A CONCEPTUAL APPROACH TO THE EARLY LEARNING OF ALGEBRA USING A COMPUTER

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I declare that the research presented in this thesis has not been previously submitted for a degree at any university. The only exception is that some of the earlier parts of Chapter 6 were used in a different form as part of a submission for the degree of M.Sc. at Warwick University.

I further declare that the work here presented is my sole unaided work.

SUMMARY

This thesis describes an investigation into the conceptual understanding of algebra by early learners (age 11-13 years) and how a computer-based approach may be used to improve such, without any consequent loss of manipulative skills.

The psychological framework for the investigation centred on the importance of the individual child's construction of a cognitive framework of knowledge and the relevance of the current state of this to the facilitating of concept acquisition. As such it incorporates elements of the developmental psychology of Piaget, Ausubel and Skemp. Furthermore, in order to assist in the synthesis of a sufficiently broad psychological theory of education it was necessary to postulate the formulation of a new integrated bi-modal model of learning. This is described, along with details of its application and significance to a theory of cognitive integration which is designed to promote versatile learning (after Brumby, 1982) in mathematics through a relational linking of global/holistic and serialist/analytic schemas.

The research comprised two initial investigations followed by the main experiment. The results of the initial investigations with early learners of algebra showed that the dynamic algebra module written for the research produced a significant improvement in the children's conceptual understanding of algebra. The main experiment sought to further clarify this improvement and to compare and contrast it with that produced by a traditional skill-based algebra module. In order to facilitate this comparison, the performance of 57 matched pairs of pupils from two groups of three parallel forms of the first year of a 12+ entry co-educational secondary school was analysed.

The results of the investigation confirmed the value of the dynamic algebra module as a generic organiser (in the sense of Tall, 1986) for the understanding of algebraic concepts, producing a significant difference in conceptual understanding, without any detrimental effect on manipulative skills. Furthermore, the beneficial effects of the programme were such that its results showed that it had provided a better base than the skill-based approach for the extension of algebraic understanding past the initial stages and into more involved areas such as linear equations and inequalities.

The findings of this research show that the use of a module based on a computer environment, with its many advantages for conceptual learning, prior to the more formal introduction of algebraic techniques, is of great cognitive value. They also provide evidence for the theoretical model of learning proposed in the thesis, and suggest that for the production of a versatile learner in mathematics, more attention should be paid to the integration of the global/holistic abilities of the individual with his/her serialist/analytic abilities. The implications for the future are that such abilities, and hence mathematical competence may well be improved in other areas of the curriculum by the use of the computer within a similar theoretical framework.

Chapter 1

Introduction

This thesis describes a method of providing a conceptual basis in the early stages of children's learning of algebra (generalised arithmetic), at around the age of eleven years old, and provides the psychological framework for its successful application.

Most teachers of mathematics are more than familiar with the difficulties which many secondary school children, throughout the world, have when faced with algebra for the first time. What they are not familiar with, and this is what this research sought to address, is what experiences they might use to help such children to successfully build a conceptual understanding of basic algebra.

When initially addressing this problem, and seeking to ascertain its root cause, it becomes clear that one of the main difficulties such children face is their failure to be able to attach any significant meaning to the symbols, the letters, used in algebra. This lack should not be too surprising when we consider that many textbooks, and indeed teachers, when introducing the subject for the first time pay scant attention to building a cognitively meaningful understanding of what such letters represent. Often they are treated, and encouraged to be viewed, as objects in their own right e.g. 3a + 5a may be spoken of as similar to 3 apples plus 5 apples and the concept of letters as standing for a range of numbers as, for example, when they are used as generalised numbers or variables, is rarely presented. Having made a cognitive link which, although not necessarily erroneous, has serious limitations with respect to understanding, the evidence is that it is not easy to persuade students to another point of view (see for example Ross 1985). Hence it became clear that the way in which algebra is introduced is of great importance, since for many children, once they have embarked on a wrong course it is difficult to turn them against the tide of their own thinking.

I was aware that the computer environment, which is rich in the use of variables, strong in visualisation and high in motivating power held great potential for assisting with just such a problem as this, given a carefully thought out approach and the right combination of programming and software experiences. The success of Booth (1983a) in improving the performance and understanding of pupils by using an imaginary machine like a computer suggested that the way ahead was indeed to produce a module of work based on a computer environment and designed to precede the techniques of algebra in the form of an advance organiser (Ausubel 1978). The finished module was entitled ′Dynamic Algebra′ since its emphasis was to be on the dynamic nature of variables as representing ranges of numbers rather than a more static picture of their treatment as objects in their own right or as standing for a single unknown. The programme of work included worksheets and a piece of software especially written to make the most of the computer′s advantages, as discussed in chapter 5, in the algebra domain. Whilst engaging in the synthesis of the module, and making sure that it was set into an appropriate theoretical framework, it became clear that although the theories of learning of Piaget, Ausubel and Skemp were a vital consideration, as discussed in chapter 2, there existed areas which were not so clearly delineated by these theories, as for example, the place of visual imagery in cognition. Chapter 3 describes the results of my investigation of these areas of psychological theory and leads to the presentation of a model of learning which, although no doubt will be further refined, helps to place visual and other imagery, reversibility of thought, mental blocks, and leaps of insight in a theoretical framework. I have called this model the Integrated Bi-Modal model. This model has implications, I believe, far beyond the context of this thesis, since it sheds light on the importance of cognitive structures and the ways in which such qualitatively different but fully integrated structures may well be organised in the mind. The ramifications of the insight are such that they suggest that there may be a need to re-think some of the pedagogical processes of mathematics education in terms of whether we may have been neglecting to teach in such a way as to maximise the potential of the whole of the individual´s mind rather than just part of it. I suggest that the fact that mathematics involves not just the ability to think logically, i.e. serially/analytically, but also requires the capacity to take the global/holistic view of a situation when profitable, and to be able to switch at will between the two as necessary, in order to be successful is an important consideration for mathematics educators. I shall use the term Cognitive Integration to represent a psychological approach which has this integration of the two thought processes as its aim. It is the application of both of these types of thinking which characterise the versatile thinker (Brumby 1982), who is more likely to be the competent mathematician with the propensity for continued progress. The model I propose raises the question of whether current practice has neglected to teach with regard for the global/holistic abilities necessary for versatile learning (Tall & Thomas 1988), since, as chapter 3 discusses, these are relatively silent and inaccessible to the conscious self. This seems an area rich for future research.

The algebra and the computer backgrounds to the research are considered in chapters 4 and 5 respectively, along with a look at the possible changes to the curriculum which the technological progress may necessitate. This background shows the sound theoretical base of the research both in algebraic practice and in successful applications in the computer paradigm.

Chapter 6 describes the initial stages of the investigation and the resulting confirmation of the cognitive value of the conceptual approach to the early learning of algebra proposed in the work. On the basis of this initial success, improvements suggested and the important result of a longer term follow up study on some of its subjects, the module of work was thoroughly revised. Chapter 7 follows the progress of the module to its final form and then describes the main experiment, where it was compared with a traditional programme of work based mainly on acquisition of algebraic skills rather than conceptual understanding. This experiment involved two sets of three parallel forms in a 12+ entry co-educational comprehensive school.

An analysis of the results of this experiment, in chapter 8, leads to the conclusion that a significant improvement in children's understanding of algebraic concepts may be effected, and the specific areas where this was noted are discussed here.

One of the results which opens up an area for further research involves the differential effects observed with respect to gender and performance between the groups. The possibility that the route to becoming a versatile learner, and hence to success in mathematics, may be different for boys than for girls is one which is raised.

Important and valuable though results in tests are, they are consolidated by the structured interview situation, since it is here that lack of understanding may surface which has been concealed previously. Chapter 9 describes the results of such interviews with some of the pupils who took part in the main experiment and consists of an in depth investigation of their understanding of some algebraic concepts through interviews designed to probe understanding. Chapter 10 then looks in more detail at some of the issues raised in the study and interview results through the use of a questionnaire. Both of these procedures confirmed the value of the dynamic algebra module in respect of conceptual understanding in algebra, without detrimental effect on skills. They also provided evidence, which is discussed, for the theory of learning based on an integrated mind which I have proposed.

Any scheme of work which is intended to be applicable to the classroom situation must undergo considerations of practicability and it should be such that it can be taken and used by classroom teachers without extensive training on their part. With regard to a computer-based programme this pedagogical consideration is a significant point since it is still the case that many teachers of mathematics are unfamiliar with the available technology. In the present work the module has been used successfully by such teachers and so their comments and suggestions, and the knowledge gained from their experience, are worthy of close scrutiny. These are included where appropriate, and include both the positive and negative aspects of the classroom application. Chapter 11 includes a brief summary of these, along with the implications of the research for the teaching of algebra, and a recommendation that the beneficial results of the research are such that a programme such as the one here presented should become more widely used. Furthermore, the principles on which the programme has been based are such that they could, and I suggest, should, be applied to other areas of mathematics both in the primary and secondary mathematics classroom, and indeed beyond. The computer paradigm offers mathematics educators advantages which, whilst not being a panacea for children's difficulties with mathematics, provides great opportunities for us to help students achieve the versatility of thought, with integration of serialist/analytic and global/holistic abilities, which success in mathematics appears to require.

Chapter 2

Psychological Perspectives

The wide ranging approach to the dynamic learning of algebra proposed in this thesis is such that it does not fit neatly into any one of the previously presented psychological models, but rather draws on aspects of several of them, as well as introducing some different concepts. Therefore the psychological framework for the research is discussed in this chapter is based on the constructivist Piagetian theory, Ausubel's theory of meaningful learning and the relational understanding of Skemp. All of these theories relate to the importance of the 'framework of knowledge' which the individual constructs in a given cognitive area. Those areas of the theory of Piaget, Ausubel and Skemp which have been the basis of the research are outlined, in context, below.

2.1 The Piagetian Model

Piaget's model of learning has been the basis of much valuable research in the last twenty to thirty years and deserves the attention of anyone interested in the processes of cognitive advancement.

His model is based on the idea that intelligence is an assimilatory activity whose functional laws are laid down as organic life and whose structures are elaborated by interaction between itself and the external environment. Thus activity is necessary on the part of the individual for the process to be possible (Piaget 1953). There are two factors affecting the individual's intelligence in his/her interaction with the environment. The first is the tendency of the individual to organise, integrating his/her psychological structures into a coherent system. The second is the tendency to adapt these psychological structures, called schemata by Piaget, in response to the environment. This process of adaptation to the environment may be considered in terms of two complementary processes which Piaget calls assimilation and accommodation. Assimilation is defined in the following terms:

> "Intelligence is assimilation to the extent that it incorporates all the given data of experience within its framework. Whether it is a question of thought which, due to judgement, brings the new into the known and thus reduces the universe to its own terms or whether it is a question of sensorimotor intelligence which also structures things perceived by bringing them into its schemata, in every case intellectual adaptation involves an element of assimilation, that is to say, of structuring through incorporation of external reality into forms due to the subject´s activity" (Piaget, 1953, p.6)

Hence an individual assimilates an experience by incorporation into his/her schemata, thus giving it meaning. Some experiences are not capable of easy assimilation and so require a change on the part of the person´s cognitive structures to give them meaning. Hence Piaget defines accommodation in the following terms :

> "....mental life is also accommodation. Assimilation can never be pure because by incorporating new elements into its earlier schemata the intelligence constantly modifies the latter in order to adjust them to new elements."
> (Piaget, 1953, pp.6,7)

This process of accommodation is accomplished by changing the schemata in such a way as to make a place for the new experience.

The power of the theory propounded by Piaget is that all individuals progress through the same series of stages with respect to mental abilities by the use of these two processes, although they do not necessarily do so at the same rate. Furthermore the order in which they pass through these stages is the same for all mental abilities, and this is called the unity of the stage construct. Each of the stages is characterised by different psychological structures or schemata. The four main stages, in order of development (with approximate ages), given by Piaget are :

a) Sensori-motor (Birth - 2 years)

b) Pre-operational (2 - 7 years)

c) Concrete operational (7 - 12 years)

d) Formal operational (12 years onward)

It is to be realised that the ages given by Piaget are intended as rough estimates, since the transition age for each stage varies from person to person and also from culture to culture. The order of the stages, however, is declared by Piaget to be invariant across these and each of the stages must be traversed by every individual, albeit at different rates, and in a gradual and continuing fashion. Piaget (1950) argues that the schemata associated with each stage, and hence the mode of reasoning available to the individual at that stage involve a change in kind and not just degree as one passes to the next stage.

During the sensori-motor eriod the child is actively seeking contact with the environment, interpreting events which occur. Development is a process of decentration, or separating self from the environment, and the production of a mature object concept, or conceiving the independent existence of objects.

The start of the appearance of the pre-operational stage is characterised by the emergence of the semiotic function, or the ability to form mental representations or symbols for objects. It continues to be characterised by attention to only a limited part of the total information available, particularly by that which is static in nature. One feature of later thought which is noticeably absent at this stage is that of reversibility, which Piaget speaks of in the following terms :

> "At the first stage, the child's thought is still irreversible, in that each perception is a particular moment in the stream of his experience, without stable means of return, since there are no operations by which one perception can be composed with another...during the third stage, the operations go beyond the field of perception, and in doing so, attain complete reversibility."
>
> (Piaget, 1952 p.202)

This important ability has been recognised by others, Krutetskii for example lists it as an important component of mathematical ability, and refers to it as "the ability to switch from a direct to a reverse train of thought" (Krutetskii, 1976 p.85). I shall return to the subject of reversibility of thought in chapter 3, when presenting some extensions to the current psychological models.

In contrast to to the pre-operational child, the individual at the concrete operational stage is capable of decentration, or being able to focus on several aspects of a situation simultaneously. They are also sensitive to transformations, are able to appreciate changes which have taken place in a situation and have the important ability of reversibility of thought mentioned above. It is this period of development when the first real mathematical thought emerges, even though still tied to the concrete, and it is toward the end of this period when most children meet algebra for the first time. Hence it is this stage of development which has been of primary importance in the formulation of the research programme outlined in this work, the need to have this programme firmly fixed in a concrete operational mode being very important, recognising that "in sum, concrete thought remains essentially attached to empirical reality." (Inhelder & Piaget, 1958, p.250), although it should be appreciated that, as Sinclair 1971) states:

> "Concrete operations...does not mean that the child can think logically only if he can at the same time manipulate objects...Concrete, in the Piagetian sense, means that the child can think in a logically coherent manner about objects that do exist and have real properties...he can perform the mental operations involved...The actual presence of the objects is no intrinsic condition."
> (Sinclair 1971,pp.5,6)

The child's experiences with algebra at this age are vital if he/she is to have a strong foundation on which to build the higher algebraic concepts during the formal operational stage which Piaget propounds.

It is during the formal operational stage Piaget says, that the child becomes capable of considering the hypothetical, the possible, rather than the actual, saying that "the most distinctive property of formal thought is the reversal of direction between reality and possibility" (Inhelder & Piaget 1958, p.251). He further states that the concrete to formal change is a 'fundamental decentering' (Piaget & Inhelder 1969, p.130), the child then working mainly with propositions rather than objects. These types of abilities enable the construction of structure in mental schemata, causing Piaget to describe a fundamental difference between this form of thinking and concrete operational as "the difference between concrete and formal thinking relates to the construction of the 'structured whole'." (Inhelder & Piaget, 1958, p.163). These abilities become important in later algebra.

The factors which produce the transition from one of the above mentioned cognitive stages to the next, the mechanisms of transition, are described by Piaget as involving both development and, in a narrow sense, learning. The process of development involves physical maturation, particularly of the brain and central nervous system, experience (or physical interaction with ´things´), social transmission and equilibration. Piaget comments on the concrete to formal change as follows

> "....it seems clear that the development of formal structures in adolescence is linked to the maturation of cerebral structures. However, the exact form of linkage is far from simple, since the organisation of formal structures must depend on the social milieu as well. The age of about 11-12 years, which in our society we found to mark the beginning of formal thinking, must be extremely relative, since the logic of the so-called primitive societies appears to be without such structures. Moreover, the history of formal structures is linked to the evolution of culture and collective representations as well as their ontogenic history.....Thus the age of 11-12 years may be, beyond the neurological factors, a product of a progressive acceleration of individual development under the influence of education, and perhaps nothing stands in the way of a further reduction of the average age in a more or less distant future." (Inhelder & Piaget,

1958, p.337).

Thus Piaget allows for the factors influencing the transition from one stage to another to lower the age at which the transition occurs, and this point is worthy of note in respect of some of the criticisms of Piaget´s theory outlined below.

From physical interaction with things around them children may engage in a simple form of abstraction, the process of extracting the common properties of the objects or events they meet. It is by means of this process of abstraction that the young child acquires a vast category of structured knowledge, or concepts. Physical experiences alone, however, are not sufficient to learn some concepts, such as the equivalence of number. Social transmission enables the child to learn these through the experience and accumulated wisdom of others through, for example, reading or instruction, provided he/she has the necessary cognitive structures. The process of equilibration as a factor influencing mental progress refers to the child´s self-regulatory processes, as a result of which he/she progressively attains a higher degree of equilibrium or relative harmony in interaction with the incoming information from the environment, at each stage of development. The four factors influencing development mentioned above may be thought of as learning in a wide sense. Adding to this, Piaget includes the process whereby new information is acquired which is only relevant to a specific situation (e.g. rote learning of the names of the planets in the solar system) although he believes that this does not contribute to true cognitive development.

Piaget thus explains to educators, it is important for them to take into account that children learn only in a minor way from verbal explanation and that concrete experience must come first. This is an important principle often neglected in the initial teaching of algebra, but one which is fundamental to the reasoning behind this research. The idea of dynamic algebra is that the child is involved in a concrete model in a very real way, interacting with it and a small group (2/3) of their peers, giving maximum opportunity for the abstraction and social transmission of concepts. Since these children are primarily at what Piaget calls the concrete operational stage of development, when, according to Piaget, concrete experience is still vital, such experiences are important in the learning process and should precede a more formal and abstract approach. This is still not the usual approach in most secondary schools.

2.1.1 The Application of Piagetian Theory to Mathematics

This approach to cognitive development of Piaget, and the stages of such which he has defined, have been considered to be well supported by evidence from research into the learning of mathematics, and have been the basis of much work into the way in which children learn mathematical concepts. For example, Davis (1978, p.38) when discussing his work in mathematics education states "we have studied Piaget's work extensively and his influence is clear." Halford (1978) and Collis (1974) too, adopting a Piagetian approach, claim to have provided evidence of levels of mathematical concept development corresponding to the Piagetian levels, the corresponding ages at which they found that children in general reach these levels and hence the types of work which children at these levels should be attempting. Another influential Piagetian worker has been

Dienes. It has been suggested that his work is a successful application of Piagetian cognitive theory to mathematics. He proposes that concepts may be learned by abstracting their properties from different examples of the same concept. Dienes (1960) has listed the stages in the formation of a mathematical concept, in this way, as follows:

1. Free play
2. Games
3. The search for communalities
4. Representation
5. Symbolisation
6. Formalisation

He has also described the types of activity which will enable these stages in concept formation to be successfully reached. They are :

1. The Play Stage.

   This is a time of undirected activity with the ingredients of the concept available as play material.
2. A great number of experiences of varying structure but all leading to the same concept. Although this stage is more directed and puposeful it is characterised by a lack of any clear realisation of what is being taught.
3. Adequate practice for the fixing and application of the concept. The games used in this stage are :
   a) Preliminary games
   b) Structured games
   c) Practice games and/or analytical games.

Since children generally learn constructive thinking before analytical thinking, Dienes (1960) argues that it is important when devising learning situations to meet the requirements of the above stages to concentrate on constructive tasks, only gradually introducing the analytical aspects. Thus concrete material should be made freely available to the children, preferably individually , or in two´s or three´s, since the learning of a concept is obviously an individual process which would be hampered by lack of individual capacity for advancement.

Dienes has also shown a principle, which he calls ´The Mathematical Variability Principle´ to be fundamental to the individual child´s understanding of a concept. This principle says that, if a concept involves variables of any kind, then these should, wherever possible, be made to vary in the experience of the child, whilst keeping the concept intact. This emphasises the need for personal experience of the varying of the numbers represented by a single letter in algebra in as many different situations as possible. Whilst this theory has appeared to be of value in the learning of algebra, where abstraction plays a fundamental role, there has been some doubt expressed as to whether it is a sufficiently broad approach to the learning of mathematics. Borasi (1984), for example, has put forward the view that several approaches to concept formation, including abstraction, should be devised and taken into account since, he argues, the in-depth study of specific instances, or models, may play a fundamental role in the acquisition of mathematics. Lesh (1987) has shown that the value of the instructional principles of Dienes may be successfully applied to mathematics education when this is approached via a computer environment.

2.1.2 Criticisms of the Model

Whilst Piaget's model of intelligence and learning, briefly discusssed above, is of undoubted value in modern mathematical education, one must be mindful that there have been some problems associated with the application of some Piagetian notions and hence there are some criticisms which have been levelled against it. Wallace (1972), points out the difficulty in arriving at definite conclusions to the objections raised, but since these are often based on sound research they deserve our attention. Hiebert and Carpenter (1982) have surveyed the available research and have concluded that the Piagetian tasks are not useful readiness measures for success in mathematics. However, there are three areas in particular where the theory has been questioned. These are :

1. The evidence for the existence of a unified stage construct across different tasks.
2. The variability of the age at which children apppear to progress from one stage to another.
3. The validity of the description of the formal operational stage.

With regard to the first point above, Brown & Desforges (1977), in a review of studies which did not support Piaget's ideas, concluded that the substantial degree of homogeneity to be expected in the appearance of operations characteristic of each stage were not in fact observed and that therefore the concept of stage structure is in doubt. Replying to this criticism, Shayer (1979) cites other studies (e.g. Bart 1971, Lovell & Shields 1967) which tend to support the existence of such a structure, providing evidence at least for the utility of such a construct, if not its validity. The evidence for the existence of such a unified stage construct appears, at least, to be mixed and it may well be that there are task related factors which are also important in their effect on the performance of the child.

Similarly on the question of the age at which transition from one stage to another occurs, some studies (e.g. Donaldson 1978, Brainerd 1978, Gelman 1969, 1972 and Bryant & Tabasso 1971) have shown that, under suitable conditions, children do demonstrate certain logical operations at a much earlier age than predicted by Piaget's model. Churchill (1958) has provided evidence to show that Piaget's stages may be too rigid to accommodate the individual differences in children and explains that the right experience for a particular child can accelerate the growth of their mental structures. Novak (1977) points out that studies have shown that :

> "a significant number of young children could demonstrate highly formal reasoning in explanations of phenomena for which they have had appropriate cognitive preparation" and that "most people, children or adults, can evidence highly formal and abstract thinking when they have acquired a sufficiently well developed framework of concepts relevant to the tasks." (Novak 1972, pp. 472,3)

Krutetskii (1976) too explains that the studies of Soviet psychologists such as Blonskii (1961) disagree with Piaget's stages and have shown that "with special teaching methods, younger pupils acquire a much greater ability for abstraction and reasoning than is commonly thought" (Krutetskii 1976, p. 331). Although Piaget himself recognises that the ages of transition are not immutably fixed, as for example in the passage quoted above (Inhelder & Piaget, 1958, p.337), the studies do call into question whether the abilities manifested by children do appear in the fixed sequence given in the Piagetian theory. Some of the studies purporting to show precocious ability, such as Mehler and Bever (1967), have been criticised with regard to procedural errors made or the question of what the training has actually achieved relative to the child's level of understanding as opposed to their ability to tackle a given task in a rote way. Attempts to establish answers by investigating the level of transfer to other tasks appear inconclusive. The question is still open but it is an important one since it is necessary for the educator to know whether theories such as that of Piaget are decribing what children are capable of achieving or what they actually are achieving, in order to ascertain the advisability of attempting a particular teaching strategy.

The validity of the formal operational stage description has been questioned by, for example, Bynum, Thomas & Weitz (1972), Brown & Desforges (1977) and Smedslund (1977). Part of the problem seems to lie in the fact that an individual who is capable of what Piaget calls formal thinking may not use such thinking in a given situation. This may be because he/she thinks that the problem does not require it (possibly mistakenly), or that the effort required is too great. For whatever reason, the clear emergence of the formal operations stage seems elusive, with even Piaget introducing the concept of decalage, or variability of performance, to cover areas where expected performance is not realised. For others, their concern has led them to make their own interpretation of what constitutes the difference between concrete and formal operations and whether the latter truly exists as a separate stage. In chapter 3 I shall propose a possible alternative view of what may constitute this stage of formal operations.

2.1.3 Summary

In this section I have outlined the parts of Piaget's theory of learning which are relevant to the conceptual learning of algebra discussed in this thesis, and have outlined some of the difficulties associated with it. Whilst the questions and problems attached to Piagetian theory outlined above need to be addressed, they do not prove to be sufficient to topple the basic theory. Piaget's model of intelligence and its prediction of the behaviour and learning pattern of an individual have proved valuable in many areas of research into the learning of mathematics and other subjects and have much supportive evidence. The positive contribution of Piaget's work to the psychological framework for this research included the need for a concrete approach for the early secondary school age child, the principles and methods of the abstraction of concepts and the accommodation of schemata and the importance of reversibility of thought. It was considered, however, that, in the present research, as others too have thought (e.g. Booth

1983), that there were other psychological models which it was important to consider when approaching the subject of children's learning of algebra, and that taking these and extending them might help to explain some of the apparent weaknesses in the Piagetian framework. Thus below, I have described briefly two of the theories, the work of Ausubel and Skemp, which I believe to be of relevance in this area of mathematical learning, and in chapter 3 I shall consider some extensions to the theories.

2.2 The Cognitive Theory of Ausubel

The main thrust of the cognitive theory of Ausubel may be considered to involve the synthesis of what constitutes meaningful learning for an individual, and under what conditions such learning may take place. He describes such meaningful learning, as opposed to learning by rote, in these terms :

> "In meaningful learning the learner has a set to relate substantive (as opposed to verbatim) aspects of new concepts, information or situations to relevant components of existing cognitive structure in various ways that make possible the incorporation of derivative, elaborative, correlative, supportive, qualifying or representational relationships." (Ausubel, 1963, p.22)

Novak (1980, p.136) emphasises the difference between rote and meaningful learning as :

> "Here we must distinguish between rote learning wherein new knowledge is arbitrarily incorporated into cognitive structure in contrast to meaningful learning wherein new knowledge is assimilated into specifically relevant existing concepts or propositions in cognitive structure."
>
> (Novak, 1980, p.136)

and he stresses that for a given individual and a given knowledge domain, the Ausubelian rote/meaningful distinction is not in the form of a dichotomy but rather is a continuum.

Since, according to the definition above, meaningful learning is subjective in nature, it is natural that one of the principles fundamental to Ausubel's (1963, 1968, 1978) theory of cognitive development is the idea that in order to facilitate learning attention should be paid to the pupil's current cognitive structure, and that new material should be learned in relation to the previously learned background of concepts etc. In fact Ausubel (1968, p.vi) has summed up the philosophy of his approach to learning as "The most important single factor influencing learning is what the learner already knows. Ascertain this and teach him accordingly". Thus attention should be paid to the 'framework of knowledge' which the child constructs in relation to an area of study. He states that 'logically meaningful material becomes incorporated most readily and stably into cognitive structure in so far as it is subsumable (able to be linked to already existing relevant concepts) under specifically relevant existing ideas' and that, therefore, one of the best ways of learning is to increase the availability of such subsumers, or existing relevant ideas. In Ausubel et al (1978) he makes it clear that subsumption is not just a simple linking of new concepts to old, but that rather, in meaningful learning,

> "the very process of acquiring information results in a modification of both the newly acquired information and the specifically relevant aspect of cognitive structure to which the new information is linked."
>
> (Ausubel, 1978, p.57)

Hence, as Novak (1977) observes, this process is not quite the same as Piget´s concept of assimilation, since

> "The process of subsumption differs from Piaget´s concept of assimilation in that: (1) new knowledge is linked to specifically relevant concepts or propositions and (2) this process is continuous and major changes in meaningful learning...occur not as a result of general stages of cognitive development but rather as a result of growing differentiation and integration of specifically relevant concepts in cognitive structure."
>
> (Novak, 1977, p.456)

Thus, as new knowledge is acquired in a meaningful way, Ausubel argues that a modification must take place in the whole structure of the relevant mental schemas, and that this re-structuring is a continual process throughout life. Novak (1977, p.457) puts it this way: "Thus any given concept is never

´acquired´ but is in the process of being differentiated." It is this accumulation of information relative to a given concept held in the mind, and in the process of being dynamically re-structured, which Tall and Vinner (1981) have defined as the concept image:

> "We shall use the term concept image to describe the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes. It is built up over the years through experiences of all kinds, changing as the individual meets new stimuli and matures."
>
> (Tall and Vinner, 1981, p.152)

Realising that the individual´s cognitive structure contains such concept images seems an important step towards helping to channel the dynamic re-structuring of schemas in a positive direction mathematically. As Tall (1981, p.319) comments, one of the advantages of Ausubel´s theory with its emphasis on the individual´s conceptual framework is that it can be applied to "all individual´s at all ages", as educators seek to assist in the promotion of favourable conditions for the re-structuring of concept images. Others such as Collis (1978) and Skemp (1977) have agreed with Ausubel´s emphasis on the importance with respect to learning, of the ´framework of knowledge´ of the individualand of its progressive re-structuring. Collis (1978, p.257) states that it seems likely that ´the development of the use of formal operations in mathematics depends on the construction of a network of relations within the child´s cognitions which will enable him to move freely about the developing structure of mathematics´. Skemp (1977 p.113) also argues that "our conceptual structures are thus a major factor of progress too", adding that "to understand a concept, group of concepts, or symbols, is to connect it with an appropriate schema." (ibid, p.148).

2.2.1 Advance Organisers

In order to make this type of learning more likely to take place, the educator must seek to increase the availability of concepts, or subsumers, in appropriate schemas, to which such links may be made. This may be done, Ausubel (1978, pp. 171,2) contends, by the use of suitably structured work, or advance organisers, presented to the child ahead of the work to be learned. The principal function of these advance organisers is to bridge the gap between what a student already knows and what he needs to know in order to successfully learn the work at hand and it is for this reason that they have also been called cognitive bridges, by Novak (1976, p.500). The criteria to be satisfied for a programme of work to be such an organiser have often been misunderstood, but are clearly described by Ausubel (1978, pp.170-173) when he says of an advance organiser:

1. It should be a properly designed instructional sequence
2. It should be more general and more abstract than the information to follow
3. It should serve to facilitate meaningful learning by linking to relevant existing concepts in the students mind
4. It should be learnable and stated in familiar terms

Commenting on the application of Ausubel´s theory to science education, Novak (1976) emphasises the value of these advance organisers, or cognitive bridges as he calls them. He claims that innovations in teaching should be based on an analysis of the conceptual framework of the discipline and that the properly designed instructional sequence, of the type outlined above, introduced prior to the new information should signal what will be the key concept in the new material. Novak, (1977, p.453) further explaining his preference for Ausubel over Piaget, claims that Ausubel´s views favour the interpretation that "since all concepts have at least some remote relevance to other concepts, the total mass of specific concepts acquired over a life span will influence the acquisition and use of other concepts". Dawson (1984) also recommends Ausubel´s theory of advance organisers, noting that it encourages teachers to make the links between concepts more evident, to structure work carefully and to attempt to assist pupils to tie concepts together into a well understood network. These areas of Ausubelian theory were also considered to be of value in the work on children´s strategies and errors in early algebra carried out by Booth (1983), and discussed more fully in chapter 5.

It is true that some (e.g. Barnes and Clawson, 1975) have been critical of studies purporting to show evidence of the value or use of advance organisers in education. Ausubel admits that:

> "Somewhat equivocal findings have been reported for studies involving advance organisers...This is due in part to failure to adhere to explicit criteria of what an

> advance organiser is (see above) and in part to various methodological deficiencies in research design."
>
> (Ausubel, 1978, p.175)

Novak too, commenting on these apparent problem with advance organisers says:

> "There are also instructional variables such as pacing and organization of the learning task that must be considered."
>
> (Novak, 1977, p.458)

The problem may well be then, not one with the soundness of the concept of an organiser, but rather with the lack of care taken over the application. This was the view taken in this particular research, and the results (see chapter 7), seem to bear out the practical value of such a position. Thus, I maintain, that the implications of Ausubelian theory of organisers are that it will be easier for a student to learn material in which the major concepts presented are disjoint from those in his/her cognitive structure if some bridging material is used first. In mathematics one meets many examples of this type of new work as one progresses through the secondary school curriculum, but there can be few cases where the need for an advance organiser is more apparent than for bridging the gap between an arithmetic mode of thinking and the algebraic mode. Here the learner is faced not just with unfamiliar concepts, but also with the conceptual problems of having to use a whole new symbol system in which to represent such concepts. Such a symbol system has I believe, as I discuss in chapter 4, been all to often far from meaningful for the majority of children, and I propose that it is partly due to the lack of an advance organiser that many students have such difficulties in the learning of algebra. They find the concepts with which they are suddenly presented very difficult to link to their existing cognitive structure and where such links are made they are often inappropriate. It was the purpose of this research study to address this need and provide a fully tested programme of such bridging material. It was recognised that any programme designed to improve this situation would need to take account of the requirement to supply links to existing concepts, such as those in the arithmetic domain, thus enabling the pupils to build a relevant framework of knowledge, and in chapter 6 I have discussed the ways in which this was done.

2.2.2 Reception and Discovery Learning

Another area of learning to which Ausubel´s theory makes a significant contribution is the discussion of the relative merits of reception and discovery learning. In reception learning, the "entire content of what is to be learned is presented to the learner in its final form." (Ausubel, 1963, p.16), compared with discovery learning, in which "the principal content of what is to be learned is not given but must be independently discovered by the learner before he can internalise it." (Ausubel, 1963, p.16). He makes the point that both types of learning may be meaningful and whether this is so depends on whether the cognitive connections are made in an arbitrary fashion or not. What has to be appreciated is that, as with the rote/meaningful distinction, the reception/discovery difference in acquiring knowledge is also to be understood as a continuum. One of the important factors governing the reception-discovery boundary is the extent of assistance which may be given in a discovery learning situation. Such assistance Ausubel (1963) calls guidance, with complete guidance amounting to reception learning and the absence of guidance to what he calls autonomous learning. Commenting on the amount of guidance which should be given in the learning situation, Ausubel states:

> "All of the foregoing methods have proved more effective than either complete discovery or reception learning, particularly in the retention and transfer of problem solving skills...A review of short-term studies of the role of guidance in meaningful discovery learning leads to the conclusion that guided or semi-autonomous discovery is more efficacious for learning, retention and transfer than is either completely autonomous discovery or the provision of complete guidance."
>
> (Ausubel, 1978, pp.336,7)

Thus the important conclusion reached on the amount of guidance which should be provided by educators in a meaningful discovery learning situation is that some guidance is more effective than either none or complete guidance. It seems to be particularly true that such guidance is valuable when the transfer of knowledge between domains is involved and since this is often a problem it is an important consideration. Thus the provision of stimuli through cues and prompts is valuable in such a learning environment. In this research this principle is one which has been considered important, and the ways in which it has been promoted in the computer learning environment are discussed in Chapter 5.

As mentioned above, it was the intention of this research to try to construct a programme of work to meet Ausubel's criteria for meaningful learning, in the area of the early learning of algebra. The work would need to act as an advance organiser, preparing pupils for a more detailed and formal study of algebra. It would need to let them know unobtrusively, in a meaningful discovery learning situation with some guidance provided, that the key concept in that future work was to be the understanding of the use of letters as variables (or as generalised numbers) and it would attempt to provide the links necessary to tie this and other concepts to concepts of number and arithmetic already known, to make a useful framework of knowledge.

2.3 The Cognitive Theory of Skemp

Skemp's theory of learning has broad agreement with many of the ideas of Piaget's theory including, for example, the abstraction of concepts from experience and their assimilation, the subsequent accommodation (extended to include expansion and re-construction in Skemp 1979 pp.126,7) of schemata, and the existence of reflective intelligence. However, Skemp has at the base of his theory a different conception of intelligence and a different model of intelligence and learning.

Skemp's model of intelligence is based on the concept of intelligence as a goal-directed activity, with learning as :

> "a goal-directed change of state of a director system towards states which, for the assumed environment, make possible optimal functioning."

(Skemp 1979c, p.89)

Thus, according to this model, we engage in a mental construction of reality by building and testing a knowledge structure (Skemp 1985). There are three modes of building and three of testing such structures given by Skemp (1979c, p.163), they are:

REALITY CONSTRUCTION

REALITY BUILDING	REALITY TESTING
Mode (i)	Mode (i)
from our own encounters with actuality: experience	against expectation of events in actuality: experiment
Mode (ii)	Mode (ii)
from the realities of others: communication	comparison with the realities of others: discussion
Mode (iii)	Mode (iii)
from within, by formation of higher-order concepts, by extrapolation, imagination, intuition: creativity	comparison with one´s own existing knowledge and beliefs: internal consistency

(Skemp, 1979c, p.163)

The acquisition of concepts in one´s knowledge structure may be by abstraction by direct sensory experience from actuality (primary concepts) or by deriving them from other concepts (secondary concepts). Skemp describes two director systems which he calls delta-one and delta-two, where delta-one is a kind of sensori-motor system which:

> "receives information...compares this with a goal state, and with the help of a plan which it constructs from available schemas, takes the operand from its present state to its goal state"
>
> (Skemp, 1979b, p.44)

Delta-two on the other hand is a goal directed mental activity, whose operands are in delta-one, and its job is to optimise the functioning of delta-one (Skemp, 1979a). The acquisition of new concepts may require expansion (Piaget´s accommodation) of the relevant schema or it may be that re-construction of the schema is necessary, that is its altering to take into account a concept for which it is relevant but not adequate (Skemp, 1979c, p.126). Some of the factors Skemp describes as affecting one´s ability to form concepts using the director systems are the frequency of contributory experiences, the existence of noise (presence of irrelevant input) and the availability of lower order concepts. He also outlines two modes of mental activity which take place, intuitive and reflective:

> "In the intuitive mode of mental activity, consciousness is centred in delta-one. In the reflective mode, consciousness is centred in delta-two. ´Intuitive thus refers to spontaneous processes, those within delta-one, in which delta-two takes part either not at all, or not consciously. ´Reflective´ refers to conscious activity by delta-two on delta-one."
>
> (Skemp, 1979b, p.48)

Thus there exist two types of ´intelligent´ activity, intuitive and reflective. It is the reflective activity which is considered vital for successful building of knowledge structures, a hypothesis of the theory being that it increases mathematical performance. Such activity involves the awareness of one's own concepts and schemas including the ability to examine them reflectively and thus improve them. The methods available for this are listed by Skemp as:

> "a) Formulating our concepts and schemas.
>
> b) Devising experiments by which to test the predictive powers of our schemas.
>
> c) Revising our schemas as necessary in the light of these (and other) events.
>
> d) Mental experiment, by which we try to optimize our plans before putting them into action.
>
> e) Examining our schemas for inconsistencies and false inferences.
>
> f) Generalising our concepts and schemas.
>
> g) Looking for connections between events and our existing schemas. This is a reflective activity in which the person is expaining to himself.
>
> h) Increasing the number of conceptual connections within a schema, as a special case of
>
> i) Improving and systematizing the knowledge we already have."

(Skemp 1979, p191)

Using tests devised by Skemp, Jurdak (1980) has provided some evidence for the value of reflective intelligence in improving mathematical ability.

One of the valuable insights provided by Skemp's model is an appreciation of the qualitative nature of understanding. He initially divided understanding into two different types called instrumental and relational (Skemp 1976). Whilst his ideas received wide acceptance as valuable insights, there were some attempts to extend and re-shape some of the ideas he presented. One such (Byers and Herscovics 1977) argued for an extension of these two types of understanding to four, to include intuitive and formal understanding. Intuitive understanding was included to cover those instances where one has an implicit perception of a problem as a whole with little awareness of the solution process and an absence of a sequence of well defined steps to it. There may also in this case be a reliance on imagery. Formal understanding, they argued, was the ability to express in conventional form the mathematics done. Skemp (1979b) has accepted this need for a category of logical understanding, akin to formal understanding but these additions had initially been considered unnecessary by others (e.g. Backhouse 1978, Tall 1978). Following discussion on this division, the number of types of understanding was increased by Skemp to three, instrumental, relational and logical. Instrumental understanding has been described as learning 'how to' and involves learning by rote, memorising facts and rules. In contrast, relational learning, or learning 'why to', consists primarily of relating a task to an appropriate schema. It is important too to note that for Skemp a schema is "a conceptual structure existing in its own right, independently of action." (Skemp 1979c, p219). This idea fits in well with those of the advocates of meaningful learning, such as Ausubel, since we may describe the goal of meaningful learning as relational understanding. Logical understanding Skemp describes in these terms:

> "Logical understanding is evidenced by the ability to demonstrate that what has been stated follows of logical necessity, by a chain of inferences, from (i) the given premises, together with (ii) suitably chosen items from what is accepted as established mathematical knowledge (axioms and theorems)."
>
> (Skemp, 1979b, p.47)

Hence, the acquiring of logical understanding means that the individual not only has a relational understanding of a matter, but is able to convey evidence of such understanding to others by means of "a valid sequence of logical inferences" (Skemp, 1979a, p.200). Skemp maintains that all three types of understanding have an intuitive and a reflective dimension, as discussed above, and in Skemp (1979b) he gives examples of each in mathematics with the exception of intuitive logical understanding. I suggest that it may be that the Jordan curve theorem can provide an example of this. A student may be able to produce an intuitively logical argument that, for a closed curve crossed once from a point outside one ends up inside, whereas crossing twice would put one outside. They may then go on to generalise this argument to include odd and even numbers of such crossings. However, only on production of the full mathematically rigorous proof of the theorem would there be a demonstration of reflective logical understanding.

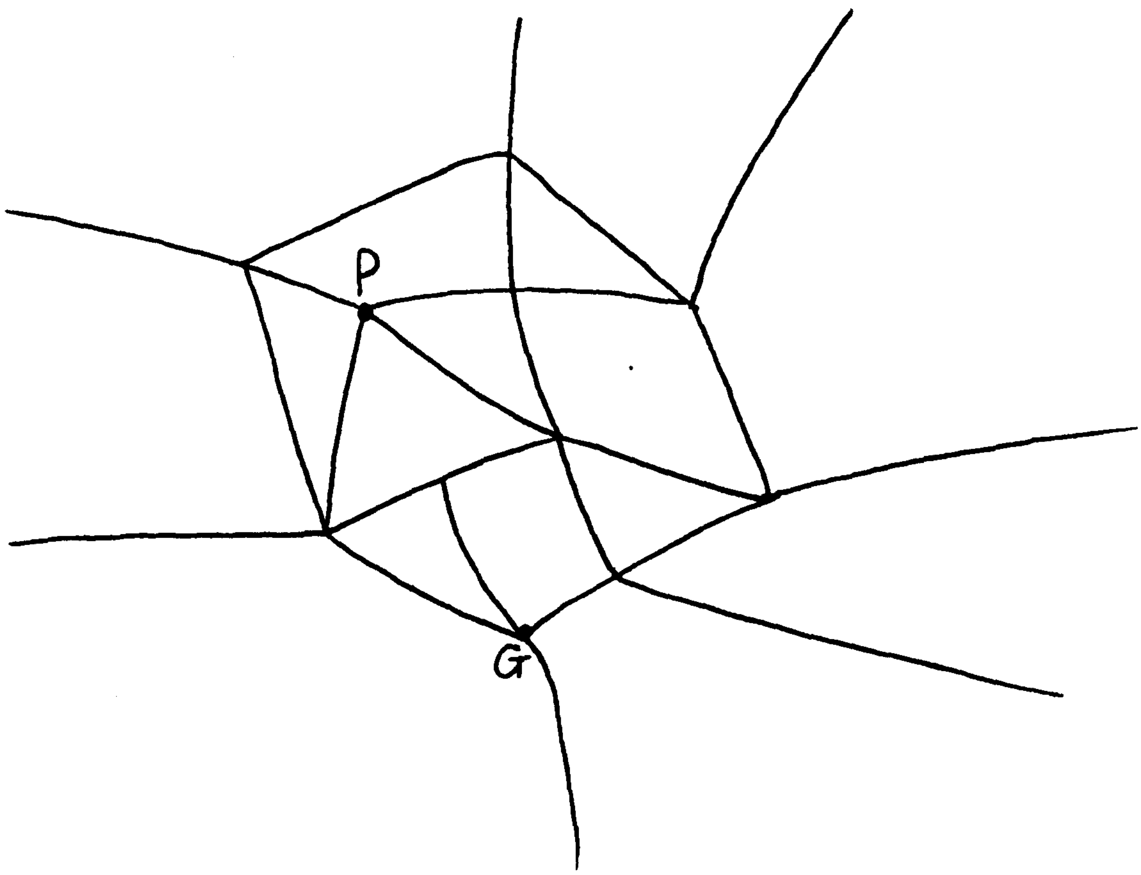


Figure 2.1

A Network Representation of A Generalised Schema

Skemp has introduced a very helpful mental image for our appreciation of the concepts in the model. It consists of a network such as Figure 2.1, which may be thought of as representing a generalised schema. It must be remembered that the diagram is not intended to picture what physically takes place in the brain, but is an aid to understanding. Each point of the diagram represents a particular state, such as a concept, although it must be appreciated that each state may have interiority (consist of a network of concepts itself). Thus one gets from the present state, P, to a goal state, G, by traversing an appropriate path, the path taken depending on the ease of activation of neighbouring concepts (their concept threshhold), the nearer the concept, the more likely its activation. Thus "to understand a concept, group of concepts, or symbols is to connect it (sic) with an appropriate schema." (Skemp 1979c, p148). Understanding therefore implies the realisation of a present state and a goal state in an appropriate schema, meaning that a path exists between them. It does not mean however that we ´know´ the appropriate path, as Skemp (1979b, p.48) says "´Goal-directed´ does not imply ´consciously goal-directed´". In fact the further apart the present and goal states are then the less likely is the route to surface and we may well just start off in the general direction which we consider to be correct. Using this model we may picture at least five possible states of understanding or lack of it :

1. Initially we cannot categorise an event or object. We say we understand it when we can link it to a schema which tells us what it is. We do not necessarily understand it itself.

2. We understand what a concept is by connecting it to an appropriate schema, but the paths to connect it to goal states in other schemas do not exist.

3. No understanding takes place, or can take place, because the relevant schemas are totally absent. There is no awareness of the lack of understanding.

4. We think we understand a concept but we mis-understand because we have connected it to an inappropriate schema or have wrongly connected it to an appropriate schema.

5. To understand a word or symbol is a special kind of understanding which means to associate it with an appropriate concept. (Based on Skemp 1979c, pp.146-148)

Therefore "our conceptual structures are a major factor of our progress" (ibid, p.113), since the schemas which exist either serve to promote or restrict the association of new concepts, and here Skemp and Ausubel are in close agreement, that what an individual already knows is a primary factor affecting their ability to understand. The qualitative difference in the links between concepts in the picture used (figure 2.1) is taken care of in the model by referring to associative or A-links for instrumental understanding and conceptual or C-links for relational understanding.

The advantages of relational teaching and learning put forward by Skemp include :

"a) The schema as a whole is more easily remembered since there is less to learn.

b) It evokes and favours the activity of reflective intelligence so necessary for constructing a knowledge structure.

c) It is more adaptable to new tasks.

d) It can be effective as a goal state in itself, motivating the learner to want to know and understand more.

e) Relational schemas are organic in quality i.e. if a learner enjoys this type of understanding they (sic) may develop a taste for it."

(Skemp 1976, p20-26)

Although this means that relational understanding is highly desirable it would seem that there is no easy way to teach it. It has been suggested (Meissner 1983) that it may be furthered through the use of many examples before systematisation; the allowing of trials and guess and test procedures; the use of estimations and freedom in using words and notations. Herscovics (1979) devised a teaching outline for linear equations and their graphs designed to investigate the types of understanding displayed by students. He found that, as predicted by Skemp's theory, intuitive and relational aspects were the ones most easily learned, and that such understanding improved best over time, again showing its value. The current research programme has encouraged the use of many examples in this way as well as the use of trial procedures in the solution of problems on the computer in the hope of stimulating and encouraging relational learning. Skemp's ideas concerning reality building and testing mode 2 (Skemp 1985) also lend themselves well to the computer setting. Tall (1986a), initially citing Skemp, comments on the application of these modes to the computer environment:

> "as currently taught, pure mathematics relies heavily on mode (iii), in varying degrees on mode (ii) and not at all on mode (i). However, if one interprets the idea of initial concrete activities being centred in mode (i), aided by explanation and discussion in mode (ii), then the computer gives us a totally new way of attacking the learning of mathematical concepts."
>
> (Tall, 1986a, p.24)

This means that the computer paradigm, properly used (see chapter 4) encourages the building and testing of individual knowledge structures in a way that promotes meaningful learning through the cultivation of relational understanding. Tall (1986b) has shown the benefits which may be obtained from such an approach in the teaching of calculus in the upper secondary school. Skemp´s comments on the value of interaction in the context of small groups are worthy of note in this connection:

> "Many of these activities take the form of games, providing shared experiences which give rise to mathematical discussion among the children. This peer group interaction is an important method of learning and children correct each others´ mistakes in a way which is much less threatening than being told they are wrong by the teacher"
>
> (Skemp 1985, p.450)

An advantage of small groups is thus seen as giving the freedom of opportunity, and encouragement, to communicate relational understanding free from the possible restrictions of teacher vocabulary etc, and it was considered that this was a benefit which would accrue from the computer building and testing environment used in this programme, if the pupils were encouraged to use tackle the work in groups of two or three. Pearson (1986), working with Logo on the computer found a close fit between Skemp´s model and children´s learning experiences and was able to say that she had experienced the value of mode (ii) in action in the classroom. Using an understanding of Skemp´s learning theory, and designing their study around it, Bye, Harrison and Brindley (1980) constructed an experiment to assess the effects of a concrete process-oriented teaching method for fractions and ratios. The results of their work showed that this approach produced a significant improvement in achievement in, and attitude towards, fractions and ratios across a wide ability range of twelve and thirteen year-olds. Studies such as this show the value of taking into account the theory of Skemp in order to produce material which is both practical and beneficial.

2.4 Chapter Summary

In this chapter I have presented some of the main ideas of the cognitive psychology of Piaget, Ausubel and Skemp which have been at the forefront of the development of the project here described. Some of the areas in which the thinking behind the work has been influenced by these theories and some of the pedagogical implications have been discussed. It has been recognised that there are still questions outstanding if we adopt Skemp´s model of learning, or indeed any of the other models, or a combination of their valuable features. How, for example, do we explain intuition in the model? Also, for a particular skill do all the parts have to be relationally understood in order to have relational understanding? Further, how do we use the model to describe unambiguously such features of cognitive activity as remembering, forgetting, mental blocks, leaps of insight etc.? (Tall 1978). These questions will be discussed in the next chapter in the context of what I propose as a slight extension of Skemp's model, and a new overall model of cognitive activity in which to place it. I feel that this may go some way to helping to promote an understanding of more of the processes involved in learning and cognition.

Chapter 3

Towards an Integrated Psychological Model of Cognition

In this chapter I shall consider the qualitative nature of the knowledge structures of human cognition and, using the theory of Skemp outlined in chapter 2, attempt to assist in the synthesis of a theory of their use through an analysis of the cognitive functions in mathematics (and probably most other) education. This theory attempts to integrate these psychological functions into a coherent system, presented by means of a model, also based on the ideas of Skemp, which will provide possible answers to some of the questions raised at the end of chapter 2.

3.1 Extending the Theoretical Model

In order to try and indicate how Skemp's model of intelligence and learning may be adapted in an attempt to answer some of these questions we shall refer frequently to the network (Figure 2.1) introduced as a mental aid for the understanding of a mental schema. It should be remembered throughout that these diagrams are not intended to be in any sense a physical representation of the cell structure of the brain or the way in which it stores data. They are an aid to our perception of the cognitive structures of the mind and hence help towards understanding the ways in which the brain stores and manipulates concepts and how this affects our thinking, learning and understanding. It is necessary to state too, that, because of the interiority of many concepts, the diagrams must be considered in a way which is described by Skemp (1979c) as vari-focal. That is to say that many of the points in the pictures which at one stage are viewed as representing concepts may, because of their interiority, be viewed (by a process akin to magnification), as schemas themselves. An example of this relevant to the current research is that of the variable in algebra. Far from being a simple concept, this has great interiority, giving rise to at least ten different, interconnected usages depending on the context in which it is found (e.g. Wagner 1981).

3.2 The Qualitative Nature of Conceptual Links

3.2.1 The Strength of the Links

In his model, Skemp rightly argues that "activation of one concept can activate, or lower the threshhold for, others." (Skemp 1979c, p131) and he uses the analogy of a net lifted up through shallow water by holding it at a point to suggest the idea that the more closely connected a concept is the more likely it is to be evoked by a neighbouring concept (those nearer the surface being more strongly evoked). It seems though that the concept of closeness is too quantitative in nature to represent the true qualitative state of the links between concepts. Thus while it is certainly true that the activation of a concept is more likely to evoke some concepts than others it is, I suggest, primarily the quality of the link which determines the concept threshhold. For example the activation of the concept of work in the mind of many people is more likely to evoke concepts such as employment, occupation etc than it is to evoke the concepts energy, force etc. This is due, I suggest, to these former concepts being more strongly linked to that of work than the latter ones in the mental structures of most people. Hence we shall introduce the concept of the strength factor of a link and say that the concept threshhold is determined by the strength factor of the link connecting it to the activated concept, such strength factors varying from weak to strong continuously. It must also be appreciated that the strength factor of the links is not necessarily immutably fixed, but rather, like the mental schemas themselves as a whole, is dynamic in nature. This dynamic nature of the learning process has been pointed out previously (e.g. Tall 1978a, Rachlin 1981). In the context of algebra, Rachlin has recognised this dynamic nature, and the construction of the Soviet´ascertaining´ methodology in order to study it, stating:

> "A student´s understanding of algebra is dynamic. It changes with time and experience. The long-term duration of the ascertaining experiment has been designed to capture the quality of the dynamic nature of learning."
>
> (Rachlin 1981 p61)

Factors affecting the dynamic strength factor of links would seem to include :

a) The use of the path as a function of time

b) The domain in which the link is established/used

and I shall now consider each of these.

Once one has achieved a measure of understanding by attaching a concept to an appropriate schema, the link will be considered to have a strength factor which has been determined by the methods used in the learning process, the mode employed when engaging in the process of reality building (Skemp 1985) and the environment in which this occurred (see below). It is reasonable to suppose that this strength factor can then be increased if the link is given use as part of paths between present and goal states as the individual uses his/her schemas. Conversely it will also be subject to decay over time if it is not so used. Skemp´s theory suggests that the rate of decay will be much greater for links which are A-links (associative links connected with instrumental understanding) compared with those which are C-links (conceptual links connected with relational understanding). This decay in strength factor partly corresponds to what is commonly referred to as loss of memory. Thus what we are here saying is that although a concept may be understood (attached to an appropriate schema) this is no guarantee that it will be activated at appropriate times (i.e. such understanding made use of), since the lowering of its threshold is dependent on the strength factor of its links, and that therefore activation may be made more likely by increasing the strength factor of its links through means such as promoting use of the concept in meaningful thought.

One of the problems associated with mathematical and other learning has been that of transfer of understanding from one knowledge domain to another. It has been shown that children who are able to cope well with a task in one environment are not always able to transfer their knowledge and skills to another. This has been noted by many who have studied children´s mathematical abilities (e.g. Rachlin 1981, Krutetskii 1976, Wagner 1977, Booth 1983a), Krutetskii listing it as one of the basic mathematical abilities. Relating this problem of transfer to the current model we see that the strength factor of the links will depend on the domain in which the concept has been introduced. For transfer to be effected the concept must be connected to concepts in other domains for which it is a relevant concept. If this is left totally for the reflective thought of the individual to accomplish then such links may not be completed. The stronger the strength factor between the concept and those in the schemas of the domain in which it was understood the less likely it may be that linking to other relevant schemas, which could facilitate transfer, will take place. In this case it would seem that concepts should be learned as far as possible in a domain independent way, by introducing examples of the concept from as many different areas as one can. An example of these ideas relevant to the current research is the use of brackets. It has been found (Booth 1983a, Thomas 1985) that pupils may use these correctly in a computer or ´Maths. Machine´ environment yet be unable to transfer this ability successfully to the more traditional setting of the mathematics classroom.

3.2.2 Directional Links

One of the four basic mathematical abilities listed by Krutetskii in his model is reversibility of thought. He describes it this way :

> Reversibility is "an ability to restructure the direction of a mental process, to change it from a direct to a reverse train of thought. A particular case of this (as will be clarified below) is the ability to form two-way (reversible) associations. We realise that from a mathematical standpoint symmetric (reciprocal) relations, direct and reverse operations, direct and converse theorems, and simple skill in ´reading´ a formula from left to right versus skill in reading in reverse are all categories of entirely different kinds. We believe however that

> their internal psychological basis in each case is a reconstruction of the direction of the mental process, a change from a direct to a reverse train of thought, and the establishment of two-way (reversible) associations."
>
> (Krutetskii 1976, p143)

Piaget too recognises this important concept in the building of the thinking process :

> "These then are the three aspects of a single process that defines the evolution of reason : transition from perception to primacy of deduction, progressive co-ordination of operations and gradual development of reversibility."
>
> (Piaget 1952, p202)

He defines reversibility as "reversibility is defined as the permanent possibility of returning to the starting point of the operation in question." (Inhelder and Piaget, 1958, p.272). Such an important part of the thinking process needs to be included in any model of cognition. The model under discussion here lends itself to such inclusion in a natural way. It becomes clear that the linking between the states may be either uni-directional or bi-directional in nature. This is not to say that the links themselves are necessarily two-way, but that there are links in both directions, and I shall use the term ´bi-directional links´ to mean this. Thus the activation of concept A may cause concept B to be evoked but it does not necessarily follow that the activation of concept B will cause concept A to be evoked. A possible example of this relevant to algebra has to do with square roots. The uni-directional links evident here may be indicated by the fact that although many children seem to appreciate the concept in that they can carry out a process such as :

Find $\sqrt{a^2}$

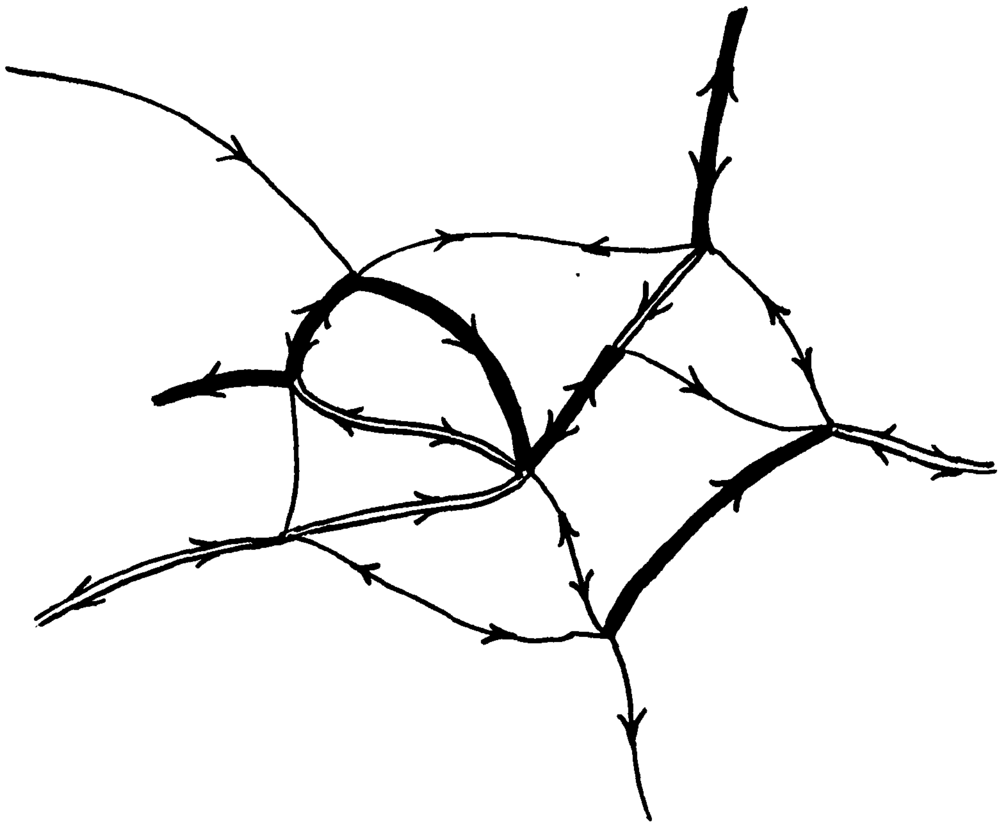
they find the understanding of the relationship when it is reversed much harder, e.g.

Simplify $\sqrt{a} \times \sqrt{a}$

This is an extension of the fact that they are able to answer arithmetic questions such as - What is $\sqrt{81}$?, but have difficulty in calculating $\sqrt{9} * \sqrt{9}$ directly, prefering to calculate and multiply rather than using an understanding of the definition and this problem continues into algebra, where it becomes more serious since the option of calculating first is no longer available. We may say that they are able to think of the process of calculating a square root, but are unable to reverse their thought to arrive at an answer from a definition, without first calculating or extracting numerical answers. We shall say then, in the terms of the model under discussion, that an individual has reversibility of thought in a given area if he/she has bi-directional links there. Krutetskii and others have provided evidence that the reversibility of thought is far less common in school-children than one might imagine (e.g. Krutetskii 1976). Thus one of the aims of mathematics educators in helping children to build their knowledge structures should be the formation of high strength factor, bi-directional C-links.

3.2.3 Modifying the Model

It is now necessary in view of the above points to modify the network diagram used as a mental model of schemas to take



Links of High Strength Factor ----

Links of Medium Strength Factor ----

Links of Low Strength Factor ----

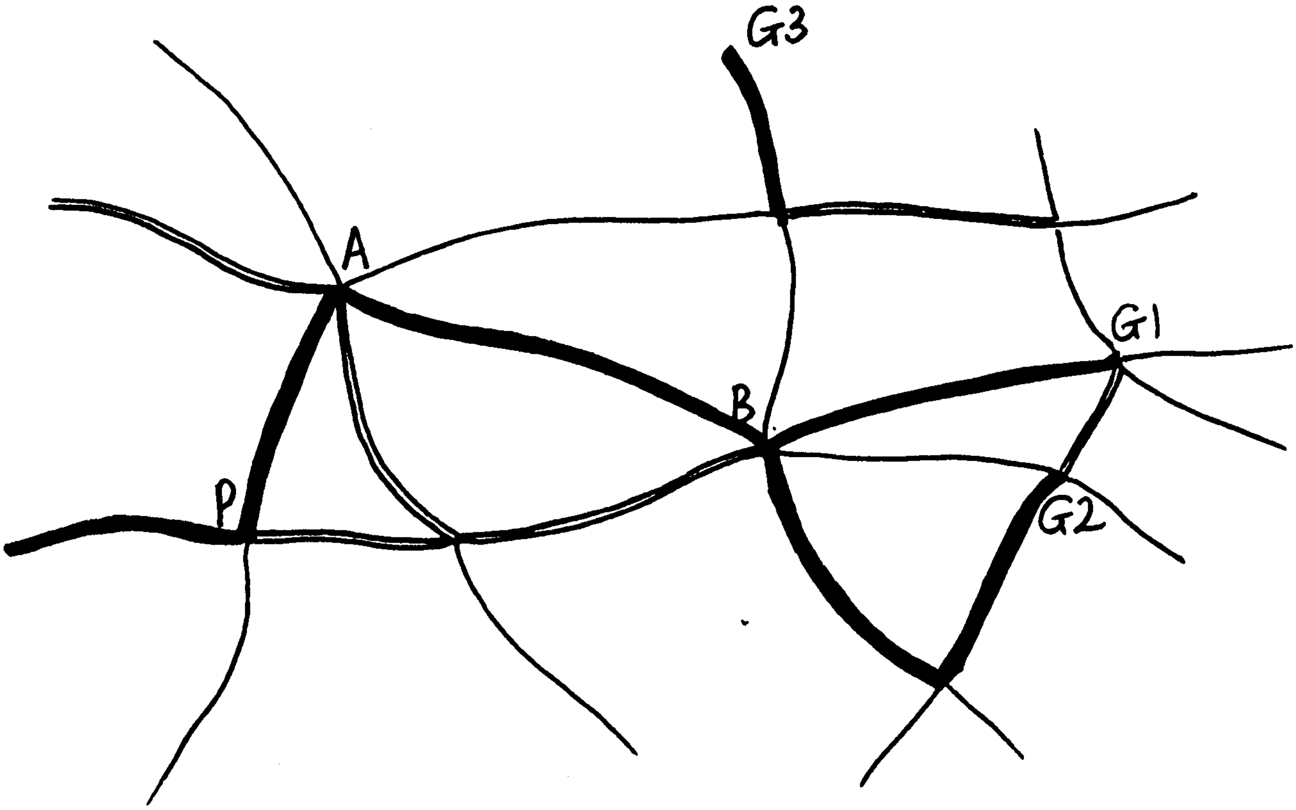
Figure 3.1

A Representation of A Generalised Schema Showing Links of Different Strength Factor And Direction account of the strength factors and bi-directional links introduced above. I shall introduce three levels of strength factor in the links in order to show the value of these ideas, but it should be remembered that strength factors will vary over a wide range. A typical schema may now be pictured as in Figure 3.1.

3.3 Consequences of The Qualitative Nature of Schematic Links

This approach leads to valuable insights including why we find some desired goals hard to achieve and yet others unexpectedly seem to thrust themselves upon our consciousness.

How may this be explained? In the mental process of finding a path from a present state P, to a goal state G, there are at virtually every stage of the path, different possibilities for the route. The ideas above indicate that, at any nodal concept, the path most likely to be taken is that which has the highest strength factor. This is so since, the activation of a concept most strongly evokes, among the concepts it is linked to, those with the highest strength factor. Thus these tend to dominate our thinking process at each stage. This is not to say that we must take this particular route, but simply that it is more prominent and thus more likely to be taken. This may be likened to travelling from one place to another in a car. If we wish to get to our destination in the most efficient way we may likely travel along the motorway. Periodically we will approach junctions, with roads of varying size leaving the motorway, ranging from small country roads to other motorways. Given a choice, and the desire for the most efficient journey, we will probably choose to use the major route at each junction. We do however have a choice, and other



Links of Low Strength Factor ----

Links of Medium Strength Factor ----

Links of High Strength Factor ----

Figure 3.2

A Generalised Schema To Illustrate A Mental Block factors may influence our decision, such as our level of tiredness, boredom, interest in a passing feature of the landscape etc. With intelligent learning, just as with this illustration, it is often the case that, in order to use the most efficient route, indeed sometimes in order to gain access to any major route at all, we must first avail ourselves of minor routes, rather than always sticking to the main ones. In the context of our thought processes, to positively choose to take such a path of lower strength factor rather than one of a higher factor would require a measure of reflective thinking and of effort.

To appreciate the cognitive problems that this may cause, and how they may be overcome, we shall consider the representation of a schema in figure 3.2, where we may consider all the links to be b-directional for the purpose of illustration. Starting from present state P there is a direct, major route comprising links of high strength factor to the goal state G1, and so if this is our goal it may be easily reached. The goal state G2 also, although it probably may not be reached by the most direct route available, can also be relatively easily reached using links of high strength factor at each stage. The problem arises though if our goal should be the state G3. It is likely that this goal would only be attained with difficulty, even though it is strongly linked to its nearest concept, since the strength factors of the links at the states A and B have a tendency to direct attention away from G3. Such a situation may be described as a mental block. We may even be aware that we have the necessary information and understanding to achieve our desired aim, but are temporarily unable to do so, to find the right cognitive path to our goal state.

3.3.1 Levels of Understanding

The discussion above leads us to consider the definition of understanding put forward by Skemp :

> "To understand a concept, group of concepts, or symbols, is to connect it (sic) with an appropriate schema. To understand an experience is to realize it within an appropriate schema."
>
> (Skemp 1979c, p148)

We need to consider whether it is enough to connect a concept to a single appropriate schema in order to say that we understand it. Although this certainly is a form of understanding, which we may call first degree understanding, it is a basic form of understanding only. No indication of the quality of the connection made is given, or implied. Is, for example, the concept to be connected to only one concept in the schema, or more than one? If we refer to such understanding, involving any appropriate linking to a schema as first degree understanding, then we may speak of second degree understanding, by defining it as existing when the concept is connected with bi-directional, conceptual or C-links of high strength factor to all appropriate concepts in an appropriate schema. Similarly, we may define third degree understanding, not easily fully achieved, although the goal for educators (since it would facilitate full transfer and reversibility) which would be to connect the concept with bi-directional C-links of high strength factor to all appropriate concepts in all appropriate schemas. The possession of a form of third degree understanding thus

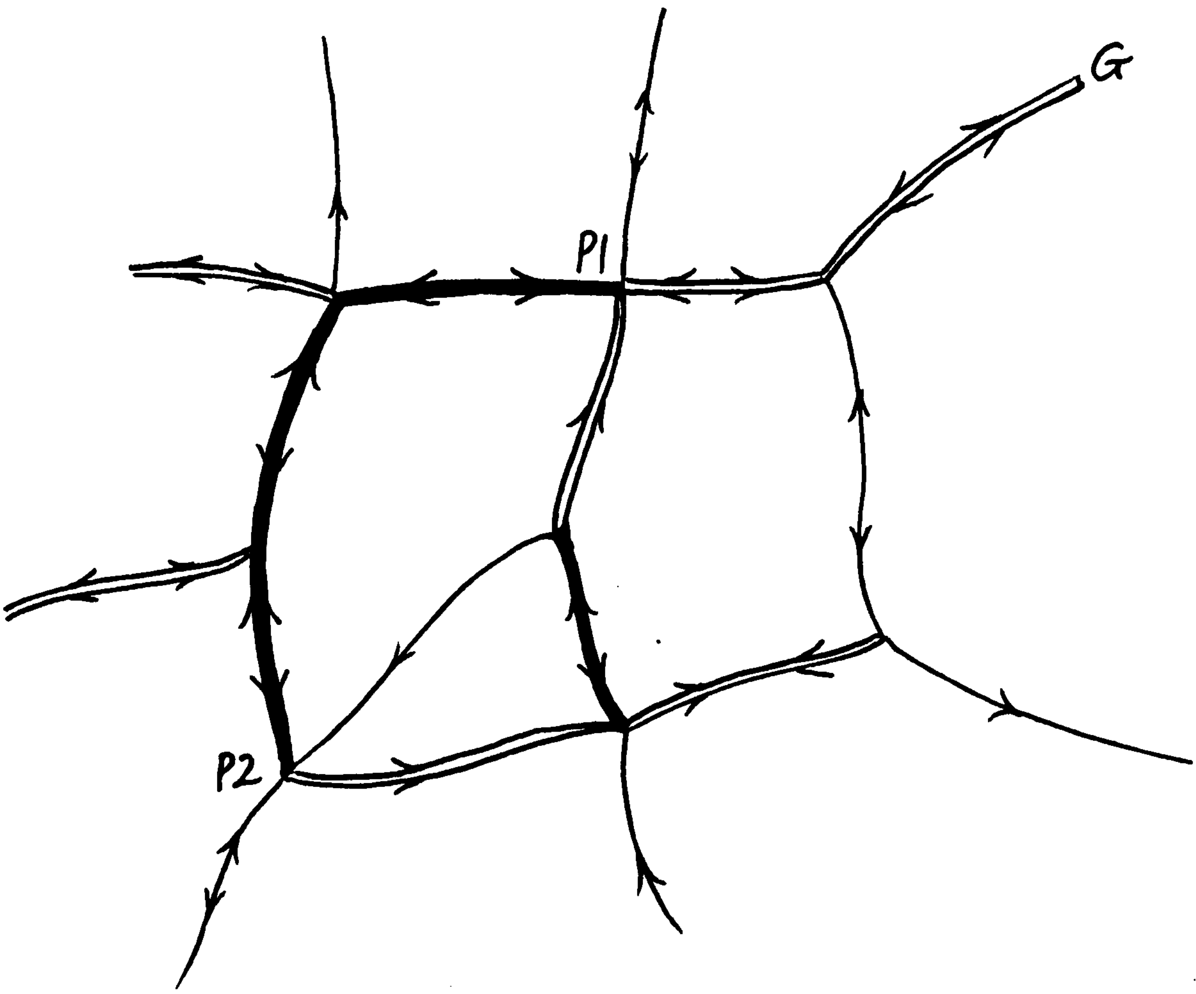


Figure 3.3

An Example of A Generalised Schema To Illustrate The

Value of A Reflective Change of Present State implies the potentiality for full transfer of knowledge from one knowledge domain to another. Although this ability may be lacking in second degree understanding, mental blocks as described above would be largely removed in the domain where this obtained, and one would possess what would in practice be described as a full understanding of a given field of knowledge. In reality of course individuals will possess a combination of first and second degree understanding with a measure of the linking to other areas characterising the third degree, and what we need is to find ways of both upgrading the level of children's understanding in a certain area and of coping with the problems that arise because of their lack. This may be done by the use of carefully researched, structured modules of work, such as described herein, which achieve a measure of success in this area.

3.3.2 Coping With Problems Of Mental Blocks

For the answer to the problem of mental blocks as described above we look to the ideas of what has been called lateral thinking; De Bono (1977) introduces the idea of mental flow in thinking and the idea that this may be more efficient along some pathways than others. We may use these ideas in the context of our diagram of a mental schema to illustrate the vital point that the present state P, or starting point in our attempt to reach our goal state G, is very important and can drastically alter our success or failure. If we consider the imaginary section of a schema pictured in Figure 3.3 where the present state is taken to be P1, then the most likely path to be taken, in view of the discussion above, is one which cycles endlessly around the centre section without ever reaching the goal, since at each point in the path the concept connected by the link of highest strength factor is the one which is most likely to be activated. However, changing the present state to P2, should this be possible, means that, not only does the most likely path end up at the goal G, but it does so very quickly.

What may we deduce from this relative to the process of teaching children? It means that part of the process of encouraging and teaching reflective thinking should be the ability to analyse in a given situation, our mental start position and to build in factors which enable us to move this in a systematic way. This may have the effect of counteracting tendencies such as the above caused by the relative strengths of the links in the schema and may result in the apparently sudden attaining of the goal. Such sudden flashes of insight far from being accidental may thus be of a type teachable to children. Ways of achieving such a moving of one's start position which could be taught include the encouraging of random input to a situation, the rotation of attention around the constituent parts of the problem and the reversal of attention etc. (De Bono 1977). This means the need to construct a mathematical environment which is conducive to this type of thinking rather than one which tries to constrain the child to a single approach, often that of the teacher, whose mental schemas may be totally different in construction and quality from those of the pupils. In the context of the present research I shall describe later the reasons why the computer environment is one in which such thinking may be encouraged. It would seem that such an approach would be most successful with those children who are not of high ability in mathematics, since their understanding is more likely to be of the first degree and thus their schema contain more of the types of problem areas outlined above.

What we have here built up then is a model of understanding, based on that of Skemp (see Chapter 2), in which the qualitative nature of the links in the framework of knowledge constructing by the individual are taken into account, and the implications of this considered. In order to extend the model further it is necessary to look in some detail at the subject of mental images and their physiological basis.

3.4 A Consideration of the Cognitive Role of Mental Images

One of the parts of mathematical thinking which has received attention over the years, without being fully integrated into learning theory is the use of mental images. Piaget admits :

> "The problem then, throughout the child's development, is to describe the relationship between imaginal symbolism and the pre-operatory or operatory mechanisms of thought."
>
> (Piaget and Inhelder 1969, p.70)

What do we mean by a mental image? There does not seem to be a general agreement on what constitutes the definition of a mental image (see e.g. Lean and Clements, 1981), but I shall take this to be a visual or auditory sensation which occurs in the 'mind's eye', without necessary recourse to a perception. Hence such mental images consist of a wide variety of mental activity including:

a) pictures of objects, people and places which one can 'see' even when they are not physically present - we could call these images of a concrete nature.

b) representations of more abstract objects such as the symbols of mathematics, the letter combinations which represent words etc. - we could call these images of an abstract nature.

c) representations of sounds which we may 'hear' even though there is no sound wave reaching our ears. This would include the common sounds of life as well as the more complex ones associated with musical arrangements - we could call these images of an auditory nature.

This is not intended to be a classification of such images, but rather an indication of their type based on content. Piaget and Inhelder (1971) have taken a different path in their attempt to classify mental images in terms of their structure. They have divided imagery into reproductive and anticipatory sub-divisions, where the former is a copy of some reality but the latter does not depend on any entity but rather pre-empts it. They further divide each of these types into static, kinetic and transformational images. From their classification, based on experimental evidence, emerges the important principle that mental images are symbolic in nature and are not simply copies of perception. It has been recognised too that their construction and use play an important part in the mathematics learning process (e.g. Piaget and Inhelder 1969, 1971, Ponte 1984), although their precise role and relationship to other types of concepts has been more difficult to ascertain. However, an inability to form an appropriate mental representation of a problem has been the subject of much research in the United States, and has been shown to be the cause of problems in mathematics for many pupils (e.g. Davies 1978, Clement et al 1981, DeCorte and Verschaffel 1985). It has also been shown that the different mental images formed by different children can affect their learning capability :

> "Different representational forms of material interact with processing activities to develop different cognitive structures. These can relate critically to students' previous experience and thus produce different learning outcomes."
>
> (Hartley 1980, p.10)

In fact Krutetskii has used the strength of an individual's visual pictorial ability as one factor in outlining the three distinct types of mathematical ability, namely the analytic, the geometric and the harmonic (Krutetskii 1976, p.315).

3.4.1 The Relationship of Mental Images to Cognitive Progress

How then are such images, whose value has been widely recognised related to other mental schemas? One of the major areas to be considered is their relationship with the schemas of logical thought and language, or as Piaget and Inhelder (1971) refer to it, the relationship between such figurative thought and operative thought. Piaget has shown that such imagery is not merely a representational copy of reality (Piaget and Inhelder 1971) but is part of a cognitive building process. This point, as Piaget argues, that mental images are not simply prolongations of perception is important, and Piaget's evidence for this includes results that show that, although they are slightly behind in the learning process (up to four years), blind children too are able to build such images (Piaget and Inhelder 1969). The same researchers have also demonstrated the images independence of spoken language by showing that deaf mutes too are able to construct them. Commenting on the schematic nature of mental imagery Piaget says:

> "If we use the term ´scheme´ (scheme) to designate a generalization instrument enabling the subject to isolate and utilize the elements common to similar successive behaviours...in this sense there also exist imaginal schemes...but if we use the term ´schema´ (schema) to designate a simplified model intended to facilitate presentation...then there can be no perceptual ´schemata´, since the ´schema´ serves only for figuration and evocation. Imaginal figuration, on the other hand, is ´schematized´ precisely in the ´schema´ sense, though at the same time it may entail ´schemes´."
>
> (Piaget and Inhelder, 1971, p.366)

The value of such an imaginal structure later in the child´s progress is indicated by Ginsburg and Opper :

> "The ability to form images of this sort does not guarantee that the child can conserve number: as we have already seen, the processes underlying conservation are not solely perceptual or imaginal. Nevertheless, the child who has a correct image of the transformation is certainly ahead of the child who does not. In other words,

> images are a useful and necessary auxiliary to thought during the concrete operational stage. By providing relatively accurate representations of the world, images assist the process of reasoning although they do not cause it."
>
> (Ginsburg and Opper 1979, pp.165,166).

Piaget too accepts the important role of images in thinking :

> "This .... implies that judgements and operations are not images, but it does not rule out the possibility that the image can play a role in thinking as a complementary symbolic auxiliary to language."
>
> (Piaget and Inhelder 1969, p.70)

> "Any representational cognition...presupposes the bringing into play of some symbolic function. It would be preferable to describe this function as semiotic...Without this semiotic function thought could not be formulated, and consequently could not be expressed intelligently to others or to oneself (internal language etc.). Now there are two fundamental reasons why the collective sign system, or language, does not fulfil the requirements of this semiotic function and why it needs to be complemented by a system of imaginal symbols."
>
> (Piaget and Inhelder, 1971, p.380)

We see then that the role of such images has been refined by Piaget and is now seen as fundamental to thought in his eyes, playing a complementary role to language. When considering the relative place of each in the cognitive process we should note that psychologists have, for many years, suggested that "language facilitates thinking by supplying verbal symbols which can represent concepts." (Hanley, 1978). However, discussing the psychological basis of cognition, Piaget says:

> "These data...indicate that language does not constitute the source of logic but is, on the contrary, structured by it."
>
> (Piaget and Inhelder, 1969, p.90)

hence arguing that language is not the source of logical thinking. Davis (1982) has also expressed the idea that there may be processes more fundamental than language involved in learning:

> "In my view there is compelling evidence that the key ideas of mathematics are not expressed in natural language while they are being processed in the human mind."
>
> (Davis, 1982, p.118)

It seems reasonable that very young children build up a considerable cognitive structure before they build language schemas which they may access, and that these early structures are therefore built up in terms of some symbolism such as mental imagery rather than language. This is not to reason that language is not vitally important in cognitive growth, the study of Curtiss et al (1974), for example, of a girl kept in isolation with no linguistic experience until the age of thirteen and a half years reporting that she scored at only 15 months on a non-verbal cognitive test. This study promoted the following conclusion regarding the value of language to cognitive growth:

> "In summary, the tragic and prolonged deprivation of all linguistic inputs has revealed the fundamental role of language in creating a human person with cognitive and creative abilities."
>
> (Popper and Ellis, 1977, p.310)

Hence it would seem that although language is vital for cognitive progress, other factors, such as the use of mental imagery are important. Evidence for the possible utilisation of such mental images comes from some studies (e.g. Klahr and Wallace 1973) which have suggested that small numbers may be subitized, or recalled direct from a mental image, without recourse to language or other processes such as a counting procedure. Although others (Gelman and Tucker 1975) disagree with their findings, their data has been collected from studies of children of three years of age and over. Children of this age have already built up considerable language schemas, and even some mathematical ones, making it difficult to know the extent to which visualisation has affected their thought processes. Another of the difficulties in dealing with a study of the processes employed by young children is that of communication. In order to find out how the young child processes information it has been necessary to resort to the use of language in many instances and this may have tended to obscure the reality of their cognition, since descriptions are often inadequate. Even the use of drawings presents problems associated with the skill needed to produce them accurately. There is here though an indication of the possibility of a direct influence on thinking by mental imagery, without recourse to language, and this would fit in well with the psychological model described here.

What we have seen then is that, although there is no general agreement on the position of mental images in cognitive advancement, many educators have considered their significance and consider them of fundamental importance in the learning process.

In order to further the synthesis of the broad based psychological model of learning which has been partly developed above, we need to be able to incorporate mental images relative to other types of thought processes, and so it is necessary now to consider some of the implications of physiological investigations of the workings of the human brain. This is the purpose of the next section.

3.5 Some Physiological Investigations of the Brain

There have been many physiological studies of the brain and its functions which have focussed on the observed difference in operation of the hemispheres of the brain in order to attempt a classification of hemispheric function. I shall now consider some of the significant findings of these studies and their relevance to the current model of cognition. It should be remembered in the discussion that follows that reference to the 'right' and 'left' brain and their functions should be viewed metaphorically (Thomas and Tall, 1988b), since although this is the way the literature refers to them this is a generalisation and the functions of each metaphorical half may be carried out physically in the left or the right brain depending on the individual. What is important is the dichotomy of function which may be described by such terminology. Prominent among such investigations of the functions of the brain hemispheres, over many years, has been the work of Sperry and his colleagues. By looking at subjects who have undergone commissurotomy, the surgical separating of the two hemispheres of the brain by partial or total cutting of the corpus collosum joining them, they have been able to specify the function of each of the hemispheres, saying :

> "Though predominantly mute and generally inferior in all performances involving language or linguistic or mathematical reasoning, the minor hemisphere is nevertheless clearly the superior cerebral member for certain types of tasks. If we remember that in the great majority of tests it is the disconnected left hemisphere that is superior and dominant, we can review quickly now some of the kinds of exceptional activities in which it is the minor hemisphere that excels. First, of course, as one would predict, these are all nonlinguistic nonmathematical functions. largely they involve the apprehension and processing of spatial patterns, relations and transformations. They seem to be holistic and unitary rather than analytic and fragmentary, and orientational more than focal, and to involve concrete perceptual insight rather than abstract, symbolic sequential reasoning."
>
> (Sperry, 1974, p.11)

This agrees well with other observations which have attributed differing abilities to the hemispheres of the brain. Fidelman (1985, p.59) quotes the collection by Bogen (1969) of the dichotomies which surfaced during studies as follows :

Author	Left Hemisphere	Right Hemisphere
Wensenberg & McBride	lingual	visual
Milner	verbal	conceptual/ non-verbal
Sommes,Weinstein,Ghent,Teuber	discrete	diffused
Zangwill	symbolic	visuo-spatial
Macaen,Ajuriaguerra,Angelergues	lingual	pre-verbal
Bogen & Gazzangia	verbal	visuo-spatial
Levi-Agresti & Sperry	logical/ analytical	synthetic conceptual

Glennon (1981, p.3) has also compiled a list of cognitive hemispheric function based on the findings of many research studies. His list is :

Left Hemisphere	Right Hemisphere
Verbal	Visuospatial (including gestural communication)
Logical	Analogical, intuitive
Analytic	Synthetic
Linear	Gestalt, holistic
Sequential	Simultaneous and multiple processing
Conceptual similarity	Structural similarity

So we see that there is strong evidence to suggest a difference in the way each hemisphere of the brain functions, with the dominant left hemisphere primarily the seat of verbal, logical cognition and the right minor hemisphere primarily a visual, spatial processor. We may go further though and say that the evidence from Sperry and others is such that not only is there a dichotomy of function, but that each hemisphere is actually a separate processing system in its own right. As Sperry (1974) and Zangwill (1974) describe it:

> "it is our own interpretation, based on a large number and variety of nonverbal tests, that the minor hemisphere is indeed a conscious system in its own right, perceiving, thinking remembering, reasoning willing and emoting, all at a characteristically human level"
>
> (Sperry, 1974, p.11)

> "It appears then, that each of the separated cerebral hemispheres is, when operating in isolation from its fellow, capable of perceptual, memory and executive performance at a level comparable to that achieved by an individual whose central nervous system is intact."
>
> (Zangwill, 1974, p.267)

> "Sperry, then, clearly holds the view that, under circumstances in which the disconnected hemispheres react to lateralised sensory input, each functions as an essentially independent processor..."
>
> (ibid, p.267)

Experiments by Seamon (1974) on response times to data presented verbally, as separate images and as relational images confirmed this picture of the brain as having two separate processors :

> "The data are consistent with the hypothesis of separate processing systems for verbally and visually coded information and suggest that the systems may be discriminated along hemispheric lines."
>
> (Seamon, 1974, p.200)

Not only does it seem that there are two processing systems, then, but the experiments of Seamon caused him to conclude that the processing in each hemisphere is qualitatively different :

> "A qualitatively different retrieval model appears necessary to account for the data of the Relational Imagery group. Since [Reaction Time] did not increase with set size, a serial retrieval process in which the probe is successively compared to the items in memory may be rejected. Instead data are consistent with a parallel process in which the probe is simultaneously compared to all items in the memory set with each comparison taking a constant amount of time."
>
> (ibid, p.191)

> "It may be that,....verbally and visually coded information was processed exclusively in the left and right hemisphere respectively."
>
> (ibid, p.199)

This agrees with the results of Levy (1969) who performed tests involving ´blind´ perception of objects by touching with each hand and concluded that the left hemisphere was applying a verbal analytic mode of reasoning whereas the right hemisphere used direct perceptual, synthetic or Gestalt processing. In a review of some other experiments designed to look for evidence of parallel processing of information by the brain, Haber & Hershenson concluded :

> "As a consequence of this paradoxical result, a simple element-by-element model of visual matching is clearly untenable. Consequently some investigators have proposed models that include separate processes for producing the ´same´ and the ´different´ judgements: a fast, global comparison process to account for the fast ´same´ responses, and a slower, analytic, element-by-element comparison process to account for the ´different´ responses..."
>
> (Haber & Hershenon, 1980, p.359)

Thus there is evidence that seems to point to two processors, the left hemisphere which is primarily a serial processor, and the right hemisphere which is primarily a parallel processor able to process coded visual images at tremendous speed and to make global decisions on them without the need to sequentially examine each subpart. This fits in well with one of the functions of the right brain which is to process visual data direct from the retinas of the eyes. Since such data emerges in parallel form from the cones and rods of the retina, there is need for a system capable of processing it (in its coded form) in parallel to preserve the data´s meaning. It is necessary to realise that these two distinct processors may be at work at the same time processing, in their own way, the same data, the effect being "that both the left and right hemisphere may be conscious simultaneously in different, even in mutually conflicting, mental experiences that run along in parallel." (Sperry, 1974, p.11). An example of how these processors work for us may be gleaned from examining the cases of individuals who do not seem to possess this ability present in most humans. In a case study of agnosia, Humphreys and Riddoch (1987) presented a series of displays of shapes where the task was to identify the shape which was out of place. For example, in a regular pattern of T´s one was inverted. Based on their findings, they concluded that for their subject :

> "He had to search each item in the display, one at a time, to decide whether or not each was the target.....This pattern of performance remained essentially unchanged even when we gave him up to 3,000 trials! John´s search for a combination of features differs qualitatively from the control subjects. This difference seems to occur because, while control subjects can combine features simultaneously at different spatial locations (to perceive the homogeneous group of T´s) John cannot."
>
> (Humphries & Riddoch, 1987, p.76)

It would appear that the holistic processing ability of a right brain parallel processor is at least partially missing in John´s case.

However, the system for processing in the brain described above would be a recipe for disaster in humans were that the whole story, since the two processors would operate autonomously, each interfering with the work of the other. This is plainly not the case in most humans and the co-ordinating factor is described by Gazzaniga (1974) :

> "....contributions to our unified behaviour are made from cerebral systems in both the left and the right half brain. Clearly, a system is necessary that would bring order to this variety of mental operations. It must coordinate all the incoming information from the disparate cerebral sites as well as dispatch information to a variety of sites for processing. Its function then is to bring unity to a diverse cognitive field. It is the ultimate arbiter of all these activities."
>
> (Gazzaniga, 1974, p.374)

Thus, he concludes, that the evidence points to a central control system co-ordinating the processing in each hemisphere of the brain, and furthermore, that it is the presence of this control system, resident in the left half of the brain which leads to its cerebral dominance. Some learning difficulties, such as dyslexia, he claims may well be the result of a poorly developed central control system rather than a poor language processor, causing a competition for control between the brain halves (Gazzaniga, 1974, p.375). Some of the elements of a processing model such as we are describing here have been described by others interested in an information processing model of perception, namely Haber & Hershenson (1980). Their model however is concerned totally with the processing of visual input rather than extending to include all mental imagery, and as such they suggest that for this type of input, parallel processing occurs automatically before any sequential or serial processing, negating the need for a control system such as that described above. In order to see how this orchestration occurs physically between the two hemispheres we must turn again to the physiological studies of commissurotomy. Sperry´s view of the interaction is that data is transmitted between the two hemispheres by the corpus callosum which has "200 million fibers cross-connecting nearly all regions of the cerebral cortex" (Sperry, 1974, p.6). Commenting on this work of Sperry and others, Popper and Eccles (1979) put forward the view that they have shown that the transmission of data between the hemispheres occurs via the corpus callosum. Speaking of this they say :

> "It must be recognised that this transmission in the corpus callosum is not a simple one-way transmission. The 200 million fibres must carry a fantastic wealth of impulse traffic in both directions. For example, a conservative estimate of the average impulse frequency in a fibre would be 20Hz, which gives a total traffic of 4x10 impulses in a second. In the normal operation of the cerebral hemispheres, activity of any part of a hemisphere is as effectively and rapidly transferred to the other hemisphere as to another lobe of the same hemisphere."
>
> (Popper and Eccles, 1979, pp. 327,328)

Following his investigations of interhemispheric transfer of data in monkeys, Butler (1979) agree that transfer is by the corpus callosum. although he showed that there is also some transfer through the anterior and hippocampal commissures. His conclusion was that "Total forebrain commissurotomy abolished transfer on all tasks." (p.353).

So far then we have a description of a unified processing system, with a control unit and extremely efficient data transfer between the two, qualitatively different, processors. Again, the case study of visual agnosia recorded by Humphreys & Riddoch (1987) is of value in appreciating the interaction between the two systems. One of the problems displayed by the subject was an inability to name objects or people in spite of being able to see them reasonably well, albeit it black and white only, and being able to describe clearly their appearance. Further, although when out of doors he was able to see and describe his surroundings he was unable to attach any meaning to them that would enable him to be able to navigate, even though he was near his home, in territory which should have been thoroughly familiar. In spite of this his reasoning ability was unimpared.

> ".....his performance on our tests of face recognition suggests that he actually doesn´t recognise the people when he misidentifies them, after all, the tests just required people to be categorised (as famous or a politicians)..."
>
> (Humphreys & Riddoch, 1987, p.62)

> "He is only impaired when the task requires him to recognise the objects." (ibid, p.63)

> "...he has good knowledge about the objects he fails to recognise, and he is well able to

articulate this knowledge......John´s verbal definitions clearly refute the idea that he has generally impaired intellectual functions..."

(ibid, pp. 63,65)

"If he was able to identify the object, he was able to draw in the missing part. He was unable to complete any of the drawings where he failed to identify the object.....These tests indicate that he does have stored knowledge about what objects look like"

(ibid, pp.65,67)

Thus his inability to ´recognise´ people, objects and landmarks and to link his store of visual knowledge with his verbal knowledge would fit well with the present model if we postulate that in his case the two processors were functioning but that either the links between them were damaged or that the control system co-ordinating them was damaged.

The next stage of the description is more surprising. Gazzaniga (1974), Popper & Eccles (1979) and Sleamon (1974) all agree that the available evidence points to the fact that in young children neither of the hemispheres of the brain is dominant and that each half develops in response to inputs more or less independent of the other.

"There is good evidence of a remarkable plasticity, the functions of the dominant hemisphere being effectively transferred up to 5 years of age. There is evidence that, at this early age, linguistic ability normally is well developed in the right as well as the left hemisphere (Basser 1962). Then, over the first few years of life cerebral dominance is established with a regression of the linguistic ability of the minor hemisphere." (Popper & Eccles, 1979, p.332)

"..there is good reason to believe that the young child up to and through the age of 8 years or so lays down engrams in both cerebral hemispheres for language and perception of all kinds.

...this data would suggest that as dominance is established in the left hemisphere, inhibitory processes develop which suppress the upper cognitive level and decision making capacity of the right hemisphere."

(Gazzaniga, 1974, pp.376,377)

The physiological evidence seems to show that this dominance of the left brain suppresses the processing of the minor hemisphere to such an extent that its processing is no longer available to the consciousness of the individual. Reporting again on Sperry´s work with commissurotomy subjects, Popper & Eccles (1979) and make this vital point :

"The outstanding discovery in the investigation of these subjects is the uniqueness and exclusiveness of the dominant hemisphere in respect of conscious experience."

"This minor hemisphere continues to perform as a very superior brain with a refined ability in stereognosis, and in pattern recognition and copying, yet none of the goings-on in that hemisphere gives conscious experiences to the patient, except by delayed and very diffuse pathways in the brain....."

"It was postulated that in normal subjects activities in the minor hemisphere reach consciousness only after transmission to the dominant hemisphere, which very effectively occurs via the immense impulse traffic in the corpus callosum..."

(Popper & Eccles, 1979, pp. 315, 316, 326 resp.)

Thus whilst the processing of the minor half of the brain is probably not accessible to the conscious mind the results of it are made rapidly available to the dominant, conscious half.

This then completes the overview of the synthesis of a model based on the physiological dichotomy of the processing of the brain. It should be stressed again at this point that it is a model and is not intended to show the exact physical workings of the brain. In the following section I shall look at the psychological implications of this processing model.

3.6 Mental Images and the Psychological Model

The implications of the physiologically based processing model outlined above for a psychological understanding of the learning process are considerable. The physiological evidence points clearly to a means of including mental imagery in a synthesis of a psychological model of cognition by placing them in the context of two qualitatively different hemispheric processors both giving rise to independent knowledge structures which are unified into an integrated whole. I shall now describe this integration as I see it.

To extend the psychological model described above to include these structures of the right hemisphere, I shall introduce what I shall call a first degree level knowledge structure, the elements of which are primarily mental images of one form or another, with these ´existing in´, and processed by the minor hemisphere. These elements or states are connected together to form schemas in a way similar to those already discussed for the sequential/serialist/analytical structures of the left hemisphere and they display similar characteristics and properties. It is these first degree knowledge structures I contend which primarily give rise to the global/holistic mental abilities. Piaget and Inhelder (1971, p.366 - quoted above) have described evidence for the schematic nature of such mental imagery.

We may glimpse the schematic nature of our own mental imagery by thinking of an image of a scene in a town such as say York, with which we are very familiar. Concentrating on the rapid sequence of images which may involuntarily appear and disappear may reveal the schematic nature of the images stored, since it appears that images which are connected to the initial image of ´York´ (or the town chosen) are linked in a structure in which each image evokes others. With such a structured sequence we may even take a tour around the town, even though it may be far away! Such a structure constitutes an example of the schemas of mental imagery. It is to be noted that the mental imagery structure we are here discussing will include the image representations of words and symbols, such as the image ´York´ in the above illustration, since these are simply that, images, and not the concept represented by the image. Hence a sequence of images such as that mentioned above may result from seeing the word ´York´ or ´Minster´ if these exist with high strength factor links to other mental images in the schemas of which they are part. On the existence of such symbolic imagery Sperry comments "with any direct visual-visual matching of shape or pattern, the right hemisphere is dominant. This applies even for words, provided that no interpretation of word meaning is involved." (Sperry, 1974, p.13). These schemas, as with all mental schemas, are dynamic in nature and are constantly being enlarged, throughout life, to provide images which relate to, and take account of, increases in understanding in all areas of cognition. One of the best examples of a developing mental imagery is given by the well known Piagetian experiment where young children, asked to draw the water level in a container held at an angle show the failure of their understanding reflected in their mental images by not drawing the water level horizontal, but rather always making it perpendicular to the sides of the container. Later when their understanding, as centred in their second degree knowledge structures improves then these mental images must be updated so that the water level is ´seen´ as horizontal throughout. The reverse effect, of mental imagery on the higher level cognitive functions, can be seen by another example from Piaget. In Piaget and Inhelder

(1971, p.366) the images of the pre-operational child are shown to have a detrimental effect on his/her concept of conservation. When water is poured from one container to another, narrower, one the water level is seen to rise. This image gives the child a false view that there is more water in this container. This illustrates both the power of the mental image and the dangers of relying on imagery alone in order to arrive at conclusions.

These schemas of mental images are so basic to the process of learning that some have even described them as holding the key ideas. In order to see how the imaginal schemas may be of value as part of the goal-directed activity mental activity of the individual we must include in the model a means of accessing these schemas from the qualitatively different logical/sequential/serialist/analytic schemas of the second degree. This will be pictured by introducing ´vertical´ links between the concepts and schemas in each level of the model so that it becomes, not two separate knowledge structures, but an integrated whole with two connected, distinct modes of operation. these ´vertical´ links correspond to the flow of data, between all parts of each hemisphere, across the corpus callosum in the brain, and it is by means of such links that the mental imagery schemas are able to influence the higher level cognitive functions of the mind.

Thus to recapitulate, in this integrative model, the knowledge structures whose elements are primarily language based are what we may call second degree/level structures, are left hemisphere associated and correspond to the sequential/analytic mental processes. These are built up in connection with and concurrent with the first degree structures

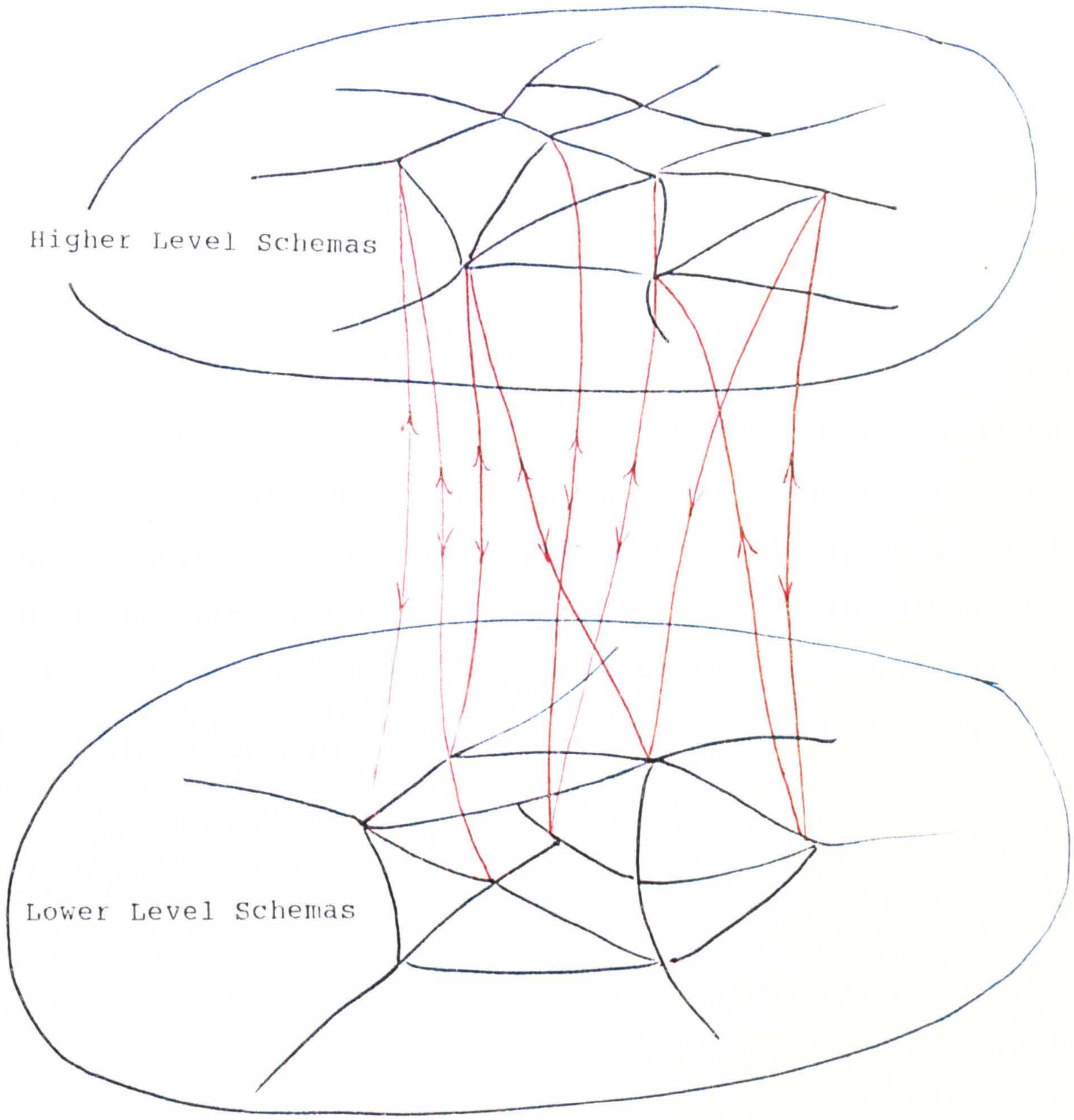


Figure 3.4

An Integrated Bi-Modal Model of Cognition of mental images, in all their forms as discusssed above, which are primarily right hemisphere associated and correspond to the global/holistic mental processes. The two qualitatively different levels of structure contain, and are connected by, links displaying all the characteristics discussed earlier, including continuously varying strength factor and uni- or bi-directional nature, such links in the psychological model paralleling the physical inter-callosal links. Each of the knowledge structures may work independently on the same or related data, each giving rise to a goal-oriented cognitive path either of which may be utilised, as and when needed, under the control of the individual´s control unit (or involuntarily due to the strength of the links established), at any stage in the path, in the attainment of the goal state. This gives us a model a very much simplified version of which may be pictured as in Figure 3.4, where I have just indicated the type of structures and links which form the model.

Although they have not been identified previously with the type of model outlined here, the existence of two distinct learning strategies described as sequential/serialist/analytic and global/holistic has been recognised for some time. The first style has been labelled analytic by e.g. Bruner (1960) and serialist by Pask (1972), whilst the second has been called global (Witkin 1977), holist (Pask 1972) and intuitive (Bruner 1960). The essential characteristics distinguishing the two learning styles have been listed by Brumby (1982) as:

"(i) Immediately breaking a problem or task into its component parts, and studying them step-by-step, as discrete entities, in isolation from each other and from their surroundings.

(ii) An overall view, or seeing the topic/task as a whole, integrating and relating its various subcomponents, and seeing them in the context of their surroundings."

(Brumby 1982, p.244)

Brumby's (1982) study attempted to observe the cognitive strategies of students in attempting to solve problems, involving biological concepts, relative to these learning styles. Her results showed that three distinct groups of students existed, namely those who consistently used only serialist/analytic strategies, those who consistently used only global/holistic strategies and those who used a combination of both, and who have been described as versatile learners. She found that overall 42% of her sample maintained purely a serialist/analytic style, 8% a global/holistic style and 50% were versatile. Further, in a gender comparison, she found that 45% of the females were serialist/analytic only compared with 35% of the males; 3% of the females were global/holistic only compared with 12% of the males, leaving 52% of females as versatile and 53% of the males. Taking up these results and the learning styles discussed Scott-Hodgetts' (1986) hypothesis is :

"...children who are predisposed to a serialistic approach are less likely to develop into versatile learners within the mathematics classroom than those who are inclined to adopt holistic strategies; this situation is held to be directly attributable to teacher behaviour"

(Scott- Hodgetts 1986, p.70)

This concept of a versatile learner is very important, since the nature of mathematics is such that it certainly suggests that the versatile learner is more likely to be successful in mastering concepts at the higher levels of the subject where the ability to alter one's viewpoint of a problem from a local analytical one to a global one in order to be able to place the details in place as part of a structured whole become more important. Hence we may describe a versatile learner in the current discussion as one who has managed to construct meaningful schemas at both the higher and lower cognitive levels described above and has constructed two-way links where appropriate between the concepts of these schemas as well as the important 'vertical' links between the two qualitatively different knowledge structures. Thus, such a learner is in the position of being able to use the higher level relational understanding of the schemas associated with the left-brain i.e. sequential/serialist/analytic strategy or the lower level relational understanding of the schemas associated with the right brain i.e. mainly global/holistic strategy, or, most importantly, to switch smoothly and easily between the two as and when appropriate i.e. a versatile learner. This is not to say that the serialist/analytic learner does not use the lower level schemas, since of course all humans must do so, but it means that such schemas are less well developed in such a learner or their links may be instrumental rather than relational in nature, with fewer links to the important higher level concepts.

I proposé to use the term integrated bi-modal model for the model which I have developed above in order to explain the relationship between the two qualitatively different modes of human thought which have been consistently described as global/holistic and sequential/serialist/analytic. I further propose to use the term Cognitive Integration for a psychological approach, based on this model, which seeks to unify the sequential/analytic and global/holistic thought of an individual in a manner consistent with the descriptions and predictions of the model, with the aim of producing a versatile learner.

3.7 <u>Gender Differences</u>

Although it had not originally been the intention of this research to look for gender differences in the understanding of algebra, during the course of the experiments certain differences thrust themselves upon the teachers and so this aspect was included in some of the results analysis (see Chapter 8). In view of this I shall review here some of the background studies which relate both to gender and the model of cognition now under discussion.

3.7.1 <u>Some Theoretical Considerations on Gender Differences</u>

The existence of differing levels of performance in mathematics based on gender and the possible reasons for them have been the subject of considerable discussion and research in recent times. Although it has generally been accepted that the level of attainment of boys in mathematics, at least by the age of 16 years and at the higher ability levels, has been superior to that of girls, the difficulty has been in explaining the reasons for this apparent difference. Reporting on the results of a five year, large scale monitoring of performance of 11 and

16 year olds in England and Wales by the Assessment of Performance Unit (APU), Shuard (1986) comments:

> "As pupils grow older and tackle more advanced mathematics, problem solving and the understanding of mathematical concepts become more important, so that they are absolutely central to progress. Somehow, by the age of ten, more boys than girls have got themselves into a position where they are able to cope with these aspects of mathematics. ...it does not seem very likely that the experiences which boys have in their primary school mathematics lessons are responsible for this fortunate bent..."
>
> (Shuard 1986, p.34)

Also commenting on the same APU survey Joffe & Foxman (1986) say:

> "There are significant differences in mean scores of boys and girls on a number of different topics but, overall, boys are best, relative to girls, in the applied and practical areas--measures and rate and ratio--while girls do best relative to boys in computation with whole numbers and decimals and some aspects of algebra. By age 15, girls are marginally behind the boys in the latter topics but more substantially so in the former."
>
> (Joffe & Foxman 1986, p.48)

Thus this APU study is consistent with some of the elements of the differential performance between girls and boys in mathematical attainment. Fennema (1974) reviewed 36 studies comparing the performance of girls and boys and concluded that while there was little evidence of differences before or during secondary school there was a trend for boys to be better at higher level cognitive tasks and girls at the lower level ones. Therefore, whilst there is some conflicting evidence about the differences which exist it would seem to be instructive and potentially beneficial to investigate the possible reasons for such differences. One such attempt to explain these has been discussed by Scott-Hodgetts (1986), based on the hypothesis that:

> "...some pupils (both boys and girls, but more girls than boys) may be adhering to a particular set of strategies which have led to their success in mathematics, but which, when used exclusively, have negative implications for these pupils´ mathenatical development."
>
> (Scott-Hodgetts 1986, p.68)

The particular ´set of strategies´ which she identifies as detrimental is based on the existence of two distinct learning strategies identified as the analytic/serialist and global/holistic discussed above. Her argument is that since more girls than boys have a predisposition to the use of a serialist/analytical approach only then this is why they do not perform as well. The nature of mathematics is such that it appears likely that the versatile learner will be more successful in mastering concepts at the higher levels of the subject where the ability to alter one´s viewpoint of a problem from a local analytical one to a global one in order to be able to place the details in place as part of a structured whole become more important.

