# NORMALIZERS AND COVERING SUBGROUPS OF FINITE SOLUBLE GROUPS

by

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# Preface

I declare that this dissertation, entitled "Normalizers and covering subgroups of finite soluble groups", is not substantially the same as any that I have submitted for a degree or diploma or any other qualification at any other University, and that no part of it has already been, or is being currently, submitted for any such degree, diploma, or other qualification. I further declare that, with the exception of passages where specific reference is made to the work of others and of results which are described as "well-known", this dissertation is my own original work.

I wish to thank my supervisor, Dr. R.W. Carter, for his unfailing advice and encouragement over the past three years. I am also grateful to Dr. T.O. Hawkes of the University of Warwick for many helpful conversations. I am indebted to the South African Council for Scientific and Industrial Research for a post - M.Sc. bursary during my first year of research, and to the University of Natal for an Emma Smith Overseas Scholarship held for the remaining two years.

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Abstract

In this dissertation we investigate two questions arising from Carter and Hawkes's generalization of P.Hall's theory of system normalizers to a theory of $\mathfrak{F}$-normalizers (i.e. normalizers of $\mathfrak{F}$-systems) of a finite soluble group, where $\mathfrak{F}$ is a saturated formation defined locally by an integrated set of formations $\{\mathfrak{F}(p)\}$. [The $\mathfrak{F}$-normalizers of a finite soluble group, J. Algebra 5 (1967), 175-202]. We first show that if, in addition, each formation $\mathfrak{F}(p)$ is subgroup-closed, then the whole of Carter's invariant theory [Chapter 2 of Nilpotent self-normalizing subgroups and system normalizers, Proc. London Math. Soc. (3) 12 (1962), 535-563.] can be extended to a theory of $\mathfrak{F}$-invariants of a finite soluble group -- the subgroup-closure of $\mathfrak{F}(p)$ is needed to enable us to define the concept of an $\mathfrak{F}$-system of the group reducing into a subgroup.

The remainder of the thesis is concerned with generalizations of Carter and Hawkes's theory. We choose, instead of the $\mathfrak{F}(p)$-residual, an arbitrary normal subgroup X(p) of the finite soluble group G for each prime p dividing the order of G, forming a normal system $\mathfrak{X} = \{X(p)\}$ of G. Then, from each Sylow system $\mathcal{S}$ of G, we obtain an $\mathfrak{X}$-system of G by intersecting X(p) with the Sylow p-complement of G appearing in $\mathcal{S}$ -- the normalizers of the $\mathfrak{X}$-systems of G are called $\mathfrak{X}$-normalizers of the group. We show that these subgroups satisfy many of the properties satisfied by $\mathfrak{F}$-

normalizers; however, they do not satisfy all the properties of $\mathfrak{F}$-normalizers unless $\mathfrak{X}$ is a so-called integrated normal system of G. We use the fact that the $\mathfrak{X}$-normalizers cover or avoid each chief factor of the group as a basis for several characterizations of $\mathfrak{X}$-normalizers, both for non-integrated and integrated normal systems $\mathfrak{X}$. We now associate with each normal system $\mathfrak{X}$ of G a further conjugacy class of subgroups, the $\mathfrak{X}$-covering subgroups of the group, which possess properties similar to those of $\mathfrak{F}$-covering subgroups and are related to the $\mathfrak{X}$-normalizers. However, when the X(p) are chosen in such a way that the $\mathfrak{X}$-normalizers become $\mathfrak{F}$-normalizers, the $\mathfrak{X}$-covering subgroups need not coincide with the $\mathfrak{F}$-covering subgroups. Nevertheless, most of Carter and Hawkes's results are paralleled in the present situation.

In our final chapter we apply our methods to B.Fischer's even more general situation. [Pronormal subgroups in finite soluble groups, To appear.] He considers sets $\mathfrak{M} = \{M(\pi_\lambda) \mid \lambda \in \Lambda\}$, where the $M(\pi_\lambda)$ are normal subgroups of the finite soluble group G, one for each $\lambda$ in the finite set $\Lambda$, and the $\pi_\lambda$ are sets of primes, rather than just single distinct primes as in a normal system $\mathfrak{X}$. He then obtains an $\mathfrak{M}$-system from each Sylow system $\mathcal{S}$ of G by intersecting $M(\pi_\lambda)$ with the Hall $\pi_\lambda$-complement of G appearing in $\mathcal{S}$. He develops several properties of the normalizers of $\mathfrak{M}$-systems -- $\mathfrak{M}$-normalizers -- and defines $\mathfrak{M}$-covering subgroups as limits of a certain type of sequence of subgroups. The $\mathfrak{M}$-

covering subgroups, too, have properties similar to those of $\mathfrak{F}$-covering subgroups and are related to the $\mathfrak{M}$-normalizers. Furthermore, in the special case of $\mathfrak{M}$ a normal system $\mathfrak{X}$, the $\mathfrak{M}$-covering subgroups are the $\mathfrak{X}$-covering subgroups. Using our $\mathfrak{X}$-theory methods in this situation, we obtain further properties of $\mathfrak{M}$-normalizers and give an alternative approach to the $\mathfrak{M}$-covering subgroups.

CONTENTS

# INTRODUCTION

This dissertation arises primarily from the work of R.W. Carter and T.O.Hawkes on a generalization of P.Hall's theory of system normalizers of a finite soluble group ([4]). In this paper they defined the $\mathfrak{F}$-normalizers of a finite soluble group for any saturated formation $\mathfrak{F}$ ( [7],[8] ), the only restriction imposed being that the system of formations $\{\mathfrak{F}(p)\}$ defining $\mathfrak{F}$ locally ([7],[13]) should be integrated i.e. each formation $\mathfrak{F}(p)$ should be contained in $\mathfrak{F}$. This type of local definition of $\mathfrak{F}$ enabled them to introduce the concept of an $\mathfrak{F}$-central ( $\mathfrak{F}$-eccentric) chief factor of G, a concept depending only on $\mathfrak{F}$ and reducing to that of a central (eccentric) chief factor when $\mathfrak{F}$ is the class of nilpotent groups. Suppose that G(p) is the $\mathfrak{F}(p)$-residual of G i.e. the smallest normal subgroup of G with factor group in the formation $\mathfrak{F}(p)$, and $\mathcal{S} = \{S^p\}$ is the Sylow system of G defined by the complete set of p-complements $S^p$ of G. The set $\{S^p \cap G(p)\}$ is called an $\mathfrak{F}$-system and its normalizer an $\mathfrak{F}$-normalizer of the group G --- these subgroups specialize to the system normalizers of G when $\mathfrak{F}$ is the class of nilpotent groups.

Carter and Hawkes proved that the $\mathfrak{F}$-normalizers satisfied properties similar to those of system normalizers viz.

they lie in $\mathfrak{F}$, are conjugate, invariant under homomorphisms of the group, cover the $\mathfrak{F}$-central chief factors and avoid the $\mathfrak{F}$-eccentric chief factors of the group and can be characterized by means of the maximal chains of subgroups joining them to the group. Furthermore, the intersection of a chief series of G with an $\mathfrak{F}$-normalizer is a chief series of the $\mathfrak{F}$-normalizer. Carter and Hawkes then went on to show that the relationship between the system normalizers and Carter subgroups (i.e. nilpotent self-normalizing subgroups, [2] ) of the group is paralleled by a relation between the $\mathfrak{F}$-normalizers and W.Gaschütz's $\mathfrak{F}$-covering subgroups ([7]) viz. every $\mathfrak{F}$-normalizer of G is contained in an $\mathfrak{F}$-covering subgroup of the group, and vice versa. In fact, when G is an extension of a nilpotent group by a group in the formation $\mathfrak{F}$, the $\mathfrak{F}$-normalizers and $\mathfrak{F}$-covering subgroups of G coincide.

Two questions arise from this generalization. The first concerns R.W.Carter's theory of invariants developed in [3]. Carter let X be any subgroup of a finite soluble group G and considered four different types of factor occurring in an X-composition series of G i.e. a series of subgroups of G in which each member is normalized by X and no further terms can be inserted. The product of the orders of the factors of the same type in a given X-composition series was in each case found to be independent of the series chosen, yielding four invariants $z(X)$, $\varpi(X)$, $z_0(X)$ and $\varpi_0(X)$ associated with X. His interpretations of the last two of these invariants show that the Carter subgroups E of G may be characterized by the equations $\varpi(E) = 1$, $z_0(E) = 1$. They also involve a result on the number of Sylow systems of G reducing into a maximal chain from X to G --- this result is considerably simplified when X is a system normalizer of G. We ask whether this theory can be extended to a theory of $\mathfrak{F}$-invariants, where $\mathfrak{F}$ is a saturated formation. We show that this is indeed possible provided that all the formations occurring in the integrated system $\{\mathfrak{F}(p)\}$ which defines $\mathfrak{F}$ locally are subgroup-closed.

We now ask whether Carter and Hawkes's $\mathfrak{F}$-normalizers themselves can be generalized. We have seen that their definition involves choosing a certain normal subgroup of the group associated with each prime divisor of the order of the group. Proceeding a stage further, suppose that we choose an arbitrary normal subgroup X(p) of the finite soluble group G for each prime p dividing the order of G, obtaining a so-called normal system $\mathfrak{X} = \{X(p)\}$ of G. We can then form an $\mathfrak{X}$-system in the same way as Carter and Hawkes obtained an $\mathfrak{F}$-system, and we are interested in knowing whether the normalizers of the $\mathfrak{X}$-systems --- the $\mathfrak{X}$-normalizers of G --- satisfy properties similar to those of $\mathfrak{F}$-normalizers. We show that for any normal system $\mathfrak{X}$, the $\mathfrak{X}$-normalizers do indeed satisfy most of the properties of $\mathfrak{F}$-normalizers. However, they do not satisfy all the properties of $\mathfrak{F}$-normalizers unless $\mathfrak{X}$ is a so-called integrated normal system. An example of this type of normal system is the case when all X(p) are the same normal subgroup --- the $\mathfrak{X}$-normalizers then become P.Hall's relative system normalizers ([10]).

Having developed a satisfactory theory of $\mathfrak{X}$-normalizers, it seemed reasonable to seek a further conjugacy class of subgroups depending on the normal system $\mathfrak{X}$ which would play the role of the $\mathfrak{F}$-covering subgroups in Carter and Hawkes's theory. This search was successful --- we show that for any normal system $\mathfrak{X}$ of a finite soluble group, one can define and prove the existence of a new conjugacy class of subgroups, the $\mathfrak{X}$-covering subgroups of the group. These subgroups possess properties similar to those of the $\mathfrak{F}$-covering subgroups and are related to the $\mathfrak{X}$-normalizers. Nevertheless, these $\mathfrak{X}$-covering subgroups need not coincide with the $\mathfrak{F}$-covering subgroups when we choose the normal system $\mathfrak{X}$ in such a way that the $\mathfrak{X}$-normalizers become $\mathfrak{F}$-normalizers for a saturated formation $\mathfrak{F}$.

B.Fischer, too, considered the possibility of generalizing Carter and Hawkes's theory. Working independently, he went even further by investigating sets $\mathfrak{M} = \{M(\pi_\lambda) \mid \lambda \in \Lambda\}$ in a finite soluble group, where the $M(\pi_\lambda)$ are arbitrary normal subgroups of the group, one for each element $\lambda$ of the finite set $\Lambda$, and the $\pi_\lambda$ are sets of primes rather than just single distinct primes as in a normal system ([5]). From each Sylow system $\mathcal{S}$ of the group he obtained an $\mathfrak{M}$-system by intersecting $M(\pi_\lambda)$ with the Hall $\pi_\lambda$-complement appearing in $\mathcal{S}$ for each $\lambda \in \Lambda$. The normalizers of the $\mathfrak{M}$-systems -- the $\mathfrak{M}$-normalizers of the group -- were then shown to be conjugate and homomorphism-invariant, though not necessarily covering or avoiding every chief factor of the group. Then, inspired by Carter's method of obtaining the Carter subgroups of A-groups as limits of certain sequences of subgroups ( [3], page 548 ), he developed a process yielding a further conjugacy class of subgroups, the $\mathfrak{M}$-covering subgroups of the group. Surprisingly enough, these turned out to be the $\mathfrak{X}$-covering subgroups in the special case of $\mathfrak{M}$ a normal system $\mathfrak{X}$ of the group. This discovery led us to apply our methods to Fischer's more general situation, with great success. We prove further properties of $\mathfrak{M}$-normalizers -- in particular that they are subabnormal subgroups -- and give an alternative approach to his $\mathfrak{M}$-covering subgroups.

Chapter one contains a section on notation and terminology followed by a summary of well-known results assumed in the main body of the dissertation. In addition, using the methods of Gaschütz ([6]), we extend a result of Carter and Hawkes on p-Frattini theory ([4], pages 179-180 ) to the case of a set of primes $\varpi$ rather than just a single prime p --- this result is used in chapter seven.

In chapter two we develop our theory of $\mathfrak{F}$-invariants, where $\mathfrak{F}$ is a saturated formation defined locally by an integrated system of subgroup-closed formations $\{\mathfrak{F}(p)\}$. We let X be any subgroup of the finite soluble group G and divide the X-composition factors of G into four types by defining $\mathfrak{F}$-central and $\mathfrak{F}$-eccentric X-composition factors. Here, too, the product of the orders of all the factors of the same type in a given X-composition series of G is independent of the series chosen, yielding four invariants $z^{\mathfrak{F}}(X)$, $\varpi^{\mathfrak{F}}(X)$, $z_o^{\mathfrak{F}}(X)$, $\varpi_o^{\mathfrak{F}}(X)$ associated with X which satisfy the relations

$$z^{\mathfrak{F}}(X).\varpi^{\mathfrak{F}}(X) = |X| , \quad z_o^{\mathfrak{F}}(X).\varpi_o^{\mathfrak{F}}(X) = |G : X| .$$

$z^{\mathfrak{F}}(X)$ is the $\mathfrak{F}$-central order of X i.e. the product of the orders of the $\mathfrak{F}$-central chief factors in a given chief series of X, and $\varpi^{\mathfrak{F}}(X)$ is the number of distint $\mathfrak{F}$-systems of X. The condition that all $\mathfrak{F}(p)$ should be subgroup-closed ensures that we can define the concept of an $\mathfrak{F}$-system of G reducing into a subgroup of G. We then determine the number of $\mathfrak{F}$-systems of G reducing into a maximal chain $\mathcal{L}$ joining X to G --- this becomes simply the product of the indices of the $\mathfrak{F}$-normal links ([4], page 179) in $\mathcal{L}$ in the special case of X an $\mathfrak{F}$-normalizer of G. It also enables us to show that $z_o^{\mathfrak{F}}(X)$ is the largest value taken by the product of the indices of the $\mathfrak{F}$-normal links in $\mathcal{L}$ as $\mathcal{L}$ runs through all possible maximal chains from X to G, and that $\varpi_o^{\mathfrak{F}}(X)$ equals the total number of $\mathfrak{F}$-systems of G divided by the number reducing into X. The $\mathfrak{F}$-covering subgroups E of G are seen to be characterized by the equations $\varpi^{\mathfrak{F}}(E) = 1$, $z_o^{\mathfrak{F}}(E) = 1$.

Chapter three introduces the basic concepts used in the following three chapters. Having defined a normal system $\mathfrak{X}$ of the finite soluble group G, we bring in the closely related concepts of an $\mathfrak{X}$-central ($\mathfrak{X}$-eccentric) chief factor and an $\mathfrak{X}$-normal ($\mathfrak{X}$-abnormal) maximal subgroup of G. This enables us to define a set $\bar{\mathfrak{X}}$ of sections of G (i.e. factor groups of subgroups of G) which in a way takes the place of the formation $\mathfrak{F}$ in Carter and Hawkes's theory, and thus plays an important part in the sequel. $\mathfrak{F}$ and $\bar{\mathfrak{X}}$ clearly differ in that $\mathfrak{F}$ is an isomorphism class whereas the set $\bar{\mathfrak{X}}$ is defined entirely within the given group G; in fact $\bar{\mathfrak{X}}$ is not even closed under isomorphisms within G. However, we show that $\bar{\mathfrak{X}}$ satisfies properties analogous to those of a saturated formation.

The properties of the $\mathfrak{X}$-normalizers of a finite soluble group G are investigated in chapter four. We show that these subgroups are homomorphism-invariant, conjugate, cover the $\mathfrak{X}$-central and avoid the $\mathfrak{X}$-eccentric chief factors of G ; furthermore, they lie in $\bar{\mathfrak{X}}$ and are minimal members of the set of so-called $\mathfrak{X}$-subabnormal subgroups of G. An arbitrary maximal subgroup of G contains an $\mathfrak{X}$-normalizer of G if and only if it is $\mathfrak{X}$-abnormal; in fact an $\mathfrak{X} \cap M$-normalizer of an $\mathfrak{X}$-abnormal maximal subgroup M of G always contains an $\mathfrak{X}$-normalizer of G (where $\mathfrak{X} \cap M$ is the normal system of M obtained by intersecting each element of $\mathfrak{X}$ with M), and these two subgroups will be equal if M is a so-called $\mathfrak{X}$-critical maximal subgroup of G. However, in contrast to the theory of $\mathfrak{F}$-normalizers, it is possible to have G not in $\bar{\mathfrak{X}}$ and yet possessing no $\mathfrak{X}$-critical maximal subgroup. Thus we cannot in general obtain every $\mathfrak{X}$-normalizer as the terminal member of an $\mathfrak{X}$-critical maximal chain. There are two further differences between $\mathfrak{X}$- and $\mathfrak{F}$-normalizers viz. a chief series of G need not intersect an $\mathfrak{X}$-normalizer D of G in a chief series of D, nor need a $\mathcal{Y} \cap D$-normalizer of an $\mathfrak{X}$-normalizer D of G be a $\mathcal{Y}$-normalizer of G (where $\mathcal{Y}$ is another normal system of G). However, all these differences fall away if we assume that our normal systems are integrated

The first part of chapter five is concerned with the existence and main properties of the $\mathfrak{X}$-covering subgroups of the finite soluble group G. Following the definition of an $\mathfrak{F}$-covering subgroup of G (see [4], page 190), we define an $\mathfrak{X}$-covering subgroup of G to be a subgroup E satisfying the conditions (i) $E \in \bar{\mathfrak{X}}$.

(ii) E covers every section $F/F_0$ of G such that $F/F_0 \in \bar{\mathfrak{X}}$ and F contains E.

Then, in spite of the fact that $\bar{\mathfrak{X}}$ is not isomorphism-closed, we are able to show that these subgroups, if they exist, are abnormal in G, homomorphism-invariant and conjugate. We prove that the $\mathfrak{X}$-covering subgroups of G always exist by exhibiting them as the terminal members of so-called $\mathfrak{X}$-crucial maximal chains of G. In the remainder of the chapter, the relation between the $\mathfrak{X}$-covering subgroups and the $\mathfrak{X}$-normalizers of G is discussed and further properties of the former are developed. We first prove that every $\mathfrak{X}$-covering subgroup of G contains an $\mathfrak{X}$-normalizer of G and vice versa. This result enables us to characterize the $\mathfrak{X}$-covering subgroups of G along the lines of Lemma5.1 of [4] ; to determine which E-composition factors of G will be covered by the $\mathfrak{X}$-covering subgroup E of G and which avoided (as in Lemma5.2 of [4]); and to give a necessary and sufficient condition for an $\mathfrak{X}$-normalizer to be an $\mathfrak{X}$-covering subgroup of G. In the special case in which the factor group of G by its Fitting group is in $\bar{\mathfrak{X}}$, the $\mathfrak{X}$-covering subgroups and $\mathfrak{X}$-normalizers of G coincide and the $\mathfrak{X}$-normalizers are precisely those subgroups of G which cover the $\mathfrak{X}$-central and avoid the $\mathfrak{X}$-eccentric chief factors; several other results of Carter and Hawkes on $\mathfrak{N}\mathfrak{F}$-groups ([4], chapter 5) also carry over into this situation with only slight modification. Our study of this special case enables us to relate the $\mathfrak{X}$-covering subgroups of G to those of certain subgroups of G as follows. Let $\mathfrak{X} = \{X(p)\}$ and X be the intersection of all the X(p). Then, if the subgroup L of G supplements the Fitting subgroup of X in G, every $\mathfrak{X} \cap L$-covering subgroup of L can be written as the intersection of an $\mathfrak{X}$-covering subgroup of G with L ; a special case of this situation arises when L is a $\mathfrak{Y}$-normalizer of G for an integrated normal system $\mathfrak{Y} = \{Y(p)\}$ of G such that Y(p) is contained in X(p) for each prime p. The following result provides a further parallel with Carter and Hawkes's theory. Suppose that $\mathfrak{X}$ and $\mathfrak{Y}$ are any two integrated normal systems of G. Then, if the $\mathfrak{X}$- and $\mathfrak{Y}$-covering subgroups of G coincide, so do the $\mathfrak{X}$- and $\mathfrak{Y}$-normalizers of G ; furthermore, as in Carter and Hawkes's theory, the converse is false. We also give examples showing that an $\mathfrak{X}$-covering subgroup of G need not be an $\mathfrak{F}$-covering subgroup of G for any saturated formation $\mathfrak{F}$, and conversely, that an $\mathfrak{F}$-covering subgroup of G need not be an $\mathfrak{X}$-covering subgroup of G for any normal system $\mathfrak{X}$ of G. We close the chapter with a brief discussion of the special case in which all X(p) are the same normal subgroup of G.

Chapter six is devoted to characterizations of the $\mathfrak{X}$-normalizers of a finite soluble group G for both non-integrated and integrated normal systems of G. We begin by proving that if, for any normal system $\mathfrak{X}$ of G, the $\mathfrak{X}$-normalizers coincide with the $\mathfrak{X}$-covering subgroups of G they can then be characterized as those subgroups of G which cover the $\mathfrak{X}$-central and avoid the $\mathfrak{X}$-eccentric chief factors of G. It is this covering and avoidance property of $\mathfrak{X}$-normalizers which forms the basis of all our characterizations. Taking the non-integrated case first, we assume that a subgroup H of G covers or avoids each chief factor of G in a certain way, and seek an additional condition on H which will ensure that it is an $\mathfrak{X}$-normalizer of G (By Example1 of [11], page 344, the covering and avoidance property alone will not in general be sufficient to ensure that H is an $\mathfrak{X}$-normalizer of G). One of the two such conditions found involves the type of chain connecting H to the whole group ; the other demands that H should commute with a Sylow p-complement of G for each prime p dividing the order of G. The two characterizations of "non-integrated $\mathfrak{X}$-normalizers" thus obtained are adapted to characterizations of "integrated $\mathfrak{X}$-normalizers" by use of the distinguishing fact that a chief series is preserved by intersection with an $\mathfrak{X}$-normalizer defined by integrated normal system $\mathfrak{X}$.

In the seventh and final chapter we discuss Fischer's $\mathfrak{M}$-normalizers and $\mathfrak{M}$-covering subgroups of a finite soluble group G. We show that, besides forming a homomorphism-invariant conjugacy class of subgroups of G, the $\mathfrak{M}$-normalizers are subabnormal in G and cover certain of the chief factors of G (though they need not avoid the remaining chief factors) ; furthermore, as in the special case of an $\mathfrak{X}$-normalizer, each Sylow system of G reduces into the $\mathfrak{M}$-normalizer of G which it defines. We then describe in detail Fischer's definition of an $\mathfrak{M}$-covering subgroup of G and state some of the properties of these subgroups. One of these properties shows that the $\mathfrak{M}$-covering subgroups satisfy conditions similar to those used to define an $\mathfrak{X}$-covering subgroup of G for a normal system $\mathfrak{X}$ of G. Taking this property as our definition of an $\mathfrak{M}$-covering subgroup of G, we are able to show that the $\mathfrak{M}$-covering subgroups exist as the terminal members of so-called $\mathfrak{M}$-crucial maximal chains, and to develop their properties anew. In conclusion we prove that each $\mathfrak{M}$-covering subgroup of G contains an $\mathfrak{M}$-normalizer of G ( and vice versa ) and does indeed satisfy Fischer's definition.

# Chapter One

## PRELIMINARIES

Notation and Terminology. Groups are denoted by capital Roman letters and their elements by small Roman letters --- all groups considered are finite and soluble. We use braces $\{\ \}$ to denote sets, $\langle g \mid \ldots \rangle$ to denote the group generated by the elements g to be specified, and $|H|$ to denote the order of H. If $\varpi$ is a set of primes, $\varpi'$ is the complementary set ; and H is an $\varpi$-group if all the prime divisors of $|H|$ lie in $\varpi$ . A Hall $\varpi$-subgroup of a group is an $\varpi$-subgroup whose order is prime to its index ; and a Hall $\varpi$-complement is a Hall $\varpi'$-subgroup. If $\varpi$ is the single prime p, we use the terms Sylow p-subgroup and (Sylow) p-complement. A Sylow system $\mathcal{S}$ of G is a complete set of (Sylow) p-complements of G together with all their intersections -- we write $\mathcal{S} = \{S^p\}$ , where $S^p$ is a p-complement of G. $\mathcal{S}$ is said to reduce into a subgroup H of G if $S^p \cap H$ is a p-complement of H for each prime p.

We frequently refer to the following subgroups of the finite soluble group G : $\phi(G)$ is the Frattini subgroup i.e. the intersection of all the maximal subgroups of G ; $F(G)$ is the Fitting subgroup i.e. the largest nilpotent normal subgroup of G ; $\phi_{\varpi}(G)$ is the $\varpi$-Frattini subgroup i.e. the intersection of every maximal subgroup of G whose index is a power of a prime in $\varpi$; $O_{\varpi'\varpi}(G)$ is the largest normal subgroup of G possessing a normal Hall $\varpi$-complement ; 1 is used to denote the identity subgroup as well as the numeral "one".

For subgroups H,K of G, $N_H(K)$, $C_H(K)$ denote respectively the normalizer of K in H and the centralizer of K in H ; $H \times K$ denotes the direct product of H and K. We take $h^k = k^{-1}hk$ , $[h,k] = h^{-1}h^k$ and define $[H,K] = \langle [h,k] \mid h \in H, k \in K \rangle$ . Further, $K \leq H$ means that K is a subgroup of H -- the index of K in H is denoted by $|H:K|$. The relations (a) $K < H$ , (b) $K \lessdot H$ , (c) $K \lhd H$ and (d) $K \lhd\lhd H$ mean, in turn, that K is a (a) proper, (b) maximal, (c) normal and (d) subnormal subgroup of H. An oblique line through these symbols denotes negation. If $K \leq H$ , the core of K in H (written $\mathrm{Core}_H K$ ) is the intersection of all the conjugates of K in H i.e. the largest normal subgroup of H contained in K. When M is a maximal subgroup of G we usually write, simply, CoreM for $\mathrm{Core}_G M$ . If $K \lhd H$ and $K \leq H \lhd G$, we call H/K a factor of G ; if H/K is a minimal normal subgroup of G/K it is then said to be a chief factor of G. If H/K is a chief factor of G, $\mathrm{Aut}_G(H/K)$ denotes the set of automorphisms induced on H/K by conjugation by elements of G. Similarly, if H/K is an X-composition factor for some subgroup X of G, we denote by $\mathrm{Aut}_X(H/K)$ the set of automorphisms induced on H/K by conjugation by elements of X. If $H \lhd G$, we say that the subgroup X of G supplements H in G if $HX = G$ and complements H in G if, in addition, $H \cap X = 1$ .

Let $\mathfrak{A}, \mathfrak{B}$ be sets of groups. Then $G \in \mathfrak{A}\mathfrak{B}$ or G is an

$\mathfrak{A}\mathfrak{B}$-group means that there exists $N \lhd G$ such that $N \in \mathfrak{A}$ and $G/N \in \mathfrak{B}$. We denote by $\mathfrak{N}$ the class of all nilpotent groups, and write $\mathfrak{N}^2$ for $\mathfrak{N}\mathfrak{N}$. Finally, we remark that the term "projector" is sometimes used instead of "covering subgroup" -- we have chosen the latter as we feel it is more suited to our method of approach.

Prerequisites. We make frequent, and often tacit, use of the standard isomorphism theorems, the operator form of the Jordan-Hölder Theorem and the well-known Dedekind relation viz. if A commutes as a subgroup with B and is contained in C, then $A(B \cap C) = AB \cap C$. The concepts of a subgroup covering and avoiding a factor also occur frequently --- see Taunt [15], page 25, for a detailed account of these concepts. The following simple lemma involving these concepts is easily verified.

LEMMA 1.1 Let H/K be a chief factor of G and L a subgroup of G satisfying $LC_G(H/K) = G$. Then L covers or avoids H/K. If H/K is covered by L, then $H \cap L/K \cap L$ is a chief factor of L operator-isomorphic to H/K, and $C_G(H/K) \cap L = C_L(H \cap L/K \cap L)$.

In addition, we assume the following well-known results from the theory of finite soluble groups. Firstly, every finite soluble group G possesses a complete set of Sylow p-complements and thus a Hall $\varpi$-subgroup for every set of primes $\varpi$; and any two Hall $\varpi$-subgroups of G are conjugate in G. Every chief factor H/K of G is elementary abelian and thus of prime power order ( $H/K$ is said to be a p-chief factor if $|H/K| = p^{\alpha}$ ) and every maximal subgroup M is of prime power index ( M is said to be p-maximal if $|G:M| = p^{\beta}$ ). Furthermore, if $H/K$ is a p-chief factor of G, then $Aut_G(H/K) \cong G/C_G(H/K)$ has no normal p-subgroup. Let M be a maximal subgroup of G and $K = \text{Core}M$ . Then $G/K$ has a unique minimal normal subgroup $N/K$ ; $N/K$ is self-centralizing and complemented by $M/K$ in $G/K$, and all the complements of $N/K$ in $G/K$ are conjugate to $M/K$. $F(G)$ is the intersection of the centralizers of all the chief factors of G, and $O_{p'p}(G)$ is the intersection of the centralizers of all the p-chief factors of G for each prime p dividing $|G|$. In addition,

$$O_{p'p}(G/\emptyset_p(G)) = O_{p'p}(G)/\emptyset_p(G) \quad \text{(see [4], pages 179,180).}$$

This result is easily extended to the case of a set of primes $\varpi$ contained in the set of prime divisors of $|G|$ :

Since $\emptyset_{\varpi}(G) = \bigcap_{p \in \varpi} \emptyset_p(G)$ and $\emptyset_p(G)$ is p-nilpotent (i.e. has a normal Sylow p-complement) for each prime p, $\emptyset_{\varpi}(G)$ has a normal Hall $\varpi$-complement ; $\emptyset_{\varpi}(G)$ is also normal in G since all the $\emptyset_p(G)$ are. Hence $\emptyset_{\varpi}(G) \leq O_{\varpi'\varpi}(G)$ .

<u>LEMMA 1.2</u> If H is a subgroup of G such that $H\emptyset_{\varpi}(G) = G$ and $|G:H| \in \varpi$, then $H = G$ .

<u>Proof</u>. Suppose, if possible, that $H < G$ and let M be a maximal subgroup of G containing H. Then $|G:M| \in \varpi$ so $\emptyset_{\varpi}(G) \leq M$ . Hence $M \geq H\emptyset_{\varpi}(G) = G$ -- a contradiction. Thus $H = G$ as required.

<u>LEMMA 1.3</u> If X,R are normal subgroups of G such that $R \leq$

$\emptyset_{\omega}(G)$ and X/R has a normal Hall $\omega$-complement, then X has a normal Hall $\omega$-complement.

Proof. (cf. Gaschutz, [6].) Let Q be a Hall $\omega$-complement of X, and $Y = N_G(Q)$. We first show that $Y\emptyset_{\omega}(G) = G$. Now QR/R is a Hall $\omega$-complement of X/R and so $QR \lhd G$. Thus, for $g \in G$, $Q, Q^g$ are Hall $\omega$-complements of QR. Hence, by Hall, there exists some $r \in R$ such that $Q^g = Q^r$. Thus $gr^{-1} \in N_G(Q)$ giving $g \in RY \leq \emptyset_{\omega}(G).Y$, as required. We now show that $|G:Y| \in \omega$. Let $\mathcal{S} = \{S^p\}$ be a Sylow system of G such that $Q = X \cap \bigcap_{p \in \omega} S^p$. Then $Y = N_G(Q) \geq \bigcap_{p \in \omega} N_G(S^p) = Z$, say, and so $|G:Y|$ divides $|G:Z|$. Since Z is the intersection of subgroups of coprime index, $|G:Z| = \prod_{p \in \omega} |G:N_G(S^p)| \in \omega$, and the result follows. Lemma1.2 now yields $Y = G$ as required.

