$ (taken to be the graph theoretic distance between v and v_∞) and the number of children $|\chi(e)|$. If the depth of $D_T(e)$ is 1 and $|\chi(e)| = 1$ then $D_T^\star(e)$ is a tree containing two edges and the inequality is obvious.

Suppose that (3.33) holds whenever the depth of $D_T(e)$ is at most n. Consider some $D_T(e)$ of depth $n + 1$ with $\chi(e) = \{f\}$. Then $D_T(e) = D_T(f) \cup f$ and any random cluster measure on $D_T^\star(e)$ is a product of a random cluster model of $D_T^\star(f)$ and a Bernoulli($\pi(p)$) random variable $\omega(f)$. So (3.33) is satisfied.

Next suppose that (3.33) holds whenever the depth of $D_T(e)$ is at most n and $|\chi(e)|$ is at most d and consider $D_T(e)$ of depth n with $|\chi(e)| = d + 1$. Choose $f \in \chi(e)$ and set $\mathcal{G}_1 = D_T^\star(f) \cup f$

and $\mathcal{G}_2 = D_T^\star(e) \setminus \mathcal{G}_1$. Let $\mathbb{Q}_{\mathcal{G}_i}^{\mathrm{H}}$ and $\bar{\mathbb{Q}}_{\mathcal{G}_i}^1$ be random cluster models on $\mathcal{G}_i$ as above for $i \in \{1,2\}$. As $\mathcal{G}_1$ and $\mathcal{G}_2$ are both edge like subgraphs of $D_T(e)$ we may construct a coupling of $\mathbb{Q}_{D_T(e)}^{\mathrm{H}}$ and $\bar{\mathbb{Q}}_{D_T(e)}^1$ using the rejection algorithm above.

By inductive assumption we have

$$\mathbb{Q}_{\mathcal{G}_i}^{\mathrm{H}}[e \downarrow_{\partial T}] \geq \bar{\mathbb{Q}}_{\mathcal{G}_i}^1[e \downarrow_{v_\infty}] \geq \mathbb{Q}_{\mathcal{G}_i}^{\mathrm{H}}[e \downarrow_{v_\infty}]. \tag{3.34}$$

Couple $\mathbb{Q}_{\mathcal{G}_i}^{\mathrm{H}}$ and $\bar{\mathbb{Q}}_{\mathcal{G}_i}^1$ by selecting $(\omega_i, \bar{\omega}_i)$ in such a way that ω_i is distributed as $\mathbb{Q}_{\mathcal{G}_i}^{\mathrm{H}}$, $\bar{\omega}_i$ is distributed as $\bar{\mathbb{Q}}_{\mathcal{G}_i}^1$ and $(v \xleftrightarrow{\omega_i} v_\infty) \Rightarrow (v \xleftrightarrow{\bar{\omega}_i} v_\infty) \Rightarrow (v \xleftrightarrow{\bar{\omega}_i} \partial \mathcal{G}_i)$ almost surely.

Now choose coupled pairs $(\omega_1, \bar{\omega}_1)$, $(\omega_2, \bar{\omega}_2)$ independently as above. If we have no more than one of $v \xleftrightarrow{\omega_1} v_\infty$ and $v \xleftrightarrow{\omega_2} v_\infty$ then accept the sample. If both $v \xleftrightarrow{\omega_1} v_\infty$ and $v \xleftrightarrow{\omega_2} v_\infty$ then reject the sample with probability $1 - q$.

If both $v \xleftrightarrow{\bar{\omega}_1} v_\infty$ and $v \xleftrightarrow{\bar{\omega}_2} v_\infty$ but not both $v \xleftrightarrow{\omega_1} v_\infty$ and $v \xleftrightarrow{\omega_2} v_\infty$ then with probability q accept both (ω_1, ω_2) and $(\bar{\omega}_1, \bar{\omega}_2)$, and with probability $1 - q$ accept (ω_1, ω_2), but reject $(\bar{\omega}_1, \bar{\omega}_2)$. Let $\omega = (\omega_1, \omega_2)$ and $\bar{\omega} = (\bar{\omega}_1, \bar{\omega}_2) \in \Omega_{D_T^\star(e)}$ be the configurations produced.

If both samples are accepted at the same time then by construction we have $[v \xleftrightarrow{\bar{\omega}} v_\infty]$ only if $[v \xleftrightarrow{\omega} \partial \mathcal{G}_i]$. If not, and $(\bar{\omega}_1, \bar{\omega}_2)$ is rejected while $\omega = (\omega_1, \omega_2)$ is accepted then $[v \xleftrightarrow{\omega} \partial \mathcal{G}_i]$.

Therefore $v \xleftrightarrow{\bar{\omega}} v_\infty$ only if $v \xleftrightarrow{\omega} \partial \mathcal{G}_i$ and we see that

$$\mathbb{Q}_{D_T(e)}^{\mathrm{H}}[e \downarrow_{\partial T}] \geq \bar{\mathbb{Q}}_{D_T(e)}^1[e \downarrow_{v_\infty}]. \tag{3.35}$$

Hence by induction the inequality (3.33) holds for all pairs $e \in T \in \mathbb{T}$.

Now let $T \uparrow \mathcal{T}$. If H is robust then $\mathbb{Q}_{D_T(e)}^{\mathrm{H}}[e \downarrow_{\partial T}]$ decreases to $\mathrm{H}(e)$ and so $\mathrm{H}(e) = \lim_{T \uparrow \mathcal{T}} \bar{\mathbb{Q}}_{D_T(e)}^1[e \downarrow_{v_\infty}] = \bar{\mathrm{H}}(e)$. Therefore there can exist at most one robust entrance law. $\qquad\square$

3.4   ON REGULAR TREES

For the remainder of this chapter we concentrate on regular trees with isotropic edge weights. For $1 < k \in \mathbb{N}$ and $\tau \in (0,1)$ we fix $\mathcal{T}$ to be the tree such that every vertex has degree $k+1$ and every edge has weight $\gamma(e) = \tau$.

As $\mathcal{T}$ is translation invariant it is natural to consider the constant entrance laws $H(e) \equiv \eta$. Recall equation (3.13) in the definition of an entrance law. To describe the constant entrance laws we may rewrite the right hand side of (3.13) as a function.

Set

$$F_{\tau,q,k}(\theta) = 1 - \left(1 - \pi^{-1}(\pi(\tau)\pi(\theta))\right)^k \qquad (3.36)$$
$$= 1 - \left(1 - \frac{\tau\theta}{1 + (1-\tau)(1-\theta)(q-1)}\right)^k$$

Then the constant function η is an entrance law if and only if η is a fixed point of $F_{\tau,q,k}$.

On the general tree $\mathcal{T}$ the question of robustness is a hurdle to our understanding of the set $\mathscr{R}_{\mathcal{T},q}$. For the regular tree the situation is simpler, the specification of an entrance law and the condition of robustness each reduce to a single equation.

LEMMA 3.14

Every non-zero constant entrance law on a regular tree is robust.

PROOF

Recall that η is a constant entrance law if and only if

$$1 - \eta = \left(1 - \pi^{-1}(\pi(\tau)\pi(\eta))\right)^k. \qquad (3.37)$$

Rearranging we have

$$\pi(\tau) = \frac{\pi\left(1 - (1-\eta)^{1/k}\right)}{\pi(\eta)} \qquad (3.38)$$
$$\leq \frac{\pi\left(\frac{\eta}{k}\right)}{\pi(\eta)} \qquad (3.39)$$

by Bernoulli's inequality.

Conversely if η is not robust then

$$\frac{1}{k} < \frac{\pi(\tau)\,(1-\pi(\eta))}{1-\pi(\tau)\pi(\eta)} \tag{3.40}$$

$$\Leftrightarrow \pi(\tau) \geq \left(k\,(1-\pi(\eta)) - \pi(\eta)\right)^{-1}. \tag{3.41}$$

From the definition (2.24), $\pi(\eta) = \frac{\eta}{\eta + (1-\eta)q}$, so we have

$$\frac{\pi\left(\frac{\eta}{k}\right)}{\pi(\eta)} = \frac{\eta + (1-\eta)\,q}{\eta + (k-\eta)\,q} \tag{3.42}$$

$$(k\,(1-\pi(\eta)) - \pi(\eta)) = \frac{k\,(1-\eta)\,q - \eta}{\eta + (1-\eta)\,q}. \tag{3.43}$$

Combining (3.39) with (3.41), if η is a constant entrance law, but not robust we have

$$k\,(1-\eta)\,q - \eta \geq \eta + (k-\eta)\,q \tag{3.44}$$

$$\Leftrightarrow \qquad 0 \geq ((k-1)\,q + 2)\,\eta \tag{3.45}$$

Therefore as $k \geq 1$, any strictly positive constant entrance law is robust. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

It remains to identify the fixed points of the function (3.36). Recall from Chapter 1 that Häggström [38] constructs translation invariant random cluster models from the roots of a particular equation (1.35). Although Häggström's derivation of equation (1.35) differs from our approach — which is similar to that of Grimmett [35, §10.10] — we may rearrange (3.36) to show that the fixed points of (3.36) correspond exactly to the roots of (1.35).

THEOREM 3.15: HÄGGSTRÖM

If $q \leq 2$ then there exists $\tau_u = \frac{q}{q+k-1}$ such that

- If $\tau \leq \tau_u$ then $F_{\tau,q,k}(\theta) < \theta$ for all $\theta > 1$.

- If $\tau > \tau_u$ there exists a unique fixed point $\bar{\eta} = F_{\tau,q,k}(\bar{\eta}) \in (0,1)$ attractive in the domain $(0,1)$.

If $q > 2$ then there exist $\tau_c < \tau_u = \frac{q}{q+k-1}$ such that

- If $\tau < \tau_c$ then $F_{\tau,q,k}(\theta) < \theta$ for all $\theta > 1$.

- If $\tau = \tau_c$ then there exists a unique fixed point $\eta = F_{\tau,q,k}(\eta) \in (0,1)$ with $F_{\tau,q,k}(\theta) < \theta$ for $\theta \in (0,\eta) \cup (\eta,1)$.

- If $\tau_c < \tau < \tau_u$ then there exist fixed points $0 < \eta < \bar{\eta} < 1$ with $F_{\tau,q,k}(\theta) > \theta$ for $\theta \in (\eta,\bar{\eta})$ and $F_{\tau,q,k}(\theta) < \theta$ for $\theta \in (0,\eta) \cup (\bar{\eta},1)$.

- If $\tau \geq \tau_u$ there exists a unique fixed point $\bar{\eta} = F_{\tau,q,k}(\bar{\eta}) \in (0,1)$ attractive in the domain $(0,1)$.

Furthermore if $k = 2$ we may express $\tau_c = \frac{2\sqrt{q-1}}{1+2\sqrt{q-1}}$.

Theorem 3.15 is summarized by the graphs in Figure 8.

PROOF

A full analysis of Equation 1.35 appears in [38] and we will not repeat it here. Instead set $\eta = 1 - x^k$, then

$$\eta \leq F_{\tau,q,k}(\eta) \tag{3.46}$$

$$\Leftrightarrow \quad x \leq \frac{\tau\left(1 - x^k\right)}{1 + (1-\tau)(q-1)x^k} \tag{3.47}$$

$$\Leftrightarrow \quad 0 \geq (q-1)x^{k+1} + \left(1 - \tfrac{\tau}{1-\tau} - q\right)x^k + \left(\tfrac{\tau}{1-\tau} + 1\right)x - 1 \tag{3.48}$$

with equality iff $\eta = F_{\tau,q,k}(\eta)$. The conclusions of Theorem 3.15 are stated in [38] in terms of the polynomial (3.48). ❑

THEOREM 3.16

If $q \leq 2$ then $\mathscr{R}^\star_{\mathcal{T}_k,q}$ is a singleton with $\mathscr{R}^\star_{\mathcal{T}_k,q} = \{\mathbb{P}_{\pi(\mathcal{T}_k)}\}$ if and only if $\tau \leq \tau_c$. If $q > 2$ then $\mathscr{R}^\star_{\mathcal{T}_k,q}$ is a singleton if and only if $\tau < \tau_c$, in which case $\mathscr{R}^\star_{\mathcal{T}_k,q} = \{\mathbb{P}_{\pi(\mathcal{T}_k)}\}$; or $\tau > \tau_u$, in which case $\mathscr{R}^\star_{\mathcal{T}_k,q} \neq \{\mathbb{P}_{\pi(\mathcal{T}_k)}\}$. If $\tau \in [\tau_c,\tau_u]$ then there exist uncountably many extremal random cluster models.

FIGURE 8
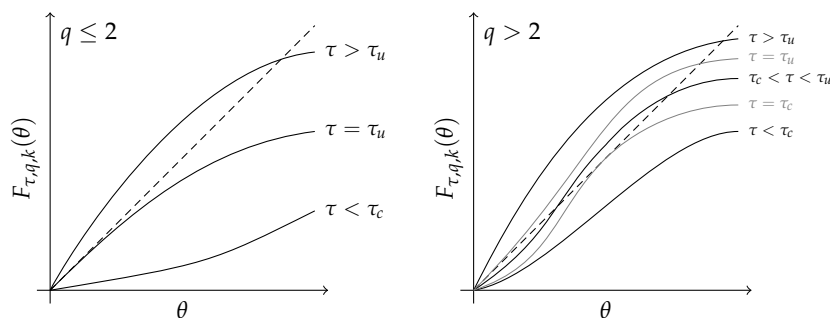
Sketch graphs of the function $F_{\tau,q,k}$ for differing values of τ. The behaviour of the function $F_{\tau,q,k}$ is qualitatively different in the two intervals $0 < q \leq 2$ and $q > 2$.

REMARK 3.17

Before we prove this theorem we remind the reader that the behaviour of $\mathscr{R}^{\star}_{\mathcal{T}_k,q}$ is described in Häggström [38] for $\tau \leq \tau_u$. It remains to prove the uniqueness of the random cluster model for $\tau > \tau_u$, Conjecture 1.9 of [38].

PROOF

Set $\overline{\eta} = \max\left\{\theta \in [0,1] : \theta = F_{\tau,q,k}(\theta)\right\}$, then $\overline{\eta} = \lim_{n \to \infty} F^n_{\tau,q,k}(1)$ and the maximal entrance law $\overline{H}(e) \equiv \overline{\eta}$. Now, if $\overline{\eta} = 0$, that is if $q \leq 2$ and $\tau \leq \tau_c = \tau_u$ or $q > 2$ and $\tau < \tau_c$, then the zero function is the only entrance law. Furthermore as $\tau_c \leq \tau_u = \frac{q}{q+k-1} = \pi_q^{-1}\left(\frac{1}{k}\right)$ and so the zero entrance law is robust. Therefore $\mathscr{R}^{\star}_{\mathcal{T}_k,q} = \left\{\mathbb{P}_{\pi(\mathcal{T}_k)}\right\}$.

If $\overline{\eta} > 0$ but $\tau \leq \tau_u$ then there exist at least two extremal entrance laws. In Section 5 of [38] Häggström constructs a continuum of random cluster models by combining the maximal and minimal random cluster models. We refer the reader to [38] for details as we do not rely on this fact for any further results.

If $\tau > \tau_u$ then $\overline{\eta} > 0$ is a robust entrance law from Lemma 3.14, and the zero function is not robust as $\pi(\tau) > \frac{1}{k}$.

If $q \leq 1$ then from Theorem 3.13 $\overline{H}(e) \equiv \begin{cases} 0 & : \text{if } \tau \leq \tau_u \\ \overline{\eta} & : \text{if } \tau > \tau_u \end{cases}$ is the only robust entrance law for $\mathcal{T}_k$ with cluster factor q. Hence the conclusions of the theorem hold for $q < 1$.

For $q > 1$ choose some $e \in \mathcal{T}_k$ and set

$$\gamma = \pi^{-1}\left(\inf_{\varphi \in \mathscr{E}^{\star}_{\mathcal{T}_k,q}} \varphi\big(e \downarrow_\infty \,\big|\, J_e^{\complement}\big) \right) \geq \mathbb{P}_{\pi(\tau)}[e \downarrow_\infty] > 0. \qquad (3.49)$$

As the set of entrance laws is closed we may choose some entrance law H with $H(e) = \gamma$. As $\mathcal{T}_k$ is translation invariant then for any $f \in \chi(e)$ we have $H(f) \geq \gamma$. Hence, by monotonicity of equation (3.13) in the definition of an entrance law, we have $\gamma = H(e) \geq F_{\tau,q,k}(\gamma) > 0$ and so we must have $\gamma \geq \overline{\eta}$.

Thus for any extremal random cluster model φ we have $\overline{\eta} \geq H_\varphi(e) \geq \gamma \geq \overline{\eta}$ and so there exists a unique robust entrance law $\overline{H}(e) \equiv \overline{\eta}$. $\qquad\square$

4 | RANDOM CONNECTIONS AND BOUNDARY CONDITIONS

The random cluster model on a tree with general boundary conditions was studied by Grimmett and Janson [36], however the construction of the model in that paper contains an error. In this chapter we provide an alternative construction of boundary conditions by defining a *random connection*.

A random connection $\leftrightarrow$ is a set of events $\{u \leftrightarrow v : u, v \in V\}$ that generalizes the usual connection events $\{u \longleftrightarrow v : u, v \in V\}$. We introduce axioms for a random connection that allow us to repeat the conventional construction of the random cluster model on an infinite graph that we saw in Chapter 2. As a random connection is a set of events we may study its properties under Bernoulli percolation.

We associate two critical probabilities with a random connection $\leftrightarrow$. The first $\tau_c^{\leftrightarrow}$ marks the appearence of infinite loops, the second $\tau_u^{\leftrightarrow}$ the onset of a single cluster. We extend results of [38] and Chapter 3 to general random connections to show that for $q > 1$, if $p < \tau_c^{\leftrightarrow}$ or $\pi_q(p) > \tau_u^{\leftrightarrow}$ the random cluster model is unique. If in addition $\leftrightarrow$ satisfies an extra measurability condition, we may show that for sufficiently large q there exists an open interval in which there exists a continuum of $\leftrightarrow$-random cluster models.

We have seen two possible definitions of the random cluster model. In both cases we characterize the random cluster through a Gibbs specification where conditional probabilities are given in terms of a *loop event*; $\mathcal{L}_e$ or $\mathcal{L}_e^\star$ respectively. In turn both loop events may be defined in terms of a "connection rule." Recall from equations (2.20) and (2.40) that for an edge $e = \langle u, v \rangle$ we may express

$$\mathcal{L}_e = [u \longleftrightarrow v]_e\,, \qquad (4.1)$$

$$\mathcal{L}_e^\star = [u \Longleftrightarrow v]_e\,. \qquad (4.2)$$

Informally we may interpret the infinity wired random cluster model as the random cluster model where clusters may be "connected at infinity". In [36] Gimmett and Janson consider more general boundary conditions. Their method is to define the *boundary* of a tree as the set of rays – half infinite paths on the tree, where two rays are considered equivalent if they differ on only a finite number of edges. The authors then consider equivalence relations on the set of rays and define random cluster models where two clusters are connected at infinity if they contain equivalent open rays.

Unfortunately the formal definition of the random cluster model in [36] contains an error. The random cluster model is defined in terms of a Gibbs specification that in general may not be consistent.

Here we take a formal approach to the definition of the general random cluster model. In Chapter 1 we considered a third set of events $\{[u \longleftrightarrow v] : u, v \in \mathbf{V}(\mathcal{Q})\}$ and suggested that we define a random cluster model using the arrow $\longleftrightarrow$ in place of either $\longleftrightarrow$ of $\Longleftrightarrow$. Our first aim is to identify key properties shared by the events $\{u \longleftrightarrow v : u, v \in \mathbf{V}(\mathcal{N})\}$ and $\{u \Longleftrightarrow v : u, v \in \mathbf{V}(\mathcal{N})\}$ used in the construction of the random cluster model on a network $\mathcal{N}$.

We may then repeat the construction for a general *random connection* $\longleftrightarrow$, to be interpreted as a connection rule that may replace either $\longleftrightarrow$ or $\Longleftrightarrow$.

4.1 RANDOM CONNECTIONS

Say a *random relation* $\sim$ on a network $\mathcal{N}$ is a set of events

$$\sim = \left\{ [u \sim v] : u, v \in \mathbf{V}(\mathcal{N}) \right\} \subset \mathscr{F}_{\mathcal{N}}. \qquad (4.3)$$

For each $\omega \in \Omega$ there is a relation

$$\overset{\omega}{\sim} = \left\{ (u, v) \in \mathbf{V}(\mathcal{N})^2 \,\middle|\, \omega \in [u \sim v] \right\}. \qquad (4.4)$$

We say $\overset{\omega}{\sim}$ is a *realization* of $\sim$.

We allow the usual terminology for binary relations (see for example [12]) to carry over to random relations. For example we say a random relation $\sim$ is reflexive if every realization of $\sim$ is reflexive, or we say $\sim$ is a random equivalence relation if $\sim$ is reflexive, symmetric and transitive. (That is if every realization of $\sim$ is an equivalence relation.)

We will define a Gibbs specification on a tree in terms of a formal object, a *random connection*. A random connection $\leftrightarrow\!\!\!\!\dashrightarrow$ will be a random equivalence relation with similar properties to the connections $\longleftrightarrow$ and $\Longleftrightarrow$ that we have already seen. The random cluster model may then be defined following the main steps in Chapter 2.

We will define a random connection, and the associated random cluster model on a general network as there is no advantage to restricting the definition to a tree at this stage. Later however we restrict our analysis to homogeneous regular trees and will not attempt to adapt our results to a general network.

DEFINITION 4.1

We say a random equivalence relation on a network $\mathcal{N}$ is a *random connection* if it satisfies the following three conditions.

I $\leftrightarrow\!\!\!\!\dashrightarrow$ extends $\longleftrightarrow$ through boundary connections:

$$[u \leftrightarrow\!\!\!\!\dashrightarrow v] \subseteq [u \Longleftrightarrow v]$$
$$= [u \longleftrightarrow v] \cup \left([u \longleftrightarrow \infty] \cap [v \longleftrightarrow \infty] \right)$$

II $[u \longleftrightarrow v]$ is an increasing event for every $u, v \in \mathbf{V}(\mathcal{N})$.

III For an edge $e = \langle \tilde{u}, \tilde{v} \rangle$

$$[u \longleftrightarrow v]^e = [u \longleftrightarrow v]_e \cup [u \longleftrightarrow \tilde{u}, \tilde{v} \longleftrightarrow v]_e$$
$$\cup \, [u \longleftrightarrow \tilde{v}, \tilde{u} \longleftrightarrow v]_e$$

for every $u, v \in \mathbf{V}(\mathcal{N})$.

For a random connection $\longleftrightarrow$ associate a *loop event*

$$\mathcal{L}_e^{\longleftrightarrow} = [u \longleftrightarrow v]_e \tag{4.5}$$

with each edge $e = \langle u, v \rangle \in \mathbf{E}(\mathcal{N})$.

Notice that both the free connection $\{[u \longleftrightarrow v] : u, v \in \mathbf{V}(\mathcal{N})\}$ and the wired connection $\{[u \Longleftrightarrow v] : u, v \in \mathbf{V}(\mathcal{N})\}$ satisfy the definition of a random connection. Our aim is to repeat the construction of the free and wired random cluster models, substituting a general random connection $\longleftrightarrow$ in place of either $\longleftrightarrow$ or $\Longleftrightarrow$. As a random connection is always a random equivalence relation we may count the number of clusters that intersect a finite subnetwork of $\mathcal{N}$ and define a Gibbs specification for a general random connection in the same way as for the free and wired random cluster models. Axioms I–III represent key properties shared by $\longleftrightarrow$ and $\Longleftrightarrow$ used in the construction.

Axiom I ensures that the behaviour of a random connection respects the local structure of the network.

Axiom II allows us to use monotonicity arguments when $q \geq 1$. In particular we will see in Theorem 4.5 below that the set of random cluster models we define is non-empty for $q \geq 1$. For $q < 1$ it is not the case that every random connection admits a random cluster model. Example 4.13 provides a counterexample.

Axiom III is a technical condition that allows us to specify single bond conditional probabilities in terms of the loop event $\mathcal{L}_e^{\longleftrightarrow}$. In particular the effect of opening a single bond $\langle u, v \rangle$ is to alter the number of equivalence classes by at most one. Lemma 4.3 below plays the role of Theorem 2.7 for random cluster models defined in terms of a random connection.

DEFINITION 4.2

For a network $\mathcal{N}$, a random connection $\longleftrightarrow$ on $\mathcal{N}$ and a subnetwork $\mathcal{G} \in \mathbb{G}_{\mathcal{N}}$ let $\kappa_{\mathcal{G}}^{\leftrightarrow}(\omega)$ be the number of equivalence classes of $\overset{\omega}{\longleftrightarrow}$ that intersect $\mathbf{V}(\mathcal{G})$. Define the $\longleftrightarrow$ random cluster measure $\mathbb{Q}_{\mathcal{G},q}^{\overset{\xi}{\longleftrightarrow}}$ to be the probability measure concentrated on $\Omega_{\mathcal{G}}^{\xi}$ with

$$\mathbb{Q}_{\mathcal{G},q}^{\overset{\xi}{\longleftrightarrow}}(\omega) = Z_{\mathcal{G},\xi}^{-1} \; \mathbb{1}_{\Omega_{\mathcal{G}}^{\xi}}(\omega) \left(\prod_{e \in \mathbf{E}(\mathcal{G})} \left(\frac{\gamma(e)}{1-\gamma(e)} \right)^{\omega(e)} \right) q^{\kappa_{\mathcal{G}}^{\leftrightarrow}(\omega)}$$

(4.6)

$$\text{where} \quad Z_{\mathcal{G},\xi} = \sum_{\omega \in \Omega_{\mathcal{G}}^{\xi}} \left(\prod_{e \in \mathbf{E}(\mathcal{G})} \left(\frac{\gamma(e)}{1-\gamma(e)} \right)^{\omega(e)} \right) q^{\kappa_{\mathcal{G}}^{\leftrightarrow}(\omega)}.$$

(4.7)

And let

$$\mathscr{R}_{\mathcal{N},q}^{\longleftrightarrow} = \bigcap_{\mathcal{G} \in \mathbb{G}_{\mathcal{N}}} \left\{ \mu \; \middle| \; (\mu \mid \mathscr{T}_{\mathcal{G}})(\xi) = \mathbb{Q}_{\mathcal{G},q}^{\overset{\xi}{\longleftrightarrow}} \right\}$$

(4.8)

be the set of $\longleftrightarrow$ random cluster models.

Now we check that the Gibbs specification above is consistent and that the set $\mathscr{R}_{\mathcal{N},q}^{\longleftrightarrow}$ is non-empty.

LEMMA 4.3

$\mathbb{Q}_{\mathcal{G},q}^{\overset{\xi}{\longleftrightarrow}}$ is the unique probability measure concentrated on $\Omega_{\mathcal{G}}^{\xi}$ that satisfies the single bond conditional specification

$$\mathbb{Q}_{\mathcal{G},q}^{\overset{\xi}{\longleftrightarrow}}(J_e \mid \mathscr{T}_e)(\omega) = \begin{cases} \gamma(e) & : \text{if } \omega \in \mathcal{L}_e^{\leftrightarrow}, \\ \pi_q(\gamma(e)) & : \text{if } \omega \notin \mathcal{L}_e^{\leftrightarrow}. \end{cases}$$

(4.9)

PROOF

By definition $\mathbb{Q}_{\mathcal{G},q}^{\overset{\xi}{\longleftrightarrow}}$ is concentrated on the finite state space $\Omega_{\mathcal{G}}^{\xi}$. First we show that there may exist at most one such probability measure satisfying (4.9).

As for the random cluster model on the finite graph we may rewrite (4.9) in the equivalent form

$$\frac{\mathbb{Q}_{\mathcal{G},q}^{\xleftarrow{\tilde{\xi}}}(\omega^e)}{\mathbb{Q}_{\mathcal{G},q}^{\xleftarrow{\tilde{\xi}}}(\omega_e)} = \frac{\gamma(e)}{1-\gamma(e)} \, q^{\mathbb{1}_{\mathcal{L}_e^{\leftrightarrow}}(\omega)-1}. \tag{4.10}$$

Therefore if μ and ν are two probability measures satisfying (4.9), and equivalently (4.10), we must have $\frac{\mu(\omega^e)}{\mu(\omega_e)} = \frac{\nu(\omega^e)}{\nu(\omega_e)}$ and therefore $\frac{\mu(\omega^e)}{\nu(\omega^e)} = \frac{\mu(\omega_e)}{\nu(\omega_e)}$. We may extend this by induction to show that $\frac{\mu(\omega)}{\nu(\omega)} = \frac{\mu(\omega')}{\nu(\omega')}$ whenever ω and ω' disagree on only finitely many bonds.

As the subgraph $\mathcal{G}$ is finite and any $\omega, \omega' \in \Omega_{\mathcal{G}}^{\tilde{\xi}}$ agree on $\mathcal{N} \setminus \mathcal{G}$; the ratio $\frac{\mu(\omega)}{\nu(\omega)}$ is constant on the finite state space $\Omega_{\mathcal{G}}^{\tilde{\xi}}$. Thus there may exist at most one probability measure concentrated on $\Omega_{\mathcal{G}}^{\tilde{\xi}}$ that satisfies (4.9).

Now from the definition of $\mathbb{Q}_{\mathcal{G},q}^{\xleftarrow{\tilde{\xi}}}$ we may write

$$\frac{\mathbb{Q}_{\mathcal{G},q}^{\xleftarrow{\tilde{\xi}}}(\omega^e)}{\mathbb{Q}_{\mathcal{G},q}^{\xleftarrow{\tilde{\xi}}}(\omega_e)} = \frac{\gamma(e)}{1-\gamma(e)} \, q^{\kappa_{\mathcal{G}}^{\leftrightarrow}(\omega^e)-\kappa_{\mathcal{G}}^{\leftrightarrow}(\omega_e)}. \tag{4.11}$$

Comparing (4.10) with (4.11) we see that $\mathbb{Q}_{\mathcal{G},q}^{\xleftarrow{\tilde{\xi}}}$ is the unique probability measure satisfying (4.9) if and only if

$$\kappa_{\mathcal{G}}^{\leftrightarrow}(\omega_e) - \kappa_{\mathcal{G}}^{\leftrightarrow}(\omega^e) = 1 - \mathbb{1}_{\mathcal{L}_e^{\leftrightarrow}}(\omega). \tag{4.12}$$

Fix $e = \langle \tilde{u}, \tilde{v} \rangle \in \mathbf{E}(\mathcal{G})$ and choose $u, v \in \mathbf{V}(\mathcal{G})$ arbitrarily. The event $[u \longleftrightarrow v]$ is increasing by Axiom II and so we must have $[u \longleftrightarrow v]^e \supset [u \longleftrightarrow v]_e$. Now suppose $\omega \in [u \longleftrightarrow v]^e \setminus [u \longleftrightarrow v]_e$. From Axiom III we may assume without loss of generality that $\omega \in [u \longleftrightarrow \tilde{u}, \tilde{v} \longleftrightarrow v]_e$.

From the definition $\mathcal{L}_e^{\leftrightarrow} = [\tilde{u} \longleftrightarrow \tilde{v}]_e$ and as any random connection is also a random equivalence relation we must have $\mathcal{L}_e^{\leftrightarrow} \cap [u \longleftrightarrow \tilde{u}, \tilde{v} \longleftrightarrow v]_e \subset [u \longleftrightarrow v]_e$.