If the above is indeed the case, then the questions which remain to be answered at this stage then are, Why is it that more girls than boys seem to adopt such a learning strategy? How can we explain the situation in terms of the theory of learning mathematics? and What experiences, if any, may be used in the teaching situation in order to maximise the opportunities of both sexes for mastering mathematical concepts?

In order to provide possible answers to these questions I shall refer again to the psychological theory above which may be deduced from a consideration of the differing capabilities of the brain´s hemispheres. A discussion of such cognitive areas has been recognised as the logical place to begin to look for an explanation of gender related differences in mathematics, since it is a cognitive activity, (e.g. Fennema 1979). One possible relevant variable put forward by Fennema is that regarding spatial visualisation skills, where she says ´male superiority on tasks that require spatial visualisation is evident beginning during adolescence´ (1979, p.392), although she concluded that her studies investigating a relationship between such abilities and mathematical achievement did not support the idea that it was a helpful variable. However, as made clear in chapter 3, when looking to the abilities of the brain hemispheres for an explanation of differential performance we must consider the superiority of the right brain in the processing of all mental images and the fact that they are constantly in use in all of our cognitive activity, and to restrict ones viewpoint merely to a subset of mental images such as spatial visualisation, with the implication that images are only used in such activities is likely to be too restrictive to be helpful in explaining gender related (or any other) differences. Rather we have postulated that a versatile learner is one who has managed to construct meaningful schemas at both the higher and lower cognitive levels described earlier and has constructed links in both directions, where appropriate, between the concepts of these schemas. Thus, such a learner is in the position of being able to use the higher level relational understanding of the schemas associated with the left-brain i.e. serialist/analytic strategy or the lower level relational understanding of the schemas associated with the right brain i.e. mainly global/holistic strategy, or, most importantly, to switch smoothly and easily between the two as and when appropriate i.e. a versatile learner. This is not to say that the serialist/analytic learner does not use the lower level schemas, since of course all humans do so, but it means that such schemas are less well developed in such a learner or their links may be instrumental rather than relational in nature, with fewer links to the important higher level concepts. Hence we are here saying that the possibility that this may be the case with girls to a greater extent than it is with boys and that this might possibly be a cause of the difference in mathematical performance between them is a hypothesis which would repay further investigation. Scott-Hodgetts describes the problems they face as follows:

> "It could be argued that even at this stage teacher exposition tends to be serialistic in style, and that serialists are therefore not

> disadvantaged, but such an argument fails to take account of the fact that pupils are expected to do more than simply reproduce items of knowledge as they have been taught. They must, for example, also be able to restructure bodies of knowledge in ways appropriate to different problems -- a difficult task for serialists because of their inclination to learn sequentially, without necessarily forming an overall picture of the relationships involved. ...whilst holists are busy speculating about relationships, and discovering the connections between initially disjoint areas of mathematics, it may not even occur to serialists to begin to look for such links;"
>
> (Scott-Hodgetts 1986, p.73)

What reason though may there be for this imbalance in approach between girls and boys? Is it, as Scott-Hodgetts proposed. ´directly attributable to teacher behaviour´ or is there another explanation? Although this is undoubtably an important factor in the difference in performance, there is also evidence of of gender related cognitive differences based on the half-brain activities. Bever (1983) describes this in the following terms:

> "men and women have different assymetries, though the literature has not settled on which sex is more lateralized for what. Explicit handedness and sex are unambiguous variables, although it is not obvious why sex should interact with neurological organisation in particular."

(Bever, 1983, p.21)

Borod et al (1983) give an example of gender differences in the processing of emotional expressions, they found that:

> "Females were more left faced than males for positive expressions, whereas males were more left faced than females for negative expressions...In the light of the interactions between sex and level of pleasantness, we reassessed the relationship between lateral dominance and facedness separately for males and females...thus our data may reflect greater left hemisphere lateralization for linguistic behaviours in males than females"
>
> (Borod, Kaff and Caron, 1983, pp.101,103)

There is in fact quite some evidence for the connection between the right hemisphere and emotional processing, for example :

> "One might argue that cognitive awareness of all all emotional expressions is mediated by the right hemisphere...My own bias is that all emotions are mediated by the right hemisphere."
>
> (Moscovitch, 1983, pp.75,77)

> "A decade of research has implicated the right cerebral hemisphere as dominant for both the expression of emotion and the appreciation of emotional situations...It may be that the right hemisphere is specialized for emotion in the same way that the left hemisphere is specialized for language."
>
> (Borod, Kaff and Caron, 1983, p.83)

It seems then that there is here the possibility of a relationship between the failure of many girls to have a predisposition to the holistic processing of the right hemisphere over the serialistic of the left and the residence of other facilities, such as emotional processing, in the right hemisphere, with differentiation along lines of gender. Whilst it is not clear if this is the case it seems an area worthy of further research.

3.8 Cognitive Integration in Action

The model which I have proposed in this chapter may be better understood by a consideration of some examples of how Cognitive Integration works. The application of the vertical links in the diagram, for example, may be illustrated by considering, say, the concept activated by hearing the word 'rigid'. The activation of this first level mental image (in the form of an auditory image) will evoke a second level concept and/or conceptual path, designed to give meaning to the symbol. These second level concepts may themselves evoke several more mental images, the first of which is likely to be the actual picture of the word itself, and this may be followed by the image which one's present environment encourages for the concept. This may be a person, or a length of wood etc. The image evoked may then lead to further processing of images, where there are links between them and hence we see the way in which they evoke other images, and in turn other second level concepts. The evoking, and processing, of these mental images is not part of our conscious experience and so usually goes unnoticed, although it continues in parallel with the cognitive processes in the second level structures. The results of the processing though thrusts itself on our consciousness and a vivid image may be recalled, for processing in the second level structures. The vividness, or otherwise, of the mental image on the consciousness, serves to illustrate something which is within the experience of us all, that the vertical links to mental images are of differing strength factors as described earlier. The speed of the processing of mental images by the right brain is extremely high and very efficient. These properties, and the lack of conscious knowledge of the process, have been recognised and used in the past by advertisers who have sought to promote their products by what has been called subliminal means. Although the individual is not consciously aware of it the brain has recorded and processed the split second image, and the image has in turn evoked concepts in the second level schemas which in turn have affected the conscious activity of individuals. The efficiency of the system has been further shown by the brain's ability to reconstruct images from a minimum of information, for example, a few dots in vital positions may be all that is necessary to construct a full image, as well as from partial data, such as the ability to 'see', and hence reason on, all of an object even when it is partially hidden from view. The efficiency of imaginal schemas has also been used to good effect by those seeking to improve straight memory recall of facts. Many such memory systems are built on the fact that it is often far easier to recall a relational image based picture, however obscure or absurd, than it is to recall a list of language based items. Thus pictures are built up to contain a reference to each item to be recalled from the fast and efficient mental imagery schemas. The directional nature of the vertical links in the model may also be illustrated by the young child learning to read. On being shown an image of the word 'dog' a child may be able to point to an example of a dog, showing that the image has evoked a second level concept and that the concept is understood (either instrumentally or relationally). This process may not though work in reverse. The child on seeing a dog may have the second level concept of 'dog' evoked but be unable to point to the correct word image from a choice in a book, and vice-versa. The example of reading may also be used to illustrate that the vertical links between the first degree and second degree structures may not even be present for certain concepts. Often when learning to read a child is shown a picture of a word and asked to repeat it. Although they may be able to do this corrrectly, the image of the word evoking an auditory image which they can repeat, there may be an inability to point to an example of the concept and this is an indication that they have not connected the image before them with a concept in their second level schemas and so have not understood it, or attached a meaning to the symbol (see e.g. Bryden and Ley, 1983).

The qualitative differences, in terms of the sequential/analytic and global/holistic dichotomy, between the working of the two levels of knowledge structures may also be illustrated by an example involving reading. When we are more proficient at reading, we gain the ability to take in words and even parts of sentences as a whole. This is a facility extended by the lower level imagery structures which, in turn are linked to the higher level concepts which attach meaning to the images. Althouth we are not always aware of this global view which we have, sometimes it is interrupted by a signal which tells us that an error or other problem has occurred, even if we are not sure what it is. Hence in the first word of the previous sentence we may have received such a message. In order to ascertain exactly what the problem is though the global processor, and its corresponding schemas in the psychological model must often hand over to the relatively slower sequential processor connected with the higher level schemas, which may make a word by word or letter by letter scan in order to find it. The common trick in which one asks someone to read the following :

A LITTLE OF WHAT YOU
YOU FANCY DOES YOU GOOD

or some such sentence, makes use of the fast global image processing associated with the first degree schemas. Taking in the sentence as a whole, and matching it with a known image pattern and its associated meaning in the second level, the mind may be fooled into ´seeing´ what is not there, or not seeing what is (see e.g. Paivio, 1975). Studies by Coltheart 1980, 1983) have led him to postulate problems of deep dyslexia as emanating from ´right brain reading´, which is prone to semantic error and inability to translate print to phonology. In mathematics we must be careful not to think that only certain mathematical topics lend themselves to the use of mental imagery, and that these are the only ones where a global/holistic approach is of value. Rather as Lean and Clements (1981, p,269) remind us "Equally important, but less obvious, is the fact that many children use visual imagery when thinking about topics which do not appear to require visual thinking". Taking this further, and seeking to explain how this is true, Ginsburg (1983) has defined the concept of visually moderated step-by-step sequences, where an imaginal input constantly cues the retrieval from memory of some algorithmic procedure until a task is accomplished. Scott-Hodgetts (1986) gives the following example from arithmetic of this process in action:

```
"A visual clue                                   392
                                                  x7
                                                 ---
triggers the retrieval                           ---
of a procedure                   (ah,yes! What are 7
                                           lots of 2s)
which produces a new
(non-visual) cue                                (=14)

which triggers retrieval from
memory of another procedure    (I must carry the 1)

which produces a new                             392
visual input                                      x7
                                                 ---
and the process continues...                       4
                                                 ---
                                                  1 "
                  (Scott-Hodgetts, 1986, p.70)
```

If we describe what are here talked about as visual inputs as being converted into mental imagery in the mind, and hence dealt with as such, then what we have here described is an example of the integration of the first and second degree schemas of the bi-modal model described here in action, with a mental image in the first degree cognitive schemas evoking concepts in the higher or second level structures and vice-versa. To further explain how this process may proceed in practice I shall now give two mathematical examples of the need for cognitive integration which relate to arithmetic and algebra, and to which

I shall return later (see Chapter 10). The first involves the interpretation of an expression like 3/4. If the individual's mental imagery does not include a mental image for this type of expression which is linked to the second level schemas for fractions, then the sequential/analytic type processing of the type associated with the second degree schemas will take over and this expression will always be interpreted as 'three divided by four' rather than as 'three-quarters', that is as the concatenation of three separate images rather than as a single entity. It is only when the images and links necessary to be able to view 3/4 as either of these are present that one can really understand the notation and make correct use of it in any context and/or answer questions which involve it. Thus cognitive integration helps to explain why this is so and why some have difficulty with it (see Chapter 10). A second example involves the type of question such as:

$$\text{Factorise} \quad 3x(2x + 1)^2 - 5(2x + 1).$$

A sequential/analytic approach to this type of question, as promoted by the type of schemas in the second level makes this a difficult question to answer since it may lead to a seemingly logical step-by-step approach such as multiplying out brackets, simplifying and then re-factorising. The spotting of the common factor (2x + 1) however requires the intervention of the global image processing capability of the lower level schemas so that, in conjunction with the sequential processes of the second level, the result may be arrived at more easily and with a level of understanding more likely to promote future progress. In general cognitive integration is beneficial in algebra since a purely sequential/analytic approach may well tend to divert attention away from a recognition of structure and towards a purely process/output mode of thinking. The versatile learner is thus more likely to be successful in algebra since he/she is able to call on global/holistic thought as appropriate, as well as analytic thinking. The work of Thompson (1987) has shown the value of providing a mental imagery in algebra, using the computer, which enhances the global/holistic abilities in a way which helps students to see structure and hence improves algebraic competence.

3.9 Psychological Implications of Cognitive Integration

The cognitive integration of the use of mental imagery schemas with the language/logical symbolism based schemas in the way outlined above, lends itself to an explanation of some ideas which have proved difficult to explain. One of these areas involves leaps of insight, where the solution to a mathematical problem seems to appear from nowhere, that is with no logical chain of reasoning. When those experiencing such leaps have been asked to explain such, then their explanation often involves much reference to imagery. What may be happening is that the path from the present state P to the goal state G passes, at some stage of the path, through vertical links to the concurrent path of the mental imagery schemas. This passing from one level to the other may be partly or totally unobtrusive to the conscious thought. Part of the route then lies in these first level schemas, where it is inaccessible to conscious thought, until it passes, again via a vertical link, to the second level, back to conscious thought, and hence to G. Thus it appears to the conscious thought that a leap of insight has occurred, the key parts of the leap, if they are consciously

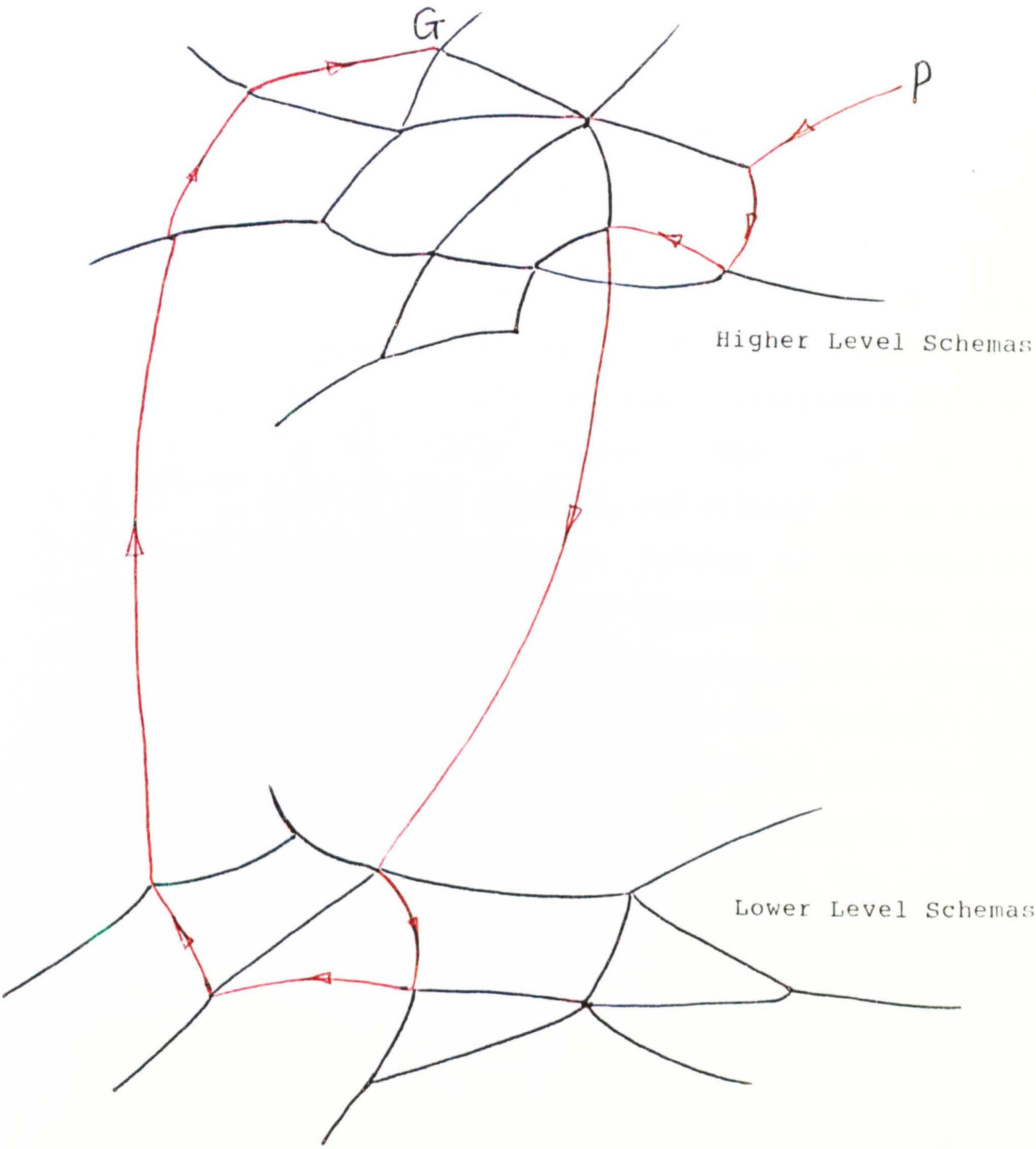


Figure 3.5

An Illustration of A leap Of Insight Produced By Cognitive Integration known, being possibly being described in terms of the mental imagery part of the path. This may be pictured as shown in Figure 3.5.

We may also use the integrated bi-modal model developed to postulate an answer to the difficult question of the purpose of sleep. Since the mind is constantly receiving data in the form of sensory input such as coded visual images during waking hours, the processing in the minor hemisphere is constant throughout this period, with vast amounts of data being processed and temporarily stored. It is not possible for all of this information to be formed into part of the first level knowledge structures with all the cognitive links, including those to the second level schemas, established while new data is constantly being input. The processes of assimilation, sorting the information and sifting out relevant, important data taking priority. The forming of the cognitive links between many of these imaginal concepts and other first level concepts would involve a reconstruction of the schemas at this level. It may also be necessary for the 'vertical' links between the first level concepts and the second level concepts to be made. It would seem then that this reconstruction process, centred in the first level schemas, cannot be properly accomplished whilst new data is entering through the visual and other sensory input systems and that the only way that such reconstruction of schemas and construction of inter-level links can be made is by shutting down sensory input to a minimum. This, I postulate then, appears to be one of the main functions of sleep, with the visual imagery nature of dreaming being an indication of the re-construction process taking place. This would seem to be an area where further research based on the current model would be of benefit.

One of the areas of Piaget´s work which has been the subject of some debate concerns the relationship between concrete and formal operational thinking. The current model of cognitive integration may help to describe the differences between these types of thought. Piaget describes the the main features of the two as follows :

> "In sum, concrete thought remains essentially attached to empirical reality."
>
> (Inhelder & Piaget, 1958, p.250)
>
> "The most distinctive property of formal thought is this reversal of direction between reality and possibility."
>
> (ibid, p.251)
>
> "Now the most prominent feature of formal thought is that it no longer deals with objects directly but with verbal elements....However this is not the whole problem, for all verbal thought is not formal and it is possible to get correct reasoning about simple propositions as early as the 7-8 year level, provided that these propositions correspond to sufficiently concrete representations. "
>
> (ibid, p.252)
>
> "But in this as in all the other cases examined up to this point, the difference between concrete and formal thinking relates to the construction of the ´structured whole´."

(ibid, p.163)

Some studies, however have questioned Piaget's division of the two types of thinking. Some have expressed doubts about the age at which formal operations arise, appearing to show such thinking at an earlier age (Blonkskii 1961). Some have even doubted the existence at all of a stage of formal operations (eg Smedslund 1977), and others have put forward the view that the difference in cognitive capacity between the child and the adult is one of quantity rather than quality (eg in Fodor 1972) - see Chapter 2 for more details. The present model may be used to give a possible explanation of the difference. We may think of the difference between concrete and formal operational thought in terms of a change of emphasis in thought, rather than a qualitative change in type as such. A child at the concrete operational level of thinking may be in the process of developing second level schematic dominance, on the basis of the left brain dominance discussed above from a physiological point of view, whilst at the same time continuing to develop equally both first and second level schemas. As also mentioned above, physiological evidence is that until about eight years of age engrams are laid down in both halves of the brain. Until such a dominance is firmly established, the first level schemas assume equal prominance for the child and so the imaginal nature of concrete objects to be able to think with are an invaluable part of the learning process. At each stage of learning such a child relies not only on connecting new concepts to others in the sequential/analytic based knowledge structures but also on the construction of vertical links to and from first level concepts, in a conscious way. Thus the concrete operational child has been well described as needing a concrete ´object´ or ´model´ to think with, such an object being internally represented in the first level schemas by mental imagery and providing a structure on which the second level concepts may be built. One who has reached the stage of formal operational thinking no longer has a need to constantly link the second level concepts at each stage to the mental imagery of the first level, although such links may be made consciously, or otherwise, and at times may even be highly desirable (see below). Hence, although both individuals are building knowledge structures at both levels, in the first case the emphasis on the conscious use of mental imagery may be greater whereas in the second the emphasis exists firmly in logical and analytical based schemas of the second degree structures. This change of emphasis from imagery based thought to analytical thought is mentioned in Piaget´s work :

> "Even if the content of the complexion problem requires nothing more than serial order operations, the fact that it cannot be solved in exclusively verbal terms until several years after the child can solve it with the aid of physical props shows that some other factor is at work here. If we consider the mental images involved in the problem we see how difficult it is for the subject to set up the data in his own mind (because only the relations are given). The result is that the subject is unable to translate the data into representational imagery and has to formulate them in exclusively hypothetical terms

if he is to see the necessary consequences."

(Inhelder and Piaget 1958, p252)

The present theory of cognitive integration may also provide an explanation of the fact that there are areas where many, even apparently mathematically capable individuals, have difficulty in understanding. Such individuals, who are at the level of formal operations (but see below), are able to extend their second level knowledge structures without the need for vertical links to the first level mental imagery in many cases. Thus they are able to reason logically and analytically without recourse to imagery or ´concrete´ objects. It may be however that there is a limit to the extent to which such extension of the second level may go without such supporting links to and from the mental imagery. Thus, although the individual is happy with his/her understanding and feels no lack, gradually the schemas approach a point where, without such imaginal underpinning there is a danger of reaching a situation where understanding is not forthcoming. In other words, new concepts may no longer be able to be linked to the second level schemas without a vertical linking. It may be likened to trying to build a model bridge by linking straws or some such items together, without providing vertical supports. Although this may continue for a time, eventually the bridge will collapse. When this happens in the mind the solution is to provide such supporting links to and from the mental imagery. This problem is likely to arise where there is a study of abstract generalised mathematics without a build up through less abstract examples. An example of this may be the student who is suddenly faced with a study of n-dimensional spaces without having knowledge of an appropriate mental image for the corresponding 2- or 3-dimensional example to which he/she can refer. In spite of having been successful in abstract algebra leading up to this point, such a student may struggle until such links with a suitable mental image can be made. Looking at this the other way round, and providing a specific example, we may say that since most mathematics students have studied 3-dimensional vector spaces at school they have a mental image to which they may refer (even if it is not entirely appropriate) when they meet the abstract case of n-dimensional vector spaces at university/college, and the ideas of cognitive integration put forward here seem to imply that it may well be desirable to refer to this imagery so that valuable supporting links may be made. A second example of the value of imagery in promoting and supporting understanding is shown by the fact that, historically, the Leibniz notation for the calculus seems to have provided a better set of mental imagery than that of Newton and this may go some way to explaining why his followers progressed better than Newton's even though the concepts were similar and a mental image does not appear to be strictly necessary. This area of the calculus and the understanding of the concepts involved has been shown to be still a problem today. By providing suitable mental imagery using a computer it has been shown that it is possible to improve pupils' understanding of concepts such as the gradient of a curve and the significance of the first and second derivative (Tall 1986d).

Another example of the problems which the lack of a suitable underpinning and integrative mental imagery can result in, and that still exists, is with the understanding of X-ray diffraction. Such diffraction occurs when "a monochromatic beam of X-radiation interacts with matter and is scattered in different directions, with no absorption of energy." (Gilbert, 1975, p.120) and it is used to help in a determination of the structure of a crystal or molecule by enabling calculation of the spacings between ions or atoms respectively. The diffraction pattern obtained on a photographic film following the incidence of such radiation on a crystal or molecule is studied so that from the pattern of scattering the positions of the ions or atoms can be deduced. The problems which many students seem to have with this method may be due to the fact that it is very difficult to picture in three dimensions what is happening due to phase problems and, further since the problem may be reduced to an analysis based on measurements from the film and standard calculations without the need to ´picture´ how these relate to the solid at each stage, this is what is often done. However, when it is necessary to interpret the results of the calculations by turning them into a physical description of structure, this involves working in ´reciprocal space´, and the necessary imaginal constructs are not there. Cognitive integration suggests that where possible imagery, even if it is not totally representative of the situation, should be promoted so that it may support the analytical understanding of the subject matter.

3.10 Relevance to the Current Research

The theory discussed above has important implications for the learning of algebra in secondary schools, in several different ways, some of which I have already considered in some detail above. In this section I shall look directly at other applications of the theory of cognitive integration proposed to the research described in this thesis.

3.10.1 A Consideration of Concept Links

Two important aspects of the cognitive links between concepts in both of the qualitatively different cognitive structures I have discussed, have to do with the relative strength factor of such links and their uni- or bi-directional nature. A consideration of these qualities of conceptual links aids us in an understanding of children's cognitive problems when first learning algebra and the ways in which they may be helped to learn it in the secondary school in a meaningful way.

When we begin an analysis of the previous mathematical experience of the beginning algebra student in the light of cognitive integration certain patterns of strongly linked concepts emerge which are handicaps to a meaningful understanding of algebra. One of the foremost examples of this is that strong links may have been formed in their mental schemas between the written use of letters and language or words, so that it is difficult for them to break out of this and see the letters as representative of some other concept. As Skemp says concerning the understanding of symbols :

> "There is a rather special kind of understanding, which applies to words and other symbols. To understand one of these means to associate the symbol with an appropriate concept. So to mis-understand means to associate it with an inappropriate concept, and not to understand

it at all means to be unable to associate it with any concept."

(Skemp 1979c, p148)

Hence it seems that many children mis-understand the use of letters in algebra by associating the symbols (letters) used with the concepts in their language schemas. If they do not do this then a second strong link which it has been recognised has sometimes been formed in the minds of many children arises when they are forced into the position of having to deal with letters in a mathematical rather than a language way, and exhibits the form of a strong predisposition to the serial coding of letters, putting for example, A=1, B=2, C=3 etc. (Wagner 1977, 1981, Thomas 1985). A second major problem which early learners of algebra have seems to be caused by the fact that in their previous work in mathematics, especially in arithmetic based questions, the concept of an ´answer´ to a question or problem has implied the finding of a numerical solution to it and so it seems that in their cognitive structures the concepts ´answer´ and ´number´ have developed a strong uni-directional link. That is, the evoking of the concept ´answer´ has a strong tendency to evoke the concept ´number´, although this may not be true the other way around. One implication of this for algebra learning is that it is difficult for them to accept as an ´answer´ anything which does not conform to this established pattern. Therefore they are often unable to accept answers which are expessed in non-numerical terms or answers which involve a lack of closure (Collis 1975), that is answers which are not solely numerical and are expressed in terms of an operation, such as a + 2, and so this can cause many problems. To aid children to overcome all these types of problems associated with their schemas, the module of work developed for this research has been able to introduce the use of letters naturally in a computer environment, initially as part of the computer's language. In this language they learn that letters are used to represent variables, and these are first introduced and encountered by the children in an environment where stress is laid on their association as the label (i.e. language based) of a location or 'box' in the computer. The laying of strong emphasis on this model and doing so in a computer environment where the mental images are re-inforced through the use of the computer screen encourages the formation of links between such images and the concept of the use of letters to be representative of variables, without producing cognitive conflict with already present strong links. Further, the tendency for thought to flow along such links to language whenever letters are seen may be circumvented too. Using these variables in a computer programming environment as a part of expressions, to 'instruct' a machine was also thought to be a useful bridging step to their acceptance in mathematics, since such a use would help build up links of high strength factor (over the course of the module) with other arithmetic concepts, and would form vital mental imagery of such a use of letters, which would eventually lead them to accept them as answers too.

Many children also have a tendency when trying to attach meaning to the use of letters, which is often encouraged by their teacher in beginning algebra, to treat letters as objects in their own right, thinking of 3a + 2a = 5a in much the same way as 3 apples plus 2 apples equals five apples (Küchemann

1980). Although this approach works well for a time, in that it provides the all important answer to a question, the cognitive links formed by such an understanding soon produce cognitive difficulties when questions such as 3a + 2b + 2a =?, are reached. Once again the fundamental picture of a variable used in this research may help to avoid problems such as these by presenting a clear definition of the way letters are being used before they are used alone, without their accompanying store box.

3.11 The Directional Nature of Conceptual Links

The model on which cognitive integration has here been based shows that it is also important to take into account the idea that when children form conceptual links, these are directional in nature. This recognition is one which this research has also addressed in the synthesis of the algebra modules. An important example of the application of this in the methods which children of early secondary school age employ in the solution of algebraic equations. Kieran (1985) has shown that, when solving equations such as 2x + 3 = 9, about 50% of the novice algebra students used methods of trial based on substitution of values into 2x + 3 and looking for a balance of the two sides of the equation. The other 50% used methods based on the transposing of terms from one side of the equation to the other. Kieran concludes that this dichotomy of approach should be taken into account when teaching methods of solution of such equations, so that two different approaches to this area of learning are necessary. In the context of the current research and theory we may describe this as recognising the necessity to build bi-directional links in this cognitive area in order to achieve the reversibility of thought needed to understand these equations rather than, as is often the case, to just be able to solve them. This lack of understanding of such equations has been noticed by Firth, who says :

> "Nonetheless there seems to be some evidence to support the contention that the solution of linear equations may have little to do with understanding. It may be that when equations such as these are first encountered they demand an understanding of the solution and the processes involved, but that once an algorism has been learned the pupil need not understand either of these aspects and may still be very successful in obtaining correct answers."
>
> (Firth, 1975, p.58)

The Dynamic Algebra Programme's approach to equation solving recognises the apparently natural inclination of many children to want to construct the solutions to equations by a substitution process, and indeed the value to understanding of such a method. Consequently it provides an environment on the computer or cardboard maths machines (see Chapter 7) which encourages and even promotes, this type of investigative method. The more analytical approach of transfering terms, also necessary as the complementary direction in the linking of concepts, is not forgotten but the ground work for this is laid by the inclusion of work on the simplification of algebraic sums which is part of the necessary basis for these methods. Thus, rather than ignoring the child's current knowledge and preferences, the programme of work helps in the synthesis into a coherent whole of the necessary parts of the understanding of linear algebraic equations.

A third area where a fuller understanding of the nature of concept links in mental schemas is useful concerns the appreciation of the idea that there may be more than one level of understanding of a given conceptual area. This, as put forward above, is due to the possibility of forming links between the given concepts and more than one possible mathematical cognitive domain. The implication of this is that, wherever possible, mathematical concepts should not be learned in the isolation of a single topic environment, but consideration should be given to connecting them with as many mathematical domains as are readily available. In Chapter 5 I shall discuss more fully the ways in which some have sought to apply this principle through the use of multiple representations in computer software. When we consider this applied to the learning of algebra, then it means that the educator should not treat algebra as a topic in its own right to be studied independently of other areas of mathematics, but that where appropriate links may be made, such as to arithmetic, graphs etc., then these should be encouraged. In this research programme this is achieved in part through the introduction of the cardboard maths. machines. Whilst they are using these machines, (whose method of working are described in Chapter 7) and obtaining active involvement with the vital concept of variable, the children are required to replace the variables in the programs by numerical values and to perform any necessary arithmetic, in imitation of the workings of the computer, themselves. Thus in the program :

```
10 INPUT a
20 b=2*(a+1)
30 PRINT b
40 GOTO 10
```

typical of those in the programme, they may decide to use the value of 7 for a. They then are faced with having to calculate the value of 2*(7+1) for themselves. In this way they are helped to appreciate that many of the operations of algebra are the same as those they have already used in arithmetic and that the concepts like brackets which appear in algebra and are so important in later work, are also the same in the much more familiar domain of arithmetic. thus concept links between concepts such as brackets and the domains of algebra and arithmetic are fostered, opening up the existent power of the arithmetic the child has learned to algebraic learning. This may also be beneficial in helping to prevent the problem which has been noticed (e.g. Booth 1983b) of a failure of children to exhibit the transfer of knowledge built up in one area of mathematics to another area.

3.11.1 Mental Blocks

The overcoming of the difficulties in achieving reflective thinking due to mental blocks is another area where the Dynamic Algebra Programme can make a useful contribution to the learning of algebra. The importance of a relatively free, although structured, problem solving approach, as adopted in the programme, is that such an environment allows for variability of input and starting point shown to be necessary for the avoidance of such problems, whilst still giving overall direction to the investigative activities. The computer program developed here acts as a generic organiser (Tall 1986), a type of mathematical microworld (Thompson, 1985b), wherein the child may explore the concepts available from a variety of different starting points. Thus, in their search for a ´solution´ to a problem presented in this microworld, the inputs made and the direction travelled are in the hands of the child. This should help to reduce the common situation where a child fails to finish a problem because it differs in some slight way from the standard example learned by rote, and which has contributed to a mental block. It is important to realise that it is the concepts abstracted from the programme which are the important things and not the ´answers´ to the exercises themselves.

3.11.2 Mental Imagery

As is well documented in the literature, it seems to be the way in which mental images are used in the learning process which is often in dispute, rather than the value or otherwise of such imagery. If, as I propose, the mental image is fundamental to the learning process, and thus holds great power for teaching programmes to exploit, then it is important to make full use of this cognitive facility, in the way that cognitive integration proposes. Tall (1986) has shown the possible value of presenting mental images in the teaching of the calculus to sixth form students and, since it is likely that the conscious usage of imagery declines as one gets older (see earlier), then it is even more likely that mental imagery would be of value in the early years of the secondary school mathematics curriculum. It must be appreciated that it is the integration of mental imagery from the computer with other cognitive processes which is the power of the approach since, for example, it has been shown that relying on mental imagery provided by the computer alone can lead to problems such as perceiving a smooth curve as stepped since this is how the picture appeared (Goldenberg 1987). Making full use of this powerful aid to understanding is achieved in this research by the constant reference to a pictorial representation of variables (Figure 6.1). This representation is used, and therefore re-inforced in all aspects of the work, including the cardboard maths. machines and the computer program maths. machine. Seeing the variability in this way of the numbers represented by the letters used at the same time as they are employing letters in algebraic processes gives the children the opportunity to build into their knowledge structures mental images which are relevant to the learning materialand so which may be linked to the serialist/analytic, non-visual second level concepts in the manner described earlier in this thesis. This stimulation of the visual knowledge systems has been lacking in most of the work on secondary school algebra, where, I maintain, the children have great need of it.

3.12 Overview of the Chapter

In this chapter I have endeavoured to extend the psychological theory of Skemp in a way which encompasses the knowledge structures of mental imagery. The result is a theory of Cognitive Integration which seeks to explain how a versatile learner in mathematics, i.e. one who is able to use both global/holistic as well as serialist/analytic strategies as and when required, may be encouraged. I have also looked at the application of this psychological theory to the understanding of mathematics and in particular of the early learning of algebra.

Chapter 4

A Review of The Algebra Background

In this chapter I shall consider the well documented difficulties which many secondary school children experience when they are presented with algebra. I shall also discuss evidence that these problems are not limited to the lower end of the secondary school (11-14 year olds) but often persist, even for the more mathematically capable students, after they have left school. The rest of the chapter consists of a review of the research which has attempted to describe and alleviate the situation. These research efforts relevant to secondary school algebra may be categorised as covering one or more of the following areas :

a) The understanding of the meaning of literal symbols

b) Expressions involving literal symbols

dc Equations involving a single variable

d) Problems involving translation from words to algebra

and therefore each of these will be considered in turn.

4.1 <u>The Nature of the Problem in the Early Learning of Algebra</u>

The general problem which exists in the early learning of algebra has been expressed as follows :

> "Results revealed and reconfirmed many common misconceptions of beginning algebra students in the areas of interpreting variables, manipulating expressions, understanding equations, and graphing of functions. That these misconceptions are so widespread, as documented by this and other research in the area of algebra learning, implies that standard classroom texts and instructional procedures are not providing

> many students with the experiences necessary for them to internalize basic algebraic concepts and to apprehend the richness of certain algebraic principles."
>
> (Wagner, Rachlin & Jensen, 1984, p.55)

It seems clear that many schoolchildren do have difficulties with algebra (e.g. see also Harper 1979, Küchemann 1980, Kaput 1987) and that the way in which algebra has been taught in many schools has contributed to these problems. Children are often presented with meaningless symbols for which they are then given a set of equally meaningless rules of manipulation, so that they may :

> "..solve linear equations (often by applying rules and not always checking to see if a proposed solution is correct); they collect like terms (sometimes to the accompaniment of grossly misleading statements from the teacher about the incompatibility of apples and bananas; they substitute numbers (often into expressions which have no obvious concrete meaning); they insert and remove brackets."
>
> (Hale 1980, p.11)

What has not been so clear is the precise nature of the difficulties and the exact demands of the subject which produce them (e.g. Wheeler 1981). The approach often adopted, seemingly devoid of meaningful learning has caused some to question the way in which algebra has been traditionally taught, with its emphasis on the syntactics of the subject, and have argued that the semantics of algebra should receive greater stress. Davis (1985), reporting on ICME5, highlighted this as an important shift necessary in the thinking of mathematics educators, namely away from meaningless manipulation and towards the construction of meaningful mental representations in the mind of the student. This change in emphasis towards semantics has also been described (Davis 1986a, p.21) as moving from a notational emphasis to a deeper level, namely the meaning of the symbols used. The provision of a suitably structured environment for the acquisition of this deep conceptual understanding of algebra has been a prime motivation of this research study. This is not to say that manipulation in algebra is not of any importance, since for example as Nantais, Herscovics and Bergeron (1984) comment, a refusal to accept skills as showing understanding would be an extreme position and the constructivist approach is to recognise the acquisition of skills as procedural understanding, part of the global process of the construction of understanding. Kaput (1987a) agrees that an important pedagogical issue is how to include the principle that neither a syntactic nor a semantic approach, alone, is of value. Hence again, although this current research is based on improving a deep understanding of the symbolic literals of algebra, it was considered important to include analysis relative to skill acquisition, to make sure that there was no detrimental effect on this, and in Chapter 8 these results are considered. Part of the change of emphasis proposed by some (e.g. Vergnaud and Cortes 1986) has included the desirability of a more practical approach for many pupils, in order to facilitate such relational understanding.

4.1.1 Some Strategies Adopted By Early Learners

A review of the relevant literature reveals that faced with a bewildering array of ideas which are totally new and often meaningless, the average secondary schoolchild adopts one or more of several strategies :

a) they invent their own procedures

b) they rely on familiar arithmetic procedures

c) they adapt previous procedures to fit the new work

One of the influences of arithmetic which remains from their previous experience is a strong disposition towards the need for a numerical answer. This may be produced by the use of coding, for example by making the substitutions a=1, b=2, c=3, etc., with the linear order of the alphabet corresponding to the linear order of numbers (Wagner 1977, 1981, Thomas 1985). This process extends to further use of arithmetic procedures which they feel comfortable with, such procedures including the invention by children of their own methods for solving algebraic problems. Ginsburg (1977, p.86) has described this tendency as follows: "Children often solve written arithmetic problems by invented procedures. Children do not simply employ standard algorithms as taught in school and instead devise their own procedures. These usually rely in part on assimilation into familiar schemes." Filloy and Rojano (1985) have also recorded the use of such intuitive strategies in their study of 12 and 13 year old algebraic novices. These methods may possibly lead to errors such as wrongly encoding results (Booth 1981). They may also include methods based on arithmetical trial, for example in the solving of equations, re-emphasising the predisposition towards the arithmetic which they feel comfortable with (Kieran

1985). One feature of algebra which may contribute to such an arithmetic tendency is the = sign. This sign seems too strongly associated in their cognitive structures (see Chapter 3) with the injunction to come up with an answer, and so when they meet equations early on in a traditional algebra course there is a tendency to latch on to this and thus associate the whole of algebra with solving equations. The later absence of the equals sign, or its use in a different way, causes conceptual problems and an inability to assign any meaning to indeterminate forms (Kieran 1981a, 1981b) - see the later section on equation solving.

Other problems and difficulties associated with the beginning of algebra have been studied, and shown to include a lack of understanding of the use of operators and the conventions of notation in algebra. This leads children to see little or no need for the use of brackets (Booth 1981, Kieran 1979, Wood 1978), and failure to appreciate the need for operators between letters, so that answers such as x + y are conjoined and written as xy (Booth 1984). It has also been shown that children have a strong tendency to want to work from left to right and that this type of thought process may well negate in their minds the need for brackets in algebraic expressions (Kieran 1979).

4.2 <u>The Persistence of the Problems</u>

One of the notable, and most worrying, features of the problems associated with the learning of algebra discussed above, is their persistence throughout the school career of many children, and even after. The investigation into children's understanding of the use of letters in algebra by Küchemann

Level	CSMS Percentages 13 years	14 years	15 years
0	10	6	5
1	50	35	30
2	23	24	23
3	15	29	31
4	2	6	9

Items at levels 0, 1, 2 can all be solved without having to operate on letters as unknowns.

Items at levels 3 and 4 involve treating the letters at least as specific unknown, and in some cases, as generalised number or variables.

Table 4.1

The CSMS Percentages of Children At Each Algebra Level

At Ages 13, 14 and 15 Years Old

(1981) showed that the four or five years of algebra teaching which most secondary school children had received had made very little difference to their understanding of the subject, and problems persisted throughout (Table 4.1). Similarly, an investigation by Kieran (1985) into the methods employed by children in solving algebra problems, showed that intermediate and older/more experienced students not only used the same methods in solving algebraic problems, but they also made the same mistakes.

Even among those who appear to have mastered the techniques of algebra better than most, there have been shown to be serious, persistent problems of understanding. Rachlin (1981) for example, investigated the processes used by college students in understanding basic algebra and found that they too possessed many of the same error patterns as less successful pupils. Rosnick and Clement (1980) also tested college students and found that :

> "Many of the college students that we have interviewed and tested have demonstrated that they do not recognise the use of letters as standing for numbers. They confuse the use of letters as variable with the use of letter as a label or unit."
>
> (Rosnick & Clement, 1980, pp.5,6)

On the basis of her detailed investigation of the errors made by students in algebra, Matz (1980) concluded:

> "The goal of this report is to provide an account for the striking unifirmity of errors students make in solving algebra

problems...30 common errors have been analyzed...Some of these errors were committed by adept problem-solvers as well as by naive problem-solvers."

(Matz, 1980, pp.154,155)

In a similar vein, Tonnessen (1980) has found that the levels of attainment of the concept of variable in algebra was low even among college mathematics students. Thus researchers agree that, even among mathematically oriented students, there exists a large number who have missed the basic notion of variable and have confusion between algebraic symbols and their meaning.

It seems reasonable to conclude that there is a serious problem with the teaching and learning of algebra in secondary schools when even the most able fail to understand fully.

4.2.1 The Understanding of the Use of Symbolic Literals

Of all the difficulties associated with algebra, it is the problem of the interpretation which the children give to the symbolic literals, or letters, used which seems to cause the most problems. As Davies (1978) says :

"The concept of 'variable', in its usual mathematical sense, is so fundamental in mathematics that we include it in Section VIII. It is also the source of a great deal of trouble for a great many students."

(Davies, 1978, p.125)

This is understandable of course, since if the letters are not cognitively meaningful themselves then this is certain to lead to other problems later when they are put to use. This difficulty in interpreting the meaning of letters is not surprising when we consider how little attention, if any, is paid to this in the beginning algebra classroom, and the wide variety of different ways in which letters are used in mathematics. Wagner has described up to ten different uses of literal symbols in mathematics :

> "Literal variable symbols are used in a multitude of ways in mathematics. Depending on the context in which they occur and the element(s) to which they refer, the role of a variable may be described as that of a name, a placeholder, an index, an unknown, a generalised number, an indeterminate, an independent or dependent variable, a constant, or a parameter. Adding to this complexity is the fact that, generally speaking, different literal symbols can be used to represent the same thing, and the same literal symbol can be used to represent different things....It is no wonder that students have so much difficulty working with literal variables."
>
> (Wagner 1981, p.165)

Usiskin (1987) too gives many examples of the possible definitions and referrents of the term variable. Matz (1979) argues that this bewildering array of usage tends to blur the distinctions in the uses of letters, causing the only likely abstraction to be the abstractness of the concept. The concept that different letters may stand for the same value is one which many beginning algebra students seem to find difficult, possibly due to the one-to-one correspondence type of thinking mentioned earlier. Wagner again describes the problem this way :

> "...many students tend to believe that different letters used in the same context represent different values. For example, even some students who have studied algebra do not seem to realise that changing the ´unknown´ in an equation does not affect the solution to the equation."
>
> (Wagner, 1979, p.216)

It has also been pointed out that even mathematicians, who would claim to understand the concept, do not necessarily agree on a definition, some viewing variable as a placeholder, some as a set representative and others as defined in terms of functional dependence (Herscovics and Bergeron 1985). The term variable (and function) has, it appears, such a subtle meaning that it is not always used in a consistent way (Tall 1985a), and this lack of consistency has added to the confusion of students. Tonnessen (1980) demonstrated the complexity of the concept variable by identifying eight criteria for its understanding, four for its acquisition and four for its use. He concluded that there was no evidence from his study that its acquisition can be used as a predictor for its use, since he found that the students were able to use variables without having a correct understanding of them. In fact he commented that "A striking characteristic of the students´ performances is the low level of correct response on concept acquisition items." (Tonnessen, 1980, p.206). Such a rich and varied use of one term should certainly be taken into account in the classroom if we are to avoid the mental confusion now evident. Harper (1978) describes the problem this way :

> "...we would be well advised to reflect on

> the meanings we give to common terms such as ´variable´. ...How many different interpretations abound in a classroom at any given time, and how often are we sure that the learners entertain ones similar to our own?"
>
> (Harper, 1978, p.237)

Matz (1979, 1980) has also suggested, on the basis of a study of errors, that the multiplicity of uses of literal symbols in algebra is one of the major problems. She says:

> "One subtle and complicating feature about the abstract character of symbolic values is that the precise nature of the abstraction varies. Symbolic values may be constrained, unconstrained, or constant, and may assume a set of values or a single value. Lumping together symbolic constants, parameters, unknowns, arbitrary symbolic values, and pattern variables as simply ´variables´ draws attention only to a single common feature - their abstractness. Such an overly general concept of a variable blurs distinctions that affect how that entity is manipulated; it obscures restrictions about exactly how and where the value it refers to varies"
>
> (Matz, 1980, p.135)

It may well be then that there is a need to separate out more clearly the various uses of letters in algebra in the secondary classroom, in order to improve understanding. Wagner (1983, p.478) agrees, saying "if we want students to gain a real appreciation for the power of literal symbols...we need to introduce these ideas gradually as different uses of literal symbols appear in the curriculum". This lack of understanding is a serious obstacle to genuine progress in mathematics and it is one to which this research is addressed. By giving a single, simple interpretation for the use of letters in algebra (see Chapter 7), it was hoped to clarify the meaning of the use of letters for the children and give them the basis on which they could progress to building other facets of their use, thus avoiding problems in the future. The need for a programme of work, such as that presented here, which takes into account the possibility of pupils' confusion over the introduction of letters in algebra and seeks to provide a clear, unambiguous initial meaning for their use is indicated by the above discussion. Realising the extent of the problems of understanding of the use of symbolic literals in algebra confirms the value that such a programme could have. For example, in an in-depth investigation of the difficulties of children with algebra, and the ways in which secondary school children understand the use of letters in algebra, as part of the research entitled Concepts in Secondary Mathematics and Science (CSMS) and based at Chelsea College, London, Küchemann (1981b) shown that different children are at different levels of understanding with regard to the concept of the variable in algebra, and that their ability to tackle algebraic problems is directly related to the level which they are at. The levels of understanding and usage of letters in algebra he has described are as follows :

LEVEL 1 - Those items which are purely numerical, or which may possibly have a simple structure which can be solved by treating the letters as objects in themselves.

LEVEL 2 - Those items which are of increased complexity, but may still be solved by evaluating the letters or viewing the letter as an object. There is, however, the first indication of a willingness to accept a lack of closure in answers by children at this level. i.e. Answers such as 3x + 2 are becoming acceptable.

LEVEL 3 - Those items which involve an understanding and acceptance of letters as specific unknowns and also acceptance of lack of closure in answers.

LEVEL 4 - Those items which are similar to level 3, but which involve more complex structure. They may also involve a knowledge of using letter as a generalised number or variable.

He also claims (Küchemann 1981b, p.117) that these are related to the following Piagetian levels :

LEVEL 1 - Below late concrete

LEVEL 2 - Late concrete

LEVEL 3 - Early formal

LEVEL 4 - Late formal

Moreover, the first two levels of understanding essentially need only involve an arithmetic whereas the third and fourth levels require the ability to think algebraically. These levels thus give a crucial test of children's conceptual understanding of the use of letters in algebra, and, along with Küchemann's

1. The letter is evaluated i.e. it is given a numerical value from the outset.

2. The letter is not used at all i.e. it is either ignored all together or is given no meaning at all.

3. The letter is used as an object in its own right i.e. it is used as a shorthand notation for an object, or as an object in its own right.

4. The letter is used as a specific unknown i.e. it is a specific, but unknown number that can be operated on directly (due to Collis, 1975).

5. The letter is used as a generalised number i.e. it is seen to be able to represent or take several values.

6. The letter is seen and used as a variable i.e. it is seen to represent a range of unspecified values and a systematic relationship can be perceived between two such sets of values.

Figure 4.1

The Six Uses of Letter in Algebra Given By Küchemann algebra test which classifies questions according to these levels, they have been used in Chapters 6 and 8 to analyse the results of the experiments. In addition, Küchemann has shown that children's interpretation of letters in algebra can pass through six different categories of usage as shown in Figure 4.1.

Unfortunately, as his work showed, many children fail to progress past the fourth of these stages, and some do not even get this far. The percentage of children found by the CSMS algebra tests to be at the levels one to four mentioned above were given in Table 4.1. Combining the descriptions of the understanding of the use of letters in algebra in Figure 4.1 with his data in Table 4.1, it may be seen that the majority of secondary school children have a naive understanding of the use of letters in algebra and are only able to cope with what are essentially arithmetic uses of letters. Their limited view, however, may often be obscured by their instrumental understanding, which may allow, for example, a child at one of the first four stages of understanding given above to perform adequately on some questions without ever really having a deep understanding of the concepts involved.

Harper (1979) similarly has recorded three different levels of usage of letters and also noted that by far the majority of the subjects held limited understanding of the usages of literal symbols. The conclusion that the majority of children of early secondary school age have a naive view of the uses of letters in algebra, is described by Wagner, Rachlin and Jensen (1984) in these terms :

"....studies seem to suggest that students'

> ability to manipulate literal symbols far exceeds their understanding of what variables are."
>
> (Wagner, Rachlin and Jensen, 1984, p.4)

Harper (1979) also analysed children's interpretation of numerical variables and concluded that, as discussed above, they were carrying over from arithmetic into algebra inappropriate concepts such as 'ordered entities'. She described the confusion she found in the interpretation of symbols as a clash of interpretative frameworks, something which can arise when, either one can entertain more than one meaning and is undecided which to select, or cannot rationalise the interpretations as distinct aspects of the same concept. Thus pupils in her study were observed to switch interpretations of letters depending on the task content. Looking at the conceptual problems involved in the transition from arithmetic to algebraic thought Harper (1980, 1981) also concluded that "algebra requires a different conceptual understanding of the usage made of letters to that demanded by an arithmetic with letter appendages." (Harper, 1980, pp. 237,238) and that children need guidance towards a non-ordered view of symbolic literals. This arithmetic carry over by children of the need for an ordering in the unknowns of algebra was also reported by McLeay (1980) following her research. The effect of arithmetic thinking on students' understanding of the use of letters in algebra is also shown by Kieran's (1981a, p.161) study, which showed that when beginning algebra students were asked to interpret letters they replied with a value. The cause of the common reversal error 6S=P observed by Rosnick (1981), when students attempted to symbolise the fact that there were six times as many students (S) as professors (P) was also considered to be a flaw in the students' concept of variable:

> "They should learn to distinguish when letters are used as labels referring to concrete entities or, alternately, as variables standing abstractly for some number or number of things."
>
> (Rosnick, 1980, p.420)

He admitted, however, that there seemed to be no quick solution to helping students to see such distinctions and that several different teaching strategies (see Rosnick and Clement 1980) had been tried, but without success.

4.3 Studies Designed To Address Understanding of Variable

One researcher who has been instrumental in helping to promote a better understanding of the concept of variable among secondary school children is Wagner (1977, 1979, 1981, 1983, 1984). One insight she has provided (1981) is that the literal symbols for variables only acquire meaning as they appear in some context and represent some numerical or geometrical referrent, and this is in agreement with Kaput (1987b) who speaks of them as only acquiring meaning relative to some representation. Wagner also reports that one of the children's misconceptions is that changing the symbol implies changing the referrent. The effect of this is that, "many students tend to believe that different letters used in the same context represent different values." (Wagner 1979, p.216). The opportunity to see the symbols as labels, whilst at the same time manipulating their varying values in an environment where comparisons between variables could be made was a contribution of the present research which was expected to help with this difficulty. Another important study which had some success in improving students understanding of the use of letters in algebra was that of Booth (1983b), and the content, results and conclusions of her study were considered to be of fundamental importance to the ethos of the present research. As part of the SESM research project which was a follow up to the CSMS research, (of which the results of Küchemann, included here, were significant), Booth interviewed children of 13, 14 and 15 known to be at the levels of understanding identified by Küchemann (1981). She found that there were three main areas of difficulty underlying the errors made in algebra. These were :

1. The difficulty of the interpretation of letters.

2. The difficulties of coping with the notation and its conventions, such as the use of brackets.

3. The difficulty in formalising and symbolising, in algebraic notation, the solution to a problem.

This is born out by the work of Küchemann (1981) who found that children used their own ideas of what letters stood for even after being taught algebraic procedures. Thus these became part of their concept image, even though erroneous. In order to try and ameliorate these areas of children´s difficulty with algebra, Booth designed a teaching experiment based on the results of her interviews. The instructional sequence consisted of six 35 minute lessons designed specifically to tackle the mis-understandings she had found. The programme was presented to one group of six 13 year-olds and two groups of five 14

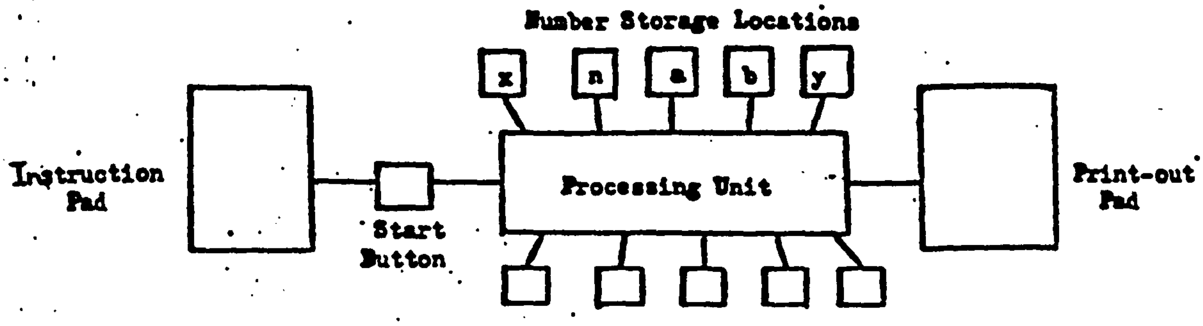


FIGURE 4.2

Booth's 'Mathematics Machine'

year-olds and was followed by paper and pencil type tests based on the algebra test devised by Küchemann. Her work made use of an imaginary 'Mathematics Machine' as shown in Figure 4.2. The idea of this was that the children should concentrate more on the concrete situation of trying to give the 'Machine' instructions to solve a problem, rather than on the more abstract process of of trying to produce an answer themselves, since the disecting of a problem and the representation of it in order to present it to a machine, even if an imagined one, would encourage an understanding of the concepts and principles involved. On the value of this she comments :

> "Also of evident value was the presentation of a schematized 'model' of the machine which gave the children a more concrete picture of the situation and in particular presented a breakdown of the procedural steps involved, from the giving of instructions through the processing or computing stage to the representation of an output or answer. It is suggested that this helped to focus the children's attention on a form of mathematical procedure with which they are not often explicitly aware."
>
> (Booth, 1984, p.45)

These benefits were incorporated into this research by taking her idea of an imaginary 'Mathematics machine' and translating it into the 'Maths. Machine' used in the teaching module presented in Chapters 6 and 7. This 'Maths Machine' has been designed to function as a working model of the processes involved in the computer's processing of BASIC statements involving variables. Booth´s research showed clearly the value of this type of approach for children, producing a significant improvement in understanding in several areas, including conjoining in algebraic addition and formalisation of method. A summary of her findings is included later. With regard to the concept of letter as a generalised number she observed that, since there was an improvement in performance between the immediate and delayed post-tests, it appeared that the ideas in this concept were not easy to assimilate and that time was required for its understanding. She also concluded that particular problems seemed to be associated with the understanding of generalised number and that further research, paying attention to the framework of reference constructed by the child with reference to it was necessary.

Other attempts to tackle the problem of children´s understanding of variable in algebra, have involved the use of a Logo computer programming environment (e.g. Sutherland 1985, 1987, Sutherland and Hoyles 1985, 1986 and Nelson 1985, 1987). Sutherland and Hoyles used longitudinal studies of four pairs of pupils programming in Logo to investigate whether such experiences would effect their understanding of variable and help them to form a conceptual framework for the use of variable in a non-programming mathematical environment. Their results showed that, although the pupils did make use of variables in several different ways in their Logo progarmming (Sutherland and Hoyles 1985) they did not do so naturally but required guidance from the teacher. Noss (1986) concludes as follows about the success of his study in helping children construct meaning for elementary algebra concepts :

> "It may be worth emphasising here what the study is not intended to illustrate : namely that children who have learned Logo for some time will necessarily have learned something about algebra in general or about the concept of variable in particular. The interpretation of the data offered here...is that children may - under the appropriate conditions - make use of the algebra they have used in a Logo environment in order to construct algebraic meaning in a non-computational context."
>
> (Noss, 1986, pp.353,4)

When Sutherland and Hoyles examined the transfer of their understanding to traditional ´paper and pencil´ type tests of understanding, based on the CSMS algebra paper, some success was reported (Sutherland, 1987) but questions involving the use of letters as generalised number and as variable proved resistant. Nelson (1985, 1987) investigated the long-term effects of programming in a Logo environment on the understanding of the concept of literal symbols by 3 average ability fourth grade students and concluded that there was some beneficial effect. Samurcay (1985), however, has pointed out some of the problems associated with using Logo to improve children´s understanding of variables and states that it tends to be the logo variables rather than the algebraic concept which is learned. It may be that, as Tall (1985a) suggests, that the need to distinguish symbolically, in Logo, between a variable´s name and its value is a further complication blurring the meaning of the notation and making the promotion of a deep understanding less amenable to the child than is the case in a language like BASIC where such a distinction does not exist, and which is therefore more akin to algebra.

4.4 The Understanding of Algebraic Expressions

As Kieran (1987, p.12) has commented "Variables are only the first step which algebra novices are faced with. The next step is making sense of these variables and operating with them in the context of algebraic expressions."

The first expressions which children are likely to meet are those of the type 2a + 5a, which they are asked to simplify. In spite of what we have seen above is a poor understanding of symbolic literals, many children initially cope well with this type of question. They develop an instrumental understanding which invokes the concept of using a letter as an object in its own right and hence may think of the example given as, for example, ‘two apples plus five apples makes seven apples’. Unfortunately this approach means that conceptual problems may soon begin to emerge when they are faced with having to simplify expressions such as 2a - 5a. They may overcome these by thinking of ‘owing three apples’ etc. but such a situation reaches its limit when questions such as: Simplify 2a x 5a, are asked, and severe cognitive difficulties may be the result. Another way in which the lack of understanding of the use of letters in algebraic expressions has been shown to manifest itself is in the difficulty in assigning any meaning to letters used in expressions such as a + 3. Kieran (1981a) has shown that 70% of the pupils she interviewed could not assign any meaning to letters used in this way. For them, giving the meaning of a letter meant finding its value. A study by Booth (1981) also showed that an algebraic expression is not regarded as a legitimate answer, but that, again, there was a strong desire to arrive at a numerical ´solution´. Davis, Jockush and McKnight (1978, p.100) maintain that this understanding is one of the hardest things for 13 year-olds to cope with in early algebra, and this inability of children´s knowledge structures to understand the use of letters in expressions such as a + 3 has been classified by Collis (1975a) as failure to accept lack of closure. McLeay (1980) studied children´s interpretations of literal expressions and found evidence that children did indeed develop an instrumental understanding which enabled them to cope with many of the questions which they encountered in their classrooms, but "Most of the subjects taking part in this study were unable to answer correctly items which required ´x´ to be considered as a variable...they also required closure of the operation." (McLeay 1980, p.56). Matz (1979, 1980) has recorded problems associated with the difference in meaning attached to juxtaposition of symbols in the algebraic framework compared with the arithmetic one. For example, in arithmetic the symbols 267 placed together imply place value and in 2 addition, whereas in the expression 2xy multiplication is implied. This implied use of operators, and often in a way different from that used in arithmetic, is a further source of problems in students´ attempts to find meaning for expressions in algebra. Saad (1957) found this to be the case when, even amongst the top 25% of 15/16 year olds from which his sample was taken, he noted that

13% of the girls wrote xy for x+y. To combat this the programme developed in this research made use of programming in BASIC where the constraints are such that one is required to make all operators explicit, thus encouraging an understanding of their importance. It was recognised though that, once again, the problem would be whether there would be a transfer of such understanding to the algebra domain. Chapter 8 discusses the results which bear on the success of this transfer. It has been noted (e.g. Kieran, 1987, p.16) that a lack of structural knowledge of expressions manifested by an inability to parse them correctly is a general cause of error amongst early learners of algebra when dealing with expressions. They have demonstrated a predisposition to working from left to right, regardless of the structure, and so, for example, may simplify 39x - 4 as 35x. Sleeman (1986), in a detailed study of the types of errors made by algebra students, recor ded that over 50% of her sample of 14 year-olds simplified 2 + 3x as 5x and she maintains that this is a very common type of error amongst such students. This being the case, special attention was paid to this error during the present study, and a discussion of the possible effects of the programme on errors of this type is contained in Chapters 9 and 10.

The failure of the beginning algebra student to see structure has caused Larkin (1987) to attempt to remedy this situation by constructing a simple model of performance in algebra based on making such implied operations explicit. Following a study of the nature of children's errors relative to algebraic expressions Resnick (1987) concluded that there was a need to anchor algebra in referential meaning, stating :

> "If, however, they understand algebra expessions as having referential as well as formal meaning, they will be in a position to use what they already know about the semantics of situations and of fundamental mathematical concepts to constrain their formal constructions."
>
> (Resnick, 1987, p.31)

The study of Putnam et al (1987) confirms the value of such a referential linked introduction of expressions and shows that it helped to facilitate the learning of the formal rules which govern the equivalence of expressions. A note of caution in the use of concrete referrents for algebraic expressions is sounded by Booth (1987), whose study of 12 and 13 year-olds led her to conclude that the use of concrete models can lead to the formation of inappropriate interpretations by children who may be limited by possessing only an informal knowledge of even arithmetic relationships and so this is a situation which one would need to be aware of.

4.4.1 Studies Aimed at Improving Understanding of Expressions

Although the problems involved in developing a relational understanding of expressions are fairly well documented, there have been far fewer attempts to improve the situation. Among those studies which have tried to do so have been those of Herscovics and Chalouh (1984) and Chalouh and Herscovics (1984, 1988). In the first of these studies Herscovics and Chalouh devised a teaching experiment which attempted to construct a meaning for algebraic expressions by representing them as answers to problems. Using arrays of crosses, they introduced the concept of a hidden quantity as a pre-cursor or Ausubelian

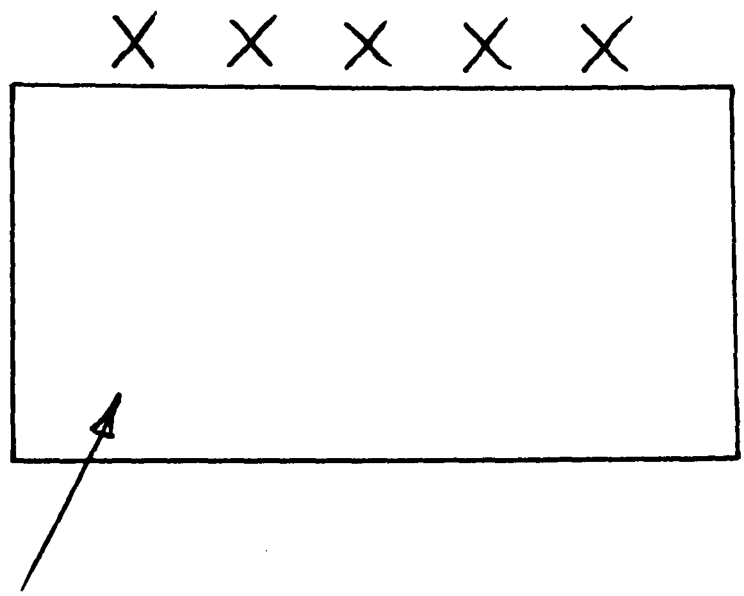


Figure 4.3

The Picture Used By Herscovics and Chalouh To Help Pupils Understand the Concept of Specific Unknown subsumer for the concept of specific unknown. They used a situation such as that in Figure 4.3 where part of an array was covered up and only the top row was visible, the rest being hidden. Pupils were encouraged to represent the 'hidden quantity', the number of rows of crosses by □ and to write 5 x □ (here) for the number of x's in the array. The idea was that, although the quantity was hidden it would be revealed in the pupils' experience, and that this would help them towards an understanding of specific unknown, where the unknown would not be revealed. In a follow up study, Chalouh and Herscovics (1984) attempted to use the same type of visual construct with the difference that now the number of rows rather than being revealed was to be viewed as unknown and instead of writing 5 x□ the students were encouraged to write 5 x a. Based on the results of these studies they concluded that although the students were quite happy with the concept of a number of hidden rows which would soon be revealed they experienced cognitive difficulties with the transition to the view that the unknown would mean no such revealing. Hence they were only really happy with a numerical answer and viewed answers such as 5 x a as somehow incomplete, not really being able to accept their lack of closure. In a further study, Chalouh and Herscovics (1988) extended their previous work to include arrays of dots, line segments and areas of rectangles to try and develop meaning for expressions such as 2a + 5a. Once again, they noted that although the geometric approach seemed successful at this level, the children were generally unable to make the cognitive transition to viewing the expression as 7a. That is the meaning which they had developed did not allow them to progress to a meaning for simplification.

Another study which has attempted to provide meaning for algebraic expressions (Thompson (1987) and Thompson and Thompson (1987)) has done so by representing them as expression trees drawn on a computer screen. The aim of the trees is to enable the structure of expressions to be represented, explicitly, in an unambiguous manner, such that students might be able to work with them in a way which made them have to reason on the expression by recognition of its structure (Thompson, 1987, p.10). In the study, a group of eight 13 year-old algebraic novices were involved in manipulating expressions, through their trees, on the computer screen, and were asked to work through a series of graduated exercises. On their reaction to the use of letters Thompson says:

> "To my amazement, students were not bothered by the introduction of letters in expressions. They quickly saw letters as placeholders for substructures in a tree...they were developing a generalized concept of variable - letters could stand for numbers, but in general they were placeholders within a structure and anything could be put in their place, including other expressions."
>
> (Thompson, 1987, pp.33,34)

This generalisation from the programme seems to be a very valuable one, and one which is not readily available to children from a traditional approach to algebra in the sxecondary school. The results of the study were such that they suggested that "mal-rules need not be a natural occurence when students operate in an environment that supports explicit attention to expression's structures, and where structure also imposes constraints on students' actions." (Thompson and Thompson, 1987, pp. 252,253). These small scale results seem very encouraging, and may offer the possibility of a way forward which will avoid many of the problems of inappropriate or mal-rules and over-generalisation commonly employed by children, although it must be said that the study did not attempt to find out the extent to which the understanding demonstrated would transfer to more traditional 'paper and pencil' exercises and this is an area which has proved problematic in other studies.

4.5 <u>A Consideration of Linear Equations in One Variable</u>

The solving of algebraic equations is one of the fundamental processes of algebra, indeed of mathematics, and it is one which the early learner of algebra soon encounters. The key difference between expressions and equations, at least in the mind of the student, is the introduction of the '=' sign. This visual key, unfortunately, corresponds to a mental image in many children's first level knowledge structures (see Chapter 3) which in turn evokes the wrong kind of concepts in their second level structures. Kieran (1980) has described how the '=' sign is viewed by many 11 and 12 year-old children as a 'do something signal'., telling them to perform some operation, to come up with an answer, and that this is related both to their arithmetic experience and the language they use (such as 'makes') when they are reading the sign. They do not see it as a relational symbol, for example as in its use in equivalence in mathematics. A year or so later, students such as these have been noted as not using the '=' sign in a consistent way and still tending to view it as a signal to operate. The theoretical notions of the power and value of mental imagery put forward in Chapter 3 would seem to indicate that since this image is so strongly linked in children's schemas to these views that a different imagery should be used for equivalence. The fact that such an image already exists, but is little used, makes it all the more surprising that this has not been the case. The use of '-' instead of '=' could be a way of avoiding the evocation of inappropriate concepts by the mental image and it seems that further research here would be useful. Most algebra novices are taught standard algorithms for the solution of algebraic equations, and appear to be able to carry these out with reasonable success on equations of the type

$$ax + b = c,$$

where the coefficients are small (Wagner, 1984). As we shall see however, it is not always the case that they are actually employing the standard algorithms, and there understanding of what they are doing is often very poor. There does appear though to be a clear distinction between the equations mentioned above and those of the type :

$$ax + b = cx + d,$$

as far as children's ability to comprehend and solve them are concerned. It has been suggested by Vergnaud and Cortes (1986), and others, that the progression from the first of these to the second is what contains the essential difficulty of basic algebraic equations. They argue that equations of the form

$$ax + b = c$$

require one to operate only on numbers whereas the other type of equation involves one in operating on unknowns such as x. Thus, in the terminology of Küchemann (1978), employed throughout this study, the former involves only an understanding of the use of letters as objects but the latter requires at least an understanding of specific unknown. Sleeman (1986, p.44) also records the difficulties of the 14 year-olds in here study with equations involving multiple x´s, since they "did not know how to compute the sum of MX and NX.". Although her phrasing of the problem puts the emphasis, wrongly I believe, on skills rather than understanding, the recognition of the serious problem that exists with these equations, labelled as non-trivial by Herscovics and Kieran (1980), is the same. The division noted here is, I maintain, the result of a problem of understanding of semantics rather than merely a process oriented problem, and the reasons for this are discussed below.

4.6 The Relationship Between Understanding and Method

It has been shown that children use a variety of techniques when solving even the simplest of equations in algebra, and these include :

a) Informal or intuitive methods

b) Arithmetic substitution based on balancing equivalent expressions

c) Transposing of terms

d) Performing the same operation on both sides

It is necessary then to look at each of these methods in turn in order to establish what they tell us about the child´s understanding of the processes of algebra as distinct from their manipulative abilities.

4.6.1 Informal or Intuitive Procedures

Examples of intuitive procedures have been recorded by some researchers, such as Bell, O´Brien and Shiu (1980) and include methods such as solving : 3x + 2 = 8, by spotting that 8 is the same as 2 + 6 and hence deducing that 3x must equal 6 and so x is 2. Petitto (1979) has described such processes as dealing only with those properties and relationships which happen to be perceived, rather than using a well-formed linear sequence of steps. Kieran describes what she calls ´equality structure´ as one type of reasoning often employed e.g. the spotting that 8 = 2 + 6 in the above example. The problem with such informal, intuitive methods, again pointed out by Petitto (1979), is that they do not generalise, and therefore a child employing them has to cope anew each time with any question which deviates even slightly from an established pattern. Unfortunately, for them, many children do not understand this and, as Firth (1975, p.68) noted in his study of rule dependence in elementary algebra "Another aspect of the behaviour observed was the tendency to make judgments on the basis of insufficient information: to infer a general rule from one or two instances." Sleeman (1986, p.52) agrees with this finding stating "Pupils have a great facility for inferring their own rules, or sometimes higher level schema, and then using them consistently and often in inappropriate situations". Booth (1984) also investigated children´s ability to formalize methods in algebra and concluded :

> "b) The procedures which children use in solving arithmetic problems are often informal methods which are difficult to symbolize concisely.
>
> c) The procedures used are often context-dependent so that they do not readily

> generalize to other examples (such as algebraic cases), and are symbolized (if at all) in an informal manner which requires reference to the particular context for interpretation."
>
> (Booth, 1984, pp.85,86)

Hence she had found that the method employed may be context specific and Bell and O´Brien (1981) also reported that children´s methods changed dependent on the particular question they were attempting.

4.6.2 Arithmetic Substitution

Although the method of arithmetic substitution into an algebraic equation may be viewed as an informal method, the fact that it is so common among early learners of algebra and that it may be applied in a systematic way, seem to justify its separate inclusion in a list of techniques employed. One of the first to identify it as a separate method, and to investigate it was Collis (1975b) and Kieran (1985) reported that about 50% of her novice equation solvers used arithmetical methods such as substitution and known number facts, with the rest transposing terms. An example of this method in solving 3x + 2 = 17 might be to say : When x = 2, 3*2 = 6 plus 2 makes 8, therefore 2 is too small. Try x = 4, 3*4 = 12 plus 2 makes 14, therfore 4 is too small. Try x = 6, 3*6 = 18 which is too big therefore try x = 5, 3*5 = 15 plus 2 makes 17, therefore x = 5. Thus the aim here is substitute values for x and to balance the value of the expression on the left hand side of the equation with the value on the right hand side. Much of this argument may be carried on mentally with possibly only the final answer written down, and it is often only successful on equations with small coefficients.

4.6.3 Transposing of Terms

The method of solving simple linear algebraic equations by transposing terms in the equation has also been described as inversing (Kieran 1984) and Change side - Change sign (e.g. O´Brien, 1980), and although to many more expert mathematicians it may look like a short cut in the method of performing the same operation on both sides of the equation, Kieran (1988) has provided evidence that they are perceived differently by early students of algebra. In her experiment, students who had initially preferred inversing as a method were taught to perform the same operation on both sides of the equation, but were generally unable to make sense of it. On this Kieran says:

> "The procedure of performing the same operation on both sides emphasizes the left-right equality structure of an equation; whereas, this emphasis is quite absent in the use of the procedure of inversing."
>
> (Kieran, 1987, p.23)

It would seem that a necessary prerequisite for students to be happy with transposing terms is a development of full reversibility of thought in the arithmetic domain, or in the language of cognitive integration (Chapter 3), to have developed bi-directional links there, since, as Wagner (1984) points out, it requires an understanding of the reversible nature of arithmetic operations such as in equating the solution of the equation $y + 4 = 7$ with that of $y = 7 - 4$. The lack of this reversibility in students using the method of transposing may well lead to errors of the type reported by Kieran (1984) and which she calls the Redistribution Error and the Switching Addends Error. In the former, equations such as y + 4 = 7 are considered to have the same solution as y = 4 + 7 and in the latter, to have the same solution as y + 4 - 2 = 7 + 2, i.e. the balancing of a subtraction on one side of the equation with an addition on the other. In his investigation of the rules employed by secondary school students in algebra Firth (1975, p.164) identified the low level of understanding of algebraic equations he found as being "closely related to rules such as ´shift terms - change signs´ when manipulating algebraic equations". That is, for many pupils, especially those who have not mastered the reversibility of arithmetic operations, this method of solution opens up the distinct possibility of manipulation without understanding, and its attendant errors. Evidence of this in the present study is described in Chapters 9 and 10, where control pupils presented such features.

4.6.4 Performing the Same Operation on Both Sides

The technique of performing the same operation on both sides of an algebraic equation was also identified by Collis (1975b) and has been described as deduction (Matz, 1979) and manipulation (Swain, 1962). This is the method favoured by many more expert solvers and Neves (1979), after analysing the methods employed by such solvers, constructed a model of the solution process based on performing the same operation on both sides. Kieran (1988) when studying early learners of algebra, found that the ability to comprehend and use this method depended on the student´s view of the equation. Those who were the most receptive to the procedure were those who focussed "on the given surface operations and the equality structure of an equation" (Kieran, 1987, p.24), and were those whose initial view of the use of letters in an equation was that they represented a specific unknown. In contrast to this group, those whose initial view of the letters used was that they had no meaning until their value was found were not so receptive to the method. This distinction may be a key point, since the evidence of Küchemann (1980) is that about 83% of 13 year-olds, 65% of 14 year-olds and 58% of 15 year-olds do not have the level of understanding which enables them to perceive letters in algebra as standing for specific unknowns. Thus, before such children will be receptive to the teaching of a procedure such as performing the same operation on both sides, one of the prerequisites is an improvement in their understanding of the use of symbolic literals so that they can accept them as, at least, specific unknowns. The use of generic organising material, such as that provided in the programme used in this research, has this as one of its aims, and in Chapter 8 the results of the study relative to this are discussed.

4.7 A Comparison of the Value of These Procedures

Several studies have attempted a direct comparison of the efficacy of the methods of solution of equations outlined above. Although there are no clear interpretations of the results of such studies, there are some patterns which do emerge. Whitman (1976), for example, taught three groups of 13 year-old students, with one group being taught only informal techniques, another only formal techniques and the third informal followed by formal. Her results showed that those taught only informal methods had performed the best, and those taught only formal methods the worst, and this led her to conclude that the formal techniques had tended to interact negatively with children's intuitive equation solving capabilities. In this particular study though this may have been due to the close proximity of the teaching of the two methods, since Petitto (1979) observed that in her study, those students who used a combination of both formal and informal methods were more successful than those who used just one in isolation. Her conclusion was :

> "either intuition or formality alone is generally inferior to a combination of the two for solving nonstandard problems containing elements in nontrivial relationships."
>
> (Petitto, 1979, p.81)

She further commented though that the two techniques should somehow be integrated and not just learned in isolation. In a similar vein, Wagner (1984) suggested that, on the basis of the results of her study of children's equation solving methods, that the introduction of informal techniques such as trial-and-error, guess-test-revise, pattern spotting etc. prior to the teaching of standard algorithms would help students to appreciate the standard techniques more and improve their ability to solve equations. Booth (1984), after studying children's ability to formalize algebraic methods, agreed with this and said :

> "Only when children become aware of the limitations of their own methods, it is suggested, will they be prepared to contemplate the value of the more formal methods which the teacher is attempting to teach...Hence as long as

childrendo obtain success by use of their own informal methods, they will be unlikely to attend to the (perhaps more obscure) procedures being developed by the teacher, with inevitable consequences when the more complex equations are eventually met. To remedy this situation, the child's method and its usefulness in solving easy equations must be recognised and the possibility of applying these procedures to complex equations must be assessed"

(Booth, 1984, p.93)

Firth (1975) also recommends this on the basis of his research, saying:

"the course of action recommended is that the material be introduced on a less formal basis and that pupils not be required to learn methods of solution for standard problem types. Instead, unstructured exploration might be encouraged through which experience of the material will be gained thus enhancing the prospects of a deep understanding when the same material is re-taught at a later stage"

(Firth, 1975, p.170)

Bell et al (1980) also make the same conclusion, that equations may be made more meaningful for the child by teaching them what they call 'meanings method' based on intuitive techniques. The current programme has included some worksheets on equation solving using the 'maths. machine' (see Chapter 7) where the children are encouraged to engage in this type of preparatory intuitive method of solving equations, with the specific aim of providing a basis for a deep understanding. Adi (1978) studied the effect of combining informal techniques with the formal process of performing the same operation on both sides. The results indicated that those students in the late concrete and early formal Piagetian stages were more likely to benefit from such a combination than those who had not yet advanced to these stages. This indicates the possibility of a certain cognitive readiness factor, identified by Booth (1984) also, for the formal methods of equation solving although, as I have described in Chapter 3, it may be that this is in fact merely due to the lack of suitable mental imagery.

4.8 Providing Cognitive Meaning for Formal Methods

The danger inherent in trying to describe the methods and errors of early learners of algebra is that one might tend to concentrate on the syntactics of algebra and a corresponding surface understanding which permits successful manipulation, rather than attempting to make algebra more meaningful. Avoiding this wrong emphasis means promoting deep understanding through through the provision of cognitive meaning for the symbols used and the methods employed.

O'Brien (1980) formulated a means of trying to do this by the use of counters and cubes as concrete materials with which equations could be modelled. Two groups of children were compared, one which had been taught using the materials and another which had been shown the formal methods of transposing terms and performing the same operation on both sides of the equation. In this study the latter group performed better on solving equations although since all the students had learned some algebra before the research the results are difficult to interpret. Herscovics and Kieran (1980) took 6 12/13 year-old algebraic novices and tried to provide meaning for performing the same operation on both sides of the equation when it had the form ax + b = cx + d. They did this by approaching the equations through the concept of a hidden number and anchoring the concept of an equation in arithmetic before the introduction of algebraic equations, which were avoided until a meaning for them could be established. They concluded that this approach was successful in helping their students retain a clear understanding of the concept of equation and the justification of the algebraic rules for them. Filloy and Rojano (1985) used a geometric approach based on representing algebraic terms by areas of rectangles to provide a meaning for the equations, however they found that the model tended to obscure the application of the students' previous knowledge to the equations. This is an example of a study which illustrates the power of visual imagery, as discussed in Chapter 3, and the need to exercise care in its integration into mathematics by doing so at all appropriate stages of a topic without placing undue emphasis on it. The aim is to integrate its use fully with serialist/analytic methods rather than to end up with the dominance of one over the other. The limited success obtained in providing meaning for non-trivial algebraic linear equations shows the difficulty of making such equations meaningful for students and the value of any approach, such as that described in this thesis, which can assist in the process.

4.9 The Conceptual Route to Meaningful Algebraic Equations

It may be the case that the way to make equations meaningful is to synthesise their meaning from their constituent concepts. Researchers have described some of the concepts associated with elementary algebraic equations which are necessary in such a synthesis of a meaningful understanding. Apart from the extremely important concepts of variable and expression, as summarised above, other important concepts associated with equations are described below.

Wagner (1977) has described the concept of conservation of equation as the fact that the solution of an equation is invariant under a change of symbolic literal. She found evidence that many of the early learners of algebra that she studied did not have this conceptual understanding, or did not conserve equation, as she described it, and this was confirmed by a later study. Ekenstaam and Nilsson (1979) also record evidence of students resolving equations which were eseentially the same except for a change of variable, and it seems that this understanding is often lacking in the early years of algebra. Although it may not seem particularly valuable to appreciate that the equations :

$$2p - 1 = 5 \quad \text{and} \quad 2s - 1 = 5$$

have the same solutions, without the need to solve the equations, spotting the connection between :

$$2p - 1 = 5 \quad \text{and} \quad 2(p + 1) - 1 = 5,$$

for example, is much more useful, and an important concept. In order to see the effect of this current research module on such conceptual understanding, this question was included in the interviews conducted after the programme of work so that it might be made clear, in the way that only such an interview can, how the students would solve such equations. The results of these interviews are described in Chapter 9.

A second important concept in the solving of algebraic equations which has been shown by Kieran (1984) to be lacking in both the 12/13 year-olds and the 14-17 year-olds in her comparison study is an awareness of the fact that the solution of an equation remains the same at every step of the solution process. None of the students in her study showed any awareness of this concept. This result was again confirmed by Ekenstaam and Nilsson (1979) who gave a series of equations to 2167 16 year-old students to solve. An example of the equations, given here, shows that they were essentially the top down steps, in the order they were given, in the solution of a single algebraic equation:

$$\frac{3x - 2}{2} = \frac{x}{3} \; ; \qquad 3(3x - 2) = 2x \; ;$$

$$9x - 6 = 2x \; ; \qquad 7x - 6 = 0 \; ; \qquad 7x = 6.$$

In spite of this they noted that "It is important to emphasize that the students who participated were not aware of the sequences and found no connection between the problems." (ibid, p.46). This somewhat surprising result is an example of what Davis (1975, p.32) refers to as "making a line-by-line solution of an equation" rather than having "the big picture", or in the language developed in Chapter 3, the student is locked in to a serialist/analytic mode of working which, when faced at each line of the solution with an equation to solve, or when faced in each question of the Ekenstaam and Nilsson study, he/she has analytic procedures for solving such equations evoked without ever getting the global/holistic view of what they are attempting to do, or what Davis calls ´the big picture´. Thus their mental imagery is such that seeing an equation, the over-riding impulse is to solve it rather than to perceive it as possibly a step in another process or in terms of its structure. This amy be a direct result of the procedures being evoked by the mental image ´=´, as discussed above.

4.10 Problems Involved in Translating Word Problems to Algebra

In the algebra reported in this this thesis, involving as it does the early learner of algebra, the main emphasis is on the importance of symbolic literals and their use in expressions and equations which do not have to be derived from word problems but are given. However, it is instructive to consider briefly some of the research which bears on the difficulties associated with such translation and its understanding.

There seems little doubt that even the more mathematically able students experience some difficulties in translating word problems into algebraic notation (e.g. Lochhead 1980, Wollman 1983), or as Clement et al describe the situation :

> "Even after taking a semester or more of calculus, many students have difficulty expressing relationships algebraically. They cannot translate reliably between algebra and other symbol systems such as English"
>
> (Clement, Lochhead and Monk, 1981, p.289)

One of the errors most discussed in the literature on word problems is that described as the variable reversal error. This error was reported following an investigation of 150 calculus-level students´ attempts to answer the students-and-professors problem :

> "Write an equation for the following statement: ´There are six times as many students as professors at this university.´ Use S for the number of students and P for the number of professors."
>
> (Clement, Lochhead and Monk, 1981, p.288)

The results of the investigation showed that 37% of the students answered incorrectly and that two-thirds of these gave the answer as 6S = P, rather than S = 6P. Lochhead (1980) gave a corresponding problem to this one to members of a university´s faculties and found that "Apart from those in the physical sciences and mathematics, the overall success rate was about 50%", with many making the reversal error. He concluded that this error is deeply ingrained and resistant to years of mathematical instruction. The possible reasons behind such an error have been categorised as "word order matching", where six times as many students as professors is translated directly to 6xS = P and "static comparison", where 6S is viewed as a formulation of the representation of a groupsix times the size of a group represented by P. Rosnick (1981) has described the underlying problem as he sees it in terms of the view of the letters :

> "They should learn to distinguish when letters are used as labels referring to concrete entities or, alternately, as variables standing abstractly for some number or number of things."
>
> (Rosnick, 1981, p.420)

He further says that such an ambiguity in the mind of the student may be enforced by the educator who writes ´P = professors´ when he/she means ´P = number of professors´. It may be then that the fundamental reason for this error is related to the students´ understanding of the use of letters as variables. Rosnick and Clement also noted :

> "Many of the college students that we have interviewed and tested have demonstrated that they do not recognise the use of letters as standing for numbers. They confuse the use of letter as a variable with the use of letter as a label or unit. These students also tend to write the reversed equation 6S = P as the answer to the students and professors problem."
>
> (Rosnick and Clement, 1980, pp.5,6)

and they concluded that one answer to the problem was to pay more attention to the fundamental concept of variable. Hence this concept, as addressed in this present research study, may be at the root of some translation problems too, and a programme which improves its understanding may have beneficial repercussions in this area also. Karplus et al (1981) have also recognised that interpretating such word problems involves an ability to view the problem in terms of operations on unknowns, but they concluded that students were able to reason with such unknowns and that the main problem was one of translation rather than conceptualisation. We have seen above, however, that the ability to operate successfully with letters viewed as specific unknowns is on which many early learners of algebra do not possess and that this lack has a tendency to persist for some years, so the difficulty may still lie with the formalisation and symbolisation of the concept. In their study designed to probe the possible reasons for errors such as the reversal error, Kaput and Sims-Knight (1983) did conclude that their results showed that :

> "This ability does not appear to involve conceptions of algebraic variables"
>
> (Kaput and Sims-Knight, 1983, p.75)

and they identified a different problem area :

> "the abilities involved...are intimately connected to our natural linguistic and imagistic representational systems whose developmental roots date from early childhood and are thus extremely potent in determining our ability to use the more formal systems of mathematics."
>
> (ibid, p.76)

Relating this again to the concepts introduced in Chapter 3, we may say that it may be then that the mental imagery schemas of many students are not sufficiently well integrated with their linguistic and mathematical schemas to enable them to cope with the demands of problems involving all three cognitive areas. This seems to indicate another reason for the positive promotion of cognitive integration as described in Chapter 3. An initial investigation of the relationship between language and the mental imagery and subsequent mathematical concepts which it evokes has been undertaken in the present study through the use of a questionnaire, and the results and conclusions relevant to the early learning of algebra are presented in Chapter 10.

4.11 Overview of the Relevance to the Research

In this chapter I have attempted to present the relevant research on the understanding of algebra by early learners and the reasons for its relevance. I have perceived the concept of the usage of letters or symbolic literals in algebra in all its guises up to their use as variables to be the key concept with regard to the novice´s understanding of algebra. The research suggests that a programme of work which could successfully make the use of letters meaningful for students would open up deep meaning for them in their use of algebraic expressions and solving of algebraic eq uations. Thus it was the aim of this research to concentrate on this key concept and present it in an environment where it could be freely manipulated by the student using procedures which were informal or intuitive in nature and also linked to their understanding of arithmetic as appropriate. This work was to be a generic organiser (Tall 1986c) for the future introduction of formal techniques for the solving of algebraic equations, the research indicating that this was the best approach, since their knowledge structures would have developed to the point where they had a cognitive readiness for them. Chapters 6 through 10 of this thesis relate how these objectives were attained and the consequent results and conclusions.

In this chapter I have not addressed the important question of the future of secondary school algebra and the effect that computers and calculators capable of handling algebraic manipulation may have. I shall consider this at the end of the next chapter, after I have looked in general at the role of computers in the teaching of algebra.

Chapter 5

A Discussion of The Role of The Computer In Mathematics Education

This chapter reviews the advantages of the computer as a tool in the learning of mathematics and assesses the relative merits of the ways in which educators have sought to apply these benefits pedagogically. As well as considering past and present practice I shall look at the potential for the future, with particular regard for the use of the computer as an aid in the conceptual teaching and learning of algebra in the secondary school. This projection into the future involves a consideration of the role of algebra in the secondary school mathematics curriculum in the light of advances in technology such as symbolic manipulator computer programs and algebraic calculators.

5.1 Some Historical Perspectives

The computer, since its conception, has been a natural and valuable tool in mathematical applications. It was somewhat later in its development when the availability of hardware became such that its use in mathematics education became a practical possibility. The early use of the computer in education generally recognised two main advantages associated with it, namely its motivating power and its ability to handle repetitious operations quickly, logically and interactively. Thus the birth of computer aided instruction (CAI) involved programs designed as 'drill and practice' sessions for students, leading them through an interactive instruction programme with a record of individual progress and achievement. The resulting student improvement in performance registered on traditional tests, with some pupils reported as making gains the equivalent of up to 3 or 4 grades in one year, (e.g. Davis 1978, Alderman, Swinton and Braswell 1979) encouraged the growth of this kind of computer use and large scale programmes such as the Time-Shared Interactive Computer Controlled Information Television (TICCIT) and Programmed Logic for Automatic Teaching Operation (PLATO) projects in the U.S.A. were instigated. PLATO alone involved about 140 sites, 8000 hours of instructional material and over 3000 authors (O´Shea and Self 1983, p.93) and this led to an emphasis amongst some educators on how to improve the efficiency of computers as question-response-correction machines (e.g. McGettrick 1979).

Such an approach, however, with its emphasis on skill acquisition rather than conceptual understanding has gradually come under inspection, since as Alderman, Swinton and Braswell explain :

> "...exposure to the computer mathematics curriculum seems to improve student´s proficiency in taking tests in as much as they omit fewer items. This result conforms with the emphasis in the computer curriculum oh drill and practice. The treatment, by design, neither teaches new concepts in mathematics nor diagnoses children´s misconceptions about mathematical processes. Simple drill on problems may not remedy students´ weaknesses in understanding mathematics, but does appear to make them more adept and efficient in answering questions."
>
> (Alderman, Swinton & Braswell, 1979, p.21)

O´Shea and Self (1983, p.121) agree, stating that whilst the use of such programs has some merit, their contribution to education is marginal and Hartley (1980), whilst again recognising the value of Computer-Aided Learning (CAL) suggests that they may not be of the right type educationally :

> "...there is a need for research studies into the process of cognition, learning and problem solving within mathematics in terms which relate to the computer medium."
>
> (Hartley, 1980, p.35)

Others, such as Lesgold (1982) have been more directly critical of the 'drill and practice' approach. From among these, has emerged the vision of Papert who encouraged educators to question the way in which computers were being used :

> "In many schools today, the phrase 'computer-aided-instruction' means making the computer teach the child. One might say, the computer is being used to program the child. In my vision, the child programs the computer and, in doing so acquires a sense of mastery...and establishes an intimate contact with some of the deepest ideas from science, from mathematics..." (Papert, 1980, p.5)

The important question was seen to be then, not 'How can we use the computer in mathematics education?', but 'What are the implications from educational theory for the use of the computer in mathematics education and how should this affect learning practice in the classroom?' The latter approach, with its theoretical base, means that, rather than leaving the way in which computers are integrated into the curriculum to chance and the individual teacher, direction and help need to be given by those with the appropriate educational knowledge and experience. Commenting on whether the computer should be viewed as a tool, albeit a useful one, whose use should be left to the discretion of the individual teacher, or whether it should be seen as a vital, integral part of every mathematics classroom, Pimm (1983) states:

> "The microcomputer changes this because it can combine the calculating power on the one hand and the continuity of image and transformation on the other in an interactive way which has not been seen before. It can thus, I claim, enable mathematics to be illustrated in a qualitatively new way. This is one sense in which I feel the computer is necessary for mathematics and not just nice."
>
> (Pimm, 1983, p.43)

Howson (1985, p.297), too, discussing the impact of computers on the teaching of mathematics says "the computer...will affect not only the mathematics we study, but also the way in which we do, teach and learn mathematics". The transition to the computer paradigm is not proving an easy one, and was listed by Freudenthal (1981) as one of the major problems of mathematics education in the eighties. Although, as I have discussed above, the use of computers in mathematics education made an inauspicious start, in recent years the picture has begun to change. In the section below I shall outline the progress which has been made in the process of effectively integrating the computer into the teaching and learning of mathematics.

5.2 A Cognitive Integration of the Computer into Mathematics Education

5.2.1 A Theoretical Overview

In recent years some groups of mathematics educators have started to formulate an educational answer to the way in which computers may be successfully integrated into the mathematics curriculum in schools. Although there may be some schools of thought which would still disagree, there seems to be a widespread agreement that the purpose of mathematics education is not just to improve acquisition of skills, but also, even especially, to help students become better at mathematics by improving their understanding of the concepts involved in its application. (see e.g. Davis (1986)). One of the aspects of educational theory which has been considered in the new paradigm is the emphasis which constructivism places on the construction of relationships between concepts and the formation of such into knowledge structures in the student´s mind. In Chapters 2 and 3 of this thesis I have discussed the importance of these structures and their synthesis in the mind. Some educators, such as Papert, have had this in mind when trying to build a cognitive theory embracing computer use. He says "I take from Jean Piaget a model of children as builders of their own intellectual structures." (Papert 1980, p.7). Jahnke (1983) commenting on the work of Papert explains that he sees Piaget´s work as an attempt to construct a knowledge-based theory of learning. What educators such as Papert have attempted to do is to marry the cognitive benefits of the computer environment with the available, accepted cognitive theories of student development, with their emphasis on understanding and the structures, facilitating this, built up in an individual´s mind.

The benefits accruing from the use of the computer have been seen to include the following :

The power to motivate students

The ability to interact logically with the student

The capability for mathematical modelling

The improvement of problem solving ability

The production of student generated feedback

The encouragement of algorithmic thinking

The improvement of reflective thinking ability

The capacity to concretise and personalise formal thinking

The power to improve the formation of mental images through concept visualisation

(See e.g. Papert 1980, Cockcroft 1982. Tall 1986c, Cooper 1986, O´Shea & Self 1983)

and that, further, it is particularly those items in the latter parts of this list wherein the power of the computer in education lies.

In the next section I shall discuss how these benefits have been, and may be, utilised in the classroom.

5.2.2 <u>The Place of Computer Programming in Mathematics Education</u>

One of the fundamental principles of Papert (1980) is that the child should feel that he/she is programming the computer, that he/she is in control, manipulating a concretisation of concepts, if relational understanding (Skemp 1976), the goal of meaningful learning (Ausubel 1968) is to result. Actually programming the computer is one way in which the child can be made to feel this way and hence to get the cognitive benefits. Engel (1976, p.266) believes that, from a problem solving point of view, "the best way to learn something is to teach it. The computer plays the role of the ´model student´." and Meissner (1976) argues that, since most thinking appears to be algorithmic in nature, the programming of a computer may serve to make our own programmed thinking more available to our consciousness and thus facilitate improvement. Mathematics educators such as Sweeten (1982), Wardle (1983) and Fletcher (1983) have added their voice to those who believe that there is value in some elementary programming as part of the curriculum of secondary school pupils. Attempting to classify the benefits of such programming, Ross and Howe (1981) have summarised what they see as claims for the cognitive advantages of computer programming as follows:

> "(1) that programming provides some justification for, and illustration of, formal mathematical rigour; (2) that programming encourages children to study mathematics through exploratory activity; (3) that programming gives key insight into certain mathematical concepts; and (4) that programming provides a context for problem solving, and a language with which the pupil may describe his own problem solving."
>
> (Ross & Howe 1981, p.143)

Lawler (1983, p.74) also gives some advantages of programming, saying that such concrete experiences help children to conceive of objects in a more formal way; may lead children to think more systematically and may enhance their reflexivity of thought

- all amounting to a significant increase in the analytical ability of such children. These proposed benefits are echoed by Ross (1985, p.5). Fletcher (1983) states that the ability to program a computer to carry out a particular task is a good test of whether an individual really understands what is involved, meaning that the computer can help to build such understanding. Other studies which have provided evidence for the beneficial effects of students programming of computers include those of Clement, Lochhead and Soloway (1980), where students were helped to solve equations by the emphasis on semantics provided by the programming, and Du Boulay (1980), where student teachers with difficulties in mathematics were taught programming in Logo, the language developed by Feurzig, Papert and Bobrow. Du Boulay concluded that it did improve the students' understanding of particular mathematical ideas but that care needed to be taken over a number of factors in the design of the teaching programmes, including :

> "(a) Programs to be written by the student should be short and should deal with the properties of the mathematical objects or processes under consideration, and not with the accidental features of their representation within the chosen programming language.
>
> ...(c) Instructions that generate visual illustrations of mathematical ideas are very useful."
>
> (Du Boulay, 1980, p.359)

Summarising their findings on the value of programming, the

Pendley Manor report (1985) concluded that :

> "Research has shown that: -embedding (selected) mathematical ideas and concepts in an algorithm (or computer program) results in better understanding. -investigating mathematical ideas and solving problems through computer programming (algorithm design) contributes significantly to the development of problem solving ability (with transfer). -exploring mathematical ideas through an algorithmic approach : a) enables children to develop a dynamic view of important concepts and relationships..."
>
> (Pendley Manor Report, 1985, p.26)

Thus we see that the argument is not that programming a computer is good per se, but that when such programming is based on the conceptual ideas of mathematics and is used to encourage and promote the type of useful visual imagery described in Chapter 3, then it can be a powerful force in aiding mathematical understanding. One of the factors about which there has been some discussion is the relative value of the language which is used by the students to program the computer. It is important to realise that, one particular language may have advantages over another in a given context, and obtaining the full value of programming as an aid to the conceptual understanding of mathematics may involve a consideration of the language to be used. This is important because, as Pea and Kurland (1984, p.145) write, it has been fashionable on the part of some to look down on some languages, for example BASIC, but they and

Lesgold (1982) speak out in favour of BASIC since it has many of the advantages mentioned above and is widely available in schools. Tall (1985b) advises taking a wide view of which language one should use, since no one language is likely to offer all that is needed in education. Hence we see that it is the context, in terms of the hardware and software available and the educational objective in view which may determine the appropriate language to use in a given situation. Although, as I have indicated above, different languages may have different advantages for the mathematics educator it is true that much of the evidence of its value has come from the popularity of the Logo language. Hoyles and Sutherland (1985, 1986) contend that their studies in a Logo environment have shown that Logo programming activity renders pupils' intuitive mathematical strategies more accessible to themselves, thus aiding the reflective thought necessary for formal operational thinking. They have also shown that there is benefit to be had from getting the children to work together in small groups when programming, since the peer group interaction aids cognitive development. Matos (1986) and Nelson (1987) have also recorded conceptual benefits obtained from programming in Logo.

It is true that there have been some who are yet to be fully convinced of the value of programming as a cognitive exercise. Ridgway (1985), for example, has expressed such doubts, although rather than totally decrying its educational value, what he has questioned is the way in which it has been implemented by some. This is a valid point and the important comments made by Du Boulay (1980), and quoted in part above, namely that attention must be paid to the design of teaching modules so that they are based on sound conceptual principles rather than ad hoc procedures are worthy of note. In an extensive discussion of the educational advantages of programming, Pea and Kurland (1984) point out that, although it appears that there may well be cognitive benefits from learning to program a computer, many of the research studies have failed to address the issues adequately. One of the difficulties involved is assessing the extent to which abilities and understanding gained in one context or domain will transfer to another, different domain. Thus understanding from a computer environment may not be applied in the mathematical domain, and they say that testing this transfer is a weakness of many studies. They concluded however,

> "Embedding computer programming activities of increasing cognitive complexity in children's goal structures may promote learning to program and support the transfer of what is learned in programming to problem solving activities in other domains."
>
> (Pea & Kurland, 1984, p.149)

Hence, in summary, the evidence is that, provided care is taken in the following areas, programming a computer can be of cognitive benefit with regard to improving mathematical conceptual understanding. The areas where care is needed are :

a) Correct building of the instruction programme based on conceptual understanding of mathematical properties and encouragement of visual imagery.

b) Ensuring that there is guidance as necessary from the teacher during the programme.

c) Building in work which will encourage transfer to the mathematical domain.

The ways in which this has been done in this research are discussed in Chapters 6 and 7.

5.2.3 Enhancing Understanding With Computer Software

Although, as discussed above, the programming of a computer may be of educational value, it is through the use of software carefully designed to encourage conceptual understanding that the full benefits of the computer paradigm may be accessed. Such software may take advantage of the full range of benefits of the computer listed above, including the important aspect of the encouragement of the formation of the lower level cognitive schemata, as stressed in the model of learning developed in Chapter 3. The promotion of such imagery and the encouragement of the formation of the conceptual or C-links of relational understanding with the higher level cognitive concepts are areas where the computer can be of great importance to the educator. Such benefits as visual imagery, and those mentioned as associated with programming can be even more important when software is used to aid mathematical understanding. Commenting on the type of software needed to accomplish such good results, Linn (1983) makes the point that higher cognitive skills can be fostered by software which lends itself to student adaptation and creativity. Cockcroft (1982, p.409) recommends that special attention be paid to developing programs which encourage problem solving abilities and logical thinking in a mathematical context. Such programs require much thought in their production, as Bajpai et al (1985) have recognised, listing some of the 'do's and don'ts' of writing such software. They also propose the name Computer Enhanced Learning (CEL) for a computer environment which uses this type of program to aid understanding rather than simply promoting skills. One significant feature of the software providing such an environment is emphasised by Phillips (1986, p.44), namely that production of educational software is not just a case of presenting information visually but "The power of ...thinking." and that further, the display should be able to change in step with the user's thinking. Papert (1980), in describing the type of software he considered beneficial, coined the term 'microworld', and spoke of:

> "a computer-based interactive learning environment where the pre-requisites are built into the system and where learners can become active, constructing architects of their own learning."
>
> (Papert, 1980, p.122)

Thompson (1985), building on this notion of a microworld introduced by Papert, has recognised the need, in mathematics, for software of the type mentioned above and has proposed the notion of a mathematical microworld. He defines such a computer program as:

> "I will use 'mathematical microworld' to mean a system composed of objects, relationships among objects, and operations that transform objects and relationships...In practice a mathematical microworld incorporates a graphical display which depicts a visualisation of the microworld's initial

objects. The display in conjunction with operations upon the microworld's objects constitutes a model of the concept or concepts being proposed to the students. In a very real sense, the microworld embodies the structure of the concept. The students' task is to internalize that structure, and make it their own."

(Thompson, 1985b, p.3)

Such a program is intended to model the concept which is to be learned and typically would have the following characteristics:

"- The concept is modelled dynamically. - There is a limited number of operations (typically three to ten) at the student's disposal. - These operations can be combined into more complex operations. - The microworld can be used at different levels. - rather than including one curriculum, the microworld can support a large number of curricula."

(Dreyfus, 1986, p.13)

Tall (1985, p.106), taking the idea of an advance organiser after Ausubel (1968) and a microworld after Papert, has put forward the idea of a generic organiser, being a microworld "which enables the learner to manipulate examples of a concept." Such programs, based on the ideas of Thompson and/or Tall, with their dynamic visual imagery under the direct control of the student, and reflecting the formal mathematics, have been put to good use in mathematics by some. Tall and Sheath (1983), for example, record positive student reaction to the use of dynamic pictures of what are dynamic concepts. Another characteristic which has been used to good effect in some of the programs under discussion has been that of ´multiple representations´ in the computer displays. The theory behind such software is that, as Kaput (1986) describes it, mathematics is:

> "a network of representational systems, which interlock not only with each other, but interact differently with different kinds of mathematical knowledge as well as with nonmathematical representation systems, such as natural language and pictures."
>
> (Kaput, 1986, p.187)

and that by interacting with a computer program which not only shows some of these different representations, such as algebraic, linguistic and graphical, but actively encourages the student to relate through investigation one to the other, this could only be of benefit to their construction of linked mathematical cognitive structures. Commenting on the value of this, Tall (1987) says:

> "Multi-representational software is especially valuable when it allows the translation of precise numerical or symbolic information into graphical form...and it is at its most powerful when the data and its visual representation may be manipulated at will to see what happens when certain parameters are changed."
>
> (Tall, 1987, p.7)

This fits well with the ideas which I have put forward in Chapter 3, namely that in order to fully understand a concept or a system of concepts it is necessary to connect it, in a relational way, not just to a relevant schema, but to as many such relevant schemas as one possibly can. The multiple representations proposed in such software may encourage such links with their inherent possibility for transfer, but it is not clear yet whether this will necessarily be the case, Kaput (1986), for example, questions whether multiple representations will support the translation of skills and it seems therefore that there is a need for more empirical evidence (see below for some encouraging reports from studies so far). Already some, such as Goldenberg (1987) and Nachmias and Linn (1987), have discovered that students may pick up misconceptions from the visual data presented. In particular, both studies found a problem with an interpretation by their students of the nature of graphs, those of Nachmias and Linn for example considering the steplike graph displayed to be the true nature of what should have been a smooth curve, and interpreting the meaning correspondingly, and Goldenberg finding that students tended to focus on inappropriate features of the visual display, lacking discrimination.

5.2.4 Some Examples of Programs Designed to Improve Understanding

I shall now consider examples of some studies which have made good use of the dynamic visual display along with some of the features of mathematical microworlds, generic organisers and multiple representations.