LEMMA 1.4 $O_{\omega'\omega}(G/\emptyset_{\omega}(G)) = O_{\omega'\omega}(G) / \emptyset_{\omega}(G)$.

Proof. Set $R = \emptyset_{\omega}(G)$ and let $O_{\omega'\omega}(G/R) = X/R$. Then, certainly, $O_{\omega'\omega}(G)/R \leq X/R$, giving $O_{\omega'\omega}(G) \leq X$. Conversely, by Lemma1.3, X has a normal Hall $\omega$-complement and so $X \leq O_{\omega'\omega}(G)$ and the result follows.

Finally, a subgroup H is abnormal in the finite soluble group G if and only if every subgroup of G containing H is self-normalizing in G -- this (unpublished) result is due to D.R.Taunt. ( In general, a subgroup H is said to be abnormal in a group G if $g \in \langle H, g^{-1}Hg \rangle$ for every element $g \in G$. i.e. if and only if every subgroup of G containing H is self-normalizing in G and H is not contained in two different conjugate subgroups of G ([2], page 136) ; H is subabnormal in G if there is a chain $H = H_0 < H_1 < \ldots < H_r = G$ of subgroups from H to G such that $H_i$ is abnormal in $H_{i+1}$ for each i. )

For completeness we now recall the definitions of the various types of formations.

DEFINITION. A formation $\mathfrak{F}$ is an isomorphism class of finite soluble groups satisfying the two conditions

(1) $G \in \mathfrak{F}$, $N \lhd G \implies G/N \in \mathfrak{F}$.

(2) $G/N_1, G/N_2 \in \mathfrak{F} \implies G/N_1 \cap N_2 \in \mathfrak{F}$.

If, in addition, $\mathfrak{F}$ satisfies the condition

(3) $G/\Phi(G) \in \mathfrak{F} \implies G \in \mathfrak{F}$ ( $\mathfrak{F}$ is "Frattini-closed"),

it is said to be a saturated formation ([7],[8]). Suppose, now, that a formation $\mathfrak{F}(p)$ is associated with each prime p. Then the class $\mathfrak{F}$ of finite soluble groups defined by

$G \in \mathfrak{F} \iff G/O_{p'p}(G) \in \mathfrak{F}(p)$ for each prime $p \mid |G|$,

is a formation --- we say that $\mathfrak{F}$ is defined locally by the set $\{\mathfrak{F}(p)\}$. For example, if we take $\mathfrak{F}(p) = \{1\}$ for each prime p, then $\{\mathfrak{F}(p)\}$ defines $\mathfrak{N}$ locally.

In 3.1 of [7], Gaschütz shows that a local formation is saturated ; and Gaschütz and Lubeseder have proved that conversely, every saturated formation may be defined locally (see [13]). In chapter two, like Carter and Hawkes, we consider a saturated formation $\mathfrak{F}$ defined locally by an integrated set of non-empty formations $\{\mathfrak{F}(p)\}$ ( i.e. $\{1\} \leq \mathfrak{F}(p)$

$\leq \mathfrak{F}$ for each prime p -- thus $\mathfrak{F} \geq \mathfrak{N}$ .). For definitions of an $\mathfrak{F}$-central chief factor and an $\mathfrak{F}$-normal maximal subgroup of a finite soluble group see [4], page 179 ; and for the definition and properties of $\mathfrak{F}$-normalizers see chapter 4 of [4].

We conclude with the concept of a group acting on a set, used in Lemma2.1.

<u>DEFINITION</u>. Let G be a multiplicative group with identity 1 and A a set. Then, if there is a mapping $\varphi : A \times G \to A$ such that (1) $(a,1)\varphi = a$

(2) $((a,g_1)\varphi , g_2)\varphi = (a,g_1g_2)\varphi$

for all $a \in A$ and $g_i \in G$ , we say that G acts on A. The stabilizer of an element $a \in A$ (written St(a) ) is defined by $St(a) = \{ g \in G \mid (a,g)\varphi = a \}$ , and is easily seen to be a subgroup of G.

We now define an equivalence relation on A. For $a,b \in A$ we say that $a \sim b$ if there exists $g \in G$ such that $b = (a,g)\varphi$ . The equivalence classes are called transitive components or orbits -- orb(a) is the equivalence class containing $a \in A$ . Thus A is a disjoint union of transitive components $A_i$ satisfying

$$|A_i| = |\operatorname{orb}(a_i)| = |G : St(a_i)| \quad (a_i \in A_i).$$

For, if $g_1,g_2 \in G$ , $(a_i,g_1)\varphi = (a_i,g_2)\varphi$ if and only if $a_i = (a_i,g_1g_2^{-1})\varphi$ i.e. $St(a_i)g_1 = St(a_i)g_2$.

Chapter Two

# THE INVARIANTS $z^{\mathfrak{F}}(X)$, $\varpi^{\mathfrak{F}}(X)$, $z_0^{\mathfrak{F}}(X)$, $\varpi_0^{\mathfrak{F}}(X)$.

Throughout this chapter we assume that $\mathfrak{F}$ is a saturated formation defined locally by an integrated set of non-empty formations $\{\mathfrak{F}(p)\}$, and that X is any subgroup of the finite soluble group G.

We begin by recalling some concepts and results from R.W. Carter's invariant theory ([3], chapter 2).

DEFINITIONS. We say that a subgroup H of G is an X-subgroup of G if $[H,X] \leq H$ i.e. X normalizes H. An X-composition series of G is a series

$$G = H_r > H_{r-1} > \dots > H_{i+1} > H_i > \dots > H_0 = 1$$

of X-subgroups of G such that $H_i \lhd H_{i+1}$ for each i, and no further terms can be inserted. We say that the factor groups $H_{i+1}/H_i$ are X-composition factors of G. An X-composition factor $H_{i+1}/H_i$ of G can clearly have no proper characteristic subgroup and is thus elementary abelian; hence there exists no X-subgroup of G between $H_i$ and $H_{i+1}$. Therefore, since $H_i \leq (H_{i+1} \cap X)H_i \leq H_{i+1}$ and $(H_{i+1} \cap X)H_i$ is normalized by X for any X-composition factor $H_{i+1}/H_i$ of G, X either covers or avoids each X-composition factor of G.

We now divide the X-composition factors of G into two classes in a different way.

DEFINITION. An X-composition factor $H_{i+1}/H_i$ of G of order a power of prime p is said to be <u>$\mathfrak{F}$-central</u> if $\mathrm{Aut}_X(H_{i+1}/H_i) \in \mathfrak{F}(p)$ i.e. $X/C_X(H_{i+1}/H_i) \in \mathfrak{F}(p)$ -- otherwise <u>$\mathfrak{F}$-eccentric</u>.

In the case $X = G$, this definition agrees with Carter and Hawkes's definition of $\mathfrak{F}$-central and $\mathfrak{F}$-eccentric chief factors of G, and if $\mathfrak{F} = \mathfrak{N}$ it reduces to Carter's definition of central and eccentric X-composition factors. Furthermore, as in Carter and Hawkes's theory, this definition is independent of the particular integrated system of formations chosen to define $\mathfrak{F}$ locally. For Carter and Hawkes show that if $\mathfrak{F}$ is defined locally by $\{\mathfrak{F}_1(p)\}$ and by $\{\mathfrak{F}_2(p)\}$ with both the formations $\mathfrak{F}_1(p)$, $\mathfrak{F}_2(p) \leq \mathfrak{F}$ for each prime p, then

$\mathfrak{P}\,\mathfrak{F}_1(p) = \mathfrak{P}\,\mathfrak{F}_2(p)$ , where $\mathfrak{P}$ is the ~~set~~ class of all p-groups. This fact, together with the following lemma ( whose proof we include for the sake of completeness ) gives the desired result.

<u>LEMMA 2.1</u> Let $H/K$ be an X-composition factor of G of order $p^\alpha$, $C = C_X(H/K)$ and $N/C$ a minimal normal subgroup of $X/C$. Then $|N/C| = q^\beta$ for some prime $q \neq p$.

<u>Proof</u>. Suppose, if possible, that $|N/C| = p^\beta$. Now N acts on $H/K$ by the action $(hK, n) \to h^nK$ ( $h \in H$, $n \in N$ ). Split $H/K$ into transitive components $H_1/K, \ldots, H_r/K$. Then

(A) --- $|H/K| = |H_1/K| + |H_2/K| + \ldots + |H_r/K|$ .

Since $|N/C| = p^\beta$, $|H_i/K|$ is a power of p for each $i = 1,..,r$. This follows from the fact that, for $h_iK \in H_i/K$,

$|H_i/K| = |\,\mathrm{orb}(h_iK)| = |\,N : \mathrm{St}(h_iK)\,|$ , where

$\mathrm{St}(h_iK) = \{\, n \in N \mid h_i^nK = h_iK \,\} \geq C$ . Thus, assuming $H_1/K = K$, (A) yields $p^\alpha = 1 + p^{\alpha_2} + \ldots + p^{\alpha_r}$. This is clearly impossible unless $\alpha_i = 0$ for some $i \geq 2$. Hence there exists $h_i \in H$ such that $h_i \notin K$ and $h_i^n K = h_i K$ for all $n \in N$. i.e. $[h_i,N] \leq K$. We show that this is impossible by proving that $\{h \in H \mid [h,N] \leq K\} = K$. Let $Y = \{h \in H \mid [h,N] \leq K\}$. Then, since K is an X-subgroup of G, $H \geq Y \geq K$. Furthermore, Y is easily seen to be an X-subgroup of G, and is strictly less than H since $N > C = C_X(H/K)$. Thus $Y = K$ since $H/K$ is an X-composition factor of G. This contradiction shows that $|N/C|$ is not a power of p.

Now any two X-composition series of G have X-composition factors operator-isomorphic in pairs, by the operator form of the Jordan-Holder theorem. Suppose that $H/K$ and $J/L$ is one such pair of X-isomorphic X-composition factors of G. Then there is an isomorphism $\varphi : H/K \to J/L$ such that $[(hK)^x]\varphi = [(hK)\varphi]^x$ for all $h \in H$ and $x \in X$. It is easily seen that this yields $C_X(H/K) = C_X(J/L)$. Thus there exists a (1-1) correspondence between the factors of any two X-composition series of G, and in this correspondence an $\mathfrak{F}$-central factor of one corresponds to an $\mathfrak{F}$-central factor of the other. Therefore the product of the orders of the $\mathfrak{F}$-central factors in an X-composition series of G is independent of the particular series chosen -- we shall use this fact later.

There are thus four different types of X-composition factor: the $\mathfrak{F}$-central factors covered by X, the $\mathfrak{F}$-eccentric factors covered by X, the $\mathfrak{F}$-central factors avoided by X, and the $\mathfrak{F}$-eccentric factors avoided by X. We now show that the product of the orders of all the factors of the same type in a given X-composition series of G is independent of the particular series chosen. We do this by considering the chain

$$G = H_r X \geq H_{r-1} X \geq .. \geq H_0 X = X = X \cap H_r \geq X \cap H_{r-1} \geq .. \geq X \cap H_0 = 1$$

of subgroups of G passing through X and derived from an X-composition series

$$G = H_r > H_{r-1} > .... > H_0 = 1 \qquad \text{of } G .$$

Suppose first that X covers $H_{i+1}/H_i$. Then (see Carter, [3]) $H_{i+1} \cap X/H_i \cap X$ is a chief factor of X isomorphic to $H_{i+1}/H_i$ and it is easily verified that

$$C_X(H_{i+1}/H_i) = C_X(H_{i+1} \cap X/H_i \cap X) .$$

Thus in this case, $H_{i+1}/H_i$ is an $\mathfrak{F}$-central X-composition factor of G if and only if $H_{i+1} \cap X/H_i \cap X$ is an $\mathfrak{F}$-central chief factor of X.

Let $z^{\mathfrak{F}}(X)$ be the $\mathfrak{F}$-central order of X i.e. the product of the orders of the $\mathfrak{F}$-central chief factors in any chief series of X, and $\varpi^{\mathfrak{F}}(X)$ be the number of $\mathfrak{F}$-systems of X. Then $\varpi^{\mathfrak{F}}(X)$ equals the index in X of an $\mathfrak{F}$-normalizer of X i.e. the product of the orders of the $\mathfrak{F}$-eccentric chief factors in any chief series of X and so $z^{\mathfrak{F}}(X)\,\varpi^{\mathfrak{F}}(X) = |X|$.

We have therefore shown that the product of the orders of the $\mathfrak{F}$-central factors covered by X in an X-composition series is equal to $z^{\mathfrak{F}}(X)$, while the product of the orders of the $\mathfrak{F}$-eccentric factors covered by X in an X-composition series is equal to $\varpi^{\mathfrak{F}}(X)$. Thus both of these products depend only on

$\mathfrak{F}$ and the structure of X.

Now suppose that X avoids the X-composition factor $H_{i+1}/H_i$ of G. Then, as in [3], $H_iX$ is maximal in $H_{i+1}X$, of index $|H_{i+1}/H_i|$ . We show that in this case $H_{i+1}/H_i$ is an $\mathfrak{F}$-central X-composition factor if and only if $H_iX$ is $\mathfrak{F}$-normal in $H_{i+1}X$ . Since $H_{i+1}/H_i$ has no proper X-subgroup it is a chief factor of $XH_{i+1}$ with centralizer $C_{XH_{i+1}}(H_{i+1}/H_i) = H_{i+1}C_X(H_{i+1}/H_i)$ . Thus $XH_{i+1}/C_{XH_{i+1}}(H_{i+1}/H_i) \cong X/C_X(H_{i+1}/H_i)$ so that $H_{i+1}/H_i$ is an $\mathfrak{F}$-central X-composition factor if and only if $H_{i+1}/H_i$ is an $\mathfrak{F}$-central chief factor of $XH_{i+1}$, giving the required result.

Denote by $z_0^{\mathfrak{F}}(X)$ the product of the orders of the $\mathfrak{F}$-central factors avoided by X and by $\varpi_0^{\mathfrak{F}}(X)$ the product of the orders of the $\mathfrak{F}$-eccentric factors avoided by X in the given X-composition series of G. Then both these products are independent of the particular series chosen. For we have seen that the product of the orders of all the $\mathfrak{F}$-central factors in an X-composition series of G is independent of the particular series, and we have proved that this is also the case for the product of the orders of the $\mathfrak{F}$-central factors covered by X in an X-composition series. Thus $z_0^{\mathfrak{F}}(X)$ is independent of the particular series chosen. Hence, since $\varpi_0^{\mathfrak{F}}(X)z_0^{\mathfrak{F}}(X) = |G : X|$ , $\varpi_0^{\mathfrak{F}}(X)$ is also independent of the X-composition series chosen.

We have shown in addition that

$z_0^{\mathfrak{F}}(X)$ = the product of the indices of the $\mathfrak{F}$-normal links in a maximal chain of subgroups from X to G derived from an X-composition series of G, and

$\varpi_0^{\mathfrak{F}}(X)$ = the product of the indices of the $\mathfrak{F}$-abnormal links in a maximal chain of subgroups from X to G derived from an X-composition series of G.

Thus we have associated with any subgroup X of G the four invariants $z^{\mathfrak{F}}(X)$, $\varpi^{\mathfrak{F}}(X)$, $z_0^{\mathfrak{F}}(X)$, $\varpi_0^{\mathfrak{F}}(X)$. These invariants satisfy the relations

$$z^{\mathfrak{F}}(X)\,\varpi^{\mathfrak{F}}(X) = |X| , \quad z_0^{\mathfrak{F}}(X)\,\varpi_0^{\mathfrak{F}}(X) = |G : X| .$$

$z^{\mathfrak{F}}(X)$ is the $\mathfrak{F}$-central order of X and $\varpi^{\mathfrak{F}}(X)$ is the number of $\mathfrak{F}$-systems of X, so these two invariants depend only on $\mathfrak{F}$ and the structure of X whereas $z_0^{\mathfrak{F}}(X)$ and $\varpi_0^{\mathfrak{F}}(X)$ also depend on the way in which X is embedded in G.

We now seek interpretations of $\varpi_0^{\mathfrak{F}}(X)$ and $z_0^{\mathfrak{F}}(X)$ along the lines of those which Carter obtained for $\varpi_0(X)$ and $z_0(X)$ in [3]. Carter's interpretations involve the concept of a Sylow system of G reducing into a subgroup of G, and so we first develop a similar concept for an $\mathfrak{F}$-system of G. For this we require that the formations $\mathfrak{F}(p)$ should in addition be subgroup-closed; then $\mathfrak{F}$ too is subgroup-closed. Denote by H(p) the $\mathfrak{F}(p)$-residual of the subgroup H of G, for each prime p. Then, for any subgroup H of G, $H(p) \leq G(p)$ for each prime p. This fact justifies the following definition

<u>DEFINITION</u>. Let $\mathcal{T} = \{ T^p = S^p \cap G(p) \}$ be an $\mathfrak{F}$-system of G ( defined by Sylow system $\mathcal{S} = \{ S^p \}$ of G ) and H any sub-

group of G. We say that $\mathcal{T}$ reduces into ( an $\mathcal{F}$-system of ) H if $T^p \cap H(p)$ is a p-complement of $H(p)$ for each prime p dividing $|G|$ .

Since $H(p) \leq G(p)$ , every $\mathcal{F}$-system of the subgroup H of G does arise from an $\mathcal{F}$-system of G in this manner, as is to be expected. For suppose that $\mathcal{K}$ is an $\mathcal{F}$-system of H defined by Sylow system $\mathcal{H}$ of H. Extend $\mathcal{H}$ to a Sylow system $\mathcal{S}$ of G. Then the $\mathcal{F}$-system of G defined by $\mathcal{S}$ reduces into $\mathcal{K}$ . However, this need not be the case if $H(p) \not\leq G(p)$ ; hence the restriction imposed on $\mathcal{H}(p)$. Furthermore, in the case of $\mathcal{F} = \mathcal{N}$ , this definition specializes to the usual definition of a Sylow system of G reducing into a subgroup H of G.

We prove two results which we shall require later in the chapter. The first concerns the number of $\mathcal{F}$-systems reducing into a maximal subgroup of G.

LEMMA 2.2 Let M be a p-maximal subgroup of G. If M is $\mathcal{F}$-normal in G, every $\mathcal{F}$-system of G reduces into M. If M is $\mathcal{F}$-abnormal in G, $\mathcal{F}$-system $\mathcal{T} = \{ T^q \}$ of G reduces into M if and only if $T^p \leq M$ . Thus the number of $\mathcal{F}$-systems of G reducing into an $\mathcal{F}$-abnormal maximal subgroup M of G is equal to $\varpi^{\mathcal{F}}(G)/|G:M|$ ( where $\varpi^{\mathcal{F}}(G)$ is the number of $\mathcal{F}$-systems of G ).

Proof. Let $\mathcal{T}$ be defined by Sylow system $\mathcal{S} = \{ S^q \}$ of G -- then $T^q = S^q \cap G(q)$ for each prime q. It is easily seen that for each prime $q \neq p$ , $T^q \cap M(q)$ is a q-complement of M(q). For in this case, $S^q M = G$ and so $S^q \cap M$ is a q-complement of M. M(q) is normal in M and contained in G(q), and thus $T^q \cap M(q) = S^q \cap M(q)$ is a q-complement of $M(q)$. Hence it only remains to consider $T^p \cap M(p)$ .

Firstly, suppose that M is $\mathfrak{F}$-normal in G. Then the unique minimal normal subgroup N/CoreM of G/CoreM is $\mathfrak{F}$-central i.e. $G/N \in \mathfrak{F}(p)$ and so $G(p) \leq N$ . Now, by Carter and Hawkes, we can assume without loss of generality that $\mathfrak{F}(p) = \mathfrak{P}\,\mathfrak{F}(p)$ ( where $\mathfrak{P}$ is the set of all p-groups ). Thus we have $G(p) \leq \mathrm{Core}M$ . Hence $M(p) \lhd G(p)$ and so $T^p \cap M(p)$ is a p-complement of $M(p)$, as required. We have thus proved the first statement of the lemma.

We now assume that M is $\mathfrak{F}$-abnormal in G, and write $K = \mathrm{Core}M$. N/K is thus $\mathfrak{F}$-eccentric i.e. $G/N \notin \mathfrak{F}(p)$ and so $G(p) \not\leq N$ . Hence $G(p)K > N$ . We first show that $(G(p) \cap M)K = M(p)K$ . Since $M(p) \leq G(p) \cap M$ , we need only prove that $G(p) \cap M \leq M(p)K$ . Now, by definition of $M(p)$, $M/M(p)K \in \mathfrak{F}(p)$ and so $G/M(p)N \cong M/M(p)K \in \mathfrak{F}(p)$ . Thus $G(p) \leq M(p)N$ giving $G(p) \cap M \leq M(p)N \cap M = M(p)K$ , as required.

Suppose now that $\mathfrak{I}$ does reduce into M i.e. $T^p \cap M(p)$ is a p-complement of $M(p)$. Then $T^pK/K$, $(T^p \cap M(p))K/K$ are p-complements of $G(p)K/K$ , $M(p)K/K$ respectively. Further, since $(G(p) \cap M)K = M(p)K$ and $G(p)K > N$ , $|G(p)K : M(p)K| = |G(p)K : (G(p) \cap M)K| = |G:M|$ , a power of p. Hence $T^pK/K$ , $(T^p \cap M(p))K/K$ are both p-complements of $G(p)K/K$ , and so $T^pK = (T^p \cap M(p))K$ . Thus $T^p \leq M$ .

Conversely, let $T^p \leq M$ . Then $T^p$ is a p-complement of $G(p) \cap M$ and so $T^p \cap M(p)$ is a p-complement of the normal subgroup $M(p)$ of $G(p) \cap M$ .

Thus, if M is an $\mathfrak{F}$-abnormal maximal subgroup of G, the number of $\mathfrak{F}$-systems $\mathfrak{J} = \{ T^p \}$ of G reducing into M is the number of $\mathfrak{F}$-systems with $T^p \leq M$. Now $T^p, (T^p)^g \leq M$ for $g \in G$ implies that $N_G(T^p) \leq M$ since M is $\mathfrak{F}$-abnormal (as in Lemma 3.2 of [4]), and also that $T^p, (T^p)^g$ are p-complements of $G(p) \cap M$. Thus, by P.Hall, $(T^p)^g = (T^p)^m$ for some $m \in G(p) \cap M$. Then $gm^{-1} \in N_G(T^p) \leq M$, giving $g \in M$. Hence the number of distinct p-complements $T^p$ of $G(p)$ contained in M equals $|M : N_G(T^p)| = |G : N_G(T^p)|/|G : M|$. Since the number of distinct q-complements $T^q$ of $G(q)$ equals $|G : N_G(T^q)|$ for all q, the number of $\mathfrak{F}$-systems $\mathfrak{J}$ of G reducing into M equals $\prod_{q \mid |G|} |G : N_G(T^q)| / |G : M| = |G : N_G(\mathfrak{J})| / |G : M| = \varpi^{\mathfrak{F}}(G)/|G : M|$ where $N_G(\mathfrak{J})$ denotes the normalizer of the $\mathfrak{F}$-system $\mathfrak{J}$ of G --- an $\mathfrak{F}$-normalizer of G.

<u>LEMMA 2.3</u> Let $\mathfrak{J} = \{ T^p \}$ be an $\mathfrak{F}$-system of G reducing into X and $G = H_r > \ldots > H_0 = 1$ be an X-composition series of G. Then $\mathfrak{J}$ reduces into $XH_i$ for $i = 0,..,r$.

<u>Proof</u>. We use induction on i. The result is true for $i = 0$, since $XH_0 = X$. We thus assume the result for $i = k$ i.e. $\mathfrak{J}$ reduces into $Z_k = XH_k$. If X covers $H_{k+1}/H_k$, then $XH_{k+1} = XH_k$ and the result is true for $i = k+1$.

Thus we assume that $H_{k+1}/H_k$ is avoided by X and so $Z_k$ is maximal in $Z_{k+1} = XH_{k+1}$ and $H_{k+1} \cap Z_k = H_k$. We show that this implies

(A) --- $H_{k+1} Z_{k+1}(q) = H_{k+1} Z_k(q)$ for each prime q.

Since $\mathfrak{F}(q)$ is subgroup-closed, $Z_k(q) \leq Z_{k+1}(q)$, so it remains to prove that $Z_{k+1}(q) \leq H_{k+1} Z_k(q)$. Now $Z_k(q) \lhd$

$Z_k$ and $H_{k+1} \lhd Z_{k+1}$, so $H_{k+1} Z_k(q) \lhd H_{k+1} Z_k = Z_{k+1}$. Furthermore, since $H_{k+1} Z_k = Z_{k+1}$ and $H_{k+1} \cap Z_k = H_k$,

$$Z_{k+1}/H_{k+1} Z_k(q) \cong Z_k/H_k Z_k(q) \in \mathfrak{F}(q)$$

giving $Z_{k+1}(q) \leq H_{k+1} Z_k(q)$ as required.

We must prove that $T^q \cap Z_{k+1}(q)$ is a q-complement of $Z_{k+1}(q)$ i.e. $|Z_{k+1}(q) : T^q \cap Z_{k+1}(q)|$ is a power of q for each prime q. Let $Q = (T^q \cap Z_{k+1}(q))(H_{k+1} \cap Z_{k+1}(q))$. We show that $|Z_{k+1}(q) : Q|$ and $|Q : T^q \cap Z_{k+1}(q)|$ are powers of q. By induction $T^q \cap Z_k(q)$ is a q-complement of $Z_k(q)$, and so $H_{k+1}(T^q \cap Z_k(q))/H_{k+1}$ is a q-complement of $H_{k+1} Z_k(q)/H_{k+1} = H_{k+1} Z_{k+1}(q)/H_{k+1}$ (using (A) ). Therefore, since $Z_k(q) \leq Z_{k+1}(q)$, $H_{k+1}(T^q \cap Z_{k+1}(q))$ is of index a power of q in $H_{k+1} Z_{k+1}(q)$. Hence Q is of index a power of q in $Z_{k+1}(q)$. Furthermore, since $H_{k+1} \lhd\lhd G$, $T^q \cap H_{k+1}$ is a q-complement of $G(q) \cap H_{k+1}$. Thus, since $Z_{k+1}(q) \cap H_{k+1} \lhd G(q) \cap H_{k+1}$, $T^q \cap H_{k+1} \cap Z_{k+1}(q)$ is a q-complement of $H_{k+1} \cap Z_{k+1}(q)$. It follows immediately that $T^q \cap Z_{k+1}(q)$ is of index a power of q in Q, and we are done.

We now consider all possible maximal chains of subgroups joining X to G. Denote such a maximal chain by $\mathcal{C}$ and the product of the indices of the $\mathfrak{F}$-normal links in $\mathcal{C}$ by $\beta^{\mathfrak{F}}(\mathcal{C})$. We have seen that if the maximal chain $\mathcal{C}$ is derived from an X-composition series of G in the way described earlier, then $\beta^{\mathfrak{F}}(\mathcal{C}) = z_o^{\mathfrak{F}}(X)$, but this will in general not be true for every maximal chain from X to G. We show that $z_o^{\mathfrak{F}}(X)$ is, in fact, the greatest value taken by $\beta^{\mathfrak{F}}(\mathcal{C})$ for all maximal chains $\mathcal{C}$ joining X to G.

As in [3], we begin by obtaining the connection between $\beta^{\mathfrak{F}}(\mathcal{C})$ and the number of $\mathfrak{F}$-systems of G reducing into $\mathcal{C}$ i.e. reducing into every subgroup in $\mathcal{C}$.

<u>THEOREM 2.4</u> The number of $\mathfrak{F}$-systems of G reducing into the maximal chain $\mathcal{C}$ from X to G is equal to $\omega^{\mathfrak{F}}(G)\beta^{\mathfrak{F}}(\mathcal{C})/|G:X|$, where $\omega^{\mathfrak{F}}(G)$ is the number of $\mathfrak{F}$-systems of G.

<u>Proof</u>. By Lemma 2.2, the number of $\mathfrak{F}$-systems of G reducing into $\mathcal{C}$ is $\omega^{\mathfrak{F}}(G)/\prod$(indices of the $\mathfrak{F}$-abnormal links in $\mathcal{C}$) = $\omega^{\mathfrak{F}}(G)\beta^{\mathfrak{F}}(\mathcal{C})/|G:X|$.

This result is considerably simplified if X is an $\mathfrak{F}$-normalizer of G, for in this case $\omega^{\mathfrak{F}}(G) = |G:X|$. Thus the number of $\mathfrak{F}$-systems reducing into a maximal chain $\mathcal{C}$ joining an $\mathfrak{F}$-normalizer of G to G is simply $\beta^{\mathfrak{F}}(\mathcal{C})$, the product of the indices of the $\mathfrak{F}$-normal links in $\mathcal{C}$.

We are now in a position to give the required interpretation of $z_o^{\mathfrak{F}}(X)$.

<u>THEOREM 2.5</u> $z_o^{\mathfrak{F}}(X)$ is the greatest value taken by $\beta^{\mathfrak{F}}(\mathcal{C})$ for all maximal chains $\mathcal{C}$ joining X to G.

<u>Proof</u>. By Theorem 2.4,

$\beta^{\mathfrak{F}}(\mathcal{C})$ = (number of $\mathfrak{F}$-systems reducing into $\mathcal{C}$)$|G:X|/\omega^{\mathfrak{F}}(G)$.

Thus the chains $\mathcal{C}$ for which $\beta^{\mathfrak{F}}(\mathcal{C})$ is as large as possible are those into which as many $\mathfrak{F}$-systems as possible reduce. Denote by $\sigma^{\mathfrak{F}}(X)$ the number of $\mathfrak{F}$-systems of G reducible into X. Then certainly the number of $\mathfrak{F}$-systems of G reducing into

$\mathcal{L}$ is at most $\sigma^{\mathfrak{F}}(X)$. However, if $\mathcal{L}$ is of the type derived from an X-composition series of G, every $\mathfrak{F}$-system of G reducing into X also reduces into $\mathcal{L}$ by Lemma 2.3. Hence, for this type of chain, $\beta^{\mathfrak{F}}(\mathcal{L})$ attains its maximal value viz. $z_o^{\mathfrak{F}}(X)$.

<u>COROLLARY</u> $\varpi_o^{\mathfrak{F}}(X) = \varpi^{\mathfrak{F}}(G)/\sigma^{\mathfrak{F}}(X)$ , where $\varpi^{\mathfrak{F}}(G)$ is the number of $\mathfrak{F}$-systems of G, and $\sigma^{\mathfrak{F}}(X)$ is the number of $\mathfrak{F}$-systems of G which reduce into X.

<u>Proof</u>. Let $\mathcal{L}$ be a maximal chain joining X to G obtained from an X-composition series of G. Then $\beta^{\mathfrak{F}}(\mathcal{L}) = z_o^{\mathfrak{F}}(X)$ and the number of $\mathfrak{F}$-systems of G reducible into $\mathcal{L}$ is $\sigma^{\mathfrak{F}}(X)$, as we have seen. Thus, by Theorem 2.4, $\sigma^{\mathfrak{F}}(X) = \varpi^{\mathfrak{F}}(G)z_o^{\mathfrak{F}}(X)/|G:X| = \varpi^{\mathfrak{F}}(G)/\varpi_o^{\mathfrak{F}}(X)$ , since $|G:X| = z_o^{\mathfrak{F}}(X)\,\varpi_o^{\mathfrak{F}}(X)$ . Hence $\varpi_o^{\mathfrak{F}}(X) = \varpi^{\mathfrak{F}}(G)/\sigma^{\mathfrak{F}}(X)$ , as required.

Thus, providing that all the formations $\mathfrak{F}(p)$ are subgroup-closed, Carter's interpretations of $z_o(X)$ and $\varpi_o(X)$ may be generalized to the present situation.