In particular we have $[u \leftrightarrow v]^e \cap \mathcal{L}_e^{\leftrightarrow} = [u \leftrightarrow v]_e \cap \mathcal{L}_e^{\leftrightarrow}$. There-fore, as u and v are arbitrary, we have $\kappa_{\mathcal{G}}^{\leftrightarrow}(\omega^e) = \kappa_{\mathcal{G}}^{\leftrightarrow}(\omega_e)$ for every $\omega \in \mathcal{L}_e^{\leftrightarrow}$.

Conversely if $\omega \notin \mathcal{L}_e^{\leftrightarrow}$ then $\tilde{u}$ and $\tilde{v}$ are in distinct $\xleftrightarrow{\omega_e}$ clusters, but are in the same $\xleftrightarrow{\omega^e}$ cluster, and so $\kappa_{\mathcal{G}}^{\leftrightarrow}(\omega_e) - \kappa_{\mathcal{G}}^{\leftrightarrow}(\omega^e) \geq 1$. However any $\xleftrightarrow{\omega_e}$ cluster that contains neither $\tilde{u}$ nor $\tilde{v}$ is also a $\xleftrightarrow{\omega^e}$ cluster by Axiom III. Hence $\kappa_{\mathcal{G}}^{\leftrightarrow}(\omega_e) - \kappa_{\mathcal{G}}^{\leftrightarrow}(\omega^e) \leq 1$ as well.

Therefore $\kappa_{\mathcal{G}}^{\leftrightarrow}(\omega_e) - \kappa_{\mathcal{G}}^{\leftrightarrow}(\omega^e) = 1 - \mathbb{1}_{\mathcal{L}_e^{\leftrightarrow}}(\omega)$ as required and we are done. $\qquad\square$

COROLLARY 4.4

1. The measures $\left\{ \mathbb{Q}_{\mathcal{G},q}^{\xleftrightarrow{\xi}} \,\middle|\, \mathcal{G} \in \mathbb{G}_{\mathcal{N}} \right\}$ form a consistent Gibbs spec-ification. That is for $\mathcal{G}' \subset \mathcal{G} \in \mathbb{G}_{\mathcal{N}}$ and $\omega \in \Omega_{\mathcal{G}}^{\xi}$

$$\left(\mathbb{Q}_{\mathcal{G},q}^{\xleftrightarrow{\xi}} \,\middle|\, \mathscr{T}_{\mathcal{G}'} \right)(\omega) = \mathbb{Q}_{\mathcal{G}',q}^{\xleftrightarrow{\omega}}. \tag{4.13}$$

2. If $q \geq 1$ then the measure $\mathbb{Q}_{\mathcal{G},q}^{\xleftrightarrow{\xi}}$ satisfies the FKG inequality.

3. If $\xi \geq \xi' \in \Omega_{\mathcal{N}}$, $q, q' \geq 1$ and γ, γ' are two weightings of $\mathcal{N}$ with $\gamma(e) \geq \gamma'(e)$ and $\pi_q(\gamma(e)) \geq \pi_{q'}(\gamma'(e))$ for every $e \in \mathbf{E}(\mathcal{N})$ then for every $\mathcal{G} \in \mathbb{G}_{\mathcal{N}}$

$$\mathbb{Q}_{\mathcal{G}_\gamma,q}^{\xleftrightarrow{\xi}} \succ \mathbb{Q}_{\mathcal{G}_{\gamma'},q'}^{\xleftrightarrow{\xi'}}. \tag{4.14}$$

PROOF

For the first statement fix configurations $\xi \in \Omega_{\mathcal{N}}$ and $\omega \in \Omega_{\mathcal{G}}^{\xi}$, and set $\mu = \left(\mathbb{Q}_{\mathcal{G},q}^{\xleftrightarrow{\xi}} \,\middle|\, \mathscr{T}_{\mathcal{G}'} \right)(\omega)$. By Lemma 4.3 we need to show only that μ satisfies (4.9). That is, it is enough to check that $\mu(J_e \,|\, \mathscr{T}_e) = \mathbb{Q}_{\mathcal{G}',q}^{\xleftrightarrow{\omega}}(J_e \,|\, \mathscr{T}_e)$ for arbitrary $e \in \mathbf{E}(\mathcal{G}')$.

From the definition of μ, for any $\mathscr{T}_e$-measurable X we have

$$\mu(\mathbb{1}_{J_e} \cdot X) = \mathrm{Q}_{\mathcal{G},q}^{\xleftrightarrow{\xi}}(\mathbb{1}_{J_e} \cdot X \,|\, \mathscr{T}_{\mathcal{G}'})(\omega) \tag{4.15}$$

$$= \mathrm{Q}_{\mathcal{G},q}^{\xleftrightarrow{\xi}}\left(\mathrm{Q}_{\mathcal{G},q}^{\xleftrightarrow{\xi}}(J_e \,|\, \mathscr{T}_e) \cdot X \,\Big|\, \mathscr{T}_{\mathcal{G}'}\right)(\omega) \tag{4.16}$$

$$= \mathrm{Q}_{\mathcal{G},q}^{\xleftrightarrow{\xi}}\left(\mathrm{Q}_{\mathcal{G}',q}^{\xleftrightarrow{\omega}}(J_e \,|\, \mathscr{T}_e) \cdot X \,\Big|\, \mathscr{T}_{\mathcal{G}'}\right)(\omega) \tag{4.17}$$

$$= \mu\left(\mathrm{Q}_{\mathcal{G}',q}^{\xleftrightarrow{\omega}}(J_e \,|\, \mathscr{T}_e) \cdot X\right). \tag{4.18}$$

The final two statements follow from the Markov chain argument of Theorem 2.7. We need only note that $\mathcal{L}_e^{\xleftrightarrow{}}$ is increasing by Axiom II and so

$$\mathrm{Q}_{\mathcal{G}_\gamma,q}^{\xleftrightarrow{\xi}}(J_e \,|\, \mathscr{T}_e)(\omega) \geq \mathrm{Q}_{\mathcal{G}_{\gamma'},q}^{\xleftrightarrow{\xi'}}(J_e \,|\, \mathscr{T}_e)(\omega') \tag{4.19}$$

for $\omega \in \Omega_{\mathcal{G}}^{\xi}$ and $\omega' \in \Omega_{\mathcal{G}}^{\xi'}$ with $\omega \geq \omega'$, and q, q', γ, γ' satisfying the conditions of part 3. ❑

For the free and wired random cluster models, with $q \geq 1$ we were able to obtain at least one random cluster model as a weak limit $\mathrm{wlim}_{\mathcal{G}\uparrow\mathcal{N}} \mathrm{Q}_{\mathcal{G},q}^{\xi_0} \in \mathscr{R}_{\mathcal{N},q}$ or $\mathrm{wlim}_{\mathcal{G}\uparrow\mathcal{N}} \overline{\mathrm{Q}}_{\mathcal{G},q}^{\xi_1} \in \mathscr{R}_{\mathcal{N},q}^{\star}$. The proof of this relies on the respective left and right continuity of the indicators $\{\mathbb{1}_{\mathcal{L}_e} : e \in \mathcal{N}\}$ and $\{\mathbb{1}_{\mathcal{L}_e^\star} : e \in \mathcal{N}\}$. However in general the loop events $\mathcal{L}_e^{\xleftrightarrow{}}$ may be neither left nor right continuous and for a typical random connection $\xleftrightarrow{}$ the limits $\mathrm{wlim}_{\mathcal{G}\uparrow\mathcal{N}} \mathrm{Q}_{\mathcal{G},q}^{\xleftrightarrow{\xi_0}}$ and $\mathrm{wlim}_{\mathcal{G}\uparrow\mathcal{N}} \mathrm{Q}_{\mathcal{G},q}^{\xleftrightarrow{\xi_1}}$ may not be random cluster measures.

Our next aim is to show that although this direct construction may fail, there still exists both a minimal and a maximal random cluster model for any random connection.

THEOREM 4.5

Let $\xleftrightarrow{}$ be a random connection on a network $\mathcal{N}$ and choose $q \geq 1$. There exist measures $\mathrm{Q}_{\mathcal{N},q}^{\xleftrightarrow{}}$ and $\overline{\mathrm{Q}}_{\mathcal{N},q}^{\xleftrightarrow{}} \in \mathscr{R}_{\mathcal{N},q}^{\xleftrightarrow{}}$ with the property that

$$\mathrm{Q}_{\mathcal{N},q}^{\xleftrightarrow{}} \prec \varphi \prec \overline{\mathrm{Q}}_{\mathcal{N},q}^{\xleftrightarrow{}} \tag{4.20}$$

for every $\varphi \in \mathscr{R}_{\mathcal{N},q}^{\xleftrightarrow{}}$.

We will only prove the existence of the minimal random cluster model. The proof for the maximal random cluster model is identical but with the stochastic ordering reversed.

Rather than consider the probability measures $\mathbb{Q}_{\mathcal{G},q}^{\overleftrightarrow{\xi}}$ for fixed ξ we integrate the function $\xi \mapsto \mathbb{Q}_{\mathcal{G},q}^{\overleftrightarrow{\xi}}$ with respect to a probability measure $\mu \in \mathscr{P}_{\mathcal{N}}$. We may identify a set of probability measures for which the resulting map $\mathcal{G} \mapsto \int \mathbb{Q}_{\mathcal{G},q}^{\overleftrightarrow{\xi}} d\mu(\xi)$ is monotonic for $\mu \in \mathcal{I}$.

LEMMA 4.6

For every $\mathcal{G} \in \mathbb{G}_{\mathcal{N}}$ there exists an operator $M_{\mathcal{G}} : \mathscr{P}_{\mathcal{N}} \to \mathscr{P}_{\mathcal{N}}$ with

$$M_{\mathcal{G}}\mu(X) = \int \mathbb{Q}_{\mathcal{G},q}^{\overleftrightarrow{\xi}}(X) d\mu(\xi) \qquad (4.21)$$

for every bounded $\mathscr{F}_{\mathcal{N}}$-measurable random variable X.

Let $\mathcal{I} = \mathcal{I}_{\mathcal{N},q}^{\overleftrightarrow{}} \subset \mathscr{P}_{\mathcal{N}}$ be the set of measures such that $M_{\mathcal{G}}\mu \succ \mu$ for every $\mathcal{G} \in \mathbb{G}_{\mathcal{N}}$. Then $\mathcal{I}$ is nonempty, it is closed under weak limits of increasing sequences and there is a well defined *increasing* operator $M : \mathcal{I} \to \mathcal{I}$ such that

$$M_{\mathcal{G}}\mu \uparrow M\mu \quad \text{as } \mathcal{G} \uparrow \mathcal{N} \qquad (4.22)$$

and $M\mu \succ \mu$ for every $\mu \in \mathcal{I}$.

PROOF

First for $\mu \in \mathscr{P}_{\mathcal{N}}$ and $\mathcal{G} \in \mathbb{G}_{\mathcal{N}}$ the right hand side of (4.21) is linear in X and is countably additive on indicator functions by the monotone convergence theorem. Therefore there is a well defined measure $M_{\mathcal{G}}\mu$ with expectation given by (4.21).

Set δ_0 to be the probability measure concentrated on ξ_0, that is $\delta_0(J_e) = 0$ for every $e \in \mathbf{E}(\mathcal{N})$. Every probability measure on $\Omega_{\mathcal{N}}$ dominates δ_0 so trivially we have $\mu_0 \in \mathcal{I}$. In particular the set $\mathcal{I}$ is nonempty.

For a fixed *increasing* continuous simple function X the random variable $\xi \mapsto \mathbb{Q}_{\mathcal{G},q}^{\overleftrightarrow{\xi}}(X)$ is increasing and $\mathscr{T}_{\mathcal{G}}$-measurable by Corollary 4.4. Therefore if $\mu \succ \nu$ we have $M_{\mathcal{G}}\mu(X) \geq M_{\mathcal{G}}\nu(X)$ by the

definition of stochastic domination. Therefore $M_{\mathcal{G}}\mu \succ M_{\mathcal{G}}\nu$ by Lemma 2.12.

Consider an increasing sequence of probability measures $\mu_n \in \mathcal{I}$ with $\mu_n \uparrow \mu$. Then $\mu \succ \mu_n$ for every $n \in \mathbb{N}$ and for $\mathcal{G} \in \mathbb{G}_N$ we have $M_{\mathcal{G}}\mu \succ M_{\mathcal{G}}\mu_n \succ \mu_n$. Therefore $M_{\mathcal{G}}\mu \succ \mu$ by Theorem 2.13.

Next consider $\mathcal{G}' \subset \mathcal{G} \in \mathbb{G}_N$. From Corollary 4.4 for every $\xi \in \Omega_N$ and $\omega \in \Omega_{\mathcal{G}}^{\xi}$ we have $\left(\mathbb{Q}_{\mathcal{G},q}^{\overleftrightarrow{\xi}}\,\middle|\,\mathscr{T}_{\mathcal{G}'}\right)(\omega) = \mathbb{Q}_{\mathcal{G}',q}^{\overleftrightarrow{\omega}}$.

Noting that $\mathbb{Q}_{\mathcal{G},q}^{\overleftrightarrow{\xi}}$ is concentrated on $\Omega_{\mathcal{G}}^{\xi}$; for every $\mu \in \mathscr{P}_N$ and bounded $\mathscr{F}_N$-measurable X we have

$$M_{\mathcal{G}'} \cdot M_{\mathcal{G}}\mu(X) = \int \mathbb{Q}_{\mathcal{G}',q}^{\overleftrightarrow{\omega}}(X)dM_{\mathcal{G}}\mu(\omega) \tag{4.23}$$

$$= \iint \mathbb{Q}_{\mathcal{G}',q}^{\overleftrightarrow{\omega}}(X)d\mathbb{Q}_{\mathcal{G},q}^{\overleftrightarrow{\xi}}(\omega)d\mu(\xi) \tag{4.24}$$

$$= \iint \mathbb{Q}_{\mathcal{G},q}^{\overleftrightarrow{\xi}}(X\,|\,\mathscr{T}_{\mathcal{G}'})(\omega)d\mathbb{Q}_{\mathcal{G},q}^{\overleftrightarrow{\xi}}(\omega)d\mu(\xi) \tag{4.25}$$

$$= \int \mathbb{Q}_{\mathcal{G},q}^{\overleftrightarrow{\xi}}(X)d\mu(\xi) \tag{4.26}$$

$$= M_{\mathcal{G}}(X). \tag{4.27}$$

So if $\mu \in \mathcal{I}$ we have $M_{\mathcal{G}'}\mu \prec M_{\mathcal{G}'} \cdot M_{\mathcal{G}}\mu = M_{\mathcal{G}}\mu$. Therefore for some increasing sequence of finite graphs $\mathcal{G}_n \uparrow N$ we may set $M\mu = \operatorname{wlim}_{n\to\infty} M_{\mathcal{G}_n}\mu$. It is easy to see that this limit is independent of the choice of subsequence hence $M_{\mathcal{G}}\mu \uparrow M\mu$ as $\mathcal{G} \uparrow N$ for every $\mu \in \mathcal{I}$. In addition for every $\mathcal{G} \in \mathbb{G}_N$ and $\mu \in \mathcal{I}$ we have $M\mu \succ M_{\mathcal{G}}\mu \succ \mu$.

It remains to show that $M\mu \in \mathcal{I}$ for all $\mu \in \mathcal{I}$.

Fix $\mu \in \mathcal{I}$, $\mathcal{G} \in \mathbb{G}_N$ and some *increasing* continuous simple function X. Consider a sequence of finite subgraphs $\mathcal{G} \subset \mathcal{G}_n \uparrow N$; then as X is continuous and increasing we have $M_{\mathcal{G}_n}\mu(X) \uparrow M\mu(X)$ as $n \to \infty$.

So for every $\varepsilon > 0$ we may choose $n \in \mathbb{N}$ such that

$$M\mu(X) - \varepsilon \le M_{\mathcal{G}_n}\mu(X) \tag{4.28}$$

$$= M_{\mathcal{G}} \cdot M_{\mathcal{G}_n}\mu(X) \tag{4.29}$$

$$\le M_{\mathcal{G}} \cdot M\mu(X) \tag{4.30}$$

Letting $\varepsilon \to 0$ we have $M\mu(X) \leq M_{\mathcal{G}} \cdot M\mu(X)$ for every increasing continuous simple function X, hence by Lemma 2.12 we have $M\mu(X) \prec M_{\mathcal{G}} \cdot M\mu(X)$ and so, as $\mathcal{G} \in \mathbb{G}_N$ is arbitrary, we have $M\mu \in \mathcal{I}$ for every $\mu \in \mathcal{I}$. $\qquad \square$

The first step in proving Theorem 4.5 will be to show that $\mathcal{I}$ contains a fixed point.

Recall that a *chain* is a totally ordered subset of a partially ordered set. We say a partially ordered set $(\mathcal{X}, \geq)$ is *chain complete* if for every nonempty chain $\mathcal{C} \subset \mathcal{X}$ there exists a least upper bound of $\mathcal{C}$ in $\mathcal{X}$.

THEOREM 4.7: BOURBAKI–WITT

If $(\mathcal{X}, \geq)$ is a chain complete partially ordered set and $F : \mathcal{X} \to \mathcal{X}$ is a function such that $F(x) \geq x$ for every $x \in \mathcal{X}$ then $\mathcal{X}$ contains a fixed point of F.

This was proved independently by Bourbaki [11] and Witt [65]. We will not attempt a proof here and refer the interested reader to Lang [48] for details.

We have only claimed that $\mathcal{I}$ is closed under the weak limits of increasing sequences, to satisfy the conditions of Bourbaki-Witt Theorem we must check that it is chain complete.

LEMMA 4.8

Let $\mathcal{X} \subset \mathscr{P}_N$ be a set of probability measures on Ω_N such that $\text{wlim}_{n \to \infty} \mu_n \in \mathcal{X}$ for every *increasing* sequence $\mu_n \in \mathcal{X}$. Then $\mathcal{X}$ is chain complete.

PROOF

Let $\mathcal{X}$ satisfy the conditions of the Lemma and consider a nonempty chain $\mathcal{C} \subset \mathcal{X}$. As $\mathscr{P}_N$ is a compact metrizable space we may nominate a countably dense subset $C = \{c_n : n \in \mathbb{N}\} \subset \mathcal{C}$.

If there exists some $\bar{c} \in C$ with $\bar{c} \succ c_n$ for every $n \in \mathbb{N}$ then $\bar{c}$ is a least upper bound for C.

If there exists no such $\bar{c} \in C$ then as C is also a chain we may choose an increasing subsequence c_{n_i} such that for every $c_m \in C$ there exists i such that $c_{n_i} \succ c_m$. From Theorem 2.13 we may set $\bar{c} = \text{wlim}_{i \to \infty} c_{n_i} \in \mathcal{X}$ to be the least upper bound of $\{c_{n_i} : i \in \mathbb{N}\}$. Then for every $c_m \in C$ we may choose $i \in \mathbb{N}$ with $c_m \prec c_{n_i} \prec \bar{c}$ hence $\bar{c}$ is an upper bound of C and $\{c_{n_i} : i \in \mathbb{N}\} \subset C$ ensures that $\bar{c}$ is the least upper bound of C.

Now C is dense in $\mathcal{C}$ so for any $c \in \mathcal{C}$ we may choose a subsequence $c_{m_i} \overset{w}{\Rightarrow} c$. Furthermore, from Theorem 2.13, there exists some sequence μ_i (not necessarily in $\mathcal{X}$) such that $\mu_i \prec c_{m_i} \prec \bar{c}$ and $\mu_{m_i} \uparrow c$ as $i \to \infty$.

Then c is the least upper bound of the subsequence μ_{m_i} and as $\bar{c}$ is an upper bound for μ_{m_i} we have $\bar{c} \succ c$.

Therefore $\bar{c}$ is an upper bound of $\mathcal{C}$ and as $\bar{c}$ is the least upper bound of $C \subset \mathcal{C}$ it must be the case that $\bar{c} \in \mathcal{X}$ is the least upper bound of $\mathcal{C}$. ❏

We are ready to prove Theorem 4.5

PROOF OF THEOREM 4.5

Recall the set $\mathcal{I}$, we claim that a measure φ is a $\leftrightarrow$ random cluster model if and only if $\varphi \in \mathcal{I}$ and $M\varphi = \varphi$.

Recall that a measure $\varphi \in \mathcal{R}_{N,q}^{\rightarrow}$ if and only if $(\varphi \mid \mathcal{T}_{\mathcal{G}})(\xi) = Q_{\mathcal{G},q}^{\overset{\xi}{\leftrightarrow}}$ for every $\mathcal{G} \in \mathbb{G}_N$ and φ-almost every $\xi \in \Omega_N$. Therefore for any $\mathcal{F}_N$-measurable random variable X and $\mathcal{G} \in \mathbb{G}_N$ we have

$$M_{\mathcal{G}}\varphi(X) = \int Q_{\mathcal{G},q}^{\overset{\xi}{\leftrightarrow}}(X)d\varphi(\xi) \tag{4.31}$$

$$= \int \varphi(X \mid \mathcal{T}_{\mathcal{G}})(\xi)d\varphi(\xi) \tag{4.32}$$

$$= \varphi(X). \tag{4.33}$$

So trivially $\varphi \in \mathcal{I}$ and $M\varphi = \varphi$.

Now if $\varphi \in \mathcal{I}$ is a fixed point of M then for every $\mathcal{G} \in \mathbf{G}_{\mathcal{N}}$ we have

$$\varphi \prec M_{\mathcal{G}}\varphi \prec M\varphi = \varphi. \tag{4.34}$$

So φ is invariant under $M_{\mathcal{G}}$ and for any bounded $\mathscr{F}_{\mathcal{N}}$-measurable X and $\mathscr{T}_{\mathcal{G}}$-measurable Y we have

$$\varphi(XY) = M_{\mathcal{G}}\varphi(XY) \tag{4.35}$$
$$= \int \mathbb{Q}_{\mathcal{G},q}^{\overleftrightarrow{\xi}}(XY)d\varphi(\xi) \tag{4.36}$$
$$= \int \mathbb{Q}_{\mathcal{G},q}^{\overleftrightarrow{\xi}}(X)Y(\xi)d\varphi(\xi) \tag{4.37}$$

as Y is $\mathbb{Q}_{\mathcal{G},q}^{\overleftrightarrow{\xi}}$-almost surely constant.

Therefore $(\varphi \,|\, \mathscr{T}_{\mathcal{G}})(\xi) = \mathbb{Q}_{\mathcal{G},q}^{\overleftrightarrow{\xi}}$ and so $\varphi \in \mathscr{R}_{\mathcal{G},q}^{\overrightarrow{}}$. In particular we have

$$\mathscr{R}_{\mathcal{N},q}^{\overrightarrow{}} = \{\varphi \in \mathcal{I} : M\varphi = \varphi\}. \tag{4.38}$$

Recall the minimal probability measure δ_0 defined by $\delta_0(J_e) = 0$ for every $e \in \mathbf{E}(\mathcal{N})$. From Lemma 4.6 we have $\delta_0 \in \mathcal{I}$ and $\mathcal{I}$ is closed under M. Furthermore from Lemma 4.8 $\mathcal{I}$ is chain complete.

Therefore the triple $(\mathcal{I}, \succ, M)$ satisfies the conditions of Theorem 4.7. Hence $\mathcal{I}$ contains a fixed point of M which must be a random cluster model. It remains to show that $\mathscr{R}_{\mathcal{N},q}^{\overleftrightarrow{}}$ contains a minimal element.

Let $\mathcal{A}$ be the set of subsets of $\mathcal{I}$ which contain δ_0 and are closed under M and under the limits of increasing sequences. It is easy to check that $\mathcal{A}$ is closed under arbitrary intersections. Let $\mathcal{X} \in \mathcal{A}$ be the intersection of every set in $\mathcal{A}$.

The triple $(\mathcal{X}, \succ, M)$ satisfies the conditions of Theorem 4.7 and so we may nominate $\mathbb{Q}_{\mathcal{N},q}^{\overrightarrow{}} \in \mathcal{X}$ to be some fixed point of M.

Therefore $\mathbb{Q}_{\mathcal{N},q}^{\overrightarrow{}} \in \mathscr{R}_{\mathcal{N},q}^{\overrightarrow{}}$ and we claim that $\mathbb{Q}_{\mathcal{N},q}^{\overrightarrow{}}$ is the minimal element of $\mathscr{R}_{\mathcal{N},q}^{\overleftrightarrow{}}$.

To see this let $\varphi \in \mathscr{R}_{\mathcal{N},q}^{\leftrightarrow}$ be a $\leftrightarrow$-random cluster measure and define a set of probability measures $A_\varphi = \{\mu \in \mathcal{I} : \varphi \succ \mu\}$. It is easy to see that $\delta_0 \in A_\varphi$ and that A_φ is closed under the limits of increasing sequences. As $\varphi \in \mathscr{R}_{\mathcal{N},q}^{\leftrightarrow}$ we have $M\varphi = \varphi$ and, as M is increasing, for any $\mu \in A_\varphi$ we have $M\mu \prec M\varphi = \varphi$ and so A_φ is closed under M and $A_\varphi \in \mathcal{A}$.

Therefore $Q_{\mathcal{N},q}^{\leftrightarrow} \in \mathcal{X} \subset A_\varphi$ and $Q_{\mathcal{N},q}^{\leftrightarrow} \prec \varphi$. $\qquad\qquad\square$

4.2 RANDOM CONNECTIONS ON TREES

We have defined a random connection as a formal object and constructed a class of models that we may describe as generalized random cluster measures. Axioms I–III of Definition 4.1 are not motivated by any example. Rather, Definition 4.1 is a list of the assumptions we have used to prove Corollary 4.4.

We have described two random connections so far, the "free connection" $\longleftrightarrow$ and the "wired" connection $\Longleftrightarrow$. In this section we first show that examples of random connections exist intermediate between $\longleftrightarrow$ and $\Longleftrightarrow$.

REMARK 4.9

It is possible to define a random connection on any network by specifying some increasing event $A \in \mathscr{T}_{\mathcal{N}}$ and setting

$$[u \overset{A}{\longleftrightarrow} v] = [u \longleftrightarrow v] \cup (A \cap [u \Longleftrightarrow v]). \tag{4.39}$$

It is easy to check that this satisfies the axioms of a random connection and we may write down the extremal elements of $\mathscr{R}_{\mathcal{N}}^{\leftrightarrow}$ as

$$\mathscr{E}_{\mathcal{N}}^{\leftrightarrow} = \{\varphi \in \mathscr{E}_{\mathcal{N}} : \varphi(A) = 0\} \cup \{\varphi \in \mathscr{E}_{\mathcal{N}}^{\star} : \varphi(A) = 1\}. \tag{4.40}$$

We have not attempted to exclude this class of models from the definition of a random connection, although it would be unsatisfying if they were the only examples of random connections. In this section we describe a method of generating random connections that excludes random connection in the form of (4.39).

One way to specify an equivalence relation is to define a neighbour relation and to consider the strictest equivalence relation that contains the described neighbour relation. We may extend this as a method to generate random equivalence relations from random neighbour relations.

Axiom III of definition 4.1 requires a random connection $\longleftrightarrow$ to have the property that $[u \longleftrightarrow v] \subset [u \dashleftarrow\dashrightarrow v]$ for every $u, v \in \mathbf{V}(\mathcal{T})$. So, given a random neighbour relation $\sim$, we may define the strictest possible random equivalence relation that contains both $\sim$ and $\dashleftarrow\dashrightarrow$. We will use this method to describe all the examples of random connections that we will encounter on the tree.

4.2.1 Random connections from random neighbour relations

DEFINITION 4.10

We say a random neighbour relation $\sim$ on the vertices of a network $\mathcal{N}$ *generates* a random connection $\dashleftarrow\dashrightarrow$ if for every $\omega \in \Omega_\mathcal{N}$ $\overset{\omega}{\dashleftarrow\dashrightarrow}$ is the strictest equivalence relation on $\mathbf{V}(\mathcal{N})$ such that $u \overset{\omega}{\dashleftarrow\dashrightarrow} v$ whenever $u \overset{\omega}{\dashleftarrow\dashrightarrow} v$ or $u \overset{\omega}{\sim} v$.

We say two random neighbour relations $\sim_1$ and $\sim_2$ are *equivalent* if for every $\omega \in \Omega_\mathcal{N}$ and $u, v \in \mathbf{V}(\mathcal{N})$

- If $u \overset{\omega}{\sim}_1 v$ then either $u \overset{\omega}{\dashleftarrow\dashrightarrow} v$ or there exist $u', v' \in \mathbf{V}(\mathcal{N})$ with $u \overset{\omega}{\dashleftarrow\dashrightarrow} u' \overset{\omega}{\sim}_2 v' \overset{\omega}{\dashleftarrow\dashrightarrow} v$.

- If $u \overset{\omega}{\sim}_2 v$ then either $u \overset{\omega}{\dashleftarrow\dashrightarrow} v$ or there exist $u', v' \in \mathbf{V}(\mathcal{N})$ with $u \overset{\omega}{\dashleftarrow\dashrightarrow} u' \overset{\omega}{\sim}_1 v' \overset{\omega}{\dashleftarrow\dashrightarrow} v$.

Of course not every random neighbour relation generates a random connection. In Definition 4.17 below we specify some sufficient conditions for a random neighbour relation to generate a random connection. We will use the symbol $\Downarrow$ for a random neighbour relation on $\mathbf{V}(\mathcal{N})$ that generates a random connection.

Before proceeding we introduce two examples of random connections generated by random neighbour relations in order to familiarize the reader with the objects we are studying.

For now we will leave the reader to convince himself that the random neighbour relations in both Example 4.11 and Example 4.13 generate random connections. In fact we shall see that both are examples of *quasi-boundary conditions* as described in Definition 4.17. Theorem 4.19 below shows that such random neighbour relations always generate a random connection.

EXAMPLE 4.11: GRIMMETT–JANSON CONNECTIONS

Let $\mathcal{T}$ be a tree and name the set $\Pi_{\mathcal{T}}$ of half infinite self avoiding paths in $\mathcal{T}$. For a vertex $v \in \mathbf{V}(\mathcal{T})$ let $\mathcal{R}_v \subset \Pi_{\mathcal{T}}$ be the set of *v-rays*, half infinite self-avoiding paths started at v. Choose a root $\rho \in \mathbf{V}(\mathcal{T})$ arbitrarily and set $\mathcal{R} = \mathcal{R}_\rho$. There exists a map $R : \Pi_{\mathcal{T}} \to \mathcal{R}$ such that for every $\Pi \in \Pi_{\mathcal{T}}$ the two rays Π and $R(\Pi)$ differ on only finitely many edges.

Now let $\sim$ be an equivalence relation on $\mathcal{R}$. Define a set of events $\{[u \Downarrow v] : u, v \in \mathbf{V}(\mathcal{T})\}$ where $[u \Downarrow v]$ is the event that there exist open rays $\Pi_u \in \mathcal{R}_u$ and $\Pi_v \in \mathcal{R}_v$ such that $R(\Pi_u) \sim R(\Pi_v)$.

We say an equivalence relation $\sim$ on $\mathcal{R}$ is *measurable* if

$$\{[u \Downarrow v] : u, v \in \mathbf{V}(\mathcal{T})\} \subset \mathscr{F}_{\mathcal{T}}. \tag{4.41}$$

For an equivalence relation $\sim$ let $\Longleftrightarrow$ be the random connection generated by $\Downarrow$.