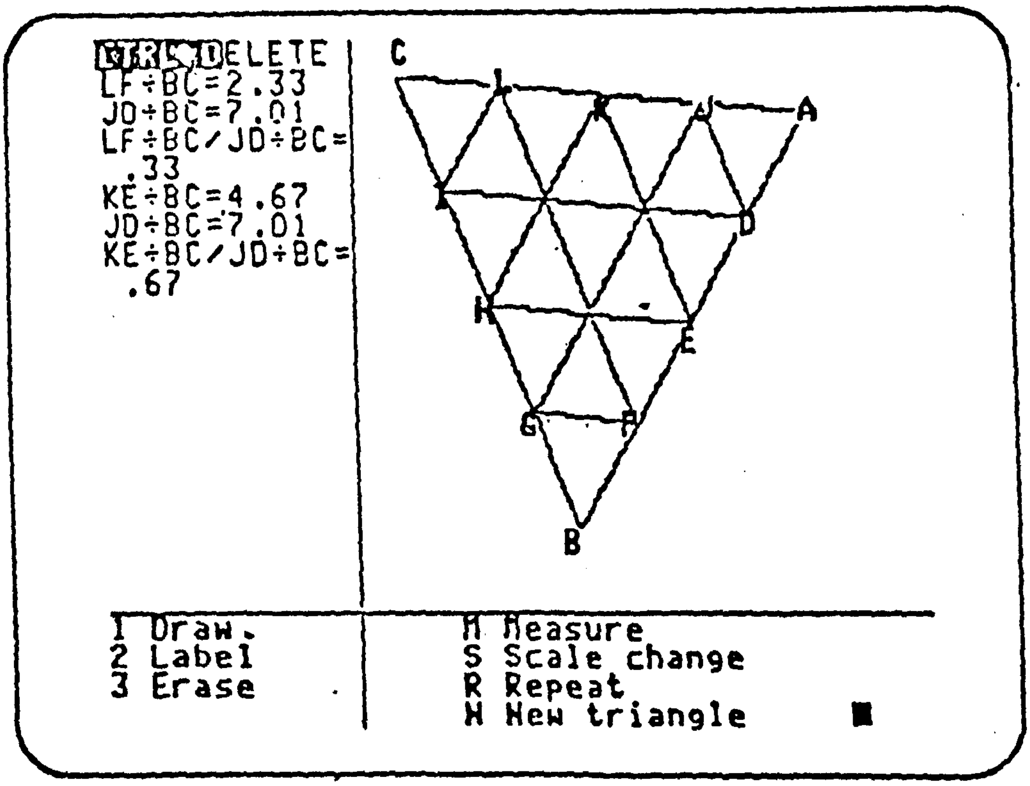
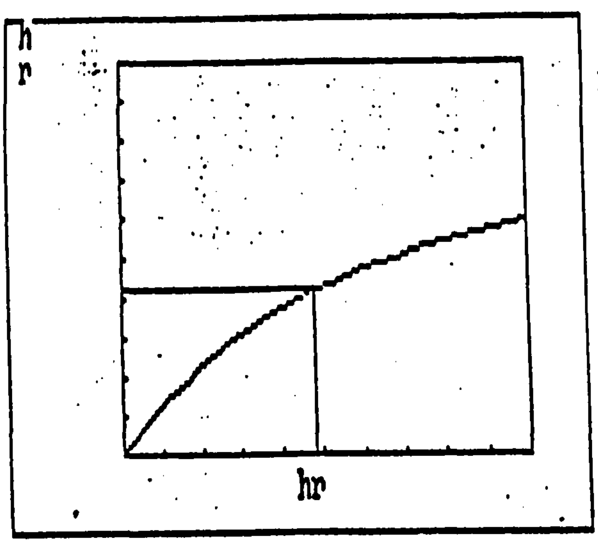


Figure 5.1

A Sample Screen From the Geometric Supposer

	HOW MANY	WHAT			NOTES
A	1	lawn		A	the job to be done
B	2	hr		B	mowing time of person 1
C	t	hr		C	mowing time of person 2
D	.75	hr		D	time for 1 & 2 to mow lawn
E	.5	lawn/hr	A/B	E	mowing rate of person 1
F	1/t	lawn/hr	A/C	F	mowing rate of person 2
G	.5+[1/t]	lawn/hr	E+F	G	combined rate of persons 1 & 2
H	1/[.5+[1/t]]	hr	A/G	H	combined time as function of t
I				I	
J				J	
K				K	
L				L	
M				M	
N				N	



[1/[.5+[1/t]]]

0 < t < 3 hr

0 < VERTICAL < 2 hr

< > ordinates

S change scale

+ - change step size

RETURN

t	FUNCTION
0	undefined
.24	.214
.48	.387
.72	.529
.96	.649
1.2	.75
1.44	.837

Figure 5.2

Sample Screens From The Algebraic Proposer

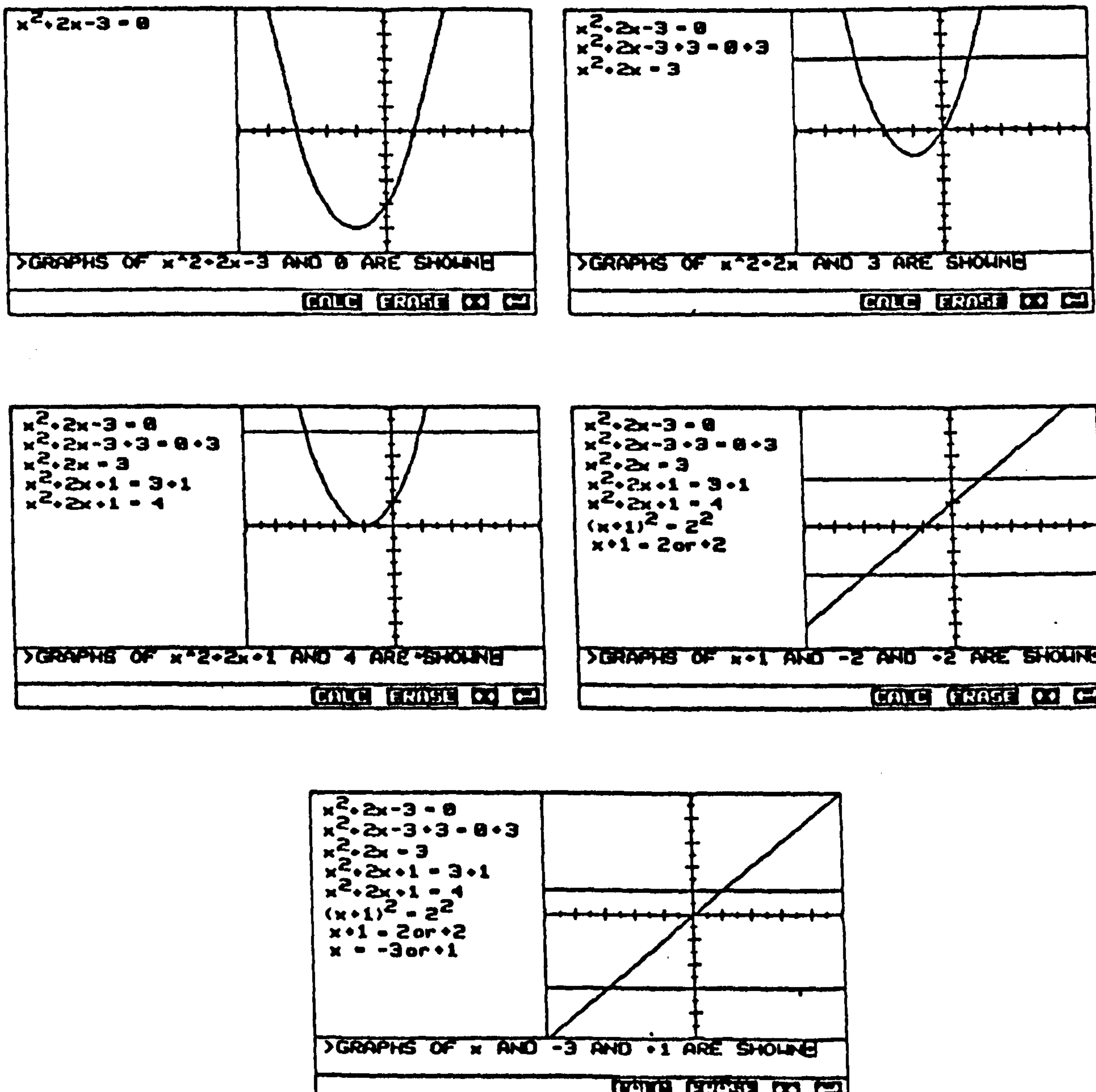


Figure 5.3

Sample Screens From The 'SAM' Software

The 'Geometric Supposer' of Schwartz (1985) allows the investigation of geometric properties and the search for generalisations. The display consists of a drawing of a geometric figure as well as calculations of numerical data relative to it, which students may manipulate (see Figure 5.1). Having used this software Kaput (1986) reports its value in that students have even been able to use it to produce new geometric theorems. In a similar way the 'Algebraic Proposer' software of Schwartz (1987) provides combinations of verbal, graphical, symbolic and tabular representations as an environment for modelling and solving algebraically problems given in linguistic format (see Figure 5.2). Lesh and Herre (1987) have produced a symbolic manipulator/function plotter, SAM, which may be used to solve algebraic equations. The display (see Figure 5.3) shows the algebraic equations resulting from operations on an original equation and the corresponding graphical picture. Using this software they have argued that the computer can be used to fit the requirements for providing concrete activity for the student in the Piagetian mode, and that the constructive, multiple-embodiment and dynamic principles of Dienes, based on the Piagetian model, can be implemented on the computer with resulting benefits. They argue :

> "The constructive principle is involved when we 'take apart and then reassemble' complex mathematical systems related to polynomials.
> The multiple embodiment principle is involved when we focus on mappings between two given models (i.e. written symbols and graphs of

equations).

> The dynamic principle can be used to show how transformations performed on algebraic equations are reflected in changes in the graphs of the equations at each step."
>
> (Lesh and Herre 1987b, p.18)

Thus they conclude; "computers make it easy for the student to manipulate one model and immediately see corresponding transformations in one or more other models." (Lesh and Herre 1987a, p.217). Their claim that multiple repesentations on the computer can improve students´ ability to translate between these different representations and hence to abstract from their structural similarities fits well with the ideas I have presented in Chapter 3. Of the five distinct representation systems in mathematics which they identify, namely ´real-world´ scripts, manipulative models, static pictures, spoken language and written symbols, at least four involve the capability, if not indeed, the necessity, of internalisation as mental imagery. The cognitive relationships between such mental imagery and the higher order concepts embodying them, as outlined in the cognitive integration I have described, thus become crucial to appreciate if one is to promote understanding, and enhance flexible manipulation from one mental representation to another. Zehavi (1986) has used a program called MaxMix which incorporates both algebraic and graphical representations on the computer screen and she reports it as evoking in students a meaningful interaction between these two representation systems. Zehavi (1987) has also used a generic organiser in a study with 3 grade 7 classes of average ability students in order to represent both graphically and symbolically the linear relationships between two variables. The software appeared to successfully create an intuitive readiness for future concepts, in the manner intended of an advance organiser, and its value was such that those using it were shown to be better able to symbolise linear relationships up to eight months after the programme. Using somewhat similar principles, but at a somewhat higher level academically, Dreyfus and Eisenberg (1987) have described their success with 11/12th grade students using software designed on the principles of the mathematical microworlds of Thompson, described above. This software consisted of a graphical representation of a set of points in two dimenasions through which the student was required to fit a polynomial curve. By relating the effect of changing the parameters in the symbolic expressions, also on the computer screen, to the corresponding changes in the graphs, the students seemed to be able to gain some significant insights into the deep structure of functions. It was true, however, that the researchers described their success as only partial, with some aspects of understanding proving difficult to improve upon. Nevertheless, in these areas where traditionally there have proven to be problems in assimilation of deep concepts, any progress is to be welcomed, and the use of multiple representations does hold out some promise. Thompson (1985a, 1985b) himself has produced two successful mathematical microworlds called ´Motions´, designed to focus on an understanding of Euclid´s transformations of the co-ordinate plane, and ´Integers´, which aimed to develop the concept of the ring of integers. Moser (1983) was another who designed software

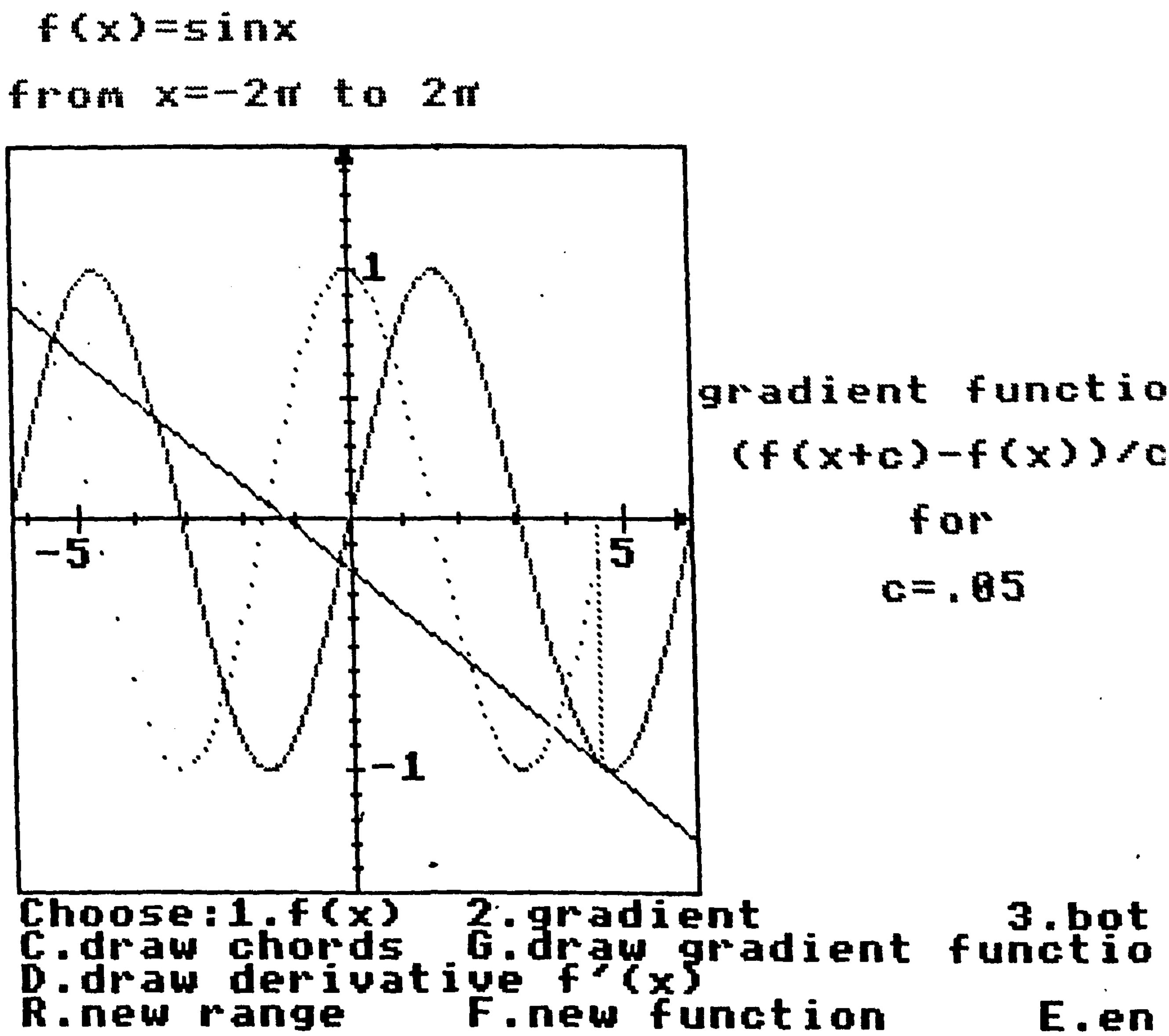


Figure 5.4

A Sample Screen From Graphic Calculus

involving multiple representations. His program utilised a pictorial and a symbolic display to enable first grade pupils to represent word problems as number sentences and hence to solve them. He concluded that this computer approach was successful and that "the children quickly grasped the concepts represented to them." (Moser, 1983, p.344). Among the most successful of the programs produced involving the principles above have been the generic organiser programs designed by Tall. The 'Graphic Calculus' package enables the user to explore graphical representations of the concepts of the calculus and thus to develop a relevant mental picture to accompany the formal mathematics. The display may contain a picture of the graph of a given function and its derived function, as well as the corresponding numerical data, all under the control of the user (see Figure 5.4). This produces an environment which, under the direct control of the student, encourages his/her experimentation with the functions and their graphs and derivatives. This approach can lead to remarkable insights, such as that reported by Tall and Sheath (1983), where they describe the reaction of one student who announced that he was able to 'see' (on the computer display) that the derivative of sin x is cos x, because he had built up the two graphs on the software. The programs have been used with sixth form pupils (16-18) and have produced a significant improvement in their understanding of the gradient of a function and its differentiability (Tall 1985e, 1986c) as well as linear and locally linear graphs (Blackett 1987), without adversely affecting their formal manipulations of the calculus. Instead it provided them with a highly valuable visualisation of the processes involved in such manipulations. Tall (1985d) has applied the same principles in software produced to help students to visualise the processes in arriving at solutions to differential equations. The programs are built around the use of the central concept of the tangent vector and Tall (1984, p.6) claims that "the theory of ordinary differential equations may be given a unified meaning that enriches and complements the collection of isolated analytic techniques."

These examples of the successful application of a computer environment acting as a generic organiser (Tall 1986c), give some idea of the current value and the potential of the computer in tackling the difficult problems associated with improving students' understanding of many of the deeper concepts of mathematics, many of which relate to one form of algebra or another, and provide a background for the use of the computer in the present research study.

5.2.5 The Structured Environment

There is building up a considerable amount of evidence that the way in which the environment of the learner is structured when he/she is using the computer in mathematics has an effect on the cognitive outcome, as to whether their abilities are enhanced or not. Initially, the argument of those in favour of an unstructured environment, such as Papert and others using Logo, was that the pupils should be given total freedom to explore the language and the concepts embedded in it, or representible by it. Thus for maximising advancement of the children it was thought that they needed to be totally in control of their own progress with no interference from outside, even from the teacher. However, it has been shown to be important to consider carefully the structuring of the pedagogical environment in which the exercises take place. There is some evidence that an environment in which the teacher places some structure on what may be tackled is more beneficial than one where individual discovery alone is used. This was shown to be the case in the short term studies looked at by Ausubel (1978), and mentioned earlier, in Chapter 2, where the benefits of some guidance in the discovery learning situation as exposited by him were discussed. The evidence from Alderman (1979) too is that a high degree of learner control on learning produced a detrimental effect, and he suggests some degree of control over the pace at which students progress is necessary. The study of Markman (1973) had shown that, as a general principle, young children need a structured learning environment in order for their cognitive performance to be improved. Similarly Ross (1985, p.12), on the basis of his studies concludes "Our experience has been that a controlled discovery learning approach is practicable.". Agreeing with this conclusion, Pea and Kurland (1983) also recommend instruction as well as discovery learning in the computer environment. Leron (1983) states that a structured learning environment is important when programming if children are to acquire the mathematical content of their work. Following their study of the effects of using structured training on children's performance on a problem-solving computer program compared with other methods, reported a significant superiority of an informed training group over others and commented :

> "The results of this study emphasize the importance of paying attention to the way in

> which the learning situation is structured for the child. Leaving children to discover the rules of a program following little or no training, as in the BT [blind-training] and control groups, did not constitute an effective learning procedure."
>
> (Simon, McShane and Radley, 1985, p.10)

Another to note the value of such structuring by the teacher is Thompson :

> "the curricula discussed...are actually more dependent on the teacher than are conventional curricula...teachers must be - for want of a better word - choreographers. They must have a structure that they can ´dance´ through as they confront the obstacles that are inevitable in the flow of classroom interactions."
>
> (Thompson, 1985, p.231)

Seeing such a structured learning environment as a vital part of the process of education in the computer paradigm, Tall (1985e, 1986b) also has recommended that the teacher provide direction for the pupils as an integral part of the process of using a software generic organiser, and when these are combined in this way he has defined the resulting mode of teaching as the ´Enhanced Socratic Mode´ (Tall 1986c). It is this mode, with its combination of some teacher guidance and a generic organiser, which has been used in this present study in order to facilitate an improvement of children´s understanding of algebra.

5.3 Algebra in the Computer Environment

In this section I shall discuss some ways in which the application of a computer environment to the meaningful learning of mathematics have been used, and may be more beneficially used, in the learning of algebra.

Many of the general benefits of the computer paradigm discussed above are, I believe, particularly relevant to improving the teaching and learning of secondary school algebra. The situation with algebra and the computer is unique in that both share a common basis in their fundamental use of variables. My thesis is that algebra is best approached through an understanding of what constitutes variables, and that the computer environment is the ideal place to explore their meaning and significance. Specifically the computer brings to the learning of algebra:

1. A context in which meaning may be given to the concept of a variable
2. A programming environment which is rich in variables and in which examples of the concept may be manipulated in a meaningful way
3. The possibility of a software environment strong in visual imagery which may be utilised to aid and promote relevant cognitive visual images and make explicit their connection with algebraic symbolism and concepts

I shall discuss below some attempts to use the computer to access these benefits in the learning of algebra, covering three main areas, namely use as a skill tutoring system, as a programming environment and finally through the use of the kind of software which makes use of linked visual imagery in the ways outlined above.

5.3.1 The Use of Software for Teaching Skills

Many of the earliest computer programs used in education placed an emphasis on the acquisition of skills. There have however been relatively few attempts to set up controlled investigations of the value of such programs as an aid in improving children's algebraic abilities. One such study is that of Saunders and Bell (1980), where, throughout an eight month study, covering an entire algebra course for 12 year-old children, the computer was used to try and enhance learning. The students ran library programs on algebra and were tutored by these accordingly. The results of the study showed that there was no significant effect on the children's achievements in algebra and that further some of the students of higher ability were slow to accept the computer as teacher, even resenting it. In another study reported by Menis (1980), students spent half an hour a week at the computer doing algebra exercises and this study showed a significant improvement in general mathematical ability, particularly amongst the weakest children.

Thus it would appear that there is little evidence for the value of such methods in improving students algebraic ability.

5.3.2 The Computer as an Aid In Conceptual Learning of Algebra

The conceptual approach to the learning of algebra, with its emphasis on the semantics of algebra has given evidence of a more successful way forward than the skill-based approach. Some work on children's understanding of the concept of variable has been carried out in a logo programming environment (e.g.

Samurcay 1985, Sutherland and Hoyles 1985, 1986, Sutherland 1987, Noss 1987, and Nelson 1985, 1987) with some measure of success in improving understanding. Thompson (1987) and Thompson and Thompson (1987) have also had some success with their mathematical microworld enabling the representation of algebraic expressions as parsing trees. Details of these studies have been included in Chapter 4. The main difficulty encountered by these researchers, as with many other studies, has been the difficulty of promoting transfer of understanding gained in the computer environment to the ´paper and pencil´ classroom situation. It has been suggested that in the case of Logo this may possibly be due to the fact that variables in Logo tend to lead to more than one conception and that the pupils tend to assimilate the Logo variables rather than the concept as used in early algebra (Samurcay 1985, Tall and Blythe 1985). The failure of many studies to address the problem of transfer (noted by Pea and Kurland (1984), and described above), and the difficulty of promoting it, even when it is considered, are demonstrated by the Logo results, highlighting an important consideration for studies in this area. The success of the present study in tackling this difficult problem is described in Chapter 7. The common use of variables between the computer and algebra was seen by some, such as Hart (1979), as the natural starting point for improving children´s understanding of algebraic concepts. The basis of Hart´s Nottingham Programming in Mathematics Project, reported in Hart (1982), was the use of programming in BASIC built on a model of a variable and involving practical applications of algebra. Mixed ability classes of 11 and 12 year-old pupils were given a course of elementary programming before they were taught algebra, with the commonalities emphasised. Although it was not possible for him to arrange for a statistically controlled experiment due to the logistics of the school involved, the results of Hart's work were encouraging. The results of those classes who had the benefit of the programming work, as measured on the CSMS algebra test, were generally better than those who had not had such a course, as well as being above the 'norm' suggested by the CSMS work. Hart postulated that the emphasis on the use of variables in the programming was having beneficial effects on children's understanding of this concept, and he found that the 11 year old pupils did indeed find the concept of variable and function easier after their programming of short programs in BASIC than they had before (Hart 1980). This study was an important one in that it provided evidence that such an approach could be valuable in improving conceptual understanding of algebra rather than just manipulative ability. Tall (1983) also described encouraging success in using short programs in BASIC as a means of introducing algebra on the computer to children with no algebraic experience. It was the aim of the present research to build on these promising beginnings and extend them to produce a complete module of pre-algebra work for the novice, and hence to provide statistically controlled evidence that the conceptual approach using the computer is indeed the way ahead in overcoming children's problems with algebra.

5.4 Implications for the Algebra Curriculum of the Future

One important question which needs constant addressing in any consideration of the secondary school curriculum, in mathematics as in all subjects, is 'What changes should be made in the light of new knowledge?' When we seek an answer to this important question relative to the teaching of algebra in the secondary school then a vital point to be considered is the possible effect of the new technology, the computer-oriented paradigm, on the traditional secondary school algebra syllabus. In view of this technology we must ask questions such as 'What computer hardware and software are appropriate for use in the algebra classroom?', 'What are the implications of this for the algebra which should be taught in schools, and to whom should it be taught?' I shall now consider briefly some considerations in providing possible answers to these questions.

The first point to realise is, that in spite of the advantages of the computer paradigm outlined above, the technology is still not effectively used in most parts of the world. Ball (1983) recognises this and says :

> "Computers are a very powerful aid to teachers and learners, and we have only just begun to think about how to use them effectively."
>
> (Ball, 1983, p.39)

Tall (1986a) agrees that the "immediate prognosis for the use of computers in the mathematics curriculum is not as bright as we may like it to be." A survey in Britain, carried out on behalf of the Mathematical Association by Jones and Green (1986), reports only 5% of computers as located in mathematics classrooms and only 12% of schools using computers in mathematics lessons ´often´, and Tall et al (1987) describe the reality of the little effective use of computers in mathematics teaching, with the Stanhope Report (1987) agreeing.

When we consider the best way forward to gain the advantages of the computer in mathematics, and in this context particularly secondary school algebra, then we need to address not only the types of software which are most effective, as described above, but also the type of language which would best meet the educational need. Discussing the semantic nature of algebra, Boileau et al (1987) propose the need for a programming language which would serve as an intermediate representation between the problem to be solved and the final coding in order to facilitate better understanding. Tall (1985b, pp. 2-4) lists a number of desirable attributes which any future language should possess if it is to aid understanding of ´major mathematical concepts and processes´ such as the use of letters as specific unknowns, symbolic algebra and symbolic function manipulation, amongst others. This concept of the ability of a language to perform symbolic manipulation is one which others have noted as a valuable future requirement (e.g. Winkelmann (1984)) with respect to algebra. Some steps in this direction have already been taken with the languages now available and there is growing use of such computer programs, called symbolic manipulators, with their emphasis on the syntactics of algebra rather than the semantics, and I shall now consider their possible future effects.

5.4.1 Symbolic Manipulators and The Facilities Available

Although symbolic manipulators are not new, having been available on mainframe computers since about 1951, their expense and the lack of publicity about their capabilities, coupled with their restriction to a select group of individuals, have precluded their more general use until quite recently. This is because they have now become available, at relatively low cost, on certain microcomputers. A symbolic manipulator such as Mumath for example, which is currently available for microcomputers, has the capability to carry out a wide range of algebraic processes. These include :

The simplification of expressions

The solution of equations

Matrix manipulation

Taylor series expansion and summation

Symbolic differentiation

Symbolic indefinite integration

Limit evaluation.

Other, similar programs which are also available for various computers include MACSYMA, REDUCE, CAMAL, Maple, SMP, ALTRAN, FORMAC, SAC-2, Picomath and others (see e.g. van Heuzen & Calmet, 1983), and many of these are interactive systems which enable the user to define expressions, apply operations to them and manipulate the output. The value of such programs is enthusiastically propogated by many of those who have used them extensively (Pavelle, Rothstein & Fitch 1981, Lane, Ollongren and Stoutmeyer, 1985, Small, Housack and Lane, 1986, MacCallum 1986, Neuwirth 1987), although there has been a report of a lack of enthusiasm on the part of students (Hodgson 1987). Some of the key benefits they quote include the accuracy and speed of the programs. For example, an algebraic calculation, of a type increasingly common in science today, which took ten years to do and another ten to check by hand, took just twenty hours on a symbolic manipulator. Accuracy also is vital in scientific calculations and the danger of relying on tables of, for example indefinite integrals, has been recognised

> "A numerical computer study of eight widely employed tables of indefinite integrals discovered that about 10 percent of the formulas are in error; one of the tables was found to have an error rate of 25 percent."
>
> (Pavelle, Rothstein & Fitch 1981, p.111)

Thus the programs are claimed to increase efficiency through the avoidance of major errors, the tedious search for minor errors and the time spent on doing lengthy manipulations by hand. However their value is expected to be even more than this hence MacCallum says about the use of such programs:

> "Like numerical programs, using computers for algebra gives not only a better way of doing what one could do before, but also opens up new ways of working, for both students and researchers."
>
> (MacCallum 1986, p.54)

Whilst these examples may seem to be far above the type of use one might envisage in the secondary school, the power available to the user at the upper end of such schools may be appreciated by an example such as finding

$$\int x \sin^2 x \; dx$$

by entering INT(x*SIN(x)^2, x) and very quickly receiving the answer:

-x*SIN(2*x)/4 + x^2/4 - COS(2*x)/8.

Given that the programs are capable of manipulations such as this which would be useful in schools, we may well ask whether it is proper or desirable, to prepare pupils lower down the secondary school for such uses, and if so, what is the best way to do so. Furthermore, since the trend is towards smaller, cheaper and more powerful versions of such manipulators, so that there is for example now available a hand held calculator, the Hewlett Packard HP28C, which can carry out symbolic manipulation, including simplification of expressions and the solution of equations of the type commonly met by the secondary school algebra student, such students could in the foreseeable future have access to such facilities outside of the classroom. The questions are then ´Are these aids educationally desirable and/or beneficial or will their use, inside or outside of the classroom, cause educational difficulties?´ and ´If such aids are deemed desirable, what is the best way to prepare the secondary school child for their use?´

5.4.2 Problems and Challenges in Using Symbolic Manipulators

On the one hand, it may be argued, that in taking over the burden of so much manipulation which is essentially instrumental in nature that such programs as those described above would make it possible for students to concentrate their efforts on the basic mathematical concepts involved in an application of mathematics, and to investigate a wider range of examples and circumstances than would otherwise be possible, making more realistic problems accessible. This might be particularly true for the student of average ability or below, since some claim that what has often characterised the performance of such students in the past has been their inability to cope with the algebraic manipulation involved in a problem even if they were able to see the underlying conceptual basis of it. Increasing the accessibility of the solution to such problems could be a very good thing since it might help to fill the increasing need in the modern technological world for mathematically competent individuals.

However, one of the foremost arguments which may be made against symbolic manipulators is that their emphasis on the syntactical side of algebra can only result in a detrimental effect on the understanding of its semantics, and that by using such aids the student´s understanding of what is involved in the processes of algebra can only be reduced. This is an important consideration since many would question whether our aim as mathematics educators is purely a functional ability in mathematics. The two sides of the coin are put well by Davis (1986) when he says, reporting on the views presented at ICME-5:

> "There are two views about mathematics and the learning of mathematics:
>
> A) ´Doing mathematics´ means following explicit instructions and knowing certain facts;
>
> B) ´Learning mathematics´ means building up REPRESENTATIONS in the student´s mind."
>
> (Davis, 1986, p.205)

Many would now agree that the deep structure of algebra, as indicated in the description B) above, is preferable to the surface structure described in A), and that therefore this is where the efforts of educators should be directed. What then may we say about the effect on algebraic understanding of the use of such programs? The first thing to realise is that paper and pencil computation by hand may well have its place in developing a relational understanding of concepts even if symbolic manipulators should be brought into use (e.g. Small D., Hosack J. & Lane K., 1986), and so the question of whether such practice should be retained, and if so, the extent to which it should be so, would be an area for investigation. The second point to appreciate, and a most important one, is that those who have used symbolic manipulators and have investigated their use seem to agree that there are cognitive prerequisites for their use including, for example, an understanding of basic algebraic concepts. Commenting on these requirements MacCallum (1986) says:

> "Just as a calculator is no help to a person who does not understand the difference between adding and multiplying, students who use computers for algebra have to understand the meaning of the processes (for example differentiation, integration, factorisation and calculating group tables) the system is carrying out."
>
> (MacCallum, 1986, p.54)

and in a similar vein Hodgson (1987) says:

> "Awareness of the sense of a computation rests on a deep understanding of the mathematics involved, not merely at the computational level itself (after all, this is exactly what the computer is doing for us!), but much more profoundly, at a conceptual level.....The availability of ready-made numerical and symbolic software will thus not result in a reduction in the mathematical sophistication of the user."

(Hodgson, 1987, p.3)

Neuwirth (1987, p.5) agrees, adding "we might say that you usually have to understand quite a lot of mathematical concepts in order to be able to use such a system sensibly." Thus what appears at first to be a distinct disadvantage of symbolic manipulators could well turn out to be a strength. As Small, Hosack, & Lane. (1986) say :

> "Too much time in undergraduate mathematics is spent carrying out routine algorithmic manipulation (which students do not remember) at the expense of conceptual understanding and appreciation of mathematical processes"
>
> (Small, Hosack. & Lane, 1986, p.423)

What is true at the undergraduate level is probably true too of many children in school, who would also benefit from a greater emphasis on meaningful understanding as opposed to routine manipulation. Some of these concepts on which time could usefully be spent, and which are necessary in order to use such programs successfully, would include the development of a meaning for variables and expressions, an appreciation of the parsing structure of expressions, an understanding of the use of implied operations in written algebra, the conservation of solution in the steps leading up to the solving of an equation, the conservation of equation under a change of variable, etc. It could well be that the use of symbolic manipulators could free some of the time needed to concentrate the efforts of the teacher and the student on gaining a meaningful understanding of these sort of concepts.

To illustrate just one way in which understanding is still of great importance we may consider the following example taken from the HP28C calculator mentioned above. In order to solve for x the equation :

$$a(x + b) + 2x = c$$

the following processes, with their associated understanding, are required:

$ax + ab + 2x = c$ (the need to expand the brackets)

$ab + ax + 2x = c$ (the commutativity of addition)

$ab + (ax + 2x) = c$ (the associativity of addition)

$ab + (a + 2)x = c$ (the process of factorisation)

before the calculator will, at this stage, solve for x. It is true that the manipulation at each stage is carried out by the program, but understanding the need for such manipulation is a prerequisite.

In the reverse situation, however, when one wishes to examine the steps taken in the solution of a problem, it should be stated that symbolic manipulators such as Mumath do contain trace packages which enable the user to see each step in an algebraic simplification rather than just the final result. Facilities such as these may well help to improve meaningful understanding in these circumstances, but this is an area for further research.

A further difficulty associated with the use of symbolic manipulators is that produced by the nature of the notation which they use. Thus in order to enter the algebraic expression

$$2y(y^2 - z) + 2z(y + z)$$

one is required to specify operators such as multiplication explicitly, viz.

```
2*y*(y^2 - z) + 2*z*(y + z).
```

Also, working the other way around, when one has to interpret the results of the processing it is necessary to interpret the result in terms of what is for most of us the more familiar mathematical notation. Tall (1986b) gives an example which well illustrates this difficulty respecting the need for translation of computer generated solutions into a form which can be easily assimilated. How many of us for example, who may be used to such notation, let alone the secondary school pupil, recognise

(((a*(x^2)) + (b*x)) + c)

to be the more familiar $ax^2 + bx + c$, or indeed prefer it? This is, I suggest, a non-trivial problem which again requires investigation before the widespread introduction of these programs into schools. An understanding of the structure of expressions is something which many children find very difficult at present, and instructional modules such as that designed by Thompson (1987) will therefore still be necessary, as will, I believe, those such as the one presented in this thesis for the understanding of the dynamic nature of the use of letters in algebra, in order to generate algebraic thinking in the pre-algebra student.

Not to be overlooked in a discussion of the introduction of symbolic manipulators into the secondary mathematics classroom are the problems that their use would pose for many teachers. Since the programs enable the pupils to tackle a wider range of more sophisticated problems, it has been suggested that they may well cause more trouble to teachers than did, for example, the introduction of calculators (Davenport, 1985). An example of the type of problem which comes within reach with such a system is given by Lane, Ollongren. & Stoutemyer. (1985):

"Experiment with your computer algebra system to form a conjecture about how the reduced form of the algebraic expression

$$(x^m-1)/(x^n-1)$$

depends on m and n."

(Lane, Ollongren. & Stoutemyer. 1985, p.144)

This may be at the same time both an exciting and a frightening prospect for teachers, for whom suitable provision should be made well in advance of the introduction of symbolic manipulators into the classroom. This means that there will soon exist a need for the training of teachers in an awareness of the capabilities and limitations of the technology as well as the educational possibilities which it opens up.

5.4.3 The Algebra Curriculum of the Future

The emergence of symbolic manipulators and other facets of the computer paradigm should lead us to at least question the role of algebra in the curriculum and in society as well as considering the validity of attempting to teach the amount of algebra currently in the secondary school curriculum, across the whole ability range. It should be remembered, that in Britain at least, only those at the top of the ability range studied any algebra at all twenty years ago. Whether the computer will, or indeed should, have a similar effect on mathematics syllabuses relative to algebra content as the electronic calculator has had on the same syllabuses relative to the teaching of logarithms, for example, is one which cannot be avoided. It is already argued by some (e.g. Usiskin 1980 etc.) that much of the complicated symbolic manipulation taught in schools is unnecessary and impractical at that level. Speaking of some specific examples in algebra Usiskin says :

> "These problems involve very complicated manipulations. In having them as standard fare, we are asking students to do the most complicated of algebraic manipulations in their first course of study of algebra. Do all students need this experience? Clearly not; most first-year algebra students do not go on to take senior mathematics and do not have occupations that require this kind of manipulative dexterity."
>
> (Usiskin, 1980, p.417)

She also adds that the algebra taught to most secondary school pupils is not the algebra that they need to know. What then should be included? Davis (1986b, p.21) asks whether there should there be an emphasis on notation and its manipulation or on the ideas which the symbols are supposed to represent, on the meanings of the symbols. He argues that there is not universal agreement on the answer to this question and other pedagogical considerations effecting the use of computers in the algebra curriculum and that these issues require resolution in the near future if effective use is to result. It is argued in this thesis that there is a need for the introduction of algebra to be meaningful to the pupil, and that such a meaningful first acquaintance has beneficial effects (see Chapter 8). Further there is a case for an introductory module of work to give intuitive readiness for the cognitive concepts associated with the dynamic nature of algebra, the building blocks of future work. I have discussed in Chapter 4 how it is necessary to provide meaning for expressions and equations, and how such understanding is a necessary pre-requisite for coping with equations which involve manipulations of variables. Thus any analysis of the curriculum should reveal the need for the inclusion of such a semantic approach. It is true that it may take longer to cover the ground than before, as has been seen in the experiments discussed here (see Chapters 6-10), but the extra time is worth the gains and the pruning from the curriculum of any unnecessarily complicated simplification etc. would provide the space. How much further than this should algebra teaching go? The answer here is not clear at the moment and further thought and research is needed.

Should the pattern suggested above be true for all pupils in the secondary school? It may well be the case that there are some at the bottom of the ability range for whom algebra will never have either any real meaning or any real practical use and for whom, therefore, we should consider the viability of spending time and effort which could be more profitably employed in improving understanding and efficiency in other areas of mathematics. What of the average ability student? Usiskin (1987) argues strongly for algebra in North American schools, saying that some algebra should, and must, be taught to students, concluding "Thus to average students, we should teach algebra in the 8th grade [13 year-olds]. To our best we should teach it in 7th grade." (ibid, p.16). The question of who should be taught algebra in the secondary school is a difficult one, but again it is one which the use of the computer will bring into sharp focus and further investigation is required.

5.5 Overview of the Chapter

In this chapter I have considered the ways in which the computer has been used in mathematics education, and given some pointers as to the ways in which it is educationally desirable that it should be used. I have looked at the resulting benefits which may accrue from a well thought out approach to the use of computers, designed to maximise their potential. In particular I have addressed the subject of the use of computers in secondary school algebra, and given the background studies for the research project which is described in the following chapters. Finally I have included a discussion of the way in which the algebra curriculum of the future may possibly be affected by the computer paradigm, with particular reference to the influence of symbolic manipulators.

Chapter 6

The Two Initial Investigations

The application of the principles in the previous chapters to the present research study is decribed in this chapter. This involves a consideration of the links between the understanding of algebraic concepts by early learners and the computer paradigm. The two initial investigations which took place are then described and their results are given, along with the conclusions which led to the form of the main experiment, presented in Chapter 7.

6.1 The Background to the Research

In Chapter 4 I have mentioned, in some detail, the problems associated with the learning of algebra by algebraic novices, and I shall review here the main points considered in the synthesis of the research module of this study. Many of children's difficulties in algebra are as a direct result of a wrong or inappropriate understanding of the use of letters in algebra. The idea that many children have difficulty with algebra because they do not understand the concept of a variable and therefore do not have the necessary base on which abstraction may be made was a conclusion of the research entitled Concepts in Secondary Mathematics and Science (CSMS) and based at Chelsea College, London. The work of Küchemann (1981b) formed the algebra part of this in-depth study of the difficulties of school children with mathematics. His research showed that different children are at different levels of understanding with regard to the concept of the variable in algebra, and that their ability to tackle algebraic problems is directly related to the level which they are at. He has described the levels of understanding and usage of letters in algebra as follows :

LEVEL 1 - Those items which are purely numerical, or which may possibly have a simple structure which can be solved by treating the letters as objects in themselves.

LEVEL 2 - Those items which are of increased complexity, but may still be solved by evaluating the letters or viewing the letter as an object. There is, however, the first indication of a willingness to accept a lack of closure in answers by children at this level. i.e. Answers such as 3x + 2 are becoming acceptable.

LEVEL 3 - Those items which involve an understanding and acceptance of letters as specific unknowns and also acceptance of lack of closure in answers.

LEVEL 4 - Those items which are similar to level 3, but which involve more complex structure. They may also involve a knowledge of using letter as a generalised number or variable.

He has further postulated (Küchemann 1981b, p.117) that there is a link between children who are performing at each of these levels of understanding with regard to the use of letters in algebra and the following Piagetian levels :

LEVEL 1 - Below late concrete

LEVEL 2 - Late concrete

LEVEL 3 - Early formal

LEVEL 4 - Late formal

In addition, Küchemann has shown that children's interpretation of letters in algebra can pass through six stages, as shown below and in Figure 4.1 :

1. The letter is evaluated i.e. it is given a numerical value from the outset.

2. The letter is not used at all i.e. it is either ignored all together or is given no meaning at all.

3. The letter is used as an object in its own right i.e. it is used as a shorthand notation for an object, or as an object in its own right.

4. The letter is used as a specific unknown i.e. it is a specific, but unknown number that can be operated on directly (due to Collis, 1975).

5. The letter is used as a generalised number i.e. it is seen to be able to represent or take several values.

6. The letter is seen and used as a variable i.e. it is seen to represent a range of unspecified values and a systematic relationship can be perceived between two such sets of values.

Unfortunately, as his work showed, many children fail to progress past the fourth of the above stages, and some do not even get this far. The percentage of children found by the CSMS algebra tests to be at the levels one to four mentioned above were given in Table 4.1. These figures indicate that the majority of 11-14 year olds in schools have a naive view of the use of letters in algebra. Their limited view may often be obscured by their instrumental understanding, which may allow, for example, a child at one of the first four stages of understanding given above to solve equations such as :

$$x + 3 = 7,$$

by viewing the x simply as a stand-in for the number 4, and using their instrumental rote recall of the number fact

$$4 + 3 = 7.$$

The idea of using a letter as an object in its own right works well too, and may even be encouraged, in many classrooms, where in questions such as :

Simplify $4a + 2a + 5a$,

pupils may think of this in terms of objects such as, four apples plus two apples plus five apples makes eleven apples all together. As Collis (1978) points out, the acceptance or encouragement of this approach is often through taking account of what is expedient rather than what is correct and of long term benefit mathematically. The way in which this may cause cognitive problems later may be illustrated, for example, by considering the re-structuring of mental schemas which might need to take place to cope with questions such as :

Simplify (i) $2b - 5b$ (ii) $2 + 3a + 4$ or (iii) $3a * 4a$

However, it is when the question requires an understanding of the use of letters as standing for a range of numbers where, for most children, their ability to stretch their instrumental understanding reaches its limit. We are reminded by Küchemann that in order even to make sense of a question like :

Which is larger $2n$ or $n + 2$?,

let alone solve it, a level of understanding of five or six in his classification is required.

Booth (1983b) undertook some research in response to these problems of understanding in algebra. She studied the types of errors made by children in secondary school algebra and then attempted to construct a programme of work to improve their conceptual understanding. She found that there were three main

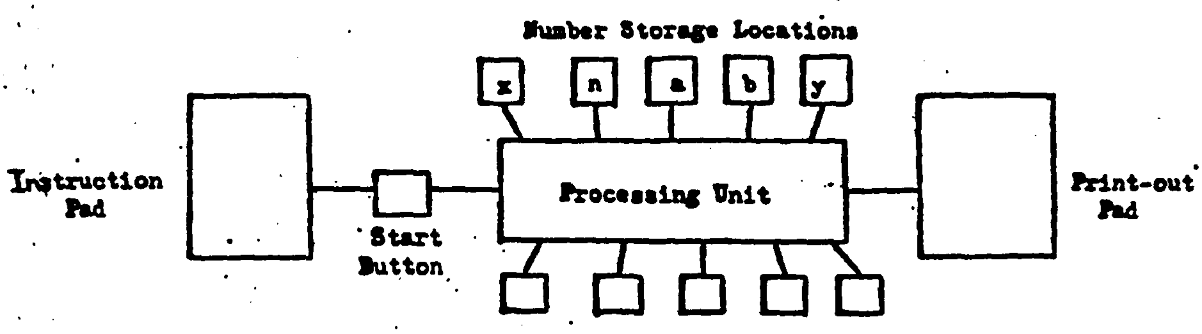


FIGURE 6.1

Booth's 'Mathematics Machine'

areas of difficulty underlying the errors made in algebra. These were :

1. The difficulty of the interpretation of letters.

2. The difficulties of coping with the notation and its conventions, such as the use of brackets.

3. The difficulty of formalising and symbolising, in algebraic notation, the solution to a problem.

In order to try and improve children´s understanding in these areas, Booth´s study made use of an imaginary ´Mathematics Machine´ as shown in Figure 6.1. The idea of this was that the children should concentrate more on the concrete situation of trying to give the ´Machine´ instructions to solve a problem, rather than on the more abstract process of of trying to produce an answer themselves. It was thought that the disecting of a problem and the representation of it in order to present it to a machine, even if an imagined one, would encourage an understanding of the concepts and principles involved.

Her research showed clearly the value of this approach, producing a significant improvement in understanding in several areas, including conjoining in algebraic addition and formalisation of method. However, with regard to the concept of letter as a generalised number she observed that the improvement in performance between the immediate and delayed post-tests on such items appeared to indicate that the ideas involved were not easy to assimilate, and that time was required for understanding to develop. She also concluded that particular problems seemed to be associated with the understanding of this concept of generalised number and that further research, paying attention to the framework of reference constructed by the child with regard to it was necessary.

6.2 The Rationale Behind the Use of the Computer in the Research

In view of the relative success of Booth´s study, it seemed that it might be possible to build on the success of her work and assist in the process of finding an answer to children´s difficulties with algebra by using a real rather than an ´imagined´ machine to enhance understanding. The computer, with its direct link to the variable in algebra through its use of letters for the names of stores, and its intrinsic interest to children, seemed ready made for this role.

The main advantages of the use of a computer in mathematics education were summarised in Chapter 5 as :

The power to motivate students
The ability to interact logically with the student
The capability for mathematical modelling
The improvement of problem solving ability
The production of student generated feedback
The encouragement of algorithmic thinking
The improvement of reflective thinking ability
The capacity to concretise and personalise formal thinking
The power to improve the formation of mental images through concept visualisation
(See e.g. Papert 1980, Cockcroft 1982. Tall 1986c, Cooper 1986, O´Shea & Self 1983)

For algebra in particular the situation with the computer is special in that both share a common basis in their fundamental use of variables. Hence the computer provides :

1. A context in which meaning may be given to the concept of a variable
2. A programming environment which is rich in variables and in which examples of the concept may be manipulated in a meaningful way
3. The possibility of a software environment strong in visual imagery which may be utilised to aid and promote relevant cognitive visual images and make explicit their connection with algebraic symbolism and concepts

Further, it was recognised, as has been shown in Chapter 5, that there is cognitive advantage to be gained, particularly in the study of early algebra, from a programme of work which includes both computer programming and the use of software generic organisers (Tall, 1986c).

An attempt to produce evidence to show the value to algebra learning of the use of computer programming in BASIC was the basis of the Nottingham Programming in Mathematics Project reported on by Hart (1979, 1982). In this project, mixed ability classes of first and second year secondary pupils (11/12 year-olds) were given a course of elementary programming in BASIC before they learned any algebra. The course was built around a pigeon-hole model of a variable and involved the children in problem solving activities using the computer. All the problems were to be solved by writing simple programs in BASIC and there was no use of any pre-written software. The results of the children involved with the computer on the CSMS algebra test were then compared with similar classes who had not had the benefit of such a course. From these Hart was able to conclude that those involved in the project had generally achieved better results than classes which were not, and that they found the concepts of variable and function easier after their programming activities. He found too that the classes using the computer also did better than the ´norm´ suggested by the CSMS work. Unfortunately it was not possible in this project to exercise proper controls over the results due to the logistics of the schools and their desire for all classes to use the computers.

The present research was intended to try and improve on the results of the studies mentioned above, extending the work to include software of the generic organiser type to help pupils get a deep understanding of the basic concepts of algebra which would in turn enable more of them to tackle algebra at a higher level than the general school population. Hence a computer program was specially written in the form of an environment in which pupils could manipulate examples of variables with the goal that through such direct experience and control they could come to understand them better. The computer program was intended to relate to the programming in the earlier part of the module and also to extend the concepts to include comparison of expressions, simple linear equations in one unknown and simple inequalities in one unknown. The program which was written is given in Appendix D and a full description of its use is given in Chapter 7.

6.2.1 <u>The Concept of Variable Used</u>

I have previously discussed how it is difficult to find agreement in the literature on a single definition of the concept of variable, and so it was necessary to formulate a definition for algebra which would be used in this research. Küchemann (1980, p.110) considered that "it would seem reasonable to argue that the concept implies, in particular, some understanding of how the values of an unknown change" and it seemed that, given the way the term is used in the computer context as a name of a location which may hold many values but only one at a time, the definition to use this research in elementary algebra was of a label representing a complete range of values from a given domain. This use of the term variable would enable differentiation between the use of letters as objects, specific unknowns, generalised numbers and variables, since, for example, in response to the question :

For what value or values of x is $3 + x > 7$?

the answer

	$x = 5$	might indicate understanding of x as a specific unknown
	$x = 5, 7, 10$	might indicate understanding of the use of x as generalised number
and	$x > 4$	gives evidence of understanding of x as a variable, able to represent a complete range of numbers.

6.2.2 The Cultivation of Relevant Mental Imagery

A fundamental aim of this research was to provide a mental model of variable as defined above which would be readily accessible to the children and around which a programme of computer programming and related instruction could be built. It was recognised that, in view of the importance of mental imagery

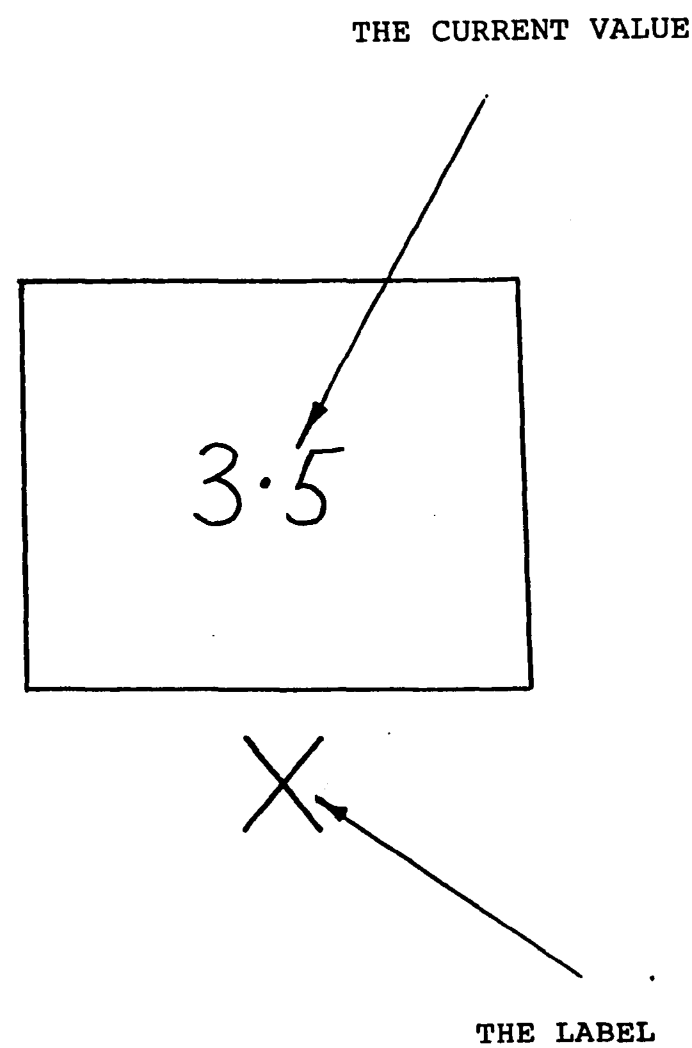


Figure 6.2

The Mental Model of a Variable

in the learning process (see Chapter 3), for the approach to promote cognitive integration the imagery promoted at each stage and its relevance to mathematical content were vital considerations. Thus the model chosen consisted of two parts, a location or ´box´ containing the current value of the variable and a label by which the variable was known. Restricting the label to a single letter was designed to enable the relationship between letters used in algebra and these locations to be more easily perceived, and to encourage the formation of links between the mental images of letters and the conceptual use of letters other than in language, something which it was known that children find very difficult. The mental image of variable promoted is given in Figure 6.2. The view was also encouraged that this ´box´ was in fact a window onto a range of values, the value seen being only one of a range of possible values. The advantage of this simple model was that, being stripped of all extraneous detail the image would be easy to assimilate. Comparing it, for example, with that used in the Nottingham project of Hart, which required the picking out from a matrix of a particular cellular value, we may see that the simplicity of the model with no distracting features is part of the power of the image, making it easier to relate to and remember, and thus able to form the basis for the acquisition of a workable concept image (Tall and Vinner, 1981) of the concept of variable in algebra. Furthermore it was expected that, being immersed in an environment rich in relevant visual imagery, on the computer screen, the pupils would be encouraged to form their own appropriate mental images. These images, according to the principles which I have set forth in Chapter 3, are a vital part of the cognitive integration necessary for the promotion of relational understanding.

6.2.3 The Capacity for Concretisation of Formal Thinking

The theories of learning mathematics on which this research is based imply that for children at the concrete operational level of development to learn a formal, abstract concept they need a concrete model as something on which they can ´hang´ the concept. I have argued that it this model which is necessary in order to produce the mental images which must underpin learning at this level. Once they have assimilated the concept the model is not as important, and the concept may be applied outside the model, with reference to the mental images formed from the model as necessary. Whilst no concrete model can probably fulfil all the requirements of the formal definition of a concept, the computer can be a valuable aid in concretising and personalising formal, abstract thinking for children, by placing them in a programming or software environment which is rich in the materials for building models which lead directly to mathematical concepts, which many would find difficult to learn in a more traditional learning situation.

Dienes (1960), following a Piagetian approach to learning, has listed the stages in the formation of a mathematical concept as follows:

1. Free play
2. Games
3. The search for communalities

4. Representation

5. Symbolisation

6. Formalisation

He has also described the types of activity which will enable these stages in concept formation to be successfully reached. They are :

1. The Play Stage - This is a time of undirected activity with the ingredients of the concept available as play material.

2. A great number of experiences of varying structure but all leading to the same concept. Although this stage is more directed and puposeful it is characterised by a lack of any clear realisation of what is being taught.

3. Adequate practice for the fixing and application of the concept. The games used in this stage are :

a) Preliminary games

b) Structured games

c) Practice games and/or analytical games.

Since it seems clear that children learn constructive thinking long before analytical thinking, in order to follow Dienes recommendations, and requirements of the above stages, it is necessary when devising learning situations to concentrate on constructive tasks, only gradually introducing the analytical aspects. Thus Dienes recommends that concrete material should be made freely available to the children, preferably individually, or in two´s or three´s, since the individual´s learning of a concept is a process which would be hampered by lack of individual capacity for advancement.

Dienes also promotes the view that if a concept involves variables of any kind, then these should, wherever possible, be made to vary in the experience of the child, whilst keeping the concept intact. Applied to elementary algebra, this emphasises the need for personal experience of the varying of the numbers represented by a single letter in as many different situations as possible. Whilst Dienes' theory has appeared to be of value in the learning of algebra, where abstraction plays a fundamental role, it is true that there has been some doubt expressed as to whether it is a sufficiently broad approach to the learning of mathematics. Borasi (1984), for example, has put forward the view that several approaches to concept formation, including abstraction, should be devised and taken into account arguing that the in-depth study of specific instances, or models, may play a fundamental role in the acquisition of mathematics. One reason why concept abstraction may be difficult in many traditional teaching environments is that the rote learning used is a dissociated model, unconnected with the concepts behind the new ideas being taught. To use Skemp's (1979c) terminology it is encouraging the formation of associative or A-links rather than the conceptual or C-links necessary for relational and thus meaningful learning. The use of the computer in the teaching of algebra provides an environment rich in variables and, when combined with an appropriate readily assimilable model of them as introduced above, namely a store location with a label and a value inside, then this gives a concretisation of the concept of variable in algebra which may aid its abstraction.

6.2.4 Encouragement of Problem Solving Abilities

Encouraging the use of computers in schools, Cockcroft (1982 p.409) says that special attention should be paid to the development of computer programs for mathematical activities which will encourage problem solving and logical thinking in a mathematical context. Similarly the Stanhope Report (1987, p.50) on the role of new technology in mathematics education encouraged the development of projects which would "illustrate a range of teaching styles which can be used to support problem solving and investigation." Problem solving on the part of the pupil is an extremely important activity for concept formation and the ways in which it may be incorporated in teaching programmes requires serious thought. Engel (1976 p.266) has argued that the computer can help in problem solving situations by acting as a student, saying "the best way to learn something is to teach it. The computer plays the role of the ´model student´". Thus the computer has tremendous potential for education both in terms of programming activity and the use of specially written software generic organisers of the type discussed in Chapter 5. It is the researcher´s view that the computer program developed as part of this research and described below, fits the criteria necessary for use as such a generic organising problem solving tool and that its use as directed in the programme produced will promote problem solving abilities in algebra.

6.2.5 The Capability for Personalisation of Formal Thought

Any approach to learning which seeks to combine elements of the theories of Piaget, Ausubel and Skemp, implies that to be successful, attention must be paid to individualised learning, since these educators agree that all children, even of the same age, are not necessarily at the same point of cognitive development nor is the route to concept acquisition necessarily the same for each individual. Tall and Vinner (1981), have emphasised the need to consider the individual's concept image, which they define as all the cognitive structure in an individual's mind that is associated with a given concept. These concept images may have aspects quite different from the formal concept definitions and from the concept images of other individuals. They may even contain erroneous ideas which may or may only come into play on certain occasions. Thus, in a consideration of algebraic concepts and their teaching, it may not necessarily be the case that one should seek to remove all of a pupil's earlier stages in the understanding of the use of letters, but rather to make available to them and their expanding concept image extra levels of understanding, such as an understanding of the use as generalised numbers or variables, which are then available for use as and when necessary. This seems to agree with what some more expert mathematicians have found to be true, that on occasion in mathematics they use procedures from their concept images which may be at a low level of understanding (or even erroneous), in the cause of pragmatism. That this is the case in early algebra is born out by the work of Küchemann (1981b) who found that children used their own ideas of what letters stood for even after being taught algebraic procedures. Thus these became part of their concept image, even though erroneous, and were often used. As Wagner et al (1984) and others have argued, the best way to imrove this situation is probably not to try and eradicate such erroneous ideas but rather to provide children with experiences and concepts which will help them to recognise the deficiencies of their methods and ideas and will cause them to readily abandon them in favour of ´correct´ ones. The idea that children may have in their concept image of the use of letters in algebra ideas from several of the levels of understanding of the use of letter as given by Küchemann (1981b), and that they may use parts of this image which may be inappropriate or even erroneous, has been considered in this study. One way in which this has been done is by recognising that any programme that is to be successful in improving relational understanding is likely to be based on individual work, combined with the opportunity to compare informally with others the contents of one´s concept image. Skemp (1979c, p.163) refers to these important processes in reality construction as the use of reality building and testing modes 1 and 2 (see Chapter 2).

Papert (1980), in his work on the LOGO programming language has shown that, when children are allowed to ´program´ a computer, using it as an object to think with, they can be brought into contact with some of the deepest ideas in mathematics. He has shown that, in effect, children can be guided to teach themselves mathematical concepts if they are put in a computer environment. His work has shown the value of the computer in personalising problems and making them into challenges. Traditional problems in algebra such as :

$$3 + 5x = 7 \text{ therefore what is } x = ?,$$

are often not seen by the children as a problem relevant to them. The lack of any obvious use to them in the problem results in a low level of motivation. In contrast, the use of the computer often means that the problem is presented as a personal challenge with the pupil against the computer. It is thus highly relevant to them in this context and they are therefore more motivated to understand the context in order to obtain a 'winning' solution.

6.2.6 Improving an Understanding of Algebraic Notation

In addition to the specific, variable related benefits from computer programming, the intrinsically high level of interest generated in children by the use of the new technology, the visual reinforcement and consequent construction of mental imagery encouraged by the representation of concepts on the screen of the computer, the need for a precision of input to the computer in order to obtain results was expected to be an advantage from the use of the computer. This last point is very valuable in helping children to understand the importance of algebraic conventions of notation since they learn to appreciate that the computer needs expressions which are precise and not just nearly right. They find that in order to be successful at all in using the computer one has to pay particular attention to the precision of notation, such as the use of brackets and the inclusion of all arithmetic operators or the computer soon reports an error of notation (syntax). The order in which instructions or operations are given is also important since this too may affect the results. This problem of the correct use of notation is one of the areas where errors are commonly made by early learners of algebra (Booth 1983a). Their lack of understanding soon becomes apparent to them in the computer environment, for example with regard to the use of brackets. Children quickly realise that the use, non-use or wrong use of brackets affects the answer from the computer e.g. when finding the average of two numbers using a wrong expression such as a + b/2 instead of (a + b)/2, when they are encouraged to check the validity of their results by using small values.

6.3 The Cognitive Instruments Developed for the Research

It was expected that the computer programming to be used in the research programme would provide the type of cognitive benefits outlined in Chapter 5. Working, and being involved in, an environment where letters were to be used abundantly to represent ranges of numbers was considered the ideal basis for an introduction to the intricacies of variables in algebra.

In Chapter 5 I have explained in some detail that, although there is great benefit to be had from such computer programming in mathematics education, the use of mathematical microworlds (Thompson 1985b) or generic organisers (Tall 1986c), where students can build their knowledge structures through intimate involvement with examples of concepts hold even greater promise. To harness this power for the better understanding of algebraic concepts is was necessary to write a program which would enable pupils to manipulate examples of variables in a problem solving environment so that they could come to abstract their properties.

Hence, an instrument designed specifically for this programme, was such a computer program allowing algebraic problem solving. An important feature of the program was that it should allow normal algebraic input, since to do otherwise would cause interference with the students' mental imagery and its links to the higher cognitive concepts. This is not a

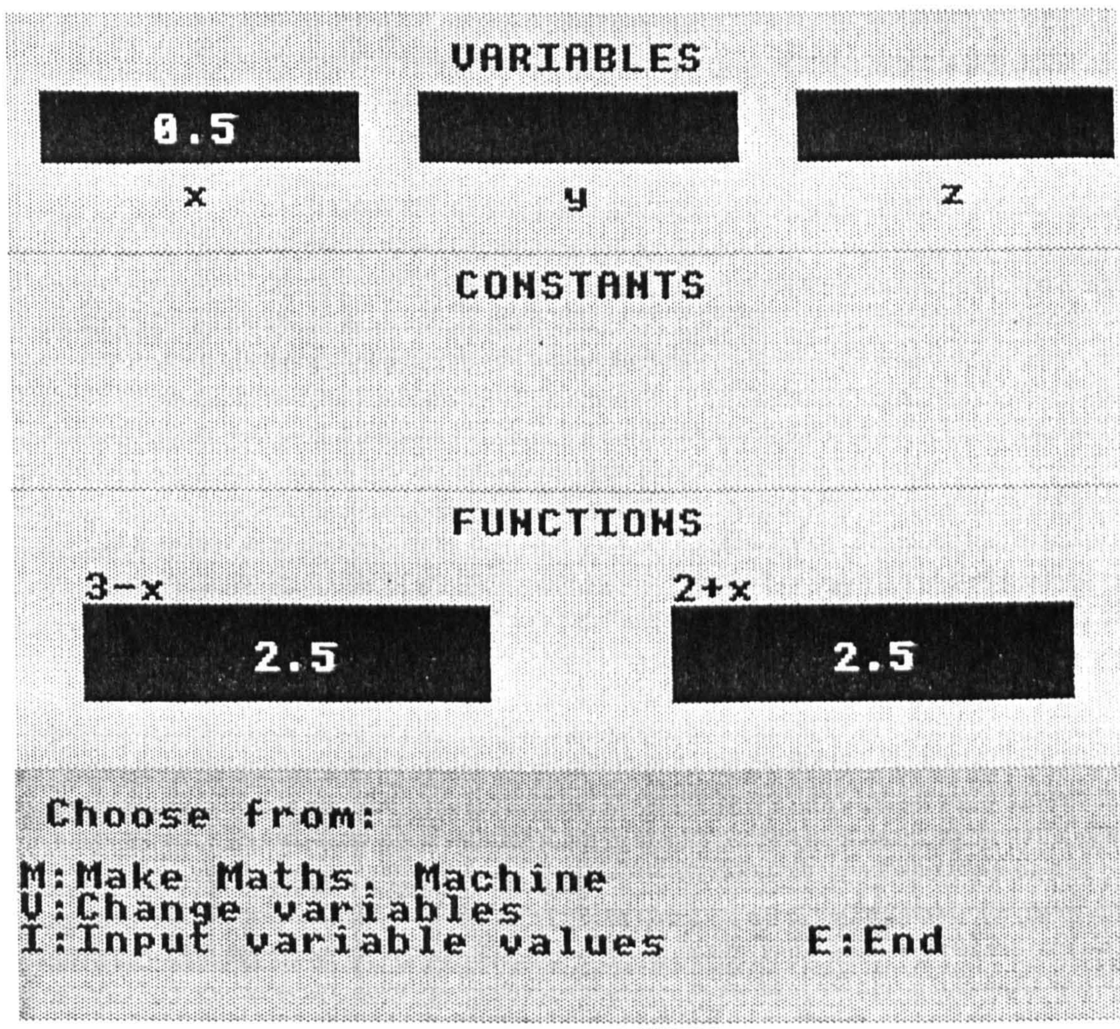


Figure 6.3

A Typical Screen From The Use Of The

´Maths.Machine´ Algebra Programme

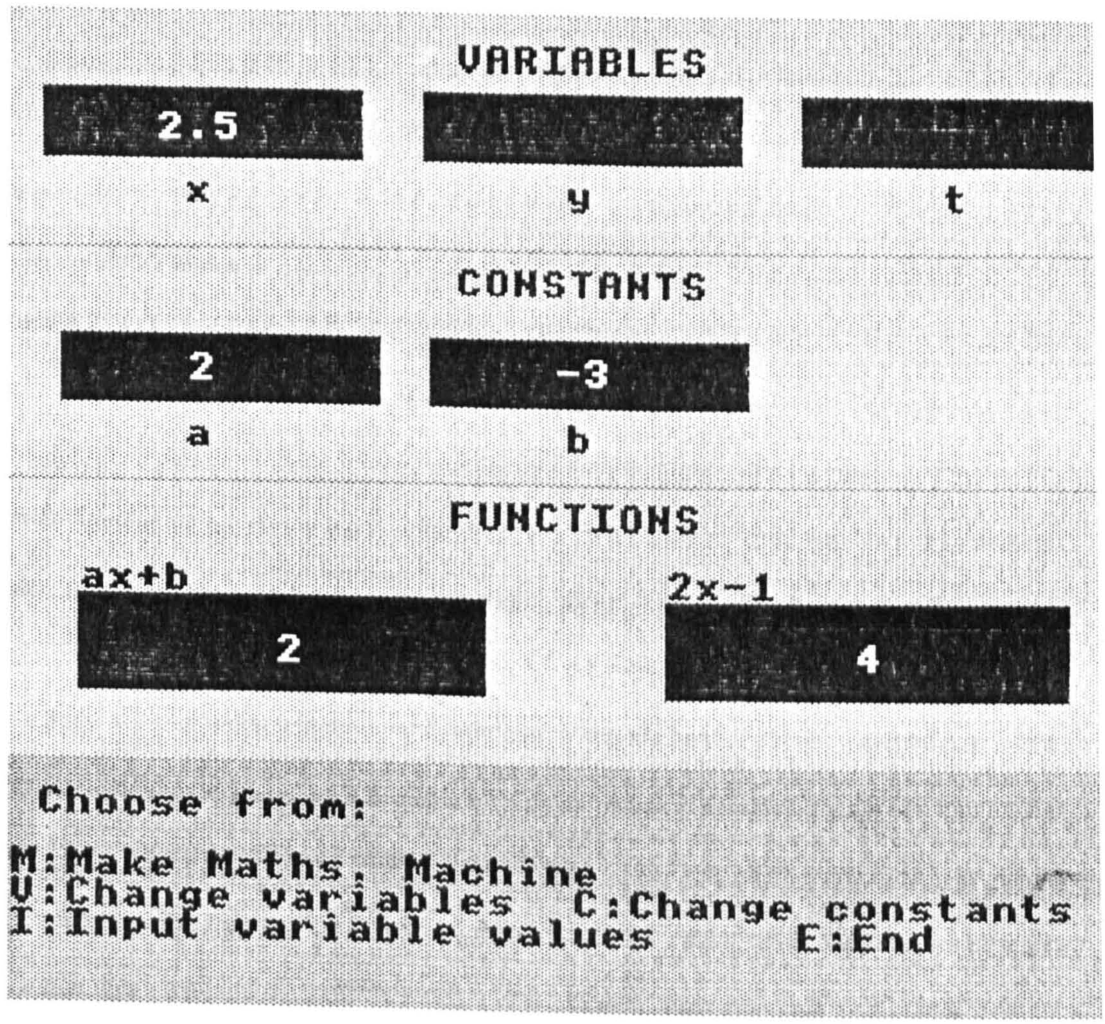


Figure 6.4

A Typical Screen From The Use Of The 'Maths.Machine' Algebra Programme trivial problem on most micro-computers since, as explained above, it is normal for computer to expect operators to be explicit in algebraic expressions and so most computer programs available are not able to emulate standard algebraic notation, requiring for example 2*a*b^2 for $2ab^2$. The machine code routine to enable this type of input was written by Tall, who kindly donated it for use in the program developed, and so its listing is not included with the program. A fundamental aim of the software was to enable children to develop their algorithmic thinking through tackling structured practical algebraic problems by a concrete process, at the same time re-inforcing the mental model of a variable. The program allows side by side comparison of the values of different expressions by displaying two boxes with their corresponding formulae written over the top, the boxes containing the current values relating to each formula. The display also allowed for the values of some variables to be inputted and for others to be fixed or constants. This computer program enabled the pupils to experiment with solving linear equations and inequalities on the computer, being able to see the current value of variables and functions on the screen in the same format as the mental image encouraged, and to be in control of them as their values were made to vary. The program, which was called a software 'Maths. Machine', is given in Appendix D and typical screens from the use of the program are given in Figures 6.3 and 6.4. The ways in which the program was actually used in the teaching module is described in detail in the notes on the main experiment in Chapter 7.

The third, and very important, part of the process of building understanding of algebraic concepts using a computer environment, and called the cardboard 'Maths. Machine', was not included until after the initial investigation. I shall therefore describe it and its inception after I have described this investigation.

6.4 <u>The Initial Investigation</u>

The school in which the first experiment was to take place was a co-educational independent school where entry depended on passing an entrance exam. It was thought that the population might not be typical of the population of schoolchildren in general and so a pilot study was undertaken in order to examine the general level of algebraic ability in the school.

In order to ascertain this level, a sample of the pupils in the school was chosen and compared with the average schoolchild by giving them a test based on the CSMS algebra test. This test, presented in Appendix A, was given to a randomly chosen class from each of the years two to four. The results of these tests are in Appendix B. To test the hypothesis that the results of the school population did not differ from those of the relevant population at large, as given by the CSMS norms, a chi-squared test of the difference in means was calculated. The value of chi-squared was 760.6 and hence it was concluded that the hypothesis must be rejected ($p<0.0001$) and the results of the school were significantly better than the general secondary school population. This meant that the results of the first experiment related specifically to pupils of higher than average algebraic ability.

It was noted from this pilot study, that the pupils,

15. Simplify :

a) $3a - b + a$ .... 4a − b ✓

b) $2p - 3q + 8q$ .... 2p + 5q

16. If $c + d = 15$ and $c = 2d$ then $d = ?$ 5

17. Simplify $(x - y) + y$ .... x [redacted]

18. Multiply $2n + 3$ by 7 .... 14n + 21

19. Which is larger $2n$ or $n + 2$? Explain. there is no way of telling because we do not know the value of n

20. Does $a + b + c = a + n + c$, sometimes, always or never ? Explain. Never because n does not equal b

21. I buy b blue pencils at 5p each and r red pencils at 6p each. Altogether they cost me 90p. What can you say about b and r ? .... b=6 r=10

22. If $(x - 1)^3 + x = 351$ when $x = 8$, what value of x makes $(4x - 1)^3 + 4x = 351$ true ? .... 2 ✓

Figure 6.5

An Example of Errors - Pilot Study

15. Simplify :

a) $3a - b + a$ .... 4a-b ✓

b) $2p - 3q + 8q$ .... 7p-11q ✗

16. If $c + d = 15$ and $c = 2d$ then d = ? .... ✗

17. Simplify $(x - y) + y$ .... x ✓

18. Multiply $2n + 3$ by 7 .... 23n ✗

19. Which is larger $2n$ or $n + 2$? Explain. 2n, because it is being multiplied

20. Does $a + b + c = a + n + c$, sometimes, always or never ? Explain. No, because there are different letters in the answer

21. I buy b blue pencils at 5p each and r red pencils at 6p each. Altogether they cost me 90p. What can you say about b and r ? .... r=40, b=50 ✗

22. If $(x - 1)^3 + x = 351$ when $x = 8$, what value of x makes $(4x - 1)^3 + 4x = 351$ true ? .... ✗

$(8-1)^3 + 8 = 351$

$7^3 + 8 = 351$

$343 + 8 = 351$

$49 \times 7 = 343$

6

Figure 6.6

An Example of Errors - Pilot Study

15. Simplify :

a) 3a − b + a   4a − b   ....4a − b... ✓

b) 2p − 3q + 8q   2p + 5q   ....2p + 5q. ✓

16. If c + d = 15 and c = 2d then d = ?   ......5...... ✓

17. Simplify (x − y) + y   x + y − y + y   ....x + y... ✗

18. Multiply 2n + 3 by 7   .... ✗

19. Which is larger 2n or n + 2 ? Explain.   2n is larger because n is double

20. Does a + b + c = a + n + c , sometimes, always or never ? Explain.   never n cannot be the same value as b

21. I buy b blue pencils at 5p each and r red pencils at 6p each. Altogether they cost me 90p. What can you say about b and r ?   b = 9 r = 7.5

22. If $(x - 1)^3 + x = 351$ when x = 8, what value of x makes $(4x - 1)^3 + 4x = 351$ true ?   ......2...... ✓

$(8 - 1)^3 + 8 = 351$

7

6 × r + b × 5 = 90

5 × 5 = 25p   6 × 6 = 36p

9 × 5 = 45

7.5 × 6 = 45.0

49 × 7 = 343, 343 + 8 = 351

4 × 2

Figure 6.7

An Example of Errors - Pilot Study

although of high algebraic ability, were also making some of the errors noted by Küchemann (1981b) with regard to the use of letter as a variable. It was also interesting to note that the errors which did occur tended to recur throughout the three years, possibly showing that traditional schooling was having little effect on these particular errors, as was a feature of the errors reported on by Küchemann (1981b). Figures 6.5, 6.6 and 6.7 show extracts from the pilot study of typical test papers of pupils in the second, third and fourth years respectively of the secondary school In each of the questions 19, 20 and 21, a misunderstanding of the use of letter in algebra is displayed. In question 19, for example, both of the third and fourth year pupils, although having spotted the obvious first order relationship $2n>n + 2$, have failed to progress to the point where they see that the difference in the value of the two expressions in this inequality decreases to zero at $n = 2$, and then becomes negative - i.e. they have failed to perceive n as a variable, according to Küchemann's definition. Their emphasis on the operations involved is possibly because of their lack of understanding of the concept of variable. In question 20, all three of the pupils have the idea, clearly expressed in their answers, that, since b and n are different letters, they cannot have the same value and therefore must stand for different numbers. Thus even if they have managed to see b and n as generalised number, able to take several values, they can not see them as having any values in common, a common misconception. In question 21, again all three of them have attempted to assign a single, fixed value to each of b and r, rather than trying to form and symbolise a

ALGEBRA TEST

NAME :   FORM :

Attempt all the questions. Write your answers in the spaces provided.

1. Find a if a − 3 = 9 ……… a = 12 ✓

2. If x + y = 31 , what does x + y + 5 = ? ……… 36 ✓

3. Simplify :

   a) 4y + 6y ……… 10y ✓

   b) 9x − 3x ……… 6x ✓

4. A square has sides of length b cm. Write down an expression for the perimeter of the square. ……… b + b + b + b = 4 ✓

5. Find b if 3b = 12 ……… b = 4 ✓

6. Write down the area of a rectangle of length p cm. and width q cm. ……… pq cm² ✓

7. Calculate p if p = 3q + 5 and q = 2 ……… p = 11 ✓

8. Calculate a if b = 2a − 4 and b = 10 ……… a = 7 ✓

9. Simplify :

   a) 3x + 9y + 2x ……… 5x + 9y ✓

   b) 5z + 3x + 2z + 5x ……… 7z + 8x ~~7x + 8~~ ✓

10. Add 6 onto a + 2 ……… a + 8 ✓

11. Add 5 onto 3m ……… 3m + 5 ✓

12. A polygon has n sides, each of length 2 cm. Write down the perimeter of the figure. ……… 2n ✓

13. If x + y = 7 then x + y + z = ? ……… 7z ✗

14. If r = s + t and r + s + t = 30 then r = ? ……… 15 ✓

Figure 6.8

An Example of Errors - Pilot Study

ALGEBRA TEST

NAME : **FORM :**

Attempt all the questions. Write your answers in the spaces provided.

1. Find a if $a - 3 = 9$	..12........ ✓
2. If $x + y = 31$, what does $x + y + 5 = ?$	..36........ ✓
3. Simplify :	
a) $4y + 6y$	.10y........ ✓
b) $9x - 3x$	..6x........ ✓
4. A square has sides of length b cm. Write down an expression for the perimeter of the square.	..b^4........ ✗
5. Find b if $3b = 12$	...4........ ✓
6. Write down the area of a rectangle of length p cm. and width q cm.	..p.q........ ✓
7. Calculate p if $p = 3q + 5$ and $q = 2$	.p=11........ ✓
8. Calculate a if $b = 2a - 4$ and $b = 10$	.a=7........ ✓
9. Simplify :	
a) $3x + 9y + 2x$	..~~11xy~~ 5x+9y........ ✓
b) $5z + 3x + 2z + 5x$	.8x+7z........ ✓
10. Add 6 onto $a + 2$	..8a........ ✗
11. Add 5 onto 3m	..8m........ ✗
12. A polygon has n sides, each of length 2 cm. Write down the perimeter of the figure.	..2n........ ✓
13. If $x + y = 7$ then $x + y + z = ?$	...7z........ ✗
14. If $r = s + t$ and $r + s + t = 30$ then $r = ?$	..15........ ✗

Figure 6.9

An Example of Errors - Pilot Study

ALGEBRA TEST

NAME :   FORM :

Attempt all the questions. Write your answers in the spaces provided.

1. Find a if a − 3 = 9 ....12 ✓

2. If x + y = 31 , what does x + y + 5 = ? ....36 ✓

3. Simplify :

   a) 4y + 6y ....10y ✓

   b) 9x − 3x ....6x ✓

4. A square has sides of length b cm. Write down an expression for the perimeter of the square. ....4b cm² ✓

5. Find b if 3b = 12 ....4 ✓

6. Write down the area of a rectangle of length p cm. and width q cm. ....pq cm ✗

7. Calculate p if p = 3q + 5 and q = 2 ....11 ✓

8. Calculate a if b = 2a − 4 and b =10 ....7 ✓

9. Simplify :

   a) 3x + 9y + 2x ....5x + 9y ✓

   b) 5z + 3x + 2z + 5x ....7z + 8x ✓

10. Add 6 onto a + 2 ....6a+2 ✗

11. Add 5 onto 3m ....15m ✗

12. A polygon has n sides, each of length 2 cm. Write down the perimeter of the figure. ....NOT POSSIBLE ✗

13. If x + y = 7 then x + y + z = ? ....9 ✗

14. If r = s + t and r + s + t = 30 then r = ? ....15 ✓

Figure 6.10

An Example of Errors - Pilot Study

relationship between them based on sets of values of the letters which they may have perceived (there is no certainty that they had not seen other sets of values for b and r but saw no need to record these). All these errors show a lack of understanding of the use of letters as generalised numbers, or as variables.

Figures 6.8, 6.9 and 6.10 are again taken from the pilot study tests of three other pupils in the second, third and fourth years of the school, respectively, and they illustrate some other types of errors made. One of the most common errors made is that of conjoining in addition (Booth 1983b), i.e. of writing, or using, for example, a + b as ab. This type of error is seen in question 13 in Figures 6.8 and 6.9, question 11 in Figures 6.9 and 6.10 and in question 10 in Figure 6.9. We see also the confusion of notation, or symbolisation of expressions, in question 4 in Figure 6.9, where b^4 has been confused with 4b, and in question 6 in Figure 6.10, where 2p has probably been confused with p^2. This confusion between additive and multiplicative notations is a common area of error (Booth 1983b).

Thus, although the population under discussion here was seen to be of higher than average algebraic understanding on the tests given, there were errors being made, particularly with regard to the understanding of the use of letters as variables. It was, therefore, considered to be worthwhile carrying out an experiment, based on a teaching programme built around programming on the computer, with a sample from this population, to see if this position with regard to understanding could be improved.

6.5 The Details of the First Investigation

In this section I shall look at the details of the first of the two experiments comprising the initial investigations. The subjects, design and procedures of the experiment are considered as well as the results and conclusions which affected the form and content of the second of the investigations.

6.5.1 The Subjects

The subjects were chosen from among ten and eleven year olds in the first year of a co-educational independent secondary school. Forty of the members of the year group were chosen on the basis of their ability to be put into matched pairs using their verbal reasoning and mathematical quotients (VRQ and MQ) obtained from the results of their entrance exam into the school (see Appendix C).

6.5.2 The Instruments

An algebra test based on the CSMS Algebra 1 test was used as a pre-test, and is given in Appendix E, and the actual CSMS algebra test was used for both the post-test and the delayed post-test, and this is given in Appendix F. This test was used since it had been specifically developed on the basis of two main criteria, namely 'the structural complexity of the items, and the meaning that can be given to the letters' (Küchemann 1981, p.102).

6.5.3 The Research Design

The research design used in the first experiment was the pre-test, post-test, control group design with matching. The pupils were divided randomly into matched pairs to form two groups, a control group and an experimental group. Both groups were then given a pre-test. The experimental group received the programme of instruction based on programming of a micro-computer and use of the software 'Maths.Machine', following which both groups were immediately given a pretest. During the period of instruction the control group continued with their normal timetable of lessons. After five months both groups were given a delayed post-test.

6.5.4 <u>The Procedure</u>

The lessons below were taught by the researcher during a single week in June 1984. All the members of the experimental group were taught together in one room, having the use of a computer between two children. The teaching module consisting of the five forty minute lessons used is given in Appendix G. All three of the tests used were administered to both groups during the pupil's normal lessons, at the same times, in order to eliminate errors due to interaction between them. The immediate post-test was given 3 days after the end of the teaching programme, and the delayed post-test was given six months later, in December 1984.

6.5.5 <u>The Results</u>

The hypothesis under test in the first experiment was that the teaching programme would produce an improvement in the understanding of the children of the uses of letters in algebra, particularly their use as variables. In order to test this hypothesis, the results of the post-test and delayed post-test were analysed using a t-test for non-independent samples (the groups being in matched pairs). The data used in this analysis are given in Appendix H. The null hypothesis was that the mean performance of the experimental group was less than or equal to

TEST (max.=50)	EXPERIMENTAL MEAN	CONTROL MEAN	MEAN DIFF.	S.D.	N	t	df	p
POST-TEST	37.33	35.50	1.83	7.94	18	0.95	17	---
DELAYED POST-TEST	38.36	36.14	2.22	8.62	18	1.06	17	---

Table 6.1

A Comparison of Experiment 1 Performance on Post-Test and Delayed Post-Test.

the mean performance of the control group. This was tested using the appropriate t-test statistic. A summary of the results for all the tests is given in Table 6.1. A probability level of $p=0.05$ was chosen for this analysis as the most appropriate level, after consideration of the possible Type I and Type II errors. It was concluded from the analysis on the immediate post-test that the null hypothesis could not be rejected and that therefore the mean performance of the experimental group was probably not better than that of the control group. It can be seen from Table 6.1, that a similar conclusion was arrived at on the basis of the analysis of the delayed post-test results. A full analysis of covariance was in fact carried out on these results mentioned above, but it confirmed this conclusion and its details are not given here.

The question remained however - Had there been any specific improvement in the experimental group's understanding of the use of letters as generalised numbers or as variables? This question still needed looking at since the tests used had included questions at all four of the levels of difficulty proposed by Küchemann (1981b), whereas as he asserts, only questions at levels 3 and 4 require an understanding of these concepts. Furthermore it was on questions of this type that the errors in the pilot study had been observed. It was decided therefore that it was necessary to calculate other statistics in order to fully answer this question.

In order to arrive at a clear idea of the answer, the results at levels 3 and 4 were compared using a t test. A summary of the results is given in Table 6.2. From these results also it was concluded that it was likely that the mean

TEST (max.=21)	EXPERIMENTAL MEAN	CONTROL MEAN	MEAN DIFF.	S.D.	N	t	df	p
POST-TEST	10.97	10.50	0.47	4.81	18	0.40	17	---
DELAYED POST-TEST	11.78	10.64	1.14	4.88	18	0.96	17	---

Table 6.2

A Comparison of Experiment 1 Performance on Level 3 and 4 Questions of the Post-Test and Delayed Post-Test.

Group	Pre-Test(%)	Post-Test(%)	Delayed Post-Test(%)	Question(a)
Experimental	10	5.6	21.2	larger, 2n
Control	10	10	10.6	or n + 2 ?
Experimental	80	66.7	58.3	Simplify
Control	60	60	58.3	(x - y) + y
Experimental	55	38.9	63.6	L + M + N
Control	30	40	42.4	= L + P + N

(a) Brief description only given. The full question is given in the post-test in appendix F.

Figure 6.11

A Comparison of Some Experimental and Control Group Facilities on Questions Involving Understanding of Variables performance of the experimental group in the understanding of the use of letters as variables, as measured by the tests, was not significantly better than that of the control group, based on both the pre-test results and the delayed post-test results.

To try and ascertain if the pupils had found the model of a variable etc. in the teaching programme beneficial in any way, the three questions, at the level 4, where errors had been noted in the pilot study were looked at. Since these also involved the most difficult aspects of the concept of variable, they were therefore the area in which, for a group of above average algebraic ability, any improvement in understanding would be likely to be evident. The results from these three questions are given in Figure 6.11. The fall in performance of the experimental group in the post-test in each instance, followed by a considerable improvement in performance in the delayed post-test in two of the cases, seemed to illustrate that the programme had affected the area of their cognitive structure associated with these concepts, but that a process of re-structuring of their schemas had been necessary resulting in a temporary fall in achievement. This would confirm the findings of Booth (1983b) that time is required for the accommmodation of these difficult concepts. Hence it may be argued then that the pupils found some of the concepts in the teaching programme useful, and benefited cognitively from them, but only after they had had time to incorporate them properly into their cognitive structure. In spite of these accommodation problems with these concepts, once their cognitive structure had been re-organised and appropriate connections in their networks made, they were able to show a significant improvement in performance over both their original level, and that of the control group.

It was encouraging that the teaching programme had produced some improvement in such a difficult area as the understanding of the use of letters as variables in algebra, and that with further refinement and slight redirection, the aim of providing a package which could produce a significant improvement in children's understanding of the use of letters in algebra could be achieved. The improvements decided upon and the reasoning behind them are looked at in the next section.

The effect of the teaching programme on errors of notation, particularly the use of brackets, and formalisation of method were looked at, but there appeared to be no benefit from the programme in these areas, for these pupils of above average algebraic ability.

6.6 Improving the Teaching Module

6.6.1 Problems Arising From the First Experiment

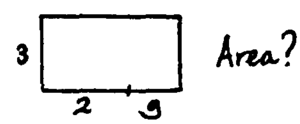
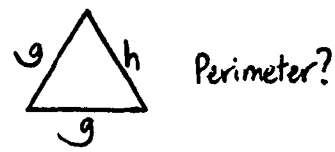
There were a number of problems which were noticeable after the first investigation and whose identification contributed to the improvement of the teaching module.

The first was that since the subjects in the experiment had already been formally taught algebra for one year before the study they had already built up a considerable cognitive structure with regard to the concepts involved. Thus they were more likely to experience problems in assimilation of the concepts presented in the teaching programme and any necessary restructuring of their schemas, since their ideas and knowledge structures were well set and proven in practice. This effect was particularly noticed with regard to items such as : Simplify (a - b) + b . Whereas the pupils had previously had great exposure to questions like this, and had coped with them, it now seemed that they experienced some difficulties. In this case the performance of the group fell, the fall averaging 12.2%, suggesting some new difficulty with questions with which they were very familiar.

The fact that the subjects were of above average ability seemed to increase the type of problem mentioned above, since it would appear that, because the pupils were able to cope well with many of the test items on the pre-test (average 75.5%) (only failing on questions requiring an understanding of letters as variables), they were probably reluctant to change their methods in favour of, to them, an as yet unproven approach. We may describe this in terms of the fact that their cognitive structure relative to many of the questions was well developed and their concept images (Tall and Vinner 1981) contained few errors or omissions and hence they felt no need to 'discover' the value of the new concepts, even in those areas where they would have benefited, such as the use of letters as variables.

Another factor which has to be considered is the fact that the experimental group did better, on average, than the control group on the pre-test (4.9% better on average), although not significantly so, and this, coupled with their relatively high average score of 75.5% gave them somewhat less room to improve than the control group. For example, 13 of the experimental group compared with 9 of the control group scored over 80% correct on the pre-test.

A third area which arose from the investigation was to do with test items such as questions 9d and 12c on the

Group	Pre-Test(%)	Post-Test(%)	Delayed Post-Test(%)	Question(a)
Experimental	30	33.3	37.1	
Control	15	15	26.5	
Experimental	75	77.8	84.1	
Control	55	85	84.1	

(a) Brief description only given. The full question is given in the post-test in appendix J.

(b) Not Given

Table 6.3

A Comparison of Some Experimental and Control Group Facilities on Questions Involving Diagrams pre-test which involved pictures of plane figures with letters written against the sides. It was possible that this may have been the source of some confusion in the minds of the experimental group due to its similarity with the basic picture or model of a variable used (see Figure 6.2), of a rectangle or 'box' with the variable letter underneath. As the facilities in Table 6.3 show, the experimental group appear to have coped somewhat worse relatively with these type of questions than the control group.

In line with the theory developed in Chapters 2 and 3, the final area identified where improvement could be made concerned the need to encourage conceptual or C-links between concepts. The programme used in the first experiment made little effort to guide and encourage the pupils in the making of such links. The discussion in Chapter 5 of the principles involved in teacher guidance demonstrates that it is important to provide pointers for the making of such links, as and when appropriate throughout the programme. It would seem possible that some concepts which the pupils had abstracted from the work were not linked to those concepts in areas of their cognitive structure to do with the understanding of algebra and that a measure of guidance could improve this.

6.6.2 The Improvements Decided Upon

In order to try and eliminate these problems and improve the teaching programme for a second investigation, the following changes in the programme and its application were made.

The concept of the work being an advance organiser (Ausubel 1968) for the more formal teaching of algebra was one which it was felt needed to come more to the fore. In order to fulfil the criteria for an advance organiser or cognitive bridge (Novak 1976), it was first of all necessary to make sure that the programme was applied to pupils who had not yet had any formal teaching of algebra, since otherwise it could obviously not precede this and hence signal its important concepts. To facilitate this, the population chosen for the second experiment consisted of the final year of a middle school, where they had not been exposed to any algebra teaching. To eliminate the problem of only dealing with the upper end of the ability range, who were less likely to benefit from the programme, it was necessary to use a comprehensive school covering a wide ability range, and the school chosen fulfilled this too. The top three-quarters of the school's fourth and final year group was used, the bottom one-quarter being unfortunately unavailable due to the internal logistics of the school.

The imbalance in the groups used in the first experiment, due to using matching based on their VRQ and MQ on entry to the school, was another problem which required attention. To overcome this it was decided to use a pre-test post-test design with matching, but the matching was to be done on the basis of the results of the pre-test, with the post-test being a similar, but different version of the pre-test. With such a relatively small sample (21 in actuality) in the experimental group this was calculated to reduce errors due to sampling.

The difficulty which the pupils seemed to experience over the test items involving pictures of plane shapes needed to be addressed. It was decided to eliminate this form of question from the tests. Several of the original questions were left in the tests, but they were presented in a wholly written format, without the diagram, as it was considered that these were not an essential part of the question. Although this problem of pictures would have to be met, it was considered that, on balance, it was better that this was done later, once the concept of letters as variables had been firmly established in the cognitive structure of the pupils and the mental model was therefore of decreasing importance to them.

6.6.3 The Introduction of the Cardboard 'Maths. Machine'

The problem of providing links to other relevant concepts in the pupils cognitive structure was considered to be very important. Therefore, in the second version of the programme a conscious effort was made to assist the children, in an unobtrusive way, in the making of such links. For example a section was included relating the expressions used in the BASIC programming to their equivalents in arithmetical notation so that a link could be made between the two systems. One of the major changes in the second programme was the introduction of a 'Mathematics Machine' which was to be an exact parallel of the work taking place on the computer, only with the pupils having to work out all the answers to the calculations themselves. The advantage of this was that they would see clearly the calculations which the computer was carrying out, and would be able to link the work to their knowledge of the concepts of arithmetic which they had instead of merely seeing the results of the computer's calculations. The 'Maths Machine' designed comprised two sheets of card, one marked with rectangles to simulate the stores of the computer, to emulate the model adopted for a variable when in use, and the other plain, to act as a screen for 'printing' results onto. Onto the first sheet

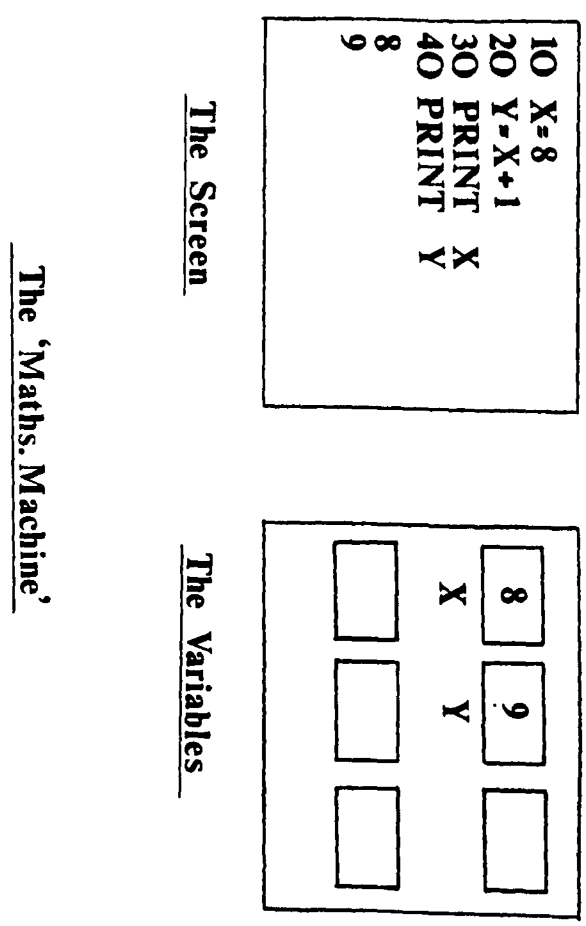


Figure 6.12

A Typical Set-Up Of The Cardboard 'Maths. Machine' squares of card with letters and numbers on were to be placed by the children to make the rectangles into variables, with labels and values. Figure 6.12 shows a typical set up of this machine in use. Instead of merely pressing keys and getting results, this ´Maths. Machine´ was designed to require the children to perform their own calculations and to be physically involved in the model, transferring answers around the room. Initially the boxes on the card were empty and without labels, but on meeting the assignment :

$$a = 3$$

the pupils were required to represent this by labelling one of the boxes "a", and putting inside it a card bearing the number 3. For another assignment :

$$b = a + 1$$

it was necessary for one pupil to label another box "b", another to look inside "a" to find its current value, another to do the calculation by adding one on to that value, to make 4, and finally for one of the group to place inside the box labelled "b" a card bearing the value 4. Thus all of the work and calculations in the ´Machine´ were carried out by the pupils, and it was set up in the room so that it required the pupils to move around the room to operate it, thus getting some feel of the dynamics of a machine. Thus they were to act out the workings of a computer, with the variables and their values constantly in front of them.

6.7 The Details of the Second Investigation

This section deals with the subjects design and procedures of thé second experiment. The improved and extended version of the teaching programme is described, along with the thinking behind the changes.

6.7.1 <u>The Subjects</u>

The subjects of the experiment were the whole of the top three-quarters of the fourth year of a comprehensive, co-educational middle school (the bottom one quarter was not available for internal logistical reasons). The forty-two members of the sample, consisting of 25 boys and 17 girls, aged approximately eleven and a half to twelve and a half years old, were divided into an experimental and a control group, as described below.

6.7.2 <u>The Instruments</u>

An algebra test based on the CSMS algebra test 1 was given as the pre-test, on the basis of which the sample was divided into two groups of matched pairs, and a similar but slightly longer test was used for both the immediate and the delayed post-tests. Care was taken to ensure an even distribution of questions at each of the levels of difficulty described by Küchemann (1981b), in order to be able to monitor progress carefully, and both tests are given in Appendices K and M respectively. The computer program 'Maths.Machine', developed for the investigation, and allowing algebraic input instead of the normal BASIC symbolisation was again prominent in this experiment and its listing is given in Appendix D. A cardboard 'Mathematics Machine', named after the work of Booth (1983b), and described above, was designed to emulate the internal working of a computer and created for this experiment.

The 'Maths. Machine', computer program and lesson plans were designed to be in the form of a kit which could be handed

Control		Experimental	
Lisa	16	Darren	9
Stuart	14	Andrew	14
Kristian	6	Michael	2
Mark	14	Kirsty	15
Debi	5	Mark	5
Jason	4	Julie	3
Brian	20	Robert	26
Lisa	11	Jonathan	11
Steven	13	Claire	12
Lorraine	11	Claire	11
Simon	12	Robert	12
Damon	11	Mark	11
Deborah	17	Stephen	15
Elizabeth	11	Emma	11
Paul	11	Jason	10
Tammy	6	Rachel	6
Gary	9	Jonathan	9
Nicholas	7	Elvis	8
Jayne	7	Victoria	7
Louise	4	Rachael	4
Matthew	6	Marc	7

Figure 6.13

Experiment 2 - The Matched Pairs and Pre-Test Scores over to the classroom teacher as a unit ready for use.

6.7.3 The Research Design

The research design used in this experiment was a pre-test post-test control group design with matching. The matching was done on the basis of the pre-test scores in order to reduce sampling errors. This design has the advantage also of reducing the standard error, since it gives a high correlation between the pre-test and the post-test results. The members of the matched pairs were then randomly assigned to the experimental and control groups. Figure 6.13 shows the pairs and their pre-test scores.

6.7.4 The Procedure

Following the pre-test, and the subsequent division of the pupils into the experimental and control groups, the 'kit' consisting of the computer program, the 'Maths. Machine' and the annotated lesson plans was given to the teacher who had volunteered to teach the programme. A week or so later, after she had familiarised herself with the contents of the lessons and had looked carefully at the materials, the researcher met with her and discussed the minor matters of procedure relating to the implementation of the programme, which she raised. At this stage even, the teacher expressed her enthusiasm for the programme, saying that she wished that she 'had learned algebra this way'.

She began the programme with the experimental group in late January 1985, and finished towards the end of February, spending all three of the groups mathematics lessons each week varying in length from forty minutes to an hour, on the programme, ther equivalent of about 12 sixty minute periods

During this time the control group continued with their normal lessons. Both groups were given the tests at the same time to try and eliminate interaction errors on these. The immediate post-test was given one week after the end of the programme, and the delayed post-test was given in the middle of May 1985. Appendix N contains the raw data from both of these tests.

6.7.5 <u>The Teaching Programme</u>

The lesson notes which were given as part of the kit for the teaching of the programme are given in full in Appendix L. They include a brief idea of the background to the investigation for the benefit of the teacher, and were annotated throughout, using capitals and comments in brackets, for the benefit of the teacher.

The teacher was asked to cover as much or as little of the programme as seemed advisable, since the school was happy that no time limit was set for the completion of the programme, although it was expected that it would cover about four weeks work. She was asked to make sure that all the pupils, who were divided into groups of three because of the number of computers available, were rotated in their groups between the 'Maths. Machines' and the computers each lesson, so that they gained experience regularly with each activity and to note the following points :

1. The point in the programme reached after each lesson.
2. Any points in the programme which the children found difficult.
3. Any pupils in the group who found particular problẻms, and where they found them.

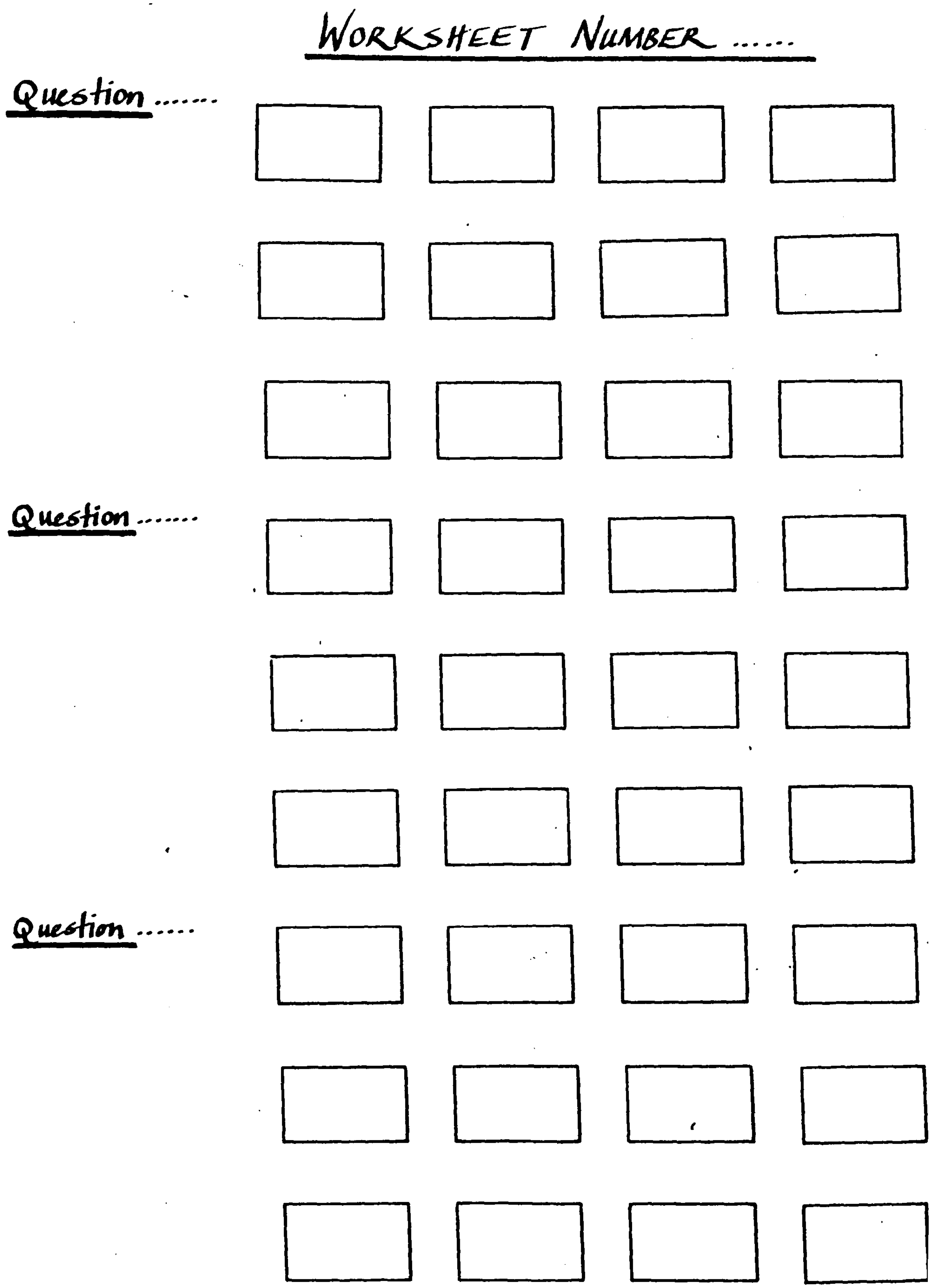
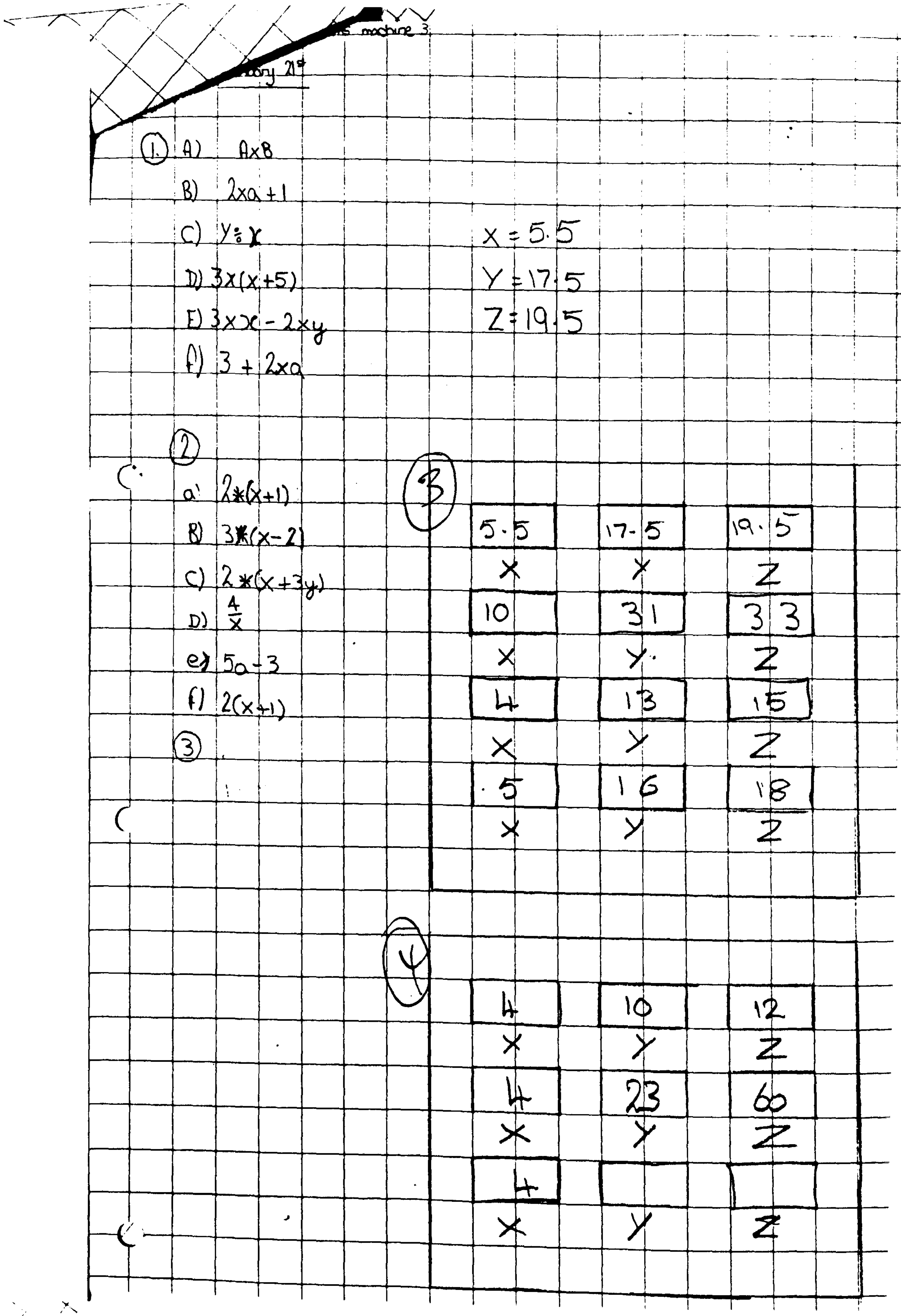


Figure 6.14

A Blank Stores Sheet


...machine 3

...ary 21st

1. A) A×B
B) 2×a+1
C) y÷x
D) 3x(x+5)
E) 3×x − 2×y
F) 3 + 2×a

x = 5.5
y = 17.5
z = 19.5

2.
a) 2*(x+1)
B) 3*(x−2)
C) 2*(x+3y)
D) 4/x
e) 5a−3
f) 2(x+1)

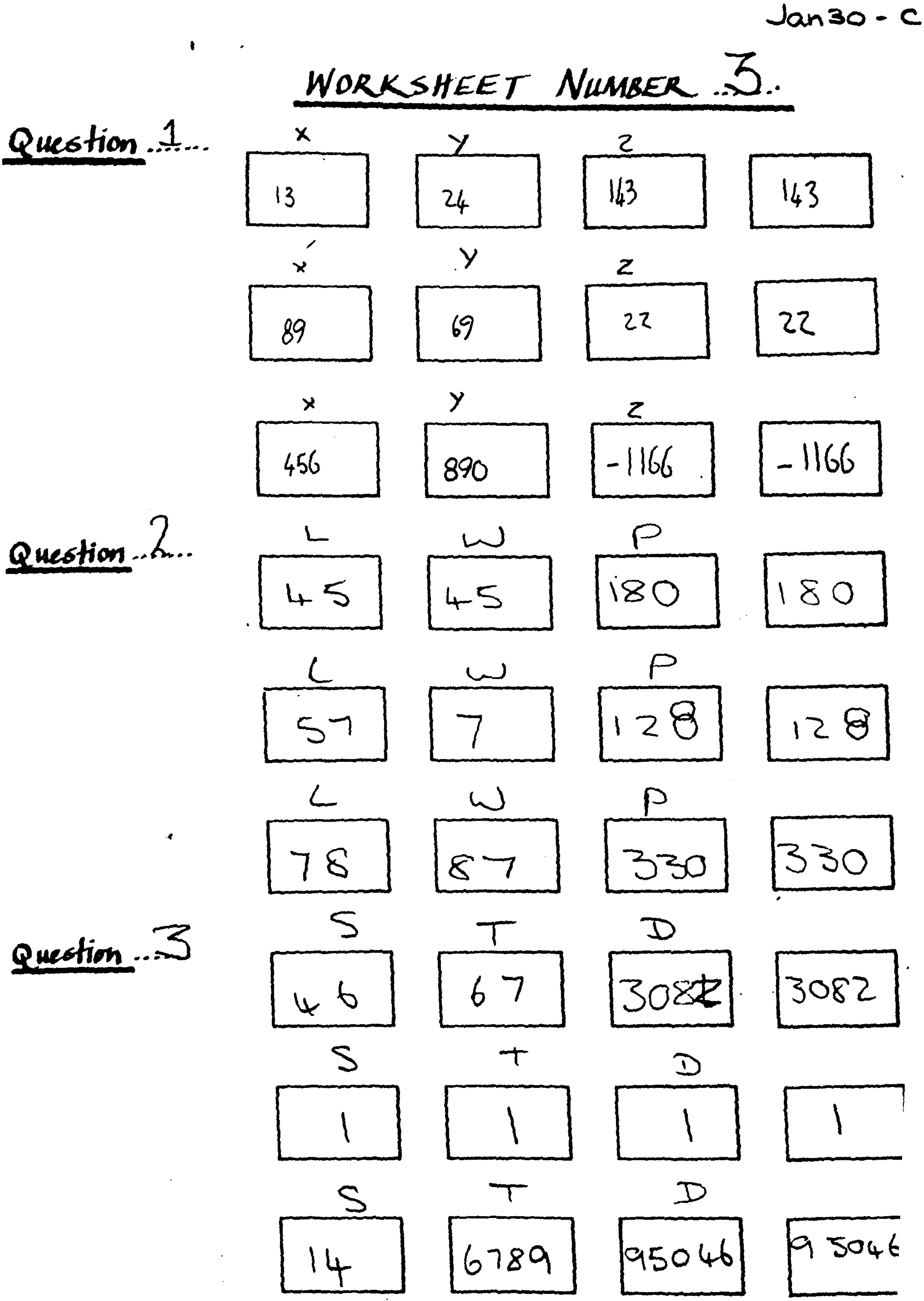
3.