We close with a brief mention of the subgroups X for which $z_o^{\mathfrak{F}}(X)$ takes its extreme values. In [3], Carter characterizes the abnormal subgroups of G as those subgroups X of G which satisfy $z_o(X) = 1$ . In line with this, we say that a subgroup X of G is $\mathfrak{F}$-abnormal in G if $z_o^{\mathfrak{F}}(X) = 1$ --- this means that every link in every maximal chain from X to G is $\mathfrak{F}$-abnormal. By Lemma 5.1 of [4], the $\mathfrak{F}$-covering subgroups E of G are $\mathfrak{F}$-abnormal in G, and may be characterized by the equations $\omega^{\mathfrak{F}}(E) = 1$ , $z_0^{\mathfrak{F}}(E) = 1$ ; for $\omega^{\mathfrak{F}}(E) = 1$ means that $E \in \mathfrak{F}$. On the other hand, $z_0^{\mathfrak{F}}(X) = |G:X|$ if and only if there is some maximal chain from X to G in which every link is $\mathfrak{F}$-normal --- we say that X is $\mathfrak{F}$-subnormal in G in this case. Since $z_0^{\mathfrak{F}}(X)\, \omega_0^{\mathfrak{F}}(X) = |G:X|$ , this implies that $\omega_0^{\mathfrak{F}}(X) = 1$ which means that every $\mathfrak{F}$-system of G reduces into X. Therefore, X is $\mathfrak{F}$-subnormal in G if and only if every $\mathfrak{F}$-system of G reduces into X.

# Chapter Three

## NORMAL SYSTEMS AND RELATED CONCEPTS

DEFINITIONS. A normal system $\mathfrak{X} = \{X(p)\}$ of a group G is a set of normal subgroups $X(p)$ of G, one for each prime p dividing $|G|$. Let G be a group with normal system $\mathfrak{X} = \{X(p)\}$. We say that a p-chief factor H/K of G is $\mathfrak{X}$-central if $X(p) \leq C_G(H/K)$, and $\mathfrak{X}$-eccentric otherwise. A p-maximal subgroup M of G is said to be $\mathfrak{X}$-normal if $X(p) \leq N$, where N/CoreM is the unique minimal normal subgroup of G/CoreM --- otherwise $\mathfrak{X}$-abnormal. Thus M is $\mathfrak{X}$-normal if and only if N/CoreM is $\mathfrak{X}$-central. Since any chief factor of G complemented by M is operator-isomorphic to N/CoreM, M is $\mathfrak{X}$-normal in G if and only if it complements an $\mathfrak{X}$-central chief factor of G.

Let H be a subgroup of G. We denote by $\mathfrak{X} \cap H$ the normal system of H obtained by intersecting with H those $X(p)$ for which p divides $|H|$. We now consider the set $\bar{\mathfrak{X}}$ consisting of all sections H/K of G such that all chief factors of H above K are $\mathfrak{X} \cap H$-central. This important set to a large extent takes the place of the formation $\mathfrak{F}$ in Carter and Hawkes's theory, though differing from $\mathfrak{F}$ in that it is not an isomorphism class and is defined entirely within the given group G. In fact, $\bar{\mathfrak{X}}$ is not even closed under isomorphisms within G, as is shown by the following example.

EXAMPLE 3.1 Let G be the direct product of two copies of $\Sigma_3$, the symmetric group on 3 elements --- say $G = \Sigma_3 \times \bar{\Sigma}_3$ . Let $S_3$, $\bar{S}_3$ be the Sylow 3-subgroup of $\Sigma_3$, $\bar{\Sigma}_3$ respectively. Take $X(2) = G$, and $X(3) = \bar{S}_3\Sigma_3$ . Then $G/\bar{\Sigma}_3 \notin \bar{\mathfrak{X}}$ since $S_3\bar{\Sigma}_3/\bar{\Sigma}_3$ is $\mathfrak{X}$-eccentric. The diagonal subgroup $\Sigma^* = \{ \sigma\bar{\sigma} \mid \sigma \in \Sigma_3\}$ of G belongs to $\bar{\mathfrak{X}}$ since $X(3) \cap \Sigma^*$ is the Sylow 3-subgroup of $\Sigma^*$; but $\bar{\Sigma}_3\Sigma^*/\bar{\Sigma}_3 = G/\bar{\Sigma}_3 \notin \bar{\mathfrak{X}}$.

However, in the situation $K_0 \lhd K \leq G$, $H_0 \lhd H \leq G$ with $H_0K = H$ and $H_0 \cap K = K_0$, $H/H_0 \in \bar{\mathfrak{X}}$ clearly implies $K/K_0 \in \bar{\mathfrak{X}}$ ; although the converse may not be true. We use this fact frequently, especially in chapter five.

<u>LEMMA 3.2</u> $\bar{\mathfrak{X}}$ has the following properties

(i) $H/K \in \bar{\mathfrak{X}}$ , $K \leq K_1 \lhd H$ implies $H/K_1 \in \bar{\mathfrak{X}}$ .

(ii) $H/K_1$, $H/K_2 \in \bar{\mathfrak{X}}$ implies $H/K_1 \cap K_2 \in \bar{\mathfrak{X}}$ and thus H possesses an $\bar{\mathfrak{X}}$-residual.

(iii) $H \in \bar{\mathfrak{X}}$ if and only if $H/\Phi(H) \in \bar{\mathfrak{X}}$ .

(iv) $H \in \bar{\mathfrak{X}}$ , $H_1 \leq H$ implies $H_1 \in \bar{\mathfrak{X}}$ .

(v) $\bar{\mathfrak{X}} = \{ H/K \mid K \lhd H \leq G,\ K(X(p) \cap H)/K$ is p-nilpotent for each prime p $\}$.

<u>Proof</u>. (i) is obvious.

(ii) Since $H/K_1 \in \bar{\mathfrak{X}}$ , each chief factor of H above $K_1$ is $\mathfrak{X} \cap H$ -central. Further, each chief factor of H between $K_1 \cap K_2$ and $K_1$ is operator-isomorphic to one between $K_2$ and $K_1K_2$ and is thus $\mathfrak{X} \cap H$ -central, since $H/K_2 \in \bar{\mathfrak{X}}$ . The result then follows, since any chief factor of H above $K_1 \cap K_2$ is operator-isomorphic to one in a chief series of H through $K_1$.

(iii) Certainly $H \in \bar{\mathfrak{X}}$ implies that $H/\Phi(H) \in \bar{\mathfrak{X}}$ .

Conversely, suppose that $H \notin \bar{\mathfrak{X}}$. Then H has an $\mathfrak{X} \cap H$ -eccentric p-chief factor for some prime p and thus one between $\bar{\phi}_p(H)$ and $O_{p'p}(H)$, since $O_{p'p}(H)$ is in fact the intersection of the centralizers of the chief factors of H between $\bar{\phi}_p(H)$ and $O_{p'p}(H)$ (all of which are p-chief factors; see [4], page 180) Hence $H/\bar{\phi}_p(H) \notin \bar{\mathfrak{X}}$ and thus $H/\phi(H) \notin \bar{\mathfrak{X}}$, since $\phi(H) \leq \phi_p(H)$.

(iv) $H \in \bar{\mathfrak{X}}$ implies $X(p) \cap H \leq O_{p'p}(H)$ for each prime p dividing $|H|$. Thus $X(p) \cap H_1 \leq O_{p'p}(H) \cap H_1 \leq O_{p'p}(H_1)$, giving $H_1 \in \bar{\mathfrak{X}}$.

(v) is obvious, since K centralizes all the chief factors of H above itself.

## Chapter Four

# $\mathfrak{X}$ - NORMALIZERS

DEFINITION. Let $\mathfrak{X} = \{X(p)\}$ be a normal system of G and $\mathcal{S} = \{ S^p \}$ a Sylow system of G. Set $X^p = S^p \cap X(p)$, a p-complement of $X(p)$, for each prime p. Then $\{ X^p \}$ is called an $\mathfrak{X}$-system of G. Since any two Sylow systems of G are conjugate ( P.Hall [9] ), so are any two $\mathfrak{X}$-systems of G. We call

$D^{\mathfrak{X}}(G) = \bigcap_{p \mid |G|} N_G(X^p)$ an $\mathfrak{X}$-system normalizer of G, or, simply, an $\mathfrak{X}$-normalizer of G. Since any two $\mathfrak{X}$-systems of G are conjugate, so are any two $\mathfrak{X}$-normalizers of G.

LEMMA 4.1 $H \in \bar{\mathfrak{X}}$ if and only if $D^{\mathfrak{X} \cap H}(H) = H$ .

Proof. First let $H \in \bar{\mathfrak{X}}$ . Then, for each prime p dividing $|H|$, $X(p) \cap H$ is p-nilpotent and thus has a normal p-complement $(X(p) \cap H)^p$. Then $(X(p) \cap H)^p$ is a characteristic subgroup of the normal subgroup $X(p) \cap H$ of H and so is normal in H. Thus

$$D^{\mathfrak{X} \cap H}(H) = \bigcap_{p \mid |H|} N_H( (X(p) \cap H)^p ) = H .$$

Conversely, $D^{\mathfrak{X} \cap H}(H) = H$ implies that $X(p) \cap H$ has a normal p-complement for each prime p dividing $|H|$. Thus $X(p) \cap H$ is p-nilpotent and so centralizes all the p-chief factors of H. Hence $H \in \bar{\mathfrak{X}}$ .

The following theorem shows the relation between the $\mathfrak{X}$-normalizers of G and the chief factors of G.

THEOREM 4.2 Let D be an $\mathfrak{X}$-normalizer of G. Then D covers each $\mathfrak{X}$-central chief factor of G and avoids each $\mathfrak{X}$-eccentric chief factor of G. The order of D is the product of the orders of the $\mathfrak{X}$-central chief factors in a chief series of G.

Proof. By P.Hall ([10]), $N_G(X^p)$ avoids each p-chief factor of G which is eccentric for $X(p)$ i.e. each $\mathfrak{X}$-eccentric p-chief factor of G, and covers all the remaining chief factors of G. Thus $|G : N_G(X^p)|$ is the product of the orders of the $\mathfrak{X}$-eccentric p-chief factors in a chief series of G. Since $D = \bigcap_{p \mid |G|} N_G(X^p)$ it avoids each p-chief factor which $N_G(X^p)$ avoids and thus each $\mathfrak{X}$-eccentric chief factor of G. Also, since $|G : N_G(X^p)|$ is a power of p, D is an intersection of subgroups of coprime index. Hence $|G:D| = \prod_{p \mid |G|} |G : N_G(X^p)|$ and so $|D|$ is the product of the orders of the $\mathfrak{X}$-central chief factors in a chief series of G. Thus, by considerations of order, D must cover each $\mathfrak{X}$-central chief factor of G.

COROLLARY 1. Every $\mathfrak{X}$-normalizer of G lies in $\bar{\mathfrak{X}}$.

Proof. Let D be the $\mathfrak{X}$-normalizer of G corresponding to Sylow system $\mathcal{S}$ of G. Then $\mathcal{S}$ reduces into D. For $|N_G(X^p) : D|$ is prime to p and $|N_G(X^p) : S^p|$ is a power of p, so that $S^pD = N_G(X^p)$ for each prime p. Thus $X^p \cap D = (S^p \cap D) \cap (X(p) \cap D)$ is a p-complement of $X(p) \cap D$ and is normal in D. Hence $X(p) \cap D$ is p-nilpotent for each prime p, and the result follows.

COROLLARY 2. If D is an $\mathfrak{X}$-normalizer of G and $N \lhd G$, then $ND/N$ is an $N\mathfrak{X}/N$-normalizer of $G/N$ (where $N\mathfrak{X}/N = \{NX(p)/N \mid X(p) \in \mathfrak{X}, \text{ p divides } |G:N| \}$). Thus the $\mathfrak{X}$-normalizers of G are invariant under homomorphisms.

Proof. Let $X^p$ be a p-complement of $X(p)$. Then $NX^p/N$ is a p-complement of $NX(p)/N$ and $N\,N_G(X^p)/N \leq N_{G/N}(NX^p/N)$ . Hence $ND/N \leq \bigcap_{p \mid |G:N|} N_{G/N}(NX^p/N) = \bar{D}$ , an $N\mathfrak{X}/N$ -normalizer of $G/N$.

Consider a chief series of G through N. Then by Theorem 4.2,

$|\bar{D}|$ = the product of the orders of the $\mathfrak{X}$-central chief factors above N in this chief series.

Further, $N \cap D$ covers the $\mathfrak{X}$-central chief factors of G below N and avoids the $\mathfrak{X}$-eccentric chief factors of G below N since D does. Hence

$|N \cap D|$ = the product of the orders of the $\mathfrak{X}$-central chief factors below N in this chief series.

Thus $|D| = |\bar{D}|.|N \cap D|$ and so $|ND/N| = |D/D \cap N| = |\bar{D}|$ and the result follows.

Remark. The intersection of an $\mathfrak{X}$-normalizer D of G with a chief series of G need not be a chief series of D as is shown by the following simple example :

Take $G = \Sigma_4$ , the symmetric group on 4 elements. Set $X(2) = V$, the normal subgroup of G of order 4, and $X(3) = G$. Then the $\mathfrak{X}$-normalizers of G are the Sylow 2-subgroups $S_2$ of G, and $V/1$ is not a chief factor of an $S_2$.

We return to this question later when we consider integrated normal systems.

We now turn to the relation of the $\mathfrak{X}$-normalizers to the maximal subgroups of G.

THEOREM 4.3 A maximal subgroup M of G contains an $\mathfrak{X}$-normalizer of G if and only if M is $\mathfrak{X}$-abnormal. In this case, every $\mathfrak{X} \cap M$ -normalizer of M contains an $\mathfrak{X}$-normalizer of G.

Proof. Let N/CoreM be the unique minimal normal subgroup of G/CoreM, and $|G:M|$ be a power of p. $M \geq D^{\mathfrak{X}}$ implies that $D^{\mathfrak{X}}$ avoids N/CoreM which is thus $\mathfrak{X}$-eccentric by Theorem 4.2. Hence M is $\mathfrak{X}$-abnormal.

Now let M be $\mathfrak{X}$-abnormal, $S^p$ be a p-complement of G contained in M, and $X^p = S^p \cap X(p)$. Then, as in [4], 3.2, with $C_p$ replaced by X(p), $N_G(X^p) \leq M$ and so M contains an $\mathfrak{X}$-normalizer of G.

Finally, let M be $\mathfrak{X}$-abnormal, $\mathfrak{X} \cap M = \{Y(q)\}$ and $\{Y^q = M^q \cap Y(q)\}$ be any $\mathfrak{X} \cap M$ -system of M. ($M^q$ a q-complement of M). If $p \nmid |M|$, define $M^p = M$. Then $M^p$ is a p-complement of G, and setting $X^p = M^p \cap X(p)$, we have $N_G(X^p) \leq M$. For $q \neq p$, $M^q = S^q \cap M$ for some q-complement $S^q$ of G. Hence, for all q dividing $|M|$, $Y^q = X^q \cap Y(q)$ ( where $X^q = S^q \cap X(q)$ ) and thus $N_M(X^q) \leq N_M(Y^q)$. Thus

$$D^{\mathfrak{X}} = \bigcap_{q \mid |G|} N_G(X^q) = \bigcap_{q \mid |G|} N_M(X^q) \leq \bigcap_{q \mid |M|} N_M(Y^q) = D^{\mathfrak{X} \cap M}(M).$$

We show that certain $\mathfrak{X}$-abnormal maximal subgroups M of G will yield equality in the last statement of this theorem.

DEFINITION. We say that M is an $\mathfrak{X}$-critical maximal subgroup of G if M is $\mathfrak{X}$-abnormal and $M\,F(X) = G$, where F(X) is the Fitting subgroup of $X = \bigcap_p X(p)$.

In the case $X(p) = C_p(G)$ for each prime p ([4],3), this last condition reduces to $M\,F(G) = G$.

LEMMA 4.4 Let L be a subgroup of G satisfying $L.F(X) = G$ . Then L covers or avoids each chief factor of G. If $H/K$ is a chief factor of G covered by L, then

(i) $H \cap L / K \cap L$ is a chief factor of L isomorphic to H/K and with $C_G(H/K) \cap L = C_L(H \cap L / K \cap L)$ .

(ii) $H \cap L / K \cap L$ is $\mathfrak{X} \cap L$ -central in L if and only if $H/K$ is $\mathfrak{X}$-central in G.

Proof. By Lemma 1.1, we need only prove (ii). Let $H \cap L / K \cap L$ be $\mathfrak{X} \cap L$ -central. Then $X(p) \cap L \leq C_L(H \cap L / K \cap L) = C_G(H/K) \cap L$ by (i), and hence $X(p) = X(p) \cap L.F(X) = F(X)(X(p) \cap L) \leq C_G(H/K)$ i.e. H/K is $\mathfrak{X}$-central. The converse is clearly true.

THEOREM 4.5 An $\mathfrak{X} \cap M$ -normalizer of the $\mathfrak{X}$-critical maximal subgroup M of G is an $\mathfrak{X}$-normalizer of G.

Proof. By Theorem 4.3 we have only to show that $|D^{\mathfrak{X} \cap M}(M)| \leq |D^{\mathfrak{X}}(G)|$. This follows immediately from Theorem 4.2 and Lemma 4 .4.

However, in general G need not possess an $\mathfrak{X}$-critical maximal subgroup. For example, Taking $G = \Sigma_4$ , $X(2) = V$ and $X(3) = G$ as before, the only $\mathfrak{X}$-abnormal maximal subgroups of G are the Sylow 2-subgroups which do not supplement $F(X) = V$ in G.

Remark. The proof of Theorem 4.5 would go through if instead we defined X to be the intersection of those X(p) which do not satisfy $X(p) \leq O_{p'p}(G)$ . However, as the above counterexample shows, an $\mathfrak{X}$-abnormal maximal subgroup M satisfying $M.F(X) = G$ still need not exist.

We show later that $\mathfrak{X}$-critical maximal subgroups of G exist if $\mathfrak{X}$ is an integrated normal system, and thus that in this case $D^{\mathfrak{X}}$ can be connected to G by an $\mathfrak{X}$-critical maximal chain i.e. a chain $H = H_0 < H_1 < \dots < H_r = G$ of subgroups of G satisfying the condition that $H_i$ is an $\mathfrak{X} \cap H_{i+1}$ - critical maximal subgroup of $H_{i+1}$ for each i. However, a similar embedding result does in fact hold for any normal system $\mathfrak{X}$ of G.

DEFINITIONS. (a) Let $\mathfrak{X} = \{X(p)\}$ and $\mathfrak{Y} = \{Y(p)\}$ be two normal systems of G. We say that $\mathfrak{X} \geq \mathfrak{Y}$ if $X(p) \geq Y(p)$ for all primes p. If there is a prime q such that $X(p) = Y(p)$ for all primes $p \neq q$, and $X(q)/Y(q)$ is a chief factor of G, we call $\mathfrak{X}$ and $\mathfrak{Y}$ consecutive normal systems of G.

Clearly, if $D^{\mathfrak{X}}$ and $D^{\mathfrak{Y}}$ are $\mathfrak{X}$- and $\mathfrak{Y}$-normalizers of G respectively, both corresponding to the same Sylow system of G, then $\mathfrak{X} \geq \mathfrak{Y}$ implies $D^{\mathfrak{X}} \leq D^{\mathfrak{Y}}$.

(b) We say that the chain $H = H_0 < H_1 < .. < H_r = G$ of subgroups of G is $\mathfrak{X}$-abnormal maximal if $H_i$ is an $\mathfrak{X} \cap H_{i+1}$-abnormal maximal subgroup of $H_{i+1}$ for each i. H is then called an $\mathfrak{X}$-subabnormal subgroup of G.

LEMMA 4.6 Let $\mathfrak{X} = \{X(p)\}$ be a normal system of G and $X^p$ a p-complement of X(p). Then $N_G(X^p)$ is $\mathfrak{X}$-subabnormal in G.

Proof. We use induction on $|G : N_G(X^p)|$. There is nothing to prove if X(p) is p-nilpotent. Thus assume $X(p) \not\leq O_{p'p}(G)$ and hence $G > H = N_G(X^p)$ . Let M be a maximal subgroup of G containing H. Then M is $\mathfrak{X}$-abnormal in G by Theorem 4.3, and the result follows if $H = M$. If $H < M$, $\mathfrak{X} \cap M$ is a normal system of M containing $X(p) \cap M$. Now $X^p \leq X(p)$, M and thus $X^p$ is a p-complement of $X(p) \cap M$. Thus, by induction, $H = N_M(X^p)$ is $\mathfrak{X} \cap M$ -subabnormal in M and thus $\mathfrak{X}$-subabnormal in G.

<u>LEMMA 4.7</u> Let $\mathfrak{X} = \{X(p)\}$ and $\mathcal{Y} = \{Y(p)\}$ be consecutive normal systems of G such that $X(q) = Y(q)$ for all primes $q \neq p$, $X(p)/Y(p)$ is a chief factor of G and $D^{\mathfrak{X}} < D^{\mathcal{Y}}$. Then, if $D^{\mathfrak{X}}$ covers $X(p)/Y(p)$, $D^{\mathfrak{X}}$ is an $\mathfrak{X} \cap D^{\mathcal{Y}}$-normalizer of $D^{\mathcal{Y}}$. In fact, $D^{\mathfrak{X}}$ is the normalizer in $D^{\mathcal{Y}}$ of a p-complement of a normal subgroup of $D^{\mathcal{Y}}$ and is thus $\mathfrak{X} \cap D^{\mathcal{Y}}$-subabnormal in $D^{\mathcal{Y}}$. If $D^{\mathfrak{X}}$ avoids $X(p)/Y(p)$, $D^{\mathfrak{X}}$ is not an $\mathfrak{X} \cap D^{\mathcal{Y}}$-normalizer of $D^{\mathcal{Y}}$.

<u>Proof</u>. Let $\mathcal{S} = \{ S^q \}$ be the Sylow system of G defining $D^{\mathfrak{X}}$ and $D^{\mathcal{Y}}$ i.e. $D^{\mathfrak{X}} = \bigcap_{q \mid |G|} N_G(X^q)$ , $D^{\mathcal{Y}} = \bigcap_{q \mid |G|} N_G(Y^q)$ where $X^q = X(q) \cap S^q$, $Y^q = Y(q) \cap S^q$, and set $Z(q) = X(q) \cap D^{\mathcal{Y}}$ and $Z^q = Z(q) \cap S^q$. Then $\mathcal{Z} = \{Z(q)\}$ is a normal system of $D^{\mathcal{Y}}$ and $Z^q$ is a q-complement of $Z(q)$, since $\mathcal{S}$ reduces into $D^{\mathcal{Y}}$ and $Z(q) \lhd D^{\mathcal{Y}}$. Since $X(q) = Y(q)$ for all primes $q \neq p$, $D^{\mathfrak{X}}$ and $D^{\mathcal{Y}}$ cover and avoid the same q-chief factors of G for $q \neq p$ and thus $|D^{\mathcal{Y}} : D^{\mathfrak{X}}|$ is a power of p. In addition, $D^{\mathfrak{X}} = N_{D^{\mathcal{Y}}}(X^p)$. Also, $Z(q)$ is a q-nilpotent normal subgroup of $D^{\mathcal{Y}}$ for $q \neq p$, and thus $N_{D^{\mathcal{Y}}}(Z^p)$ is an $\mathfrak{X} \cap D^{\mathcal{Y}}$-normalizer of $D^{\mathcal{Y}}$.

We show that if $D^{\mathfrak{X}}$ covers $X(p)/Y(p)$, then $X^p = Z^pY^p$ and thus $N_{D^{\mathcal{Y}}}(Z^p) = N_{D^{\mathcal{Y}}}(X^p)$. Since $D^{\mathcal{Y}} > D^{\mathfrak{X}}$, $D^{\mathcal{Y}}$ covers $X(p)/Y(p)$ i.e. $X(p) = (D^{\mathcal{Y}} \cap X(p))Y(p) = Z(p)Y(p)$ . Now $Z^pY^p$ is a subgroup of $X^p$ since $Z^p$ normalizes $Y^p$, and

$$|X(p) : Z^pY^p| = |Z(p)Y(p) : Z^pY(p)|.|Z^pY(p) : Z^pY^p|$$
$$= |Z(p) : Z(p) \cap Z^pY(p)|.|Y(p) : Y(p) \cap Z^pY^p|$$

which is a power of p. Hence $X^p = Z^pY^p$ and thus $N_{D^{\mathcal{Y}}}(Z^p) \leq N_{D^{\mathcal{Y}}}(X^p)$, since $D^{\mathcal{Y}}$ normalizes $Y^p$. The converse inequality is certainly true. Thus $D^{\mathcal{X}}$ is an $\mathcal{X} \cap D^{\mathcal{Y}}$-normalizer of $D^{\mathcal{Y}}$ and is $\mathcal{X} \cap D^{\mathcal{Y}}$-subabnormal in $D^{\mathcal{Y}}$ by Lemma 4.6.

Now let $D^{\mathcal{X}}$ avoid $X(p)/Y(p)$. We show that in this case $N_{D^{\mathcal{Y}}}(Z^p) = D^{\mathcal{Y}} \neq D^{\mathcal{X}}$. Let H/K be a p-chief factor of G covered by $D^{\mathcal{Y}}$ and avoided by $D^{\mathcal{X}}$. Then $Y(p) \leq X(p) \cap C_G(H/K) < X(p)$ and so $Y(p) = X(p) \cap C_G(H/K)$, since $X(p)/Y(p)$ is a chief factor of G. Hence $X(p)/Y(p)$ must be a q-chief factor of G for $q \neq p$, since $G/C_G(H/K)$ has no minimal normal subgroups of order a power of p. Thus, since $D^{\mathcal{X}}$ avoids $X(p)/Y(p)$, so does $D^{\mathcal{Y}}$. Hence $Z(p) = D^{\mathcal{Y}} \cap X(p) = D^{\mathcal{Y}} \cap Y(p)$ which is a p-nilpotent normal subgroup of $D^{\mathcal{Y}}$, so that $Z^p$ is characteristic in Z(p) and thus normal in $D^{\mathcal{Y}}$.

<u>LEMMA 4.8</u> Let $\mathcal{X} = \{X(p)\}$ be any normal system of G and D an $\mathcal{X}$-normalizer of G. Then there exists a prime p and a chief factor $X(p)/Y(p)$ of G covered by D.

<u>Proof</u>. If $X(p) = X$ for all primes p, then any chief factor X/Y of G is centralized by X and thus covered by D. Thus assume all X(p) are not equal, and let $Z = \prod_q X(q)$. Then there exist primes $p, q_1, \dots, q_r$ ( $r \geq 1$ ) such that

$$Z = X(p).\bar{Z} \quad \text{and} \quad Z > \bar{Z} = \prod_{i=1}^{r} X(q_i) .$$

Since $Z > \bar{Z}$, $X(p) \cap \bar{Z} < X(p)$ and thus we can choose $Y(p) \lhd$

$G$ such that $X(p)/Y(p)$ is a chief factor of $G$ and $Y(p) \geq X(p) \cap \bar{Z}$. Then $X(p)/Y(p)$ is operator-isomorphic to chief factor $Z/Y(p)\bar{Z}$ which is centralized by $Z$ and thus by all $X(q)$. Hence $X(p)/Y(p)$ is centralized by $X(q)$ for all primes $q$ and is thus covered by $D$.

<u>THEOREM 4.9</u> $D^{\mathfrak{X}}$ is $\mathfrak{X}$-subabnormal in $G$.

<u>Proof</u>. We use induction on $|G:D^{\mathfrak{X}}|$. There is nothing to prove for $D^{\mathfrak{X}} = G$. Thus assume $D^{\mathfrak{X}} < G$ and let $\mathfrak{Y} = \{Y(p)\}$ be a normal system minimal with respect to the conditions $\mathfrak{Y} \leq \mathfrak{X}$ and $D^{\mathfrak{Y}} = D^{\mathfrak{X}}$. By Lemma 4.8 there exists a prime $p$ and chief factor $Y(p)/Z(p)$ of $G$ covered by $D^{\mathfrak{Y}}$. Denote by $\mathfrak{Z}$ the consecutive normal system obtained by setting $Z(q) = Y(q)$ for all primes $q \neq p$. Then, by the definition of $\mathfrak{Y}$, $D^{\mathfrak{Z}} > D^{\mathfrak{Y}}$. Hence $D^{\mathfrak{Y}}$ is $\mathfrak{Y} \cap D^{\mathfrak{Z}}$-subabnormal in $D^{\mathfrak{Z}}$ by Lemma 4.7. $D^{\mathfrak{Z}}$ is $\mathfrak{Z}$-subabnormal in $G$ by induction, and thus $D^{\mathfrak{X}} = D^{\mathfrak{Y}}$ is $\mathfrak{X}$-subabnormal in $G$, since $\mathfrak{X} \geq \mathfrak{Y} > \mathfrak{Z}$.

This gives us a characterization of the $\mathfrak{X}$-normalizers of $G$ corresponding to [4], 4.8, for $\mathfrak{F}$-normalizers.

<u>THEOREM 4.10</u> Let $\mathfrak{M}$ be the set of all $\mathfrak{X}$-subabnormal subgroups of $G$. Then the $\mathfrak{X}$-normalizers of $G$ are the minimal members of $\mathfrak{M}$.

<u>Proof</u>. By repeated application of Theorem 4.3, every $\mathfrak{X}$-subabnormal subgroup of $G$ contains an $\mathfrak{X}$-normalizer of $G$. ( In fact, if $H$ is $\mathfrak{X}$-subabnormal in $G$, every $\mathfrak{X} \cap H$-normalizer of $H$ contains an $\mathfrak{X}$-normalizer of $G$.) The result follows by Theorem4.9.

Before turning to integrated normal systems, we give a necessary and sufficient condition for the $\mathfrak{X}$- and $\mathfrak{Y}$-normalizers of G to coincide, for $\mathfrak{X}, \mathfrak{Y}$ any two normal systems of G.

<u>LEMMA 4.11</u> Let $\mathfrak{X} = \{X(p)\}$, $\mathfrak{Y} = \{Y(p)\}$ be two normal systems of G. Then the $\mathfrak{X}$- and $\mathfrak{Y}$-normalizers of G coincide if and only if X(p) and Y(p) centralize the same p-chief factors of G for each prime p.

<u>Proof</u>. If the $\mathfrak{X}$- and $\mathfrak{Y}$-normalizers coincide, the result follows by Theorem 4.2. Conversely, define $Z(p) = X(p)Y(p)$ for each prime p. Then $Z(p) \geq X(p)$ and thus $D^{\mathfrak{Z}} \leq D^{\mathfrak{X}}$, where $D^{\mathfrak{Z}}$ and $D^{\mathfrak{X}}$ correspond to the same Sylow system of G. But X(p) and Z(p) centralize the same p-chief factors for each prime p, and thus, by Theorem 4.2, $|D^{\mathfrak{Z}}| = |D^{\mathfrak{X}}|$. Hence $D^{\mathfrak{Z}} = D^{\mathfrak{X}}$ and similarly $D^{\mathfrak{Z}} = D^{\mathfrak{Y}}$.

For the remainder of this chapter we confine ourselves to the case of $\mathfrak{X}$ an integrated normal system.

<u>DEFINITION</u>. A normal system $\mathfrak{X} = \{X(p)\}$ of G is said to be <u>integrated</u> if X(p) centralizes all p-chief factors of G above X(q) for all primes p,q.

e.g. (i) $X(p) = X$ for all primes p.

(ii) $X(p) = C_p(G)$ for each prime p. ( Notation of [4], chapter 3, where $\mathfrak{F}$ is a saturated formation defined locally by an integrated set of formations $\{ \mathfrak{F}(p)\}$.)

<u>LEMMA 4.12</u> The following conditions on a normal system $\mathfrak{X} = \{X(p)\}$ of G are equivalent.

(i) $\mathfrak{X}$ is integrated.

(ii) $G/X \in \bar{\mathfrak{X}}$.

(iii) $D^{\mathfrak{X}}.X = G$ where $X = \bigcap_p X(p)$.

