SUBIDEALS of LIE ALGEBRAS

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ABSTRACT

We study infinite-dimensional Lie algebras, with particular regard to their subideal structure.

Chapter 1 sets up notation.

Chapter 2 gives an algebraic treatment of Mal'cev's correspondence between complete locally nilpotent torsion-free groups and locally nilpotent Lie algebras over the rational field. This enables us to translate certain of our later results into theorems about groups. As an application we prove a theorem about bracket varieties.

Chapter 3 considers Lie algebras in which every subalgebra is an n-step subideal and shows that such algebras are nilpotent of class bounded in terms of n. This is the Lie-theoretic analogue of a theorem of J.E.Roseblade about groups.

Chapter 4 considers Lie algebras satisfying certain minimal conditions on subideals. We show that the minimal condition for 2-step subideals implies Min-si, the minimal condition for <u>all</u> subideals, and that any Lie algebra satisfying Min-si is an extension of a $\mathfrak{T}$-algebra by a finite-dimensional algebra (a $\mathfrak{T}$-algebra is one in which every subideal is an ideal.)

We show that algebras satisfying Min-si have an ascending series of ideals with factors simple or finite-dimensional abelian, and that the type of such a series may be made any given ordinal number by suitable choice of Lie algebra. We show that the Lie algebra of all endomorphisms of a vector space satisfies Min-si. As a by-product we show that every Lie algebra can be embedded in a simple Lie algebra. Not every Lie algebra can be embedded as a subideal of a perfect Lie algebra.

Chapter 5 considers chain conditions in more specialised classes of Lie algebras. The results are applied to groups.

Chapter 6 develops the theory of $\mathfrak{J}$-algebras, and in particular classifies such algebras under conditions of solubility (over any field) or finite-dimensionality (characteristic zero). We also classify locally finite Lie algebras, every subalgebra of which lies in $\mathfrak{J}$, over algebraically closed fields of characteristic zero.

Chapter 7 concerns various radicals in Lie algebras. We show that not every Baer algebra is Fitting answering a question of B.Hartley. As a consequence we can exhibit a torsion-free Baer group which is not a Fitting group (previous examples have all been periodic). We show that under certain circumstances Baer implies Fitting (for groups or Lie algebras). The last section considers Gruenberg algebras.

Chapter 8 is an investigation parallelling those of Hall and Kulatilaka for groups. We ask: when does an infinite-dimensional Lie algebra have an infinite-dimensional abelian subalgebra? The answer is: not always. Under certain conditions of generalised solubility the answer is 'yes' and we can make the abelian subalgebra in question have additional properties (e.g. be a subideal). The answer is also shown to be 'yes' if the algebra is locally finite (over a field of characteristic zero). This enables us to prove a theorem concerning the minimal condition for subalgebras.

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Chapter One

Notation and Terminology

Throughout this thesis we shall be dealing mainly with infinite-dimensional Lie algebras. Notation and terminology in this area is non-standard; the basic concepts we shall need are dealt with in this preliminary chapter. In any particular situation all Lie algebras will be over the same fixed (but arbitrary) field k; though on occasion we may impose further conditions on k.

1.1 Subideals

Let L be a Lie algebra (of finite or infinite dimension) over an arbitrary field k. If $x,y \in L$ we use square brackets $[x,y]$ to denote the Lie product of x and y. If H is a (Lie) subalgebra of L we write $H \leq L$, and if H is an ideal of L we write $H \triangleleft L$. The symbol $\subseteq$ will denote set-theoretic inclusion.

A subalgebra $H \leq L$ is an <u>ascendant subalgebra</u> if there exists an ordinal number σ and a collection $\{H_\alpha : 0 \leq \alpha \leq \sigma\}$ of subalgebras of L such that $H_0 = H$, $H_\sigma = L$, $H_\alpha \triangleleft H_{\alpha+1}$ for all $0 \leq \alpha < \sigma$, and $H_\lambda = \bigcup_{\alpha<\lambda} H_\alpha$ for limit ordinals $\lambda \leq \sigma$. If this is the case we write

$H \triangleleft^{\sigma} L$. H asc L will denote that $H \triangleleft^{\sigma} L$ for some σ. If $H \triangleleft^{n} L$ for a finite ordinal n we say H is a subideal of L and write H si L. If we wish to emphasize the role of the integer n we shall refer to H as an n-step subideal of L.

If $A,B \leq L$, $X \subseteq L$, and $a,b \in L$ we define $\langle X \rangle$ to be the subalgebra of L generated by X; $[A,B]$ to be the subalgebra generated by all products $[a,b]$ $(a \in A,\ b \in B)$; $[A,{}_nB] = [[A,{}_{n-1}B],B]$ and $[A,{}_0B] = A$; $[a,{}_nb] = [[a,{}_{n-1}b],b]$ and $[a,{}_0b] = a$. We let $\langle X^A \rangle$ denote the ideal closure of X under A, i.e. the smallest subalgebra of L which contains X and is invariant under Lie multiplication by elements of A.

1.2 Derivations

A map $d: L \to L$ is a derivation of L if it is linear and, for all $x,y \in L$,

$$[x,y]d = [xd,y] + [x,yd].$$

The set of all derivations of L forms a Lie algebra under the usual vector space operations, with Lie product $[d_1,d_2] = d_1d_2 - d_2d_1$. We denote this algebra by der(L) and refer to it as the derivation algebra of L. If $x \in L$ the map $\mathrm{ad}(x): L \to L$ defined by

$$y.\mathrm{ad}(x) = [y,x] \qquad (y \in L)$$

is a derivation of L. Such derivations are called <u>inner derivations</u>. The map $x \rightarrow ad(x)$ is a Lie homomorphism $L \rightarrow der(L)$.

A derivation d of L is a <u>nil derivation</u> if for any $x \in L$ there exists an integer $n > 0$ such that $xd^n = 0$. If d is nil and k has characteristic 0 then

$$\exp(d) = \sum_{n=0}^{\infty} \frac{1}{n!} d^n$$

is a well-defined linear transformation of L, and is in fact an automorphism of L (see Hartley [14] p.262). If $x_1,\ldots,x_r \in L$ are such that $ad(x_i)$ is nil $(i = 1,\ldots,r)$ then the map

$$\exp(ad(x_1))\ldots\exp(ad(x_r))$$

is an <u>inner automorphism</u> of L.

1.3 Central and Derived Series

L^n will denote the n-th term of the lower central series of L, so that $L^1 = L$, $L^{n+1} = [L^n,L]$. $L^{(\alpha)}$ (for ordinals α) will denote the α-th term of the (transfinite) derived series of L, so that $L^{(0)} = L$, $L^{(\alpha+1)} = [L^{(\alpha)},L^{(\alpha)}]$, and $L^{(\lambda)} = \bigcap_{\alpha<\lambda} L^{(\alpha)}$ for limit ordinals λ.

$\zeta_\alpha(L)$ will denote the α-th term of the (transfinite) upper central series of L, so that $\zeta_1(L)$ is the centre of L, $\zeta_{\alpha+1}(L)/\zeta_\alpha(L) = \zeta_1(L/\zeta_\alpha(L))$, $\zeta_\lambda(L) = \bigcup_{\alpha<\lambda} \zeta_\alpha(L)$ for limit ordinals λ.

L^n, $L^{(\alpha)}$, and $\zeta_\alpha(L)$ are all <u>characteristic ideals</u> of L in the sense that they are invariant under derivations of L. We write I ch L to mean that I is a characteristic ideal of L. The important property of characteristic ideals is that I ch K $\triangleleft$ L implies I $\triangleleft$ L (see Hartley [14] p.257).

L is <u>nilpotent</u> (of <u>class</u> $\leq$ n) if $L^{n+1} = 0$, and is <u>soluble</u> (of <u>derived length</u> $\leq$ n) if $L^{(n)} = 0$.

1.4 Classes of Lie Algebras

We borrow from group theory the very useful 'Calculus of Classes and Closure Operations' of P.Hall [10].

By a <u>class of Lie algebras</u> we shall understand a class $\mathfrak{X}$ in the usual sense, whose elements are Lie algebras, with the further properties

C1) $\{0\} \in \mathfrak{X}$,

C2) $L \in \mathfrak{X}$ and $K \cong L$ implies $K \in \mathfrak{X}$.

Familiar classes of Lie algebras are:

$\mathfrak{D}$ = the class of all Lie algebras

$\mathfrak{A}$ = abelian Lie algebras

$\mathfrak{N}$ = nilpotent Lie algebras

$\mathfrak{N}_c$ = nilpotent Lie algebras of class $\leq$ c

$\mathfrak{F}$ = finite-dimensional Lie algebras

$\mathfrak{F}_m$ = Lie algebras of dimension $\leq$ m

$\mathfrak{G}$ = finitely generated Lie algebras

$\mathfrak{G}_r$ = Lie algebras generated by $\leq$ r elements.
We shall introduce other classes later on, and will maintain a fixed symbolism for the more important classes. The symbols $\mathfrak{X}$, $\mathfrak{Y}$ will be reserved for arbitrary classes of Lie algebras. Algebras belonging to the class $\mathfrak{X}$ will often be called <u>$\mathfrak{X}$-algebras</u>.

A (non-commutative non-associative) binary operation on classes of Lie algebras is defined as follows: if $\mathfrak{X}$ and $\mathfrak{Y}$ are any two classes let $\mathfrak{X}\mathfrak{Y}$ be the class of all Lie algebras L having an ideal I such that $I \in \mathfrak{X}$ and $L/I \in \mathfrak{Y}$. Algebras in this class will sometimes be called <u>$\mathfrak{X}$-by-$\mathfrak{Y}$-algebras</u>. We extend this definition to products of n classes by defining

$$\mathfrak{X}_1 \ldots \mathfrak{X}_n = ((\mathfrak{X}_1 \ldots \mathfrak{X}_{n-1})\mathfrak{X}_n).$$

We may put all $\mathfrak{X}_i = \mathfrak{X}$ and denote the result by $\mathfrak{X}^n$. Thus in particular $\mathfrak{A}^n$ is the class of soluble Lie algebras of derived length $\leq$ n.

(0) will denote the class of 0-dimensional Lie algebras.

<u>1.5</u> <u>Closure Operations</u>

A <u>closure operation</u> A assigns to each class $\mathfrak{X}$ another class $A\mathfrak{X}$ (the <u>A-closure</u> of $\mathfrak{X}$) in such a way that for all classes $\mathfrak{X}$, $\mathfrak{Y}$ the following axioms are satisfied:

01) $A(0) = (0)$

02) $\mathfrak{X} \leq A\mathfrak{X}$

03) $A(A\mathfrak{X}) = A\mathfrak{X}$

04) $\mathfrak{X} \leq \mathfrak{Y}$ implies $A\mathfrak{X} \leq A\mathfrak{Y}$.

($\leq$ will denote ordinary inclusion for classes of Lie algebras). $\mathfrak{X}$ is said to be <u>A-closed</u> if $\mathfrak{X} = A\mathfrak{X}$. It is often easier to define a closure operation A by specifying which classes are A-closed. Suppose $\mathcal{S}$ is a collection of classes such that $(0) \in \mathcal{S}$ and $\mathcal{S}$ is closed under arbitrary intersections. Then we can define, for each class $\mathfrak{X}$, the class

$$A\mathfrak{X} = \bigcap \{\mathfrak{Y} \in \mathcal{S} : \mathfrak{X} \leq \mathfrak{Y}\}$$

(where the empty intersection is the universal class $\mathfrak{O}$). It is easily seen that A is a closure operation, and that $\mathfrak{X}$ is A-closed if and only if $\mathfrak{X} \in \mathcal{S}$. Conversely if A is a closure operation the set $\mathcal{S}$ of all A-closed classes contains (0), is closed under arbitrary intersections, and determines A.

Standard examples of closure operations are S, I, Q, E, N_0, L defined as follows: $\mathfrak{X}$ is S-closed (I-closed, Q-closed) according as every subalgebra (ideal, quotient algebra) of an $\mathfrak{X}$-algebra is always an $\mathfrak{X}$-algebra. $\mathfrak{X}$ is E-closed if every extension of an $\mathfrak{X}$-algebra by an $\mathfrak{X}$-algebra is an $\mathfrak{X}$-algebra, equivalently if $\mathfrak{X} = \mathfrak{X}^2$. $\mathfrak{X}$ is N_0-closed if $I, J \triangleleft L$, $I, J \in \mathfrak{X}$ implies $I+J \in \mathfrak{X}$. Finally $L \in L\mathfrak{X}$ if and only if every finite subset of L is contained in an $\mathfrak{X}$-subalgebra of L. $L\mathfrak{X}$ is the class of <u>locally $\mathfrak{X}$-algebras</u>.

Clearly $S\mathfrak{X}$ consists of all subalgebras of $\mathfrak{X}$-algebras, $I\mathfrak{X}$ consists of all <u>sub</u>ideals of $\mathfrak{X}$-algebras, and $Q\mathfrak{X}$ consists of all epimorphic images of $\mathfrak{X}$-algebras; while $E\mathfrak{X} = \bigcup_{n=1}^{\infty} \mathfrak{X}^n$ and consists of all Lie algebras having a finite series of subalgebras

$$0 = L_0 \leq L_1 \leq \cdots \leq L_n = L$$

with $L_i \triangleleft L_{i+1}$ $(0 \leq i \leq n-1)$ and $L_{i+1}/L_i \in \mathfrak{X}$ $(0 \leq i \leq n-1)$.

Thus $E\mathfrak{A}$ is the class of soluble Lie algebras, $L\mathfrak{N}$ the class of locally nilpotent Lie algebras, and $L\mathfrak{F}$ the class of locally finite (-dimensional) Lie algebras.

Suppose A and B are two closure operations. Then the product AB defined by $AB\mathfrak{X} = A(B\mathfrak{X})$ need not be a closure operation - O3 may fail to hold. We can define $\{A,B\}$ to be the closure operation whose closed classes are those classes $\mathfrak{X}$ which are both A-closed and B-closed. If we partially order operations on classes by writing $A \leq B$ if and only if $A\mathfrak{X} \leq B\mathfrak{X}$ for any class $\mathfrak{X}$, then $\{A,B\}$ is the smallest closure operation greater than both A and B. It is easy to see (as in Robinson [30] p.4) that $AB = \{A,B\}$ (and is consequently a closure operation) if and only if $BA \leq AB$. From this it is easy to deduce that ES, EI, QS, QI, LS, LI, EQ, LQ are closure operations.

1.6 Ascending Series

Let σ be any ordinal number. An ascending series of type σ of a Lie algebra L is a set $(L_\alpha)_{\alpha\leq\sigma}$ of subalgebras of L such that $L_0 = 0$, $L_\sigma = L$, $L_\alpha \triangleleft L_{\alpha+1}$ $(0 \leq \alpha < \sigma)$, $L_\lambda = \bigcup_{\alpha<\lambda} L_\alpha$ for limit ordinals $\lambda \leq \sigma$. The Lie algebras $L_{\alpha+1}/L_\alpha$ are the factors of the series; if every factor lies in the class $\mathfrak{X}$ then the series (L_α) is an $\mathfrak{X}$-series for L. If further $L_\alpha \triangleleft L$ for each $\alpha \leq \sigma$ then (L_α) will be called an ascending series of ideals of L.

(Note: we could define more general types of series, as in Robinson [30] p.5ff. - but we restrain ourselves from doing so.)

We may now define another closure operation $\acute{E}$; $\acute{E}\mathfrak{X}$ consists of all Lie algebras having an ascending $\mathfrak{X}$-series.

Chapter Two

A Correspondence between Complete Locally Nilpotent Torsion-free Groups and Locally Nilpotent Lie Algebras

In [26] A.I.Mal'cev proves the existence of a connection between locally nilpotent torsion-free groups and locally nilpotent Lie algebras over the rational field, which relates the normality structure of the group to the ideal structure of the Lie algebra. This connection is essentially the standard Lie group - Lie algebra correspondence in an infinite-dimensional situation. Mal'cev's treatment is of a topological nature, involving properties of nilmanifolds; but since the results can be stated in purely algebraic terms, it is of interest to find algebraic proofs. In [24,25] M.Lazard outlines an algebraic treatment of Mal'cev's results, using 'typical sequences' (suites typiques) in a free group. Here we present a third approach, via matrices.

2.1 The Campbell-Hausdorff Formula

Let G be a finitely generated nilpotent torsion-free group. It is well-known (Hall [11] p.56 lemma 7.5, Swan [41]) that G can be embedded in a group of (upper) unitriangular n x n matrices over the integers $\mathbb{Z}$ for some integer n > 0. This in turn embeds in the obvious manner in the group T of unitriangular n x n matrices over the rational field $\mathbb{Q}$. Let U denote the set of n x n zero-triangular matrices over $\mathbb{Q}$. With the usual operations U forms an associative $\mathbb{Q}$-algebra, and this is nilpotent; indeed $U^n = 0$.

For any $t \in T$ we may use the logarithmic series to define

$$\begin{aligned} \log(t) &= \log(1+(t-1)) \\ &= (t-1) - \frac{(t-1)^2}{2} + \frac{(t-1)^3}{3} - \dots \end{aligned} \qquad (1)$$

for if $t \in T$ then $t-1 \in U$ so $(t-1)^n = 0$, and the series (1) has only finitely many non-zero terms. If $t \in T$ then $\log(t) \in U$.

Conversely if $u \in U$ we may use the exponential series to define

$$\exp(u) = 1 + u + \frac{u^2}{2!} + \frac{u^3}{3!} + \dots \qquad (2)$$

and $\exp(u) \in T$ if $u \in U$.

Standard computations reveal that the maps $\log: T \to U$ and $\exp: U \to T$ are mutual inverses; in particular they are bijective.

U can be made into a Lie algebra over $\mathbb{Q}$ by defining a Lie product

$$[u,v] = uv-vu \qquad (u,v \in U). \qquad (3)$$

As usual we define $[u_1,\ldots,u_m]$ ($u_i \in U$, $i = 1,\ldots,m$) inductively to be $[[u_1,\ldots,u_{m-1}],u_m]$ ($m \geq 2$).

<u>Lemma 2.1.1</u> (Campbell-Hausdorff Formula)

If $x,y \in U$ then

$$\log(\exp(x).\exp(y)) = x + y + \tfrac{1}{2}[x,y] + \tfrac{1}{12}[x,y,y] + \ldots$$

where each term is a rational multiple of a Lie product $[z_1,\ldots,z_m]$ of length m such that each z_i is equal either to x or to y, and such that only finitely many products of any given length occur.

The proof is well-known, and can be found in Jacobson [17] p.173.

<u>Corollary</u>

1) If $a,b \in U$ and $ab = ba$ then $\log(\exp(a)\exp(b)) = a + b$.

2) If $t \in T$, $n \in \mathbb{Z}$ then $\log(t^n) = n.\log(t)$.

These may also be proved directly.

A group H is said to be <u>complete</u> (in the sense of Kuroš [23] p.233) if for every $n \in \mathbb{Z}$, $h \in H$ there exists $g \in H$ with $g^n = h$.

H is an <u>R-group</u> (Kuroš [23] p.242) if $g, h \in H$ and $n \in \mathbb{Z}$, together with $g^n = h^n$, imply $g = h$.

If H is a complete R-group, $h \in H$, and $q \in \mathbb{Q}$, then it is easy to see that we may define h^q as follows: if $q = m/n$, $m, n \in \mathbb{Z}$, then h^q is the unique $g \in H$ for which $g^n = h^m$. Further, if $h \in H$, $q, r \in \mathbb{Q}$, we can show that $(h^q)^r = h^{qr}$, $h^{q+r} = (h^q)(h^r)$.

<u>Lemma 2.1.2</u>

T is a complete R-group.

Proof:

1) T is complete: let $t \in T$, $n \in \mathbb{Z}$. Define $s = \exp(\frac{1}{n}\log(t))$ and use corollary to lemma 2.1.1 to show that $s^n = t$.

2) T is an R-group: suppose $s, t \in T$, $n \in \mathbb{Z}$, and $s^n = t^n$. Then $n.\log(s) = n.\log(t)$ so $s = t$.

This gives us easy proofs of two known results:

<u>Proposition 2.1.3</u>

Let H be a finitely generated nilpotent torsion-free group. Then H is an R-group, and can be embedded in a complete R-group (which may be taken to be a group of unitriangular matrices over $\mathbb{Q}$).

Proof:

It suffices to note that a subgroup of an R-group is itself an R-group.

2.2 The Matrix Version

Suppose T is as above, and let G be a complete subgroup of T. Let U be equipped with the Lie algebra structure defined by (3). Define two maps $\flat$, $\#$ as follows:

$$\flat : G \to U \quad , \quad g^{\flat} = \log(g) \qquad (g \in G). \tag{4}$$

Let $L = G^{\flat} = \{g^{\flat} : g \in G\}$:

$$\# : L \to G \quad , \quad \ell^{\#} = \exp(\ell) \qquad (\ell \in L). \tag{5}$$

The aim of this section is to prove

Theorem 2.2.1

With the above notation,

1) The maps $\flat$, $\#$ are mutual inverses.

2) If H is a complete subgroup of G then $H^{\flat}$ is a Lie subalgebra of L. In particular L is a Lie algebra.

3) If M is a subalgebra of L then $M^{\#}$ is a complete subgroup of G.

4) If H is a complete normal subgroup of a complete subgroup K of G, then $H^{\flat}$ is an ideal of $K^{\flat}$.

5) If M is an ideal of a subalgebra N of L, then $M^{\#}$ is a complete normal subgroup of $N^{\#}$.

The proof requires several remarks:

Remark 2.2.2

L is contained in a nilpotent Lie algebra, since U is nilpotent as an associative algebra and hence as a Lie algebra.

Remark 2.2.3

Let $g \in G$, $\lambda \in \mathbb{Q}$, and define g^λ as suggested immediately before lemma 2.1.2. Then $(g^\lambda)^b = \lambda g^b$. For let $\lambda = m/n$, $m,n \in \mathbb{Z}$. By definition $(g^\lambda)^n = g^m$. Taking logs and using part 2 of the corollary to lemma 2.1.1 we find $n.\log(g^\lambda) = m.\log(g)$. Thus we have $(g^\lambda)^b = \log(g^\lambda) = \frac{m}{n}\log(g) = \lambda g^b$.

Remark 2.2.4

Denoting group commutators by round brackets (to avoid confusion with Lie products) thus:

$$(x,y) = x^{-1}y^{-1}xy$$

and inductively $(x_1,\ldots,x_m) = ((x_1,\ldots,x_{m-1}),x_m)$ then the Campbell-Hausdorff Formula implies that for $g_1,\ldots,g_m \in G$,

$$(g_1,\ldots,g_m)^b = [g_1^b,\ldots,g_m^b] + \sum_w P_w$$

where each P_w is a rational linear combination of products $[g_{i_1}^b,\ldots,g_{i_w}^b]$ with $w > m$ and $i_\lambda \in \{1,\ldots,m\}$ for $1 \leq \lambda \leq w$, such that each of $1,\ldots,m$ occurs at least once among the i_λ $(1 \leq \lambda \leq w)$. The exact form of the P_w is determined by the Campbell-Hausdorff Formula.

The proof is by induction on m and can be found in Jennings [19] 6.1.6.

Remark 2.2.5

We now describe a special method of manipulating expressions with terms of the form h^b , where h lies in some subset H of G, which will be needed in the sequel. Suppose we have an expression

$$h^b + \Sigma \lambda_j C_j \qquad (\lambda_j \in \mathbb{Q}) \tag{6}$$

where each C_j is a Lie product of length $\geq$ r of elements of H^b . We can write this as

$$h^b + \Sigma \mu_j D_j + \Sigma \nu_i E_i \quad (\mu_j, \nu_i \in \mathbb{Q})$$

where the D_j are of length r, the E_i of length $\geq$ r+1. Take one of the terms D_j, say

$$D = D_1 = [h_1^b, \dots, h_r^b].$$

By remark 2.2.4 we may replace D by the expression

$$(h_1, \dots, h_r)^b + \Sigma \alpha_k F_k \qquad (\alpha_k \in \mathbb{Q})$$

where each F_k is a product of length $\geq$ r+1 of elements of H^b . Let $(h_1, \dots, h_r) = g \in G$. By the Campbell-Hausdorff Formula and remark 2.2.3

$$(hg^\lambda)^b = h^b + \lambda g^b + \Sigma \beta_\ell G_\ell \quad (\lambda, \beta_\ell \in \mathbb{Q})$$

where the G_ℓ are products of length $\geq$ 2 of elements equal either to h^b or to g^b . But $g^b = D - \Sigma \alpha_k F_k$, each term of which is a product of $\geq$ r elements of H^b .

Thus we may remove the terms D_j one by one to obtain a new expression for (6), of the form

$$(hg_1^{\lambda_1}\ldots g_s^{\lambda_s})^{\flat} + \Sigma\, \gamma_i H_i \qquad (\lambda_j, \gamma_i \in \mathbb{Q})$$

where the g_j are group commutators of length r in elements of H, and the H_i are products of length $\geq$ r+1 in elements of $H^{\flat}$.

We are now ready for the

Proof of theorem 2.2.1

1) Follows from the definitions of $\flat$, $\#$.

2) Any element of the Lie algebra generated by $H^{\flat}$ is of the form (6) with r = 1, h = 0. Using remark 2.2.5 over and over again, we can express this element as

$$(h')^{\flat} + \Sigma\, \delta_i J_i \qquad (\delta_i \in \mathbb{Q})$$

where, since H is a subgroup of G and is complete, $h' \in H$; and the J_i are products of length > c, the class of nilpotency of U. But then $J_i = 0$, and the element under consideration has been expressed as an element of $H^{\flat}$. Thus $H^{\flat}$ is a Lie algebra. In particular so is $L = G^{\flat}$.

3) Let $m, n \in M$, $\lambda \in \mathbb{Q}$. We must show that $(m^{\#})^{\lambda}$ and $m^{\#}n^{\#}$ are elements of $M^{\#}$. Now $(m^{\#})^{\lambda} = (\lambda m)^{\#} \in M^{\#}$. Further, the Campbell-Hausdorff Formula implies that $(m^{\#}n^{\#})^{\flat} = m + n + \frac{1}{2}[m,n] + \ldots \in M$. By part (1) of this theorem $m^{\#}n^{\#} \in M^{\#}$.

4) Let $h \in H$, $k \in K$. We must show that $[h^\flat, k^\flat] \in H^\flat$. We prove, using descending induction on r, that any product of the form $[a_1^\flat, \ldots, a_r^\flat]$ with $a_j \in K$ for all j and at least one $a_i \in H$ is a member of $H^\flat$. This is trivially true for $r > c$, the class of nilpotency of U. The transition from r+1 to r follows from remark 2.2.4, noting that if a group commutator $(k_1, \ldots, k_m)$ with all $k_j \in K$ has some element $k_i \in H$, then the whole commutator lies in H (since H is a normal subgroup of K). The case $r = 2$ gives the result required.

5) Let $m \in M$, $n \in N$. Then $(m^\#, n^\#)^\flat = [m, n]$ + products of length ≥ 3 of elements of M and N, each term containing at least one element of M (Remark 2.2.4). Since M is an ideal of N each such term lies in M, so that $(m^\#, n^\#)^\flat \in M$. By part (1) $(m^\#, n^\#) \in M^\#$, whence $M^\#$ is normal in $N^\#$.

2.3 Inversion of the Campbell-Hausdorff Formula

A given finitely generated nilpotent torsion-free group can in general be embedded in a unitriangular matrix group in many ways. In order to extend our results to locally nilpotent groups and Lie algebras we need a more 'natural' correspondence. This comes from a closer examination of the matrix situation; the method used is to effect what Lazard [25] refers to as 'inversion of the Campbell-Hausdorff formula'. To express the result concisely we must briefly discuss infinite products in locally nilpotent groups. The set-up is analogous to that in Lie algebras with regard to infinite sums (such as the right-hand side of the Campbell-Hausdorff formula) which make sense provided the algebra is locally nilpotent; for then only finitely many terms of the series are non-zero.