As an aid to understanding we may construct simple equiva-
lence relations as follows. Choose a subtree $T \in \mathbb{T}_\rho$ and arbi-
trarily partition the leaves of T by choosing some equivalence
relation $\sim_T$ on ΛT. As $\rho \in T$ each $\Pi \in \mathcal{R}$ passes through exactly
one leaf $e_\Pi \in \Lambda T$. Let $\Pi_1 \sim \Pi_2$ whenever $e_{\Pi_1} \sim_T e_{\Pi_2}$. This class
of equivalence relations forms the *open* equivalence relations de-
scribed in [36] and we will return to them in Chapter 5.

Informally we may think of $[u \Downarrow v]$ as the event that there exists
an infinitely long "indirect" path from u to v that passes through
the boundary of the tree. Extending to an equivalence in the ob-
vious way, $[u \iff v]$ is the event that there exists an indirect path
from u to v that is allowed to pass through the boundary finitely
many times. It is easy to check that this informal definition sat-
isfies the axioms of a random connection.

REMARK 4.12

The idea of constructing a random cluster model based on an
equivalence relation as in Example 4.11 was first proposed in
Grimmett and Janson [36]. Definition 4.17 may be seen as a
generalization of this idea. We have mentioned that there is an
error in [36] on which we will elaborate here. For a finite sub-
tree $T \subset \mathcal{T}$, and $\xi \in \Omega_\mathcal{T}$ let $\xi_T = \bigwedge_{e \in T} \xi_e$ be the configuration
obtained from ξ by switching off all edges in T. Given a ran-
dom connection $\longleftrightarrow$ we may define a graph $\mathcal{G}_{T,\xi}^{\longleftrightarrow}$ by identifying
vertices $u, v \in T$ whenever $\xi_T \in [u \longleftrightarrow v]$.

It may be shown that the cylinder measure $\mathbb{Q}_{\mathcal{T},q}^{\overset{\xi}{\longleftrightarrow}} = \mathbb{Q}_{\mathcal{G}_{T,\xi}^{\longleftrightarrow},q}$, using
the obvious identification $\Omega_T^{\xi} \leftrightarrow \Omega_{\mathcal{G}_{T,\xi}^{\longleftrightarrow}}$.

Grimmett and Janson [36] use this approach to define the ran-
dom cluster model generated by an equivalence relation on $\mathcal{R}$.
However, rather than identifying vertices $u, v \in T$ when $\xi_T \in$
$[u \longleftrightarrow v]$; u and v are identified only when $\xi_T \in [u \Downarrow v]$. For
general measurable equivalence relations the set of cylinder ran-
dom cluster models defined does not form a consistent Gibbs
specification.

For the open boundary conditions as described above this speci-
fication holds as long as we restrict it to large enough trees. For

boundary conditions that are not open the specification cannot be recovered.

Theorem 6.2 of [36] claims that for any closed equivalence relation the limit of the sequence $\lim_{T \uparrow \mathcal{T}} \overset{\sim,\xi_1}{\overleftarrow{Q_{T,q}}} \in \overset{\sim}{\mathcal{R}_{\mathcal{T},q}}$.

We shall see that the random connection $\longleftrightarrow$ described in Chapter 1 may be described in terms of a closed equivalence relation. The statement of [36, Theorem 6.2] is false for this connection and for $\tau \in \left(2^{-d}, \pi^{-1} \left(2^{\frac{1-d}{2}} \right) \right)$ the above limit does not satisfy the appropriate DLR conditions.

EXAMPLE 4.13

Let $\mathcal{T}$ be a regular 3-tree directed with respect to an arbitrary root ρ. For an edge $e = |u, v\rangle$ we say $e \Downarrow_\infty$ or $v \Downarrow_\infty$ if v is the root of some open 2-tree contained within $D(e)$. (Recall a rooted k-tree is a tree such that every vertex of a tree has degree $k + 1$ with the exception of the root which has degree k.)

We may define a random neighbour relation

$$[u \Downarrow v] = [u \Downarrow_\infty] \cap [v \Downarrow_\infty]. \tag{4.42}$$

We claim that $\Downarrow$ generates a random connection $\longleftrightarrow$ such that if $\mathcal{T}$ is weighted homogeneously with $\gamma(e) \equiv \tau$, the set of $\longleftrightarrow$ random cluster measures is empty for sufficiently small τ and q.

Before proving this claim we will establish two facts about the events $[e \Downarrow_\infty]$.

LEMMA 4.14

Choose $p \in \left[\frac{8}{9}, 1 \right)$ and let $\mathbb{P}$ be Bernoulli bond percolation on a regular 3-tree $\mathcal{T}$ with $\mathbb{P}(J_e) \equiv p$. Then $\mathbb{P}[e \Downarrow_\infty] > 0$ for every directed edge e.

PROOF

Notice that the event

$$[e\Downarrow_\infty] = \bigcup_{f,f'\in\chi(e)} J_f \cap [f\Downarrow_\infty] \cap J_{f'} \cap [f'\Downarrow_\infty] \qquad (4.43)$$

Thus from standard branching process theory, $\mathbb{P}[e\Downarrow_\infty] \equiv s$ where $s \in (0,1)$ is the largest solution to

$$s = p^3 s^3 + 3p^2 s^2 (1-p)(1-s). \qquad (4.44)$$

It is easy to check that if $p = \frac{8}{9}$ then $s = \frac{27}{32}$ solves (4.44). ❑

LEMMA 4.15

Let $\mathcal{T}$ be a regular 3-tree with edge weights $\gamma(e) \equiv \tau \leq \frac{1}{3}$ then for every $q > 0$, $\xi \in \Omega_{\mathcal{T}}$ and $e \in \mathbf{E}(\mathcal{T})$ the probability $\overline{\mathbb{Q}}_{T,q}^{\xi}[e\Downarrow_\infty] \to 0$ as $T \uparrow \mathcal{T}$.

PROOF

As $\mathcal{T}$ is homogeneous we may choose e arbitrarily. Set

$$s = \sup_{\xi\in[e\downarrow_\infty]} \limsup_{T\uparrow\mathcal{T}} \overline{\mathbb{Q}}_{T,q}^{\xi}(e\Downarrow_\infty \mid e\downarrow_\infty) \qquad (4.45)$$

Here we are taking the supremum over $\xi \in [e\downarrow_\infty]$ to ensure that $\overline{\mathbb{Q}}_{T,q}^{\xi}[e\downarrow_\infty]$ is strictly positive.

From the definition of s, as $\mathcal{T}$ is homogeneous, for every $\varepsilon > 0$ and $\xi \in [e\downarrow_\infty]$ we may choose some $T \in \mathbb{T}$ such that for every $f \in \chi(e)$ we have

$$\beta(f) = \overline{\mathbb{Q}}_{T,q}^{\xi}(f\Downarrow_\infty \mid f\downarrow_\infty) < s + \varepsilon. \qquad (4.46)$$

Now consider the graph $T^\star$ and the map $\psi_T : \Omega_{\mathcal{T}} \to \Omega_{T^\star}$. Recall that the push forwards measure $\psi_T\left(\overline{\mathbb{Q}}_{T,q}^{\xi}\right)$ is the random cluster measure on $T^\star$ where the edge weights of ∂T are given by $\gamma(e^\star) = \mathbb{1}_{[e\downarrow_\infty]}(\xi)$.

Now let S be the tree containing e and $\chi(e)$. From Theorem 2.23 the push forwards measure $\psi_s\left(\overline{\mathbb{Q}}_{T,q}^{\xi}\right) = \psi_{s,T}\psi_T\left(\overline{\mathbb{Q}}_{T,q}^{\xi}\right)$ is a random cluster model on $S^\star$ and for $f \in \chi(e)$ the $D_T(f)$-measurable event $[f\Downarrow_\infty]$ is conditionally independent of ψ_s given the indicator $\mathbb{1}_{[f\downarrow_\infty]}$.

Next let $\widetilde{\Omega}_e$ be the state space $\Omega_{S^\star} \times \Omega_{\partial\chi(e)}$ We interpret $(\tilde{\omega}, \sigma) \in \widetilde{\Omega}_e$ as a configuration ω of $S^\star$ and a colouring σ of the edges in $\partial\chi(e)$ where the bond $f^\star$ is *blue* if $\sigma(f^\star) = 1$. A blue edge will represent the event $[f\Downarrow_\infty]$.

As $\widetilde{\Omega}_e$ is finite, set $\tilde{\mathscr{F}}$ to be the set of subsets of $\widetilde{\Omega}_e$

Let $\tilde{\psi}_s : \Omega_{\mathcal{T}} \to \widetilde{\Omega}_e$ be the map defined by $\omega \mapsto (\psi_s(\omega), \sigma)$ where $\sigma(f^\star) = \mathbb{1}_{[f\Downarrow_\infty]}$ and let $\tilde{\varphi}$ be the push forwards measure $\tilde{\varphi} = \tilde{\psi}_s\left(\mathbb{Q}_{T^\star}^{\xi} \mid [e\downarrow_\infty]\right)$.

Introduce a Markov kernel $M : \tilde{\mathscr{F}} \times \widetilde{\Omega}_e \to [0,1]$ by setting $M(A, \tilde{\omega})$ to be the probability that $\tilde{\omega}' \in A$, where we select $\tilde{\omega}' = (\omega', \sigma') \in \widetilde{\Omega}_e$ as follows.

- Set $\omega'(f) = \omega(f)$ for every edge $e \in S^\star \setminus \chi(e)$.

- Let the events $\{[\omega'(f) = 1] : f \in \chi(e)\}$ occur independently with probability $\pi(\tau) + (\tau - \pi(\tau))\,\omega(f^\star)$.

- For each $f \in \chi(e)$ with $\omega(f^\star) = 1$ colour $f^\star$ blue independently of everything else and with probability $\beta(f)$.

Now define $\tilde{\mathscr{G}} \subset \tilde{\mathscr{F}}$ to be the σ-algebra generated by the events $\{J_f : f \in f^\star \setminus \chi(e)\}$.

We claim that for any event $A \in \tilde{\mathscr{F}}$ and $\tilde{\omega} \in [e\downarrow_{v_\infty}]$ we have

$$\frac{M(A \cap [e\downarrow_{v_\infty}], \tilde{\omega})}{M([e\downarrow_{v_\infty}], \tilde{\omega})} = \tilde{\varphi}\left(A \mid \tilde{\mathscr{G}}\right)(\tilde{\omega}). \tag{4.47}$$

First choose $\tilde{\omega}$ and $\tilde{\omega}'$ with $\omega, \omega' \in [e\downarrow_{v_\infty}]$ such that ω and ω' agree on $S^\star \setminus \chi(e)$ and $\omega(e) = \omega(\tilde{e}^\star) = 1$, where $\tilde{e}^\star$ is the single member of $\partial S \setminus \partial\chi(e)$. The colourings σ and σ' may be chosen arbitrarily so long as $\sigma(f) = \sigma'(f) = 0$ whenever $\omega(f^\star) = 0$.

As we have fixed $\omega(e) = \omega'(\tilde{e}^\star) = 1$ we have $\mathbb{1}_{\mathcal{L}_f}(\omega) = \omega(f^\star)$ for every $f \in \chi(e)$. Furthermore for $f \in \chi(e)$ the event $\{f \downarrow_\infty\}$ depends on the map ψ_s only through the edge state $\omega(f^\star)$ and occurs with probability $\beta(f) \cdot \omega(f^\star)$. Therefore from the recipe above we have $M(\{\tilde{\omega}'\}, \tilde{\omega}) = \tilde{\psi}_s\left(\mathbb{Q}_{T^\star}^{\tilde{\xi}}\right)\left(\{\tilde{\omega}'\} \,\middle|\, \mathcal{T}_{\chi(e)}\right)(\omega)$.

As $\tilde{\varphi}$ is defined by conditioning $\mathbb{Q}_{T^\star}^{\tilde{\xi}}$ on some $\tilde{\psi}_s$-measurable event then for $\tilde{\omega}$ and $\tilde{\omega}'$ chosen as above we have

$$\frac{M(\{\tilde{\omega}'\}, \tilde{\omega})}{M(\{\tilde{\omega}\}, \tilde{\omega})} = \frac{\tilde{\varphi}(\tilde{\omega}')}{\tilde{\varphi}(\tilde{\omega})} \tag{4.48}$$

$$= \frac{\tilde{\varphi}(\tilde{\omega}' \,|\, \tilde{\mathcal{G}})(\tilde{\omega})}{\tilde{\varphi}(\tilde{\omega} \,|\, \tilde{\mathcal{G}})(\tilde{\omega})}. \tag{4.49}$$

We argue that (4.49) holds for any choice of $\tilde{\omega}$ and $\tilde{\omega}'$ with $\omega, \omega' \in [e \downarrow_{v_\infty}]$. Firstly if ω and ω' do not agree on $S^\star \setminus \chi(e)$ then both sides of (4.49) are zero. We have insisted above that $\omega(e) = \omega'(\tilde{e}^\star) = 1$ however the choices in the description of M above are not affected by the states of these edges. Furthermore as $\tilde{\varphi}$ is conditioned on the event $[e \downarrow_\infty]$, the state of these bonds does not affect $\tilde{\varphi}$ on the edge like subgraph $\chi^\star(e)$ by the generalized series and parallel laws of Theorem 2.23.

Therefore (4.49) holds for any choice of $\tilde{\omega}, \tilde{\omega}'$ with $\omega, \omega' \in [e \downarrow_{v_\infty}]$ and for any event $A \in \tilde{\mathcal{F}}$ we have

$$\frac{M(A \cap [e \downarrow_{v_\infty}], \tilde{\omega})}{M(\{\tilde{\omega}\}, \tilde{\omega})} = \frac{\tilde{\varphi}(A \,|\, \tilde{\mathcal{G}})(\tilde{\omega})}{\tilde{\varphi}(\{\tilde{\omega}\} \,|\, \tilde{\mathcal{G}})(\tilde{\omega})}. \tag{4.50}$$

Therefore

$$\frac{M(A \cap [e \downarrow_{v_\infty}], \tilde{\omega})}{M([e \downarrow_{v_\infty}], \tilde{\omega})} = \frac{\tilde{\varphi}(A \,|\, \tilde{\mathcal{G}})(\tilde{\omega})}{\tilde{\varphi}\left(\tilde{\Omega}_e \,\middle|\, \tilde{\mathcal{G}}\right)(\tilde{\omega})} \tag{4.51}$$

$$= \tilde{\varphi}(A \,|\, \tilde{\mathcal{G}})(\tilde{\omega}). \tag{4.52}$$

Now for $\tilde{\varphi}$-almost every $\tilde{\omega} \in \tilde{\Omega}_e$ at least one $f^\star \in \partial\chi(e)$ is open and we must have $M([e \downarrow_{v_\infty}], \tilde{\omega}) > \tau$. Similarly there are at most three open pairs $\{f_1, f_2\} \subset \chi(e)$ and as $\beta(f) < s + \varepsilon$ we have $M(e \downdownarrows_\infty, \tilde{\omega}) < 3\tau^2 (s + \varepsilon)^2$.

In particular letting $\varepsilon \to 0$ and integrating (4.47) with respect to $\tilde{\varphi}$ we have $s < 3\tau s^2$. Trivially $s < 1$ and so for $\tau \leq \frac{1}{3}$ we have $s = 0$ and we are done. $\qquad\qquad\square$

Lemma 4.14 shows that if we choose $\pi(\tau)$ large enough then Bernoulli-$\pi(\tau)$ percolation cannot satisfy DLR conditions for $\longleftrightarrow$. An easy corollary of Lemma 4.15 is that we may choose τ small enough that the wired model random cluster model on $\mathcal{T}$ does not satisfy DLR conditions either.

Combining these two facts we may choose τ and q such that there can exist no $\longleftrightarrow$ random cluster model at all on the homogeneous 3-tree.

THEOREM 4.16

Let $\mathcal{T}$ be a regular 3 tree with homogeneous bond weights $\gamma(e) \equiv \tau \leq \frac{1}{3}$ and let $\longleftrightarrow$ be the random connection generated by the boundary condition $\Downarrow$ in Example 4.13. Choose $q \leq \frac{\tau}{9(1-\tau)}$.

Then the set $\mathscr{R}^{\longleftrightarrow}_{\mathcal{T},q}$ of $\longleftrightarrow$ random cluster models is empty.

PROOF

Notice that we have chosen (τ, q) in such a way that $\pi_q(\tau) \geq \frac{8}{9}$. This fact will be crucial to our argument.

Choose an arbitrary root $\rho \in \mathbf{V}(\mathcal{T})$, for each vertex v let $e_v \in \mathbf{E}_\rho$ be the unique edge directed away from ρ in the form $e_v = |v', v\rangle$.

Next we colour each open directed edge $e \in \mathbf{E}_\rho(\mathcal{T})$

- *blue* if $e\Downarrow_\infty$,

- *yellow* if e is not blue but there exists some $w \in D(e)$ with $v \longleftrightarrow w$ and $e_w \Downarrow_\infty$,

- *red* if e is neither blue nor yellow.

We say an edge is green if it is either blue or yellow, and white if it is closed. Now for $\xi \in \Omega_{\mathcal{T}}$ let $G(\xi) \in \Omega_{\mathcal{T}}$ be the configuration with $G(\xi) \in J_e$ if and only if e is green.

We leave it to the reader to convince himself that the random neighbour relation $\Downarrow$ of Example 4.13 generates a random connection $\leftrightarrow$ and that for any $T \in \mathbb{T}_\rho$ we have $\mathbb{Q}_{T,q}^{\overleftrightarrow{\xi}} = \overline{\mathbb{Q}}_{T,q}^{G(\xi)}$.

Now suppose $\varphi \in \mathscr{R}_{\mathcal{T},q}^{\leftrightarrow}$ and choose an edge $e \in \mathbf{E}_\rho(\mathcal{T})$. For every $\xi \in \Omega_{\mathcal{T}}$ and sequence $T_n \uparrow \mathcal{T}$ we have $\mathbb{Q}_{T_n,q}^{\overleftrightarrow{\xi}}[e \text{ is blue}] = \overline{\mathbb{Q}}_{T_n,q}^{G(\xi)}[e \text{ is blue}] \to 0$ as $n \to \infty$ by Lemma 4.15. Hence by the dominated convergence theorem we have

$$\varphi[e \text{ is blue}] = \lim_{n \to \infty} \int \mathbb{Q}_{T_n,q}^{\overleftrightarrow{\xi}}[e \text{ is blue}]d\varphi(\xi) = 0. \qquad (4.53)$$

In particular there are φ-almost surely no blue edges. Furthermore if an edge e is yellow we may find some blue edge in $D(e)$, hence there are φ almost surely no green edges in $\mathbf{E}_\rho(\mathcal{T})$.

So for every edge e we have $\varphi(\mathcal{L}_e^{\leftrightarrow}) = 0$ and so we must have $\varphi = \mathbb{P}_{\pi_q(\mathcal{T})}$ but we have chosen τ and q such that $\pi_q(\tau) \geq \frac{8}{9}$ and so from Lemma 4.14 there exist infinitely many blue edges φ almost surely.

Therefore there can exist no $\varphi \in \mathscr{R}_{\mathcal{T},q}^{\leftrightarrow}$ for $\mathcal{T}, q$ as in the statement of the theorem. ❏

4.2.2 Random connections from boundary conditions

We have invited the reader to convince himself that the random neighbour relations in Examples 4.11 and 4.13 above generate random connections. We would like a more reliable method of ascertaining whether a particular random neighbour relation generates a random connection.

Next we specify a class of random neighbour relations that always generate a random connection. This definition expresses

more naturally the concept of vertices being connected at the boundary and it may be easier to construct examples of this type than constructing random connections directly.

First, with each pair of vertices $u, v \in \mathbf{V}(\mathcal{T})$, we associate two σ-algebras. For fixed u and v there exists a unique directed path $\Pi_{u,v}$ from u to v. If we remove the path $\Pi_{u,v}$ from $\mathcal{T}$ we are left with a forest. We are interested in two components of this forest, that containing u and that containing v.

Formally for a directed edge e recall the set $D(e)$ of *descendants* of e and set

$$D(u,v) = \bigcap_{e \in \Pi_{u,v}} D(e) \tag{4.54}$$

to be the descendants of the path $\Pi_{u,v}$.

Now define two σ-algebras

$$\mathscr{G}_{u,v} = \sigma\left\{ J_e : e \in (D(u,v) \cup D(v,u)) \right\}, \tag{4.55}$$

$$\mathscr{G}_{u,v}^{\star} = \sigma\left\{ J_e \wedge \mathbb{1}_{[e\downarrow\infty]} : e \in (D(u,v) \cup D(v,u)) \right\}. \tag{4.56}$$

A *boundary condition* on a tree will be a *random neighbour relation* $\Downarrow$ where $[u \Downarrow v]$ is interpreted as the event that u and v are connected "through the boundary" one condition of which is that each event $[u \Downarrow v]$ is $\mathscr{G}_{u,v}$-measurable. This measurability condition may be interpreted informally as a connection from u to the boundary using edges in $D(v,u)$ and back to v through edges in $D(u,v)$.

We will also define a *strong boundary condition* where the events $[u \Downarrow v]$ are required to be $\mathscr{G}_{u,v}^{\star}$ measurable. This assumption is used in the proof of Theorem 4.23. We leave open the question of whether there exist any examples (or more subjectively any interesting examples) of boundary conditions which are not strong boundary conditions.

DEFINITION 4.17

We say a random neighbour relation $\{[u \Downarrow v] : u, v \in \mathbf{V}(\mathcal{T})\}$ is a *boundary condition* if

I $[u \Downarrow v]$ is an increasing event for every $u, v \in \mathbf{V}(\mathcal{T})$.

II $[u \Downarrow v] \in \mathscr{G}_{u,v}$ for every $u, v \in \mathbf{V}(\mathcal{T})$.

III For any finite subtree T and vertices $u, v \in \mathbf{V}(T)$ then if $u \Downarrow v$ there exist vertices $u', v' \in \mathbf{V}(\mathcal{T} \setminus T)$ such that

$$u \longleftrightarrow u' \Downarrow v' \longleftrightarrow v$$

We say $\Downarrow$ is a *strong boundary condition* if in addition each event $[u \Downarrow v] \in \mathscr{G}_{u,v}^{\star}$.

A random neighbour relation $\Downarrow$ is a (strong) *quasi-boundary condition* if it is equivalent, in the sense of Definition 4.10, to some (strong) boundary condition $\Downarrow'$.

Before we prove that random neighbour relations of this type always generate random connections we check that our two examples above are included in the class of boundary conditions.

LEMMA 4.18

Both random relations $\Downarrow$ of Example 4.11 (where $\sim$ is some measurable equivalence relation) and $\Downarrow$ of Example 4.13 are strong quasi-boundary conditions.

PROOF

It is easy to check that both random relations satisfy Axiom I and III. However neither $[u \Downarrow v]$ nor $[u \Downarrow v]$ are $\mathscr{G}_{u,v}$-measurable. Therefore we must construct strong boundary conditions $\Downarrow'$ and $\Downarrow'$ equivalent to $\Downarrow$ and $\Downarrow'$ which satisfy the measurabillity condition of Axiom II.

For vertices $u, v \in \mathbf{V}(\mathcal{T})$ define a map $G_{u,v} : \Omega_{\mathcal{T}} \to \Omega_{\mathcal{T}}$ as follows.

- For edges $e \in D_{u,v} \cup D_{v,u}$ set $G_{u,v}^{-1}(J_e) = J_e \cap [e \downarrow_\infty]$.

- For $e \in \mathbf{E}_v(\mathcal{T}) \setminus (D_{u,v} \cup D_{v,u})$ set $G_{u,v}^{-1}(J_e) = \emptyset$.

The σ-algebra $\mathscr{G}_{u,v}^\star$ is exactly the σ-algebra generated by the map $G_{u,v}$.

Now set $[u \Downarrow' v] = G_{u,v}^{-1}[u \Downarrow v]$ and $[u \Downdownarrow' v] = G_{u,v}^{-1}[u \Downdownarrow v]$. Both $[u \Downarrow' v]$ and $[u \Downarrow' v]$ are increasing $\mathscr{G}_{u,v}^\star$-measurable events.

It remains to check that $\Downarrow'$ is equivalent to $\Downarrow$, $\Downdownarrow'$ is equivalent to $\Downdownarrow$ and that both $\Downarrow'$ and $\Downdownarrow'$ satisfy Axiom III.

Choose $u, v \in \mathbf{V}(\mathcal{T})$ and let $T \in \mathbb{T}$ be any finite subtree, we assume without loss of generality that T contains both u and v. If $u \Downarrow v$ then we may choose open equivalent rays $\Pi_u \in \mathcal{R}_u$ and $\Pi_v \in \mathcal{R}_v$.

Now let u' and v' be the two leaf vertices of T on the paths Π_u and Π_v respectively. Then $u \longleftrightarrow u'$ and $v \longleftrightarrow v'$, and there exist open subpaths $\Pi_u \supset \Pi_u' \in \mathcal{R}_{u'}$ and $\Pi_v \supset \Pi_v' \in \mathcal{R}_{v'}$. As u' is a leaf of T then $\Pi_u' \subset D(w, u')$ for every w in T. Furthermore if Π_u' is open in ω it is open in $G_{w,u'}(\omega)$ and so setting $w = v'$ and arguing similarly for Π_v' we have $[u' \Downarrow' v']$.

Similarly if $u \Downdownarrow v$ we may find leaves u' and v' of T on the open 2-trees rooted at u and v respectively. Then there must be some open 2-tree rooted at u' contained in $D(u, u')$ and some open 2-tree rooted at v' contained in $D(v, v')$

In particular we have $u \longleftrightarrow u' \Downdownarrow' v' \longleftrightarrow v$.

Therefore $\Downarrow$ is equivalent to $\Downarrow'$ and $\Downdownarrow$ is equivalent to $\Downdownarrow'$ and both satisfy Axiom III. ❏

THEOREM 4.19

Let $\Downdownarrow$ be a quasi-boundary condition. Write $u \longleftrightarrow\!\!\!\!\!\rightarrow v$ if there exist finite sequences of vertices $u_0, \ldots, u_n$ and $v_0, \ldots, v_n$ such that

$$u = u_0 \longleftrightarrow v_0 \Downdownarrow u_1 \longleftrightarrow v_1 \Downdownarrow \ldots \Downdownarrow u_n \longleftrightarrow v_n = v.$$

Then $\longleftrightarrow\!\!\!\!\!\rightarrow$ is a random connection and is generated by $\Downdownarrow$.

Note: we allow $n = 0$ so that $[u \longleftrightarrow v] \subset [u \longleftrightarrow\!\!\!\!\!\rightarrow v]$ and we allow $u = v_0, v = v_1$ so that $[u \Downdownarrow v] \subset u \longleftrightarrow\!\!\!\!\!\rightarrow v$.

PROOF

It is immediate from the definition that $\longleftrightarrow$ is a random equivalence relation. Furthermore if two boundary conditions $\Downarrow$ and $\Downarrow'$ are equivalent it is easy to check that both random relations generate $\longleftrightarrow$ and so we may assume without loss of generality that $\Downarrow$ is a boundary condition.

Each event $[u_i \longleftrightarrow v_i]$ and $[v_i \Downarrow u_{i+1}]$ is increasing hence the event $[u \longleftrightarrow v]$ is increasing also. If $u \longleftrightarrow v$ but not $u \longleftrightarrow v$ then there exist pairs v_0, u_1 and v_{n-1}, u_n with $v_0 \Downarrow u_1$ and $v_{n-1} \Downarrow u_n$. It follows from Axiom III of the definition of a boundary condition that $u \longleftrightarrow v_0 \longleftrightarrow \infty$ and $v \longleftrightarrow u_n \longleftrightarrow \infty$.

It remains to check Axiom III of Definition 4.1.

Suppose $u \longleftrightarrow v$. If $u \longleftrightarrow v$ then for any edge $e = \langle \tilde{u}, \tilde{v} \rangle$ either e is not on the direct path from u to v in which case $u \overset{\omega_e}{\longleftrightarrow} v$ or e is on the path from u to v in which case either $u \overset{\omega_e}{\longleftrightarrow} \tilde{u}$ and $v \overset{\omega_e}{\longleftrightarrow} \tilde{v}$ or $u \overset{\omega_e}{\longleftrightarrow} \tilde{v}$ and $v \overset{\omega_e}{\longleftrightarrow} \tilde{u}$ depending on the orientation of e.

So suppose $u \longleftrightarrow v$ but not $u \longleftrightarrow v$ and choose sequences $u = u_0, \ldots, u_n$ and $v_0, \ldots, v_n = v$ as in the statement of the theorem. Notice that we may assume without loss of generality that $u_i \longleftrightarrow v_j$ if and only if $i = j$, for if not we may find shorter sequences of vertices $u_0, \ldots, u_i, u_j, \ldots u_n$ and $v_0, \ldots, v_i, v_j, \ldots, v_n$ that form a path from u to v.

Now let T be the smallest subtree of $\mathcal{T}$ that contains every vertex, $u_0, v_0, \ldots, u_n, v_n$ and the edge e.

Suppose each vertex u_i, v_i is a leaf of T, then the descendants $D(v_i, u_{i+1})$ and $D(u_{i+1}, v_i)$ of the path Π_{u_{i+1}, v_i} lie outside of T and hence the events $[u_{i+1} \Downarrow v_i] \in \mathscr{G}_{u_{i+1}, v_i} \subset \mathscr{T}_T$ do not depend on the state of the edge $e \in T$. Therefore if $\omega_e \notin [u \longleftrightarrow v]$ then e must lie on exactly one of the paths from u_i to v_i. in particular we may assume without loss of generality that $u_i \overset{\omega_e}{\longleftrightarrow} \tilde{u}$ and $\tilde{v} \overset{\omega_e}{\longleftrightarrow} v_i$. As none of the other events in our sequence are affected by the state of e we have $u \overset{\omega_e}{\longleftrightarrow} \tilde{u}$ and $\tilde{v} \overset{\omega_e}{\longleftrightarrow} v$.

Of course it may not be the case that each u_i, v_i is a leaf of T. To complete the proof we will use Axiom III of Definition 4.17

to choose sequences of vertices $u'_0, \ldots, u'_n$ and $v'_0, \ldots v'_n$ such that $u_i \longleftrightarrow u'_i \Downarrow v'_{i+1} \longleftrightarrow v_{i+1}$ and all vertices u'_i, v'_i are leaves of the smallest subtree containing T and every vertex $u'_0, v'_0, \ldots, u'_n, v'_n$.

Set $T_0 = T$ and suppose inductively we have constructed $T_i \supset T$ with leaves $v'_0, u'_1 \ldots, u'_{i-1}, v'_{i-1}, u'_i$ such that $v_j \longleftrightarrow v'_j \Downarrow u'_{j+1} \longleftrightarrow u_{j+1}$ for every $j < i$.

Now from Axiom III in Definition 4.17, as T_i is finite we may choose $v'_i, u'_{i+1} \notin T$ with $v_i \longleftrightarrow v'_i \Downarrow u'_{i+1} \longleftrightarrow u_{i+1}$. Set T_{i+1} to be the smallest subtree of $\mathcal{T}$ that contains T_i, u'_{i+1} and v'_i. We claim that for every $0 \leq j < i$ both v'_j and u'_{j+1} are leaves of T_{i+1}.

Let $w \in T_{i+1} \setminus T_i$ be a leaf of T_{i+1}. If $v_1 \neq w \neq u_{i+1}$ then let f be the single edge of T_{i+1} that contains w. Then $f \notin T_i$ so $T'_{i+1} = T_{i+1} \setminus f$ contains v'_i, u'_{i+1} and T_{i-1} contradicting the assumption that T_{i+1} is the smallest tree satisfying this condition.