Proof. $\mathfrak{X}$ is integrated if and only if $G/X(p) \in \bar{\mathfrak{X}}$ for all primes p, and thus if and only if $G/X \in \bar{\mathfrak{X}}$ (Lemma 3.2(i) and (ii)); and $G/X \in \bar{\mathfrak{X}}$ if and only if $D^{\mathfrak{X}}$ covers $G/X$ i.e. $D^{\mathfrak{X}}.X = G$, by Theorem 4.2.

We first consider the intersection of $D^{\mathfrak{X}}$ for integrated $\mathfrak{X}$ with a chief series of G.

THEOREM 4.13 If $\mathfrak{X} = \{X(p)\}$ is an integrated normal system of G, the intersection of any $\mathfrak{X}$-normalizer D of G with a chief series of G is a chief series of D with corresponding chief factors operator-isomorphic.

Proof. By Theorem 4.2, we need only consider the $\mathfrak{X}$-central chief factors of G. If H/K is an $\mathfrak{X}$-central p-chief factor of G, $X(p) \leq C_G(H/K)$ and thus, by Lemma 4.12, $D.C_G(H/K) = G$. The result follows by Lemma 1.1.

We now turn to the existence of $\mathfrak{X}$-critical maximal subgroups of G.

THEOREM 4.14 If $\mathfrak{X} = \{X(p)\}$ is an integrated normal system of G and $D^{\mathfrak{X}} < G$, then there exists an $\mathfrak{X}$-critical maximal subgroup M of G containing $D^{\mathfrak{X}}$ as an $\mathfrak{X} \cap M$-normalizer.

Proof. Since $\mathfrak{X}$ is integrated, $D^{\mathfrak{X}}X = G$ where $X = \bigcap_p X(p)$. Thus, if $X \in \mathfrak{N}$, every maximal subgroup of G containing $D^{\mathfrak{X}}$ supplements F(X) and so is $\mathfrak{X}$-critical in G. Hence we can assume that $X \notin \mathfrak{N}$ and so $\Phi(G) \cap X \leq F(X) < X$. Write $\bar{G} = G/\Phi(G) \cap X$, $\bar{X} = X/\Phi(G) \cap X$, etc. We now apply the results of Gaschütz ([6]). Firstly, $\Phi(\bar{X}) = \bar{1}$. For $\Phi(\bar{X}) \leq \Phi(\bar{G})$ since $\bar{X} \lhd \bar{G}$, and $\Phi(\bar{G}) = \overline{\Phi(G)}$ since $X \cap \Phi(G) \leq \Phi(G)$ ( [6], Theorem 2.). Thus $\Phi(\bar{X}) \leq \bar{X} \cap \overline{\Phi(G)} = \bar{1}$. Secondly, $F(\bar{X}) = \overline{F(X)}$. For suppose $F(\bar{X}) = R/\Phi(G) \cap X$. Then clearly $F(X) \leq R$, so it remains to prove that $R \leq F(X)$. Now $\bar{R}$ is a characteristic subgroup of the normal subgroup $\bar{X}$ of $\bar{G}$, and so $\bar{R} \lhd \bar{G}$ giving $R \lhd G$. Further, $\Phi(G) \cap X = \Phi(G) \cap R$. Thus we have $R/\Phi(G) \cap R \in \mathfrak{N}$ with $R \lhd G$, and so $R \in \mathfrak{N}$ by [6], Theorem 10, and we are done. Thus $\bar{X} \notin \mathfrak{N}$ and we can write

$$F(\bar{X}) = \bar{A}_1 \times \ldots \times \bar{A}_k = \bigcap_{i=1}^{k} C_{\bar{X}}(\bar{A}_i) ,$$

where the $\bar{A}_i$ are minimal normal subgroups of $\bar{X}$. Let $\bar{H}_1, \ldots, \bar{H}_k$ be minimal normal subgroups of $\bar{G}$ such that $\bar{A}_i \leq \bar{H}_i \leq \overline{F(X)}$ for each i. Since $\bar{X} \notin \mathfrak{N}$, all $\bar{H}_i$ are not centralized by $\bar{X}$. Thus there exists some $\bar{X}$-eccentric minimal normal subgroup $\bar{H} = H/\Phi(G) \cap X$ of $\bar{G}$ which is contained in $\overline{F(X)}$. $\bar{H} \not\leq \Phi(\bar{G})$, and thus there exists a maximal subgroup M of G which complements $H/\Phi(G) \cap X$. Since this is an $\mathfrak{X}$-eccentric chief factor of G, and $H \leq F(X)$, M is $\mathfrak{X}$-critical. By Theorem 4.5 and the conjugacy of $\mathfrak{X}$-normalizers, if D(M) is an $\mathfrak{X} \cap M$ -normalizer of M, $D^{\mathfrak{X}} = D(M)^g$ for some $g \in G$ and is thus an $\mathfrak{X} \cap M^g$ -normalizer of the $\mathfrak{X}$-critical maximal subgroup $M^g$ of G.

<u>LEMMA 4.15</u> If $\mathfrak{X} = \{X(p)\}$ is an integrated normal system and M an $\mathfrak{X}$-abnormal maximal subgroup of G, then $\mathfrak{X} \cap M$ is an integrated normal system of M.

Proof. Since M is $\mathfrak{X}$-abnormal it contains an $\mathfrak{X}$-normalizer of G (Theorem 4.3). Thus, by Lemma 4.12, $MX = G$ where $X = \bigcap_p X(p)$. $G/X \in \bar{\mathfrak{X}}$ since $\mathfrak{X}$ is integrated, and hence $M/M \cap X \in \bar{\mathfrak{X}}$. Thus $M/M \cap X \in \overline{\mathfrak{X} \cap M}$ and the result follows by Lemma 4.12.

Repeated application of Theorem 4.14 together with this lemma gives

THEOREM 4.16 If $\mathfrak{X}$ is an integrated normal system of G, any $\mathfrak{X}$-normalizer $D^{\mathfrak{X}}$ of G can be connected to G by an $\mathfrak{X}$-critical maximal chain.

Our final result in this chapter concerns the relationship between the $\mathfrak{X}$- and $\mathfrak{Y}$-normalizers of G for $\mathfrak{X}, \mathfrak{Y}$ normal systems of G with $\mathfrak{X} \geq \mathfrak{Y}$. In Lemma 4.7 we have seen that an $\mathfrak{X} \cap D^{\mathfrak{Y}}$-normalizer of $D^{\mathfrak{Y}}$ is not always an $\mathfrak{X}$-normalizer of G. However, if we assume that $\mathfrak{Y}$ is integrated, we do get equality.

THEOREM 4.17 Let $\mathfrak{X}$ and $\mathfrak{Y}$ be two normal systems of G such that $\mathfrak{X} \geq \mathfrak{Y}$ and $\mathfrak{Y}$ is integrated. Then every $\mathfrak{X} \cap D^{\mathfrak{Y}}$-normalizer of $D^{\mathfrak{Y}}$ is an $\mathfrak{X}$-normalizer of G.

Proof. We use induction on $|G:D^{\mathfrak{Y}}|$. There is nothing to prove for $G = D^{\mathfrak{Y}}$, so assume $D^{\mathfrak{Y}} < G$, and let M be a $\mathfrak{Y}$-critical maximal subgroup of G containing $D^{\mathfrak{Y}}$ as a $\mathfrak{Y} \cap M$-normalizer (Theorem 4.14). $\mathfrak{X} \cap M \geq \mathfrak{Y} \cap M$ and $\mathfrak{Y} \cap M$ is an integrated normal system of M by Lemma 4.15. Thus, by induction, every $\mathfrak{X} \cap M \cap D^{\mathfrak{Y}}$-normalizer of $D^{\mathfrak{Y}}$ i.e. every $\mathfrak{X} \cap D^{\mathfrak{Y}}$-normalizer of

$D^y$ is an $\mathfrak{X} \cap M$-normalizer of M and thus an $\mathfrak{X}$-normalizer of G (Theorem 4.5) since M is $\mathfrak{X}$-critical.

## Chapter Five

# $\mathfrak{X}$ - COVERING SUBGROUPS

We assume throughout this chapter that $\mathfrak{X} = \{X(p)\}$ is a normal system of the finite soluble group G.

DEFINITION. A subgroup $E^{\mathfrak{X}}$ of G is called an $\mathfrak{X}$-covering subgroup of G if (i) $E^{\mathfrak{X}} \in \bar{\mathfrak{X}}$

(ii) $E^{\mathfrak{X}} \leq H \leq G$, $H/H_0 \in \bar{\mathfrak{X}}$ implies $H_0E^{\mathfrak{X}} = H$.

Our first concern is to show the existence, conjugacy and homomorphism-invariance of these subgroups. On account of the fact that $\bar{\mathfrak{X}}$ is not isomorphism-closed, homomorphism-invariance is not immediate, neither can Gaschütz's proof of the existence of $\mathfrak{F}$-covering subgroups ([7], Theorem 2.1) be carried over to the present situation. We first prove some simple results on $\mathfrak{X}$-covering subgroups.

LEMMA 5.1 Let E be an $\mathfrak{X}$-covering subgroup of G. Then

(a) If $E \leq H \leq G$, then E is an $\mathfrak{X} \cap H$-covering subgroup of H.

(b) $E^g$ is an $\mathfrak{X}$-covering subgroup of G for each $g \in G$.

(c) E is abnormal in G.

(d) If $NE/N \in \bar{\mathfrak{X}}$ for normal subgroup N of G, then NE/N is an $N\mathfrak{X}/N$-covering subgroup of G/N --- this is clearly true if $N \leq E$.

Proof. (a) $E \in \bar{\mathfrak{X}}$ implies $E \in \overline{\mathfrak{X} \cap H}$ since $(X(p) \cap H) \cap E = X(p) \cap E$. Further, let $E \leq F \leq H$, $F_0 \lhd F$ and $F/F_0 \in \overline{\mathfrak{X} \cap H}$.

We must show that $EF_0 = F$. Now

$$F_0(X(p) \cap F)/F_0 = F_0(X(p) \cap H \cap F)/F_0 \leq O_{p'p}(F/F_0)$$

since $F/F_0 \in \overline{\mathfrak{X} \cap H}$. Thus $F/F_0 \in \bar{\mathfrak{X}}$ and the result follows immediately.

(b) Since $E \in \bar{\mathfrak{X}}$, $X(p) \cap E^g = (X(p) \cap E)^g$ is p-nilpotent. Hence $E^g \in \bar{\mathfrak{X}}$. Let $E^g \leq F \leq G$, $F_0 \lhd F$ and $F/F_0 \in \bar{\mathfrak{X}}$; and write $F^* = F^{g^{-1}}$, $F_0^* = F_0^{g^{-1}}$. Then $E \leq F^* \leq G$, $F_0^* \lhd F^*$ and $F^*/F_0^* \in \bar{\mathfrak{X}}$, since $F_0(F \cap X(p))/F_0$ is p-nilpotent implies that $F_0^*(F^* \cap X(p))/F_0^*$ is p-nilpotent. Hence $EF_0^* = F^*$ and thus $E^g F_0 = F$.

(c) By Taunt, it is sufficient to show that every subgroup of G containing E is self-normalizing. Let $H \geq E$ and $Q = N_G(H)$. Suppose, if possible, that $Q > H$. Choose a maximal normal subgroup M of Q containing H. Then $Q/M \in \bar{\mathfrak{X}}$ since it is nilpotent. Hence, by the definition of E, $Q = EM = M$ --- a contradiction. Thus $Q = H$.

(d) Since $NE/N \in \bar{\mathfrak{X}}$, $NX(p)/N \cap NE/N = N(X(p) \cap NE)/N$ is p-nilpotent and so $NE/N \in \bar{\mathcal{Y}}$ ( where $\mathcal{Y} = N\mathfrak{X}/N$ ). Let $NE/N \leq F/N \leq G/N$, $F_0/N \lhd F/N$ and $F/N \,/\, F_0/N \in \bar{\mathcal{Y}}$. Then $E \leq F \leq G$, $F_0 \lhd F$ and $F/F_0 \in \bar{\mathfrak{X}}$, since

$$F_0(X(p) \cap F)/F_0 \cong F_0/N\ (\ NX(p)/N \cap F/N\ )\ /\ F_0/N .$$

Thus $EF_0 = F$ and the result then follows.

<u>LEMMA 5.2</u> If M is a maximal subgroup of G satisfying $M \in \bar{\mathfrak{X}}$ but $G/\mathrm{Core}M \notin \bar{\mathfrak{X}}$, then M is an $\mathfrak{X}$-covering subgroup of G.

<u>Proof</u>. We need only show that M supplements the $\bar{\mathfrak{X}}$-residual of G. If this were not the case, we would have $G^{\bar{\mathfrak{X}}} \leq \mathrm{Core}M$ and

hence $G/\mathrm{Core}M \in \bar{\mathfrak{X}}$ by Lemma 3.2(i) --- a contradiction.

<u>THEOREM 5.3</u> If $E$ is an $\mathfrak{X}$-covering subgroup of $G$ and $N \lhd G$, then $NE/N$ is an $N\mathfrak{X}/N$ -covering subgroup of $G/N$.

<u>Proof</u>. We prove that $NE/N \in \bar{\mathfrak{X}}$ and the result then follows by Lemma 5.1(d). Suppose this is not the case, and let $G$ be a counterexample of minimal order. Then there exists a normal system $\mathfrak{X} = \{X(p)\}$, an $\mathfrak{X}$-covering subgroup $E$ and a normal subgroup $N$ of $G$ such that $NE/N \notin \bar{\mathfrak{X}}$ . Since $E \in \bar{\mathfrak{X}}$ , we must have $N > 1$. Let $N_o$ be a minimal normal subgroup of $G$ contained in $N$. Then $N_oE = G$ . For $E$ is an $(\mathfrak{X} \cap N_oE)$-covering subgroup of $N_oE$ and thus, by induction, $N_oE < G$ implies $N_oE/N_o \in \overline{\mathfrak{X} \cap N_oE}$ . Thus $N_oE/N_o \in \overline{\mathfrak{X} \cap NE}$ and so is an $N_o(\mathfrak{X} \cap NE)/N_o$ -covering subgroup of $NE/N_o$. Thus, by induction, $NE/N_o \,/\, N/N_o \in \overline{N_o(\mathfrak{X} \cap NE)/N_o}$ and so $NE/N \in \overline{\mathfrak{X} \cap NE}$ , contradicting $NE/N \notin \bar{\mathfrak{X}}$ . Hence $N_oE = G$ , and so $E$ is a maximal subgroup of $G$. $G/N_o \notin \bar{\mathfrak{X}}$ since $G/N = NE/N \notin \bar{\mathfrak{X}}$ . Thus there exists an $\mathfrak{X}$-eccentric p-chief factor $H/K$ of $G$ above $N_o$ for some prime p. Let $C = C_G(H/K)$ . Then

(1) --- $N_o \leq C$

(2) --- $X(p) \nleq C$

(3) --- $X(p) \cap E \leq C_E(H \cap E \,/\, K \cap E) = C \cap E$ (since $E \in \bar{\mathfrak{X}}$).

(2) and (3) imply $X(p) \nleq E$ and thus $X(p)E = G$ since $E$ is maximal. We show that

(4) --- $X(p) \cap E = X(p) \cap N_o(X(p) \cap E)$ which is normal in $G$.:

$X(p) \cap N_o(X(p) \cap E) = (X(p) \cap N_o)(X(p) \cap E) = (X(p) \cap N_o)E \cap X(p)$

and $(X(p) \cap N_\circ)E = G$ implies $X(p) = (X(p) \cap N_\circ)(X(p) \cap E) \leq C$, by (1) and (3). This contradicts (2), so that $(X(p) \cap N_\circ)E = E$ and the result follows.

By the definition of E, $G/X(p) \cap E \notin \bar{\mathfrak{X}}$ since $E < G$. We obtain a contradiction to this by showing that both $G/X(p)$ and $G/N_\circ(X(p) \cap E)$ lie in $\bar{\mathfrak{X}}$, and then using (4) and Lemma 3.2(ii). By Lemma 5.1(d), $E/X(p) \cap E$ is a $\mathfrak{Y}$-covering subgroup of $G/X(p) \cap E$ (where $\mathfrak{Y} = (X(p) \cap E)\mathfrak{X}/(X(p) \cap E)$ ). Also $X(p) \cap E > 1$. For $X(p) \cap E = 1$ implies that $X(p)$ is a minimal normal subgroup of G, and so $X(p) \leq F(G) \leq C$, contradicting (2). By induction we therefore have $G/N_\circ(X(p) \cap E) = N_\circ E/N_\circ(X(p) \cap E) \in \bar{\mathfrak{X}}$ and $G/X(p) = X(p)E/X(p) \in \bar{\mathfrak{X}}$, as required. This proves the theorem.

<u>THEOREM 5.4</u> Any two $\mathfrak{X}$-covering subgroups of G are conjugate.

<u>Proof</u>. We use induction on $|G|$ as in [7],2.1. The result is trivial for $G = 1$ and for $G \in \bar{\mathfrak{X}}$. Thus assume $G > 1$ and $G \notin \bar{\mathfrak{X}}$, and consider separately the following two cases :

Case (a). $G/N \notin \bar{\mathfrak{X}}$ for some minimal normal subgroup N of G. In this case, if $E_1$, $E_2$ are two $\mathfrak{X}$-covering subgroups of G, then $NE_1/N$, $NE_2/N$ are $N\mathfrak{X}/N$-covering subgroups of G/N by Theorem 5.3. Thus, by induction, $NE_1 = NE_2^g$ for some $g \in G$. $E_1$, $E_2^g$ are $(\mathfrak{X} \cap NE_1)$-covering subgroups of $NE_1$ and $NE_1 < G$, since $G/N \notin \bar{\mathfrak{X}}$. The result thus follows by induction.

Case (b). $G/N \in \bar{\mathfrak{X}}$ for all minimal normal subgroups N of G. Then the $\bar{\mathfrak{X}}$-residual $\bar{G}$ of G is the unique minimal normal subgroup of G, since $G \notin \bar{\mathfrak{X}}$ (Lemma 3.2(ii)). $\bar{G}$ is complemented in G by Lemma 3.2(iii), and its complements are the $\mathfrak{X}$-covering subgroups of G. Since they are maximal in G with trivial core, they are conjugate in G.

We are even now not in a position to prove the existence of $\mathfrak{X}$-covering subgroups along the lines of Theorem 5.4. For in case (a) above, we can assume by induction that G/N has an $N\mathfrak{X}/N$ -covering subgroup $E_1/N < G/N$. Again by induction we can find an $\mathfrak{X} \cap E_1$ -covering subgroup E of $E_1$. Then $E \in \bar{\mathfrak{X}}$, but we are unable to show that it is in fact an $\mathfrak{X}$-covering subgroup of G. For suppose $E \leq F \leq G$, $F_0 \lhd F$ and $F/F_0 \in \bar{\mathfrak{X}}$. Then $E \leq F \cap E_1 \leq E_1$, $F_0 \cap E_1 \lhd F \cap E_1$ and $F \cap E_1 / F_0 \cap E_1 \cong F_0(F \cap E_1)/F_0 \in \bar{\mathfrak{X}}$ by Lemma 3.2(iv). Hence $E(F_0 \cap E_1) = F \cap E_1$ by definition of E, and thus $EF_0 = (F \cap E_1)F_0$. To prove that $(F \cap E_1)F_0 = F$ as required, we need to utilize the fact that $E_1/N$ is an $N\mathfrak{X}/N$ -covering subgroup of G/N, and this is where the proof breaks down. Application of Theorem 5.3 to E in $E_1$ yields $E_1 = NE$, since $E_1/N \in \bar{\mathfrak{X}}$. Thus $E_1 \leq NF \leq G$ and $NF_0 \lhd NF$, but $NF/NF_0$ need not lie in $\bar{\mathfrak{X}}$, since $\bar{\mathfrak{X}}$ is not isomorphism-closed.

Now for $\mathfrak{F}$ a saturated formation, the $\mathfrak{F}$-covering subgroups are the terminal members of $\mathfrak{F}$-crucial maximal chains ([4],5.4). We approach the existence of $\mathfrak{X}$-covering subgroups along these lines viz. by defining $\mathfrak{X}$-crucial maximal chains in the natural way, showing that these always exist for any normal system $\mathfrak{X}$ of G and that their terminal members, which lie in $\bar{\mathfrak{X}}$, are in fact $\mathfrak{X}$-covering subgroups of G.

DEFINITION. A maximal subgroup M of G is said to be $\mathfrak{X}$-crucial if M is $\mathfrak{X}$-abnormal and $G/N \in \bar{\mathfrak{X}}$, where $N/\mathrm{Core}M$ is the unique minimal normal subgroup of $G/\mathrm{Core}M$. We say that the chain $H = H_0 < H_1 < \ldots < H_r = G$ of subgroups of G is an $\mathfrak{X}$-crucial maximal chain if $H_i$ is an $\mathfrak{X} \cap H_{i+1}$-crucial maximal subgroup of $H_{i+1}$ for each i.

LEMMA 5.5 The following conditions on a maximal subgroup M of G are equivalent

(i) M is $\mathfrak{X}$-crucial.

(ii) $G/N \in \bar{\mathfrak{X}}$ but $G/\mathrm{Core}M \notin \bar{\mathfrak{X}}$.

(iii) $G^{\bar{\mathfrak{X}}}/G^{\bar{\mathfrak{X}}} \cap M$ is a chief factor of G, where $G^{\bar{\mathfrak{X}}}$ is the $\bar{\mathfrak{X}}$-residual of G.

Proof. (i) and (ii) are trivially equivalent. We show that (i) is equivalent to (iii). If M is $\mathfrak{X}$-crucial, $G^{\bar{\mathfrak{X}}} \leq N$ and thus $G^{\bar{\mathfrak{X}}} \cap M = G^{\bar{\mathfrak{X}}} \cap \mathrm{Core}M \lhd G$. Also, by Theorems 4.2 and 4.3, $G^{\bar{\mathfrak{X}}}M = G$ and the result follows, since M is maximal. Conversely, if $G^{\bar{\mathfrak{X}}}/G^{\bar{\mathfrak{X}}} \cap M$ is a chief factor of G, it is complemented by M. Since this is an $\mathfrak{X}$-eccentric chief factor, M is $\mathfrak{X}$-abnormal; and $G/N \in \bar{\mathfrak{X}}$ since $G^{\bar{\mathfrak{X}}}.\mathrm{Core}M = N$.

LEMMA 5.6 If $G \notin \bar{\mathfrak{X}}$, it possesses an $\mathfrak{X}$-crucial maximal subgroup. Thus, for $G \notin \bar{\mathfrak{X}}$, we can construct an $\mathfrak{X}$-crucial maximal chain of G whose terminal member lies in $\bar{\mathfrak{X}}$.

Proof. Since $G \notin \bar{\mathfrak{X}}$, $1 < \bar{G}$, the $\bar{\mathfrak{X}}$-residual of G. Let $\bar{G}/K$ be a chief factor of G. Then $G/K \notin \bar{\mathfrak{X}}$ and so $\bar{G}/K$ is $\mathfrak{X}$-eccentric and complemented in G/K. For $\bar{G}/K \leq \Phi(G/K)$ implies that $G/K\,/\,\Phi(G/K) \in \overline{K\mathfrak{X}/K}$. Lemma 3.2(iii) then gives $G/K \in \overline{K\mathfrak{X}/K}$

i.e. $G/K \in \bar{\mathfrak{X}}$, a contradiction. Thus there exists a maximal subgroup M of G complementing $\bar{G}/K$. M is $\mathfrak{X}$-crucial by Lemma5.5.

The following lemma is the crux of the proof that an $\mathfrak{X} \cap M$-covering subgroup of an $\mathfrak{X}$-crucial maximal subgroup M of G is an $\mathfrak{X}$-covering subgroup of G.

<u>LEMMA 5.7</u> Let E be an $\mathfrak{X} \cap M$-covering subgroup of the $\mathfrak{X}$-crucial maximal subgroup M of G. Then, if $E < F < G$ and $F \not\leq M$, $F/F \cap \mathrm{Core}M \notin \bar{\mathfrak{X}}$.

<u>Proof</u>. Suppose this is not so, and let G be a counterexample of minimal order. Thus we have $E < F < G$, $F \not\leq M$ and $F/F \cap \mathrm{Core}M \in \bar{\mathfrak{X}}$ for some subgroup F of G --- choose F to be of maximal order subject to these conditions. Let $N/\mathrm{Core}M$ be the unique minimal normal subgroup of $G/\mathrm{Core}M$.

Now since M is $\mathfrak{X}$-crucial, $G/N \in \bar{\mathfrak{X}}$ and so $M/\mathrm{Core}M \in \bar{\mathfrak{X}}$. Further, $MG^{\bar{\mathfrak{X}}} = G$ giving $M/M \cap G^{\bar{\mathfrak{X}}} \in \bar{\mathfrak{X}}$. Thus, by the definition of E, $E\mathrm{Core}M = M$ and $E(G^{\bar{\mathfrak{X}}} \cap M) = M$ so that $F\mathrm{Core}M \geq M$ and $FG^{\bar{\mathfrak{X}}} \geq M$. Since $F \not\leq M$, we therefore have

(1) --- $F\mathrm{Core}M = G$ and $FG^{\bar{\mathfrak{X}}} = G$.

We now show that F is a maximal subgroup of G. Let $F < \hat{F} \leq G$. We prove that $\hat{F} \cap M$ is then an $\mathfrak{X} \cap \hat{F}$-crucial maximal subgroup of $\hat{F}$. By (1), $\hat{F}\mathrm{Core}M = G$. Thus $\hat{F} \cap N / \hat{F} \cap \mathrm{Core}M$ is a chief factor of $\hat{F}$ and is in fact complemented by $\hat{F} \cap M$. For $(\hat{F} \cap M)\mathrm{Core}M = \hat{F}\mathrm{Core}M \cap M = M$, giving $(\hat{F} \cap M)(\hat{F} \cap N) = \hat{F} \cap (\hat{F} \cap M)N = \hat{F}$. Thus $\hat{F} \cap M$ is maximal in $\hat{F}$, and it is easily verified that

(2) --- $\mathrm{Core}_{\hat{F}}(\hat{F} \cap M) = \hat{F} \cap \mathrm{Core}M$.

Now, by the definition of F, $\hat{F}/\hat{F} \cap \mathrm{Core}M \notin \bar{\mathfrak{X}}$. On the other hand, $\hat{F}/\hat{F} \cap N \in \bar{\mathfrak{X}}$ since $\hat{F}N = G$ and $G/N \in \bar{\mathfrak{X}}$. The result thus follows by Lemma 5.5(ii). F is now easily seen to be maximal in G. For suppose, if possible, that $\hat{F} < G$. In $\hat{F}$ we have E an $\mathfrak{X} \cap \hat{F} \cap M$-covering subgroup of the $\mathfrak{X} \cap \hat{F}$-crucial maximal subgroup $\hat{F} \cap M$, $E < F < \hat{F}$ and $F \nleq \hat{F} \cap M$. Thus, by the definition of G, $F / F \cap \mathrm{Core}_{\hat{F}}(\hat{F} \cap M) \notin \bar{\mathfrak{X}}$ i.e. $F/F \cap \mathrm{Core}M \notin \bar{\mathfrak{X}}$, using (2). This contradicts the definition of F and so $\hat{F} = G$ and F is maximal in G.

Let $|G:M| = p^{\alpha}$ for some prime p. Then

(3) --- $X(p) \nleq N$ since $N/\mathrm{Core}M$ is $\mathfrak{X}$-eccentric, and

(4) --- $X(p) \cap F \leq C_F(F \cap N / F \cap \mathrm{Core}M) = F \cap N$ since $F/F \cap \mathrm{Core}M \in \bar{\mathfrak{X}}$ and thus $F \cap N / F \cap \mathrm{Core}M$ is $\mathfrak{X} \cap F$-central. (3) and (4) together imply that $X(p) \nleq F$, and thus

(5) --- $X(p)F = G$ since F is maximal in G.

We now show that $F \cap \mathrm{Core}M = \mathrm{Core}M \cap X(p)(F \cap \mathrm{Core}M)$ and so is normal in G by (5). As in 5.3, $\mathrm{Core}M \cap X(p)(F \cap \mathrm{Core}M) = (\mathrm{Core}M \cap X(p))F \cap \mathrm{Core}M$, and $(\mathrm{Core}M \cap X(p))F = G$ implies that $X(p) = (\mathrm{Core}M \cap X(p))(F \cap X(p)) \leq N$ by (4). This contradicts (3), so $(\mathrm{Core}M \cap X(p))F = F$ and the result follows.

Write $G^* = G/F \cap \mathrm{Core}M$, $F^* = F/F \cap \mathrm{Core}M$ etc., and $\mathfrak{Y} = (F \cap \mathrm{Core}M)\mathfrak{X}/(F \cap \mathrm{Core}M)$. Then $F^* \in \bar{\mathfrak{Y}}$, $F^*$ is maximal in $G^*$ and $G^*/\mathrm{Core}F^* \notin \bar{\mathfrak{Y}}$. This last statement follows from the fact that $G^*/\mathrm{Core}F^* \in \bar{\mathfrak{Y}}$ implies $G/\mathrm{Core}F \in \bar{\mathfrak{X}}$, since $\mathrm{Core}F^* = (\mathrm{Core}F)^*$. This then gives $G^{\bar{\mathfrak{X}}} \leq \mathrm{Core}F$, contradicting (1). Thus $F^*$ is, in fact, a $\mathfrak{Y}$-covering subgroup of $G^*$ by Lemma 5.2. Hence, by the homomorphism-invariance of $\mathfrak{Y}$-covering subgroups,

$G^*/(\text{Core}M)^* = F^*(\text{Core}M)^*/(\text{Core}M)^* \in \bar{\mathfrak{Y}}$ i.e. $G/\text{Core}M \in \bar{\mathfrak{X}}$, contradicting the fact that M is $\mathfrak{X}$-abnormal. This last contradiction proves the lemma.

<u>THEOREM 5.8</u> An $\mathfrak{X} \cap M$ -covering subgroup of an $\mathfrak{X}$-crucial maximal subgroup M of G is an $\mathfrak{X}$-covering subgroup of G.

<u>Proof</u>. Let E be an $\mathfrak{X} \cap M$ -covering subgroup of M. Then $E \in \bar{\mathfrak{X}}$. Let $E \leq F \leq G$ and $F/F_0 \in \bar{\mathfrak{X}}$. We must show that $EF_0 = F$. Certainly if $F \leq M$ this is true, by the definition of E. Also, if $F = G$ it is again true, since $EG^{\bar{\mathfrak{X}}} = E(G^{\bar{\mathfrak{X}}} \cap M)G^{\bar{\mathfrak{X}}} = MG^{\bar{\mathfrak{X}}} = G$. Thus we need only consider the case $E < F < G$ with $F \not\leq M$. We show that $F_0(F \cap M) = F$. As in 5.7, $F\text{Core}M = G$ so that $F_0\text{Core}M \lhd G$ and thus $F_0 M = F_0\text{Core}M.M$ is a subgroup of G. If $F_0 \leq M$, we have $F_0\text{Core}M \leq \text{Core}M$ and thus $F_0 \leq F \cap \text{Core}M$ so that $F/F \cap \text{Core}M \in \bar{\mathfrak{X}}$. This contradicts Lemma 5.7, and so $F_0 \not\leq M$ and thus $F_0 M = G$. Hence $F_0(F \cap M) = F \cap F_0 M = F$. Thus $F \cap M / F_0 \cap M \cong F/F_0 \in \bar{\mathfrak{X}}$ and so $F \cap M / F_0 \cap M \in \overline{\mathfrak{X} \cap M}$. By the definition of E, $E(F_0 \cap M) = F \cap M$. Hence $EF_0 = (F \cap M)F_0 = F$.

By Lemma5.6 and repeated application of Theorem5.8, we have

<u>THEOREM 5.9</u> G possesses an $\mathfrak{X}$-covering subgroup for any normal system $\mathfrak{X}$.