Suppose we have a finite set of variables $\{x_1, \dots, x_f\}$. A formal infinite product

$$\omega(x_1, \dots, x_f) = \prod_{i=0}^{\infty} K_i^{\lambda_i}$$

is said to be an extended word in these variables if

E1) $\lambda_i \in \mathbb{Q}$ for all i,

E2) Each K_i is a commutator word $K_i(x_1, \dots, x_f) = (x_{j_1}, \dots, x_{j_r})$ (r depending on i) in the variables $x_1, \dots, x_f$,

E3) Only finitely many terms K_i have any given length r.

Suppose G is a complete locally nilpotent torsion-free group, and $g_1, \dots, g_f \in G$. G is a complete R-group (Proposition 2.1.3) so that

$$(K_i(g_1, \dots, g_f))^{\lambda_i} = (g_{j_1}, \dots, g_{j_r})^{\lambda_i}$$

is defined in G. The group H generated by $g_1, \dots, g_f$ is nilpotent of class c (say) so if K_i has length > c $K_i(g_1,\ldots,g_f) = 1$. Thus only finitely many values of $(K_i(g_1,\ldots,g_f))^{\lambda_i} \neq 1$ and we may define $\omega(g_1,\ldots,g_f)$ to be the product (in order) of the non-1 terms. Thus if $\omega(x_1,\ldots,x_f)$ is an extended word, and G is any complete locally nilpotent torsion-free group, then we may consider ω to be a function $\omega: G^f \to G$.

Similarly we may define an <u>extended Lie word</u> to be a formal sum

$$\zeta(w_1,\ldots,w_e) = \sum_{i=0}^{\infty} \mu_j J_j$$

where

D1) $\mu_j \in \mathbb{Q}$ for all j,

D2) Each J_j is a Lie product $J_j(w_1,\ldots,w_e) = [w_{i_1},\ldots,w_{i_s}]$ (s depending on j) in the variables $w_1,\ldots,w_e$,

D3) Only finitely many terms J_j have any given length s.

Then if L is any locally nilpotent Lie algebra over $\mathbb{Q}$, we may consider ζ to be a function $\zeta: L^e \to L$.

Let us now return to the matrix group / matrix algebra correspondence of section 2.2. Suppose we 'lift' the Lie operations from L to G by defining

$$\lambda g = (\lambda g^{\flat})^{\#}$$
$$g+h = (g^{\flat}+h^{\flat})^{\#}$$
$$[g,h] = [g^{\flat},h^{\flat}]^{\#}$$

($g,h \in G$, $\lambda \in \mathbb{Q}$). Then G with these operations forms a Lie algebra which we shall denote by $\mathcal{L}(G)$. Similarly we may 'drop' the group operations from G to L by defining

$$\ell m = (\ell^{\#} m^{\#})^{\flat}$$
$$\ell^{\lambda} = (\ell^{\#\lambda})^{\flat}$$

($\ell, m \in L$, $\lambda \in \mathbb{Q}$). L with these operations forms a complete group $\mathcal{G}(L)$. $\mathcal{L}(G)$ is isomorphic to L and $\mathcal{G}(L)$ is isomorphic to G.

The crucial observation we require is that these operations can be expressed as extended words (resp. extended Lie words). This is Lazard's 'inversion'.

Lemma 2.3.1

Let G be a complete subgroup of T, and let $L = G^{\flat}$ as described in section 2.2. Then there exist extended words $\varepsilon_{\lambda}(x)$ ($\lambda \in \mathbb{Q}$), $\sigma(x,y)$, $\pi(x,y)$ such that for $g,h \in G$, $\lambda \in \mathbb{Q}$,

$$\lambda g = \varepsilon_{\lambda}(g)$$
$$g+h = \sigma(g,h)$$
$$[g,h] = \pi(g,h)$$

(where the operations on the left are those defined above).

Further there exist extended Lie words $\delta_\lambda(x)$ $(\lambda \in \mathbb{Q})$, $\mu(x,y)$, $\gamma(x,y)$ such that

$$\ell^\lambda = d_\lambda(\ell)$$
$$\ell m = \mu(\ell,m)$$
$$(\ell,m) = \gamma(\ell,m)$$

$(\ell,m \in L,\ \lambda \in \mathbb{Q})$ (operations on left as above).

<u>These words can be taken to be independent of the particular G, L chosen</u>.

Proof:

1) ε_λ:

$$(\lambda g^\flat)^\# = \exp(\lambda.\log(g)) = g^\lambda, \text{ so } \varepsilon_\lambda(x) = x^\lambda \text{ has}$$

the required properties.

2) σ:

Here we must do more work. We show that there exist words $\sigma_i(x,y)$ satisfying

$$\sigma_{i+1}(x,y) = \sigma_i(x,y)\gamma_{i+1}(x,y)$$
$$\sigma_o(x,y) = 1$$

where γ_{i+1} is a word of the form

$$K_1^{\lambda_1} \ldots K_u^{\lambda_u} \quad (\lambda_j \in \mathbb{Q}\ 1 \leq j \leq u)$$

with each K_j a commutator word $(z_{j_1}, \ldots, z_{j_{i+1}})$ of length $i+1$ with $z_{j_k} = x$ or y $(1 \leq k \leq i+1)$; such that if G is a complete subgroup of the group of c x c unitriangular matrices over $\mathbb{Q}$ $(c \geq 1)$ then

$$g+h = \sigma_c(g,h) \qquad (g,h \in G).$$

The existence of these words is a consequence of the manipulation process described in remark 2.2.5. This enables us to take an expression of the form

$$h^\flat + \Sigma\, \lambda_j C_j \qquad (\lambda_j \in \mathbb{Q}) \tag{7}$$

where h lies in some subset H of G, and the C_j are Lie products of length $\geq$ r in elements of $H^\flat$, and replace it by an expression

$$(hg_1^{\mu_1}\ldots g_m^{\mu_m}) + \Sigma\, \gamma_i H_i \qquad (\mu_j,\ \gamma_i \in \mathbb{Q})$$

where the g_j are commutator words in elements of H of length r, and the H_i are Lie products of elements of $H^\flat$ of length $\geq$ r+1.

We obtain the σ_i by systematically applying this procedure to the expression $g^\flat + h^\flat$. We choose a total ordering $\ll$ of the left-normed Lie products in x,y in such a way that the length is compatible with the ordering. Next we apply the process of section 2.2.5 to the expression $g^\flat + h^\flat$ (with g playing the role of h in (7), $\lambda_1 = 1$, $C_1 = h^\flat$) and at each stage in the process

1) Express all Lie products in $g^\flat, h^\flat$ as sums of left-normed commutators (using anticommutativity and the Jacobi identity),

2) Collect together all multiples of the same

left-normed product,

3) Operate on the term D (in the notation of Remark 2.2.5) which is smallest in the ordering $\ll$.

At the i-th stage we will have expressed $g+h$ in the form

$$(\sigma_i(g,h))^{b} + \Sigma\, \theta_k I_k \qquad (\theta_k \in \mathbb{Q})$$

where σ_i is a word in g, h and the terms I_k are Lie products in g^{b}, h^{b} of length > i. At the (i+1)-th stage this will have been modified to

$$(\sigma_i(g,h).g_1^{\lambda_1}\ldots g_m^{\lambda_m})^{b} + \Sigma\, \phi_\ell J_\ell \qquad (\phi_\ell \in \mathbb{Q})$$

where the g_i are group commutators in g,h of length i+1, the $\lambda_i \in \mathbb{Q}$, and the J_ℓ are Lie products in g^{b}, h^{b} of length > i+1.

We put

$$\gamma_{i+1}(g,h) = g_1^{\lambda_1}\ldots g_m^{\lambda_m},$$
$$\sigma_{i+1}(g,h) = \sigma_i(g,h)\gamma_{i+1}(g,h)$$
$$\sigma_0(g,h) = 1.$$

It is clear from the way that the process 2.2.4 operates that the form of the words σ_i, γ_i depends only on the ordering $\ll$ (and the Campbell-Hausdorff formula) so that we can define the required words $\sigma_i(x,y)$ and $\gamma_i(x,y)$ independently of G.

Now if G consists of c x c matrices, then at the c-th stage we have

$$g^\flat + h^\flat = (\sigma_c(g,h))^\flat + \Sigma\, \psi_p K_p \quad (\psi_p \in \mathbb{Q})$$

where the terms K_p are of length > c so are 0. Thus

$$g+h = (g^\flat + h^\flat)^\# = \sigma_c(g,h)$$

as claimed.

We now define

$$\sigma(x,y) = \prod_{i=0}^{\infty} \sigma_i(x,y).$$

If G is a complete group of unitriangular c x c matrices over $\mathbb{Q}$, then G is nilpotent of class $\leq$ c, so for all j > 0 $\sigma_{c+j}(g,h) = 1$, so $\sigma(g,h) = \sigma_c(g,h)$. Hence for any such G we have $g+h = \sigma(g,h)$ as required.

3) π:

Similar proof. Work on the expression

$$1^\flat + [g^\flat, h^\flat]$$

(which equals $[g^\flat, h^\flat]$) with 1 playing the role of h in (7), $\lambda_1 = 1$, $C_1 = [g^\flat, h^\flat]$.

4) δ_λ:

$$\ell^\lambda = (\ell^{\#\lambda})^\flat = \log(\exp(\ell)^\lambda) = \lambda\ell \quad (\ell \in L) \text{ so}$$

$\delta_\lambda(x) = \lambda x$ will do.

5) μ:

Put $\mu(x,y) = x + y + \frac{1}{2}[x,y] + \ldots$ as in the Campbell-Hausdorff formula.

6) γ:

Follows at once from the existence of δ_λ and μ. The lemma is proved.

To illustrate the method, we calculate the function σ up to terms of length 3. To this length the Campbell-Hausdorff formula becomes

$$(gh)^\flat = g^\flat + h^\flat + \frac{1}{2}[g^\flat,h^\flat] + \frac{1}{12}([g^\flat,h^\flat,h^\flat] + [h^\flat,g^\flat,g^\flat])$$

and thus

$$(x,y)^\flat = [x^\flat,y^\flat] + \frac{1}{2}([x^\flat,y^\flat,x^\flat] + [x^\flat,y^\flat,y^\flat]).$$

We choose left-normed commutators as follows:

$$a^\flat \ll b^\flat \ll [a^\flat,b^\flat] \ll [a^\flat,b^\flat,a^\flat] \ll [a^\flat,b^\flat,b^\flat].$$

Now $(a+b)^\flat = a^\flat+b^\flat$ by definition

$$= (ab)^\flat - \frac{1}{2}[a^\flat,b^\flat] - \frac{1}{12}([a^\flat,b^\flat,b^\flat] - [a^\flat,b^\flat,a^\flat])$$

$$= (ab)^\flat - \frac{1}{2}\{(a,b)^\flat - \frac{1}{2}([a^\flat,b^\flat,a^\flat] + [a^\flat,b^\flat,b^\flat])\} + \frac{1}{12}([a^\flat,b^\flat,a^\flat] - [a^\flat,b^\flat,b^\flat])$$

$$= (ab(a,b)^{-1/2})^\flat - \frac{1}{2}([(ab)^\flat\ ,(a,b)^{-1/2\ \flat}]) + \frac{1}{4}([a^\flat,b^\flat,a^\flat] + [a^\flat,b^\flat,b^\flat]) + \frac{1}{12}([a^\flat,b^\flat,a^\flat] - [a^\flat,b^\flat,b^\flat])$$

$$= (ab(a,b)^{-1/2})^\flat - \frac{1}{2}([a^\flat+b^\flat,\ -\frac{1}{2}[a^\flat,b^\flat]]) + \frac{1}{4}([a^\flat,b^\flat,a^\flat] + [a^\flat,b^\flat,b^\flat]) + \frac{1}{12}([a^\flat,b^\flat,a^\flat] - [a^\flat,b^\flat,b^\flat])$$

$$= (ab(a,b)^{-1/2})^\flat + \frac{1}{12}[a^\flat,b^\flat,a^\flat] - \frac{1}{12}[a^\flat,b^\flat,b^\flat]$$

$$= (ab(a,b)^{-1/2}(a,b,a)^{1/12}(a,b,b)^{-1/12})^\flat .$$

Thus up to terms of length 3

$$\sigma(a,b) = ab(a,b)^{-1/2}(a,b,a)^{1/12}(a,b,b)^{-1/12}.$$

Similarly we find

$$\pi(a,b) = (a,b)(a,b,a)^{-1/2}(a,b,b)^{-1/2}.$$

2.4 The General Version

As remarked in section 2.3, if $\omega(x_1,\ldots,x_f)$ is an extended word and G any complete locally nilpotent torsion-free group, then ω can be considered as a function $G^f \to G$. Similarly for extended Lie words and locally nilpotent Lie algebras over $\mathbb{Q}$. On this basis we can establish a general version of Mal'cev's correspondence as follows:

Theorem 2.4.1

Let G be a complete locally nilpotent torsion-free group. Define operations on G as follows:

If $\lambda \in \mathbb{Q}$, $g,h \in G$ set

$$\lambda g = \varepsilon_\lambda(g)$$
$$g+h = \sigma(g,h)$$
$$[g,h] = \pi(g,h).$$

With these operations G becomes a Lie algebra over $\mathbb{Q}$, which we denote by $\mathcal{L}(G)$. $\mathcal{L}(G)$ is a locally nilpotent Lie algebra.

Conversely, let L be a locally nilpotent Lie algebra over $\mathbb{Q}$. Define, for $\lambda \in \mathbb{Q}$, $\ell, m \in L$, operations:

$$\ell^\lambda = \delta_\lambda(\ell)$$
$$\ell m = \mu(\ell,m).$$

With these operations L becomes a complete locally nilpotent torsion-free group, which we denote by $\mathcal{G}(L)$.

Proof:

The axioms for a Lie algebra can be expressed as certain relations between the functions ε_λ, σ, π involving at most 3 variables. Thus if these relations can be shown to hold in any 3-generator subgroup of G, they hold throughout G. But, as remarked earlier, any finitely generated nilpotent torsion-free group can be embedded in a group of unitriangular c x c matrices over $\mathbb{Q}$ for some integer c > 0 (Hall [11], Swan [41]). But the required relations certainly hold in this situation, since by the construction of ε_λ, σ, π they express the fact that the logarithms of these matrices form a Lie algebra under the usual operations - a fact which is manifest.

Any finitely generated subalgebra of $\mathcal{L}(G)$ is the image under $\mathcal{L}$ of the completion $\bar{H}$ of some finitely generated subgroup H of G. H is nilpotent, so by Kuroš [23] p.258, $\bar{H}$ is also nilpotent. The form of the words ε_λ, σ, π now ensures that the original finitely generated subalgebra of $\mathcal{L}(G)$ is nilpotent. Hence $\mathcal{L}(G)$ is locally nilpotent.

In a similar way the axioms for a complete group hold in L if they hold in any finitely generated subalgebra.

Now a finitely generated nilpotent Lie algebra is finite-dimensional (Hartley [14] p.261) and any finite-dimensional nilpotent Lie algebra over $\mathbb{Q}$ can be embedded in a Lie algebra of zero-triangular matrices over $\mathbb{Q}$ (Birkhoff [3]). We may therefore proceed analogously to complete the proof.

We next consider the relation between the structure of G and that of $\mathcal{L}(G)$; also L and $\mathcal{G}(L)$.

<u>Theorem 2.4.2</u>

Let G, H be complete locally nilpotent torsion-free groups; let L be a locally nilpotent Lie algebra over $\mathbb{Q}$. Then

1) $\mathcal{G}(\mathcal{L}(G)) = G, \quad \mathcal{L}(\mathcal{G}(L)) = L.$

2) H is a subgroup of G if and only if

$$\mathcal{L}(H) \leq \mathcal{L}(G).$$

3) H is a normal subgroup of G if and only if

$$\mathcal{L}(H) \lhd \mathcal{L}(G).$$

4) $\phi: G \to H$ is a group homomorphism if and only if $\phi: \mathcal{L}(G) \to \mathcal{L}(H)$ is a Lie homomorphism. The kernel of ϕ is the same in both cases.

5) If H is a normal subgroup of G, then

$$\mathcal{L}(G/H) = \mathcal{L}(G)/\mathcal{L}(H).$$

(Note: using part (1) we can easily recast parts (2), (3), (4), (5) in a '$\mathcal{G}$' form instead of an '$\mathcal{L}$' form.)

Proof:

1) Let $g, h \in G$. We must show that for $\lambda \in \mathbb{Q}$

$$g^{\lambda} = \delta_{\lambda}(g)$$
$$gh = \mu(g,h)$$

where δ_{λ}, μ are defined in terms of the Lie operations of $\mathcal{L}(G)$. Now $\delta_{\lambda}(g) = \lambda g = \varepsilon_{\lambda}(g) = g^{\lambda}$. To show that $gh = \mu(g,h)$ we may confine our attention to the completion of the group generated by g and h. Thus without loss of generality G is a group of unitriangular matrices over $\mathbb{Q}$.

Now by definition

$$\mu(g,h) = g + h + \tfrac{1}{2}[g,h] + \cdots$$

and +, [,] are defined in $\mathcal{L}(G)$ by

$$g+h = (g^{\flat}+h^{\flat})^{\#}$$
$$[g,h] = [g^{\flat},h^{\flat}]^{\#}$$

so

$$\mu(g,h)^{\flat} = g^{\flat} + h^{\flat} + \tfrac{1}{2}[g^{\flat},h^{\flat}] + \cdots$$
$$= (gh)^{\flat} \quad \text{by Campbell-Hausdorff}$$

so $\mu(g,h) = gh$ as required.

The converse is similar and will be omitted.

2) and 3) are clear from the form of the functions $\varepsilon_{\lambda}, \pi, \sigma, \delta_{\lambda}, \mu, \gamma$.

4) Follows from the observation that group homomorphisms (resp. Lie homomorphisms) preserve extended words (resp.

extended Lie words). The kernels are the same since the identity element of G is the zero element of $\mathcal{L}(G)$.

5) We first show that H-cosets in G are the same as $\mathcal{L}(H)$-cosets in $\mathcal{L}(G)$.

Let $x \in G$, $z \in Hx$. Then $z = hx$ for some $h \in H$, and $hx = h + x + \frac{1}{2}[h,x] + \ldots \in \mathcal{L}(H) + x$ since $h \in \mathcal{L}(H)$ which is an ideal of $\mathcal{L}(G)$. Thus $Hx \subseteq \mathcal{L}(H) + x$.

Now let $y \in \mathcal{L}(H) + x$. Then $y = h+x$ for some $h \in H$, and $h + x = h.x.(h,x)^{-1/2}\ldots \in Hx$ since H is a normal subgroup of G. Therefore $\mathcal{L}(H) + x \subseteq Hx$.

Hence $Hx = \mathcal{L}(H) + x$. The operations on the cosets are defined by the same extended words, and the result follows.

<u>Remark</u>

In categorical guise, let $\mathcal{C}_{\mathcal{G}}$ denote the category of complete locally nilpotent torsion-free groups and group homomorphisms, $\mathcal{C}_{\mathcal{L}}$ the category of locally nilpotent Lie algebras over $\mathbb{Q}$ and Lie homomorphisms. Then

$$\mathcal{L} : \mathcal{C}_{\mathcal{G}} \to \mathcal{C}_{\mathcal{L}}$$
$$\mathcal{G} : \mathcal{C}_{\mathcal{L}} \to \mathcal{C}_{\mathcal{G}}$$

are covariant functors, defining an isomorphism between the two categories.

Observe, however, that our definition of $\mathcal{L}$ and $\mathcal{G}$

is stronger than a purely category-theortetic one - as far as the underlying <u>sets</u> are concerned they are both identity maps.

We shall now develop a few more properties of the correspondence, which we need later. But first let us recall the definition of a <u>centraliser</u> in a Lie algebra: suppose $H \subseteq X \subseteq L$, $H \leq L$, and $H \triangleleft X$. Then

$$C_L(X/H) = \{c \in L : [c,X] \leq H\}.$$

There is a similar definition for groups.

<u>Lemma 2.4.3</u>

Let G, H be complete locally nilpotent torsion-free groups, with $H \leq G$, $H \triangleleft X \subseteq G$. Then

$$\mathcal{L}(C_G(X/H)) = C_{\mathcal{L}(G)}(\mathcal{L}(X)/\mathcal{L}(H))$$

(where the notation $\mathcal{L}(X)$ indicates the set X considered as a subset of $\mathcal{L}(H)$).

Proof:

Let $c \in C = C_G(X/H)$. Then for any $x \in X$, $[c,x] = (c,x)(c,x,c)^{-1/2}\ldots \in H$ (from the definition of C and since $H \triangleleft X$). Consequently $c \in C_{\mathcal{L}(G)}(\mathcal{L}(X)/\mathcal{L}(H))$. The converse inclusion is similar.

<u>Corollary 1</u>

1) $\mathcal{L}(C_G(X)) = C_{\mathcal{L}(G)}(\mathcal{L}(X))$ (put $H = 0$)

2) $\mathcal{L}(N_G(H)) = I_{\mathcal{L}(G)}(\mathcal{L}(H))$ (put $X = H$).

(Here N_G denotes the normaliser in G, and $I_{\mathcal{L}(G)}$ the idealiser in $\mathcal{L}(G)$ (also called the normaliser in Jacobson [17] p.57, but we prefer the alternative terminology)).

Corollary 2

Letting $\zeta_\alpha(G)$ denote the α-th term of the upper central series of G, then

$$\mathcal{L}(\zeta_\alpha(G)) = \zeta_\alpha(\mathcal{L}(G)).$$

Proof:

Use transfinite induction on α and lemma 2.4.3.

Corollary 3

The upper central series of G and $\mathcal{L}(G)$ become stationary at the same ordinal α. In particular if either G or $\mathcal{L}(G)$ is nilpotent then so is the other and their classes of nilpotency are equal.

Proof:

Immediate from Corollary 2.

Suppose G is a complete locally nilpotent torsion-free group, and H is any subgroup. Then the completion $\bar{H}$ of H in G is the smallest complete subgroup of G which contains H. The next lemma collects some known facts about completions.

Lemma 2.4.4

Suppose G is a complete locally nilpotent torsion-free group, and $H \leq K \leq G$.

1) If $H \lhd K$ then $\overline{H} \lhd \overline{K}$.

2) $\overline{K}$ is equal to the <u>isolator</u> of K in G, which is the <u>set</u> of all $g \in G$ such that $g^n \in K$ for some $n \in \mathbb{Z}$.

Proof:

1) see Kuroš [23] p.254.

2) see Kuroš [23] pp. 249, 255.

<u>Lemma 2.4.5</u>

Let G be a complete locally nilpotent torsion-free group, H a complete subgroup of G. Then $H \lhd^{\alpha} G$ if and only if $\mathcal{L}(H) \lhd^{\alpha} \mathcal{L}(G)$.

Proof:

There is a normal series

$$H = H_0 \lhd H_1 \lhd \ldots \quad H_\beta \lhd H_{\beta+1} \lhd \ldots \quad H_\alpha = G$$

from H to G, such that $H_\lambda = \bigcup_{\beta<\lambda} H_\beta$ at limit ordinals λ. Let $L_\beta = \mathcal{L}(\overline{H}_\beta)$ (bars denoting completions in G). Then $\mathcal{L}(H) = L_0$, $\mathcal{L}(G) = L_\alpha$. By lemma 2.4.4.1 and theorem 2.4.2.3 we have $L_\beta \lhd L_{\beta+1}$ for all $\beta < \alpha$. Lemma 2.4.4.2 easily shows that at limit ordinals λ $L_\lambda = \bigcup_{\beta<\lambda} L_\beta$. The result follows.

In particular H is subnormal in G if and only if $\mathcal{L}(H)$ is a subideal of $\mathcal{L}(G)$; and H is ascendant in G if and only if $\mathcal{L}(H)$ is an ascendant subalgebra of $\mathcal{L}(G)$.

As an application of these results we will give a generalisation of a result of Yu.G.Fedorov (see Kuroš [23] p.257) which states that a nilpotent torsion-free group and its completion have the same class of nilpotency. Our generalisation (proved in the next section) does not seem to have appeared in the literature.

Other applications of the Mal'cev correspondence will be given in later chapters. It seems possible to enumerate properties of the correspondence *ad nauseam* - but we shall avoid this. Any further attributes of the correspondence will be developed as and when they are required.

2.5 Bracket Varieties

Let $\phi = \phi(x_1,\ldots,x_n)$ and $\psi = \psi(y_1,\ldots,y_m)$ be two group words. Following P.Hall we define the *outer commutator* word $(\phi,\psi)_o$ to be the word

$$(\phi,\psi)_o(x_1,\ldots,x_{n+m}) = (\phi(x_1,\ldots,x_n))^{-1}(\psi(x_{n+1},\ldots,x_{n+m}))^{-1}(\phi(x_1,\ldots,x_n))(\psi(x_{n+1},\ldots,x_{n+m})).$$

We define *bracket words* inductively: the identity word $\iota(x_1) = x_1$ is a bracket word of *height* $h(\iota) = 1$. If ϕ, ψ are bracket words then $(\phi,\psi)_o$ is a bracket word of height $h(\phi)+h(\psi)$.

Thus for example (x,y), $((x,y),z)$ and $((x,y),(z,t))$ are bracket words.

Analogous definitions can be made for Lie algebras. In this case we denote the outer commutator by $[\phi,\psi]_o$, and the height again by h. To each group bracket word ϕ there corresponds in a natural way a Lie bracket word ϕ^* defined inductively by

$$\iota^* = \iota$$

$$(\phi,\psi)^*_o = [\phi^*,\psi^*]_o \ .$$

Clearly $h(\phi) = h(\phi^*)$, and ϕ^* is obtained from ϕ by changing all round brackets to square ones.

If G is a group and ϕ a group bracket word, the <u>verbal subgroup</u> corresponding to ϕ is

$$\phi(G) = \langle \phi(g_1,\ldots,g_n) \ : \ g_i \in G \quad 1 \leq i \leq n \rangle$$

and the <u>variety</u> $\mathfrak{V}_\phi$ determined by ϕ is the class of all groups G for which $\phi(G) = 1$; equivalently those G for which the relation $\phi(g_1,\ldots,g_n) = 1$ holds identically in G.

Similarly we define the <u>verbal subalgebra</u> $\phi^*(L)$ of a Lie algebra L determined by a Lie bracket word ϕ^*, and the variety $\mathfrak{V}_{\phi^*}$.

If G is a group and ϕ a group bracket word, then a <u>ϕ-value</u> in G is an element expressible as $\phi(g_1,\ldots,g_n)$ $(g_i \in G \quad 1 \leq i \leq n)$. Similarly for Lie algebras.

Lemma 2.5.1

Let ϕ, ψ be Lie bracket words, L any Lie algebra (over an arbitrary field). Then

1) $\phi(L)$ is the vector subspace of L spanned by the ϕ-values in L.

2) $\phi(L) \triangleleft L$.

3) $[\phi,\psi]_o(L) = [\phi(L), \psi(L)]$.

Proof:

We prove (1) and (2) simultaneously by induction on the height of ϕ.

If $h(\phi) = 1$ then $\phi = \iota$ and (1) and (2) are trivial. If $h(\phi) > 1$ then there are bracket words ψ, χ such that $\phi = [\psi,\chi]_o$ and $h(\psi)$, $h(\chi) < h(\phi)$. Inductively we may suppose that (1) and (2) hold for ψ and χ. Let x be a ϕ-value in L. Then there exist $\underline{y} = (y_1,\dots,y_n)$ and $\underline{z} = (z_1,\dots,z_m)$ $(y_1,\dots,y_n,z_1,\dots,z_m \in L)$ such that $x = \phi(\underline{y},\underline{z}) = [\psi(\underline{y}),\chi(\underline{z})]$. If $t \in L$ then $[x,t] = [[\psi(\underline{y}),\chi(\underline{z})],t] = [[\psi(\underline{y}),t],\chi(\underline{z})] + [\psi(\underline{y}),[\chi(\underline{z}),t]]$ by Jacobi. By part (2) inductively $[\psi(\underline{y}),t]$ lies in $\psi(L)$; by part (1) it is a linear combination of ψ-values. Similarly for $[\chi(\underline{z}),t]$. Thus $[x,t]$ is a linear combination of $[\psi,\chi]_o$ -values. Hence the subspace spanned by the ϕ-values is an ideal of L, and so is equal to $\phi(L)$. This proves parts (1) and (2).