5.5	17.5	19.5
x	y	z
10	31	33
x	y	z
4	13	15
x	y	z
5	16	18
x	y	z

4

4	10	12
x	y	z
4	23	66
x	y	z
4		
x	y	z

Figure 6.15
Experiment 2 - Some Answers to Worksheet 1



Jan30 - C

WORKSHEET NUMBER 3.

Question 1

x	y	z	
13	24	143	143
89	69	22	22
456	890	-1166	-1166

Question 2

L	W	P	
45	45	180	180
57	7	128	128
78	87	330	330

Question 3

S	T	D	
46	67	3082	3082
1	1	1	1
14	6789	95046	95046

Figure 6.16

Experiment 2 - Some Answers to Worksheet 3

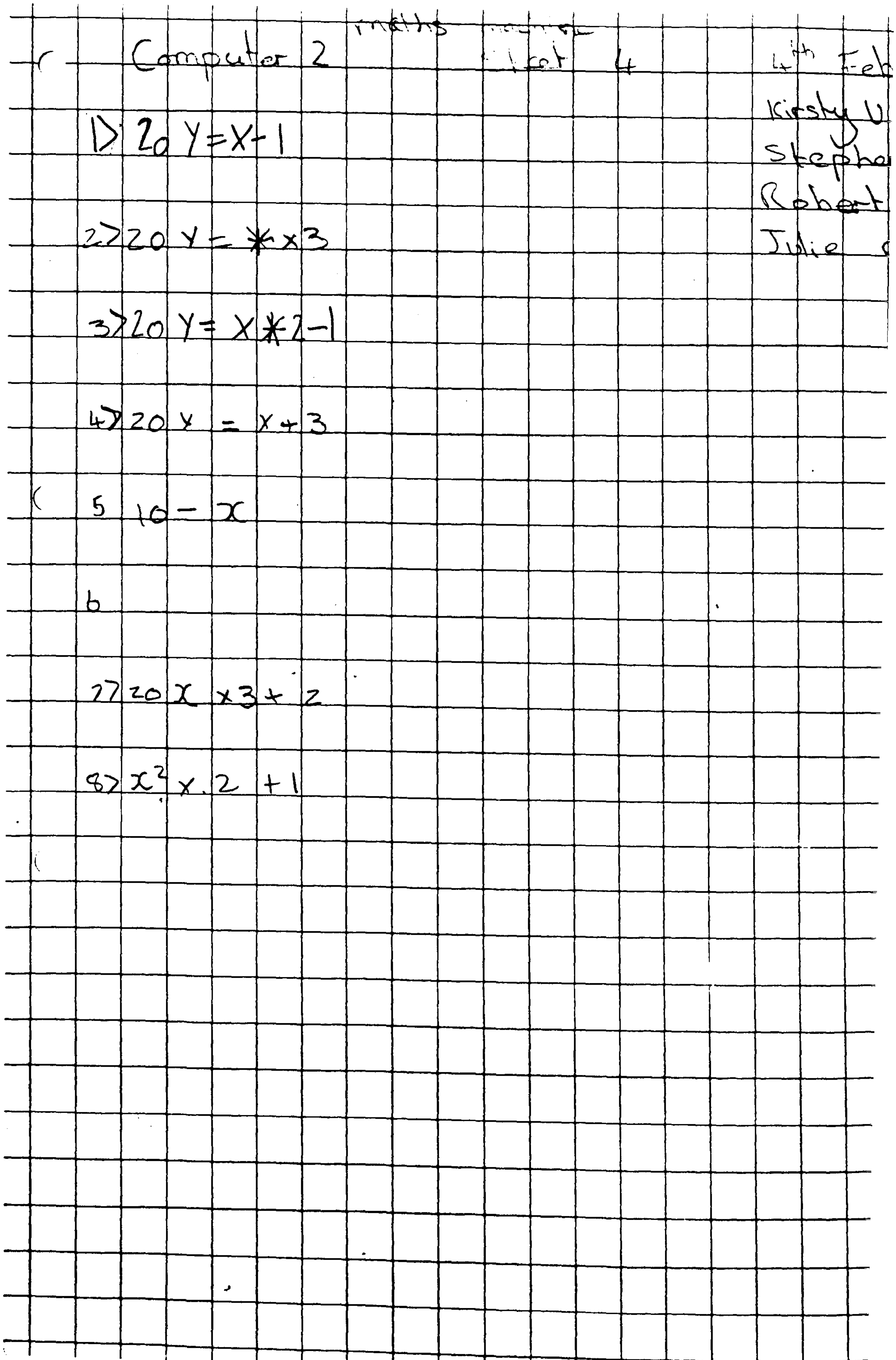


Figure 6.17
Experiment 2 - Some Answers to Worksheet 4

WORKSHEET NUMBER 5 Doz, Emma, Claire

Use the Maths. Machine program in front of you to try and solve the following problems. Write your answers in the spaces given.

1. a) When does $3 + x = 8$?	When x=5
b) When does $x + 2 = 7$?	When x=5
c) When does $14 - 2y = 6$?	When Y=4.
d) When does $3x - 4 = 8$?	When X=4.
2. a) When does $x + y = x + z$?	When x & Y have son
b) When does $x + y = x$?	When Y=0
3. Which is bigger, and by how much ?	
a) $x - 2$ or $x + 1$	X+1, bigger by
b) $x - 3$ or $x - 2$	X-2, bigger by
c) $x + y$ or $x + y + 2$	X+Y+2, bigger by
4. For what value(s) of x does :	
a) $x + 2 = 2x + 1$	When X=1
b) $3 + x = 7 - x$	When x=2
c) $3x + 1 = x + 9$	When x=4
5. For what values of x is :	
a) $2x + 1 > 5$	When x>3
b) $3x - 2 < 7$	When x<2
c) $13 - 2x < 3$	When x>6
6. For what values of y is :	
a) $3y > y + 2$	When Y=2
b) $2y < y + 3$	When Y=2
7. For what values of t is :	
$3t + 6 = 3(t + 2)$	Any Number
8. a) When does $2a + b = 2(a + b)$?	When B=C
b) When does $2a + 2b = 2(a + b)$?	Any Numbe

Figure 6.18

Experiment 2 - Some Answers to Worksheet 5

4. Any areas where she thought that the programme could be improved.

5. Any areas in the programme which seemed particularly beneficial.

A record of the pupils work on the worksheets was kept by them using the stores sheets provided. Figure 6.14 shows one of these sheets, and the worksheets are given in Appendix M.

6.7.6 <u>Observations From The Work In The Programme</u>

In order to give some idea of how well the group coped with the work in the programme, and also some of the sort of mistakes that they made, highlighting their difficulties, some of the work done by the pupils is examined here.

Figures 6.15, 6.16, 6.17 and 6.18 show sets of answers to some of the worksheets used (see Appendix M). They illustrate several important points about the way the children successfully tackled the work :

1. The pupils were happy to adopt the model of the variable given, even, when necessary, to draw their own versions of the boxes and to label them with the variable names.

2. They were making use of elementary checking procedures, such as trying large or unusual numbers (in question 1 with little regard for the context) to test if the procedures held good for all values before accepting their findings.

3. That, using the techniques of the programme, they have been successful in solving questions as difficult as numbers 5, 7 and 8 on sheet 4, and the questions on sheet 5, which involve inequalities as well as quite complex algebraic expressions. These are questions of a type which many older children find very difficult, since they require a level of understanding of

TEST (max.=71)	EXPERIMENTAL MEAN	CONTROL MEAN	MEAN DIFF.	S.D.	N	t	df	p
POST-TEST	32.55	19.98	12.57	10.61	21	5.30	20	<0.0005
DELAYED POST-TEST	34.70	25.73	8.47	11.81	20	3.13	19	<0.005

Table 6.4

A Comparison of Experiment 2 Performance on Post-Test and Delayed Post-Test.

the use of letters as generalised numbers or variables.

4. The mixing of the algebraic and BASIC notations in the answers was a sign that they were successfully making links with their arithmetic schemas.

6.8 The Results of the Second Experiment

The hypothesis was that the teaching programme would produce an improvement in the understanding of the use of letters in algebra. In order to test this hypothesis the results of the post-test and delayed post-test were analysed using a t-test for non-independent samples (the groups being in matched pairs on the basis of the results of the pre-test). The null hypothesis was that the mean result of the experimental group was less than or equal to the mean result of the control group. A summary of the results is given in Table 6.4.

In view of these results it was concluded that the null hypothesis should be rejected and that, in the case of both the post-test and the delayed post-test the results of the experimental group were probably significantly better than those of the control group.

Table 6.4 shows also that the experimental group's performance increased, on average, by 2.15 marks between the post-test and the delayed post-test. This increasing level of performance with time is consistent with previous findings that time is often needed in order for children to fully re-structure schemas to accommodate concepts in this area. (The improvement in the control group's performance is attributed mainly to interaction between the groups - see later)

Whilst this was considered to be an important and encouraging result, the study was concerned not just with

TEST (max.=25)	EXPERIMENTAL MEAN	CONTROL MEAN	MEAN DIFF.	S.D.	N	t	df	p
POST-TEST	5.19	1.50	3.69	3.83	21	4.30	20	<0.0005
DELAYED POST-TEST	5.40	3.05	2.35	4.42	20	2.32	19	<0.025

Table 6.5

A Comparison of Experiment 2 Performance on Level 3 and 4 Questions of the Post-Test and Delayed Post-Test.

Group	Pre-Test(%)	Post-Test(%)	Delayed Post-Test(%)	Question(a)
Experimental	0	33.3	20	a + b
Control	0	23.8	7.5	= a + c ?
Experimental	0	47.6	37.5	x + y
Control	0	14.3	10	= x ?
Experimental	0	42.9	22.5	L + M + N
Control	0	9.5	5	= L + P + N ?

(a) Brief description only given. The full question is given in the post-test in appendix J.

Figure 6.19

A Comparison of Some Experimental and Control Group Facilities on Questions Involving Understanding of Variables overall success, but with measuring improvements in conceptual understanding. In particular this involved considering how they had fared on those questions involving the understanding of the use of letters as specific unknowns, generalised numbers or as variables. As Küchemann (1981b, p.116) in his study of levels of understanding reported, "The items at levels 1 and 2 can all be solved without having to operate on letters as unknowns, whereas at levels 3 and 4 the letters have to be treated at least as specific unknowns and in some cases as generalised numbers or variables.". Hence it was felt that a comparison of the performances of the two groups restricted to those questions of difficulty levels 3 and 4 only would indicate the extent of the success of the programme in improving deep conceptual understanding.

The results of this comparison are given in Table 6.5. They show that in both the case of the post-test and the delayed post-test, that the experimental groups performance was probably significantly better than that of the control group. Thus they not only had a better understanding of algebra following the programme, but more importantly in some ways, they had a better conceptual understanding of the use of letters as specific unknowns, generalised numbers and variables. Since this is a difficult area of understanding and one which has been the cause of problems with many children, this was considered to be a very encouraging result. It showed that the approach used, namely the use of BASIC programming on a microcomputer, coupled with a generic organiser and a cardboard 'Maths.Machine' was a way to promote better understanding in algebra.

In order to see even more clearly the improvement which

1. When does $M + P + N = N + M + R$? Always , never , sometimes when.....

Pre-test	0%
Post-test	42.9%
CSMS 15 year olds	27%

2. Write down the area of a rectangle whose length is X centimetres and whose width is W centimetres.

Pre-test	4.8%
Post-test	71.4%
CSMS 14 year olds	68%

3. If $x = 4$ and $y = x + 3$ then $y = ?$

Pre-test	(Not Given)
Post-test	81%
CSMS 15 year olds	70%

4. A figure has p sides each of length 3 centimetres. What is the perimeter of the figure ?

Pre-test	0%
Post-test	42.9%
CSMS 15 year olds	41%

5. If $x + y = 5$ then $x + y + z = ?$

Pre-test	4.8%
Post-test	28.6%
CSMS 13 year olds	25%

6. If $p = 2$ and $m = 3p + 2$, then $m = ?$

Pre-test	(Not Given)
Post-test	47.6%
CSMS 13 year olds	44%

n.b. The CSMS facilities given are for the equivalent question on algebra test 1.

Figure 6.

Experiment 2 - a Comparison With Some CSMS Results

Name	Pre-Test (Max.=51)	Post-Test (Max.=71)	Delayed Post-Test (Max.=71)
Stuart	14	19	32
Andrew	14	47	48
Mark	14	21	20.5
Kirsty	15	54	60
Jason	4	10	18
Julie	3	24	35
Deborah	17	19	23
Stephen	15	39.5	38
Gary	9	12	18
Jonathan	9	40.5	37
Nicholas	7	11	11
Elvis	8	25	26
Jayne	7	18	17
Victoria	7	29.5	42
Louise	4	25	25
Rachael	4	31	38
Matthew	6	15	17.5
Marc	7	26	33.5

Figure 6.2

A Comparison of Some Scores of Matcned Pairs on Pre-Test, Post-Test and Delayed Post-Test had been obtained, Figure 6.19 contains a comparison of the results of the groups on some specific questions requiring one of the highest levels of understanding of the use of letters in order to correctly answer them. The improvement over the control group on questions such as these is clearly seen, such improvement holding up well considering the difficulty of these questions.

6.8.1 A Comparison With CSMS Results

The question arose as to just how well had the experimental group performed? How did their understanding compare with other pupils of this age? In order to try and get some objective measure to answer these questions a comparison was made on some specific questions with the results of the CSMS project, as given by Küchemann (1981b). Figure 6.20 gives a comparison of performance on some questions.

These extremely favourable comparisons, especially considering the fact that the average age of the group was only 12.3 years and they had not received any formal instruction in algebraic techniques, shows the progress made by the group, not just relative to the control group, but also relative to pupils who had received through their schooling up to 4 years of algebraic instruction and practice in techniques.

6.8.2 Individual Progress

The next stage in the analysis of the results was to look at the results and progress of individuals in the group relative to their mathematical ability. Although no direct measure of this was available, other than the pre-test, the results indicate that there was benefit to pupils across the ability range. Also, in Figure 6.21 it may be seen that four pupils of

Robert :

Before :
What is the total cost of x pencils at 5 pence each and y crayons at 7 pence each ?

Answer - 40p, 72p

After :
What is the total cost of x pencils at 8 pence each and y crayons at 9 pence each ?

Answer - 8x + 9y

Andrew :

Before :
If John has J marbles and Peter has P marbles, write down how many marbles they have got altogether.

Answer - 23

After :
If John has G jigsaws and Peter has H jigsaws, write down how many jigsaws they have got altogether.

Answer - G + H

Before :
Does x + y = x : always, never, sometimes, when.............

Answer - never

After :
Does y + x = x : always, never, sometimes, when.............

Answer - sometimes when y = 0

Jonathan :

Before :
Multiply 4a by 3.

Answer - L

After :
Multiply 3b by 5.

Answer - 15b

Figure 6.22

Some Specific Examples of Individual Improvements in Understanding

Darren :

Before :
When does $L + M + N = L + P + N$? Always , never , sometimes when....

Answer - never

After :
When does $M + P + N = N + M + R$? Always , never , sometimes when....

Answer - sometimes when $R = P$

Before :
For what value or values of a is $a + 5 > 8$?

Answer - 2

After :
For what value or values of a is $a + 3 > 7$?

Answer - 5 and over

Mark :

Before :
If $x = 3$ and $y = 5$ then $x + y + w = ?$

Answer - 72

After :
If $x = 2$ and $y = 6$ then $x + y + w = ?$

Answer - $8 + w$

Before :
Add 3 onto $x + 2$.

Answer - 29

After :
Add 4 onto $x + 5$.

Answer - $x + 9$

Stephen :

Before :
A figure has n sides each of length 2 centimetres. What is the perimeter of the shape ?

Answer - 8

After :
A figure has p sides each of length 3 centimetres. What is the perimeter of the shape ?

Answer - 3p

Figure 6.23

Some Specific Examples of Individual Improvements in Understanding

Formalisation of Method

Pre-Test(%)	Post-Test(%)	Delayed Post-Test(%)	Question(a)
4.8	71.4	60	Area W by X
0	42.9	45	Perimeter 3p
4.8	28.6	15	e + f = 8, e + f + g = ?

Use of Brackets

Pre-Test(%)	Post-Test(%)	Delayed Post-Test(%)	Question(a)
9.5	33.3	45	y + 2 ---x2-->
0	0	0	Multiply b-3 by 5

(a) Brief description only given. The full question is given in the post-test in appendix H.

Figure 6.24

Experiment 2 - Some Facilities on Errors relatively low ability (as measured on the pre-test), Julie, Rachael, Victoria and Marc seem to have done particularly well on the delayed post-test compared with the post-test. This may indicate that for those who initially found the concepts of algebra difficult, the programme may have benefit over a period of time as the process of re-structuring of their schemas and the consequent formation of concept links proceeds.

6.8.3 Specific Questions

One important indicator of individual progress is an examination of answers to corresponding questions on the tests to see if there is any evidence of an improvement in understanding. Figures 6.22 and 6.23 contain a comparison of the answers of some of the pupils in the experimental group, between the pre-test and the immediate post-test.

As can be seen, the improvement in understanding shown here is often considerable. Prior to the teaching programme many of the common misconceptions can be seen. For example, letters aré treated as objects, are replaced by numbers according to some code such as a=1, b=2, c=3, etc. or are apparently totally ignored in a desperate attempt to arrive at a numerical answer, whereas after the programme these ideas are often replaced by answers which indicate a quite clear understanding of the concepts of the use of letters in algebra. Of course, not all pupils displayed such progress to the same extent, nevertheless there was sufficient progress for the results to be very encouraging.

6.8.4 The Effect On Errors

Whilst it was not the purpose of this research to tackle the problem of the types of errors in algebra thoroughly researched by Booth (1983b), the aim being to attack the problems early before such errors had evolved, it was considered worthwhile to consider the effect of the programme on those questions where common errors might have been displayed. This included questions where problems with formalisation of method and the correct use of brackets were involved. As discussed earlier, it was hoped that the process of ´teaching´ a computer or a ´Maths. Machine´ would be the stimulus necessary for the children to focus their thoughts more clearly on the methods which they were going to employ in the solution of problems, and that the need to express clearly and unambiguously algebraic expressions for the computer would help with the use of brackets. During the work in the programme, they certainly coped well, recognising the need for brackets in order to communicate correctly with the computer. It was not clear, however, that they would transfer this understanding to mathematics in general since transfer is often a major problem.

Figure 6.24 gives some facilities for those questions given by Booth (1983b, p.229) where such errors are exhibited.

On the formalisation of method questions we see a marked improvement in performance in each case in the post-test, sustained into the delayed post-test. Although it was not intended, or possible, to analyse the occurrence of the errors, for the reason given above, these proportions of correct answers do indicate that the proportions of error answers found by Booth in her pre-test results were not present in this group after the programme. it would seem, therefore, that the process of structuring a method of solution in order to present it to the computer had been of value.

A second important error identified by Booth was that associated with the failure to correctly use or evaluate expressions involving brackets. The results of the programme, when looked at in the light of this problem, made it apparent that the way in which a question is expressed can have a marked effect on the outcome in this area. Figure 6.24 shows that the facilities on questions which are essentially the same vary considerably and hence they were answered with totally different levels of success, one question being tackled well, and the other badly. One possible explanation for this is that the form of the question is important and that wrong answers may often be the result, not of an inability to answer the question, but of a failure to understand what it is that the question is requiring them to do.

6.9 The Longer Term Effects of the Programme

One of the major tests of a claimed improvement in conceptual understanding as opposed to rote understanding or purely manipulative skills is its decay over a period of time. The local school situation in the district where the second investigation took place was such that when, at the end of the school year in which the experiment had taken place the pupils transferred to other schools for their secondary education, a number of them attended the same school. This gave the opportunity to conduct a follow-up study to investigate the longer-term effects of the programme.

It turned out that pupils comprising eleven of the original matched pairs had all attended the same school, and had been put into comparable mathematics sets in a co-educational, comprehensive school. Thus for the whole of that year the

TEST (max.=66)	EXPERIMENTAL MEAN	CONTROL MEAN	MEAN DIFF.	S.D.	N	t	df	p
POST-TEST	32.55	19.98	12.57	10.61	21	5.30	20	<0.0005
DELAYED POST-TEST	34.70	25.73	8.47	11.81	20	3.13	19	<0.005
ONE YEAR LATER	44.10	37.40	6.70	7.76	10	2.59	9	<0.025

Table 6.6

A Comparison of Experiment 2 Performance on Post-Test, Delayed Post-Test and Test One Year After the End of the Experiment.

TEST (max.=25)	EXPERIMENTAL MEAN	CONTROL MEAN	MEAN DIFF.	S.D.	N	t	df	p
POST-TEST	5.19	1.50	3.69	3.83	21	4.30	20	<0.0005
DELAYED POST-TEST	5.40	3.05	2.35	4.42	20	2.32	19	<0.025
ONE YEAR LATER	7.60	4.55	3.05	1.63	10	1.87	9	<0.05

Table 6.7

A Comparison of Experiment 2 Performance on Level 3 and 4 Questions of the Post-Test, Delayed Post-Test and One Year After the End of the Experiment.

	Control			Experimental	
	Total (max=71)	Level 3/4 (max=25)		Total (max=71)	Level 3/4 (max=25)
Lisa	39	8	Darren	49	9
Kristian	37.5	3.5	Michael	44.5	5.5
Debi	34	2	Mark	35	3
Brian	45	7.5	Robert	51	14
Lisa	33.5	2.5	Jonathan	58	15
Lorraine	47.5	9.5	Claire	46	7
Simon	40	8	Robert	53	11
Tammy	abs	abs	Rachel	27	2
Jayne	31	1	Victoria	40.5	5.5
Louise	38.5	3.5	Rachael	37	3
Matthew	28	0	Marc	27	3

Figure 6.25

Experiment 2 - The Matched Pairs and Scores

One Year After The Experiment

1. When does M + P + N = N + M + R ? Always , never , sometimes when.....

Pre-test	0%
Post-test	42.9%
One year later	22.7%
CSMS 13 year olds	11%

2. Write down the area of a rectangle whose length is X centimetres and whose width is W centimetres.

Pre-test	4.8%
Post-test	71.4%
One year later	81.8%
CSMS 15 year olds	76%

3. If x = 4 and y = x + 3 then y = ?

Pre-test	(Not Given)
Post-test	81%
One year later	72.7%
CSMS 15 year olds	70%

4. A figure has p sides each of length 3 centimetres. What is the perimeter of the figure ?

Pre-test	0%
Post-test	42.9%
One year later	72.7%
CSMS 15 year olds	41%

5. If x + y = 5 then x + y + z = ?

Pre-test	4.8%
Post-test	28.6%
One year later	36.4%
CSMS 13 year olds	25%

6. If p = 2 and m = 3p + 2, then m = ?

Pre-test	(Not Given)
Post-test	47.6%
One year later	72.7%
CSMS 15 year olds	67%

n.b. The CSMS facilities given are for the equivalent question on algebra test 1.

Figure 6.26

Experiment 2 - a Comparison With Some CSMS Results pupils, who were now aged 12 to 13, received equivalent mathematics teaching, including some algebra work. This gave an ideal opportunity to examine the effect of the algebra module with repect to its function as an advance organiser for formal algebra teaching, since one group would have received formal teaching alone for a year and the other would have had such teaching preceded by the algebra module.

At the end of this year, in July 1986, all of these pupils were again given the algebra test used in the original study (see Appendix J).

The results of these tests are given in Figure 6.25, and include a breakdown of the results on the questions at levels 3 and 4. An analysis of the results using the same null hypothesis and t-test as in the original study revealed that the experimental group was still, over one year later, performing significantly better than the control group on the test as a whole. A summary of the results of the analysis giving a comparison with their earlier results is given in Table 6.6. When the results of the questions at levels 3 and 4, which require deeper conceptual understanding, were compared it was also the case that the experimental group performed significantly better here too. Again a summary of their results including a comparison with their post-test and delayed post-test is given in Table 6.7. Not only were the results of the experimental group significantly better, but they had increased the percentage of questions which, on average, they were able to answer at this level to just over 30%. This compares very favourably with the CSMS results where, at this conceptually difficult level, 13 year-olds managed an average facility of 16.4%, 14 year-olds 27.2% and 15 year-olds 32%. Figure 6.26 compares the performance of the group with the general school population, as described by the CSMS results, on a few specific questions. We see, although the sample size is smaller than previously, that the results have held up well in comparison and in most of these questions they have in fact moved ahead. Hence we may say that the children who had experienced the Dynamic Algebra module had attained a level of understanding considerably above the average for their age (13 years) and comparable to that of pupils up to two years older.

These results lend strong support to the idea that the introduction of a module of work, giving a dynamic view of variables in algebra, and emphasizing conceptualisation and the use of mental imagery rather than skill acquisition, prior to the teaching of formal skills, can provide significant longer-term conceptual benefits. I would suggest that this is because such an approach promotes the cognitive integration, described in Chapter 3, which will lead to a versatile learner able to apply both a global/holistic and a serialist frame of reference as appropriate.

6.10 Overview of the Chapter

In this chapter I have looked at those studies which formed the immediate background to the present research, in particular the work of Küchemann, Booth and Hart, and their relationship to its content. I have described the procedures and results of the first two investigations in this study of a conceptual approach to the early learning of algebra. The results of the experiments are significant in themselves in providing evidence for the value of the computer paradigm applied to this area, and the longer-term advantage of this. The two experiments also provided valuable insights into improvements which could be made to the Dynamic Algebra module and pointers as to how to compare it, in a statistically controlled way, with a more traditional skill-based approach. This experiment is the subject of the next chapter.

Chapter 7

A Description of the Main Teaching Experiment

In the investigation described in the previous chapter I have explained how the dynamic algebra module, as it was now called (Thomas and Tall, 1988), had been compared with a traditional skill-based approach in the longer term investigation carried out one year after the second experiment. However it was decided that it was important to compare directly its effectiveness in promoting conceptual understanding with a more traditional skill-based algebra course in a larger scale experiment. Hence, although its value in promoting conceptual understanding had already been clearly demonstrated, it was considered necessary to obtain data from such an investigation in order to try and establish its superiority as a means of teaching algebraic concepts. This chapter describes the revision of the module in the light of the second experiment and the methodology, form and content of this larger scale main teaching experiment, along with some observations arising from it.

7.1 Areas for Further Consideration

A close analysis of the second study, involving a detailed consideration of the teaching module, the results of the experiment and the comments of the teacher who taught the lessons illuminated several areas where the form and content could be improved upon for the main experiment, and these are described in this section.

7.1.1 Improvements to the Dynamic Algebra Teaching Module

This third experiment was to take place in a secondary school which, as is usual in Britain, was tied to an inflexible daily timetable, compared with the more flexible arrangement of a junior or middle school where a single teacher takes each class and may apportion time as appropriate. In view of this it was necessary to adjust the segmentation of the module to fit this format, with sections which would allow for better organisation of classroom time. A second consideration was the order of the work in the module. The section on linking the BASIC notation to the mathematical notation of algebra was considered to need more explanation and to be better placed before the section on the use of notation such as brackets, and this was done. The teacher of the module in the second experiment had pointed out that in section 5b of those lesson plans she actually had to get the children to type in :

x = 1

y = x + 2

PRINT y for <u>each</u> value of x before they could accept that y = x + 2 would work for <u>every</u> line in the table. This demonstrated a deficiency in their cognitive structure and so it was decided to expand the explanation at this point to include a process like the above. The final module includes such an expanded explanation and a worked example.

The finished version of the module, with a discussion of some of its theoretical implications, is given below.

7.1.2 Improvements to the Worksheets

Since the module had been re-written as outlined above, the worksheets needed to be re-organised to correlate more closely with the lesson plans so that at the end of each section of the module there would be a worksheet. Hence the worksheets were changed and the number of them increased from 5 to 8.

After the end of the second experiment the class teacher had commented on the structural form of the worksheets. She described the difficulty which her pupils had found due to the fact that the questions on the worksheets were structured in the form : text - question - more text. They were used to being given all the required information in a question before the question itself. In view of this valid observation the worksheets were rearranged so that all the text preceded the question itself.

Totally new worksheets one and two were written to give a gentler introduction to the use of letters. Sheet one made no use of programs but only of direct commands and sheet two of simple programs with no use of INPUT and loops. Worksheets one, two, three and four from the second experiment, in their amended forms as described above, were used as the third, fourth fifth and sixth worksheets. Worksheet five from the second experiment was amended and divided into two worksheets, seven and eight, with worksheet eight containing some new, harder questions to be solved using the software maths. machine.

The eight worksheets in their final form are given in Appendix P.

7.1.3 Improvements to the Test Instrument Used in the Study

In the light of the experience of the initial investigations it was found that the following improvements to the algebra test would be beneficial.

It was recognised, and confirmed by the results of the second experiment that on, for example, questions involving the use of brackets or diagrams, that the form and content of the questions can be crucial to children's success (or failure) in answering them. The test used in the two previous investigations was therefore scrutinised at this stage to look for examples where the wording or form of the question may have contributed to misunderstandings. Several examples were found and altered for the test used in this main experiment. They were :

(i) It was considered that, in the wording of question 14, trying to encourage the children to give all answers that they might have been able to spot, by saying 'For what value or values of a is', may have had the adverse effect of making those who were capable of finding several answers, and who would have been happy to do so, content to find only one value which satisfied the inequality. Unfortunately in the nature of what was being tested here this could not gain them any credit, and we were left without knowing what they might have been capable of. Hence for the test used here the wording was altered to 'Write down those values of a for which'. Although the difference may at first appear minor, the omission of any reference to a single value changes the emphasis totally to a problem of finding as many values as one can, rather than being content with finding just a single one.

(ii) In the previous tests the letter x had been freely used as a variable, as it often is in mathematics classrooms. For the purposes of a test instrument however it was decided that it would remove any possibility of confusion arising over mis-reading the x as a 'times' if x was avoided altogether. This was done for the test used in this experiment.

(iii) Communication with Küchemann, who had designed the original CSMS algebra test on which all the tests were based, suggested that question 11 of the test had been made easier than in the original by the reversal of the equations and the given variable values. For the final test version these were therefore reversed to correspond to his original test.

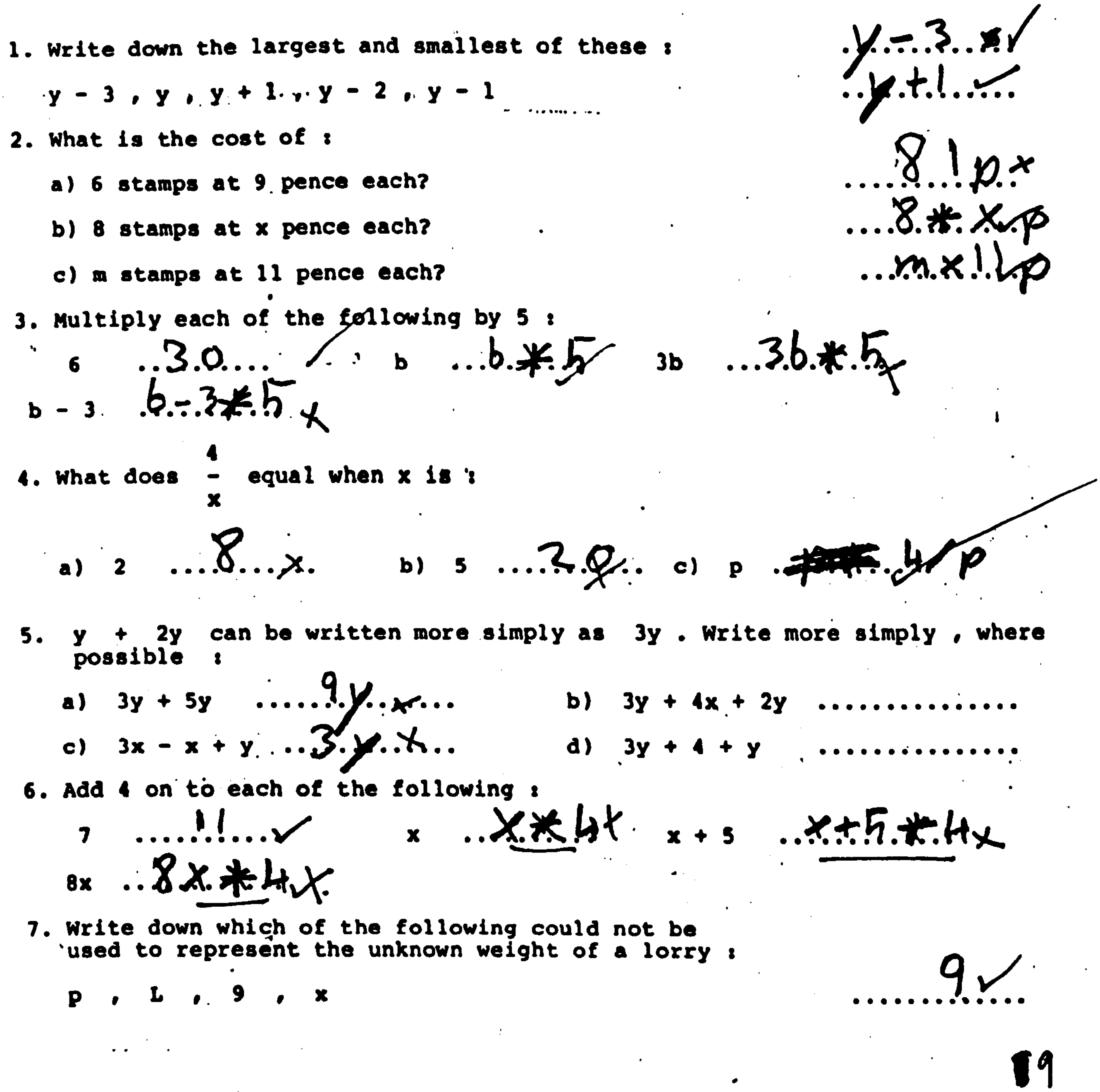
(iv) The use of the word equation in question 26, namely ´write down an equation connecting C and W´ was considered to have evoked in many pupils the concept of finding a value, and this they attempted to do. This question was so badly answered by all pupils that it was decided that a change of wording to ´write down an expression for the total cost C´ would help make an understanding of the type of answer required clearer without substantially changing the difficulty level of the question.

(v) Questions 7 and 21, which were added to Küchemann´s original test were found to correlate poorly and so were removed from the final version.

(vi) Question 13 on the final test was re-inserted from Küchemann´s original test in order to extend the number and type of questions involving the higher levels of understanding of the use of letters as specific unknown, generalised number and variable.

(vii) The variable z used in question 24 was changed to n in order to avoid any confusion which might arise from the similarity between the coefficient 2 in the question (which had to be kept) and the variable.

Following these changes to the algebra test to be used in the experiment it was apparent that it would be necessary to check the validity of the test against Küchemann´s original CSMS



1. Write down the largest and smallest of these :

y - 3 , y , y + 1 , y - 2 , y - 1 .... y - 3 x ✓ ...... y + 1 ✓

2. What is the cost of :

a) 6 stamps at 9 pence each? ......81p x

b) 8 stamps at x pence each? ....8 * x p

c) m stamps at 11 pence each? ...m x 11p

3. Multiply each of the following by 5 :

6 ..30.... ✓   b ...b * 5 x   3b ...3b * 5 x

b - 3 b - 3 * 5 x

4. What does $\frac{4}{x}$ equal when x is :

a) 2 ....8...x.   b) 5 ....20.. x   c) p 4/p

5. y + 2y can be written more simply as 3y . Write more simply , where possible :

a) 3y + 5y .....9y..x...   b) 3y + 4x + 2y ....

c) 3x - x + y ...3y..x..   d) 3y + 4 + y ....

6. Add 4 on to each of the following :

7 .....11...✓   x ..x * 4 x   x + 5 ..x + 5 * 4 x

8x ..8x * 4 x

7. Write down which of the following could not be used to represent the unknown weight of a lorry :

p , L , 9 , x ......9 ✓

9

Figure 7.1

An Answer Sheet Illustrating The Possibility

Of Group Interaction

algebra test in order to be confident in its use as a measure of algebraic understanding comparable to the original. Hence both tests were given to a sample of 26 pupils taken at random from the target population of 13 year-olds in the co-educational comprehensive secondary school to be used for the experiment, and the results are given in full in Appendix O. A Pearson r correlation coefficient was calculated and its value found to be 0.92. From this high level of correlation it was concluded that the test to be used as an instrument in the study was able to measure well the same type of algebraic understanding and capabilities as Küchemann's algebra test, which was specifically designed to test appreciation of the uses of letters in algebra as well as algebraic skills.

7.1.4 Avoidance of Pupil Interaction

Another variable which required some thought was that of the interaction between the children during the programme. It was noticeable in the second experiment that the children who had taken part in the programme were so enthusiastic about the work that they were keen to share with their friends what they had learned. Whilst it is not possible to tell to what extent this happened, at least one of the girls in the interviews freely volunteered the fact that she had been teaching a friend in the control group! Figure 7.1 also shows the answer paper of a member of the control group. It will be seen that his answers, like some of those of one of the experimental group, were given in BASIC notation as used in the programme, although this had not been taught in the schools and was not likely to appear from the context. In order to try and reduce the effects of such interaction the final experiment was to use whole classes from a school which was also divided into separate sections, or halls, providing a degree of physical separation of some of the classes. Further, restricting the programme to the usual mathematics lessons was expected to reduce the novelty of it and give it an appearance of ´another mathematics module´, of the type the pupils were used to.

7.2 The Main Teaching Experiment

7.2.1 The Subjects

The subjects of the experiment were taken from six mixed ability forms in the first year of a co-educational 12-plus entry comprehensive school. The school is divided into two halls with children assigned to each in order to provide identical cross sections of the population. The teaching in the school is done by a unified team of teachers working under the umbrella of a single head of department, and this allowed direct comparison of the effectiveness of the different teaching methods employed. On the basis of the algebra pre-test, given in Appendix Q, it proved possible to arrange the pupils into 57 matched pairs covering the full spectrum of ability in the classes.

7.2.2 The Instruments

The algebra test described above, and given in Appendix Q, was used as the pre-test, post-test and delayed post-test for this experiment, since it had been established as a reliable instrument and one which would discriminate well in the area of conceptual understanding associated with the use of letters in algebra. The computer program, with slight technical improvements, called the software ´Maths.Machine´ was again used, along with the cardboard ´Maths.Machine´, and both have been described above. The use of the computer program is described in the lesson notes which follow. A set of interview questions was collected together with the intention of probing deeper the student's conceptual understanding in areas known to cause problems. The questions chosen and the reasons for their choice are givem in Chapter 9, along with a discussion of the results of the interviews. Similarly, a questionnaire was also devised, in order to test whether the conceptual understanding developed in the programme afforded a better preparation for the more conceptually advanced equations etc. which the students would meet in the future in their school algebra. It was also designed to investigate whether there was any evidence of the increased use of mental imagery in the children who had been involved in the programme and consequent improvement in versatility of approach. The questionnaire is given in Appendix S, and the reasons for the questions used along with the results from it are given in Chapter 10.

7.2.3 The Research Design

The same research design which had been successfully implemented in the previous experiment was considered appropriate here too. The design was a pre-test, post-test control group design with matching.

7.2.4 The Procedures Followed

The researcher met with the three teachers who had volunteered to teach the algebra module in early October 1986. They comprised two men and a woman, with one of the men and the woman never having used a computer in the classroom previously. This enabled an insight into the sort of problems which might occur if the module was given general release to teachers, many of whom are in the same position. They had all been

Control Total (max=67)		Experimental Total (max=67)		Control Total (max=67)		Experimental Total (max=67)	
SB	3	SB	3	RW	29	NT	29
AF	3	MF	3	JC	31	GW	31
AW	5	ZG	5	RW	32	GJ	32
NU	6	KR	6	MC	32	JM	32
VK	7	MR	5.5	FB	32.5	GY	33
JE	7	PS	5.5	SC	33	MB	33
MC	7	PR	6.5	LO	33	DT	34
SO	7.5	SJ	8	AM	33	KS	33
NT	9	CE	8.5	AM	33	CE	34
DF	10	SL	10	HF	33.5	MB	34
MD	10	JF	9.5	MH	34.5	MH	34.5
KR	11	PC	11	DT	35	MB	35.5
PS	11	NR	11	NH	35	CE	34
HP	12	SP	12	KS	36	ZK	37
SM	13	SH	12.5	SB	36	AD	37
AM	13	NJ	15	WC	38	AG	38
DS	14	EM	15.5	SL	42	CF	42
SC	14.5	JL	16	RE	42	KA	40
AG	17	AB	16.5	ST	42	SR	40
AW	18	JC	19	AC	43	RT	43
SJ	19.5	LC	20	KH	43	ES	43
NT	21	CP	21	SW	44.5	AB	44.5
AB	24	LS	24	AW	45	MW	45
JT	25	LN	26	CB	48	AC	47
GT	25	NW	26	JS	48.5	RB	48.5
JH	25	TM	26	CC	49	KR	49.5
NP	27	MH	26.5	MB	51	JJ	51.5
RV	28	LT	29	AJ	57	FS	56
				SD	59	GH	59

Figure 7. 2

Experiment 3 - The Matched Pairs and Scores In The Pre-Test

MARK (max.=67)	EXPERIMENTAL GROUP F	CONTROL GROUP F
0 - 5	0	6
6 - 10	6	0
11 - 15	6	0
16 - 20	2	1
21 - 25	0	3
26 - 30	0	6
31 - 35	2	0
36 - 40	2	0
41 - 45	2	0
46 - 50	3	0
51 - 55	0	2
56 - 60	0	2
61 - 65	0	1
>65	0	0
Absent	4	3
Totals	27	24

Table 7.1

The Frequency Distribution of Experimental and Control Group Pre-Test Marks for Pupils Not Used in the Matched Pairs previously given a copy of the 'kit' which included the dynamic algebra module, the worksheets and a copy of the computer program 'Maths.Machine' to study. At the meeting, both of the teachers with no previous experience of using computers in the classroom understandably expressed their apprehension about what was to be, for them, the unknown territory of using the computer. At the end of this meeting, however, following a discussion of the module and its intentions, they were very happy about the programme and expressed their keen anticipation of teaching their classes.

The classes chosen to form the control group in the school also had teachers comprising two men and a woman and this balance in the teaching staff of the two groups, coupled with the unified nature of the teaching in the school as represented in this instance by the fact that the control classes (and later the experimental classes) would all be working through the same algebra module, should have minimised as much as possible the effect of variation in the results due to teaching style.

Later in October 1986 the pupils were all given the pre-test on which they were to be divided into matched pairs. The results of the pre-test are given in full in Appendix R. From the 84 pupils in the experimental group and the 81 pupils in the control group it was possible to form 57 matched pairs on the basis of this test and the pairings are given in Figure 7.2. The frequency distribution in Table 7.1 shows the relative distribution of the algebraic ability of the other pupils in the two groups as measured by the test.

During January and February 1987, for about four weeks, the experimental group received the dynamic algebra teaching module in place of their normal mathematics lessons (about 12 hours teaching) and during this same time the control group continued with their normal mathematics lessons, working through the school's standard algebra module (see Appendix T for a copy of this module). The work of one group of three or four pupils was tape recorded during this time by other researchers and these tapes were made available for this research. At the end of this time, on the 16th February 1987, all the pupils were given a post-test consisting of the test given in Appendix Q in one of their mathematics lessons.

During April and May 1987, the experimental group were given, in their normal mathematics lessons, the same algebra module which the control group had received earlier, only covering it in just two weeks instead of four. While they followed this course the control group spent the same time revising the content of this module, which they had previously worked through. Following this, during June 1987, both groups were again given the algebra test which had been designed for the experiment and is in Appendix Q.

In order to obtain a clearer picture of the understanding gained by the pupils from each method of instruction than is possible from a test alone it was decided to give a sample of the pupils from each group a semi-structured interview. During the interview their understanding of the concepts of algebra, with particular reference to the use of letters was investigated. These interviews took place on the 20th and 21st of May 1987 and were conducted on an individual basis by the researcher. Each of the interviews lasted about 15 to 20 minutes and the whole of each interview was tape recorded. The questions which were used in the interviews, with the reasons for their choice and the results of the interviews form the discussion in Chapter 9.

Whilst the above procedure gave a clear indication of the success of the programme (see Chapter 8) in improving children's conceptual understanding of algebra compared with a traditional skill-based approach, the extent to which there had been interaction with mental imagery schemas or whether it had prepared them for future concepts in algebra were considerations which it was intended to investigate too. To facilitate this a questionnaire was devised (see Appendix S) and given to all the pupils in both groups in the middle of July 1987, at the end of their school year.

Following the end of the experiment, the three teachers who had taught the experimental classes were asked to write down their thoughts on the way the work had gone. Some of these are included in here and in Chapter 11.

7.2.5 The Dynamic Algebra Teaching Module Used

The final version of the teaching module used in this experiment is reproduced on the following pages in the form in which it was provided as part of the kit given to the teachers.

Algebra Programme Lesson Notes

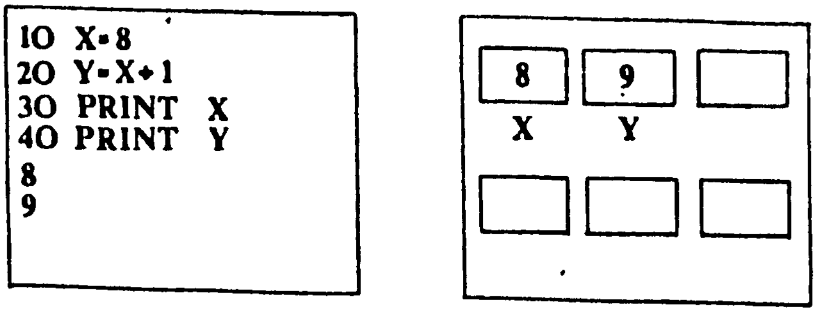
Aim of the Programme

The aim of this module of work is to give secondary school pupils who have had no experience of algebra, the necessary base in order to be able to understand the subject. It has been found that the programming of a computer in BASIC, including the use of a 'picture' for the variables used, can help a child's understanding of the concept of a letter as a variable in algebra. Children who went through such a programme in the last year of their middle school found algebra easier in the secondary school than a comparable group who did not have the experience, so much so that the difference was maintained after they had all gone through the same mathematics syllabus for a year.

The breakdown of the lessons into sections is done as a rough guide only. The amount of material covered per session is left to the discretion of the teacher, and will depend on the length of the session, the ability of the children and the facilities available. In the pilot study the work took twelve hours to complete over a three week period.

Equipment Required

The pupils will use either "cardboard maths. machines" or computers, in small groups. The cardboard maths machines consist of two large sheets of card, one representing the screen of a computer, the other, with rectangles drawn on it representing the variable stores. There should be a plentiful supply of smaller pieces of card marked with a variety of upper and lower case letters to act as labels for the stores, and others marked with digits to represent the contents of a store.



The Screen

The Variables

Figure 1

The computers should have a version of the BASIC language. They need not all be BBC computers for the first six sessions, but the final two sessions use a "computer maths machine" program which is at present available only for the BBC computer. If other computers are used, however, it is essential that you make sure that you are aware of any minor operational differences between them.

In trials the lessons have worked well in classes of twenty or so pupils with three BBC computers rotating round the class as others use cardboard maths machines and perform paper and pencil exercises. If a network of BBC computers is available, then it is possible to organise the programme so that some lessons take place in the computer room, with the experiences on the "cardboard maths machines" done in other lessons. However, one of the features of the lessons is to organise competitions between computer programs and pupils operating the cardboard maths machine and this should be carried if possible.

Teacher Preparation

It is hoped that teachers with little experience of computers will use these lessons to gain confidence in working with computers in the mathematics classroom. However, if you have never used a computer before, it is important to know how to set it up and switch it on. Then read through the lesson plans with a computer available and try out the commands as you come to them. Initially there is only simple typing to be done. But there may be minor differences if you use a computer other than the BBC. (For example, the Spectrum computer has "single stroke keywords" and the "LET" keyword is essential in giving a variable a value in a statement such as LET x=3.)

Only in the final stages will it be necessary to load a program from disc into a BBC computer and, provided you make sure that the disc-drive is the right type for the disc (40 or 80 track), all you will have to do is to put the disc in the disc-drive, close the door of the drive, and press SHIFT-BREAK. (The precise technique is to hold down the SHIFT key, then press and release the BREAK key.) You should practice this before you go into the classroom. Remember, there is no embarrassment if some young pupil knows more about the operation of the machine than you do, the worst that can happen is when nobody knows!

Teaching Style

The lessons which follow are best done beginning with teacher-lead class discussion on the concepts. It will be found that discussion comes much more naturally when there is a computer for both teacher and pupils to focus their attention. It is much easier to ask questions like "what do you think the computer will do if we type so-and-so", and encourage a pupil to try out their ideas to see if they work, than it is to have a question and answer session in which the teacher is the focus of activity. It is important in discussion to listen carefully to what the pupils say and to try to understand the reasons behind their thinking. This way you will help them form their ideas in a way which is consonant with the mathematical community.

The pupils understanding will be enhanced by allowing them to carry through the algebraic activities using the "cardboard maths machine" or the computer. It is best done in mixed ability groups or friendship groups rather than consciously banding or streaming within the class. Whilst the children are carrying out their investigations there will be opportunity to talk with individual groups and to LISTEN once more to their ideas as to what they are doing and why. This will help you to find subtle misconceptions and the discussion can help them to a richer understanding. The idea is to help them to participate in the dynamic assignment of values to algebraic variables and to participate in algebraic manipulation. As well as helping the pupils with their understanding, the exercise will give teachers valuable experience in investigational techniques appropriate for G.C.S.E. coursework.

The Lesson Plans

(1) Introducing the computer (possibly two lessons)

The first thing to do is to gain familiarity with the computer keyboard. The pupils (and the teacher!) need to know how the SHIFT and RETURN keys are used to get a + sign, an = sign and to enter instructions.

(In class this may be done as one group around the computer(s) with various pupils used in turn to do any typing.)

Use of PRINT

A good way to start is to ask the pupils how they might get the computer to print things. If no suggestions are forthcoming, suggest that the pupils may like to try the following:

```
PRINT "Hello there" <R>

PRINT"I am great"   <R>
     etc.
```

(where <R> means press the key marked RETURN, which may be marked as "ENTER" on some computers.) Some pupils may have met the language Logo before, which uses square brackets instead of speech marks, so it may be necessary to emphasise that the language now being used is BASIC.

The experience is intended to show them that PRINT is a command to put the symbols ('words') in speech marks on the screen.

But what happens when there are no speech marks? What happens when one types:

```
PRINT Good idea      <R>
```

On the BBC computer the response will be the message

```
NO SUCH VARIABLE
```

(Other computers may give a slightly different error message.)

Try again :

PRINT A <R>

The same message! Why won't it print out the words?

What does the computer mean by NO SUCH VARIABLE?

(A move away from the computer to a blackboard is advisable here.)

Variables

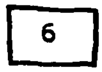
This is a crucial point in the development of children's understanding. Their main insight will probably come through playing with the equipment in a few minutes, but it is worth beginning with some kind of explanation like the following:

In the absence of speech marks, the computer looks in its memory for a thing called a "variable" or a "store" labelled Good idea, A etc. Variables are how a computer stores its information. What are they?

We can think of a variable as a store with a label. Inside the store is a number. So a variable is in two parts:

(i) Label - FRED
(ii) Current value- 6

and we can draw it like this:



FRED

as a rectangle with a label underneath and the current value as a number inside.

N.B. THIS IS THE FUNDAMENTAL MENTAL PICTURE FOR THE VARIABLE CONCEPT.

When we type:

PRINT FRED <R>,

the computer looks in its memory for a location/box/variable labelled FRED so that it can put a copy of the value in it onto the screen.

The response NO SUCH VARIABLE means it failed to find one with the name given.

Making Variables

Ask the pupils:

How do you think we might tell the computer to make a variable called FRED?

Listen to their suggestions and have volunteers try them on the computer using PRINT FRED <R> to see if they work i.e. if we get a value.

(Be careful here. Some of the children may have used Logo before, where the command is in the form:

```
MAKE "FRED 6
```

Logo distinguishes between a variable FRED, its name "FRED (preceding it with a speech mark), and its value :FRED (preceding it with a colon. We are not using Logo in this work because the distinction between the name "x and the value :x will complicate the issue when we come to doing ordinary algebra. However, it may open up a useful conversation on the fact that different computer languages are designed to do things in different ways....)

If the children are unsuccessful then tell them to try :

```
LET FRED = 3     <R>
PRINT FRED       <R>
```

(From now on, we will omit the RETURN key symbol <R>)

and

```
LET A = 6
PRINT A
```

(On most computers, but not the Spectrum, the "LET" command is optional, so you can miss it out. We will miss it out later, but Spectrum owners, beware!)

Explain that typing

```
LET A = 6
```

means to the computer:

Put 6 in a box labelled A.

The command

```
PRINT A
```

means take a copy of the number in A and put it on the screen.

Given that A = 6, ask the pupils what will happen if

```
PRINT A + 1
```

is typed. Get someone to type it in to see.

Try the same kind of thing with other expressions. For instance,

```
PRINT A - 1
```

or

```
PRINT A + 3

PRINT A + A.
```

Introduce the * sign meaning multiply and the / sign meaning divide. For instance, ask what does 5*7 mean to the computer?

Answer?

What would 36/9 be?

Try :

```
A = 7
B = 3 * A
PRINT B
```

The "Cardboard Maths Machine"

The pupils can make themselves act as parts of a simple computer, which we shall call a 'Maths Machine' by using the card sheets and the letters and numbers with them which make up a "cardboard maths machine". (Remember that each such machine consists of two sheets of cardboard, one empty: the screen, and one with square boxes on it, to label with letters and to place number values inside, made out of card.)

Method:

The children share out the jobs of operating the maths machine. For example, one child puts commands such as

```
B = 7
```

on the cardboard "screen".

The children operating the maths machine must do all the "computer operations".

Someone is assigned to put the label B on one of the boxes set aside on the other sheet of cardboard for variables. Then someone in charge of the numbering takes a piece of cardboard marked with the number 7 and puts it in the box with the label B.

If the command

```
PRINT B
```

is to be carried out, someone is assigned to go (the 'Machine' could be across the room) and look in the box labelled B and bring back a copy of the number in it to put on the blank 'screen' card.

This should first be done with a small group carrying out the instructions, before the class is divided up into several smalle groups. Some groups can work on real computers, others on "cardboar maths machines". Depending on the resources it should be possible t have 3 or 4 children working in each group.

If only a few computers are available, it is best to organise rotation of the groups so that every group gets to use a computer an a maths machine at regular intervals.

The beginning of group activities

In their groups get the pupils to investigate:

```
X = 5
Y = X + 1
PRINT X
PRINT Y
```

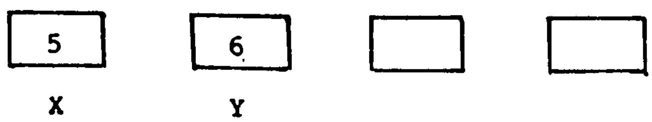
An explanation of the mechanics of the second line is necessary fo those on the 'Machines'. Ask them : How do we carry it out?

Answer : Find out the value in X; add 1 to this; make a variable wit label Y; put a copy of this value in Y.

Do it, possibly as a race. Do the computers agree with the 'Machines'

The final values of the variables may be recorded on a "result sheet", which is just a layout of empty rectangles ready for fillin in. Each problem should be recorded on a separate horizontal line using as many boxes as is necessary. The children using the cardboar maths machine will be recording the final state of their variabl card, those using the computer will have to think about their menta representation of variables.

For example, when X=5 and Y=X+1, the final result can be recorded t labelling two adjacent boxes X and Y, writing the value 5 inside the box and 6 inside the Y box. Each set of final results for a particula example should be recorded on a separate line, ready for marking.



Try the examples on SHEET 1 in this way.

(2) **Programs**

To make a program in BASIC we do the same as before, but we number each line of the instructions. The numbers don't have to be 1,2,3,... But they will be carried out in sequence with the lowest numbered line first.

Ask the pupils what they think the following program does:

```
 1 X = 2
 2 Y = X + 1
10 PRINT Y
```

This should be typed into a computer to see if it behaves as expected. To get the computer to carry out the program we must type RUN <R> , after we have typed it all in.

Get them to try (in their rotating groups) to try these problems from sheet 2:

```
(i)  10 X = 3          (ii)  10 X = 3
     20 Y = 5                20 Y = 5
     30 Z = X + Y            30 Z = Y + X
     40 PRINT Z              40 PRINT Z
```

[N.B. For those using the computers, the ability to change a program, by retyping one line only (here 30), is now useful. It is also useful to explain that it doesn't matter what order the lines are typed in, and a line can be deleted simply by typing the line number. The command LIST will list the program in the correct order.]

Do the two programs give the same answers? Why? Try changing the values of X and Y by retyping lines 10 and 20. Are the answers still the same? This means that the order in which we add the values in the variables does not matter.

(This is the idea of commutativity, that the order of addition is immaterial.)

Try these (which are also on sheet 2):

```
(iii) 10 A = 7              (iv) 10 A = 7
      20 B = 9                   20 B = 9
      30 C = 15                  30 C = 15
      40 D = A + B + C           40 D = B + A + C
      50 PRINT D                 50 PRINT D
```

What do you notice? Try different values for A,B,C. Why are the answers the same?

Revision

```
Try : (i) 10 X = 8        (ii) 10 X = 8        (iii) 10 X = 8
          20 Y = 2 * X         20 Y = X * 2          20 Y = X + 2
          30 PRINT Y           30 PRINT Y            30 PRINT Y
```

What do you notice? What does this mean? (Perhaps you would like to discuss the fact that multiplication is commutative.)

(3) Links with algebra

(There are two different ideas in this lesson: algebraic notation for division and implicit multiplication, and the use of brackets. The ideas should be discussed carefully with the children, and the material extended to give them more opportunity to explore and practise, as necessary.)

What does 3*x mean to the computer? - take (a copy of!) the value in x and multiply it by 3. In algebraic notation in ordinary maths this is often written as 3x, with the multiplication sign missing. It is traditional in algebra to miss out the multiplication sign between a number and a letter, so that 2*a is written as 2a. (The number is usually written first, so that it is written as 2a and not a2.)

It would be very foolish to miss out the multiplication sign between two numbers. There is a world of difference between 2*3 and 23...!

When typing commands in the BASIC language into the computer, the multiplication sign must always be included.

Division on the computer is written as a/b, meaning "a divided by b". This means "take the value in store a and divide it by the value in store b". If a is 6 and b is 3 then a/b is 2.

Try the program:

```
10 a=12
20 b=3
30 PRINT b
```

on the computer. When it is RUN, what happens?

What happens in the following case?

```
10 a=10
20 PRINT a/2
```

On the computer we must type the symbols one after another in a line but in mathematics notation we sometimes write a/b as

$$\frac{a}{b}.$$

In mathematics we might write 1 divided by 4 as

$$\frac{1}{4}$$

But on the computer we would type this in as 1/4.

Write on the board (?), and ask what these would be in Maths. :

2*x + 1, 5 - 2*y, a*b, 2*a + 2*b, 3*x - 2, 5/x, 7 + 8/y .

What would these be in the BASIC computer language?

$\frac{2}{x}$, 4y - 1, 3x + 2y, 4a + 3b, xy, 7 - 5x, 6 + $\frac{y}{4}$.

The use of brackets

How would we tell the computer to add 1 to x first, and then multiply it by 3?

The answer may be given as 3*x + 1 or 3x + 1, you may also get x+1*3.

Try the process suggested in a program, for example, try:

```
10 x = 6
20 y = 3*x + 1      What answer should we get?
30 PRINT y
```

Now RUN. Does it give the expected answer? Why? Suggestions? If none come forward, suggest the use of brackets around the x + 1.

Try it, by typing in the new line

```
20 y=3*(x+1)
```

This time it works, as expected. In computer notation 3*x + 1 and 3*(x + 1) are not the same.

i.e. In mathematical notation 3x + 1 and 3(x + 1) are not the same. What does the bracket mean? It means do the calculation inside the bracket first.

Pupils can now try SHEET 3 making sure that they :

a) Use both the computers and the 'Maths Machines'
b) Know how to record their results on the store sheets.

4. INPUT

Explain that the instruction

```
10 INPUT X
```

means that the computer will put a ? on the screen and wait for us to type in a value for X. When we do so, and press RETURN the value is stored in the location/box X.

To check this try :

```
10 INPUT x
20 PRINT x
```

so that they see it works.

N.B.THE MIXING OF UPPER AND LOWER CASE LETTERS IS IMPORTANT IN ORDER TO BE FULLY CONFIDENT IN WORKING WITH EITHER. THE NORM IN ALGEBRA IS THE LOWER CASE, OF COURSE.

We can put in lots of values for x by adding just one line to this program:

```
30 GOTO 10
```

This tells the computer to go back to line 10 and start again. What does INPUT mean for our 'cardboard maths machine'?

It means that we check and see if there is a box labelled x. If not then we label one x. We are then free to choose any number to put in this box.

N.B. WE HAVE HERE THE START OF THE IDEA OF LETTER AS GENERALISED NUMBER.IE THAT A SINGLE LETTER CAN BE USED TO REPRESENT MORE THAN ONE NUMBER AT DIFFERENT TIMES; IN FACT A SET OF NUMBERS.

Using INPUT

Try this program to see if the pupils have the idea.

```
10 INPUT A
20 B = 2*A
30 PRINT B
40 GOTO 10
```

Put in 3 values for A

N.B. THIS IS AN OPPORTUNITY FOR A COMPETITIVE RACE BETWEEN GROUPS, EVEN BETWEEN THOSE WITH COMPUTERS AND THOSE WITH MATHS MACHINES.

Now they have the idea they can try these two programs, putting any three, BUT THE SAME THREE, values for x into each.

(i)
```
10 INPUT X
20 Y = X + X
30 PRINT Y
40 GOTO 10
```

(ii)
```
10 INPUT X
20 Y = X * 2
30 PRINT Y
40 GOTO 10
```

Then ask what is noticed. It means that X + X and X*2 give the same answers; they must be the same. Does 2*X give the same? Try it if it is not obvious.

Get them to work through SHEET 4, making sure that :

a) They rotate where possible, computer to 'Machine'
b) They record their results on the stores sheets.
c) They put their names on the stores sheets.

5. Application to problems

HERE WE WANT TO CEMENT MORE FIRMLY THE RELATIONSHIP BETWEEN THE WORK WITH VARIABLES ON THE COMPUTER AND THE USE OF VARIABLES IN MATHEMATICS, PARTICULARLY IN ALGEBRA.

a) An example

If we know the length and width of a rectangle,how do we find its area? How can we use the computer/'Maths. Machine' to help us do this for any rectangle?

The answer we want is that we use a box/location/variable called L to hold the length and another called W to hold the width.

Can they write the program to work out the area of any rectangle, using INPUT?

We want :

```
10 INPUT L
20 INPUT W
30 A = L*W
40 PRINT A
50 GOTO 10
```

This should be tried two or three times with different (whole number?) values for L and W.

b) A second example

If the computer has a box labelled S, in which is stored the number of sweets we are to buy, and these sweets are 5 pence each, how can we tell the computer how much they cost?

IT MAY WELL BE NECESSARY TO USE NUMERICAL EXAMPLES HERE TO ARRIVE AT 'THE FORMULA'

We want the answer:

5*S

(or 5S making the point that these are equivalent in mathematical notation, but the computer needs the explicit * sign.)

If also, the computer has a box labelled C, containing the number of comics we are to buy, and these are 12 pence each, what would we use for the cost of the comics?

Answer - 12*C (or 12C)

So, can we write a program to get the computer to work out the total cost of the sweets and the comics, using INPUT?

Let them try, and help them towards :

```
10 INPUT S
20 INPUT C
30 P = 5*S + 12*C
40 PRINT P
50 GOTO 10
```

- run this 2/3 times (I suggest here that you give them values for S and C, and have someone checking to see if the computer is right)

They can now try sheet 5. They may need some assistance with the formulae.

6. a) **Notation**

Suppose we have a variable X, and we calculate Y as X+3. Then we get a table of values like this:

X	Y
1	4
2	5
3	6
4	7
etc.	

To get an answer in the second column, we add three to the value in the first column. We can see this as a rule for changing the values which we write as

```
1 ---------> 4
2 ---------> 5
3 ---------> 6
4 ---------> 7
etc.
```

where the arrow sign ---------> in this case means "take the value in the first column and add three to get the value in the second column.

We can use the notation :

```
X ---------> Y
```

to represent the 'rule' of a 'Maths Machine', suggesting that if we input the value of X, then the machine outputs the value of Y. The following program:

```
10 INPUT X
20 Y= X + 3
30 PRINT Y
40 GOTO 10
```

is such a program. Whatever value is input for X, it gives the variable Y the value X+3.

What would the numbers be for this rule in the following cases?

```
3 ---------> ?
7 ---------> ?
```

b) Problems Based On This.

The part of the program below which corresponds to X------> Y. is missing. However the results the computer gave are given as a list. What is the missing line?

```
10 INPUT X
20 Y =
30 PRINT X,Y
40 GOTO 10
```

X	Y
1	3
2	4
3	5
4	6
5	7

EXPLAIN LINE 30 MEANS PUT THE VALUE IN Y ALONGSIDE THE VALUE IN X

Get some suggestions and let them try them to see if they work. n.b. They need to realise that they must get the right answer for **every** value of X.

Then try :

```
10 INPUT X
20 Y =
30 PRINT Y
40 GOTO 10
```

X	Y
1	3
2	5
3	7
4	9
5	11

as above.

They need only change line 20 in their program of course to try this.

Answers : Y = X + 2 and Y = 2*X + 1.

They can now try SHEET 6 in the same way.

THIS IS ANOTHER WAY OF GETTING THEM TO APPRECIATE THAT BEHIND THE LETTERS IN A FORMULA, I.E.BEHIND THE LETTERS IN ALGEBRA LIE A RANGE OF NUMERICAL VALUES (A VARIABLE) OR AT LEAST THAT SEVERAL VALUES MAY RELATE TO THE SAME LETTER (GENERALISED NUMBER). WE ALSO HAVE HERE, IN THE 2 LISTS, THE VALUES FOR THE INDEPENDENT VARIABLE (X) AND THE DEPENDENT VARIABLE (Y).

7. The "Computer Maths Machine" Program

This is a piece of software, written to operate in a similar way to the cardboard maths machine. The computer maths machine has up to three variables for input, and up to two formulae which may be used to calculate values from these variables. (There is also a provision for the formulae to include "constants" that are defined at the outset and not allowed to have variable inputs, but these need not be used in the initial stages.)

The main commands to be used are:

M: make maths machine

and

I:input variable values.

(The command:

V:Change variables

simply changes the letters for the variable labels and need not be used in the first instance.)

The M command is used to put in the function expressions which we are looking at.The I is used to put in the values of the variables.

The "Computer Maths Machine" may be used to solve algebraic problems. For example:

1. When does $5 + x = 9$? (THE PROGRAM WORKS IN LOWER CASE)

a) Press M to select a function
b) Choose 1, and press RETURN
c) Type in 5 + x <R>

Ask how can we use the 'Machine' to find when $5 + x = 9$?
Answer put in values of x until the value in the 'function' box is 9

d) Press I to input values of x

e) Put in a suggested value <R> Ask how can we tell what value to try next? If the answer was too big we try a smaller value. If it was too small we try a bigger one!

f) We answer Y <R> to the question of any more values, until we find the correct answer. We then note down the value in the x box as this is our answer.

2. When does $3 - x = 2 + x$? (A LITTLE HARDER!)

a) Press M and choose 2 functions
b) Put 3 - x into one and 2 + x into the other
c) Press I
d) Put in values of x, as before, until when? Until the function boxes both contain the same value. The value in the x box is then the answer we want to write down. n.b. They will need a little direction here as to why we choose bigger or smaller values of x. Experimentation is to be encouraged, the secret being which of the 2 boxes has the bigger value for any given x.

3. When is $y - 2 > 5$? (A VERY DIFFICULT QUESTION)

n.b. We are happy here to end up with a string of suitable values for y. It is not expected that many (any ?) of them will arrive at $y > 7$.

a) Choose 1 function.

b) Put y - 2 into it.

c) Press I

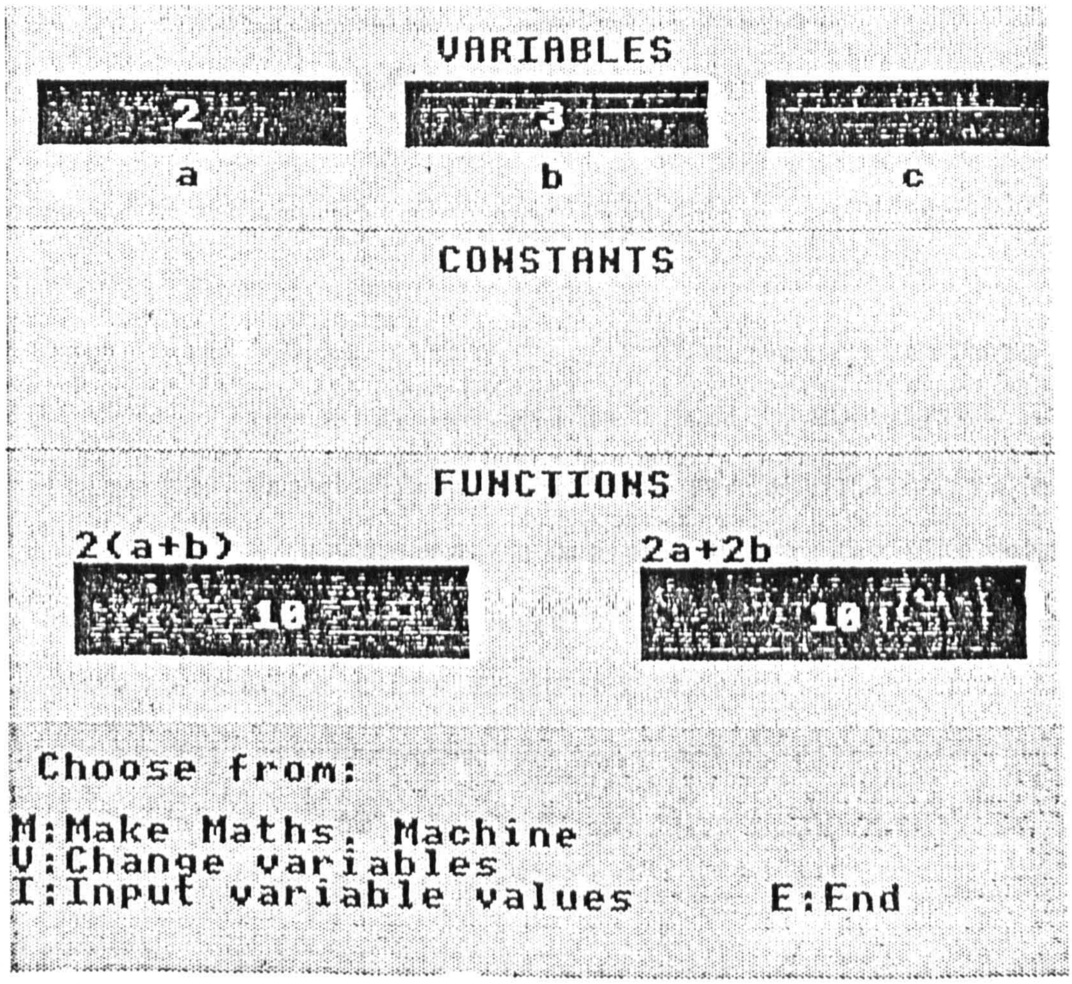
d) Put in values for y until we get an answer in the function box which is bigger than 5. We note the value in the y box for ANY such value. They can be asked if there is a rule for the answers, someone may spot it.

Pupils can now try SHEET 7. With ingenuity this can be done with the "cardboard maths machines", but it is better on the computer.

The classroom organisation here will take a little thought.

THE VALUE OF THE 'COMPUTER MATHS MACHINE' IS THAT IT CONTINUES THE 'PICTURE' OF A VARIABLE WHICH WE HAVE PREVIOUSLY USED, BUT NOW THEY CAN ACTUALLY SEE THE VALUES IN THE VARIABLES AS THEY VARY I.E. THE VARYING WILL BE IN THEIR EXPERIENCE. FURTHERMORE THEY ARE USING THIS VARIABILITY TO TACKLE 'REAL' PROBLEMS AND THUS IT IS HOPED THAT IT WILL HELP THEM TO HAVE ENOUGH CONFIDENCE IN THE MENTAL PICTURE TO USE IT IN THE FUTURE, PARTICULARLY IN THEIR ALGEBRA LESSONS.

Further problems for the pupils are given on SHEET 8.



7.2.6 The Cognitive Value of This Approach

This experiment was to aim an algebra teaching programme at pupils with no previous experience of algebra, except that very occasionally a letter may have been used in their mathematics lessons, with the intention that the programme would be, not only an introduction to the basic concepts of algebra, but would also act as an advance organiser for the later formal teaching of the techniques and ideas of algebra. The module needed to point to the key concept of the use of letters as variables and to pay close attention to the individual's framework of knowledge which he/she would construct with regard to the topic, so that subsumption leading to meaningful learning could more easily take place. Some of the reasons why the structure of the module of work using programming, cardboard 'Maths.Machines' and software, as set out above, was considered valuable, are explained below.

Although the great value of individual learning is recognised, there is also value in pupil discussion, in order to permit, as Skemp describes it, reality building mode 2. In practice, due to resources it was necessary for the pupils to work in small groups, although it was hoped that these would be small enough to enable individualised learning, as well as having the benefit of encouraging team work, with members of the groups encouraging and trying to help each other. In the second experiment the teacher had found the small groups of three pupils had operated very successfully. A small element of competition was encouraged between the computers and the

'Maths. Machines' and was felt to be a good motivation factor, although the logistics of the school, having as it did a computer network in a single room made this difficult in the third experiment.

The first section of the module was designed to introduce in a personal way, (without their knowledge of its importance or significance), the word variable. This happens in a natural way on the BBC computer through the error 'NO SUCH VARIABLE'. A principle fundamental to the research was to provide a mental model of a variable which was readily accessible to the children and around which a programme of computer programming related instruction could be built and so an early introduction to the mental image 'variable' as well as the concept itself was beneficial. The model chosen has been described as consisting of two parts, a location or 'box' containing the current value of the variable and a label by which the variable was known. Restricting the label to a single letter was designed to enable the relationship between letters used in algebra and these locations to be more easily perceived. The model as used in picture form was given in Figure 6.2. The view was also encouraged that this 'box' was in fact a window onto a range of values, the present value seen being only one of a range of possible values. The establishment of this link between the mental image of the word variable and the mental model of a variable would ensure that the word variable could be freely used throughout the programme in the knowledge that a mental picture existed and had been established and was likely to be evoked by the use of the word variable, in the manner discussed in Chapter 3. This also has the diversionary effect of directing attention away from the first three, somewhat limiting, possible interpretations of the use of letters in algebra, as given by Küchemann (1981b), since whenever the letter A, for example, is mentioned the children are enabled to 'see' it as a label with its accompanying 'box' containing the current value. They have no need therefore to assign a value to the letter from the start, nor should they want to treat the letter as an object in its own right.

Having established the fundamental imagery on which the versatile approach to algebra was to be built, the second section was able to be used to introduce operations on variables in the computer programming domain. There can be a plentiful supply of such variables on hand in programming, and they can, by the use of the PRINT statement in BASIC, easily be seen to vary in the experience of the pupils. Even more important, they can be made to vary by the children themselves giving them a sense of involvement in the model which is so necessary, and even a feeling of control over it, the computer unerringly obeying their correctly entered commands. The introduction of the cardboard 'Maths.Machines' at this stage enabled activities which encourage concept formation, since they involve a measure of undirected activity with the ingredients being the variables previously introduced with all the activities leading unsuspectedly to the same concept, namely the use of letter as a variable. The 'Maths. Machine' had been introduced to help the children see that the computer was in fact doing the arithmetic with which they were familiar, 'behind' the screen. The machines made them carry out all of the stages of the calculation process themselves, but doing so in connection with the ever present mental model of the variable. It was considered that the concepts of arithmetic, which were already quite firmly fixed in their cognitive structures, would then act as subsumers for the use of algebraic manipulations with variables. Also the 'Maths. Machines' would further concretise and bring into the direct experience of the children the processes involved in practical problem solving. They would be physically, in their movements around the room, or model, as well as mentally involved in the problem solving process. The use of computer programming in working through the first two worksheets produces the added benefit of the appreciation that correct symbolisation and representation, is vital, since the computer only accepts correctly written statements, providing an error message for incorrect ones. This need for a precision of input to the computer in order to obtain results is very valuable in helping children to understand the importance of algebraic conventions of notation since they can appreciate that the computer needs expressions which are precise and not just nearly right. The problem of the correct use of notation is one of the areas where errors are commonly made by early learners of algebra. These also include the appreciation of the need for correct and unambiguous symbolisation of expressions in algebra and the need for an appropriate formal method of solution to a problem, (Booth 1983a). One area where this soon becomes apparent is in the use of brackets. Children soon realise that the use, non-use or wrong use of brackets affects the answer from the computer e.g. when finding the average of two numbers using a wrong expression such as a + b / 2 instead of (a + b)/2.

Following such familiarisation with the computer notation, the module contained a directional section encouraging the formation of cognitive links between the newly acquired operators of the computer and the much more familiar arithmetic operators. Such cross schematic linking has been described in Chapter 3 as being vital in the promotion of a high level of conceptual understanding, so that concepts are not learned in the isolation of a single knowledge domain but are linked to other existing, cognitively relevant, concepts in other domains.

It was hoped that subsumption with arithmetic concepts would take place through the addition of a section on the connection between the computer operators and the arithmetic operations. Since the computer notation for multiplication and division, to be used throughout the programme, were * and / respectively, the pupils might have felt that the letters and what they represented could only be used with these symbols in a computer context. To avoid this, and to promote links with arithmetic schemas, the section on notation attempted to show the connection with the arithmetic notation familiar to them, in a numerical context, thus enabling subsumption. An extension of this was the inclusion in the applications section of the transition from a problem stated in English, again familiar to them, to its symbolisation in a computer environment. e.g. If 5 apples cost x pence each, how can we tell the computer how to work out the cost once we know x? These problems are known to be found very difficult by many school students of algebra (see Chapter 4). The third worksheet aids in the cementing of such relational understanding between the two systems through a process of structured practice. Throughout the programme, the constant and continued use of the stores sheets (see Figure

6.14) for the recording of answers to the exercises weaves the mental imagery established into the sequential processes of the solutions, fostering and promoting the cognitive integration which, I have postulated, will lead to valuable versatility in mathematics.

The structure of the material in these early stages of the work is such that there is a gentle progression for the students through the levels of understanding of letter in algebra. The use of INPUT in the programs highlights the difference between the allocation of a single fixed value to a letter, as in A = 3, (corresponding to a constant in algebra) and the use of letter as a generalised number through several RUNs of a program using INPUT A. The programs also contain the ideas (although this was left unsaid for the pupils but mentioned for the teachers) of dependent and independent variables.

An effort was made too to make available concepts which, although not specifically highlighted during the programme, could usefully be abstracted from the work and extend the linking to other concept areas. These included the introduction of questions involving commutativity in the programs, the equivalence, or not, of expressions using different notations, particularly the use of brackets as in 2(x + 1) and 2x + 2, and the problem of conjoining in addition, or the writing of expressions such as a + b as ab. The abstraction of any of these concepts from the programme would have the effect of increasing the number of subsumers available for the future teaching of algebra.

It is important that pupils gain first hand experience of variables actually varying in value and, what is more, doing so in relation to concepts such as area, distance, speed, percentage etc. which are already firmly fixed in their schemata, thus aiding accommodation of new concepts, and, in Ausubelian terms, enabling their framework of knowledge relative to this area to be constructed, via subsumption. In section 5 of the module notes, and on the accompanying worksheet 5 dealing with applications to problems, these type of concepts were utilised to promote the formation of as many conceptual or C-links between the variables in use and the well established knowledge structures. It was also hoped that the practice of breaking down the word problems in order to formalise the solution and present it as a step-by-step program for the computer would encourage the type of structured approach to solving word problems which would be necessary in the future.

Section six was designed to enable the children to see the reverse side of the concept, namely that a range of values may be represented by a single letter. It thus aimed to give practice at the process of representing a relationship between two such sets of numbers in terms of the variables which represent them. This means that this relationship has to be correctly symbolised in order to give the right values when the program is RUN. What they are playing at this stage is a structured analytical game, using the computer to help them solve a problem. This reduces the problem from an abstract symbolisation problem to a concrete one of persuading the computer to emulate the list of values given in the question. The giving of a list or range of values as the starting point rounds out the concept by illustrating its two-way nature. This aspect of genuine problem solving, using the concepts and the framework of knowledge constructed containing them is an area considered of importance in this investigation.

Section seven extends and completes the experiences considered necessary to firmly fix the conceptual understanding of the uses of letters in elementary algebra. The software 'Maths.Machine' is introduced for the first time in this section and provides a generic organiser (Tall, 1986c) for studying the concept of variable. Such an environment allows the children to tackle equations and inequalities, which involve important, and relatively difficult, aspects of algebraic variables, through problems in the form of structured games which, although analytical in nature themselves, may be solved by a concrete process. It has been recognised (e.g. Dienes (1960)), that some children can only generalise one variable at a time and hence it was expected that some of the children would find some of the questions at this level difficult. Particular attention was paid at this stage to extending the boundaries of the application of the model and to helping the pupils to apply the concept to actual algebraic problems solved on the computer, which they would be able to recognise as algebra once they had been formally introduced to it in their mathematics lessons. Thus for the first time they were encouraged to solve 'practical' algebraic problems whose solution involved an understanding of variables themselves. It was felt that this personal challenge, using the computer as a tool, and approaching the problem in concrete terms by typing in values and looking on the screen for results, then adjusting the values by a process of structured trial and error, would stimulate interest and produce good results.The program, acting as a generic organiser, also makes use of on-screen multiple representations in the form of algebraic representation coupled with the visual image of a variable. The benefits of such a combination have been described in Chapter 5, and, in algebraic terms, such a combination serves to divert attention away from a purely sequential view of algebraic processes and toward a versatile view by adding a global/holistic dimension in the form of the mental imagery of a changing value in a ´box´. This is the promotion of cognitive integration aimed at the production of a more versatile learner of mathematics.

Success in such practical applications as those outlined above, involving as it does much problem solving, should re-inforce the value of the mental imagery and stimulate the links in the children´s knowledge frameworks which would give them the confidence to apply the model in other areas in the future, eventually rendering the mental model unnecessary.

7.3 Observations From The Programme in Practice

A consideration of the practical implications of any new module of work is an important aspect of its introduction and so I shall now briefly indicate some observations of the classroom practicability of the module which arose from the third experiment. Overall, the programme of work in the third experiment went well, with one of the teachers who had never used a computer in a mathematics lesson before commenting: "The lessons proved popular with the pupils and I, too, found the work surprisingly pleasant." There were, however, some useful observations which came out of the use of the module in this

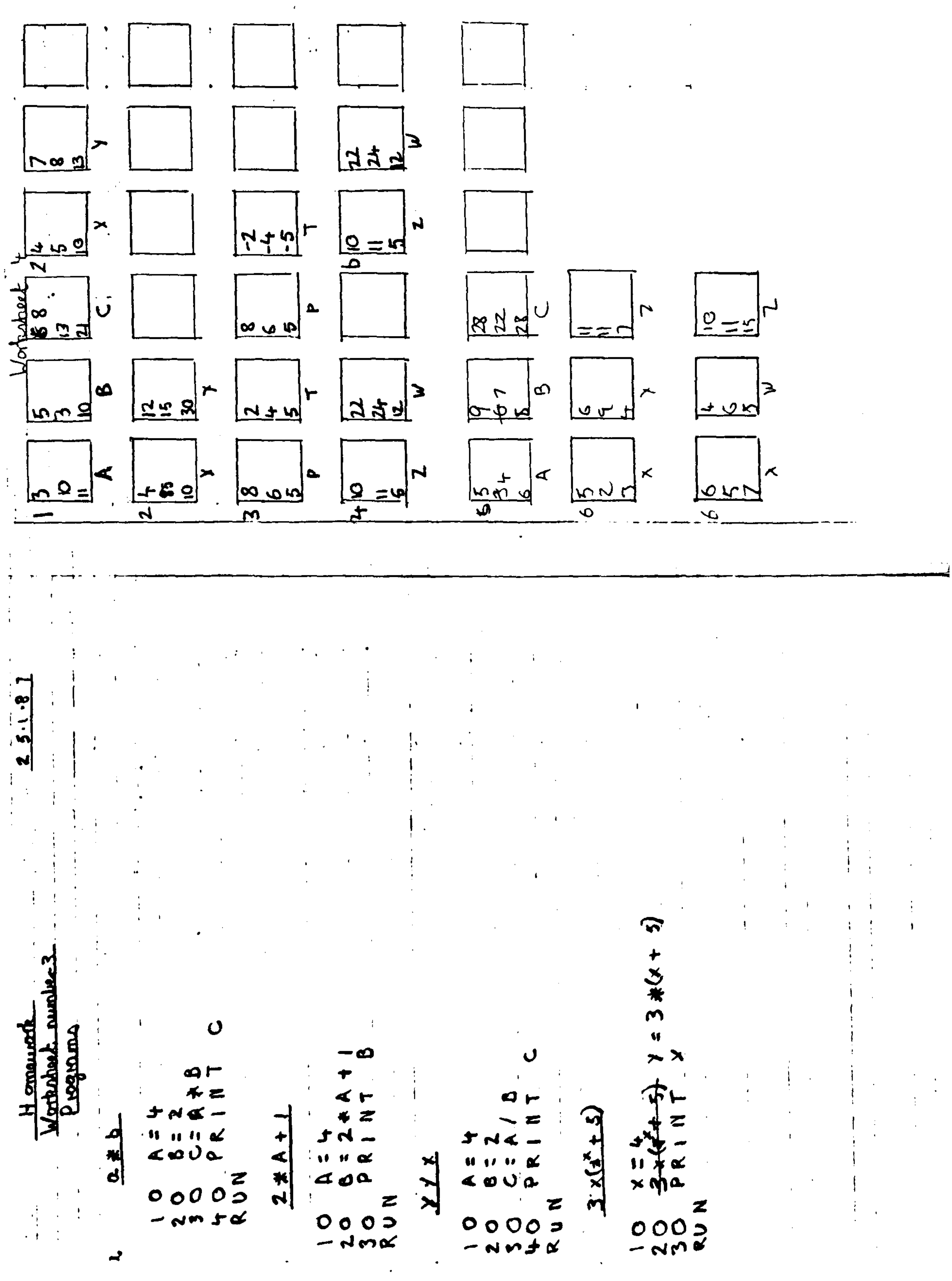


Figure 7.3

An Example of a Pupil's Work From The Third Experiment

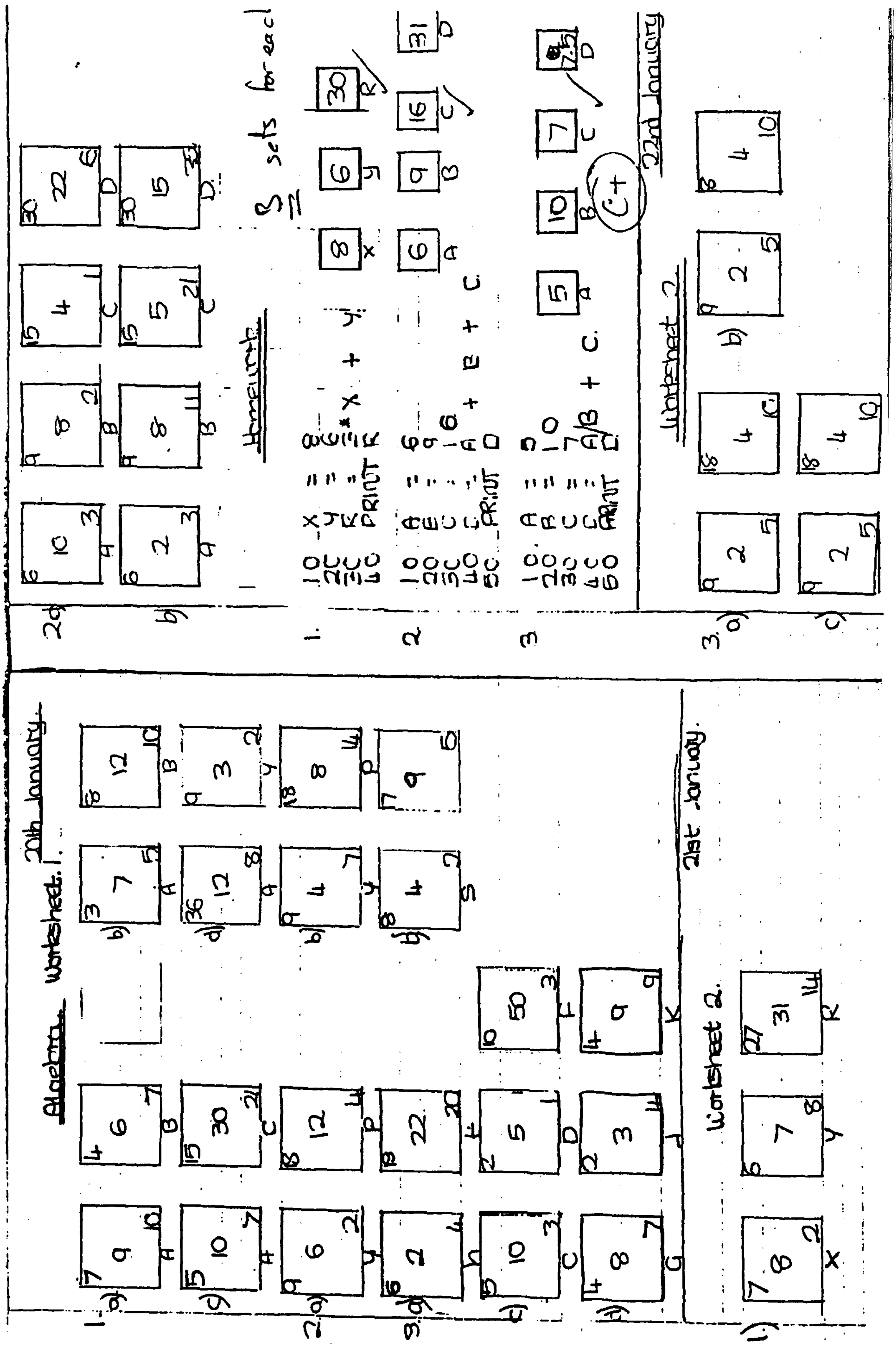


Figure 7.4

An Example of a Pupil's Work From The Third Experiment

type of secondary school, and these are discussed below.

7.3.1 Positive Benefits Resulting From the Programme

One encouraging and positive aspect of the way in which the children took to the work was in the use of the mental image provided for variables. They all were more than happy to use the imagery, and thus, I feel, to gain its benefit. Figures 7.3 and 7.4 show examples of its use from pupils in the experiment. We can see how they have been encouraged to adapt the image by putting all three values of each variable in the same box, but keeping them clearly delineated so that they are still able to tell corresponding values. This interesting adaptation may even be an improvement in that it may further emphasise that the letter may stand for several values at the same time rather than only one. These two figures also show that in the programme they have learned that variables may stand for values other than positive integral ones. In Figure 7.3 for example we see the use of negative values in question three and in Figure 7.4 the value of D in question three is recorded as 7.5. This appreciation of variables as being able to take fractional or decimal values is important and, with the computer often giving its answers as decimals, the opportunity to abstract this feature of the concept from the programme is clearly evident - the comments on the interviews later show further evidence of such understanding. Figures 7.5 and 7.6 show the way in which the pupils were able to arrive at some of the desired conclusions by using the worksheets in the programme. We can see that this pupil has been enabled to see clearly the relationship between 2xa, a + a and ax2, namely as different notations for the same entity whereas pxq and p + q must be different because they

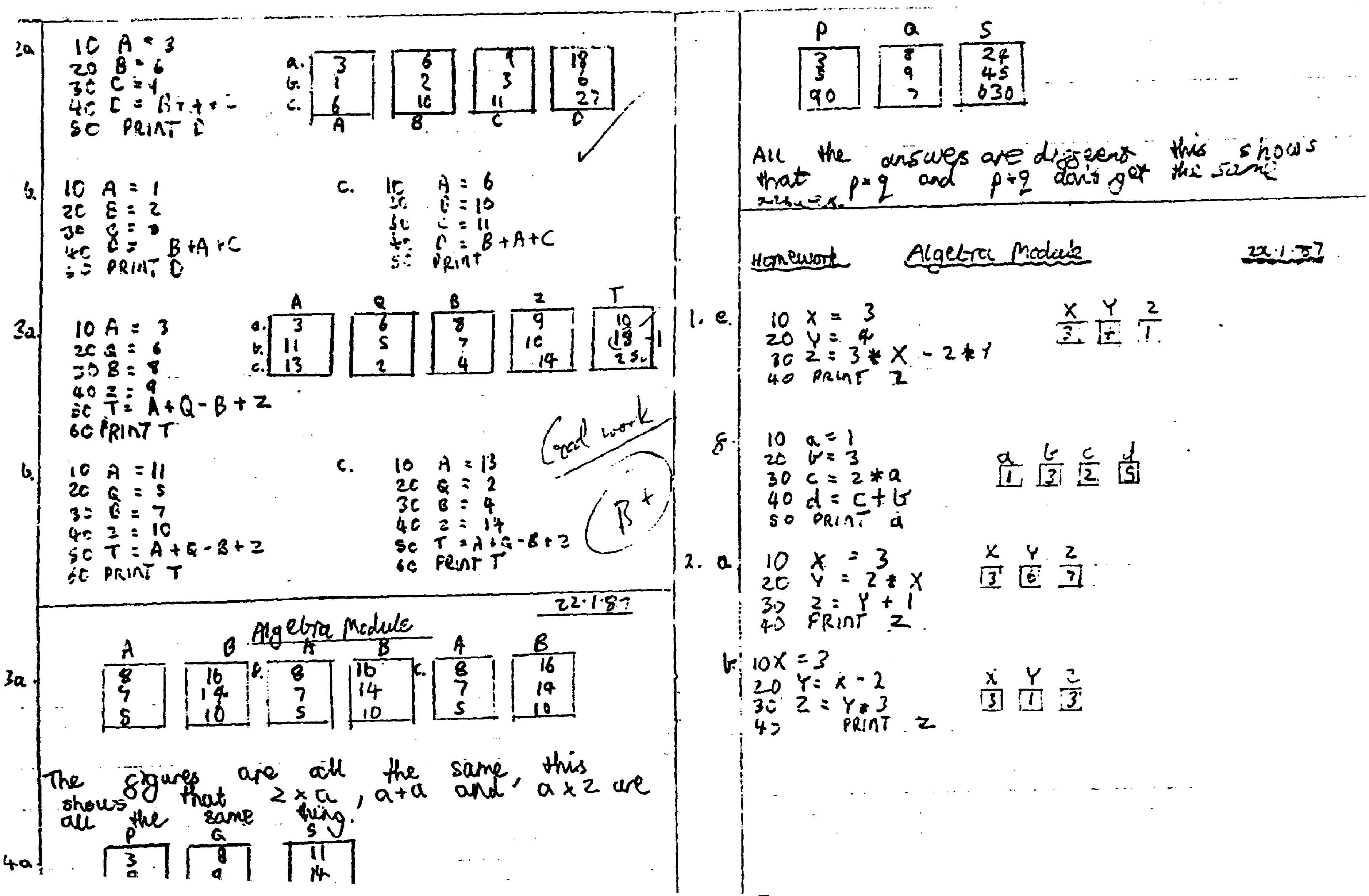


Figure 7.5

An Example of a Pupil's Work From The Third Experiment

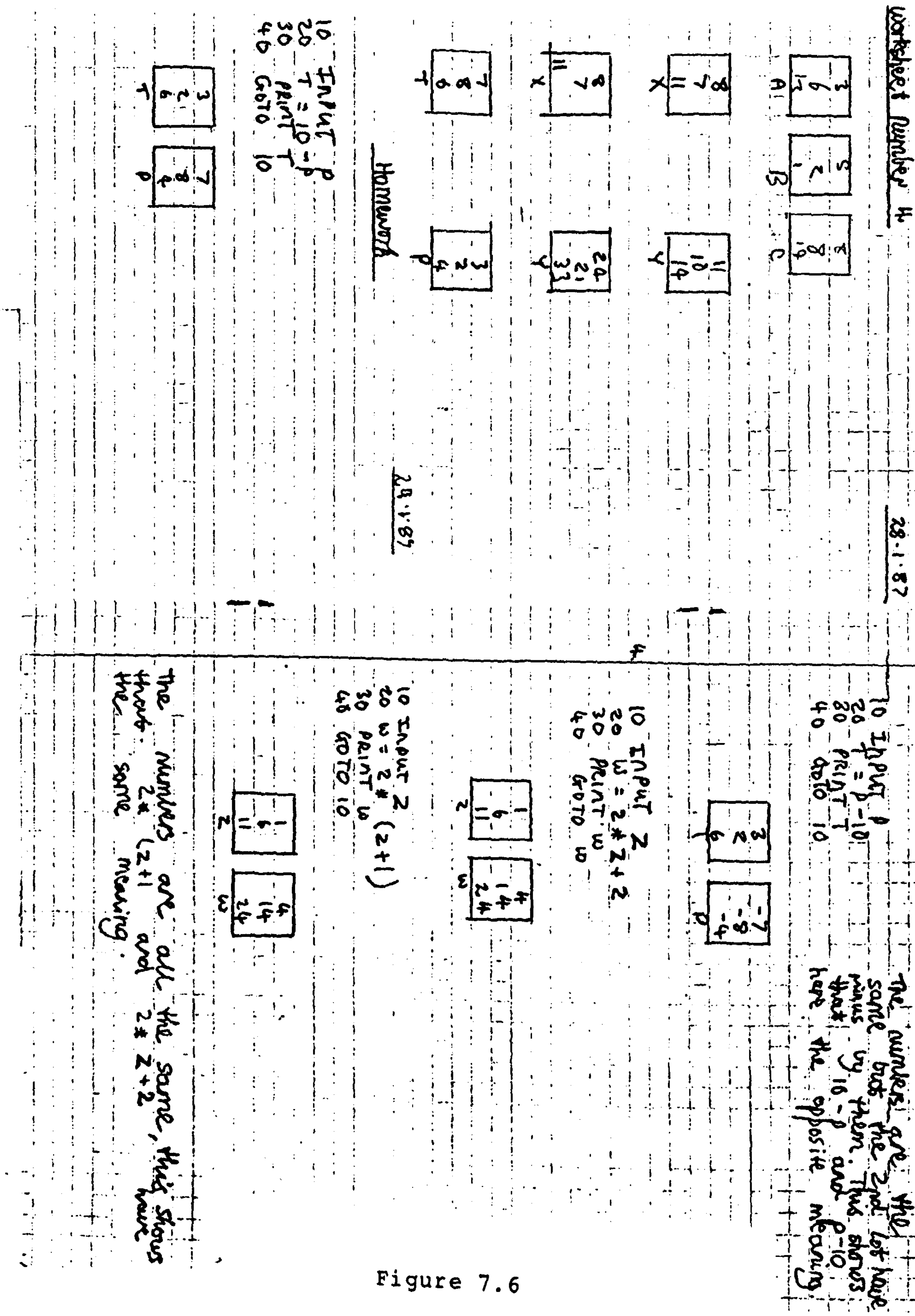


Figure 7.6

An Example of a Pupil's Work From The Third Experiment

"don´t get the same answers" for several differnet values of the variables. One of the most difficult concepts associated with the application of variables in algebra seems to be the use of brackets (Booth 1984). Figure 7.6 shows how the programme has helped the pupil arrive at conclusions such as 2*(z + 1) and 2*z + 2 are essentially the same through a practical involvement in a testing and problem solving environment.

The understanding of the use of brackets, and the value of interaction with others in a small group to help towards an understanding of it, is one of the points well illustrated by the excerpts from the tape recordings made of some of the pupils during the programme. Figures 7.7 to 7.10 contain extracts from the transcripts of these tapes. The first two of these figures show the two pupils using each other and the computer environment in their efforts to come to grips with the role of brackets in expressions, testing out their ideas on the computer, examining its answers and adjusting their view accordingly. For example : "A..I think you´ve got your answer slightly wrong! B. I don´t think those brackets work somehow." They also help each other, as in "B.Where´s the brackets. I can´t find the brackets. A. It´s on the 89." and "B. I don´t think those brackets work somehow...Do you think I should change this? I´ll just do this. A. Delete this. You have to use the arrows.". Although this may not seem as if one is imparting conceptual ideas to the other in these exchanges the common use of mathematical language helps to establish its meaning and purpose and the interchanges required to formulate agreement as to what action to take constitute an important part of the learning process described by Skemp (1979c) as reality building

A. You did six boxes!

B. Then I throw one of mine?

A. Yes.

B. Has it got brackets in it? I wonder if brackets work.

A. Yes, they do.

B. Do they? That's all right then.

A. There's the brackets.

B. Hang on! I don't want brackets yet! Ah, this'll do.. b..

A. You want 'close brackets'.

B. Hold on!

A. It's all right, I know how brackets work.

B. What was it? b plus a ...

IJ: Which one are you on now?

B. We've just finished the sheet ...

IJ You've finished sheet 2?

A and B: Yes.

IJ Oh, splendid. You're just trying one out?

B. We're just trying the one I did last night.

IJ Oh, good!

You might find, James, that you need another bracket there, because you've opened brackets and closed brackets, and you've opened brackets ...

B. That's a mistake. I don't know (*unintelligible*).

A. I'll try mine out.

Figure 7.7

An Extract From The Transcript of The Pupils' Conversation During The Third Experiment

IJ Oh no, you do need that in. If I were you ... Do you want to divide by all of that?

B Well, it's e *plus* a, divided by c, minus that in the bracket.

IJ So you want another bracket *there*. So that *that* bracket closes *that* bracket. OK? You've always got to close brackets once you've opened them. Yes. Try that.

B. Fifteen thousand divided by ...

IJ Have you predicted what the answer's going to be?

B. Yes, it's that.

IJ Good. You'll be able to check it.

B. Where's the brackets? I can't find the brackets.

A. It's on 89.

B. Ah, there. ... brackets d, minus ... got to open brackets.

Look, this is amazing. It's double brackets.

A. You've got your brackets the wrong way round.

B. No, it's double brackets. Look, Mr Jones has just put it on. ... c times d ... never worked out ...

A. More brackets?

B. More brackets, plus ...

A. What? I bet it's ....

B. Return and 70, p dot, 60 or less, p dot ..

A. Now what? I think you've got your answer slightly wrong!

B. I don't think those brackets work, somehow. Minus two, minus ... Do you think I should change this? I'll just do this ...

A. Delete this. You have to use the arrows.

B. Oh, right ...

A. Now, go along ... didididdidididi ... Now, I'll tell you how to start. Go on, keep going. Now stop. Go, go back, go delete.

B. Don't want that. Just one, and that's it. Return.

Figure 7.8

A Extract From The Transcript of The Pupils' Conversation During The Third Experiment

A. What is it then?

B. Three plus, two times a,

A. Two twos. Two times two.

B. Two times two is four. Three plus four equals seven.

A. Two times two. Plus four.

(Unintelligible)

B. Two times x. It'll be x ... right. x equals ...

A. No.

B. x equals 2, then it'll be .... x ...

Sounds of typing.

B. Two times x equals one.

C. So it's two times ten plus one.

A. Where did you get the 'times' from, James?

C. Where did you get the 'times' from James? Probably the (*sounds like* copiot).

A. Where did you get the 'times' from?

B. 2x is 'two *times* x'. Look, 2x, two xes, is two times x, isn't it?

C. No, *(unintelligible, but something like)* but it can't be the numbers.

A. Instead of representing two things, you're representing two exes,

B. When you've got '3y' and things like that ... x is 2 there.

C. How can x be 2?

A. Well it's not, it's 10.

B. Two times x plus 1 equals ... 21.

A. You don't do that, though. Because it doesn't matter.

C. I've just put two times ten, plus 1.

B. You've got to do it in Basic, how the computer would write it.

C. I'll do it at home, tonight.

B. Did he tell us to stop?

A. Yes.

Figure 7.9

A Extract From The Transcript of The Pupils' Conversation During The Third Experiment

B. a times 3 minus 2 times y.

A. And now what?

B. Now you've got to work it out.

A. It's 50!

B. No.

A. Ten fives are 50, take away 2y. What's the value of y?

C. He hasn't written the value of y down.

B. Course he's written the value of y down. b is y and a is x. This boffin's just mucked it up.

A. Which boffin?

B. Why did you put a and b instead of x and y?

A. Because I'd already written the boxes to do that. It doesn' make any difference what *they* are - a and b, or b and c, or d and x and q and r and p and z.

Everyone talking together about what it should be.

C. That should be a.

A. Ten fives ... Look ...

B. It's thirty, minus ...

A. a.

B. Ten. Equals twenty.

Figure 7.10

A Extract From The Transcript of The Pupils' Conversation During The Third Experiment

mode 2. Sometimes the concept is more clearly seen, as in Figure 7.9, where we have : A. Where did you get the times from James? ...B. 2x is two times x. Look 2x, two xes, is two times x, isn´t it?". This concept of implicit multiplication in expressions is one which many children mis-understand and gives a good example of how the informal interplay in the computer environment with the opportunity to demonstrate ´the answer´ on the computer can be invaluable in improving such conceptual appreciation. This figure also shows how A and B, having overcome themselves the left to right tendency of many children of this age in parsing expressions such as 3 + 2a are able to help convey this understanding to C, who is the pupil lacking it. The exchange recorded in Figure 7.10 is very interesting and shows a discussion about the variables themselves. It shows that the value of y is not pre-determined in the minds of the pupils, as if it was viewed as an object its own right, or as a replacement for some specific number, but that it is able to take different values, and that here it is even equated with another variable : "B...b is y and a is x". The insight by A that "It doesn´t make any difference what they are - a and b, or b and c, or d and x and q and r and p and z." demonstrates a measure of the understanding described by Wagner (1977) as conservation of expression or function, understanding that the essential nature of the expression is unchanged by a variable substitution. Such insights seem to arise naturally in the computer environment and with the small groups the opportunity for discussion is available and can prove very helpful.

7.3.2 <u>Difficulties With the Programme</u>

As I have mentioned above, one of the major considerations of any new approach to the learning of a topic in school mathematics must be the practical nature of the programme from the point of view of the classroom teacher. In this experiment I had the assistance of three experienced secondary school teachers, two of whom had never used a computer in mathematics lessons before. Their experience and observation of the difficulties involved from their point of view are therefore extremely valuable for the future application of the module.

One of the difficulties of a practical nature which they encountered at the start of the work with the computers was the tendency of some of the boys who had had previous experience of computers to want to display their knowledge by filling the screen of the computer with comments such as ´Man Utd are ace´ (the name of a football team!) in different colours. As one of the teachers remarked "They needed at least one lesson to get it out of their system". This is an unfortunate by-product of the enthusiasm which the new technology generates in the children, and they desire to show their understanding of the computer to all. As the computer becomes more widely available in the home and is used more and more in the primary and junior schools then all pupils will become more conversant with the computer and its use as an educational tool and so this should diminish this desire in pupils at the start of the secondary school. A second difficulty of a minor nature, and briefly mentioned above, was the logistics of organising the module around the facilities of the school. The module had been prepared with the possibility in view of having a room containing a few computers, say one to three or four children, as well as table tops where the children could write and work and be able to manipulate the cardboard

Maths. Machines. In the school used the computers were available in a room as a network and this caused two slight problems. Firstly the class had to be taken from the room in which they were timetabled to the computer room in order to use the computers. Although this was not a major problem it did result in some loss of time each lesson when added to the time for clearing away at the end of the lesson and over the course of the module this loss of time could be significant and should be taken into account. Secondly, the room used was such that it was not feasible to use both the computers and the cardboard Maths.Machines at the same time, as intended. Instead, as one of the teachers said: "We alternated between Computer Network and classroom using cardboard Maths Machine, so that all children were doing the same type of work. This worked well." Hence, although this was not the intended pattern of use it worked well in practice with the possible exception that, since all the pupils had to work together on the same section rather than at their own pace so that the class was kept together , some of the better boys were noticed to exhibit boredom on occasion. These difficulties show that considerable thought as to the logistical problems that each particular school setup might produce are necessary before embarking on a programme of this nature. There is a suggestion too that, comparing the attitudes shown with that of the second experiment, that the pupils (particularly the boys) were just a little bit too old for the programme and that the more usual first year of secondary school, at the age of 12 is probably the right time for this work.