Using the conjugacy of the $\mathfrak{X}$-covering subgroups, Theorem 5.8 also yields the following lemma

LEMMA 5.10 If $G \notin \bar{\mathfrak{X}}$, every $\mathfrak{X}$-covering subgroup of G is an $\mathfrak{X} \cap M$ -covering subgroup of some $\mathfrak{X}$-crucial maximal subgroup M of G.

This gives us a characterization of the $\mathfrak{X}$-covering subgroups similar to that of the $\mathfrak{F}$-covering subgroups viz.

THEOREM 5.11 The $\mathfrak{X}$-covering subgroups of G are the terminal members of the $\mathfrak{X}$-crucial maximal chains of G.

We are now in a position to obtain the desired relationship between the $\mathfrak{X}$-normalizers and $\mathfrak{X}$-covering subgroups.

THEOREM 5.12 Every $\mathfrak{X}$-covering subgroup of G contains an $\mathfrak{X}$-normalizer of G, and conversely.

Proof. By Theorem 5.11, an $\mathfrak{X}$-covering subgroup of G is a member of the set $\mathfrak{M}$ of Theorem4.10 and thus contains an $\mathfrak{X}$-normalizer of G. The converse follows by the conjugacy of the $\mathfrak{X}$-covering subgroups and of the $\mathfrak{X}$-normalizers of G.

COROLLARY 1 The $\mathfrak{X}$-covering subgroups of G can be characterized by the conditions

(i) $E \in \bar{\mathfrak{X}}$

(ii) If $E \leq U \lessdot V \leq G$, then U is $\mathfrak{X} \cap V$ -abnormal in V.

Proof. Firstly, let E be an $\mathfrak{X}$-covering subgroup of G and $E \leq U \lessdot V \leq G$. Then E is an $\mathfrak{X} \cap V$ -covering subgroup of V and so contains an $\mathfrak{X} \cap V$ -normalizer of V, by Theorem5.12. U is then $\mathfrak{X} \cap V$ -abnormal in V by Theorem4.3. Conversely, let E satisfy (i) and (ii), and $E \leq V \leq G$, $V_\circ \lhd V$ with $V/V_\circ \in \bar{\mathfrak{X}}$. We must show that $V_0E = V$. Suppose this is not the case, and let U be a maximal subgroup of V containing $V_0E$. Then $V_0 \leq \mathrm{Core}_V U$ and so $V/\mathrm{Core}_V U \in \bar{\mathfrak{X}}$ by Lemma3.2(i). U is thus $\mathfrak{X} \cap V$ -normal in V, contradicting (ii). Hence $V_0E = V$.

<u>COROLLARY 2</u> Let E be an $\mathfrak{X}$-covering subgroup of G, K/L an E-composition factor of G of order a power of prime p, and $K \leq X(p)$. Then E covers K/L if $X(p) \cap E \leq C_E(K/L)$ and avoids K/L if $X(p) \cap E \not\leq C_E(K/L)$ .

<u>Proof</u>. This proof follows closely that of Lemma5.2 of [4]. As we have seen in Chapter 2, E either covers or avoids K/L. If E covers K/L, $K \cap E/L \cap E$ is a p-chief factor of E with $C_E(K/L) = C_E(K \cap E/L \cap E)$. Since $E \in \bar{\mathfrak{X}}$ this gives $X(p) \cap E \leq C_E(K/L)$. On the other hand, if E avoids K/L, LE is a maximal subgroup of KE and so is $\mathfrak{X} \cap KE$ -abnormal in KE, by Corollary 1. The p-chief factor K/L of KE is complemented by LE and so is $\mathfrak{X} \cap KE$ -eccentric i.e. $X(p) \cap KE \not\leq C_{KE}(K/L) = KC_E(K/L)$ . Thus $X(p) \cap E \not\leq C_E(K/L)$.

<u>COROLLARY 3</u> An $\mathfrak{X}$-normalizer D of G is an $\mathfrak{X}$-covering subgroup of G if and only if $D \lessdot H \leq G$ implies that D is $\mathfrak{X} \cap H$ -abnormal in H.

<u>Proof</u>. Assume first that $D \lessdot H \leq G$ implies that D is $\mathfrak{X} \cap H$ -abnormal in H. Then D is a maximal $\bar{\mathfrak{X}}$-subgroup of G. For if $D < H$ with $H \in \bar{\mathfrak{X}}$, and $D \lessdot H_1 \leq H$, we have $H_1 \in \bar{\mathfrak{X}}$ by Lemma 3.2(iv). D is then $\mathfrak{X} \cap H_1$-normal in $H_1$ --- a contradiction. Now $D \leq E$, an $\mathfrak{X}$-covering subgroup of G, by Theorem5.12. Since $E \in \bar{\mathfrak{X}}$ we must have $D = E$. The converse follows

immediately from Corollary 1.

In [12] T.O.Hawkes gives a similar condition for the $\mathfrak{F}$-normalizers and $\mathfrak{F}$-covering subgroups of a group to coincide, and an example of abnormal supersoluble normalizers which are not supersoluble covering subgroups. We describe his example briefly as it shows that an abnormal $\mathfrak{X}$-normalizer need not be an $\mathfrak{X}$-covering subgroup.

<u>EXAMPLE 5.13</u> Let $W = C_5 \wr \Sigma_4$, the wreath product of a cyclic group of order 5 with the symmetric group on 4 elements. W is the semidirect product of an elementary abelian group $N = < a_1, \ldots, a_4 >$ of order $5^4$ with $\Sigma_4$. Let $\alpha$ be the automorphism of N mapping $a_i \to a_i^2$ ( i=1,..,4 ). Then $\alpha$ is of order 4 and commutes elementwise with $\Sigma_4$. Let $G^* = \Sigma_4 \times < \alpha >$ and G be the splitting extension of N by G*. For each prime p let $\mathfrak{F}(p)$ be the formation of abelian groups of exponent dividing p-1. Then, by Theorem6.1 of [4], $\{ \mathfrak{F}(p) \}$ is a set of integrated formations defining locally the formation of supersoluble groups. Set X(p) equal to the $\mathfrak{F}(p)$-residual of G for p= 2,3,5 i.e. $X(2) = G$, $X(3) = NA_4 < \alpha^2 >$, $X(5) = NA_4$ where $A_4$ is the alternating group on 4 elements. Then the supersoluble normalizer $D = < a_1 a_2 a_3 a_4 > H < \alpha >$ (where H is the subgroup of $\Sigma_4$ leaving $a_4$ invariant) is an $\mathfrak{X}$-normalizer of G. We show that the supersoluble covering subgroup

$E = < a_1 a_2 a_3 > < a_4 > H < \alpha >$ is also an $\mathfrak{X}$-covering subgroup of G. The chain $E \lessdot NH < \alpha > \lessdot G$ is easily seen to be an $\mathfrak{X}$-crucial maximal chain, so E contains an $\mathfrak{X}$-covering subgroup of G by Theorem5.11. The required equality then follows, since $E \in \bar{\mathfrak{X}}$ . D is abnormal in G since the only proper subgroups of G containing it are E, NH$< \alpha >$ and $< a_1 a_2 a_3 a_4 >$. $\Sigma_4 < \alpha >$ , all of which are self-normalizing in G.

In the above example, the $\mathfrak{F}$- and $\mathfrak{X}$-covering subgroups of G coincided when we took X(p) equal to $G^{\mathfrak{F}(p)}$ for each prime p. We now show that this is not always the case, nor even for suitable choice of X(p) between $G^{\mathfrak{F}(p)}$ and $C_p(G)$ (Notation of [4]). The following, due to R.W.Carter, is an example of an $\mathfrak{F}$-covering subgroup which is not an $\mathfrak{X}$-covering subgroup for any normal system $\mathfrak{X}$.

<u>EXAMPLE 5.14</u> Let $G = C_5 \wr \Sigma_4/Z$ where Z is the centre of the wreath product $C_5 \wr \Sigma_4$ of Example5.13. $|Z| = 5$ and thus $|G| = 24.5^3$. Now take $\Gamma = C_7 \wr G$, the wreath product of a cyclic group of order 7 with G. Then $\Gamma$ is the semidirect product of an elementary abelian group N of order $7^{|G|}$ with the group G, and thus $|\Gamma| = 24.5^3.7^{|G|}$. Let $\mathfrak{F} = \mathfrak{N}^2$ ---- this saturated formation can be defined locally by taking $\mathfrak{F}(p) = \mathfrak{N}$ for each prime p. We first determine the $\mathfrak{N}^2$-covering subgroups of $\Gamma$ .

An $\mathfrak{N}^2$-covering subgroup $\bar{E}$ of G is a direct product of a symmetric group on 3 elements and a cyclic group of order 5, and has thus order 2.3.5. $\bar{E}^{\mathfrak{N}}$ is cyclic of order 3. Let E be an $\mathfrak{N}^2$-covering subgroup of $\Gamma$ . Then, by the homomorphism-invariance, NE/N is an $\mathfrak{N}^2$-covering subgroup of $\Gamma$/N and so NE/N

$= N\bar{E}/N$. Hence $E \leq N\bar{E} = K$, say, and thus E is an $\mathfrak{N}^2$-covering subgroup of K. Also, since G complements $F(\Gamma) = N$, we can choose E such that $\bar{E} = G \cap E$ (Theorem5.12 of [4]). Thus $E = (N \cap E)\bar{E}$ and it only remains to determine $N \cap E$. By Maschke, since $7 \nmid |\bar{E}|$, we can write $N = N_1 \times \ldots \times N_k$ as the direct product of subgroups $N_i$ such that each $N_i$ is normalized by $\bar{E}$ and no proper subgroup of $N_i$ is normalized by $\bar{E}$. Then each $N_i$ is an E-composition factor of K. By Lemma5.2 of [4], E covers $N_i$ if $Aut_E(N_i) \in \mathfrak{N}$ and avoids $N_i$ if $Aut_E(N_i) \notin \mathfrak{N}$. Since $N \cap E$ centralizes each $N_i$ and $E = (N \cap E)\bar{E}$, $Aut_E(N_i) \cong Aut_{\bar{E}}(N_i)$ for each i. Thus E covers $N_i$ if $Aut_{\bar{E}}(N_i) \in \mathfrak{N}$ i.e $\bar{E}^{\mathfrak{N}} \leq C_{\bar{E}}(N_i)$ or $N_i \leq C_N(\bar{E}^{\mathfrak{N}})$, and E avoids $N_i$ if $Aut_{\bar{E}}(N_i) \notin \mathfrak{N}$ i.e. $N_i \not\leq C_N(\bar{E}^{\mathfrak{N}})$. Thus $N \cap E = C_N(\bar{E}^{\mathfrak{N}})$ and $E = \bar{E}C_N(\bar{E}^{\mathfrak{N}})$. We now show that $O_{7',7}(E) = \bar{E}^{\mathfrak{N}}C_N(\bar{E}^{\mathfrak{N}})$, and so the order of $O_{7',7}(E)$ is not divisible by 5. $\bar{E}^{\mathfrak{N}}C_N(\bar{E}^{\mathfrak{N}})$ is certainly a 7-nilpotent normal subgroup of E, so $\bar{E}^{\mathfrak{N}}C_N(\bar{E}^{\mathfrak{N}}) \leq O_{7',7}(E)$. Suppose, if possible, that this inclusion is proper. Then $\bar{E}^{\mathfrak{N}} < O_{7',7}(E) \cap \bar{E} = P$, say, and so $C_N(P) < C_N(\bar{E}^{\mathfrak{N}})$. However, P is the 7-complement of $O_{7',7}(E)$ and is thus normal in E. Hence $[\, C_N(\bar{E}^{\mathfrak{N}}), P\,] \leq P \cap C_N(\bar{E}^{\mathfrak{N}}) = 1$ giving $C_N(\bar{E}^{\mathfrak{N}}) \leq C_N(P)$ --- a contradiction.

To enable us to prove that E is not an $\mathfrak{X}$-covering subgroup of $\Gamma$ for any normal system $\mathfrak{X}$ of $\Gamma$ we need more information about the structure of $\Gamma$. Let

$G = G_0 \underset{2}{>} G_1 \underset{3}{>} G_2 \underset{2^2}{>} G_3 \underset{5^3}{>} G_4 = 1$ be the unique chief series of G. Since $7 \nmid |G|$, we can write (by Maschke) $N = A_1 \times \ldots \times A_\ell$ where the $A_j$ are G-composition factors of $\Gamma$ i.e. minimal normal subgroups of $\Gamma$. Since $N \leq C_\Gamma(A_j) \lhd \Gamma$ for each j, $C_\Gamma(A_j)$ is either $N$, $NG_3$, $NG_2$, $NG_1$ or $\Gamma$. Denote by $\bar{A}_{i+1}$ the direct product of those $A_j$ with centralizer equal to $NG_i$ ( $i=0,..,4$ ). Then

$\bar{A}_1 \times \ldots \times \bar{A}_{i+1} = C_N(G_i)$ and thus has order $7^{|G:G_i|}$ ( $i=0,..,4$ ). Set $\begin{cases} N_i = \bar{A}_{i+1} \times \ldots \times \bar{A}_5 & \text{for } i=0,..,4 \\ N_5 = 1. \end{cases}$

Then $N = N_0 \underset{7}{\geq} N_1 \underset{7}{\geq} N_2 \underset{7^4}{\geq} N_3 \underset{7^{18}}{\geq} N_4 \underset{7^{24(5^3-1)}}{\geq} N_5 = 1$

is a series of normal subgroups of $\Gamma$ satisfying, for $i=0,.,4$,

(i) $[N, G_i] \leq N_{i+1}$

(ii) $|N : N_{i+1}| = 7^{|G:G_i|}$

(iii) the centralizer of every chief factor of $\Gamma$ between $N_{i+1}$ and $N_i$ is equal to $NG_i$.

$N_{i+1}G_i \lhd \Gamma$ ( $i=0,..,4$ ). For $G_i \lhd G$ and $N_{i+1} \lhd \Gamma$ implies $N_{i+1}G_i \lhd N_{i+1}G$, and (i) implies that N normalizes $N_{i+1}G_i$. The only central chief factors of $\Gamma$ are those operator-isomorphic to $\Gamma/N_1G$ and $\Gamma/NG_1$ and thus $\Gamma^{n} = N_1G_1$. Thus no chief factor of $\Gamma$ below $N_2$ is centralized by $\Gamma^{n}$, and so $\Gamma^{n^2} = N_2G_2$.

Now suppose, if possible, that E is an $\mathfrak{X}$-covering subgroup of $\Gamma$ for some normal system $\mathfrak{X} = \{X(p)\}$ of $\Gamma$. Then $\Gamma^{\bar{\mathfrak{X}}} = \Gamma^{n^2}$. For $\Gamma = \Gamma^{n^2}E$ implies that $\Gamma/\Gamma^{n^2} \in \bar{\mathfrak{X}}$ by the homomorphism-invariance of $\mathfrak{X}$-covering subgroups. Thus $\Gamma^{\bar{\mathfrak{X}}} \leq \Gamma^{n^2}$ and similarly $\Gamma^{n^2} \leq \Gamma^{\bar{\mathfrak{X}}}$. We obtain a contradiction by showing that this implies that $X(7) \cap E \not\leq O_{7',7}(E)$ and so $E \notin \bar{\mathfrak{X}}$. Since $\Gamma^{\bar{\mathfrak{X}}} = N_2G_2$, the 7-chief factor $N_1G_2/N_2G_2$ is $\mathfrak{X}$-central and every chief factor of $\Gamma$ between $N_3G_2$ and

$N_2G_2$ is $\mathfrak{X}$-eccentric i.e. by (iii),

(iv) $X(7) \leq C_T(N_1/N_2) = NG_1$

(v) $X(7) \not\leq C_T$(any chief factor of T between $N_3$ and $N_2$) $= NG_2$

Now $X(7) = (X(7) \cap N)(X(7) \cap G)$ since these two subgroups are of coprime index in $X(7)$. Thus (iv) and (v) imply $X(7) \cap G \leq G_1$ and $X(7) \cap G \not\leq G_2$. Hence $X(7) \cap G = G_1$, since G has a unique chief series, and so $X(7) \cap E \geq X(7) \cap \bar{\bar{E}}$ $= G_1 \cap \bar{E}$. $|G_1 \cap \bar{E}|$ is divisible by 5 and hence so is $|X(7) \cap E|$. Thus $X(7) \cap E \not\leq O_{7',7}(E)$, and we are done.

We have thus seen that an $\mathfrak{F}$-covering subgroup need not be an $\mathfrak{X}$-covering subgroup. The converse is also true --- the following simple example shows that an $\mathfrak{X}$-covering subgroup need not be an $\mathfrak{F}$-covering subgroup for any saturated formation $\mathfrak{F}$. Take $G = \Sigma_3 \times \Sigma_3$, $X(2) = G$ and $X(3) = \Sigma_3\bar{\Sigma}_3$ as in Example3.1. Then the $\mathfrak{X}$-normalizers of G are of the form $C_2 \times \bar{\Sigma}_3$ where $C_2$ is a cyclic group of order 2. Since these are maximal in G and $G \notin \bar{\mathfrak{X}}$, they are the $\mathfrak{X}$-covering subgroups E of G (Theorem5.12). Let $\mathfrak{F}^*$ be the smallest saturated formation containing $C_2 \times \bar{\Sigma}_3$. Then $\mathfrak{F}^*$ contains $\Sigma_3$ and thus G. Hence E is not an $\mathfrak{F}^*$-covering subgroup of G and is thus not an $\mathfrak{F}$-covering subgroup of G for any saturated formation $\mathfrak{F}$.

We now turn our attention to $\mathfrak{N}\bar{\mathfrak{X}}$-groups i.e. groups G in which $G/F(G) \in \bar{\mathfrak{X}}$.

<u>LEMMA 5.15</u> Let $G \in \mathfrak{N}\bar{\mathfrak{X}}$, $G \notin \bar{\mathfrak{X}}$ and M be an $\mathfrak{X}$-abnormal

maximal subgroup of G. Then

(i) M is $\mathfrak{X}$-crucial in G.

(ii) The intersection of a chief series of G with M is a chief series of M, corresponding factors being operator-isomorphic.

(iii) If H/K is a chief factor of G covered by M, $H\cap M/K\cap M$ is $\mathfrak{X}\cap M$-central in M if and only if H/K is $\mathfrak{X}$-central in G.

(iv) The $\mathfrak{X}\cap M$-normalizers of M are $\mathfrak{X}$-normalizers of G.

<u>Proof</u>. The hypotheses imply $MF(G) = G$ and so $M\cap F(G) \lhd G$ and (ii) holds, by Lemma 1.1. $MG^{\bar{\mathfrak{X}}} = G$ since M is $\mathfrak{X}$-abnormal and $G^{\bar{\mathfrak{X}}} \leq F(G)$ since $G \in \mathfrak{N}\bar{\mathfrak{X}}$. Thus $M\cap G^{\bar{\mathfrak{X}}} \leq M\cap F(G) \leq \text{Core}M$ giving $M\cap G^{\bar{\mathfrak{X}}} = \text{Core}M \cap G^{\bar{\mathfrak{X}}} \lhd G$. $G^{\bar{\mathfrak{X}}}/G^{\bar{\mathfrak{X}}}\cap M$ is a chief factor of G since M is maximal in G, and so, by Lemma 5.5, M is $\mathfrak{X}$-crucial and (i) is proved.

We now prove (iii). Certainly H/K $\mathfrak{X}$-central implies $H\cap M/K\cap M$ is $\mathfrak{X}\cap M$-central, so it remains to prove the converse. Suppose, if possible, that this is not the case. Then, for some $\mathfrak{X}$-abnormal maximal subgroup M of G, there exists an $\mathfrak{X}$-eccentric p-chief factor H/K covered by M such that $H\cap M/K\cap M$ is $\mathfrak{X}\cap M$-central. Thus

(1) --- $X(p) \not\leq C$

(2) --- $X(p)\cap M \leq C\cap M$ where $C = C_G(H/K)$.

By (1) and (2), $X(p) \not\leq M$ and thus $X(p)M = G$ since M is maximal in G. As in 5.3 we obtain

(3) --- $M\cap C = (M\cap C)X(p) \cap C$ which is normal in G.

Since $C \geq F(G)$, we have $G/C \in \bar{\mathfrak{X}}$ and $CM = G$ so that $M/M\cap C \in \bar{\mathfrak{X}}$. Also, $M/M\cap C$ is $\mathfrak{Y}$-abnormal in $G/M\cap C$ (where

$\mathcal{Y} = (M \cap C)\mathfrak{X}/(M \cap C)$ ) and so is a $\mathcal{Y}$-covering subgroup of $G/M \cap C$ by Lemma5.2. Hence, by Theorem5.3, $G/X(p)(M \cap C) \in \bar{\mathcal{Y}}$ i.e. $G/X(p)(M \cap C) \in \bar{\mathfrak{X}}$. Thus, by (3) and Lemma3.2(ii), $G/M \cap C \in \bar{\mathfrak{X}}$, contradicting the fact that M is $\mathfrak{X}$-abnormal.

To prove (iv) we just have to show that $|D^{\mathfrak{X} \cap M}(M)| \leq |D^{\mathfrak{X}}(G)|$ (Theorem4.3) --- this follows immediately from (iii).

THEOREM 5.16 If $G \in \mathfrak{N}\bar{\mathfrak{X}}$, the $\mathfrak{X}$-normalizers and $\mathfrak{X}$-covering subgroups of G coincide.

Proof. The result is trivial if $G \in \bar{\mathfrak{X}}$. Thus assume $G \notin \bar{\mathfrak{X}}$ and let $D^{\mathfrak{X}} = H_0 < H_1 < \ldots\ldots < H_{r-1} < H_r = G$ be an $\mathfrak{X}$-abnormal maximal chain connecting $D^{\mathfrak{X}}$ to G (Theorem4.9). We show that in this case $H_i$ is in fact $\mathfrak{X} \cap H_{i+1}$ -crucial in $H_{i+1}$ for each i. $H_iF(G) = G$ for each i, since $H_i \geq D^{\mathfrak{X}}$ and $G/F(G) \in \bar{\mathfrak{X}}$ (Theorem4.2). Thus $H_i/H_i \cap F(G) \in \bar{\mathfrak{X}}$ for each i. Since $F(H_i) \geq H_i \cap F(G)$, this implies that $H_i \in \mathfrak{N}(\overline{\mathfrak{X} \cap H_i})$ for each i. The result then follows by (i) of Lemma5.15.

COROLLARY If $G \in \mathfrak{N}\mathfrak{N}\bar{\mathfrak{X}}$ (i.e. $G/F(G) \in \mathfrak{N}\bar{\mathfrak{X}}$ ) then each $\mathfrak{X}$-normalizer of G is contained in exactly one $\mathfrak{X}$-covering subgroup of G.

Proof. We omit the proof since it follows word for word that of Theorem5.9 of [4] (replacing $\mathfrak{F}$ by $\mathfrak{X}$ or $\bar{\mathfrak{X}}$ as appropriate).

THEOREM 5.17 If $G \in \mathfrak{N}\bar{\mathfrak{X}}$ and the subgroup H covers all $\mathfrak{X}$-central chief factors of G, then H contains an $\mathfrak{X}$-normalizer of G. In particular, if H also avoids the $\mathfrak{X}$-eccentric chief factors of G it is an $\mathfrak{X}$-normalizer of G.

Proof. Here, too, we follow closely along the lines of Theorem 5.7 of [4]. If $H = G$ the result is trivial. We thus assume that $G \notin \bar{\mathfrak{X}}$ and $H < G$. Let M be a maximal subgroup of G containing H. Then the hypothesis implies that M complements an $\mathfrak{X}$-eccentric chief factor and so is $\mathfrak{X}$-abnormal. By Lemma 5.15(ii) and (iii), H covers all the $\mathfrak{X} \cap M$-central chief factors of M and so contains an $\mathfrak{X} \cap M$ -normalizer of M by induction. The result then follows by (iv) of Lemma5.15. Finally, if H also avoids the $\mathfrak{X}$-eccentric chief factors of G, it will have the same order as the $\mathfrak{X}$-normalizers and will thus be one of them.

Our next result corresponds to Theorem5.15 of [4].

THEOREM 5.18 If $G^{\bar{\mathfrak{X}}}$ is abelian it is complemented in G and any two complements are conjugate. The complements are the $\mathfrak{X}$-normalizers of G.

Proof. This proof follows word for word that of Theorem5.15 of [4] (after replacing $\mathfrak{F}$ by $\mathfrak{X}$ or $\bar{\mathfrak{X}}$ as appropriate) up to the choice of B. Let $B = AX(p)$. Then N $\mathfrak{X}$-central implies $B \leq C_G(N)$. Since $G/A \in \bar{\mathfrak{X}}$ , B/A is p-nilpotent, so we can define Q as in the above-mentioned proof. Then $QA \not\leq C_G(A/N)$. For A/N $\mathfrak{X}$-eccentric implies $B \not\leq C_G(A/N)$. Thus, if $QA \leq C_G(A/N)$, we would have a non-trivial normal p-subgroup $BC_G(A/N)/C_G(A/N)$ of $G/C_G(A/N)$ which is impossible. So $QA \not\leq C_G(A/N)$. The remainder of Carter and Hawkes's proof then goes through unaltered.

In [4] Carter and Hawkes also show that if $G \in \mathfrak{N}\mathfrak{F}$ (where $\mathfrak{F}$ is a saturated formation) and L is a subgroup of G satisfying $L \in \mathfrak{F}$ and $LF(G) = G$, then $N_G(L)$ is contained in an $\mathfrak{F}$-covering subgroup of G. (Theorem5.8). However, this result does not carry over into the present situation merely by replacing $\mathfrak{F}$ by $\mathfrak{X}$ or $\bar{\mathfrak{X}}$ as appropriate, as the following example shows.

<u>EXAMPLE 5.19</u> Take $G = \Sigma_3 \times \bar{\Sigma}_3/\bar{S}_3$, $X(2) = G$, $X(3) = \Sigma_3\bar{S}_3/\bar{S}_3$ and $L = \Sigma^*\bar{S}_3/\bar{S}_3$ (see Example3.1). Then $G \in \mathfrak{N}\bar{\mathfrak{X}}$, $L \in \bar{\mathfrak{X}}$, $LF(G) = G$ and $G \notin \bar{\mathfrak{X}}$. L is normal in G and so $N_G(L)$ is not contained in an $\mathfrak{X}$-covering subgroup of G, since $G \notin \bar{\mathfrak{X}}$.

However, if in addition we replace $F(G)$ by $F(X)$ (where $X = \bigcap_p X(p)$ ), this theorem is then valid.

<u>LEMMA 5.20</u> Let L be a subgroup of G satisfying $L \in \bar{\mathfrak{X}}$ and $LF(X) = G$. Then $NL/N \in \bar{\mathfrak{X}}$ for all normal subgroups N of G.

<u>Proof</u>. Let H/K be a p-chief factor of NL above N. Then $H \cap L/K \cap L$ is a p-chief factor of L above $L \cap N$, and thus $X(p) \cap L \leq C_L(H \cap L/K \cap L) = L \cap C_{NL}(H/K)$, since $L \in \bar{\mathfrak{X}}$. We must show that this implies $X(p) \cap NL \leq C_{NL}(H/K)$. We first show that $L.F(X(p) \cap NL) = NL$. Now $F(X) \leq F(X(p))$ since $X \lhd X(p)$, and so $LF(X(p)) = G$. Thus $L(NL \cap F(X(p)) = NL$ and the result follows since $F(X(p)) \cap NL \leq F(X(p) \cap NL)$. Hence $X(p) \cap NL = X(p) \cap L.F(X(p) \cap NL) = (X(p) \cap L)F(X(p) \cap NL) \leq C_{NL}(H/K)$, since $F(X(p) \cap NL) \leq F(NL)$.

The proof of Theorem5.8 of [4], together with this lemma, gives us the required theorem viz.

<u>THEOREM 5.21</u> If $G \in \mathfrak{N}\bar{\mathfrak{X}}$ and L is a subgroup of G satisfying $L \in \bar{\mathfrak{X}}$ and $LF(X) = G$, then $N_G(L) \leq E$, an $\mathfrak{X}$-covering subgroup of G.

This theorem enables us to relate the $\mathfrak{X}$-covering subgroups of G to the $\mathfrak{X}$-covering subgroups of certain subgroups of G.

<u>THEOREM 5.22</u> Let L be a subgroup of G satisfying $LF(X) = G$. Then every $\mathfrak{X}\cap L$-covering subgroup of L is of the form $L \cap E$ for some $\mathfrak{X}$-covering subgroup E of G.

<u>Proof</u>. We consider separately the two cases $G/F(X) \in \bar{\mathfrak{X}}$ and $G/F(X) \notin \bar{\mathfrak{X}}$.

(a) $G/F(X) \in \bar{\mathfrak{X}}$. Then $L/L \cap F(X) \in \bar{\mathfrak{X}}$ since $LF(X) = G$. Thus an $\mathfrak{X}\cap L$-covering subgroup $E_1$ of L satisfies $E_1(L \cap F(X)) = L$ and thus $E_1F(X) = G$. Also $E_1 \in \bar{\mathfrak{X}}$ and $G \in \mathfrak{N}\bar{\mathfrak{X}}$. Therefore, by Theorem5.21, $E_1 \leq L \cap E$ for a suitable $\mathfrak{X}$-covering subgroup E of G. We show that $|L \cap E| \leq |E_1|$, giving equality. Now $L \in \mathfrak{N}(\overline{\mathfrak{X}\cap L})$ since $L \cap F(X) \leq F(L)$. Thus $E_1$ is an $\mathfrak{X}\cap L$-normalizer of L and E is an $\mathfrak{X}$-normalizer of G (Theorem5.16). Also, by Lemma1.1, if H/K is an $\mathfrak{X}$-central chief factor of G covered by L, then $H \cap L/K \cap L$ is an $\mathfrak{X}\cap L$-central chief factor of L. Thus, using Theorem4.2,

$|L \cap E| \leq$ the product of the orders of the $\mathfrak{X}$-central chief factors covered by L in a given chief series of G

$\leq$ the product of the orders of the $\mathfrak{X}\cap L$ -central chief factors in a given chief series of L

$= |E_1|$ .

(b) $G/F(X) \notin \bar{\mathfrak{X}}$ . Here, too, we follow closely the proof of 5.12 in [4]. Since $G/F(X) \notin \bar{\mathfrak{X}}$ , G has an $\mathfrak{X}$-crucial maximal subgroup M containing F(X). Let K = CoreM and H/K be the minimal normal subgroup of G/K. $F(X) \leq K$ and so L covers G/K. Thus, by Lemma1.1, $H\cap L/K\cap L$ is a chief factor of L, easily seen to be $\mathfrak{X}\cap L$ -eccentric (as in Lemma5.20). Also $L/L\cap H \in \bar{\mathfrak{X}}$ since $G/H \in \bar{\mathfrak{X}}$ . $M\cap L$ complements $H\cap L/K\cap L$ in L and so is an $\mathfrak{X}\cap L$ -crucial maximal subgroup of L. Now $(M\cap L)F(X) = M$ and $F(X) \leq F(M\cap X) \leq M$ . Thus $(M\cap L)F(M\cap X) = M$. Hence, applying induction to M, every $\mathfrak{X}\cap M\cap L$ -covering subgroup of $M\cap L$ is of the form $M\cap L\cap E$ for some $\mathfrak{X}\cap M$ -covering subgroup E of M i.e. every $\mathfrak{X}\cap L$ -covering subgroup of L is of the form $M\cap L\cap E = L\cap E$ for some $\mathfrak{X}$-covering subgroup E of G (Theorem5.8).

<u>COROLLARY</u> Let $\mathfrak{X}$ and $\mathfrak{Y}$ be two normal systems of G such that $\mathfrak{X} \geq \mathfrak{Y}$ and $\mathfrak{Y}$ is integrated, and let D be a $\mathfrak{Y}$-normalizer of G. Then every $\mathfrak{X}\cap D$ -covering subgroup of D is of the form $D\cap E$ for some $\mathfrak{X}$-covering subgroup E of G.

<u>Proof</u>. Since $\mathfrak{Y}$ is integrated and $\mathfrak{X} \geq \mathfrak{Y}$ , D can be connected to G by an $\mathfrak{X}$-critical maximal chain. The result then follows by repeated application of Theorem5.22.