Part (3) now follows at once from part (1).

Results analogous to parts (2) and (3) are well known for groups.

Let G be a locally nilpotent torsion-free group. Then it is known that G has a unique completion $\bar{G}$, that is a complete locally nilpotent torsion-free group containing G and such that the completion of G in $\bar{G}$ is the whole of $\bar{G}$. Note that we cannot use Mal'cev's work on completions to establish the existence of $\bar{G}$ since we are trying to produce algebraic proofs of our theorems. The whole of Mal'cev's theory of completions has been developed in a purely algebraic setting by Kargapolov [20,21]; and a method is outlined in Hall [11] p.46.

Under the Mal'cev correspondence $\bar{G}$ can also be considered to be a Lie algebra over $\mathbb{Q}$. Denote completions (in $\bar{G}$) of subgroups of $\bar{G}$ by overbars. Temporarily denote by i<X> the ideal of $\bar{G}$ generated by X (considering $\bar{G}$ as a Lie algebra) and let n<X> denote the normal subgroup of $\bar{G}$ generated by X, for any subset X of G.

Lemma 2.5.2

Let G be a locally nilpotent torsion-free group, A, B $\triangleleft$ G.

Then $\overline{(A,B)} = \overline{(\bar{A},\bar{B})} = [\bar{A},\bar{B}]$

(where in the third expression $\bar{A}$ and $\bar{B}$ are considered as subalgebras of $\bar{G}$).

Proof:

Throughout let a run through A, b through B, and α,β through $\mathbb{Q}$. Then

$$\overline{(A,B)} = \overline{n<(a,b)>}$$

$= i<[a,b]>$ since from the form of the words π, γ of lemma 2.3.1 it is clear that $(a,b) \in i<[a,b]>$ and $[a,b]$ $\overline{n<(a,b)>}$

$$= i<[\alpha a,\beta b]>$$

$$= i<[a^{\alpha},b^{\beta}]> \qquad (*)$$

$= [\bar{A},\bar{B}]$ using lemma 2.4.4.2

But also

$(*) = \overline{n<(a^{\alpha},b^{\beta})>}$ (as above)

$= \overline{(\bar{A},\bar{B})}$ using lemma 2.4.4.2.

The promised generalisation of Fedorov's result:

<u>Theorem 2.5.3</u>

Let G be any locally nilpotent torsion-free group, $\bar{G}$ its completion (viewed also as a Lie algebra over $\mathbb{Q}$). Let ϕ be any group bracket word. Then

1) $\overline{\phi(G)} = \overline{\phi(\bar{G})} = \phi^*(\bar{G})$

2) $G \in \mathfrak{V}_{\phi} \Leftrightarrow \bar{G} \in \mathfrak{V}_{\phi} \Leftrightarrow \bar{G} \in \mathfrak{V}_{\phi^*}$.

Proof:

1) Use induction on $h(\phi) = h(\phi^*)$. If $h(\phi) = 1$ the result is clear. If not, then $\phi = (\psi,\chi)_0$ and so $\phi^* = [\psi^*,\chi^*]_0$ where all of $h(\psi)$, $h(\chi)$, $h(\psi^*)$, $h(\chi^*)$ are less than $h(\phi)$. Thus

$$
\begin{aligned}
\overline{\phi(G)} &= \overline{(\psi,\chi)_0(G)} \\
&= \overline{(\psi(G),\chi(G))} && \text{(lemma 2.5.1.3 for groups)} \\
&= \overline{(\overline{\psi(G)},\overline{\chi(G)})} && \text{(lemma 2.5.2)} \\
&= \overline{(\overline{\psi(\overline{G})},\overline{\chi(\overline{G})})} && \text{(induction hypothesis)} \quad (*) \\
&= \overline{(\psi(\overline{G}),\chi(\overline{G}))} && \text{(lemma 2.5.2)} \\
&= \overline{(\psi,\chi)_0(\overline{G})} \\
&= \overline{\phi(\overline{G})}.
\end{aligned}
$$

Also,

$$
\begin{aligned}
(*) &= [\overline{\psi(\overline{G})},\overline{\chi(\overline{G})}] && \text{(lemma 2.5.2)} \\
&= [\psi^*(\overline{G}),\chi^*(\overline{G})] && \text{(induction hypothesis)} \\
&= [\psi^*,\chi^*]_0(\overline{G}) && \text{(lemma 2.5.1.3)} \\
&= \phi^*(G)
\end{aligned}
$$

which proves part (1).

2)
$$
\begin{aligned}
G \in \mathfrak{V}_\phi &\iff \phi(G) = 1 \\
&\iff \overline{\phi(G)} = 1 \\
&\iff \overline{\phi(\overline{G})} = 1 && (**) \\
&\iff \phi(\overline{G}) = 1 \\
&\iff \overline{G} \in \mathfrak{V}_\phi .
\end{aligned}
$$

Also

$$(**) \iff \phi^*(\bar{G}) = 0$$
$$\iff \bar{G} \in \mathfrak{V}_{\phi^*}.$$

<u>Corollary</u>

Let $\mathfrak{X}$ be a union of bracket varieties of groups, $\mathfrak{X}^*$ the union of the corresponding Lie bracket varieties. Then

$$G \in \mathfrak{X} \Leftrightarrow \bar{G} \in \mathfrak{X} \Leftrightarrow \bar{G} \in \mathfrak{X}^* .$$

In particular we may take for $\mathfrak{X}$ the classes (using P.Hall's notation [10]):

$$\mathfrak{N}_c, \mathfrak{N} , \mathfrak{A}^d, \text{E}\mathfrak{A} , \mathfrak{A}\mathfrak{N}.$$

(The case $\mathfrak{X} = \mathfrak{N}_c$ is Fedorov's theorem.)

Chapter Three

Lie algebras, all of whose subalgebras are n-step subideals

A theorem of J.E.Roseblade [33] states that if G is a group such that every subgroup K of G is subnormal in at most n steps, i.e. there exists a series of subgroups

$$K = K_0 \triangleleft K_1 \triangleleft \dots \triangleleft K_n = G,$$

then G is nilpotent of class $\leq f(n)$ for some function $f: \mathbb{Z} \to \mathbb{Z}$.

This chapter is devoted to a proof of the analogous result for Lie algebras over fields of arbitrary characteristic.

3.1 Subnormality and completions

It might be thought that we could prove the theorem for Lie algebras over $\mathbb{Q}$ by a combination of Roseblade's result and the Mal'cev correspondence, as follows:

Suppose L is a Lie algebra over $\mathbb{Q}$, such that every subalgebra $K \leq L$ satisfies $K \triangleleft^n L$. By a theorem of Hartley [14] p.259 (cor. to theorem 3) $L \in L\mathfrak{N}$.

We may therefore form the corresponding group $\mathcal{G}(L)$. Clearly every complete subgroup H of G satisfies $H \lhd^n G$. If we could show that every subgroup of G is boundedly subnormal in its completion, we could use Roseblade's theorem to deduce the nilpotence (of bounded class) of G, hence of L.

This approach fails, however - we shall show that a locally nilpotent torsion-free group need not be subnormal in its completion, let alone boundedly so.

Let $T_n(\mathbb{Q})$ denote the group of (n+1) x (n+1) unitriangular matrices over $\mathbb{Q}$, $U_n(\mathbb{Q})$ the Lie algebra of all (n+1) x (n+1) zero-triangular matrices over $\mathbb{Q}$. Similarly define $T_n(\mathbb{Z})$, $U_n(\mathbb{Z})$.

If H is a subnormal subgroup of G let d(H,G) be the least integer d for which (in an obvious notation) $H \lhd^d G$. d is the defect of H in G.

Lemma 3.1.1

$$d(T_n(\mathbb{Z}), T_n(\mathbb{Q})) = n.$$

Proof:

Let $T = T_n(\mathbb{Q})$, $S = T_n(\mathbb{Z})$, $d = d(S,T)$. Then $d \leq n$ since T is nilpotent of class n. We show that $S \lhd^{n-1} T$ is false. Suppose, if possible, that $S \lhd^{n-1} T$. Then for all $s \in S$, $t \in T$ we would have

$$(t,{}_{n-1}s) \in S$$

(where $(a,{}_m b)$ denotes $(\ldots(a,\underbrace{b),b),\ldots,b}_{m})$.)

Taking logarithms,

$$\log(t,{}_{n-1}s) \in \log(S).$$

By the Campbell-Hausdorff formula, remembering that T is nilpotent of class n, this means that

$$[\log(t),{}_{n-1}\log(s)] \in \log(S).$$

We choose $s \in S$ in such a way as to prevent this happening.

Consider the matrix $X = \begin{bmatrix} 0 & x & 0 & \ldots & 0 \\ & \ddots & \ddots & \ddots & \\ & 0 & & & \end{bmatrix}$

Then $\exp(X) = \begin{bmatrix} 1 & x & x^2/2! & x^3/3! & \ldots & x^n/n! \\ & \ddots & \ddots & \ddots & \ddots & \\ & 0 & & & & \end{bmatrix}$

So if we put $s = \exp \begin{bmatrix} 0 & n! & 0 & \ldots & 0 \\ & \ddots & \ddots & \ddots & \\ & 0 & & & \end{bmatrix}$

then $s \in S$.

Let $$t = \exp \left[\begin{array}{ccccc} 0 & \lambda & 0 & \ldots & 0 \\ \hline & & & & \\ & & O & & \\ & & & & \end{array}\right]$$

where for the moment λ is an arbitrary element of $\mathbb{Q}$.

An easy induction shows that

$$[\log(t),_{n-1}\log(s)] = \left[\begin{array}{cccc} 0 & \ldots & 0 & \alpha \\ \hline & & & \\ & O & & \\ & & & \end{array}\right] = A \text{ (say)},$$

where $\alpha = \lambda.(n!)^{n-1}$.

Now $$\exp(A) = \begin{bmatrix} 1 & 0 & \ldots & 0 & \alpha \\ & \ddots & & O & \\ & O & & \ddots & \\ & & & & 1 \end{bmatrix}$$

and we can choose $\lambda \in \mathbb{Q}$ so that $\alpha \notin \mathbb{Z}$. Thus $\exp(A) \notin S$, so $A \notin \log(S)$, a contradiction. This shows $d \geq n$, so that $d = n$ as claimed.

<u>Corollary 1</u>

There is no bound to the defect of a nilpotent torsion-free group in its completion.

Proof:

$T_n(\mathbb{Q})$ is easily seen to be the completion of $T_n(\mathbb{Z})$.

Corollary 2

A locally nilpotent torsion-free group need not be subnormal in its completion.

Proof:

Let $$V = \operatorname*{Dr}_{n=1}^{\infty} T_n(\mathbb{Z}).$$

Then $$\bar{V} = \operatorname*{Dr}_{n=1}^{\infty} T_n(\mathbb{Q}).$$

If V were subnormal in $\bar{V}$ then $V \triangleleft^m \bar{V}$ for some $m \in \mathbb{Z}$, so that $T_{m+1}(\mathbb{Z}) \triangleleft^m T_{m+1}(\mathbb{Q})$ contrary to lemma 3.1.1.

3.2 Analogue of a theorem of P.Hall

We prove the theorem we want directly for Lie algebras, using methods based on those of Roseblade. Throughout the chapter all Lie algebras will be over a fixed but arbitrary field k (of arbitrary characteristic). We introduce 3 new classes of Lie algebras:

$$L \in \mathfrak{D} \iff (H \leq L \Rightarrow H \text{ si } L)$$
$$L \in \mathfrak{D}_n \iff (H \leq L \Rightarrow H \triangleleft^n L)$$
$$L \in \mathfrak{I} \iff (H < L \Rightarrow I_L(H) > H).$$

(The last condition is known as the idealiser condition).

Throughout this chapter $\mu_i(m,n,\ldots)$ will denote a positive-integer valued function depending only on those arguments explicitly shown.

Our first aim is to show that if $H \triangleleft L$, $H \in \mathfrak{N}_c$, and $L/H^2 \in \mathfrak{N}_d$, then $L \in \mathfrak{N}_{\mu_1(c,d)}$ for some function μ_1. For the purposes of this chapter it is immaterial what the exact form of μ_1 is; but it is of independent interest to obtain a good bound. The group-theoretic version, with $\mu_1(c,d) = \binom{c+1}{2}d - \binom{c}{2}$, is due to P.Hall [12]; the result for Lie algebras with this bound is proved by Chong-Yun Chao [5] (stated only for finite-dimensional algebras). In [40] A.G.R.Stewart improves Hall's bound in the group-theoretic case to $cd+(c-1)(d-1)$ and shows this is best possible. We add a fourth voice to the canon by showing that similar results hold for Lie algebras (using essentially the same arguments). A few preliminary lemmas are needed to set up the machinery.

Lemma 3.2.1

If L is a Lie algebra and A, B, C $\leq$ L then

$$[[A,B],C] \leq [[B,C],A] + [[C,A],B].$$

Proof:

From the Jacobi identity.

Lemma 3.2.2

If L is a Lie algebra and A, B, C $\leq$ L then

$$[[A,B],{}_nC] \leq \sum_{\substack{i+j=n \\ i,j\geq 0}} [[A,{}_iC],[B,{}_jC]].$$

Proof:

Use induction on n. If n = 1 lemma 3.2.1 gives the result. Suppose the lemma holds for n. Then

$$[[A,B],_{n+1}C] = [[[A,B],_n C],C]$$

$$\leq \sum_{i+j=n} [[[A,_i C],[B,_j C]],C] \text{ by hypothesis}$$

$$\leq \sum_{i+j=n} [[A,_{i+1}C],[B,_j C]] + [[A,_i C],[B,_{j+1}C]]$$

by lemma 3.2.1

$$= \sum_{i+j=n+1} [[A,_i C],[B,_j C]]$$

and the induction step goes through.

<u>Theorem 3.2.3</u>

Let L be a Lie algebra, $H \triangleleft L$, such that $H \in \mathfrak{N}_c$ and $L/H^2 \in \mathfrak{N}_d$. Then $L \in \mathfrak{N}_{\mu_1(c,d)}$ where

$$\mu_1(c,d) = cd + (c-1)(d-1).$$

Further, this bound is best possible.

Proof:

Induction on c. If c = 1 the result is obvious. If c > 1, then for any r with $1 \leq r \leq c$ we have $M_r = H/H^{r+1} \triangleleft N_r = L/H^{r+1}$. $M_r \in \mathfrak{N}_r$ and $N_r/M_r^{\,2} \in \mathfrak{N}_d$ so inductively we may assume

$$L^{2rd-r-d+2} \leq H^{r+1} \qquad 1 \leq r \leq c-1.$$

Now $$L^{2rd-c-d+2} \leq [H^2,_{2cd-2d-c+1}L]$$

$$\leq \sum_i [[H,_i L],[H,_{2cd-2d-c+1-i}L]]$$

summed over the interval $0 \le i \le 2cd-2d-c+1$ (by lemma 3.2.2). Each such i belongs to an interval

$$2(j-1)d-d-(j-1)+1 \le i < 2jd-d-j+1 \qquad (1 \le j \le c).$$

Consider an arbitrary j. By induction if $j \neq 1$, and since $H \lhd L$ if $j = 1$, we have

$$[[H,{}_iL],[H,{}_{2cd-2d-c+1-i}L]]$$
$$\le [H^j, L^{2d(c-j)-d-(c-j)+2+2dj-d-j-i} \cap H]$$

(also using the fact that $[H,{}_tL] \le L^{t+1}$)

$$\le [H^j, L^{2d(c-j)-d-(c-j)+2} \cap H] \quad \text{since } 2dj-d-j \ge i$$
$$\le [H^j, L^{c-j+1} \cap H] \quad \text{by induction if } c-j \neq 0, \text{ and}$$

obviously if $c-j = 0$

$$\le H^{c+1}$$
$$= 0.$$

Thus $L^{2cd-c-d+2} = 0$ and the induction hypothesis carries over. The result follows.

Next we show that this value of μ_1 is best possible, in the sense that for all $c,d > 0$ there exist Lie algebras L, H satisfying the hypotheses of the theorem, such that L is nilpotent of class <u>precisely</u> $cd + (c-1)(d-1)$.

Now in [40] A.G.R.Stewart constructs a nilpotent <u>torsion-free</u> group G having a normal subgroup N with N nilpotent of class c, G/N' nilpotent of class d, and G nilpotent of class precisely $cd + (c-1)(d-1)$. Let $\bar{G}$ be the completion of G, $\bar{N}$ the completion of N. Put $L = \mathcal{L}(\bar{G})$,

$H = \mathcal{L}(\bar{N})$. Using the results of chapter 2 it is easily seen that these have the required properties.

3.3 The class $\mathfrak{X}_n$

Write $L \in \mathfrak{X}_n \iff \langle H^L \rangle^n \leq H$ for all $H \leq L$.

Lemma 3.3.1

$\mathfrak{X}_n = QS\mathfrak{X}_n$.

Proof:

Trivial.

Lemma 3.3.2

$\mathfrak{D}_n \cap \mathfrak{A}^2 \leq \mathfrak{X}_n$.

Proof:

Let $H \leq L \in \mathfrak{D}_n \cap \mathfrak{A}^2$, so that $L^{(2)} = 0$. We show by induction on m that

$$\langle H^L \rangle^m \leq H^m + \sum_{i=1}^{\infty} [[H,{}_iL],{}_{m-1}H].$$

m=1:

$$\langle H^L \rangle = H + \sum_{i=1}^{\infty} [H,{}_iL] \quad \text{obviously.} \qquad (*)$$

m=2:

$$\langle H^L \rangle^2 = [H+\Sigma[H,{}_iL],\ H+\Sigma[H,{}_jL]] \quad \text{from } (*)$$
$$\leq [H,H] + \Sigma\ [[H,{}_iL],H]$$

since L^2 .

m>2:

$$\langle H^L \rangle^m \leq [H^{m-1}+\Sigma[[H,{}_iL],{}_{m-2}H],\ H+\Sigma[H,{}_jL]]$$
$$\leq H^m + \Sigma\ [[H,{}_iL],{}_{m-1}H]$$

since $L^2 \in \mathfrak{A}$.

Now if $L \in \mathfrak{D}_n$ it is clear that $[L,_nH] \leq H$, and consequently $\langle H^L \rangle^n \trianglelefteq H^n + H = H$, which shows that $L \in \mathfrak{X}_n$ as claimed.

Lemma 3.3.3

If K is a minimal ideal of $L \in L\mathfrak{N}$ then $K \leq \zeta_1(L)$.

Proof:

See Hartley [14] lemma 10 p.269.

Lemma 3.3.4

If $K \triangleleft L \in L\mathfrak{N}$ and $K \in \mathfrak{F}_h$, then $K \leq \zeta_h(L)$.

Proof:

Induction on h. If $h = 0$ the result is clear. Let $0 = K_0 < K_1 < \ldots < K_\alpha = K$ be a series of ideals $K_i \triangleleft L$ $(i = 0,\ldots,\alpha)$ such that the series cannot be refined (this exists since K is finite-dimensional). Then K_{i+1}/K_i is a minimal ideal of L/K_i. By our induction hypothesis $K_{\alpha-1} \leq \zeta_{h-1}(L)$, and $K_\alpha + \zeta_{h-1}(L)/\zeta_{h-1}(L)$ is a minimal ideal of $L/\zeta_{h-1}(L)$, so by lemma 3.3.3 it is contained in $\zeta_1(L/\zeta_{h-1}(L))$ which implies $K \leq \zeta_h(L)$. The result follows.

Lemma 3.3.5

If $H \leq L \in \mathfrak{N}_r \cap \mathfrak{G}_s$ then $H \in \mathfrak{F}_{\mu_2(r,s)}$ where

$$\mu_2(r,s) = s + s^2 + \ldots + s^r.$$

Proof:

It is sufficient to show $L \in \mathfrak{F}_{\mu_2}(r,s)$. Now L is spanned (qua vector space) by commutators of the form $[g_1,\dots,g_i]$ $(i \leq r)$ where the g_j are chosen from the given set of s generators. This gives the result.

Next we need an unpublished theorem of B.Hartley:

Theorem 3.3.6 (Hartley)

$\mathfrak{J} \leq L\mathfrak{N}$.

Proof:

Let $L \in \mathfrak{J}$, and let M be maximal with respect to $M \leq L$, $M \in L\mathfrak{N}$ (such an M exists by a Zorn's lemma argument). Let $u \in I = I_L(M)$. Then $K = M + \langle u\rangle \leq L$. $L \in \mathfrak{J}$ so $K \in \mathfrak{J}$, from which it is easy to deduce that K has an ascending series $(U_\alpha)_{\alpha \leq \sigma}$ with $U_1 = \langle u\rangle$. Then

$$\begin{aligned} U_\alpha &= (M \cap U_\alpha) + (\langle u\rangle \cap U_\alpha) \\ &= (M \cap U_\alpha) + \langle u\rangle, \end{aligned}$$

so

$$U_{\alpha+1} = (M \cap U_{\alpha+1}) + U_\alpha . \qquad (*)$$

We show by transfinite induction on α that $U_\alpha \in L\mathfrak{N}$. $U_1 = \langle u\rangle \in \mathfrak{A} \leq L\mathfrak{N}$. $M \cap U_{\alpha+1} \triangleleft U_{\alpha+1}$ (since $M \triangleleft K$) and $M \cap U_{\alpha+1} \in L\mathfrak{N}$; also $U_\alpha \triangleleft U_{\alpha+1}$ and $U_\alpha \in L\mathfrak{N}$. By Hartley [14] lemma 7 p.265 and (*) $U_{\alpha+1} \in L\mathfrak{N}$. At limit ordinals the induction step is clear. Hence

$U_\sigma = K \in L\mathfrak{N}$. By maximality of M we have K = M, so I = M. But $L \in \mathfrak{J}$ so M = L. Therefore $L \in L\mathfrak{N}$ which finishes the proof.

Lemma 3.3.7

$$\mathfrak{D}_n \leq \mathfrak{D} \leq L\mathfrak{N}.$$

Proof:

Clearly $\mathfrak{D}_n \leq \mathfrak{D} \leq \mathfrak{J}$. Now use theorem 3.3.6.

Lemma 3.3.8

If $x \in L \in \mathfrak{X}_n$, then $\langle x^L\rangle \in \mathfrak{N}_n$.

Proof:

$\langle x^L\rangle^n \leq \langle x\rangle$ since $L \in \mathfrak{X}_n$. If $\langle x^L\rangle^n = 0$ we are home. If not, then $\langle x\rangle = \langle x^L\rangle^n$ ch $\langle x^L\rangle \triangleleft L$, so $\langle x\rangle \triangleleft L$. Thus $x \in C_L(x) \triangleleft L$, so $\langle x^L\rangle \leq C_L(x)$ and $\langle x^L\rangle^{n+1} = 0$ as claimed.

Lemma 3.3.9

$$\mathfrak{A}^2 \cap \mathfrak{D}_n \leq \mathfrak{N}_{\mu_3(n)}.$$

Proof:

Let $L \in \mathfrak{A}^2 \cap \mathfrak{D}_n$. $L^n = \langle [x_1,\ldots,x_n]^L : x_i \in L\rangle$. Let $X = \langle x_1,\ldots,x_n\rangle$. By lemma 3.3.2 $L \in \mathfrak{X}_n$, so if $x \in L$, then $\langle x^L\rangle \in \mathfrak{N}_n$ by lemma 3.3.8. Let $T = \langle X^L\rangle = \langle x_1{}^L\rangle + \ldots + \langle x_n{}^L\rangle$, a sum of n $\mathfrak{N}_n$-ideals of L. By Hartley [14] lemma 1 (iii) p.261 $T \in \mathfrak{N}_{n^2}$. Thus $X \in \mathfrak{N}_{n^2} \cap \mathfrak{G}_n$, so by lemma 3.3.5 every subalgebra of X has dimension $\leq r = \mu_2(n^2,n)$. $L \in \mathfrak{X}_n$ so $T^n \leq X$.

$Y = \langle [x_1,\ldots,x_n]^L \rangle \leq T^n \leq X$ so $\dim(Y) \leq r$. By lemma 3.3.7 $\mathfrak{D}_n \leq L\mathfrak{N}$, and $Y \triangleleft L$; consequently lemma 3.3.4 applies and $Y \leq \zeta_r(L)$. Thus $L^n \leq \zeta_r(L)$, and $L \in \mathfrak{N}_{n+r}$.

We may therefore take $\mu_3(n) = n + \mu_2(n^2,n)$.

Lemma 3.3.10

$$\mathfrak{A}^d \cap \mathfrak{D}_n \leq \mathfrak{N}_{\mu_4(n,d)}.$$

Proof:

Induction on d. If $d = 1$ we may take $\mu_4(n,1) = 1$. If $d = 2$, then by lemma 3.3.9 we may take $\mu_4(n,2) = \mu_3(n)$. If $d > 2$, let $M = L^{(d-2)}$. Then $M \in \mathfrak{A}^2 \cap \mathfrak{D}_n \leq \mathfrak{N}_{\mu_3(n)}$ by lemma 3.3.9, and $L/M^2 \in \mathfrak{A}^{d-1} \cap \mathfrak{D}_n \leq \mathfrak{N}_{\mu_4(n,d-1)}$ by induction. By theorem 3.2.3

$$L \in \mathfrak{N}_{\mu_4(n,d)}$$

where

$$\mu_4(n,d) = ab + (a-1)(b-1),$$
$$a = \mu_3(n), \quad b = \mu_4(n,d-1).$$

Lemma 3.3.11

If $0 \neq A \triangleleft L \in \mathfrak{N}$ then $A \cap \zeta_1(L) \neq 0$.

Proof:

See Schenkman [35] lemma 8.

Define $\mathcal{L}(L) = \{x \in L : \langle x^L \rangle \in \mathfrak{A}\}$

$$\mathcal{L}_n(L) = \{x \in L : \langle x^L \rangle \in \mathfrak{A} \cap \mathfrak{F}_n\}.$$

Lemma 3.3.12

If $L = \langle \mathcal{L}_n(L) \rangle$ then $L \in \mathfrak{N}_n$.

Proof:

L is generated by abelian ideals, so by lemma 1 (iii) of Hartley [14] p.261 $L \in L\mathfrak{N}$. Let the abelian ideals which generate L and are of dimension $\leq n$ be $\{A_\lambda : \lambda \in \Lambda\}$. By lemma 3.3.4 $A_\lambda \leq \zeta_n(L)$ so $L = \zeta_n(L)$ as required.

Lemma 3.3.13

If $H = \langle \mathcal{L}(H) \rangle$ and $H \in \mathfrak{X}_n$ then $H \in \mathfrak{A}^{\mu_5(n)}$.

Proof:

It is easily seen that $H^n = \langle [x_1, \ldots, x_n]^H : x_i \in \mathcal{L}(H) \rangle$. Let $X = \langle x_1, \ldots, x_n \rangle$. $\langle X^H \rangle = T = \langle x_1{}^H \rangle + \ldots + \langle x_n{}^H \rangle \in \mathfrak{N}_n$ by Hartley [14] lemma 1 (iii) p.261. Since $H \in \mathfrak{X}_n$ $T^n \leq X \in \mathfrak{G}_n \cap \mathfrak{N}_n$. Therefore if $Y = \langle [x_1, \ldots, x_n]^H \rangle$ then $Y \leq T^n \leq X$ so by lemma 3.3.5 $Y \in \mathfrak{F}_{\mu_2(n,n)}$. $Y \leq \langle x_1{}^H \rangle \in \mathfrak{A}$ so $Y \in \mathfrak{A} \cap \mathfrak{F}_{\mu_2(n,n)}$. Therefore $H^n \leq \langle \mathcal{L}_{\mu_2(n,n)}(H) \rangle = D$, say, and $D = \langle \mathcal{L}_{\mu_2(n,n)}(D) \rangle$. Thus $H/D \in \mathfrak{A}^{n-1}$, and by lemma 3.3.12 $D \in \mathfrak{N}_{\mu_2(n,n)} \leq \mathfrak{A}^{\mu_2(n,n)}$. Therefore $H \in \mathfrak{A}^{\mu_5(n)}$ where

$$\mu_5(n) = n - 1 + \mu_2(n,n).$$

Lemma 3.3.14

$\mathfrak{X}_n \leq \mathfrak{A}^{\mu_6(n)}$.