7.4 Overview of the Chapter

This chapter has reviewed the improvements made to the dynamic algebra teaching module in the light of the first two experiments. It has also included the continued development of the Dynamic Algebra module to its final form, as well as discussing the methodology, form and content of the third and main teaching experiment. The reasons for the approach finally chosen for this experiment, and some practical observations from the programme itself which indicate areas of benefit from its use in the classroom, have also been decsribed. In the next chapter I shall look at the results of this third experiment.

Chapter 8

An Analysis of The Results of The Third Experiment

In this chapter the results of the third and final experiment are presented, including an analysis of both individual and group performance. The analysis includes both qualitative and quantitative aspects of the students performance on the algebra test and the conclusions from these results will also be discussed.

8.1 The Overall Group Results of the Third Experiment

8.1.1 The Post-Test Results

The hypothesis under test was that the programme of instruction based on the use of the computer would produce an improvement in understanding of algebra concepts, as measured by the test instrument used, which had been shown to correlate well with the standardised CSMS algebra test.

The full results of the matched pairs in the immediate post-test are given in Appendix U, along with a separate list of the results of those pupils not in the matched pairs. The facilities of the experimental and control matched pair groups are recorded in Appendix V. The summary of the immediate post-test results, in Table 8.1 shows the mean difference in the scores of the two groups and the corresponding statistical t-test analysis. There was no significant difference between the total scores achieved by the two groups, and therefore the null hypothesis that the mean score of the experimental group was less than or equal to that of the control group cannot be rejected. However, a deeper, qualitative analysis based on the conceptual difficulty of the questions presented a different picture. On the skill-based type of question, which was closely related to the content of the module which the control group had been given, Table 8.2 shows that they performed significantly

TEST (max.=67)	EXPERIMENTAL MEAN	CONTROL MEAN	MEAN DIFF.	S.D.	N	t	df	p
POST-TEST	36.0	35.9	0.10	10.46	48	0.06	47	---
DELAYED POST-TEST	42.0	39.3	2.70	8.92	47	2.08	46	<0.025

Table 8.1

A Comparison of Experiment 3 Performance on Post-Test and Delayed Post-Test.

Question	Experimental %	Control %	z	p
Multiply 3c by 5	14	41	3.07	<0.005
Simplify 3a+4b+2a	50	73	2.46	<0.01
Simplify 3b-b+2a	29	61	3.36	<0.0005
Simplify 3a+4+a	38	78	1.60	n.s.
G jigsaws and J jigsaws =?	55	78	2.39	<0.01

Table 8.2

A Comparison of Experimental and Control Performance

On Some Questions Involving Manipulative Type Skills

Question	Experimental %	Control %	z	p
For what values of a is a+3>7 ?	31	12	2.33	<0.01
For what values of a is 6 > a+3 ?	22	6	2.33	<0.01
a+b=b, always, never, sometimes ... when?	31	17	1.65	<0.05
M+P+N=N+M+R, always, never, sometimes ... when?	38	28	1.08	n.s.
Perimeter of rectangle D by 4	50	27	2.46	<0.025
Perimeter of rectangle 5 by F	50	29	2.24	<0.025
Larger of 2n and n + 2 ?	7	0	1.91	<0.05

Table 8.3

A Comparison of Experimental and Control Performance

On Some Questions Involving Higher Order Understanding

These included the manipulative simplification type of question which forms the basis of much of the traditional teaching of algebra at the start of the secondary school. However, looking at the other side of the situation, consisting of those questions which are considered to be more conceptually demanding, according to Küchemann's levels of understanding (see Appendix Q for the question levels) then Table 8.3 shows that the the experimental group performed significantly better than the control group. Some of the questions, such as those involving the solution of a linear inequality and the comparison of expressions in more than one variable involve the highest levels of understanding, including the concept of the use of letters as generalised numbers or variables. Although this direct comparison between the two groups is the most appropriate one, the data in Tables 8.4 and 8.5 shows that the experimental group made significant gains in more questions, and hence a wider range, than the control pupils, including conceptually important questions involving an understanding of the use of letters, such as questions 8, 15c), 16b) and 17d).

The apparent differential nature of the results from the two approaches to algebra may be explained in terms of a skills/conceptual understanding dichotomy. Considering the levels of understanding described by Küchemann (see Chapter 7), we may identify levels one and two as involving only arithmetic skills and the use of letters as objects, and it is only when children exhibit the understanding of levels 3 and 4 that they are really involved in algebraic thinking. For example, level 3 involves an appreciation of letters as specific

Question and number	Question level	Pre-Test Proportion	Post-Test Proportion	z	p
5c) 3b - b + a = ?	2	0.14	0.29	1.89	<0.05
5d) 3a + 4 + a = ?	3	0.19	0.38	2.15	<0.01
6b) Add 4 to p	2	0.37	0.61	2.54	<0.01
6c) Add 4 to p + 5	2	0.26	0.45	2.04	<0.025
6d) Add 4 to 8p	3	0.12	0.30	2.35	<0.01
8a. b - c = a - c, when?	4	0.09	0.34	3.12	<0.005
8d) M + P + N = N + M + R, when?	4	0.07	0.37	3.80	<0.0005
9. r pencils at 8 pence and s crayons at 9 pence?	3	0.27	0.47	2.26	<0.025
10b) Area G by 9	2	0.48	0.65	1.83	<0.05
10c) Area 6 by H	2	0.46	0.65	1.98	<0.025
10d) Area W by Y	2	0.48	0.65	1.83	<0.05
11a) b = a + 3, a = 4	2	0.50	0.79	3.22	<0.001
11b) m = 3p + 2, p = 2	2	0.30	0.49	2.04	<0.025
11c) t = 3s + b, s = 5, then t = ?	3	0.14	0.35	2.57	<0.025
12a) a + 3 = 7, a = ?	1	0.65	0.86	2.56	<0.025
12b) a when a + 3 > 7?	4	0.11	0.31	2.71	<0.005
12d) a when 6 > a + 3 ?	4	0.04	0.22	2.99	<0.005
14d) Double 7	1	0.23	0.38	1.70	<0.05
15c) v = 2, y = 6, v + y + w = ?	3	0.33	0.61	2.53	<0.025
16a) a + b = 5, a + b + c = ?	3	0.30	0.59	2.71	<0.005
16b) p + q = 3, p + q + r =10, r = ?	2	0.53	0.77	2.68	<0.005
17b) Perimeter, rectangle D by 4 ?	3	0.33	0.50	1.80	<0.05
17c) Perimeter, rectangle 5 by F ?	3	0.33	0.50	1.80	<0.05
20a) p - 189 = 675, p - 190 = ?	2	0.23	0.59	4.30	<0.0001
20b) y + 279 = 978, y + 277 = ?	2	0.18	0.50	3.65	<0.0005
21b) b + 5 = b + 2, always, never, sometimes?	1	0.54	0.70	1.67	<0.05
21c) y + y = 2y, always, never, sometimes?	1	0.40	0.70	3.13	<0.005

n.b. Questions are in abbreviated form. See Appendix Q for the Full Question

Table 8.4

Experimental Group Gains From Pre-test to Post-Test On Individual Questions

Question and number	Question level	Pre-Test Proportion	Post-Test Proportion	z	p
3b) Multiply b by 5	2	0.18	0.41	2.65	<0.005
5b) 3a + 4b + 2a	2	0.28	0.73	4.66	<0.0001
5c) 3b - b + a = ?	2	0.18	0.61	4.63	<0.0001
5d) 3a + 4 + a = ?	3	0.16	0.53	4.07	<0.0001
6c) Add 4 to p + 5	2	0.26	0.55	3.02	<0.005
6d) Add 4 to 8p	3	0.09	0.31	2.87	<0.005
9. r pencils at 8 pence and s crayons at 9 pence?	3	0.35	0.57	2.27	<0.025
10b) Area G by 9	2	0.49	0.82	3.48	<0.001
10c) Area 6 by H	2	0.49	0.82	3.48	<0.001
10d) Area W by Y	2	0.49	0.80	3.24	<0.001
11b) m = 3p + 2, p = 2	2	0.32	0.63	3.26	<0.001
11c) t = 3s + b, s = 5, then t = ?	3	0.17	0.35	2.14	<0.025
11d) If 3h = c + 3 and h = 2, c = ?	2	0.28	0.50	2.21	<0.025
15a) v = 2 and y = 6 v + y = ?	1	0.48	0.96	5.35	<0.0001
18a) Paper costs 12 pence, delivery 5 pence, total?	1	0.47	0.73	2.73	<0.005
18b) Paper costs 20 pence, delivery 5 pence, total?	1	0.47	0.73	2.73	<0.005
21b) b + 5 = b + 2, always, never, sometimes?	1	0.49	0.76	2.78	<0.005
21c) y + y = 2y, always, never, sometimes?	1	0.47	0.73	2.73	<0.005

n.b. Questions are in abbreviated form. See Appendix Q for the Full Question

Table 8.5

Control Group Gains From Pre-test to Post-Test

On Individual Questions

or variables. Kuchemann has shown that few children reach this level of understanding (17% at age 13) and so any approach which is able to improve this situation is a significant advance (see Figure 8.12). When all of the questions at levels 3 and 4 on the algebra post-test were considered then, as Table 8.6 shows, the experimental group's performance was significantly better than that of the control students. The better performance of the experimental group at these levels then is noteworthy, especially since Table 8.6 also shows that when all of the level 1 and 2 questions on the post-test were included in the analysis then there was no significant difference in the performance of the two groups, the experimental group on average answering only 3.6% fewer questions correctly. Thus the improvement in conceptual understanding of the experimental group pupils had not been to the detriment of their manipulative skills even though, at this stage of the programme, they had not been specifically taught such skills.

8.1.2 The Delayed Post-Test Results

The design of the programme was such that its effect would be seen properly after it had been combined, as a generic organiser (Tall, 1986c), with a skill-based module. The full results of the delayed post-test, enabling such a comparison, are given in Appendices W (matched pairs) and X (all). The summary in Table 8.1 contains the details of the overall results of the matched pairs on the delayed post-test, given after the experimental group had been through the content of school's traditional algebra module. It shows that the experimental group performed significantly better than the

Level 1 and 2 Questions (Lower order skills)

TEST (max.=41)	EXPERIMENTAL MEAN	CONTROL MEAN	MEAN DIFF.	S.D.	N	t	df	p
POST-TEST	26.7	28.2	1.48	6.92	48	1.48	47	n.s.
DELAYED POST-TEST	30.9	29.7	1.22	5.95	47	1.41	46	n.s.

Level 3 and 4 Questions (Higher order skills)

TEST (max.=26)	EXPERIMENTAL MEAN	CONTROL MEAN	MEAN DIFF.	S.D.	N	t	df	p
POST-TEST	9.3	7.8	1.50	4.81	48	2.14	47	<0.025
DELAYED POST-TEST	11.0	9.5	1.51	4.73	47	2.17	46	<0.025

Table 8.6

A Comparison of Experiment 3 Performance on Level 1 and 2 as well as Level 3 and 4 Questions of the Post-Test and Delayed Post-Test.

control group on the test as a whole, and therefore the null hypothesis that the mean mark of the experimental group was less than or equal to that of the control group must be rejected. Looking again at the skills/understanding dichotomy described above, then Table 8.6 reveals that the experimental group maintained their significantly better results on these conceptually harder questions whilst at the same time improving their standard on the questions involving manipulative skills to the extent that they caught up, and indeed overtook, the control group. Although they were, on average, able to answer 2.9% more of the questions correctly than the control group, this was not a statistically significant better performance. Whilst appreciating that the appropriate comparisons statistically here are between the experimental and the control group matched pairs, it is worth noting the change in the mean mark of the experimental group on these type of questions from the post-test to the delayed post-test. Their mean percentage mark increased from 26.7% to 30.9%, a significant gain in score ($t = 7.05$, $p<0.001$) but the control gain of 1.5%, from 28.7% to 29.7% was not significant. Further, Table 8.7 shows those questions where the experimental group made significant gains in correct answers over the post-test. In contrast the control pupils only made significant improvement in six parts covering three different questions. This illustrates, I think, both the good preparation provided by the computer module for the traditional work, and the level of algebraic understanding, namely the lower level skills, provided by this type of skill-based module.

From the above we may say that at the end of the experiment, the results indicate that the experimental group was performing significantly better than the control group in algebra and also had a significantly better understanding of the most difficult algebraic concepts than the controls, without any detrimental effect on their manipulative skills. One noteworthy feature of these results is that the understanding promoted by the computer work appears to have transferred successfully to the 'paper and pencil' algebra questions. This is often a problem experienced with teaching programmes and has been an area where, in previous studies in algebra, such as that of Booth (1983b) and the Logo studies of Sutherland and Hoyles (see Chapters 4 and 5), researchers have found problems arising. Thus this study constitutes an important step in the improvement of conceptual understanding of algebra in the secondary mathematics classroom.

These results are all the more impressive in the light of a disclosure by one of the control group teachers, which came to light after the experiment. Like all good teachers she had been concerned about the performance of her class on the test which they had been given and admitted to the researcher that, since it contained some questions (e.g. inequalities) which they had not seen before, between the post-test and delayed post-test she had taught them a lesson on each of the topics they needed! Although it is not possible to know the effects of this teaching for the test which one class out of three of the controls received, the superior results of the experimental group are further highlighted by it.

8.2 A Comparison With the Standardised CSMS Results

The work by Küchemann (1980) in testing some 3000+ pupils, including some at ages 13 years, 14 years and 15 years,

Question and number	Question level	Pre-Test Proportion	Post-Test Proportion	z	p
3b) Multiply b by 5	2	0.64	0.81	2.02	<0.025
3c) Multiply 3b by 5	3	0.14	0.44	3.48	<0.0005
3d) Multiply b - 3 by 5	4	0.11	0.24	1.85	<0.05
5b) 3a + 4b + 2a = ?	2	0.50	0.74	2.60	<0.005
5c) 3b - b + a = ?	2	0.29	0.67	4.00	<0.0001
5d) 3a + 4 + a = ?	3	0.38	0.54	1.71	<0.05
11b) m = 3p + 2, p = 2	2	0.49	0.70	2.18	<0.025
11d) 3h = c + 3, h = 2, then c = ?	2	0.35	0.69	3.63	<0.0005
14d) Double 7	1	0.38	0.65	2.86	<0.005
14e) Double p	2	0.41	0.61	2.10	<0.025
18c)Paper costs t pence, delivery 5 pence, total?	3	0.57	0.74	1.87	<0.05

n.b. Questions are in abbreviated form. See Appendix Q for the Full Question

Table 8.7

Experimental Group Gains From Post-Test to Delayed Post-Test On Individual Questions on his algebra test, has provided the opportunity to compare the understanding of the experimental group not just with the controls but with the broader population from which they are drawn, namely pupils in secondary schools in Britain. The following section contains an analysis of the results of the comparison between the results in this experiment and the CSMS results.

Since the test used was adapted from the CSMS algebra test, not all the questions could be directly compared. However, of the questions set 16 out of the 51 level 1 or 2 questions and 9 out of the 26 level 3 or 4 questions corresponded directly with questions in the CSMS test, and of these the facilities for 7 and 9 questions respectively were available from the CSMS study, and allow a direct comparison. Table 8.8 contains a summary of the level 1 and 2 question results for the post-test and shows that, on these questions which primarily involve manipulative and arithmetic skills and no real algebraic understanding, the experimental group performed better than the general secondary school population of 13 year-old children in 6 of the 9 questions, and significantly better in 2 of them. When we consider that they had not been formally instructed in such techniques at this stage of the experiment since the teaching module had been designed to increase conceptual understanding as an advance organiser for the later introduction of such techniques, then these results are as one would expect at this stage, perhaps even somewhat better. When we look at Table 8.9 however, which gives the corresponding situation at the delayed post-test stage then we see that now, having had the combined effect of both the preparatory conceptual work and the

Question and number	Question level	Experimental %	CSMS %	z	p
5a) 3a + 5a =?	1	79	77	0.27	ns
5b) 3a + 4b + 2a =?	2	50	40	1.48	ns
10a) Area 5 by 7	1	84	79	0.89	ns
10d) Area W by Y	2	66	54	1.77	<0.05
11a) b = a + 3, a = 4	2	80	49	4.57	<0.0001
11b) m = 3p + 2, p = 2	2	50	44	0.88	ns
12a) a + 3 = 7, a = ?	1	86	86	---	ns

n.b. Questions are in abbreviated form. See Appendix Q for the Full Question.

Table 8.8

A Comparison of Performance on Experiment 3 Questions at Levels 1 or 2 on the Post-Test With the CSMS Facilities For 13 Year-Olds

Question and number	Question level	Experimental %	CSMS %	z	p
5a) 3a + 5a =?	1	87	77	1.72	<0.05
5b) 3a + 4b + 2a =?	2	74	40	4.95	<0.0001
10a) Area 5 by 7	1	81	79	0.44	ns
10d) Area W by Y	2	80	54	3.69	<0.0005
11a) b = a + 3, a = 4	2	81	49	4.65	<0.0001
11b) m = 3p + 2, p = 2	2	70	44	3.79	<0.0001
12a) a + 3 = 7, a = ?	1	96	86	2.16	<0.025

n.b. Questions are in abbreviated form. See Appendix Q for the Full Question.

Table 8.9

A Comparison of Performance on Experiment 3 Questions at Levels 1 or 2 on the Delayed Post-Test With the CSMS Facilities For 13 Year-Olds

Question and number	Question level	Experimental %	CSMS %	z	p
3d) 5 times b - 3	4	11	8	0.57	ns
6d) 8p add 4	3	30	22	1.46	ns
8d) M + P + N = N + M + R, when?	4	38	11	5.85	<0.0001
13. r = s + t, r + s + t = 30, r=?	3	32	30	0.34	ns
14f) 2 times y + 2	4	9	8	0.25	ns
16a) a + b = 5, a + b + c = ?	3	59	25	5.58	<0.0001
19.7m + 5n represents?	4	18	14	0.80	ns
22.Greater, 2n or n + 2?	4	7	4	1.15	ns
23. p sides of 3 cm. Perimeter = ?	3	48	24	4.06	<0.0001

n.b. Questions are in abbreviated form. See Appendix Q for the Full Question

Table 8.10

A Comparison of Performance on Experiment 3 Questions at Levels 3 or 4 on the Post-Test With the CSMS Facilities For 13 Year-Olds

Question and number	Question level	Experimental %	CSMS %	z	p
3d) 5 times b - 3	4	24	8	4.06	<0.0001
6d) 8p add 4	3	33	22	1.94	<0.05
8d) M + P + N = N + M + R, when?	4	32	11	4.70	<0.0001
13. r = s + t, r + s + t = 30, r=?	3	41	30	1.67	<0.05
14f) 2 times y + 2	4	15	8	1.76	<0.05
16a) a + b = 5, a + b + c = ?	3	59	25	5.54	<0.0001
19.7m + 5n represents?	4	22	14	1.68	<0.05
22.Greater, 2n or n + 2?	4	11	4	2.49	<0.01
23. p sides of 3 cm. Perimeter = ?	3	61	24	6.06	<0.0001

n.b. Questions are in abbreviated form. See Appendix Q for the Full Question

Table 8.11

A Comparison of Performance on Experiment 3 Questions at Levels 3 or 4 on the Delayed Post-Test With the CSMS Facilities For 13 Year-Olds skill-based module, then the experimental group performed better in all 9 of the questions and significantly so in 8 of them. This is evidence that, far from being adversely affected in any way by the algebra programme, their manipulative skills were now significantly better than those of the average secondary school pupil of the same age.

Looking now at the level 3 and 4 questions, which involve algebraic reasoning for the first time, require a high degree of understanding and are conceptually much more demanding, Table 8.10 contains a summary of the comparison of the post-test results on the 9 questions at this level. Again the experimental group outperformed the general school population of 13 year-olds in every one of the questions, significantly so in 3 of them. However, as Table 8.11 shows, it is after the the teaching module has been combined with the more traditional skill-based approach, giving both the benefit of conceptual improvement and skill acquistion that the real superiority of the experimental group is seen. These results based on the delayed post-test show that the group performed significantly better than the population of 13 year-olds in all of the 9 questions. This is a significant achievement, and is further evidence of the value of combining the conceptual Dynamic Algebra module with the skill-based one, in order to promote versatile learning through the cognitive integration which the combination enhances.

Although there were no figures available for direct comparison of the experimental group´s performance on 17 of the level 3 or 4 questions on the algebra test, it was possible to obtain an idea of how their results compared on these important and difficult questions by comparing their results on these questions with the maximum recorded CSMS facility for any question at level 3 and level 4 respectively. These maximum facilities for 13 year-old children were 34% for level 3 and 15% for level 4 respectively. Table 8.12 shows the results of this comparison for the 15 questions on which the experimental group did better than the CSMS maximum facilities in the post-test. Even here, after the Dynamic Algebra module alone, the group performed significantly better than the highest recorded CSMS facility in 10 of the 15 questions. In the two remaining questions, where they did not do as well as these maximum levels, questions 3c) and 24, both were misunderstood, 3c) in terms of the form of the answer required, with 24 pupils leaving the answer incomplete as 5x3b or 3bx5 instead of 15b and in 24 the question was mis-read, with 5 pupils giving Wx25 + C as the answer. Good though this performance in the post-test was, it was, again, in the delayed post-test where the superiority of the experimental group´s results over the general population was most striking. Here they improved their performance to outperform the CSMS facilities in 16 of the 17 questions (with only question 24 being mis-understood again, this time 10 pupils giving the answer Wx25 + C) and significantly so in 13 of these 16. Table 8.13 contains a summary of what constitutes a considerable achievement. Looking collectively at the comparison of the results at levels 3 and 4 in the 26 questions covered in Tables 8.11 and 8.13 it seems clear that the addition of the Dynamic Algebra module before the study of the traditional skill-based one has raised the standard of performance to a level significantly above that to be expected in, the understanding of the use of letter as a

Question and number	Question level	Experimental %	CSMS max. %	z	p
4c) 4/c when c=p?	3	53.6	34	2.99	<0.005
5d) 3a + 4 + a =?	3	37.5	34	0.54	ns
8a. b - c = a - c, when?	4	34.8	15	3.93	<0.0001
8b) a + b = b, when?	4	31.3	15	3.24	<0.001
9. r pencils at 8 pence and s crayons at 9 pence?	3	48.2	34	2.17	<0.025
11c) t = 3s + b, s = 5, t = ?	3	34.8	34	0.13	ns
12b) a when a + 3 > 7?	4	31.3	15	3.24	<0.001
12d) a when 6 < a + 3 ?	4	22.3	15	1.48	ns
14c) 3n add 5 = ?	3	35.7	34	0.26	ns
15c) v = 2, y = 6, v + y + w = ?	3	60.7	34	4.07	<0.0001
17b) Perimeter, rectangle D by 4 ?	3	50.0	34	2.45	<0.01
17c) Perimeter, rectangle 5 by F ?	3	50.0	34	2.45	<0.01
17d) Perimeter, rectangle P by Q ?	3	44.6	34	1.63	ns
18c) Paper costs t pence, delivery 5 pence, total?	3	57.1	34	3.53	<0.0005
21a) 2a + 2 = 2(a + 1), when ?	4	25.0	15	2.01	<0.025

n.b. Questions are in abbreviated form. See Appendix Q for the Full Question

Table 8.12

A Comparison of Performance on Experiment 3 Questions at Levels 3 or 4 on the Post-Test With the CSMS Maximum Facilities For 13 Year-Olds

Question and number	Question level	Experimental %	CSMS max. %	z	p
3c) 5 times 3b	3	44.4	34	1.57	ns
4c) 4/c when c=p?	3	61.1	34	4.06	<0.0001
5d) 3a + 4 + a =?	3	53.7	34	2.96	<0.005
8a. b - c = a - c, when?	4	31.5	15	3.23	<0.001
8b) a + b = b, when?	4	31.5	15	3.23	<0.001
9. r pencils at 8 pence and s crayons at 9 pence?	3	61.6	34	4.06	<0.0001
11c) t = 3s + b, s = 5, t = ?	3	44.4	34	1.57	ns
12b) a when a + 3 > 7?	4	35.2	15	3.94	<0.0001
12d) a when 6 > a + 3 ?	4	25.9	15	2.16	<0.025
14c) 3n add 5 = ?	3	44.4	34	1.57	ns
15c) v = 2, y = 6, v + y + w = ?	3	70.4	34	5.34	<0.0001
17b) Perimeter, rectangle D by 4 ?	3	57.4	34	3.51	<0.0005
17c) Perimeter, rectangle 5 by F ?	3	59.3	34	3.78	<0.0001
17d) Perimeter, rectangle P by Q ?	3	59.3	34	3.78	<0.0001
18c) Paper costs t pence, delivery 5 pence, total?	3	74.1	34	5.97	<0.0001
21a) 2a + 2 = 2(a + 1), when ?	4	31.5	15	3.23	<0.001

n.b. Questions are in abbreviated form. See Appendix Q for the Full Question

Table 8.13

A Comparison of Performance on Experiment 3 Questions at Levels 3 or 4 on the Delayed Post-Test With the CSMS Maximum Facilities For 13 Year-Olds specific unknown (questions 4c), 5d), 6d), 11c), 13, 14c), 15c), 16a), 17b), c), d), 18c) and 23) and as a generalised number or variable (questions 8a), b), d), 12b), d), 21a), 22), as well as in questions involving these in a difficult structure or form (e.g. 9, 13, 17d), 19) or possibly involving the use of brackets (e.g. 3d), 14f), 21a)).

In many of these questions the experimental group were performing at a level one or even two years above that which might be expected from their age group (13 years-old). Table 8.14 contains a summary of those questions where, on their delayed post-test results they achieved above either the level recorded in the CSMS study for 14 or 15 year-olds or the maximum facility recorded for these age groups. One would not expect them to be significantly better than these standards (although in 5 of the 26 questions they were) but merely to be obtaining results at or above these levels on 21 out of the 26 questions is a considerable achievement.

8.3 <u>The Individual Performances</u>

In order to obtain some idea of how the programme had affected the algebraic understanding of individuals in the experimental group, it was necessary to look in more detail at their results as individuals and to interview some of them to ascertain the extent of the improvement in their understanding. The results of the interviews are contained in a later section, here I shall discuss the test results on an individual basis.

Comparing the proportions of the experimental group attaining each of the four levels of understanding described by Kuchemann, we see from Table 8.15 that the proportions at the higher levels of understanding are significantly above the

Question and number	Question level	Experimental %	CSMS %	Level in CSMS	z	p
3d) 5 times b - 3	4	24	17	a	1.32	ns
4c) 4/c when c=p?	3	61	52	b	1.29	ns
5d) 3a + 4 + a =?	3	54	52	b	0.29	ns
8a. b - c = a - c, when?	4	31	25	b	0.99	ns
8b) a + b = b, when?	4	31	25	b	0.99	ns
8d) M + P + N = N + M + R, when?	4	32	25	a	1.15	ns
9. r pencils at 8 pence and s crayons at 9 pence?	3	61	52	b	1.29	ns
12b) a when a + 3 > 7?	4	35	25	b	1.64	<0.05
12d) a when 6 > a + 3 ?	4	26	25	b	---	ns
13. r = s + t, r + s + t = 30, r=?	3	41	41	a	---	ns
15c) v = 2, y = 6, v + y + w = ?	3	70	56	d	2.02	<0.025
16a) a + b = 5, a + b + c = ?	3	59	41	a	2.61	<0.005
17b) Perimeter, rectangle D by 4 ?	3	57	52	b	0.72	ns
17c) Perimeter, rectangle 5 by F ?	3	59	52	b	1.00	ns
17d) Perimeter, rectangle P by Q ?	3	59	52	b	1.00	ns
18c) Paper costs t pence, delivery 5 pence, total?	3	74	56	d	2.60	<0.005
19.7m + 5n represents?	4	22	22	a	---	ns
21a) 2a + 2 = 2(a + |), when ?	4	32	25	b	0.99	ns
22.Greater, 2n or n + 2?	4	11	6	a	1.48	ns
23. p sides of 3 cm. Perimeter = ?	3	61	41	c	2.90	<0.005

a - actual 14 year-old facility b - maximum 14 year-old facility
c - actual 15 year-old facility d - maximum 15 year-old facility

n.b. Questions are in abbreviated form. See Appendix Q for the Full Question

Table 8.14

A Comparison of Performance on Experiment 3 Questions at Levels 3 or 4 on the Delayed Post-Test With the CSMS Maximum Facilities For 14 Or 15 Year-Olds corresponding CSMS figures.

Looking at the results of the 79 pupils who formed the whole experimental group with regard to their ability band in algebra as measured by the pre-test, there were 35 of lower ability (less than or equal to 20 out of 67 on the test), 26 of middle ability (between 20 and 41 on the test) and 18 of higher ability (more than 40 on the test). Of these, 23 of the bottom band, 23 of the middle band and 13 of the top band improved noticeably in their understanding after the programme, as measured by the post-test and delayed post-test. Table 8.15 shows that the progress in understanding of the group was such that they reached much higher levels of understanding than the general school population. The majority of those who made little or no improvement were at the bottom of the ability range and it would seem that for some pupils there may be little in the way of a general class module of work which can help them to improve their understanding, but rather they need more specialised individual treatment, the CSMS results showing that 37% of pupils are still not displaying level 3 understanding (specific unknown) at age 15 years.

To illustrate the kind of advancement made by some of these individuals in each of the bands of ability/understanding I shall now include some detailed profiles of the ways in which they progressed.

8.3.1 Pupils of Lower Level Understanding

Figures 8.1 to 8.4 inclusive are profiles of the changes in understanding across the three tests of Lisa, Ellen, Peter, and Nigel and show clearly the typical dramatic improvements which can be effected by the combination of the Dynamic Algebra

Level	Experimental Cumulative Percentage	CSMS Cumulative Percentages 13 years	14 years	15 years
0,1	100	100	100	100
2	85.2	40	58	63
3	68.5	17	34	40
4	27.7	2	6	9

Items at levels 0, 1, 2 can all be solved without having to operate on letters as unknowns.

Items at levels 3 and 4 involve treating the letters at least as specific unknown, and in some cases, as generalised number or variables.

Table 8.15

A Comparison of The Experimental Group and CSMS Figures For Cumulative Percentages of Children at Each Level Of Algebraic Understanding

Question and number	Question level	Answers Pre	Post	Delayed Post
3d) 5 times b - 3	4	-	-	5x(b - 3)
5c) 3b - b + a = ?	2	-	-	2b + a
5d) 3a + 4 + a = ?	3	8a	8	4a + 4
6b) Add 4 to p	2	-	p + 4	5p
6c) Add 4 to p + 5	2	-	p + 9	5p + 5
10b) Area G by 9 ?	2	63	Gx9	9G
10c) Area 6 by H ?	2	48	6xH	H6
10d) Area W by Y ?	2	-	WxY	WY
12c) a when 9 = a - 4 ?	2	13	5	13
14b) Add 5 to m	2	-	m + 5	m + 5
14c) Add 5 to 3n	3	-	na	3n + 5
15c) v = 2, y = 6, v + y + w = ?	3	9	8 + w	8 + w
16a) a + b = 5, a + b + c = ?	3	9	5 + c	9
20a) p - 189 = 675, p - 190 = ?	2	676	674	664
22.Greater, 2n or n + 2?	4	-	2n	2n, n might be 3 which means it =(but if it was n+2 it would be 5

(na = not attempted)

n.b. Questions are in abbreviated form. See Appendix Q for the Full Question

Figure 8.1

A Profile of the Change of Understanding On Experiment 3 Questior in Pre-Test, Post-Test and Delayed Post-Test of Lisa, a Pupil of Lower Initial Understanding

Question and number	Question level	Answers Pre	Post	Delayed-Post
3d) 5 times b - 3	4	5	(b-3)x5	(b - 3)x5
5b) 3a + 4b + 2a = ?	2	13a	6a + 4b	5a + 4b
7.G jigsaws and H jigsaws	2	15	G + H	G,H
9. r pencils at 8 pence and s crayons at 9 pence?	3	315	rx8 sx9	rx8 + sx9
10d) Area W by Y	2	598	WxY	WxY
11a) b = a + 3, a = 4	2	na	a + 3 and a=4	B = 7
11b) m = 3p + 2, p = 2	2	na	m = 5	m = 8
12b) a when a + 3 > 7?	4	4	5	A = 5 and more
12d) a when 6 > a + 3 ?	4	2	3	A = 0,1,2
15c) v = 2, y = 6, v + y + w = ?	3	na	8 + w	8 + w
16a) a + b = 5, a + b + c = ?	3	9	5 + c	5c
20a) p - 189 = 675, p - 190 = ?	2	na	674	674
20b) y + 279 = 978, y + 277 = ?	2	na	279	976

(na = not attempted)

n.b. Questions are in abbreviated form. See Appendix Q for the Full Question

Figure 8.2

A Profile of the Change of Understanding On Experiment 3 Questior in Pre-Test, Post-Test and Delayed Post-Test of Ellen, a Pupil oi Lower Initial Understanding

Question and number	Question level	Pre	Post	Delayed-Post
3b) Multiply b by 5	2	10	5xb	5b
3c) Multiply 3b by 5	3	160	5x3b	15b
5b) 3a + 4b + 2a = ?	2	9a	9 + b	5a + 4b
5c) 3b - b + a = ?	2	6b	4b + a	2b + a
6b) Add 4 to p	2	20	4 + p	4 + p
6c) Add 4 to p + 5	2	25	9 + p	9p
6d) Add 4 to 8p	3	12p	8xp + 4	12p
7.G jigsaws add H jigsaws	2	15	GxH	G + H
8a. b - c = a - c, when?	4	never	never	sometimes when a = b
8d) M + P + N = N + M + R, when?	4	never	sometimes when P = R	sometimes when P = R
9. r pencils at 8 pence N + M + R, when? and s crayons at 9 pence?	3	na	rx8 + sx9	8r + 9s
10d) Area W by Y	2	757	WxY	WxY
11a) b = a + 3, a = 4	2	2	7	7
11b) m = 3p + 2, p = 2	2	13	7	8
11c) t = 3s + b, s = 5, then t = ?	3	20	t = 8 + b	15 + b
13. r = s + t, r + s + t = 30, r=?	3	na	r = 15	r = 15
16a) a + b = 5, a + b + c = ?	3	na	5 + c	5 + c

(na = not attempted)

n.b. Questions are in abbreviated form. See Appendix Q for the Full Question

Figure 8.3

A Profile of the Change of Understanding On Experiment 3 Question in Pre-Test, Post-Test and Delayed Post-Test of Peter, a Pupil of Lower Initial Understanding

Question and number	Question level	Pre	Post	Delayed-Post
3b) Multiply b by 5	2	30	5b	5b
3c) Multiply 3b by 5	3	168	15b	5x3b
5b) 3a + 4b + 2a = ?	2	9a	9a	5a + 4b
5c) 3b - b + a = ?	2	3a	3b	2b + a
6b) Add 4 to p	2	na	4p	4 + p
6c) Add 4 to p + 5	2	na	9p	p + 9
9. r pencils at 8 pence and s crayons at 9 pence?	3	na	8xr, sx9	8r + 9s
10b) Area G by 9	2	na	9g	G9
10c) Area 6 by H	2	na	6h	6h
10d) Area W by Y	2	na	WxY	wy
12c) a when 9 = a - 4?	2	5	13	a = 13
14b) Add 5 to m	2	2m	m + 5	m + 6
14c) Add 5 to 3n	3	na	3n + 5	3xn
21a) 2a + 2 = 2(a + 1), when ?	4	na	always	always
23. Perimeter of p sides each of 3cm	3	na	Px3	Px3

(na = not attempted)

n.b. Questions are in abbreviated form. See Appendix Q for the Full Question

Figure 8.4

A Profile of the Change of Understanding On Experiment 3 Question in Pre-Test, Post-Test and Delayed Post-Test of Nigel, a Pupil of Lower Initial Understanding module and traditional skill-based teaching in pupils of lower initial understanding.

In Figure 8.1 we see a profile of Lisa, whose scores in the three tests were 19.5, 31.5 and 35 (maximum 67). We see that at the pre-test stage she has a desire for numerical answers that is so strong that she makes the one-to-one correspondence A=1, B=2, C=3 etc., described in Chapter 5, in questions 10b) and c) and another type in questions 15c) and 16a). There is no evidence of acceptance of lack of closure or understanding of letters as specific unknowns, with the only letter used, in question 5d), exhibiting a standard parsing error described by Booth (1984) as conjoining. However, in the post-test questions 6b), c), 14b), 15c) and 16a) all reveal an appreciation of specific unknowns with the later two involving more complex structure as well. On the delayed post-test the correct use of brackets in 3d) and the contraction of multiplication from explicit to implicit in question 10 are added to this, showing an appreciation of the need for correct symbolisation of answers and in question 22 we see the first signs of an understanding of letters as generalised numbers with a consideration of the effect of the size of n upon the answer. Figure 8.2 also shows that for Ellen, whose scores were 14, 29 and 40 in the three tests, initially her answers are numerical and that she is prepared to carry the coding to its limits (question 10d) e.g. where 598 = 23 x 26, Y taken as the 26th letter), as well as displaying the standard parsing error in question 5b). Her progress is seen to be very similar to Lisa's except that in her answers to questions 12b) and d) in the delayed post-test she has appeared to understand that a

Question and number	Question level	Answers Pre	Post	Delayed-Post
5c) 3b - b + a = ?	2	3b	2b + a	2b + a
5d) 3a + 4 + a = ?	3	8a	4a + 4=8a	4a + 4
6b) Add 4 to p	2	p + 4 = 4p	P + 4	P + 4
6c) Add 4 to p + 5	2	10p	P + 9	P + 9
6d) Add 4 to 8p	3	12p	8p + 4	8p + 4
8b) a + b = b, when?	4	always	sometimes, when b=6, c=2 a=6	sometimes, when they all equal 2
8d) M + P + N = N + M + R, when?	4	always	sometimes, when R and P equal the same	sometimes when R is the same value as P
9. r pencils at 8 pence and s crayons at 9 pence?	3	44	R*8 + s*9	rx8 + sx9
11c) t = 3s + b, s = 5, then t = ?	3	1	t = 15 + b	15 + b = t
12b) a when a + 3 > 7?	4	5	a = 5 or more eg 6,7,8	a = 5 and over
12c) a when 9 = a - 4?	2	9	a = 5	13
12d) a when 6 > a + 3 ?	4	1	a = 4 or more eg 5,6 7 etc	a = 2 and less
13. r = s + t, r + s + t = 30, r=?	3	r=10, s=10, t=10	r = 15	r = 15
14c) Add 5 to 3n	3	2m	m + 5	m + 5
14f) Double y + 2	4	y + 4	y + 2 * 2	2y + 2
15c) v = 2, y = 6, v + y + w = ?	3	18	8 + w	8 + w
16a) a + b = 5, a + b + c = ?	3	8	5 + c	5 + c
19.7m + 5n represents?	4	na	42 + 25 = the area of the wood	The pieces and lengths of the wood
22.Greater, 2n or n + 2?	4	na	2n is larger because it is 2xn and n + 2	2n because it is 2xn but not always except when n=1

(na = not attempted)

n.b. Questions are in abbreviated form. See Appendix Q for the Full Question

Figure 8.5

A Profile of the Change of Understanding On Experiment 3 Questions in Pre-Test, Post-Test and Delayed Post-Test of Lucy, a Pupil of Middle Initial Understanding

Question and number	Question level	Answers Pre	Post	Delayed-Post
5c) 3b - b + a = ?	2	3 + a	4b + a	2b + a
6d) Add 4 to 8p	3	4p	p + 4	p + 4 = 4p
8a. b - c = a - c, when?	4	na	sometimes, when a & b are equal	sometimes, when a & b are the same otherwise never
8d) M + P + N = N + M + R, when?	4	na	sometimes, when R=P and P=R	sometimes, when p & r are equal
11c) t = 3s + b, s = 5, then t = ?	3	na	15 + b	15 + b
12d) a when 6 > a + 3 ?	4	a = 2,3,..	1 or 2	a = 0,1,2
14c) Add 5 to 3n	3	8n	m + 5	m + 5
14f) Double y + 2	4	y + 4	2y + 2	2y + 4
15c) v = 2, y = 6, v + y + w = ?	3	9	8 + w	8 + w
16a) a + b = 5, a + b + c = ?	3	9	anything above 5	5 + c
17d)Perimeter, rectangle P by Q ?	3	na	P + Q x2	2P + 2Q
19.7m + 5n represents?	4	na	The lengths of wood	All the wood
22.Greater, 2n or n + 2?	4	na	If number is larger than 2, 2n will be larger	2n if n is 2 or more as 2n is x & n + 2 is +
24b)Cost of parcel weight W kg at 25 pence/kg and 60 pence basic	4	na	60 + (Wx25)	Wx25 + C

(na = not attempted)

n.b. Questions are in abbreviated form. See Appendix Q for the Full Question

Figure 8.6

A Profile of the Change of Understanding On Experiment 3 Questions in Pre-Test, Post-Test and Delayed Post-Test of Nicholas, a Pupil of Middle Initial Understanding

Question and number	Question level	Pre	Post	Delayed-Post
5b) 3a + 4b + 2a = ?	2	9	5a + 4b	5a + 4b
5c) 3b - b + a = ?	2	3	2b + a	2b + a
5d) 3a + 4 + a = ?	3	8a	4a + 4	4a + 4
6b) Add 4 to p	2	4p	p + 4	p + 4
6d) Add 4 to 8p	3	12p	12p	8p + 4
8b) a + b = b, when?	4	na	never	sometimes, when a = 0
8d) M + P + N = N + M + R, when?	4	na	never	sometimes, when R = P
11c) t = 3s + b, s = 5, then t = ?	3	22	15 + B	15 + 7 = 22
12b) a when a + 3 > 7?	4	5	5,6 etc	5,6,7,8 etc
12d) a when 6 > a + 3 ?	4	1	1,2	1,2,3
15c) v = 2, y = 6, v + y + w = ?	3	12	8 + w	8 + w
16a) a + b = 5, a + b + c = ?	3	6.5	5 + c	5 + c
18c) Paper costs t pence, delivery 5 pence, total?	3	t	t + 5	t + 5
19.7m + 5n represents?	4	the pieces of would	everything	The length and amount of wood
20a) p - 189 = 675, p - 190 = ?	2	704	674	674
20b) y + 279 = 978, y + 277 = ?	2	978	976	1534
21a) 2a + 2 = 2(a + 1), when ?	4	na	never	always

(na = not attempted)

n.b. Questions are in abbreviated form. See Appendix Q for the Full Question

Figure 8.7

A Profile of the Change of Understanding On Experiment 3 Questions in Pre-Test, Post-Test and Delayed Post-Test of Katharine, a Pupil of Average Initial Understanding

Question and number	Question level	Pre	Post	Delayed-Post
5b) 3a + 4b + 2a = ?	2	na	5a + 4b	5a + 4b
5c) 3b - b + a = ?	2	na	2b + a	2b + a
5d) 3a + 4 + a = ?	3	na	4a + 4	8a
6d) Add 4 to 8p	3	12p	8p + 4	12p
8a. b - c = a - c, when?	4	never	sometimes, when a and b are worth the same value	sometimes, when a and b have the same value
8b) a + b = b, when?	4	na	sometimes, when a is worth nothing	sometimes, when a is worth nothing
8d) M + P + N = N + M + R, when?	4	never	sometimes, when P and R are worth the same value	sometimes, when P and R have the same value
10d) Area W by Y	2	na	2W + 2Y	WxY
11a) b = a + 3, a = 4	2	1	7	b = 7
11b) m = 3p + 2, p = 2	2	na	34	m = 8
11c) t = 3s + b, s = 5, then t = ?	3	na	35 + b	t = 15 + b
12d) a when 6 > a + 3 ?	4	na	a = _5	a = 0,1,2
14c) Add 5 to 3n	3	na	3xn + 5	3n + 5
17d) Perimeter, rectangle P by Q ?	3	na	Px2 + Qx2	2P + 2Q

(na = not attempted)

n.b. Questions are in abbreviated form. See Appendix Q for the Full Question

Figure 8.8

A Profile of the Change of Understanding On Experiment 3 Questions in Pre-Test, Post-Test and Delayed Post-Test of Jennifer, a Pupil of Average Initial Understanding letter may represent a set of numbers i.e. is a generalised number. Peter and Nigel whose test scores were 7, 31, 43 and 13, 32, 38 respectively display similar progression to understanding of specific unknown from a numerical base and a mis-understanding of parsing, as shown in Figures 8.3 and 8.4. Peter even answers correctly the conceptually demanding questions 8a) and d) which involve the understanding of the use of letters as variables in the comparison of two expressions in order to establish a condition for equivalence. It is true that there is some slight regression shown in the answers from post-test to delayed post-test for some of these pupils, but this only emphasises that time is necessary for the cognitive re-structuring of schemas necessary for the fixing of these concepts, and that this seems to be especially so for pupils of the lower ability range.

8.3.2 <u>Pupils of Middle Level Understanding</u>

For the pupils of average ability, the typical improvements in understanding which can accrue are represented by profiles of Lucy, Nicholas, Katharine and Jennifer´s answers in Figures 8.5 to 8.8 inclusive. We see that they started from a level of understanding of the use of letters in algebra not far from that of the lower ability children. Although there is the first indication of a desire to accept lack of closure, question 6b) of Figure 8.5 and the answers to 6d) from all of them show that they are still conjoining and do not have the concept of letter as a specific unknown at that stage. The results of these children in the three tests were 24, 50, 52.5; 29, 56, 56; 33, 53.5, 56 and 32, 47, 52 respectively. Following the Dynamic Algebra module alone, however, we see that the concept of specific unknown is firmly present in all of these pupils and that at the end of the experiment they can even cope with some of the complications of structure presented in questions 9, 13, 14f), 15c), 16a) and 17d). These pupils though have also all progressed in their understanding to the point where they can accept letters as generalised numbers and variables, as shown by their answers to questions 8b) and d), 12b) and d), 21a) and for Lucy and Nicholas the difficult question 22.

8.3.3 Pupils Of Higher Level Understanding

The question of the effect on the understanding of children who displayed a high ability on the pre-test is addressed by a consideration of the answers in Figures 8.9 to 8.12 inclusive. These answers are from the papers of Rebecca, Elizabeth, Jenny and Andrew whose test scores were 43, 53.5, 59.5; 43, 54, 57; 51, 56, 60 and 48, 54, 61 respectively. Even pupils with scores as high as these on the pre-test (maximum 67) still show in their answers to questions 5, 6 and 9 a reluctance to accept lack of closure (demonstrated by parsing errors involving conjoining), which indicates their lack of understanding of letters as specific unknowns, as well as a failure to appreciate some of the formalities of algebraic notation such as representing the answer in its simplest form by completing operations wherever possible (e.g. question 3c) Figure 8.12 and 6c) Figures 8.10 and 8.12). They also display no understanding of the use of letters as generalised numbers or as variables. After the programme their understanding of specific unknown is firmly set and their answers to questions 8, 12 and 22 show that these pupils have abstracted from the programme the concept of generalised number and, sometimes, variable,

Question and number	Question level	Pre	Post	Delayed-Post
3d) Multiply b - 3 by 5	4	5b - 3	(b - 3)x5	5b - 3
5b) 4a + 3b + 2a = ?	2	5a + 4b = 9ab	5a + 4b	5a + 4b
5d) 3a + 4 + a = ?	3	8a	4a + 4	4a + 4
6b) Add 4 to p	2	4p	p + 4	4 + p
6c) Add 4 to p + 5	2	9p	p + 9	p + 9
6d) Add 4 to 8p	3	12p	8p + 4	8p + 4
7.G jigsaws add H jigsaws	2	G,H	G + H	G + H
8a. b - c = a - c, when?	4	never	sometimes, when b & a are the same number	sometimes, when b and c are the same number
8b) a + b = b, when?	4	never	sometimes, when a = 0	sometimes, when a = 0
8d) M + P + N = N + M + R, when?	4	always	sometimes, when P & R are the same number	sometimes, when P and R are the same number
9. r pencils at 8 pence and s crayons at 9 pence?	3	8r + 9s = 17rs	8r + 9s	8r + 9s
12b) a when a + 3 > 7?	4	5	a + 3 > 7	a = 5 and over
12d) a when 6 > a + 3 ?	4	1	6 > a + 3	a = 0,1,2
13. r = s + t, r + s + t = 30, r=?	3	0	na	r = 15
14f) Double y + 2	4	2y + 2	2(y + 2)	2y + 2
23. p sides of 3 cm. Perimeter = ?	3	4p	p3	3p

(na = not attempted)

n.b. Questions are in abbreviated form. See Appendix Q for the Full Question

Figure 8.9

A Profile of the Change of Understanding On Experiment 3 Question in Pre-Test, Post-Test and Delayed Post-Test of Rebecca, a Pupil o High Initial Understanding

Question and number	Question level	Pre	Post	Delayed-Post
3c) Multiply 3b by 5	3	36x5	15b	15b
3d) Multiply b - 3 by 5	4	b - 3x5	5b - 3	5b - 15
5b) 4a + 3b + 2a = ?	2	5a + 4b or 9ab	5a + 4b	5a + 4b
5d) 3a + 4 + a = ?	3	4a + 4 or 8a	4a + 4	4a + 4
6c) Add 4 to p + 5	2	p + 5 + 4	p + 9	p + 9
6d) Add 4 to 8p	3	12p	12p	8p + 4
8a. b - c = a - c, when?	4	never	sometimes, when a is the same number as b	sometimes, when b and a are the same number
8d) M + P + N = N + M + R, when?	4	never	sometimes, when R is the same as P	sometimes, when P & R are the same numbers
11b) m = 3p + 2, p = 2	2	4	8	8
12b) a when a + 3 > 7?	4	A = 5	5 and over	a = 5,6,7,8...
12d) a when 6 > a + 3 ?	4	A = 1	2	a = 2,1,0...
14f) Double y + 2	4	Y + 4	2y + 2	2y + 4
16a) a + b = 5, a + b + c = ?	3	9	5 + c	5 + c

(na = not attempted)

n.b Questions are in abbreviated for. See Appendix Q for the Full Question

Figure 8.10

A Profile of the Change of Understanding On Experiment 3 Questions in Pre-Test, Post-Test and Delayed Post-Test of Elizabeth, a Pupil of High Initial Understanding

Question and number	Question level	Answers Pre	Post	Delayed-Post
3d) Multiply b - 3 by 5	4	b - 15	b - 3x5	5b - 15
5d) 3a + 4 + a = ?	3	8a	4a + 4	4 + 4a
6d) Add 4 to 8p	3	12p	8p + 4	8p + 4
8b) a + b = b, when?	4	never	sometimes, when a and b are the same	sometimes, when b = a
8d) M + P + N = N + M + R, when?	4	never	sometimes, when P and R are the same	sometimes, when R = P
14c) Add 5 to 3n	3	8n	3n + 5	3n + 5
14f) Double y + 2	4	y + 4	2y + 4	2y + 4
15c) v = 2, y = 6, v + y + w = ?	3	8w	8 + w	8 + w
24b) Cost of parcel weight W kg at 25 pence/kg and 60 pence basic	4	C + W = T	C = 60 + 25xW	C + 25W

n.b. Questions are in abbreviated form. See Appendix Q for the Full Question

Figure 8.11

A Profile of the Change of Understanding On Experiment 3 Questions in Pre-Test, Post-Test and Delayed Post-Test of Jenny, a Pupil of High Initial Understanding

Question and number	Question level	Answers Pre	Post	Delayed-Post
3c) Multiply 3b by 5	3	(3b)x5	15b	15b
6c) Add 4 to p + 5	2	p + 5 + 4	p + 5 + 4	p + 9
8a. b - c = a - c, when?	4	never	sometimes, when b = a	never
8d) M + P + N = N + M + R, when?	4	never	sometimes, when R = P	never
12d) a when 6 $>$ a + 3 ?	4	1	a$<$3	$<$3
19.7m + 5n represents?	4	nx5 + mx7	length of wood	length of wood
22.Greater, 2n or n + 2?	4	2n because 2n = 2xn	If n$>$2 yes	2n if n$>$1
23. p sides of 3 cm. Perimeter = ?	3	6x2p	px3	3p
24b)Cost of parcel weight W kg at 25 pence/kg and 60 pence basic	4	CxW	C + 25xW	60 + Wx25

n.b. Questions are in abbreviated form. See Appendix Q for the Full Question

Figure 8.12

A Profile of the Change of Understanding On Experiment 3 Questions in Pre-Test, Post-Test and Delayed Post-Test of Andrew, a Pupil of High Initial Understanding something which, according to Küchemann's (1981b) study only about 2% of 13 year-olds in the general school population achieve.

8.3.4 Summary

In summary we have seen that on an individual level, many of the children in the experimental group were displaying in their answers to the test questions at the post and delayed post stages a level of understanding which commonly included specific unknown, reached by only 17% of 13 year-olds in general (Küchemann 1980), with evidence of understanding of generalised numbers and variables well above the level of the 2% of schoolchildren in general who attain this. These encouraging results indicate that the computer paradigm as described herein has much to offer in the improvement of children's conceptual understanding of algebra.

8.4 Gender and Ability Related Considerations

The results outlined above with regard to the advances in understanding of the pupils in different bands of ability as measured by the algebra pre-test and the reactions of the boys and girls to the programme as described in Chapter 7, indicated the importance of analysing the results of the experiment in order to ascertain whether ability or gender are affective variables in the outcome of this computer-based method of approach to conceptual algebra learning.

8.4.1 A Factor Analysis of the Results of Experiment 3

In order to investigate further the ways in which the deep conceptual approach of the computer work might have had a differential effect on the pupils' understanding it was necessary to carry out at this stage a factor analysis of the results and method of teaching (computer module or traditional module), algebraic ability (as measured by the pre-test) and gender were used as the variables. A random sample of 11 boys and girls (the number in the smallest group) from each of two ability levels (so that the class sizes would be acceptably large) in each programme, was selected and their results analysed.

The summary of the factor analysis results in Table 8.16 show that there was a significant interaction between the method of instruction and gender. In order to ascertain exactly where this significant difference in the mean scores had occured it was necessary to test the differences in the means. The mean scores of the pupils in each group, in each of the three tests, are given in Table 8.17 along with a comparative analysis. This shows that the boys of higher ability in the control group had performed somewhat better than the corresponding girls in the control group and the boys in the computer group. The girls of higher ability in the experimental group however had significantly outperformed the corresponding boys in that group and the girls in the control group. Taking into account where each of the pupils had started from, by considering their gain in scores, as given in Table 8.18, then we see that the girls of higher ability in the experimental group improved somewhat more than the boys of higher ability in the same group and significantly more than the corresponding girls in the control group. Also the experimental group girls of lower initial ability improved more than the corresponding boys in their group and the girls in the control group, as well as

Source of Variation	df	Sum of Squares	Mean Square	F	p
Error (Within Cells)	80	22616.2	282.7	---	---
Method x Ability x Sex	1	232.4	232.4	0.82	ns
Method x Ability	1	189.1	189.1	0.67	ns
Method x Sex	1	102873.3	102873.3	363.9	<.001
Ability x Sex	1	0.6	0.6	0	ns
Method	1	1.4	1.4	0	ns
Ability	1	12409.4	12409.4	43.9	<.001
Sex	1	36.9	36.9	0.1	ns

Table 8.16

Experiment 3 Factor Analysis Results With Method, Algebraic Ability and Gender as Variables

	Experimental Group				Control Group			
	Boys		Girls		Boys		Girls	
	Low	High	Low	High	Low	High	Low	High
Pre-Test	13.0	38.2	14.0	44.2	14.5	43.0	15.6	40.9
Post-Test	21.4	44.6	23.6	53.7	25.7	49.6	25.7	43.4
Delayed Post-Test	23.8	49.3	28.5	55.4	28.1	54.1	30.1	47.9

Statistical Comparison of Some of the 8 Post-test Means Above :

Groups	Group 1 Mean	Group 2 Mean	Mean Diff.	S.D.	N	df	t	p
EGH v CGH	53.7	43.4	10.3	8.48	22	20	2.85	<0.01
EGH v EBH	53.7	44.6	9.1	5.76	22	20	3.70	<0.002
EGL v CGL	23.6	25.7	2.1	11.9	22	20	0.36	ns
EGL v EBL	23.6	21.4	2.2	13.2	22	20	0.40	ns
CBH v CGH	49.6	43.4	6.2	10.0	22	20	1.45	ns

E = Experimental C = Control G = Girls B = Boys
H = Higher Ability L = Lower Ability

Table 8. 17

A Comparison and Analysis of the Mean Scores on Pre- Post- and Delayed Post-Tests For Random Samples of Boys and Girls on Experiment 3

	Experimental Group				Control Group			
	Boys		Girls		Boys		Girls	
	Low	High	Low	High	Low	High	Low	High
Pre-Test to Post-Test	8.36	2.41	9.50	9.50	11.0	6.55	9.18	2.55
Post-Test to Delayed Post-Test	2.41	4.64	4.25	1.86	0.33	4.45	6.40	6.05

Statistical Comparison of Some of the 8 Post-test Mean Gains Above :

Groups	Group 1 Mean Gain	Group 2 Mean Gain	Mean Diff. Gain	S.D.	N	df	t	p
EGH v CGH	9.50	2.55	6.95	7.28	22	20	2.24	<0.05
EGH v EBH	9.50	6.45	3.05	6.56	22	20	1.09	ns
EGL v CGH	9.50	2.55	6.95	7.54	22	20	2.16	<0.05
EGL v EBH	9.50	6.45	3.05	6.84	22	20	1.05	ns
CBH v CGH	6.55	2.55	4.0	6.88	22	20	1.36	ns

E = Experimental C = Control G = Girls B = Boys
H = Higher Ability L = Lower Ability

Table 8.18

An Analysis of the Comparison of Some Mean Post-test Score Gains

For Random Samples of Boys and Girls on Experiment 3 significantly more than the girls of higher ability in the control group. The boys in the control group improved slightly more than those in the experimental group in each case, but not significantly so. The figures for the delayed post-test do not admit a direct comparison between the two distinct methods of concept acquisition, but there were no significant differences in performance there. These results show that the significant interaction between the method used and gender was in the form of, particularly, a significantly better performance among the girls of higher ability who used the Dynamic Algebra computer module compared with those who did not. However, it must be borne in mind when considering the differences in means that these results are from random samples of children in broad bands of ability, comprising about 50% of the pupils in the study. This means that the pupils in each group were not directly matched as to ability on the pre-test, giving a less appropriate method of mean gain comparability than the results obtained from the matched pairs design, since the pupils start from different places.

Although then there appears to be some interaction between gender and method, it is not clear from these results what the reasons for this might be, as to whether they are socially or cognitively based, since it was noted from the experiment that the boys were more likely to be the ones who were bored more easily whereas the girls took the work very seriously and attentively. However, it is possible to find in the literature indications of gender difference which bears on the theory of Cognitive Integration outlined in Chapter 3, and these have been included in that chapter.

Applying these concepts to the experiment in this research, it may have been that the computer-based algebra module had provided the mental imagery necessary for the construction of a global/holistic viewpoint in such a way that it could be linked to the serialist/analytic type of higher level schemas and produce versatile learners. Thus, through the computer module, some girls may have been able to add to their serialist/analytic tendency the means for producing a global viewpoint. In the context of algebra this requires suitable mental images being produced which enable an individual to have the global view of the use of all letters in algebra as representing a range of numbers i.e. as a generalised number or as a variable whilst at the same time being able to use the letters, in different contexts, in some of the other mathematical ways listed by Wagner (1984) and to understand the relationship between the uses. Whilst I am not suggesting that many of the girls, if any, would accomplish all of this, the superior performance of the girls instructed using the computer-based work indicated by the factor analysis, compared with both the boys on this module and the girls taught in the traditional classroom manner, immediately after the module, suggests that they had benefited from the work in this sort of way.

This would appear to be an area where further research is necessary in order to establish any such differential pre-dispositions in the cognition of girls and boys.

8.4.2 The Role of the Computer Environment

If it should be the case that this is a relevant factor in the performance of girls in mathematics, then we must address the question of how the situation might be improved, so that girls may make fuller use of the cognitive lower level schemas asociated with the right hemisphere of the brain, and whether there are any implications for improving the teaching for boys. The indication of the research described above is that what might be needed are methods of instruction which have been designed to improve versatility by encouraging the formation of the relational lower level schemas and links to appropriate higher level concepts. I suggest that, in view of the results outlined above that the computer environment, which is conducive to the production of mental images can be used to improve the global/holistic viewpoint capabilities of the pupil, is one which should be examined in more detail. Such an environment, with its benefits for both boys and girls, is to be found through the computer paradigm by carefully structuring work using a computer to encourage a propensity for the construction of (particularly generic) mental images and combining this with the best of the more traditional methods of teaching with their predominantly serialistic/analytic emphasis.

8.5 The Effect of the Programme on Errors

As explained in Chapter 6, one of the aims of presenting the module of work on the computer before the pupils had any experience of algebra was to try and avoid some of the common errors which early learners have. It was recognised however that many of these errors are actually a result inappropriate application of arithmetic concepts to algebra, and therefore might be more difficult to circumvent (e.g. Matz, 1980; Booth, 1983b). Hence an analysed of the errors made by the

Question and	Error	Facilities Pre-Test	Post-Test	Delayed Post-Test
3c) Multiply 3b by 5	3bx5	36.7	41.0	21.3
3d) Multiply b - 3 by 5	b-3x5	31.6	37.3	15.0
5b) 3a + 4b + 2a = ?	9a, 9b	31.6	24.1	12.5
5c) 3b - b + a = ?	2ab	1.3	2.4	0
	3a	16.5	18.1	18.8
5d) 3a + 4 + a = ?	8a	55.7	41.0	36.3
6b) Add 4 to p	4p	20.2	13.3	36.3
6c) Add 4 to p + 5	9p	1.3	10.8	31.3
	p + 5 + 4	17.7	14.5	7.5
6d) Add 4 to 8p	12p	63.3	50.6	47.5
7.G jigsaws add H jigsaws	GH	3.8	9.6	7.5
9. r pencils at 8 pence and s crayons at 9 pence?	8r, 9s	10.1	7.2	7.5
	8 + r + 9 + s	7.6	3.6	3.8
	17rs	3.8	2.4	1.3
11c) t = 3s + b, s = 5, then t = ?	22	6.3	3.6	6.3
	8 + b	2.5	4.8	3.8
	15b	3.8	3.6	6.3
12b) a when a + 3 > 7?	Incorrect range	1.3	3.6	0
12d) a when 6 > a + 3 ?	Incorrect range	0	7.2	5.0
13. r = s + t, r + s + t = 30, r=?	r = 10	15.2	9.6	15.0
	r = 20	0	2.4	8.8
	r = 30	2.5	7.2	3.8

n.b. Questions are in abbreviated form. See Appendix Q for the Full Question

Figure 8.13

Some Experimental Group Error Facilities For Experiment 3 Questions on Pre-Test, Post-Test and Delayed Post-Test

Question and	Error	Facilities Pre-Test	Post-Test	Delayed Post-Tes
14c) Add 5 to 3n	8n	10.1	6.0	12.5
14f) Double y + 2	2y + 2	3.8	6.0	13.8
	y + 4	8.9	3.6	7.5
	y + 2x2	6.3	8.4	5.0
15c) v = 2, y = 6, v + y + w = ?	8w	6.3	2.4	5.0
16a) a + b = 5, a + b + c = ?	5c	3.8	4.8	5.0
18c) Paper costs t pence, delivery 5 pence, total?	5t	5.1	4.8	3.8
20a) p - 189 = 675, p - 190 = ?	676	7.6	4.8	2.5
20b) y + 279 = 978, y + 277 = ?	980	6.3	9.6	6.3
21a) 2a + 2 = 2(a + l), when ?	Never	41.8	51.8	55.0
22. Greater, 2n or n + 2?	Same	10.1	8.4	10.0
	2n	27.8	57.8	55.0
24b) Cost of parcel weight W kg at 25 pence/kg and 60 pence basic	Wx25 + C	7.6	8.4	17.5
	CxW	3.8	18.1	16.3
	C + W	17.7	15.7	17.5

n.b. Questions are in abbreviated form. See Appendix Q for the Full Question

Figure 8.14

Some Experimental Group Error Facilities For Experiment 3 Questions on Pre-Test, Post-Test and Delayed Post-Test experimental group was carried out to find the effect of the programme on the errors which those in the experimental group were making. Figures 8.13 and 8.14 contain a breakdown of the proportions making many of the common errors.

One of the most striking features of this analysis is the apparent resistance to change of the proportions across the three tests, with some questions even having slight rises in occurrence of the errors. This does not mean however that the programme had no effect on these errors for individuals because, as has been discussed above, many made considerable improvements in understanding, avoiding previous errors. These error facilities could well be evidence for the idea that for some children time is needed to work out the cognitive conflict which results from the meeting of the new concepts with their previous, erroneous, ideas. During this time a reconstruction of their schemas takes place, and so we see, initially, a rise in the error level followed by a fall at the delayed post-test stage. Examples of this occured in questions 3c), 3d), 7, 11c), 13 and 14f). All of these are examples of wrong understanding of the use of notation, which agrees with the idea above that it is conflict with arithmetic concepts which is the cause (e.g. the answer 8 + b in question 11c) involves calculating 3s as 3 + s when s = 5). It should also be noted then that for many pupils there were a lot of the questions on the test that, particularly in the pre-test, they did not attempt and therefore made no error on. The addition of the attempts of these pupils on some questions in the two later tests has affected the error facilities. This was most noticeable in questions 21a), 22 and 24b) in the test, where the error facilities rise drastically as seen in Figure 8.14. It should be noted that for these questions the proportions obtaining the correct answer and those not attempting the question at all in each of pre-test, post-test and delayed post-test respectively were as follows :

21a) Correct - 11.4, 20.5, 23.8.
Not attempted - 39.2, 20.0, 7.5.

22 Correct - 4.4, 6.6, 7.5.
Not attempted - 44.3, 28.9, 15.0.

24b) Correct - 0 (7.6), 3.6 (12.0), 1.3 (18.8).
Not attempted - 58.2, 37.3, 25.

The figures in brackets for question 24b) include those who mis-read the question as the cost is C pence (rather than 60 pence) instead of the total cost is C pence and correctly gave Wx25 + C rather than Wx25 + 60 for their understanding of the question. Thus it seems that many who had not previously attempted these questions were now giving error answers as their attempts as well as an increase in the number of correct answers. This would appear to be evidence that the understanding (or lack of it) which gives rise to these types of answers is a stage which many pupils go through on their way to correct understanding. This was not the conclusion of Küchemann, who emphasised that his levels of understanding were not a hierarchy but merely attainable levels. The reasons for such errors, across the questions, were the subject of many of the interview questions described in Chapter 9, and the questionnaire discussed in Chapter 10.

In spite of the above observation, there were some areas where the number of errors did decrease, and noticeably so in some. One of these involved conjoining in addition, writing for example 15b instead of 15 + b. This error has been discussed above relative to individuals in the experiment and described as being due to a failure to accept the use of letters as specific unknowns. Figures 8.13 and 8.14 show that in questions 5b), 5d) and 6d) there were considerable decreases in the occurrence of this type of error throughout the programme, and lower ones in questions 5c) and 9. In questions 14c) and 15c) the proportion fell after the computer module only to rise again following the traditional one. This is again partly due to the number of pupils attempting the questions. One way pupils try to avoid the problem is by ignoring letters which they cannot incorporate in their answers. This is shown in question 5b), where either a or b is sometimes ignored in answers. Again though there is a significant decrease in the occurrence of this error after the programme.

A common phenomenon which was technically considered an error in this study involved the failure to complete operations, leaving answers such as p + 5 + 4 instead of p + 9, and 3bx5 for 15b. Figure 8.13 shows that again there was a significant decrease in this type of answer following the modules in questions 3c), 3d) and 6c), as well as question 9. In questions 6b) and 6c), there is a sharp increase from post-test to delayed post-test (13.3% to 36.3% and 10.8% to 31.3% respectively), in those making the conjoining errors 4p and 9p along with corresponding decreases in the number of pupils correctly answering the questions (61% to 44% and 45% to 43%). The reasons for this are not clear, but there seems to have been some conflict raised in the pupils minds by the traditional module, so that their relatively good performance following the computer module was adversely affected. Another example of an error of notational use involved the answer y + 2x2 for what should have been 2(y + 2) or equivalent in question 14f). The continued use of this throughout the tests prompted a specific investigation of it in the interviews and the questionnaire. The results are described in the next two chapters.

Other miscellaneous errors of note included question 11c) where some of the pupils were, sensibly in many ways, affected by the context of the question and used the value of b = 7 which they had found in part a) of the same question and substituted it into 15 + b to get 22 as their answer. It is difficult to know whether this was because they were trying to avoid the use of a specific unknown, or were simply alert enough to spot the value and use it or misunderstood that once letters have been assigned a value in a context then they must keep this value. The question 12 error included, namely of giving an incorrect range of values for the variables is also of not since, although the answer is wrong, it does give evidence of an understanding of the use of letters as generalised number or variable since it is understood by these pupils that the letter may represent several values and not just one. The slight increase in this type of answer across the tests then is an indication of an improvement in such understanding.

8.6 Overview of the Chapter

In this chapter I have presented an analysis of the results of the third experiment from an overall point of view, both quantitatively and qualitatively. These results show clearly that the deep conceptual understanding produced by the computer-based programme of teaching used for the experimental pupils had successfully transferred to the ´paper and pencil´ algebra domain to produce a significantly better performance than the traditional programme. I have also looked at the results from the perspective of the individual child at differing levels of ability to demonstrate clearly the areas in which the improved understanding lay, and have compared the results with the general school population as represented by the CSMS results. The latter gave evidence of a very favourable comparison with the general school population. An analysis by gender has been included which indicates the possibility of differential outcomes along gender lines, requiring further research in this area. Finally the errors made by the experimental group have been briefly discussed with indications of their importance. All the results in the chapter have contributed to the view that the module of work produced, and the way in which it has been used in this research, have contributed to a significant improvement in conceptual understanding of algebra on the part of the pupils involved.

Chapter 9

The Interviews Following The Third Experiment

The interviews in the study were considered important as a means of analysing clearly the reasons behind some of the errors made by the two groups, as well as comparing the level of conceptual understanding attained by the subjects of the experiment with that of the control pupils. Such interviews provide some useful data on whether it had been possible for the pupils to obtain correct answers in the algebra test from erroneous reasoning, or indeed to write down a symbolically incorrect answer from correct reasoning. They also gave the opportunity to discover the extent of the understanding of the concepts by seeing how far such understanding would permit application of the concepts, particularly for the experimental group, with their better conceptual understanding. The rationale behind the interviews and the results from them are described in this chapter.

9.1 The Interviews

9.1.1 The Methods Applied in the Interviews

The intention was that the interview should be a semi-structured one. The schedule used for the interview was that given in Figure 9.1, and was based on that used by Booth (1983b). The separation of the processes involved in the interview facilitated the spotting of where any error in the child's sequence of activities resulting in their answer had arisen. The schedule was intended to be a framework for the interview and so it was not rigorously adhered to but rather each process was included or not as appropriate for each child. In addition to this schedule the interview strategies given in Figure 9.2, and taken from the suggestions of Wagner et al (1984), were applied to the interviews. The aims here were to

Process	Question Asked (Example)
1. Reading	Please read the question out loud.
2. Comprehension Interpretation	What is the question asking you to do? What does .... mean?
3. Strategy Selection Skills Selection	How will you do this question? Why?
4. Process	Work out the question and tell me what you are doing as you go.
5. Memory	(Check for recollection of intermediate step).
6. Encoding	Now write down the answer.
7. Consolidation	What does the answer mean?
8 Verification	Is there any way you can check to make sure your answer is right? Suppose I said I didn't believe you, could you prove to me that your answer is right?
9. 'Conflict'	Suppose I put 3 for x ...... Another student said ......
10 Similarity	Can you make up a question like that one? Which of these questions is like that one Why? Which is different? Why?
11 Generalisation	(Solving problem of more complex or more abstract nature, or one set in different context).

Figure 9.1

The Semi-Structured Interview Schedule

Used For the Experiment 3 Interviews

Specific Strategies: 1. First, try to help the student understand.

Can you restate the problem in your own words?

What is the problem asking for?

What are you given in the problem?

2. If necessary, make the problem more understandable.

a) Generalization

Have you seen other problems like this?

How could you make the problem simpler?

Suppose the problem were . . . (suggest a simpler form)

b) Reversibility

What should your answer look like?

What do you need to know in order to get the answer?

If the student is silent for some time, ask what he/she is thinking.

If the student is writing but not talking, ask him/her to tell you what he/she is writing.

Never interrupt the student while he/she is talking to paraphrase what is being said. Wait until the student finishes and then ask him/her to repeat or paraphrase the statement if it is not clear.

Ask the student to explain a procedure even if you "know" what was done. We might be surprised!

Ask neutral queations -- questions that could be asked with regard to any task: What did you do? Why? What are you thinking? Avoid leading questions that suggest a particular answer or process.

Make comments as neutral and nondirective as possible. Again, comments should be general and not task-specific.

Be uniformly positive with all students. Avoid phrases like "why you're confused," "hanging you up," "causing you trouble," "why you're having difficulty," etc. Do give reinforcement wherever possible -- fine, good job, nice solution, etc.

Make nonevaluative comments with regard to correctness of answers (How can you tell?), whether a method is different (Fine, can you give me still a different method?), whether something is obvious, easy, similar (Here is another problem; Let's take a look at this one; etc).

Don't bug students unnecessarily. There's a fine line between giving them a chance to think and constantly asking them what they are thinking. You just have to use judgment here. In particular, wait quite a while before concluding a student is stuck and giving hints. If hints come too quickly, not only will students feel that they can't solve problems themselves, but they will begin relying on you for solutions.

Figure 9.2

The Interview Strategies Used in

The Experiment 3 Interviews

make sure that what was recorded was the student's thinking, that he/she had not been led in any way by the interviewer and that the experience was a positive one for the student.

9.1.2 The Procedures

The interviews were all conducted by the researcher at the school used for the experiment on the 20th and 21st of May 1987. The dates were arranged to follow the presentation of the traditional algebra module to the computer group so that the full effects of the two programmes might be evaluated directly. Each interview lasted for about 20 minutes and the whole of the interview was tape-recorded and subsequently transcribed. The transcriptions of the interviews are given in full in Appendix Y1. The students were shown the questions one at a time and were asked to write down their answers to each of the questions in ink so that a permanent record of these was also kept, and these are given in Appendix Y2.

9.1.3 The Subjects

The subjects for the interviews were chosen from the experimental and control groups so as to be a cross section of pupils from the different bands of algebraic ability who were observed to be displaying some of the common errors of understanding described in Chapter 4. Particular emphasis was placed on the experimental group since the analysis of the extent of the difference in their conceptual understanding was a primary aim. Thus 13 pupils from this group were chosen compared with 8 from the control group. On the days of the interviews two of the experimental group and one of the control group pupils were absent from school and could not be interviewed. Figure 9.3 shows the test results for the pupils selected for the

Pupil Designation	Pre-Test	Post-Test	Delayed Post-Test
Experimental :			
P1	46	52.5	54
P2	33	55	58
P3	14.5	28	31.5
P4	7	31	43
P5	21	29	37
P6	33	53.5	56
P7	24	50	52.5
P8	13	32	38
P9	14	28	26
--- (not recorded)	15.5	19.5	25.5
--- (not recorded)	35.5	19	25.5
			Mean = 40.6
Control :			
P10	12	31	42
P11	4	24	33
P12	33	53	53
P13	16.5	31	32
P14	21	42	abs
P15	26	51	47
P16	19	44	35
			Mean = 40.3

Figure 9.3

The Algebra Test Scores of the Experiment 3 Interviewees

INTERVIEW QUESTIONS

Sheet one - Standard Questions

1. What is the value of $3 + 2m$ when $m = 5$?

2. If $2p - 1 = 5$ and $2s - 1 = 5$, can you say anything about p and s?

   What is the value of p?

3. Multiply $d - 4$ by 3.

4. Which is bigger $3y$ or $3 + y$?

5. Add 3 onto $6g$.

6. What is the value of $3 + 2 \times c$ when $c = 5$?

Figure 9.4

The Interview Questions

Additions To Questions

Standard Questions

1. What does 2m mean here?

2. For what values of p does 2(p + 1) - 1 = 5 ?

3. What if 5d - 4 multiplied by 3 ?

4. Are 3y and 3 + y ever the same ?

5. Add 3 onto g.

6. Value of 3 x c + 2 , when c = 5 ?

Figure 9.5

The Interview Questions

INTERVIEW QUESTIONS

Sheet two - Extra Questions

1. Write $2(a + b)$ without brackets.

2. What is the value of $3s + b$ when $s = 5$?

3. Write more simply $2d + 3 + d$.

4. Add 6 onto $e + 4$.

5. If $p + q = 3$ then $p + q + r =$?

6. When does $p - b = p - c$? Always, never or sometimes? If sometimes, say when.

7. For what value of c is $7 = c - 2$?

8. Write more simply $3f + 5g + 2f$.

9. What is the value of $6/h$ when $h = 2$?

10. Write more simply $(2m - n) + 3n$.

11. Solve $3x - 5 = 2x + 1$.

12. For what values of d is $17 - 3d > 2$?

Figure 9.6

The Interview Questions

Extra Questions

1. Write $2 \times (a + b)$ without brackets.

2. Value of $3 \times s + b$ when $s = 5$?

3. What is $2d + d + 3$?

4. What if add 6 onto $3e + 4$?

5. What does $r + p + q = ?$

6. When does $b - p = c - p$?

7. What about $7 = 3c - 2$?

8. Brackets around $3f + 5g$ make any difference? ie $(3f + 3g) + 2f$

9. $\frac{6}{h}$?

10. What about $2m - (n + 3n)$?

11. What about $2x + 1 = 3x - 5$?

12. What about $17 + 3d > 2$?

Figure 9.7

The Interview Questions

interviews, and designated P1 to P16 below (three of the tape recordings were technically too poor to use).

9.1.4 The Questions Used

The questions were divided into two groups, with the first group of 6 questions being asked of all the pupils. The second group of 12 questions were arranged to cover a range of difficulty and each of the interview subjects was asked only 4 or 5 questions from this second group, depending on the level of their understanding on the pre-test. For every question in each of the two groups an extension question was prepared on a separate sheet so that the extent of their understanding on a particular concept could be determined. The questions used and the extensions for each are given in Figures 9.4 to 9.7.

The questions which were chosen for the interview were selected in order to analyse understanding in areas which were well known either to contain problems and errors of understanding for many children or to give rise to a variety of different ways of 'solving' the problem. Booth (1984) has categorised these areas as :

1. The meaning of the letters used
2. The process of operating with the letters
3. Notation and convention in algebra

The classification of the questions used for the interviews in this research was:

1. The tendency of children to work from left to right regardless of the parsing structure of an expression.

Standard questions 1 and 6
Extra question 7

2. Misunderstandings of notation. These included :

a) The appreciation of implied multiplication

Standard questions 1 and 4

Extra questions 1, 2 and 11

b) The ability to use brackets correctly

Standard questions 3

Extra questions 1 and 10

c) Conjoining in addition

Standard questions 4 and 5

Extra questions 2, 4 and 5

d) The use of / for division

Extra question 9

3. An understanding of the use of letters as specific unknown.

Standard question 5

Extra questions 2, 3, 4, and 5

4. The understanding of the use of letters as generalised numbers or variables, including the fact that different letters may take the same value.

Standard questions 2 and 4

Extra questions 6 and 12

5. The processes used in equation solving.

Standard question 2

Extra questions 7 and 11

6. A comparison of the use of formal and invented procedures

Standard question 2

Extra questions 1, 3, 5, 7, 8, 10, 11, 12

Question	Response	Experimental Number Giving	Control Number Giving
S1 - Given to 11 experimental and 7 control pupils			
3 + 2m when m = 5?	Correct - 13	9	4
	25 i.e. (3 + 2)m	2	0
	10 i.e. 3 + 2 + m	0	2
	13m ´13 lots of m´	0	1
S6 - Given to 11 experimental and 7 control pupils			
3 + 2xc when c = 5?	Correct - 13	2	1
	25 i.e. (3 + 2)m	9	5
	10 i.e. 3 + 2 + m	0	1
E1 - Given to 8 experimental and 5 control pupils			
2(a + b) without brackets	Correct - 2a + 2b	3	0
	2a + b	2	2
	2xa + b = a + bx2	2	0
	2xa + b only	0	2
	2c where c = a + b	1	0
	Unable to answer	0	1

Figure 9.8

An Analysis of Answers to Some Experiment 3 Interview Questions Involving Understanding of Notation

9.2 Results of the Interviews

I shall now consider the results of the interviews relative to the broad areas above into which the questions were designed to be grouped, and describe the insights into their understanding which were obtained from the pupils´ comments.

9.2.1 The Left/Right Tendency and Other Mis-Understandings of Notation

Since these turned out to be closely interrelated I shall consider them together. Questions 1 and 6 of the standard questions were intended to be essentially the same except for the change of notation (the multiplication being implicit in question 1, but explicit in question 6) and variable. Hence they were separated as widely as possible on the questions to allow for their treatment as separate questions. Figure 9.8 contains a summary of the results of the 18 pupils on these two questions. It shows, perhaps surprisingly, that 9 of the 11 experimental group pupils and 4 of the 7 control group pupils correctly answered question 1, but only 2 and 1 respectively successfully answered question 6. This is interesting in that previous evidence has been that many pupils of this age tend to conjoin 3 + 2m type expressions into 5m when there is no value for m, yet here, when a value is given they have coped with this far better than when the multiplication is also given. This seems to illustrate an important point concerning most of the pupils´ view of notation, and one which is connected to the way in which the imagery in the mathematics affects our understanding of it, as described in Chapter 3. It would appear that in the introduction of the x into 3 + 2m, making it 3 + 2xm, (actually 3 + 2xc), the imaginal concept associated with the x in the lower level schemas causes the linking to a different conceptual area of the higher level schemas than when it is absent and hence evokes a different mathematical process of solution. In question 1, I would say that the proximity of the 2 and the m in 2m enable it to be viewed as a single image or entity which is then decoded via the conceptual understanding which this single image evokes, namely ´2 times m´ or ´2 lots of m´ or erroneously as ´2 + m´, (all the pupils read the question in this way, with no operator between the 2 and the m). However, with the operator explicit the question is viewed in a totally different way. Now there is no part of the image of the question which requires ´decoding´ because each part is explicitly given, including the value of m. What strategy was often applied in this case then?

INT : How do you decide which bit to do first on those two questions?

P7 : It´s always the first bit unless it´s got brackets around

----

INT : How did you decide on these, which bit to do first, can you tell me?

P5 : Because it´s not in brackets, so you go along this first.

INT : So you start from where then if it´s not in brackets?

P5 : The 3

INT : So from the left. And the same here is it?

P5 : Yes

INT : So if there aren´t any brackets you start from the left and work along?

P5 : Yes

From those pupils interviewed here it would seem that it is their mis-understanding of the use of brackets which is at the cause of their problem. They understand the purpose of brackets as being a sign that what is in them must be done first but mistakenly conclude that if there are no brackets then there is no priority and one starts at the left (as in the English language) and works through from left to right, or may choose where to start. This also avoided, in some of their minds, the problem of holding an operation in abeyance, having read it, until some other operation further along has been carried out :

P13 : Well first you add those two, then times that [in 2 + 3xc], but with that one [3xc + 2] you times that and then just add that at the end.

INT : And why do you do them in that order, can you tell me?

P13 : Well, you don't sort of do the 3 times c and then go back.

One of the control pupils had an ingenious way around the problem of deciding where to start :

INT : On questions like this which bit do you do first then?

P16 : Well I do the easiest bit, the first bit first. If I find that that bit is easier then the 3 times c bit is easier then I do that first.

INT : So that is the easier bit there then is it? What if the easier bit was further along the question, would you do that first still or would you start from the left?

P16 : yes I would probably start further along and then do the bit at the back and then add them together.

INT : ...could you give me an example of a question that would be like that where you would do the easier bit first or not, if it was further along? Can you think of one or not?

P16 : 7 times 15 plus 3

INT : ...so where would you start with that then?

P16 : There that bit. I´d do 3 plus 15 is 18 times 7 and then I would take my time about that.

Thus it seems that for some pupils it is a matter of expediency as to where they start in an expression like this which has no brackets and their pragmatic solution is, sometimes, to do it in the way which is easiest for them. It is somewhat disappointing that the experimental pupils did not cope better than they did with calculating 3 + 2xc when c = 5 since the computer experience emphasised the importance of the order of operations. It may be that the connection between the * used for multiplication by the computer in BASIC and the x of arithmetic have not been sufficiently connected as images in the minds of these children. This is an area for further consideration.

That this problem is initiated by the absence of brackets causes our attention to focus further now on the children´s understanding of these. The idea that they are to make one ´carry out the brackets first´ was common and caused some conflict in question 1 of the extra questions - Write 2(a + b) without brackets. As Figure 9.8 shows, the only correct answers to this question came from 3 of the experimental group pupils and it was in this group, who seemed to have a better understanding of brackets, that their inability to do the addition first, because they had no values for a and b, caused problems. This is illustrated by the following comments :

Pl : Got to have a and b two separate numbers times 2. You always times first.

INT : ...

Pl : Well it isn´t a plus b times 2 ´cos it´ll be a plus b times 2. a plus b over 2. That way you plus those before dividing them. $\left[\frac{a+b}{2}\right]$

INT : I see. It doesn´t matter that it´s divided now.

Pl : Oh, yes.

INT : ...can you explain to me the difficulty?

Pl : Well its brackets, so you´ve got to add these two numbers before times it and the problem is you always multiply first.

Here we see the pupil is at first almost ready to abandon the operation of multiplication in favour of division because it would resolve the conflict of finding a notation whose imagery fits in with his dilemma of doing both the bracket and the multiplication first and he can then see no easy way around his dilemma. His final solution required a deep insight of replacing the expression a + b by the variable c, thus avoiding the conflict :

Pl : Unless you went along and put a plus b equals c and then put 2 times c, but that´s a long way round.

Two of the control group pupils showed in their comments that they did not appear to understand the brackets as having any purpose at all :

Question	Response	Experimental Number Giving	Control Number Giving
S3 - Given to 11 experimental and 7 control pupils			
Multiply d - 4 by 3	Correct - 3(d - 4) or (d -4)3	1	1
	3xd - 4 alone	1	0
	3xd - 4 = 3d - 4	1	0
	d - 4x3	2	0
	d - 4x3 = 5d	0	1
	d - 12	1	0
	3d - 4	0	1
	12d	0	1
	3x where x = d - 4	1	0
	Unable to answer	4	3
S5 - Given to 11 experimental and 7 control pupils			
Add 3 onto 6g	Correct - 3 + 6g or 6g + 3	6	1
	9g	3	4
	3 + 6g = 9g	1	2
	63 [i.e. g = 10]	1	0

Figure 9.9

An Analysis of Answers to Some Experiment 3 Interview Questions Involving Understanding of Notation

P16 : It would be 2a plus b.

INT : ...Can you tell me how you got it?

P16 : It says without brackets so I just took the brackets away.

INT : ..That answer you have written down [2a + b] there is that the same as this here only without the brackets?

P16 : Yes.

INT : It's the same is it? Good, so if you worked it out you would get the same answer. [Persisting to see if the problem is one of understanding the English or the Mathematics]

P16 : Yes.

Question 3 of the standard questions gave an opportunity to analyse the way in which the students were prepared to use brackets to record their answers as opposed to their interpretation of pre-written notation. The problem was to symbolise the result of multiplying d - 4 by 3. As Figure 9.9 shows, only one pupil from each of the groups was successful on this question. Two experimental and one control group pupils, consistent with question 6, used d - 4x3 to represent what they understood to mean (d - 4)x3 i.e. they had translated the answer into their left to right precedence notation in a 'correct' way. As one of them explains:

"P4 : 2 times a and then you would plus b.[2xa + b]

INT : Good. And that's the same is it?

P4 : Yes. No. No it's not the same as that. [2(a + b)]

INT : Why isn't it the same?

P4 : Because you would have to have brackets.

INT :...Is there any way of writing this [2(a + b)] without brackets so that it is the same?

P4 : Yes, you could have a plus b and then times by 2.

INT :...So that one [a + b x2] is the same as this one [2(a + b)] but without the brackets.

P4 : Yes.

INT : But what about this one? `2xa + b§

P4 : No you do 2 times a and then you plus b.

This consistency of use of the left to right precedence is one which it would appear that many children may be using as one of their invented procedures which is masking to some extent their understanding of basic algebraic procedures. Both of the pupils with the correct answer used brackets in their answer i.e. no-one wrote 3d - 12, since ´the d minus 4 has to be done first´. However only the pupil from the computer group wrote 3(d - 4) as the answer, the other recording d - 4x3 = (d - 4)x3. These comments on the use of brackets based on the interviews show why there is such a problem in helping children to improve in their understanding of their use. It partly confirms, for the control pupils, the difficulty with understanding brackets, expressed by the findings of Booth (1984) that :

> "In the case of the use of brackets items, the marked drop in performance between immediate and delayed post-tests indicates that the ideas relating to this aspect were generally not assimilated by the children."
>
> (Booth, 1984, p.79)

> "Children ignore the use of brackets, mainly because they consider them unnecessary."
>
> (ibid, p.86)

Question	Experimental Propn. Right	Control Propn. Right	χ^2 [From Raw Data]	p
S5 Add 3 onto 6g	0.55	0.17	2.91	<0.1
E10 Simplify (2m - n) + 3n	1.00	0.0	8.00	<0.005
E11 Solve 3x - 5 = 2x + 1	0.75	0.0	3.00	<0.1
S5 and E2 - E5 Conjoin errors in specific unknown	0.20	0.50	4.06	<0.05
S4, E6 and E12 Generalised number and variable	0.37	0.07	3.86	<0.05

Table 9.1

A Comparison of Experiment 3 Performance on Some of The Interview Questions

For the experimental pupils however there is some evidence that the problem with the use of brackets is not that they consider them unnecessary but that their full understanding is impeded by the conflict of their perceived use and how it interacts with other arithmetic operations. This is an area requiring further investigation.

It was noticeable on the extra question 1, discussed above, and the other questions, that there was a far greater tendency on the part of the pupils who had followed the traditional skill-based module to conjoin their answers when they involved addition. In standard question 3 for example 2 of their number wrote 12d or 5d for the answer, but none of the computer group did. Likewise on question 5 of the standard questions, Figure 9.9 shows that 5 out of 7 of the controls wrote 9g compared with only 4 out of 11 of the others, 6 of the experimental group getting the correct answer 3 + 6g or 6g + 3, but only one in the controls. As Table 9.1 shows, this is a significantly better performance by the computer group and is an indication that the computer module has had a valuable effect on the problem of conjoining in addition. The explanation of two of the control pupils suggests that the problem of conjoining may have some connection with a mis-understanding of the implied operator as addition and/or the strength of the arithmetic schemas, as well as their understanding of the use of letters, (see later) :

P11 : Add 3 and 6 and then put a g next to it.

INT : ...Good, can I just ask you how you would do this one over here? [Add 3 to g]

P11 : Put the 3 next to the g which is 3g.

INT : Ah, good, ok. So how did you get that answer?

P11 : On the other question if there is a number before it you write down that number anyway, so I thought if you put the g down, if the g´s there you just, well put it next to it because it´s adding on.

P13 : 3g

INT :...Can you tell me how you got the answer there?

P13 : Well it´s just 3 plus g, so you regard g as a number instead of a letter.

There is also an element of the mis-understanding being language based here since one of them seems to equate ´added to´ with being placed next to. Finding the extent to which language may influence pupils´ understanding was part of the intention of the questionnaire described in Chapter 10.

9.2.2 Equation Solving and Invented/Formal Procedures

One of the characteristics of children´s equation solving attempts described in Chapter 4 is the use of invented or ad hoc procedures rather than those which they have been taught. This can be seen from the interviews in the children´s attempts to solve 2p - 1 = 5 in standard question 2. The expressions of two of the control pupils, who had been taught only formal equation solving techniques, show the lack of understanding resulting from the instrumental way in which they had been taught them. The resulting confusion and the way in which the method can be forgotten may be seen from :

P11 : p and s equal 6.

INT :...Can you tell me how you got that?

P11 : 2p minus 1 equal 5. If you add the 1 to the 5 that's 6, so because there's no other minus p I forget the p and do the 2p minus 1 equals. If you do [pause] add the 1 to the 5 which is 6 and then you take 1 from the 6 [pause] No, I don't get that, I know how I've done it but...

INT :...What would the value of p be did you say?

P11 : Six.

INT : How do you set about doing something like that?

P14 : I've forgotten.

INT : Can you tell me what you are thinking?

P14 : To go backwards, add the 1 onto the 5 and then divide p and 2 in 6, so it's 2 times 3 equals 6, minus 1 is 5. So it's 3. [Using substitution as a checking procedure]

Two of the control students also abandoned the formal methods altogether and used only a method based on substitution into the left hand side of the equation:

P12 : Is it that p equals a certain number and if you were to minus 1 from it, say p was 3 you would be timesing that by 2 so you would have a 6 and minus 1 that would give you 5.

P16 : 2 of whatever minus 1 has got to be that, so I did 2 what makes 6, so that has got to be 2 threes.

It was interesting that the pupils, from both groups, used informal methods based on substitution to try and solve the equation. There may again be an imagery aspect to this, since the variable was p, rather than x, which still seems to be the

Question	Response	Experimental Number Giving	Control Number Giving
S2 - Given to 11 experimental and 7 control pupils			
2p - 1 = 5, p = ?	Correct - p = 3	9	5
	p = 6	1	0
	p = 4	0	1
	Unable to answer	1	1
Extension - Given to 4 experimental and 2 control pupils			
2(p + 1) - 1 = 5, p = ?	Correct - p = 2	3	1
	p = 5	1	0
	p = 1	0	1
E10 - Given to 4 experimental and 2 control pupils			
Solve 3x - 5 = 2x + 1	Correct - x = 6	3	0
	1x	0	1
	Unable to answer	1	1

Figure 9.10

An Analysis of Answers to Some Experiment 3 Interview Questions Involving Understanding of Equations dominant letter used in teaching, and it may be that the image 'x' evokes manipulative methods in the higher level schemas more than some other letters, and that equations in other variables are not viewed as the same type as those in x (or say y). As one pupil remarked :

INT : Is there any reason why you chose x?

P1 : Well it's a letter we are always using.

INT : You use x a lot?

P1 : Yes.

Figure 9.10 shows that 9 of the experimental pupils correctly answered the question, compared with 5 of the controls. One important concept associated with understanding equations, and described by Wagner (1977), is the ability to conserve equation following a transformation of variable, and she has described how pupils who have this understanding also avoid the common error of thinking that different letters must have different values. The extension to question 1 was intended to analyse the extent to which the children were able to conserve the equation $2p - 1 = 5$ when the variable was transformed to $p + 1$, i.e. $2(p + 1) - 1 = 5$. In the event, 4 of the 5 experimental group pupils who were asked this difficult extension question answered it correctly compared with only one of the 2 controls. Two of the 4 pupils from the computer section, however, showed that they had deep conceptual insight by not resolving the equation again but using the transformation. They had, in Wagner's words, conserved the equation, they explained :

INT :...can you tell me anything about the value of p in this question?' [$2(p + 1) - 1 = 5$]

P1 : Yes, p equals 2.

INT : Right, that´s very interesting. How did you work that out?

P1 : Well it´s the same but then its plus 1, so minus 1 add 3.

P2 :...Oh, it would be 2.

INT : Can you tell me why?

P2 : Because p plus 1 if that´s 3 its the same as the last one only the p is less because you´ve got to add 1 to the sum.

The possibility of children gaining such powerful insights from their work in a computer environment must be another good reason for its use in developing a real conceptual understanding of algebra.

Question 11 of the extra questions, involving the solution of the equation : $3x - 5 = 2x + 1$, was included because neither set of pupils had been taught how to solve questions like it and so there would be a chance to see if the conceptual basis of the computer group was more likely to permit an extension of their understanding to include such conceptually demanding equations than that of the controls. It was encouraging to see that, as Figure 9.10 records, 3 of the 4 pupils from the computer module correctly answered this question whereas neither of the 2 from the controls did. It was interesting that the two pupils who had succeeded above also managed to work out, more or less, the formal method of solution of this equation, since it involves algebraic conceptualisation to deal with the transference of x´s rather than simply arithmetic thought. The third pupil attempted a trial and error method of substitution, based on the computer module work, before getting it correct. The first two said :

P1 : Well I´m thinking that maybe take x some number away from both sides so you´ll be able to take a x from both sides. That wouldn´t leave anything in there to go on. You´d have nothing there if you take 2x away and 1x minus 5 equals plus 1.

INT : Would it help to write that down?

P1 : [Writes x - 5 = +1]

INT : So how might you do it now?

P1 : I was thinking maybe get rid of this and forget about that 4 by putting, adding 5 to both sides - that should do it - so it would be 3x equals 2x plus 6 [pause] try to take x away [pause, writes 3x = 2x + 6]

P1 : It would be x equals 6.

INT : How did you get that?

p1 : Well if you go on about the plus I suppose I should have taken 2x away from both sides - so that you´ve got nothing there and 1x there.

P2 : You would add 5 to that to get rid of the minus 5 and then that plus 6, so that it would be 3x equals 2x plus 6.[writes this]

INT :...just tell me what you are doing.

P2 : I´m not too sure how to get rid of the plus 6. I think you should get rid of that.

and...

P2 : Well that plus 6 has got a bigger x because 2x plus 6 equals 3x, that means another 6 would equal x, so make 3x as well.

INT Good, can you write that down?

P2 : Shall I just put x equals 6?

This second solution shows an interesting combination of the formal procedure of transferring numerical terms from one side to the other combined with a balancing strategy which equates the deficit in x´s on one side to the numerical value on the other. Again these examples show that the computer module produced a deep conceptual understanding which allowed them to successfully attempt this conceptually difficult question which was beyond anything they have seen before. One very surprising point arising from this equation was how few of the pupils seemed to realise what form the answer would be in. Some comments which illustrate this were :

INT : What sort of answer will you get?

P7 : I don´t know [this pupil solved the equation]

P12 : You would have to find the value of x before you could start.

INT :...What does the word solve mean to you in that question?

P12 : Find the answer. It could be find the value of x.

INT : What does the word solve mean?

P15 : I´d say figure out or get the answer to.

INT : I see, and does it tell you anything about what sort of answer you would expect from that?

P15 : No not really just find out.

This is particularly surprising from the last two, who were control group students, since it means that although they had worked on equations very similar looking to the one set, although not as conceptually difficult, they had not obtained a clear conceptual picture of what they were required to do, namely to find the value of x. This suggests the need for further investigation of the type of mental imagery and associated concepts connected with equations which are built up in the minds of students, to see how cognitive integration might improve this situation.

Two questions were set on the extra sheet, questions 8 and 10, involving a somewhat difficult examination of manipulative skills as well as conceptual understanding, of which one, question 10 (simplify (2m - n) + 3n), was particularly demanding. One might have expected the control group to have done at least as well on this type of question, but as Table 9.1 shows, they actually did significantly worse. All 5 of the experimental pupils asked this question answered correctly whereas none of the 3 controls did. Their explanation shows the failure of the instrumental understanding resulting from the skill-based approach to cope with such a difficulty level :

P12 : 4n minus 2m...Well it's 1 times n so if you added the m and n together you would have 4n and you are still going to be taking that away from it.

P15 : First add 3n plus n which I would say is 4n and then the answer I would put 2n, no 4n minus 2m.

P16 : Well it's got 2m minus n in brackets and if it was 3n minus m then it would be easier. I don't know whether that is the same as that or not.

This ability of the computer group to keep the correct operation with the variables disjointed by the use of brackets may well have arisen from the typing of such expressions into the computer and using it as an environment in which to conceptualise their treatment in algebra. One final insight into notation which one of the computer group expressed in the interviews was to do with commutativity. he said :

P8 : Would it be e plus 10?

INT :...And is that the only way to write that answer?

P8 : You could have 10 plus e.

INT :...are they the only ways?

P8 : Yes.

9.3 <u>The Understanding of the Use of Letters in Algebra</u>

The understanding of the use of letters as specific unknowns, generalised numbers and variables has already been shown to be one of the major failings of the conceptual algebraic understanding of secondary school children. Chapter 8 has highlighted the extent to which the programme in experiment three was able to improve this understanding as shown by the test questions. It was hoped that the interviews would further illuminate the differential nature of the conceptual understanding between the two groups.

9.3.1 <u>Letter as Specific Unknown</u>

Küchemann's (1980) study describes the use of letter as

Question	Response	Experimental Number Giving	Control Number Giving
E2 - Given to 2 experimental and 4 control pupils			
3s + b when s = 5	Correct - 15 + b	1	2
	35 + b = 35b	0	1
	13 i.e. 3 + 5 + 5	0	1
	Unable to answer	1	0
E3 - Given to 2 experimental and 0 control pupils			
Simplify 2d + 3 + d	Correct - 3d + 3 etc.	1	0
	Unable to answer	1	0
E4 - Given to 6 experimental and 3 control pupils			
Add 6 to e + 4	Correct - e + 10 etc.	3	2
	6 + e + 4 alone	1	0
	6 + 4	1	0
	10e	0	1
	Unable to answer	1	0
E5 - Given to 4 experimental and 2 control pupils			
p + q = 3, p + q + r = ?	Correct - 3 + r etc.	1	1
	3r	0	1
	6 i.e. p=1, q=2, r=3	1	0
	Unable to answer	2	0

Figure 9.11

An Analysis of Answers to Some Experiment 3 Interview Questions Involving Understanding of Specific Unknown specific unknown as the first steps in algebraic reasoning because until children are willing to write answers such as 3n + 4, they have sought ways of avoiding generalised arithmetic. As Küchemann (1981b, p.108) describes it, they "have to recognise that this is all that can be done to combine the elements. Many children were unable (or willing) to do this and instead gave the answer 7n". Thus the evidence described above for the way in which the experimental group were far less likely to conjoin their answers in addition is also evidence that they had a better understanding of the use of letters as specific unknowns. Table 9.1 shows that on question 5 of the standard questions, add 6 to 3g, the results of the experimental group were significantly better than those of the control group. Taking all the interview questions in which a reluctance to accept specific unknown could be displayed, namely standard question 5 and extra questions 2, 3, 4 and 5 then, as Table 9.1 shows, the occurence of conjoined answers for the control group, at 8 out of 16 possible answers, was significantly higher than that of the computer group, with 5 out of 25. The results of the pupils on the individual questions are given in Figure 9.11.

One of the computer group made an interesting comment in the interviews as to how he felt he had benefited from the work on the computer :

INT : Can you think of any particular way that it did or what?

P4 : Didn´t understand when you had d like plus b, I would always put a number in and then it was plus b or whatever.

INT : Very good. So you don´t have to put a number?

P4 : No.

INT : And you're happy with that now are you?

P4 : Yes.

Here this pupil is describing the way in which the computer module has helped him to an understanding of specific unknown in algebra. This may be compared with one of the control group pupils, of similar ability level on the pre-test (scores 7 and 4 respectively), who commented on the extra question 5 :

P11 : One of those is either 2 or 3 or the other way round, so I'd put 3, I don't know what r would equal.

INT :...Can you write that down? [Writes 3r] Good and what does that answer mean, can you tell me?

P11 : If p plus q equals 3 then because you don't know what the r is you don't know how to add it to that, so you just put [3r]

Thus, unable to accept the lack of closure of an answer such as 3 + r, 'you don't know how to add it', the typical response, as here, for one without this level of conceptual understanding is to write 3r. These results and comments are an indication of the greater difficulty in moving towards an understanding of specific unknown from a traditional skill-based route than from the conceptually rich computer paradigm.

9.3.2 Generalised Number and Variable

Although a very important step on the road to full conceptual understanding of the use of symbolic literals in algebra, specific unknown is still at a relatively low level compared with many of the uses of such symbols in elementary algebra. Remembering that only 2%, 6% and 9% of 13, 14 and 15 year-olds respectively (Küchemann, 1981b) reach such an

Question	Response	Experimental Number Giving	Control Number Giving
S4 - Given to 11 experimental and 7 control pupils			
Are 3y and 3 + y ever the same?	Correct - range	3	1
	Always same	0	2
	3 + y always bigger	0	2
	3y since multiply	6	2
	Unable to answer	2	0
E6 - Given to 4 experimental and 5 control pupils			
p - c = p - b, when?	Correct - when b = c	2	0
	Never	1	1
	Always	1	1
	Sometimes, when you add something to b	0	1
	Unable to answer	0	2
E12 - Given to 4 experimental and 2 control pupils			
Values of d when 17 - 3d > 2 ?	Correct - range	2	0
	Single wrong value	1	2
	Unable to answer	1	0

Figure 9.12

An Analysis of Answers to Some Experiment 3 Interview Questions Involving Understanding of Generalised Number And Variable understanding, it is here that the deeper conceptual understanding of the computer group should have become evident on the interviews. Figure 9.12 shows that on standard question 4 and extra questions 6 and 12, which required such understanding, there were 7 out of 20 and 1 out of 14 correct answers from the experimental and control groups, respectively, and Table 9.1 shows that this is a significantly better performance by the computer module pupils.

The comments in the interviews however reveal the extent to which the understanding is better. For example, even among those pupils who did not get question 6 of the extra questions right (when does p - b = p - c?), they still showed some movement toward a conception of generalised number or variable :

P3 : Always.

INT : What always the same?

P3 : Yes, because they could be anything, the value could be anything.

INT : So if the value could be anything will they always be the same?

P3 : I think so.

P8 : Never.

INT : Never. Can you tell me how you get that answer?

P8 : Well if you didn´t know what p and b and c were then you wouldn´t know what to put. Wouldn´t know the answer.

INT : I see. Does that mean that they are never going to be the same then?

P8: Yes.

INT : So there aren't any values when they could be the same?

P8 : No.

On standard question 4 too (Which is bigger, 3y or 3 + y?), this was displayed :

P5 : Depending on what the value of y is it could be 3 plus whatever that is or 3y, so it is 3y that is bigger.

INT :...Now when you said depending on what the value of y is, what did you mean by that?

P5 : Say it was 10, 3 plus y would be 13 but 3y would be 30.

INT : So there aren't any values of y for which 3y isn't bigger?

P5 : No.

P7 : Well 3 times y is.

INT :...can you tell me why?

P7 : Because 3 plus y is 3 plus something and 3 times y is more.

INT : Is it always more?

P7 : No. If that was 2, 3 times 2, that could be 3 plus 6. `takes a different value for y in each expressions

...INT : So 3y would always be bigger no matter what y is?

P7 : Yes.

INT : What if y is 1?

P7 : Then 3 + y would be more.

INT : So is 3y always bigger?

P7 : No.

INT : What does it depend on then? Can you tell me?

P7 : Everything apart from 1 and 0.

Compare these with the control pupils answers to the same questions. For when does p - b = p - c? :

P10 : Well on the test I didn´t quite get it and I still don´t.

INT :...Can you tell me what you think it means then. Does it mean anything to you at all?

P10 : It probably does but I don´t know it.

INT : Understand what that is asking?

P13 : No, not really.

INT : No, does it make any sense at all to you?

P13 : I´ve seen this type of question further on in our algebra module but I´ve never got that far.

INT : Ah, I see, so you can´t answer that one at all?

P13 : No.

INT : Have you any idea at all what it might be getting at?

P15 : I think it´s saying like is p minus b equal to p minus c, get to the end. That would be minus pb or minus pc, I´m not sure. [Gives a skill-based, manipulative type answer]

and for Which is bigger 3y or 3 + y? :

P11 : the same...Well if it´s 3y it looks as if it has already been added or whatever and its´s being added there. [3 + y].

P15 : I would s say that the second sum 3 plus y would be more.

INT : Would you, can you tell me why?

P15 : Because you have already got one y there plus another 3 would make 4y. [Again a skill-type, instrumental answer]

Added to this, when the experimental pupils got the answer right they showed deep insight again into the nature of use of letters in algebra :

P1 : Well that depends on what y is, because you can have 3 times 1 which is 3, 3 plus 1 is 4.

...P1 : When you get bigger numbers it will be `pause§ that one will be bigger [points to 3y]

INT : You can´t tell me...how big and why you can get it?

P1 : It´s got to be over a certain number. 2 times the 6 [pause] 3 plus 2 is 5, it will be 2 then. That´s the only one.

INT : I see. Are there any other numbers at all?

P1 :...3 times 0 is nought, 3 plus 0 is 3...So it´s got 2 if you have decimals and things stuck in between.

P2 : unless the y was 1. 3 times 1 would be 3 and 3 plus 1 would be 4. Well no just the very low ones.

INT :...Can you give me any idea of what they might be?

P2 : Well 1 and maybe fractions.

...INT : So what does it depend on?

P2 : Whether y is a fraction or a very low number like 0 to 1.

This understanding that a letter may stand for a range of values, and even include decimal or fractional values is indicative of the type of deep conceptual understanding which may result from the computer approach, compared with the surface type reasoning and understanding of the control pupils.

Question 12 of the extra questions (When is 17 - 3d ¨ 2?) was again set to see if the concept of a range of values was present in the pupils schemas. This question was only given to the pupils who had scored over 50 out of 67 in the post-test since it is so difficult. Of the 4 experimental and 2 control pupils, Figure 9.12 shows that 2 of the experimental group pupils correctly answered the question, compared with neither of the controls, who had little idea of a solution to the question. One of the computer group, although not getting the question right, showed understanding of trying to find a range of numbers for the answer :

> INT : Do you understand what you have got to try and find?
>
> P7 : All the numbers which are higher than d.

Once again the pupils getting the question right showed deep understanding:

> Pl : Um, well it can be anything really. IT's got to be anything greater than 2 in the end, so it will be 17 minus 3 that will be 14 and that's greater than 2. Anything up to [pause] Ah that's 2. So it's anything less than 2, d can be anything less than 5 I mean.
>
> ----------
>
> P2 : It's going to be quite a small number, if its bigger than 2. I mean it wouldn't be say 6 because they

would make it minus 1, so it would have to be less than 6. It could be 5, that would be equal to 2, so it would probably be right for 4 and under...4, 3, 2, 1 and fractions.