<u>Remarks</u>. 1. Example5.19 shows that the condition $LF(X) = G$ in Theorem5.22 cannot be relaxed to $LF(G) = G$ as in the

corresponding Theorem5.12 of [4]. For in this example, L is not contained in an $\mathfrak{X}$-covering subgroup of G.

2. Corollary 2 of Theorem5.12 and Theorem5.22 (like the corresponding 5.2 and 5.12 of [4]) are useful in the determination of $\mathfrak{X}$-covering subgroups of wreath products of the form $C_p \wr G$ where $C_p$ is a cyclic group of order p, $p \nmid |G|$ and all X(q) contain the base group.

<u>THEOREM 5.23</u> Let $\mathfrak{X} = \{X(p)\}$ and $\mathfrak{Y} = \{Y(p)\}$ be two normal systems of G such that $\mathfrak{X} \geq \mathfrak{Y}$ and $\mathfrak{Y}$ is integrated. Then, if the $\mathfrak{X}$- and $\mathfrak{Y}$-covering subgroups of G coincide, so do the $\mathfrak{X}$- and $\mathfrak{Y}$-normalizers of G.

<u>Proof</u>. Since the $\mathfrak{X}$- and $\mathfrak{Y}$-covering subgroups coincide, $G^{\bar{\mathfrak{X}}} = G^{\bar{\mathfrak{Y}}} = \bar{G}$, say. We use induction on $|\bar{G}|$. If $\bar{G} = 1$, then $D^{\mathfrak{X}} = D^{\mathfrak{Y}} = G$ and the result is trivially true. Thus assume $\bar{G} > 1$. Then a chief factor $\bar{G}/L$ of G will be complemented by a maximal subgroup M which is both $\mathfrak{X}$- and $\mathfrak{Y}$-crucial in G. Thus the $\mathfrak{X} \cap M$ - and $\mathfrak{Y} \cap M$ -covering subgroups of M coincide, and hence, by induction, so do the $\mathfrak{X} \cap M$ - and $\mathfrak{Y} \cap M$ -normalizers of M. Thus, by Lemma4.11, $X(p) \cap M$ and $Y(p) \cap M$ centralize the same p-chief factors of M for each prime p. We show that this implies that X(p) and Y(p) centralize the same p-chief factors of G for each prime p. All the chief factors of G above $\bar{G}$ are both $\mathfrak{X}$- and $\mathfrak{Y}$-central and $\bar{G}/L$ is both $\mathfrak{X}$- and $\mathfrak{Y}$-eccentric, so it remains to consider the chief factors of G below L. Let H/K be a $\mathfrak{Y}$-central p-chief factor of G below L. Then H/K is a p-chief factor of M. For suppose that $K < J \leq H$ for some normal subgroup J of M. Then $[\, J, Y(p)\,] \leq K$

since Y(p) centralizes H/K, and so J is normalized by Y(p). But $\mathcal{Y}$ is integrated and thus $\bar{G} \leq Y(p)$ for each prime p, yielding $G = MY(p)$. Thus $J \lhd G$ and so $J = H$. $Y(p) \cap M \leq C_M(H/K)$ and so $X(p) \cap M \leq C_M(H/K)$. Thus, since $\mathcal{Y} \leq \mathcal{X}$, $X(p) = X(p) \cap MY(p) = Y(p)(X(p) \cap M) \leq C_G(H/K)$. Hence every $\mathcal{Y}$-central chief factor of G is $\mathcal{X}$-central and the converse is certainly true, since $\mathcal{Y} \leq \mathcal{X}$. By Lemma4.11, the $\mathcal{X}$- and $\mathcal{Y}$-normalizers of G thus coincide.

It is easily seen that the same conclusion would hold if $\mathcal{X}$ and $\mathcal{Y}$ were any two integrated normal systems of G. For in this case we would have (notation as above) $\bar{G} \leq X(p) \cap Y(p)$ and so $M(X(p) \cap Y(p)) = G$. Then $X(p) \cap M \leq C_M(H/K)$ would imply $X(p) = X(p) \cap M(X(p) \cap Y(p)) = (X(p) \cap Y(p))(X(p) \cap M) \leq C_G(H/K)$ as required. However, it need not hold for all normal systems $\mathcal{X}$ and $\mathcal{Y}$ of G, as the following example shows.

Take $G = \Sigma_4$, the symmetric group on 4 elements, $X(2) = X(3) = G$ and $Y(2)$ = the normal subgroup of order 4 of G, $Y(3) = G$. Then $\mathcal{X} \geq \mathcal{Y}$ but $\mathcal{Y}$ is not integrated as the 3-chief factor is $\mathcal{Y}$-eccentric. The $\mathcal{X}$- and $\mathcal{Y}$-covering subgroups of G are the Sylow 2-subgroups of G, but the $\mathcal{X}$- and $\mathcal{Y}$-normalizers of G are of order 2 and 8 respectively, and so do not coincide.

The following example shows that, as in Carter and Hawkes's theory, the converse of Theorem5.23 does not hold, even with $\mathcal{X} \geq \mathcal{Y}$ and $\mathcal{Y}$ an integrated normal system.

<u>EXAMPLE 5.24</u> Let $G = C_5 \wr A_4$, the wreath product of a cyclic group of order 5 with the alternating group on 4 elements (in its natural presentation). Then G is the semidirect product of an elementary abelian group N of order $5^4$ with $A_4$, and so $|G| = 2^2.3.5^4$. Let Z be the centre of G and H/N the normal subgroup of order 4 of G/N. Then $|Z| = 5$ and G has a chief series $G \underset{3}{>} H \underset{2^2}{>} N \underset{5^3}{>} Z \underset{5}{>} 1$ with $C_G(N/Z) = N$. Let $X(p) = G$ for $p=2,3,5$ ; $Y(p) = G$ for $p=2,3$ and $Y(5) = H$. Then $\mathfrak{X} \geq \mathfrak{Y}$ and $\mathfrak{Y}$ is integrated, and the $\mathfrak{X}$- and $\mathfrak{Y}$-normalizers are the system normalizers of G. However, the $\mathfrak{X}$-covering subgroups do not coincide with the $\mathfrak{Y}$-covering subgroups. For let E be an $\mathfrak{X}$-covering subgroup of G. Then, since $G/N \in \mathfrak{N}\bar{\mathfrak{X}}$, NE/N is an $N\mathfrak{X}/N$ -normalizer of G/N (Theorem5.16). Thus we can write $NE = N\Gamma$ where $\Gamma$ is a Sylow 3-subgroup of $A_4$. E is thus an $\mathfrak{X} \cap N\Gamma$ -covering subgroup of $N\Gamma$. Now $F(X \cap N\Gamma) = N$ so $\Gamma$ satisfies the conditions of Theorem5.22 in $N\Gamma$. Hence we can assume that $E \geq \Gamma$ and thus $E = (E \cap N)\Gamma$. We show that $E \cap N = C_N(X(5) \cap \Gamma)$.

As in Example5.14, since $5 \nmid |\Gamma|$ we get, by Maschke, $N = N_1 \times \ldots \times N_k$ where the $N_i$ are $\Gamma$-composition factors and thus E-composition factors of $N\Gamma$. Since $X(5) \cap N\Gamma \geq N$, we can apply Corollary2 of Theorem5.12, to get E covers $N_i$ if $X(5) \cap E \leq C_E(N_i)$ and avoids $N_i$ if $X(5) \cap E \not\leq C_E(N_i)$. Since $X(5) \cap E = (X(5) \cap \Gamma)(N \cap E)$, E covers $N_i$ if $N_i \leq C_N(X(5) \cap \Gamma)$ and avoids $N_i$ if $N_i \not\leq C_N(X(5) \cap \Gamma)$, and thus $N \cap E = C_N(X(5) \cap \Gamma)$ as required.

Hence $E = \Gamma C_N(X(5) \cap \Gamma) = \Gamma C_N(\Gamma)$, of order $3.5^2$. The same method also yields $E^{\mathfrak{Y}} = \Gamma C_N(Y(5) \cap \Gamma) = \Gamma N$, of order $3.5^4$.

We conclude the present chapter with two examples of $\mathfrak{X}$-covering subgroups for special normal systems and necessary and sufficient conditions for $\mathfrak{X}$- and $\mathfrak{Y}$-covering subgroups to coincide in one of these two special cases.

THEOREM 5.25 (i) If $X(p) = X$ for all primes p, then $E^{\mathfrak{X}} = N_G(C(X))$ where $C(X)$ is a Carter subgroup of X.

(ii) If $X(p) = X$ and $X(q) = 1$ for all primes $q \neq p$, then $E^{\mathfrak{X}} = D^{\mathfrak{X}} = N_G(X^p)$.

Proof. (i) $G/X \in \bar{\mathfrak{X}}$ and thus $XE^{\mathfrak{X}} = G$. Let $E = E^{\mathfrak{X}} \cap X$. We show that $N_G(E) = E^{\mathfrak{X}}$ and E is a Carter subgroup of X. Since $E^{\mathfrak{X}} \in \bar{\mathfrak{X}}$, $E = E^{\mathfrak{X}} \cap X \leq F(E^{\mathfrak{X}})$ and so is nilpotent. Suppose, if possible, that E is not self-normalizing in X, and let $H = N_X(E) > E$. Now $E \lhd E^{\mathfrak{X}}$ so $N_G(E) \geq E^{\mathfrak{X}}$. $H = X \cap N_G(E)$ and $HE^{\mathfrak{X}} = N_G(E)$. Let $H/K$ be a chief factor of $N_G(E)$ such that $K \geq E$. Then $N_G(E)/K \in \bar{\mathfrak{X}}$ and thus $N_G(E) = KE^{\mathfrak{X}}$. This gives $H = K(H \cap E^{\mathfrak{X}}) = K$, a contradiction. Thus $H = E$ and $E^{\mathfrak{X}} = N_G(E)$, giving the required result.

(ii) Let $D = N_G(X^p)$ be an $\mathfrak{X}$-normalizer of G. Then $DX = G$, since $G/X \in \bar{\mathfrak{X}}$. Let $D \lessdot H \leq G$ and suppose, if possible, that D is $\mathfrak{X} \cap H$-normal in H. Then, if $N/\mathrm{Core}_H D$ is the minimal normal subgroup of $H/\mathrm{Core}_H D$, $X \cap H \leq N$. Thus $X \cap D \leq \mathrm{Core}_H D$ and so is normal in H. i.e. $X \cap D < X \cap H \leq$

$N_X(X \cap D)$. This contradicts the fact that $X \cap D = N_X(X^p)$ which is abnormal in X. Hence $D \lessdot H \leq G$ implies that D is $\mathfrak{X} \cap H$ -abnormal in H. The result follows by Corollary 3 of Theorem5.12.

<u>COROLLARY</u> Let $\mathfrak{X} = \{X(p)\}$ and $\mathcal{Y} = \{Y(p)\}$ be two normal systems of G satisfying $X(p) = X$, $Y(p) = Y$ for all primes p, and $X \geq Y$. Then $E^{\mathfrak{X}} = E^{\mathcal{Y}}$ if and only if $X/Y$ is nilpotent and $C(Y) = Y \cap C(X)$.

<u>Proof</u>. Suppose $E^{\mathfrak{X}} = E^{\mathcal{Y}}$. Then $YE^{\mathfrak{X}} = G$ and so $G/Y \in \bar{\mathfrak{X}}$ by the homomorphism-invariance of $\mathfrak{X}$-covering subgroups. Thus $X/Y \leq F(G/Y)$ and so is nilpotent. Also, by Theorem5.25, $C(Y) = Y \cap E^{\mathcal{Y}} = Y \cap E^{\mathfrak{X}} = Y \cap C(X)$. Conversely, $C(Y) = Y \cap C(X)$ implies $E^{\mathfrak{X}} = N_G(C(X)) \leq N_G(C(Y)) = E^{\mathcal{Y}}$, by Theorem5.25. Also, since $X/Y$ is nilpotent, $G/Y \in \bar{\mathfrak{X}}$ and so $YE^{\mathfrak{X}} = G$. Hence $YC(X) = Y(X \cap E^{\mathfrak{X}}) = X$ and thus $|G : E^{\mathfrak{X}}| = |X : C(X)| = |Y : C(Y)| = |G : E^{\mathcal{Y}}|$ and the result follows.

## Chapter Six

# CHARACTERIZATIONS OF $\mathfrak{X}$ - NORMALIZERS

The property of $\mathfrak{X}$-normalizers basic to all our characterizations is the covering and avoidance property of Theorem 4.2. In general, this property does not characterize $\mathfrak{X}$-normalizers, for T.O.Hawkes has shown (Example1 of [11]) that a subgroup covering the central chief factors and avoiding the eccentric chief factors of a group need not be a system normalizer of the group. However, we have seen that in an $\mathfrak{N}\bar{\mathfrak{X}}$-group, the $\mathfrak{X}$-normalizers are those subgroups which cover the $\mathfrak{X}$-central and avoid the $\mathfrak{X}$-eccentric chief factors ( Theorem 5.17 ). We can extend this result to the case of any group G in which the $\mathfrak{X}$-normalizers and $\mathfrak{X}$-covering subgroups coincide.

THEOREM 6.1 Let G be a group with normal system $\mathfrak{X}$ in which the $\mathfrak{X}$-covering subgroups and $\mathfrak{X}$-normalizers coincide. Then the $\mathfrak{X}$-normalizers of G are those subgroups which cover the $\mathfrak{X}$-central and avoid the $\mathfrak{X}$-eccentric chief factors in a given chief series of G.

Proof. We use induction on $|G|$. Suppose H is a subgroup of G which covers the $\mathfrak{X}$-central and avoids the $\mathfrak{X}$-eccentric chief factors in a given chief series of G. Let N be the minimal normal subgroup of G appearing in this chief series of G. Then NH/N covers all $N\mathfrak{X}/N$ -central and avoids all $N\mathfrak{X}/N$ -eccentric chief factors in a chief series of G/N. Also, by the homomorphism-invariance of $\mathfrak{X}$-normalizers and $\mathfrak{X}$-covering subgroups, the $N\mathfrak{X}/N$ -covering subgroups and $N\mathfrak{X}/N$ -normalizers of G/N coincide. Hence, by induction, NH/N is an $N\mathfrak{X}/N$ -normalizer of G/N i.e. $NH/N = ND^{\mathfrak{X}}/N$ for some $\mathfrak{X}$-normalizer $D^{\mathfrak{X}}$ of G. Thus $NH = ND^{\mathfrak{X}} = NE^{\mathfrak{X}}$ for some $\mathfrak{X}$-covering subgroup $E^{\mathfrak{X}}$ of G. If N is $\mathfrak{X}$-central in G, this implies $H = D^{\mathfrak{X}}$. Thus we may assume that N is $\mathfrak{X}$-eccentric and thus avoided by $D^{\mathfrak{X}}$ and H. Now $NH/N = ND^{\mathfrak{X}}/N \in \bar{\mathfrak{X}}$ and thus $NH \in \mathfrak{N}(\overline{\mathfrak{X} \cap NH})$. By Theorem5.16, we then have $D^{\mathfrak{X} \cap NH}(NH) = E^{\mathfrak{X} \cap NH}(NH)$. But $E^{\mathfrak{X}} \leq NH$ and so is an $\mathfrak{X} \cap NH$ -covering subgroup of NH. Thus $D^{\mathfrak{X}} = E^{\mathfrak{X}} = D^{\mathfrak{X} \cap NH}(NH)$ and so $D^{\mathfrak{X}}$ covers the $\mathfrak{X} \cap NH$ -central and avoids the $\mathfrak{X} \cap NH$ -eccentric chief factors of NH (Theorem 4.2). Therefore, since $D^{\mathfrak{X}}$ complements N in NH, the chief factors of NH above N are $\mathfrak{X} \cap NH$ -central while those below N are $\mathfrak{X} \cap NH$ -eccentric. H also complements N in NH and so satisfies our hypotheses in NH. Hence, if $NH < G$, $H = D^{\mathfrak{X} \cap NH}(NH) = D^{\mathfrak{X}}$ as required. Thus we can assume that $NH = G$ and so H is maximal in G ( if $H = G$, $G \in \bar{\mathfrak{X}}$ and so $D^{\mathfrak{X}} = G = H$ ). $N = G^{\bar{\mathfrak{X}}}$ and so $HG^{\bar{\mathfrak{X}}} = G$. Also $H \in \bar{\mathfrak{X}}$ since G/N does. Thus H is an $\mathfrak{X}$-covering subgroup of G and so an $\mathfrak{X}$-normalizer of G.

We now return to the general case of an $\mathfrak{X}$-normalizer $D^{\mathfrak{X}}$ defined by a non-integrated normal system $\mathfrak{X} = \{X(p)\}$. For the sake of brevity we make the following definition.

<u>DEFINITION</u>. Let H be a subgroup of G which covers or avoids

each chief factor of G. We say that H satisfies <u>the centralizer condition</u> if, for each prime p dividing $|G|$, the intersection of the centralizers of the p-chief factors of G covered by H is not contained in the centralizer of any p-chief factor of G avoided by H ( where the intersection is understood to be G if H avoids all the p-chief factors of G ).

Every $\mathfrak{X}$-normalizer $D^{\mathfrak{X}}$ of G satisfies the centralizer condition, for X(p) centralizes precisely those p-chief factors of G covered by $D^{\mathfrak{X}}$ (Theorem4.2). Hence this is a necessary condition for a subgroup to be an $\mathfrak{X}$-normalizer. We thus consider a subgroup H of G which covers or avoids each chief factor of G and satisfies the centralizer condition. Then, if $\mathfrak{X} = \{X(p)\}$ is the normal system of G obtained by setting

$$6.2 \quad \begin{cases} X(p) = \text{the intersection of the centralizers of the p-chief factors of G covered by H} & \text{if there is at least one p-chief factor of G covered by H.} \\ \quad\;\; = G & \text{if H avoids all the p-chief factors of G.} \end{cases}$$

X(p) centralizes precisely those p-chief factors of G covered by H, and so $|H| = |D^{\mathfrak{X}}|$. However, by the above-mentioned example of T.O.Hawkes, H need not equal $D^{\mathfrak{X}}$. We therefore seek a further condition on H which will force these subgroups to coincide --- in fact we find two possible conditions.

The first involves the way in which H is embedded in the group G. We recall the proof of Theorem4.9 in which any non-integrated $\mathfrak{X}$-normalizer $D^{\mathfrak{X}}$ (i.e. an $\mathfrak{X}$-normalizer defined by non-integrated normal system $\mathfrak{X}$) of G is shown to be $\mathfrak{X}$-subabnormal in G. We first choose a normal system $\mathfrak{X}_0$ of G minimal with respect to the conditions $\mathfrak{X}_0 \leq \mathfrak{X}$ and $D^{\mathfrak{X}_0} = D^{\mathfrak{X}}$. We then select a chief factor $X_0(p_0)/Y_0(p_0)$ of G which is covered by $D^{\mathfrak{X}_0}$ ( at least one such chief factor exists ). Setting $Y_0(q) = X_0(q)$ for all primes $q \neq p_0$, we obtain, with $Y_0(p_0)$, a normal system $\mathfrak{Y}_0$ of G. Then $D^{\mathfrak{X}_0}$ is of index a power of $p_0$ in $D^{\mathfrak{Y}_0}$ (where $D^{\mathfrak{X}_0}$ and $D^{\mathfrak{Y}_0}$ are defined by the same Sylow system of G). We choose a minimal normal system $\mathfrak{X}_1$ such that $\mathfrak{X}_1 \leq \mathfrak{Y}_0$ and $D^{\mathfrak{Y}_0} = D^{\mathfrak{X}_1}$, and repeat the whole process, obtaining a chain

$$D^{\mathfrak{X}} = D^{\mathfrak{X}_0} < D^{\mathfrak{X}_1} < \ldots . < D^{\mathfrak{X}_r} = G$$

of normalizers in which $D^{\mathfrak{X}_i}$ is of index a prime power in $D^{\mathfrak{X}_{i+1}}$ for each i. Thus each member of the above chain covers or avoids the chief factors of G and satisfies the centralizer condition. Also, the prime $p_i$ dividing $|D^{\mathfrak{X}_{i+1}} : D^{\mathfrak{X}_i}|$ is that appearing in the selected chief factor $X_i(p_i)/Y_i(p_i)$ covered by $D^{\mathfrak{X}_i}$; and $X_i(p_i)$, $Y_i(p_i)$ centralize precisely those $p_i$-chief factors of G covered by $D^{\mathfrak{X}_i}$, $D^{\mathfrak{X}_{i+1}}$ respectively for each i. We prove that if H is embedded in G in this manner, it is an $\mathfrak{X}$-normalizer of G.

<u>THEOREM 6.3</u> Let H be a subgroup of G satisfying the following three conditions

(i) H covers or avoids each chief factor of G

(ii) the centralizer condition

(iii) there exists a prime power chain

$H = H_0 < H_1 < \ldots < H_r = G$ of subgroups of G such that

(a) each $H_i$ satisfies (i) and (ii)

(b) for each i there exists a chief factor $X_i(p)/Y_i(p)$ of G such that $X_i(p)$, $Y_i(p)$ centralize precisely those p-chief factors of G covered by $H_i$, $H_{i+1}$ respectively and $H_i$ covers $X_i(p)/Y_i(p)$. (where p is the prime dividing $|H_{i+1} : H_i|$ ).
Then H is an $\mathfrak{X}$-normalizer of G, where $\mathfrak{X} = \{X(p)\}$ and $X(p)$ centralizes precisely those p-chief factors of G covered by H ( e.g. $\mathfrak{X}$ as defined in 6.2 ).

Proof. Let $|H_1 : H_0|$ be a power of prime p and define, for all primes $q \neq p$, $X_0(q)$ as the intersection of the centralizers of the q-chief factors of G covered by $H_0$ (or G if $H_0$ avoids all the q-chief factors of G). We thus obtain, with $X_0(p)$, a normal system $\mathfrak{X}_0$ of G such that $X_0(q)$ centralizes precisely those q-chief factors of G covered by H for each prime q. We show that $H_0$ is an $\mathfrak{X}_0$-normalizer of G.

Define $Y_0(q) = X_0(q)$ for all primes $q \neq p$, obtaining, with $Y_0(p)$, a normal system $\mathcal{Y}_0$ of G. Since $|H_1 : H_0|$ is a power of p, $H_1$ and $H_0$ cover the same q-chief factors of G for all primes $q \neq p$ . Thus, by induction, $H_1$ is a $\mathcal{Y}_0$-normalizer of G ---- say $H_1 = \bigcap_q N_G(Y_0^q)$ , where $Y_0^q = Y_0(q) \cap S^q$ and $\mathcal{S} = \{ S^q \}$ is a Sylow system of G. If we now set $X_0^q = X_0(q) \cap S^q$ for all primes q, we have

$$\bigcap_q N_G(X_0^q) = H_1 \cap N_G(X_0^p) = N_{H_1}(X_0^p) .$$

We show that $H_0$ is a conjugate of $N_{H_1}(X_0^p)$ and is thus an $\mathfrak{X}_0$-normalizer of G.

Since $H_0$ covers precisely those p-chief factors of G centralized by $X_0(p)$, $X_0(p) \cap H_0$ centralizes all the p-chief factors of $H_o$. Thus $X_o(p) \cap H_o \leq O_{p'p}(H_o)$ and so $X_o(p) \cap H_o$ has a normal p-complement Q. Q is then characteristic in the normal subgroup $X_o(p) \cap H_o$ of $H_o$ and so $Q \lhd H_o$. Thus

(1) --- $H_o \leq N_{H_1}(Q)$ .

Now $\mathcal{S} = \{ S^q \}$ reduces into $H_1$, the $\mathcal{Y}_o$-normalizer of G defined by it. Hence $X_o^p \cap H_1$ is a p-complement of $X_o(p) \cap H_1$. But $|H_1 : H_o|$ is a power of p, so Q is also a p-complement of $X_o(p) \cap H_1$. Hence, by Hall, $Q^g = X_o^p \cap H_1$ for some $g \in X_o(p) \cap H_1$. Thus

(2) --- $N_{H_1}(X_o^p) \leq N_{H_1}(Q^g)$ .

On the other hand, $H_o$ covers $X_o(p)/Y_o(p)$ and so

$$X_o(p) = (H_o \cap X_o(p))Y_o(p) = (H_o \cap X_o(p))^g.Y_o(p) .$$

This implies that $X_o^p = Q^g Y_o^p$ , giving

(3) --- $N_{H_1}(Q^g) \leq N_{H_1}(X_o^p)$

since $H_1$ normalizes $Y_o^p$. (2) and (3) give $N_{H_1}(X_o^p) = N_{H_1}(Q^g) = ( N_{H_1}(Q) )^g$ , and so $|H_o| = |D^{\mathfrak{X}_o}| = |N_{H_1}(X_o^p)| = |N_{H_1}(Q)|$ . Thus, by (1), $H_o = N_{H_1}(Q) = ( N_{H_1}(X_o^p) )^{g^{-1}}$ as required.

<u>COROLLARY</u>. Conditions (i),(ii),(iii) of Theorem6.3 characterize non-integrated $\mathfrak{X}$-normalizers.

T.O.Hawkes's example shows that (b) cannot be omitted from (iii) in the above theorem :

Take G as in Example 1 of [11] (page 344). Let $H = \langle s \rangle \times \langle zc_3 \rangle$ , the subgroup of G which is shown to cover the central and avoid the eccentric chief factors of G but is not a system normalizer of G. Then $H < \langle s \rangle.K < G$ is a prime power chain connecting H to G, for $| \langle s \rangle K : H| = 5^6$

and $|G : \langle s\rangle K| = 3$. H certainly satisfies (i) and (ii), and so does $\langle s\rangle K$ as it is an $\mathfrak{X}$-normalizer of G for the normal system $\mathfrak{X} = \{X(p)\}$ defined by $X(2) = X(3) = G$, $X(5) = K$. Thus H satisfies (i), (ii) and (iii)(a) but is not a system normalizer of G.

The second condition is much simpler. In the proof of Corollary1 of Theorem4.2 we saw that if $D^{\mathfrak{X}}$ is an $\mathfrak{X}$-normalizer of G defined by Sylow system $\mathcal{S} = \{ S^p \}$ of G, then $D^{\mathfrak{X}}S^p = N_G(S^p \cap X(p))$ . i.e. $D^{\mathfrak{X}}$ commutes with a Sylow p-complement of G for each prime p dividing $|G|$. We show that if a subgroup H which covers or avoids the chief factors of G and satisfies the centralizer condition has this property, then it is an $\mathfrak{X}$-normalizer of G.

<u>LEMMA 6.4</u> Let H be a subgroup of G. Then H commutes with a Sylow p-complement of G for each prime p dividing $|G|$ if and only if H is an intersection of subgroups of prime power index in G.

<u>Proof</u>. Assume firstly that H commutes with the complete set $\{ S^p \}$ of Sylow p-complements of G. Then $HS^p$ is a subgroup of G for each prime p, and $H \leq L = \bigcap_{p \mid |G|} HS^p$ . We show that these two subgroups are in fact equal. Suppose, if possible, that $H < L$ , and let q be a prime dividing $|L:H|$. Then q divides $|HS^q : H| = |S^q : S^q \cap H|$ which is clearly impossible. Thus $H = L$ and the result follows since $|G : HS^p|$ is a power of p for each prime p.

Conversely, let $H = \bigcap_{p \in \omega} X^p$ where $|G:X^p|$ is a power of p, and $\omega \leq \pi$, the set of prime divisors of $|G|$. Let $S^p$ be a p-complement of $X^p$ and thus of G. Then $X^p = HS^p$ since $|X^p : S^p|$ is a power of p and $|X^p : H|$ is prime to p. Hence H commutes with a Sylow p-complement of G for all $p \in \omega$. Let $p \in \pi \setminus \omega$. Then $p \nmid |G:H|$ and so $HS^p = G$ for any p-complement $S^p$ of G, and we are done.

LEMMA 6.5 Let H be a subgroup of prime power index in G satisfying the two conditions

(i) H covers or avoids each chief factor of G

(ii) the centralizer condition.

Then H is an $\mathfrak{X}$-normalizer of G. More precisely, H is the normalizer in G of a p-complement of the normal subgroup X(p) defined in 6.2.

Proof. Let $|G:H|$ be a power of prime p, say, and define $\mathfrak{X} = \{X(p)\}$ as in 6.2. Then, as we have already seen, $|H| = |D^{\mathfrak{X}}|$ Let $S^p$ be a p-complement of H and thus of G, and set $X^p = S^p \cap X(p)$. Then $X^p$ is in fact a p-complement of $X(p) \cap H$. Now $X(p) \cap H$ is p-nilpotent since X(p) centralizes precisely the p-chief factors of G covered by H. Thus $X^p$ is a characteristic subgroup of the normal subgroup $X(p) \cap H$ of H and so is normal in H. Hence $H \leq N_G(X^p)$, an $\mathfrak{X}$-normalizer of G ( since $X(q) = O_{q'q}(G)$ for all primes $q \neq p$ ). Thus $H = N_G(X^p)$.

Our second characterization of non-integrated $\mathfrak{X}$-normalizers follows easily from these two lemmas.

THEOREM 6.6 Non-integrated $\mathfrak{X}$-normalizers of G can be characterized as those subgroups H of G which satisfy the following three conditions

(i) H covers or avoids each chief factor of G

(ii) the centralizer condition

(iii) H commutes with a Sylow p-complement of G for each prime p dividing $|G|$.

Proof. Let H satisfy (i) and (ii) and commute with set $\{ S^p \}$ of p-complements of G, and let $\mathfrak{X} = \{X(q)\}$ be defined as in 6.2. Then, by Lemma6.4, $H = \bigcap_p HS^p$. We show that $HS^p = N_G(S^p \cap X(p))$ for each prime p, so that H is an $\mathfrak{X}$-normalizer of G. Since $|HS^p : H|$ is prime to p, $HS^p$ covers the p-chief factors of G covered by H and avoids the remaining p-chief factors. Also, $HS^p$ covers all q-chief factors of G for $q \neq p$, since $|G : HS^p|$ is a power of p. Thus $HS^p$ covers or avoids each chief factor of G and satisfies the centralizer condition for prime p since H does and for all other primes $q \neq p$ trivially. Let $\mathfrak{Y} = \{Y(q)\}$ be the natural normal system of G arising from $HS^p$ as in 6.2. Then $Y(p) = X(p)$. By Lemma6.5, $HS^p = N_G(Y(p) \cap S^p)$ and the result follows.

Since any $\mathfrak{X}$-normalizer of G satisfies (i), (ii) and (iii), the theorem is proved.

Two characterizations of integrated $\mathfrak{X}$-normalizers (i.e. $\mathfrak{X}$-normalizers for integrated normal systems $\mathfrak{X}$ ) are now easily obtained. If $D^{\mathfrak{X}}$ is an integrated $\mathfrak{X}$-normalizer it has, in addition, the following property :

"The intersection of a chief series of G with $D^{\mathfrak{X}}$ is a chief series of $D^{\mathfrak{X}}$ with corresponding factors operator-isomorphic". ( Theorem4.13 )

We say that if a subgroup H of G has this property, it satisfies the chief series condition (in G). It is this additional condition which gives us the required characterizations.

LEMMA 6.7 Let H be a subgroup of G satisfying the three conditions (i) H covers or avoids each chief factor of G.

(ii) the centralizer condition.

(iii) the chief series condition.

and let the normal system $\mathfrak{X} = \{X(p)\}$ be defined as in 6.2. Then $\mathfrak{X}$ is integrated and $HX(p) = G$ for each prime p.

Proof. Let K/L be a p-chief factor of G covered by H. Then $H \cap C_G(K/L) = C_H(K \cap H/L \cap H)$ and $H \cap K/H \cap L$ is a chief factor of H operator-isomorphic to K/L (by (iii) ). Thus we have $G/C_G(K/L) \cong HC_G(K/L) / C_G(K/L)$ with $HC_G(K/L) \leq G$. Hence $G = HC_G(K/L)$. Now (ii) shows that if K/L, $\bar{K}/\bar{L}$ are operator-isomorphic chief factors of G and H covers K/L, then H also covers $\bar{K}/\bar{L}$. Thus every chief factor of G above X(p) is covered by H. For such a chief factor is operator-isomorphic to one above $C_G(K/L)$ for some p-chief factor K/L covered by H, and $G = HC_G(K/L)$. Hence $HX(p) = G$ for each prime p. $\mathfrak{X}$ is integrated since any q-chief factor of G above X(p) is covered by H and thus centralized by X(q) for all primes p and q.