Proof:

Let $H \leq L \in \mathfrak{X}_n$. Then $H \geq \langle H^L \rangle^n \triangleleft L$. $\langle H^L \rangle / \langle H^L \rangle^n \in \mathfrak{N}_{n-1}$, so by Hartley [14] lemma 1 (ii) p.261 $H/\langle H^L \rangle^n \triangleleft^{n-1} \langle H^L \rangle / \langle H^L \rangle^n$, so $H \triangleleft^{n-1} \langle H^L \rangle \triangleleft L$. Thus $H \triangleleft^n L$ and $L \in \mathfrak{D}_n$. Hence $\mathfrak{X}_n \leq \mathfrak{D}_n \leq L\mathfrak{N}$ by lemma 3.3.7.

By lemma 3.3.8 $x \in L \Rightarrow \langle x^L \rangle \in \mathfrak{N}_n$. So if we define

$$L_1 = \Sigma \{A: A \triangleleft L, A \in \mathfrak{A}\}$$

then $L_1 > 0$ (since e.g. $0 \neq \zeta_1(\langle x^L \rangle) \leq L_1$). Similarly let

$$L_{i+1}/L_i = \Sigma \{A: A \triangleleft L/L_i, A \in \mathfrak{A}\}.$$

Then

$$0 < L_1 \leq L_2 \leq \dots \quad .$$

Let $y \in L$. Then $Y = \langle y^L \rangle \triangleleft L$ and $Y \in \mathfrak{N}_n$. An easy induction shows $\zeta_i(Y) \leq L_i$ so $y \in L_n$. Therefore $L_n = L$. By lemma 3.3.1 $L_{i+1}/L_i \in \mathfrak{X}_n$, and clearly we have $L_{i+1}/L_i = \langle \mathcal{L}(L_{i+1}/L_i) \rangle$, so by lemma 3.3.13

$$L_{i+1}/L_i \in \mathfrak{A}^{\mu_5(n)}.$$

Thus $L \in \mathfrak{A}^{\mu_6(n)}$ where $\mu_6(n) = n\mu_5(n)$.

We have now set up most of the machinery needed to prove the main result by induction; this is done in the next section.

3.4 The Induction Step

Lemma 3.4.1

$$\mathfrak{D}_n = QS\,\mathfrak{D}_n.$$

Proof:

Trivial.

Lemma 3.4.2

$$\mathfrak{D}_1 = \mathfrak{N}_1 = \mathfrak{A}.$$

Proof:

Let $x, y \in L \in \mathfrak{D}_1$. Then $\langle x\rangle, \langle y\rangle \triangleleft L$. If x and y are linearly independent then $[x,y] \in \langle x\rangle \cap \langle y\rangle = 0$. If x and y are linearly dependent then $[x,y] = 0$ anyway. Thus $L \in \mathfrak{A} = \mathfrak{N}_1$.

We now define the ideal closure series of a subalgebra of a Lie algebra. Let L be a Lie algebra, $K \leq L$. Define $K_0 = L$, $K_{i+1} = \langle K^{K_i}\rangle$. The series

$$K_0 \geq K_1 \geq \dots \geq K_n \geq \dots$$

is the ideal closure series of K in L.

Lemma 3.4.3

1) If $K = L_n \triangleleft L_{n-1} \triangleleft \dots \triangleleft L_0 = L$ then $L_i \geq K_i$ for $i = 0,\dots,n$.

2) $K \triangleleft^n L$ if and only if $K_n = K$.

Proof:

1) By induction. For $i = 0$ we have equality. Now

$K_{i+1} = \langle K^{K_i} \rangle \leq \langle K^{L_i} \rangle \leq L_{i+1}$ so the induction step goes through.

2) Clearly $K_{i+1} \triangleleft K_i$, so that if $K_n = K$ then

$$K = K_n \triangleleft K_{n-1} \triangleleft \ldots \triangleleft K_o = L.$$

On the other hand, if $K \triangleleft^n L$ then

$$K = L_n \triangleleft L_{n-1} \triangleleft \ldots \triangleleft L_o = L,$$

and by part (1) $K \leq K_n \leq L_n = K$.

<u>Lemma 3.4.4</u>

Let $H \leq L \in \mathfrak{D}_n$, H_i the i-th term of the ideal closure series of H in L. Then $H_i/H_{i+1} \in \mathfrak{D}_{n-i}$.

Proof:

$$H = H_n \triangleleft H_{n-1} \triangleleft \ldots \triangleleft H_{i+1} \triangleleft H_i \triangleleft \ldots \triangleleft H_o = L.$$

Suppose $H_{i+1} \leq K \leq H_i$. If $j \leq i$ then $K_j \leq H_j$ by lemma 3.4.3.1, so $K_i \leq H_i$. But $H \leq H_{i+1} \leq K$ so an easy induction on j shows that $H_j \leq K_j$. Thus $H_i = K_i$. But $L \in \mathfrak{D}_n$ so $K \triangleleft^n L$, and K has ideal closure series

$$K = K_n \triangleleft K_{n-1} \triangleleft \ldots \triangleleft K_i \triangleleft \ldots \triangleleft K_o = L.$$

Therefore

$$K = K_n \triangleleft K_{n-1} \triangleleft \ldots \triangleleft K_i = H_i, \text{ and } K \triangleleft^{n-i} H_i.$$

Thus $K/H_{i+1} \triangleleft^{n-i} H_i/H_{i+1}$ and the lemma is proved.

It is this result that provides the basis for an induction proof of our main result in this chapter, which follows:

Theorem 3.4.5

$$\mathfrak{D}_n \leq \mathfrak{N}_{\mu(n)}.$$

Proof:

As promised, by induction on n.

If n = 1 then by lemma 3.4.2 we may take $\mu(1) = 1$. If $n > 1$ let $L \in \mathfrak{D}_n$, $H \leq L$. By lemma 3.4.4, if $i \geq 1$ $H_i/H_{i+1} \in \mathfrak{D}_{n-i} \leq \mathfrak{D}_{n-1} \leq \mathfrak{N}_{\mu(n-1)}$ by inductive hypothesis. Let $m = \mu(n-1)$. Then certainly $H_i/H_{i+1} \in \mathfrak{A}^m$, and so $H_1^{(m(n-1))} \leq H$ for all $H \leq L$. Let $Q = H_1/H_1^{(m(n-1))} \in \mathfrak{D}_n \cap \mathfrak{A}^{m(n-1)}$. By lemma 3.3.10 $Q \in \mathfrak{N}_c$, where $c = \mu_4(n, m(n-1))$. Thus $Q^{c+1} = 0$ so $H_1^{c+1} \leq H_1^{(m(n-1))} \leq H$, so that $L \in \mathfrak{X}_{c+1}$. By lemma 3.3.14 $L \in \mathfrak{A}^d$ where $d = \mu_6(c+1)$. Finally therefore $L \in \mathfrak{A}^d \cap \mathfrak{D}_n \leq \mathfrak{N}_{\mu(n)}$ by lemma 3.3.10, where

$$\mu(n) = \mu_4(n, \mu_6(1 + \mu_4(n, (n-1).\mu(n-1)))).$$

The theorem is proved.

Remark

The value of $\mu(n)$ so obtained becomes astronomical even for small n, and is by no means best possible. However, without modifying the argument it is hard to improve it significantly.

Using the Mal'cev correspondence we can prove

<u>Theorem 3.4.6</u>

Let G be a complete torsion-free R-group (in the sense of lemma 2.1.2) such that if H is a complete subgroup of G then $H \triangleleft^n G$. Then G is nilpotent of class $\leq \mu(n)$.

Proof:

Let $x \in G$, $X = \{x^\lambda : \lambda \in \mathbb{Q}\}$. Since G is a complete R-group $X \cong \mathbb{Q}$ (under addition) so X is abelian and complete. Therefore $\langle x \rangle \triangleleft X \triangleleft^n G$, so $\langle x \rangle$ is subnormal in G and G is a Baer group (see chapter 7 - Baer calls them <u>nilgroups</u>) so is locally nilpotent (Baer [1] §3 Zusatz 2). G is also complete and torsion-free so we may form the Lie algebra $\mathcal{L}(G)$ over $\mathbb{Q}$. If $K \leq \mathcal{L}(G)$ then $\mathcal{G}(K)$ is a complete subgroup of G (theorem 2.4.2) so $\mathcal{G}(K) \triangleleft^n G$. By lemma 2.4.5 $K \triangleleft^n \mathcal{L}(G)$. By theorem 3.4.5 $\mathcal{L}(G) \in \mathfrak{D}_n \leq \mathfrak{N}_{\mu(n)}$. By theorem 2.5.4 G is nilpotent of class $\leq \mu(n)$.

We may also recover Roseblade's original result for the case of torsion-free groups. Suppose G is a torsion-free group, every subgroup of which is subnormal of defect $\leq n$. Then G is a Baer group so is locally nilpotent. Let $\overline{G}$ be the completion of G (Note: we must again avoid Mal'cev and appeal either to Kargapolov or

Hall in order to maintain algebraic purity). Then every complete subgroup of $\bar{G}$ is the completion of its intersection with G (Kuroš [23] p.257) which is $\triangleleft^n$ G. By lemma 2.4.4 we deduce that every complete subgroup of $\bar{G}$ is $\triangleleft^n \bar{G}$. $\bar{G}$ is a complete R-group, so theorem 3.4.6 applies.

We have not been able to decide whether or not $\mathcal{D} = \mathcal{N}$. The corresponding result for groups is now known to be false (Heineken and Mohamed [15]) but their counterexample is a p-group; so we cannot use the Mal'cev correspondence to produce a counterexample for the Lie algebra case.

Chapter Four

The Minimal Condition for Subideals

"From Nature's chain whatever link you strike,
Tenth or ten thousandth, breaks the chain alike."

Alexander Pope

In [31] D.J.S.Robinson proves a theorem implying that any group G satisfying the minimal condition for subnormal subgroups of defect ≤ 2 must also satisfy the minimal condition for all subnormal subgroups; further any such group is a finite extension of a $\mathfrak{T}$-group (i.e. a group in which all subnormal subgroups are normal).

In this chapter we prove two Lie-theoretic analogues of these results. We construct non-trivial examples of Lie algebras satisfying the minimal condition for subideals. In particular we show that the Lie algebra of all endomorphisms of a vector space is such an algebra. As a by-product we show that any Lie algebra can be embedded in a simple Lie algebra. However, in contrast to the situation for groups, not every Lie algebra can be embedded as a subideal of a perfect Lie algebra.

4.1 The Minimal Condition for 2-step Subideals

A Lie algebra L satisfies the minimal condition for subideals if every non-empty collection of subideals of L has a least element under inclusion; equivalently if L has no infinite properly descending chain

$$H_1 > H_2 > H_3 > \dots$$

of subideals.

We denote by Min-si both this condition and the class of Lie algebras which satisfy it. The minimal condition for n-step subideals is defined in a similar manner; both this condition and the class of Lie algebras satisfying it will be denoted by Min-$\triangleleft^n$. (We write Min-$\triangleleft$ for Min-$\triangleleft^1$).

Note first that Min-$\triangleleft$ does not imply Min-si. In [14] p.269 §7 B.Hartley constructs a Lie algebra L with the following properties:

L is a split extension (Jacobson [17] p.18) $P \oplus Q$ where P is infinite-dimensional abelian, Q is 3-dimensional nilpotent, and P is a minimal ideal of L. It follows that any ideal of L is either of dimension ≤ 3 or of codimension ≤ 3. Thus $L \in$ Min-$\triangleleft$. But P, being infinite-dimensional abelian, has an infinite properly descending chain of ideals, and these are 2-step subideals of L. So $L \notin$ Min-si.

Lemma 4.1.1

1) Min-si is $\{Q,E,I\}$-closed.

2) Min-$\triangleleft^n$ is $\{Q,E\}$-closed.

3) If $K \triangleleft^m L \in$ Min-$\triangleleft^n$ and $m \leq n$ then $K \in$ Min-$\triangleleft^{n-m}$.

Proof:

1) $\{Q,I\}$-closure is clear. Suppose now that $K \triangleleft L$, such that $K, L/K \in$ Min-si. Let

$$I_1 \geq I_2 \geq I_3 \geq \dots$$

be a descending chain of subideals of L. Then

$$I_1 \cap K \geq I_2 \cap K \geq I_3 \cap K \geq \dots$$

is a descending chain of subideals of $K \in$ Min-si, so for some integer N $(I_n \cap K) = (I_N \cap K)$ for all $n \geq N$.

$$(I_1+K)/K \geq (I_2+K)/K \geq (I_3+K)/K \geq \dots$$

is a descending chain of subideals of $L/K \in$ Min-si, so for some integer M $(I_m+K)/K = (I_M+K)/K$ for all $m \geq M$. If $r \geq R = \max(M,N)$ we have $I_r \cap K = I_R \cap K$, $I_r+K = I_R+K$, $I_r \leq I_R$. Thus (using the modular law) $I_r = I_r \cap (K+I_R)$ $= (I_r \cap K)+I_R = I_R$, so the chain breaks off and $L \in$ Min-si.

2) Q-closure is clear, E-closure follows as for Min-si.

3) If $H \triangleleft^{n-m} K$ then $H \triangleleft^n L$. Result follows.

A result we shall make extensive use of, which is peculiar to the Lie-theoretic case, is proved in Schenkman [35,36]; it is also given as an exercise in Jacobson [17] p.29 ex.9:

Lemma 4.1.2

If L is a Lie algebra and A si L then

$$A^{\omega} = \bigcap_{i=1}^{\infty} A^{i}$$

is an ideal of L.

The other basic result we need is due to Hartley ([14] cor. to theorem 3 p.259):

Lemma 4.1.3

Let L be a Lie algebra over a field of characteristic zero. Then L possesses a unique maximal locally nilpotent ideal $\rho(L)$; the join $\beta(L)$ of all nilpotent subideals of L is an ideal of L, contained in $\rho(L)$.

$\rho(L)$ is the Hirsch-Plotkin radical of L, $\beta(L)$ the Baer radical.

Let $\mathfrak{Z}$ denote the class of Lie algebras L such that $L = \zeta_{\alpha}(L)$ for some ordinal α. (These are the Lie-theoretic analogues of the ZA-groups of Kuroš [23] p.218). It is easy to see that $\mathfrak{Z}$ is S-closed.

Lemma 4.1.4

Let $L \in \mathfrak{Z}$. Then $L^{(\alpha)} = 0$ for some ordinal α.

Proof:

First we require a variant of Grün's lemma (see Kuroš [23] p.227). Let K be any Lie algebra such that

$\zeta_2(K) > \zeta_1(K)$. We show that $K^{(1)} < K$. For let $a \in \zeta_2(K) \setminus \zeta_1(K)$, and consider the map $\phi: K \to \zeta_1(K)$ defined by $x\phi = [x,a]$ $(x \in K)$. ϕ turns out to be a homomorphism, and since $a \notin \zeta_1(K)$ $x\phi \neq 0$ for some $x \in K$. Hence K has a non-zero abelian homomorphic image and $K^{(1)} < K$.

Now let $L \in \mathfrak{Z}$, and put $P = \bigcap_{\beta > 0} L^{(\beta)}$. Then $P = L^{(\alpha)}$ for some ordinal α. Since $P \leq L$ it follows that $P \in \mathfrak{Z}$. Thus either $P = 0$, $P = \zeta_1(P)$, or $\zeta_2(P) > \zeta_1(P)$. The second and third cases imply that $P^{(1)} < P$ (directly for the second, and by the variant of Grün's lemma for the third) whence $L^{(\alpha+1)} < L^{(\alpha)}$ contradicting the definition of P. Thus $P = 0$ as claimed.

<u>Lemma 4.1.5</u>

$L\mathfrak{N} \cap \text{Min-}\triangleleft \leq E\mathfrak{A}$.

Proof:

Let $L \in L\mathfrak{N} \cap \text{Min-}\triangleleft$, $U = \bigcup_{\beta \geq 0} \zeta_\beta(L)$. Then $U = \zeta_\alpha(L)$ for some ordinal α. Suppose if possible that $U \neq L$. Then $L/U \neq 0$, and $L/U \in L\mathfrak{N} \cap \text{Min-}\triangleleft$ (by lemma 4.1.1.2). Let M/U be a minimal ideal of L/U. By lemma 3.3.3 $M/U \leq \zeta_1(L/U)$. But this means that $\zeta_{\alpha+1}(L) > \zeta_\alpha(L)$ contrary to the definition of U. Thus $U = L$ so $L \in \mathfrak{Z}$.

By lemma 4.1.4 $L^{(\alpha)} = 0$ for some ordinal α. Now each term $L^{(\beta)}$ of the derived series of L is an ideal of L, and $L^{(\beta+1)} \leq L^{(\beta)}$. $L \in \text{Min-}\triangleleft$ so $L^{(\beta+1)} = L^{(\beta)}$ for some <u>finite</u> β. Then $L^{(\beta)} = L^{(\alpha)} = 0$ so $L \in E\mathfrak{A}$.

<u>Lemma 4.1.6</u>

If $L \in \text{Min-}\triangleleft^2$ then $\rho(L) \in \mathfrak{F} \cap \mathfrak{N}$.

Proof:

$R = \rho(L) \in L\mathfrak{N}$, and satisfies Min-$\triangleleft$ by lemma 4.1.1.3. By lemma 4.1.5 $R \in E\mathfrak{A}$. $R^{(n)}$ ch $R \triangleleft L$ so $R^{(n)} \triangleleft L$. By lemma 4.1.1.3 $R^{(n)} \in \text{Min-}\triangleleft$, so that $R^{(n)}/R^{(n+1)} \in \text{Min-}\triangleleft \cap \mathfrak{A}$. Now an ideal of an abelian Lie algebra is precisely a vector subspace, so $R^{(n)}/R^{(n+1)} \in \mathfrak{F}$. Thus $R \in E\mathfrak{F} = \mathfrak{F}$. Since we know $R \in L\mathfrak{N}$ this implies $R \in \mathfrak{N}$.

We now have the machinery to prove the main theorem of this section:

<u>Theorem 4.1.7</u>

If L is a Lie algebra over a field of characteristic zero, satisfying Min-$\triangleleft^2$, then L satisfies Min-si.

Proof:

Assume the contrary. Then there exists M minimal with respect to $M \triangleleft L$ and $M \notin$ Min-si. Let N be any proper ideal of M. For any integer $i > 0$ we have N^i ch $N \triangleleft M \triangleleft L$ so $N^i \triangleleft^2 L$. Since $L \in \text{Min-}\triangleleft^2$ it follows that $N^{\omega} = \bigcap_{i=1}^{\infty} N^i = N^c$ for some integer $c > 0$. By lemma 4.1.2 $N^c \triangleleft L$. Now N/N^c si L/N^c, and $N/N^c \in \mathfrak{N}$, so by lemma 4.1.3 $N/N^c \leq \beta(L/N^c) \leq \rho(L/N^c)$. By lemma 4.1.6 $\rho(L/N^c) \in \mathfrak{F}$, so $N/N^c \in \mathfrak{F}$. But $N^c < M$, $N^c \triangleleft L$, so by minimality of M $N^c \in$ Min-si. Thus $N \in$ (Min-si)$\mathfrak{F} \leq$ (Min-si)^2 = Min-si by lemma 4.1.1.1.

Thus any proper ideal of M satisfies Min-si.

If $I_1 > I_2 > \ldots$ is a properly descending chain of subideals of M, then $I_2 \leq I \triangleleft M$ for some $I \neq M$. Thus by the above $I \in$ Min-si. But $I_2 > I_3 > \ldots$ is an infinite properly descending chain of subideals of I, which is a contradiction.

Thus $L \in$ Min-si and the theorem is proved.

For the case where the field has characteristic $p \neq 0$, $\beta(L)$ is not well-behaved (see Hartley [14] §7.2 or Jacobson [17] p.75) and the best we have been able to prove is

<u>Proposition 4.1.8</u>

If L is a Lie algebra over a field of arbitrary characteristic, satisfying Min-$\triangleleft^3$, then L satisfies Min-si.

Proof:

Imitate theorem 4.1.7, except that we now show directly that $N/N^c \in \mathfrak{F}$ as follows:

N^i ch $N \lhd M \lhd L$ so $N^i \lhd^2 L$. By lemma 4.1.1.3 $N^i \in$ Min-$\lhd$. Thus $N^i/N^{i+1} \in$ Min-$\lhd \cap \mathfrak{A} \leq \mathfrak{F}$, so $N/N^c \in E\mathfrak{F} = \mathfrak{F}$.

4.2 The Minimal Condition for Subideals

We now investigate in more detail the structure of Lie algebras (over fields of characteristic zero) which satisfy Min-si (equivalently, by theorem 4.1.7, Min-$\lhd^2$). First an elementary property of centralisers:

Suppose L is a Lie algebra (any field) and $I \lhd L$. It is easy to see that $C_L(I) \lhd L$. For any $x \in L$ the map $\phi_x: I \to I$ defined by

$$i.\phi_x = [i,x] \qquad (i \in I)$$

is a derivation of I. (Note: $\phi_x = ad(x)|_I$). The map

$$\phi: L \to der(I)$$

sending $x \in L$ to ϕ_x is a Lie homomorphism, with kernel $C_L(I)$. Hence $L/C_L(I) \cong D \leq der(I)$. In particular

Lemma 4.2.1

If $I \lhd L$ and $I \in \mathfrak{F}$ then $L/C_L(I) \in \mathfrak{F}$.

Proof:

$der(I) \in \mathfrak{F}$.

Let $\mathfrak{J}$ denote the class of Lie algebras in which the relation of being an ideal is transitive; i.e. $L \in \mathfrak{J}$ if and only if $H \text{ si } L \Rightarrow H \triangleleft L$. (We study such algebras further in chapter 6).

Suppose $L \in \text{Min-}\triangleleft$. Then the $\mathfrak{F}$-residual of L is defined to be the unique subalgebra F of L minimal with respect to $F \triangleleft L$, $L/F \in \mathfrak{F}$ (uniqueness and existence are clear). We denote it by $\delta(L)$.

Warning

In group theory it is well-known that every subgroup of finite index contains a normal subgroup of finite index. It is not true in general that for Lie algebras every subalgebra of finite codimension contains an ideal of finite codimension - to see this let L be the Lie algebra $P \oplus Q$ described just before lemma 4.1.1. $P \in \mathfrak{A}$ so P contains a proper subalgebra S of finite codimension in P, so S is of finite codimension in L. But P is a minimal ideal of L, so S contains no ideal of finite codimension.

This means that $\delta(L)$ may itself have proper ideals of finite codimension. However,

Lemma 4.2.2

If $L \in \text{Min-si}$ then $\delta(\delta(L)) = \delta(L)$ so $\delta(L)$ has no proper ideals of finite codimension.

Proof:

Let $F = \delta(L)$, $I = \delta(F)$. By Min-si $I^c = I^{c+1}$ for some $c > 0$, so $I^c \lhd L$ by lemma 4.1.2. By Min-si each factor $I^i/I^{i+1} \in \mathfrak{F}$ so $F/I^c \in \mathfrak{F}$. Thus $L/I^c \in \mathfrak{F}$, and $I^c \leq \delta(L) = F \geq I \geq I^c$. Thus $I = F$.

We may now prove an analogue of lemma 3.2 of Robinson [31] p.36:

<u>Theorem 4.2.3</u>

Let L be a Lie algebra over a field of characteristic zero, satisfying Min-si. Then $\delta(L) \in \mathfrak{J}$, so that L is an extension of a $\mathfrak{J}$-algebra by a finite-dimensional algebra.

Proof:

Let $F = \delta(L)$. We show $F \in \mathfrak{J}$. Assume the contrary. Then there exists K minimal with respect to K si F but $K \ntriangleleft F$. If $K = K^2$ then by lemma 4.1.2 $K \lhd L$, which is impossible. So $K^2 < K$. But $K^2 \lhd K$ si F so by minimality of K, $K^2 \lhd F$. K/K^2 si F/K^2 and $K/K^2 \in \mathfrak{A}$, so $K/K^2 \leq B/K^2 = \beta(F/K^2)$. By lemma 4.1.3 $B/K^2 \lhd F/K^2$ and by lemma 4.1.6 $B/K^2 \in \mathfrak{F}$. If $C/K^2 = C_{F/K^2}(B/K^2)$ then $F/C \in \mathfrak{F}$ by lemma 4.2.1. By lemma 4.2.2 $F = C$. Therefore $B/K^2 \leq \zeta_1(F/K^2)$, so $K/K^2 \leq \zeta_1(F/K^2)$, so $K/K^2 \lhd F/K^2$, and $K \lhd F$. This is a contradiction.

Hence $F \in \mathfrak{J}$. Since $L/F \in \mathfrak{F}$ (by definition of F) the theorem follows.

Theorem 4.2.4

Let L be a Lie algebra over a field of characteristic zero, satisfying Min-si. Then L has an ascending series of ideals whose factors are either simple or finite-dimensional abelian; and $\delta(L)$ has an ascending series of ideals whose factors are either infinite-dimensional simple or 1-dimensional and central.

Proof:

First let K be any Lie algebra over a field of characteristic zero, satisfying Min-si. We show that every minimal ideal of K is either simple or lies in $\mathfrak{A} \cap \mathfrak{F}$. For suppose M is a minimal ideal of K. If M is not simple then there exists $I \triangleleft M$, $0 \neq I \neq M$. By Min-si $I^c = I^{c+1}$ for some $c > 0$, and by lemma 4.1.2 $I^c \triangleleft K$. By minimality of M $I^c = 0$ so $I \in \mathfrak{N}$. I si K so by lemma 4.1.3 $R = \rho(K) \neq 0$. Minimality of M implies $M \leq R$. $R \in \mathfrak{N}$ by lemma 4.1.6, so by lemma 3.3.11 $M \cap \zeta_1(R) \neq 0$. Minimality again implies $M \leq \zeta_1(R) \in \mathfrak{A} \cap \mathfrak{F}$ so $M \in \mathfrak{A} \cap \mathfrak{F}$ as claimed.

We now return to the Lie algebra L and define ideals M_α of L inductively as follows:

$M_0 = 0$. $M_{\alpha+1}/M_\alpha$ is some minimal ideal of L/M_α

provided $M_\alpha \neq L$, and $M_\lambda = \bigcup_{\alpha<\lambda} M_\alpha$ for limit ordinals λ. Clearly the sequence $\{M_\alpha\}$ ascends until some $M_\sigma = L$. Then $(M_\alpha)_{\alpha\leq\sigma}$ is an ascending series of ideals of L. Each factor $M_{\alpha+1}/M_\alpha$, being a minimal ideal of $K = L/M_\alpha \in$ Min-si, is either simple or $\mathfrak{A} \cap \mathfrak{F}$, by the observation above.

Now $F = \delta(L) \in$ Min-si so F has a series $(G_\alpha)_{\alpha\leq\sigma}$ with factors either simple or $\mathfrak{A} \cap \mathfrak{F}$. We show how to deal with finite-dimensional factors. Suppose that $G_{\alpha+1}/G_\alpha \in \mathfrak{F}$. Let $C/G_\alpha = C_{F/G_\alpha}(G_{\alpha+1}/G_\alpha)$. By lemma 4.2.1 $F/C \in \mathfrak{F}$. By lemma 4.2.2 $C = F$ so that $G_{\alpha+1}/G_\alpha \leq \zeta_1(F/G_\alpha) \in \mathfrak{A} \cap \mathfrak{F}$. Thus we may interpolate new terms in the series:

$$G_\alpha/G_\alpha = H_0/G_\alpha < H_1/G_\alpha < \ldots < H_n/G_\alpha = G_{\alpha+1}/G_\alpha$$

in such a way that $\dim(H_{i+1}/H_i) = 1$. Since $G_{\alpha+1}/G_\alpha$ is central, $H_i \triangleleft L$ and H_{i+1}/H_i is central.