I have mentioned earlier too, that pupil 1 in the interviews twice replaced an expression in his answers by a variable during the interviews, showing that he understood that variables may be replaced by expressions. In the extension to standard question 2, which 3 of the experimental group successfully answered, two of them used the transformation p ---> p + 1 to answer it, not only mentally replacing a variable by an expression, but happily doing so with an expression containing the original variable.

The combination of the evidence above from the interviews confirms, I believe, that the pupils who had used the computer module as a generic organiser for their introduction to manipulative algebra, had gained a deeper conceptual understanding of algebra than those pupils who had simply been through the skill-based module.

9.4 An Overview Of the Chapter

In this chapter I have looked at the results of the interviews with 18 of the pupils involved in experiment 3, the main teaching experiment. I have analysed the results of these interviews, and shown how they support the claim that a superior level of conceptual understanding existed among the experimental pupils after the study. Several points of note arose from these which required further consideration, and that was the purpose of the questionnaire which I developed, and which is described in the next chapter.

Chapter 10

The Questionnaire

An explanation of the reasons for the production of the questionnaire which was given to all the pupils in the third experiment, and a description of its form and content are presented in this chapter. The results, and conclusions drawn from them, are discussed, along with their relevance to the present research and cognitive theory of Chapter 3.

10.1 The Rationale Behind the Questionnaire

The algebra test answers of some of the pupils and the comments in some of the interviews had uncovered certain misunderstandings of a conceptual and notational nature. One aim of the questionnaire was to test to see how widespread these misunderstandings were and to what extent, if any, the computer algebra programme had exerted a beneficial influence on them. A second major aim was to measure any superiority of the conceptual approach to algebra developed herein in allowing students to extend their mental schemas to include an understanding of the solution of the conceptually difficult linear equations of the form :

$$ax + b = cx + d,$$

and to the solution of more difficult linear inequalities. A secondary consideration, given the connection of some of the misunderstandings to symbolic notation, was to analyse them for evidence supporting the cognitive theory encompassing mental imagery described in Chapter 3.

The questionnaire used is given in Appendix S , and the reasons for the choice of each of the questions is given below.

Question one offered two expressions, such as 3 + 2m and 3 + 2xm, which represented two answers supposedly written by a pupil as equivalent, and they were asked to explain whether this constituted a correct statement, or not, and why. The purpose of question one was to investigate understanding in the following areas :

a) The use of brackets - The interviews had shown that some of the pupils were experiencing cognitive conflict over their understanding of brackets as an instruction to 'do the bracket first'. Question 1c) was designed to test how widespread this was by asking them to compare (n + 3) + 6 with n + 9, to see if (i) the concept of doing the bracket first would override the possibility of arithmetic addition, (ii) the bracket would just be ignored, and for whom these would happen. The interviews had also raised the reluctance of pupils to write 2(a + b) as 2a + 2b, since, again, it had revealed the desire to perform the addition of a and b first, before multiplying by 2 and the use of invented notation, such as a + bx2 as an alternative. Question 1g) repeated this question to see how common a problem this was, and whether the computer group were more or less likely to have this difficulty.

b) Conjoining in addition - The desire of many children to 'simplify' expressions such as 2 + 3c to 5c because they cannot accept lack of closure due to not understanding the concept of specific unknown is well known (see Chapter 4). The interviews had raised the question of, not just how common such conjoining is, but under what circumstances do children understand it to be correct? Parts a), d) and e) of question 1 explored the difference between language based questions - 3g add 4 - and symbolic questions - 2 + 3c - as well as the effect of explicit versus implicit multiplication in 3 + 2xm and 3 + 2m, where the interviews had shown mainly the use of the former to mean (3 + 2)xm by some pupils. It seemed from the test results that some pupils were using conjoining in addition because they had misunderstood the notation to mean just that. Part f) of the question investigated the extent of this misconception.

c) Notation - Part b) was included to test the understanding of the equivalence of use of operators which overlap from arithmetic into algebra. The understanding of division has been shown to be a particular problem for children (e.g. Slesnick, 1982), especially when it involves division of a smaller number by a larger one (e.g. Bell et al., 1981, Bell et al., 1984). The tape-recording of some of the pupils during the computer work had illuminated the type of misunderstanding which exists in this area, even on the fundamental concepts, and Figure 10.1, which is taken from the transcript of the recording, demonstrates the lack of understanding in connecting fractions, for example, and division. Here three pupils, labelled A, B and C, are involved in a discussion while using the computer, and their difficulties with division are exemplified by statements or comments such as 'you

B. y divided by x is ... no, I'm putting the sum, as well.

A. Yes, I have. You haven't put the answers.

B. y divided by x ...

A. What is it?

B. That'll be 0.2.

A. No, it's 2.

B. y is divided by ... yes, 2. I always do that.

A. It's because you're back to front.

B. This is terrible. I wish it were back to front.

A. Well is it? It's 5, divided by 10.

B. If it were 10 divided by 5 ... No, it's not, it's 0.2 or something.

A. No it's not. Look, do it on the thingy. Print.

B. Print. Yes. 5 divided by ...

A. No, that won't work.

B. Not 5. 5 divided by 10.

A. It won't work.

C. 1.5

B. 5 divided by 10 ... 25 .... 3 times x plus 5 ... x plus ... just put ...

A. x plus ... that's 15. Three fifteens ...

B. Just put ten plus five ...

C. Well, what is three divided by four?

B. You can't have three divided by four!

C. Yes you can!

B. One, and about two hundred million (*sounds like*) little bits.

C. Hey, Chris, what's three divided by four?

A. It's a number!

C. Be serious!

A. Minus 1 ...

B. It's one minus one. It's minus one, one.

A. Dot. Four into thirty goes ... seven (general agreement on 7). Two left over. Four into twenty goes ... five. Seventy five.

B. I knew that!

A. Except it's seven point five ... nought point seven five. Three-quarters, isn't it?

B. Nought point seven five.

Figure 10.1

Two Discussions on Division Taken From The Transcript of the Pupils' Conversations During the Third Experiment can't have three divided by four', or 'it's minus 1', for 3 divided by 4, and the attempt to avoid the problem with 5 divided by 10 by saying 'I wish it were back to front' and 'If it were 10 divided by 5'.

Question 2 on the questionnaire again was prepared in a form which encouraged the writing of explanations for the pupils understanding. They were asked to explain to a visitor from outer space the meaning of some algebraic (and one arithmetic) symbols. The purpose of this was again to investigate their perceptions of the notation, including misunderstandings, and any differential nature of these relative to the two courses of instruction. It was hoped that the improved conceptual understanding of algebra described in Chapter 9 would have given the experimental pupils an improved view of notation which would show through in their answers as relational explanations rather than instrumental ones. Finally it was considered that there might arise from the children's explanations of the notation some evidence for the power and use of imagery in the way described in the cognitive theory in Chapter 3.

Questions 3 and 5 of the questionnaire involved the conceptually difficult areas of elementary algebra of linear equations with letters on both sides of the equation (see Chapter 4) and linear inequalities. The aim was to see which of the groups would find the extension of their cognitive structures into these conceptually more demanding areas easier. This is an important consideration since some (e.g. Herscovics and Kieran 1980, Vergnaud and Cortes 1986) have expressed the view that it is only at this level of manipulation that one is really involved in algebraic thought. Similarly question 4

Question	Experimental Proportion	Control Proportion	z	p
1a)	0.84	0.77	0.91	ns
1b)	0.76	0.44	3.38	<0.0005
1c)	0.61	0.69	0.82	ns
1d)	0.47	0.42	0.54	ns
1e)	0.41	0.31	1.24	ns
1f)	0.76	0.69	1.20	ns
1g)	0.57	0.31	2.69	<0.005
3a)	0.31	0.29	0.24	ns
3b)	0.31	0.27	0.47	ns
3c)	0.39	0.33	0.61	ns
3d)	0.43	0.27	1.83	<0.05
4a)	0.39	0.38	0.18	ns
4b)	0.24	0.08	2.16	<0.025
5	0.31	0.13	2.37	<0.01

Table 10.1

A Comparison of Some Experimental and Control Matched Pair Proportions On The Questionnaire involved two difficult manipulation type questions: (2m - n) + 3n and 5h - (3g + 2h); with the added complication of the use of brackets, incorporating the type of cognitive conflict arising from these, and described above.

10.2 Quantitative Results From the Questionnaire

The proportions of experimental and control pupils from the matched pairs giving correct responses to the questions where there was a correct answer (1, 3, 4 and 5) are given in Table 10.1. The results show that in all but one part, question 1c) (considered later), the experimental group performed better than the control group, significantly so in 5 cases. Of these, the result of question 1g) shows that the experimental group had a significantly better understanding of the meaning of brackets in this question, involving the equivalence of 2(a + b) and 2a + 2b, than the control group, with a significantly higher proportion of them understanding this. The understanding of the use of brackets in algebra has been found to be resistant to teaching methods (e.g. Booth, 1984) and so this result confirms the value of the conceptual approach over the skill-based one for this type of conceptually difficult question.

Similarly, the results of the experimental group on questions 3d), 4b) and 5, demonstrate their significantly better performance, compared with the control group on these new, conceptually difficult questions. These are important results since they show that, when both groups were asked to attempt questions of a conceptual difficulty outside the scope of what they had been taught thus far, such as solving $13 - y = 2y + 7$ and $17 - 3e < 2$ and simplifying 5h - (3g + 2h), the conceptual base of the computer pupils proved

Question	Explanation	Experimental Number Giving	Control Number Giving
Explain the notation 4/5	4 5	10	10
	4 fifths	1	5
	4 out of 5	9	12
	4 divided by 5	5	1
	4 parts of 5 parts	0	1
	0.8	1	0
	a fraction	3	4
	4 5	0	1
	4 but not 5	11	8
	between 4 and 5	0	2
	not the intersection of 4 and 5	1	0
	factors of 4 but not 5	1	0
	4 is the same as 5	1	1
	4 is not the same as 5	0	1
	how many 4´s in 5	0	1
	4 into 5	0	1
	4 + 5	1	0
	4 more than 5	1	0
	4 and 5	1	0
	4 and then 5	1	0
	4 - 5	3	0
	subtract 4 from 5	1	0
	4 x 5	2	1
	4 lots of 5	0	1
	4 or 5	2	2
	from 4 to 5	1	1

Figure 10.2

An Analysis of The Answers Given to Question 2a)
Of the Questionnaire By all Pupils Attempting It

Question	Explanation	Experimental Number Giving	Control Number Giving
Explain the notation a3	ax3	37	27
	3 lots of a	3	1
	3xa	3	0
	multiply a by 3	2	3
	a multiplied three times	1	0
	a times 3	4	2
	3a	0	2
	a lots of 3	0	1
	a number times 3	0	2
	any number times 3	4	6
	a´s value multiplied by 3	2	1
	3 lots of anything	0	1
	mystery number with a number	1	0
	a for the value of 3	1	0
	another number before 3	0	1
	three of the symbols a	0	1
	a + 3	8	5
	a + 2a	1	0
	any given number added to 3	0	1
	add them up	0	1
	1 + 3	0	1
	1 x 3	0	1
	a size of paper	0	2

Figure 10.3

An Analysis of The Answers Given to Question 2b) Of the Questionnaire By all Pupils Attempting It

Question	Explanation	Experimental Number Giving	Control Number Giving
Explain the notation 3 + m	m + 3	1	2
	3 plus m	3	0
	add 3 to m	2	5
	add m to 3	2	1
	3 add m	8	3
	m plus 3	0	1
	add the value of m to 3	3	0
	3 plus what m equals	13	8
	add 3 to m´s value	1	1
	3 plus a mystery number	1	0
	add a certain number to 3	0	1
	3 plus a number we call m	0	1
	3 plus a letter which means a number	1	0
	3 add any number	6	6
	3 more than m	1	1
	3 plus whatever number the key says	0	1
	3 plus any whole number	0	2
	3 + 1m	1	0
	3 added to letter m	1	1
	3m	13	14
	4m	1	0
	3 lots of m	0	1
	a + m	1	0
	4	2	1
	3 of one thing and one m	0	1
	3 m´s add m	1	0
	3 somethings plus an m	2	0
	find what m means	1	0

Figure 10.4

An Analysis of The Answers Given to Question 2c) Of the Questionnaire By all Pupils Attempting It

Question	Explanation	Experimental Number Giving	Control Number Giving
Explain the notation b - 2xc	minus 2c from b	0	1
	b - 2c	5	1
	2xc and take the answer from b	5	0
	2 lots of any number from any number but c	1	0
	b minus the answer to 2 lots of c	1	0
	2c - b	0	1
	2 times c then take away b	2	0
	2xc - b	0	1
	2 times what c equals take away b	0	1
	b minus 2 times c	7	6
	what b equals minus 2 times what c equals	6	5
	b minus 2 multiplied by c	3	1
	value of b - 2 and x by c	0	1
	certain number - 2 multiplied by another number	1	0
	any number minus 2 multiplied by any number	7	4
	calculation as (b - 2)c	2	4
	(b - 2)xc	0	1
	unknown - 2 x unknown	1	0
	any number minus 2 times a different number	1	0
	value of b minus 2 multiplied by value of c	2	0
	letter b minus 2 times letter c	1	1
	b - 2xc = 0	1	0
	1 - 2x1	1	0
	bxc - 2	1	1
	multiply b by c and take away 2	0	1
	-1bc	0	1
	2 parts of b x c	0	1
	add 2 to b then times by c	1	0

Figure 10.5

An Analysis of The Answers Given to Question 2d) Of the Questionnaire By all Pupils Attempting It

Question	Explanation	Experimental Number Giving	Control Number Giving
Explain the notation ab	axb	26	26
	a times b	6	1
	a multiplied by b	7	4
	a lots of b	1	0
	b lots of a	1	0
	times a by b	0	1
	a certain number times another number	0	1
	value of a times b	0	1
	whatever a equals multiplied by whatever b equals	7	2
	any 2 numbers timesed together	4	3
	any number times a different number	2	0
	any whole number multiplied by another whole number	0	1
	letter a multiplied by letter b	1	1
	two mystery numbers	1	0
	whatever a equals and b put together	1	0
	one a and one b	0	1
	a + b	6	8
	add a and b together	0	1
	sections a and b	0	1
	1x1 = 1	0	1
	first letters of the alphabet	0	1

Figure 10.6

An Analysis of The Answers Given to Question 2e)

Of the Questionnaire By all Pupils Attempting It

Question	Explanation	Experimental Number Giving	Control Number Giving
Explain the notation : 2c - 1 = 3	c = 2	16	18
	c = 3	0	1
	c = 4	3	3
	c = 1.5	1	0
	c is 2	1	0
	2c = 4	0	2
	2c = 2	0	1
	c contains 2	1	0
	c stands for 1	1	0
	c stands for 2	3	0
	c could be anything but it is 2 in this case	1	0
	you have to find the value of c	0	1
	2c minus 1 equals 3	0	2
	2xc - 1 = 3	9	4
	2 times a number - 1 = 3	0	4
	2 times unknown - 1 = 3	1	0
	2 multiplied by c minus 1 equals 3	4	2
	2 lots of number - 1 = 3	2	0
	2 times what c represents - 1 = 3	2	0
	2 times any number - 1 = 3	1	0
	any number multiplied by 2 - 1 must equal 3	1	0
	2 times letter c minus 1 = 3	0	2
	2x2 - 1 = 3	3	0
	Work out the value of c	2	0
	2 + 1 - c	0	1

Figure 10.7

An Analysis of The Answers Given to Question 2f) Of the Questionnaire By all Pupils Attempting It superior to that developed by the other group in its extensibility to encompass the understanding of such questions. This, I would suggest, is an indication of their superior relational understanding, since this type of understanding should be more open to extension than instrumental, which is more likely to require the input of rules to enable expansion of schemas.

The significantly better results of the experimental group on the type of questions described above indicates then that the method of instruction which they had received, using the computer Dynamic Algebra module as an advance organiser for the skill-based work, had better equipped them for a future study of algebraic simplification, equations and inequalities than the skill-based module alone had done for the control pupils. A study of the long term effects of the programme on such understanding will no doubt show this to be the case, as it did for the pupils involved in the second experiment.

10.3 A Qualitative Analysis of the Questionnaire Answers

10.3.1 The Relationship to the Cognitive Theory

Looking at question 2, which required pupils to explain their understanding of the symbolic imagery presented in each part of the question, Figures 10.2 to 10.7 provide an analysis of the explanations given. One striking feature of the figures is the diversity of response by the children to common symbolisations such as 4/5, ab, 3 + m etc. and the variety of ways in which language impinges on understanding. This is, I would claim, evidence not only for the individual nature of mental schemas, but also for some of the theoretical proposals in Chapter 3. For example it seems that :

a) The image of / has evoked in the minds of the pupils many differing concepts, many of which are linked to other imaginal concepts, such as , x, +, and - (in); a3 is explained as ax3, a + 3, 3xa; 3 + m as m + 3, 3m, 4m etc. and ab as axb, a + b. These explanations are often given purely in terms of another symbolism, or image, by many pupils even though their path to the goal state may have passed through higher level concepts. This indicates the schematic nature of such imagery as described in Chapter 3.

b) There is here too, I think, some evidence of the varying strength factors of schematic links. For 23 of the 147 pupils the symbol / was strongly linked in their schemas to set theoretic concepts such as in one set but not in the other, and 5 different such explanations involving sets were given (see Figure 10.2). Since this type of module had recently been completed before the questionnaire was given it shows that time is a variable in the strength factor of schematic links and that such strong links from the image (/ here) may even over-ride other concepts and cause inappropriate application of the concept. In this case for example the set theory ideas were still applied for / even though in all its previous usage the symbol would have occured between letters, representing sets, rather than numbers. This power of an image to evoke an inappropriate application of a concept is also seen in the answer given by two pupils in the control group to question 2b), involving the symbol a3. They both replied that this represented a size of paper. It shows that this link in their mental schemas is so strong that it over-rules the algebraic/mathematical nature of the context in which the question is put (i.e. the questionnaire) to return a correct, but inappropriate explanation of the imagery. Kieran (1981b) has described this phenomenon in respect of the power of the image ´=´ to evoke the concept of ´solve´ on the part of children. Question 2f) confirmed the high strength factor of this linking when 25 experimental group pupils (32.1%) and 21 controls (30.4%) responded to the equation symbolism by solving it and giving the solution as their explanation, 18 of the former and 14 of the latter group correctly giving c = 2.

c) There is some evidence in the pupil responses too for the directional nature of conceptual linking in schemas. Figures 10.8 and 10.9 show the explanations given by two pupils in question 2 of the questionnaire. It can be seen in each case that their conception of addition in algebra is such that an expression such as 3 + m evokes the expression 3m as equivalent. However, working in the reverse direction, from letters which are already juxtapositioned, the expression ab evokes, not a + b, but axb. Thus the linking here appears to be uni-directional rather than bi-directional. Thus in one direction the pupils have a correct understanding but in the other a common misunderstanding. This may be compared with Figure 10.10 where the answers of Nick

2. Imagine that a visitor from outer space has asked you to explain what the following Mathematics symbols mean. What would you say to it? Write your explanation in the space provided underneath each part.

a) 4/5

Well...you see the line means divide so...you would put it a nother way as 4÷5

b) a3

This one...you have added together so to get a m... front of 3...the sum would be...a+3.

c) 3 + m

To get...the answer to this one...you add it together...so you could get a smaller answer on

d) b - 2 × c

It's nearly the same as the bottom because then you have taken 2 away from then xc you get k because the k stands for 3

e) ab

Well...it...means...a x b...and when comes together...you...get ab...................

f) 2c - 1 = 3

This one is that to get to 3 the c must stand for 1 so when take one you take the c away

3. Can you say for which values of y the following equations are true? Try to explain your working in the space provided.

a) 3y - 5 = 2y + 1

yes because when...
taking 2 from five
you get 2 and they...
stand for 1.......... X

Figure 10.8

An Example Of A Pupil's Answers To the Questionnaire Showing a Misunderstanding Of Addition in Algebra

2. Imagine that a visitor from outer space has asked you to explain what the following Mathematics symbols mean. What would you say to it? Write your explanation in the space provided underneath each part.

a) 4/5

This symbol means in set A But not in set B.

b) a3

This means ax3 But it has been shorted to a3

c) 3 + m

This means 3m.

d) b - 2 × c

e) ab

This means axb. But it has been Shorted to ab

f) 2c - 1 = 3

3. Can you say for which values of y the following equations are true? Try to explain your working in the space provided.

a) 3y - 5 = 2y + 1

Figure 10.9

An Example Of A Pupil's Answers To the Questionnaire Showing a Misunderstanding Of Addition in Algebra

2. Imagine that a visitor from outer space has asked you to explain what the following Mathematics symbols mean. What would you say to it? Write your explanation in the space provided underneath each part.

a) 4/5

....what...is...in-between...four...and..
...five,...answer = Ø : empty...set..

b) a3

....a3 = a + 3 ...which ...would....
....equal...3a...

c) 3 + m

....3+m...would..mean...add....together
....to....give = 3m..................................

d) b - 2 × c

...b-2×c...means (2×c)-b...to...get...
..................answer,...find what...c+b are

e) ab

.....ab....means......a+b....,...find...out
...............what....numbers....a+b...represent

f) 2c - 1 = 3

...2c-1=3...;...2+c-1=3,...c=2.......

....

3. Can you say for which values of y the following equations are true? Try to explain your working in the space provided.

a) 3y - 5 = 2y + 1

....y=8..,...true.....
.....equation...y... ✗
..represents....the..
...number....eight.

Figure 10.10

An Example Of A Pupil's Answers To the Questionnaire Showing a Misunderstanding Of Addition in Algebra

1. A girl wrote the following in a Mathematics test at her school. Write underneath each part in the space provided whether she was right or not and explain why you so answer.

a) 3 + 2m is the same as 3 + 2 × m

✓ because 2×m = 2m ✓

b) $\frac{6}{7}$ is the same as 6÷7

× because $\frac{6}{7}$ is 6 out of 7 and 6÷7 is 6 divided by 7 ×

c) (n + 3) + 6 is the same as n + 9

✓ ✓

d) 3g add 4 equals 7g

× because g could be anything ✓

e) 2 + 3c is the same as 5c

× because c could be anything ✓

f) 2n is the same as n + 2

× " n " " ✓

g) 2(a + b) is the same as 2a + 2b

× " a and B " " " ×

Figure 10.11

An Example Of A Pupil's Answers To the Questionnaire Showing a Misunderstanding Of Division reveal that the misunderstanding of 3 + m as meaning 3m is complemented by the corresponding understanding that ab represents a + b. Here then the linking appears to be bi-directional in nature, although founded upon the erroneous understanding that added to means placed next to rather than a mathematical operation. A second example of this phenomenon is given in Figure 10.11 where, in explaining why, in his view, $\frac{6}{7}$ and 6÷ 7 are not the same Alan says that the former means 6 out of 7 whereas the latter is 6 divided by 7, and for him these are not equivalent. This conflict between the meaning of words in language and their usage in mathematics and the linking in schemas of words with specific concepts only is an area for careful consideration and research since it seems that often the understanding of the use of common words in, say English, by pupils in the classroom may be different from that understood by the teacher who has the experience and schemas to discriminate between the meanings and to link equivalent formulations. Comparing questions 1d) and 1e) to see if the use of the word add instead of the symbol + made any difference to the conjoining in addition, the results showed that there was no significant difference between them, with 46.3% and 40.1% respectively of the pupils correctly replying to these questions. Similarly there was no significant difference between the proportions in each group correctly responding to these two parts.

d) The results of question 1d), involving an

b) $\frac{6}{7}$ is the same as $6 \div 7$

No because 6/7 is a fraction and if you divided 6÷7 it would be a decimal. It would be differ ×

No it is not correct because $\frac{6}{7}$ is a fraction. 6÷7 is a sum ×

She is wrong because $\frac{6}{7}$ is a fraction and not a sum ×

Figure 10.12

An Example Of A Pupil's Answers To the Questionnaire Showing a Failure to Connect Fractions With Division explanation of whether $\frac{6}{7}$ and $6 \div 7$ are equivalent or not, provided a surprisingly strong example of the power of an image, through a link of high strength factor to evoke a higher order concept which prevents, rather than improving, understanding. Of the 147 pupils given the questionnaire, 22.4% of them did not consider them to be equivalent, because, as they explained it, the former is ´a fraction´ or ´6 sevenths´ but the latter is ´a sum´ or ´a divided by´ (see Figure 10.12). Thus a high proportion of children at this age see $\frac{6}{7}$ as an indivisible whole, an entity, propogated by the holistic mode of processing which very strongly evokes the concept of fraction for this image. However, for the imagery $6 \div 7$, many pupils process this sequentially, since in their schemas the image is strongly linked to the concept of a process, or ´doing a sum´, rather than an entity, and they view the symbolism in three distinct parts, as 6 (value), divided by (process or operation) and 7 (second value). What these pupils lack is the versatility, described in Chapter 3, which would enable them to view either or both of these symbolisms or images in a global/holistic or a serialistic/sequential way. Looking at the differences between the two groups however, there is some evidence that the work on the computer had had a beneficial effect on this situation, as predicted by the theory previously described. The proportion of experimental pupils in the matched pairs displaying this misunderstanding was 11.8%, significantly lower (z

= 2.77, $p<0.005$) than the 35.4% of the control pupils who exhibited it. One of the reasons for this versatility which appears to have come from the computer work in this area is probably that, when entering a fraction such as into the computer one has to enter it sequentially as 6/7, i.e. as number, divided by, number. Furthermore, once it has been entered and is displayed by the computer it is repesented as a decimal value. This combination of the processes in which the pupil is closely involved, and the imagery which is constantly present, is a powerful force in the production of versatility. Turning again to the discussions from the programme reproduced in Figure 10.1, there is a clear example of the group starting from a low level of understanding about the sum three divided by four and, using a combination of arithmetic and group discussion around the computer, arriving at the equivalence of 0.75, three-quarters and three divided by four. The value of using group discussion in a computer problem-solving environment in this way to encourage mathematical understanding agrees with the conclusions of Bell (1979) :

> "The key concepts in the three areas we have studied are, in problem solving the ability to maintain the tension between productive and reflective modes of proof, the need for a group in which to communicate and defend newly discovered propositions, and in mathematisation,

Question and Answer	Experimental Proportion	Control Proportion	z	p
2c) 3 + m = 3m	0.08	0.27	2.54	<0.01
2a) ab = a + b	0.06	0.23	1.71	<0.05
2b) a3 = a + 3	0.04	0.06	0.53	ns
2d) b - 2xc = (b - 2)c	0.09	0.23	1.77	<0.05
2d) b - 2xc = b - 2c	0.18	0.02	2.57	<0.01
1a) 3 + 2m = 3 + 2xm = 5m	0.04	0.13	1.57	ns
1a) 3 + 2xm = 5m	0.02	0.02	0.04	ns

Table 10.2

A Comparison of Some Experimental and Control Matched Pair Proportions On Questionnaire Answers the need for material which promotes work which inter-relates the particular and the general and the situation and its representation."

(Bell, 1979, p.385)

Although this understanding relative to division is a benefit in the arithmetic scheme rather than the algebraic, the use of the same operators and symbolisms in algebra as in arithmetic make it important that children understand these before they will be able to make a success of algebra. The computer paradigm has much to offer in this area by its promotion of versatility through the cognitive integration I have described.

10.3.2 The Extent of Some Misunderstandings

Table 10.2 gives an analysis of the results respecting some of the common misconceptions for those pupils in the matched pairs, so that the effect of the programme on these could be ascertained, and others which had emerged from the algebra test answers and the interviews. The implications of these will now be discussed.

a) Conjoining in Addition

Overall, amongst all the pupils, 18% understood 3m to be another form of 3 + m, 10% thought that ab and a + b were equivalent, 9% that a3 and a + 3 were and 8% considered both 3 + 2m and 3 + 2xm to be forms of 5m. These results seem to confirm that the misunderstandings associated with conjoining in algebra are indeed widespread among pupils of this age (13 years).

Looking at the effect of the computer module on the understanding related to the error of conjoining in addition in algebra, Table 10.2 shows that significantly fewer of the experimental group understood 3 + m to be equivalent to 3m than did the control pupils, and there were also fewer of each of the other misunderstandings present in this group. These are encouraging results relative to the benefits of the programme on such misunderstandings, but the extent of the problem is such that there is still much to be done to avoid it.

b) Non-Conventional Use of Notation

The interviews had uncovered a misunderstanding relating to the way in which some pupils had been using notation in a non-conventional way. They had used expressions such as b - 2xc to stand for (b - 2)c, the use of the explicit multiplication signifying to them implied brackets, so that b - 2c was taken as a different expression. This was included in the questionnaire to see how common such a use was. In the event 13.6% of the pupils clearly understood b - 2xc to be (b - 2)xc (the explanation of some did not make clear their interpretation) whereas only 1.4% explained that 3 + 2m and 3 + 2xm were not the same because the latter equalled 5m but the former did not and only 2.0% gave a + bx2 as another form of 2(a +b) in 1g). These results and those in the interviews appear somewhat contradictory then, and it may be that the misunderstanding only comes into focus when pupils are asked to evaluate expressions, as they were in the interview, it being primarily a product of their arithmetic schemas rather than their algebraic ones. Table 10.2 shows that the effect of the programme on this mis-understanding was such that significantly fewer of the experimental pupils treated b - 2xc as (b - 2)xc and that further, the proportion of them correctly understanding it to mean b - 2c was also significantly higher than that of the control group. Thus again, the computer group pupils were performing at a level of understanding significantly better than those who had not had the benefit of the programme.

c) The Use of Brackets

Question 1c) of the questionnaire turned out to be the only question on which a higher proportion of control pupils than experimental pupils gave the correct answer, 69% compared with 61%. This question, as to whether (n + 3) + 6 is the same as n + 9 and why, was one of two questions of this type involving brackets, the other being 1g). Careful analysis of the reasons given by the children for their answers revealed the differential nature of their conceptual understanding of these. Most of the control group pupils gave as their reason for the equivalence of the two expressions the fact that 3 + 6 = 9, with no mention of the brackets at all. This agrees with both Kieran (1979) and Booth (1984) who record that most children of this age ignore the use of brackets, mainly because they find them unnecessary or lacking meaning. In contrast to this the experimental group had formed a strong conceptual understanding of the use of brackets as marking what has to be done first, to the extent that 29.4% of them gave this as a reason for the lack of equivalence of the two expressions, only 4.2% of the controls doing so. Although at first sight it might appear that their limited conceptual understanding of the use of brackets had been a hindrance, producing a detrimental effect on their answers, I suggest that the cognitive conflict resulting from this is more likely to lead to a proper understanding of the use of brackets

c) (n + 3) + 6 is the same as n + 9

Yes because the second defenition is a short way of writing the first ✓

g) 2(a + b) is the same as 2a + 2b

Yes this is always the same ✓

c) (n + 3) + 6 is the same as n + 9

False because [illegible] ✗

the same [illegible]

g) 2(a + b) is the same as 2a + 2b

True because [illegible] ✓

[illegible]

c) (n + 3) + 6 is the same as n + 9

~~No I dont think~~ She's right here because if n = 3 in the 1st one the answer would be 12 the same as the [illegible] ✓

g) 2(a + b) is the same as 2a + 2b

Yes I think she's right here because if a = 1 and b = 2 you get the same answer for both. ✓

Figure 10.13

An Example Of A Pupil's Answers To the Questionnaire Showing Elementary Checking Procedures in Questions With Brackets in algebra than the approach which just ignores them because they lack any meaning. One of the ways in which pupils were noticed to be working out a solution to their conflict in the experimental group was through the use of numerical substitution for the variables as an elementary checking procedure. Figure 10.13 shows the papers of pupils who have used this method to arrive at the conclusion that they are equivalent, with one of them stating that they understand this to be so for all values (although he had not proved this of course). Further evidence to support the hypothesis that the experimental group pupils will have a better understanding of the use of brackets is provided by the results to question 1g), mentioned earlier. Here a significantly higher proportion ($z = 2.69$, $p<0.005$) of the computer group children (57%) than the controls (31%) understood $2(a + b)$ to be equivalent to $2a + 2b$. Furthermore, in this question 7.8% of the experimental pupils and 10.3% of the controls ($z = 0.77$, no significant difference) considered the expressions not to be equivalent because the bracket had to be done first. This tends to indicate that, for the computer group, the presence of two variables in the bracket, rather than only one in the $n + 3$, tended to move them away from this idea and this would also perhaps support the idea that their conceptual understanding in algebraic terms is benefiting.

10.3.3 Evidence of Understanding of the Use of Letters

The occurence of answers to question 2 which showed an understanding of the use of the letters in the notation as standing for specific unknown or generalised numbers was analysed. Figures 10.2 to 10.7 show that a considerable number of pupils gave explanations which referred to 'adding a certain number to 3′ or ′a′s value multiplied by 3′, indicating that they viewed the letter as specific unknown. Others included descriptions such as ′any number times 3′, ′any two numbers timesed together′ and ′3 add any number′ giving evidence of understanding of generalised number. The only significant difference between the two groups of pupils in these areas was on question 2d), where, in explaining the meaning of b - 2xc, a significantly higher proportion of experimental pupils (17.6%) compared with the controls (2.1%) used expressions which implied an understanding of generalised number ($z = 2.57$, $p<0.01$). Another problem, worthy of mention by its absence, is the view which some pupils have that different letters must take different values (e.g. Wagner, 1977). On question 2 parts d) and e) there was an opportunity to display such a misunderstanding. In the event, only 3 experimental pupils, and no controls did so on part d), and 2 experimental pupils and 1 control on part e), showing that although it exists, it did not seem widespread. It is true that the form of the questions asked was such that many may have given an explanation of the notational symbols without feeling the need to refer to their understanding of the letters involved, prefering to concentrate on other factors such as the operations involved and their order (as many did). In view of this care must be taken when interpreting the above results on the understanding of the use of letters.

10.4 Overview

This chapter has considered the form and content of the questionnaire given to all the pupils involved in the third experiment. The results of the questionnaire have been analysed and they have provided evidence of the beneficial effect of the programme on mis-understandings of notation and advance preparation for conceptually more difficult areas of secondary school algebra, including the conceptually harder linear equations and inequalities. The benefits have further highlighted the power of mental imagery in promoting versatility in mathematics through its underpinning of, and linking with, the higher level cognitive schemas. The results have also supported the cognitive theory advanced in Chapter 3, including the directional nature of such cognitive links and the way in which links of high strength factor may evoke inappropriate higher level concepts and hence lead to error in mathematics. The computer environment used as the basis for the programme presented here has again been shown, through the questionnaire results, to have great value in the learning of mathematical concepts by providing an environment in which the building and testing of both lower and higher level concepts may be encouraged, leading to a versatile learner of mathematics.

Chapter 11

Implications for Teaching and Recommendations For Further Research

In this chapter I shall review the main results of the research, alongside the comments made by the teachers concerning its pedagogical suitability and describe the implications for the teaching of algebra in secondary schools resulting from these. The research has also highlighted areas where future research would be beneficial and so recommendations as to research based on the present study are given.

11.1 An Overview of the Results

The results of the experiments comprising this research have clearly demonstrated the following points relative to using the computer paradigm as an approach to the conceptual learning of algebra in the secondary school mathematics classroom :

a) The use of a module of work, such as the Dynamic Algebra module used here, carefully designed to make full use of the benefits of the visual and problem-solving nature of the computer environment, before any formal teaching of algebra in a skill-based way produced in the early learner of algebra a significantly better performance in algebra than the traditional skill-based instruction alone. Moreover this better better performance was most marked in those areas of algebra requiring conceptual understanding. The improved conceptual understanding gained was not in any way marred by a corresponding loss of lower level manipulative skills, but rather there was no difference between the two groups in this area. When used in this way the computer module acts as an advance organiser or cognitive bridge in the sense of Ausubel (1968) or a generic organiser, after Tall (1986), for the concepts of algebra.

b) One important feature of the better conceptual understanding produced by the programme was in the understanding of the use of letters in algebra. Those pupils who had followed the computer module before more formal algebra teaching ended up with a significantly better understanding of both letter as specific unknown, and letter as generalised number or variable. This improved understanding of specific unknown also resulted in a decrease in the occurrence of conjoining errors in addition due to a failure to accept lack of closure in the operation (Collis, 1975).

c) The longer term nature of the improved conceptual understanding was demonstrated in that one group of pupils was still performing significantly better than their corresponding group over one year after the experiment. This is a feature of relational understanding, in that it is less likely to decay over time than instrumental understanding, and so tends to indicate that it was this type of understanding which the programme had encouraged.

d) The nature of the relational understanding developed by the computer approach was such that the pupils were in a better position to extend their knowledge frameworks to include an understanding of other parts of algebra, such as equations with variables on both sides, inequalities and more difficult simplification. This was a result of the powerful insights gained by the pupils who used the conceptually rich computer environment, enabling the construction of high strength factor conceptual links, in comparison with the greater difficulty in constructing such schemas from the more traditional approach.

e) The computer programming, combined with the use of the computer as a mathematical microworld (Thompson, 1985b) and the re-inforcement provided by the cardboard 'Maths.Machines', had encouraged, through the promotion of visual imagery, a global/holistic view of algebraic concepts and symbolisms which, when combined with the serialistic/analytical approach, produced a more versatile algebra student than the traditional serialist approach alone. This student is in a position, through the cognitive integration (see Chapter 3) developed in his/her schemas to alter his/her viewpoint from one to the other as and when appropriate. This seems to be a vital constituent of success in mathematics in general, and it was seen to be improved by the methodology adopted in this research.

11.1.1 Some Pedagogical Considerations

I have stated previously that there are at least two important aspects to the synthesis of a successful teaching programme. The programme should accomplish its cognitive aims but it should also be pedagogically suitable. In previous chapters I have indicated the success or otherwise of the present programme in the classroom, and I include here a summary of this aspect.

The module of work has now been used by at least six teachers from differing backgrounds and in different types of schools. These range from the top class of a middle school to the first year of a 12+ entry comprehensive secondary school, and the teachers have included those with no experience at all

The lessons proved popular with pupils and I, too, found the work surprisingly pleasant. The novelty was, for many, stimulating and a change from 'normal' lessons.

On the whole, I feel the computational algebra was successful, quite enjoyable,

I/We certainly feel that they have a different (more correct) feel for the nature of a variable than a traditional introduction would have produced

Many pupils thoroughly enjoyed working through the programs of the kind on sheets 2 -> 4.

They were eager from the outset and did not wane in any way right through to the last session.

This was a very worthwhile project which proved to be very pupil orientated, allowing the teacher to 'survey' the scene and help sort out any minor irritations experienced by any of the groups. It was also nice to circulate around the room and listen in to the reasoning between group members.

Figure 11.1

Some Teachers' Comments on the Success of The Programme of using computers in the mathematics classroom (three) to those with considerable experience of such work (one). There has been throughout a basic agreement that the module has been beneficial and interesting to the pupils, and even to most of the staff and that, with the necessary preparation (e.g. in the making of the 'Maths.Machines' and the logistics of the computer availability), works very well in the classroom. Figure 11.1 gives a selection of the comments of some of the teachers summarising their feelings about the success of the module in the classroom. Although, as may be seen, many of the pupils thoroughly enjoyed the work and were motivated by it, there was an element of loss of interest after a while on the part of some in the main experiment, usually the more able of the 13 year-olds.

11.2 The Implications For The Teaching of Algebra

The major implication for the teaching of algebra in the secondary school arising from this research is, given that, in the light of technological advances such as symbolic manipulators (see Chapter 5), algebra is to remain an important part of the secondary school curriculum, then a programme such as that described herein should become more widely used in algebra teaching. The effect of this would be to improve both the conceptual understanding of early learners of algebra in the secondary school and to promote versatility in their cognitive frameworks which could have far reaching benefits.

At present children are introduced to algebra by a wide variety of different means, many of which are encouraging the formation of concepts, such as letter as object, which will cause conceptual problems later in algebra as well as producing the errors which accompany such understanding. Replacing the present situation by a system where algebra is introduced in the way described in this thesis would produce an environment in which children start with a view of letters as generalised number, standing for a range of values (only one of which might be displayed at any one time). This starting point has many advantages, and is the one recommended by the research of Booth (1983b, 1984) as well as the present research. Furthermore the computer environment provides structure for the use of the letters so that concepts connected with them may be built and tested and methods formalised. Combining all this with the promotion of a versatile approach to mathematics through the integration of a global/holistic view with a serialist one will circumvent many of the problems which children currently experience with algebra. The need for algebra in the secondary school will still exist in the future, with pupils still needing the basic concepts of algebra, and the implications of this research are that the computer-based approach is the way forward.

It may well be the case that, initially at least, this approach will require that a little more time be assigned to algebra than is presently done in many schools. However, the resulting benefits of such extra time are, I believe, so clear that such an investment in the future understanding of mathematics of so many pupils who now fail to gain such understanding at a very early age outweigh any disadvantage. Adopting the methods put forward in this research does not imply a turning against the acquisition of skills in algebra, but rather recognises the importance of a sound conceptual understanding on which to base such skills, so that they are relationally linked rather than instrumentally. In the semantic/syntactic dichotomy debate in algebra, this research has shown the value of building syntactically on the foundation of semantic understanding.

Such an introduction of the work would have widespread implications for teachers of mathematics in the secondary school, since the majority of them, according to the surveys carried out (e.g. Jones and Green 1986, Tall et al 1987, Stanhope 1987) have little experience of using the computer in mathematics. This should be seen as a positive benefit from introducing modules such as the one described in this research. One area they would have to consider is that of keeping the interest of their pupils. The intrinsic motivating ability of the computer is an important factor, but seems to have more effect on the pupils of 12 than it does at 13, the extra year making some difference. It may well be that the implication is that the best time to introduce the module, all factors considered, is at the first year of secondary school in many areas, where the pupils are 11+.

The main reason for the problem of loss of interest described above was probably that the module had been written in a format designed to allow a progression through the concepts at a speed appropriate to the average learner and in a way in which the average teacher who has little or no experience of computers in mathematics would appreciate. In view of this, it would seem prudent to recommend that the individual teacher might undertake a careful rationalisation of some parts of the material in the module, keeping the order and content intact, if he/she feels that this would be appropriate either to himself/herself or to the standard of his/her pupils.

11.3 Recommendations For Further Research

The present research has uncovered several areas where further research would be valuable, and these are presented below.

One of the aspects of any improvement in understanding which a teaching method purports to instil is the extent to which it decays over time. Concepts which have been fixed with relational understanding should stand the test of time better than those learned by rote, or with instrumental understanding. Therefore, it would be useful to have a long-term research study which would evaluate the success of the computer-based approach to algebra learning. The later algebraic development and understanding of pupils taught algebra in this way would be monitored throughout their school life and changes relating to the method of instruction analysed. There are many examples (see Chapter 4) of even the better mathematics students at college or university who display many of the error patterns and mis-understandings of younger pupils. Data describing the effect of the conceptual approach to algebra described here on these would be very valuable.

This research has shown, in agreement with Booth (1984), that some errors, for example involving the use of brackets, are resistant to change among pupils of this age. The reasons for this are still not totally clear, but it may be that there are powerful images at work in the minds of the children of the type described in Chapter 10, and predicted by the theory Cognitive Integration, which are set at a very young age.

cognitive imagery with its links of high strength factor may well be directing children away from an understanding of some aspects of algebra. What these images might be and the ways in which the situation could be improved would be subjects for further research. Another aspect of this identified in this research concerns the way in which the use and understanding of notation by children differs from that accepted by the mathematical community at large. An investigation of the reasons behind this with the possible aim of seeing if there is an imaginal factor in the notation which evokes the wrong conceptual areas could be carried out. Further research to discover how widespread these personalised notations, and the use of informal methods, are in algebra is still needed (c.f. Booth 1984) as well as investigations of other areas of mathematics where this may also be the case. This would assist in classifying the reasons for children's use and understanding of notation in mathematics and its relationship to cognitive theory.

The research in this study has indicated, through one of the results relating to gender, the possibility of differential factors based on gender in the understanding of algebra. It is by no means clear that this is the case and that it is not simply the result of classroom interactions. However, it has raised the possibility that the route to versatility in mathematics may be different for boys from that for girls and this is an area where, again, research could establish whether this is the case or not. Such research could be aimed at investigating the global/holistic and serialist/analytic abilities relating to each gender and directly comparing them in an attempt to measure objectively the contribution of each to understanding.

The theory of Cognitive Integration, based on the bi-modal model of learning described in Chapter 3 could, I believe, have far reaching implications for the teaching of mathematics. There is need of research to ascertain whether there is evidence which supports this model as the route to versatility, and hence success in mathematics, in other branches of the subject. Applying the cognitive model to these branches of mathematics is only the first step however, since it would then be necessary to use this knowledge to improve understanding for the average child. Here the evidence is that the computer has much to offer (see Chapter 5). Research which analyses areas of the mathematics curriculum for the best way to utilise the many benefits of the computer paradigm for promotion of the global/holistic understanding which may hold the key to relational conceptual understanding would need to follow. The eventual aim of such research would be the production of a curriculum for mathematics fully integrating the computer in the way which best promotes in the individual both the global/holistic and the serialist/analytical base for concept acquisition which will enable many more to attain the versatility of thought necessary for success in this subject.

The psychological model outlined in the research provides two other interesting recommendations for research. The first, in psychology, would involve the setting up of experiments to attempt to find evidence that one of the purposes of sleep is the re-construction of mental schemas. The research would look, particularly in the region of mental imagery, I suggest, for concept linkings which did not exist before sleep, but appeared after. The second area involves artificial intelligence. In order to build a machine which would be capable of this, it is probably best to model it on biological intelligence of the highest known form, namely humans. Basing such a machine on the model of cognition given in Chapter 3 would mean the construction of a computer which employed both serial and parallel processing, sometimes on the same data, with a control unit managing the flow of data and results. Research into the possibility of such a machine is indicated from this research.

In this chapter I have briefly outlined the main results of the research and the implications which they have for the teaching of algebra in the secondary mathematics classroom. The results and theoretical proposals have also given rise to other questions which require answers, and some suggestions as to possible productive avenues of future research have been briefly outlined.

BIBLIOGRAPHY

ALDERMAN D.L., SWINTON S.S., BRASWELL J.S., Assessing Basic Arithmetic Skills and Understanding Across Curricula : Computer Assisted Instruction and Compensatory Education, Journal of Children´s Mathematical Behaviour, 2(2), 1979, 3-28.

ALDERMAN D.L., Evaluation of the TICCIT Computer-Assisted Instructional System in the Community College, SIGCUE Bulletin, 13(3), 1979, 5-17.

AUSUBEL D.P., The Psychology of Meaningful Verbal Learning, Grune and Stratton, New York, 1963

AUSUBEL D.P., Educational Psychology - A Cognitive View, Holt, Reinhart and Winston Inc., 1968.

AUSUBEL D.P., NOVAK J.D. & HANESIAN H., Educational Psychology : A Cognitive View, Holt, Reinhart and Winston, New York, 1978.

BACKHOUSE J.K., Understanding School Mathematics - A Comment, Mathematics Teaching, 82, March 1978, 39-41.

BAJPAI A.C., FAIRLEY J.A., HARRISON M.C., MUSTOE L.R., WALKER D. & WHITFIELD A.H., Mathematics and The Micro : Some Hints on Software Development, International Journal of Mathematical Education in Science and Technology, 16(3), 1985, 407-412.

BALL D., What Do You Really Think of Logo?, Mathematics Teaching, 105, Dec. 1983, 38-39.

BARNES B.R. & CLAWSON E.U., Do Advance Organisers Facilitate Learning? Recommendations for Further Research Based on an Analysis of 32 Studies, Review of Educational Research, 45(4), 1975, 637-659.

BART W.M., The Factor Structure of Formal Operations, British Journal of Educational Psychology, 41, 1971, 70-77.

BELL A.W., The Learning of Process Aspects of Mathematics, Educational Studies in Mathematics, 10, 1979, 361-387.

BELL A.W. & O´BRIEN D., Solving Equations - A Teaching Experiment, Unpublished Working Paper, Shell Centre for Mathematical Education, University of Nottingham, England, 1981.

BELL A.W., O´BRIEN D. & SHIU C., Designing Teaching in the Light of Research on Understanding, Proceedings of the 4th IGPME Conference, Berkeley, California, 1980, 119-125.

BELL A.W., SWAN M. & TAYLOR G., Choice of Operations In Verbal Problems With Decimal Numbers, Educational Studies in Mathematics, 12, 1981, 399-420.

BELL A.W., FISCHBEIN E. & GREER B., Choice of Operation in Verbal Arithmetic Problems : The Effects of Number Size, Problem Structure and Context, Educational Studies in Mathematics, 15, 1984, 129-147.

BEVER T.G., Cerebral Lateralization, Cognitive Asymmetry and Human Consciousness, in E.Perecman (Ed.), Cognitive Processing in the Right Hemisphere, Academic Press, New York, 1983, 19-40.

BISHOP A., Visual Abilities and Mathematics Learning, Proceedings of the 3rd IGPME Conference, Warwick, 1979, 21-26.

BLACKETT N., Computer Graphics and Children's Understanding of Linear and Locally Linear Graphs, Unpublished M.Sc. Thesis, University of Warwick, 1987.

BLONSKII P.P., Selected Pedagogical Works, APN Press, Moscow, 1961.

BOILEAU A., KIERAN C. & GARANCON M., La Pensee Algorithmique dans L'Initiation A L'Algebre, Proceedings of the 11th IGPME Conference, Montreal, Canada, 1987, I, 183-189.

BOOTH L.R., Strategies and Errors in Generalised arithmetic, Proceedings of the 5th IGPME Conference, Grenoble, France, 1981, 140-146.

BOOTH L.R., A Diagnostic Teaching Programme In Elementary Algebra: Results and Implications, Proceedings of the 7th IGPME Conference, Shoresh, Israel, 1983, 307 -312.

BOOTH L.R., Misconceptions Leading To Error in Elementary Algebra (Generalised Arithmetic), Doctoral Dissertation, Chelsea College, London, 1983.

BOOTH L.R., Algebra: Children's Strategies and Errors. A Report of the Strategies and Errors in Secondary Mathematics Project, NFER-Nelson, Windsor, England, 1984.

BOOTH L.R., Equations Revisited, Proceedings of the 11th IGPME Conference, Montreal, Canada, 1987, Vol. 1, 282-288.

BORASI, R., Some Reflections On and Criticisms of the Principle of Learning Concepts by Abstraction, For the Learning of Mathematics 4, FLM Publishing Association, Montreal, 1984.

BOROD J.C., KOFF E. & CARON H.S., Right Hemisphere Specialization for the Expression and Appreciation of Emotion : A Focus on the Face, in E.Perecman (Ed.), Cognitive Processing in the Right Hemisphere, Academic Press, New York, 1983, 83-110.

BRAINERD C.J., Learning Research and Piagetian Theory, in S. Brainerd (Ed.), Alternatives to Piaget, Academic Press, New York, 1978.

BROPHY J. & HANNON P., On the Future of Microcomputers in the Classroom, The Journal of Mathematical Behaviour, 4, 1985, 47-67.

BROWN G. & DESFORGES C., Piagetian Psychology and Education: Time For Revision, British Journal of Educational Psychology, 47, 1977, 7-17.

BRUMBY M.N., Consistent Differences in Cognitive Styles Shown for Qualitative Biological Problem Solving, British Journal of Educational Psychology, 52, 1982, 244-257.

BRUNER J.S., GOODNOW J.J. & AUSTIN G.A., A Study of Thinking, J. Wiley & Sons, New York, 1956.

BRUNER J.S., The Process of Education, Harvard University Press, Cambridge, Mass., 1960.

BRYANT P.E. & TRABASCO T., Transitive Inferences and Memory in Young Children, Nature, 232, 1971, 456-458.

BRYDEN M.P. & LEY R.G., Right Hemispheric Involvement in Imagery and Affect, in E.Perecman (Ed.), Cognitive Processing in The Right Hemisphere, Academic Press, London, 1983, 111-123.

BUTLER S.R., Interhemispheric Transfer of Visual Information Via the Corpus Callosum and Anterior Commissure in the Monkey, in I.S. Russell, M.W. Van Hof & G. Berlucchi (Eds.), Structure and Function of Cerebral Commissures, MacMillan, London, 1979, 343-357.

BYE M.P., HARRISON B. & BRINDLEY S., Calgary Junior High School Mathematics Project : Final Report, Planning and Research, Alberta Education and the Calgary Board of Education, 1980.

BYERS V. & HERSCOVICS N., Understanding School Mathematics, Mathematics Teaching, 81, Dec. 1977, 24-27.

BYNUM T.W., THOMAS J.A. & WEITZ L.J., Truth Functional Logic in Formal Operational Thinking: Inhelder and Piaget´s Evidence, Developmental Psychology, 7, 1972, 129-132.

CHALOUH L. & HERSCOVICS N., From Letter Representing a Hidden Quantity to Letter Representing an Unknown Quantity, Proceedings of the 6th Annual Meeting of PME-NA, 1984, 71-76.

CHALOUH L. & HERSCOVICS N., Teaching Expressions in a Meaningful Way, in A. Coxford (Ed.), NCTM 1988 Yearbook : Algebra, Reston, Va, 1988, In Press.

CHURCHILL E.M., The Number Concepts of the Young Child, Leeds University Res. and Stud., 17 & 18, 1958, 34-39 & 28-46.

CLEMENT J., LOCHHEAD J. & SOLOWAY E., Positive Effects of Computer Programming on Students´ Understanding of Variables and Equations, Proceedings of the Annual Conference of the American Society for Computing Machinery, 1980, 467-474.

CLEMENT J., LOCHHEAD J. & MONK G.S., Translation Difficulties In Learning Algebra, American Mathematics Monthly 88, 1981, 286-290.

COCKCROFT W.H., Mathematics Counts, London, H.M.S.O., 1982.

COLLIS K.F., Cognitive Development and Mathematics Learning, Paper, Chelsea College, London, 1974.

COLLIS K.F., A Study of Concrete and Formal Operations In School Mathematics - A Piagetian Viewpoint, Melbourne, ACER, 1975.

COLLIS K.F., Operational Thinking in Elementary Mathematics, in J.A. Keats, K.F. Collis, G.S. Halford (Eds.), Cognitive Development - Research Based on a Neo-Piagetian Approach, New

York, Wiley, 1978.

COLTHEART M., Deep Dyslexia : A Right Hemisphere Hypothesis, in M.Coltheart, K.Patterson & J.C.Marshall (Eds.), Deep Dyslexia, Routledge and Kegan Paul, London, 1980.

COLTHEART M., Right Hemisphere and Disorders of Reading, in A.W.Young (Ed.), Functions of The Right Cerebral Hemisphere, Academic Press, London, 1983.

DAVENPORT J.H., Mathematics of Computer Algebra Systems, in A.G. Howson & J.P. Kahane (Eds.), The Influence of Computers and Informatics on Mathematics and Its Teaching, ICMI, Strasbourg, CUP, Cambridge, 1985, 133-146.

DAVIS R.B., Cognitive Processes in Solving Simple Algebraic Equations, Journal of Children's Mathematical Behaviour, 1(3), 1975, 7-35.

DAVIS, R.B., JOCKUSH, E., McKNIGHT, C. Cognitive Processes in Learning Algebra. Journal of Children's Mathematical Behaviour, 2(1), 1978, 10-320.

DAVIS R.B. & McKNIGHT C., Modelling the Processes of Mathematical Thinking, Journal of Children's Mathematical Behaviour, 2(2), 1979, 99-113.

DAVIS R.B., ICME5 Report: Algebraic Thinking in the Early Grades, Journal of Mathematical Behaviour, 4, 1985, 195-208.

DAVIS R.B., What 'Algebra' Should Students Learn?, Journal of Mathematical Behaviour, 5, 1986, 21-24.

DAVIS R.B., The Convergence of Cognitive Science and Mathematics Education, Journal of Mathematical Behaviour, 5, 1986, 321-333.

DAWSON C.J., Pupils' Difficulties : What Can the Teacher Do?, School Science Review, 1984, 120 - 122.

DE BONO E., The Use of Lateral Thinking, Jonathan Cape, London, 1973.

DE BONO E., The Mechanism of Mind, Penguin, 1977.

DE CORTE E. & VERSCHAFFEL L., Beginning First Graders' Initial Representation of Arithmetic Word Problems, Journal of Mathematical Behaviour, 4(1), 1985, 3-21.

DIENES Z.P., Building Up Mathematics, Hutchinson, London, 1960.

DONALDSON M., Children's Minds, Croom Helm, London, 1978.

DREYFUS T., How to Use a Computer to Teach Mathematical Concepts, Proceedings of the 6th PME-NA Conference, 1984, 239-244.

DREYFUS T., Computer Use in Mathematics Education and PME, Discussion Paper, in New Insights into Mathematics Education With the Microcomputer, 10th IGPME Conference, London, England, 1986, 13-15.

DREYFUS T. & EISENBERG T., On the Deep Structure of Functions, Proceedings of the 11th IGPME Conference, Montreal, Canada, 1987, I, 190-196.

DU BOULAY J.B.H., Teaching Teachers Mathematics Through Programming, International Journal of Mathematical Education in Science and Technology, 11(3), 1980, 347-360.

EISENBERG T. & DREYFUS T., On Visual Versus Analytical Thinking in Mathematics, Proceedings of the 10th IGPME Conference, London, England, 1986, 153-158.

EKENSTAAM A.A. & NILSSON M., A New Approach to the Assessment of Children´s Competence, Educational Studies in Mathematics, 10, 1979, 41-66.

ENGEL A., The Role of Algorithms and Computers In Teaching Mathematics At School, Proceedings of 3rd International Congress on Mathematical Education, Karlsruhe Universitat, Germany, 1976.

FENNEMA E., Mathematics Learning and the Sexes: A Review, Journal for Research in Mathematics Education, 5, 1974, 126.

FENNEMA E., Women and Girls in Mathematics - Equity in Mathematics Education, Educational Studies in Mathematics, 10, 1979, 389-401.

FERRY G., Parallel Learning in Brains and Machines, New Scientist, March 13, 1986, 36-38.

FIDELMAN U., Hemispheric Basis for Schools in Mathematics, Educational Studies in Mathematics, 16, 1985, 59-74.

FILLOY E. & ROJANO T., From an Arithmetical to An Algebraic Thought (A Clinical Study With 12-13 Year Olds), Proceedings of 6th PME-NA Conference, 1984, 51-56.

FILLOY E. & ROJANO T., Obstructions to the Acquisition of Elemental Algebra Concepts and Teachng Strategies, Proceedings of the 9th IGPME Conference, Utrecht, Netherlands, 1985, 154-158.

FIRTH D.E., A Study of Rule Dependence in Elementary Algebra, Unpublished M.Phil. Thesis, University of Nottingham, 1975.

FLETCHER T.J., Micro-Computers and Mathematics In Schools, Department of Education and Science, London, 1977.

FODOR J.A., Some Reflections on L.S. Vygotsky´s Thought and Language, Cognition, 1, 1972, 83-95.

FREUDENTHAL H., Major Problems of Mathematics Education, Educational Studies in Mathematics, 12, 1981, 133-150.

GAZZANIGA M.S., Cerebral Dominance Viewed as a Decision System, in S.J. Dimond & J.G. Beaumont (Eds.), Hemisphere Function in the Human Brain, Elek Science, London, 1974, 367-382.

GELMAN R., Conservation Acquisition: A Problem of Learning to Attend to Relevant Attributes, Journal of Experimental Child Psychology, 7, 1969, 167-186.

GELMAN R., Logical Capacity of Very Young Children: Number Invariance Rules, Child Development, 1972, 43, 75-90.

GELMAN R. & TUCKER M.F., Further Investigations of the Young Child´s Conception of Number, Child Development, 46, 1975, 167-175.

GELMAN R. & GALLISTEL C.R., The Child´s Understanding of Number, Harvard University Press, 1978.

GINSBURG, H., The Psychology of Arithmetic Thinking, Journal of Children´s Mathematical Behaviour, 1977, 1(4), 1-90.

GINSBURG H. (Ed.), The Development of Mathematical Thinking Academic Press, London, 1983.

GINSBURG H. & OPPER S., Piaget´s Theory of Intellectual Development, Prentice-Hall, New Jersey, U.S.A., 1979.

GLENNON V.J., Neuropsychology and the Instructional Psychology of Mathematics, Paper for Research Council for Diagnostic and Prescriptive Mathematics, Ohio, U.S.A., April 1980.

GOLDENBERG E.P., Believing is Seeing: How Preconceptions Influence the Perception of Graphs, Proceedings of the 11th IGPME Conference, Montreal, Canada, 1987, I, 197-203.

HABER R.N. & HERSHENSON M., The Psychology of Visual Perception (2nd Edition), Holt Reinhart & Winston, New York, 1980.

HALE D., Algebra Teaching Today, Maths in Schools, 9(1), Jan. 1980, 11-12.

HALFORD G.S., An Approach To the Definition of Cognitive Developmental Stages In School Mathematics, British Journal of Educational Psychology, 1978, 298 - 313.

HANLEY A., Verbal Mathematics, Mathematics in Schools, 7(4), 1978, 27-30.

HARPER E.W., Sketch for a Theory of Confusion in Mathematics Learning, International Journal of Mathematical Education in Science and Technology, 9(2), 1978, 231-237.

HARPER E.W., The Child´s Interpretation of a Numerical Variable, Unpublished Ph.D. Thesis, University of Bath, 1979.

HARPER E.W., The Boundary Between Arithmetic and Algebra: Conceptual Understanding in Two Language Systems, International Journal of Mathematical Education in Science and Technology, 11(2), 1980, 237-243.

HARPER E.W., Psychological Changes Attending a Transition From Arithmetical to Algebraic Thought, Proceedings of the 5th IGPME Conference, Grenoble, 1981, 171-176.

HART K., The Step to Formalisation, Proceedings of the 10th IGPME Conference, London, England, 1986, 159-164.

HART M., Using Computers to Understand Maths., Mathematics Teaching, 80, 1977, 40-42.

HART M., Microprocessors : What Shall We Do With Them?, Mathematics in Schools, 8(2), March 1979, 10,11.

HART M., Computer Programming in the Maths. Classroom as an Aid to Understanding, Trent Polytechnic, Nottingham, 1980.

HART M., Nottingham Programming In Mathematics Project Report No.2, Trent Polytechnic, Nottingham, 1982.

HARTLEY J.R., Using the Computer to Study and Assist the Learning of Mathematics, Occasional Paper No.2, in Proceedings of the British Society for the Psychology of Learning Mathematics Conference, University of Nottingham, January 1980.

HERSCOVICS N., Comprehension de la droite et de Son Equation au Niveau Secondaire, Unpublished Doctoral Dissertation, Universite de Montreal, 1979.

HERSCOVICS N. & BERGERON J.C., A Constructivist vs A Formalist Approach in the Teaching of the Even-Odd Number Concept at the Elementary Level, Proceedings of the 9th IGPME Conference, Utrecht, Netherlands, 1985, 459-464.

HERSCOVICS N. & CHALOUH L., Using Literal Symbols to Represent Hidden Quantities, Proceedings of the 6th PME-NA Conference, 1984, 64-70.

HERSCOVICS N. & KIERAN C., Constructing Meaning for the Concept of Equation, Mathematics Teacher, 73, 1980, 572-580.

HIEBERT J. & CARPENTER T.P., Piagetian Tasks as Readiness Measures in Mathematics Instruction: A Critical Review, Educational Studies in Mathematics, 13, 1982, 329-345.

HIGGO J.R., The Effect Of Micro-Computers in the Mathematics Curriculum, Mathematical Association Discussion Document, 1984.

HODGSON B.R., Symbolic and Numerical Computation: The Computer as a Tool in Mathematics, Proceedings of IFIP on Informatics and the teaching of Mathematics, Sofia, Bulgaria, 1987.

HOWSON G., The Impact of Computers on Mathematics Education, Journal of Mathematical Behaviour, 4, 1985, 295-303.

HOYLES C. & SUTHERLAND R., Children Learning Mathematics - Insights from Within a Logo Environment, Proceedings of the 9th IGPME, Utrecht, 1985, 30-39.

HOYLES C. & SUTHERLAND R., Peer Interaction in a Programming Environment, Proceedings of the 10th IGPME Conference, London, England, 1987, 354-359.

HUMPHREYS G.W. & RIDDOCH M.J., To See But Not to See: A Case Study of Visual Agnosia, Lawrence Erlbaum, London, 1987.

ICMI, The Influence of Computers and Informatics on Mathematics and its Teaching, A Discussion Document, 1984.

INHELDER B. & PIAGET J., The Growth of Logical Thinking, Routledge and Kegan Paul, London, 1958.

JAHNKE H.N., Technology and Education: The Example of the Computer, Educational Studies in Mathematics, 14, 1983, 87-100.

JANVIER C., Difficulties Related to the Concept of Variable Presented Graphically, Proceedings of the 5th IGPME Conference, Grenoble, France, 1981, 189-192.

JOFFE L. & FOXMAN D., Attitudes and Sex Differences - Some APU Findings, in L. Burton (Ed.), Girls Into Maths Can Go, Holt, Reinhart & Winston, London, 1986, 38-50.

JURDAK M., Reflective Intelligence and Mathematics Performance, Proceedings of the 4th IGPME Conference, Berkeley, California, 1980, 198-205.

KAPUT J. & SIMS-KNIGHT J., Errors in Translations to Algebraic Equations: Roots and Implications, in M. Behr & G. Bright (Eds.), Focus on Learning Problems in Mathematics, 5, 1983, 63-78.

KAPUT J., Information Technology and Mathematics: Opening New Representational Windows, Journal of Mathematical Behaviour, 5, 1986, 187-207.

KAPUT J., Representation in Algebra, Paper presented at the NCTM Research Agenda Project Algebra Conference, Athens, Georgia, 1987.

KAPUT J., PME Xl Algebra Papers : A Representational Framework, Proceedings of the 11th IGPME Conference, Montreal, Canada, 1987, I, 345-354.

KARPLUS R., PULOS S. & STAGE E.K., Early Adolescents' Reasoning With Unknowns, Proceedings of the 5th IGPME Conference, Grenoble, 1981, 147-152.

KIERAN C., Children's Operational Thinking Within the context of Bracketing and the Order of operations, Proceedings of the 3rd IGPME Conference, Warwick, England, 1979, 128-133.

KIERAN C., The Interpretation of the Equal Sign: Symbol for an Equivalence Relation Versus an Operator Symbol, Proceedings of the 4th IGPME Conference, Berkeley, California, 1980, 163-169.

KIERAN C., Pre-Algebraic Notions Among 12 and 13 Year-Olds, Proceedings of the 5th IGPME Conference, Grenoble, 1981, 158-164.

KIERAN C., Concepts Associated With the Equality Symbol, Educational Studies in Mathematics, 12, 1981, 317-326.

KIERAN C., A Comparison Between Novice and More Expert Algebra Students on Tasks Dealing With the Equivalence of Equations, Proceedings of the 6th Annual Meeting of PME-NA, 1984, 83-91.

KIERAN C., The Equation Solving Errors of Novice and Intermediate Algebra Students, Proceedings of the 9th IGPME Conference, Utrecht, Netherlands, 1985, 141-146.

KIERAN C., The Early Learning of Algebra, Paper Presented at the NCTM Research Agenda Project Algebra Conference, Athens, Georgia, 1987.

KIERAN C., Two different Approaches Among Algebra Learners, in A. Coxford (Ed.), NCTM 1988 Yearbook : Algebra, Reston, VA, NCTM, 1988, in press.

KLAHR D. & WALLACE J.G., The Role of Quantification Operators in the Development of Conservation, Cognitive Psychology, 1973, 301-327.

KRUTETSKII V.A., The Psychology of Mathematical Abilities in School Children, University of Chicago Press, 1976.

KUCHEMANN D.E., Children´s Understanding of Numerical Variables, Maths in Schools, 7(4), 1978, 23-26.

KUCHEMANN D.E., The Understanding of Generalised Arithmetic by Secondary School Children, Unpublished Doctoral Dissertation, Chelsea College, London, 1980.

KUCHEMANN D.E., Cognitive Demands of Secondary School Mathematics Items, Educational Studies in Mathematics, 12, 1981, 301-316.

KUCHEMANN D.E., in K.M. Hart (Ed.), Children´s Understanding of Mathematics : 11-16, John Murray, 1981.

LANE K.D., OLLONGREN A. & STOUTEMEYER D.R., Computer Based Symbolic Mathematics for Discovery, The Influence of Computers and Informatics on Mathematics and Its Teaching, in A.G. Howson & J-P. Kahane (Eds.), Proceedings of ICMI, CUP, Cambridge, 1985.

LARKIN J.H., Robust Performance in Algebra : The Role of Problem Representation, Paper Presented at the NCTM Research Agenda Project Conference in Algebra, Athens, Georgia, 1987.

LAWLER R.W., Computer Experience and Cognitive Development : A Child´s Learning in a Computer Culture, Ellis Horwood, Chichester, England, 1985.

LEAN G.A. & CLEMENTS M.A., Spatial Abilty, Visual Imagery and Mathematical Performance, Educational Studies in Mathematics, 12, 1981, 267-299.

LERON U., Some Problems in Children´s Logo Learning, Proceedings of the 7th IGPME Conference, Shoresh, Israel, 346-351.

LESGOLD A.M., REIF F. (Eds.), Computers in Education : Realising the Potential, Report of a Research Conference, Pittsburgh, Pennsylvania, Nov., 1982.

LESH R. & HERRE J., Dienes Revisited: Multiple Embodiments in Computer Environments, Proceedings of the 11th IGPME Conference, Montreal, Canada, 1987, I, 211-220.

LEVY J., Information Processing and Higher Psychological Functions in the Disconnected Hemispheres of Human Commissurotomy Patients, Unpublished Ph.D. Thesis, California Institute of Technology, Pasadena, California, 1969.

LINN M.C., Fostering Equitable Consequences From Computer Learning Environments, Paper Presented at Conference of American Educational Research Association, California University, Berkeley, 1983.

LOCHHEAD J., Faculty Interpretations of Simple Algebraic Statements: The Professors Side of the Equation, Journal of Mathematical Behaviour, 3(1), 1980, 29-37.

MacCALLUM M., Computer Algebra - Tomorrow's Calculator?, New Scientist, Oct. 23 1986, 52-55.

MARKMAN E.M., Factors Affecting the Young Child's Ability to Monitor His Memory, Unpublished Ph.D. Dissertation, University of Pennsylvania, 1973.

MATOS J., The Construction of the Concept Variable in a Logo Environment: A Case Study, Proceedings of the 10th IGPME Conference, London, England, 1986, 271-276.

MATZ M., Towards a Process Model for High School Algebra Errors, Unpublished Working Paper No.181, MIT, 1979.

MATZ M., Towards a Computational Theory of Algebraic Competence, Journal of Mathematical Behaviour, 3(1), 1980, 93-166.

McGETTRICK A.D., Mathematics by Computer-Assisted Learning (CAL) - The Real Problems, International Journal of Mathematical Education in Science and Technology, 10(1), 1979, 65-73.

McLEAY H., Some Aspects of Children's Interpretation of Literal Expressions, Unpublished M.Ed. Dissertation, Bath, 1980.

MEHLER J. & BEVER T.G., Cognitive Capacity of Very Young Children, Science, 158, 1967, 141-1

MEISSNER H., How to Prove Relational Understanding, Proceedings of the 7th IGPME Conference, Shoresh, Israel, 1983, 76-81.

MEISSNER H., in Proceedings of the 3rd IGPME Conference, Karlsruhe Universitat, Germany, 1976, 294.

MENIS Y., Improving Achievement in Algebra by Means of the Computer, Educational Technology, 20(8), 1980, 19-22.

MEVARECH Z.R., The Effects of CAI on Affective Variables in Mathematics, Proceedings of the 9th IGPME Conference, Noordwijkerhout, Netherlands, 1985, 59-64.

MOSCOVITCH M., The Linguistic and Emotional Functions of the Normal Right Hemisphere, in E. Perecman (Ed.), Cognitive Processing in the Right Hemisphere, Academic Press, New York, 1983, 57-82.

MOSER J.M., Using a Microcomputer to Teach Representational Skills, Proceedings of the 7th IGPME Conference, Shoresh, Israel, 1983, 339-345.

NANTAIS N., HERSCOVICS N. & BERGERON J.C., The Skills - Understanding Dilemma in Mathematics Education, Proceedings of the 6th Annual Meeting of PME-NA, 1984, 229-235.

NELSON G.T., Development of Fourth Graders7 Concept of Literal Symbols Through Computer-Oriented Problem-Solving Activities, Unpublished Doctoral Dissertation, Georgia State University, 1985.

NELSON G.T., Using Microcomputer-Assisted Problem Solving to Explore the Concept of Literal Symbols, Proceedings of the 11th IGPME Conference, Montreal, Canada, 1987, I, 221-227.

NEUWIRTH E., The Impact of Computer Algebra for the Teaching of Mathematics, Proceedings of the Conference of IFIP on Informatics and the Teaching of Mathematics, Sofia, Bulgaria, 1987.

NEVES D., Learning Algebra From a Textbook, Paper Presented at the Conference on Cognitive Processes in Algebra, University of Pittsburgh, 1979.

NOSS R., Constructing a Conceptual Framework for Elementary Algebra Through Logo programming, Educational Studies in Mathematics, 17, 1986, 335-357.

NOVAK J.D., Understanding The Learning Process and Effectiveness of Teaching Methods in the Classroom , Laboratory and Field, Science Education, 1976 , 493 - 512.

NOVAK J.D., An Alternative to Piagetian Psychology for Science and Mathematics Education, Science Education, 1977, 61(4), 453-477.

NOVAK J.D., Methadological Issues in Investigating Meaningful Learning, Cognitive Development Research in Science and Maths., University of Leeds, 1980, 129-148.

O´BRIEN D.J., Solving Equations, Unpublished Master´s Thesis, University of Nottingham, England, 1980.

O´SHEA T. & SELF J., Learning and Teaching With Computers, Harvester Press, 1983.

PAIVIO A., Perceptual Comparisons Through the Mind´s Eye, Memory and Cognition, 3, 1975, 635-647.

PAPERT S., Mindstorms, Harvester Press, 1980.

PAPERT S., Beyond the Cognitive, the Other Face of Mathematics, Plenary Paper in Proceedings of the 10th IGPME Conference, London, England, 1986.

PASK G. & SCOTT B., Learning Strategies and Individual Competence, International Journal of Man-Machine Studies, 4, 1972, 217-253.

PAVELLE R., ROTHSTEIN M. & FITCH J., Computer Algebra, Scientific American, 245(6), Dec. 1981, 102-113.

PENDLEY MANOR REPORT, Maths in Schools, 14(2), March 1985, 26-28.

PEA R.D. & KURLAND D.M., Logo Programming and the Development of Planning Skills, Technical Report No.16, New York Centre for Children and Technology, Bank Street College of Education, 1983.

PEA R.D. & KURLAND D.M., On the Cognitive Effects of Learning Computer Programming, New Ideas in Psychology, 2(2), 1984, 137-168.

PEARSON J., Learning About Mathematics Through Logo, Proceedings of the First International Conference for Logo and Mathematics Education, London, 1985.

PETITTO A., The Role of Formal and Non-Formal Thinking in Doing Algebra, Journal of Children's Mathematical Behaviour, 2(2), 1979, 69-82.

PHILLIPS R.J., Computer Graphics as a Memory Aid and a Thinking Aid, Journal of Computer Assisted Learning, 2, 1986, 37-44.

PIAGET J., The Child's Conception of Number, Routledge and Kegan Paul, London, 1952.

PIAGET J., The Origin of Intelligence in the Child, Routledge and Kegan Paul, London, 1953.

PIAGET J. & INHELDER B., The Psychology of the Child, Routledge and Kegan Paul, London, 1969.

PIAGET J. & INHELDER B., Mental Imagery in the Child : A Study of the Development of Imaginal Representation, Routledge and Kegan Paul, London, 1971.

PIMM D., Self Images for a Computer, Mathematics Teaching, 105, Dec. 1983, 40-43.

PLOEGER F.D., The Effectiveness of Microcomputers in Education, Literature Review, Austin, Texas, 1983.

PONTE J., Functional Reasoning and The Interpretation of Cartesian Graphs, Unpublished Doctoral Dissertation, University of Georgia, Athens, Georgia, 1984.

POPPER K.R., & ECCLES J.C., The Self and Its Brain, Springer, Berlin, 1977.

PUTNAM R.T., LESGOLD S.B., RESNICK L.B. & STERRETT S.C., Understanding Sign Change Transformations, Proceedings of the 11th IGPME Conference, Montreal, Canada, 1987, I, 338-344.

RACHLIN S.L., Processes Used by College Students in Understanding Basic Algebra, Unpublished Ph.D. Dissertation, University of Athens, Georgia, 1981.

REGGINI H.C., A Revision of Learning and Teaching, Literature Review, Revista del Instituto de Investigaciones Educativas, Buenos Aires, Argentina, 1983.

RESNICK L.B., Understanding Algebra, Paper Presented at the NCTM Research Agenda Project Conference, Athens, Georgia, 1987.

RIDGWAY J., Problem Solving and Programming, Micromath, 1(3), Winter 1985, 28-31.

ROGALSKI J., Mental Representations in Programming by 15-16 Year-Old Students, Proceedings of the 10th IGPME Conference, London, England, 1986, 283-288.

ROSE S., Memories and Molecules, New Scientist, Nov. 27, 1986, 40-44.

ROSNICK P. & CLEMENT J., Learning Without Understanding: The Effect of Tutoring Strategies on Algebra Misconceptions, Journal of Mathematical Behaviour, 3(1), 1980, 3-27.

ROSNICK P., Some Misconceptions Concerning the Concept of Variable, Mathematics Teacher, 74, 1981, 418-420,450.

ROSS P., Modelling as a Method of Learning Physical Science and Mathematics, Occasional Paper, University of Edinburgh, 1985.

SAAD L.G., Understanding in Mathematics : Factorial and Qualitative Studies, Unpublished Ph.D. Thesis, University of Birmingham, England, 1957.

SAMURCAY R., Learning Programming: Constructing the Concept of Variable by Beginning Students, Proceedings of the 9th IGPME Conference, Utrecht Netherlands, 1985, 77-82.

SAUNDERS J. & BELL F.H., Computer Enhanced Resources: Their Effects on Achievement and Attitudes, International Journal of Mathematical Education in Science and Technology, 11(4), 1980, 465-473.

SCNEIDERMAN E.I., Hemisphere Involvement in Language Acquisition, in J. Vaid (Ed.), Language Processing in Bi-Linguals: Psycholinguistic and Neurophysiological Perspectives, Lawrence Erlabaum, new Jersey, 1986, 233-253.

SCHWARTZ J.L. & KOSSLYN S.M., A Computer Simulation Approach to Studying Mental Imagery, in J.Mehler, E.C.T.Walker & M.Garrett (Eds.), Perspectives on Mental Representations, Lawrence Erlbaum, New Jersey, 1982, 69-85.

SCHWARTZ J.L. & YERUSHALMY M., The Geometric Supposer : Triangles (for the Apple Computer), Sunburst Communications, Pleasantville, New York, 1985.

SCHWARTZ J.L., The Representation of Function in the Algebraic Proposer, Proceedings of the 11th IGPME Conference, Montreal, Canada, 1987, I, 235-240.

SCOTT-HODGETTS R., Girls and Mathematics: The Negative Implications of Success, in L.Burton (Ed.), Girls into Maths Can Go, Holt, Reinhart and Winston, London, 1986, 61-76.

SEAMON J.G., Coding and Retrieval Processes and the Hemispheres of the Brain, in S.J.Dimond & J.G.Beaumont (Eds.), Hemisphere Function in the Human Brain, Elek Science, London, 1974, 184-203.

SHAYER M., Has Piaget´s Construct of Formal Operational Thinking Any Utility?, British Journal of Educational Psychology, 49, 1979, 265-276.

SHUARD H., The Relative Attainment of Girls and Boys in Mathematics in the Primary Years, in L.Burton (Ed.), Girls into Maths Can Go, Holt, Reinhart and Winston, London, 1986, 22-37.

SILVER E.A., BRANCA N.A. & ADAMS V.M., Metacognition: The Missing Link in Problem Solving?, Proceedings of the 4th IGPME Conference, Berkeley, California, 1980, 213-221.

SIMON T., McSHANE J. & RADLEY S., Learning With Microcomputers: Training Primary School Children on a Problem-Solving Program, Project Report, London School of Economics, London, 1985.

SINCLAIR H., Piaget´s Theory of Development : The Main Stages, in M.F. Rosskopf, L.P. Steffe & S. Taback (Eds.), Piagetian Cognitive Development Research and Mathematical Education, Washington D.C., NCTM, 1971, 5,6.

SKEMP R.R., The Psychology of Learning Mathematics, Penguin, 1971.

SKEMP R.R., Relational Understanding and Instrumental Understanding, Mathematics Teaching, 77, Dec. 1976, 20-26.

SKEMP R.R., A Revised Model for Reflective Intelligence, Proceedings of the 2nd IGPME Conference, Osnabruck, Germany, 1978, 286-295.

SKEMP R.R., Goals of Learning and Qualities of Understanding, Proceedings of the 3rd IGPME Conference, Warwick, England, 1979, 197-202.

SKEMP R.R., Goals of Learning and Qualities of Understanding, Mathematics Teaching, 88, 1979, 44-49.

SKEMP R.R., Intelligence Learning and Action - A Foundation for Theory and Practice in Education, Wiley, 1979.

SKEMP R.R., PMP: A Progress Report, Proceedings of the 9th IGPME Conference, Utrecht Netherlands, 1985, 447-452.

SLEEMAN D., Introductory Algebra: A Case Study of Student

Misconceptions, Journal of Mathematical Behaviour, 5, 1986, 25-52.

SLESNICK T., Algorithmic Skills vs. Conceptual Understanding, Educational Studies in Mathematics, 13, 1982, 143-154.

SMALL D., HOSACK J. & LANE K., Computer Algebra Systems in Undergraduate Instruction, The College Mathematics Journal, 17(5), Nov. 1986, 423-433.

SMEDSLUND J., Piaget´s Psychology in Practice, British Journal of Educational Psychology, 47, 1977, 1-6.

SPERRY R.W., GAZZANIGA M.S. & BOGEN J.E., Interhemispheric Relationships: The Neocortical Commissures; Syndromes of Hemisphere Disconnection, in P.J. Vinken & G.W.Bruyn (Eds.), Handbook of Clinical Neurology, Vol 4, North Holland, Amsterdam, 1969, 273-290.

SPERRY R.W., Forebrain Commisurotomy and Conscious Awareness, in J.Orbach (Ed.), Neuropsychology After Ashley: Fifty Years Since the Publication of Brain Mechanisms and Intelligence, Lawrence Erlbaum, New Jersey, 1982, 497-522.

SPERRY R.W., Lateral Specialisation in Surgically Separated Hemispheres, in F.O.Schmitt & F.G.Worden (Eds.), The Neurosciences Third Study Program, MIT Press, Cambridge, Massachusetts, 1974, 5-19.

STANHOPE REPORT, Will Mathematics Count?, The Report of The Seminar at Stanhope on Computers in Mathematics, Mathematics Association, England, 1986.

SUTHERLAND R., A Study of the Use and Understanding of Algebra Related COncepts Within a Logo Environment, Proceedings of the 11th IGPME Conference, Montreal, Canada, 1987, I, 241-247.

SUTHERLAND R. & HOYLES C., Children´s Understanding of the Concept of Variable in a Logo Context, Occasional Paper, London University Institute of Education, 1985.

SUTHERLAND R. & HOYLES C., Logo As a Context for Learning About Variable, Proceedings of the 10th IGPME Conference, London, England, 1986, 301-306.

SWAIN R.L., The Equation, Mathematics Teacher, 55, 1962, 226-236.

SWEETEN C., Mathematics and Micro-Computers, ISMEC, Discussion Document, 1982.

TALL D.O., The Dynamics of Understanding Mathematics, Mathematics Teaching, 84, 1978, 50-52.

TALL D.O., Generic Thinking - A Central Topic for Investigation, Occasional Paper, Mathematics Institute, University of Warwick, Aug. 1978.

TALL D.O., Mathematical Thinking and the Brain, Proceedings of the 2nd IGPME Conference, Osnabruck, Germany, 1978, 333-343.

TALL D.O., The Mutual Relationship Between Higher Mathematics and a Complete Cognitive Theory for Mathematical Education, Proceedings of the 5th IGPME Conference, Grenoble, 1981, 316-321.

TALL D.O., Introducing Algebra on the Computer, Mathematics In Schools, 12, 1983, 37 - 40.

TALL D.O., Visualising Calculus Concepts Using a Computer, Mathematics Education Research Centre, Warwick University, 1984.

TALL D.O., Whither Calculus, Mathematics Teaching, 1986, 25-29.

TALL D.O., Understanding Variables and Functions Using a Computer, Occasional Paper, Mathematics Education Research Centre, Warwick University, 1985.

TALL D.O., Computing Languages for the Mathematics Classroom, Mathematical Association Discussion Document, England, Dec. 1985.

TALL D.O., Tangents and the Leibniz Notation, Mathematics Teaching, 113, Dec. 1985, 48-51.

TALL D.O., Using computer Graphics as Generic Organisers for the Concept Image of Differentiation, Proceedings of the 9th IGPME Conference, Utrecht, Netherlands, 1985, 105-110.

TALL D.O., Using the Computer in the Secondary Mathematics Classroom, Mathematical Association Discussion Document, England, 1986.

TALL D.O., Using the Computer as an Environment for Building and Testing Mathematical Concepts: A tribute to Richard Skemp, in Papers in Honour of Richard Skemp, Mathematics Education Research Centre, University of Warwick, 1986.

TALL D.O., Building and Testing a Cognitive Approach to the Calculus Using Interactive Computer Graphics, Ph.D. Dissertation, Mathematics Education Research Centre, University of Warwick, 1986.

TALL D.O., Graphical Packages for Mathematics Teaching and Learning, Mathematics Education Research Centre, University of Warwick, 1987.

TALL D.O. & BLYTHE K., Procedures, Functions and Variables, Paper for Leicester Mini-Conference on Primary Education, Dec. 1985.

TALL D.O. & SHEATH G., Visualizing Higher Level Mathematical Concepts Using Computer Graphics, Proceedings of the 7th IGPME Conference, Shoresh, Israel, 1983, 357-362.

TALL D.O. and THOMAS M.O.J., Playing Algebra With the Computer, in Microelectronics Programme Reader Number 8 : Exploring Mathematics with Micro-Computers, 1986, 59-74.

TALL D.O. and THOMAS M.O.J., Versatile Learning and the Computer, Focus Special Issue, 1988, to appear.

TALL D.O. and VINNER S., Concept Image and Concept Definition in Mathematics With Particular Reference to Limits and Continuity, Educational Studies in Mathematics 81, 1981, 151 - 169.

THOMAS M.O.J., The Effect of BASIC Computer Programming on Children´s Understanding of the Use of Letters as Variables in Algebra, Unpublished M.Sc. Thesis, University of Warwick, 1985.

THOMAS M.O.J., Algebra With the Aid of a Computer, Maths in Schools, 16/1, January 1987, 36-38.

THOMAS M.O.J. & TALL D.O., The Value of the Computer in Learning Algebra Concepts, Proceedings of the 10th IGPME Conference, London, England, 1986, 313-318.

THOMAS M.O.J. and TALL D.O., Dynamic Algebra, A Mathematics Association Special Report on Using the Computer in the Mathematics Classroom, 1988, to appear.

THOMAS M.O.J. & TALL D.O., Longer-Term Conceptual Benefits From Using a Computer in Algebra Teaching, Proceedings of the 12th IGPME Conference, Veszprem, Hungary, 1988, to appear.

THOMPSON P.W., Experience, Problem Solving and Learning Mathematics: Considerations in Developing Mathematics Curricula, in F.A.Silver (Ed.), Learning and Teaching Mathematical Problem Solving, Erlbaum, Nilledale, New Jersey, 1985, 189-236.

THOMPSON P.W., Mathematical Microworlds and Intelligent Computer Assisted Instruction, in G.E.Kearsley (Ed.), Artificial Intelligence and Instruction: Applications and Methods, Addison-Wesley, 1985.

THOMPSON P.W., Artificial Intelligence, Advanced Technology and Learning and Teaching Algebra, Paper Presented at the ICTM Research Agenda Project Conference in Algebra, Athens, Georgia, 1987.

THOMPSON P.W. & THOMPSON A. G., Computer Presentations of Structure in Algebra, Proceedings of the 11th IGPME Conference, Montreal, Canada, 1987, I, 248-254.

THWAITES G.N., Why Do Children Find Mathematics Difficult?, Mathematics in Schools, September 1982, 16,17.

TONNESSEN L.H., Measurement of the Levels of Attainment by College Mathematics Students of the Concept of Variable, Unpublished Ph.D. Dissertation, University of Wisconsin-Madison, 1980.

USISKIN Z., What Should Not Be in the Algebra and Geometry Curricula of Average Students?, The Mathematics Teacher, Sept.

1980, 413-424.

USISKIN Z., Conceptions of Algebra and Uses of Variables, Paper Presented at the NCTM Research Agenda Conference on Algebra, Athens, Georgia, 1987.

VAN HULZEN J.A. & CALMET J., Computer Algebra Systems, in B.Buchberger, G.E.Collins & R.Loos (Eds.), Computer Algebra Symbolic and Algebraic Computation (2nd Edition), Springer-Verlag, 1983, 221-243.

VERGNAUD G. & CORTES A., Introducing Algebra to 'Low Level' 8th and 9th Graders, Proceedings of the 10th IGPME Conference, London, England, 1986, 319-324.

WAGNER S., Conservation of Equation, Conservation of Function and their Relationship to Formal Operational Thinking, Unpublished Doctoral Dissertation, New York University, 1977.

WAGNER S., Mathematical Variables and Verbal Variables : An Essential Difference, Proceedings of the 3rd IGPME Conference, Warwick, England, 1979, 215-216.

WAGNER S., An Analytical Framework for Mathematical Variables, Proceedings of the 5th IGPME Conference, Grenoble, France, 1981, 165-170.

WAGNER S., What Are These Things Called Variables?, Mathematics Teacher, 76, 1983, 474-479.

WAGNER S., RACHLIN S.L. & JENSEN R.J., Algebra Learning Project - Final Report, Department of Mathematics Education, University of Georgia, Canada, 1984.

WALLACE J.G., Stages and Transition in Conceptual Development, An Experimental Study, NFER, 1972.

WARDLE M., Computing in Mathematics 1. Back to Basics, Mathematics in Schools, 12, 1983, 34 - 36.

WHEELER D., Awareness of Algebra, Mathematics Teaching, 96, Sept.1981, 29-34.

WHITMAN B.S., Intuitive Equation Solving Skills and the Effects on Them of Formal Techniques of Equation Solving, Unpublished Doctoral Dissertation, Florida State University, U.S.A., 1975.

WINKELMANN B., The Impact of the Computer on the Teaching of Analysis, International Journal of Mathematical Education in Science and Technology, 15(6), 1984, 675-689.

WINSTON P.H., Learning Structural Descriptions From Examples, in P.H. Winston (Ed.), The Psychology of Computer Vision, New York, McGraw-Hill, 1975.

WITKIN H.A., MOORE C.A., GOODENOUGH D.R. & COX P.W., Educational Implications of Cognitive Styles, Review of Research in Education, 4, 1977, 1-64.

WOOD M., Formulae and an Initial Teaching Algebra, Maths in Schools, 7(1), Jan.1978, 27.

YOUNG R.M., Seriation by Children: An AI Analysis of a Piagetian Task, Birkhauser Verlag, Basel, 1976.

ZANGWILL O.L., Consciousness and the Cerebral Hemispheres, in S.J.Dimond & J.G.Beaumont (Eds.), Hemisphere Function in the Human Brain, Elek Science, London, 264-278.

ZEHAVI N., Interaction Between Graphical and Algebraic Representations in the Use of Micro-Computer Software, Proceedings of the 10th IGPME Conference, London, England, 1986, 217-222.

ZEHAVI N., GONEN R., OMER S. & TAIZI N., The Effects of Microcomputer Software on Intuitive Understanding of Graphs of Quantitative Relationships, Proceedings of the 11th IGPME Conference, Montreal, Canada, 1987, I, 255-261.

A CONCEPTUAL APPROACH TO THE EARLY LEARNING OF ALGEBRA

USING A COMPUTER

Volume II

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Ph.D. IN SCIENCE EDUCATION

WARWICK UNIVERSITY

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A Conceptual Approach To The Early Learning of Algebra

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Volume II

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Appendices

Appendix A

The Pilot Study Test

ALGEBRA TEST

NAME : .... FORM : ....

Attempt all the questions. Write your answers in the spaces provided.

1. Find a if $a - 3 = 9$ ....
2. If $x + y = 31$, what does $x + y + 5 = ?$ ....
3. Simplify :
   a) $4y + 6y$ ....
   b) $9x - 3x$ ....
4. A square has sides of length b cm. Write down an expression for the perimeter of the square. ....
5. Find b if $3b = 12$ ....
6. Write down the area of a rectangle of length p cm. and width q cm. ....
7. Calculate p if $p = 3q + 5$ and $q = 2$ ....
8. Calculate a if $b = 2a - 4$ and $b = 10$ ....
9. Simplify :
   a) $3x + 9y + 2x$ ....
   b) $5z + 3x + 2z + 5x$ ....
10. Add 6 onto $a + 2$ ....
11. Add 5 onto $3m$ ....
12. A polygon has n sides, each of length 2 cm. Write down the perimeter of the figure. ....
13. If $x + y = 7$ then $x + y + z = ?$ ....
14. If $r = s + t$ and $r + s + t = 30$ then $r = ?$ ....

15. Simplify :

a) $3a - b + a$ ....

b) $2p - 3q + 8q$ ....

16. If $c + d = 15$ and $c = 2d$ then $d = ?$ ....

17. Simplify $(x - y) + y$ ....

18. Multiply $2n + 3$ by 7 ....

19. Which is larger $2n$ or $n + 2$? Explain. ....

20. Does $a + b + c = a + n + c$, sometimes, always or never ? Explain. ....

21. I buy b blue pencils at 5p each and r red pencils at 6p each. Altogether they cost me 90p. What can you say about b and r ? ....

22. If $(x - 1)^3 + x = 351$ when $x = 8$, what value of x makes $(4x - 1)^3 + 4x = 351$ true ? ....

Appendix B

The Pilot Study - Raw Data Facilities

The Pilot Study Raw Data Facilities

Question	Second Year	Third Year	Fourth Year
1.	96.6	92.3	96.6
2.	96.6	96.2	96.6
3.a)	93.1	100	75.9
b)	100	100	79.3
4.	75.9	76.9	79.3
5.	100	96.2	100
6.	89.7	88.5	93.1
7.	100	100	100
8.	79.3	61.5	89.7
9.a)	100	100	86.2
b)	93.1	92.3	86.2
10.	100	80.8	86.2
11.	100	88.5	82.8
12.	93.1	84.6	79.3
13.	89.7	92.3	58.6
14.	86.2	76.9	65.5
15.a)	93.1	100	89.7
b)	79.3	88.5	82.8
16.	79.3	96.2	89.7
17.	48.3	88.5	48.3
18.	79.3	84.6	89.7
19.	3.5	11.5	27.6
20.	41.4	57.7	69.0
21.	34.5	26.9	17.2
22.	62.1	57.7	82.8

Appendix C

Experiment 1 - The VRQ and MQ Used For Matching

Experiment 1 - MQ and VRQ

Name	Mathematics	Verbal Reasoning
Emma	124	115
Louisa	124	116
Helen	122	133
Elizabeth	122	131
Julia	116	104
Sarah	116	108
Neil	133	140+
Mark	132	140+
Richard	112	119
Chris	113	119
Warren	119	112
Sarah	119	112
Anna	121	112
Victoria	121	109
Stephen	115	110
Emma	117	110
Gareth	122	125
Jeremy	121	127
David	110	124
Sophia	109	126
Helen	118	113
Robert	118	114

Name	Mathematics	Verbal reasoning
Richard	120	118
Craig	120	116
Elaine	116	117
Simon	116	115
Jake	105	119
Lawrence	106	113
Chris	121	121
Marcus	121	121
Martyn	114	109
Lorna	115	109
Melissa	111	105
Anne-Marie	111	110
Nicola	119	106
Christina	118	108
Ian	119	125
Benjamin	118	126
Joanne	118	116
Craig	117	115

Appendix D

The Software Maths. Machine

Computer Program

```
0Z%=&7000:GOTO30000:C-MOJ THOMAS
 90 REM
100 REM Program to aid in the understanding of algebra
110 REM
120DIMfn$(3),var$(4),cst$(4),vv(3),valc(3),F(3),P(26)
140A$="":  cc$="": var$(1)="x":  var$(2)="y":  var$(3)="t":
cst$(1)="": cst$(2)=""
150cst$(3)="":  CV=0:   valc(1)=0:   valc(2)=0:  valc(3)=0:
fnnumber%=0: D%=1: I%=&900: T%=&940
160!&84=I%:!&86=T%:?&83=16:F%=&995:$F%="S":$T%=""
240MODE1
250VDU23;8202;0;0;0;
260PROCscl
270  ON ERROR PROCe :VDU4 :CLS :PRINT :PRINT" Don´t be silly!"
:FORT=1 TO 7000 :NEXT:*FX15
272 REM
274 REM MAIN INPUT LOOP
276 REM
280REPEAT
320IFINKEY(-100)PROCsc2
340IFINKEY(-83)PROCsc3
350IFINKEY(-102)PROCsc4
360IFINKEY(-38)PROCsc5
380IFINKEY(-35)MODE7:PAGE=&3B00:END
420CLS:PROCscbl
440UNTILFALSE
450 REM
460
640DEFPROCscl
```

```
660VDU24,0;256;1279;1023;
680VDU28,0,31,39,24
700COLOUR129:COLOUR0:CLS
720VDU19,2,4,0,0,0
740GCOL0,130:GCOL0,0:CLG
760VDU5
780GCOL0,3
800MOVE510,989:PRINT"VARIABLES"
820MOVE510,760:PRINT"CONSTANTS"
840MOVE500,520:PRINT"FUNCTIONS"
860GCOL0,1
880MOVE0,780:DRAW1280,780:MOVE0,540:DRAW1280,540
900VDU4
920PROCvclr
1020MOVE80,422:MOVE80,326:PLOT81,460,96:PLOT81,0,-96
1040PLOT0,210,96:PLOT0,0,-96:PLOT81,460,96:PLOT81,0,-96
1060PROClprt
1080PROCscbl
1090ENDPROC
1100
1160DEFPROCscbl
1170*FX15
1180COLOUR129:COLOUR3:VDU31,2,2
1195CLS
1200VDU31,1,1:COLOUR3:PRINT"Choose from:":PRINT
1220COLOUR3:PRINT:PRINT"V:Change variables  "cc$
1230PRINT"I:Input variable values     E:End"
1250PRINT: VDU31,0,3: IF((TIME DIV 200) MOD 60)/2=
INT(((TIME DIV 200) MOD 60)/2) COLOUR0 ELSE COLOUR3
```

```
1260PRINT"M:Make Maths. Machine"
1300IFINKEY(5)=-1THEN1250
1310COLOUR3
1320*FX15,1
1340ENDPROC
1350
1400DEFPROCsc2
1420CLS:VDU31,1,2:INPUT"How many variables shall I change?"vch
1430IFvch<>1ANDvch<>2ANDvch<>3THEN1420
1440CLS
1460FORK=1TOvch
1480CLS
1500VDU31,4,2:INPUT"Which variable shall I change?"chv$
1520VDU31,4,4:INPUT"To what?"newv$
1540FORI=1TO3
1560IF CHR$(ASC(chv$) OR 32) = var$(I) THEN var$(I)=
CHR$(ASC(newv$) OR 32)
1580NEXTI
1600NEXTK
1610FORX=200TO1064STEP432:PROClclr(X,852):NEXT
1612PROClprt
1640ENDPROC
1650
1680DEFPROCsc3
1700CLS:VDU31,1,2:INPUT"How many constants shall I change?"cch
1710IFcch<>CVANDcch<>CV-1ANDcch<>CV-2THEN1700
1720CLS
1740FORK=1TOcch
1760CLS
```

```
1780VDU31,1,2:INPUT"Which constant´s value shall I change?"chc$
1800VDU31,1,4:INPUT"TO what value?"newc$
1810FORI=1TO3
1814IFCHR$(ASC(chc$)OR32)=cst$(I)THENvalc(I)=VAL(newc$)
1816NEXTI
1819AA=CV:FORCV=1TOAA
1820PROCcvprt
1830NEXTCV:CV=AA
1835NEXTK
1840ENDPROC
1850
1880DEFPROCsc4
1890cst$(1)="":cst$(2)="":cst$(3)="":cc$=""
1895GCOL0,2:MOVE80,490:MOVE80,425:PLOT81,1200,65:PLOT81,0,-65
1896PROCclearc
1898PROCfnclear
1899cc$="":PROCvclr
1900CLS:PRINT:PRINT:INPUT"How many functions  do  you want,1
or 2?"fnn%
1907IFfnn%<>1ANDfnn%<>2THEN1900
1915CV=0:R=0:QA=0:$T%=""
1920IFfnn%=1THENfn$(fnn%+1)=""
1940FORG=0TO31:?(&9A0+G)=0:NEXT
1950FORP=1TOfnn%
1955REPEAT
1960CLS:VDU31,8,2  :PRINT"Tell   me   your  function  "  :PRINT
:PRINTTAB(8)" "; :PROCK(1):CALL&A46:PROCK(0)
1962N=1
1963LP=0:N=1
```

```
1966REPEAT
1967av$=MID$($T%,N,1):  IF ASC(av$)¨96 AND  ASC(av$)  ˜  123
THEN PROCreplace
1970N=N+1
1972UNTILN=LEN$T%+1 OR LP=1
1973IFLP=1  THEN  $T%="":  FORT=0TO15: ?(&9A0+16*(P-1)+T)=0:
NEXT: IF P<>1 FORCV=1TOAA:PROCcvprt:NEXTCV:CV=AA
1974 UNTIL LP<>1
1975fn$(P)=$T%
1980PROCfnprint(P):AA=CV:NEXTP
2020ENDPROC
2030
2060DEFPROCsc5
2080CLS
2085PROCvclr
2090FORI=1TO3:vv(I)=0:NEXT
2100PROCvarin
2130PROCfnvpt
2135CLS:VDU10:VDU10:PRINT"Do you  want  some  more  variable
values?"
2137IFGET=89THEN2080
2139[%=10
2140ENDPROC
2150
2300DEFPROCvvpt
2330V=41
2335VDU5:  GCOL0,0:  I=V+432*(LA-1):  v1=vv(I DIV 432+1): IF
LEFT$(FNn(v1),2)="0." deco=8 ELSE deco=6
2340QQ=32*(deco-INT(LEN(FNn(v1))/2+0.6)):MOVEI+QQ,916:IF F(I
```

```
DIV432+1)=1THENPRINTFNn(v1)
2360VDU4
2370@%=&10
2380ENDPROC
2390
2400DEFPROCcvprt
2410IFCV¨3PRINT" Too many constants!":GOTO1950
2420PROCconstclear
2450VDU5: FORI=73+(CV-1)*432 TO 73+(CV-1)*432  STEP 432:cl=I
DIV 432+1: GCOL0,0
2452SC=32*(4-INTLEN(FNn(valc(cl)))/2+0.6):     MOVEI+SC,670:
PRINT FNn(valc(cl))
2454GCOL0,3: MOVE200+432*(CV-1),600: PRINTcst$(cl): NEXT
2460VDU4
2480ENDPROC
2490
2500DEFPROCfnprint(X)
2520VDU5:GCOL0,3:MOVE670*(X-1)+80,455:PRINT$I%
2540VDU4
2560ENDPROC
2600DEFPROCvclr
2610z=68
2620GCOL0,3:MOVE40,940:MOVE40,872
2640FORN=1TO3
2660PLOT81,360,z:PLOT81,0,-z
2680PLOT0,72,z:PLOT0,0,-z
2682NEXT
2685ENDPROC
2686
```

```
2687DEFPROCconstclear
2688GCOL0,3
2690MOVE0+(CV-1)*420,620
2694PLOT0,60,z:PLOT0,0,-z:PLOT81,360,z:PLOT81,0,-z
2696ENDPROC
2697
2698DEFPROCfnclear
2699y=96:GCOL0,3
2700MOVE80,422:MOVE80,326:PLOT81,460,y:PLOT81,0,-y
2720PLOT0,210,y:PLOT0,0,-y:PLOT81,460,y:PLOT81,0,-y
2740ENDPROC
2750
2760DEFPROCfnvpt
2780PROCfnclear
2800VDU5
2810FORY=1TOfnn%
2817GCOL0,0:fg$=FNn(EVAL(fn$(Y)))
2818YZ=80+32*(7-INTLEN(fg$)/2+0.6)
2820MOVEYZ+(Y-1)*660,380:PRINTfg$
2830NEXTY
2840VDU4
2850@%=&10
2860ENDPROC
2870
3300DEFPROClclr(X,Y)
3320GCOL0,2: VDU5: MOVEX,Y-25: MOVEX,Y+15: PLOT85,X+40,Y-25:
PLOT85,X+40,Y+15: VDU4
3330ENDPROC
3335
```

```
3340DEFPROClprt
3345VDU5
3350GCOL0,3:    MOVE200,852:    FORI=1TO3:    PRINTvar$(I):
MOVE200+432*I,852: NEXTI
3360VDU4
3370ENDPROC
3380
3500DEFPROCvarin
3505LA=0:F(1)=0:F(2)=0:F(3)=0
3510FORG=0TO31
3512NB=?(&9A0+G)
3515IFNB=0ORNB˜97ORNB¨122THEN3550
3520FORS=1TO3
3530IF  ASC(var$(S))=?(&9A0+G)  AND  F(S)˜¨1  THEN  PRINT:
PRINT"What  is the  value  of  "var$(S):  VDU11:  FORT=1TO22:
VDU9:NEXT: INPUTvv(S)
3540NEXTS
3545PROCvvpt
3547LA=0
3550NEXTG
3560ENDPROC
3570
4000DEFFNn(K)
4005IFK=0Q$="0"ELSEQ$=STR$K
4007@%=&1020709: IFABSK>1E-9  AND ABSK<.1
@%=&1020009+256*(4-INT LOG ABS K)
4008IF  K 1000   AND  LEN(Q$)<>1  AND  RIGHT$(Q$,1)="0"  AND
K<>INT(ABSK)    REPEAT     Q$=LEFT$(Q$,LENQ$-1):     UNTIL
RIGHT$(Q$,1)<>"0"
```

```
4009IFRIGHT$(Q$,1)="."Q$=LEFT$(Q$,LENQ$-1)

4010IFK=INTK @%=&10

4020=Q$

4030

4150DEFPROCclearc

4160GCOL0,2:MOVE0,696:MOVE0,560:PLOT81,1280,146:PLOT81,0,-146

4170ENDPROC

4180

4200DEFPROCe

4210 IF ERR=17 THEN 1160 ELSE ENDPROC

4220ENDPROC

4230

4400DEFPROCreplace

4405LP=0:QA=CV

4407FORY=1TO3

4410IF  av$=var$(Y)THEN$T%=LEFT$($T%,N-1)+"vv("+STR$(Y)+")"+
RIGHT$($T%,LEN$T%-N): N=N+4:  ?(&9A0+16*(P-1)+R) = ASC(av$):
R=R+1: ENDPROC

4420IF  av$  =  cst$(Y)  THEN $T% = LEFT$ ($T%,N-1) + " valc
("+STR$(Y)    +    ")"    +RIGHT$($T%,LEN    $T%-N):   N=N+6:
?(&9A0+16*(P-1)+R)= ASC(av$): R=R+1:

4425NEXT

4430CLS: PRINT:  PRINT  "Do  you  want";  :COLOUR2:  PRINTav$;:
COLOUR3: PRINT" as a constant?"

4440PRINT:AG=GET:IF  AG<>89  AND  P<>1  A$="Please  use  the
variables    and    constants":    B$="you    have    chosen":
CV=CV+1:PROCNEWC:ENDPROC

4450PRINT::CV=CV+1:IFCV>3    AND    P<>1    A$="    Too    many
constants!":B$="":PROCNEWC:ENDPROC  ELSE  IF  CV>3  A$="  Too
```

```
many constants!":B$="":
 4455$T%=LEFT$($T%,N-1)+"valc("+STR$(CV)+")"+RIGHT$($T%,
LEN$T%-N):N=N+6:cc$="C:Change constants"
 4457PROCcvprt
 4460ENDPROC
 4470
 8000DEFPROCNEWC
 8010 CLS: VDU10: PRINTA$: PRINT: PRINTSPC12;B$:FORI=1TO5000:
NEXT:  PROCclearc:  FORI=1TOCV-QA:cst$(4-I)="":  valc(4-I)=0:
NEXTI: CV=Q
 8030DEFPROCNEWC2
 8040CLS:  VDU10: PRINTA$: PRINT:PRINTSPC12;B$: FORI=1TO5000:
NEXT:PROCclearc: FORI=1TOCV-1:  cst$(I)="": valc(I)=0: NEXTI:
CV=0: JG=1
 9000DEFPROCK(Q):VDU23;:IFQ=0VDU8202;ELSEVDU10,96
 9010VDU0;0;0;:ENDPROC
 9020
 9030 REM INSTRUCTIONS
 9040
30000MODE7:VDU23;8202;0;0;0;:M%=&816:CLS:PRINTTAB(12,8);
CHR$141"ALGEBRA       TUTOR":PRINTTAB(12,9);CHR$141"ALGEBRA
TUTOR"
30005PRINTTAB(4,20)"Press RETURN for instructions";
30007*FX15
30008IF       GET<>13       THEN      30060
30010CLS:PRINTTAB(1,4);"This program has been designed to aid
pupils  who  are starting to learn     algebra.It is written
in the form of  a 'Maths. Machine' which the pupils  can use
to   investigate   a   wide    range     of   problems."
```

```
30020PRINTTAB(1,12);"The full range of such algebraic
problems which may be looked at is given in the
accompanying manual."
30025PRINTTAB(4,20)"Press RETURN to continue";
30027*FX15
30028REPEAT UNTIL GET=13
30030CLS:PRINTTAB(1,4);"The facilities available are:"
30035PRINTTAB(1,7);"M:Make Maths. Machine - this is used
to enter the algebraic expression(s) into the ´Machine´.A
choice of 1 or 2 expressions is then given.The choice of 2
expressions allows direct comparisons"
30038PRINTTAB(14,12);"to be made."
30040PRINTTAB(1,14);"V: Change Variables - the program
starts with the variables x,y,t as resident variables for
use in your expressions, but any or all of these may be
changed by using this facility"
30045PRINTTAB(6,20)"Press RETURN to continue";:*FX15
30047REPEATUNTILGET=13
30050CLS:PRINTTAB(1,4);"I:Input Variable Values - this is
used to put in values for the chosen variables.Sets of
values may be input until the investigation is finished."
30051PRINTTAB(1,10);"C:Change Constants - the program has
the facility to work with constants as well as
variables.Although there is a space on the screen for these,
the option to change them is not given until a
letter"
30052PRINTTAB(16,15);"which is not a variable is used.This
optionallows anycurrent constantsto be changed invalue."
30060PRINTTAB(4,20);"TouchSPACEto run program";:*FX15
```

```
30070REPEAT UNTIL GET=32

30080$Z%="uprqst£pr"

30090!&70=!&12:!&72=&900:        !&74=&400:CALLZ%:E%=TOP-M%+2:

B%=PAGE+?(PAGE+3):          L%=E%-B%:T%=&E00+L%:         !&70=B%:

!&72=&E00:!&74=L%:        CALLZ%:

30110      !&70=279511202:!&74=-401932131:      !&78=-59966768:

?&7C=&FF:I%=&900:    T%=&922:    K%=1:B%=1E4:    D%=0:   ?&83=33:

!&84=I%:!&86=T%: RUN
```

Appendix E

Experiment 1 - The Pre-Test

ALGEBRA TEST

NAME : .... FORM :..................