Now suppose v'_j is not a leaf of T_{i+1} for some $0 \leq j \leq i$, then there must exist some leaf $w \in T_{i+1} \setminus T_i$ of T_{i+1} such that u'_j lies on the direct path from T_i to w; but as $w \in \{v'_i, u'_{i+1}\}$ and $\{v_i, u_{i+1}\} \subset T_i$ there must be an open path from w into T_i and so $v'_j \longleftrightarrow w$.

If $w = u'_{i+1}$ then $v_j \longleftrightarrow v'_j \longleftrightarrow u_{i+i} \longleftrightarrow u_{i+1}$ and if $w = v'_i$ we have $v_j \longleftrightarrow v'_i \longleftrightarrow v_i \longleftrightarrow u_i$, in both cases contradicting our assumption that $u_j \longleftrightarrow v_i$ only if $i = j$.

We may argue similarly that u'_j is a leaf of T_{i+1} whenever $1 \leq j \leq i+1$ and so continuing the construction we may find a tree T_{n-1} containing e, with leaves $v'_0, u'_1, \ldots v_{n-1}, u_n$ and arguing as above either $\omega_e \in [u \longleftrightarrow v]$ or e lies on the open path between u_j and v_j for exactly one j and we have $\omega_e \in [u \longleftrightarrow \tilde{u}, \tilde{v} \longleftrightarrow v] \cup [u \longleftrightarrow \tilde{v}, \tilde{u} \longleftrightarrow v]$.

Thus $\longleftrightarrow$ satisfies Axioms I–III of Definition 4.1 and is a random connection. Furthermore it is easy to see from the definition that $\longleftrightarrow$ is the random connection generated by $\Downarrow$ described in Definition 4.10. $\qquad\square$

4.3 THE PHASE DIAGRAM FOR REGULAR TREES

So far in this chapter we have constructed a class of random cluster models and shown that, on a tree at least, there exist non trivial examples of random connections. We have not yet discussed how these models behave.

There is a well established connection between the random cluster model and Bernoulli bond percolation on a general network. On a regular k-tree we have seen in Chapter 3 that the behaviour of the wired random cluster model is related to the critical point of Bernoulli percolation on the tree.

In this section we establish a link between Bernoulli bond percolation and the generalized random cluster model on the regular tree by considering the behaviour of the events $[u \leftrightarrow v]$ under homogeneous bond percolation. In particular we define two critical percolation probabilities $\tau_c^{\leftrightarrow} \leq \tau_u^{\leftrightarrow}$ associated with a random connection, we may then identify both a free phase, $\mathscr{R}_{\mathcal{T},q}^{\leftrightarrow} = \left\{ \mathbb{P}_{\pi_q(\mathcal{T})} \right\}$ whenever $\tau < \tau_c^{\leftrightarrow}$ and a wired phase, $\mathscr{R}_{\mathcal{T},q}^{\leftrightarrow} = \{ \mathbb{Q}_{\mathcal{T}}^{\star} \}$ whenever $\pi(\tau) > \tau_u^{\leftrightarrow}$.
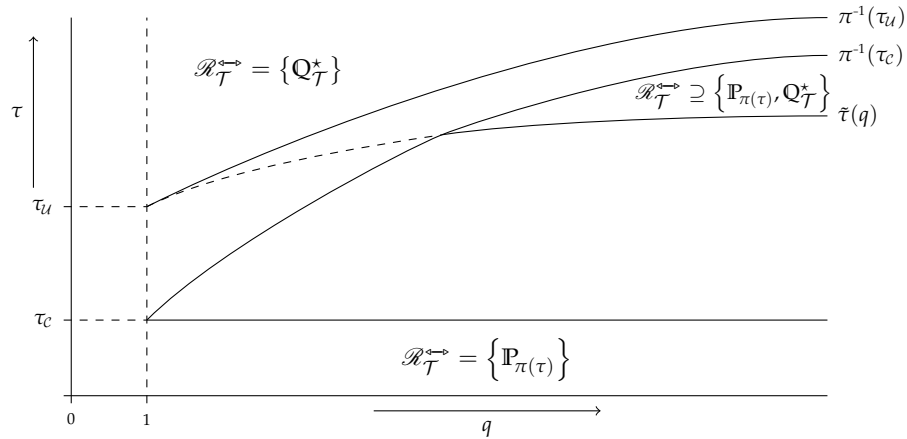
If $\leftrightarrow$ is the random connection generated by some strong boundary condition with $\tau_u^{\leftrightarrow} < 1$ then in addition, for sufficiently large q there exists some $\tilde{\tau} = \tilde{\tau}(q)$ such that the set $\mathscr{R}_{\mathcal{T},q}^{\leftrightarrow}$ is uncountable whenever τ is in the interval $(\tilde{\tau}, \pi^{-1}(\tau_u^{\leftrightarrow}))$.

The remainder of this chapter is devoted to establishing the phase diagram in Figure 9.

4.3.1   The uniqueness phases

Fix $2 \leq k \in \mathbb{N}$, $\tau \in (0, 1)$ and $q > 0$ and let $\mathcal{T}$ be a regular k-tree with homogeneous bond weights $\gamma(e) \equiv \tau$. Let $\bar{\eta} = \bar{\eta}(k, \tau, q) \in [0, 1)$ be the largest solution to equation (3.36) (that is the largest constant entrance law). Recall that $\mathbb{Q}_{\mathcal{T}}^{\star} = \mathbb{Q}_{\mathcal{T}}^{\bar{\eta}}$ is the maximal wired random cluster model for $(\mathcal{T}, q)$.

FIGURE 9

Phase diagram for the set of $\leftrightsquigarrow$ random cluster models when $\frac{1}{k} < \tau_c < \tau_u < 1$. In this case the random cluster model exhibits a free phase and a wired phase. If the random connection is generated by a *strong boundary condition* then the random cluster model exhibits a nonuniqueness phase when q is large.

DEFINITION 4.20

We say a measure μ has the $\leftrightsquigarrow$-*loopless property* if we have $\mu(\mathcal{L}_e^{\leftrightsquigarrow}) = 0$ for every edge $e \in \mathbf{E}(\mathcal{T})$.

We say μ has the $\leftrightsquigarrow$-*single cluster property* if for every pair of vertices $u, v \in \mathbf{V}(\mathcal{T})$ we have $\mu[u \leftrightsquigarrow v] = \mu[u \Leftrightarrow v] > \mu[u \leftrightarrow v]$.

We say a random connection is *connected* if every measure concentrated on the event that only finitely many bonds are closed has the $\leftrightsquigarrow$ single cluster property and *strongly connected* if there exists some $p < 1$ such that $\mathbb{P}_p$ has the $\leftrightsquigarrow$ single cluster property.

We will mainly be interested in strongly connected random connections, in particular we are interested in two questions, When is $\mathbb{P}_{\pi(\tau)} \in \mathscr{R}_{\mathcal{T}}^{\leftrightsquigarrow}$ and when is $\mathbb{Q}_{\mathcal{T}}^{\star} \in \mathscr{R}_{\mathcal{T}}^{\leftrightsquigarrow}$?

Now we define critical probabilities for Bernoulli percolation on $\mathcal{T}$ corresponding to the loopless and single cluster properties.

For a strongly connected random connection we define two critical probabilities.

$$\tau_c = \sup \left\{ p \in (0,1) : \mathbb{P}_p \text{ has the } \longleftrightarrow \text{ loopless property} \right\},$$

$$\tau_u = \inf \left\{ p \in \left(\frac{1}{k}, 1 \right) \,\middle|\, \mathbb{P}_p \text{ has the } \longleftrightarrow \text{ single cluster property} \right\}.$$

Our first task is to check that both the loopless and single cluster properties represent critical phenomena for Bernoulli percolation on the regular tree..

LEMMA 4.21

If $p < \pi_c$ then $\mathbb{P}_p$ has the $\longleftrightarrow$ loopless property and if $p > \tau_u$ then $\mathbb{P}_p$ has the single cluster property.

PROOF

The first statement follows directly from the monotonicity of $\mathcal{L}_e$. From the definition of τ_c for any $p < \tau_c$ we may choose $p < p' \leq \tau_c$ such that $\mathbb{P}_{p'}$ has the $\longleftrightarrow$ loopless property. Hence $\mathbb{P}_p(\mathcal{L}_e^{\leftrightarrow}) \leq \mathbb{P}_{p'}(\mathcal{L}_e^{\leftrightarrow}) = 0$ for every e.

The single cluster property may not be expressed in terms of monotonic events.

However as $\mathcal{T}$ is a regular k-tree it has *unimodular* symmetry group. A theorem of Häggström and Peres [40] states that for any $p > p'$ there exists a coupling $\omega = (\omega_{p'}, \omega_p)$ with $\omega_{p'} \sim \mathbb{P}_{p'}$ and $\omega_p \sim \mathbb{P}_p$ such that almost every ω has the property that $\omega_p \geq \omega_{p'}$ and every infinite cluster of ω_p contains an infinite cluster of $\omega_{p'}$.

So given $p > \tau_u$ we may choose some $p' < p$ such that p' has the $\longleftrightarrow$ single cluster property. Coupling as above if $\omega_p \in [u \downarrow_\infty, v \downarrow_\infty]$ then with probability one there exist vertices u', v' with $u \xleftrightarrow{\omega_p} u'$, $v \xleftrightarrow{\omega_p} v'$ and $\omega_{p'} \in [u' \downarrow_\infty, v' \downarrow_\infty]$.

As $\mathbb{P}_{p'}$ has the $\longleftrightarrow$ single cluster property we have $u' \xleftrightarrow{\omega_{p'}} v'$ almost surely and as $\omega_p \geq \omega_{p'}$ and $[u \longleftrightarrow v]$ is increasing we must have $u \xleftrightarrow{\omega_p} u' \xleftrightarrow{\omega_p} v' \xleftrightarrow{\omega_p} v$ and so $u \xleftrightarrow{\omega_p} v$. □

So Bernoulli bond percolation exhibits either two or three phases with respect to a strongly connected random connection. If $p < \tau_c$ then $\leftarrow\!\!\!\dashrightarrow$ is indistinguishable from $\longleftrightarrow$ and if $p > \tau_u$ then $\leftarrow\!\!\!\dashrightarrow$ is indistinguishable from $\Longleftrightarrow$. We have insisted that $\tau_u < 1$ and trivially $\tau_c \geq \frac{1}{k}$ so these two phases are nontrivial. In addition we may have $\tau_c < \tau_u$, in which case there exists an intermediate phase where the behaviour of $\leftarrow\!\!\!\dashrightarrow$ is not adequately described by either $\longleftrightarrow$ or $\Longleftrightarrow$.

We may translate this observation into a statement about the random cluster model using the comparison inequalities of Theorem 2.7.

THEOREM 4.22

Let $\mathcal{T}$ be a homogeneous regular tree with $\gamma(e) \equiv \tau$ and fix $q > 1$.

- If $\tau < \tau_c$ then $\mathscr{R}_{\mathcal{T}}^{\leftrightarrow} = \{\mathbb{P}_\tau\}$.

- If $\pi(\tau) < \tau_c$ then $\mathscr{R}_{\mathcal{T}}^{\leftrightarrow} \supseteq \{\mathbb{P}_\tau\}$.

- If $\pi(\tau) > \tau_u$ then $\mathscr{R}_{\mathcal{T}}^{\leftrightarrow} = \{\mathbb{Q}_{\mathcal{T}}^\star\}$.

PROOF

First if $\pi(\tau) < \tau_c$ then $\mathbb{P}_{\pi(\tau)}(\mathcal{L}_e^{\leftrightarrow}) = 0$ for every e and so $\mathbb{P}_{\pi(\tau)} \in \mathscr{R}_{\mathcal{T}}^{\leftrightarrow}$ by Corollary 4.4. If in addition $\tau < \tau_c$ then for any $\varphi \in \mathscr{R}_{\mathcal{T}}^{\leftrightarrow}$ we have $\varphi(\mathcal{L}_e^{\leftrightarrow}) \leq \mathbb{P}_\tau(\mathcal{L}_e^{\leftrightarrow}) = 0$ for every $e \in \mathbf{E}(\mathcal{T})$. Again by Corollary 4.4 we must have $\varphi = \mathbb{P}_{\pi(\tau)}$.

Suppose $\pi(\tau) > \tau_u$. Set $\theta = \mathbb{P}_{\pi(\tau)}[e \downarrow_\infty] > 0$. Recall the measure $\mathbb{Q}_T^\theta$ on the graph $T^\star$, we aim to show that for every $\varphi \in \mathscr{R}_{\mathcal{T}}^{\leftrightarrow}$ and $T \in \mathbb{T}$ we have $\psi_T(\varphi) \prec \mathbb{Q}_T^\theta$.

First $\psi_T\left(\mathbb{P}_{\pi(\tau)}\right)$ is independent bond percolation on $T^\star$, with bonds $e^\star \in \partial T$ open with probability θ. So for every increasing $\mathscr{F}_{\partial T}$-measurable random variable X, $\psi_T\left(\mathbb{P}_{\pi(\tau)}\right)(X) > \mathbb{Q}_T^\theta(X)$ and in particular for any increasing event A

$$\int \mathbb{Q}_T^\theta(A \mid \mathscr{F}_{\partial T})(\psi_T(\xi)) d\mathbb{P}_{\pi(\tau)}(\xi) \geq \int \mathbb{Q}_T^\theta(A \mid \mathscr{F}_{\partial T})(\omega) d\mathbb{Q}_T^\theta(\omega)$$

Next, as $\mathbb{P}_{\pi(\tau)}$ has the $\longleftrightarrow$ single cluster property, $Q_T^{\overset{\xi}{\leftrightarrow}} = \bar{Q}_T^\xi$ for $\mathbb{P}_{\pi(\tau)}$ almost every ξ.

Now let A be some increasing ψ_T-measurable event and let $\varphi \in \mathscr{R}_T^{\leftrightarrow}$ be any random cluster measure. As $\varphi \succ \mathbb{P}_{\pi(\tau)}$ we may calculate

$$\varphi(A) = \int Q_T^{\overset{\xi}{\leftrightarrow}}(A)d\varphi(\xi) \tag{4.57}$$

$$\geq \int Q_T^{\overset{\xi}{\leftrightarrow}}(A)d\mathbb{P}_{\pi(\tau)}(\xi) \tag{4.58}$$

$$= \int \bar{Q}_T^\xi(A)d\mathbb{P}_{\pi(\tau)}(\xi) \tag{4.59}$$

$$= \int Q_T^\theta(A\,|\mathscr{F}_{\partial T})(\psi_T(\xi))d\mathbb{P}_{\pi(\tau)}(\xi) \tag{4.60}$$

$$\geq \int Q_T^\theta(A\,|\mathscr{F}_{\partial T})(\omega)dQ_T^\theta(\xi) \tag{4.61}$$

$$= Q_T^\theta(A). \tag{4.62}$$

So we have $\psi_T(\varphi) \succ Q_T^\theta$ for some $\theta > 0$. As $\pi(\tau) > \tau_u \geq \frac{1}{k}$ we have $Q_T^\theta \to Q_T^\star$ as $T \uparrow \mathcal{T}$ by Theorem 3.15. Hence $\varphi \succ Q_T^\star$. Similarly by considering $Q_T^{\pi^{-1}(\theta)}$ it is clear that $\varphi \prec Q_T^1 \to Q_T^\star$ as $T \uparrow \mathcal{T}$.

Therefore $\varphi = Q_T^\star$ and as φ is arbitrary we have $\mathscr{R}_T^{\leftrightarrow} = \{Q_T^\star\}$ as required. ❑

4.3.2 The nonuniqueness phase

We have been able to use the comparison with percolation to establish the existence of both a "high temperature phase" where $\mathscr{R}_T^{\leftrightarrow} = \left\{\mathbb{P}_{\pi(\tau)}\right\}$ and a "low temperature phase" where $\mathscr{R}_T^{\leftrightarrow} = \{Q_T^\star\}$.

Jonasson [44] characterized the property of nonamenability for a network $\mathcal{N}$ by demonstrating that $\mathcal{N}$ is nonamenable if and only if for every sufficiently large q there is an open interval (p_1, p_2)

such that the wired random cluster model in not unique on $\mathcal{N}$ whenever $\gamma(e) \equiv p \in (p_1, p_2)$.

The tree is certainly not amenable, and we have seen that this phenomenon holds for wired boundary conditions on the regular tree. What can we say about more general boundary conditions?

We conclude this chapter by demonstrating that if a random connection on a regular tree is generated by a *strong boundary condition* and has $\tau_u < 1$ then, as with the wired model, we may always find q sufficiently high such that the random cluster model is not uniqueness for an entire interval of bond strengths.

THEOREM 4.23

Let $\longleftrightarrow$ be a strongly continuous random connection generated by a strong boundary condition $\Downarrow$. For fixed τ, q and some η satisfying $\eta = F_{\tau,q,k}(\eta)$ set

$$\tilde{p} = \frac{1 - (1 - \pi(\overline{\eta}))^{1/k}}{\pi(\overline{\eta})}. \tag{4.63}$$

If $\tilde{p} > \tau_u$ then $\mathbb{Q}_{\mathcal{T}}^{\eta} \in \mathcal{R}_{T_k,q}^{\longleftrightarrow}$.

PROOF

We aim to show that if τ, q and k are such that $\tilde{p} > \tau_u$ then $\mathbb{Q}_{\mathcal{T}}^{\star}$ has the $\longleftrightarrow$ single cluster property.

Choose a root ρ arbitrarily and direct all edges away from ρ. For an edge $e \in \mathbf{E}_\rho(\mathcal{T})$ *green* if e is open and $e \downarrow_\infty$, colour e *red* if e is open but not green and *white* if e is closed.

For a subtree $T \in \mathbb{T}_\rho$ and a configuration $\omega \in \Omega_{T^\star}$ colour directed edges $e \in \mathbf{E}_\rho(T^\star)$ green if $e \downarrow_{v_\infty}$ and red or white as above. For edges $e^\star \in \partial T$ interpret $e^\star$ as green if open and white if closed.

Now choose a subtree $S \in \mathbb{T}_\rho$ and a configuration $\omega \in \Omega_{S^\star}$ arbitrarily. Our strategy is to create a coupling $(\omega_\mathbb{P}, \omega_\mathbb{Q})$ with

$\omega_{\mathbb{P}} \sim (\mathbb{P}_{\tilde{p}} \, | \psi_s)(\omega)$ and $\omega_{\mathbb{Q}} \sim (\mathbb{Q}_{\mathcal{T}}^{\star} \, | \psi_s)(\omega)$ in such a way that every $\omega_{\mathbb{P}}$-green edge is also $\omega_{\mathbb{Q}}$-green.

Suppose for some $T \supset S$ we have constructed a coupling $\omega^T = (\omega_{\mathbb{P}}^T, \omega_{\mathbb{Q}}^T)$ with $\omega_{\mathbb{P}}^T \sim (\psi_T(\mathbb{P}_{\tilde{p}}) \, | \psi_s)(\omega)$ and $\omega_{\mathbb{Q}} \sim (\psi_T(\mathbb{Q}_{\mathcal{T}}^{\star}) \, | \psi_s)(\omega)$ such that every $\omega_{\mathbb{P}}$-green edge is also $\omega_{\mathbb{Q}}$-green. This is trivial for $T = S$.

We have chosen $\tilde{p}$ so that $1 - \pi(\bar{\eta}) = (1 - \tilde{p} \cdot \pi(\bar{\eta}))^k$ and in particular that $\mathbb{P}_{\tilde{p}}[e \downarrow_{\infty}] = \pi(\bar{\eta}) = \mathbb{Q}_{\mathcal{T}}^{\star}(e \downarrow_{\infty} | J_e^{\complement})$. Notice that by stochastic domination we have

$$\mathbb{P}_{\pi(\tau)}[e \downarrow_{\infty}] \leq \mathbb{Q}_{\mathcal{T}}^{\star}(e \downarrow_{\infty} | J_e^{\complement}) \leq \mathbb{P}_{\tau}[e \downarrow_{\infty}]. \tag{4.64}$$

Therefore $\pi(\tau) \leq \tilde{p} \leq \tau$.

Now choose an edge $e \in \Lambda T$. We use the generalized series and parallel laws of Theorem 2.23 to construct ω^{T+e} with the properties above.

Recall the graphs $\mathcal{D}_{k,\tau,\eta}$ from Subsection 2.4.4. Set $\mathcal{D}_{\mathbb{P}} = \mathcal{D}_{k,\tilde{p},\pi(\eta)}$ and $\mathcal{D}_{\mathbb{Q}} = \mathcal{D}_{k,\tau,\bar{\eta}}$. Then $\mathcal{D}_{\mathbb{P}}$ and $\mathcal{D}_{\mathbb{Q}}$ are isomorphic to $\chi^{\star}(e)$, weighted appropriately for $\mathbb{P}_{\tilde{p}}$ and $\mathbb{Q}_{\mathcal{T}}^{\star}$ respectively.

Next choose independent uniform random variables $x_1, \ldots, x_k$ and $y_1, \ldots, y_k$ from $(0,1)$ and N from $[\![1,k]\!]$. We will choose ω^{T+e} according to the recipe in Subsection 2.4.4.

Set

$$\begin{aligned}
\alpha &= \bar{\eta} & &\geq & \alpha' &= \pi(\bar{\eta}), & (4.65) \\
\beta &= \frac{1 - \pi(\tau)}{1 - \pi(\tau)\pi(\eta)} & &\geq & \beta' &= \frac{1 - \tilde{p}}{1 - \tilde{p}\pi(\eta)}, & (4.66) \\
\gamma &= \frac{(1 - \pi(\tau))\,\pi(\eta)}{1 - \pi(\tau)\pi(\eta)} & &\geq & \gamma' &= \frac{(1 - \tilde{p})\,\pi(\eta)}{1 - \tilde{p}\pi(\eta)} &
\end{aligned}$$
$$(4.67)$$

and recall the function

$$\theta(\alpha, x) = \left(1 - \frac{1-\alpha}{1-\alpha x}\right)^{1/k}. \tag{4.68}$$

Now ω^T is such that every $\omega_{\mathbb{P}}^T$-green edge is $\omega_{\mathbb{Q}}^T$-green. In particular, as $e^\star$ is green if and only if it is open, we have $\omega_{\mathbb{Q}}^T(e^\star) \geq \omega_{\mathbb{P}}^T(e^\star)$.

For $f \in T^\star \cap (T+e)^\star$ set $\omega_{\mathbb{P}}^{T+e}(f) = \omega_{\mathbb{P}}^T(f)$ and $\omega_{\mathbb{Q}}^{T+e}(f) = \omega_{\mathbb{Q}}^T(f)$.

For the remaining edges number $\chi(e) = \{f_1, \ldots, f_k\}$ and following Subsection 2.4.4 set

$$\omega_{\mathbb{Q}}^{T+e}(f_N) = \mathbb{1}_{[x_i > \beta]} \vee \omega_{\mathbb{Q}}^T(e) \tag{4.69}$$

$$\omega_{\mathbb{Q}}^{T+e}(f_i) = \mathbb{1}_{[x_i > \beta]} \vee \left(\omega_{\mathbb{Q}}^T(e) \wedge \mathbb{1}_{[y_i < \theta(\alpha, y_N)]} \right) \tag{4.70}$$

$$\omega_{\mathbb{Q}}^{T+e}(f_N^\star) = \mathbb{1}_{[x_i < \gamma]} \vee \omega_{\mathbb{Q}}^T(e) \tag{4.71}$$

$$\omega_{\mathbb{Q}}^{T+e}(f_i^\star) = \mathbb{1}_{[x_i < \gamma]} \vee \left(\omega_{\mathbb{Q}}^T(e) \wedge \mathbb{1}_{[y_i < \theta(\alpha, y_N)]} \right) \tag{4.72}$$

and similarly

$$\omega_{\mathbb{P}}^{T+e}(f_N) = \mathbb{1}_{[x_i > \beta']} \vee \omega_{\mathbb{Q}}^T(e) \tag{4.73}$$

$$\omega_{\mathbb{P}}^{T+e}(f_i) = \mathbb{1}_{[x_i > \beta']} \vee \left(\omega_{\mathbb{Q}}^T(e) \wedge \mathbb{1}_{[y_i < \theta(\alpha', y_N)]} \right) \tag{4.74}$$

$$\omega_{\mathbb{P}}^{T+e}(f_N^\star) = \mathbb{1}_{[x_i < \gamma']} \vee \omega_{\mathbb{Q}}^T(e) \tag{4.75}$$

$$\omega_{\mathbb{P}}^{T+e}(f_i^\star) = \mathbb{1}_{[x_i < \gamma']} \vee \left(\omega_{\mathbb{Q}}^T(e) \wedge \mathbb{1}_{[y_i < \theta(\alpha', y_N)]} \right) \tag{4.76}$$

So $\omega_{\mathbb{P}}^{T+e} \sim \left(\psi_{T+e}(\mathbb{P}_{\tilde{p}}) \mid \psi_T \right)(\omega_{\mathbb{P}})^T$ and $\omega_{\mathbb{Q}}^{T+e} \sim \left(\psi_{T+e}(\mathbb{Q}_{\mathcal{T}}^\star) \mid \psi_T \right)(\omega_{\mathbb{Q}})^T$ and it is easy to check that every $\omega_{\mathbb{P}}$-green edge of $\chi^\star(e)$ is $\omega_{\mathbb{Q}}$-green. Furthermore $\omega_{\mathbb{P}}^T(e) = 1$ if and only if $\omega_{\mathbb{P}}^{T+e}(f_N) = \omega_{\mathbb{P}}^{T+e}(f_N^\star) = 1$ (and similar for $\omega_{\mathbb{Q}}^T$, $\omega_{\mathbb{Q}}^{T+e}$) therefore the colours of edges in $T^\star \cap T + e^\star$ agree under ω^T and ω^{T+e} so we have constructed ω^{T+1} as required.

Now letting $T \uparrow \mathcal{T}$ we have a coupling $(\omega_{\mathbb{P}}, \omega_{\mathbb{Q}})$ with $\omega_{\mathbb{P}} \sim \left(\mathbb{P}_{\tilde{p}} \mid \psi_s \right)(\omega)$ and $\omega_{\mathbb{Q}} \sim (\mathbb{Q}_{\mathcal{T}}^\star \mid \psi_s)(\omega)$ in such a way that every $\omega_{\mathbb{P}}$-green edge is also $\omega_{\mathbb{Q}}$-green.

For vertices $u, v, \rho \in S$ suppose $\omega_{\mathbb{Q}} \in [u \Longleftrightarrow v]$. The event $[u \Longleftrightarrow v]$ is ψ_s-measurable and so $\omega_{\mathbb{P}} \in [u \Longleftrightarrow v]$ as well. As $\mathbb{P}_{\tilde{p}}$ has the $\longleftrightarrow$ single cluster property, $\omega_{\mathbb{P}} \in [u \longleftrightarrow v]$ and arguing as in the proof

of Theorem 4.19 we may find a tree $T \in \mathbb{T}_\rho$ containing S with leaf vertices $u_0, \ldots u_{n-1}$ and $v_1, \ldots, v_n$ such that

$$\omega_\mathbb{P} \in [u \longleftrightarrow u_0 \Downarrow v_1 \longleftrightarrow u_1 \Downarrow \ldots \Downarrow v_n \longleftrightarrow v]. \tag{4.77}$$

We argue that the above path is also open in $\omega_\mathbb{Q}$.

First $\omega_\mathbb{P} \in [u \longleftrightarrow u_0]$. Consider the path between u and u_0. We may split this path into two components, those edges in on the path from ρ to u and those edges on the path from ρ to u_0. As $u \in S \in \mathbb{T}_\rho$ the first group of edges are all in S hence are open in $\omega_\mathbb{Q}$ as $\omega_\mathbb{Q}$ agrees with $\omega_\mathbb{P}$ on S. The second group of edges are directed away from ρ and as $\omega_\mathbb{P} \in u_0 \Downarrow v_1$ there is an $\omega_\mathbb{P}$-green path from v to infinity, hence all edges in the second part of the path are $\omega_\mathbb{P}$-green so open in $\omega_\mathbb{Q}$. Therefore $\omega_\mathbb{Q} \in [u \longleftrightarrow u_0]$. Similarly we have $\omega_q \in [v_n \longleftrightarrow v]$ and for each pair u_i, v_i the path between u_i and v_i comprises green edges on the path from ρ to u_i and green edges on the path from ρ to v_i and so $\omega_\mathbb{Q} \in [u_i \longleftrightarrow v_i]$.

It remains to argue that $\omega_\mathbb{Q} \in [v_i \Downarrow u_{i+1}]$ for each i. As we have chosen v_i and u_{i+1} to be leaf vertices of T, the path between v_i and u_{i+1} is contained in T hence the sigma algebra $\mathscr{G}^\star_{v_i, u_{i+1}}$ is contained in the sigma algebra generated by events $[e \text{ is green}]$ for $e \in \mathbf{E}_\rho(\mathcal{T}) \setminus \mathbf{E}_\rho(T)$.

As the event $[v_i \Downarrow v_{i+1}]$ is increasing and $\mathscr{G}^\star_{v_i, u_{i+1}}$-measurable for the strong boundary condition $\Downarrow$ and as every $\omega_\mathbb{P}$-green edge is $\omega_\mathbb{Q}$-green we must have $\omega_\mathbb{Q} \in [v_i \Downarrow u_{i+1}]$. Therefore $\omega_\mathbb{Q} \in [u \longleftrightarrow v]$ whenever $\omega_\mathbb{Q} \in [u \Longleftrightarrow v]$.

As $\omega_\mathbb{Q} \sim (\mathbb{Q}^\star_\mathcal{T} \,|\, \psi_s)(\omega)$ for arbitrary $\omega \in \Omega_{S^\star}$ so $\mathbb{Q}^\star_\mathcal{T}$ has the $\longleftrightarrow$-single cluster property and in particular the events $\mathcal{L}^\star_e$ and $\mathcal{L}^{\longleftrightarrow}_e$ are $\mathbb{Q}^\star_\mathcal{T}$-indistinguishable.

Therefore

$$\mathbb{Q}^\star_\mathcal{T}(J_e \,|\, \mathscr{T}_e)(\omega) = \pi(\tau) + (\tau - \pi(\tau)) \mathbb{1}_{\mathcal{L}^\star_e}(\omega) \tag{4.78}$$

$$= \pi(\tau) + (\tau - \pi(\tau)) \mathbb{1}_{\mathcal{L}^{\longleftrightarrow}_e}(\omega) \tag{4.79}$$

for $\mathbb{Q}^\star_\mathcal{T}$-almost every ω therefore $\mathbb{Q}^\star_\mathcal{T} \in \mathscr{R}^{\longleftrightarrow}_\mathcal{T}$. ☐

COROLLARY 4.24

If $\longleftrightarrow$ is a strongly connected random connection generated by some strong boundary condition $\Downarrow$ such that $\frac{1}{k} < \tau_u < 1$ then there exists some $q_c \in \mathbb{R}^+$ such that whenever $q > q_\mathbf{c}$ there exists $\tilde{\tau}_u < \tau_u$ where $\mathscr{R}_{\mathcal{T},q}^{\leftrightarrow} \supseteq \left\{\mathbb{P}_{\pi(\tau)}, \mathbb{Q}_{\mathcal{T}}^{\star}\right\}$ whenever $\tau \in (\tilde{\tau}_u, \tau_u)$.