This completes the proof.

In the next section we shall construct, for any ordinal σ, Lie algebras $\in$ Min-si having such a series of type σ. To do this we require a partial converse of theorem 4.2.4. First:

Lemma 4.2.5

Let L be a Lie algebra (any field) having two subideals H, K such that K is simple and not abelian.

Suppose that $K \cap H = 0$. Then $[K,H] = 0$.

Proof:

Lemma 4.1.2 immediately shows that $K \triangleleft L$. Let $H \triangleleft^n L$ and use induction on n. If $n = 1$ then $H \triangleleft L$ and $[K,H] \leq K \cap H = 0$. If not, then for some J we have $H \triangleleft J \triangleleft^{n-1} L$. If $K \cap J = 0$ then $[K,H] \leq [K,J] = 0$ by induction. Otherwise, since $K \cap J$ si K and K is simple, we must have $K \cap J = K$, so $K \leq J$. Thus $K,H \triangleleft J$ so that $[K,H] \leq K \cap H = 0$.

(This is a Lie-theoretic analogue of a theorem of Wielandt [42].

Now the partial converse to theorem 4.2.4:

<u>Lemma 4.2.6</u>

Suppose a Lie algebra L has an ascending series of ideals $(G_\alpha)_{\alpha \leq \sigma}$ such that for all $\alpha < \sigma$

1) $G_{\alpha+1}/G_\alpha$ is non-abelian and simple,

2) $C_{L/G_\alpha}(G_{\alpha+1}/G_\alpha) = G_\alpha/G_\alpha$.

Then the only subideals of L are the G_α. Consequently $L \in$ Min-si $\cap\ \mathfrak{I}$.

Proof:

Let M be a proper subideal of L and let α be the least ordinal such that $G_\alpha \nleq M$. It is easy to see that α is not a limit ordinal, so $\alpha = \beta+1$ for some β, and $(M+G_\beta)/G_\beta$ is a subideal of L/G_β which does not contain

$G_{\beta+1}/G_\beta$. As the latter is a simple non-abelian ideal of L/G_β we have

$$(M+G_\beta)/G_\beta \cap G_{\beta+1}/G_\beta = G_\beta/G_\beta$$

so by lemma 4.2.5 M centralises $G_{\beta+1}/G_\beta$. By part (2) of the hypotheses, $M \leq G_\beta$. Thus $M = G_\beta$.

This shows that every subideal is G_β for suitable β; this is an ideal so $L \in \mathfrak{I}$. $L \in$ Min-si since the ordinals are well-ordered.

4.3 An example of a Lie algebra satisfying Min-si

Theorem 4.2.4 shows that a Lie algebra over a field of characteristic zero, satisfying Min-si, has an ascending series of ideals with factors either simple or $\mathfrak{A} \cap \mathfrak{F}$. In this section we show that for any ordinal σ there exists a Lie algebra satisfying Min-si possessing such a series of type σ.

Let k be any field, V a vector space of infinite dimension over k. Let S be the set of all linear transformations of V, regarded as a Lie algebra under the usual Lie multiplication $[s,t] = st-ts$ $(s,t \in S)$. An element $a \in S$ is said to be <u>of trace zero</u> if

1) Its image Va is of finite dimension,

2) a restricted to Va has trace zero in the usual sense of linear algebra.

Let A be the set of elements of trace zero in S.

Lemma 4.3.1

A is an infinite-dimensional simple ideal of S. $C_S(A) = k$, where k is as usual identified with the scalar multiplications of V.

Proof:

Note first that if $a \in S$ and U is a finite-dimensional subspace of V containing Va, then $Ua \leq U$ and the traces of the restrictions of a to U and to Va are equal. Now let $a_1, a_2 \in A$ and let $U = Va_1+Va_2$. Then if $\lambda_1, \lambda_2 \in k$ the image of $\lambda_1 a_1+\lambda_2 a_2$ is contained in U. Since each of a_1 and a_2 has trace zero on U it follows that $\lambda_1 a_1+\lambda_2 a_2 \in A$.

Now let $s \in S$, $a \in A$, and let $x = [s,a] = sa-as$. Clearly $Vx \leq Va + (Va)s = W$, say, a finite-dimensional subspace of V. Choose a basis $(v_\lambda)_{\lambda \in \Lambda}$ of V such that $(v_\lambda)_{\lambda \in \Lambda_o}$ is a basis of W. Let $(\sigma_{\lambda\mu})$, $(\alpha_{\lambda\mu})$ be the matrices of s and a respectively with respect to this basis. Then for $\lambda \in \Lambda$ we have

$$v_\lambda(sa-as) = \sum_{\mu,\nu} (\sigma_{\lambda\mu}\alpha_{\mu\nu} - \alpha_{\lambda\mu}\sigma_{\mu\nu})\, v_\nu .$$

The trace of x on W is thus $\sum_{\lambda,\mu} \sigma_{\lambda\mu}\alpha_{\mu\lambda} - \alpha_{\lambda\mu}\sigma_{\mu\lambda}$ where, since terms corresponding to $\lambda \not\in \Lambda_o$ are zero, we may suppose that λ and μ each range over the whole of Λ. Hence x has trace zero on W and $A \triangleleft S$.

If $e_{\lambda\mu}$ is the linear transformation which sends v_λ to v_μ and every other basis vector v_ν to zero, then an elementary calculation shows that the only elements of S which centralise $e_{\lambda\mu}$ ($\lambda \neq \mu$) are the elements of k. Hence $C_S(A) = k$.

Now suppose $a_1,\ldots,a_n$ are finitely many elements of A. The kernel K_i of a_i has finite codimension in V and hence $K = \bigcap_{i=1}^{n} K_i$ also has finite codimension in V. Let U be a finite-dimensional subspace of V containing $\sum_{i=1}^{n} Va_i$ and such that $K+U = V$. If K_0 is a complement for $U \cap K$ in K then $V = K_0 \oplus U$. Let B be the set of all linear transformations a of V such that $K_0a = 0$, $Ua \leq U$, and a has trace zero on U. Then $a_i \in B$ for $i = 1,\ldots,n$ and B is a Lie subalgebra of A. B is clearly isomorphic to the Lie algebra of all linear transformations of trace zero of U. It is well-known and easy to prove that this is simple unless k has prime characteristic p which divides dim(U) (see Jacobson [17] p.136 for the case char(k) = 0; Seligman [38] p.66 for char(k) = p $\neq$ 2,3. The result can be established in all cases by elementary calculations). We may thus choose U so that B is simple. It follows that every finite set of elements of A lies in a simple subalgebra of A, and hence that A is simple.

Clearly A has infinite dimension.

<u>Theorem 4.3.2</u>

Let δ be any ordinal number, k any field. Then there exists a Lie algebra L over k such that

1) $L \in \text{Min-si} \cap \mathfrak{J}$,

2) L has an ascending series of ideals of type δ, each factor of which is isomorphic to a certain infinite-dimensional simple Lie algebra over k.

We carry out the proof in stages, using a construction similar to one employed in the group-theoretic situation (see Robinson [31]).

We may clearly assume $\delta > 0$. Choose an ordinal γ such that for each $\alpha < \delta$

$$\alpha + \gamma = \gamma.$$

Then $\delta \leq \gamma$ and γ is infinite. (As in [31] we could take γ to be the first prime component $\geq \delta$. See Sierpiński [39] theorem 1 p.282 and cor. to theorem 10 p.308). Let X be the set of all sequences of type γ with co-ordinates in $\mathbb{Z}$; that is, functions from γ to $\mathbb{Z}$. If $x \in X$ and $\alpha < \delta$, we denote by $^*x_\alpha$ the sequence of type α formed from the co-ordinates x_β of x with $\beta < \alpha$, and by $x_\alpha{}^*$ the sequence formed by the co-ordinates x_β with $\beta \geq \alpha$. We write

$$x = (^*x_\alpha, x_\alpha{}^*)$$

and notice that, since $\alpha + \gamma = \gamma$, $x_\alpha{}^*$ may be viewed as an element of X.

Let V be a vector space over k with basis v_x $(x \in X)$. If $\alpha < \delta$ and $x \in X$, then we have an epimorphism j_α and a monomorphism $i_{x,\alpha}$ of V defined by

$$v_y j_\alpha = v_{y_\alpha *} \qquad (y \in X) \tag{1}$$

$$v_y i_{x,\alpha} = v_{(*x_\alpha, y)} . \tag{2}$$

Evidently

$$i_{x,\alpha} j_\alpha = 1 \tag{3}$$

$$i_{x,\alpha} = i_{t,\alpha} \quad \text{if} \quad *x_\alpha = *t_\alpha . \tag{4}$$

In particular, (4) holds for all t such that v_t lies in $Vi_{x,\alpha}$.

As before, let S denote the set of all linear transformations of V. If $s \in S$ and $\alpha < \delta$ we define $s^\alpha \in S$ by

$$v_x s^\alpha = v_x j_\alpha s i_{x,\alpha} . \tag{5}$$

Clearly $s \to s^\alpha$ is a linear transformation of S. If $s,t \in S$ then $v_x(st)^\alpha = v_x j_\alpha st i_{x,\alpha} = v_x j_\alpha s i_{x,\alpha} j_\alpha t i_{x,\alpha}$ $= v_x s^\alpha t^\alpha$ since $v_x s^\alpha$ is a linear combination of elements v_y for which $*y_\alpha = *x_\alpha$. Thus $s \to s^\alpha$ is an associative algebra endomorphism of S and therefore also a Lie endomorphism of S. It follows from the fact that $i_{x,\alpha}$ is a monomorphism and j_α an epimorphism, together with (5), that $s^\alpha = 0$ if and only if $s = 0$. Thus $s \to s^\alpha$ is a monendomorphism of S.

Lemma 4.3.3

Let $s \in S$. Then $s \in S^{\alpha} = \{s^{\alpha}: s \in S\}$ if and only if $\ker(s) \geq \ker(j_{\alpha})$ and $v_x s \in \operatorname{im}(i_{x,\alpha})$ for all $x \in X$.

Proof:

The necessity of the conditions is obvious.

To see that they are sufficient, let $s \in S$ and suppose that $\ker(s) \geq \ker(j_{\alpha})$ and $v_x s \in \operatorname{im}(i_{x,\alpha})$ for all $x \in X$. Choose an arbitrary sequence $z \in X$ and consider

$$t = i_{z,\alpha} s j_{\alpha}. \tag{6}$$

Now it follows from (3) that for any $u \in X$ $(i_{u,\alpha} - i_{z,\alpha}) j_{\alpha} = 0$, so since $\ker(s) \geq \ker(j_{\alpha})$ we have $(i_{u,\alpha} - i_{z,\alpha}) s = 0$. Hence (6) is independent of the particular sequnce z chosen. Thus for any $x \in X$

$$\begin{aligned} v_x t^{\alpha} &= v_x j_{\alpha} i_{x,\alpha} s j_{\alpha} i_{x,\alpha} \\ &= v_x s j_{\alpha} i_{x,\alpha} \\ &= v_x s \end{aligned}$$

since $j_{\alpha} i_{x,\alpha}$ clearly acts as the identity on $\operatorname{im}(i_{x,\alpha})$ and this contains $v_x s$. Thus $s = t^{\alpha} \in S^{\alpha}$ as claimed.

Corollary

$S^{\beta} \leq S^{\alpha}$ if $\beta \geq \alpha$.

For clearly $\ker(j_{\beta}) \geq \ker(j_{\alpha})$ and $\operatorname{im}(i_{x,\alpha}) \geq \operatorname{im}(i_{x,\beta})$ whenever $\beta \geq \alpha$.

Now let A be the subalgebra of S consisting of all elements of trace zero in the sense previously defined, and for $\alpha < \delta$ let $A^{\alpha} = \{a^{\alpha} : a \in A\}$, $L_{\alpha} = \sum_{\beta<\alpha} A^{\beta}$, $L = L_{\delta}$. By the above corollary we find that for $\sigma \geq \alpha$ $[A^{\sigma}, A^{\alpha}] \leq [S^{\alpha}, A^{\alpha}] \leq [S, A]^{\alpha} \leq A^{\alpha}$ by lemma 4.3.1. Consequently $L_{\alpha} \triangleleft L$ for all $\alpha < \delta$. Clearly if τ is a limit ordinal $\leq \delta$ then $L_{\tau} = \sum_{\beta<\tau} A^{\beta} = \bigcup_{\beta<\tau} L_{\beta}$. Also $L_{\alpha+1} = L_{\alpha} + A^{\alpha}$. The next result shows that $L_{\alpha+1}/L_{\alpha} \cong A$ for $\alpha < \delta$. Hence L satisfies condition (2) of theorem 4.3.2.

<u>Lemma 4.3.4</u>

$L_{\alpha} \cap A^{\alpha} = 0.$

Proof:

As A^{α} is isomorphic to A so is simple, and $L_{\alpha} \triangleleft L$, it is enough to show that $A^{\alpha} \nleq L_{\alpha}$. Now if $t \in L_{\alpha}$ then $t \in \sum_{\sigma\leq\beta} A^{\sigma}$ for some $\beta < \alpha$. Suppose

$$t = \sum_{i=1}^{n} a_i^{\sigma_i}$$

where $a_i \in A$ and $\sigma_1 < \sigma_2 < \dots < \sigma_n \leq \beta$. Each a_i has finite-dimensional image, and (5) shows that

$$(a_i^{\sigma_i}) j_{\sigma_i}$$

has finite-dimensional image. Hence

$$t j_{\beta} = \sum_{i=1}^{n} a_i^{\sigma_i} . j_{\sigma_i} j_{\beta-\sigma_i}$$

has finite-dimensional image. However, choose $x \neq x'$ in X and let $e_{x,x'}$ be the transformation which sends v_x to $v_{x'}$, and sends every other basis vector to zero.

Then for any sequence $*y_\alpha$ of type α, and any $\beta < \alpha$, we have

$$v_{(*y_\alpha, x)} e^\alpha_{x,x'} j_\beta = v_{(*y_\alpha, x')} j_\beta .$$

Now by allowing the β-component of $*y_\alpha$ to range over all integer values we see that infinitely many basis vectors v_z belong to the image of $e^\alpha_{x,x'} j_\beta$. This image is thus of infinite dimension for any $\beta < \alpha$. Hence $e^\alpha_{x,x'} \notin L_\alpha$. But $e_{x,x'} \in A$ so $e^\alpha_{x,x'} \in A^\alpha$. This proves the lemma.

<u>Lemma 4.3.5</u>

$$C_{L/L_\alpha}(L_{\alpha+1}/L_\alpha) = L_\alpha/L_\alpha \text{ for all } \alpha < \delta.$$

Proof:

Let C_α/L_α denote the centraliser in question. If $C_\alpha > L_\alpha$ then $C_\alpha \cap (\sum_{\delta > \beta \geq \alpha} A^\beta) \neq 0$ and so by lemma 4.3.3 corollary, $C_\alpha \cap S^\alpha \neq 0$. Let $0 \neq s^\alpha \in C_\alpha \cap S^\alpha$. Then using lemma 4.3.3 we have $[A^\alpha, s^\alpha] \leq L_\alpha \cap [A, s]^\alpha \leq L_\alpha \cap A^\alpha = 0$. Thus by lemma 4.3.1 s is a scalar multiplication. The definition shows that $t = s^\alpha$ is also a scalar multiplication. Choose $\sigma < \delta$ such that $t \in L_{\sigma+1} \setminus L_\sigma$. Then $t + L_\sigma$ is a non-trivial central element of the infinite-dimensional simple algebra $L_{\sigma+1}/L_\sigma$, a contradiction. This establishes the lemma.

We have thus demonstrated that L, with its ascending series $(L_\beta)_{\beta \leq \delta}$, satisfies the hypotheses of lemma

4.2.6. Therefore $L \in$ Min-si $\cap\ \mathfrak{J}$, which proves theorem 4.3.2.

4.4 The full Endomorphism Algebra of a Vector Space

Another interesting class of Lie algebras satisfying Min-si emerges from a study of the Lie algebra of all linear transformations of an infinite-dimensional vector space (for finite-dimensional spaces our main result is trivially true). A special case gives us some information on the status of theorem 4.2.4.

If c is a cardinal number, we shall denote the successor cardinal by c^+.

Let k be any field (of arbitrary characteristic), c and d any infinite cardinals with $d \leq c^+$, and V a vector space of dimension c over k. Let E(c,d) denote the set of all linear transformations $\alpha: V \rightarrow V$ such that $\dim_k(\mathrm{im}(\alpha)) < d$. Note that the set of all linear transformations of V is $E(c,c^+)$.

Since d is infinite, E(c,d) is an associative k-algebra. Under the usual Lie multiplication $[\alpha,\beta] = \alpha\beta - \beta\alpha$ E(c,d) becomes a Lie algebra over k, which we shall distinguish by the symbol L(c,d).

We shall show among other things that L(c,d) satisfies Min-si. We attack the problem indirectly via the associative ideal structure of E(c,d) (which is easily determined), and then use the following theorem of Herstein [16] (see also Baxter [2]):

<u>Lemma 4.4.1</u> (Herstein)

If A is an associative simple ring, and U is a Lie ideal of A, then with one exception either $U \leq Z(A)$ or $[A,A] \leq U$.

In the exceptional case A is 4-dimensional over $Z(A) \cong$ a field of characteristic 2 ~~GF(2), so A is finite (with 16 elements)~~.

(A <u>Lie ideal</u> of an associative ring A is a subring I of A such that if $i \in I$, $a \in A$ then $ia-ai \in I$; equivalently it is a Lie ideal of the Lie ring obtained from A in the usual manner. Z(A) is the centre of A. [A,A] is the set of all finite sums of elements of the form ab-ba ($a,b \in A$). Note that Z(A) and [A,A] are always Lie ideals of A (though not necessarily associative ideals)).

Our first step is to put this into an 'algebra' form rather than a 'ring' form:

<u>Lemma 4.4.2</u>

If A is a simple associative k-algebra and $[A,A] = A$ then any proper Lie algebra ideal of the Lie algebra associated with A is contained in Z(A), with the same single exception.

Proof:

By Jacobson [18] p.108 §5 A is simple as an associative algebra if and only if it is a simple ring. Algebra ideals are certainly ring ideals, so the lemma follows from lemma 4.4.1.

In what follows we shall apply lemma 4.4.2 only in the case where A is infinite-dimensional over its centre, so the exceptional situation will never arise.

The associative ideal structure of E(c,d) is fairly transparent:

<u>Theorem 4.4.3</u>

Let c, d be infinite cardinals with $d \leq c^+$. Then any non-zero associative ideal of E(c,d) is of the form E(c,e) with $\aleph_0 \leq e \leq d$.

Proof:

We show that if I is an associative ideal of E(c,d) and some $\alpha \in I$ has $\dim(\mathrm{im}(\alpha)) = f$, then $E(c,f^+) \leq I$. This clearly implies the result.

Let $J = \mathrm{im}(\alpha)$, so $\dim(J) = f$. Let $(v_\lambda)_{\lambda \in \Lambda_0}$ be a basis of J extending to a basis $(v_\lambda)_{\lambda \in \Lambda}$ of V. For each $\lambda \in \Lambda_0$ there exists $w_\lambda \in V$ such that

$$w_\lambda \alpha = v_\lambda \qquad (1)$$

since $J = \mathrm{im}(\alpha)$. Define a linear transformation β of V:

$$v_\lambda \beta = w_\lambda \quad (\lambda \in \Lambda_o)$$
$$= 0 \quad (\lambda \in \Lambda \setminus \Lambda_o) . \qquad (2)$$

Let $\gamma \in E(c,f^+)$. Then $\dim(\mathrm{im}(\gamma)) \leq f$ so we can find a subset M_o of Λ_o and a basis $(x_\mu)_{\mu \in M_o}$ for $\mathrm{im}(\gamma)$ which extends to a basis $(x_\mu)_{\mu \in \Lambda}$ for V. Define $\delta : V \to V$ and $\mathcal{E} : V \to V$ by

$$x_\mu \delta = v_\mu \quad (\mu \in M_o)$$
$$= 0 \quad (\mu \in \Lambda \setminus M_o) \qquad (3)$$

$$v_\mu \mathcal{E} = x_\mu \quad (\mu \in M_o)$$
$$= 0 \quad (\mu \in \Lambda \setminus M_o). \qquad (4)$$

If λ is any element of Λ it follows from (1),(2),(3),(4) that

$$v_\lambda \cdot \gamma \delta \beta \alpha \mathcal{E} = v_\lambda \cdot \gamma$$

so that $\gamma = \gamma \delta \beta \alpha \mathcal{E} \in I$ (since $\alpha \in I$ and I is an associative ideal) which is what we wanted to prove.

(A weaker version of this lemma is proved by Jacobson in [18] using similar methods.)

<u>Corollary</u>

If $c \geq d$ are infinite cardinals, then

$$E(c,d^+)/E(c,d)$$

is a simple non-commutative associative algebra.

To facilitate calculations we shall represent linear transformations in some 'matrix-like' fashion. We will index bases of vector spaces by <u>ordinals</u>, so that a vector space of dimension c will have a basis of the form $(v_\alpha)_{\alpha<\sigma}$ where σ is an ordinal of cardinality c. (For even greater convenience we take σ to be the least ordinal with cardinality c, so when c is infinite σ is a limit ordinal.)

Let $e_{\alpha\beta}$ be the linear transformation defined by

$$v_\alpha \to v_\beta$$
$$v_\gamma \to 0 \quad (\alpha \neq \gamma < \sigma)$$

when $\alpha, \beta < \sigma$. Suppose we have a linear transformation $a: V \to V$. Then

$$v_\alpha a = \Sigma\, a_{\alpha\beta} v_\beta$$

where all but a finite number of the $a_{\alpha\beta}$ are zero. Thus we may write a as the formal sum

$$a = \Sigma\, a_{\alpha\beta} e_{\alpha\beta}$$

where for a given value of α only finitely many $a_{\alpha\beta}$ are non-zero. It is easily checked that such fromal sums can be manipulated in a way formally identical with the usual operations on finite sums. From now on any sum $\Sigma\, a_{\alpha\beta} e_{\alpha\beta}$ will be understood to be of this special type.

<u>Lemma 4.4.4</u>

Suppose k is any field; c, d are cardinals such that $\aleph_0 < d \leq c^+$; and $E = E(c,d)$ (over k).

Then $[E,E] = E$.

Proof:

Let $a \in E$, $I = \text{im}(a)$. $\dim(I) < d$ so we can choose a basis $(v_\lambda)_{\lambda<\sigma}$ for I with σ of cardinality $< d$, extending to a basis $(v_\lambda)_{\lambda<\rho}$ of V (ρ of cardinality c). With respect to this basis

$$a = \Sigma\, a_{\alpha\beta} e_{\alpha\beta} \quad (\alpha, \beta < \rho).$$

Since $I = \text{im}(a)$ $a_{\alpha\beta} = 0$ if $\beta \geq \sigma$, so we have

$$a = \Sigma\, a_{\alpha\beta} e_{\alpha\beta} \quad (\alpha < \rho,\ \beta < \sigma).$$

We will express a in the form $[b, t]$ where $b, t \in E$. Let $t = \sum_{\alpha<\sigma} e_{\alpha,\alpha+1} \in E$. For any $b \in E(c, c^+)$ a simple calculation shows that

$$[b, t] = [\Sigma\, b_{\alpha\beta} e_{\alpha\beta},\ \sum_{\gamma<\sigma} e_{\gamma,\gamma+1}]$$

$$= \sum_{\substack{\mu<\rho \\ \nu<\sigma}} b_{\mu,\nu-1}\, e_{\mu\nu} - \sum_{\substack{\mu<\sigma \\ \nu<\rho}} b_{\mu+1,\nu}\, e_{\mu\nu} \qquad (*)$$

where the apparently meaningless symbol $b_{\mu,\nu-1}$ will be given the conventional meaning 0 if ν is a limit ordinal.

We can make (*) equal to a if we can solve the infinite system of equations

$$\begin{aligned} b_{\mu,\nu-1} - b_{\mu+1,\nu} &= a_{\mu\nu} & (\mu, \nu < \sigma) \\ - b_{\mu+1,\nu} &= a_{\mu\nu} & (\mu<\sigma, \nu \geq \sigma) \\ b_{\mu,\nu-1} &= a_{\mu\nu} & (\mu \geq \sigma, \nu < \sigma) \end{aligned} \qquad (**)$$

(note that in the second equation $a_{\mu\nu} = 0$ since $\nu \geq \sigma$).

We solve (**) by defining:

$$
\begin{aligned}
b_{\mu\nu} &= 0 && (\mu, \nu \geq \sigma) \\
&= 0 && (\mu < \sigma, \nu \geq \sigma) \\
&= a_{\mu,\nu+1} && (\mu \geq \sigma, \nu < \sigma)
\end{aligned}
$$

and, if both $\mu, \nu < \sigma$, set

$$
\begin{aligned}
b_{\mu\nu} &= 0 \quad \text{if } \mu \text{ is a limit ordinal} \\
&= -a_{\mu-1,\nu} \quad \text{if } \nu \text{ is a limit ordinal,}
\end{aligned}
$$

and use the first equation of (**) to determine inductively the values of $b_{\mu+1,\nu}$, $b_{\mu+2,\nu+1}$, ... , $b_{\mu+n+1,\nu+n}$, ... It is clear that the values so determined are well-defined since a given $b_{\mu\nu}$ can be reached in precisely one way (the induction step moves 'down diagonals'). It is also clear that for a given value of μ $b_{\mu\nu}$ is non-zero for only a finite number of values of ν . So b is a well-defined linear transformation. Since $d > \aleph_0$ $b \in E$, (If $d = \aleph_0$ b may have infinite-dimensional image and so lie outside E(c,d).)

Thus $a = [b,t] \in [E,E]$. Since a was an arbitrary element of E, $E = [E,E]$.

(Note that the case $d = \aleph_0$ represents a genuine exception, for in this case $[E,E]$ is the ideal of all linear transformations of trace zero (in the sense of section 4.3) which is not the whole of E.)

Lemma 4.4.5

If $c \geq d$ are infinite cardinals, then

$$Z(E(c,d^+)/E(c,d))$$

is trivial unless $c = d$, when it is 1-dimensional and consists of scalar maps (modulo $E(c,d)$).

Proof:

By lemma 4.4.7 which we have found it more convenient to state and prove later on.

Theorem 4.4.6

If c and d are infinite cardinals with $c \geq d$, and k is any field, then the Lie algebra

$$L(c,d^+)/L(c,d)$$

is simple except when $c = d$. In this case its only ideal other than 0 or the whole algebra is its centre, which is 1-dimensional and consists of scalar maps (modulo $L(c,d)$).

Proof:

$L(c,d^+)/L(c,d)$ is the Lie algebra corresponding to the associative algebra $E(c,d^+)/E(c,d)$. Lemmas 4.4.2, 4.4.4, 4.4.5 complete the proof.

We have now found inside $L(c,d)$ a system of ideals, many of the factors of which are simple. This in itself is not sufficient to ensure that $L(c,d)$ satisfies Min-si.

Eventually this will follow using lemma 4.2.6. The presence of trace zero maps and scalar maps introduces an additional complication, so instead of looking at L(c,d) we study a suitable quotient.

Let S = the set of scalar maps, $F = L(c,\aleph_0)$, T = the set of trace zero maps, L = L(c,d), I = F+S. Then $L^* = L/I$ has an ascending series of ideals

$$0 = L_0^* \leq L_1^* \leq \dots \leq L_\alpha^* \leq \dots \leq L_\delta^* = L^*$$

where δ is a suitable ordinal, and the L_α^* are the ideals $(L(c,e)+S)/I$ arranged in ascending order as e varies.