This lemma, together with Lemma6.5, gives us a characterization of integrated $\mathfrak{X}$-normalizers similar to Theorem6.3.

<u>THEOREM 6.8</u> Integrated $\mathfrak{X}$-normalizers of G can be characterized as those subgroups H of G which satisfy the following four conditions

(i) H covers or avoids each chief factor of G

(ii) the centralizer condition

(iii) the chief series condition

(iv) there exists a prime power chain $H = H_0 < H_1 < \ldots$ $\ldots < H_r = G$ of subgroups of G such that each $H_i$ satisfies conditions (i),(ii) and (iii).

<u>Proof</u>. An integrated $\mathfrak{X}$-normalizer $D^{\mathfrak{X}}$ of G certainly satisfies (i),(ii) and (iii), and is easily seen to satisfy (iv). For we can modify very slightly the process used in Theorem4.9 (and described earlier in this chapter) to yield a chain $D^{\mathfrak{X}} = D^{\mathfrak{X}_0} < \ldots < D^{\mathfrak{X}_r} = G$ of normalizers in which every normal system $\mathfrak{X}_i$ is, in fact, integrated : Starting with the normal system $\mathfrak{X}$, we choose normal system $\mathfrak{X}_0$ minimal with respect to the 3 conditions $D^{\mathfrak{X}} = D^{\mathfrak{X}_0}$, $\mathfrak{X} \geq \mathfrak{X}_0$ and $\mathfrak{X}_0$ is integrated. As before, we can now select a chief factor $X_0(p_0)/Y_0(p_0)$ covered by $D^{\mathfrak{X}_0}$ for some prime $p_0$. Then the normal system $\mathfrak{Y}_0$ obtained by taking $Y_0(q) = X_0(q)$ for all primes q $\neq p_0$, together with $Y_0(p_0)$, is integrated. Thus $D^{\mathfrak{X}_0} < D^{\mathfrak{Y}_0}$ and we can continue in this manner until the whole group is reached.

Thus we assume that the subgroup H of G satisfies (i),.., (iv). Let $\mathfrak{X} = \{X(q)\}$, $\mathfrak{Y} = \{Y(q)\}$ be the natural normal systems of G arising from $H, H_1$ respectively as in 6.2. Then $\mathfrak{X} \geq \mathfrak{Y}$ and, by Lemma6.7, $\mathfrak{X}$ and $\mathfrak{Y}$ are integrated. Let $|H_1 : H|$

be a power of prime p. We show that H is an $\mathfrak{X} \cap H_1$ -normalizer of $H_1$. Let $K \cap H_1/L \cap H_1$ be a chief factor of $H_1$. Then K/L is a chief factor of G covered by $H_1$. If H avoids K/L it certainly avoids $K \cap H_1/L \cap H_1$, and if H covers K/L we have

$$(K \cap H_1 \cap H)(L \cap H_1) = (K \cap H)(L \cap H_1) = (K \cap H)L \cap H_1 = K \cap H_1$$

i.e. H covers $K \cap H_1/L \cap H_1$. Thus H covers or avoids each chief factor in a chief series of $H_1$ arising by intersection with a given chief series of G.

Now define normal system $\mathfrak{Z} = \{Z(p)\}$ of $H_1$ as follows. For $q \neq p$, let $Z(q) = O_{q'q}(H_1)$ --- then $Z(q) = H_1 \cap Y(q)$, by definition of Y(q) and (iii) for $H_1$. Since $|H_1 : H|$ is a power of p, $Y(q) = X(q)$ for all primes $q \neq p$. Hence, for $q \neq p$, $Z(q) = H_1 \cap X(q)$. Let Z(p) be the intersection of the centralizers (in $H_1$) of the p-chief factors $K \cap H_1/L \cap H_1$ of $H_1$ covered by H in the above-mentioned chief series of $H_1$ (or G if H avoids all the p-chief factors of $H_1$ in this chief series ). Then Z(p) is the intersection of $H_1$ with the intersection of the centralizers (in G) of the p-chief factors K/L of G covered by H in the given chief series of G. Hence $Z(p) = H_1 \cap X(p)$ by the Jordan-Hölder Theorem and (ii) for H. Thus $\mathfrak{Z} = \mathfrak{X} \cap H_1$.

We show that H satisfies the centralizer condition within the chief series of $H_1$ under consideration. This is trivially true for all primes $q \neq p$. Suppose, if possible, that $Z(p) \leq C_{H_1}(K \cap H_1/L \cap H_1)$ for some p-chief factor $K \cap H_1/L \cap H_1$ of $H_1$ avoided by H. Then $H_1 \cap X(p) \leq H_1 \cap C_G(K/L)$ where K/L is a p-chief factor of G covered by $H_1$ but avoided by H --- thus

$Y(p) \leq C_G(K/L)$ . By Lemma6.7, $H_1Y(p) = G$. Hence

$X(p) = X(p) \cap H_1Y(p) = Y(p)(H_1 \cap X(p)) \leq Y(p)(H_1 \cap C_G(K/L)) = H_1Y(p) \cap C_G(K/L) = C_G(K/L)$ . This contradicts the fact that H satisfies (ii), giving the required result.

Thus, in $H_1$, H satisfies conditions (i) and (ii) of Lemma 6.5 within a given chief series of $H_1$ and is of index a prime power. The proof of Lemma6.5 goes through in this situation, giving H a $\mathfrak{Z}$-normalizer of $H_1$ i.e. an $\mathfrak{X} \cap H_1$ -normalizer of $H_1$. By induction $H_1$ is a $\mathfrak{Y}$-normalizer of G, and thus H is an $\mathfrak{X}$-normalizer of G by Theorem4.17. The theorem is thus proved.

Our second characterization of integrated $\mathfrak{X}$-normalizers is an immediate consequence of Theorem6.6 and Lemma6.7.

<u>THEOREM 6.9</u> Integrated $\mathfrak{X}$-normalizers of G can be characterized as those subgroups H of G satisfying the following four conditions

(i) H covers or avoids each chief factor of G

(ii) the centralizer condition

(iii) the chief series condition

(iv) H commutes with a Sylow p-complement of G for each prime p dividing $|G|$.

## Chapter Seven

# FISCHER'S $\mathfrak{M}$-NORMALIZERS AND $\mathfrak{M}$-COVERING SUBGROUPS

In this final chapter we apply our methods of Chapters 3, 4 and 5 to the following even more general situation considered by B.Fischer ([5]). G is assumed to be a fixed finite soluble group, $\Lambda$ a non-empty finite set, and $\pi_\lambda$ a set of primes for each $\lambda \in \Lambda$ --- without loss of generality $\pi_\lambda \leq \pi$, the set of prime divisors of $|G|$, for each $\lambda$. Then normal subgroups $M(\pi_\lambda)$ of G are chosen, one for each $\lambda \in \Lambda$, forming a set $\mathfrak{M} = \{ M(\pi_\lambda) \mid \lambda \in \Lambda \}$. This set $\mathfrak{M}$ takes the place of the normal system $\mathfrak{X}$ in our theory --- we denote by $\mathfrak{M} \cap H$ the set $\{ M(\pi_\lambda) \cap H \mid \lambda \in \Lambda \}$ for H any subgroup of G, and by $N\mathfrak{M}/N$ the set $\{ NM(\pi_\lambda)/N \mid \lambda \in \Lambda \}$ for N any normal subgroup of G. We prove some additional properties of $\mathfrak{M}$-normalizers and give an alternative approach to Fischer's $\mathfrak{M}$-covering subgroups.

We begin with Fischer's definition of an $\mathfrak{M}$-normalizer of G.

<u>DEFINITION</u>. Let $\mathcal{S} = \{ S^p \}$ be a Sylow system of G. Then $S^{\pi_\lambda} = \bigcap_{p \in \pi_\lambda} S^p$ is a Hall $\pi_\lambda$-complement of G for each $\lambda \in \Lambda$. Write $M^{\pi_\lambda} = S^{\pi_\lambda} \cap M(\pi_\lambda)$ for each $\lambda \in \Lambda$. Then $M^{\pi_\lambda}$ is a Hall $\pi_\lambda$-complement of $M(\pi_\lambda)$, since $M(\pi_\lambda) \lhd G$. We call $\{M^{\pi_\lambda} \mid \lambda \in \Lambda\}$ the <u>$\mathfrak{M}$-system</u> of G defined by $\mathcal{S}$ and $D^{\mathfrak{M}}(G) = \bigcap_{\lambda \in \Lambda} N_G(M^{\pi_\lambda})$ the <u>$\mathfrak{M}$-system normalizer</u> or, simply, the <u>$\mathfrak{M}$-normalizer</u> of G defined by $\mathcal{S}$.

e.g. $\Lambda = \{\lambda\}$, $\pi_\lambda = \pi \setminus \{p\}$ (for some prime $p \in \pi$), $M(\pi_\lambda) = G$. Then $D^{\mathfrak{M}}(G) = N_G(S_p)$ where $S_p$ is a Sylow p-subgroup of G.

By P.Hall, all the $\mathfrak{M}$-systems and hence all the $\mathfrak{M}$-normalizers of G are conjugate, and every $\mathfrak{M}$-normalizer clearly contains a system normalizer of G. In addition, Fischer shows that $ND^{\mathfrak{M}}(G)/N = D^{N\mathfrak{M}/N}(G/N)$ for any normal subgroup N of G i.e. the $\mathfrak{M}$-normalizers are homomorphism-invariant. However, unlike $\mathfrak{X}$-normalizers, $\mathfrak{M}$-normalizers need not cover or avoid each chief factor of G, as the following example shows.

<u>EXAMPLE 7.1</u> Take G to be the primitive soluble group of order 168 mentioned on page525 of [10]. Let $\Lambda = \{\lambda\}$, $\pi_\lambda = \{2,7\}$ and $M(\pi_\lambda) = G$. Then $M^{\pi_\lambda}$ is a Sylow 3-subgroup of G with normalizer of order 6. Thus $D^{\mathfrak{M}} = N_G(M^{\pi_\lambda})$ does not cover or avoid the minimal normal subgroup of G of order $2^3$.

We show first of all that an $\mathfrak{M}$-normalizer nevertheless covers certain chief factors of G.

<u>THEOREM 7.2</u> $D^{\mathfrak{M}}$ covers each p-chief factor of G which is centralized by all $M(\pi_\lambda)$ with $p \in \pi_\lambda$.

<u>Proof</u>. Let N be a minimal normal subgroup of G of order a power of prime p, and let $M(\pi_\lambda) \leq C = C_G(N)$ for all $\lambda$ such that $p \in \pi_\lambda$. N splits into central p-chief factors in C and thus is covered by $\bigcap_{\substack{\lambda\in\Lambda \\ p\in\pi_\lambda}} N_C(M^{\pi_\lambda})$, since this subgroup contains a system normalizer of C. Hence $\bigcap_{\substack{\lambda\in\Lambda \\ p\in\pi_\lambda}} N_G(M^{\pi_\lambda}) \geq N$. Also, for $p \notin \pi_\lambda$, $N \leq S^{\pi_\lambda} \leq N_G(M^{\pi_\lambda})$. Thus

$N \leq \bigcap_{\lambda\in\Lambda} N_G(M^{\pi_\lambda}) = D^{\mathfrak{M}}$. The result then follows by the homomorphism-invariance of $\mathfrak{M}$-normalizers.

A further contrast to the theory of $\mathfrak{X}$-normalizers appears in the fact that whereas $|N_G(X^p) : D^{\mathfrak{X}}|$ is prime to p, $|N_G(M^{\pi_\lambda}) : D^{\mathfrak{M}}|$ need not be prime to $\pi_\lambda$. This is due to the fact that the same prime p may occur in several different $\pi_\lambda$, and is easily illustrated. Take G as in Example7.1, and let $\Lambda = \{\lambda, \mu\}$, $\pi_\lambda = \{2,7\}$, $\pi_\mu = \{2\}$ and $M(\pi_\lambda) = M(\pi_\mu) = G$. Then $N_G(M^{\pi_\mu})$ is of order 21 and $N_G(M^{\pi_\lambda})$ is of order 6. Hence $|D^{\mathfrak{M}}| = 3$, giving $|N_G(M^{\pi_\lambda}) : D^{\mathfrak{M}}| = 2 \in \pi_\lambda$.

We now turn to the problem of showing that $\mathfrak{M}$-normalizers are subabnormal in G. Our approach is similar to that used in Chapter4 to prove the $\mathfrak{X}$-normalizers are $\mathfrak{X}$-subabnormal --- in fact, all but one of the proofs are so similar to the corresponding proofs in Chapter4 that they are only briefly sketched. It is only in the proof that a Sylow system of G reduces into the $\mathfrak{M}$-normalizer of G which it defines that a little ingenuity is needed. For our proof of the corresponding result for $\mathfrak{X}$-normalizers depended on the fact that $|N_G(X^p) : D^{\mathfrak{X}}|$ was prime to p ( Corollary1 of Theorem4.2 ) whereas we have just seen that in the present situation $|N_G(M^{\pi_\lambda}) : D^{\mathfrak{M}}|$ need not be prime to $\pi_\lambda$. We start with this result.

<u>THEOREM 7.3</u> A Sylow system of G reduces into the $\mathfrak{M}$-normalizer of G which it defines.

<u>Proof</u>. Let $\mathcal{S} = \{ S^p \}$ be a Sylow system of G and $S^{\pi_\lambda} =$

$\bigcap_{p \in \pi_\lambda} S^p$, $M^{\pi_\lambda} = S^{\pi_\lambda} \cap M(\pi_\lambda)$ for each $\lambda \in \Lambda$. We must show that $\mathcal{S}$ reduces into $H = \bigcap_{\lambda \in \Lambda} N_G(M^{\pi_\lambda})$ . Let $H^p$ be a p-complement of H for each prime p dividing $|G|$. Now $Z(p) = \langle M^{\pi_\lambda} \mid p \in \pi_\lambda \rangle$ is a subgroup of $S^p$ by the definition of $M^{\pi_\lambda}$, and so is a p'-group. By the definition of H, $H^p$ normalizes each $M^{\pi_\lambda}$ and thus Z(p). Hence $H^pZ(p)$ is a p'-subgroup of G for each prime p $\in \pi$ --- let $T^p$ be a p-complement of G containing $H^pZ(p)$. Then $\mathcal{J} = \{ T^p \}$ is a Sylow system of G reducing into H, and there exists $g \in G$ such that $\mathcal{J} = \mathcal{S}^g$ ( by P.Hall ). We show that $g \in H$. For $\lambda \in \Lambda$, $(M^{\pi_\lambda})^g = M(\pi_\lambda) \cap (S^{\pi_\lambda})^g$ = $M(\pi_\lambda) \cap \bigcap_{p \in \pi_\lambda} (S^p)^g = M(\pi_\lambda) \cap \bigcap_{p \in \pi_\lambda} T^p \geq M^{\pi_\lambda}$. Thus $(M^{\pi_\lambda})^g = M^{\pi_\lambda}$ for all $\lambda \in \Lambda$ and so $g \in H$. Hence $S^p = (T^p)^h$ for all $p \in \pi$ and some $h \in H$. Thus $S^p \cap H = (T^p)^h \cap H = (T^p \cap H)^h = (H^p)^h$, a p-complement of H, for each $p \in \pi$ i.e. $\mathcal{S}$ reduces into H.

The following simple lemma takes the place of Lemma4.6.

LEMMA 7.4 $N_G(M^{\pi_\lambda})$ is abnormal in G for each $\lambda \in \Lambda$ .

Proof. By definition of $M^{\pi_\lambda}$, $N_G(M^{\pi_\lambda}) \geq N_G(S^{\pi_\lambda})$. A Frattini argument shows that every subgroup of G containing $N_G(S^{\pi_\lambda})$ and hence every subgroup of G containing $N_G(M^{\pi_\lambda})$ is self-normalizing, and the result follows immediately by Taunt.

We now consider the variation of the $\mathfrak{M}$-normalizers with $\mathfrak{M}$. Suppose, now, that $\mathcal{L} = \{ L(\pi_\lambda) \mid \lambda \in \Lambda \}$ is another set of normal subgroups of G with $L(\pi_\lambda) \leq M(\pi_\lambda)$ for each $\lambda \in \Lambda$. Then, if we write $L^{\pi_\lambda} = S^{\pi_\lambda} \cap L(\pi_\lambda)$ , we have $L^{\pi_\lambda} = M^{\pi_\lambda} \cap L(\pi_\lambda)$

for all $\lambda \in \Lambda$ and so $D^{\mathcal{M}} \leq D^{\mathcal{L}}$. As before, we say that $\mathcal{M}$, $\mathcal{L}$ are <u>consecutive</u> if $M(\pi_\mu)/L(\pi_\mu)$ is a chief factor of G for some $\mu \in \Lambda$, and $L(\pi_\lambda) = M(\pi_\lambda)$ for all $\lambda \neq \mu$. In this case, $D^{\mathcal{M}} = D^{\mathcal{L}} \cap N_G(M^{\pi_\mu}) = N_{D^{\mathcal{L}}}(M^{\pi_\mu})$.

We use these ideas in the following lemma (which corresponds to Lemma4.7).

<u>LEMMA 7.5</u> Suppose that $\mathcal{L} < \mathcal{M}$ are consecutive as above, the chief factor $M(\pi_\mu)/L(\pi_\mu)$ is covered by $D^{\mathcal{M}}$ and $D^{\mathcal{M}} < D^{\mathcal{L}}$. Then $D^{\mathcal{M}}$ is an $\mathcal{M} \cap D^{\mathcal{L}}$-normalizer of $D^{\mathcal{L}}$ and is abnormal in $D^{\mathcal{L}}$.

<u>Proof</u>. Write $K(\pi_\lambda) = M(\pi_\lambda) \cap D^{\mathcal{L}}$ and $K^{\pi_\lambda} = M^{\pi_\lambda} \cap D^{\mathcal{L}}$ for each $\lambda \in \Lambda$. Since $\mathcal{S}$ reduces into $D^{\mathcal{L}}$ (Theorem7.3), $K^{\pi_\lambda}$ is then a Hall $\pi_\lambda$-complement of $K(\pi_\lambda)$. As in 4.7, we obtain $N_{D^{\mathcal{L}}}(K^{\pi_\mu}) = N_{D^{\mathcal{L}}}(M^{\pi_\mu})$, using the fact that $D^{\mathcal{M}}$ covers $M(\pi_\mu)/L(\pi_\mu)$. This gives the desired result, by Lemma7.4.

<u>LEMMA 7.6</u> If $\mathcal{M} = \{ M(\pi_\lambda) \mid \lambda \in \Lambda \}$ is a set of normal subgroups of G, one for each $\lambda \in \Lambda$, there exists $\mu \in \Lambda$ and a normal subgroup $L(\pi_\mu)$ of G such that $M(\pi_\mu)/L(\pi_\mu)$ is a chief factor of G covered by $D^{\mathcal{M}}$.

<u>Proof</u>. The chief factor obtained as in 4.8 with the $X(p)$ replaced by the $M(\pi_\lambda)$ is centralized by all the $M(\pi_\lambda)$ and is thus covered by $D^{\mathcal{M}}$ using Theorem7.2.

These last two lemmas yield the required result.

<u>THEOREM 7.7</u> $D^{\mathcal{M}}$ is subabnormal in G.

<u>Proof</u>. We assume that $D^{\mathcal{M}} < G$ and also, without loss of gen-

erality, that no consecutive set $\mathfrak{L}$ contained in $\mathfrak{M}$ defines $D^{\mathfrak{M}}$. Lemma7.6 then enables us to choose a consecutive set $\mathfrak{L} < \mathfrak{M}$ which satisfies the conditions of Lemma7.5. $D^{\mathfrak{M}}$ is thus abnormal in $D^{\mathfrak{L}}$ and the result follows, since $D^{\mathfrak{L}}$ is subabnormal in G by induction.

We conclude this discussion of the properties of $\mathfrak{M}$-normalizers with a possible definition of an $\mathfrak{M}$-central chief factor and an $\mathfrak{M}$-normal maximal subgroup of G.

DEFINITION. A p-chief factor of G is said to be $\mathfrak{M}$-central if it is centralized by all $M(\pi_\lambda)$ with $p \in \pi_\lambda$, and $\mathfrak{M}$-eccentric otherwise. We then say that a maximal subgroup of G is $\mathfrak{M}$-normal if it complements an $\mathfrak{M}$-central chief factor, and $\mathfrak{M}$-abnormal otherwise. It is clear that when $\mathfrak{M}$ is actually a normal system $\mathfrak{X}$, these definitions reduce to those of an $\mathfrak{X}$-central chief factor and an $\mathfrak{X}$-normal maximal subgroup.

Theorem7.2 can now be restated to read " $D^{\mathfrak{M}}$ covers all the $\mathfrak{M}$-central chief factors of G " (in line with 4.2). This gives us the further result

" A maximal subgroup of G containing an $\mathfrak{M}$-normalizer of G is $\mathfrak{M}$-abnormal in G "

However, in contrast to the theory of $\mathfrak{X}$-normalizers, the converse of this result is patently false. For, in Example7.1, the maximal subgroups of order 21 complementing the $\mathfrak{M}$-eccentric chief factor of order $2^3$ clearly do not contain an $\mathfrak{M}$-normalizer of G. There thus seems little point in pursuing these concepts further.

We now turn our attention to the other conjugacy class of subgroups obtained by Fischer in the present situation viz. the $\mathfrak{M}$-covering subgroups of G. As we have already remarked, these subgroups are defined as the limit of a certain sequence of subgroups of G. However, before we can go into this definition in detail, we need the concept of an $\mathfrak{M}$-reducer of a subgroup of G.

<u>DEFINITION</u>. We say that the $\mathfrak{M}$-system $\mathcal{B} = \{ M^{\pi_\lambda} \mid \lambda \in \Lambda \}$ of G <u>reduces</u> into a subgroup H of G if $\mathcal{B} \cap H$ is an $\mathfrak{M} \cap H$-system of H i.e. $M^{\pi_\lambda} \cap H$ is a Hall $\pi_\lambda$-complement of $M(\pi_\lambda) \cap H$ for each $\lambda \in \Lambda$. This is merely an extension of the usual definition of a Sylow system of G reducing into a subgroup of G. Suppose, then, that the $\mathfrak{M}$-system $\mathcal{B}$ of G reduces into H. Then the subgroup

$$R_G^{\mathfrak{M}}(H) = \langle g \in G \mid \mathcal{B}^g \text{ reduces into } H \rangle$$

is called the <u>$\mathfrak{M}$-reducer of H in G</u> --- this subgroup is easily seen to be independent of the particular $\mathfrak{M}$-system $\mathcal{B}$ used in the definition, since all $\mathfrak{M}$-systems of G are conjugate. If, in particular, $\mathcal{B}$ is a Sylow system $\mathcal{S}$ of G reducing into H, the subgroup

$$R_G(H) = \langle g \in G \mid \mathcal{S}^g \text{ reduces into } H \rangle$$

is simply called the <u>reducer of H in G</u>.

Clearly, if a Sylow system $\mathcal{S}$ of G reduces into H, then so will the $\mathfrak{M}$-system defined by $\mathcal{S}$ and thus

$$R_G^{\mathfrak{M}}(H) \geq R_G(H) \geq H \qquad \text{for any } \mathfrak{M}.$$

$R_G(H)$ is easily shown to be abnormal in G, and hence so is $R_G^{\mathfrak{M}}(H)$. In addition, it is not difficult to see that any $\mathfrak{M}$-system $\mathcal{B}$ of G reducing into H also reduces into every subgroup V of G containing $R_G^{\mathfrak{M}}(H)$. We give a proof of this statement as an example of the methods used.

Suppose that the $\mathfrak{M}$-system $\mathcal{B} = \{ M^{\pi_\lambda} \}$ of G reduces into H and is defined by Sylow system $\mathcal{S} = \{ S^p \}$ of G. Choose a p-complement $H^p$ of H containing $S^p \cap H$ for each prime p. Then the Sylow system $\mathcal{H} = \{ H^p \}$ of H defines the $\mathfrak{M} \cap H$-system $\mathcal{B} \cap H$ of H. Extend $\mathcal{H}$ through a Sylow system $\mathcal{V} = \{ V^p \}$ of V to Sylow system $\mathcal{T} = \{ T^p \}$ of G. Then, by P.Hall, $\mathcal{T} = \mathcal{S}^g$ for some $g \in G$. In fact, $g \in R_G^{\mathfrak{M}}(H)$, for the $\mathfrak{M}$-system $\mathcal{B}^g$ of G is defined by $\mathcal{T}$ and so reduces into H with $\mathcal{T}$. Thus we can write $\mathcal{S} = \mathcal{T}^v$ for some $v \in V$, and therefore $\mathcal{B} \cap V = \{ (V^{\pi_\lambda} \cap M(\pi_\lambda))^v \}$ is an $\mathfrak{M} \cap V$-system of V i.e. $\mathcal{B}$ reduces into V.

Fischer proves, in addition, that $\mathfrak{M}$-reducers are homomorphism-invariant i.e. for any normal subgroup N of G,

$$R_{G/N}^{N\mathfrak{M}/N}(NH/N) = N.R_G^{\mathfrak{M}}(H)/N .$$

Now suppose that H is an $\mathfrak{M}$-normalizer D of G. Let D normalize the $\mathfrak{M}$-system $\mathcal{B} = \{M^{\pi_\lambda}\}$ of G --- we write $D = N_G(\mathcal{B})$ to distinguish between the $\mathfrak{M}$-normalizers of G in this way. Then $\mathcal{B}$ reduces into D since $|M(\pi_\lambda) \cap D : M^{\pi_\lambda} \cap D| = | M^{\pi_\lambda}(M(\pi_\lambda) \cap D) : M^{\pi_\lambda} | \in \pi_\lambda$ for every $\lambda \in \Lambda$. ( In fact, we have seen in Theorem7.3 that the Sylow system of G defining $\mathcal{B}$ reduces into D.) Therefore, as we have seen above, $\mathcal{B}$ reduces into the $\mathfrak{M}$-reducer of $D = N_G(\mathcal{B})$ in G. It is this fact which enables us to make the following definition of an $\mathfrak{M}$-covering subgroup.

DEFINITION. (Fischer) Let $\mathcal{B}$ be any $\mathfrak{M}$-system of G. We define the following sequence of subgroups inductively. Let $R_0 = G$ and $D_0 = N_G(\mathcal{B})$. Then, assuming that $R_i$ and $D_i$ are defined, let

$$R_{i+1} = R_{R_i}^{\mathfrak{M} \cap R_i}(D_i) \quad \text{and} \quad D_{i+1} = N_{R_{i+1}}(\mathcal{B} \cap R_{i+1}) .$$

( $\mathcal{B} \cap R_{i+1}$ is an $\mathfrak{M} \cap R_{i+1}$-system of $R_{i+1}$, by repeated application of the above remarks.) The $R_i$ clearly form a descending sequence of subgroups --- the last member of this series ( which certainly exists ) is called an $\mathfrak{M}$-covering subgroup $E^{\mathfrak{M}}$ of G.

Several interesting facts about this definition are fairly easily proved. Firstly, since the same $\mathfrak{M}$-system $\mathcal{B}$ of G is used throughout the definition, the $D_i$ form an ascending sequence of subgroups --- the last member of this sequence is in fact equal to $E^{\mathfrak{M}}$. This is a simple consequence of the following lemma whose proof we sketch briefly.

LEMMA 7.8 (Fischer) Let $D = N_G(\mathcal{B})$ for some $\mathfrak{M}$-system $\mathcal{B} = \{M^{\pi_\lambda}\}$ of G, and $R = R_G^{\mathfrak{M}}(D)$. Then $R = G$ if, and only if $D = G$.

Proof. Certainly $D = G$ implies $R = G$. Suppose that the converse is false, if possible, and let G be a counterexample of minimal order. Then $D < G$ and $R = G$. By the homomorphism-invariance of $\mathfrak{M}$-reducers and $\mathfrak{M}$-normalizers, and the definition of G, $ND = G$ for any minimal normal subgroup N of G. Thus D is maximal in G --- let $|G:D| = p^{\alpha}$. Then, for $p \notin \pi_{\lambda}$, $M^{\pi_{\lambda}} \triangleleft G$. On the other hand, if $p \in \pi_{\lambda}$, $M^{\pi_{\lambda}} \leq D$. For otherwise we would have $|M^{\pi_{\lambda}} : M^{\pi_{\lambda}} \cap D|$ divisible by $p \in \pi_{\lambda}$, contradicting the definition of $M^{\pi_{\lambda}}$. We show that this implies $R = D$, giving the required contradiction. Suppose $\mathcal{B}^g$ reduces into G for some $g \in G$. Then $\mathcal{B}^g \cap D = \mathcal{B} \cap D$, the unique $\mathfrak{M} \cap D$-system of D i.e. $(M^{\pi_{\lambda}})^g \cap D = M^{\pi_{\lambda}} \cap D$ for each $\lambda \in \Lambda$. Thus, for $p \in \pi_{\lambda}$, $(M^{\pi_{\lambda}})^g \cap D = M^{\pi_{\lambda}}$, giving $(M^{\pi_{\lambda}})^g = M^{\pi_{\lambda}}$. Since this relation is trivially true for $p \notin \pi_{\lambda}$, we have $g \in N_G(\mathcal{B}) = D$ and the result follows.

Suppose, then, that the sequence of $R_i$ terminates at the j-th place. i.e. $R_{j+1} = R_j = E^{\mathfrak{M}}$. Application of this lemma to $R_j$ then yields $D_j = R_j$, and so both sequences converge to the $\mathfrak{M}$-covering subgroup $E^{\mathfrak{M}}$. Thus $E^{\mathfrak{M}}$, being equal to $D_j$, is its own $\mathfrak{M} \cap E^{\mathfrak{M}}$-normalizer.

Secondly, using induction on i, it is a simple matter to verify that in fact $R_{i+1} = R_G^{\mathfrak{M}}(D_i)$ for each i. This shows immediately that $E^{\mathfrak{M}} = R_j$ is abnormal in G.

Two further properties to be expected of $\mathfrak{M}$-covering subgroups viz. conjugacy and homomorphism-invariance follow straight from the definition. For each $\mathfrak{M}$-system $\mathcal{B}$ of G gives rise to a single $\mathfrak{M}$-covering subgroup of G. All the $\mathfrak{M}$-systems of G are conjugate, and it is easily verified that the sequence obtained by using $\mathcal{B}^g$ in the above definition is simply $R_0^g, R_1^g, \dots$ . It is therefore clear that all the $\mathfrak{M}$-covering subgroups of G are conjugate. The homomorphism-invariance of both $\mathfrak{M}$-normalizers and $\mathfrak{M}$-reducers shows that $\mathfrak{M}$-covering subgroups are also invariant under homomorphisms.

Fischer now proves the deeper result that an $\mathfrak{M}$-covering subgroup of G is an $\mathfrak{M} \cap V$ -covering subgroup of any subgroup V of G in which it is contained. This result enables him to show that an $\mathfrak{M}$-covering subgroup $E^{\mathfrak{M}}$ of G satisfies the following two conditions

7.9 $\begin{cases} \text{(i)} & E^{\mathfrak{M}} \text{ is its own } \mathfrak{M} \cap E^{\mathfrak{M}} \text{-normalizer} \\ \text{(ii)} & \text{If } E^{\mathfrak{M}} \leq F \leq G,\ F_0 \lhd F \text{ and } F/F_0 \text{ is its own} \end{cases}$

$F_0(\mathfrak{M} \cap F)/F_0$ -normalizer, then $E^{\mathfrak{M}} F_0 = F$ .

For, in (ii), we can assume by induction that $F = G$. Then $G/F_0 = N_{G/F_0}(F_0 \mathfrak{B}/F_0) = F_0 N_G(\mathfrak{B}) / F_0$ for any $\mathfrak{M}$-system $\mathfrak{B}$ of G. The result then follows since $E^{\mathfrak{M}} \geq N_G(\mathfrak{B}) = D_0$.