PROOF

Set $\zeta = \mathbb{P}_{\tau_u}[e \downarrow_\infty]$ for arbitrary $e \in \overline{\mathbb{E}}$ and recall from (3.36) the function

$$F_{\tau,q,k}(\eta) = 1 - \left(1 - \pi^{-1}(\pi(\tau)\pi(\eta))\right)^k \tag{4.80}$$

From Theorem 4.23 we have that $\mathbb{Q}_{\mathcal{T}}^{\star} \in \mathscr{R}_{\mathcal{T}}^{\leftrightarrow}$ whenever $\pi(\bar{\eta}) > \zeta$; where $\bar{\eta}$ is the largest fixed point of $\bar{\eta} = f(\bar{\eta})$. From Theorem 4.22 $\mathbb{P}_{\pi(\tau)} \in \mathscr{R}_{\mathcal{T},q}^{\rightarrow}$ whenever $\pi(\tau) < \tau_c$.

Now as $F_{\tau,q,k}$ is continuous and $F_{\tau,q,k}(1) < 1$ then if $F_{\tau,q,k}(\pi^{-1}(\zeta)) > \pi^{-1}(\zeta)$ for some $\tau < \pi^{-1}(\tau_c)$ then we must have $\pi(\bar{\eta}) > \zeta$ and so $\mathscr{R}_{\mathcal{T},q}^{\leftrightarrow} \supseteq \left\{\mathbb{P}_{\pi(\tau)}, \mathbb{Q}_{\mathcal{T}}^{\star}\right\}$.

The function $\tau \mapsto F_{\tau,q,k}(\pi^{-1}(\zeta))$ is continuous and increasing hence if for sufficiently large q we have $F_{\pi^{-1}(\tau_c),k,q}(\pi^{-1}(\zeta)) > \pi^{-1}(\zeta)$ then we may find an interval $I_q^{\leftrightarrow} = (\tilde{\tau}_u, \pi^{-1}(\tau_c))$ such that $\mathscr{R}_{\mathcal{T},q}^{\leftrightarrow} \supseteq \left\{\mathbb{P}_{\pi(\tau)}, \mathbb{Q}_{\mathcal{T}}^{\star}\right\}$ for any $\tau \in I_q^{\leftrightarrow}$.

Rearranging (4.80) we must find q large enough so that

$$1 - \pi^{-1}(\zeta) > \left(1 - \pi^{-1}(\zeta\tau_c)\right)^k. \tag{4.81}$$

To see that such q exists observe that

$$\left(1 - \pi^{-1}(p)\right)q = \frac{(1-p)q}{pq + (1-p)} \tag{4.82}$$

$$\rightarrow \frac{1-p}{p} \quad \text{as } q \rightarrow \infty. \tag{4.83}$$

Therefore the left hand side of (4.81) decays as q^{-1} and the right as q^{-k} and (4.81) is satisfied for sufficiently large q. $\qquad\qquad\square$

5 GRIMMETT AND JANSON'S BOUNDARY CONDITIONS

In this chapter we return to the boundary conditions described by Grimmett and Janson [36]. In that paper the authors categorized equivalence relations according to their topological properties. Here we take an alternative but equivalent view by categorizing equivalence relations according to their quotient spaces.

Using this approach we consider two examples of canopies that are not connected, the *open* boundary conditions of [36] and a new "paired tree model." We continue the work of [36] by providing a complete description of the behaviour of random cluster models with open boundary conditions.

Lastly we introduce a random connection $\longleftrightarrow$ inspired by the weak limit in Chapter 1 on a d-dimensional quad tree. This may be described by the map from the set of rays to the "canopy" $[0,1]^d$ associated with the "canonical curdling" process of Mandelbrot [51].

We are able to calculate the critical probabilities for this random connection exactly and by combining this with the results in Chapter 4 we demonstrate that the $\longleftrightarrow$-random cluster model exhibits all three possible phases for every $q > 1$.

CONTENTS

We have a partially complete phase picture of the random cluster model for strongly connected random connections. In this chapter we take a closer look at boundary conditions described by equivalence relations.

Recall that $\mathcal{T}$ is some infinite tree and $\mathcal{R} = \mathcal{R}_\rho$ is the set of half infinite paths on $\mathcal{T}$ started at some nominated root ρ. We may define $\mathcal{R}_v$ for any vertex $v \in \mathbf{V}(\mathcal{T})$ and there is a natural bijection $R : \mathcal{R}_u \to \mathcal{R}$ such that Π and $R(\Pi)$ differ on only finitely many edges.

Recall also that for an equivalence relation $\sim$ on $\mathcal{R}$ we may define $[u \Downarrow v]$ to be the event that there exists some pair of open rays $\Pi_u \in \mathcal{R}_u$ and $\Pi_v \in \mathcal{R}_v$ with $R(\Pi)_u \sim R(\Pi)_v$.

We say an equivalence relation is *measurable* if the event $[u \Downarrow v] \in \mathscr{F}_\mathcal{T}$ for every pair $u, v \in \mathbf{V}$. From Lemma 4.18 every measurable equivalence relation is a strong boundary condition and so generates a random connection $\overset{\sim}{\Leftrightarrow}$.

5.1 EQUIVALENCE RELATIONS AND TOPOLOGY

Grimmett and Janson [36] classify random equivalence relations according to the topology on the set of *rays* $\mathcal{R}$. We may describe this topology by representing a ray as a configuration and inheriting the topology of $\Omega_\mathcal{T}$.

Identify a ray $\Pi \in \Pi_\mathcal{T}$ with a configuration $\omega_\Pi \in \Omega_\mathcal{T}$ where $\omega_\Pi(e) = 1$ if and only if e is on the path Π. Let $\Omega_\mathcal{R} \subset \Omega_\mathcal{T}$ be the set of configurations $\{\omega_\Pi : \Pi \in \mathcal{R}\}$ and let $\mathcal{R}$ have the topology of $\Omega_\mathcal{R}$ under the subspace topology. It is easy to check that the set $\Omega_\mathcal{R}$ is closed, hence compact and Hausdorff.

We say an equivalence relation is *open* (respectively *closed*, *Borel*) if the set $S^\sim = \{(\Pi, \Pi') : \Pi \sim \Pi'\}$ is open (respectively closed, Borel). The classification of equivalence relations as open, closed, Borel and measurable is due entirely to Grimmett and Janson [36]. Furthermore it is proved in [36] that these classifications form a hierarchy as follows.

THEOREM 5.1: GRIMMETT–JANSON

- Every open equivalence relation is closed.

- Every closed equivalence relation is measurable.

- Every measurable equivalence relation is Borel.

We include a proof of this statement for completion and as a vehicle to introduce some new terminology. The underlying argument is unchanged from that in [36]

PROOF

The first statement will follow directly from the forthcoming Theorem 5.4.

For a pair of rays $\Pi_1, \Pi_2 \in \mathcal{R}$ define the *thread* $\vartheta(\Pi_1, \Pi_2) \in \Omega_{\mathcal{T}}$ to be the configuration such that

$$\mathbb{1}_{J_e}(\vartheta(\Pi_1, \Pi_2)) = |\omega_{\Pi_1}(e) - \omega_{\Pi_2}(e)| . \tag{5.1}$$

That is $\vartheta(\Pi_1, \Pi_2) \in J_e$ if and only if e is on exactly one of the paths Π_1, Π_2. Let $\Theta = \vartheta(\mathcal{R}^2)$ be the set of threads and notice that the set Θ does not depend on our choice of root vertex ρ.

Say a thread $\vartheta(\Pi_1, \Pi_2)$ is a *stitch* if $\Pi_1 \sim \Pi_2$ and a *darn* otherwise. Let Σ be the set of stitches and $\Sigma^{\complement}$ the set of darns. Now ϑ is a continuous (hence closed and measurable) map $\mathcal{R}^2 \to \Omega_{\mathcal{T}}$. Therefore if $\sim$ is a closed equivalence relation the set of stitches Σ is closed and if $\Sigma \in \mathscr{F}_{\mathcal{T}}$ then $\sim$ is Borel.

First notice that for each $u, v \in \mathbf{V}(\mathcal{T})$ the set $[u \leftrightarrow v] \cap \Theta$ is exactly the set of bi-infinite paths that pass through both u and v. Now for any ray Π the set of stitches Σ contains $\vartheta(\Pi, \Pi)$ which is the empty configuration. The set of darns however contains only bi-infinite paths and so for any darn $\vartheta \in \Sigma^{\complement}$ there is some pair of vertices $u, v \in \mathbf{V}(\mathcal{T})$ with $\vartheta \in [u \leftrightarrow v]$. Furthermore we have

$$\Sigma^{\complement} \cap [u \leftrightarrow v] = \Theta \cap ([u \leftrightarrow v] \setminus [u \Downarrow v]) . \tag{5.2}$$

Thus if $[u \Downarrow v] \in \mathscr{F}_\mathcal{T}$ we may take a union over the countable set of pairs (u, v) and $\Sigma^\complement \in \mathscr{F}_\mathcal{T}$. Therefore for any measurable equivalence relation $\sim$ on $\mathcal{R}$ the set $\Sigma \in \mathscr{F}_\mathcal{T}$ hence $\sim$ is Borel.

Now for a finite tree $T \in \mathbb{T}_\rho$ let $A^T_{u,v}$ be the event that there exists some stitch $\vartheta_T \in \Sigma$ that passes through both u and v such that every bond $e \in (T \setminus \Pi_{u,v})$ on the thread ϑ_T is open. We claim that if Σ is closed then $[u \Downarrow v] = \bigcap_{T \in \mathbb{T}_\rho} A^T_{u,v}$.

First if $u \Downarrow v$ then we may set $\vartheta_T = \vartheta(\Pi_u, \Pi_v)$ for some pair of open rays $\Pi_u \in \mathcal{R}_u$ and $\Pi_v \in \mathcal{R}_v$, hence $[u \Downarrow v] \subseteq \bigcap_{T \in \mathbb{T}_\rho} A^T_{u,v}$. Conversely if we have $A^T_{u,v}$ for every $T \in \mathbb{T}_\rho$ then if Σ is closed we may choose some limit point $\vartheta \in \Sigma$ such that for any $T \in \mathbb{T}$ there exists $T' \supset T$ such that $\vartheta_{T'}$ agrees with ϑ on T, hence every $e \in (T \setminus \Pi_{u,v})$ on ϑ is open and as T is arbitrary we have $\vartheta = \vartheta(\Pi_u, \Pi_v)$ for some pair of open rays $\Pi_u \in \mathcal{R}_u$ and $\Pi_v \in \mathcal{R}_v$ hence $[u \Downarrow v] = \bigcap_{T \in \mathbb{T}_\rho} A^T_{u,v} \in \mathscr{F}_\mathcal{T}$.

Therefore $[u \Downarrow' v] = \bigcap_{T \in \mathbb{T}_\rho} A^T_{u,v} \in \mathscr{F}_\mathcal{T}$ for any closed equivalence relation $\sim$. ❏

Now we take an alternative view of Grimmet-Janson type boundary conditions. Rather than considering the equivalence relations themselves we focus on the quotient spaces.

DEFINITION 5.2

Let $\sim$ be some equivalence relation on the set $\mathcal{R}_\mathcal{T}$ define the *canopy* of $\sim$ to be the set

$$\mathcal{C}^\sim = \left\{ \left\{ \Pi' \in \Pi_\mathcal{T} : R(\Pi') \sim \Pi \right\} : \Pi \in \mathcal{R} \right\}.$$

There is an obvious bijection between $\mathcal{C}^\sim$ and the quotient space $\mathcal{R}/\sim$ and we equip $\mathcal{C}^\sim$ with a topology such that the two spaces are homeomorphic.

The canopy gives us an alternative view of the random connection. For a vertex $v \in \mathbf{V}(\mathcal{T})$ and a point $c \in \mathcal{C}^\sim$ write $u \downarrow_c$ if there

exists an open ray $\Pi \in c \cap \mathcal{R}_v$. Notice that we may express the event $[u \Downarrow v] = \bigcup_{c \in \mathcal{C}} [u \downarrow_c, v \downarrow_c]$.

We may extend the random connection $\longleftrightarrow$ to the canopy.

First we define a random neighbour relation on $\mathcal{C}$ by defining events $[c_1 \curvearrowright c_2]$ to be the event that there exist rays $\Pi_1 \in c_1$ and $\Pi_2 \in c_2$ such that the thread $\vartheta(\Pi_1, \Pi_2)$ is open.

We wish to extend the relation $\curvearrowright$ to an equivalence relation on $\mathcal{C}$. This idea is easiest to visualize when $\mathcal{C}$ is a discrete space and we explore this idea in more detail below. First we write down the equivalence relation generated by $\curvearrowright$.

LEMMA 5.3

For $c_1 \neq c_2 \in \mathcal{C}$ set

$$[c_1 \longleftrightarrow c_2] = \bigcup_{u,v \in \mathbf{V}(\mathcal{T})} \left([u \downarrow_{c_1}] \cap [u \longleftrightarrow v] \cap [v \downarrow_{c_2}] \right) \qquad (5.3)$$

with $c \longleftrightarrow c$ for every $c \in \mathcal{C}$.

Then $\longleftrightarrow$ is the smallest random equivalence relation on $\mathcal{C}$ such that $c_1 \longleftrightarrow c_2$ whenever $c_1 \curvearrowright c_2$.

PROOF

First if $c_1 \curvearrowright c_2$ then there is some open thread $\vartheta(\Pi_1, \Pi_2)$ with $\Pi_1 \in c_1$ and $\Pi_2 \in c_2$. Then for any pair of vertices u, v on the thread we have $u \downarrow_{c_1}$, $u \longleftrightarrow v$ and $v \downarrow_{c_2}$ hence $c_1 \longleftrightarrow c_2$. It is easy to see that the relation $\longleftrightarrow$ is symmetric, for transitivity suppose $c_1 \longleftrightarrow c_2$ and $c_2 \longleftrightarrow c_3$. Then we may find vertices $u \downarrow_{c_1}$, $v \downarrow_{c_2}$, $v' \downarrow_{c_2}$ and $w \downarrow_{c_3}$ with $u \longleftrightarrow v$ and $v' \longleftrightarrow w$. In this case we have $v \Downarrow v'$ and so $u \longleftrightarrow w$. Therefore $\longleftrightarrow$ is indeed a random equivalence relation on $\mathcal{C}$.

Now suppose there is some smaller equivalence relation $\overset{\star}{\longleftrightarrow}$ on $\mathcal{C}$, satisfying the above. Then we may describe a second equivalence on $\mathbf{V}(\mathcal{T})$ by setting $u \overset{\star}{\longleftrightarrow} v$ if $u \longleftrightarrow v$ or there exist $c_1 \overset{\star}{\longleftrightarrow} c_2$ with $u \downarrow_{c_1}$ and $v \downarrow_{c_2}$.

It is easy to check that $\overset{*}{\leftrightarrow}$ is an equivalence relation on $\mathbf{V}(\mathcal{T})$ and that $u \overset{*}{\leftrightarrow} v$ whenever $u \Downarrow v$. Therefore as $\longleftrightarrow$ is the smallest such equivalence relation on $\mathbf{V}(\mathcal{T})$ we see that if for any $c_1 \neq c_2 \in \mathcal{C}$ we have

$$[c_1 \longleftrightarrow c_2] \subseteq [u \downarrow_{c_1}] \cap [v \downarrow_{c_2}] \cap [u \longleftrightarrow v] \tag{5.4}$$

$$\subseteq [u \downarrow_{c_1}] \cap [v \downarrow_{c_2}] \cap [u \overset{*}{\leftrightarrow} v] \tag{5.5}$$

$$\subseteq [c_1 \overset{*}{\leftrightarrow} c_2] . \tag{5.6}$$

So $\longleftrightarrow$ is indeed the smallest random equivalence relation on $\mathcal{C}$ that satisfies the conditions of the lemma. ❏

A canopy gives an alternative method of specifying an equivalence relation, and hence a random connection. Rather than defining an equivalence relation and then determining the canopy and its topology we may describe the canopy directly as a topological space and specify a continuous map $\mathcal{R} \to \mathcal{C}$. This defines an equivalence relation and if that equivalence relation is measurable we may define a random cluster model on the pair $(\mathcal{T}, \mathcal{C})$. We use this method in the next section where we define a random cluster model based on the map from $\mathcal{R} \to [0, 1]^d$ implicit in the definition of the QuadTree.

First we state a relationship between the topological properties of the canopy and the equivalence relation.

THEOREM 5.4

i). An equivalence relation $\sim$ is open if and only if $\mathcal{C}^{\sim}$ is a discrete topological space.

ii). An equivalence relation $\sim$ is closed if and only if $\mathcal{C}^{\sim}$ is Hausdorff.

iii). The random connection $\overset{\sim}{\leftrightarrow}$ is connected if and only if $\mathcal{C}^{\sim}$ is connected.

PROOF OF I.

Let $q : \mathcal{R} \to \mathcal{C}^{\sim}$ be the quotient map, combined with an appropriate homeomorphism $(\mathcal{R}/\!\sim) \to \mathcal{C} \sim$.

If $\mathcal{C}^{\sim}$ is a discrete space then the set $\{\Pi' : \Pi' \sim \Pi\} = q^{-1} \cdot q(\Pi)$ is the pre-image of a single point in the canopy which is open in the discrete topology. Therefore $\{(\Pi_1, \Pi_2) : \Pi_1 \sim \Pi_2\}$ is a union of open rectangles and is open.

Alternatively if $\{(\Pi_1, \Pi_2) : \Pi_1 \sim \Pi_2\}$ is open then for any $\Pi \in \mathcal{R}$ we may find some open rectangle $a \times b \subset \{(\Pi_1, \Pi_2) : \Pi_1 \sim \Pi_2\}$ with $(\Pi, \Pi) \in a \times b$. In particular $\Pi \in a \cap b \subset \{\Pi' \in \mathcal{R} : \Pi' \sim \Pi\}$ thus each equivalence class is open. As the equivalence classes form a disjoint open cover of the compact set $\mathcal{R}$ there may be only finitely many of them, thus each equivalence class is both open and closed and $\mathcal{C}^{\sim}$ is a finite discrete space. ❑

PROOF OF II.

If $\mathcal{C}^{\sim}$ is Hausdorff then for any pair $\Pi_1 \nsim \Pi_2$ we may choose a pair of disjoint open sets $\mathcal{O}_1 \ni q(\Pi_1)$ and $\mathcal{O}_2 \ni q(\Pi_2) \subset \mathcal{C}^{\sim}$. As the sets $\mathcal{O}_1$ and $\mathcal{O}_2$ are disjoint then $q^{-1}(\mathcal{O}_1) \times q^{-1}(\mathcal{O}_2)$ is an open rectangle with

$$(\Pi_1, \Pi_2) \in q^{-1}(\mathcal{O}_1) \times q^{-1}(\mathcal{O}_2) \tag{5.7}$$

$$\subset \left\{ (\Pi_1', \Pi_2') : \Pi_1' \nsim \Pi'2 \right\}. \tag{5.8}$$

Therefore $\{(\Pi_1', \Pi_2') : \Pi_1' \nsim \Pi_2'\}$ is open in $\mathcal{R}^2$ and in particular $\{(\Pi_1', \Pi_2') : \Pi_1' \sim \Pi_2'\}$ is closed.

Alternatively suppose the set $\{(\Pi_1', \Pi_2') : \Pi_1' \nsim \Pi_2'\}$ is closed and choose $x_1 = q(\Pi_1), x_2 = q(\Pi_2)$.

The equivalence class $q^{-1}(x_1)$ is the projection of the compact set $(\{\Pi_1\} \times \mathcal{R}) \cap \{(\Pi_1', \Pi_2') : \Pi_1' \sim \Pi_2'\}$ and is closed.

So for any ray $\Pi_2' \sim \Pi_2$ the rectangle $q^{-1}(x_1) \times \{\Pi_2'\}$ is a compact subset of the open set $\{(\Pi_1', \Pi_2') : \Pi_1' \nsim \Pi_2'\}$ and we may choose a finite cover of $q^{-1}(x_1) \times \{\Pi_2'\}$ by some finite set of open rectangles

$\{a_i \times b_i : i = 1 \dots n\}$ with $a_i \times b_i \subset \{(\Pi_1', \Pi_2') : \Pi_1' \approx \Pi_2'\}$. Then we may choose a open rectangle

$$q^{-1}(x_1) \times \{\Pi_2'\} \subset \left(\bigcup_{i=1\dots n} a_i \right) \times \left(\bigcap_{i=1\dots n} b_i \right) \tag{5.9}$$

$$\subset \{(\Pi_1', \Pi_2') : \Pi_1' \approx \Pi_2'\}. \tag{5.10}$$

The rectangles above are open and cover the closed rectangle $q^{-1}(x_1) \times q^{-1}(x_2)$ thus we may choose a finite subcover and, combining in the same fashion as above, we may find a open rectangle

$$q^{-1}(x_1) \times q^{-1}(x_2) \subset A \times B \subset \{(\Pi_1', \Pi_2') : \Pi_1' \approx \Pi_2'\}. \tag{5.11}$$

We finish the proof by arguing that q is a closed map. Let S be any closed set in $\mathcal{R}$ and assume without loss of generality that $q(S)$ contains x_1 but not x_2. From above we may choose disjoint open sets A, B with $q^{-1}(x_1) \subset A$ and $q^{-1}(x_2) \subset B$. As x_1 is an arbitrary point of S we may choose a finite set of such pairs A_i, B_i with $S \subset \bigcup_{i=1}^{n} A_i$ and $q^{-1}(x_2) \subset \bigcap_{i=1}^{n} B_i$. The set $\bigcap_{i=1}^{n} B_i$ is open and as x_2 is an arbitrary point of $\mathcal{R} \setminus S$ the set $q^{-1}(\mathcal{C}^{\sim} \setminus q(S))$ is open in $\mathcal{R}$, hence $q(S)$ is closed.

So for arbitrary $x_1 \neq x_2 \in \mathcal{C}^{\sim}$ we may choose disjoint open sets $A \supset q^{-1}(x_1)$ and $B \supset q^{-1}(x_2)$. As q is a closed map the sets $\mathcal{C}^{\sim} \setminus q(\mathcal{R} \setminus A) \ni x_1$ and $\mathcal{C}^{\sim} \setminus q(\mathcal{R} \setminus B) \ni x_2$ are disjoint open sets and $\mathcal{C}^{\sim}$ is Hausdorff. ❑

Before we prove the last statement we state and prove a preliminary lemma.

LEMMA 5.5

Subsets $A, B \subset \mathcal{C}^{\sim}$ have disjoint closures if and only if there exists a tree $T \in \mathbb{T}$ such that any thread $\vartheta \in \vartheta(q^{-1}(A) \times q^{-1}(B))$ passes through T.

PROOF

First assume A and B are closed sets, then $\vartheta\big(q^{-1}(A) \times q^{-1}(B)\big)$ is a continuous image of a compact set, hence is compact.

If in addition there exists some sequence $T_n \uparrow \mathcal{T}$ and threads $\vartheta_n \in \vartheta(A \times B)$ such that ϑ_n does not intersect T_n. then the sequence ϑ_n converges weakly to the empty configuration, $\xi_0 \in \Sigma$. Thus there is some ray $\Pi \in q^{-1}(A) \cap q^{-1}(B)$ and A and B cannot be disjoint. Conversely if A and B are not disjoint then then there is some ray Π with $q(\mathcal{R}) \in A \cap B$ and so $\vartheta\big(q^{-1}(A) \times q^{-1}(B)\big)$ contains the empty configuration. ❑

PROOF OF THEOREM 5.4(III)

Let $T \in \mathbb{T}$ be a finite subtree and let ω_T be that configuration with $\omega_T(e) = 0$ iff $e \in T$. Now consider the equivalence relation $\xleftrightarrow{\omega_T}$ on $\mathcal{C}^\sim$ and write $\Pi_1 \overset{T}{\sim} \Pi_2$ if $q(\Pi_1) \xleftrightarrow{\omega_T} q(\Pi_2)$. Notice that the measure concentrated on ω_T has the single cluster property if and only if the quotient space $\mathcal{C}^\sim / \xleftrightarrow{\omega_T}$ consists of a single point.

Now suppose $\Pi_1 \overset{T}{\sim} \Pi_2$ and let Π_1' and Π_2' be any two rays such that neither Π_1 and Π_1' nor Π_2 and Π_2' differ on T. Then the threads $\vartheta(\Pi_1, \Pi_1')$ and $\vartheta(\Pi_2, \Pi_2')$ do not intersect T and so are open in ω_T. Therefore $q(\Pi_1) \xleftrightarrow{\omega_T} q(\Pi_1')$ and $q(\Pi_2) \xleftrightarrow{\omega_T} q(\Pi_2')$ so we have $\Pi_1' \overset{T}{\sim} \Pi_2'$ and in particular the relation $\overset{T}{\sim}$ is open.

Therefore the quotient space $\mathcal{C}^\sim / \xleftrightarrow{\omega_T}$ is a discrete space. Hence if $\mathcal{C}$ is connected $\mathcal{C}^\sim / \xleftrightarrow{\omega_T}$ is a single point and the measure concentrated on ω_T has the single cluster property. As T is arbitrary the random equivalence relation $\xleftrightarrow{}$ is connected whenever $\mathcal{C}^\sim$ is connected.

Alternatively if $\mathcal{C}^\sim$ is not connected we may partition $\mathcal{C}^\sim$ into two disjoint closed sets A and B. From Lemma 5.5 we may choose $T \in \mathbb{T}$ such that any thread $\vartheta \in \vartheta(A \times B)$ passes through T. Therefore the configuration ω_T does not contain any open thread $\vartheta \in \vartheta(A \times B)$ and the equivalence relation $\overset{T}{\sim}$ is contained within $(A \times A) \cup B \times B$. Therefore the measure concentrated on ω_T does not have the single cluster property and $\xleftrightarrow{}$ is not connected. ❑

5.2 NONCONNECTED RANDOM CONNECTIONS

Our analysis of random connections in the last section was restricted to strongly connected random connections. In particular we have shown that for any strongly connected random connection $\longleftrightarrow$ the wired random cluster model satisfies DLR conditions for $\longleftrightarrow$ when τ is sufficiently high. If a random connection is not strongly connected then the wired random connection will not satisfy DLR conditions for high τ and we must take an alternative approach. In this section we will look at two examples of disconnected canopies where we may still describe the phase diagram.

5.2.1 Open Boundary Conditions

If $\sim$ is an open equivalence relation then the canopy $\mathcal{C}^\sim$ is a finite discrete space. Therefore the random equivalence relation $\longleftrightarrow$ divides $\mathcal{C}^\sim$ into finitely many "boundary clusters." Each boundary cluster corresponds to exactly one infinite cluster of the tree, informally then the random cluster model on a pair $(\mathcal{T}, \mathcal{C}^\sim)$ has in some sense only finitely many more clusters the the wired random cluster model.

Can we then weight the wired random cluster model according to the number of boundary clusters in the same way that we weight percolation on a finite graph according to the number of percolation clusters to obtain a $\longleftrightarrow$ random cluster model?

Our next theorem gives a positive answer to this question.

THEOREM 5.6

Suppose $\mathcal{C} = \mathcal{C}^\sim$ for some *open* equivalence relation $\sim$ Let $\kappa_c :$ $\Omega_\mathcal{T} \to \mathbb{N}$ be the number of equivalence classes of $\mathcal{C}$ under the equivalence relation $\overset{\omega}{\longleftrightarrow}$ defined above.

- A measure φ on $\Omega_{\mathcal{T}}$ is a $\dashleftarrow\dashrightarrow$ random cluster model if and only if there exists $\varphi^\star \in \mathscr{R}_{\mathcal{T}}^\star$ such that

$$\varphi(A) = Z_\varphi^{-1} \int \mathbb{1}_A(\omega) q^{\kappa_C(\omega)} d\varphi^\star(\omega) \tag{5.12}$$

for the normalizing constant $Z_\varphi = \varphi(q^{\kappa_C})$.

- For a configuration $\xi \in \Omega_{\mathcal{T}}$ we have $\mathrm{wlim}_{T\uparrow\mathcal{T}} \mathsf{Q}_{\mathcal{T}}^{\overset{\xi}{\dashleftarrow\dashrightarrow}} = \varphi \in \mathscr{R}_{\mathcal{T}}^{\dashleftarrow\dashrightarrow}$ if and only if $\mathrm{wlim}_{T\uparrow\mathcal{T}} \bar{\mathsf{Q}}_{\mathcal{T}}^{\xi} = \varphi^\star \in \mathscr{R}_{\mathcal{T}}^\star$ with φ and $\varphi^\star$ related as above.

PROOF

First let $c_1 \neq c_2$ be any two distinct points of the canopy. As $\mathcal{C}$ is a discrete space the points c_1 and c_2 are closed disjoint sets hence there is some tree $S \in \mathbb{T}$ such that any $\vartheta \in \vartheta(c_1 \times c_2)$ passes through S.

As there are only finitely many such pairs then we may assume that S above is large enough that any darn $\vartheta \in \Sigma^{\mathcal{C}}$ must pass through S.

Now let T be any finite tree with $S \subset T \in \mathbb{T}$ and recall the map $\psi_T : \Omega_{\mathcal{T}} \to \Omega_{T^\star}$. We claim that for any vertex $v \in T$ and for any pair $c_1 \neq c_2 \in \mathcal{C}$ the events $[v \downarrow_{c_1}]$ and $[c_1 \curvearrowright c_2]$ are ψ_T-measurable.

As the root ρ is arbitrary we may assume without loss of generality that $\rho = v$ so that $\mathcal{R} = \mathcal{R}_v$.

For a ray $\Pi \in \mathcal{R}$ recall the configuration $\omega_\Pi \in \Omega_{\mathcal{T}}$. Let $R \subset \Omega_{T^\star}$ be the set of configurations

$$R = \{\psi_T(\omega_\Pi) : \Pi \in \mathcal{R}\}. \tag{5.13}$$

Now as $R \subset \Omega_{T^\star}$ is finite we may choose finitely many representatives $\{\Pi_1^T, \ldots, \Pi_n^T\}$ such that for any $\Pi \in \mathcal{R}$ we have $\psi_T(\omega_\Pi) = \psi_T\left(\omega_{\Pi_i^T}\right)$ for exactly one $i \in [\![1, n]\!]$.

Notice that for any ray $\Pi \in \mathcal{R}$ if $\psi_T(\omega_\Pi) = \psi_T\left(\omega_{\Pi_i^T}\right)$ then ω_Π and $\omega_{\Pi_i^T}$ agree on T and so $\vartheta(\Pi, \Pi_i^T) \in J_e^\complement$ for every edge $e \in T$. Hence the thread $\vartheta(\Pi, \Pi_i^T)$ does not intersect T and so $\Pi \sim \Pi_i^T$.

In particular we may partition R into subsets

$$R = \bigsqcup_{c \in \mathcal{C}} R_c \tag{5.14}$$

where

$$R_c = \left\{ \psi_T\left(\omega_{\Pi_i^T}\right) \,\middle|\, \Pi_i^T \in c \right\}. \tag{5.15}$$

Furthermore $\omega \in [v \downarrow_c]$ if and only if $\psi_T(\omega) \geq \psi_T(\omega_i)$ for some $\omega_i \in R_c$. Therefore $[v \downarrow_c]$ is ψ_T-measurable and as every darn intersects T we have

$$[c_1 \curvearrowright c_2] = \bigcup_{v \in T} [v \downarrow_{c_1}, v \downarrow_{c_2}] \tag{5.16}$$

is ψ_T-measurable.

Now the equivalence relation $\longleftrightarrow$ is generated by $\curvearrowright$ and is ψ_T-measurable. In particular for $u, v \in \mathbf{V}(T)$ the event $[u \longleftrightarrow v]$ is ψ_T-measurable also.