I has a series

$$0 \leq T \leq F \leq I$$

of ideals. T is simple (lemma 4.3.1) and F/T and I/F are 1-dimensional. Thus $I \in (\text{Min-si})(\mathfrak{I})(\mathfrak{I}) \leq \text{Min-si}$. To prove $L \in$ Min-si it is sufficient to show $L^* \in$ Min-si since Min-si is E-closed (lemma 4.1.1.1). This will follow by lemma 4.2.6 provided we can show that

$$C_{L^*/L_\alpha^*}(L_{\alpha+1}^*/L_\alpha^*) = L_\alpha^*/L_\alpha^*.$$

Equivalently we must prove

<u>Lemma 4.4.7</u>

If $c \geq d$ are infinite cardinals, and $z \in L(c,c^+)$ satisfies

$$[z,\ L(c,d^+)] \leq L(c,d) + S$$

then $z \in L(c,d) + S$.

The proof, which is more intricate than one might hope, will be made in several steps. To simplify the notation, let $L = L(c,c^+)$, $E = L(c,d)$, $G = L(c,d^+)$. Suppose $z \in L$ and $[z,G] \leq E+S$. We must show $z \in E+S$.

Lemma 4.4.8

Let V be a vector space with basis $(v_\lambda)_{\lambda\in\Lambda}$ where Λ is infinite. Let a be a linear transformation of V such that $\dim(\mathrm{im}(a)) = e$ is infinite. If we let

$$B = \{\beta : a_{\alpha\beta} \neq 0 \text{ for some } \alpha \in \Lambda\}$$

and denote cardinalities by vertical bars thus: $|B|$, then $|B| = e$.

Proof:

Let $W = \sum_{\lambda\in B} kv_\lambda$. By definition $\dim(W) = |B|$, and clearly $\mathrm{im}(a) \leq W$, so $e \leq |B|$.

Now let $(i_\mu)_{\mu\in M}$ be a basis for $\mathrm{im}(a)$. Then $|M| = e$. For each $\mu \in M$ we have

$$i_\mu = \sum_{j=1}^{n(\mu)} k_j v_{\lambda_{j,\mu}}$$

where $k_j \in k \quad (j = 1,\ldots,n(\mu))$ and $\lambda_{j,\mu} \in \Lambda$.

By definition if $\lambda \in B$ then $\lambda = \lambda_{j,\mu}$ for some j,μ so that $|B| = |\{\lambda_{j,\mu}\}| \leq |\mathbb{Z} \times M| = \aleph_0|M| = e$ since e is infinite.

This completes the proof.

Let $(v_\lambda)_{\lambda \in \Lambda}$ be a basis for the vector space V under consideration, so that λ has cardinality c.

Lemma 4.4.9

Let z be as above. Then there exists z' such that $z'_{\alpha\alpha} = 0$ $(\alpha \in \Lambda)$, $[z', G] \leq E+S$, and $z - z' \in E+S$.

Proof:

Let $\mathcal{M}$ be the set of all pairs $(M, <)$ where M is a subset of Λ and < is a well-ordering on M, such that if $\alpha \in M$ then $z_{\alpha\alpha} \neq z_{\alpha+1,\alpha+1}$ (where $\alpha+1$ denotes the successor to α in the ordering <). Order $\mathcal{M}$ by $\ll$, where $(M_1, <_1) \ll (M_2, <_2)$ if and only if $M_1 \subseteq M_2$ and $<_2|_{M_1} = <_1$. Then it is easy to see that $\mathcal{M}$ is not empty, and that $(\mathcal{M}, \ll)$ satisfies the hypotheses of Zorn's lemma. Let $(M, <)$ be a maximal element of $\mathcal{M}$. Suppose if possible that $|M| \geq d$. Take an initial segment I of M with $|I| = d$, and look at

$$t = [z, \sum_{\alpha \in I} e_{\alpha,\alpha+1}].$$

By hypothesis $t \in E+S$. But

$$t = \Sigma\, z_{\alpha\beta} e_{\alpha\beta} e_{\beta,\beta+1} - \Sigma\, z_{\alpha\beta} e_{\alpha-1,\alpha} e_{\alpha\beta}$$
$$= \Sigma\, (z_{\alpha,\beta-1} - z_{\alpha+1,\beta}) e_{\alpha\beta}.$$

The coefficient of $e_{\alpha,\alpha+1}$ is $z_{\alpha\alpha} - z_{\alpha+1,\alpha+1} \neq 0$ for d values of α. By lemma 4.8.8 $t \notin E+S$ which is a contradiction.

Thus after choosing fewer than d values of α all the remaining $z_{\alpha\alpha}$ are equal. Thus $\Sigma\, z_{\alpha\alpha}e_{\alpha\alpha} \in E+S$. Put $z' = z - \Sigma\, z_{\alpha\alpha}e_{\alpha\alpha}$.

Now we work on z'.

<u>Lemma 4.4.10</u>

Suppose $z' \notin E+S$. Then there exist subsets A, A' of Λ such that

1) $A \cap A' = \emptyset$,

2) There is a bijection $\phi: A \to A'$ (write $\phi(\alpha) = \alpha'$),

3) $z'_{\alpha\alpha'} \neq 0$ if $\alpha \in A$,

4) $|A| = |A'| = d$.

Proof:

Let $\mathcal{S}$ be the collection of all triples (A, A', ϕ) satisfying (1), (2), and (3). Partially order $\mathcal{S}$ by $\ll$ where $(A, A', \phi) \ll (B, B', \psi)$ if and only if $A \subseteq B$, $A' \subseteq B'$, and $\psi|_A = \phi$. It is easily checked that $\mathcal{S}$, ordered in this way, satisfies the hypotheses of Zorn's lemma. Let (A, A', ϕ) be a maximal element of $\mathcal{S}$, and write $\phi(\alpha) = \alpha'$ $(\alpha \in A)$.

We claim that $|A| \geq d$.

Suppose not. Then $|A| = d' < d$. Let

$$D = \{\delta : z'_{\gamma\delta} \neq 0,\ \gamma \in A \cup A'\}.$$

Since d is infinite it is clear that $|D| < d$. By lemma 4.4.8 there must exist $\gamma' \notin (A \cup A' \cup D)$ with $z'_{\gamma\gamma'} \neq 0$ for some $\gamma \neq \gamma'$ (since $z' \notin E+S$). Then $\gamma \notin (A \cup A')$ (or

else $\gamma' \in D$). Therefore $\gamma \neq \gamma'$, $\gamma \notin (A \cup A')$, $\gamma' \notin (A \cup A')$.
Define

$$B = A \cup \{\gamma\}$$
$$B' = A' \cup \{\gamma'\}$$
$$\psi(\beta) = \beta' \quad (\beta \in A)$$
$$= \gamma' \quad (\beta = \gamma).$$

Then $(B, B', \psi) \in \mathcal{S}$ and is greater than (A, A', ϕ) under the ordering $\ll$. This contradicts the choice of (A, A', ϕ). Therefore $|A| \geq d$ as claimed.

If S is a subset of A with $|S| = d$ then the triple $(S, \phi(S), \phi|_S)$ satisfies the conclusions of the lemma.

We may now derive the final contradiction required to prove lemma 4.4.7.

Suppose if possible that $z' \not\subseteq E+S$. Then there exists (A, A', ϕ) as in lemma 4.4.10. Define π: $V \to V$ by

$$v_\alpha \pi = v_{\alpha'} \qquad (\alpha \in A)$$
$$v_{\alpha'} \pi = v_{\alpha'} \qquad (\alpha' \in A')$$
$$v_\beta \pi = 0 \qquad (\beta \in \Lambda \setminus (A \cup A')).$$

By definition $\pi \in G$. So by hypothesis $u = [z', \pi] \in E+S$. But for $\alpha \in A$ we have

$$v_\alpha(z'\pi - \pi z') = \Sigma\, z'_{\alpha\beta} v_\beta \pi - \Sigma\, z'_{\alpha'\beta} v_\beta.$$

The coefficient of $v_{\alpha'}$ is

$$z'_{\alpha\alpha} + z'_{\alpha\alpha'} - z'_{\alpha'\alpha'} = z'_{\alpha\alpha'} \neq 0$$

(bearing in mind that $z'_{\beta\beta} = 0$ and $\alpha \in A$). Thus $u_{\alpha\alpha'} \neq 0$ if $\alpha \in A$. But $|A| = d$ and $\alpha \neq \alpha'$ so $u \notin E+S$.

This contradiction shows that $z' \in E+S$, so $z \in E+S$, whence lemma 4.4.7 is proved.

Application of lemma 4.2.6 now proves

Theorem 4.4.11

If c and d are infinite cardinals with $d \leq c^+$, then $L(c,d) \in$ Min-si.

(we can also easily show $L(c,d) \in \mathfrak{I}$ using theorem 4.2.3. Suppose $L = L(c,d)$ has a proper ideal I with $L/I \in \mathfrak{F}$. L has an ascending series, the finite-dimensional factors of which are abelian, the rest simple, so L/I must be soluble. Then $[L,L] < L$ contrary to lemma 4.4.4. Thus $L = \delta(L) \in \mathfrak{I}$. The special case of $L(c, \aleph_0)$ can be handled easily by other methods.)

Remarks 4.4.12

1) Let $L = L(\aleph_0, \aleph_0^+)$. L has a series of ideals

$$0 < T < F < S+F < L$$

(S,T,F as before). L/F is an extension of the 1-dimensional algebra S+F/F by the infinite-dimensional simple algebra L/S+F. We claim this is not a split extension.

Let $M = L/F$, $J = S+F/F$, and suppose there were a subalgebra K with $J+K = M$, $J \cap K = 0$. Let $C = C_M(J)$. $C \triangleleft M$ and M/C has dimension ≤ 1 (by the remarks preceding lemma 4.2.1). $M = [M,M]$ by 4.4.4 so $C = M$. Thus M is the direct sum $J \oplus K$, and $[M,M] \leq K < M$, a contradiction.

Thus M does not split over its radical (either the soluble radical or the nil radical or any sensible generalisation thereof), in contrast to the Levi splitting theorem for finite-dimensional Lie algebras (see Jacobson [17] p.91).

2) $M \in$ Min-si $\cap \mathfrak{J}$, and any ascending series of ideals with simple factors contains a 1-dimensional factor which cannot be moved to the top. Thus the 1-dimensional central factors mentioned in the second part of theorem 4.2.4 cannot in general be dispensed with.

3) Similar remarks apply to L(c,d) in general. It has a series with two 1-dimensional factors, which may occur in various places, but not at the top.

4.5 An Embedding Theorem

A result of an entirely different kind which falls out of the previous analysis with very little prodding makes up

Theorem 4.5.1

Let k be any field, K any Lie algebra over k. Then K can be embedded in a simple Lie algebra over k.

Proof:

By Jacobson [17] p.162 cor. 4 K has a faithful representation by linear transformations (of a vector space V of dimension c (say) over k). By enlarging V if necessary we may take c to be infinite; further enlargement enables us to assume $K \leq L(c^+,c^+)$. Since c is infinite $c.c^+ = c^+$, so if Γ is a set with $|\Gamma| = c^+$ and Λ is a set with $|\Lambda| = c$ we can find two bases $(v_\gamma)_{\gamma\in\Gamma}$, $(w_{\gamma\delta})_{\gamma\in\Gamma,\ \delta\in\Lambda}$ of V. Let $\alpha \in L(c^+,c^+)$. Then

$$v_\gamma\alpha = \Sigma\ a_{\gamma\gamma'}v_{\gamma'}$$

and $\dim(\mathrm{im}(\alpha)) \leq c$. Define $\alpha^*\colon V \to V$ by

$$w_{\gamma\delta}\alpha^* = \Sigma\ a_{\gamma\gamma'}w_{\gamma'\delta}.$$

(Roughly speaking we split V into c subspaces of dimension c^+ and copy the action of α on each.)

Clearly the map $*\colon \alpha \to \alpha^*$ is a monomorphism of $L(c^+,c^+) \to L(c^+,c^+)$. But $\mathrm{im}(\alpha^*)$ has dimension $\geq c$ unless $\alpha = 0$, so $\mathrm{im}(*) \cap L(c^+,c) = 0$. Consequently $\mathrm{im}(*)$ is mapped isomorphically by the natural quotient map $L(c^+,c^+) \to L(c^+,c^+)/L(c^+,c)$. The composite embedding

$$K \to L(c^+,c^+) \to L(c^+,c^+)/L(c^+,c)$$

embeds K in a simple algebra (by theorem 4.4.6).

Using the corollary to theorem 4.4.3 we could perform a similar trick with associative algebras. The theorem also holds for groups, proved by essentially the same trick in Scott [37] p.316 11.5.4.

Not all known embedding theorems for groups carry over to the Lie case. For example, Dark [8] has proved that every group can be embedded as a subnormal subgroup of a perfect group. Strangely, the analogue of this result fails for Lie algebras - does this indicate the absence of a wreath product for Lie algebras? (L is <u>perfect</u> if $L = L^2$.) More specifically:

<u>Theorem 4.5.2</u>

Let K be a Lie algebra with the following properties:

1) $K^{\omega} = \bigcap_{i=1}^{\infty} K^i \neq 0$,

2) $K^{\omega} \not\leq \zeta_1(K)$,

3) $\mathrm{der}(K^{\omega}) \in E\mathfrak{A}$.

Then K cannot be embedded as a subideal of a perfect Lie algebra.

(Note: Condition (3) is most easily satisfied if $\dim(K^{\omega}) = 1$. A concrete example of K satisfying these hypotheses is the 2-dimensional soluble algebra

$$K = \langle a,b : [a,b] = a \rangle$$

for which $K^{\omega} = \langle a \rangle$ has dimension 1 and is not central.)

Proof:

Suppose there exists $L = L^2$ with K si L. Then by lemma 4.1.2 $K^{\omega} \lhd L$. Then $C = C_L(K\ ) \lhd L$. By the remarks before lemma 4.2.1 $L/C \cong D \leq \mathrm{der}(K^{\omega}) \in E\mathfrak{A}$. If $C \neq L$ then $L \neq L^2$, so $C = L$. Then $[K^{\omega}, L] = 0$ so $[K^{\omega}, K] = 0$ contrary to (2). This contradiction establishes the non-embeddability of K in a perfect Lie algebra.

(Note: It is not hard to state a rather more general non-embedding theorem based on the same proof.)

Chapter Five

Chain Conditions in special classes of Lie algebras

We now investigate the effect of imposing chain conditions (both maximal and minimal) on more specialised classes of Lie algebras, with particular regard to locally nilpotent Lie algebras. Application of the Mal'cev correspondence then produces some information on chain conditions for complete subgroups of complete locally nilpotent torsion-free groups.

5.1 Minimal Conditions

Lemma 4.1.6 immediately implies

Proposition 5.1.1

$$L\mathfrak{N} \cap \text{Min-}\triangleleft^2 = \mathfrak{N} \cap \mathfrak{F}.$$

If we relax the condition to Min-$\triangleleft$ lemma 4.1.5 shows that $L\mathfrak{N} \cap \text{Min-}\triangleleft \leq E\mathfrak{A} \cap \mathfrak{Z}$. But in contrast to proposition 5.1.1 we have

Proposition 5.1.2

$$L\mathfrak{N} \cap \text{Min-}\triangleleft \not\leq \mathfrak{N} \cup \mathfrak{F}.$$

Proof:

Let k be any field. Let A be an abelian Lie algebra of countable dimension over k, with basis $(x_n)_{0 < n \in \mathbb{Z}}$. There is a derivation σ of A defined by

$$x_i\sigma = x_{i-1} \qquad (i > 1)$$
$$x_1\sigma = 0.$$

Let L be the split extension (Jacobson [17] p.18) $A \oplus \langle\sigma\rangle$. Clearly $L \in L\mathfrak{N} \setminus (\mathfrak{N} \cup \mathfrak{F})$. Let $A_i = \langle x_1, \ldots, x_i \rangle$. We show that the only ideals of L are 0, $A_i (i>0)$, A, or L. For let $I \triangleleft L$, and suppose $I \nleq A$. Then there exists $\lambda \neq 0$, $\lambda \in k$, and $x \in A$, such that $\lambda\sigma + x \in I$. Then $x_i = [\lambda^{-1}x_{i+1}, \lambda\sigma + x] \in I$ so $A \leq I$. Thus $x \in I$, so $\sigma \in I$, and $I = L$.

Otherwise suppose $0 \neq I \leq A$. For some $n \in \mathbb{Z}$ we have

$$x = \lambda_n x_n + \lambda_{n-1}x_{n-1} + \cdots + \lambda_1 x_1 \in I$$

where $0 \neq \lambda_n$, $\lambda_i \in k$ $(i = 1, \ldots, n)$. Then $[\lambda_n^{-1}x, {}_{n-1}\sigma] = x_1 \in I$. Suppose inductively that $A_m \leq I$ for some $m < n$. Then $[\lambda_n^{-1}x, {}_{n-m-1}\sigma] \in I$, and this equals $x_{m+1} + y$ for some $y \in A_m$. Thus $x_{m+1} \in I$ and $A_{m+1} \leq I$. From this we deduce that either $I = A_n$ for some n or $I = A$.

Thus the set of ideals of L is well-ordered by inclusion, so $L \in \text{Min-}\triangleleft$.

For Lie algebras satisfying Min-si we may define a <u>soluble radical</u> (which has slightly stronger properties when the underlying field has characteristic zero).

<u>Theorem 5.1.3</u>

Let L be a Lie algebra over a field of characteristic zero, satisfying Min-si. Then L has a unique maximal soluble ideal $\sigma(L)$. $\sigma(L) \in \mathfrak{F}$ and contains every soluble subideal of L.

Proof:

Let $F = \delta(L)$ be the $\mathfrak{F}$-residual of L, $\beta(L)$ the Baer radical. Let $\dim(L/F) = f$, $\dim(\beta(F)) = b$. Both f and b are finite. Define $B_1 = \beta(L)$, $B_{i+1}/B_i = \beta(L/B_i)$. By lemma 4.1.3 and 4.1.6 $B_i \in E\mathfrak{A} \cap \mathfrak{F}$. $B_i \cap F \triangleleft F$ and by lemma 4.2.2 F has no proper ideals of finite codimension, so by the usual centraliser argument $B_i \cap F$ is central in F, so $B_i \cap F \leq \beta(F)$. $\dim(B_i) = \dim(B_i \cap F) + \dim(B_i + F/F) \leq b+f$. Consequently $B_{i+1} = B_i$ for some i. Let $\sigma(L) = B_i$. Then $\sigma(L) \triangleleft L$, $\sigma(L) \in E\mathfrak{A} \cap \mathfrak{F}$. $L/\sigma(L)$ contains no abelian subideals, and hence no soluble subideals, other than 0. Thus $\sigma(L)$ contains every soluble subideal of L as claimed.

For the characteristic $p \neq 0$ case we prove rather less:

Theorem 5.1.4

Let L be a Lie algebra over a field of characteristic > 0, and suppose $L \in$ Min-si. Then L has a unique maximal soluble ideal $\sigma(L)$, and $\sigma(L) \in \mathfrak{F}$.

Proof:

Let $F = \delta(L)$. Suppose $S \triangleleft L$, $S \in E\mathfrak{A}$. Then $S \in E\mathfrak{A} \cap \text{Min-si} \leq \mathfrak{F}$, so $F \cap S \in \mathfrak{F}$. The usual argument shows $F \cap S \leq \zeta_1(F) \in \mathfrak{F} \cap \mathfrak{A}$. Let $\dim(\zeta_1(F)) = z$, $\dim(L/F) = f$. Then $\dim(S) = \dim(F \cap S) + \dim(S+F/F)$ $\leq z+f$. Clearly the sum of two soluble ideals of L is a soluble ideal; the above shows that the sum of all the soluble ideals of L is in fact the sum of a finite number of them, so satisfies the required conclusions for $\sigma(L)$.

Suppose now that $\mathfrak{W}$ denotes the class of Lie algebras L such that every non-trivial homomorphic image of L has a non-trivial abelian subideal; and let $\mathfrak{V}$ denote the class of all Lie algebras L such that every non-trivial homomorphic image of L has a non-trivial abelian ideal. Then immediately we have

Theorem 5.1.5

1) For fields of characteristic zero

$$\mathfrak{W} \cap \text{Min-si} = E\mathfrak{A} \cap \mathfrak{F}.$$

2) For arbitrary fields

$$\mathfrak{V} \cap \text{Min-si} = \text{E}\mathfrak{A} \cap \mathfrak{F}.$$

Proof:

If L satisfies the hypotheses then we must have $L = \sigma(L) \in \text{E}\mathfrak{A} \cap \mathfrak{F}$ as required. The converse is clear.

Digression

It is not hard to find alternative characterisations of the classes $\mathfrak{V}$, $\mathfrak{W}$. $\mathfrak{V}$ is clearly the class of all Lie algebras possessing an ascending $\mathfrak{A}$-series of ideals. These are the Lie analogues of the SI*-groups of Kuroš [23] p.183. $\mathfrak{W}$ is the Lie analogue of Baer's subsoluble groups (see [1]), which Phillips and Combrink [28] show to be the same as SJ*-groups (same reference for notation). A simple adaptation of their argument shows that $\mathfrak{W}$ consists precisely of all Lie algebras possessing an ascending $\mathfrak{A}$-series of subideals. We omit the details.

A useful corollary of theorem 5.1.5 follows from

Lemma 5.1.6

A minimal ideal of a locally soluble Lie algebra is abelian.

Proof:

Let N be a minimal ideal of $L \in LE\mathfrak{A}$ and suppose $N \notin \mathfrak{A}$. Then there exist $a,b \in N$ such that $[a,b] = c \neq 0$. By minimality $N = \langle c^L \rangle$ so there exist $x_1, \ldots, x_n \in L$ such that $a,b \in \langle c, x_1, \ldots, x_n \rangle = H$, say. $L \in LE\mathfrak{A}$ so $H \in E\mathfrak{A}$. Now $C = \langle c^H \rangle \triangleleft H$, and $a,b \in C$, so $c = [a,b] \in C^2 \text{ ch } C \triangleleft H$, so $c \in C^2 \triangleleft H$, and $C = C^2$. But $C \leq H \in E\mathfrak{A}$, a contradiction. Thus $N \in \mathfrak{A}$.

<u>Corollary</u>

$ELE\mathfrak{A} \cap \text{Min-si} = E\mathfrak{A} \cap \mathfrak{F}$.

Proof:

It is sufficient to show $LE\mathfrak{A} \cap \text{Min-si} \leq E\mathfrak{A} \cap \mathfrak{F}$. By lemma 5.1.6 $LE\mathfrak{A} \cap \text{Min-si} \leq \mathfrak{V}$ (since $LE\mathfrak{A}$ is Q-closed). Theorem 5.1.5 finishes the job.

<u>5.2</u> <u>Maximal Conditions</u>

Exactly as in section 4.1 we may define maximal conditions for subideals, namely Max-si, Max-$\triangleleft^n$, and Max-$\triangleleft$. We do not expect any results like theorem 4.1.7, and confine our attention mainly to Max-$\triangleleft$.

<u>Lemma 5.2.1</u>

$E\mathfrak{A} \cap \text{Max-}\triangleleft \leq \mathfrak{G}$.

Proof:

We show by induction on d that $\mathfrak{A}^d \cap \text{Max-}\triangleleft \leq \mathfrak{G}$. If d = 1 then $L \in \mathfrak{A} \cap \text{Max-}\triangleleft \leq \mathfrak{F} \leq \mathfrak{G}$. Suppose

$L \in \mathfrak{A}^d \cap \text{Max-}\triangleleft$, and let $A = L^{(d-1)}$. $L/A \in \mathfrak{A}^{d-1}$ and $L/A \in \text{Max-}\triangleleft$, so $L/A \in \mathfrak{G}$ by induction. $A \in \mathfrak{A}$. There exists $H \in \mathfrak{G}$ such that $L = A+H$ (Let H be generated by coset representatives of A in L corresponding to generators of L/A.) By Max-$\triangleleft$ there exist $a_1, \ldots, a_n \in A$ such that $A = \langle a_1^L \rangle + \ldots + \langle a_n^L \rangle$. But if $a \in A$, $h \in H$, then $[a_i, a+h] = [a_i, h]$ so $A+H = \langle a_1^H \rangle + \ldots + \langle a_n^H \rangle$ $= \langle a_1, \ldots, a_n, H \rangle \in \mathfrak{G}$.

Remark

It is not true that $E\mathfrak{A} \cap \text{Max-}\triangleleft \leq \mathfrak{F}$. The example discussed immeditately before lemma 4.1.1 shows this - indeed it shows that even $E\mathfrak{A} \cap \text{Max-}\triangleleft \cap \text{Min-}\triangleleft$ is not contained in $\mathfrak{F}$. This contrasts with a well-known theorem of P.Hall which states that a soluble group satisfying maximal and minimal conditions for normal subgroups is necessarily finite.

It is easy to show that $E\mathfrak{A} \cap \text{Max-}\triangleleft^2 = E\mathfrak{A} \cap \mathfrak{F}$.

Lemma 5.2.2

Let $H \triangleleft L \in LE\mathfrak{A} \cap \text{Max-}\triangleleft$. Then $H = 0$ or $H^2 < H$.

Proof:

Let $P = \bigcap H^{(\alpha)}$. Then $P \text{ ch } H \triangleleft L$ so $P \triangleleft L$. Suppose if possible $P \neq 0$. Then there exists K maximal with respect to $K \triangleleft L$, $K < P$. P/K is a minimal ideal of $L/K \in \mathrm{LE}\mathfrak{A}$, so by lemma 5.1.6 $P/K \in \mathfrak{A}$, so that $P^2 < P$ contradicting the definition of P. Thus $P = 0$ (so $H^2 < H$) or $H = 0$.

Lemma 5.2.3

If $H \leq L \in \mathfrak{N}$ and $L = H + L^2$, then $H = L$.

Proof:

We show by induction on n that $H + L^n = L$. If $n = 2$ this is our hypothesis. Now $H + L^n = H + (H+L^2)^n$ $= H + H^n + L^{n+1} = H + L^{n+1}$, so $L = H + L^{n+1}$ as required. For large enough n $L^n = 0$ so $L = H$.

Lemma 5.2.4

Let L be any Lie algebra with $P \triangleleft L$, $H \leq L$, such that $L = H + P^2$. Then $L = H + P^n$ for any integer n.

Proof:

We show $P = (H \cap P) + P^n$. Now $P = (H \cap P) + P^2$. Modulo P^n we are in the situation of lemma 5.2.3, so $P \equiv (H \cap P) \pmod{P^n}$, which provides the result.

Let $\mathfrak{Y}$ be any class of Lie algebras, L any Lie algebra. Define

$$\lambda(L, \mathfrak{Y}) = \bigcap \{N : N \triangleleft L,\ L/N \in \mathfrak{Y}\}.$$

Lemma 5.2.5

If $L \in \mathrm{LE}\mathfrak{A} \cap \text{Max-}\triangleleft$ and $L_k = \lambda(L, \mathfrak{N}^k)$, then $L/L_k \in \mathrm{E}\mathfrak{A}$.

Proof:

Induction on k. If k = 0 the result is trivial. If $k \geq 0$ assume $L/L_k \in E\mathfrak{A}$. Then $L/L_k{}^2 \in E\mathfrak{A} \cap \text{Max-}\triangleleft \leq \mathfrak{G}$ (by lemma 5.2.1). Thus there exists $H \leq L$, $H \in \mathfrak{G}$, such that $L = H + L_k{}^2$ (coset representatives again). Since $L \in LE\mathfrak{A}$ $H \in \mathfrak{A}^d$ for some d. Let $Q \triangleleft L$ with $L/Q \in \mathfrak{N}^{k+1}$. Then there exists $P \triangleleft L$ with $Q \leq P$, $P/Q \in \mathfrak{N}$, $L/P \in \mathfrak{N}^k$. By definition $L_k \leq P$ so $L_k{}^2 \leq P^2$ and $L = H + P^2$. By lemma 5.2.4 $L = H + P^n$ for any n, so $L = H + Q$ $(P/Q \in \mathfrak{N})$. $L/Q \cong H/(H \cap Q) \in \mathfrak{A}^d$. L_{k+1} is the intersection of all such Q, so by standard methods L/L_{k+1} is isomorphic to a subalgebra of the direct sum of all the possible L/Q, all of which lie in $\mathfrak{A}^d$. Therefore $L/L_{k+1} \in \mathfrak{A}^d$ as claimed.