Attempt all the questions. Write your answers in the spaces provided.

1. a) Find a if $a - 3 = 9$ ....

   b) Find b if $3b + 2 = 5b$ ....

2. a) If $x + y = 31$, what does $x + y + 5 = ?$ ....

   b) If $p + 187 = 365$, what does $p + 184 = ?$ ....

   c) If $m + n = 6$, what does $m + n + t = ?$ ....

3. $4x + 3x$ can be written more simply as $7x$.

   Write more simply :

   a) $4y + 6y$ .... f) $4x + 3y - 2x$ ....

   b) $9x - 3x$ .... g) $3x - (x + 2v)$ ....

   c) $(x + y) + x$ .... h) $3x + 7 + 2x - 3$ ....

   d) $3y + 2x + 4y$ .... i) $6x + 7v$ ....

   e) $(y - x) + x$ .... j) $5y + 3x + 2y + 5x$ ....

4. Write down the largest and smallest of the following :

   $m + 4$, m , $m - 1$, $m + 3$, $m - 5$ ....

5. Find b if $3b = 12$ ....

6. Write down the area of a rectangle of length p cm. and width q cm. ....

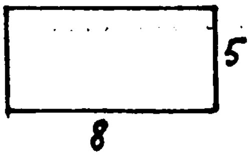
7. a) Calculate v if $v = w + 5$ and $w = 2$ ....

   b) Calculate p if $p = 3q + 5$ and $q = 2$ ....

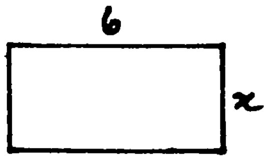
8. Calculate a if $b = 2a - 4$ and $b = 10$ ....

9. What is the area of each of the following :

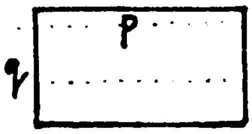
P.T.O.



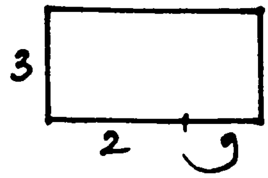
....



....



....



....

10. Add 6 onto :

a) 7 ....

b) a + 2 ....

c) 3a ....

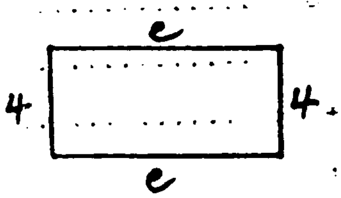
11. Multiply by 3 :

a) 9 ....

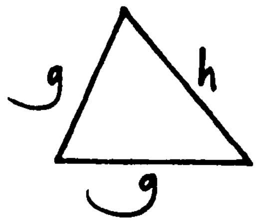
b) a + 3 ....

c) 4a ....

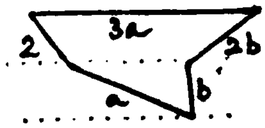
12. Find the perimeter of each of the following figures :



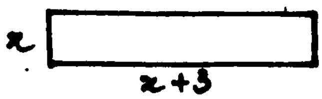
....



....



....



....

13. If x + y = 7 then x + y + z = ? ....

14. a) If h = 2g + f and 2g + f + h = 20 then h = ? ....

b) If c + d = 15 and c = 2d then d = ? ....

15. Fill in the gaps in the following :

x ------> 3x	x - ------> x - 2
5 ------>	13 ------> ....
2r ------>	p + 5 ------> ....

16. If Jane has N stamps and James has M stamps, how many stamps do they have altogether? ....

17. Which is larger : 2p or p + 1 ? Explain. ....

18.a) Does a + b + c = a + n +c , sometimes, always or never ? Explain. ....

b) Does P + Q − R = P − R + Q , sometimes, always or never ? Explain. ....

19. I buy g blue pencils at 7p each and h red pencils at 4p each. Altogether they cost me 84p. What can you say about g and h ? ....

20. The cost of sending a parcel through the post is 80p plus 25p for every 1 KG it weighs. What is the cost of sending a 6 KG parcel ? ....

If the cost of a particular parcel is C pence and it weighs W KG, write down an equation giving C in terms of W. ....

21. If $(x - 1)^3 + x = 351$ when x = 8, what value of x makes $(4x - 1)^3 + 4x = 351$ true ? ....

Appendix F

The CSMS Algebra Test

Algebra 1

Name .... *School* .... *Class* ......

Date .... *Date of Birth* ....
day *month* *year*

Boy or Girl ....

---

1. Fill in the gaps:

$x \longrightarrow x + 2$ $\qquad$ $x \longrightarrow 4x$

$6 \longrightarrow .$ $\qquad$ $3 \longrightarrow .$

$r \longrightarrow .$

---

2. Write down the smallest and the largest of these: smallest largest

n + 1, n + 4, n − 3, n, n − 7. ....

---

3. Which is the larger, 2n or n + 2 ? ....

Explain: ....

---

4. <u>4 added to n</u> can be written as <u>n + 4</u>. Add 4 onto each of these:

8 n + 5 3n

....

<u>n multiplied by 4</u> can be written as <u>4n</u>. Multiply each of these by 4:

8 n + 5 3n

....

---

5. If a + b = 43
a + b + 2 = ...

If n − 246 = 762
n − 247 = ...

If e + f = 8
e + f + g = ...

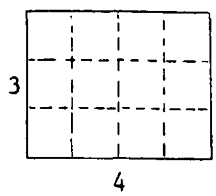
---

6. What can you say about a if a + 5 = 8 ....

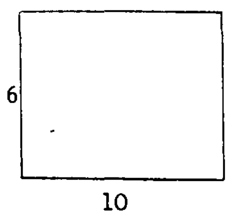
What can you say about b if b + 2 is equal to 2b ....

---

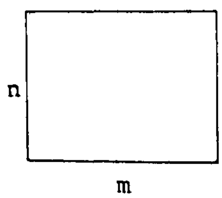
7. What are the areas of these shapes?



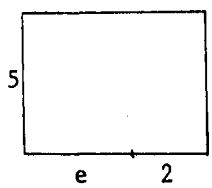
A = ...



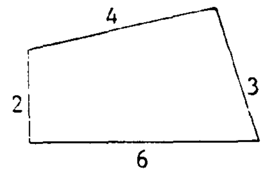
A = ...

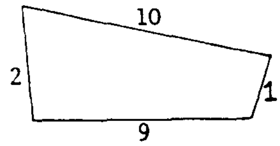


A = ...



A = ...

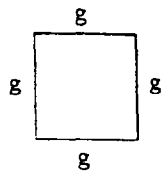




8. The perimeter of this shape is equal to 6 + 3 + 4 + 2, which equals 15.

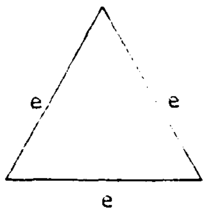
Work out the perimeter of this shape. p = ...

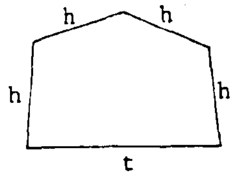
9.

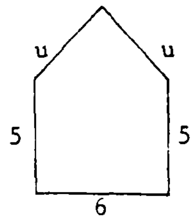


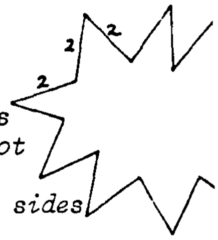
This square has sides of length g.
So, for its perimeter, we can write p = 4g.

What can we write for the perimeter of each of these shapes?









Part of this figure is not drawn. There are n *sides altogether, all of length 2.*

p = ...   p = ...   p = ...   p = ...

---

10. Cabbages cost 8 pence each and turnips cost 6 pence each.

If c stands for the *number* of cabbages bought and t stands for the *number* of turnips bought, what does

8c + 6t stand for? ....

What is the total number of vegetables bought? ....

---

11. What can you say about u if u = v + 3 and v = 1 ....

What can you say about m if m = 3n + 1 and n = 4 ....

---

12. If John has J marbles and Peter has P marbles, what could you write for the number of marbles they have altogether? ....

13. a + 3a can be written more simply as 4a.

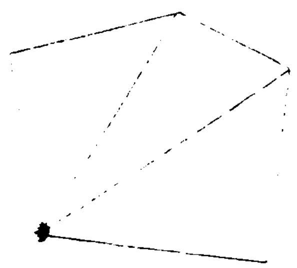
Write these more simply, where possible:

2a + 5a =	....		
2a + 5b =	....	3a − (b + a) =	....
(a + b) + a =	....	a + 4 + a − 4 =	....
2a + 5b + a =	....	3a − b + a =	....
(a − b) + b =	....	(a + b) + (a − b) =	....

---

14. What can you say about r if r = s + t
and r + s + t = 30 ....

15.



In a shape like this
you can work out the number of diagonals by
taking away 3 from the number of sides.

So, a shape with 5 sides has 2 diagonals;

a shape with 57 sides has .... diagonals;

a shape with k sides has .... diagonals.

---

16. What can you say about c if c + d = 10
and c is less than d ....

---

17. Mary's basic wage is £20 per week.
She is also paid another £2 for each hour of overtime that she works.

If h stands for the number of hours of overtime that she works, and
if W stands for her total wage (in £'s)
write down an equation connecting W and h: ....

What would Mary's total wage be if she
worked 4 hours of overtime? ....

18. When are the following true -always, never, or sometimes?
Underline the correct answer:

A + B + C = C + A + B   Always.   Never.   Sometimes, when ....

L + M + N = L + P + N   Always.   Never.   Sometimes, when ....

---

19. a = b + 3. What happens to a if b is increased by 2? ....

f = 3g + 1. What happens to f if g is increased by 2? ....

---

20. Cakes cost c pence each and buns cost b pence each.
If I buy 4 cakes and 3 buns,
what does

4c + 3b stand for? ....

---

21. If this equation ⟶ $(x + 1)^3 + x = 349$
is true when $x = 6$,

then
what value of x
will make this equation ⟶ $(5x + 1)^3 + 5x = 349$
true?

$x =$ ....

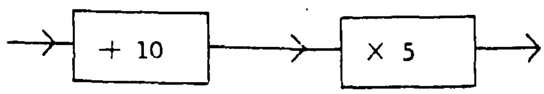
---

22. Blue pencils cost 5 pence each and red pencils cost 6 pence each.
I buy some blue and some red pencils and altogether it costs me 90 pence.

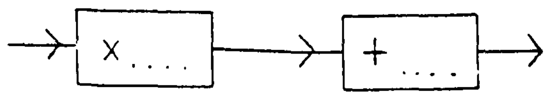
If b is the number of blue pencils bought, and
if r is the number of red pencils bought,
what can you write down about b and r? ....

---

23. You can feed any number into this machine:



Can you find another machine that has the same overall effect?



Appendix G

Experiment 1 - The Teaching Module

1. Introduction of the keyboard with particular attention to the shift key. The word PRINT and its meaning to the computer. Practical investigation of the following :

```
PRINT"HELLO"
PRINT"I AM FRED"
PRINT FRED
PRINT A
```

A discussion of the error ´No such variable´ produced by the last two commands.

The idea that the computer uses variables to store information.What are they? We shall use the following picture of one. (See figure 4.2) i.e. They consist of a location in the computer which has a label to identify it and a current value inside it. This value will vary or change.

How do we get the value in a variable?

Type : FRED = 3

PRINT FRED

What has happened?

Now try : A = 5

B = A + 1   Ask : What is the value now in B?

Try :   PRINT B   to find out.

Explain the use of +,-,*,/.

```
Try :  x = 4
       y = 3*x - 2
```

Ask : What is the value in y?

Encourage experimentation with creating variables and finding the values in them.

2. Explain that a program is a set of instructions to the computer. In BASIC (the language we are using) the lines of a program have numbers to identify them.

We can put values into locations in a program :

```
e.g. 10 A = 5
     20 B = A + 2          i.e.      5          7
     30 PRINT B

                                     A          B
```

To get the computer to obey this we type RUN and press RETURN.To change a line we re-type it.

Try changing line 10 by typing :

```
10 A = 11.5
```

and RUN again.

Explain the command INPUT, which enables us to put values into variables while a program is running.

Explain the statement GOTO, which sends execution of the program to the line given. e.g. GOTO 10

Now type in the programs on sheet 1 (See appendix E) and fill in the labels and values on the store sheet provided.(See figure 4.3).

3. We can use the ideas of the previous lesson to get the computer to work things out for us.

e.g. a) If we know how far a car has travelled and we know how long it took , how would we work out its average speed? Answer, we divide the distance by the time, making sure the units are correct. We can get the computer to do all the work for us by saying :

S = D/T   as follows :

```
10 INPUT D
20 INPUT T       c.f.     60          2          ?
30 S = D/T
40 PRINT S                D           T          S
```

RUN this program and choose your own values for D and T.

b) If we know the length, L and the width, W of a rectangle we can get the computer to work out the are as follows :

```
10 INPUT L
20 INPUT W
30 A = L*W       c.f.     2           7          14
40 PRINT A
                          L           W          A
```

Now try worksheet 2 (See appendix E). Don't forget to fill in the store labels and values.

4. Part of the following program is missing, but the values in the locations labelled X and Y which correspond to each other are given. What is the missing line in the program? We can think of this as a ´machine´ where we feed in numbers one end (X) and the ´machine´ feeds out the results the other (Y).

	X	Y
10 INPUT X	-	-
20 Y =	1	3
30 PRINT X,Y	2	5
	3	7
	4	9
	5	11

What is the rule for getting Y from X? Try your answer in the program.

Now try the same with the sets of values on worksheet 3 (See appendix E).

5. Run the program written for the research (See

appendix K for the listing).

In front of you is a 'Maths.Machine' program. Across the top are some store locations. What are their labels? We can change these labels by typing V. Try it. Change t to z. We can use this 'machine', which will show us the values in the variables at any stage, to solve some problems.

e.g. a) When does 5 + Y = 9?

The method? : Press M to select a Maths. Machine.
Choose 1 function.
Put in 5 + Y -- Why? What are we going to look for? A value of 9.
Press I.
Try values of Y until you get the answer we want , which is 9.

b) When does 3 - X = 2 + X ?

The method? : Press M.
Choose 2 functions.
Put 3 - X into one and 2 + X into the

other.

Press I.

Put in values of X until when ? They both have the same answer.Try to work out what value of X will work.

Now try worksheet 4 in the same way (See appendix E).

Appendix H

Experiment 1 - Results

Experiment 1 - The Results

Name	Pre-Test(%)	Post-Test(%)	Delayed Post-Test(%)
Emma	76	62	61
Louisa	82	82	76
Helen	73	65	81
Elizabeth	84	66	76
Julia	38	52	46
Sarah	80	a	76
Neil	58	63	61
Mark	90	90	95
Richard	70	71	86
Chris	76	75	85
Warren	74	70	72
Sarah	80	72	a
Anna	66	71	70
Victoria	62	a	71
Stephen	66	74	72
Emma	58	65	55
Gareth	75	74	76
Jeremy	98	93	97
David	80	78	83
Sophia	51	62	75
Helen	74	82	81
Robert	72	69	76

Name	Pre-Test(%)	Post-Test(%)	Delayed Post-Test(%)
Richard	68	73	78
Craig	78	83	87
Elaine	78	84	81
Simon	80	82	86
Jake	89	88	a
Lawrence	66	70	62
Chris	68	71	70
Marcus	84	76	73
Martyn	60	68	63
Lorna	56	60	65
Melissa	72	64	81
Anne-Marie	76	68	64
Nicola	50	44	50
Christina	88	91	91
Ian	84	76	82
Benjamin	70	74	61
Joanne	84	71	79
Craig	70	66	72

Experiment 1 - Question Facilities

Question	The Experimental Group Post Test (%)	The Experimental Group Delayed Post Test (%)	The Control Group Post Test (%)	The Control Group Delayed Post Test (%)
1.a)	94.4	89.4	80	89.4
b)	100	94.7	90	94.7
c)	94.4	100	85	73.5
2.	100	100	100	89.4
3.	5.6	21.2	10	10.6
4.a)	94.4	100	75	89.4
b)	94.4	100	85	94.7
c)	83.3	78.8	80	94.7
d)	88.9	100	80	94.7
e)	61.1	44.7	40	37.1
f)	88.9	73.5	60	73.5
5.a)	100	100	100	100
b)	83.3	73.5	85	89.4
c)	88.9	84.1	85	78.8
6.a)	100	94.7	95	100
b)	61.1	58.3	55	53
7.a)	100	100	100	94.7
b)	100	100	100	94.7
c)	100	100	100	100
d)	33.3	37.1	15	26.5
8.	94.4	100	100	100

Question	Experimental Group Post Test (%)	Experimental Group Delayed Post Test (%)	Control Group Post Test (%)	Control Group Delayed Post Test (%)
9.a)	88.9	100	95	100
b)	88.9	89.4	85	78.8
c)	77.8	84.1	85	84.1
d)	55.6	94.7	70	63.6
10.	--	--	--	--
11.a)	94.4	100	85	94.7
b)	100	94.7	90	94.7
12.	77.8	78.7	80	84.7
13.a)	94.4	94.7	95	100
b)	77.8	84.1	80	78.8
c)	77.8	78.8	70	84.1
d)	88.9	89.4	100	84.1
e)	66.7	58.3	60	58.3
f)	38.9	58.3	65	37.1
g)	100	89.4	90	84.1
h)	88.9	89.4	95	89.4
i)	88.9	73.5	70	84.1
14.	44.4	37.1	45	53
15.a)	94.4	100	90	100
b)	88.9	94.7	80	89.4
16.	55.6	47.7	40	21.2
17.a)	27.8	15.9	5	15.9
b)	100	84.1	80	78.8
18.a)	94.4	94.7	95	100
b)	38.9	63.6	40	42.4
19.a)	50	58.3	55	58.3
b)	22.2	15.9	0	15.9
20.	44.4	47.7	20	37.1
21.	5.6	10.6	15	5.3
22.	5.6	15.9	15	21.2

Appendix I

Experiment 2 - The Pre-Test

ALGEBRA INVESTIGATION

NAME : .... CLASS :.................

Attempt all the questions. Write your answers in the spaces provided.

1. Write down the largest and smallest of these : ....

   x - 1 , x , x + 4 , x - 3 , x + 1 ....

2. What is the cost of :

   a) 5 stamps at 8 pence each? ....

   b) 5 stamps at x pence each? ....

   c) n stamps at 8 pence each? ....

3. Multiply each of the following by 3 :

   7 .... a .... 4a ....

   a + 2 ....

4. Add 3 on to each of the following :

   8 .... x .... x + 2 ....

   3x ....

5. Write down which of the following could not be used to represent the unknown height of a tree :

   a , t , x , 7 ....

6. If John has J marbles and Peter has P marbles, write down how many marbles they have got altogether. ....

7. Underline the correct answers in this question. If sometimes is the correct answer, then write in when as well.

   a) Does a + b = a + c :

   always, never, sometimes - when.......................

   b) Does x + y = x :

   always, never, sometimes - when........................

c) Does a + b + c = b + c + a :

always, never, sometimes - when........................

d) Does L + M + N = L + P + N :

always, never, sometimes - when........................

8. What is the total cost of x pencils at 5 pence each, and y crayons at 7 pence each? ....

9. Find the area of the rectangles, whose sides measured in centimetres, are :

a) 8 by 3 ....

b) X by 5 ....

c) 4 by Y ....

d) P by Q ....

10. For what value or values of a is :

a) a + 5 = 8 ....

b) a + 5 > 8 ....

c) 6 = a - 2 ....

d) 6 > b - 2 ....

11. a) In the Maths. Machine :

x --------- x + 2

eg 6 --------- 8

Fill in the following :

9 --------- ....

a --------- ....

3a --------- ....

b) In the Maths. Machine :

x ---------- 3x

eg 3 ---------- 9

Fill in the following :

5 ---------- ....

y ---------- ....

y + 1 --------- ....

12. If x = 3 and y = 5 then :

a) x + y = ...

b) x + y + 2 = ...

c) $x + y + w =$ ....

13. If $x + y = 7$ then $x + y + z = ?$ ....

If also $a + b + c = 10$ then $c = ?$ ....

14. Find the perimeter of rectangles whose sides, measured in centimetres, are :

a) 5 by 4 ....

b) X by 2 ....

c) 3 by Y ....

d) G by H ....

15. If I have x pieces of wood each of length 5 metres and another y pieces each of length 3 metres, then what does $5x + 3y$ represent?

....

16. Underline the correct answer in this question. If sometimes is the correct answer, then write in when as well.

a) Does $2x + 2 = 2(x + 1)$:

always, never, sometimes - when.............................

b) Does $x + 2 = x + 3$:

always, never, sometimes - when.............................

c) Does $x + x = 2x$:

always, never, sometimes - when.............................

17. Which is larger : $2p$ or $p + 2$? Explain. ....

18. A figure has n sides each of length 2 centimetres. What is the perimeter of the figure?

....

19. The cost of sending a parcel through the post is 80 pence, plus 25 pence for every 1 kilogram it weighs. What is the cost of sending a 6 kilogram parcel through the post? ....

The total cost of sending another parcel is C pence. If it weighs W kilograms, write down an equation connecting C and W.

Appendix J

Experiment 2 - The Post-Test And Delayed Post-Test

ALGEBRA INVESTIGATION

NAME : ....   CLASS :.................

Attempt all the questions. Write your answers in the spaces provided.

1. Write down the largest and smallest of these : ....

   $y - 3$, y , $y + 1$, $y - 2$, $y - 1$ ....

2. What is the cost of :

   a) 6 stamps at 9 pence each? ....

   b) 8 stamps at x pence each? ....

   c) m stamps at 11 pence each? ....

3. Multiply each of the following by 5 :

   6 ....   b ....   3b ....

   b - 3 ....

4. What does $\frac{4}{x}$ equal when x is :

   a) 2 ....   b) 5 ....   c) p ....

5. $y + 2y$ can be written more simply as $3y$. Write more simply , where possible :

   a) $3y + 5y$ ....   b) $3y + 4x + 2y$ ....

   c) $3x - x + y$ ....   d) $3y + 4 + y$ ....

6. Add 4 on to each of the following :

   7 ....   x ....   x + 5 ....

   8x ....

7. Write down which of the following could not be used to represent the unknown weight of a lorry :

   p , L , 9 , x ....

8. If John has G jigsaws and Peter has H jigsaws, write down how many jigsaws they have got altogether. ....

9. Underline the correct answers in this question. If sometimes is the correct answer, then write in when as well.

a) Does $b - c = a - c$:

always, never, sometimes - when.......................

b) Does $y + x = x$:

always, never, sometimes - when.......................

c) Does $p - m + n = n + p - m$:

always, never, sometimes - when.......................

d) Does $M + P + N = N + M + R$:

always, never, sometimes - when.......................

10. What is the total cost of x pencils at 8 pence each, and y crayons at 9 pence each? ....

11. Find the area of the rectangles, whose sides measured in centimetres, are :

a) 5 by 7 ....

b) X by 9 ....

c) 6 by Y ....

d) W by X ....

12. a) If $x = 4$ and $y = x + 3$ then $y = ?$ ....

b) If $p = 2$ and $m = 3p + 2$ then $m = ?$ ....

c) If $s = 5$ and $t = 3s + b$ then $t = ?$ ....

d) If $h = 2$ and $3h = c + 3$ then $c = ?$ ....

14. For what value or values of a is :

a) $a + 3 = 7$ ....

b) $a + 3 > 7$ ....

c) $9 = a - 4$ ....

d) $5 > a + 3$ ....

15. a) In the Maths. Machine :

x ---------> x + 5

eg 4 ---------> 9

Fill in the following :

8 ---------> ....

m ---------> ....

3n ---------> ....

b) In the Maths. Machine :

x ----------> 2x

eg 4 ----------> 8

Fill in the following :

7 ----------> ....

y ----------> ....

y + 2 ---------> ....

16. If x = 2 and y = 6 then :

a) x + y = ...

b) x + y + 1 = ...

c) x + y + w = ...

17. a) If x + y = 5 then x + y + z = ? ....

b) If a + b = 3 and a + b + c = 10 then c = ? ....

18. Find the perimeter of rectangles whose sides, measured in centimetres, are :

a) 3 by 7 ....

b) X by 4 ....

c) 5 by Y ....

d) X by Y ....

19. If I have to pay 5 pence to have a paper delivered to my house, how much will I have to pay altogether if the paper costs :

a) 12 pence .... b) 20 pence .... c) x pence ....

20. If I have x pieces of wood each of length 7 metres and another y pieces each of length 5 metres, then what does 7x + 5y represent?

....

21. Write down the next term in the following series of numbers :

a) 2 , 5 , 8 , 11 , ...

b) x + 1 , x + 2 , x + 3 , x + 4 , ...

c) 2y , 3y , 4y , 5y , ...

d) 9z + 5 , 8z + 4 , 7z + 3 , 6z + 2 , ...

22.a) If $x - 189 = 675$ then $x - 190 = ?$ ....

b) If $y + 279 = 978$ then $y + 277 = ?$ ....

23. Underline the correct answer in this question. If sometimes is the correct answer, then write in when as well.

a) Does $2x + 2 = 2(x + 1)$:

always, never, sometimes - when...............................

b) Does $x + 5 = x + 2$:

always, never, sometimes - when...............................

c) Does $x + x = 2x$:

always, never, sometimes - when...............................

24. Which is larger : $2z$ or $z + 2$? Explain. ....

25. A figure has p sides each of length 3 centimetres. What is the perimeter of the figure?

....

26. The cost of sending a parcel through the post is 60 pence, plus 25 pence for every 1 kilogram it weighs. What is the cost of sending a 4 kilogram parcel through the post? ....

The total cost of sending another parcel is C pence. If it weighs W kilograms, write down an equation connecting C and W.

....

Appendix K

Experiment 2 - The Individual Results

Experiment 2 - The Results

Name	Pre-Test (Max.=51)	Post-Test (Max.=71)	Delayed Post-Test (Max.=71)
Lisa	16	15	32
Darren	9	31	32
Stuart	14	19	32
Andrew	14	47	48
Kristian	6	14	a
Michael	2	21	a
Mark	14	21	20.5
Kirsty	15	54	60
Debi	5	17	22
Mark	5	32	23
Jason	4	10	18
Julie	3	24	35
Brian	20	29	39.5
Robert	26	40	41
Lisa	11	26	27
Jonathan	11	35	32
Steven	13	23	33
Claire	12	51	36
Lorraine	11	22	33
Claire	11	34	33.5
Simon	12	26.5	43
Robert	12	35	38

Experiment 2 - The Results (Cont.)

Name	Pre-Test (Max.=51)	Post-Test (Max.=71)	Delayed Post-Test (Max.=71)
Damon	11	22	25
Mark	11	17	29
Deborah	17	19	23
Stephen	15	39.5	38
Elizabeth	11	27	29
Emma	11	35	26
Paul	11	17	20
Jason	10	12	20
Tammy	6	31	39
Rachel	6	24	26
Gary	9	12	18
Jonathan	9	40.5	37
Nicholas	7	11	11
Elvis	8	25	26
Jayne	7	18	17
Victoria	7	29.5	42
Louise	4	25	25
Rachael	4	31	38
Matthew	6	15	17.5
Marc	7	26	33.5

Appendix L

Experiment 2 - The Teaching Module

Algebra Programme Lesson Notes

Aim of the Programme

To investigate whether the use of elementary programming of a computer in BASIC, including the use of a ´picture´ for the variables used ,can help a child´s understanding of the concept of a letter as a variable in algebra.

Outline of Programme For The Teacher

A group of middle school children is given a test (pre-test), and on the basis of the results they are divided into two matched groups of equal numbers.

The control group continue with their normal lessons receiving no special tuition. The test group are given a series of lessons built around use of the BBC computer and a ´Maths. Machine´, as described below.

The breakdown of the lessons into sections is done as a rough guide only. The amount of material covered per session is left to the discretion of the teacher, and will depend on the ability of the children and the facilities available.

The Lesson Plans 1. The first thing to do is to introduce the computer keyboard to them.They need to know how the SHIFT and RETURN keys are used to get a + sign, an = sign and to enter instructions.

(I envision this done as one group around the computers with various pupils used in turn to do any typing)

Use of PRINT

Get them to try the following :

PRINT"Hello there" <R> (Press RETURN)

PRINT"I am great" <R>

etc.

Hence we can 'discover' that PRINT is a command to put the symbols ('words') in "" on the screen.

What happens when we try :

PRINT Good idea <R>

We get the message NO SUCH VARIABLE

Try again : PRINT A <R>

The same message!

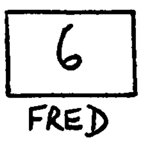
What is different? There are no "". What does the computer mean by NO SUCH VARIABLE?

(I think a move away from the computer to a blackboard is advisable here)

<u>Variables</u>

The message means that the computer looked in its memory for a variable called Good idea, A etc., because there were no "" around the words. Variables are what it uses to store its information. What are they?

We can think of them as :



ie 2 parts : (i) Label - FRED

(ii) Current value- 6

N.B. THIS IS THE FUNDAMENTAL MENTAL PICTURE FOR THE VARIABLE CONCEPT.

When we type : PRINT FRED <R>, the computer looks in iots memory for a location/box/variable called FRED so that it can put <u>a copy</u> of the value in it onto the screen. Hence NO SUCH VARIABLE means it failed to find one with the name given.

<u>Making Variables</u>

Ask them : How do you think we might make a variable called FRED?

Get the suggestions and have someone try them on the computer using PRINT FRED <R> to see if they work i.e. if we get a value.

If they are unsuccessful then tell them to try :

FRED = 3 <R>

PRINT FRED <R> (n.b. I shall drop the RETURN key symbol now.)

and A = 6

PRINT A

Explain that A = 6 means to the computer : Put 6 in a box labelled A.

PRINT A means take a copy of the number in A and put it on the screen.

<u>The 'Maths. Machine'</u>

The pupils can make themselves into a simple computer, which we shall call a 'Maths. Machine' by using the card sheets and the letters and numbers with them.

Method :

B = 7 - someone is assigned to put the label B on one of the boxes.

- someone else takes the number 7 and puts it in the box with the label B. (n.b. If a box already has a label then we do not need to give it another !)

PRINT B - someone is assigned to go (the 'Machine' could be across the room) and look in the box labelled B and bring back a copy of the number in it to put on the blank 'screen ' card.

a) <u>Revision</u>

Brief recap on :

(i) Variable picture

(ii) Mechanics of the 'Maths. Machine'

At this point it becomes necessary to divide the group up into several smaller groups. Depending on the total number of them it should be possible to have 3/4 to a computer working together and 3/4 to each of the 'Maths. Machines'.

I suggest that there is a rotation of the groups so that every group gets to use a computer every other time.

In their groups get them to investigate :

```
X = 5
Y = X + 1
PRINT X
PRINT Y
```

An explanation of the mechanics of the second line is necessary for those on the 'Machines'. Ask them : How do we carry it out?

Answer : Find out the value in X; add 1 to this; make a variable with label Y; put this value in Y.

Do it, possibly as a race. Do the computers agree with the 'Machines'?

Try this the same way :

```
A = 7
B = A - 3
PRINT A
PRINT B
```

(Note: THIS IS THE BEGINNING OF DEPENDENT AND INDEPENDENT VARIABLES)

b)<u>New Symbols</u>

Introduce the * sign meaning multiply and the / sign meaning divide. i.e. Ask what does 5*7 mean to the computer?

Answer? What would 36/9 be?

Try :

```
A = 7
B = 3 * A
PRINT B
```

c) <u>Programs</u>

To make a program on a computer we do the same as before, but we introduce numbers for each line of instructions. (Called line numbers)

To get the computer to carry out the program we must type RUN <R> , after we have typed in the whole program.

Get them to try (in their rotating groups) :

d)

```
10 X = 3
20 Y = 5
30 Z = X + Y
40 PRINT Z
```

(ii)

```
10 X = 3
20 Y = 5
30 Z = Y + X
40 PRINT Z
```

n.b. For those using the computers, the ability to edit a line, or at least to change a program by retyping one line only (30 here), is now useful.

Do these give the same answers? Why? Try changing the values of X and Y. Are the answers still the same? This means that the order in which we add the values in the variables does not matter.

(This is the idea of commutativity)

(iii)

```
10 A = 7
20 B = 9
30 C = 15
40 D = A + B + C
50 PRINT D
```

(iv)

```
10 A = 7
20 B = 9
30 C = 15
40 D = B + A + C
50 PRINT D
```

What do you notice? Try different values for A,B,C. Why are the answers the same? - because the same variables/boxes are used!

a) <u>Revision</u>

Try : (i)
```
10 X = 8
20 Y = 2 * X
30 PRINT Y
```
(ii)
```
10 X = 8
20 Y = X * 2
30 PRINT Y
```
(iii)
```
10 X = 8
20 Y = X + 2
30 PRINT Y
```

What do you notice? What does this mean? (Multiplication is also commutative. Also the notation we use is important as it changes the answer)

b) <u>Link to algebra</u>

What does 3*x mean to the computer? - take the value in x (a copy of!) and multiply it by 3. In Maths. we write this as 3x.

How would we tell the computer to add 1 to x first, and then multiply it by 3?

Expect - 3*x + 1 as the answer - in Maths. this is 3x + 1

Try it in a program:

```
10 x = 6
20 y = 3*x + 1
30 PRINT y
```

What answer should we get?

It does not work! Why? Suggestions? Use brackets around the x + 1.

Try it.This time it works. So 3*x + 1 and 3*(x + 1) are not the same.

i.e. In Maths. 3x + 1 and 3(x + 1) are not the same. The bracket means what? Do this bit first.

Write on the board (?), and ask what these would be in Maths.

2*x + 1, 5 - 2*y, a*b, 2*a + 2*b, 3*x - 2, 5/x, 7 + 8/y .

What would these be in the BASIC computer language?

$\frac{2}{x}$, 4y - 1, 3x + 2y, 4(a + b), xy, 7 - 5x, 6 + $\frac{y}{4}$.

a) <u>INPUT</u>

Explain that the instruction 10 INPUT X means that the computer will put a ? om the screen and wait for us to type in a value for X. When we do so, and press RETURN the value is stored in the location/box X.

To check this try :

```
10 INPUT x
20 PRINT x
```

so that they see it works.

N.B.THE MIXING OF UPPER AND LOWER CASE LETTERS IS IMPORTANT IN ORDER TO BE FULLY CONFIDENT IN WORKING WITH EITHER. THE NORM IN ALGEBRA IS THE LOWER CASE, OF COURSE.

We can put in lots of values for x by adding just 1 line to this program:

```
30 GOTO 10
```

This tells the computer to go back to line 10 and start again. What does INPUT mean for our ´Maths. Machine´?

It means that we check and see if there is a box labelled x.If not then we label one x.We are then free to choose any number to put in this box.

N.B. WE HAVE HERE THE START OF THE IDEA OF LETTER AS GENERALISED NUMBER.IE THAT A SINGLE LETTER CAN BE USED TO REPRESENT MORE THAN 1 NUMBER; IN FACT A SET OF NUMBERS.

b) <u>Using INPUT</u>

Try this program to see if they have the idea.

```
10 INPUT A
20 B = 2*A          Put in 3 values for A
30 PRINT B
```

```
40 GOTO 10
```

N.B. THE RACE BETWEEN GROUPS MAY BE CONTINUED THROUGHOUT.

Now they have the idea they can try these 2 programs, putting any 3, BUT THE SAME 3, values for x into each.

(i)		(ii)	
	10 INPUT X		10 INPUT X
	20 Y = X + X		20 Y = X * 2
	30 PRINT Y		30 PRINT Y
	40 GOTO 10		40 GOTO 10

Then ask what is noticed.Means that X + X and x*2 give the same answers; they must be the same. Does 2*X give the same? Try it if it is not obvious.

Get them to work through sheet 1, making sure that :

a) They rotate where possible, computer to 'Machine'

b) They record their results on the stores sheets.

c) They put their names on the stores sheets.

<u>Application to problems</u>

HERE WE WANT TO CEMENT MORE FIRMLY THE RELATIONSHIP BETWEEN THE WORK WITH VARIABLES ON THE COMPUTER AND THE USE OF VARIABLES IN MATHEMATICS, PARTICULARLY IN ALGEBRA.

a) An example

If we know the length and width of a rectangle,how do we find its area? How can we use the computer/'Maths. Machine' to help us do this for any rectangle?

The answer we want is that we use a box/location/variable called L to hold the length and another called W to hold the width.

Can they write the program to work out the area of any rectangle, using INPUT?

We want :

```
10 INPUT L
20 INPUT W
30 A = L*W        - now try this 2/3 times ( use
40 PRINT A           integral values for L and W )
50 GOTO 10
```

b) A second example

If the computer has a box labelled S, in which is stored the number of sweets we are to buy, and these sweets are 5 pence each, how can we tell the computer how much they cost?

IT MAY WELL BE NECESSARY TO USE NUMERICAL EXAMPLES HERE TO ARRIVE AT ´THE FORMULA´

We want the answer : 5*S (or 5S making the point that these are equivalent)

If also, the computer has a box labelled C, containing the number of comics we are to buy, and these are 12 pence each, what would we use for the cost of the comics?

Answer - 12*C (or 12C)

So, can we write a program to get the computer to work out the total cost of the sweets and the comics, using INPUT?

Let them try, and help them towards :

```
10 INPUT S
20 INPUT C
30 P = 5*S + 12*C     - run this 2/3 times ( I suggest
40 PRINT P              here that you give them values
50 GOTO 10              for S and C, and have someone
                        checking to see if the computer
                        is right )
```

They can now try sheet 3. They may need some assistance with the formulae.

a) Notation

We can use the notation :

X ------→ X + 3   to represent the 'rule' of a 'Maths Machine'. ie we would have a line

30 Y = X + 3

in the computer program.

What would 3 ------→ ? 5 ------→ ?

b) Problems Based On This.

The part of the program below which corresponds to X----→ ? is missing. However the results the computer gave are given as a list. What is the missing line?

	X	Y
10 INPUT X	1	3
20 Y =	2	4
30 PRINT X,Y	3	5
40 GOTO 10	4	6
	5	7

EXPLAIN LINE 30 MEANS PUT THE VALUE IN Y ALONGSIDE THE VALUE IN X

Get some suggestions and let them try them to see if they work.

n.b. They need to realise that they must get the right answer for every value of X.

Then try :

	X	Y
10 INPUT X	1	3
20 Y =	2	5
30 PRINT Y	3	7
40 GOTO 10	4	9
	5	11

as above.

They need only change line 20 in their program of course to try this.

Answers : X ------> X + 2 and X ------> 2*X + 1 (2X + 1)

They can now try sheet 4 in the same way.

THIS IS ANOTHER WAY OF GETTING THEM TO APPRECIATE THAT BEHIND THE LETTERS IN A FORMULA, I.E.BEHIND THE LETTERS IN ALGEBRA LIE A RANGE OF NUMERICAL VALUES (A VARIABLE) OR AT LEAST THAT SEVERAL VALUES MAY RELATE TO THE SAME LETTER (GENERALISED NUMBER). WE ALSO HAVE HERE, IN THE 2 LISTS, THE VALUES FOR THE INDEPENDENT VARIABLE (X) AND THE DEPENDENT VARIABLE (Y).

The 'Maths. Machine' Program

Since we are only using the 'Machine' in a simple way, it should not be necessary to use the V:Change Variables command.We shall need to know the workings of the M and the I. The M is used to put in the function expressions which we are looking at. The I is used to put in the values of the variables. Some examples :

1. When does 5 + x = 9? (THE PROGRAM WORKS IN LOWER CASE)

a) Press M to select a function

b) Choose 1, and press RETURN

c) Type in 5 + x <R>

Ask how can we use the 'Machine' to find when 5 + x = 9? Answer put in values of x until the value in the 'function' box = 9

d) Press I to input values of x

e) Put in a suggested value <R> Ask how can we tell what value to try next? If the answer was too big we try a smaller value. If it was too small we try a bigger one!

f) We answer Y <R> to the question of any more values, until we find the correct answer.We then note down the value in the x box as this is our answer.

2. When does $3 - x = 2 + x$? (A LITTLE HARDER!)

a) Press M and choose 2 functions

b) Put $3 - x$ into one and $2 + x$ into the other

c) Press I

d) Put in values of x, as before, until when? Until the function boxes both contain the same value.The value in the x box is then the answer we want to write down. n.b. They will need a little direction here as to why we choose bigger or smaller values of x. Experimentation is to be encouraged, the secret being which of the 2 boxes has the bigger value for any given x.

3. When is $y - 2 > 5$? (A VERY DIFFICULT QUESTION)

n.b. We are happy here to end up with a string of suitable values for y. It is not expected that many (any ?) of them will arrive at $y > 7$.

a) Choose 1 function.

b) Put $y - 2$ into it.

c) Press I

d) Put in values for y until we get an answer in the function box which is bigger than 5. We note the value in the y box for ANY such value. They can be asked if there is a rule for the answers, someone may spot it.

They can now try sheet 5. With ingenuity this could be done with the cardboard 'Machines', but it is better on the computer.The organisation of this will take a little thought.

THE VALUE OF THIS PROGRAM IS THAT IT CONTINUES THE 'PICTURE' OF

A VARIABLE WHICH WE HAVE PREVIOUSLY USED, BUT NOW THEY CAN ACTUALLY SEE THE VALUES IN THE VARIABLES AS THEY VARY I.E. THE VARYING WILL BE IN THEIR EXPERIENCE. FURTHERMORE THEY ARE USING THIS VARIABILITY TO TACKLE ´REAL´ PROBLEMS AND THUS IT IS HOPED THAT IT WILL HELP THEM TO HAVE ENOUGH CONFIDENCE IN THE MENTAL PICTURE TO USE IT IN THE FUTURE, PARTICULARLY IN THEIR ALGEBRA LESSONS.

Appendix M

Experiment 2 - The Worksheets

WORKSHEET NUMBER 1

1. Write in the Maths. version of the following BASIC expressions :

a) a*b ....

b) 2*a + 1 ....

c) y / x ....

d) 3*(x + 5) ....

e) 3*x - 2*y ....

f) 3 + 2*a ....

2. Write in the BASIC version of the following Maths. expressions :

a) 2x + 1 ....

b) 3(x - 2) ....

c) 2x + 3y ....

d) $\frac{4}{x}$ ....

e) 5a - 3 ....

f) 2(x + 1) ....

3.Use the following program to show that 3(x + 1) and 3x + 1 are not the same :

```
10 x = 4
20 y = 3*x + 1
30 z = 3*(x + 1)
40 PRINT y
50 PRINT z
```

Run this program 3 more times, changing the value of x in line 10.

4.Change the program lines 20 and 30 above to see if 2x + 2 is the same as 2(x + 2). Run this program 3 times too.

WORKSHEET NUMBER 2

Type in each of the following programs if you are using a computer. If you are using the 'Maths. Machine' then work through each program. . Run each of them 3 times. Fill in your results in the store locations on the sheet with the boxes. Do not forget to put the label for each variable under its box.

1.
```
10 A = 3
20 B = 5
30 C = A + B
40 PRINT C
```

Now change the values of A and B in lines 10 and 20 and RUN the program another twice.

2.
```
10 INPUT X
20 Y = X + 3
30 PRINT Y
40 GOTO 10
```

now try this program with the same X values :

```
10 INPUT X
20 Y = 3*X
30 PRINT Y
40 GOTO 10
```

Run the first program 3 times with different values for X. Then run the second program using the same 3 values for X. Put your results on the stores sheet.

3.
```
10 INPUT p
20 t = 10 - p
30 PRINT t
40 GOTO 10
```

now try this :

```
10 INPUT p
20 t = p - 10
30 PRINT t
40 GOTO 10
```

Put 3 different values into the location p for the first program and record your results on the sheet again. Then run the second program using the same p values.

4.
```
10 INPUT z
20 w = 2*z + 2
30 PRINT w
40 GOTO 10
```

now try this :

```
10 INPUT z
20 w = 2*( z + 1)
30 PRINT w
40 GOTO 10
```

Put 3 different values in for z. and record your results again. Now run the second program using the same values for z. What do you notice? What does this mean?

5.
```
10 INPUT a
20 INPUT b
30 c = 2*( a + b )
40 PRINT c
50 GOTO 10
```

now try this :

```
10 INPUT a
20 INPUT b
30 c = 2*a + 2*b
40 PRINT c
50 GOTO 10
```

Put 3 different sets of values for a and b in the first program and record your results. Now put the same sets of values into the second program. What do you notice? What does this mean?

6.
```
10 INPUT x
20 INPUT y
30 z = x + y
40 PRINT z
50 GOTO 10
```

now try this :

```
10 INPUT x
20 INPUT w
30 z = x + w
40 PRINT z
50 GOTO 10
```

Put 3 different sets of values for x and y into the first program and record your results. Then put 3 sets of values into the second program using the same x values. Do you get any answers the same? What would you need to do to get the same answers?

WORKSHEET NUMBER 3

Work through the following questions, filling in your answers on the sheet with the variable boxes.

Do not forget to put the label for each variable under its box.

1. If we know the size of 2 angles of a triangle then we can find the 3rd angle by taking each of the 2 angles away from 180. Use this program to find the 3rd angle in some triangles.

```
10 INPUT x
20 INPUT y
30 z = 180 - x - y
40 PRINT z
50 GOTO 10
```

Run this program 3 times with different values for x and y. Record your answers on the stores sheet.

2. The perimeter of a rectangle ,length L and width W is given by :

$$P = 2L + 2W$$

Fill in the correct line 30 in the program below and then use it to find the perimeter of some rectangles.

```
10 INPUT L
20 INPUT W
30 P=
40 PRINT P
50 GOTO 10
```

Run this program 3 times with different values for L , and W. Put your results on the stores sheet.

3. The distance D travelled by a car in time T when it is moving at speed S is given by :

$$D = S \times T$$

```
10 INPUT S
20 INPUT T
30 D=
40 PRINT D
50 GOTO 10
```

Run this with 3 different sets of values for S and T. Record your results again.

4. The following program will help us wok out the V.A.T. on an article using the formula : V = 15% of P (ie V = 0.15P) Notice the BASIC version of this in line 20.

```
10 INPUT P
20 V = 0.15 * P
30 PRINT V
40 GOTO 10
```

Now run this program with 3 different values for the variable P and reoord the results on the variables sheet.

WORKSHEET NUMBER 4

The following program produced the lists of variable values shown. Fill in what you think is the correct line 20 and RUN the program to check it. If it is wrong try again. If it is right then write your answer on your sheet of paper.

```
10 INPUT x
20 y =
30 PRINT x
40 PRINT y
50 GOTO 10
```

1.

x	y
1	0
2	1
3	2
4	3
5	4

2.

x	y
1	3
2	6
3	9
4	12
5	15

2.

x	y
1	1
2	3
3	5
4	7
5	9

4.

x	y
1	4
2	5
3	6
4	7
5	8

5.

x	y
1	9
2	8
3	7
4	6
5	5

6.

x	y
1	1
2	4
3	9
4	16
5	25

7.

x	y
1	5
2	8
3	11
4	14
5	17

8.

x	y
1	3
2	9
3	19
4	33
5	51

WORKSHEET NUMBER 5

Use the Maths. Machine program in front of you to try and solve the following problems. Write your answers in the spaces given.

1. a) When does $3 + x = 8$? ....

   b) When does $x + 2 = 7$? ....

   c) When does $14 - 2y = 6$? ....

   d) When does $3x - 4 = 8$? ....

2. a) When does $x + y = x + z$? ....

   b) When does $x + y = x$? ....

3. Which is bigger , and by how much ?

   a) $x - 2$ or $x + 1$ ....

   b) $x - 3$ or $x - 2$ ....

   c) $x + y$ or $x + y + 2$ ....

4. For what value(s) of x does :

   a) $x + 2 = 2x + 1$ ....

   b) $3 + x = 7 - x$ ....

   c) $3x + 1 = x + 9$ ....

5. For what values of x is :

   a) $2x + 1 > 5$ ....

   b) $3x - 2 < 7$ ....

   c) $13 - 2x < 3$ ....

6. For what values of y is :

   a) $3y > y + 2$ ....

   b) $2y < y + 3$ ....

7. For what values of t is :

   $3t + 6 = 3(t + 2)$ ....

8. a) When does $2a + b = 2 (a + b)$? ....

   b) When does $2a + 2b = 2 (a + b)$? ....

Appendix N

Experiment 2 - The Question Facilities

Experiment 2 - Question Facilities

Question	The Experimental Group: Post Test (%)	The Experimental Group: Delayed Post Test (%)	The Control Group: Post Test (%)	The Control Group: Delayed Post Test (%)
1.a)	90.5	90	71.4	85
b)	66.7	95	52.4	80
2.a)	81	75	66.7	95
b)	81	100	9.5	55
c)	66.7	90	9.5	50
3.a)	95.2	95	95.2	95
b)	81	90	19.1	50
c)	38.1	80	9.5	45
d)	0	0	0	0
4.a)	38.1	50	38.1	75
b)	4.8	25	14.3	20
c)	9.5	25	14.3	20
5.a)	85.7	90	71.4	75
b)	33.3	35	4.8	5
c)	14.3	10	0	0
d)	23.8	10	0	0
6.a)	100	95	95.2	95
b)	14.3	10	9.5	0
c)	23.8	30	9.5	5
d)	9.5	15	0	0
7.a)	57.1	80	38.1	65
8.	42.9	25	14.3	10

Experiment 2 - Question Facilities (Cont.)

Question	Experimental Group Post Test (%)	Experimental Group Delayed Post Test (%)	Control Group Post Test (%)	Control Group Delayed Post Test (%)
9.a)	33.3	20	23.8	7.5
b)	47.6	37.5	14.3	10
c)	71.4	75	66.7	50
d)	42.9	22.5	9.5	5
10.	42.9	55	4.8	15
11.a)	90.5	80	71.4	70
b)	81	65	0	40
c)	81	65	0	40
d)	71.4	60	19.1	45
12.a)	81	65	42.9	55
b)	47.6	20	0	0
c)	4.8	20	0	0
d)	23.8	30	0	20
13.	---	---	---	---
14.a)	90.5	90	95.2	100
b)	9.5	5	0	0
c)	66.7	45	38.1	75
d)	4.8	0	0	0
15.a)	33.3	30	9.5	30
b)	33.3	25	0	15
c)	19.1	20	0	15
d)	47.6	45	4.8	25
e)	33.3	45	9.5	25
f)	0	0	0	5

Experiment 2 - Question Facilities (Cont.)

Question	Experimental Group Post Test (%)	Experimental Group Delayed Post Test (%)	Control Group Post Test (%)	Control Group Delayed Post Test (%)
16.a)	95.2	95	95.2	95
b)	81	95	66.7	80
c)	38.1	15	9.5	10
17.a)	28.6	15	9.5	10
b)	85.7	95	76.2	85
18.a)	19.1	50	14.3	0
b)	4.8	5	0	0
c)	0	5	0	0
d)	0	10	0	0
19.a)	57.1	85	52.4	55
b)	57.1	80	52.4	55
c)	23.8	15	9.5	0
20.	9.5	15	4.8	10
21.a)	76.2	95	81	90
b)	85.7	95	71.4	90
c)	85.7	95	76.2	90
d)	85.7	85	66.7	80
22.a)	52.4	40	33.3	40
b)	28.6	40	33.3	30
23.a)	23.8	10	14.3	50
b)	57.1	65	57.1	80
c)	66.7	60	66.7	50
24.	0	0	0	0
25.	42.9	45	4.8	45
26.a)	61.9	75	57.1	60
b)	0	0	0	0

Appendix O

Experiment 3 - The Raw Data For the Correlation of Test With CSMS Algebra Test

Pupil	CSMS Score (max.= 51)	Algebra Test Score (max.= 67)
SB	8	3
RB	5	7
JC	36.5	46
AC	41	48
CD	9	14.5
CE	14	9
MF	3	3
ZG	9	5
SG	34.5	42.5
SJ	7	7.5
CK	26	35
JC	24	18
CE	33	33
JL	16	14.5
SL	13	10
JM	28.5	32
EM	17	14
CP	24	21
SP	15	11.5
MR	10	7
PR	27	7
KR	8	6
AS	21	14
LS	27.5	24
NT	33	29
RW	6	7
GJ	a	32
SR	a	42

Appendix P

Experiment 3 - Worksheets

WORKSHEET NUMBER 1

Type in each of the following commands if you are using a computer. If you are using the 'Maths. Machine' then work through each set of commands.
Fill in your results in the store locations on the sheet with the boxes.
Do not forget to put the label for each variable under its box.

1. Work through the commands below and fill in your results on the stores sheet. Now change the value of A and work through each set of commands 2 more times.

a)
```
A = 7
B = A - 3
PRINT A
PRINT B
```

b)
```
A = 3
B = 5 + A
PRINT A
PRINT B
```

c)
```
A = 5
C = 3*A
PRINT C
```

d)
```
A = 36
Y = A/4
PRINT Y
```

2. Work through the commands below, recording your values again on the stores sheets. Now change the value of y, keeping the same value for each of a) and b). Work through the commands again, for 2 new values of y.

a)
```
y = 9
p = 2*y
PRINT p
```

b)
```
y = 9
p = y*2
PRINT p
```

3. Choose 3 different values of your own for each of the letters at the start of the following commands. Work through the commands for each value and put your results on the stores sheets.

a)
```
h =
t = 24 - h
PRINT t
```

b)
```
s =
w = 2*s + 1
PRINT w
```

c)
```
C =
D =
F = C*D
PRINT F
```

d)
```
G =
J =
K = G + J - 2
PRINT K
```

WORKSHEET NUMBER 2

Type in each of the following programs if you are using a computer. If you are using the ´Maths. Machine´ then work through each program. Run each of them 3 times.
Fill in your results in the store locations on the sheet with the boxes.
Do not forget to put the label for each variable under its box.

1. Run the program below and fill in your results on the sheet. Now change the values of X and Y in lines 10 and 20 and RUN the program 2 more times.

```
10 X = 7
20 Y = 6
30 R = 3*X + Y
40 PRINT R
```

2. Run the two programs below and fill in your results on the stores sheets. Put different values into the locations A, B and C and run the programs twice more. Record your results on the sheet again.

```
10 A = 6
20 B = 9
30 C = 15
40 D = A + B + C
50 PRINT D
```

```
10 A = 6
20 B = 9
30 C = 15
40 D = B + A + C
50 PRINT D
```

3. RUN the first program below 3 times with different values for a. Then RUN the other programs using the same 3 values for a. Put your results on the stores sheet.

```
10 a =
20 b = 2*a
30 PRINT b
```

now try these programs with the same X values.

```
10 a =
20 b = a + a
30 PRINT b
```

```
10 a =
20 b = a*2
30 PRINT b
```

Do you notice anything? What does this mean?

4. Put 3 different values for p and q in the first program below and record you results. Now RUN the second program using the same values for p and q. What do you notice? What does this mean?

```
10 p =
20 q =
30 s = p + q
40 PRINT s
```

```
10 p =
20 q =
30 s = p*q
40 PRINT s
```

WORKSHEET NUMBER 3

1. Write in the Maths. version of the following BASIC expressions :

a) a*b ....

b) 2*a + 1 ....

c) y / x ....

d) 3*(x + 5) ....

e) 3*x - 2*y ....

f) 3 + 2*a ....

2. Write in the BASIC version of the following Maths. expressions :

a) 2x + 1 ....

b) 3(x - 2) ....

c) 2x + 3y ....

d) $\frac{4}{x}$ ....

e) 5a - 3 ....

f) 2(x + 1) ....

3. Use the following program to show that 3(x + 1) and 3x + 1 are not the same by running this program 3 times, changing the value of x in line 10 each time.

```
10 x = 4
20 y = 3*x + 1
30 z = 3*(x + 1)
40 PRINT y
50 PRINT z
```

4. Change the program lines 20 and 30 above to see if 2x + 2 is the same as 2(x + 2). Run this program 3 times too.

WORKSHEET NUMBER 4

Type in each of the following programs if you are using a computer. If you are using the 'Maths. Machine' then work through each program. Run each of them 3 times.
Fill in your results in the store locations on the sheet with the boxes.
Do not forget to put the label for each variable under its box.

1. Run the program below and fill in your results on the sheet. Now change the values of A and B in lines 10 and 20 and RUN the program 2 more times.

```
10 A = 3
20 B = 5
30 C = A + B
40 PRINT C
```

2. RUN the first program below 3 times with different values for X. Then RUN the second program using the same 3 values for X. Put your results on the stores sheet.

```
10 INPUT X
20 Y = X + 3
30 PRINT Y
40 GOTO 10
```

now try this program with the same X values :

```
10 INPUT X
20 Y = 3*X
30 PRINT Y
40 GOTO 10
```

3. Put 3 different values into the location p for the first program and record your results on the sheet again. Then RUN the second program using the same p values and record your results. What do you notice? What does this mean about 10 - p and p - 10 ?

```
10 INPUT p
20 t = 10 - p
30 PRINT t
40 GOTO 10
```

```
10 INPUT p
20 t = p - 10
30 PRINT t
40 GOTO 10
```

4. Put 3 different values for z in the first program below and record your results. Now RUN the second program using the same values for z.
What do you notice? What does this mean?

```
10 INPUT z
20 w = 2*z + 2
30 PRINT w
40 GOTO 10
```

```
10 INPUT z
20 w = 2*(z + 1)
30 PRINT w
40 GOTO 10
```

5. Put 3 different sets of values for a and b in the first program below and record your results. Now put the same sets of values into the second program. What do you notice? What does this mean?

```
10 INPUT a
20 INPUT b
30 c = 2*( a + b )
40 PRINT c
50 GOTO 10
```

```
10 INPUT a
20 INPUT b
30 c = 2*a + 2*b
40 PRINT c
50 GOTO 10
```

6. Put 3 different sets of values for x and y into the first program below and record your results. Then put 3 sets of values into the second program using the same x values Do you get any answers the same? What would you need to do to get the same answers?

```
10 INPUT x
20 INPUT y
30 z = x + y
40 PRINT z
50 GOTO 10
```

```
10 INPUT x
20 INPUT w
30 z = x + w
40 PRINT z
50 GOTO 10
```

WORKSHEET NUMBER 5

Work through the following questions, filling in your answers on the sheet with the variable boxes.

Do not forget to put the label for each variable under its box.

1. If we know the size of 2 angles of a triangle then we can find the 3rd angle by taking each of the 2 angles away from 180. Use this program to find the 3rd angle in some triangles by running it 3 times with different values for x and y. Record your answers on the stores sheet.

```
10 INPUT x
20 INPUT y
30 z = 180 - x - y
40 PRINT z
50 GOTO 10
```

2. The perimeter of a rectangle ,length L and width W is given by :

$$P = 2L + 2W$$

Fill in the correct line 30 in the program below and then use it to find the perimeter of some rectangles by running it 3 times with different values for L, and W. Put your results on the stores sheet.

```
10 INPUT L
20 INPUT W
30 P=
40 PRINT P
50 GOTO 10
```

3. The distance D travelled by a car in time T when it is moving at speed S is given by :

$$D = S \times T$$

Run this program with 3 different sets of values for S and T. Record your results again.

```
10 INPUT S
20 INPUT T
30 D=
40 PRINT D
50 GOTO 10
```

4. The following program will help us wok out the V.A.T. on an article using the formula : V = 15% of P (ie V = 0.15P). Notice the BASIC version of this in line 20. Now run this program with 3 different values for the variable P and record the results on the variables sheet.

```
10 INPUT P
20 V = 0.15 * P
30 PRINT V
40 GOTO 10
```

WORKSHEET NUMBER 6

The following program produced the lists of variable values shown. Fill in what you think is the correct line 20 and RUN the program to check it. If it is wrong try again. If it is right then write your answer on your sheet of paper. Remember that the program must work for every line in the list.

```
10 INPUT x
20 y =
30 PRINT x
40 PRINT y
50 GOTO 10
```

1.

x	y
1	0
2	1
3	2
4	3
5	4

2.

x	y
1	3
2	6
3	9
4	12
5	15

2.

x	y
1	1
2	3
3	5
4	7
5	9

4.

x	y
1	4
2	5
3	6
4	7
5	8

5.

x	y
1	9
2	8
3	7
4	6
5	5

6.

x	y
1	1
2	4
3	9
4	16
5	25

7.

x	y
1	5
2	8
3	11
4	14
5	17

8.

x	y
1	3
2	9
3	19
4	33
5	51

WORKSHEET NUMBER 7

Use the Maths. Machine program in front of you to try and solve the following problems. Write your answers in the spaces given.

1. a) When does $3 + x = 8$? ....

   b) For what value of x does $x + 2 = 7$? ....

   c) Find a value of x for which $9 = x + 3$. ....

   d) When does $14 - 2y = 6$? ....

   e) For what value of x is $3x - 4 = 8$? ....

2. a) When does $x + y = x + z$? ....

   b) When does $x + y = x$? ....

3. Is one of the following always bigger than the other? If so by how much? If not, then say when one of them is bigger.

   a) $x - 2$ and $x + 1$ ....

   b) $x - 3$ and $x - 2$ ....

   c) $x + y$ and $x + y + 2$ ....

   d) $3x$ and $x + 3$ ....

   e) $x + 10$ and $2x + 1$ ....

4. For what value(s) of x does :

   a) $x + 2 = 2x + 1$ ....

   b) $3 + x = 7 - x$ ....

   c) $3x + 1 = x + 9$ ....

   d) $3 - x = 2x - 3$ ....

5. For what values of t is :

   $3t + 6 = 3(t + 2)$ ....

WORKSHEET NUMBER 8

Use the Maths. Machine program in front of you to try and solve the following problems. Write your answers in the spaces given.

1. For what values of x is :

   a) $2x + 1 > 5$ ....

   b) $3x - 2 < 7$ ....

   c) $13 - 2x < 3$ ....

   d) $7 < 2x - 1$ ....

2. For what values of y is :

   a) $3y > y + 2$ ....

   b) $2y < y + 3$ ....

   c) $2y - 3 > y + 1$ ....

3. a) When does $2a + b = 2(a + b)$? ....

   b) When does $2a + 2b = 2(a + b)$? ....

   c) Is $a + b$ ever equal to ab ? When? ....

4. A challenge for the more daring!

   (Note : to enter x^2 on the Maths.Machine, press the keys x, ↑, 2 and in that order)

   a) Can you find any values of x for which $2x = x^2$? ....

   b) Are there any values of x and y which make

   $$(x + y)^2 = x^2 + y^2 \ ?$$ ....

Appendix Q

Experiment 3 - The Algebra Test Used and Question Levels

ALGEBRA INVESTIGATION

SCHOOL: .... CLASS: ....

NAME: .... DATE OF BIRTH: ....

Attempt all the questions. Write your answers in the spaces provided.

1. Write down the largest and smallest of these : ....

   y - 3 , y , y + 1 , y - 2 , y - 1 ....

2. What is the cost of :

   a) 6 stamps at 9 pence each? ....

   b) 8 stamps at n pence each? ....

   c) m stamps at 11 pence each? ....

3. Multiply each of the following by 5 :

   6 .... b .... 3b ....

   b - 3 ....

4. What does $\frac{4}{c}$ equal when c is :

   a) 2 .... b) 5 .... c) p ....

5. a + 2a can be written more simply as 3a . Write more simply , where possible :

   a) 3a + 5a .... b) 3a + 4b + 2a ....

   c) 3b - b + a .... d) 3a + 4 + a ....

6. Add 4 on to each of the following :

7 .... p .... p + 5 ....

8p ....

7. If John has G jigsaws and Peter has H jigsaws, write down how many jigsaws they have got altogether. ....

8. Underline the correct answers in this question. If sometimes is the correct answer, then write in when as well.

a) Does b - c = a - c :

always, never, sometimes - when........................

b) Does a + b = b :

always, never, sometimes - when........................

c) Does p - m + n = n + p - m :

always, never, sometimes - when........................

d) Does M + P + N = N + M + R :

always, never, sometimes - when........................

9. What is the total cost of r pencils at 8 pence each, and s crayons at 9 pence each? ....

10. Find the area of the rectangles, whose sides measured in centimetres, are :

a) 5 by 7 ....

b) G by 9 ....

c) 6 by H ....

d) W by Y ....

11. a) If b = a + 3 and a = 4 then b = ? ....

b) If m = 3p + 2 and p = 2 then m = ? ....

c) If t = 3s + b and s = 5 then t = ? ....

d) If 3h = c + 3 and h = 2 then c = ? ....

12. Write down those values of a for which :

a) a + 3 = 7 ....

b) a + 3 > 7 ....

c) 9 = a − 4 ....

d) 6 > a + 3 ....

13. What can you say about r if r = s + t
and r + s + t = 30 ? ....

14. a) In the Maths. Machine :

w ---------> w + 5

eg 4 ---------> 9

Fill in the following :

8 ---------> ....

m ---------> ....

3n ---------> ....

b) In the Maths. Machine :

y ----------> 2y

eg 4 ----------> 8

Fill in the following :

7 ----------> ....

p ----------> ....

y + 2 ---------> ....

15. If v = 2 and y = 6 then :

a) v + y = ...

b) v + y + 1 = ...

c) v + y + w = ...

16. a) If a + b = 5 then a + b + c = ? ....

b) If p + q = 3 and p + q + r = 10 then r = ? ....

17. Find the perimeter of rectangles whose sides, measured in centimetres, are :

a) 3 by 7 ....

b) D by 4 ....

c) 5 by F ....

d) P by Q ....

18. If I have to pay 5 pence to have a paper delivered to my house, how much will I have to pay altogether if the paper costs :

a) 12 pence .... b) 20 pence .... c) t pence ....

19. If I have m pieces of wood each of length 7 metres and another n pieces each of length 5 metres, then what does $7m + 5n$ represent?

....

20.a) If $p - 189 = 675$ then $p - 190 = ?$ ....

b) If $y + 279 = 978$ then $y + 277 = ?$ ....

21. Underline the correct answer in this question. If sometimes is the correct answer, then write in when as well.

a) Does $2a + 2 = 2(a + 1)$:

always, never, sometimes - when...............................

b) Does $b + 5 = b + 2$:

always, never, sometimes - when...............................

c) Does $y + y = 2y$:

always, never, sometimes - when...............................

22. Which is larger : $2n$ or $n + 2$? Explain. ....

23. A figure has p sides each of length 3 centimetres. What is the perimeter of the figure?

....

24. The cost of sending a parcel through the post is 60 pence, plus 25 pence for every 1 kilogram it weighs. What is the cost of sending a 4 kilogram parcel through the post? ....

The total cost of sending another parcel is C pence. If it weighs W kilograms, write down an expression for the total cost C .

....

<u>Algebra Test Question Levels</u>

Question Number	Level	Question Number	Level	Question Number	Level	Question Number	Level
1a)	1	6a)	1	11c)	3	16b)	2
1b)	1	6b)	2	11d)	2	17a)	1
2a)	1	6c)	2	12a)	1	17b)	3
2b)	2	6d)	3	12b)	4	17c)	3
2c)	2	7	2	12c)	2	17d)	3
3a)	1	8a)	4	12d)	4	18a)	1
3b)	2	8b)	4	13	3	18b)	1
3c)	3	8c)	2	14a)	1	18c)	3
3d)	4	8d)	4	14b)	2	19	4
4a)	1	9	3	14c)	3	20a)	2
4b)	1	10a)	1	14d)	1	20b)	2
4c)	3	10b)	2	14e)	2	21a)	4
5a)	1	10c)	2	14f)	4	21b)	1
5b)	2	10d)	2	15a)	1	21c)	1
5c)	2	11a)	2	15b)	1	22	4
5d)	3	11b)	2	16a)	3	23	3
						24a)	1
						24b)	4

Appendix R

Experiment 3 - The Pre-Test

Results

The Matched Pairs and Their Pre-Test Scores

	Control Total (max=67)	Experimental	Total (max=67)	Control	Total (max=67)	Experimental	Total (max=67)
SB	3	SB	3	RW	29	NT	29
AF	3	MF	3	JC	31	GW	31
AW	5	ZG	5	RW	32	GJ	32
NU	6	KR	6	MC	32	JM	32
VK	7	MR	5.5	FB	32.5	GY	33
JE	7	PS	5.5	SC	33	MB	33
MC	7	PR	6.5	LO	33	DT	34
SO	7.5	SJ	8	AM	33	KS	33
NT	9	CE	8.5	AM	33	CE	34
DF	10	SL	10	HF	33.5	MB	34
MD	10	JF	9.5	MH	34.5	MH	34.5
KR	11	PC	11	DT	35	MB	35.5
PS	11	NR	11	NH	35	CE	34
HP	12	SP	12	KS	36	ZK	37
SM	13	SH	12.5	SB	36	AD	37
AM	13	NJ	15	WC	38	AG	38
DS	14	EM	15.5	SL	42	CF	42

The Matched Pairs and Their Pre-Test Scores

	Control Total (max=67)		Experimental Total (max=67)		Control Total (max=67)		Experimental Total (max=67)
SC	14.5	JL	16	RE	42	KA	40
AG	17	AB	16.5	ST	42	SR	40
AW	18	JC	19	AC	43	RT	43
SJ	19.5	LC	20	KH	43	ES	43
NT	21	CP	21	SW	44.5	AB	44.5
AB	24	LS	24	AW	45	MW	45
JT	25	LN	26	CB	48	AC	47
GT	25	NW	26	JS	48.5	RB	48.5
JH	25	TM	26	CC	49	KR	49.5
NP	27	MH	26.5	MB	51	JJ	51.5
RV	28	LT	29	AJ	57	FS	56
				SD	59	GH	59

Appendix S

Experiment 3 - The Questionnaire

Mathematics Questionnaire

School:.................................. Class: ....

Name:................................. Date of Birth:...............

1. A girl wrote the following in a Mathematics test at her school. Write underneath each part in the space provided whether she was right or not and explain why you so answer.

a) $3 + 2m$ is the same as $3 + 2 \times m$

....

....

b) $\frac{6}{7}$ is the same as $6 \div 7$

....

....

c) $(n + 3) + 6$ is the same as $n + 9$

....

....

d) 3g add 4 equals 7g

....

....

e) $2 + 3c$ is the same as $5c$

....

....

f) $2n$ is the same as $n + 2$

....

....

g) $2(a + b)$ is the same as $2a + 2b$

....

....

2. Imagine that a visitor from outer space has asked you to explain what the following Mathematics symbols mean. What would you say to it? Write your explanation in the space provided underneath each part.

a) 4/5

....

....

b) a3

....

....

c) 3 + m

....

....

d) b - 2 × c

....

....

e) ab

....

....

f) 2c - 1 = 3

....

....

3. Can you say for which values of y the following equations are true? Try to explain your working in the space provided.

a) 3y - 5 = 2y + 1

....

....

....

....

b) $y + 3 = 3y - 5$

....

....

....

....

c) $2y + 11 = 5y + 2$

....

....

....

....

d) $13 - y = 2y + 7$

....

....

....

....

4. Can you write more simply?:

a) $(2m - n) + 3n$ ....

b) $5h - (3g + 2h)$ ....

5. For what values of e is $17 - 3e > 2$?

....

Appendix T

Experiment 3 - The Traditional Algebra Module

You know that $6+6+6+6+6$ is 5×6
and $3+3+3+3+3$ is 5×3

In general, $\square+\square+\square+\square+\square$ is $5 \times \square$ no matter what number is used for $\square$.

RULES LIKE THIS CAN BE WRITTEN USING LETTERS IN PLACE OF NUMBERS:

$x+x+x+x+x$ is $5 \times x$

This is true whatever number is used instead of x.

We do not need the $\times$ sign between numbers and letters.

$5x$ means $5 \times x$

So $a+a+a = 3a$

$b+b+b+b+b+b = 6b$

These statements are true whatever numbers are used for a and b

$2a$ is a simpler way of writing $a+a$

$4x$ is a simpler way of writing $x+x+x+x$

1. Write these in a simpler way. The first one has been done for you. Write the whole statement in your book, not just the answer.

a) $x+x+x-x = 2x$
b) $y+y+y+y$
c) $p+p+p-p$
d) $s+s+s+s-s-s$
e) $x+x+x+2x-x$
f) $y+y-y-y$
g) $m+m+3m+4m-2m$
h) $n+n+3n-2n+5n$
i) $p+p+3p-2p-p$
j) $x+3x+2x-5x$

2. Write these in a simpler way. The first one has been done for you.

a) $3x + 2x - 4x = x$
b) $4p + 3p - 2p + p$
c) $2s + 4s + 5s - 2s$
d) $5q + 3q + 2q - 3q - 4q$
e) $7a + 3a + 4a$
f) $6b + 3b - 2b + 4b$
g) $5r + 3r + 6r + 8r - 10r - 2r$
h) $8y + 3y - 2y - 4y + y$

$2x + 2x + 2x + 2x$ is $8x$ no matter what number is used instead of x

So 4 lots of $2x$ is $8x$

---

3. Write these in a simpler way. Write the whole statement in your book, not just the answer.

a) $5 \times 2x = 10x$
b) $3 \times 4a =$
c) $7 \times 4p =$
d) $6 \times 2p =$
e) $5 \times 6m =$
f) $3 \times 4m =$
g) $5 \times 7m =$
h) $8 \times 2m =$

Look at this example:

$4 \times 3x + 2 \times 5x = 12x + 10x = 22x$

$4 \times 3x + 2 \times 5x$

is the same as

$\boxed{4 \times 3x} + \boxed{2 \times 5x}$

or

$12x + 10x = 22x$

cont: sheet 2, side 1

Do these problems in the same way that the example on the last page was done.

4. a) $7 \times 3p + 4 \times 2p$
   b) $6 + 8p + 3 \times 5p$
   c) $5 \times 4p - 2 \times 3p$
   d) $4 \times 6x + 3 \times 2x$
   e) $8 \times 5x + 4 \times 2x + 3 \times 6x$
   f) $5 \times 4y + 3 \times 8y - 2 \times 3y$
   g) $2 \times 5p + 3 \times 6p$
   h) $3 \times 5m + 2 \times 8m$

Your statements should be true no matter what number is used instead of the letter.

In the example, if x was 5

$$12x = 60$$
$$10x = 50$$
$$60 + 50 = 110$$
$$22x = 22 \times 5 = 110$$

Check that your answers are correct for question 4 by making

$x = 5, \quad p = 2, \quad y = 10, \quad m = 6$

Sometimes more than one letter is used in a statement.
This is because different letters may stand for different numbers.

$3x + y + 2x + 2y$ can be written more simply as $5x + 3y$.

It cannot be simplified any more because x and y may be different numbers.

5. Make the following statements simpler.
   a) $4x + 3x + 2y + 4y$
   b) $3p + 2q + 4p + 6q$
   c) $7r + 4s + 3r - 2s$
   d) $2m + 3n - m - n$
   e) $4x + 2y - y - 2x$
   f) $7x + 3y + 5z - 2x + 4y - 2z$
   g) $5p + 3q - 2p - q$
   h) $3x + 2y - x - y + 2x$
   i) $4m + 3n + 2p - p - 3m - n$
   j) $5p + 2q - 3p + q + 4p$

Second Year

ALGEBRA: Using Brackets Sheet 2 Side 2

When we write $3 \times 4 + 6$ we mean 3 lots of 4, THEN add 6.

The answer is 18

6. Find the answers to the following

a) $4 \times 3 + 15$

b) $3 \times 8 - 6$

c) $5 \times 10 - 3 \times 2$

d) $4 \times 6 - 2 \times 4$

e) $4 \times 8 + 3 \times 2 + 5 \times 2$

f) $6 \times 10 - 5$

g) $3 \times 4 - 2 \times 2$

h) $4 \times 8 - 3 \times 6$

i) $5 \times 12 - 4 \times 3 + 2 \times 6$

j) $4 \times 16 + 3 \times 5 - 2 \times 10$

When we want the addition or subtraction done first,

we put BRACKETS around this part of the calculation.

$3 \times (4+6)$ means ADD 4 and 6, then multiply 10×3

The answer is 30.

$5 \times (8-3)$ is $5 \times 5 = 25$.

7. Write the answers to these after you have copied the statement.

For example, $6 \times (5+3) = 48$.

a) $4 \times (3+10)$

b) $6 \times (12-4)$

c) $5 \times 8 - 3$

d) $5 \times (8-3)$

e) $3 \times (6-1) + 4 \times (8-2)$

f) $10 \times (4-2) + 3 \times (6-2)$

g) $5 \times (6+1) + 3 \times (7-1)$

h) $6 \times 8 + 10$

i) $6 \times (8+10)$

j) $5 \times (12-3) + 5 \times (12+3)$

Continued on Sheet 3, side 1

8. Copy these statements into your book.
Check to see if they are true.
If they are not true, put brackets in the statement to make them true.

a) $4 \times 3 - 2 = 10$
b) $5 \times 6 - 4 = 10$
c) $8 \times 5 + 4 = 72$
d) $5 \times 6 - 1 = 29$
e) $4 \times 8 + 3 = 35$
f) $6 \times 10 - 2 = 48$
g) $5 \times 12 + 1 = 61$
h) $6 \times 12 - 1 = 66$
i) $5 \times 8 - 4 + 1 = 21$
j) $3 \times 5 + 2 = 21$

We can use brackets whenever we want one part of the calculation done before another.

$8 - 3 + 2 = 7$ but $8 - (3 + 2) = 3$

9. Find the answers to these. Write the whole statement in your book.

a) $8 - (6 + 1)$
b) $10 - 3 + 4$
c) $10 - (3 + 4)$
d) $5 \times 6 - 4 \times 2$
e) $5 \times (6 - 4) \times 2$
f) $10 + 6 + 3 - 4 - 4$
g) $10 + 6 + 3 - (4 - 1)$
h) $5 + 8 - (2 + 6)$
i) $5 + 8 - 2 + 6$
j) $10 + 2 - 5 + 1$
k) $10 + 2 - (5 + 1)$
l) $5 + 8 + 3 - (6 + 4)$

10. You are given that $a = 5$ $b = 6$ $p = 4$ $q = 8$ $r = 1$
You can use these to find the value of the expressions below.
Write the whole statement in your book. The first one has been done for yo

a) $3(p + q) = 3 \times 12 = 36$
b) $2(q - p) =$
c) $3(a + b) =$
d) $4(a + b + r) =$
e) $3(p + a - b) =$
f) $3(q + r) - 2(b - a)$
g) $2(b + p) - (q + r)$
h) $3(q + p + a) + 2(a + b + r)$
i) $3(p + q - r) - 2(r + a + b)$
j) $2(p + q) + a(b + q)$

5

continued on sheet 3 side 2

When two numbers are multiplied together we put a $\times$ sign between the numbers.

If letters are used to represent numbers, we do not need a $\times$ sign.

$3a$ means $3 \times a$ ab means $a \times b$ bc means $b \times c$

11. You are given that $a = 4$ $b = 3$ $c = 2$ $d = 1$

Find the value of these expressions.

The first one has been done for you.

a) $3c + 2d + ab = 6 + 2 + 12 = 20$
b) $4b + 3d - bc$
c) $3d + bc + ac$
d) $abc - bd$
e) $ab - (c + d)$
f) $bd + (a - c) + cd$
g) bcd
h) $a + b + c + d - ab$
i) $abc + abd$
j) $ac + bd + ad$

Look at this example.

$3(4+6)$ is $3 \times 10 = 30$
$3 \times 4 + 3 \times 6 = 12 + 18 = 30$

So $3(4+6) = 3 \times 4 + 3 \times 6$

Try doing this for the examples below.

Set your work out like this:

For $2(3+2)$
$2(3+2) = 2 \times 5 = 10$
$2 \times 3 + 2 \times 2 = 6 + 4 = 10$

12. a) $5(4+3)$
b) $4(8+1)$
c) $5(4+2)$
d) $7(6+3)$
e) $4(8+1)$
f) $6(2+1)$
g) $3(5+1)$
h) $4(6+2)$
i) $3(8+2)$
j) $4(6+3)$

In the same way $2(4-2) = 2 \times 2 = 4$
$2 \times 4 - 2 \times 2 = 8 - 4 = 4$.

Try this for the expressions over the page.

6 continued on sheet 4 side 1

13. a) $5(6-2)$
b) $7(4-1)$
c) $6(7-3)$
d) $12(12-4)$
e) $8(9-1)$
f) $8(3-1)$
g) $2(4-2)$
h) $8(5-3)$
i) $7(6-1)$
j) $4(3-2)$

We can write these rules using letters.

$$a(b+c) = ab + ac$$
$$a(b-c) = ab - ac$$

These work no matter what letters are used for a, b, c.

Inside some brackets you may find a mixture of numbers and letters.

$$3(x+2) = 3x + 3\times 2 = 3x+6$$

Write the examples below without brackets.

The first one has been done for you.

14. a) $4(y+5) = 4y + 20$
b) $3(x+4)$
c) $5(x-2)$
d) $5(y+3)$
e) $7(x-1)$
f) $12(y+2)$
g) $6(2x+1)$
h) $3(y+4)$
i) $4(y-2)$
j) $5(y+6) - 3y$ (Make your answer as simple as you can.)

Equations

Look at this simple problem:

Think of a number. If you add 8 to the number you get 21.

What is the number?

We can write this problem as a simple statement using algebra.

Call the number you are trying to find x.

So $x + 8 = 21$.

This is an <u>equation</u>. Only one value for x makes the statement true.

So x must be 13.

Continued on sheet 4, side 2

15. Find the value of the letter which makes these statements true.
The first one has been done for you. Set your work out the same way.

a) $x + 3 = 8$
so $x = 5$ b) $x - 6 = 6$ c) $x - 10 = 35$ d) $x + 40 = 50$

e) $x + 6 = 15$ f) $x + 20 = 32$ g) $x - 20 = 32$ h) $x - 40 = 50$

i) $x - 15 = 40$ j) $x + 32 = 52$ k) $x + 40 = 51$ l) $x - 31 = 47$

Look at this problem:

Think of a number. Add 2 to the number, then multiply by 6. The answer is 42. What is the number?

If x is the number, when 2 is added you have $x + 2$.
When you multiply by 6 you then have $6(x + 2) = 42$.

You can see that when $x = 5$ the statement is true.

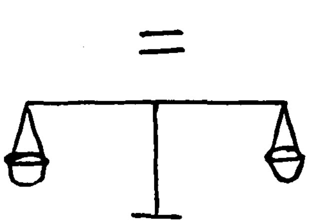
16. Find the value of the letter which makes these statements true.

a) $2(x+1) = 6$
b) $3(x+4) = 15$
c) $2(x+3) = 40$
d) $3(x-1) = 21$
e) $5(x+10) = 100$
f) $3(x-2) = 30$
g) $5(x+2) = 25$
h) $6(x+4) = 48$
i) $8(x-3) = 32$
j) $3(x+2) = 27$

When you solve a problem, you answer it correctly. A solution is an answer to a problem.

Some equations are not so easy to solve. We need a method for finding the solution.

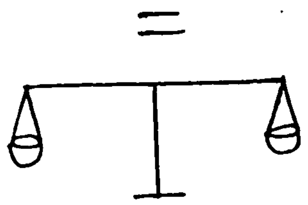
A balance scale remains balanced only if we add or subtract the same amount from both sides. The = sign is like the centre of the balance scale. The equation stays balanced only if we add or take the same amount from both sides of the equation. A true statement remains true only if we add or subtract the same amount from both sides.



continued on sheet 5 side 1

Remember the balance scale! If $x + 3 = 8$

Then if we take 3 from <u>both</u> sides of the equation, we get $x = 5$.



If $x + 12 = 20$

Then if we take 12 from <u>both</u> sides of the equation, we get $x = 8$

If $x - 6 = 40$, then if we <u>add</u> 6 to both sides of the equation, we get $x = 46$. Note that $x - 6 + 6 = 40 + 6$

so $x = 46$

17. Solve these equations by setting your work out like a) below.

a) $x - 5 = 20$
Add 5 to both sides.
$x = 25$

b) $x - 6 = 6$
c) $x - 8 = 20$
d) $x + 4 = 16$
e) $x - 3 = 13$
f) $x + 21 = 27$
g) $x - 15 = 31$
h) $x + 6 = 40$
i) $x + 12 = 31$
j) $p + 3 = 8$
k) $m - 6 = 9$

A true statement will still be true if we multiply or divide <u>both</u> sides by the same amount.

So $3x = 9$

When both sides are divided by 3, we get $x = 3$

$\frac{1}{2}x = 4$

If we multiply <u>both</u> sides by 2, we get $x = 8$

18. Solve these equations, setting out your work as in the example (a) below.

a) $3x = 27$
divide both sides by 3
$x = 9$

b) $2x = 14$
c) $\frac{1}{2}x = 6$
d) $\frac{1}{4}x = 5$
e) $11x = 88$
f) $12x = 132$
g) $\frac{1}{3}x = 6$
h) $8p = 40$
i) $7r = 42$
j) $6p = 66$
k) $11m = 33$
l) $\frac{1}{4}x = 2$
m) $\frac{1}{3}x = 2$

continued on sheet 5 side 2

Remembering the balance scale, we can use this idea to solve equations like $3x + 6 = 24$

$$3x + 6 = 24$$

Take 6 from both sides

$$3x = 18$$

Divide both sides by 3

$$x = 6$$

---

19. 19. Do these problems the same way.

a) $2x + 5 = 13$
b) $3x - 4 = 14$
c) $8p - 3 = 37$
d) $4y + 11 = 59$
f) $\frac{1}{2}x - 12 = 3$

g) $\frac{1}{4}y + 12 = 19$
h) $12y \div 8 = 136$
i) $5m + 6 = 26$
j) $\frac{1}{2}p - 20 = 2$
k) $3x - 4 = 17$

---

20. Look at this example: $2(x + 4) = 16$

Divide both sides by 2

$$x + 4 = 8$$

Take 4 from both sides

$$x = 4$$

20. Try to do the next questions in the same way.

a) $3(x+4) = 27$
b) $4(x-1) = 16$
c) $5(x+2) = 25$
d) $6(x-1) = 36$

e) $4(x+1) = 20$
f) $6(x-1) = 36$
g) $4(x+2) = 20$
h) $7(x-3) = 21$

ALGEBRA

Try to solve these 'think of a number' problems by writing an equation then solving it.

Start by writing 'Let x be the number.'

Example: Think of a number. Double it, then add 5.
The answer is 21.
What was the number?

Let x be the number.

$$2x + 5 = 21$$

Take 5 from both sides.

$$2x = 16$$

Divide both sides by 2.

$$x = 8$$

21. (a) Think of a number. Multiply it by 4, then take 6.
The answer is 22. What was the number?

(b) Think of a number. Halve it, then add 4.
You get 15. What was the number?

(c) Think of a number. Add 4 to it, then double that.
You get 32. What was the number?

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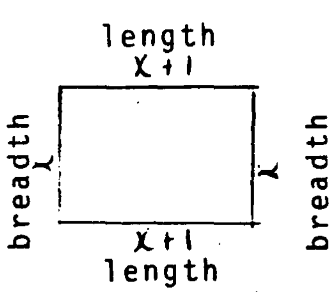
22. The PERIMETER is the distance all the way round an object.



The PERIMETER of the rectangle on the left is 34 centimetres (cm.)

The length of the rectangle is $x + 1$. How many centimetres is that?

The breadth of the rectangle is x. How many centimetres is that?

Remember, all four sides must add up to 34 cm!

Appendix U

Experiment 3 - The Post-Test

Individual Results

Experiment 3 - The Matched Pairs and Scores

In The Post-Test

	Control Total (max=67)		Experimental Total (max=67)		Control Total (max=67)		Experimental Total (max=67)
SB	8.5	SB	3	RW	30	NT	56
AF	6	MF	10	JC	a	GW	31
AW	13	ZG	18	RW	a	GJ	35.5
NU	10	KR	17.5	MC	47	JM	47
VK	13	MR	11	FB	44	GY	46.5
JE	18.5	PS	13	SC	42	MB	30.5
MC	37	PR	31	LO	38	DT	46
SO	a	SJ	7.5	AM	53	KS	53.5
NT	25	CE	17	AM	33	CE	55
DF	19.5	SL	24	HF	37.5	MB	38.5
MD	19	JF	26.5	MH	36	MH	a
KR	32	PC	11	DT	48.5	MB	46
PS	31	NR	18	NH	44	CE	47
HP	31	SP	9	KS	47	ZK	43
SM	15	SH	17.5	SB	41.5	AD	49
AM	a	NJ	32	WC	50	AG	46
DS	33.5	EM	29	SL	a	CF	47
SC	29	JL	28	RE	45	KA	52.5

Experiment 3 - The Matched Pairs and Scores

In The Post-Test

	Control Total (max=67)	Experimental Total (max=67)		Control Total (max=67)		Experimental Total (max=67)	
AG	31	AB	26.5	ST	26	SR	59
AW	44	JC	27	AC	49	RT	53.5
SJ	27	LC	31.5	KH	52	ES	54
NT	42	CP	29	SW	a	AB	31.5
AB	40	LS	50	AW	44	MW	48.5
JT	34	LN	21	CB	44	AC	54
GT	a	NW	28.5	JS	49	RB	46.5
JH	43	TM	30	CC	55	KR	50.5
NP	a	MH	30	MB	50	JJ	56
RV	36	LT	30	AJ	56	FS	58
				SD	62	GH	62

Experiment 3 - The Scores

In The Post-Test of Experimental Group

Pupils Not in Matched Pairs

Pupil	Pre-Test Score (max.= 67)	Post-Test Score (max.= 67)	Pupil	Pre-Test Score (max.= 67)	Post-Test Score (max.= 67
SL	12	11	SK	18	28
RW	43	51	SR	36	42
MM	35.5	19	SO	48	46
NS	10.5	17.5	MG	7	5
AJ	15.5	19.5	CS	a	28
CH	31	36	RB	7	18
NN	48	51	JC	46	52.5
RF	a	17	CD	14.5	17.5
CT	a	53	SG	42.5	54
CT	7	9	CK	35	54
HR	9.5	27	SP	11.5	23
KT	14	13	AS	14	28
DM	7.5	12	RW	7	7
			DS	a	18

Experiment 3 - The Scores

In The Post-Test of Control Pupils Not in

Matched Pairs

Pupil	Pre-Test Score (max.= 67)	Post-Test Score (max.= 67)	Pupil	Pre-Test Score (max.= 67)	Post-Test Score (max.= 67
IH	30.5	40	HR	25.5	25.5
CB	22	a	NK	4.5	12
IW	4	10	CH	4	12
LS	4	10	CR	26	51
SM	29.5	31	HG	a	37
NW	55.5	a	CT	57	60.5
LC	29	41	MP	0	6
NH	25.5	37	SB	4	24
LC	29	36	LM	20	a
RE	56	65.5	NB	52	54
EK	29.5	33	ND	a	26
JB	62	60	RA	a	26

Appendix V

Experiment 3 - The Post-Test

And Delayed Post-Test Question

Facilities

Experiment 3 - Question Facilities

	The Experimental Group		The Control Group	
Question Number	Post Test (%)	Delayed Post-Test (%)	Post Test (%)	Delayed Post-Test (%)
1.a)	80	94	84	88
b)	96	94	88	86
2.a)	79	72	96	86
b)	71	83	73	74
c)	71	83	73	78
3.a)	91	94	100	92
b)	64	81	76	66
c)	14	84	41	42
d)	11	24	10	26
4.a)	54	72	59	56
b)	36	46	45	46
c)	54	61	47	60
5.a)	79	87	98	88
b)	50	74	73	84
c)	29	67	61	72
d)	38	54	53	58
6.a)	98	94	96	98
b)	61	44	51	56
c)	45	43	55	42
d)	30	33	31	48

	The Experimental Group		The Control Group	
Question Number	Post Test (%)	Delayed Post-Test (%)	Post Test (%)	Delayed Post-Test (%)
7.	55	65	78	72
8.a)	35	31	26	29
b)	31	31	17	14
c)	53	63	47	58
d)	38	32	28	29
9.	48	61	57	68
10.a)	84	81	86	90
b)	66	80	82	84
c)	66	80	82	84
d)	66	80	80	84
11.a)	80	81	67	78
b)	50	70	63	68
c)	35	44	35	44
d)	34	69	49	46
12.a)	86	96	88	88
b)	31	35	12	24
c)	61	70	69	72
d)	22	26	6	20
13.	32	41	35	32
14.a)	52	63	47	66
b)	26	30	47	60
c)	36	44	31	52
d)	38	65	31	60
e)	41	61	35	56
f)	9	15	10	14

	The Experimental Group		The Control Group	
Question Number	Post Test (%)	Delayed Post-Test (%)	Post Test (%)	Delayed Post-Test (%)
15.a)	89	89	96	90
b)	71	83	76	76
c)	61	70	65	62
16.a)	59	59	61	54
b)	77	87	82	68
17.a)	55	67	37	46
b)	50	57	27	44
c)	50	59	29	42
d)	45	59	35	44
18.a)	77	87	73	70
b)	75	87	73	66
c)	57	74	63	56
19.	18	22	8	18
20 a)	59	46	55	56
b)	50	56	45	40
21.a)	25	31	14	20
b)	70	81	76	80
c)	70	81	73	82
22.	7	11	0	1
23.	48	61	38	58
24.a)	66	78	78	82
b)	5	2	0	0

Appendix W

Experiment 3 - The Delayed Post-Test

Individual Results

Experiment 3 - The Matched Pairs and Scores In The Delayed Post-Test

	Control Total (max=67)		Experimental Total (max=67)		Control Total (max=67)		Experimental Total (max=67)
SB	13	SB	4	RW	45	NT	56
AF	11	MF	3	JC	45	GW	48
AW	20	ZG	a	RW	a	GJ	45
NU	25	KR	20	MC	54	JM	52
VK	8	MR	17	FB	38	GY	47
JE	17	PS	20	SC	52	MB	47
MC	9	PR	43	LO	47	DT	45
SO	41	SJ	18	AM	53	KS	56
NT	29	CE	31	AM	45	CE	58
DF	a	SL	18	HF	37	MB	46
MD	18	JF	20	MH	50	MH	a
KR	25	PC	19.5	DT	48	MB	60
PS	26	NR	22	NH	45	CE	51.5
HP	42	SP	29	KS	48.5	ZK	48
SM	22	SH	31.5	SB	53	AD	51
AM	22	NJ	38	WC	53	AG	55
DS	29	EM	40	SL	43	CF	49.5
SC	32.5	JL	31.5	RE	57	KA	a

<u>Experiment 3 - The Matched Pairs and Scores</u>

<u>In The Delayed Post-Test</u>

	Control Total (max=67)	Experimental Total (max=67)		Control Total (max=67)		Experimental Total (max=67)	
AG	32	AB	33	ST	46.5	SR	54
AW	35	JC	34	AC	57	RT	59.5
SJ	a	LC	35	KH	51	ES	57
NT	a	CP	37	SW	a	AB	47
AB	38	LS	52.5	AW	43	MW	57
JT	40	LN	35	CB	48	AC	61
GT	45	NW	44	JS	a	RB	49
JH	48	TM	47	CC	58	KR	56
NP	37	MH	33	MB	58	JJ	60
RV	a	LT	46	AJ	63	FS	62
				SD	63	GH	65

Experiment 3 - The Scores

In The Delayed Post-Test of Experimental Group

Pupils Not in Matched Pairs

Pupil	Delayed Post-Test Score (max.= 67)	Pupil	Delayed Post-Test Score (max.= 67)
SL	15	SK	27
RW	54	SR	48
MM	23	SO	52
NS	28	MG	3
AJ	25.5	CS	29
CH	31	RB	13
NN	50	JC	54
RF	27	CD	18
CT	53	SG	56
CT	11	CK	43
HR	8	SP	21
KT	a	AS	26
DM	10.5	RW	8
		DS	39.5

Experiment 3 - The Scores

In The Delayed Post-Test of Control Group

Pupils Not in Matched Pairs

Pupil	Delayed Post-Test Score (max.= 67)	Pupil	Delayed Post-Test Score (max.= 67)
IH	48	HR	35
CB	a	NK	13
IW	13	CH	16.5
LS	16	CR	47
SM	37	HG	47
NW	60	CT	45
LC	49	MP	a
NH	44	SB	33
LC	36	LM	40
RE	65	NB	52
EK	35	ND	32
RA	49	JB	62

Appendix X

Abbreviations

Abbreviations

CSMS - Concepts in Secondary Mathematics and Science

ICMI - The International Commission of Mathematical Instruction

BASIC - Beginners´ All-Purpose Symbolic Instruction Code

VRQ - Verbal Reasoning Quotient

MQ - Mathematical Quotient

ISMEC - The Independent Schools´ Micro-Electronics Centre

IGPME - The International Group for the Psychology of Mathematical Education

APU - The Assessment of Performance Unit

TICCIT- Time-Shared Interactive Computer Controlled Information Television

PLATO - Programmed Logic For Automatic Teaching Operation

Appendix Y

Experiment 3 - The Interviews : Transcripts and Written Answers

1. $3 + 25 = 28$

2. The value of P and S = 3

3. $3 \times d - 4 =$

5. $3 + 6g = 9g$

5. $3 + g = 3g$

6. $3 + 2 = 5 \times 5 = 25$
   $3 \times 5 = 15 + 2 = 17$

1. $2a + b$

1. $2 \times a + b =$

4. $6 + c + 4 =$

5. $p + q = 3 + r =$

8. $5f + 5g =$

1. 3 + 25

2. –

3. —

4

5. 63.

6. 3 + 2 × 5 = 25.

4. 6 + 4

P1

1. $2 \times 5 = 10 + 3 = 13$

2 $p = 3$

2 $p = 2$

3 $x \times 3$

$3x = Y$

4 It depends on the number (Y)

eg $3 \times 1 = 3$ $3 + 1 = 4$

5 $6g + 3 =$

6 $2c + 3 = 13$

$c = 5$

1 $a + b \times 2$

~~a*~~

~~a+b~~

~~2~~

$a + b = c$

$2c$

10 $(2m - n) + 3n$

$(2m + 2n$

11

~~3~~$x - 5 =$ ~~3x~~ $+1$ x

$3x = 2x + 6$

$x = 6$

12. $d = < 5$

P2

Do 2m first & then add the 3

1. 2m +3 = 13

2 p = 3

3 3(D - 4)

4 3y is bigger if it is a higher number e.g.

5 6g+3

6 ~~3~~+ 2 x 5 = 25

i 2a + 2b

~~2~~ii 2m + 2n

~~i~~ii 3x -5 = 2x+1

3x = 2x+6

x = 6

2 D = 4 or below

P3

1 $3 + 2 \times m = 5 = 10 = 13$.

2 $2 \times 3 - 1 = 5$. and $2 \times 3 - 1 = 5$,

~~3~~ P+S both equal 3.

~~4~~

2.~~B~~. $3 \times 2 = 5$ $p = 2$.

3. $d - 4 \times 3$

~~B~~.

4. $3y = 3 \times y$.

5. $6g + 3$.

6. $3 + 2 \times c = 5$. = $5 \times 5 = 25$.

6. $c = 5 = c + 2 = 5 + 2 \times 3 = 21$.

3

4. $10 + e$.

7~~6~~. $7 = c - 2 = 7 = 9 - 2$.

P4

1. 3+ 2x5 = 13

2. 2x3-1=5 The vaule of p = 3 S=P S=3

3. 10-4 = 6 3X6 = 18 3 x d-4 = 3d-4

4. 3y

5. 6g +3

6. 3 + 2 x5 = 25

6. 3X5 + 2 = 17

1. 2 x a +b a+b x 2

2. 3x5 = 15 15+b

3. 3d + 3

6. If b had the same vaule as C, p-b would equal

P5

1. 13

2. P=3 S=3 They are both the same

~~8~~. P=2

3. —

4. 3y

5. 9g
4g

6. 25
17

1. 2a+b

4. E+10

6. Sometimes, when b e c are both the same valu

7. C=9

10. ~~12m~~ 2n+2m

P6

1. ~~2orx~~ 13.
2. 6. 6.
3. D - 4 × 3 .
4. 3y.
5. ~~6a~~ 6g + 3.
6. 5 × 5 = 25.

21. 5 + 2 = 7 × 3 = 21.

1. 2a + 2b.
10 2m + 2n.
11
12

P7

1. 2 x m = 10 + 3 = 13
2. they both have the Same value.
The value of P is 3.
3. 3y
4. 3y
5. 9g
6. 3 + 2 x 5 = 25
3 x 5 + 2 = 17

1. a + b x 2 , 2 x a + b
10. (2m + 2n)
11. 3x6 - 5 = 13 = 2 x 6 + 1 = 13
12. 17 - 3xd = 6 > 2

P8

1. 13

2. P+S=3

3. D-12

4. 3y

5. 9g   3g

6. 25   17

4. e + 10   10 + e

5. p+q+r = 6

6. never

8 3xf + 5xg + 2xf

P9

1. 13

2. s and p are both the same

~~3.~~ 3 is the value of p p = 3

3. –

4. 3y apart from when y is 1

5. 6g + 3 , g + 3

6. 13 ; 21

1. 2a + 2b

2. -

4. -

5. -

P10

1. 10. =13.

2. 2p = 2×3. 2q = 2×3.
p or q are the same.
p = 3. 2(p+1)−1 = 5. p = 2.

3. ~~#~~ –
4. 3y is bigger.
5. 9g. 3g.
6. 25. 17.

1. 2 × a+b.
2. 15+b.
6. –
7. 7 = c−2. c = 9.

P11

q1 3+2m m=5 5+2=7 3+7=10

q.2 2p-1=5 2S-1=5 2P=6 2S=6 2=6

q.3. —

q4 3y or 3+y are equal

q5 90 30

q.6. 3+2=5 5+5=10
3×C(=5)+2 =10

q.1 2a+b

q.2. 3S+b (b=5) 8+5=13

q5 P+q=3 then P+q+r=3r

q6. ~~Never~~ Sometimes if you add Something to

P12

1. 13

2. p ~~p~~ and s could be 3. p = 3

3. 3d − 4

4, 3y

5. 6g + 3

6. 13

7. —

8.

4 E + 10

10. 4N − 2m

11. —

12. d = 8

P13

1. 13
2. Both P and S could equal 4
3. —
4. Both equations equal the same amount.
5. 9g, 3g
6. 25, 17
4. 10e
6. —
8. 5f + 5g

P14

① 13

② 3 3

③ 12d

④ 3y

⑤ 9g 3+g 3 x 6+g

⑥ 25 17

① 2ab 2 x a + b

② 15 + b

④ 6 + a + 4 e + 10

⑤ 3 + r

P15

1. $3 + 2m$ when $m = 5$? 13 m
2. $2p - 1 = 5$ $2s - 1 = 5$? They are both the s[illegible]
   $p = 3$.
2. $2(p + 1) - 1 = 3$

3. $d - 4 \times 3 = 5d$
3. $5d - 4 \times 3 = 3d$
4. $3y$ or $3 + y$ = $3 + y$ is greater.
5. $3 + 6g = 9g$
6. $3 + 2 \times c$ when $c = 5$ = 25
6. $3 \times c + 2$ when $c = 5$ = 17
6. $p - b = p - c$? Always, never or sometim[illegible]
10. $(2m - n) + 3n = 3n + n - 2m$
11. $3x - 5 = 2x + 1 = 1x$
12. $17 - 3d > 2$ = 14

P16

1. 5+2+3=10
2. P and S are the same. Two P=6-1=5. P=3
3. d-4 × 3=(d-4)×3
4. (3y and 3+y are the same) 3+y is the biggest
5. 3+6g=9g. 3+g=3g
6. 3+2×5=25 3×5+2=17.
   7×15+3=

1. 2a+b=
2. 35+b=35b
6. Never.
10. 2m-(n+3n)=2m-4n

TAPE K1 - 20.5.87
PUPIL 1

INT: All I want to do is first of all if you could tell me what you thought about it really. Have you ever done anything like that before?

P1: No.

INT: No. What did you think about it? Did you enjoy it or did you hate it?

P1: Well, I liked doing the algebra, but I don't particularly like computers.

INT: Had you done any algebra before?

P1: No.

INT: That was the first you had done.

P1: We'd done a bit, but I didn't really understand it then.

INT: You think you understand it now.

P1: Yes.

INT: So you like maths do you?

P1: Yes.

INT: Quite good at it?

P1: Well, I wasn't but I think I'm better now.

INT: You're getting better? All I'd like to do is to get you to do one or two questions for me. It's not a test remember. Have you got a pen there? What I am really interested in is how you do them, not whether you get them right or not so you don't have to worry. There's not going to be any marks or anything. OK? If we could take this first one then. What I would like you to do is to just read the question out for me.

P1: What is the value of 3 plus 2m when m is 5?

INT: Do you understand the question? Do you think you can do it? Could you do it and write anything you want to down, but I'd like you to tell me what you are doing as you write it down.

P1; 2 times m, which is 5, makes 10 plus 3.

INT: Can you put a little 1 by that for question 1. Can I ask you what this 2m means to you then. What do you understand by it.

P1: 2 times m.

INT: That's 2 times m. You're certain about that? Good. You seem very confident. It's the right answer, which helps. Good, o.k. that's fine. Just read out this second one for me.

P1: 2p minus 1 equals 5 and 2s minus 1 equals 5. Can you say anything about p and s?

INT: Leave it there for now. Can you understand what it is asking?

P1: When I'm reading it I don't, but I'm going to look at it.

INT: OK. Have another look at it and then tell me if you can understand it.

P1: I can say that p and s have the same value.

INT: Ah, you can say that straight away can you?

P1: Yes.

INT: How did you come to that conclusion?

P1: It's 2 times a number which has to be 3 minus 1 makes 5, and 2 times a number - it's the same sum.

INT: I see. Did you work it out twice or did you spot that it was the same sign.

P1: Well I kind of worked it out automatically on that one.

INT: It was so easy was it.

P1: Yes.

INT: Good, and then did you work this one out as well? Can't remember.

P1: Well you can just see it's the same.

INT: It's obvious to you is it?

P1: Yes.

INT: Good. So the value of p is?

P1: p is 3.

INT: If I was to give you this one here which is a similar question. You can look at the two together and you can see probably that they are similar, can you see any connection between them and can you tell me anything about the value of p in this question?

P1: Yes, p equals 2.

INT: Right, that's very interesting. How did you work that out?

P1: Well it's the same but then it's plus 1, so minus 1 add 3.

INT: Excellent. You do understand this work then, don't you? Very good that is. I tell you not many people get that answer right, so that's very good. That should put you at ease. Good. Can we write that down then for No. 2. What did you say the value of P was, can you remember?

P1: 3.

INT: How would you write that down? For this one here. It's No. 2, just put it underneath it's the same question. Does it matter that it's in both?

P1: No.

INT: And they have got different values? It doesn't matter?

P1: Oh, they're different sums - I don't really know.

INT: Not sure, but yiou're not unhappy with that?

P1: No.

INT: No, right good that's fine.

P1: They're different sums so p can be anything really.

INT: Ok that's fine. Can you read out No. 3 for me.

P1: Multiply d minus 4 by 3.

INT: Understand that one?

P1: Well, no you can't do it.

INT: You can't do that one.

P1: No you haven't got a number for d.

INT: Oh I see so you don't know what d is and that is a problem is it. C you write anything down at all for that question?

P1: Well it's just a number by 3, so it's any number by 3.

INT: How would you write it down? Could you write anything for that?

INT: Can we put a question 3 by that. Lovely.

P1: 3x Well you can't have a number can you?

INT: Is there any reason why you chose x?

P1: Well it's a letter we are always using.

INT: You use x a lot.

P1: Yes.

INT: That's fine. Would it make any difference if it was this one? 5d minus 4 multiplied by 3. Would your answer be the same?

P1: It would still be a number minus 4 multiplied by 3.

INT: Would it make any difference to the way you wrote the answer?

P1: No I don't think so.

INT: OK. That's fine. Let's try No. 4 then. Can you read that one out f· me?

P1: Which is bigger 3y or 3 plus y?

INT: Do you understand the question?

P1: Yes. Well that depends on what y is, because you can have 3 times which is 3, 3 plus 1 is 4.

INT: Very good. Can you write that down then for me? How would you answe that then if this was a test which it isn't really.

P1: Do you want that written down?

INT: No, not the question, just how you would answer it. Don't worry abo the spelling. I'll understand. Good.

P1: When you get bigger numbers it will be.... that one will be bigger.

INT: You can't tell me - I suppose it's a bit difficult - how big and wh you can get it.

P1: Its got to be over a certain number. 2 times the 6....3 plus 2 is 5 it will be 2 then. That's the only one.

INT: I see. Are there any other numbers at all?

P1: ..... that's still bigger isn't it. 3 times 0 is nought, 3 plus 0 is 3.

INT: Good. So that one's bigger than nought isn't it and that one's bigger for ...?

P1: Oh it is again, yes.

INT: The 2 you said this was bigger. Good.

P1: So it's got 2 if you have decimals and things stuck in between.