Say a point $c \in \mathcal{C}$ is *T-isolated* if there exists no $v \in T$ with $T \downarrow_c$. Define a random variable ℓ_T to be the number of T-isolated points in the canopy.

Notice that from (5.16) if c is T-isolated for $T \supset S$ there can be no c' with $c \curvearrowright c'$. Furthermore for any $\xi \in \Omega_T$ the random variable ℓ_T is constant on Ω_T^ξ.

Recall that $\kappa_T^\star(\omega)$ is the number of *finite* $\longleftrightarrow$ clusters that intersect T. Now as each $\longleftrightarrow$-equivalence class of $\mathcal{C}$ is either a T isolated point or corresponds to exactly one $\stackrel{\omega,\sim}{\longleftrightarrow}$-cluster that intersects T we have

$$\kappa_T^{\leftrightarrow} + \ell_T = \kappa_T^\star + \kappa_c. \tag{5.17}$$

In particular for every $\xi \in \Omega_{\mathcal{T}}$, from the definition of the cylinder random cluster models $\mathbb{Q}_T^{\overset{\xi}{\leftrightarrow}}$ and $\overline{\mathbb{Q}}_T^{\xi}$ we have

$$\frac{\mathbb{Q}_T^{\overset{\xi}{\leftrightarrow}}(\omega)}{\overline{\mathbb{Q}}_T^{\xi}(\omega)} = \frac{\left(Z_T^{\overset{\xi}{\leftrightarrow}}\right)^{-1}\mathbb{P}_T(\omega) \cdot q^{\kappa_T^{\leftrightarrow}(\omega)}}{\left(Z_T^{\xi\star}\right)^{-1}\mathbb{P}_T(\omega) \cdot q^{\kappa_T^{\star}(\omega)}} \tag{5.18}$$

$$= \left(\frac{Z_T^{\xi\star}}{Z_T^{\overset{\xi}{\leftrightarrow}}}q^{-\ell_T(\omega)}\right)q^{\kappa_C(\omega)} \tag{5.19}$$

$$= \left(\overline{\mathbb{Q}}_T^{\xi}(q^{\kappa_C})\right)^{-1}q^{\kappa_C(\omega)} \tag{5.20}$$

where the last line follows from the observation that the quantity $\left(\frac{Z_T^{\xi\star}}{Z_T^{\overset{\xi}{\leftrightarrow}}}q^{-\ell_T(\omega)}\right)$ is $\mathscr{T}_T$-measurable and hence is a normalizing constant.

We may express (5.20) in the form

$$\overline{\mathbb{Q}}_T^{\xi}(X \cdot q^{\kappa_C}) = \mathbb{Q}_T^{\overset{\xi}{\leftrightarrow}}(X)\overline{\mathbb{Q}}_T^{\xi}(q^{\kappa_C}) \tag{5.21}$$

for any $\mathscr{F}_T$-measurable random variable X.

Now suppose $\varphi^{\star} \in \mathscr{R}_{\mathcal{T},q}^{\star}$ is a wired random cluster model and set

$$\varphi(A) = \varphi^{\star}(q^{\kappa_C})^{-1}\int \mathbb{1}_A(\omega)q^{\kappa_C(\omega)}d\varphi^{\star}(\omega). \tag{5.22}$$

We claim that φ is a $\leftrightarrow$-random cluster model. To see this let X and Y be $\varphi^{\star}$- integrable random variables, with Y $\mathscr{T}_T$-measurable. We have

$$\varphi(XY) = \varphi^{\star}(q^{\kappa_C})^{-1}\int X(\xi)Y(\xi)q^{\kappa_C(\xi)}d\varphi^{\star}(\xi) \tag{5.23}$$

$$= \varphi^{\star}(q^{\kappa_C})^{-1}\int \varphi^{\star}(Xq^{\kappa_C}\mid\mathscr{T}_T)(\xi)Y(\xi)d\varphi^{\star}(\xi) \tag{5.24}$$

$$= \varphi^{\star}(q^{\kappa_C})^{-1}\int \overline{\mathbb{Q}}_T^{\xi}(X \cdot q^{\kappa_C})Y(\xi)d\varphi^{\star}(\xi) \tag{5.25}$$

$$= \varphi^{\star}(q^{\kappa_C})^{-1}\int \mathbb{Q}_T^{\overset{\xi}{\leftrightarrow}}(X)\overline{\mathbb{Q}}_T^{\xi}(q^{\kappa_C})Y(\xi)d\varphi^{\star}(\xi) \tag{5.26}$$

$$= \varphi^{\star}(q^{\kappa_C})^{-1}\int \mathbb{Q}_T^{\overset{\xi}{\leftrightarrow}}(X)q^{\kappa_C(\xi)}Y(\xi)d\varphi^{\star}(\xi) \tag{5.27}$$

$$= \int \mathbb{Q}_T^{\overset{\xi}{\leftrightarrow}}(X)Y(\xi)d\varphi(\xi). \tag{5.28}$$

So $(\varphi \,|\, \mathscr{T}_T)(\xi) = Q_T^{\overleftarrow{\xi}}$ for sufficiently large $T \in \mathbb{T}$, hence $\varphi \in \mathscr{R}_{\mathcal{T}}^{\rightharpoonup}$. The converse is identical replacing $q^{\kappa c}$ with $q^{-\kappa c}$ and so $\varphi \in \mathscr{R}_{\mathcal{T}}^{\rightharpoonup}$ if and only if $\varphi^\star \in \mathscr{R}_{\mathcal{T}}^\star$.

Now given $\xi \in \Omega_{\mathcal{T}}$ such that $\overline{Q}_T^{\xi} \overset{w}{\Rightarrow} \varphi^\star$ as $T \uparrow \mathcal{T}$ from Lemma 3.12 $\varphi^\star$ is a Markov chain and $\psi_{T'}\left(\overline{Q}_T^{\xi}\right) \to \psi_{T'}(\varphi^\star)$ as $T \uparrow \mathcal{T}$ for every $T' \in \mathbb{T}$.

In particular for any continuous simple function X we have

$$Q_T^{\overleftarrow{\xi}}(X) = \left(\overline{Q}_T^{\xi}(q^{\kappa c})\right)^{-1} \overline{Q}_T^{\xi}(X \cdot q^{\kappa c}) \tag{5.29}$$

$$\overset{w}{\Rightarrow} (\varphi^\star(q^{\kappa c}))^{-1} \varphi^\star(X \cdot q^{\kappa c}) \quad \text{as } T \uparrow \mathcal{T} \tag{5.30}$$

$$= \varphi(X). \tag{5.31}$$

Conversely if $\overline{Q}_T^{\xi}$ does not converge then by compactness we may choose sequences of trees T_n and T_n' such that $\overline{Q}_{T_n}^{\xi} \overset{w}{\Rightarrow} \varphi^\star$ and $\overline{Q}_{T_n}^{\xi} \overset{w}{\Rightarrow} \tilde{\varphi}^\star$ with $\varphi^\star \neq \tilde{\varphi}^\star$. Then setting φ and $\tilde{\varphi}$ as above we have $\text{wlim}\,Q_{T_n}^{\overleftarrow{\xi}} = \varphi \neq \tilde{\varphi} = \text{wlim}\,Q_{T_n'}^{\overleftarrow{\xi}}$. $\qquad\square$

5.2.2 The paired tree model

Open boundary conditions concern discrete canopies containing only finitely many points. For our next example we consider a totally disconnected infinite canopy.

We will consider a random connection on a forest containing a pair of identical trees. We have not defined the apparatus of random connections on a forest. However we may consider such a forest as a single tree with one edge removed, either by conditioning it to be closed or by setting the bond strength to be very low.

Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be a pair of trees such that there exists a graph homomorphism $\zeta : \mathbf{V}(\mathcal{T}_1) \leftrightarrow \mathbf{V}(\mathcal{T}_2)$. For simplicity, assume that both $\mathcal{T}_1$ and $\mathcal{T}_2$ have isotropic bond weights $\gamma(e) \equiv \tau$. Set $\mathcal{F} =$

$\mathcal{T}_1 \cup \mathcal{T}_2$, we interpret ζ as a reflection on $\mathcal{F}$ by setting $\zeta = \zeta^{-1}$. We abuse notation by allowing $\zeta : \mathbf{E}(\mathcal{F}) \to \mathbf{E}(\mathcal{F})$ to be the induced map on the set of edges.

We introduce a random neighbour relation $\{[u \Downarrow v] : u = \zeta(v)\}$ on $\mathcal{F}$ where $u \Downarrow v$ is the event that there exists a ray $\Pi \in \mathcal{R}_u$ such that both Π and $\zeta(\Pi)$ are open and let $\longleftrightarrow$ be the random connection generated by $\Downarrow$.

We leave it to the reader to convince himself that the random relation $\Downarrow$ generates a random connection. This may easily be seen by noting that by nominating a root $\rho \in \mathbf{V}(\mathcal{T}_1)$ arbitrarily and adding an extra edge $\langle \rho, \zeta(\rho) \rangle$. We may describe the random connection generated by $\Downarrow$ in terms of an equivalence relation on the new tree.

THEOREM 5.7

Set $\widetilde{\tau} = \pi^{-1}\left(\pi(\tau)^2\right) = \frac{\tau^2}{\tau(2-\tau)+(1-\tau)^2 q}$ and let $\widetilde{\mathcal{T}}$ be the tree isomorphic to $\mathcal{T}_1$ but with edge weights $\gamma(e) \equiv \widetilde{\tau}$.

Then the sets $\mathscr{R}_{\mathcal{F}}^{\longleftrightarrow}$ and $\mathscr{R}_{\widetilde{\mathcal{T}}}^{\star}$ are homomorphic.

Before proving Theorem 5.7 we will examine the random connection $\longleftrightarrow$ more closely. In particular we wish to show that for a bond $e = \langle u, v \rangle \in \mathbf{E}(\mathcal{T}_1)$

$$\mathcal{L}_e^{\longleftrightarrow} = J_{\zeta(e)} \cap [u \Downarrow \zeta(u)] \cap [v \Downarrow \zeta(v)]. \tag{5.32}$$

Choose $u, v \in \mathbf{V}(\mathcal{T}_1)$ and suppose $u \longleftrightarrow v$. We may find a sequence of vertices $u_0, \ldots, u_n \in \mathbf{V}(\mathcal{T}_1)$ such that

$$u \longleftrightarrow u_0 \Downarrow \zeta(u_0) \longleftrightarrow \zeta(u_1) \Downarrow \ldots \Downarrow u_n \longleftrightarrow v. \tag{5.33}$$

Now suppose there exists some i such that u_i above is not on the direct path from u_{i-1} to u_{i+1}. Let u_i' be the vertex closest to

u_i on the direct path from u_{i-1} to u_{i+1}. Assume without loss of generality that i is even, then we have

$$u_{i-1} \longleftrightarrow u_i \Downarrow \zeta(u_i) \longleftrightarrow \zeta(u_{i+1}) \tag{5.34}$$

$$\Leftrightarrow \quad u_{i-1} \longleftrightarrow u_i' \longleftrightarrow u_i \Downarrow \zeta(u_i) \longleftrightarrow \zeta(u_i') \longleftrightarrow \zeta(u_{i+1}) \tag{5.35}$$

$$\Leftrightarrow \quad u_{i-1} \longleftrightarrow u_i' \Downarrow \zeta(u_i') \longleftrightarrow \zeta(u_{i+1}). \tag{5.36}$$

So we may assume without loss of generality that each u_i is on the direct path from u to v. Now if $\langle u, v \rangle \in \mathbf{E}(\mathcal{T}_1)$ then there are only two vertices (u and v) on this path. Therefore if we have $u \longleftrightarrow v$ then either $u \longleftrightarrow v$ or $u \Downarrow \zeta(u) \longleftrightarrow \zeta(v) \Downarrow v$.

Now we may use this observation, specifically in the form of equation (5.32) to describe the random cluster model on the paired tree.

PROOF OF THEOREM 5.7

Let $\widetilde{\mathcal{T}}$ be the tree isomorphic to $\mathcal{T}_1$ but with edge weights $\gamma(e) \equiv \widetilde{\tau}$ as in the statement of the theorem.

Ignoring edge weights, $\widetilde{\mathcal{T}}$ is isomorphic both to $\mathcal{T}_1$ and $\mathcal{T}_2$. For an edge $e \in \widetilde{\mathcal{T}}$ let e_1 and e_2 be the corresponding edges in $\mathcal{T}_1$ and $\mathcal{T}_2$ respectively.

Now we define a second tree $\mathcal{S}$ by replacing each edge $e \in \mathbf{E}(\widetilde{\mathcal{T}})$ with two edges $\tilde{e}_1$ and $\tilde{e}_2$ in series. Specifically we let $\mathbf{V}(\mathcal{S}) = \mathbf{V}(\widetilde{\mathcal{T}}) \cup \{v_e : e \in \mathbf{E}(\widetilde{\mathcal{T}})\}$. For definiteness we direct $\widetilde{\mathcal{T}}$ arbitrarily. For each $e = |u, v\rangle \in \mathbf{E}(\widetilde{\mathcal{T}})$ we set $\tilde{e}_1 = \langle u, v_e \rangle$ and $\tilde{e}_2 = \langle v_e, v \rangle$ and let $\mathbf{E}(\mathcal{S}) = \{\tilde{e}_i : e \in \mathbf{E}(\widetilde{\mathcal{T}}), i \in \{1, 2\}\}$. Assign $\mathcal{S}$ isotropic edge weights $\gamma(\tilde{e}_i) \equiv \tau$.

Now we create homomorphisms between the sets of random cluster models.

Define $\psi : \Omega_{\mathcal{F}} \to \Omega_{\mathcal{S}}$ and $\widetilde{\psi} : \Omega_{\mathcal{S}} \to \Omega_{\widetilde{\mathcal{T}}}$ by setting

$$\psi^{-1}(J_{\tilde{e}_i}) = J_{e_i}, \tag{5.37}$$

$$\widetilde{\psi}^{-1}(J_e) = J_{\tilde{e}_1} \cap J_{\tilde{e}_2}. \tag{5.38}$$

The theorem follows from two claims

- φ is a random cluster model on $\mathcal{S}$ if and only if $\widetilde{\psi}(\varphi)$ is a random cluster model on $\widetilde{\mathcal{T}}$ and

$$\varphi = \int \left(\mathbb{P}_{\pi(\mathcal{S})} \, \middle| \widetilde{\psi} \right)(\omega) d\varphi(\omega). \tag{5.39}$$

- φ is a random cluster model on $\mathcal{F}$ if and only if $\psi(\varphi)$ is a random cluster model on $\mathcal{S}$.

The first claim follows from the generalized series and parallel laws. For $T \in \mathbb{T}_{\widetilde{\mathcal{T}}}$ let $S \in \mathbb{T}_{\mathcal{S}}$ be the tree containing $\tilde{e}_1$ and $\tilde{e}_2$ for every $e \in T$ and let $\widetilde{\psi}_T : \Omega_{S^\star} \to \Omega_{T^\star}$ be the map defined by setting $\widetilde{\psi}_T^{-1}(J_e) = J_{\tilde{e}_1} \cap J_{\tilde{e}_2}$ for $e \in \mathbf{E}(T)$ and $\widetilde{\psi}_T(J_{e^\star}) = J_{\tilde{e}_i^\star}$ for $\tilde{e}_i \in \Lambda(S)$. (Recall only one of $\tilde{e}_1, \tilde{e}_2$ is a leaf of S.)

From Theorem 2.23 (applied once to each edge $e \in T$) for every $\widetilde{\mathcal{T}}$ entrance law H there is a coupling $\omega = (\omega_S, \omega_T) \sim \mu_\mathrm{H}$ such that $\omega_S \sim \mathbb{Q}_S^\mathrm{H}$, $\omega_T = \widetilde{\psi}_T(\omega_S) \sim \mathbb{Q}_T^\mathrm{H}$ and $\mu_\mathrm{H}(\omega_S \,|\, \omega_T) \sim \left(\mathbb{P}_{S^\star} \, \middle| \widetilde{\psi}_T \right)(\omega_S)$.

Letting $T \uparrow \widetilde{\mathcal{T}}$ the first statement holds for all extremal random cluster models and hence $\widetilde{\psi}$ is a homomorphism between $\mathscr{R}_S^\star$ and $\mathscr{R}_S^\star$.

For the second claim choose some $u \in \mathbf{V}(\widetilde{\mathcal{T}}) \subset \mathbf{V}(\mathcal{S})$. Then there is a pair of vertices $v_1 = \zeta(v_2) \in \mathcal{F}$ corresponding to v. Now $v \downarrow_\infty$ in $\mathcal{S}$ if and only if there is some v-ray Π in $\widetilde{\mathcal{T}}$ such that both $\tilde{e}_1$ and $\tilde{e}_2$ are open for every edge e on Π. Therefore we have $\psi^{-1}[v \downarrow_\infty] = [v_1 \Downarrow v_2]$.

From (5.32) we have

$$\psi^{-1}\left(\mathcal{L}_{\tilde{e}_1}^\star \right) = \psi^{-1}[u \downarrow_\infty] \cap \psi^{-1}(J_{\tilde{e}_2}) \cap \psi^{-1}[v \to \infty] \tag{5.40}$$

$$= [u_1 \Downarrow u_2] \cap J_{e_2} \cap [v_2 \Downarrow v_1] \tag{5.41}$$

$$= \mathcal{L}_{e_1}^{\leftrightarrow}. \tag{5.42}$$

Therefore if $\psi(\varphi) \in \mathscr{R}_S^\star$ we have

$$\left(\varphi \,|\, \mathscr{T}_{e_1} \right) = \left(\psi(\varphi) \,|\, \mathscr{T}_{\tilde{e}_1} \right)(\omega) \tag{5.43}$$

$$= \pi(\tau) + (\tau - \pi(\tau)) \mathbb{1}_{\mathcal{L}_{\tilde{e}_1}^\star}(\omega) \tag{5.44}$$

$$= \pi(\tau) + (\tau - \pi(\tau)) \mathbb{1}_{\mathcal{L}_{e_1}^{\leftrightarrow}}(\omega) \tag{5.45}$$

and so $\varphi \in \mathscr{R}_{\mathcal{F}}^{\leftrightarrow}$.

It is easy to see that the induced maps $\psi : \mathscr{R}_{\mathcal{F}}^{\leftrightarrow} \to \mathscr{R}_{\mathcal{S}}^{\star}$ and $\widetilde{\psi} : \mathscr{R}_{\mathcal{S}}^{\star} \to \mathscr{R}_{\mathcal{T}}^{\leftrightarrow}$ are homomorphisms hence $\mathscr{R}_{\mathcal{F}}^{\leftrightarrow}$ is homomorphic to $\mathscr{R}_{\mathcal{T}}^{\star}$ as claimed. ❑

5.3 PERCOLATION WITH MANDELBROT BOUND– ARY CONDITIONS

Recall from Chapter 1 we defined a informally a random connection $\longleftrightarrow$ on a rooted 2^d tree by considering the small λ limit of the QuadTree.

The QuadTree is a graph whose vertices are a set of *pixels*, dyadic subcubes of $[0,1]^d$. For clarity we will use $\mathcal{P}_v$ to represent the pixel associated with a vertex v.

Now we recall some terminology from Chapter 1. For a set of pixels $\mathcal{P}$, $\mathbf{T}(\mathcal{P})$ is the set of tree edges in the form $e = |\mathcal{M}(v), v\rangle$ where $\mathcal{M}(v)$ is the unique mother pixel of v. $\mathbf{L}(\mathcal{P})$ is the set of lattice edges of the QuadTree, these will not be bonds in our probability space, but will be useful is navigating the underlying tree. Recall a screen S is a rectangle in $\mathbb{R}^d$ that may be partitioned up to a set of measure zero by a set of pixels $\mathcal{P}_S^0 \subset \mathcal{P}_{\mathbb{R}^d}^n$ where n is the *resolution* of the screen.

Now let $\Pi = (v_1, v_2, \dots)$ be a directed path on $\mathbf{T}(\mathcal{Q})$. For each $i < j \in \mathbb{N}$ we have $v_j \in D(v_i)$ and so $\mathcal{P}_{v_i} \supset \mathcal{P}_{v_j}$, furthermore the sidelengths of $\mathcal{P}_{v_i}$ decrease to zero, so by compactness of $[0,1]^d$ there exists some point

$$\Psi(\Pi) = \bigcap_{i=1}^{\infty} \mathcal{P}_{v_i} \in [0,1]^d . \tag{5.46}$$

Set $\mathcal{T} = \left(\mathcal{P}_{[0,1]^d}^{[\![0,\infty]\!]}, \mathbf{T}\left(\mathcal{P}_{[0,1]^d}^{[\![0,\infty]\!]} \right) \right)$ with bond weights $\gamma(e) \equiv \tau$. We have defined a continuous map $\Psi : \mathcal{R}(\mathcal{T}) \to [0,1]^d$ this defines

FIGURE 10
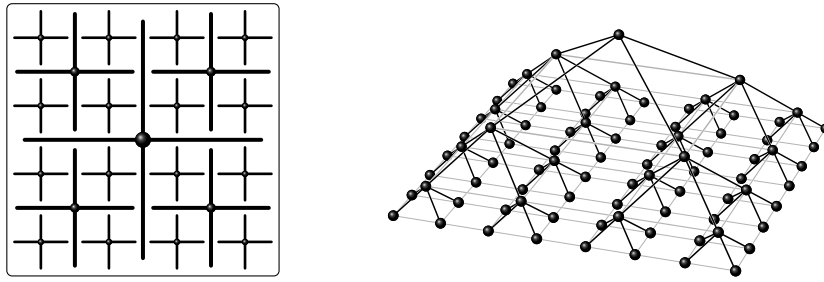
The QuadTree structure from Chapter 1, this time we are interested in the underlying tree together with a random connection defined by the canopy $[0,1]^d$.

an equivalence relation on $\mathcal{R}$ which, as $[0,1]^d$ is Hausdorff, is closed by Theorem 5.4 hence measurable from Theorem 5.1.

The results in Chapter 4 concern regular unrooted trees. $\mathcal{T}$ is a rooted tree, that is it has one vertex with only 2^d adjacent vertices. However the difference between rooted and unrooted trees is in this case only cosmetic. We have used the symmetry of the regular tree only once, in the proof of Theorem 4.21. It is easy to check that this theorem holds for the rooted tree as well as for the unrooted tree.

NOTATION

Define the canopy of $\mathcal{T}$, $\mathcal{C} = [0,1]^d$. Let $\overset{M}{\sim}$ be the equivalence relation on $\mathcal{R}$ generated by the map $\Psi : \mathcal{R} \to \mathcal{C}$ described in (5.46) and let $\longleftrightarrow$ be the random connection generated by $\overset{M}{\sim}$.

Now define a strong boundary condition $\Downarrow$ that generates $\longleftrightarrow$. Recall the subnetworks $\mathcal{P}_{\mathcal{C}}^n$ of pixels of resolution n. For $u, v \in \mathcal{P}_{\mathcal{C}}^n$ let $[u \Downarrow v]$ be the event that there exists some $c \in \mathcal{P}_u \cap \mathcal{P}_v$ such that $u \downarrow_c$ and $v \downarrow_c$.

It is clear that $\Downarrow$ is a strong boundary condition and generates the random connection $\longleftrightarrow$. To see this notice that for $u, v \in \mathcal{P}_{\mathcal{C}}^n$ we have $u \Downarrow v$ if and only if $u \Downarrow' v$ where $u \Downarrow' v$ is the strong boundary condition in Lemma 4.18. It is easy to see that if $u \in$

$\mathcal{P}_c^m$ and $v \in \mathcal{P}_c^n$ with $m < n$ then there exists some $u' \in \mathcal{P}_c^n$ with $u \longleftrightarrow u'$ and $u' \Downarrow\!\!\!\!/\ v$.

The map Ψ is clearly surjective and the canopy $\mathcal{C} = [0,1]^d$ is connected. Therefore the random connection $\longleftrightarrow$ is connected also.

The aim of this section is to show

THEOREM 5.8

- If $d = 1$ the random connection $\longleftrightarrow$ is connected but not strongly connected.

- If $d > 1$ the random connection $\longleftrightarrow$ is strongly connected.

- The critical percolation probabilities are

$$\tau_c^{\longleftrightarrow} = \tau_u^{\longleftrightarrow} = 2^{\frac{1-d}{2}}. \tag{5.47}$$

The first two statements follow directly from the third. We will prove the third statement as two lemmata, a lower bound in Lemma 5.9 and an upper bound Lemma 5.14.

5.3.1 Mandelbrot's Percolation process and τ_c

The map Ψ is not new. The image of the set of open ρ-rays $\{c \in \mathcal{C} : \rho \downarrow_c\}$ corresponds exactly to the famous *canonical curdling* process of Mandelbrot [51]; often refered to simply as Mandelbrot's percolation process. The tree was not explicitly mentioned in [51] and Mandelbrot's arguments were in places non-rigorous.

Chayes et al. [16] formalized the study of Mandelbrot's percolation process relying heavily on the percolation process $\mathbb{P}_{\mathcal{T}}$. The authors were especially interested in connectivity properties of the image $\{c \in \mathcal{C} : \rho \downarrow_c\}$, which are not a concern here. However the paper contains the quantity $2^{\frac{d-1}{2}}$ as a lower bound for

the connectivity threshold of Mandelbrot percolation and their argument may easily be adapted for our purposes.

LEMMA 5.9

$\mathbb{P}_{\mathcal{T}}$ has the $\longleftrightarrow$-*loopless* property if and only if $\tau \le 2^{\frac{1-d}{2}}$, that is

$$\tau_c^{\longleftrightarrow} = 2^{\frac{1-d}{2}}.$$

PROOF

We follow Chayes et al. [16]. We will be brief as we are only adapting an existing argument to a new setting. A more detailed version of this argument appears in [16].

Suppose $\tau < 2^{\frac{1-d}{2}}$ now choose $u, v \in \mathcal{P}_c^n$. If $\mathcal{P}_u \cap \mathcal{P}_v$ is empty then the event $[u \Downarrow v] = \varnothing$. If $\mathcal{P}_u \cap \mathcal{P}_v$ is not empty then it must be some $\tilde{d}$-dimensional subcube of $\mathcal{C}$ with $\tilde{d} < d$.

Now choose a point $c \in \mathcal{P}_u \cap \mathcal{P}_v$, and let $\Pi_u = (u = u_0, u_1, \dots) \in \mathcal{R}_u$ and $\Pi_v = (v = v_0, v_1, \dots) \in \mathcal{R}_v$ be directed rays with $\Psi(\Pi_u) = \Psi(\Pi_v) = c$.

Now for each $i \in \mathbb{N}$ we have $\left(\mathcal{P}_{u_i} \cap \mathcal{P}_{v_i} \right) \supset \left(\mathcal{P}_{u_{i+1}} \cap \mathcal{P}_{v_{i+1}} \right) \ni c$. Therefore there is some decreasing sequence of integers d_i such that $\mathcal{P}_{u_i} \cap \mathcal{P}_{v_i}$ is a d_i dimensional cube.

As d_i is decreasing it is eventually constant. Let $\left[u \Downarrow^{\tilde{d}} v \right]$ be the event that there exist open directed rays Π_u, Π_v as above with $d_i = \tilde{d}$ for every $i \ge 0$.

In particular if $u \Downarrow^{\tilde{d}} v$ then for each $i \in \mathbb{N}$, $\mathcal{P}_{u_i}$ is the orthogonal reflection of $\mathcal{P}_{v_i}$ in the $\tilde{d}$-dimensional hyperplane containing $\mathcal{P}_u \cap \mathcal{P}_v$. Now the subtree $\mathcal{T}_{u \cap v}$ with vertices $\{\tilde{u} : \mathcal{P}_u \supset \mathcal{P}_{\tilde{u}} \supset \mathcal{P}_u \cap \mathcal{P}_v\}$ is a rooted $2^{\tilde{d}}$-tree and $u \Downarrow^{\tilde{d}} v$ if and only if there exist some path $\Pi_u = (u = u_0, u_1, \dots)$ such that both Π_u and the orthogonal reflection of Π_u are open. Therefore $\mathbb{P}_{\mathcal{T}}\left[u \Downarrow^{\tilde{d}} v \right] > 0$ if and only if $\tau^2 > 2^{-\tilde{d}}$.

As $\tilde{d} \le d - 1$ then $\mathbb{P}_{\mathcal{T}}[u \Downarrow v] > 0$ if and only if $\tau > 2^{\frac{1-d}{2}}$. $\qquad \square$

5.3.2   One–dependent percolation and jungles

We move on to the upper bound. Our aim is to use renormalization techniques to construct a giant cluster far from the root that is large enough to intersect any infinite cluster with high probability.

For any screen $S \subset \mathbb{R}^d$ we may define a forest $\left(\mathcal{P}_S^{[\![0,\infty]\!]}, \mathbf{T}\left(\mathcal{P}_S^{[\![0,\infty]\!]} \right) \right)$. We interpret the rays of this forest $\mathcal{R}(S)$ to be the set of half infinite paths started at some $\mathcal{P} \in \mathcal{P}_S^0$.

Now associate with each screen a *jungle* $\mathcal{J}_S$ consisting of the forest $\mathcal{J}_S = \left(\mathcal{P}_S^{[\![0,\infty]\!]}, \mathbf{T}\left(\mathcal{P}_S^{[\![0,\infty]\!]} \right) \right)$ together with the canopy map $\Psi : \mathcal{R}(S) \to S$ and induced random connection $\longleftrightarrow$ on $\mathcal{J}_S$.

We will be particularly interested in the jungles induced by lattice edges of the QuadTree. For an edge $e = \langle u, v \rangle \in \mathbf{L}$ set $\mathcal{J}_e = \mathcal{J}_{\mathcal{P}_u \cup \mathcal{P}_v}$. It will be convenient to stage our argument not on $\mathcal{T}$ but on a jungle consisting of two trees isomorphic to $\mathcal{J}_e$ for arbitrary $e \in \mathbf{L}$. So denote the cuboid $2\mathcal{C} = [-1, 1] \times [0, 1]^{d-1}$ and set $\mathcal{J} = \mathcal{J}_{2\mathcal{C}}$.

We name the two root vertices of the forest $\mathcal{J}$, ρ and ρ' where $\mathcal{P}_\rho = [0, 1]^d$ and $\mathcal{P}_{\rho'} = [-1, 0] \times [0, 1]^{d-1}$.

Now for any edge $e \in \mathbf{L}\left(\mathcal{P}_{2\mathcal{C}}^{[\![0,\infty]\!]} \right)$ we have $\mathcal{J}_e \subset \mathcal{J}$ We may consider $\mathscr{F}_{\mathcal{J}_e}$ to be a subset of $\mathscr{F}_{\mathcal{J}}$.

We wish to define block events, that is a set of similar events $\left\{ A_e \,\middle|\, e \in \mathbf{L}\left(\mathcal{P}_{2\mathcal{C}}^{[\![0,\infty]\!]} \right) \right\}$ with $A_e \in \mathscr{F}_{\mathcal{J}_e}$. These should be interpreted as copies of an event $A \in \mathscr{F}_{\mathcal{J}}$ occurring on the subgraph $\mathcal{J}_e$.

We will construct such events using a recursive renormalization argument due to Balister et al. [3]. To this end it will be convenient to define our block events formally in such a way as to make use of the recursive structure of the QuadTree.