<u>Lemma 5.2.6</u>

If $L \in L(\mathfrak{N}^k) \cap \text{Max-}\triangleleft$, then $L/L_k \in \mathfrak{N}^k$. Thus L_k is the unique minimal ideal I of L with $L/I \in \mathfrak{N}^k$.

Proof:

By lemma 5.2.5 (since $\mathfrak{N}^k \leq E\mathfrak{A}$) $L/L_k \in E\mathfrak{A}$. But $L/L_k \in \text{Max-}\triangleleft$ so by lemma 5.2.1 $L/L_k \in \mathfrak{G}$. The usual argument shows that there exists $X \leq L$, $X \in \mathfrak{G}$, $L = L_k + X$. Then $L/L_k \cong X/(L_k \cap X)$. $X \in \mathfrak{N}^k$ since $L \in L(\mathfrak{N}^k)$ so $L/L_k \in \mathfrak{N}^k$.

Theorem 5.2.7

$$L(\mathfrak{N}^k) \cap \text{Max-}\triangleleft \leq \mathfrak{G} \cap \mathfrak{N}^k.$$

Proof:

Clearly all we need show is that if $L \in L(\mathfrak{N}^k) \cap \text{Max-}\triangleleft$ then $L \in \mathfrak{G}$. Define L_k as above. Suppose if possible that $L_k \neq 0$. Then $L_k \triangleleft L$, so by lemma 5.2.2 $L_k^2 < L_k$. By definition and lemma 5.2.6, $L_{k+1} \leq L_k^2$, so that $L_{k+1} < L_k$. But $L/L_{k+1} \in E\mathfrak{A} \cap \text{Max-}\triangleleft$ (lemma 5.2.5) $\leq \mathfrak{G}$ (lemma 5.2.1). The usual argument now shows $L/L_{k+1} \in \mathfrak{N}^k$, so that $L_k \leq L_{k+1}$, a contradiction. Thus $L_k = 0$, and $L \cong L/L_k \in E\mathfrak{A} \cap \text{Max-}\triangleleft$ (lemma 5.2.5) $\leq \mathfrak{G}$ (lemma 5.2.1).

Corollary

$$L\mathfrak{N} \cap \text{Max-}\triangleleft = \mathfrak{F} \cap \mathfrak{N}.$$

Proof:

Put $k = 1$ and note that $\mathfrak{G} \cap \mathfrak{N} = \mathfrak{F} \cap \mathfrak{N}$.

Compare this with Proposition 5.1.2.

5.3 Mal'cev Revisited

In order to apply the results of chapter 2 to obtain corresponding theorems for locally nilpotent torsion-free groups, we must find what property of the complete locally nilpotent torsion-free group G corresponds to the condition $\mathcal{L}(G) \in \mathfrak{F}$.

Lemma 5.3.1

Let G be a complete locally nilpotent torsion-free group. Then $\mathcal{L}(G) \in \mathfrak{F}$ if and only if G is nilpotent and of finite rank (in the sense of the Mal'cev special rank, see Kuroš [23] p.158).

Proof:

If $\mathcal{L}(G) \in \mathfrak{F}$ then $\mathcal{L}(G) \in \mathfrak{F} \cap \mathfrak{N}$ so has a series

$$0 = L_0 \triangleleft L_1 \triangleleft \dots \triangleleft L_n = \mathcal{L}(G)$$

such that $\dim(L_{i+1}/L_i) = 1$ ($i = 0, \dots, n-1$). Thus G has a series

$$1 = G_0 \triangleleft G_1 \triangleleft \dots \triangleleft G_n = G$$

with $G_i = \mathcal{G}(L_i)$. By lemma 2.4.2.5 $G_{i+1}/G_i \cong \mathcal{G}(L_{i+1}/L_i) = \mathbb{Q}$ (additive group). $\mathbb{Q}$ is known to be of rank 1, and it is also well-known that extensions of groups of finite rank by groups of finite rank are themselves of finite rank. Thus G is of finite rank. G is nilpotent since $\mathcal{L}(G)$ is.

Conversely suppose G is nilpotent of finite rank. Let

$$1 = Z_0 \leq Z_1 \leq \dots \leq Z_s = G$$

be the upper central series of G. From lemma 2.4.3 corollary 2 each term Z_i is complete, so is isolated in G. Therefore Z_{i+1}/Z_i is complete, torsion-free, abelian, and of finite rank (since G is of finite rank). By standard abelian group theory, Z_{i+1}/Z_i is isomorphic to a finite direct sum of copies of $\mathbb{Q}$. Hence $\mathcal{L}(Z_{i+1}/Z_i) \in \mathfrak{F}$, so $\mathcal{L}(G) \in \mathfrak{F}$ as required.

This proves the lemma.

Remark

Let rr(G) denote the rational rank of G as defined in the Plotkin survey [29] p.69. Then under the above circumstances we easily see that $\dim(\mathcal{L}(G)) = rr(G)$. According to [29] p.72 Gluškov [9] has proved that for locally nilpotent torsion-free groups G the rank of G = rr(G). Consequently $\dim(\mathcal{L}(G)) = \operatorname{rank}(G)$, a stronger result than lemma 5.3.1 (which, however, is sufficient for our purposes and easier to prove).

Applying the correspondence of chapter 2 and using the results of the present chapter, we clearly have

Theorem 5.3.2

Let G be a complete locally nilpotent torsion-free group. Then the following conditions are equivalent:

1) G is nilpotent of finite rank.

2) G satisfies the minimal condition for complete subnormal subgroups.

3) G satisfies the minimal condition for complete subnormal subgroups of defect ≤ 2.

3) G satisfies the maximal condition for complete normal subgroups.

On the other hand G may satisfy the minimal condition for complete normal subgroups without being either nilpotent or of finite rank.

(Some of these results have been obtained by Gluškov in [9]).

Chapter Six

Lie algebras in which every subideal is an ideal

Recall from section 4.2 that $L \in \mathfrak{I}$ if and only if H si L implies $H \lhd L$. Thus $L \in \mathfrak{I}$ if and only if $H \lhd K \lhd L$ implies $H \lhd L$. Further define the class $\overline{\mathfrak{I}}$ to consist of all Lie algebras L such that $H \leq L$ implies $H \in \mathfrak{I}$. Thus $L \in \overline{\mathfrak{I}}$ if and only if $H \lhd J \lhd K \leq L$ implies $H \lhd K$.

In this chapter we obtain the complete classification of:

1) Soluble $\mathfrak{I}$-algebras (over any field)

2) Finite-dimensional $\mathfrak{I}$-algebras (over any field of characteristic zero)

3) Locally finite $\overline{\mathfrak{I}}$-algebras (over any algebraically closed field of characteristic zero).

It will appear from case (1) that $E\mathfrak{A} \cap \mathfrak{I} = E\mathfrak{A} \cap \overline{\mathfrak{I}}$.

The corresponding problems for groups (which are considerably harder) have been partially solved by D.J.S.Robinson [32]. We will occasionally indicate how the Lie-theoretic and group-theoretic results compare.

6.1 Soluble $\mathfrak{J}$-algebras

For any Lie algebra L let $\nu(L)$ denote the Fitting radical of L, that is, the sum of all the nilpotent ideals of L (see chapter 7 for more information).

Lemma 6.1.1

1) Let $0 \neq H \triangleleft L \in E\mathfrak{A}$. Then H contains a non-zero abelian ideal of L.

2) Let $L \in E\mathfrak{A}$, $N = \nu(L)$. Then $C_L(N) \leq N$.

Proof:

1) Let n be the largest integer for which $H \cap L^{(n)} \neq 0$. Then if $A = H \cap L^{(n)}$ we have $[A,A] \leq H \cap L^{(n+1)} = 0$ so A is an abelian ideal of L, contained in H, and $A \neq 0$.

2) Let $C = C_L(N)$ and suppose $C \nleq N$. Then $0 \neq C+N/N \triangleleft L/N$ so by part (1) there exists $A \triangleleft L$ such that $N \leq A \leq C+N$ and $A/N \in \mathfrak{A}$. Now $A = A \cap (C+N) = A_0 + N$ where $A_0 = A \cap C$. $A_0{}^3 = [A_0{}^2, A_0] \leq [N,C] = 0$ so $A_0 \in \mathfrak{N}$. Thus $A = A_0 + N = N$, a contradiction.

Lemma 6.1.2

$\mathfrak{N} \cap \mathfrak{J} = \mathfrak{A}$.

Proof:

Let $L \in \mathfrak{N} \cap \mathfrak{J}$. Then $H \leq L$ implies H si L since $L \in \mathfrak{N}$, so $H \triangleleft L$ since $L \in \mathfrak{J}$. Thus $L \in \mathfrak{D}_1 = \mathfrak{A}$ by lemma 3.4.2. Clearly $\mathfrak{A} \leq \mathfrak{N} \cap \mathfrak{J}$.

Now suppose $L \in E\mathfrak{A} \cap \mathfrak{I}$. Suppose $L \notin \mathfrak{A}$, and let $N = \nu(L)$. Every nilpotent ideal of L lies in $\mathfrak{I}$, so by lemma 6.1.2 we must have $N \in \mathfrak{A}$.

Let U be a vector space complement for N in L. If $n \in N$, $u \in U$ then $\langle n \rangle \triangleleft N \triangleleft L$ so $\langle n \rangle \triangleleft L$ so

$$[n,u] = \lambda(n,u)n$$

where $\lambda(n,u)$ is in the underlying field k. If m,n are linearly independent elements of N, then

$$\begin{aligned}\lambda(m+n,u)(m+n) &= [m+n,u] \\ &= [m,u] + [n,u] \\ &= \lambda(m,u)m + \lambda(n,u)n\end{aligned}$$

so that $\lambda(m,u) = \lambda(m+n,u) = \lambda(n,u)$. Thus for any m we have $\lambda(m,u) = \mu(u)$ (say), independent of m. Thus

$$[n,u] = \mu(u)n$$

where $\mu : U \to k$ is linear.

Now $\ker(\mu) = C_U(N) \leq N \cap U$ (by lemma 6.1.1.2) $= 0$, and $\mathrm{im}(\mu) = k$ is 1-dimensional ($\mathrm{im}(\mu) = 0$ implies $L \in \mathfrak{A}$) so U is 1-dimensional. Consequently L is of the form

$$L = N \oplus U$$

where $N \triangleleft L$, $N \in \mathfrak{A}$, $U = \langle u \rangle$ is 1-dimensional, and u can be chosen so that $[a,u] = a$ for all $a \in N$. This determines L as a split extension, and gives part of

Theorem 6.1.3

$L \in E\mathfrak{A} \cap \mathfrak{J}$ if and only if one of the following hold:

1) $L \in \mathfrak{A}$.

2) $L = N \oplus U$, where $N \lhd L$, $N \in \mathfrak{A}$, $U = \langle u \rangle$, $N \neq 0$, $[a,u] = a$ for all $a \in N$.

A precise classification of these algebras up to isomorphism is given by the ordered pair $(\dim(L), \dim(L^2))$.

Proof:

$L \in E\mathfrak{A} \cap \mathfrak{J}$ implies (1) or (2) by the above analysis.

(1) implies $L \in E\mathfrak{A} \cap \mathfrak{J}$ trivially. Suppose L has the structure (2). We show $L \in \mathfrak{J}$ ($L \in E\mathfrak{A}$ is clear).

Let $I \lhd L$, and suppose $I \not\leq N$. Then there exists $i \in I$, $i = a + \delta u$ for some $a \in A$, $0 \neq \delta \in k$. For any $b \in N$,

$$[\delta^{-1}b, i] = b \in I.$$

Thus $N \leq I$, so $u \in I$, so $I = L$.

Now let J si L. $I = \langle J^L \rangle \lhd L$, so either $I = L$ so $J = L$ and $J \lhd L$, or $I \leq N$. Therefore $J \leq N$. If $j \in J$ then $[j,u] = j \in J$ so $J \lhd L$.

Clearly $(\dim(L), \dim(L^2))$ is an isomorphism invariant. If L, M are in $E\mathfrak{A} \cap \mathfrak{J}$ and $\dim(L) = \dim(M)$, $\dim(L^2) = \dim(M^2)$, then either L and M are abelian so isomorphic, or $L \simeq N \oplus U$, $M = N' \oplus U'$, and $N = L^2$, $N' = M^2$ so $\dim(N) = \dim(N')$. The structure indicated by (2) then shows $L \cong M$.

Remarks 6.1.4

1) $E\mathfrak{A} \cap \mathfrak{I} \leq \mathfrak{A}^2$ (this can also be proved directly as for groups, see Robinson [32] p.23).

2) $L \in (E\mathfrak{A} \cap \mathfrak{I}) \setminus \mathfrak{A}$ implies $\dim(L/L^2) = 1$. (This remark is of much use later on).

3) $E\mathfrak{A} \cap \mathfrak{I} \leq L\mathfrak{F}$ (proof immediate).

6.2 Finite-dimensional $\mathfrak{I}$-algebras

Throughout this section the characteristic of the field k will be assumed to be zero.

First we remark that the classical structure theory of finite-dimensional Lie algebras shows that any semisimple Lie algebra lies in $\mathfrak{I}$ (Jacobson [17] p.73). Let $\mathcal{S}$ denote the class of semisimple Lie algebras.

Suppose $L \in \mathfrak{F} \cap \mathfrak{I}$. By Levi's theorem (Jacobson [17] p.91) L is a split extension

$$L = R \oplus F$$

where $R \triangleleft L$, $R \cap F = 0$, $R \in E\mathfrak{A}$, and $F \in \mathcal{S}$. Now $R \in E\mathfrak{A} \cap \mathfrak{I}$ so is of the form stated in theorem 6.1.3. Let $A = \nu(R)$ ch R (in this case $\nu(R)$ reduces to the classical nil radical and Jacobson [17] p.51 shows this is a characteristic ideal. The result is true in general, cf. chapter 7.) Therefore $A \triangleleft L$ so $[A,F] \leq A$. As in the proof of theorem 6.1.3 F acts diagonally on A. $[A,F] \neq 0$ would imply that F has a non-trivial representation by diagonal matrices, so that $F^2 \neq F$. But $F \in \mathcal{S}$ so this is a contradiction (Jacobson [17] p.72). Thus $[A,F] = 0$.

If $A = R$ then $[R,F] = 0$. Otherwise $A \neq R$ so by theorem 6.1.3 $R = A \oplus U$ where $U = \langle u \rangle$ and $[a,u] = a$ for all $a \in A$. A is the nil radical of L so $[R,F] \leq A$ (Jacobson [17] p.51). Thus if $f \in F$ $[u,f] \in A$. Let $e,f \in F$. By Jacobi

$$[[u,e],f] + [[e,f]\,u] + [[f,u],e] = 0$$

so that $0 + [[e,f],u] + 0 = 0$. Thus $C_F(u) \geq F^2 = F$ since $F \in \mathcal{S}$.

Thus again $[R,F] = 0$ and L is the algebra <u>direct</u> sum $L = R \oplus F$.

This proves the first part of

<u>Theorem 6.2.1</u>

Over fields of characteristic zero, $L \in \mathfrak{E} \cap \mathfrak{I}$ if and only if L is a direct sum $R \oplus F$ where $R \in E\mathfrak{A} \cap \mathfrak{I} \cap \mathfrak{F}$ (classified in theorem 6.1.3) and $F \in \mathcal{S}$.

Proof:

$L \in \mathfrak{F} \cap \mathfrak{T}$ implies $L = R \oplus F$ by the above analysis.

Suppose $I \lhd R \oplus F$, $S = I \cap R$. Then S is the soluble radical of I and by Levi's theorem I is a split extension $S \oplus G$ where $G \in \mathfrak{S}$. By the theorem of Mal'cev - Harish-Chandra (Jacobson [17] p.92) $G \leq F^{\alpha}$ for some inner automorphism α of L (see section 1.2). But $F \lhd L$ so $F^{\alpha} = F$. Thus $G \leq F$.

$[F,G] \leq F \cap I$. Let $s+g \in F \cap I$, $s \in S$, $g \in G$. Then $s \in F \cap S \leq F \cap R = 0$ so $F \cap I = G$. Thus $G \lhd F$, and $[G,S] = 0$.

Thus $I \lhd L$ if and only if I is the direct sum $S \oplus G$, where $S \lhd R$ and $G \lhd F$. If $J \lhd I$ then by the same reasoning $J = T \oplus H$, where $T \lhd S$, $H \lhd G$. Then $T \lhd S \lhd R$ so $T \lhd R$ (since $R \in \mathfrak{T}$) and similarly $H \lhd F$. Consequently $J \lhd L$ as required.

6.3 $\overline{\mathfrak{T}}$-algebras

Theorem 6.3.1

$$E\mathfrak{A} \cap \mathfrak{T} = E\mathfrak{A} \cap \overline{\mathfrak{T}} .$$

Proof:

$\geq$ is clear. We use the classification theorem 6.1.3 to show $\leq$.

Let $L \in E\mathfrak{A} \cap \mathfrak{T}$. $L \in \mathfrak{A}$ implies $L \in \overline{\mathfrak{T}}$ so we may assume $L = N \oplus U$ etc. as usual. Let $K \leq L$. $K/(K \cap N)$ has dimension 0 or 1. If 0 then $K \leq N$ so $K \in \mathfrak{A}$ so $K \in \mathfrak{T}$. If not then there exists $t \in L$ such that $K = (K \cap N) + \langle t \rangle$. $t \notin N$ so $t = a + \delta u$, $a \in N$, $0 \neq \delta \in k$. Then if $v = \delta^{-1}t$ we have $[b,v] = b$ for all $b \in K \cap N$. Thus K is a split extension $(K \cap N) \oplus \langle v \rangle$ with v acting as the identity on $(K \cap N)$, so by theorem 6.1.3 $K \in \mathfrak{T}$.

Consequently $L \in \overline{\mathfrak{T}}$.

The same result holds for groups. Robinson [30] has shown that every finite $\overline{\mathfrak{T}}$-group is soluble. This is false for Lie algebras, but only just:

<u>Theorem 6.3.2</u>

Over algebraically closed fields of characteristic zero, $L \in \mathfrak{F} \cap \overline{\mathfrak{T}}$ if and only if one of the following hold:

1) $L \in E\mathfrak{A} \cap \mathfrak{T} \cap \mathfrak{F}$,

2) $L \cong A_1$, the <u>3-dimensional split simple algebra</u> defined by

$$A_1 = \langle e,f,h : [e,h] = 2e,\ [f,h] = -2f,\ [e,f] = h \rangle.$$

Proof:

First let $F \in \mathfrak{S} \cap \overline{\mathfrak{T}}$. Let H be a Cartan subalgebra of F. Then the subalgebra

$$B = H + \sum_{\alpha>0} F_\alpha$$

of F is soluble. Here the F_α are root-spaces corresponding to roots α (See Jacobson [17] or Carter [4] for terminology and details). By the classical theory $[H,H] = 0$ and $[H,B] \leq \sum_{\alpha>0} F_\alpha$ so that $\dim(B/B^2) \geq \dim(H)$. $B \notin \mathfrak{A}$ since by definition H is self-idealising. $F \in \overline{\mathfrak{I}}$ so $B \in \mathfrak{I}$ so by remark 6.1.4.2 $\dim(B/B^2) = 1$. Thus $\dim(H) = 1$. The only semisimple Lie algebra with a Cartan subalgebra of dimension 1 is the simple algebra A_1 (from the classification theorem for semisimple Lie algebras) so $F \cong A_1$.

Now let $L \in \mathfrak{I} \cap \overline{\mathfrak{I}}$. By theorem 6.2.1 $L = R \oplus F$ (direct) with $R \in E\mathfrak{A}$, $F \in \mathcal{S}$. $R, F \in \mathfrak{I}$ so by the above $F = 0$ or $F \cong A_1$. If $F = 0$ we are home. Otherwise $F \cong A_1$, which contains a soluble subalgebra $Q = \langle e,h \rangle \notin \mathfrak{A}$. If $R \neq 0$ then $D = R \oplus Q$ is in $(E\mathfrak{A} \cap \mathfrak{I}) \setminus \mathfrak{A}$ but has $\dim(D/D^2) \geq 2$ contrary to remark 6.1.4.2. Thus $R = 0$ and $L = F \cong A_1$.

On the other hand, $A_1 \in \overline{\mathfrak{I}}$ since $A_1 \in \mathfrak{I}$, and any proper subalgebra of A_1 has dimension ≤ 2. Such algebras are classified in Jacobson [17] p.11 and are easily seen to be $\mathfrak{I}$-algebras, and lie in $E\mathfrak{A}$.

Corollary

Over algebraically closed fields of characteristic zero, $L \in L\mathfrak{F} \cap \overline{\mathfrak{J}}$ if and only if one of the following holds:

1) $L \in E\mathfrak{A} \cap \mathfrak{J}$.

2) $L \cong A_1$.

Proof:

Either $L \in LE\mathfrak{A}$ or L contains a subalgebra $K \cong A_1$, by theorem 6.3.2. In the first case by remark 6.1.4.1 $L \in L\mathfrak{A}^2 = \mathfrak{A}^2$ so $L \in E\mathfrak{A}$. In the second case suppose $K \neq L$. Then there exists $x \in L \setminus K$. Then $\langle x,K \rangle \in \overline{\mathfrak{J}} \cap \mathfrak{F}$, is not soluble since $K \notin E\mathfrak{A}$, and is not isomorphic to A_1 since its dimension is too big. This contradicts theorem 6.3.2 and shows $L = K \cong A_1$.

The converse is clear using remark 6.1.4.3.

This completes the proof.

Chapter Seven

Baer, Fitting, and Gruenberg algebras

7.1 Summary of Group-theoretical Results

Let G be any group. The Fitting radical $\nu(G)$ is the join of the nilpotent normal subgroups of G. The Baer radical $\beta(G)$ is the join of all nilpotent subnormal subgroups of G. The Gruenberg radical $\gamma(G)$ is the join of all nilpotent ascendant subgroups of G. Clearly $\nu(G) \leq \beta(G) \leq \gamma(G)$, and it is well-known that each of the three is a locally nilpotent characteristic subgroup of G, so they all lie inside the Hirsch-Plotkin radical $\rho(G)$.

We will call G a Fitting group if $G = \nu(G)$, a Baer group if $G = \beta(G)$, and a Gruenberg group if $G = \gamma(G)$. It is easily seen that a Gruenberg group need not be a Baer group. The following problems are harder to dispose of:

G1) Is every Baer group a Fitting group?

G2) Is every locally nilpotent group a Gruenberg group?

In both cases the answer is in the negative. (G1) is answered in Robinson [30] p.107, and Dark [7] has shown that there exists a Baer group $G \neq 1$ with $\nu(G) = 1$. (G2) has been answered by Kovács and Neumann (unpublished, but see Robinson [30] p.110 for a proof). All of the groups so far constructed to answer these questions are p-groups for various primes p. The wealth of evidence (e.g. Kuroš [23]) that locally nilpotent <u>torsion-free</u> groups are on the whole better behaved than their periodic counterparts leads us to pose the following problems:

T1) Is every torsion-free Baer group a Fitting group?

T2) Is every locally nilpotent torsion-free group a Gruenberg group?

We shall show in a moment that these problems are equivalent to analogous questions about Lie algebras over $\mathbb{Q}$, and we will answer (T1) in the negative by constructing a suitable Lie algebra. This example has a number of other interesting properties: it also answers in the negative a question raised by B.Hartley in [14] p.260, and it provides alternative examples to one in [14] of Lie algebras in which the join of two subideals is not a subideal.

7.2 The three radicals in a Lie algebra

In what follows we restrict our attention to the case of Lie algebras over fields of characteristic zero.

Let L be such a Lie algebra. Following Hartley [14] we define

$$\beta(L) = \langle N : N \text{ si } L,\ N \in \mathfrak{N} \rangle,$$

$$\gamma(L) = \langle N : N \text{ asc } L,\ N \in \mathfrak{N} \rangle,$$

whence it is natural to define

$$\nu(L) = \langle N : N \lhd L,\ N \in \mathfrak{N} \rangle.$$

These will be referred to respectively as the Baer, Fitting, and Gruenberg radicals of L. Clearly for any L we have $\nu(L) \leq \beta(L) \leq \gamma(L)$. We define the classes $\mathfrak{Ft}$ (curly Ft), $\mathfrak{B}$, $\mathfrak{Gr}$ (curly Gr) of Fitting, Baer, and Gruenberg algebras by

$L \in \mathfrak{Ft}$ if and only if $\nu(L) = L$,

$L \in \mathfrak{B}$ if and only if $\beta(L) = L$,

$L \in \mathfrak{Gr}$ if and only if $\gamma(L) = L$.

As regards the status of these radicals we have:

Lemma 7.2.1

Let L be a Lie algebra over a field of characteristic zero. Then

1) $\nu(L)$ ch L, and $\nu(L) \in L\mathfrak{N}$.

2) $\beta(L)$ ch L, and $\beta(L) \in L\mathfrak{N}$.

3) $\gamma(L)$ need not even be an ideal of L, but $\gamma(L) \in L\mathfrak{N}$. On the other hand, if further $L \in L^{c}\mathfrak{N}$ then $\gamma(L) \lhd L$.

Proof:

All the statements follow from Hartley [14]:

1) Follows from theorem 1* p.267 and from lemma 1 (ii) p.261.

2) Is corollary to theorem 3, p.259.

3) For the first parts see corollary 1 to theorem 4, p.259; also p.270. For the last part use lemma 3 p.262.

We now ask the companion questions to (G1) and (G2).

L1) Is every Baer algebra a Fitting algebra?

L2) Is every locally nilpotent Lie algebra a Gruenberg algebra?

The connection between questions (Ti) and (Li) follows from

Theorem 7.2.2

Let G be a locally nilpotent torsion-free group, with completion $\bar{G}$, and let L be the Lie algebra $\mathcal{L}(G)$. Then

1) $\overline{\nu(G)} = \nu(\bar{G}) = \nu(L)$

2) $\overline{\beta(G)} = \beta(\bar{G}) = \beta(L)$

3) $\overline{\gamma(G)} = \gamma(\bar{G}) = \gamma(L)$.

Thus if any one of G, $\bar{G}$, L is Fitting (Baer, Gruenberg) so are the other two.

Proof:

Let $x \in \overline{\nu(G)}$. By lemma 2.4.4 there exists $n \in \mathbb{Z}$ such that $x^n \in \nu(G)$. Thus $x^n \in N \triangleleft G$ for some $N \in \mathfrak{N}$. Therefore $x \in \bar{N}$. $\bar{N} \triangleleft \bar{G}$ by lemma 2.4.4, and $\bar{N} \in \mathfrak{N}$ by theorem 2.5.3. Thus $x \in \nu(\bar{G})$ and $\overline{\nu(G)} \leq \nu(\bar{G})$. Now let $y \in \nu(\bar{G})$. Then $y \in M \triangleleft \bar{G}$, where $M \in \mathfrak{N}$. By theorem 2.5.3 $\bar{M} \in \mathfrak{N}$, and $\bar{M} \triangleleft \bar{G}$ by lemma 2.4.4. By Kuroš [23] p.257 $\bar{M} = \overline{\bar{M} \cap G}$. So for some $m \in \mathbb{Z}$ $y^m \in \bar{M} \cap G$. But $\bar{M} \cap G \triangleleft G$, and lies in $\mathfrak{N}$. Thus $y^m \in \nu(G)$, so $y \in \overline{\nu(G)}$. Thus $\nu(\bar{G}) \leq \overline{\nu(G)}$. Combining the two inequalities $\overline{\nu(G)} = \nu(\bar{G})$.

$$\text{Now } \nu(\bar{G}) = \langle M:\ M \triangleleft \bar{G},\ M \in \mathfrak{N} \rangle$$
$$= \langle \bar{M}:\ M \triangleleft \bar{G},\ M \in \mathfrak{N} \rangle$$

since lemma 2.4.4 and theorem 2.5.3 apply as above. By theorems 2.4.2 and 2.5.4 this equals

$$\langle H:\ H \triangleleft L,\ H \in \mathfrak{N} \rangle$$
$$= \nu(L).$$

Parts (2) and (3) are proved similarly, with 'si' or 'asc' replacing '$\triangleleft$', and using lemma 2.4.5.