It is this property of $\mathfrak{M}$-covering subgroups which forms our starting point. For in the special case of $\mathfrak{M}$ a normal system $\mathfrak{X}$ of G, these two conditions are easily seen to be the defining conditions of an $\mathfrak{X}$-covering subgroup of G. We therefore use 7.9 as our definition of an $\mathfrak{M}$-covering subgroup $E^{\mathfrak{M}}$ of G. Our methods of Chapter5, applied to this more general situation, then enable us to show that these subgroups do indeed exist and are conjugate and invariant under homomorphisms. It is clear that once our $\mathfrak{M}$-covering subgroups are proved conjugate, they coincide with those of Fischer, giving an alternative approach to the whole situation. However, we give an additional proof of this fact by showing that our $\mathfrak{M}$-covering subgroups do indeed satisfy Fischer's definition.

Our first step is to define a set $\overline{\mathfrak{M}}$ corresponding to the set $\bar{\mathfrak{X}}$ defined in Chapter 3. Rather than bring in the concept of $\mathfrak{M}$-central chief factors, we use the form of $\bar{\mathfrak{X}}$ given in Lemma 3.2(v).

<u>DEFINITION</u>. Let $\overline{\mathfrak{M}}$ be the set consisting of sections $H/K$ of $G$ in which $K(H \cap M(\pi_\lambda))/K$ has a normal Hall $\pi_\lambda$-complement for each $\lambda \in \Lambda$.

Then $\overline{\mathfrak{M}}$, like $\bar{\mathfrak{X}}$, is defined entirely within $G$ and clearly has just the same disadvantage as $\bar{\mathfrak{X}}$ viz. it is not closed under isomorphism within $G$ (Example 3.1). However, in the following lemma we show that in spite of the added generality, $\overline{\mathfrak{M}}$, like $\bar{\mathfrak{X}}$, satisfies properties analogous to those of a saturated formation. It is this fact which enables us to adapt the proofs in Chapter 5 to the present situation.

<u>LEMMA 7.10</u> Let $H$ be a subgroup of $G$. Then

(i) $H/K \in \overline{\mathfrak{M}}$, $K \leq K_1 \lhd H$ implies $H/K_1 \in \overline{\mathfrak{M}}$.

(ii) $H/K_1$, $H/K_2 \in \overline{\mathfrak{M}}$ implies $H/K_1 \cap K_2 \in \overline{\mathfrak{M}}$, and thus $H$ has an $\overline{\mathfrak{M}}$-residual $H^{\overline{\mathfrak{M}}}$.

(iii) $H \in \overline{\mathfrak{M}}$ if and only if $H/\emptyset(H) \in \overline{\mathfrak{M}}$.

(iv) $H \in \overline{\mathfrak{M}}$, $K \leq H$ implies $K \in \overline{\mathfrak{M}}$.

(v) $H/K \in \overline{\mathfrak{M}}$ if and only if $H/K$ is its own $K(\mathfrak{M} \cap H)/K$-normalizer.

<u>Proof</u>. (i) Since $H/K \in \overline{\mathfrak{M}}$, $K(M(\pi_\lambda) \cap H)/K$ has a normal Hall

$\pi_\lambda$-complement $Q^{\pi_\lambda}/K$ for each $\lambda \in \Lambda$. Thus $K_1(M(\pi_\lambda) \cap H)/K_1$ has a normal Hall $\pi_\lambda$-complement $K_1 Q^{\pi_\lambda}/K_1$ for each $\lambda \in \Lambda$, and so $H/K_1 \in \mathfrak{M}$.

(ii) We first prove the following general result for a finite soluble group $G$ :

" If $G/N_1$ and $G/N_2$ have normal Hall $\pi$-complements for some set of primes $\pi$, then so does $G/N_1 \cap N_2$. "

Let $Q_i/N_i$ be the normal Hall $\pi$-complement of $G/N_i$ (i=1,2). Then $Q_1N_2/N_1N_2$, $Q_2N_1/N_1N_2$ are normal Hall $\pi$-complements of $G/N_1N_2$ and so, by Hall, $Q_1N_2 = Q_2N_1$. We show that $Q_1 \cap Q_2/N_1 \cap N_2$ is a Hall $\pi$-complement of $G/N_1 \cap N_2$ and the result then follows.

$|G : Q_1 \cap Q_2| = |G : Q_1|.|Q_1 : Q_1 \cap Q_2| = |G : Q_1|.|Q_1Q_2 : Q_2| \in \pi$ and $|Q_1 \cap Q_2 : N_1 \cap N_2| = |Q_1 \cap Q_2 : Q_1 \cap N_2|.|Q_1 \cap N_2 : N_1 \cap N_2|$. Since $Q_1N_2 = Q_2N_1$, $N_2(Q_1 \cap Q_2) = N_2Q_1 \cap Q_2 = Q_2$ so that $|Q_1 \cap Q_2 : Q_1 \cap N_2| = |Q_2 : N_2| \in \pi'$. Also, $|Q_1 \cap N_2 : N_1 \cap N_2| = |Q_1 \cap N_1N_2 : N_1| \in \pi'$.

The proof of (ii) now follows easily. For each $\lambda \in \Lambda$ we have $N_i(M(\pi_\lambda) \cap H)/N_i$ with a normal Hall $\pi_\lambda$-complement (i=1, 2). Thus $M(\pi_\lambda) \cap H / M(\pi_\lambda) \cap H \cap N_i$ has a normal Hall $\pi_\lambda$-complement for each $\lambda \in \Lambda$, i=1,2. Hence $M(\pi_\lambda) \cap H / M(\pi_\lambda) \cap H \cap N_1 \cap N_2$ has a normal Hall $\pi_\lambda$-complement, and thus so does $(N_1 \cap N_2)(M(\pi_\lambda) \cap H)/(N_1 \cap N_2)$ for each $\lambda \in \Lambda$, as was required.

(iii) Assume that $H/\emptyset(H) \in \mathfrak{M}$. Then $H/\emptyset_{\pi_\lambda}(H) \in \mathfrak{M}$ for all $\lambda \in \Lambda$. Thus, for each $\lambda \in \Lambda$,

$\emptyset_{\pi_\lambda}(H)(M(\pi_\lambda) \cap H)/\emptyset_{\pi_\lambda}(H) \leq O_{\pi_\lambda' \pi_\lambda}(H/\emptyset_{\pi_\lambda}(H)) = O_{\pi_\lambda' \pi_\lambda}(H)/\emptyset_{\pi_\lambda}(H)$ ( Lemma1.4 ). Hence $M(\pi_\lambda) \cap H \leq O_{\pi_\lambda' \pi_\lambda}(H)$ and so has a normal Hall $\pi_\lambda$-complement, for each $\lambda \in \Lambda$. Thus $H \in \bar{\mathfrak{M}}$. The converse follows immediately from (i).

(iv) Choose Hall $\pi_\lambda$-complements $R^{\pi_\lambda}, Q^{\pi_\lambda}$ of $M(\pi_\lambda) \cap K$, $M(\pi_\lambda) \cap H$ respectively such that $Q^{\pi_\lambda} \cap K = R^{\pi_\lambda}$ for each $\lambda \in \Lambda$. Then $H \in \bar{\mathfrak{M}}$ implies $Q^{\pi_\lambda} \lhd H \cap M(\pi_\lambda)$ and so $R^{\pi_\lambda} \lhd K \cap M(\pi_\lambda)$ for each $\lambda \in \Lambda$. Hence $K \in \bar{\mathfrak{M}}$.

(v) is obvious, and the lemma is proved.

We also note that although $\bar{\mathfrak{M}}$ need not be closed under every isomorphism within the group, it is, like $\bar{\mathfrak{X}}$, closed under a certain type of isomorphism within G viz. the isomorphism $\varphi : H/K \to \bar{H}/\bar{K}$ where $\bar{H}K = H \leq G$ and $\bar{H} \cap K = \bar{K}$. For in this situation, $H/K \in \bar{\mathfrak{M}}$ implies that $K(H \cap M(\pi_\lambda))/K$ and hence $K(\bar{H} \cap M(\pi_\lambda))/K$ has a normal Hall $\pi_\lambda$-complement for each $\lambda \in \Lambda$. Since $\bar{K}(\bar{H} \cap M(\pi_\lambda))/\bar{K} \cong K(\bar{H} \cap M(\pi_\lambda))/K$, this yields $\bar{H}/\bar{K} \in \bar{\mathfrak{M}}$. This fact will be used frequently in the sequel.

Lemma7.10(v) enables us to restate our definition of an $\mathfrak{M}$-covering subgroup as follows.

<u>DEFINITION</u>. We say that $E^{\mathfrak{M}}$ is an <u>$\mathfrak{M}$-covering subgroup</u> of G if (i) $E^{\mathfrak{M}} \in \bar{\mathfrak{M}}$

(ii) $E^{\mathfrak{M}} \leq F \leq G$, $F_0 \lhd F$ and $F/F_0 \in \bar{\mathfrak{M}} \implies E^{\mathfrak{M}} F_0 = F$.

This definition of an $\mathfrak{M}$-covering subgroup yields immediately the following properties of $\mathfrak{M}$-covering subgroups ---

we omit the proofs as they follow so closely along the lines of the corresponding statements in Lemmas5.1 and 5.2.

LEMMA 7.11 Let E be an $\mathfrak{M}$-covering subgroup of G. Then

(i) $E \leq H \leq G$ implies E is an $\mathfrak{M} \cap H$ -covering subgroup of H

(ii) $E^g$ is an $\mathfrak{M}$-covering subgroup of G for all $g \in G$

(iii) E is abnormal in G

(iv) If $NE/N \in \bar{\mathfrak{M}}$ for $N \lhd G$, then $NE/N$ is an $N\mathfrak{M}/N$ -covering subgroup of G/N --- this is certainly the case if $N < E$ .

LEMMA 7.12 If M is a maximal subgroup of G satisfying $M \in \bar{\mathfrak{M}}$ and $G/\mathrm{Core}M \notin \bar{\mathfrak{M}}$ , then M is an $\mathfrak{M}$-covering subgroup of G.

Here, too, since $\bar{\mathfrak{M}}$ is not isomorphism-closed, Lemma7.11 (iv) does not at once yield the homomorphism-invariance of $\mathfrak{M}$ -covering subgroups. We also have to approach the existence of these subgroups by the less straightforward means of so-called $\mathfrak{M}$-crucial maximal chains. As in Chapter 5, we first prove the homomorphism-invariance of $\mathfrak{M}$-covering subgroups, obtaining the conjugacy as an immediate corollary.

THEOREM 7.13 If E is an $\mathfrak{M}$-covering subgroup of G, then NE/N is an $N\mathfrak{M}/N$ -covering subgroup of G/N for any $N \lhd G$.

Proof. By Lemma7.11(iv), it is sufficient to prove that $NE/N \in \bar{\mathfrak{M}}$ . Suppose this is not the case and let G be a counter-example of minimal order. Then there exists $\mathfrak{M} = \{M(\pi_\lambda) \mid \lambda \in \Lambda\}$ an $\mathfrak{M}$-covering subgroup E and a normal subgroup N of G such that $NE/N \notin \bar{\mathfrak{M}}$ . Since $E \in \bar{\mathfrak{M}}$, $N > 1$ . Let $N_0$ be a minimal normal subgroup of G contained in N. Then, as in Theorem 5.3, $N_0E = G$. Hence E is maximal in G and $G/N_0 \notin \overline{\mathfrak{M}}$. Thus there exists $\lambda \in \Lambda$ such that $N_0M(\pi_\lambda)/N_0$ does not have a normal Hall $\pi_\lambda$-complement. Now $E \in \overline{\mathfrak{M}}$ implies $M(\pi_\lambda) \cap E$ has a normal Hall $\pi_\lambda$-complement, and thus so does $N_0(M(\pi_\lambda) \cap E)/N_0$. Thus we must have $M(\pi_\lambda) \nleq E$ and also $N_0 \nleq M(\pi_\lambda)$, since $N_0 \leq M(\pi_\lambda)$ would imply $N_0(M(\pi_\lambda) \cap E) = M(\pi_\lambda) \cap N_0E = M(\pi_\lambda)$. Therefore

(a) --- $M(\pi_\lambda)E = G$ since E is maximal in G, and

(b) --- $M(\pi_\lambda) \cap N_0 = 1$ since $N_0 > M(\pi_\lambda) \cap N_0$ and $M(\pi_\lambda) \cap N_0 \lhd G$. Hence

(c) -- $M(\pi_\lambda) \cap E = (M(\pi_\lambda) \cap E)(M(\pi_\lambda) \cap N_0) = M(\pi_\lambda) \cap (M(\pi_\lambda) \cap E)N_0$

and so is normal in G. Since $E < G$, the definition of E gives

(d) --- $G/(M(\pi_\lambda) \cap E) \notin \overline{\mathfrak{M}}$.

As in Theorem5.3, we show that $G/M(\pi_\lambda)$ and $G/(M(\pi_\lambda) \cap E)N_0$ both lie in $\overline{\mathfrak{M}}$, giving a contradiction to (d) by (c) and Lemma7.10(ii).

$E/(M(\pi_\lambda) \cap E)$ is an $(M(\pi_\lambda) \cap E)\mathfrak{M}/(M(\pi_\lambda) \cap E)$ -covering subgroup of $G/(M(\pi_\lambda) \cap E)$, by Lemma7.11(iv). In addition, $M(\pi_\lambda) \cap E > 1$. For $M(\pi_\lambda) \cap E = 1$ implies that $M(\pi_\lambda)$ is a minimal normal subgroup of G. $M(\pi_\lambda)$ thus posses a normal Hall $\pi_\lambda$-complement and hence so does $N_0M(\pi_\lambda)/N_0$ --- a contradiction. We can thus apply induction to $G/(M(\pi_\lambda) \cap E)$, obtaining $G/M(\pi_\lambda) \in \overline{\mathfrak{M}}$ and $G/(M(\pi_\lambda) \cap E)N_0 \in \overline{\mathfrak{M}}$ as required. This final contradiction proves the theorem.

<u>THEOREM 7.14</u> Any two $\mathfrak{M}$-covering subgroups of G are conjug-

ate.

Proof. The proof uses the argument of Theorem5.4 with $\mathfrak{X}$ replaced by $\mathfrak{M}$.

With these two properties of $\mathfrak{M}$-covering subgroups behind us, we go on to obtain the $\mathfrak{M}$-covering subgroups as terminal members of $\mathfrak{M}$-crucial maximal chains.

DEFINITION. We say that the maximal subgroup M of G is *$\mathfrak{M}$-crucial* if (i) $G/N \in \bar{\mathfrak{M}}$

(ii) $G/\text{Core}M \notin \bar{\mathfrak{M}}$

--- where $N/\text{Core}M$ is the unique minimal normal subgroup of $G/\text{Core}M$. This definition is clearly a simple extension of the concept of an $\mathfrak{X}$-crucial maximal subgroup (and an $\mathfrak{M}$-crucial maximal subgroup will certainly be $\mathfrak{M}$-abnormal). The chain $H = H_0 < H_1 < \ldots < H_r = G$ of subgroups is called an *$\mathfrak{M}$-crucial maximal chain* if $H_i$ is an $\mathfrak{M} \cap H_{i+1}$ -crucial maximal subgroup of $H_{i+1}$ for each i.

We give several equivalent conditions for a maximal subgroup to be $\mathfrak{M}$-crucial and then show that $\mathfrak{M}$-crucial subgroups always exist whenever $G \notin \bar{\mathfrak{M}}$.

LEMMA 7.15 The following three statements are equivalent

(i) M is an $\mathfrak{M}$-crucial maximal subgroup of G

(ii) $G^{\bar{\mathfrak{M}}} / G^{\bar{\mathfrak{M}}} \cap M$ is a chief factor of G

(iii) M complements a chief factor H/K of G with $G/H \in \bar{\mathfrak{M}}$ and $G/K \notin \bar{\mathfrak{M}}$

Proof. We first prove that (i) and (ii) are equivalent. Let M be $\mathfrak{M}$-crucial. Then $G^{\bar{\mathfrak{M}}} \leq N$ and so $G^{\bar{\mathfrak{M}}} \cap M = G^{\bar{\mathfrak{M}}} \cap \mathrm{Core}M \lhd G$. Further, $G^{\bar{\mathfrak{M}}} \not\leq \mathrm{Core}M$ and hence $MG^{\bar{\mathfrak{M}}} = G$. $G^{\bar{\mathfrak{M}}} / G^{\bar{\mathfrak{M}}} \cap M$ is thus a chief factor since M is maximal in G. Conversely, if $G^{\bar{\mathfrak{M}}} / G^{\bar{\mathfrak{M}}} \cap M$ is a chief factor of G, then $G^{\bar{\mathfrak{M}}} \cap M = G^{\bar{\mathfrak{M}}} \cap \mathrm{Core}M$ and $N = G^{\bar{\mathfrak{M}}}\,\mathrm{Core}M$. Thus $G/N \in \bar{\mathfrak{M}}$. Now $G^{\bar{\mathfrak{M}}} / G^{\bar{\mathfrak{M}}} \cap M \notin \bar{\mathfrak{M}}$ by the definition of $G^{\bar{\mathfrak{M}}}$. This implies that $G/\mathrm{Core}M \notin \bar{\mathfrak{M}}$ by Lemma7.10(ii), since $G^{\bar{\mathfrak{M}}} \cap \mathrm{Core}M = G^{\bar{\mathfrak{M}}} \cap M$.

A similar proof shows that (iii) implies (i). Thus, since (i) implies (iii) trivially, these two conditions are also equivalent and the lemma is proved.

<u>LEMMA 7.16</u> If $G \notin \bar{\mathfrak{M}}$ it possesses an $\mathfrak{M}$-crucial maximal subgroup. Thus $G \notin \bar{\mathfrak{M}}$ implies that G has an $\mathfrak{M}$-crucial maximal chain whose terminal member lies in $\bar{\mathfrak{M}}$.

<u>Proof</u>. As in Lemma5.6, this proof relies on the fact that $\bar{\mathfrak{M}}$ is Frattini-closed. $G^{\bar{\mathfrak{M}}} > 1$ so we can choose a chief factor $G^{\bar{\mathfrak{M}}}/K$ of G. Then $G/K \notin \bar{\mathfrak{M}}$ and so, by Lemma7.10(iii), there exists a maximal subgroup M of G complementing $G^{\bar{\mathfrak{M}}}/K$. M is $\mathfrak{M}$-crucial by Lemma7.15.

We prove that the terminal member of an $\mathfrak{M}$-crucial maximal chain is an $\mathfrak{M}$-covering subgroup of G by showing that an $\mathfrak{M} \cap M$-covering subgroup of an $\mathfrak{M}$-crucial maximal subgroup M of G is, in fact, an $\mathfrak{M}$-covering subgroup of G. The crux of the proof lies in the following lemma which corresponds to Lemma5.7.

<u>LEMMA 7.17</u> Let E be an $\mathfrak{M} \cap M$-covering subgroup of the $\mathfrak{M}$-

crucial maximal subgroup M of G. Then, if $E < F < G$ and $F \not\leq M$, $F/F \cap \mathrm{Core}M \notin \overline{\mathfrak{M}}$.

Proof. We only give a brief sketch of this proof, apart from the section which deviates slightly from the proof of Lemma5.7. We assume the lemma to be false, and let G be a counterexample of minimal order and F a subgroup of G maximal with respect to the conditions $E < F < G$, $F \not\leq M$ and $F/F \cap \mathrm{Core}M \in \overline{\mathfrak{M}}$. Then, as in 5.7,

(1) --- $F\mathrm{Core}M = G$ and $FG^{\overline{\mathfrak{M}}} = G$

and F is maximal in G. Since $G/\mathrm{Core}M \notin \overline{\mathfrak{M}}$, there exists some $\lambda \in \Lambda$ such that $M(\pi_\lambda)\mathrm{Core}M/\mathrm{Core}M$ does not have a normal Hall $\pi_\lambda$-complement. On the other hand, $F/F \cap \mathrm{Core}M \in \overline{\mathfrak{M}}$ implies that $(F \cap M(\pi_\lambda))(F \cap \mathrm{Core}M)/(F \cap \mathrm{Core}M)$ and hence $(F \cap M(\pi_\lambda))\mathrm{Core}M/\mathrm{Core}M$ has a normal Hall $\pi_\lambda$-complement. Thus

(2) --- $M(\pi_\lambda)\mathrm{Core}M \neq (F \cap M(\pi_\lambda))\mathrm{Core}M$

and so $M(\pi_\lambda) \not\leq F$. Therefore, since F is maximal in G,

(3) --- $FM(\pi_\lambda) = G$.

(2) also enables us to show that $F(M(\pi_\lambda) \cap \mathrm{Core}M) = F$. For, suppose, if possible, that $F(M(\pi_\lambda) \cap \mathrm{Core}M) = G$ ( F is maximal in G ). This would imply that

$M(\pi_\lambda) = (M(\pi_\lambda) \cap F)(M(\pi_\lambda) \cap \mathrm{Core}M)$ and hence $M(\pi_\lambda)\mathrm{Core}M = (M(\pi_\lambda) \cap F)\mathrm{Core}M$, contradicting (2). Thus $F(M(\pi_\lambda) \cap \mathrm{Core}M) = F$ and so $F \cap \mathrm{Core}M = (F \cap \mathrm{Core}M)M(\pi_\lambda) \cap \mathrm{Core}M$ which is normal in G by (3).

Now, setting $G^* = G/F \cap \mathrm{Core}M$, $F^* = F/F \cap \mathrm{Core}M$ etc. and $\mathfrak{L} = (F \cap \mathrm{Core}M)\mathfrak{M}/(F \cap \mathrm{Core}M)$, we obtain $F^* \in \overline{\mathfrak{L}}$, $F^*$ maximal in $G^*$ and $G^*/\mathrm{Core}F^* \notin \overline{\mathfrak{L}}$ as in 5.7. Lemma7.12 then yields F* an $\mathcal{L}$-covering subgroup of G* and so $G^*/(\mathrm{Core}M)^* \in \bar{\mathcal{L}}$ by the homomorphism-invariance of $\mathcal{L}$-covering subgroups. i.e. $G/\mathrm{Core}M \in \bar{\mathfrak{M}}$ , the required contradiction.

The proof of the following theorem now follows immediately along the lines of 5.8.

THEOREM 7.18 An $\mathfrak{M}\cap M$-covering subgroup of an $\mathfrak{M}$-crucial maximal subgroup M of G is an $\mathfrak{M}$-covering subgroup of G.

COROLLARY. $\mathfrak{M}$-covering subgroups of G always exist.

The following characterization of $\mathfrak{M}$-covering subgroups is now easily proved using the conjugacy of $\mathfrak{M}$-covering subgroups.

THEOREM 7.19 The $\mathfrak{M}$-covering subgroups of G are the terminal members of the $\mathfrak{M}$-crucial maximal chains of G.

Our concluding results concern the relationships between $\mathfrak{M}$-covering subgroups and $\mathfrak{M}$-normalizers and the $\mathfrak{M}$-reducers of both these subgroups. Suppose that $\mathcal{S}$ is a Sylow system of G reducing into the $\mathfrak{M}$-covering subgroup E of G and defining the $\mathfrak{M}$-system $\mathcal{B}$ of G. We show that

$$D \leq E = R_G^{\mathfrak{M}}(E) \leq R_G^{\mathfrak{M}}(D) , \quad \text{where } D = N_G(\mathcal{B}) .$$

It is then easily seen that E is indeed the limit of a sequence of subgroups of the type described by Fischer. We begin by proving the first of the above inequalities --- our method of proof was inspired by the corresponding theorem for $\mathcal{F}$-covering subgroups and $\mathcal{F}$-normalizers, due to A.Mann([14]).

<u>THEOREM 7.20</u> Let the Sylow system $\mathcal{S} = \{S^p\}$ of G reduce into the $\mathfrak{M}$-covering subgroup E of G and define the $\mathfrak{M}$-system $\mathcal{B} = \{ M^{\pi_\lambda} = M(\pi_\lambda) \cap S^{\pi_\lambda}\}$ of G. Then $N_G(\mathcal{B}) \leq E$ .

<u>Proof</u>. Write $D = N_G(\mathcal{B}) = \bigcap_{\lambda \in \Lambda} N_G(M^{\pi_\lambda})$ . We use induction on $|G|$. If $E = G$ there is nothing to prove; so we can assume that $E < G$ and thus that $G \notin \bar{\mathfrak{M}}$ . Let N be a minimal normal subgroup of G. Then NE/N is an $N\mathfrak{M}/N$ -covering subgroup of G/N and $N\mathcal{S}/N$ is a Sylow system of G/N reducing into NE/N . Hence, by induction,

$$\bigcap_{\lambda\in\Lambda} N_{G/N}( NS^{\pi_\lambda}/N \cap NM(\pi_\lambda)/N ) \leq NE/N .$$

Now $NS^{\pi_\lambda}/N \cap NM(\pi_\lambda)/N = NM^{\pi_\lambda}/N$ , so

$$ND/N \leq \bigcap_{\lambda\in\Lambda} N.N_G(M^{\pi_\lambda})/N \leq \bigcap_{\lambda\in\Lambda} N_{G/N}(NM^{\pi_\lambda}/N) \leq NE/N .$$

Hence $D \leq NE$ --- (1).

Now E is an $\mathfrak{M}\cap NE$ -covering subgroup of NE and $\mathcal{S}\cap NE$ a Sylow system of NE ( Corollary2.8 of [1] ) reducing into E Thus, if $NE < G$ , we get by induction

$\bigcap_{\lambda\in\Lambda} N_{NE}(M^{\pi_\lambda}\cap NE) \leq E$ , since $\{M^{\pi_\lambda}\cap NE\}$ is the $\mathfrak{M}\cap NE$ -system defined by $\mathcal{S}\cap NE$. Then, using (1),

$$D = \bigcap_{\lambda\in\Lambda} N_{NE}(M^{\pi_\lambda}) \leq \bigcap_{\lambda\in\Lambda} N_{NE}(M^{\pi_\lambda}\cap NE) \leq E ,$$

as required.

We can thus assume that $NE = G$ for all minimal normal subgroups N of G. Then E is a maximal subgroup of G , and $\mathrm{Core}E = 1$ . Let N be the unique minimal normal subgroup of G, of order $p^\alpha$ say. Then either $M(\pi_\lambda) = 1$ or $M(\pi_\lambda) \geq N$ for each $\lambda \in \Lambda$ . Since $M(\pi_\lambda) = 1$ implies $M^{\pi_\lambda} \lhd G$ , we have

$$D = \bigcap_{\substack{\lambda\in\Lambda \\ M(\pi_\lambda)\geq N}} N_G(M^{\pi_\lambda}) .$$

Furthermore, since $G/N = NE/N \in \bar{\mathfrak{M}}$ , $M(\pi_\lambda)/N$ has a normal

Hall-$\pi_\lambda$-complement for all $\lambda \in \Lambda$ -- this is then a normal Hall $\pi_\lambda$-complement of $M(\pi_\lambda)$ for all $\lambda$ with $p \notin \pi_\lambda$. Thus, in fact,

(2) --- $D = \bigcap_{\left\{\substack{\lambda \in \Lambda \\ M(\pi_\lambda) \geq N \\ p \in \pi_\lambda}\right\}} N_G(M^{\pi_\lambda})$ .

Now since $\mathcal{S}$ reduces into E, we must have $S^p \leq E$ and thus $S^{\pi_\lambda} \leq E$ for all $\lambda$ with $p \in \pi_\lambda$. Hence, for $p \in \pi_\lambda$, $M^{\pi_\lambda} \leq M(\pi_\lambda) \cap E$ and so is a Hall $\pi_\lambda$-complement of $M(\pi_\lambda) \cap E$. Thus $M^{\pi_\lambda} \lhd E$ since $E \in \bar{\mathfrak{M}}$ and so $N_G(M^{\pi_\lambda}) = E$ or $G$ for $p \in \pi_\lambda$. But $G \notin \bar{\mathfrak{M}}$, so we cannot have $N_G(M^{\pi_\lambda}) = G$ for all $\lambda$ with $p \in \pi_\lambda$ as this would imply $D = G$ . Thus, for some $\lambda$ with $p \in \pi_\lambda$, $N_G(M^{\pi_\lambda}) = E$ and then $D \leq E$ by (2), and the theorem is proved.

<u>LEMMA 7.21</u> $R_G^{\mathfrak{M}}(E) \leq R_G^{\mathfrak{M}}(D)$ in the notation of Theorem7.20.

<u>Proof</u>. Since $\mathcal{S}$ reduces into E and $E \in \bar{\mathfrak{M}}$, $\mathcal{B} \cap E$ is the unique $\mathfrak{M} \cap E$ -system of E. Suppose that $\mathcal{B}^g$ reduces into E for some $g \in G$. Then $\mathcal{B}^g \cap E = \mathcal{B} \cap E$ and so $\mathcal{B}^g \cap D = \mathcal{B} \cap D$, the unique $\mathfrak{M} \cap D$ -system of D. i.e. $g \in R_G^{\mathfrak{M}}(D)$ , giving the required result.

<u>LEMMA 7.22</u> $R_G^{\mathfrak{M}}(E) = E$ for any $\mathfrak{M}$-covering subgroup E of G.

<u>Proof</u>. We use induction on $|G|$. There is nothing to prove if $E = G$, since $E \leq R_G^{\mathfrak{M}}(E)$ . We thus assume that $E < G$ and let N be a minimal normal subgroup of G. Then, by induction, $NE/N = R_{G/N}^{N\mathfrak{M}/N}(NE/N) = N.R_G^{\mathfrak{M}}(E)/N$ and so $NE = N.R_G^{\mathfrak{M}}(E)$. If $NE < G$, we can again use induction to obtain $E = R_{NE}^{\mathfrak{M} \cap NE}(E) = R_G^{\mathfrak{M}}(E)$ as required (since $R_G^{\mathfrak{M}}(E) \leq NE$ ).

We thus assume that $NE = G$ and so E is maximal in G. Therefore $R_G^{\mathfrak{M}}(E) = G$ or $E$ . Suppose, if possible, that $R_G^{\mathfrak{M}}(E) = G$ Then, by Lemma7.21, $R_G^{\mathfrak{M}}(D) = G$ where D is the $\mathfrak{M}$-normalizer of G defined by a Sylow system of G which reduces into E. Therefore, by Lemma7.8, $D = G$, contradicting $D \leq E < G$ . Thus $R_G^{\mathfrak{M}}(E) = E$ and the lemma is proved.

Finally, let $\mathfrak{B}$ be the $\mathfrak{M}$-system of G defined by a Sylow system $\mathcal{S}$ of G which reduces into a given $\mathfrak{M}$-covering subgroup E of G. Define, as before, $R_0 = G$ , $D_0 = N_G(\mathfrak{B})$ , and recursively, $R_{i+1} = R_{R_i}^{\mathfrak{M} \cap R_i}(D_i)$ , $D_{i+1} = N_{R_{i+1}}(\mathfrak{B} \cap R_{i+1})$ for $i \geq 0$. It is now a simple matter to show that this sequence converges to E. For, by Theorem7.20 and Lemma7.21, $D_0 \leq E \leq R_1$. $\mathcal{S}$ reduces into $D_0$ (Theorem7.3) and so into $R_1 \geq R_G(D_0)$, yielding a Sylow system $\mathcal{S} \cap R_1$ of $R_1$ which reduces into the $\mathfrak{M} \cap R_1$ -covering subgroup E of $R_1$ and defines the $\mathfrak{M} \cap R_1$ -system $\mathfrak{B} \cap R_1$ of $R_1$. Applying 7.20 and 7.21 to $R_1$ we then obtain $D_1 \leq E \leq R_2$, and we can continue this process, obtaining in general $D_i \leq E \leq R_{i+1}$ ( $i = 0, 1, \ldots$ ). If $D_i = E$ then $R_{i+1} = E$ , by Lemma7.22 ; and if $R_{i+1} = E$ then $D_{i+1} = E$ , since $E \in \bar{\mathfrak{M}}$ . This proves that E is the limit of the sequence and hence is an $\mathfrak{M}$-covering subgroup in Fischer's sense.

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