For any isometry $f : 2\mathcal{C} \to 2\mathcal{C}$ there is an induced homomorphism $\tilde{f} : \Omega_{\mathcal{J}} \to \Omega_{\mathcal{J}}$ which preserves the random connection

↤↦. Let $\mathscr{G} \subset \mathscr{F}_{\mathcal{J}}$ be σ-algebra of events invariant under all such homomorphisms.

For an edge $e \in \mathbf{L}\left(\mathcal{R}_{2\mathcal{C}}^{[\![0,\infty]\!]}\right)$ choose an arbitrary similitude g : $2\mathcal{C} \to (\mathcal{P}_u \cup \mathcal{P}_v)$ and extend to a map $\mathcal{J} \to \mathcal{J}_e$ in the obvious way. We may then define $\tilde{g} : \Omega_{\mathcal{J}} \to \Omega_{\mathcal{J}}$ by setting $\tilde{g}^{-1}(J_{e'}) = J_{g(e')}$.

Now define a map $T_e : \mathscr{G} \to \mathscr{F}_{\mathcal{J}}$ by setting $T_e(A) = \tilde{g}^{-1}(A)$. As we have restricted ourselves to the invariant σ-algebra $\mathscr{G}$ the map T_e does not depend on our choice of similitude.

To prove the remaining part of Theorem 5.8 we will consider events defined recursively by mapping $\Omega_{\mathcal{J}}$ onto the configuration space of a smaller graph $\mathcal{G}$.

Let $\mathcal{G} = \left(\mathcal{R}_{2\mathcal{C}}^{[\![0,1]\!]}, \mathbf{T}\left(\mathcal{R}_{2\mathcal{C}}^{[\![0,1]\!]}\right) \cup \mathbf{L}\left(\mathcal{R}_{2\mathcal{C}}^1\right)\right)$ be the graph $\mathcal{G}$ with vertices ρ, ρ' and the 2^{d+1} vertices of $\mathcal{R}_{2\mathcal{C}}^1$ together with the associated tree edges and the lattice of the bottom layer.

For an event $A \in \mathscr{G}$ we define a map $\Phi_A : \Omega_{\mathcal{J}} \to \Omega_{\mathcal{G}}$ by setting

$$\Phi_A^{-1}(J_e) = \begin{cases} J_e & : \text{if } e \in \mathbf{T}(\mathcal{G}), \\ T_e(A) & : \text{if } e \in \mathbf{L}(\mathcal{G}). \end{cases} \tag{5.48}$$

Informally then the states of the *tree* edges of $\mathcal{G}$ are preserved. The remaining lattice edges are opened if the event A occurs in the jungle $\mathcal{J}_e$.

Now consider a second event $B \in \mathscr{F}_{\mathcal{G}}$. If B is invariant under the group of symmetries of $2\mathcal{C}$ then we may define a map $W_B : \mathscr{G} \to \mathscr{G}$ by setting

$$W_B(A) = \left\{\omega \in \Omega_{\mathcal{J}} : \Phi_A(\omega) \in B\right\}. \tag{5.49}$$

It will be crucial to our argument that if two lattice edges $e = \langle u, v \rangle$ and edge $e' = \langle u', v' \rangle$ do not share a common vertex then the forests $\mathcal{J}_e$ and $\mathcal{J}_{e'}$ are disjoint and so the events $T_e(A)$ and $T_{e'}(A)$ are independent under $\mathbb{P}_{\mathcal{J}}$.

Our proof of the upper bound for Theorem 5.8 will involve recursive applications of maps in the form W_B, in particular we

FIGURE 11
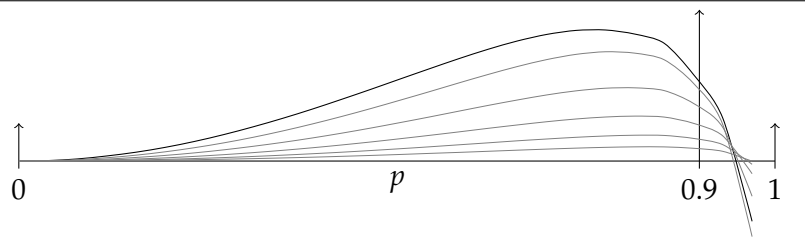
Plots of the function $(1-p)^{\tau^2} - \left(1 - \tau^2 p \left(1 + (1-\tau)\, p\right)^2\right)$ with $\tau = 2^{\frac{d-1}{2}}$ for integer d. The darker line is the case when $d = 2$.

will establish lower bounds on the probabilities $\mathbb{P}_{\mathcal{J}}(W_B(A))$ as functions of $\mathbb{P}_{\mathcal{J}}(A)$.

Now let us consider the event $[\rho \longleftrightarrow \rho']$.

We have seen that when $\tau \leq 2^{\frac{d-1}{2}}$ we have $\mathbb{P}_{\mathcal{T}}[\rho \longleftrightarrow \rho'] = 0$. Next we show that this probability as a function of τ displays a dramatic discontinuity at the critical point.

LEMMA 5.10

If $\tau > 2^{\frac{1-d}{2}}$ then $\mathbb{P}_{\mathcal{T}}[\rho \longleftrightarrow \rho'] > 0.9$.

The heart of the proof is to observe that for any $\omega \in \Omega_{\mathcal{J}}$ we have $\rho \overset{\omega}{\longleftrightarrow} \rho'$ if and only if $\Phi_{[\rho \longleftrightarrow \rho']}(\omega) \in [\rho \longleftrightarrow \rho']$. Therefore the event $[\rho \longleftrightarrow \rho']$ is invariant under the transformation $W_{[\rho \longleftrightarrow \rho']}$.

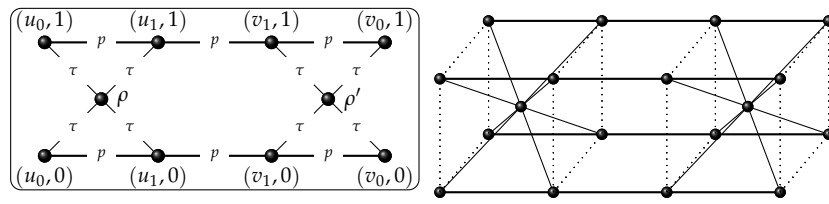Before proceeding we state a numerical inequality.

LEMMA 5.11

If $\tau = 2^{\frac{1-d}{2}}$ for some $d \in [\![2, \infty]\!]$ then for every $p \in (0, 0.9]$

$$1 - \tau^2 p \left(1 + (1-\tau)\, p\right)^2 < (1-p)^{\tau^2}.$$

Figure 11 shows plots of $(1-p)^{\tau^2} - \left(1 - \tau^2 p \left(1 + (1-\tau)\, p\right)^2\right)$ for a few values of $\tau \leq \frac{1}{\sqrt{2}}$. We may see that for every value of τ pictured there is a single root larger than 0.9. The bound of 0.9 is

FIGURE 12

A subgraph of $\mathcal{G}$ for $d = 2$ and $d = 3$. The inequality in Lemma 5.11 comes from considering independent bond percolation with the probabilities indicated on the left hand graph.

necessary for the next part of our argument and is not tight. We will delay the proof of Lemma 5.11 until the end of this section.

PROOF OF LEMMA 5.10

Set $p = \mathbb{P}_{\mathcal{J}}[\rho \longleftrightarrow \rho']$. We estimate the probability $\mathbb{P}_{\mathcal{J}}\left(W_{[\rho \leftrightarrow \rho']}(A)\right)$ in terms of $\mathbb{P}_{\mathcal{J}}(A)$ and show that p must satisfy the inequality in Lemma 5.11.

Consider the set of pixels $\mathcal{P}^1_{2c}$. The pixels have a grid structure isomorphic to $[\![0, 3]\!] \times [0, 1]^{d-1} \subset \mathbb{Z}^d$. We relabel the vertices

$$\mathcal{P}^1_{2c} = \{u_0, u_1, v_1, v_0\} \times \{0, 1\}^{d-1}. \tag{5.50}$$

With the convention that the vertices u_0, u_1, v_1, v_0 run from left to right.

For each $x \in \{0, 1\}^d$ we name the *left,middle* and *right* edges connecting the pixels $\{u_0, u_1, v_1, v_0\} \times \{x\}$

$$\ell_x = \langle (u_0, x), (u_1, x) \rangle, \tag{5.51}$$

$$m_x = \langle (u_1, x), (v_1, x) \rangle, \tag{5.52}$$

$$r_x = \langle (v_1, x), (v_0, x) \rangle. \tag{5.53}$$

We consider only subgraphs of $\mathcal{G}$ in particular we include only tree edges $\mathbf{T}(\mathcal{G})$ and the named edges $\bigcup_{x \in \{0,1\}^{d-1}} \{\ell_x, m_x, r_x\}$ parallel to the edge $\langle \rho, \rho' \rangle$.

The subgraphs in Figure 12 may be divided into 2^{d-1} independent "arms", we calculate first the probability of an open path across each arm.

Name events L_x, M_x and R_x in $\mathscr{F}_{\mathcal{J}}$

$$L_x = J_{\langle \rho, (u_1, x) \rangle} \cup \left(J_{\langle \rho, (u_0, x) \rangle} \cap \Phi_{\ell_x} \left[\rho \longleftrightarrow \rho' \right] \right), \tag{5.54}$$

$$M_x = \Phi_{m_x} \left[\rho \longleftrightarrow \rho' \right] \tag{5.55}$$

$$R_x = J_{\langle \rho', (v_1, x) \rangle} \cup \left(J_{\langle \rho', (v_0, x) \rangle} \cap \Phi_{r_x} \left[\rho \longleftrightarrow \rho' \right] \right), \tag{5.56}$$

$$E_x = L_x \cap M_x \cap R_x \tag{5.57}$$

Now the tree bonds $J_{\langle \rho, (u_0, x) \rangle}$ and $J_{\langle \rho, (u_1, x) \rangle}$ are independent of $\mathscr{F}_{\mathcal{J}_{\ell_x}}$ and so we have

$$\mathbb{P}_{\mathcal{J}}(L_x) = \tau + \tau \left(1 - \tau \right) p. \tag{5.58}$$

By symmetry we have $\mathbb{P}_{\mathcal{J}}(R_x) = \mathbb{P}_{\mathcal{J}}(L_x)$ and as the events L_x, M_x and R_x are increasing then by the FKG inequality we have

$$\begin{aligned} \mathbb{P}_{\mathcal{J}}(E_x) &\geq \mathbb{P}_{\mathcal{J}}(L_x) \cdot \mathbb{P}_{\mathcal{J}}(M_x) \cdot \mathbb{P}_{\mathcal{J}}(R_x) \\ &= \tau^2 p \left(1 + (1 - \tau) p^2 \right) \\ &\geq \tau_c^2 p \left(1 + (1 - \tau_c) p^2 \right) \end{aligned}$$

where $\tau_c = 2^{\frac{1-d}{2}}$.

Now for each $x \in \{0, 1\}^{d-1}$

$$E_e \subseteq \left[\rho \longleftrightarrow u_1 \longleftrightarrow v_1 \longleftrightarrow \rho' \right] \subseteq \left[\rho \longleftrightarrow \rho' \right]. \tag{5.59}$$

Furthermore the set of events $\left\{ E_x \,\middle|\, x \in \{0, 1\}^d \right\}$ are independent and so we have

$$p \geq \mathbb{P}_{\mathcal{J}} \left(\bigcup_{x \in \{0,1\}^{d-1}} E_x \right) \tag{5.60}$$

$$1 - p \leq \left(1 - \tau_c^2 p \left(1 + (1 - \tau_c) p^2 \right) \right)^{2^{d-1}} \tag{5.61}$$

$$(1 - p)^{\tau_c^2} \leq 1 - \tau_c^2 p \left(1 + (1 - \tau_c) p^2 \right). \tag{5.62}$$

And so by Lemma 5.11 we have $\mathbb{P}_{\mathcal{J}} \left[\rho \longleftrightarrow \rho' \right] > 0.9$. $\qquad \Box$

It remains to prove Lemma 5.11. We have seen in Figure 11 that the inequality holds for a variety of choices of τ, and we would expect that the lemma holds for every $\tau \in \left(0, \frac{1}{\sqrt{2}}\right)$, not just for integer d. However by restricting ourselves to a sequence of values we my use an induction argument which avoids the exponent τ^2 on the right hand side.

PROOF OF LEMMA 5.11

First consider the case when $d = 2$, then $\tau^2 = \frac{1}{2}$ and by squaring both sides of the inequality we need only check that

$$0 < (1 - p) - \left(1 - \frac{p}{2}\left(1 + \left(1 - \frac{1}{\sqrt{2}}\right)p\right)^2\right)^2 \qquad (5.63)$$

$$= G(p)p^2 \qquad (5.64)$$

for some polynomial G. Careful expansion gives us

$$G(p) = \frac{17 - 12\sqrt{2}}{16}\left[92 + 64\sqrt{2} - \left(56 + 40\sqrt{2}\right)p\right.$$
$$\left. - \left(36 + 24\sqrt{2}\right)p^2 - \left(8 + 4\sqrt{2}\right)p^3 - p^4\right]. \qquad (5.65)$$

Now $\frac{17 - 12\sqrt{2}}{16} > 0$ so G is decreasing in p we need only check that $G(0.9) > 0$ and we may calculate

$$G(0.9) = \frac{17 - 12\sqrt{2}}{16} \times \frac{59519 + 56440\sqrt{2}}{10000} > 0. \qquad (5.66)$$

Now we argue inductively in d, set $\tau = 2^{\frac{1-d}{2}}$ and $\tilde{\tau} = 2^{\frac{2-d}{2}}$, assume that

$$(1 - p)^{\tilde{\tau}^2} > 1 - \tilde{\tau}^2 p \left(1 + (1 - \tilde{\tau})p\right)^2 \qquad (5.67)$$

and notice that $\tilde{\tau}^2 = 2\tau^2$.

Hence

$$(1-p)^{\tau^2} = \left(\left(1-p\right)^{\tilde{\tau}^2}\right)^{\frac{1}{2}} \tag{5.68}$$

$$> \left(1 - \tilde{\tau}^2 p \left(1 + (1 - \tilde{\tau})\, p\right)^2\right)^{\frac{1}{2}} \tag{5.69}$$

$$= \left(1 - 2\tau^2 p \left(1 + \left(1 - \sqrt{2}\tau\right) p\right)^2\right)^{\frac{1}{2}}. \tag{5.70}$$

So combining (5.70) with (5.67) and squaring it is enough to show

$$1 - 2\tau^2 p \left(1 + \left(1 - \sqrt{2}\tau\right) p\right)^2 > \left(1 - \tau^2 p \left(1 + (1 - \tau)\, p\right)^2\right)^2 \tag{5.71}$$

for $0 < \tau \leq 2^{\frac{3-1}{2}} = 0.5$ and $0 < p \leq 0.9$.

Set

$$f(\tau, p) = \frac{\left(1 - 2\tau^2 p\left(1 + \left(1 - \sqrt{2}\tau\right)p\right)^2\right) - \left(1 - \tau^2 p\left(1 + (1-\tau)p\right)^2\right)^2}{\tau^2 p} \tag{5.72}$$

We prove (5.71) by first showing $f(0.5, p) > 0$ for $0 < p \leq 0.9$ and then showing that f is decreasing in τ for $0 < \tau < 0.5$, $0 < p < 0.9$.

First we have

$$f(0.5, p) = 4\sqrt{2} - \frac{9}{2} - \left(6 - 4\sqrt{2}\right)p - \frac{3p^2}{4} - \frac{p^3}{4} - \frac{p^4}{32}. \tag{5.73}$$

The right hand side is decreasing in p and we have

$$f(0.5, 0.9) = \frac{2432000\sqrt{2} - 3427281}{320000} > 0. \tag{5.74}$$

Hence $f(0.5, p) > 0$ for every $p \leq 0.9$.

Next

$$\frac{\partial f}{\partial \tau}(0.5, p) = -1 - 2p + \frac{3p^2}{2} + p^3 + \frac{3p^4}{16} \tag{5.75}$$

The right hand side is convex in p and we have

$$\frac{\partial f}{\partial \tau}(0.5, p) \leq p\frac{\partial f}{\partial \tau}(0.5, 1) + (1 - p)\frac{\partial f}{\partial \tau}(0.5, 0) \qquad (5.76)$$

$$= -p - \frac{5}{16}(1 - p) \qquad (5.77)$$

$$< 0 \qquad (5.78)$$

And so $\frac{\partial f}{\partial \tau}(0.5, p) < 0$ for every $p \in [0, 1]$.

Lastly if $0 < \tau \leq 0.5$ and $0 < \eta < 1$ we have

$$\frac{\partial^2 f}{\partial \tau^2}(\tau, p) = 4p\left(2(1 + p) - 5p\tau\right)(1 - (1 - \tau)p)^2 \qquad (5.79)$$

$$\geq 4p\left(2 - \frac{p}{2}\right)(1 - (1 - \tau)p)^2 \qquad (5.80)$$

$$> 0. \qquad (5.81)$$

Therefore $\frac{\partial f}{\partial \tau}$ is increasing and so by (5.78) we see that f is decreasing in τ whenever $0 < \tau < \frac{1}{2}$ and $0 < p \leq 0.9$. This competes the inductive step. ❑

5.3.3 The Balister, Bollobás and Walters Recursion

We have shown that if $\tau > \tau_c^{\longleftrightarrow}$ the event $[\rho \longleftrightarrow \rho']$ occurs with probability much higher than the critical probability for independent percolation on the lattice. Consider the finite grid $\mathcal{P}_{2c}^N$ for large N. Setting $J_e = T_e[\rho \longleftrightarrow \rho']$ we would expect $\mathcal{P}_{2c}^N$ to contain a single "giant" cluster that is an order of magnitude larger than the second largest cluster. That is we might expect there to exist a cluster C_N so large that if $\rho \longleftrightarrow \infty$ there must be some vertex $u \in C$ with $\rho \longleftrightarrow u$.

Such phenomena are known to occur for independent bond percolation, and even for the random cluster model, see for example Barsky et al. [4], Pisztora [54]. However the percolation process above is not independent as if two edges $e, e' \in \mathbf{L}(\mathcal{P}_{2c}^N)$ share a

common vertex the bonds J_e and $J_{e'}$ are clearly dependent and we do not know how the dependence operates.

However the process above is *one-dependent* in the sense that if e and e' do not share a common vertex then the jungles $\mathcal{J}_e$ and $\mathcal{J}_{e'}$ do not intersect an the events $T_e[\rho \longleftrightarrow \rho']$ and $T_{e'}[\rho \longleftrightarrow \rho']$ are independent.

Balister et al. [3] have studied one-dependent bond percolation on $\mathbb{Z}^2$ and were able to prove that any one dependent size percolation with the property that each edge is open with probability at least 0.9 exhibits an infinite cluster. Clearly then any such one dependent percolation process in $\mathbb{Z}^d$ exhibits an infinite cluster and if the process is translation invariant the argument of Burton and Keane [15] shows that the infinite cluster is unique.

As we are considering clusters of finite graphs it is not enough simply to state the result. However we may apply the argument directly to the tree setting and construct large finite clusters that are regular enough to intersect any infinite cluster in $\mathcal{J}$.

We note that the bound on the critical probability in [3] is better than 0.9. The 0.9 bound in fact comes from a warm up argument which is simpler and better suits the structure of the QuadTree.

We say a subgraph G of $\mathcal{P}_{2c}^N$ is a (depth-N) *k-net* of a vertex v if G is connected and the vertices $\mathbf{V}(G)$ form the leaves of some regular k-tree rooted at v.

For $d = 2$ our aim is to show that for any $n \in \mathbb{N}$ we may choose some N such that with high probability there exists some subgraph $C \subset \mathcal{P}_{2c}^{N+n}$ such that we have $T_e[u \longleftrightarrow v]$ for every $e = \langle u, v \rangle \in \mathbf{E}(C)$ and that for every $v \in \mathcal{P}_{2c}^n$ C contains some 3-net of v.

For $d = 2$ we have $2^{\frac{1-d}{2}} > \frac{1}{3}$ hence this observation will be enough to construct a path from u to v for any pair of vertices $u \longleftrightarrow \infty$ and $v \longleftrightarrow \infty$. For $d > 2$ a 3-net will not suffice for this argument and as we must construct a larger cluster the bound

FIGURE 13
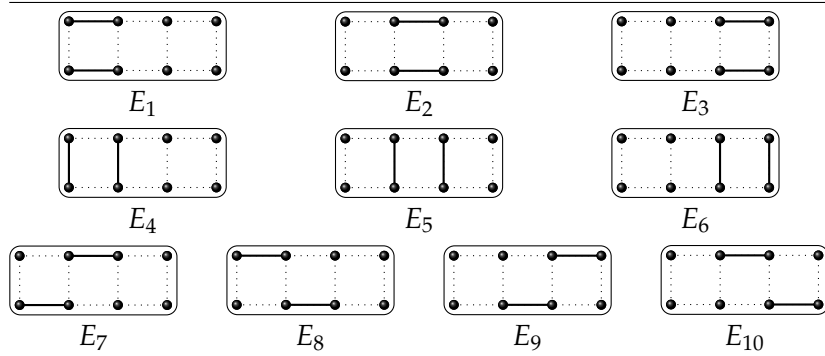
The ten pairs of bonds from [3].

of 0.9 will not be sufficient. For higher dimensions we will con-
struct a block event $K \in \mathcal{G}$ which occurs with very high proba-
bility. We may then argue as for the two dimensional case but
with the event K in place of $[\rho \longleftrightarrow \rho']$.

Define a function

$$
N(d) = \begin{cases} 10 & : \text{ if } d = 2, \\ \left(4d^2 + 4d + 1\right) 2^{2d-3} - \left(4d^2 + 2d + 1\right) 2^{d-2} & : \text{ otherwise.} \end{cases}
$$

(5.82)

We note that for $d \geq 3$ $N(d)$ is the number of pairs of disjoint
lattice edges of the graph $\mathcal{P}^1_{2c}$. It is an easy exercise to calculate
the formula in (5.82) although the exact value will not matter. It
will be convenient in the argument below that $N(d)$ is not an
overestimate.

LEMMA 5.12

Let $\Theta_n(A)$ be the event that there exists some subgraph C of $\mathcal{P}^n_{2c}$
such that $T_e(A)$ occurs for every $e \in \mathbf{E}(C)$ and C contains both a
$\left(2^d - 1\right)$-net of ρ and a $\left(2^d - 1\right)$-net of ρ'.

If $\mathbb{P}_{\mathcal{J}}(A) > 1 - \frac{1}{N(d)}$ we have $\sum_{n=1}^{\infty} \left(1 - \Theta_n(A)\right) < \infty$.

PROOF

We define a recursion as for the proof of Lemma 5.10. Let $B \subset \Omega_{\mathcal{G}}$ be the event that the largest connected component of the graph $\left(\mathcal{P}^1_{2\mathcal{C}}, \left\{e \in \mathbf{L}\left(\mathcal{P}^1_{2\mathcal{C}}\right) \mid J_e = 1\right\}\right)$ contains at least $2^d - 1$ vertices of $\mathcal{P}^1_{\mathcal{C}}$ and $\mathcal{P}^1_{-\mathcal{C}}$, with $\mathcal{C} = [0,1]^d$ and $-\mathcal{C} = [-1,0] \times [0,1]^{d-1}$.

Now for $e = \langle u, v \rangle$ and $e' = \langle v, u' \rangle \in \mathbf{L}\left(\mathcal{P}^1_{2\mathcal{C}}\right)$ if both $T_e(\Theta_n(A))$ and $T_{e'}(\Theta_n(A))$ occur then there is some large cluster which contains a depth n $\left(2^d - 1\right)$-net of u and of v and some large cluster which contains a depth n $\left(2^d - 1\right)$-net of v and of u'. As any two depth n $\left(2^d - 1\right)$-nets of v must intersect then there exists one large cluster that contains depth n $\left(2^d - 1\right)$-nets of u, v and u'.

So if $W_B(\Theta_n(A))$ occurs there is some large cluster that contains depth n $\left(2^d - 1\right)$-nets of $2^d - 1$ vertices in each of $\mathcal{P}^1_{\mathcal{C}}$ and $\mathcal{P}^1_{-\mathcal{C}}$. That is to say that cluster contains both a depth $n+1$ $\left(2^d - 1\right)$-net of ρ and of ρ'.

In particular we have

$$W_B(\Theta_n(A)) \subseteq \Theta_{n+1}(A). \tag{5.83}$$

So we are interested in the probability of $W_B(A)$ for $A \in \mathcal{G}$.

First suppose $d = 2$. Figure 13 shows ten copies of $\mathcal{P}^1_{2\mathcal{C}}$ with highlighted pairs of disjoint bonds.

Notice that if at least one bond from each pair is open then there must exist a cluster as described above containing at least three pixels from each side of $\mathcal{P}^1_{2\mathcal{C}}$. We leave it to the reader to convince himself of this. The full argument may be found in [3].

So, let E_i be the event that at least one of the bonds in the ith pair is open. Then $B \supset \bigcap_{i=1}^{10} E_i$.

For $d > 2$ we will not attempt to choose pairs in an intelligent way. Suppose that at least one bond out of each of the $N(d)$ pairs of disjoint pairs of edges is open. Then, as $\mathcal{P}^1_{2\mathcal{C}}$ contains no triangles; any set of mutually disjoint edges, and hence the set of all closed edges must have a common vertex. Therefore there

must exist a giant cluster containing either all or all but one of the vertices of $\mathcal{P}_e^1$.

In general we may order disjoint pairs of edges arbitrarily and we have

$$B \supset \bigcap_{i=1}^{N(d)} E_i \qquad (5.84)$$

If a pair of edges $e, e' \in \mathbf{L}\left(\mathcal{P}_{2c}^1\right)$ are disjoint then the events $T_e(A)$ and $T_{e'}(A)$ are independent and so

$$1 - \mathbb{P}_{\mathcal{J}}(W_B(A)) \leq N(d) \left(1 - \mathbb{P}_{\mathcal{J}}(A)\right)^2. \qquad (5.85)$$

In particular if $\mathbb{P}_{\mathcal{J}}(A) > 1 - \frac{1}{N(d)}$ then $1 - \mathbb{P}_{\mathcal{J}}\left(W_B^{n+1}(A)\right) \leq \frac{1 - \mathbb{P}_{\mathcal{J}}(A)}{N(d)}\left(W_B^n(A)\right)$ and so

$$\sum_{n=1}^{\infty} 1 - \mathbb{P}_{\mathcal{J}}(\Theta_n(A)) = \sum_{n=1}^{\infty} 1 - \mathbb{P}_{\mathcal{J}}(W_B^n(A)) \qquad (5.86)$$

$$\leq \left(1 - \frac{1 - \mathbb{P}_{\mathcal{J}}(A)}{N(d)}\right)^{-1} \left(1 - \mathbb{P}_{\mathcal{J}}(A)\right). \qquad (5.87)$$

❏

For the case when $d = 2$ we are nearly done. If $\tau > \tau_c^{\leftrightarrow}$ then for any pair of vertices u, v there must exist some large $\leftrightarrow$ cluster which contains a 3-net of u and a 3-net of v. As $\tau_c^{\leftrightarrow} > \frac{1}{3}$ then with positive probability both u and v are connected to this cluster, and this probability is bounded strictly above zero for all pairs u, v. It is easy to complete the proof that $u \leftrightarrow v$ whenever $u \leftrightarrow \infty$ and $v \leftrightarrow \infty$. For $d \geq 3$ however the estimate that $\mathbb{P}_{\mathcal{J}}[\rho \leftrightarrow \rho'] > 0.9$ is not high enough and as $N(d)$ grows so quickly in d we would not expect to satisfy $\mathbb{P}_{\mathcal{J}}[\rho \leftrightarrow \rho'] > 1 - \frac{1}{N(d)}$ for large d.

Instead we will construct a block event $K \in \mathscr{G}$ in such a way that if $e = \langle u, v \rangle$ and $e' = \langle v, u' \rangle \in \mathbf{L}\left(\mathcal{P}_{2c}^1\right)$ and both $T_e(K)$ and

$T_{e'}(K)$ occur then there is some (identifiable) large cluster that intersects all three pixels $\mathcal{P}_u$, $\mathcal{P}_{u'}$ and $\mathcal{P}_v$. We will then ensure that K may be chosen with high enough probability to apply Lemma 5.12.

Consider the cube $\mathcal{C}$. For a face f of $\mathcal{C}$ let $\mathcal{P}_f^n$ be the subset of $\mathcal{P}_{\mathcal{C}}^n$ consisting of cubes that intersect f.

Now let K_n be the event that there exists some subgraph C_e of $\mathcal{P}_{2\mathcal{C}}^n$ such that for every edge $e \in C$ we have $\Phi_e[\rho \longleftrightarrow \rho']$ and that $\mathcal{P}_f^n$ contains some 3-canopy of ρ for every 2-face f of $\mathcal{C}$ and $\mathcal{P}_f^n$ contains some 3-canopy of ρ' for every 2-face f of $-\mathcal{C}$.

LEMMA 5.13

If $\mathbb{P}_{\mathcal{J}}[\rho \longleftrightarrow \rho'] > 0.9$ then $\mathbb{P}_{\mathcal{J}}(K_n) \to 1$ as $n \to \infty$.

PROOF

First let μ be any one-dependent percolation measure on $\mathbb{Z}^2$ such that for any edge e we have $\mu(J_e) > 0.9$. We say a subgraph of the grid $[\![a, a + 2^n]\!] \times [\![b, b + 2^n]\!]$ is a 3-net if it is isomorphic to some 3-net of the pixel set $\mathcal{P}_{\mathcal{C}}^n$.

Notice that a 3-net must contain both a left–right crossing and an up–down crossing of the grid $[\![a, a + 2^n]\!] \times [\![b, b + 2^n]\!]$.

The original argument of Balister et al. [3], which we have adapted for Lemma 5.12 states that for any $\varepsilon > 0$ there is some N large enough so that if $n > N$ then with probability at least $1 - \varepsilon$ there exists some large cluster C such that

- $C \cap [\![0, 2^{n+1} - 1]\!] \times [\![0, 2^n - 1]\!]$ is connected,

- C contains some 3-net of $[\![0, 2^n - 1]\!]^2$,

- C contains some 3-net of $[\![2^n, 2^{n+1} - 1]\!] \times [\![0, 2^n - 1]\!]$.

As both 3 nets contain left–right crossings of their respective grids this cluster must contain a left–right crossing of the grid $[\![0, 2^{n+1} - 1]\!] \times [\![0, 2^n - 1]\!]$.

Similarly there is some cluster C' such that $C' \cap [\![1, 2^n]\!] \times [\![0, 2^n - 1]\!]$ contains a 3-net and this cluster contains some up–down crossing of $[\![0, 2^{n+1} - 1]\!] \times [\![0, 2^n - 1]\!]$.

Therefore the cluster $C \cap [\![0, 2^{n+1} - 1]\!] \times [\![0, 2^n - 1]\!]$ and the cluster $C' \cap [\![0, 2^{n+1} - 1]\!] \times [\![0, 2^n - 1]\!]$ intersect hence with probability at least $1 - 2\varepsilon$ there is some cluster $\tilde{C}$ such that

- $\tilde{C} \cap [\![1, 2^{n+1} - 1]\!] \times [\![0, 2^n - 1]\!]$ is connected,

- $\tilde{C}$ contains some 3-net of $[\![1, 2^n]\!] \times [\![0, 2^n - 1]\!]$,

- $\tilde{C}$ contains some 3-net of $[\![2^n, 2^{n+1} - 1]\!] \times [\![0, 2^n - 1]\!]$.