INT: Very good. So it's all the numbers in between as well?

P1: Yes.

INT: Very good. Not many people get that one right either so you're doi very well. Could you read out No. 5 for me now and try that.

P1: Plus 3 into 6g.

INT: Do you understand what that is asking you to do?

P1: Mm.

INT: Can you put a question 4 there and a 5 underneath.

P1: I don't get this into.

INT: Onto that says.

P1: Oh so it's just 6g plus 3.

INT: Good. Can you write that down for me.

P1: It hasn't got an answer.

INT: What do you mean by an answer.

P1: Well you've got to have a value for g before you can get an answer.

INT: I see, so how would you write this one if that was the question?

P1: ....

INT: Can you tell me what you're thinking?

P1: Its 6g plus 3 equals 6g plus 3 that's all.

INT: So is that the answer then? It's not a number is it?

P1: No, it can't be a number.

INT: Can you have an answer like that or do answers have to be numbers?

P1: No they don't - I'm thinking you might be able to cancel it down. I can't think how you do that.

INT: That's fine.

P1: I'm thinking that one is wrong.

INT: Yes, what's wrong with that one then? Have another go at it. No. Well leave that for now. If you think of it you can come back and c it. Let's try the last one. Can you read it out for me.

P1: What is the value of 3 plus 2 times c when c = 5? So you can do tha because it will be 10 plus 3. It's the same as the first one.

INT: It's the same is it as the first one.

P1: Yes. You've just got to times it by that.

INT: Good, can you write that down for me? Just how you would write the answer. .... Excellent. Now as you are so good at maths you get to do one or two extra ones for me. You didn't know you were good. Well some of those answers there I can tell you are very good. Could you just try one or two of these for me. You're not going to do all these. Just the hard ones. Question 1 can you read that out for me.

P1: Write 2 times a plus b without brackets.

INT: Do you understand what that is asking you to do?

P1: Yes.

INT: How are you going to do it?

P1: 2 times ....

INT: Write something down and tell me what you are thinking of as you write it down.

P1: Got to have a and b two separate numbers times 2. You always times first.

INT: That's all right you just write down what you think it is. It doesn' matter if ...

P1: Well it isn't a plus b times 2 'cos it'll be a plus b times 2. a plu b over 2. That way you plus those before dividing them.

INT: I see. It doesn't matter that it's divided now.

P1: Oh, yes.

INT: It does. Can you tell me the problem you're trying to work out? I ca see what you are trying to do - can you explain to me the difficulty

P1: Well it's brackets, so you've got to add these two numbers before times it and the problem is you always multiply first.

INT: You can't see any way round that problem?

P1: I know there is one but I can't find it.

INT: OK that's fine. Don't worry about it. If you think of it we'll come back to it.

P1: Unless you went along and put a plus b equals c and then put 2 times c, but that's a long way round.

INT: Well can you write that down.

Good, now it's got rid of the brackets and it means the same. Can we put a No. 1 by that so that I can just remember. Let's jump down to No. 10 now. Can you just read that out for me?

P1: Write more simply 2 times m minus 3 - no - 2 times m minus n plus 3

INT: Plus?

P1: Minus 3m plus 2 times n.

INT: Do you understand what it is asking you to do?

P1: I could say the brackets.

INT: OK. How are you going to set about that then?

P1: Well you can't

INT: No way of doing them then?

P1: Yes because it can then ......

INT: Can you tell me what you are doing?

P1: I'm trying to take that n away. You take an n away from both sides so that will be 2m plus 2n.

INT: Good. That's fine.

P1: I think you need the brackets still. You don't though.

INT: You don't need the brackets. Can you tell me why?

P1: Because they are both multiplications and you do them before anyway.

INT: I see. So the brackets are for what then?

P1: They are to make sure you do it first.

INT: So what sort of things would you put them round?

P1: Well if you've got a sum inside a sum like, well, it's got to be inside so you've got a sum 3 plus 2 and then you've got to times it by 5 you've got to do the 3 plus 2 first before you times it by 5.

INT: Lovely. Can you put a 10 by that, and get you to read No. 11.

P1: Solve 3x minus 5 equals 2x plus one.

INT: Do you understand that?

P1: Yes.

INT: Can you do it and tell me what you are doing. Put an 11 down. You ca turn over.

P1: You've got to find ... well the value of x must be the same because its in the same sum.

INT: How are you trying to do it?

P1: I'm trying to guess the numbers but I know that is not the right way

INT: Can you tell me why you're trying to guess then?

P1: Because I might hit the right number. ....

INT: Can you tell me what you are thinking?

P1: Well I' m thinking that maybe take x some number away from both side so you'll be able to take a x away from both sides. That wouldn't leave anything in there to go on. You'd have nothing there if you take 2x away and 1x minus 5 = plus 1.

INT: Would it help to write that down?

P1: So that's x - ......

INT: So how might you do it now?

P1: I was thinking maybe get rid of this and forget about that 4 by putting, adding 5 to both sides - that should do it - so it would be 3x equals 2x plus 6. .... try to take away x .....

INT: Is there a problem with that?

P1: If you take the x away - I'm sure you can't do it. You subtract 6.......

INT: Can you tell me what you're thinking yet?

P1: Try 3 times x thinking if I get that number I'll be able to take 6 away to get it. To get that.

INT: Why did you reject the idea of taking the x's away?

P1: Well it's leaving just plus 6 unless I take the plus away.

INT: Why is that a problem?

P1: It would be x equals 6.

INT: How did you get that?

P1: Well if you go on about the plus I suppose I should have taken 2x away from both sides - so that you've got nothing there and 1x there

INT: Why was the plus a problem? Can you tell me?

Pl: Well it isn't really. It isn't really there. Cos when you do it somewhere else like you take that minus away when you plus 5 it goes to 0 and I'm thinking here that I don't know whether I can find an answer.

INT: Have you got the answer now?

Pl: Yes. 3 times 6 is 18 minus 5 is 13. 2 times 6, yes 13, it's right

INT: It is right?

Pl: Yes because 2 times 6 is 12 plus 1. 13.

INT: Excellent. Well done. Can we just try No. 12 then finally and the you can get back to your maths lesson. Can you just read that one out and see if you can do that. Very hard one this.

Pl: For what values of d is 17 minus 3d is greater than 2?

INT: Do you understand it?

Pl: Yes.

INT: Can you tell me how you are going to try and do it then?

Pl: ...... Um, well it can be anything really. It's got to be anything greater than 2 the end, so it will be 17 minus 3 that will be 14 an that's greater than 2. Anything up to ... Ah that's 2. So it's anything less than 2, d can be anything less than 5 I mean.

INT: Good so how would you write that answer?

Pl: ....

INT: Good. Well done.

Pl: I think that's right.

INT: Excellent. You did very well there.

PUPIL 2

INT: Can I just start off by asking you what you thought of about the algebra work.

P2: Well it's quite good, it's interesting but I found a lot of it a bit easy.

INT: That's the computer work is it? Easy in what way?

P2: Oh, the computer work?

INT: Yes.

P2: No, I quite enjoyed the computer work.

INT: Anything in particular that you liked or didn't like?

P2: I liked the programme.

INT: What, the writing of the computer programmes?

P2: Yes.

INT: Have you done anything like that before?

P2: No not at this school.

INT: But you have done something before?

P2: Well we've got a computer at home I done it on.

INT: Very good. Did you learn anything about letters before that you didn't know before?

P2: Yes quite a bit I learnt. I learnt quite a lot.

INT: Anything in particular, nothing that stands out.

P2: No, not really.

INT: OK.

P2: I hadn't done much algebra before at my last school. I had done a little bit but not much.

INT: OK. Right so you're Clare Evans, let me just get that on the tape. Want I want you to do is to try these. Don't worry about all this long list. Have you got a pen. What I want you to do is just read out the question.

P2: What is the value of 3 plus 2m when m equals 5?

INT: Now do you understand what that is asking you to do?

P2: Yes.

INT: Can you try and do it and tell me how you are going to do it and then if you can write down what you can actually do and how you do it.

P2: Yes, well I do it in my head, I can't really explain how I do it.

INT: Will you try to explain as best you can because I am more interested in how you do it than the answers today because I know you can do the questions because you scored so well in the test. What I am interested in is if you can give me some idea of what you are thinking about?

P2: Well if m equals 5 that's got to be 2 times 5, so that would be 10 and 3 plus 10 would be 13.

INT: So can you write some of that down for me........ - you needn't write the question just how you've worked it out. Just write what you've told me.....You don't have to write it in English, I think perhaps that's what you have misunderstood. Just write the maths down, and can you actually write the answer for me, what you actually get, that's what I'm interested in - just the maths bit instead of the bit you worked out first......Good, lovely. OK now you've swopped that round haven't you. Does that make any difference?

P2: I think you're supposed to do the times first.

INT: So that is why you've written it first. Lovely, OK. Would you like to read out the second one for me.

P2: If 2p minus 1 = 5 and 2s minus 1 = 5, can you say anything about p and s?
What is the value of p?

INT: OK. Well let's leave that bit for a second. Do you understand what that is asking you?

P2: Yes. You've got to sort of work back to find out the value of p.

INT: Can you say anything about the values of p and s straight away from that information?

P2: Well they are both the same.

INT: Are they? Is that obvious?

P2: Yes, because they are both the same sum but different letters.

INT: So what would the value of p be then? Can you tell me how you are doing it again if you can, I know it's hard.

P2: Well find out what minus 1 so you would add 1 to that so you get rid of the 1, so that would be 6 and then its obvious that 2 times 3 equals 6 so the p would be 3.

INT: Good. Can you put a question 1 by that and underneath a 2 and then if you could just write down what the value of p was. Good, ok. I wonder if you can do this one for me. Now this is a question connected with that one, I don't know whether you can see any similarity. Can you just read that one out?

P2: For what values of p does 2 p plus 1 minus 1 equal 5? The brackets means that you've got to do that first. So it would be ...

INT: If you were to compare these two would that help you to say what p was in this one?

P2: Well it would be more, p would be more in this one.

INT: Would it, why is that?

P2: Because the p might be less wouldn't it. p would be.... p would be 3 again.

INT: It would be 3 again would it? Can you tell me why?

P2: Well 2 times p plus 1 if p is 3 would be 4 and 2 times 4 is 8 - no it wouldn't be 3. Oh, it would be 2.

INT: Can you tell me why?

P2: Because p plus 1 if that's 3 it's the same as the last one only the p is less because you've got to add 1 to the sum.

INT: Good. Excellent. Let's try question 3. Can you read that one out for me?

P2: Multiply d minus 4 by 3.

INT: Do you understand what that is asking you to do?

P2: Um, well yes it would be 3d minus 4.

INT: I see, can you write it down for me. How did you get the answer?

P2: Oh, it would be brackets.

INT: Why do you want brackets?

P2: The d minus 4 has got to be done first.

INT: So that the brackets mean what then?

P2: Well they mean the part in the brackets has got to be done before any other.

INT: What was it that made you decide that it had to have brackets.

P2: Because that came first. d minus 4 then you times it by 3. So if you hadn't got the brackets it would be 3 times d and then minus 4.

INT: Ah I see. So if the question said 3 times d minus 4 what is that?

P2: That would be 3 times d, say d was 2. 3 times d which was 6 minus 4 would be 2.

INT: Oh I see. Good. Thankyou. Fine. Question 4 then would you like to read.

P2: Which is bigger 3y or 3 plus y?

INT: Do you understand that? How would you do that?

P2: Well 3 plus y is smaller because 3y means 3 times y so if y was say 2 that would be 5 and that would be 6.

INT: Oh, I see, good and is that always the case that 3y is bigger?

P2: Most of the time.

INT: Most of the time?

P2: Well if it was 2y...

INT: Yes what if it is as there, 3y and 3+y, 3y is always bigger is it?

P2: Yes, well,.. Yes... unless the y was 1. 3 times 1 would be 3 and 3 plus 1 would be 4.

P2: Well no, just in the very low ones.

INT: In the very low ones. Can you give me any idea of what they might be?

P2: Well 1 and maybe fractions.

INT: Ah, so fractions as well. Um, very good. So if we were asked that question, which is bigger out of those two....

P2: It depends, 3y can be bigger sometimes and 3 plus 1 can be bigger.

INT: So what does it depend on?

P2: Whether the y is a fraction or a very low number like 0 to 1.

INT: Jolly good. Can you read out No. 5.

P2: Add 3 onto ...

INT: Oh, I'm sorry on No. 4 can we just write what would be your answer to that question then. How you would answer it. Lovely, Ok good would you like to read out No. 5.

P2: Add 3 onto 6g.

INT: Understand that?

P2: Mm.

INT: Can you tell me how you would do it then.

P2: It would be 6g plus 3.

INT: Good. As easy as that is it?

P2: Mm.

INT: Can you write that down? Lovely. What if it was this question instead?

P2: That would be g plus 3 or 3 plus g.

INT: So it doesn't matter

P2 Not if it was plus. If it were just those 2 numbers it wouldn't matter.

INT Good, ok. Can you read out No. 6 then.

P2 What is the value of 3 plus 2, 3 plus 2x c when c equals 5?

INT Sorry that's a times it's just the printing. Multiplication sign. Do you understand how to do that one?

P2 Mm

INT Can you tell me what you are thinking.

P2 Well if that would be 5, so it would be 3 plus 2 which is 5 times 5 which is 25.

INT Can you write that down for me? No. 6.

P2 Shall I just put 25 down?

INT No if you can put down how you get it please. Just the working. Good. Would it make any difference if you had this question here for example. Could you read that one out please.

P2 Value of 3 times c plus 2 when c equals 5.

INT How would you do that one?

P2 Well you would put the 5 in again and it would be 3 times 5 is 15 plus 2 is 17.

INT Good. I've got another sheet here which asks some difficult questions because you're quite good at maths. You don't have to do all of them. Question 1 could you just read that out please.

P2 Write 2 a plus b without brackets.

P2 Um,

INT What are you thinking, is there a problem?

P2 Well I'm not too sure about that one because if that was a number I could do it.

INT Ah, if the b was a number.

P2 Yes, if that was say 3, you would times the 2 times 3 and it would be 2a plus 6.

INT Why does it make a difference if it is a b then?

P2 Because, oh it would be 2a plus 2b. So shall I put 1 there?

INT Yes, put 1 please. Why was the b a problem, can you explain to me why you were happier if it was a number than letters.

P2 Because you can times a number and it's easier, 2b doesn't really sound like a number.

INT But you were happy with the a at the front? 2a doesn't matter.

P2 Not really, no.

INT Why were you happy with the a at the front but not with the b?

P2 Because you don't have to times it.

INT Ah, I see. Because it was near the 2.

P2 It's already there.

INT Good, let's go down to some difficult ones. Can you read out No. 10 for me.

P2 Write more simply 2m minus n plus 3n.

INT Do you understand what that is asking you to do?

P2 Yes

INT Can you tell me what you are thinking of again as you go through it....What are you thinking, is there a problem?

P2 I don't think I can do it. Would you have to take 1n off that because it is minus n there, so make that 2m, so then it would be 2m, 2 times m plus 2n.

INT Good, can you write that down for me? Why were you not sure about it?

P2 Well, it just looked a hard sum. It looked a bit complicated.

INT What made it look complicated, can you tell me?

P2 It was because it's a long sum. We haven't done much of this.

INT You haven't. Do the brackets make any difference?

P2 Well, yes I suppose they do because minus n it looks as if it's a separate thing so you can't really take it off there.

INT Good, ok, can you read no. 11 now for me.

P2 Solve 3x minus 5 equals 2x plus one.

INT Understand what you have got to do there?

P2 Yes

INT What does the word solve mean?

P2 It means do the sum. You would add 5 to that to get rid of the minus 5 and then that plus 6, so it would be 3x equals 2x plus 6.

INT Can you write it down for me as you go on this one.

And just tell me what you are doing.

P2 ....I'm not too sure how to get rid of the plus 6. I think you should get rid of the plus 6.

INT You think you should get rid of the plus 6. Why do you have to get rid of that.

P2 Well cos it's hard to do the sum, it's like a hazard.

INT Why is it a hazard, can you explain that to me.

P2 Because it sort of, I think it means 3x equals 2x, well 3x wouldn't equal 2x would it. I did know how to do it but ....

INT These are hard questions aren't they.

P2 We haven't done algebra for quite a time, we've just gone on to sets and numbers.

INT Can you just think how you might do it, it doesn't matter if you don't do it.

P2 Well that plus 6 has got a bigger x because 2x plus 6 equals 3x, that means another 6 would equal x, so make that 3x as well.

INT Good, can you write that down?

P2 Shall I just put x equals 6?

INT Does that help you get the answer?

P2 Shall I just put x equals 6?

INT Does that help you get the answer?

P2 Yes.

INT What's the answer to the question?

P2 Well, x equals 6.

INT And that's it. Good. Can you just try No. 12.

P2 For what value of d is 17 minus 3d is bigger than 2.

INT Understand that one? What is it asking you to do?

P2 Well, you've got to sort of find out the value of d.

INT How are you going to do that then?

P2 It's going to have to be quite a small number, if it's bigger than 2 I mean it wouldn't be say 6 because they would make it minus 1, so it would have to be less than 6. It could be 5, that would be equal to 2, so it would probably be right for 4 and under.

INT Good, can you write that down for me...... and what sort of values could you have for d?

P2 4, 3, 2, 1 and fractions.

INT Fractions as well. Good, ok, that's fine. You did very well there. It wasn't too bad was it. Good. Well I'm glad you are doing so well with your algebra.

PUPIL 3

INT Well, first thing I want to do is to put you at your ease and tell you that this isn't a test that is going to affect your school work at all, it's not going to go on any report, no letter to your parents, not even your maths teacher is going to be told. All I want to do is to find out what you thought of the algebra work that you did on the computer and then just have a look at one or two questions and see whether you learned anything or and what you are sort of thinking about. What did you think about that work?

P3 It was interesting, because I haven't done much algebra before in Middle School so it was all very new to me really. I found it something fresh to do, it was quite good.

INT What did you sort of learn from it, was there anything in particular that you didn't know before?

P3 Um, I don't think so, I think it was quite .. it wasn't easy cos it was new it was quite sort of like an adventure. Everyone was doing it at the same level because no-one had done much before.

INT And that was good, was it?

P3 Yes.

INT So nobody stood out.

P3 No.

INT Excellent. Well let's just give you one or two of these questions then. See if you can do them for me. I'm not interested in how well you do so you don't have to worry about that, I'm more interested in the way that you try to do the questions, not the answers, so I'm not going to keep a score or anything. Could you just put the question numbers down the left hand side and if you could read out the first question for me.

P3 What is the value of 3 plus 2m when m equals 5?

INT Do you understand what it is asking you to do?

P3 The value of m is 5 and you are adding 3 to the 2, the value of m is 5.

INT So can you actually do it for me and tell me what you do and write it down then. Can you tell me what you are actually doing as you work it out.

P3 It's 2 times 5, because it is 2 times m, so that's 3 plus 2 times m, which equals 5, so it is 2 times 5 equals 10 plus 3 equals 13.

INT Good. Right. So the 2m in that question means what to you?

P3 You need to know the value before you work out the actual sum.

INT That's good, would you like to read out No. 2 for me.

P3 If 2p minus 1 equals 5 and 2s minus 1 equals 5, can you say anything about p and s? What is the value of p?

INT Let's leave that bit for now. Do you understand what it is asking you?

P3 If that's minus 1 that's equals 4 2p, that equals, that equals 6 and then it's minus 1 plus 5 and 2s minus 1 equals 5, so that is obviously 6 as well. And then can you say anything about p and s, they are both equal 3.

INT Can you write that down....Good, so what can we write about p and s, could you just write that for me. What did you say about p and s?

P3 They are both ... p and s both equal 3.

INT Good, can you just write that down for me. ..... So you've answered this next bit. I don't know whether you can answer this one, it's actually connected to that question. Would you like to read that No. 2 out.

P3 For what values of p does 2 p plus 1 brackets minus 1 equals 5? So 2 whatever is in brackets is times that, so you do the brackets first and times it by that.

INT Does it help to look at this question in order to do this one? Can you tell me what you are thinking.

P3 I'm trying to work out the value of p to give the answer 5. That must equal 2 if it's 2 twos......

INT Can you tell me what you are trying.

P3 So that I can get to that answer.

INT And how are you trying to get there.

P3 That lot must equal 6 so if it is minus 1 it equals 5, so 2 times p plus 1 is 6 minus 1 is 5.

INT Good. So what would be the value of p then?

P3 3......2. So that equals 3 times 2 minus 1 equals 5.

INT Good, can you write that down for me. We'll still call it question 2 it's a sort of extra bit. OK Now we go on to question 3, can you rea that out for me.

P3 Multiply d minus 4 by 3.

INT Understand what it is asking you to do?

P3 Just times d by 3....The answer of d minus 4 by 3.

INT Can you actually do that and write it down for me.... No problem witl that.

P3 I'm trying to find the value of d.

INT You need to have a value do you.

P3 ....You work out what 4 no ......

INT Can you tell me what you are thinking again?

P3 I'm trying to think what that is and then times it by 3, so I can ge an answer.

INT How might you do that?

P3 I've got to know what the value of d is really.

INT You can't do it without it. OK That's alright. Let's leave that as the answer. This is 4 would you like to read that one out for me.

P3 Which is bigger 3y or 3 plus y?

INT Do you understand what that is asking you?

P3 Yes. 3y is bigger because it is 3 times y.

INT Good. Is that the answer is it. Can you write that down for me. And is that always the case.

P3 Yes.

INT So it doesn't matter what y is?

P3 No.

INT Good. Let's try No. 5.

P3 Add 3 onto 6g.

INT Understand what it is asking you to do?

P3 Yes. 6 times g whatever that is plus 3.

INT Is that the answer?

P3 Yes.

INT Happy with that answer? OK Let's try No. 6.

P3 What is the value of 3 plus 2 , is that times, times c when c equals 5?

INT Understand what that is asking you to do?

P3 That's 5 times c, 5 times 5 25

INT Good, can you write that down for me. Can you tell me this one whic is a similar question, you can put this under 6 as well. How would you do that one? Can you read that one out?

P3 Value of 3 times c plus 2, when c equals 5? c plus 2 that's 7 because that is 5.

INT Can you write that down. Good. How do you decide which bit to do first on these questions?

P3 It helps to know what c is first, so you can do that part first and then times it by that because it doesn't really matter which part yo do first when you times it.

INT And is that the same here? Does it matter which you do first there?

P3 No.

INT You can either.

P3 Yes.

INT Right, ok. There are just one or two other questions, you don't have to do all these so don't worry about all those questions there. Question 3 there can you read that one out for me?

P3 Write more simply 2d plus 3 plus d.

INT Understand what that is asking you to do?

P3 2 times d plus 3 plus d.

INT How are you going to do that one? Can you tell me what you are thinking there?

P3 ....

INT Why would you want to put the 3d? Can you just explain that.

P3 Because it looks simpler.

INT Ah, I see, so you want to make the question simpler by putting 3d. What would the answer be then.

P3 5d.

INT But you can't do that because why did you say

P3 Because it is more, it's actually 3 times d not 3 plus d.

INT So how are you going to do this one? Can you think of any way round that problem? You can't. So you can't do that one. What if it was written slightly differently. What if it was written say there was this sum. Can you read that for me.

P3 2d plus d plus 3.

INT Could you do that one? Can you tell me what you are thinking about.

P3 So you are adding on another d to that plus a 3

INT So what would you get.

P3 2d plus, No that would be the same as having 2 plus d.

INT It doesn't help to do this other question then and writing it like that.

P3 No

INT What is the particular problem. Can you tell me which bit is the big problem in the question. Is there any one bit that causes the problem or is it the whole thing.

P3 It's just this particular thing to do with these two because you can't put it like that.

INT Well ok, that's fine. Let's go onto the next one. Try Question 4. Would you like to read that out for me.

P3 Add 6 onto e plus 4.

INT Understand what you have got to do there?

P3 Yes. 10 plus e.

INT Can you write that down please. ...Good happy with that answer. No problem with it. Good. Question 6 next.

P3 When does p minus b equal p minus c? Always, never or sometimes? If sometimes, say when.

INT Understand that? Understand what it is asking you to do.

P3 When do those letters p minus b equal p minus c when they have the same value.

INT When what have the same value?

P3 Those two.

INT Which two?

P3 Both of those, p minus b and p minus c.

INT Good and can you answer that? Can you tell me when that would be.

You've got three alternatives here, are they always the same value, never or sometimes.

P3 Always.

INT What always the same.

P3 Yes, because they could be anything, the value could be anything.

INT So if the value could be anything will they always be the same?

P3 I think so.

INT You think so. OK that's fine. Let's just try Question 7 then. Would you like to read that out for me.

P3 For what value of c is 7 equal c minus 2?

INT Understand what you've got to do there?

P3 It's what equals value of c is which gives c minus 2 which is 7.

INT Good. Can you do that one then? Tell me what you are doing as you do it.

P3 I am adding onto the 2 until I get to 7. That must be 9 if it is 9 minus 2.

INT Good, can you write that down for me. That is actually question 7 so if you can put that number. Good. That's fine. Well that wasn't too bad was it. You did very well. Excellent.

P3 I found from doing the computer work it was much easier than just going straight on to it. I remembered quite a few things.

INT You found it helped you then.

P3 Yes.

INT You can't think of anything in particular where it helped you. Was there anything or not?

P3 No.

INT It just helped. Excellent. Good. Well thank you very much for your time. I hope you haven't missed an important lesson.

PUPIL 4

INT It says Peter Rixon on the top already for you, how about that. It knew you were coming you see. You're not worried about this are you?

P4 I don't know what it is.

INT You don't know what it is. You might be a bit worried then. Let me put you at ease for this because it's not a test that is going to go on any reports or get sent home to your parents, even given to your maths teacher, so you don't have to worry about it. All I want to do is to find out - do you remember the computer algebra work you did - I just want to find out what you thought of it and whether it has helped you at all basically. OK. What did you think of it? Can you tell me.

P4 It was a bit boring when you did it on the computer because it carried on for quite a long time. It was better to get on with your own book work.

INT I see, so you would have preferred it a bit shorter.

P4 Yes.

INT Do you think it helped you at all?

P4 Yes.

INT Can you tell me any particular way that it might have helped or not, I mean if there isn't anything .... Can't think of anything specific. OK. I'm going to ask you to do one or two questions for me, now it's not a test, there aren't going to be any marks, I'm not going to write down how many you get right or anything like that. I'm more interested in how you do the question. That's what I would like to see. If you could put the question numbers down the left hand side for me. Could you just read out this first question for me.

P4 What is the value of 3 plus 2m when m is 5?

INT Good. Do you understand what that is asking you to do?

P4 Yes.

INT Could you do it for me and tell me what you are doing as you do it.

P4 And write it down.

INT Yes, please. Can you tell me what you are thinking as you do it.

P4 Well I write the 3 less the 2 and then that will be times 5.

INT Good.

P4 I did it that way. 2 times 5 is ten and then plus 3.

INT Good. Is there any reason why you do it that way?

P4 Just quicker I suppose.

INT What would be another way of doing it? Is there another way of doing it?

P4 3 plus 2 times 5 but that is 2 times 5.

INT Good. That's fine. Can you read out the second one.

P4 If 2p minus 1 equals 5 and 2s minus 1 equals 5, can you say anything about p and s?

INT Do you understand what that is asking you?

P4 Find out what p and s equals.

INT Good. That's how you would answer that question is it about what you can say about p and s?

P4 Yes.

INT Can you actually do it? Can you tell me what you are doing as you do it.

P4 It will be 2 times something minus 1 equals 5, so you've got to find out what the something is.

INT Good. How would you do that?

P4 It'll be 2 times 3 minus 1 equals 5.

INT So what does that actually tell you?

P4 That the value of.p is 3.

INT Good. So what about the question that says can you say anything about p and s. Does that help you to do that?

P4 Well I could just assume that it would be asking what the value is.

INT So what are you going to do now then. Can you tell me what you are thinking.

P4 Be the same.

INT Be the same, would it? Why is that?

P4 It's just a different letter but it would have to be 2 times 3 minus 1 to equal 5.

INT So what can we say about that sum.

P4 Its equal to p.

INT Good, can you write that down. So what is the value of s then? Good. Well you've actually answered this next bit haven't you. Does it help you, there's a difficult question here that's sort of connected with that one. Can you read out that question. Its also No. 2.

P4 For what values of p does 2 and then p plus 1 minus 1 equal 5?

INT Does doing this question that you have just done help you to do this one at all? How would you do this question?

P4 It would be p plus 1 whatever p is....

INT So how would you work out this one?

P4 2 .. that would be 3

INT What would be 3?

P4 And that would be 4 plus 1 which would equal 5.....

INT It doesn't help at all that you have done this question here? You've got to start again on this one have you or what?

P4 Yes. Apart from p could be 5.

INT Why could it be 5.

P4 Because then you could plus 1 which would make it 6 and then you would minus 1 and then you would have the 5.

INT I see. Good. OK. Let's go on to the next one over here shall we No. 3 can you read that one out for me.

P4 Multiply d minus 4 by 3.

INT Do you understand what that is asking you to do?

P4 Multiply d by 3 and then minus 4. You've got to find out what d is.

INT Can you actually do that then and tell me what you are doing as you do it.

P4 Could you put any number in?

INT You want to put a number in do you? Do you need to put a number in, well you do it, put No. 3 and perhaps you could write it down for me........ Can you write down an answer for that? .... Can you tell me how you got that.

P4 I just put any number in.

INT What number did you pick?

P4 10. It took me minus the 4 and I got 6 and then I did 3 times 6.

INT Why did you pick a number for d? Can you tell me why?

P4 No reason.

INT No reason. You couldn't without a number.

P4 Yes.
You can do say ... it would be 3 times d minus 4 ....

INT Is that the answer or have you got to do something to that?

P4 You'd have to do d minus ....

INT Minus what?

P4 You could have the 3d minus 4 but that would just be another way.

INT Could you write that down? Good. So that would be the answer would it if you didn't know what d was?

P4 Yes.

INT OK, good. Let's go on to No. 4. Can you read that one out for me.

P4 Which is bigger 3y or 3 plus y?

INT Understand what it is asking you to do?

P4 Yes.

INT How would you do that then?

P4 It's 3y because 3 times whatever is y and that is only 3 plus whatever is y.

INT So that's bigger, so can you write that down. And is that true no matter what's in y?

P4 Yes.

INT Good. OK. No. 5 then. Can you read that out.

P4 Add 3 onto 6g.

INT Understand that one?

P4 Yes.

INT How would you do that one?

P4 You would add 6g plus 3.

INT Good and is that the answer?

P4 Yes.

INT Happy with that answer

P4 Yes.

INT Let's try No. 6 then. Can you read that one out?

P4 What is the value of 3 plus 2x c when c equals 5?

INT Yes, that's a times actually. It's the printing. Can you read it again.

P4 What is the value of 3 plus 2 times c when c equals 5?

INT Do you understand what that is asking you to do? How would you do that then?

P4 2 times 5 plus 3.

INT Good and what does that give?

P4 15 or you could say,..yes it would be the same as well.

INT OK can you write that down for me then. How you work it out. Tell me what you are doing as you write it down.

P4 Sorry that would be 13 and that would be 15 if you did it the other way.

INT Which other way?

P4 3 plus 2 and then times 5.

INT Ah, I see. Does it matter which way you do it then?

P4 Yes.

INT Which one is right?

P4 3 plus 2 times 5 because it should be in brackets otherwise.

INT What should be in brackets?

P4 The 2 .. um 2 times c.

INT It would have to be in brackets to be done first?

P4 Yes to be done first.

INT OK can you do the question for me then....and tell me what you are doing as you do it.

P4 3 and 2 are 5 and times 5 is 25.

INT Can you do this one which is a similar question, its also a No. 6. Can you read that one. That means find.

P4 Find the value of 3 times c plus 2 when c equals 5.

INT Its a similar question. Do you understand it?

P4 Mm.

INT Can you do that one and tell me how you do it.

P4 You would have 3 times 5 then you would add 2 to it.

INT Good, can you actually do that one and tell me what you get as you do it.

P4 3 5s are 15 and then add your 2.

INT How did you decide what to do first this time.

P4 I just went through it 3 times 5 and then you add 2 because you would have to have brackets round there if you wanted to do...around the c and 2 if you were going to do that bit first. 5 plus 2 and then you would times it all that by 3.

INT Excellent. Thank you very much. I've just got one or two others on this sheet here. Now you don't have to worry about this one too much because you don't have to do all these, we'd be here all day. Question 1, would you just like to put a 1 down. Would you like to read that one out for me.

P4 Write 2 a plus b without brackets.

INT Understand what that is asking you to do?

P4 Um.

INT Can you do it and tell me how you do it.

P4 2 times a and then you would plus b.

INT Good. And that's the same is it?

P4 Yes...No. No it's not the same as that.

INT It's not the same as that?

P4 No.

INT Why isn't it the same?

P4 Because you would have to have a brackets.

INT Is that the only way to write it the same by having the brackets in. Is there any way of writing this without brackets so that it is the same?

P4 Yes, you could have a plus b and then times by 2.

INT Could you write that for me? So that one is the same as this one but without the brackets.

P4 Yes.

INT But what about this one.

P4 No you do 2 times a and then you plus b.

INT So that is not the same?

P4 No.

INT Ah, right, ok. Good. Question 2 could you read that one out for me then.

P4 What is the value of 3s plus b when s equals 5?

INT Understand that one? Can you do that for me? ....Can you tell me what you are doing?

P4 Three times five is 15 and then plus b, that would be it. That's the answer.

INT Good. Lovely. Would you like to do No. 3 then. Just read that one out.

P4 Write more simply 2d plus 3 plus d.

INT Understand what that is asking you to do?

P4 Yes.

INT Can you do it? Tell me how you do it as you go.

P4 You write 2d plus 3d or, no, 3d plus 3.

INT Good can you actually write that down. Is that the right answer? Happy with that answer?

P4

INT Good

P4 Be 3.

INT That's right isn't it?

P4 Yes.

INT And just finally then No. 6 if you could just read that one out.

P4 When does p minus b equal p minus c? Always, never or sometimes? If sometimes, say when.

INT Understand what that is asking you?

P4 Yes

INT Tell me how you would do that then and why.

P4 It could be the same if b equalled the same as c, had the same value.

INT Oh, good, could you write that down for me. Just abbreviate it, you needn't write it in full. Just write down what you told me was the answer........Good, so is the answer always, never or sometimes.

P4 Sometimes.

INT Good. Sometimes. Fine it wasn't too bad was it. Good, you did very well. So you've learned quite a lot doing your algebra. Cos you've done some algebra since you did computer work haven't you.

P4 First of all we had a test before we knew anything about it and then we did all the algebra work and then had another test.

INT That was for me that test. I've got all the results. I think you did quite well and you've done some more algebra since that test haven't you?

P4 Yes we did it out of the book and had a sheet.

INT Did you find that you were able to understand it. Did the computer algebra help at all, do you think?

P4 Yes it did.

INT You think it did. Can you think of any particular way that it did or what.

P4 Didn't understand when you had d like plus b, I would always put a number in and then it was plus b or whatever.

INT Very good. So you don't have to put a number.

P4 No.

INT And you're happy with that now are you?

P4 Yes.

INT Good. That's fine. Thank you very much for your help.

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PUPIL 5

INT What I would like you to do is just have a go at some of these questions for me. I'm more interested not in the score but how you d them so I'm not going to keep a record of how many you get right. Ca you just read out question 1 for me.

P5 What is the value of 3 plus 2m whem m equals 5?

INT Understand that?

P5 Yes.

P5 Because 5 equals m we add two lots of m which is 10 and then plus 3.

INT Good, can you write that down. That's the answer then.

P5 Yes.

INT Good. OK. Happy with that answer. So the 2m means what then?

P5 2 lots of 5.

INT Good. Can you read out question 2.

P5 If 2p take away 1 equals 5 and 2s take away 1 equals 5, can you say anything about p and s?

INT Understand that one?

P5 Sort of.

INT Sort of, what's the problem with that one then?

P5 Saying something about p and s.

INT Just answering it, but you understand what it is asking you to do.

P5 Yes

INT Good, how would you set about answering it?

P5 That 2p equals 3.

INT What is 3?

P5 The p.

INT p is 3 is it?

P5 Yes

INT How did you get that?

P5 Two lots of 3 equals 6 take away 1 equals 5.

INT Right. can you write that down for me? Good, what about s?

P5 The same.

INT It's the same, is it? Why is that?

P5 It's just using a different letter.

INT Good so can you write that down. So in this question it says what can you say about p and s.

P5 They are both the same.

INT Good. There's a question here which is very similar to that one, I don't know having done that does it help you to do this question at all? Would you just like to read that.

P5 For what values of p does 2 bracket p plus 1 bracket take away 1 equal 5?

INT Does having done this one help you to do that question at all? And how would you do this one?

P5 Add p to 1 first.

INT How would you do that?

P5 p plus 1 ---- 2 plus 1 times 2, 3 - p plus 1 is 3 times 2 is 6 take away 1 is 5.

INT Good, so what does that tell you about p then in that case?

P5 2

INT Good, can you write that down.... Good that's still question 2 actually. The bell's just gone are we alright for a minute or so?

P5 Yes

INT Just do one more shall we and then you can come back later. Would you like to read out No. 3.

P5 Multiply d take away 4 by 3.

INT Understand that one?

P5 Whatever the value of d is take away 4 and times 4 by 3.

INT Can you write down what you would get. Can you write that down or not?

P5 Multiply the d ..

INT Oh, you need a value for d do you? Yes? You do and you can't write anything down at all unless you know what d is. OK That's fine. Good. Shall we leave it there then...... Question 4 we got to then, would you like to read it out for me.

P5 Which is bigger 3y or 3 plus y?

INT Understand what that is asking you?

P5 Yes.

INT Can you tell me how to do it then.

P5 3y is 3 lots of y, 3 plus y is 3 plus whatever y is.

INT So how will you set about answering that then.

P5 Depending on what the value of y is it could be 3 plus whatever that is or 3y, so it is 3y that is bigger.

INT Right, ok, can you write that down. Now when you said depending on what the value of y is what did you mean by that.

P5 Say it was 10, 3 plus y would be 13 but 3y would be 30.

INT So are there any values of y for which 3y isn't bigger?

P5 No.

INT There aren't.

P5 No.

INT No, ok. No. 5 then. Like to read it out for me.

P5 Add 3 onto 6g.

INT Understand what you've got to do?

P5 9g.

INT Good, can you write that down. How did you get that answer?

P5 It was just adding 3 onto the 6 and making it 9 more lots of g.

INT Right, ok. There's another question here on that one which is quite similar. This one here. Can you just read that out.

P5 Add 3 onto g.

INT Good, what would you do in that case.

P5 Make it 3g.

INT You'd make it 3g would you. How did you get that answer?

P5 4g.

INT Make it 4g. How did you get 4.

P5 Because it's g there and another 3 make it 4.

INT Right, ok, just write that underneath then. Like to read out No. 6 then.

P5 What is the value of 3 plus 2x c when c equals 5?

INT That is a multiplication. How would you read that?

P5 3 plus 2x c when c equals 5? Oh 3 times.

INT Yes, read it as times. Know what you've got to do there then?

P5 Yes.

INT Can you tell me how you'd do it.

P5 Add 3 and 2 and times it by 5.

INT Can you do it then. Write it down and tell me what you are doing as you do it. That was quick. How did you get that?

P5 Add 3 and the 2 which is 5 and then times it by c which is 5.

P5 3x... 3 times c plus 2 when c equals 5.

INT How would you do that then?

P5 It would be 3 times 5, 15, plus 2.

INT Good, can you write that down? How did you decide on these, which bit to do first, can you tell me.

P5 Because it's not in brackets, so you go along this first.

INT So you start from where then if it's not in brackets?

P5 The 3.

INT So from the left. And the same here is it?

P5 Yes

INT So if there aren't any brackets you start from the left and work along.

P5 Yes

INT Good. Don't get worried about this sheet, it's got all the questions on but you're not going to do them all. Just one or two. No.1 can you read out that please.

P5 Write 2 a plus b without brackets.

INT Understand what that is asking you to do?

P5 Is it 2a plus b?

INT What we want ... well alright you're telling me what the answers are, but we want something that's the same as that only without any brackets in it. If that's what you think the answer is then write it down and you can tell me how you got it... What are you thinking now?

P5 I'm working out what to put.

INT What to put. Are you not sure about the answer.

P5 I'm not sure.

INT Well put what you think it is. As I say the answers aren't important so you don't have to worry about whether it's right or not. I want to know what you think it is and then why... right ok so how did you get that answer?

P5 Because you can have any value for a and b. a plus b times 2 would be say 10 so 2 plus 2 times 2 or it could be 2 lots of a which is say 4 and then plus b which equals 2, well b could be the same value.

INT Did you get the same answer both ways by putting your values in. Will you get the same from there as you will from there. Did you check that?

P5 No you wouldn't.

INT What does that tell you?

P5 It's wrong.

INT Have you any idea how you might correct it? Doesn't matter if you haven't.

P5 No

INT No, ok, that's fine, let's move on to the next one. Question 4 can you read that one out for me.

P5 Add 6 onto e plus 4.

INT Understand that? Understand what it is asking you to do?

P5 Add 6 onto e plus 10

INT Good, can you write that down, it's question 4. ....Good and that's the answer is it. Happy with answer?

P5 Yes

INT Good, Question 6 then would you like to read that one out.

P5 When does p take away b equals p take away c? Always, never or sometimes? If sometimes, say when.

INT Understand what that is asking you to do?

P5 Yes, it depends what the values are.

INT Yes, alright. Good. So how does it depend, can you tell me and then you can write it down. What does it depend on?

P5 The value of these two.

INT Which two.

P5 These.

INT Which are they, can you tell me.

P5 b and c.

INT Depends on those does it?

P5 It's different from there.

INT In this particular sum, I'm not trying to catch you out or anything I just want to make sure that I understand what you are trying to say. So what would you say the answer is then? Are they always the same, never the same or sometimes first of all.

P5 Sometimes.

INT Sometimes. Good. Let's put that then and then we come to the bit - the question is well when then in that case. When will they be the same?

P5 When b and c are both the same value.

INT Good. Let's try No. 7 then. Can you read that out.

P5 For what value of c is 7 equals c take away 2?

INT Understand that one?

P5 You put a value there, 7 take away that take away .. 7 equals when c take away 2 so 9 take away 2 equals 7.

INT Good, can you write that down then. OK and I think just one more, question 10 if you could just try that one. It's a little bit harder this one.

P5 Write more simply 2m take away n plus 3n.

INT Understand what you have got to do?

P5 Yes.

INT So what are you thinking? Is there a problem?

P5 You can take away n there's only 2m left over ....

INT So what would the answer be?

P5 2m

INT Can you write that down then? Is that the whole of the answer?

P5 No

INT What else is there then? ..

P5 Just 2m plus 2n.

INT Good, why was there a problem with that then?

P5 Well I was thinking about taking that one away. The brackets were there so there would be nothing to take away.

INT Ah I see. Do the brackets make that question hard then?

P5 Yes

INT Can you tell me why. What is there about the brackets that make it hard?

P5 Confusing.

INT Just confusing is it? Good, ok, that's fine. That wasn't too bad was it? You coped very well with that I thought. Got some of the difficult ones right. That last one is quite difficult. Can I just ask you as I didn't have a chance at the start because we were a bit rushed what you actually thought about the algebra work that you did on the computer.

P5 It could have explained it a bit more.

INT In what way?

P5 Tells you how to do it a bit more.

INT You found it a bit hard to know what to do did you?

P5 Yes

INT Good. Did you enjoy any of it or did you hate any of it.

P5 It was alright.

INT Just alright was it. Good. Think it helped at all with algebra or not.

P5 Some parts.

PUPIL 6

INT Remember the computer algebra that you did, what I want to know to start with is what you thought of it basically.

P6 I didn't think it taught me much. I think it was too easy. It had too easy sums in it.

INT Um that's interesting. Have you ever done anything like it before?

P6 No. It was fun.

INT It was fun was it, so you didn't mind doing it, but you didn't think you'd learnt much.

P6 No. A bit I learnt.

INT A bit. What if I told you your marks on the test went up by 20 before and after, did you know that?

P6 No.

INT So you must have learnt a bit.

P6 My mum taught me it at home.

INT Did she. How did she do that?

P6 She knows maths. My dad taught me as well.

INT Jolly good. That's handy isn't it. What I'd like to do now is just to see what sort of things you've learned. I'm just going to ask you to do one or two questions. Now I want to emphasise it's not important how well you do them, there's no marks for this, there is no report, I'm not going to send a letter home or..I'm not going to tell Mr. Jones or anything like that. I just want to know how you think about questions, so it's not the answers so much as to how you think about them that's important. So can read out question 1 for me.

P6 What is the value of 3 plus 2m when m equals 5?

INT Good. Do you understand what you have got to do? Can you do it for me and tell me how you are doing it....Can you tell me what you are thinking.

P6 m is 5, so 2m is 10 plus 3 equals 13.

INT Good, ok. So 2m, what does that means then?

P6 2 times 5

PUPIL 7

INT All I want to do is try to get some idea - you know the computer algebra you did, just want to know what you thought about it basically and then get you to do one or two questions for me to see if you can do them now. What did you think of it when you did it?

P7 Quite interesting, but I didn't see the point - do we need it when we leave school.

INT Ah, right, that's interesting. Good. So you wondered whether it was worth doing. Do you think that about a lot of things you do at school or not?

P7 Some things we do need but algebra is one which we don't.

INT You don't see any use in it? A lot of people say that - it's probably true isn't it. Depends really on what you are going to do I suppose is the answer when you leave school. Have you got any ideas.

P7 No.

INT You've done some more algebra since haven't you?

P7 Yes, book work.

INT Did you find that what you did on the computer helped at all with that.

P7 Yes it did help.

INT Can you think of any specific way or just in general. Is there anything you had learned from the first work which helped with the second work... Can't think of anything specific. What I want to do is just to see how you do these questions basically. It's not a test or anything in the sense that it's going to be marked. There isn't any report or nothing goes home to your parents or anyone in the school even. I just want to see how you do. Can you do these for me. If you can read out Question 1 for me.

P7 What is the value of 3 plus 2m when m equals 5?

INT Understand what that is asking you to do?

P7 I've got to do the sum.

INT How are you going to set about doing it?

P7 Well I do the 2 times m first which would equal 10.

INT Can you write it down for me? Good. So when it's written as 2m like that that means what then?

P7 2 times.

P7 If 2 times p minus 1 equals 5 and 2 times s minus 1 equals 5, can say anything about p and s? They're equal, the same amount.

INT Is that obvious? It's not obvious to everybody. It's obvious to you.........Do you know what that value is, have you worked it ou yet or not? You've just looked at it and said they are the same. C you tell me what value they have then.

P7 3.

INT How did you do that?

P7 2 times 3 minus 1 is 5. 2 threes are 6.

INT Good. Ok. Can you just write down the value of p then. Read out 3 then.

P7 Multiply d minus 4 by 3.

INT Understand what you've got to do there.

P7 Well whatever d is you've got to take 4 away from and then times i by 3.

INT Can you actually write down what answer you would get.

P7 It depends what d is.

INT Can't do it unless you know what d is?

P7 No.

INT And there's no way of writing it down at all without knowing d.

P7 I don't think so.

INT You don't think so. OK Lovely. Let's try the next one then. Can yo read that one out.

P7 Which is bigger 3 times y or 3 plus y?

INT Understand what it is asking you?

P7 Well 3 times y is.

INT 3 times y is. Good. Again you decided very quickly on that can you tell me why?

P7 Because 3 plus y is 3 plus something and 3 times y is more.

INT Is it always more?

P7 No. If that was 2, 3 times 2, that could be 3 plus 6.

INT Sorry could you say that again if it was

P7 If y is the same number yes it would be the same, and if it were different numbers it wouldn't be.

INT And because it is one question would y be the same number?

P7 Yes

INT It would be the same number. So 3y would always be bigger no matter what y is.

P7 Yes.

INT What if y is 1.

P7 Then 3+y would be more.

INT So is 3y always bigger?

P7 No.

INT What does it depend on then? Can you tell me.

P7 Everything apart from 1 and 0.

INT Yes and would there be any other values or what?

P7 I don't think so.

INT No other ones. Good. OK. That's a difficult question. That is probably the hardest one of them all that question. No. 5 would you like to read that one out.

P7 Add 3 onto 6 times g.

INT Understand what you've got to do there?

P7 I just calculate 6 times g plus 3.

INT Can you write down the answer for that one?

P7 Would it be 9g?

INT Can you write it down for me and then you can tell me how you go it. How did you get that?

P7 I just added 3 onto the 6 because you still don't know what the value of g is.

INT But you are able to write the answer down although you don't know what g is on that one.

P7 It's still 9 times g not 6 times g.

INT OK. What about the last one? No. 6.

P7 What is the value of 3 plus 2 times c when c equals 5?

INT Understand how to do that?

P7 It's 3 plus 2 then times 5.

INT Can you do it?

P7 Yes.

INT Good. OK. If I gave you this question which is a very similar one how would you do that one?

P7 It would be 3 times 5 plus 2.

INT Good. How do you decide which bit to do first on those two questions.

P7 It's always the first bit unless it's got brackets round.

INT Ah I see, so you do the first bit first unless there is brackets round the rest of it. Good. OK. Only one or two more now, you're doing well. I think I'll pick some hard ones for you because you're good at maths. Could you do No. 1 here for me. Can you read that one for me.

P7 Write 2, no, write a plus b times 2 without brackets.

INT Do you understand what you have got to do? How are you going to do that one? Ah, right I see and that is the same is it as this only without the brackets in. Good. Ok. Is there any other way of writing it or not?

P7 2 times a plus b.

INT That's the same is it? Could you write that one for me as well. Good. OK. Let's just try these last three then, No. 10. Could you read that one out.

P7 Write more simply 2m minus n plus 3n.

INT Understand what you have got to do?

P7 Well I don't know the value of m and n.

INT Does that mean you can't do it then, because it says write more simply doesn't it. Can you tell me how you get it.

P7 I'm stuck about taking 1 n away and then adding 2n you might as well just add 2n instead.

INT Good. Does it make any difference that there are brackets in that question or not.

P7 No.

INT You've put some around your answer is that? - you don't need them.

P7 You don't need them.

INT OK Would you like to try No. 11 then.

P7 Solve 3x minus 5 equals 2x plus 1.

INT Do you understand what you have got to do?

P7 You've got to 3 times x minus 5 is the same as

INT What does the word solve mean to you?

P7 Try and work it out.

INT Try and work it out. What sort of answer will you get?

P7 I don't know.

INT You don't know. Can you work it out or what. It doesn't matter if you can't do it. Can you tell me what you are thinking. How you are trying to do it.

P7 I'm just trying a number on this side

INT Have you done anything like those in algebra before?

P7 Yes.

INT You have. Is that how you always do them?

P7 Yes.

INT Can you tell me what numbers you are trying.

P7 Quite high numbers and some low ones.

INT What sort of numbers for example.

P7 6 and 5.

INT Have you found anything?

P7 I think it's 6.

INT OK, good. Can you write the answer down....Good, so you've checked your answer have you. It works. Good. Just try the last one.

P7 For what values of d is 17 minus 3d is greater than 2?

INT Do you understand what you have got to try and find?

P7 All the numbers which are higher than d.

INT How would you set about doing that then?

P7 17 minus... I would find out what that is first, the 3 times d, and then I would take 17 away from it.

INT Good, can you do that then?

P7 It's 4 more than 6. Yes you can't take 17 away from it.

INT I see so could you write the answer down then. Good. OK. Thank you very much that's fine.

PUPIL 8

INT Remember the computer algebra you did, I want to know first of all what you thought of it.

P8 It was quite good but I prefer to do it on paper. I think I learnt more then.

INT Because you've done on paper as well haven't you? Can you tell me what you learnt more.

P8 Well on the computer you were just typing it in and it gave you the answer without you thinking, so when you was doing it on paper you could think about the actual answer.

INT Good. That's interesting. Do you think you learned anything at all from doing it on a computer.

P8 I learnt very basic things.

INT What would you say were very basic things?

P8 Well just the very first sort of like sums you do. The very simple ones.

INT Nothing about letters at all?

P8 No, not really.

INT No, ok. Well we'll see when we have a look at these, see what you've learned shall we. What I would like you to do is just to try some of these for me if you wouldn't mind on this bit of paper. I'm not interested in how many you get right, because that is not important, I'm not going to keep a count even, but I'm interested in how you do them, that's what I'm interested in. OK. Would you just like to read the question for me.

P8 What is the value of 3 plus 2m when m equals 5?

INT Good. Do you understand what you have got to do?

P8 Yes.

INT Can you tell me how you are going to do it then.

P8 You say like 2m is 2 times 5 because m equals 5 and you add 3 to that.

INT Good, so can you write that down for me? Done it in your head have you? Good. So that is easy. So the 2m means what then?

P8 2 times 5.

INT Good can you read No. 2

P8 If 2p take 1 equals 5 and 2s take 1 equals 5 can you say anything about p and s? What is the value of p?

INT Can you do this bit first then. What can you say about p and s, anything? You understand what you've got to do?

P8 Yes. It is 3 the p and s.

INT Is it. Good. How did you get that?

P8 Well if you say like 2 times 3 and you minus 1 it equals 5.

INT So are they both equal to that?

P8 Yes.

INT Why are they both equal to that?

P8 Because they are basically the same sum, but are different letters.

INT Good, ok, can you write that down. The answer. Good. Ok. Let's try No. 3 then.

P8 Multiply d minus 4 by 3.

INT Understand what it is asking you to do? Is there a problem?

P8 Yes.

INT What is the problem. Can you tell me what you are thinking about that.

P8 I can't understand the ......

INT Ah, do you need to know what the d is?

P8 Yes

INT Is there anything that you can write down for that answer if you don't know what d is?

P8 d minus 12.

INT Ah, I see. Can you write that down? How did you get that?

P8 Well like d and then times the 3 by the 4.

INT And that's the answer then is it?

P8 I think so. I'm not quite sure.

INT OK that's fine. Can you read out No. 4.

P8 Which is bigger 3y or 3 plus y?

INT Understand what that is asking?

P8 Yes. 3y.

INT 3y. Are you sure?

P8 Yes.

INT How do you get that answer?

P8 Well that 3 times y, that's 3 plus.

INT So which one is bigger then?

P8 3y.

INT Can you write that down. Is 3y always bigger.

P8 Yes... No, sometimes you could have like a figure ....

INT Could it, when would you get that?

P8 If it was a bigger number it would be like more, it would be more because say if that was 2 it would be 3 times 2 it would be 6, and if that was 4 it would be 7.

INT Oh, I see so if y had a different value. Will y have a different value in a question like this?

P8 It could do.

INT Could do?

P8 Yes I think so.

INT Right, ok, good. Like to read out No. 5.

P8 Add 3 onto 6g.

INT You understand what that is asking you to do?

P8 Well no not really.

INT No. You know what add 3 means. So you've got to start with 6g and add 3 to it.

P8 9g.

INT 9g. Good. Can you write that down for me then. How did you get that answer?

P8 Just add 3 to the 6.

INT You add the 3 to the 6. OK. What if you had this question here which is a very similar one. Add 3 onto g. How would you do that one?

P8 Put just like 3g.

INT Good, can you write that down. And how did you get that this time?

P8 Well as you don't know what g is I just put it on to the front of g.

INT Just put it in front. And 3g means what?

P8 3 times g.

INT Read out No. 6 for me?

P8 What is the value of 3 plus 2 times c when c equals 5?

INT Understand what that is asking you to do?

P8 Yes.

INT So how are you going to do that?

P8 3 plus 2 is 5 and then times it by 5 which would be 25.

INT Good, can you write that down. What if I gave you that question there which is very similar, how would you do that one?

P8 3 times 5 plus 2.

INT So what answer would you get?

P8 17.

INT Good can you write that down. Can you tell me how you decide on these questions which bit to do first.

P8 Well I just put the 3 plus 2 and then times it by c, so I know what the number is to times by.

INT But you didn't do that here, you did the multiplication first didn't you.

P8 Yes.

INT So how do you decide which bit to do first?

P8 I think it depends whether it is first or, whether it comes first or not.

INT So you do the bit that comes first. OK. Well there are only one or two others on this sheet here. You don't have to do all these. No. 4 would you like to try that one.

P8 Add 6 onto e plus 4.

INT Understand what you have got to do there?

P8 Would it be e plus 10?

INT Good, can you write that down? And is that the only way to write that answer?

P8 You could have 10 plus e.

INT Yes alright, can you write that down. But they are the only ways are they?

P8 Yes.

INT Question 5 then could you try that one for me.

P8 If p plus q equals 3 then p plus q and r equals?

INT What is that asking you to do?

P8 Work out the p, q and r.

INT How are you going to do that then?

P8 If p equals 1 and q equals 2 then it would be 3 then if r equals 3 it would be 6.

INT So your answer is?

P8 p plus q plus r equals 6.

INT Can you write that down then? How did you decide what values to take?

P8 Well it was like you had 3 there, so I thought that one of them must be 1 and the other one must be 2 so if they go up in order then r could be 3.

INT I see. Is there any way of writing the answer if you don't know what values p and q have?..If you don't know what values p, q and r have. Is there any way of writing the answer?

P8 No.

INT Can't see one at all?

P8 Right, ok, that's fine. Question 6 then would you like to read that one.

P8 When does p minus b equals p minus c? Always, never or sometimes? If sometimes, say when.

INT Understand what it is asking?

P8 Yes I think so.

INT How are you going to do that then?

P8 Well it means the letters and you don't know the value it's just likely to be the same.

INT So what is the answer then?

P8 Never.

INT Never. Can you tell me how you get that answer.

P8 Well if you just didn't know what p and b and c were then you wouldn't know what to put. Wouldn't know the answer.

INT I see, does that mean that they are never going to be the same then.

P8 Yes.

INT So there aren't any values where they could be the same.

P8 No.

INT Right, could you put the answer down then. Good and the last one is question 8. Would you just like to read that.

P8 Write more simply 3f plus 5g plus 2f.

INT Good, understand what you have got to do?

P8 Is it 3 times f plus 5 times g plus 2 times f.

INT Good, can you write that down. And is there any way you could write that more simply?

P8 No I don't think so.

INT No, can't do anything with that.

P8 No.

INT Good, ok, well thank you very much for your help. Wasn't too bad was it. You did very well.

PUPIL 9

INT I just wondered what you thought of it first of all. That is the first question. Did you enjoy it, was it horrible?

P9 It was better than the other work.

INT Was it? Can you tell me why. What was there that you preferred about it.

P9 Don't know.

INT Not sure.

P9 Just liked it.

INT You liked it did you. Good. Do you think you learned anything from it or not.

P9 Yes.

INT Anything in particular that you can think of.

P9 No.

INT No. You've done some algebra since haven't you?

P9 Yes.

INT Do you think helped you with doing that algebra?

P9 Yes.

INT Can you think of anyway that it helped at all.

P9 Because I could understand it better.

INT Could you. Good. OK. Well what I am interested in is your answers to some of these questions, not so much the answers as to how you do them, so I'm not going to keep a record of your marks or anything, but I want to know how you do them really and what you think about them. So if we could put No. 1 down the left hand side and could you read the question out for me please.

P9 What is the value of 3 plus 2m when m makes 5?

INT Right you understand what you have got to do.

P9 Yes.

INT Can you tell me how you would do it then.

P9 You work out what m is which is 5 and then you times the 5 by 2 and then you add 3.

INT Good, can you do it then and write it down.

P9 So it's 13.

INT Good. So 2m means what then in that question?

P9 2 times m.

INT Good. Easy wasn't it. Would you like to read out No. 2.

P9 If 2p minus 2 makes 5 and 2s minus 1 makes 5, can you say anything about p and s? What is the value of p?

INT Good, let's take this first bit then. Do you understand what it is asking?

P9 Yes.

INT How are you going to do it?

P9 They are both the same.

INT They're both the same are they? Good. Can you write that down for me. Can you tell me how you got that answer?

P9 It's the same apart from the letters, exactly the same except the letters.

INT Good, ok. You haven't worked out what the value is then?

P9 No.

INT So can you work out what the value of p is then?

P9 3

INT How did you do that?

P9 Because I added 1 to 5 which makes 6 and divided by the 2.

INT Ah good, so can you write that answer down for me. And that's the value of...?

P9 p

INT So can we write that as well alongside.

P9 Can I put 3 is the value of p.

INT Yes if you want to. Good. Is there anyway of writing that that you have just written shorter.

P9 p makes 3.

INT How would you write that? Good ok thankyou. Question 3 would you like to read that out.

P8 Multiply d minus 4 by 3.

INT Do you understand what that is asking you to do?

P9 I think so.

INT Not quite sure. Can you tell me what the problem is there.

P9 It doesn't tell you what d is.

INT Ah I see so you don't know what d is. Is there any way you can write the answer down if you don't know what d is.

P9 No.

INT None at all. So I would have to give you a value would I for d? Right ok let's just put a line by No. 3 which means we left that one. No. 4 now.

P9 Which is bigger 3y or 3 plus y.

INT Understand what it is asking?

P9 Yes.

INT How are you going to do that then?

P9 Thinking what y is, but probably that one.

INT It's probably that one is it. Good.

P9 Unless it's 1.

INT Ah.

P9 And then they are the same.

INT They are the same when y is 1. What do they both equal?

P9 Oh it's not, that one is bigger when it is one.

INT So what is the answer to which is bigger then.

P9 3y apart from when y is 1.

INT Right, can you write that down. Good, is that the only value of y for which it is not true?

P9 Yes.

INT Good. ok. Like to read out No. 5 then

P9 Add 3 onto 6g.

INT Good, do you understand that?

P9 Yes.

INT What is the answer to that then?

P9 6g plus 3.

INT Good can you write it down. Is there any other way of writing that answer.

P9 Don't know. If g is 3 it's 7g.

INT But if we don't know what g is is that the only way of writing the answer.

P9 Think so.

INT OK, good. No. 6 then.

P9 What is the value of 3 plus 2 times c when c is 5?

INT Understand what that is asking you to do?

P9 Yes.

INT Good. How would you do that then?

P9 Times 2 by c and then add 3.

INT Good, can you do it then. Do them in your head do you. Nice and easy. How would you do this one which is a similar question 6. Can you read that one and then tell me how you would do that.

P9 It's the same.

INT Can you just read it out for me because it's not quite the same.

P9 The value of 3 times c plus 2, when c makes 5? 17.

INT How did you get that?

P9 It's 21 - because 5 plus 2, 7, 21.

INT It's 21, are you sure? Can you write that down for me? Good. Now the first answer you gave me was 17 wasn't it. What made you decide it was wrong.

P9 Because I had 3 times c plus 2.

INT And why is that wrong?

P9 You're meant to put brackets in.

INT So if you had brackets there round the c and the 2 which bit would you do first then.

P9 The c plus 2.

INT But you did that bit first anyway. How do you decide in these questions which bit to do first.

P9 Because they've got brackets round.

INT Ah, I see, so if there aren't any brackets where do you start?

P9 Where it starts.

INT At the left hand side? But you didn't on this one, you decided not to. It was just a mistake was it.

P9 Yes

INT OK, that's fine. That's interesting. Good. Can I just give you one or two others, you don't have to do all these, so don't worry about all these questions on this sheet. Would you like to do No. 1 for me.

P9 Write 2 a plus b without brackets.

INT Understand what that is asking?

P9 Yes.

INT Can you do that, how would you do that?

P9 2a plus 2b.

INT How did you get that? It's the right answer, can you just tell me how you got it or are you not sure.

P9 Not sure.

INT You just sort of know it do you. Good, ok that's fine. Let's read No 2 then.

P9 What is the value of 3s plus b when s makes 5? I can't do that because I don't know what b is.

INT Ah, I see is there no way you can write the answer down at all. You can't. What about if we went back to this question here. Does that help at all. That was add 3 onto 6g wasn't it. But you wrote the answer down on that one. You can't do it on this one. Can you tell m what you are thinking, what is the problem with that one then?

P9 I don't know what b is.

INT You don't know what b is, and so we can't write anything down. OK. Let's put a line along that one then. No. 4.

P9 Add 6 onto e plus 4.

INT Can you do that one?

P9 Where do you add the 6, onto both of them or just one?

INT Well that's what I would like to know what you think, rather than me telling you. Which one would you think you would add it on to?

P9 Both of them.

INT Both of them? So what would you get it you did that?

P9 Don't know, depends what e is. Might get 10.

INT Alright, so how could we write the answer down then. Can you write anything down for me?

P9 You can put it on that bottom one.

INT Alright, so is it... you can't actually write anything down because you don't know what that one is.

P9 Yes.

INT Right, ok. Let's try No. 5 then. Let's put a line opposite 4 as well.

P9 If p plus q makes 3 then p plus q plus r makes? Don't know because don't know what r is.

INT It's the same problem in all of these isn't it? So you can't do anything to that one either? Right, ok, good. Can I just go back to this one here, can you read that one for me.

P9 Add 3 onto g.

INT Can you write anything down for that?

P9 g plus 3.

INT Can we put that - that actually goes with this one - can we put a comma and put it alongside there. You don't know what g is there do you? How can you write an answer down for that one?

P9 You can't.

INT But you seem quite happy with that. Because that is the right answer you see. g plus 3. I'm just trying to see why - can you explain to me why you can write the answer down for that one but you can't write an answer down for say this one. Can you tell me what the difference is between them or not? Don't worry if you can't I just want to know if you can..

P9 Can't.

INT Can't explain the difference between them. So you are happy with g plus 3 there and on this one if s is 5, 3s would be?

P9 15

INT So we've got 15 plus something and because we don't know what that something is we can't write anything down. Do you think we could write 15 plus b.

P9 Yes.

INT Would that be alright as an answer.

P9 Yes. Probably not.

INT Not sure. No. OK. Good. Right thanks for your help. It wasn't too painful was it. You did quite well on those, you got some interesting questions right there, this one here you know is a very difficult question, you got that right, so that's good isn't it. That is actually the hardest question of all, I don't know if you know that. You got that right.

PUPIL 10

INT Would you like to read No. 1.

P10 What is the value of 3 plus 5m when 5, I mean when m is 5?

INT Right, do you understand what it is asking you to do?

P10 Yes

INT How would you set about doing that question?

P10 First I would multiply 2 by m, m is 5, so it's 2 times 5 which is 10 and then add 3 to the 10 which is 13.

INT Good, that's right, so this 2m here means ?

P10 2 times 5.

INT 2 times 5 or 2 times m, good, that's ok. That wasn't too bad was it. Like to read No. 2 then.

P10 If 2p take 1 is 5 and 2s take 1 is 5 can you say anything about p and s? What is the value of p?

INT Good, can you say anything about p and s?

P10 You time them obviously.

INT Let's concentrate on the first bit first and leave that bit there.

P10 2 times p ...

INT Can you tell me what you are thinking when you try and do it.

P10 2 times p, p could be 3, so 2 times 3 is 6 take 1 is 5. s could be 3 as well.

INT Why is that?

P10 Well because 2 times 3 is 6 take 1 is 5.

INT OK, good, write that down then.

P10 p and s is the same.

INT Right, if I give you this one here which is very similar to the one that you've just done, this question which is also a No. 2 because it is connected with it. See that equation there and the one that you've just one what would be the value of p in this equation? Can you do that? Just tell me what you are thinking, that is what I want to know? How you do it.

P10 It needs to be 2, 2 add 1 is 3 and 2 threes are 6, take 1, is 5.

INT Good. OK. Like to write that down. Good, how did you decide that it was going to be 2?

P10 Well if it was 3, 3 and 1 is 4, 2 fours are 8 take 1 is 7 and that isn't the answer so it's got to be 2.

INT OK, lovely. Like to try No. 3 then. Like to read that one out for me.

P10 Multiply d take 4 by 3.

INT Understand what that is asking you to do?

P10 Yes, take 4 from d, it depends what d is really and then multiply that by 3.

INT I see, good. Can you write anything down for the answer?

P10 I probably could but I can't remember at the moment.

INT You can't remember at the moment. Can you tell me what the problem is with writing something down with that one.

P10 Well if we knew what d was it would probably help more.

INT So because you don't know what d is then there is a problem and you can't write an answer down then.

P10 Well you probably could but I just can't remember how.

INT You can't remember how to do it, that's fine. Just leave that. Let's do No. 4. Would you like to read it out for me.

P10 Which is bigger 3y or 3 plus y?

INT Do you understand what that is asking?

P10 3y is 3 times y and 3 plus y is 3 plus y. So probably 3y is bigger than 3 plus y.

INT All right, can you write that down for me. Now you said probably 3y is, can you tell me why?

P10 Well, if y was 4, 3 fours are 12, but 3 plus 4 is 7.

INT So why did you choose the value 4.

P10 Well it just came into my head.

INT You just chose that particular one. And is that always true that 3y is bigger.

P10 Well, not necessarily. Sometimes it is smaller.

INT When?

P10 If that was 1, 3 times 1 is 3 but 3 plus 1 is 4.

INT Right, ok, good. Like to read No. 5 for me then.

P10 Add 3 onto 6g and that would be 6g add 3, 9g I think.

INT Can you write that down for me? Good. What if the question was this one instead which is a similar question. Can you read that one.

P10 Add 3 onto g. Now that would be 3g.

INT That would be 3g would it. Would you like to write that alongside for me. Can you tell me how you got the answer for this one.

P10 Well there is nothing there, 3 g's. I'm not quite sure really, but what you have to do is just put the number there.

INT Just put the number in front of the g. Right, good. No. 6 then would you like to read that out.

P10 What is the value of 3 add 2 x c when c is 5?

INT That is actually a multiplication sign, it's not clear is it.

P10 I work from left to right, that's what the book says.

INT That's what your teacher says?

P10 Yes, but sometimes I do it different.

INT Why do you do it different sometimes?

P10 Well it according to BODMATHS, multiplies before add. It's 3 add 2 is 5, 5 times c which is 5, that's 25.

INT Good, can you write that down. So when do you decide to use your BODMATHS and do it differently?

P10 Well it really depends on what the question is really because you could use BODMATHS there, I'm not quite sure if it fits in, it really depends on how you do it.

INT So is there any way of telling which question to use it in and which to not?

P10 Not really, it's just the way it's set out.

INT I see, lovely. What about this one then, this is another similar one, not quite the same. Can you read that one.

P10 Value of 3 times c add 2, when c is 5?

INT How would you do that? Do you understand what that one is telling you.

P10 Well, here you could use your BODMATHS but it's just the same. 3 times c which is 5, that is 15, 15 add 2 is 17.

INT Good, let's write that down. Just one or two others then on this sheet here. Don't worry about all these you don't have to do all these. I've just picked a few out for you. No. 1. like to read that one out.

P10 Write 2 brackets a add b, close brackets, without brackets.

INT Do you understand what that is asking you to do?

P10 Yes, you just have to write the question without the brackets, so that would be 2 times a add b.

INT Right ok, can you write that down. And is that, that you have just written, the same as this then only without the brackets.

P10 Well it's nearly the same except for here and so they put the number next to the bracket. It saves times putting the multiplication sign.

INT But this means the same does it. Is there any other way of writing that without brackets or is that the only way.

P10 I'm not quite sure, I think that is the only way. Probably find there is a different way altogether.

INT But not one you know. Lovely, ok. Question 2 would you like to read that one.

P10 What is the value of 3s add b when s is 5. Well 3 times 5 is 15, 15 add b is 15 add b because we don't know what b is so that is how we write it.

INT Good, can you write that down. Are you quite happy with that answer?

P10 Yes

INT Good, question 6 would you like to try that one.

P10 When does p take b equals p take c? Always, never or sometimes? If sometimes, say when.

INT Understand what that is asking you?

P10 Well, on the test I didn't quite get it and I still don't.

INT I see, right. Can you tell me what you think it means then. Does it mean anything to do at all?

P10 It probably does but I don't know it.

INT No, it doesn't really mean anything to you. That's alright, well we'll leave that one then. Let's put a line alongside and try Q7, like to read that one.

P10 For what value of c is 7 equals c take 2?

INT Do you understand that one, what it is asking you to do?

P10 Well this is just the other way round. Because if equals 7 was there it would be c take 2 equals 7, so we've got to find out what the value of c is and that is 9, because 9 take 2 is 7.

INT Can you tell me how you do this No. 7 which is connected with it. Would you like to read through that one.

P10 What about 7 equals 3c take 2?

INT It's very similar to the other one isn't it.

P10 7 that's left. 3 times c take 2 equals 7. Find out what 3 times c is....

INT What are you thinking now?

P10 I'm just trying to find out what c is. I can't work that one out.

INT You can't do that one. No. How are you trying to do it, can you tell me?

P10 Well I was trying to find out a number that would fit into c and take 2 makes 7. I couldn't think of one.

INT What were you trying to do first with the number.

P10 Well I was trying to multiply 3 by any number and then take 2 to see if it ended up as 7.

INT I see, right, but you couldn't find one.

P10 No.

INT That's lovely, that's fine. Thank you very much for your help. It wasn't too bad was it. You coped very well I thought. Your algebra is quite reasonable isn't it.

PUPIL 11

INT What I'm trying to do as I told you is not to keep a count of how well you do or anything like that, I'm not going to tell anybody the scores,or things like that, I'm more interested in how you do the questions. How you think about them. So I want you to sort of tell you what you are thinking as you do them. OK. So if you could just read the first question for me.

P11 What is the value of 3 plus 2m when m equals 5?

INT Good, understand what they are asking you to do? How would you do that question then?

P11 Because the m is 5 you add 5 to 2 and then 3 plus 7.

INT Good, could you actually do that then and tell me what you are writing down as you write it down. Put question 1........ good, so this 2m here means what then?

P11 7

INT Yes, so it means 2... If we didn't know what m was for a minute what would you say 2m meant?

P11 I don't get what you mean.

INT When we write 2m like that in maths what do we mean by that? What is it telling us to do?

P11 I don't know.

INT You don't know, right, but you add them together do you. Right, ok.

P11 Well what we were told to do is put a box and then it would add plus box equals

INT Oh, right, good, so you think of it like a box do you?

P11 Yes

INT Lovely, ok, that's fine. Question 2 then would you like to read that one out for me.

P11 If 2p minus 1 equals 5 and 2s minus 1 equals 5, can you say anything about p and s?

INT Do you understand what that is asking you?

P11 No.

INT You dont, would you like to read it again then and see if you can work out what it is asking you.

P11 If 2p minus 1 equals 5, 2p equals 6 and 2s minus 1 equals 5, 2s is 6, can you say anything about p and s?

INT Understand it now?

P11 Yes.

INT Good, how would you set about doing that then?

P11 2p minus 1 equals 5....

INT What are you thinking now?

P11 I'm not, I'm just looking at it. 2s minus 1 equals 5 ......

INT How are you going to try and do that one then? Any ideas?

P11 Well what I would have put is 2p equals 6 and 2s equals 6.

INT Good, like to write that down. Does that help you say anything about p and s?

P11 p and s equal 6.

INT Both equal 6. Like to write that down. Good. Can you tell me how you got that.

P11 2p minus 1 equals 5. If you add the 1 to the 5 that's 6, so because there's no other minus p I forget the p and do the ... 2p minus 1 equals . If you do... add the 1 to the 5 which is 6 and then you take 1 from the 6 ... No, I don't get that, I know how I've done it but...

INT Yes, ok that's fine, but you can't quite remember now.

P11 Yes, when you look at it and read it you can see but you can't really say it.

INT Right, ok, that's fine. Let's go onto the next one then. I think you've done the bit here, you told me the value of p. What would the value of p be did you say?

P11 6

INT Good, question 3 then. Like to read that one out for me.

P11 Multiply d minus 4 by 3.

INT Understand what that is asking you to do?

P11 No.

INT Can you tell me what the problem with it is.

P11 There's nothing that says what d equals.

INT Ah right, that's a problem is it? Is it possible to write anything down for that answer if you don't know what d is?

P11 I wouldn't have thought so.

INT You can't, no. Ok, that's alright, let's put a line by that one then. Let's go on to the next one. Question 4.

P11 Which is bigger 3y or 3 plus y?

INT Do you understand what that is asking?

P11 Which is bigger, having just y or having a plus next to y.

INT Alright, how are you going to try that one then? How will you decide which is bigger?

P11 I would have thought because ... the same.

INT The same, why do you say that?

P11 Well if it's 3y it looks as if it has already been added or whatever and it's being added there.

INT Right, so you reckon they're the same. Good, would you like to write that down then. When you said 3y has already been added can you explain to me how.

P11 Well it looks as if you are doing 3 plus y and you have written that down you would take away the plus and put the y next to the 3.

INT Oh, right, ok. Good, fine. Let's try question 5 then.

P11 Add 3 onto 6g.

INT Understand what that is asking?

P11 Yes.

INT How would you do that then?

P11 Add 3 and 6 and then put a g next to it.

INT Right, ok. Can you write that down. Good, can I just ask you how you would do this one over here. It's a similar question to that. It's also question 5 so we can just write the answer alongside. Like to read that one and tell me how you'd do that.

P11 Put the 3 next to the g, which is 3g.

INT Ah, good, ok. So how did you get that answer?

P11 On the other question if there is a number before it you write down that number anyway, so I thought if you put the g down, if the g's there you just, well put it next to it because it's adding on.

INT Yes, ok. Question 6, would you like to read that out for me.

P11 What is the value of 3 plus 2 times c when c equals 5?

INT Understand what that is asking?

P11 Yes. Add the 3 and the 2. 3 plus 2 equals 5 by 5.

INT Right, ok, can I ask you another one which is very similar to that here. How would you do this one?

P11 What is the value of 3 times c ....

INT How would you do that one, do you understand it?

P11 It's what the c with the 5, so it's .....

INT What answer does that give you?

P11 10.

INT Good, can you write that down. So how do you decide in questions like this which bit to do first then?

P11 Well I would have thought you would have done the start first.

INT Just start from the left hand side and work through. Lovely, that's fine. Just a couple more. Not too bad is it. You haven't got to do all these so don't get worried when you see all these. Would you just like to read No. 1 for me.

P11 Write 2 bracket a plus b bracket without brackets.

INT Good, do you understand what that is asking you to do?

P11 Take the brackets away.

INT Yes, can you do that. Good. No equals in that one.

P11 No

INT Right, ok. Now what you have written down is that the same as this only without the brackets. Does it mean the same. How do you decide. You don't know how to decide that. Do you think it is the same or not, or are you not sure? You think it is. Yes, ok, lovely. Let's try question 2 then would you like to read that.

P11 What is the value of 3s plus b when s equals 5?

INT Understand what that is asking?

P11 Do 3 plus 5 and then put an s by it.

INT Like to write that down for me. Can you tell me what you are thinking as you write it down.

P11 3 plus 5, 8, ... 8 plus 5.

INT Right, ok, good. What about question 5 can you read that out for me.

P11 If p plus q equals 3 then p plus q plus r equals?

INT Understand what that is asking you?

P11 I think so.

INT How are you going to do that one then?

P11 One of those is either 2 or 3 or the other way round, so I'd put 3, I don't know what r would equal.

INT So you'd leave that like that would you. Can you write that down........ Good and what does that answer mean, can you tell me.

P11 If p plus q equals 3 then because you don't know what the r is you don't know how to add it to that, so you put it as ...

INT Lovely. Just question 6 then finally if you would like to read that one out.

P11 What does p minus b equals p minus c? Always, never or sometimes? If sometimes, say when.

INT Understand what that is asking you?

P11 No

INT No, you don't. Would you just like to look at it again.

P11 When does p minus b equal p minus c? Always, never or sometimes? If sometimes, say when. I would have said never.

INT You would have said never would you. Good, would you like to write that down. Can you tell me how you get that answer.

P11 Well if p minus b equals p .... p minus b equals p minus c ... I think I read that wrong.

INT Like to read it again then.

P11 When does p minus b equal p minus c? p minus b. Never because if p minus b equals, hang on, no, that's wrong. Must be.

INT Why is that?

P11 When does p minus b equals p minus c? I would have thought when you added anything to the c.

INT Alright good, can you write that down...... Good, can you give me any idea of what the something might be or not, does it matter?

P11 If there was something like where it says p equals b, p equals 2 or whatever, you would know that the number in there you would be able to add to.

INT I see, whatever is necessary to make it up. Lovely. OK. Good. Well thanks very much for your help.

PUPIL 12

INT Can you read out that first question for me.

P12 What is the value of 3 plus 2m when m is 5?

INT Understand what that is asking you?

P12 Yes, times 3. If m equals 5 that is going to be 2 times 5, that's 10 plus 3 so that's going to be 13.

INT Good, can you write that down for me. OK, so the 2m in that question means what then?

P12 2m means 3 plus 2 times 5.

INT Lovely. Like to read out No 2 then.

P12 If 2p minus 1 equals 5 and 2s minus 1 equals 5, can you say anything about p and s?

INT Understand what that is asking you?

P12 Um,

INT Is there something you don't understand?

P12 Is it that p equals a certain number and if you were to minus 1 from it, say p was 3 you would be timing that by 2 so you would have a 6 and minus 1 that would give you 5. So the same with s. So s could be 3 as well and you're 2 timing that, so p and s could be 3.

INT Good, would you like to write that down... Now, you've written could be 3 are there any other values they could be. Some you haven't found yet or not.

P12 No, I don't think so.

INT You don't think so. Alright. What about the value of p then.

P12 I think it's probably 3.

INT You think it's probably 3. Good. So would you like to write that down. Lovely, ok. Just read No. 3. then.

P12 Multiply d minus 4 by 3.

INT Understand what that is asking you.

P12 Well first you have to find out what d is and then you've got to.... no, first you multiply that by 3 and then minus 4 from it.

INT I see, and is there a way of writing that answer down.

P12 Yes.

INT Can you write that down. I see right, good. Lovely, ok. Would you like to read out No. 4.

P12 Which is bigger 3y or 3 plus y.

INT Understand that one?

P12 Yes. 3y is 3 times y and 3 plus y you are adding it not multiplying it.

INT So how would you answer the question then?

P12 Well if y was 5 that would be 8 and that would be 15, so that one is going to be the bigger.

INT OK, alright, good. How did you get the value 5 in that.

P12 Well any number really. Because it could have 3, could have been 2, could have been anything. I just took a number.

INT You just fancied 5. You didn't have any way of deciding to take 5.

P12 No.

INT Would you like to try No. 5 then.

P12 Add 3 onto 6g. Well 6g is 6 times g and if you are adding 3 onto it, so you are going to be adding, so when you've multiplied the 6g you are going to be adding 3 onto it.

INT Good, can you write the answer down for that one or not. ... Good, happy with that answer?

P12 Mm.

INT What are you thinking about now?

P12 I'm trying to think could you add that onto the 6 and make it 9g.

INT What do you think?

P12 No.

INT You don't think so. Can you tell me why.

P12 Because then that would be, you would be adding 3 onto it if you made that a 9 and you times g by 9.

INT And why is that not allowed and why is it wrong to do it.

P12 Because you're meant to be adding 3 to the final answer.

INT Ah right, ok, lovely. Put No. 6. down.

P12 What is the value of 3 plus 2 times c when c is 5?

INT Understand what that is asking?

P12 Yes, if you turn it all into numbers 3 plus 2 times 5. You do the times first so you've got 10 plus 3, that's 13.

INT Good, can you write that down. Why did you decide to do the times first?

P12 Because if you refer back to BODMATHS it goes brackets, of, divided by, multiplication, add, subtraction.

INT Oh, right, good. That's fine. There's just one or two others on this sheet here. I'd be interested in seeing how you do. Would you like to try No. 1 for me.

P12 Write 2 bracket a plus b bracket without the brackets.

INT Understand what that is asking you to do?

P12 Yes. You add what is in brackets, you do that first, so that a plus b whatever they could be and then you times it by 2.

INT Right so how would you write that down. Can you tell me what you are thinking.

P12 You can't if you refer to BODMATHS you have to multiply first before you can add, so you really need the brackets.

INT So that is a problem is it? Can you think of any way round the problem. Is there any way of writing it without the brackets or not.

P12 I can't.

INT You can't see any way round it, ok, that's fine. Let's just put a line beside that one. Try question 4.

P12 Add 6 onto e plus 4.

INT Understand what that is asking?

P12 Yes. First find the value of e which is impossible because it doesn't give you any indication and if you were going to add 4 to it already you just add them together so you would have 10 and then you can add 10 to e.

INT Good, is there any way of writing that answer down.

P12 e plus 10.

INT Good, No. 10 next then.

P12 Write more simply bracket 2m minus n bracket plus 3n.

INT Understand that one?

P12 Yes.

INT How would you do that then?

P12 First multiply 2m, so you would have .. if you could find out what m is it would be a help but let's just say m and n random pick 3, so we've got 2 times 3 that's 6, minus n, I'll take to be 2, so you've got 6 minus 2 that's 4, then plus another... then you times 2 by 3 and add that on to that.

INT So what answer would you get?

P12 You would get minus 2.

INT Is there any way of writing down the answer to that question then when it says write more simply.

P12 You could put..... 4n minus 2m.

INT Could you write that down for me. Good can you tell me how you got that.

P12 Well it's 1 times n so if you added the m and n together you would have 4n and you are still going to be taking that away from it.

INT Lovely, ok.

P12 And then you can put the sign to add them together.

INT Good, like to try.No. 11 then.

P12 Solve 3x minus 5 equals 2x plus 1.

INT Understand that? Got a problem. How would you set about that? Can you tell me what you are thinking.

P12 I'm trying to work out how you could take 5 from that to leave that.

INT Can you see any way of doing it?

P12 You would have to find the value of x before you could start.

INT You have to find x before you can start? What does the word solve mean to you in that question?

P12 Find the answer. It could be find the value of x.

INT What else do you think it could be?

P12 Main problem is find the value of x.

INT But you said you need that to start.

P12 Yes, well, you can experiment with different numbers.

INT And that is how you would do it, is it? But you can't come up with one now that would work.

P12 No, I was just trying 3. That's 9 minus 5, so you've got 4. If that was 3 already it would be 6 plus 1, 7, so 3 doesn't work. Try 2. 6 that's 1, 6 minus 5 is 1. 2 times 2, 4, plus 1, 5, that doesn't work either. Can't do it.

INT Can't do it then? OK, let's just put a line by the side. Let's just try No. 12.

P12 For what values of d is 17 minus 3d bigger than 2?

INT Good, understand what that is asking?

P12 Yes, if you've got 17 - well first you times d by 3 and take it from 17 and that would be bigger than 2.

INT How would you set about doing that then?

P12 First I'd times d by 3 minus it by 7 so that it would be bigger than 2.

INT Can you do that?

P12 Well, let's just say d is 3 so that would be three 3s, 9. So that's 8 you've got left and 8 is bigger than 2. So d could equal 8.

INT Is that the answer? Right. Happy with the answer?

P12 Yes.

INT Good, ok, thank you very much for your help.

PUPIL 13

P13 What is the value of 3 plus 2m when m equals 5?

INT Good, understand what that is asking you to do?

P13 Uh, yes, I think its 2m equals 2 times m which is 5 - no 2 times m would be 10, so 10 add 3 is 13.

INT Good. Could you just write that down then. Question 1. Like to read the second one out.

P13 If 2p take away 1 is 5 and 2s take away 1 equals 5, can you say anything about p and s?

INT Understand that?

P13 Not at first look.

INT No, like to read it again then and see if you can work out what it is asking.

P13 Well, p could equals 4 and s could equals 6 - no - 4. They could both equal 4.

INT Could you write that down for me then? Can you tell me how you got it?

P13 Well because if 2 take away 1 equals 5 that's not right, so if p equalled another number which is added with the 2 and 2 it could be that.

INT Good, so have you answered the next bit then?

P13 Yes, it could equal 4.

INT Lovely, ok. Like to read out No. 3.

P13 Multiply d take away 4 by 3.

INT Understand what that is asking?

P13 Well no because we don't know what d equals. So it would be - no I don't.

INT Can you write anything down for that one or not? You can't, ok, let's put a line by that one. Try No. 4.

P13 Which is bigger 3y or 3 plus y?

INT Understand that one?

P13 Yes, neither or them are bigger. They are both the same.

INT Right, ok, can you write that down. Good, can you tell me how got the answer.

P13 Because 3 add y -- you don't really need 3 plus y, just 3y is alright.

INT Good, ok, like to read out No. 5.

P13 Add 3 onto 6g. That would be 9g.

INT That was quick. Like to write it down and tell me how you got that.

P13 Well you just add the 6 and the 3 and the g is still there so 9g.

INT If it was like this, this question here, which is another No. 5, can you tell me what that one would be.

P13 3g.

INT That would be 3g would it. Like to write that alongside. Can you tell me how you got the answer there?

P13 Well it's just 3 plus g, so you regard g as a number instead of a letter.

INT Right, ok, like to read out No. 6 then.

P13 What is the value of 3 plus 2 times c when c equals 5?

INT Understand that one?

P13 So that would be 5 times 5. It would equal 25.

INT Can you tell me how you got that?

P13 Well first you do the 3 and the 2 which equals 5, c equals 5 so times c is 25.

INT Right, ok, again there is another No. 6 here, how would you work that out, it's very similar but not quite the same.

P13 Well, you do 3 times c which would be 15 then plus 2 is 17.

INT Good, can you write that alongside. And can you tell me which bit you decide to do first in the question.

P13 Well first you add those two, then times that, but with that one you times that and then just add that at the end.

INT And why do you do them in that order, can you tell me?

P13 Well, you don't sort of do 3 times c then go back.

INT So you start from the left and work through. Good, ok, one or two more then, it's not too bad is it? Question 4 could you try that one?

P13 Add 6 onto e plus 4.

INT Understand what that is asking?

P13 Yes

INT How did you do that?

P13 First I add 6 onto 4 which would be 6e, then the 4, so that would be 10e.

INT Good, can you write that down and put No. 4. Right, question 6.

P13 When does p take away b equals p take away c? Always, never or sometimes? If sometimes, say when.

INT Understand what that is asking?

P13 No, not really.

INT No, does it make any sense at all to you.

P13 I've seen this type of question further on in our algebra module but I've never got that far.

INT Ah, I see, so you can't answer that one at all.

P13 No

INT OK let's put a line by that one. Question 8 like to read that.

P13 Write more simply 3f plus 5g plus 2f.

INT Understand what that is asking?

P13 Mm, make the sum smaller and easier to add up. So you could have 5f plus 5g instead.

INT Good, can you write that down. Good and is that the only way of writing that answer or is there any other way?

P13 No, I don't think so.

INT That's it is it? Lovely. I think that will do actually. Thank you very much.

PUPIL 14

INT Just put No. 1 and we'll go through these so we know which is which. If you could read out the question.

P14 What is the value of 3 plus 2m when m equals 5?

INT Understand what that is asking you to do?

P14 Value of 3 plus 2 times 5.

INT How would you do that?

P14 I would first times the 2 and 5 which is 10 then add 3.

INT And the answer?

P14 13.

INT Would you write that down please. Good, so the 2m in a question like that means?

P14 Times.

INT Good. Would you like to read out the second one.

P14 If 2p minus 1 equals 5 and 2s minus 1 equals 5, can you say anything about p and s?

INT Understand what that is asking you?

P14 What is the value of p? 2p minus 1 all those equal 5 and all those equal 5.

INT The question says can you say anything about p and s, how would you set about doing that? What do you think it is asking you to do?

P14 Well it's telling you to multiply the 2 and the p and take away 1 to get the answer and multiply 2 and the s and take away 1 to get the answer.

INT And how would you set about doing something like that?

P14 I've forgotten.

INT That's fine, it doesn't matter, so you wouldn't be able to find the value of p either, or would you?

P14 Oh, 6....

INT Can you tell me what you are thinking.

P14 To go backwards, add the 1 onto the 5 and then divide p and 2 in

6, so it's 2 times 3 equals 6, minus 1 is 5. So it's 3.

INT Good, can you write that down for me then.

P14 3 for that one. 2s.... add the 1 and 5, 6, er 2 and 2, 6, 3 times, so s is 3 as well.

INT Good, can you write that down. Did you notice anything about those two.

P14 They're the same.

INT That wasn't obvious though. Like to read out No. 3 then.

P14 Multiply d minus 4 by 3.

INT Do you understand what that is asking you to do?...Is there a problem with that?

P14 Can't remember.

INT Can't remember how to do it, no. Can you try and explain to me what the problem might be with it.

P14 Would you add the d onto the 4 to get the answer?

INT What would the answer be then?

P14 4d, no, that would be ... would it actually be 4d?

INT Well I'm not going to tell you the right answer, I want to know what you think.

P14 There's 4 add d, 4 plus d multiplied by 3.

INT Can you write anything down for the answer.

P14 I suppose you could do 4 multiplied by 3 which is 12. 12d.

INT OK, can you write that down for me. Let's try No. 4 now.

P14 Which is bigger 3y or 3 plus y. 3y would be bigger. 3 times y.

INT And why is it bigger.

P14 Because 3 times y...3y would be 3 times y and 3 add y would be just adding them up.

INT OK like to try No. 5 then.

P14 Add 3 onto 6g.

INT Understand what that is asking?

P14 Add 3 onto 6 times g. Would you just add the 3 onto the 6 to make it 9g.

INT That's what you would do is it?

P14 Yes.

INT Good, can you write that down. Can you tell me how you would do that one then which is a similar sort of question.

P14 Add 3 onto g. That would be 3 add g.

INT Can you write that down for me. Good. Can you tell me what the difference is between those two questions then.

P14 It might be 3 add 6g, the answer to that one, or 9 add g.

INT Not quite sure which?

P14 No

INT Well it doesn't matter, that's fine. Good. Is there anyway of deciding which is right do you think? Any way you can sort of work it out.

P14 Well it's 3 add 6 - oh is it 9 times - no - 3 add 6 times g.

INT How would you write that down?

P14 Well 3 plus 6 times g.

INT Could you write that down for me alongside. Good, ok, thankyou. Try No. 6 now, can you read that out.

P14 What is the value of 3 plus 2 times c when c is 5?

INT Understand what that is asking you?

P14 Yes, 3 add 2 times 5.

INT So what would that become?

P14 25.

INT Can you write that down for me. What about this one, again it's a similar sort of question. What would that be?

P14 3 times c plus 2. 3 times 5 plus 2. So that would be 3 times 5 which is 15 plus 2, 17.

INT Good, can you write that alongside then. How do you decide in questions like this which bit you are going to do first, where to start?

P4 Well you just put in the place of the c.

INT Good so you put the number in place of the c, and where do you start working out?

P14 At the beginning.

INT Just start at the left hand side and work through. Seems a reasonable place to start. OK. Just one or two others then. You haven't got to do all these so don't get worried by that. If you'd like to do No. 1 for me.

P14 Write 2 bracket a add b bracket without brackets.

INT Good, understand what that is asking you to do?

P14 It is asking me to add a add b and then put it by the 2... just after the 2.

INT And how would you do that?

P14 a add b would be ab and then put it after the 2.

INT Could you write that down for me. So what you've written there is that the same as this only without the brackets?

P14 Yes. Oh, no, because that would be a 2 times a times b wouldn't it.

INT So how would you write it. So that it was the same only without the brackets. Could you do that or not?

P14 2 add b yes.

INT And that's the same now only without the brackets.

P14 2 times - just change that -

INT OK, happy with that?

P14 Yes. happy.

INT OK, lovely. Question 2 like to try that one.

P14 What is the value of 3s add b when s equals 5?

INT Understand what that is asking?

P14 Yes. 3 times 5 plus b.

INT How would you write that down?

P14 3 times 5, 15, plus b would be 15 plus b.

INT Good, can you write that down? Are you happy with that answer?

P14 Yes

INT Question 4 would you just like to try that.

P14 Add 6 onto e plus 4.

INT Understand what that is asking?

P14 Is it 6 add e plus 4?

INT How would you write that down?

P14 To write e plus 10 - could

INT Are you happy with that one?

P14 I'm not sure whether to add it on to that or not?

INT Oh I see onto the e or the 4.

P14 Yes

INT How do you think you might decide?

P14 Well e plus 4 would stay the same because you couldn't put it 4e because that would be multiplying it.

INT Can you tell me what the two answers would be that you would get doing it each way.

P14 Well if you added 6 onto the e then it would be 6 plus e plus 4, you wouldn't put 6e. Add it onto 4 that would be alright because that would be 10.

INT Can you write those two answers down for me and perhaps we could have a look at them. And which of those are you happier with as the answer? Or are you happy with both of them?

P14 That one.

INT You prefer the e plus 10.

P14 Yes. Saying you are actually including the adding the 6 plus something.

INT Good, ok, fine. Let's just try finally No. 5.

P14 If p plus q equals 3 then p plus q plus r equals?

INT Understand what that is asking?

P14 Um, if p and q equals 3 then that would be 3 add r. That would be 3 add r because you couldn't put 3r.

INT Can you write that down then. Good, are you happy with that answer. Good, that's it then. Thank you very much.

PUPIL 15

INT Could you read out the first question.

P15 What is the value of 3 plus 2m when m equals 5?

INT Good, do you understand what that is asking?

P15 Yes.

INT How would you do that?

P15 Say 3 plus 2m, that's 5m, that's 2m, 5 equals... that's 13. Obviously like that's m so there's 2 m's so that's 10 plus 3 is 13.

INT OK, can you write that down that. Good and what does the answer actually mean. Can you tell me?

P15 13 lots of m.

INT 13 lots of m. Lovely. Can you read out No. 2.

P15 If 2p minus 1 equals 5 and 2s minus 1 equals 5, can you say anything about p and s?

INT Good, just stop there for now. Can you understand what that is asking you to do?

P15 Yes, find out what the value of p is and s.

INT How would you set about doing that?

P15 2p minus 1 equals 5, so p equals 3, take 1 is 5, so there's 2 p's take 1 is 5. 2s is 6, 2 s's take 1 is 5 again.

INT Good, so what can'you say .. what is the answer to the question then?

P15 p and s are the same - equal.

INT Good, can you write that down. What about this part here?

P15 Value of p? 3.

INT Can you write that down. Good. ok. On this question here which is a similar one it's connected with the question here this No. 2 can you read that.

P15 For what values of p does 2 p plus 1 minus 1 equals 5?

INT How would you do that question having done this one?

P15 p plus 1 is 2. Say p plus 1, there is already one p plus another one, I'd say that was 2p, and then outside plus another 2 that is 4 minus 1 is 3 I would say.

INT So what is the answer?

P15 p equals 1 I would say, but..

INT Are you happy with that?

P1

INT Can you tell me why there is a problem.

P15 Well I would say there is one p there plus another one would be 2 and outside there is another 2 so that would be 4 minus 1 would make it 3.

INT So it doesn't seem to work? But you think the answer would be?

P15 Would be 3.

INT Could you write that down.

P15 Do I put 3 or 5 there?

INT Well whatever you think it is. Good, so that it your answer is it, lovely, ok that's fine. Let's go onto the next one. Read Out No. 3 then.

P15 Multiply 4 minus d by 3. I would say that you don't know what d is so I would say that d is minus 4 times it by 3 would be -1, I would say that that was 5d.

INT Can you write that down. Again there is a similar question here can you tell me what you think the answer to this one would be, it's very similar I just changed the beginning slightly.

P15 What are 5d minus 4 multiplied by 3? Well 5d minus 4 would be 1d multiplied by 3 ...3d.

INT Good, can you write that down. Can you read question 4.

P15 Which is bigger 3y or 3 plus y?

INT Good, understand what that is asking?

P15 Yes, I would say that the second sum 3 plus y would be more.

INT Would you, can you tell me why?

P15 Because you have already got 1 y there plus another 3 would make 4y.

INT OK can you write that down then. Good, can you read No. 5.

P15 Add 3 onto 6g.

INT Understand what that is asking you to do.

P15 Yes, I'd say that that was 9g.

INT Good, can you write that down. Can you tell me how you got the answer.

P15 I just added the 3 to the 6g, which would make it 9g.

INT Good, ok, read No. 6 then.

P15 What is the value of 3 plus 2 times c when c equals 5?

INT Understand that?

P15 Add the 3 and the 2 together, then c equals 5, so 5 times 5 is 25.

INT Good, can you write that down. We've got this one, again this is a similar sort of question, again it is asking for a value, can you just do that one for me.

P15 Value of 3 times c plus 2 when c equals 5? 3 c's would be 15 plus 2, it would be 17.

INT Good, can you write that down. On a question like this can you tell me how you decide where to start with these sort of questions?

P15 Well on this one I would start by the 3 plus 2 and I would try some way to find out what c equals. It says here it equals 5, so 5 times 5 would be 25. On that one I would say 3 times something plus 2, try and find out again what c was, and then add 3 times 5 plus 2.

INT Good, so you would start at the left hand side.

P15 Yes.

INT Good just one or two others on this other sheet, we're nearly there now. You don't have to do all these so don't get worried by the look of it. We'll just do one or two, could you do No. 6.

P15 When does p minus b equals p minus c? Always, never or sometimes? If sometimes, say when.

INT Understand what that is asking?

P15 No not really, I wouldn't..

INT Don't really understand that one? Have you any idea at all what it might be getting at?

P15 I think it's saying like is p minus b equal to p minus c, get to the end.

I

P15 That would be minus pb or minus pc, I'm not sure.

INT Not too sure on that one then. If you had to say if it whether always equal, never or sometimes, which do you think it would be.

P15 I would say always.

INT Always, would you. Alright, can we write that down then, No. 6. .... Can you explain to me where you get that answer or not?

P15 No, well there isn't a number on either of the letters so just assume that that is 1. 1p or 1b so I don't really know ....

INT That's fine, good. Let's try No. 10.

P15 Write more simply 2m minus n plus 3n.

INT Understand what that is asking?

P15 First add 3n plus n which I would say is 4n and then the answer I would put 2n, no, 4n minus 2m.

INT Good, can you write that down for me. Good I see where you get the 4n from, can you tell me where you get the minus 2m from.

P15 Just say that you can't really take m from n so just put it at the end.

INT I see, right, good, like to try No. 11 then please.

P15 Solve 3x minus 5 plus 2x plus 1.

INT Do you understand what that is asking?

P15 Yes I'd said it was minus 2x and here you've got 3x, 2x plus 1x, so I'd put that as 1x.

INT Can you write that down then? And is that the answer or?

P15 Yes.

INT What does the word solve mean?

P15 I'd say figure out or get the answer to.

INT I see and does it tell you anything about what sort of answer you would expect from that?

P15 No not really just find out.

INT OK. Good. Like to do No. 12 for me then.

P15 For what values of d is 17 minus 3d larger than 2?

INT Understand what that is asking?

P15 Yes.

INT How would you do that?

P15 What values of d is ...... No I wouldn't understand that.

INT You wouldn't understand that, not at all.

P15 No.

INT Can you give me any idea of what you might think it might be.

P15 What d is worth. It is bigger than 17 so it must be at least 13 or 14, it would be about 14, if it is bigger than 17.

INT Can you write down what you think it might be then. Good, is there any way of checking your answer or how would you check it to see if it is right?

P15 Don't know, no.

INT OK that's fine. It's very helpful, that wasn't too bad was it. Good.

PUPIL 16

INT So can you read out Question 1 for me.

P16 What is the value of 3 plus 2m when m equals 5?

INT Do you understand what that is asking you to do? How would you do that then?

P16 Well I'd say 5 plus 2 equals 7 plus 3

INT Makes?

P16 10.

INT Good, can you write that down..... So when you have something like 2m in a question like this what does that mean then?

P16 m stands for a certain number.

INT And the 2m, what does that mean to you?

P16 2 plus whatever number m stands for.

INT Ah good, ok. Would you like to read No. 2 for me now.

P16 If 2p minus 1 equals 5 and 2s minus 1 equals 5, can you say anything about p and s?

INT Good, do you understand what that is asking? Do you think you know how to set about that one? How would you do it?

P16 Well, put that p and s are the same.

INT Would you? Good. Could you write that down for me..... How did you get that answer?

P16 Well if 2p minus 1 equals 5 and 2s minus 1 equals 5 then they have got to be the same.

INT Ah, can you say why. It's obvious to you is it? Can you explain why it is obvious?

P16 Well if s was a different number to p then the answer would be different.

INT Ah, I see, right and what about the value of p then? This next bit, what would that be?

P16 3.

INT How did you work that out?

P16 I got 1 plus 2 equals 3 and then just doubled it.

INT Sorry, could you say that again.

P16 Well I got 1, I did it backwards, 1 plus 2 equals 3 and then I said if that is 3 then you would need 2 of whatever that is which, that's got to be 3. Hang on, yes.

INT Can you write that answer down for me?

P16 I said that one wrong somewhere.

INT Have a think about it then.

P16 Yes I did. 2 of whatever minus 1 has got to be that, so I did 2 what makes 6, so that has got to be 2 threes.

INT Good, can you write down the answer then... Good so what is the value p.

P16 3.

INT How would we write that down. Can you write that down alongside. Good, ok that's fine, can you read No. 3 for me then.

P16 Multiply d minus 4 by 3.

INT Do you understand what that is asking you to do?

P16 Yes.

INT How would you set about that?

P16 d minus 4 times 3 equals ... d times 3 minus 4 ... hang on

INT Confusing is it, can you tell me what you are thinking then now.

P16 I'm thinking how I would do it really.

INT And how would you?

P16 I'd go d minus 4 times by 3 would be d 3 minus 4.

INT How would you write that down, would it help to write it down?

P16 Don't know, might be.

INT Try writing it down then... Happy with that now?

P16 Mm.

INT You seem a bit confused, what is the problem with that can you tell me?

P16 I don't know how to do it really.

INT Not quite sure, can you tell me what the problem is with it?

P16 Well that is the same as that which is a bit confused.

INT So why is that a problem if they are the same. You mean what you have written down there, you have written the same thing down twice have you. Then why is that a problem? Why did you put the brackets in the second one then?

P16 Because you do that bit first, and it would be d minus 4 whatever that is times 3.

INT Good, does it matter that there aren't any brackets in this first one then?

P16 Yes, really, because you could go 4 times 3 minus d.

INT And that would be wrong would it?

P16 Yes.

INT OK, good, let's try No. 4 then. like to read that out.

P16 Which is bigger 3y or 3 plus y?

INT Understand what that is asking?

P16 Yes.

INT How would you set about that?

P16 I would say 3y is y .. is like um it's 3, 1, so that's 1y and that's 3y sort of thing so that would be 4y and 3 plus y would be the same, because they are the same.

INT I see, right, so what is the answer to the question then?

P16 Neither, they are both the same size.

INT Good, can you write that down for me then.... Is there any way of checking your answer to make sure that is right? Think of any way or not?

P16 No. Well I think that is bigger now because that is 3y and that is 3 plus 1 which would make that 4 and that just 3.

INT Ah, you'vbe changed your mind have you? OK do you want to write that answer alongside.... Happier with that one now?

P16 Mm.

....

INT On questions like this which bit to do first then.

P16 Well I do the easiest bit, the first bit first. If I find that that bit is easier then the 3 times c bit is easier then I do that first.

INT So that is the easier bit there then is it? What if the easier bit was further along the question, would you do that first still or would you start at the left?

P16 Yes I would probably start further along and then do the bit at the back and then add them together.

INT Ah, I see, could you give me an example of a question that would be like that where you would do the easier bit first or not, if it was further along. Can you think of one or not?

P16 7 times 15 plus 3.

INT Could you write that down for me just underneath. Good, so where would you start with that then.

P16 There, that bit. I'd do 3 plus 15 is 18 times 7 and then I would take my time about that.

INT That's a bit hard isn't it. Picked yourself a difficult one. Good, ok, that's fine. Just one or two more then. You needn't bother to work that one out. You don't have to do all these so don't get worried about this long sheet of questions. I'll just pick out one or two for me, would you like to try No. 1 for me.

P16 Write 2 a plus b without brackets.

INT Understand what that is asking you to do?

P16 ....

INT Don't you understand it?

P16 No.

INT Have you any idea what it might be asking you to do?

P16 It would be 2a plus b.

INT OK can you write that down. And how did you get that answer? Can you tell me how you got it?

P16 It says without brackets so I just took the brackets away.

INT Took the brackets away. Good. That answer you have written down there is that the same as this here only without the brackets.

P16 Yes.

INT It's the same is it. Good, so if you worked it out you would get the same answer.

P16 Yes.

INT Good, ok. Would you like to try No. 2 then please.

P16 What is the value of 3s plus b when s is 5?

INT Do you understand what that is asking you to do?

P16 Yes.

INT How would you do that?

P16 Go 35 plus b.

INT So what would the answer be.

P16 35b.

INT Could you write that down. Good and what does that answer mean.

P16 That that plus that equals that.

INT Does the answer actually mean anything, the 35b. How would you explain to someone what that meant. Be hard would it?

P16 Mm.

INT Ok, that's alright, don't worry too much about that. Would like to try No. 6 for me.

P16 When does p minus b equal p minus c? Always, never or sometimes? If sometimes, say when.

INT Understand what that is asking you to do?

P16 Yes.

INT How would you set about that then?

P16 I would try and find out what c and b were and then if c and b were the same then it would equal the same.

INT OK good and how would you try and find out?

P16 Ask the teacher.

INT Would you, there's no way you could find out on your own.

P16 Don't think so.

INT I see, ok, and if you were forced to make a choice here between one of these, always, never or sometimes which do you think it would be then?

P16 Never.

INT You would put never would you. Would you like to write that down then, No. 6.... Good, and just one more then No. 10 if you would like to try that.

P16 Write more simply 2m minus n plus 3n.

INT Understand what that is asking you to do? How would you do that?

P16 ....2m minus, I mean plus 2n.

INT Can you write that down for me.... Are you happy with that?

P16 No.

INT Not happy.

P16 If they are in the brackets.... so if would be take it away from that.

INT So the brackets are a problem are they or not. Can you explain why.

P16 Well it's got 2m minus n in brackets and if it was 3n minus m then it would be easier. I don't know whether that is the same as that or not.

INT What do you think the answer is then? What would be your best working out for it, or can't you write anything down for it.

P16 I wouldn't know that.

INT You couldn't do it. OK.

P16 I'd have to ask the teacher.

INT Can you tell me if this one is easier then. It's again a similar question. Would you like to read that out.

P16 2m minus n plus 3n.

INT Could you do that one, how would you do that one?

P16 I'd go 4n, n plus 3n equals 4n, and then it would be 2m minus 4n.

INT Good, can you write that one down then.... Good, can we put a little line in front to show that you didn't do the other one. Lovely. OK. Well thank you very much.