Note: it is not the case that any 3-net of $[\![1, 2^n]\!] \times [\![0, 2^n - 1]\!]$ intersects any 3-net of $[\![2^n, 2^{n+1} - 1]\!] \times [\![0, 2^n - 1]\!]$.

Now let f and f' be two 2-faces of $\mathcal{C}$ such that $f \cap f'$ is a line.

Then the pixels $\mathcal{P}_f^n \cap \mathcal{P}_{f'}^n$ are exactly those pixels that intersect the line $f \cap f'$. It is easy to see that the set $\mathcal{P}_f^n \cup \mathcal{P}_{f'}^n$ is a grid isomorphic to $[\![1, 2^{n+1} - 1]\!] \times [\![0, 2^n - 1]\!]$. So if $n > N$ then with probability at least $1 - 2\varepsilon$ there exists some cluster $D_{f,f'}$ such that $D_{f,f'} \cap \left(\mathcal{P}_f^n \cup \mathcal{P}_{f'}^n \right)$ is connected and both $D_{f,f'} \cap \mathcal{P}_f^n$ and $D_{f,f'} \cap \mathcal{P}_{f'}^n$ contain 3-nets of ρ.

As there are only $d \cdot 2^{d-1}$ edges of $\mathcal{C}$ then with probability at least $1 - d\varepsilon \cdot 2^d$ there is some cluster D such that $D \cap \mathcal{P}_f^n$ contains a 3-net of ρ for every 2-face f of $\mathcal{C}$. Similarly with probability at least $1 - d\varepsilon 2^d$ there is some cluster D' such that $D' \cap \mathcal{P}_f^n$ contains a 3-net of ρ' for every 2-face f of $-\mathcal{C}$.

Now let ℓ be some 1-face of the cube $\{0\} \times [0, 1]^{d-1}$, then $[-1, 1] \times \ell$ is a two-face of the cuboid $2\mathcal{C}$ and with probability $1 - \varepsilon$ there is a third cluster D'' such that $D'' \cap \mathcal{P}_{[0,1]\times\ell}^n$ contains a 3-net of ρ and $D'' \cap \mathcal{P}_{[-1,0]\times\ell}^n$ contains a 3-net of ρ'.

If all three clusters exist they must intersect and so $\mathbb{P}_{\mathcal{J}}(K_n) \geq 1 - \varepsilon \left(1 + d \cdot 2^{d+1} \right)$ and as ε is arbitrary we have $\mathbb{P}_{\mathcal{J}}(K_n) \to 1$. ❑

Now we are ready to finish the proof of Theorem 5.8.

LEMMA 5.14

$$\tau_{u}^{\leftrightarrow} \leq 2^{\frac{1-d}{2}}.$$

PROOF

Assume $\tau > 2^{\frac{1-d}{2}}$.

For $d \geq 3$ we have described an event K_n with $\mathbb{P}_{\mathcal{J}}(K_n) \to 1$ as $n \to \infty$. In particular we may choose $K = K_{M_1}$ where M_1 is so large that $\mathbb{P}_{\mathcal{J}}(K) > 1 - \frac{1}{N(d)}$.

Now for every $\omega \in K$ there exists some cluster $C(\omega)$ which contains depth M_1 3-nets on every 2-face of the cube $[0,1]^d$ and we may set

$$\delta = \inf_{\omega \in K} \mathbb{P}_{\mathcal{J}}[\rho \longleftrightarrow C(\omega)] > 0. \tag{5.88}$$

In the case $d = 2$ we may set $K = [\rho \longleftrightarrow \rho']$ with $M_1 = 0$ and $\delta = 1$.

Next set $F_\tau(\lambda) = (\tau\lambda + 1 - \tau)^{2^d - 1}$ and let $p \in (0,1)$ be the largest root of $F_\tau(p) = p$. Then we may choose M_2' such that $F_\tau^{M_2'}(1-\delta) > \frac{p+1}{2}$.

In particular we have chosen M_2' such that for any $M_2 \geq M_2'$ and $\omega \in W_{M_2}(K)$ there exists some large set $C_2(\omega) \subset \mathcal{P}_c^{M_2+M_3}$ with $u \longleftrightarrow^{\omega} v$ for every $u, v \in C_2$ and $\mathbb{P}_{\mathcal{J}}\left(\bigcup_{v \in C_2(\omega)} [\rho \longleftrightarrow v]\right) > \frac{1-p}{2}$.

Now choose arbitrary vertices $u, v \in \mathbf{V}(\mathcal{T})$. We need to show that $\mathbb{P}_{\mathcal{J}}[u \longleftrightarrow v] \geq \mathbb{P}_{\mathcal{J}}[u \Longleftrightarrow v]$.

It is a well known and obvious result of Bernoulli percolation theory that for any sequence of finite trees $T_n \uparrow \mathcal{T}$ if we set $X_{n,v}(\omega)$ to be the number of leaf vertices v' of T_n with $v \longleftrightarrow^{\omega} v'$ then the sequence $X_{n,v}(\omega) \to \infty$ as $n \to \infty$ and only if $v \longleftrightarrow^{\omega} \infty$.

So choose $\varepsilon > 0$. We may choose some M_3 so large that with probability $1 - \varepsilon$ one of the following occurs

- $u \longleftrightarrow v$, $u \not\longleftrightarrow \infty$ or $v \not\longleftrightarrow \infty$

- There exist at least $\frac{\log \varepsilon}{\log 1 + p - \log 2}$ vertices u' in $\mathcal{P}_c^n$ with $u \longleftrightarrow u'$ and at least $\frac{\log \varepsilon}{\log 1 + p - \log 2}$ vertices v' in $\mathcal{P}_c^n$ with $v \longleftrightarrow v'$.

To simplify the argument later we colour a vertex $v' \in \mathcal{P}_c^n$ *green* if $u \longleftrightarrow v'$ and *blue* if $v \longleftrightarrow v'$. If it is the case that there are fewer than $\frac{\log \varepsilon}{\log 1 + p - \log 2}$ green (respectively blue) vertices colour some arbitrary extra vertices *green'* (respectively *blue'*) so that there are always at least $\frac{\log \varepsilon}{\log 1 + p - \log 2}$ green or green' edges.

Let

$$E(M) = \bigcap_{e \in \mathbf{L}\left(\mathcal{P}_c^{M_3}\right)} T_e(\Theta_M(K)) \tag{5.89}$$

and choose $E = E(M_2)$ with $M_2 > M_2'$ such that $\mathbb{P}_{\mathcal{T}}(E) > 1 - \varepsilon$.

So for $\omega \in E$ there is some large set of vertices $\tilde{C}(\omega) \subset \mathcal{P}_c^{M_1 + M_2 + M_3}$ such that $u \overset{\omega}{\longleftrightarrow} v$ for every $u, v \in \tilde{C}$ and for every $v \in \mathcal{P}_c^{M_3}$ we have $\mathbb{P}_{\mathcal{T}}\left[v \longleftrightarrow \left(\tilde{C} \cap \mathcal{P}_{\mathcal{P}_v}^{M_1 + M_2}\right)\right] > \frac{1-p}{2}$.

Hence we may colour vertices of $\mathcal{P}_c^{M_3}$ red independently with probability $\frac{1-p}{2}$ in such a way that if $\omega \in E$ we have $v \overset{\omega}{\longleftrightarrow} \tilde{C}(\omega)$ for every red vertex v. Notice that the event that a vertex is red is independent of the event that it is blue or green.

So if we have $[u \Longleftrightarrow v] \setminus [u \longleftrightarrow v]$ one of the following must occur.

- The event E does not occur.

- There are fewer than $\frac{\log \varepsilon}{\log 1 + p - \log 2}$ green vertices or fewer than $\frac{\log \varepsilon}{\log 1 + p - \log 2}$ blue vertices $\mathcal{P}_c^n$.

- No green or green' vertex is red.

- No blue or blue' vertex is red.

Each of these event occurs with probability at most ε and so we have $\mathbb{P}_{\mathcal{T}}[u \Longleftrightarrow v] - \mathbb{P}_{\mathcal{T}}[u \longleftrightarrow v] < 4\varepsilon$. $\qquad \Box$

6 DISCUSSION AND CONCLUSIONS

We conclude by returning to the problem in Chapter 1. We show that when the random cluster model with Mandelbrot boundary conditions is unique then the small λ limit on the QuadTree agrees with the random cluster model on the tree.

We consider some techniques that may shed light on the small λ limit in the nonuniqueness phase and set out a research agenda to address some questions we have encountered but not answered in previous chapters.

CONTENTS

6.1 BACK TO THE QUADTREE

The motivation for this work was to investigate the random cluster model on a tree with boundary conditions defined by the QuadTree. We have carried out our investigation in as general terms as possible, in particular the construction in Chapter 4 is general enough to cover a variety of objects that may be called random cluster models.

The Mandelbrot boundary conditions of Chapter 5 were motivated by the limiting random cluster model on the QuadTree. We have a picture of the set $\mathscr{R}_{\tau,q}^{\leftarrow\rightarrow}$ for different values of τ and $q \geq 1$, but what may we say about the weak limit $\mathbb{Q}_{\mathcal{Q}(\tau,\lambda)}^q$ as $\lambda \to 0$?

Recall the measure $\mathbb{Q}_{\mathcal{Q}(\tau,\lambda)}^q$ on the QuadTree and the unique wired random cluster model $\mathbb{Q}_{\tau,q}^{\star}$ on the tree edges of $\mathcal{Q}$.

For those values of τ, q for which $\mathscr{R}_{\tau,q}^{\leftarrow\rightarrow}$ is a singleton it can be shown that that the small λ limit of the QuadTree must agree with the random cluster model on the tree.

THEOREM 6.1

- If $\tau < 2^{\frac{1-d}{2}}$ then the random cluster models $\mathbb{Q}_{\mathcal{Q}(\tau,\lambda)}^q$ converge weakly to Bernoulli percolation with bond probability $\pi(\tau)$ as $\lambda \to 0$.

- If $\pi(\tau) \geq 2^{\frac{1-d}{2}}$ then the random cluster models $\mathbb{Q}_{\mathcal{Q}(\tau,\lambda)}^q$ converge weakly to the unique wired random cluster model on the tree as $\lambda \to 0$.

PROOF

For a configuration $\omega \in \Omega_\mathcal{Q}$ say $u \xleftrightarrow{\omega} v$ if there is an open QuadTree path and $u \xleftrightarrow{\omega} v$ if there is a "Mandelbrot" path on the tree $\mathbf{T}(\mathcal{Q})$.

If $\tau < 2^{\frac{1-d}{2}}$ then the random cluster model is stochastically dominated by independent percolation with probabilities τ and λ.

Therefore it is enough to show that $\mathbb{P}_{\mathcal{Q}(\tau,\lambda)}(\mathcal{L}_e) \to 0$ as $\lambda \to 0$. We have already seen that $\mathbb{P}_\tau(\mathcal{L}_e^{\leftrightarrow}) = 0$, but this is not quite enough.

To see what may go wrong consider the critical case $\tau = 2^{\frac{1-d}{2}}$. For any $\lambda >$ and some edge $e = \langle u, v \rangle \in \mathbf{L}$ the probability $\mathbb{P}_\tau[u \longleftrightarrow v] \geq \lambda > 0$. From Lemma 5.10 the probability $\mathbb{P}_\tau[u \longleftrightarrow v]$ must satisfy the inequality (5.67) of Lemma 5.11.

In particular if $\mathbb{P}_\tau[u \longleftrightarrow v]$ is non zero it must be at least 0.9 and we may continue the proof that there is a unique infinite cluster as for the case where $\tau > 2^{\frac{1-d}{2}}$.

To control the probability $\mathbb{P}_{\mathcal{Q}(\tau,\lambda)}(\mathcal{L}_e)$ we must use a further result from Kendall and Wilson [46]. For vertices $u, v \in \mathbf{V}(\mathcal{Q})$ and an edge $e \in \mathbf{L}$ let $A_{u,v}^e$ be the event that e is on some open, self avoiding path from u to v.

Notice that for any edge $f = \langle u, v \rangle \in \mathbf{T}(\mathcal{Q})$ we must have

$$\mathbb{P}_{\mathcal{Q}(\tau,\lambda)}(\mathcal{L}_e) \leq \sum_{f \in \mathbf{L}(\mathcal{Q})} \mathbb{P}_{\mathcal{Q}(\tau,\lambda)}\left(A_{u,v}^f\right). \qquad (6.1)$$

Theorem 2.4 of [46] shows that if $\tau < 2^{\frac{1-d}{2}}$ and λ is sufficiently small then the right hand side of (6.1) is finite.

This was used in [46] to show that there is no unique single cluster for $\tau < 2^{\frac{1-d}{2}}$ and small λ. For our purposes we must go a step further.

Given $\varepsilon > 0$ we may simulate $\mathbb{P}_{\mathcal{Q}(\tau,\varepsilon\cdot\lambda)}$ by choosing ω according to $\mathbb{P}_{\mathcal{Q}(\tau,\lambda)}$ and switching off each lattice edge $e \in \mathbf{L}(\mathcal{Q})$ with probability $1 - \varepsilon$. Hence, trivially we have

$$\mathbb{P}_{\mathcal{Q}(\tau,\varepsilon\cdot\lambda)}\left(A_{\tau,\varepsilon\lambda}^e\right) \leq \varepsilon \cdot \mathbb{P}_{\mathcal{Q}(\tau,\lambda)}\left(A_{\tau,\varepsilon\lambda}^e\right) \qquad (6.2)$$

for every $e \in \mathbf{L}(\mathcal{Q})$ and the sum in (6.1) decreases to zero as $\lambda \to 0$.

Now suppose $q > 1$ and $\pi(\tau) > 2^{\frac{1-d}{2}}$. If $\lambda > 0$ then we must have $\mathbf{Q}_{\mathcal{Q}(\tau,\lambda)}^q(u \longleftrightarrow v \mid u \longleftrightarrow v) = 1$. In particular we must have

$\mathbb{Q}^q_{\mathcal{Q}(\tau,\lambda)}(J_e \mid \mathcal{T}_e)(\omega) \geq \mathbb{Q}^\star_{\tau,q}(J_e \mid \mathcal{T}_e)(\omega)$ for almost every ω. Therefore we must have $\mathbb{Q}^q_{\mathcal{Q}(\tau,\lambda)} \succ \mathbb{Q}^\star_{\tau,q}$ for every $\lambda > 0$.

Conversely for a finite subnetwork $\mathcal{G} \subset \mathcal{Q}$ the cylinder measure $\mathbb{Q}^{\xi_1}_{\mathcal{G}(\tau,\lambda)} \succ \mathbb{Q}^q_{\mathcal{Q}(\tau,\lambda)}$. Therefore we must have

$$\text{wlim}_{\lambda \to 0} \mathbb{Q}^q_{\mathcal{Q}(\tau,\lambda)} \prec \inf_{\lambda > 0} \mathbb{Q}^{\xi_1}_{\mathcal{G}(\tau,\lambda)}$$
$$\xrightarrow{w} \mathbb{Q}^\star_{\tau,q} \qquad \text{as } \mathcal{G} \uparrow \mathcal{Q}.$$

So if $\pi(\tau) > 2^{\frac{1-d}{2}}$ the random cluster models $\mathbb{Q}^q_{\mathcal{Q}(\tau,\lambda)}$ converge weakly to the unique wired random cluster model on the tree. ❏

This result represents an advance on the work of Kendall and Wilson. In addition to improving the bound on the onset of the single cluster phase for Bernoulli percolation The results on the tree that we have considered allow us to describe in detail the behaviour of the random cluster model when λ is close to 0.

We may summarize Theorem 6.1 by identifying four behavioural phases of the weak limit; pictured in Figure 14 on the next page.

I There exist no infinite clusters and $\widetilde{\mathbb{Q}}_{\tau,q} = \mathbb{P}_{\pi(\tau)}$.

II There exist many infinite clusters and $\widetilde{\mathbb{Q}}_{\tau,q} = \mathbb{P}_{\pi(\tau)}$.

III The behaviour of $\widetilde{\mathbb{Q}}_{\tau,q}$ is unknown.

IV There exists a unique infinite cluster and $\widetilde{\mathbb{Q}}_{\tau,q} = \mathbb{Q}^\star_{\tau,q}$.
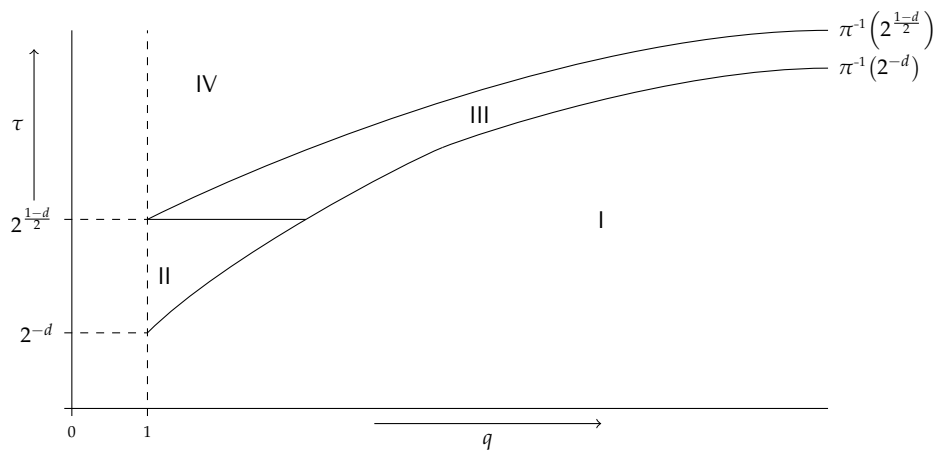
For the sake of completeness we make a conjecture concerning the behaviour in phase III

CONJECTURE 6.2

For $q > 1$ and $\tau < 2^{\frac{1-d}{2}}$ we have $\mathbb{Q}^q_{\mathcal{Q}(\tau,\lambda)} \xrightarrow{w} \mathbb{P}_{\pi(\tau)}$ as $\lambda \to 0$.

At present we have no evidence for this conjecture and are prepared to be proved wrong. Methods discussed in 6.2.2 below may shed some light on this question.

FIGURE 14

Phase diagram for the small λ limit of the random cluster model on the QuadTree.

6.2 RESEARCH AGENDA

Finally we discuss topics that have not been covered in this thesis and which would constitute interesting and significant avenues for further research.

6.2.1 Further problems on the QuadTree

Our initial focus in preparing this thesis was on the random cluster model on the QuadTree. By considering the random cluster model on the tree we have made progress towards describing the behaviour of the random cluster model on the QuadTree, particularly when λ is small. However there remain unanswered questions about the behaviour of the random cluster model on the QuadTree when λ is not small.

We have conjectured in Chapter 1 that when τ is small the random cluster model behaves in a similar way to percolation. That is if λ is greater than the critical bond strength for the random cluster model on $\mathbb{Z}^d$ then the random cluster model exhibits a

single cluster phase from every value of τ. and conversely if λ is less than the critical value then we may always find τ small enough that the random cluster model exhibits only finite clusters.

There are two obstacles to adapting the percolation results of Kendall and Wilson [46] to the random cluster model in this situation. Firstly, even when τ is very small, the layers $\mathcal{P}^n_{[0,1]^d}$ grow large enough to lose independence. Secondly not as much is known about the random cluster model on $\mathbb{Z}^d$ as is known about Bernoulli percolation.

We believe that for subcritical λ we may recover the finite cluster property for sufficiently small τ under the assumption that the distribution of the size of cluster at the origin decays exponentially, this is conjectured to hold for the random cluster model on $\mathbb{Z}^d$ for all $\lambda < p_c(q)$.

For supercritical λ we may have some success adapting the 1-dependent recursion argument of [3]. Pisztora [54] adapted the block renormalization arguments of Grimmett [33] to the random cluster model on $\mathbb{Z}^d$ under the assumption that λ was greater than the *slab* critical point. and it may be possible to adapt these methods to fit the argument of Lemma 5.12 to small τ situation.

An original motivation for this work was the following question concerning the Ising model on the QuadTree. Specifically suppose that for some large N we condition on the colours of the pixels $\mathcal{P} \in \mathcal{P}^N_{[0,1]^d}$ to be black when $\mathcal{P} \in \mathcal{P}^N_{[0,0.5] \times [0,1]^{d-1}}$ and white for $\mathcal{P} \in \mathcal{P}^N_{[0.5,1] \times [0,1]^{d-1}}$. what can we say about the distribution of the colours of some fixed layer $\mathcal{P}^n_{[0,1]^d}$ under this conditioning as $N \to \infty$?

This question has been considered on $\mathbb{Z}^3$ for the Ising model by Dobrushin [19] and in more generality by Gielis and Grimmett [30]. Both methods rely heavily on the structure of $\mathbb{Z}^d$ and we have been unable to adapt them to the QuadTree. The recursion of Lemma 5.12 may be used to describe the structure of the infinite cluster in the uniqueness phase and so may provide an alternative approach to this problem.

6.2.2 Non–uniqueness of the Mandelbrot random cluster model and the Worm's Eye QuadTree

The arguments in Section 6.1 both use stochastic domination to calculate the small λ weak limit on the QuadTree. When the random cluster model may not be dominated effectively by Bernoulli percolation we have no argument to suggest that the small λ limit satisfies DLR conditions for the Mandelbrot boundary.

One possible approach is to consider the *local weak convergence* technique of Aldous and Steele [1]. Consider a large truncated QuadTree $\mathcal{Q}^{[\![0,N]\!]}$ and choose a pixel $\mathcal{P}$ uniformly at random from the pixel set $\mathcal{P}_{[0,1]^d}^{[\![0,N]\!]}$. If N is large we may factorize the distribution of v into two parts. Choose $x \in [0,1]^d$ uniformly and chose K independently with $\mathbb{P}[K = k] = 2^{-kd}\left(1 - \frac{2^d-1}{2^{d+1}-1}\right)$.

It is easy to check that if we set $\mathcal{P}$ to be the almost surely unique pixel with $x \in \mathcal{P} \in \mathcal{P}_{[0,1]^d}^{N+1-K}$ then $\mathcal{P}$ is chosen uniformly from $\mathcal{P}_{[0,1]^d}^{[\![0,N]\!]}$. As $N \to \infty$ the random variable K converges in probability to a geometric random variable with mean $\frac{1}{1-2^d}$.

What does the QuadTree look like from the point of view of the random pixel $\mathcal{P}$? In Remark 1.6 we constructed a random pixellation $\mathcal{P}_{\mathbb{R}^d}^{-n}$ to extend the QuadTree to negative resolutions while preserving some form of translation invariance. It may be shown that the QuadTree viewed from a random pixel as above converges in the *local weak sense* to the random graph defined by adding tree and lattice edges to the random pixel set $\mathcal{P}_{\mathbb{R}^d}^{[\![-\infty,K]\!]}$.

As K is almost surely finite and independent of the pixellation we may factor it out and consider instead the random graph defined by $\mathcal{P}_{\mathbb{R}^d}^{[\![-\infty,0]\!]}$ which we name the *Worm's Eye QuadTree*. Let $\mathbb{W}$ be the distribution of the random graph $\mathcal{F}_{\mathbb{R}^d}^{[\![-\infty,0]\!]}$; we use w to represent an instance of the worm's eye QuadTree.

We may define a random cluster model on any instance w of the worms eye QuadTree. Furthermore $\mathbb{W}$-almost every w is amenable. Does this mean that the random cluster model is

unique on almost every instance w? Grimmett [32] proved that for any $q > 0$ the (p, q) random cluster model on $\mathbb{Z}^d$ is unique for all but countably many q. The key techniques in that paper relied on the amenability and transitivity of $\mathbb{Z}^d$.

The worm's eye QuadTree recovers these properties and a first reading of [32] does not reveal any reason why the techniques in that paper may not be applied, mutatis mutandis, to the worm's eye QuadTree.

If it is the case that the random cluster model is unique on $\mathbb{W}$ almost every w what may we conclude about the random cluster model on the QuadTree?

Here is a bold conjecture.

CONJECTURE 6.3

$\mathbb{Q}^q_{\mathcal{Q}(\tau,\lambda)}$ exhibits a unique infinite cluster if and only if the (τ, λ, q) random cluster model exhibits an infinite cluster on $\mathbb{W}$ almost every w.

It is almost the definition of a nonamenable graph that if we choose a vertex uniformly from a large ball the vertex chosen will lie close to the boundary with high probability. If Conjecture 6.3 can be resolved on the QuadTree, might a similar phenomenon occur on other nonamenable graphs?

For the case of the small λ limit we note that if the random cluster model in unique on each w then we may consider the wired random cluster model on the worm's eye QuadTree rather than the free. The small λ limit for the wired model may be defined with respect to two decreasing sequences; rather than one increasing and one decreasing for the free model. Therefore we may reverse the order of the limits and the small λ random cluster model on the worms eye QuadTree converges weakly to Bernoulli-$\pi(\tau)$ percolation on $\mathbf{T}(w)$ for $\mathbb{W}$-almost every w. This argument alone is not enough to conclude that $\mathbb{Q}^q_{\mathcal{Q}(\tau,\lambda)}$ converges to Bernoulli percolation as $\lambda \to 0$ but some progress could be made by considering problems in this area.

6.2.3   Anti–ferromagnetic models and robust entrance laws

We have only partially considered the case where $q < 1$. For the wired model on the tree Häggström [38] constructed measures on the homogeneous regular tree satisfying wired DLR conditions for all $q > 0$. We have seen that Häggström's measures correspond to measures defined by constant entrance laws and in Theorem 3.13 we have shown that this is unique.

For the general tree we have not been able to show that the maximal entrance law is always robust. For a homogeneous tree we were able to show in Lemma 3.14 that any constant entrance law is robust.

Where a constant entrance law is a number that satisfies a particular equation 3.13 a non-constant entrance law is a more complicated object which bears some similarities to a *flow* on the tree. There are a variety of powerful techniques on trees which may be used to investigate such objects. See for example [53, 64] and references therein. It may be possible to adapt the proof of Lemma 3.14 to the general case and we would conjecture the following.

CONJECTURE 6.4

For any weighted tree and any $q > 0$ the maximal entrance law is robust and there exists at least one wired random cluster model.

For general random connections we cannot guarantee the existence of a random cluster model when $q < 1$. Example 4.13 provides a counterexample. In fact it is easy to come up with counterexamples using the method discussed in Remark 4.9, for if $q < 1$ we may choose some increasing tail measurable event A with $\mathbb{P}_{\pi(\tau)}(A) = 1$ and $\mathbb{Q}^{\star}_{\tau,q}(A) = 0$ then set $[u \leftrightarrow v] = [u \leftrightarrow v] \cup (A \cap [u \Longleftrightarrow v])$.

Then for any extremal random cluster model μ we must have $\mu(A) \in \{0,1\}$. If $\mu(A) = 1$ then μ satisfies wired boundary conditions and $\mu(A) = \mathbb{Q}^{\star}_{\tau,q}(A) = 0$ and if $\mu(A) = 0$ then μ

satisfies free boundary conditions and $\mu(A) = \mathbb{P}_{\pi(\tau)}(A) = 1$. Hence there can be no extremal random cluster measures and hence no random cluster measures at all.

Example 4.13 uses this idea and effectively sets A above to be the event that there exist infinitely many open 2-trees. A little more work was involved to set up $\longleftrightarrow$ to be defined by a boundary condition.

Can we find a Grimmett-Janson equivalence relation for which some pair τ, q with $q < 1$ admits no random cluster model? A candidate may be the Mandelbrot boundary conditions discussed in Chapter 5.

Suppose we fix $\pi(\tau) > 2^{\frac{d-1}{2}}$, then the free random cluster model (that is Bernoulli percolation with probability $\pi(\tau)$) has the single cluster property. The partial domination of percolation from Theorem 4.23 may be reversed for $q < 1$ and we may choose q small enough so that the wired random cluster model has $\mathbb{Q}^{\star}_{\tau,q}[u \longleftrightarrow v] = \mathbb{Q}^{\star}_{\tau,q}[u \longleftrightarrow v]$.

In this case then neither the free nor the wired random cluster models may satisfy DLR conditions for the Mandelbrot connections and it is not obvious that any random cluster model exists.

If it is the case that no such measure exists what can we say about small λ limits for the QuadTree? (When $q < 1$ it may not be the case that $\mathbb{Q}^{q}_{\mathcal{Q}(\tau,\lambda)}$ converges weakly as $\lambda \to 0$, but by compactness we may choose some sequence $\lambda_n \to 0$ for which a weak limit exists.) It is an intriguing possibility that there may be some λ so small that no free random cluster model exists on the QuadTree.

6.2.4 Density of Open Boundary Conditions

Theorems 4.22 and 4.23 allow us to describe the behaviour of a random connection if it is "strongly connected."

Loosely a random connection is strongly connected if it is indistinguishable from $\iff$ for high probability Bernoulli percolation. If $\sim$ is some open boundary condition it is possible that some random connection $\leftrightarrow$ is weaker than $\overset{\sim}{\leftrightarrow}$ but for sufficiently high bond percolation $\leftrightarrow$ and $\overset{\sim}{\leftrightarrow}$ are indistinguishable. Say if $\leftrightarrow$ is defined with respect to a canopy consisting of finitely many connected components. Theorems 4.22 and 4.23 may easily be adapted to this case.

Conversely for the Mandelbrot boundary connections when $d = 1$ the canopy of the random connection $\twoheadleftrightarrow$ is a line, hence $\twoheadleftrightarrow$ is connected, but we have seen that in this case $\twoheadleftrightarrow$ is not strongly connected. This may easily be seen by noting that as the canopy is a line we may disconnect the canopy by removing a countable set of rays. As the probability of any single ray being open is zero we may conclude directly that when $d = 1$ the random connection $\twoheadleftrightarrow$ is indistinguishable from $\leftrightarrow$ for any Bernoulli percolation measure.

In both cases we may identify an "upper bound" for the random connection by considering the behaviour under high probability Bernoulli percolation. Is it possible to generalise this the results of Theorems 4.22 and 4.23 and specify the behaviour of a random connection for high bond strength in terms of some simpler bounding random connection?

We have seen that the question of when a sequence of random cluster models converges weakly to another random cluster model is far from obvious. Let us consider this question in a different form.

Suppose $\overset{n}{\sim}$ is a sequence of equivalence relations on $\mathcal{R}$. There are at least two ways in which $\overset{n}{\sim}$ may be said to converge to a limiting equivalence relation $\overset{\infty}{\sim}$. Firstly the equivalence relations may converge pointwise, that is $\Pi_1 \overset{\infty}{\sim} \Pi_2$ if and only if $\Pi_1 \overset{n}{\sim} \Pi_2$ eventually. Secondly the canopies may converge, in the sense that if $\mathcal{O}$ is an open set of rays in the quotient topology of $\overset{\infty}{\sim}$ then $\mathcal{O}$ is eventually open in the quotient topology of $\overset{n}{\sim}$.

Can we choose a topology on the set of equivalence relations whereby if μ_n is a $\overset{n}{\longleftrightarrow}$ random cluster model then any weak limit is a $\overset{\infty}{\longleftrightarrow}$ random cluster model?

Of particular interest would be the closure of the set of open equivalence relations under such a topology. An interesting aspect of this would be the case where $\mathcal{T}$ is a regular k-tree and $\pi(\tau) > \frac{1}{k}$. In this case any open equivalence relation specifies a random cluster model uniquely, but a sequence of open equivalence relations may converge to an equivalence relation that is not open and not have a unique random cluster model.

In this case may we specify the set of random cluster models as limits of different convergent sequences of equivalence relations that converge to the same limit, but specify sequences of random cluster models with different weak limits?

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