Remark 7.2.3

As a consequence of theorem 7.2.2, we see that for i = 1, 2 the answer to question (Ti) is the same as that to question (Li) for Lie algebras over the field $\mathbb{Q}$.

And with this observation in mind, let's go hunting for Baer...

7.3 A Baer algebra which is not Fitting

Let k be any field, not necessarily of characteristic zero - the Lie algebra we shall construct has some interesting properties even for characteristic $p > 0$.

Theorem 7.3.1

There exists a Lie algebra L over k such that

1) L is a split extension $V \oplus J$, $V \triangleleft L$, $V \cap J = 0$.

2) $V \in \mathfrak{A}$.

3) $J = \langle H,K \rangle$ where $H,K \leq L$, $H,K \in \mathfrak{A}$, K is 1-dimensional, and H is infinite-dimensional.

4) $H \triangleleft^5 L$, $K \triangleleft^5 L$.

5) $J = I_L(J)$ so J si L.

6) $J \in \mathfrak{N}_4$.

7) $\langle K^L \rangle \notin \mathfrak{N}$, so $L \notin \mathfrak{F}$.

Proof:

We proceed by analogy with a group-theoretic construction of Roseblade and Stonehewer [34] §1.3.

Let A be an infinite-dimensional vector space over k and let R be the exterior algebra generated by A over k. (First form the <u>tensor algebra</u>

$$T = k \oplus A \oplus A\otimes A \oplus A\otimes A\otimes A \oplus \ldots$$

and factor out by the ideal I generated by all elements $a\otimes a$ $(a \in A)$. Put $R = T/I$.)

R is well-known to have the following properties (see Chevalley [6]):

R is an associative k-algebra, containing isomorphic copies of k and A. Making the obvious identifications $k \cap A = 0$. R has a natural structure as a graded k-algebra in which the homogeneous elements of degree i are products of i elements of A (or elements of k when i = 0). Further

E1) $a\lambda = \lambda a$ $(a \in A, \lambda \in k)$

E2) $a^2 = 0$ $(a \in A)$

E3) If $x \in R$ then $xA = 0$ if and only if $x = 0$.

(Note: (E3) fails when A is finite-dimensional).

(E2) implies that for all $a,b \in A$ $(a+b)^2 = 0$ so that $ab = -ba$. Hence for any $a,b,c,d \in A$ we have

$$abc = cab \ , \quad abcd = cdab. \tag{1}$$

First we construct J as a Lie algebra of 2 x 2 matrices over A (but considered as a Lie algebra over k) under the usual Lie multiplication $[M,N] = MN-NM$.

Let K be the set of all matrices of the form

$$\begin{pmatrix} 0 & 0 \\ \lambda & 0 \end{pmatrix} \qquad (\lambda \in k)$$

and let H be the set of all matrices of the form

$$\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \qquad (a \in A).$$

Clearly H and K are abelian Lie algebras. K is 1-dimensional with basis $\{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\}$ and $H \cong A$ (under addition) so is infinite-dimensional. Put $J = \langle H,K \rangle$ and part (3) of the theorem holds.

<u>Lemma 7.3.2</u>

$\langle H^J \rangle$ and $\langle K^J \rangle$ both lie in $\mathfrak{N}_2$.

Proof:

Let Z be the subalgebra of J generated by all matrices of the form

$$\begin{pmatrix} ab+c & 0 \\ d & ab-c \end{pmatrix} \qquad (a,b,c,d \in A). \qquad (2)$$

Direct calculation shows

$$\left[\begin{pmatrix} ab+c & 0 \\ d & ab-c \end{pmatrix}, \begin{pmatrix} pq+r & 0 \\ s & pq-r \end{pmatrix}\right] = \begin{pmatrix} \alpha & 0 \\ \beta & \gamma \end{pmatrix} \qquad (p,q,r,s \in A)$$

where

$$\alpha = (ab+c)(pq+r)-(pq+r)(ab+c)$$

$$\beta = d(pq+r)+(ab-c)s-s(ab+c)-(pq-r)d$$

$$\gamma = (ab-c)(pq-r)-(pq-r)(ab-c).$$

Using (1) this reduces to

$$\begin{pmatrix} 2cr & 0 \\ 0 & 2cr \end{pmatrix} \qquad (3)$$

which is of the form (2) with a = 2c, b = r, c=d=0. Thus Z is spanned by all matrices of the form (2).

Hence $[Z,H]$ is spanned by all products

$$[\begin{pmatrix} ab+c & 0 \\ d & ab-c \end{pmatrix}, \begin{pmatrix} 0 & e \\ 0 & 0 \end{pmatrix}] \qquad (a,b,c,d,e \in A)$$

which equals

$$\begin{pmatrix} -ed & (ab+c)e-e(ab-c) \\ 0 & de \end{pmatrix}$$

and using (1) this becomes

$$\begin{pmatrix} de & 0 \\ 0 & de \end{pmatrix} \tag{4}$$

which lies in Z. Thus $[Z,H] \leq Z$.

$[Z,K]$ is spanned by all products

$$[\begin{pmatrix} ab+c & 0 \\ d & ab-c \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ \lambda & 0 \end{pmatrix}] \qquad (a,b,c,d \in A, \ \lambda \in k)$$

and this is

$$\begin{pmatrix} 0 & 0 \\ \lambda(ab-c)-(ab+c)\lambda & 0 \end{pmatrix}$$

which, using (E1), is

$$\begin{pmatrix} 0 & 0 \\ -2\lambda c & 0 \end{pmatrix} \tag{5}$$

which is in Z. Thus $[Z,K] \leq Z$.

$[H,K]$ is generated by all products

$$[\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ \lambda & 0 \end{pmatrix}] \qquad (a \in A, \ \lambda \in k)$$

which equals

$$\begin{pmatrix} \lambda a & 0 \\ 0 & -\lambda a \end{pmatrix} \in Z.$$

Consequently Z+H and Z+K are idealised by both H and K, so are idealised by J. Thus $\langle H^J \rangle \leq Z+H$, $\langle K^J \rangle \leq Z+K$. (It is not hard to show that we may replace these inequalities by equalities, but we don't need to do so). To prove the lemma it is sufficient to show that each of Z+H, Z+K $\in \mathfrak{N}_2$.

Now $[Z+H, Z+H] \leq [Z,Z] + [Z,H]$ (since $H \in \mathfrak{A}$).
Matrices in $[Z,Z]$ are sums of matrices of the form

$$\begin{pmatrix} pq & 0 \\ 0 & pq \end{pmatrix} \qquad (p,q \in A)$$

by (3). Matrices in $[Z,H]$ are also of this form by (4).
Further,

$$\left[\begin{pmatrix} ab+c & 0 \\ d & ab-c \end{pmatrix}, \begin{pmatrix} pq & 0 \\ 0 & pq \end{pmatrix}\right] = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

by (3), and

$$\left[\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} pq & 0 \\ 0 & pq \end{pmatrix}\right] = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \qquad \text{by (4).}$$

Thus

$[Z+H, Z+H, Z+H] = 0$ and $Z+H \in \mathfrak{N}_2$.

Similarly $[Z+K, Z+K] \leq [Z,Z] + [Z,K]$. By (5) $[Z,K]$ is spanned by matrices of the form

$$\begin{pmatrix} 0 & 0 \\ x & 0 \end{pmatrix} \qquad (x \in A).$$

Let Y denote the subalgebra of J generated by all matrices of the form

$$\begin{pmatrix} uv & 0 \\ w & uv \end{pmatrix} \qquad (u,v,w \in A)$$

then $[Z+K, Z+K] \leq Y$.

But by (3) $[Y,Z] = 0$ and by (5) $[Y,K] = 0$. Hence $[Z+K, Z+K, Z+K] = 0$ and $Z+K \in \mathfrak{N}_2$.

This establishes the lemma.

J acts in a natural fashion as linear transformations of the k-vector space $R \times R = V$ (say), so J can be considered as a Lie algebra of derivations of the abelian Lie algebra V. Let L be the split extension

$$L = V \oplus J, \quad V \lhd L, \quad V \cap J = 0.$$

Then parts (1) and (2) of the theorem hold.

If $(x,y) \in V$ then

$$[(x,y), \begin{pmatrix} 0 & 0 \\ \lambda & 0 \end{pmatrix}] = (\lambda y, 0) \qquad (6)$$

$$[(x,y), \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}] = (0, xa) \qquad (7)$$

by the definition of split extension.

Let $V_2 = \{(x,0): x \in R\}$, $V_1 = \{(0,y): y \in R\}$. From (6) $C_V(K) = V_2$, and from (7) and (E3) $C_V(H) = V_1$. Now $[V,H] \leq V_1$ so $[V,H,H] = 0$. Since $V \lhd L$ and $V,H \in \mathfrak{A}$ we see that $V+H \in \mathfrak{N}_2$. Thus, since any subalgebra of a Lie algebra in $\mathfrak{N}_c$ is a c-step subideal (Hartley [14] p.261) we have

$$H \lhd^2 V+H \lhd^2 V+\langle H^J \rangle \lhd L$$

so $H \lhd^5 L$. Similarly $K \lhd^5 L$ and part (4) of the theorem holds.

On the other hand, suppose $i \in I_L(J)$. Then $i = v+j$ $(v \in V,\ j \in J)$ so $[v,J] \leq J$. But $V \lhd L$ so $[v,J] \leq V$. Hence $[v,J] \leq J \cap V = 0$ so $v \in C_V(J) = C_V(H) \cap C_V(K) = V_1 \cap V_2 = 0$. Thus $i = j \in J$, and $I_L(J) = J$. Thus J cannot be a subideal of L (nor even an ascendant subalgebra of L). This proves part (5) of the theorem.

J is the sum of $\langle H^J \rangle$ and $\langle K^J \rangle$, which are nilpotent ideals of class 2. By [14] lemma 1 (iii) p.261 $J \in \mathfrak{N}_4$ proving part (6).

Note that L is the join of K and V+H, both of which are nilpotent subideals, yet L is not nilpotent (since J is self-idealising). However $L \in E\mathfrak{A}$, indeed $L \in \mathfrak{A}\mathfrak{N}_4$.

To show L is not a Fitting algebra it suffices to show $\langle K^L \rangle \notin \mathfrak{N}$. For if L were Fitting, the generator $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ of K would be contained in the sum of a finite number of nilpotent ideals of L, which would also be a nilpotent ideal of L. Thus the ideal closure of K would be nilpotent.

$\langle K^L \rangle$ contains $\langle K^J \rangle$, which contains the matrices

$$\left[\begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right] = \begin{pmatrix} c & 0 \\ 0 & -c \end{pmatrix} \qquad (c \in A)$$

and it also contains $\langle K^V \rangle$, which contains all vectors of the form $(\lambda y, 0)$ $(\lambda \in k,\ y \in A)$ by (6), so contains $(a,0)$ for any $a \in A$. Hence $\langle K^L \rangle^{n+1}$ contains any element

$$\left[(a,0), \begin{pmatrix} c_1 & 0 \\ 0 & -c_1 \end{pmatrix}, \ldots, \begin{pmatrix} c_n & 0 \\ 0 & -c_n \end{pmatrix}\right]$$

which is easily seen to equal $(ac_1c_2\ldots c_n, 0)$. From (E3) we know that if $0 \neq x \in R$ then $xA \neq 0$, so that $AA\ldots A$ (n+1 terms) $\neq 0$. Thus we may choose $a, c_1, \ldots, c_n$ from A to make $ac_1\ldots c_n \neq 0$. Thus $\langle K^L \rangle^{n+1} \neq 0$ for any n so that $\langle K^L \rangle \notin \mathfrak{N}$. Thus the last part of the theorem (part (7)) is proved.

Corollary 1

For any field k of characteristic zero there exists a Baer algebra over k which is not a Fitting algebra.

Proof:

$L = \langle H,K,V\rangle$ and each of H,K,V is an abelian subideal of L. Thus $L \in \mathfrak{B}$. But $L \notin \mathfrak{F}$.

Thus question (L1) has the answer 'no'. By remark 7.2.3 (T1) has the same answer, i.e:

Corollary 2

There exists a torsion-free Baer group which is not a Fitting group.

(See also §7.4.)

Corollary 3

For any field k there exists a Lie algebra over k having two abelian subideals H, K with dim(K) = 1 such that $J = \langle H,K\rangle$ is not a subideal, and indeed J can be made self-idealising.

By Mal'cev (with the usual trappings) we deduce

Corollary 4

There exists a torsion-free complete group G having two abelian subnormal subgroups H,K with K isomorphic to $\mathbb{Q}$, but such that the join of H and K is not subnormal in G.

Corollary 5

In a Lie algebra the join of two nilpotent subideals need not be nilpotent (open question: need it be soluble? it is here.)

For what it's worth:

Corollary 6

There exists a torsion-free non-nilpotent group generated by two nilpotent subnormal subgroups (the analogous query regarding solubility is dealt with by recent unpublished work of S.E.Stonehewer.)

The only other example in the literature where the join of subideals of a Lie algebra is not a subideal can be found in Hartley [14] p.271. In his example both subideals are infinite-dimensional.

In the same paper the following question is raised (p.260):

If B is a finite-dimensional subideal of a Lie algebra L, does there always exist $J \triangleleft L$ with $J^n \leq B \leq J$ for some integer $n > 0$?

The answer is no.

For let L be as above, and put B = K. Then if such J existed, we would have $J^n \triangleleft L$, $J^n \leq K$. Therefore either $J^n = 0$ or J^n is a minimal ideal of $L \in L\mathfrak{N}$. By lemma 3.3.3 $J^n \leq \zeta_1(L)$ so $J^{n+1} = 0$. Either way K is contained in a nilpotent ideal of L, contradicting theorem 7.3.1.7.

7.4 A torsion-free Baer group which is not Fitting

Corollary 2 to theorem 7.3.1 is perhaps a little unsatisfactory, since the group is not exhibited in any tangible form. In fact our whole procedure is a trifle curious. Starting with the Roseblade-Stonehewer group ([34]) we have constructed an analogous Lie algebra and then appealed to Mal'cev. Now the Roseblade-Stonehewer group is Baer but not Fitting (this is not stated explicitly by them, but follows as for the Lie algebra). In view of this it is natural to ask whether this group might, under suitable circumstances, be torsion-free. If so we might bypass the Lie algebra approach, as far as question (T1) is concerned.

Now it turns out that if k is a field of characteristic zero, then the Roseblade-Stonehewer group over k is indeed torsion-free. However, the easiest way to prove this is to resurrect the Lie algebra (though it ought to be possible to provide a direct proof, say by calculating the factors in a central series) as follows:

If k is a field, A an infinite-dimensional vector space over k, then the Roseblade-Stonehewer group RS(k,A) is defined as a split extension of a vector space V (2-dimensional over the exterior algebra R generated by A over k) by a group $\overline{J}$ of 2 x 2 matrices over R, generated by

$$\begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} \quad (\lambda \in k) \quad \text{and} \quad \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \quad (a \in A).$$

If char(k) = 0 V is torsion-free, so all we need to show is that $\overline{J}$ is torsion-free.

Use the same notation for the Lie algebra as above. Local nilpotence of L immediately implies that for any finite subset $\{v_1,\ldots,v_s\}$ of V and any finite subset $\{j_1,\ldots,j_t\}$ of J there is a finite-dimensional subspace U of V such that $v_i \in U$ $(i = 1,\ldots,s)$ and U is j_r-invariant $(r = 1,\ldots,t)$. Further $\{j_r|_U : r = 1,\ldots,t\}$ generates a nilpotent associative algebra, since its action on U is given by commutation in L.

Thus for any $j \in J$ we may define $\exp(j) = j^*$ to be the map from V to V given by

$$vj^* = v(1 + j + \frac{j^2}{2!} + \frac{j^3}{3!} + \ldots) \quad (v \in V).$$

The remark about invariant subspaces implies that j^* is a linear transformation of V. It has an inverse, namely $(-j)^*$, so $j^* \in \mathrm{Aut}(V)$. We show that $J^* = \{j^* : j \in J\}$ is a subgroup of Aut(V).

Let $j_1, j_2 \in J$. $(j_1)^{*-1} = (-j_1)^* \in J^*$.

Let $j = \mu(j_1, j_2) = j_1 + j_2 + \frac{1}{2}[j_1, j_2] + \ldots$ (as in lemma 2.3.1), which is defined since $\langle j_1, j_2 \rangle$ is a nilpotent Lie algebra. Then for any $v \in V$ there exists a finite-dimensional subspace U of V with $v \in U$, such that U is $\langle j_1, j_2 \rangle$-invariant. $\langle j_1, j_2 \rangle$ acts as a nilpotent associative algebra on U so the Campbell-hausdorff formula applies:

$$v(j_1{}^* j_2{}^*) = v(\mu(j_1, j_2))^*.$$

As v was arbitrary $j_1{}^* j_2{}^* = j^* \in J^*$ so J^* is a subgroup of Aut(V).

J^* is torsion-free, for if $(j^*)^n = 1$ then $nj = 0$ so $j = 0$ so $j^* = 1$. On the other hand, for any $v \in V$ direct calculation shows that

$$v\begin{pmatrix} 0 & 0 \\ \lambda & 0 \end{pmatrix}^* = v\begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} \qquad (\lambda \in k)$$
$$v\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}^* = v\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \qquad (a \in A)$$

so the generators of the group $\overline{J}$ lie inside J^*. Hence $RS(k,A) = \overline{J}$ is torsion-free, and we have proved

<u>Theorem 7.4.1</u>

If k is a field of characteristic zero, and A is an infinite-dimensional vector space over k, then the Roseblade-Stonehewer group RS(k,A) is a torsion-free Baer non-Fitting group).

7.5 Conditions under which Baer implies Fitting

Theorem 7.3.1 shows that an abelian-by-nilpotent Baer algebra need not be Fitting. In contrast to this we will show that any nilpotent-by-abelian Baer algebra *is* Fitting. We work under rather more general hypotheses.

We consider a class $\mathcal{E}$ of Lie algebras satisfying a type of Engel condition:

$L \in \mathcal{E}$ if and only if for all $x, y \in L$ there exists $n = n(x)$ independent of y for which $[y, _n x] = 0$.

$\mathcal{E}$ enters the reckoning because of

Lemma 7.5.1

$\mathcal{B} \leq \mathcal{E}$.

Proof:

Let $x, y \in L \in \mathcal{B}$. Then $\langle x \rangle \triangleleft^m L$ for some $m = m(x)$. Thus $[y, _m x] \in \langle x \rangle$ so that $[y, _{m+1} x] = 0$.

Lemma 7.5.2

$\mathfrak{A}^2 \cap \mathcal{E} \leq \mathfrak{F}$.

Proof:

Let $x \in L \in \mathfrak{A}^2 \cap \mathcal{E}$. Then $A = L^2 \in \mathfrak{A}$. We must show $\langle x^L \rangle \in \mathfrak{N}$. Now $[L, _n x] = 0$ for some integer n since $L \in \mathcal{E}$. By bilinearity if $X = \langle x \rangle$ then $[L, _n X] = 0$. Now clearly

$$\langle x^L \rangle = \sum_{i=0}^{\infty} [X, _i L]$$

so

$$\langle x^L\rangle^m = \Sigma \quad [[X,_{i_1}L],[X,_{i_2}L], \ldots , [X,_{i_m}L]]$$

summed over all $\{i_1,\ldots,i_m\}$ with $i_j \geq 0$ $(j = 1,\ldots,m)$. If $i_j > 0$ for 2 distinct values of J, then since $L^2 \triangleleft L$ and $L^2 \in \mathfrak{A}$ the corresponding term is 0. If $i_j = 0$ for n consecutive values of j then again the corresponding term is 0, since $[L,_nX] = 0$. But if $m > (n-1)+1+(n-1)$ $= 2n-1$ one or other of these situations must occur. Thus $\langle x^L\rangle^{2n} = 0$ and $\langle x^L\rangle \in \mathfrak{N}_{2n-1}$.

(Note: a refinement of this argument will prove that $\mathfrak{N}\mathfrak{A} \cap \mathfrak{E} \leq \mathfrak{F}$. Because of the way we intend to prove a corresponding theorem for groups, we proceed in a different manner.)

Lemma 7.5.3

Let $\mathfrak{X}$, $\mathfrak{Y}$ be classes of Lie algebras (over any field k) such that

1) $\mathfrak{Y} = Q\mathfrak{Y}$,
2) $\mathfrak{A}\mathfrak{X} \cap \mathfrak{Y} \leq \mathfrak{F}$.

Then

$$\mathfrak{N}\mathfrak{X} \cap \mathfrak{Y} \leq \mathfrak{F}.$$

Proof:

Let $L \in \mathfrak{N}\mathfrak{X} \cap \mathfrak{Y}$. By definition there exists $N \triangleleft L$ such that $N \in \mathfrak{N}$ and $L/N \in \mathfrak{X}$. Let $D = N^2$. Then $L/D \in \mathfrak{A}\mathfrak{X} \cap \mathfrak{Y} \leq \mathfrak{F}$. Thus $L/D = \langle N_\lambda/D : \lambda \in \Lambda\rangle$ where

$N_\lambda/D \in \mathfrak{N}$ and $N_\lambda/D \triangleleft L/D$. Thus $N_\lambda \triangleleft L$. Since N and N_λ/N^2 lie in $\mathfrak{N}$, theorem 3.2.3 tells us that $N_\lambda \in \mathfrak{N}$. Thus $L = \langle N_\lambda \rangle \in \mathfrak{F}$ as required.

Theorem 7.5.4

$\mathfrak{N}\mathfrak{A} \cap \mathfrak{E} \leq \mathfrak{F}$. In particular $\mathfrak{N}\mathfrak{A} \cap \mathfrak{B} \leq \mathfrak{F}$.

Proof:

Set $\mathfrak{X} = \mathfrak{A}$, $\mathfrak{Y} = \mathfrak{E}$ (which is clearly Q-closed) in lemma 7.5.3, and use lemma 7.5.2.

An appeal to Comrade Mal'cev easily implies that any nilpotent-by-abelian torsion-free Baer group is Fitting. In fact we may drop the condition that the group be torsion-free. Again we consider the metabelian case first.

Lemma 7.5.5

A metabelian Baer group is a Fitting group.

Proof:

Let G be any metabelian Baer group. Denote the commutator (G,H) by γGH, and write $\gamma^n A_1 \ldots A_{n+1}$ for $\gamma(\gamma^{n-1} A_1 \ldots A_n) A_{n+1}$. We prove by induction on n that for $H \leq G$

$$\gamma^{n-1} \langle H^G \rangle^n = \gamma^{n-1} H^n . \gamma^n G H^n. \qquad (*)$$

$n = 1$:

$\langle H^G \rangle = H.\gamma GH$ as required.

$n = 2$:

$$(\langle H^G\rangle, \langle H^G\rangle) = (H.\gamma GH,\ H.\gamma GH).$$

Consider $(h_1\gamma_1, h_2\gamma_2)$, where $h_1, h_2 \in H$, $\gamma_1, \gamma_2 \in \gamma GH$ $\leq G'$ which is abelian.

$$\begin{aligned}(h_1\gamma_1, h_2\gamma_2) &= (h_1, h_2\gamma_2)^{\gamma_1}(\gamma_1, h_2\gamma_2)\\ &= (h_1,\gamma_2)^{\gamma_1}(h_1,h_2)^{\gamma_2\gamma_1}(\gamma_1,\gamma_2)(\gamma_1,h_2)^{\gamma_2}\\ &= (h_1,\gamma_2)(h_1,h_2).1.(\gamma_1,h_2)\end{aligned}$$

since G' is abelian, $\gamma_i \in G'$, and any commutator $(x,y) \in G'$. Since all commutators (x,y) commute in G, this is a member of $H'.\gamma^2GH^2$ as required.

$n > 2$:

Let $A_n = \gamma^nGH^n$.

By induction we know that

$$\gamma^{n-1}\langle H^G\rangle^n = \gamma^{n-1}H^nA_n,$$

and we must prove (*) with n replaced by n+1. We have

$$\gamma^n\langle H^G\rangle^{n+1} = (\gamma^{n-1}H^nA_n, H.\gamma HG) \qquad \text{by definition}$$

and induction. Now let $\beta \in \gamma^{n-1}H^n$, $a \in A_n$, $h \in H$, $\gamma \in \gamma GH$. Then

$$\begin{aligned}(\beta a, h\gamma) &= (\beta, h\gamma)^a(a, h\gamma)\\ &= (\beta,\gamma)^a(\beta,h)^{\gamma a}(a,\gamma)(a,h)^{\gamma}\\ &= 1.(\beta,h).1.(a,h)\\ &\in \gamma^nH^{n+1}.\gamma^{n+1}GH^{n+1}\end{aligned}$$

as required. Thus (*) is established.

We may now complete the proof of the lemma. Let

$x \in G$, a metabelian Baer group. Put $H = \langle x\rangle$. Then $H \triangleleft^n G$ for some n. Then

$$\gamma^{n+1}\langle x^G\rangle^{n+2} = 1.\gamma G\langle x\rangle^{n+1} = 1$$

so $\langle x^G\rangle$ is nilpotent and G is a Fitting group.

A group-theoretic version of lemma 7.5.3 now yields the more general

Theorem 7.5.6

A nilpotent-by-abelian Baer group is a Fitting group.

7.6 A property of Gruenberg algebras

In this section we establish a property of Gruenberg algebras which will be of use in the next chapter, and which is probably a necessary preliminary to any attack on problem (L2) of section 7.2.

Lemma 7.6.1

For $L\mathfrak{N}$ Lie algebras L over fields of characteristic zero, the following are equivalent:

1) L has an ascending $\mathfrak{A}$-series ($L \in \acute{E}\mathfrak{A}$).

2) Every non-trivial homomorphic image of L has ~~non-~~trivial Gruenberg radical.

In particular for characteristic zero $\mathfrak{Gr} \leq \acute{E}\mathfrak{A}$.

Proof:

We may suppose $L \neq 0$. $\gamma(L) \neq 0$ so there exists

H asc L with $0 \neq H \in \mathfrak{A}$.

First we show how to construct an ascending $\mathfrak{A}$-series from H to $\langle H^L \rangle$. H asc L so there is a series

$$H = H_0 \triangleleft H_1 \triangleleft \dots \triangleleft H_\alpha \triangleleft \dots H_\sigma = L.$$

Let $H^* = \langle H^L \rangle$, $H_\alpha^* = \langle H^{H_\alpha} \rangle$ $(0 < \alpha \leq \sigma)$. Now $H \leq H_\alpha \triangleleft H_{\alpha+1}$ so $H_{\alpha+1}^* \leq H_\alpha$. By definition $H_\alpha^* \triangleleft H_\alpha$ so $H_\alpha^* \triangleleft H_{\alpha+1}^*$. It is easy to see that for limit ordinals λ $H_\lambda^* = \bigcup_{\alpha<\lambda} H_\alpha^*$. Therefore we have an ascending series

$$0 = H_0^* \triangleleft H = H_1^* \triangleleft H_2^* \triangleleft \dots \triangleleft H_\alpha^* \triangleleft \dots H_\sigma^* = H^*.$$

We show by induction on β that there exists an ascending $\mathfrak{A}$-series from H_β^* to $H_{\beta+1}^*$. Now $H_1^* = H \in \mathfrak{A}$ so let $\beta > 0$ and suppose the assertion is true for all ordinals $< \beta$. Now clearly $(H_\beta^*)^{H_{\beta+1}} = H_{\beta+1}^*$ so

$$H_{\beta+1}^*/H_\beta^* = \Sigma\ (H_\beta^* + [H_\beta^*, x_1, \dots, x_n])/H_\beta^*$$

summed over all possible sequences $x_1, \dots, x_n \in H_{\beta+1}$. Now $L \in L\mathfrak{N}$ and the characteristic is zero, so as in section 1.2 we may define $e(x) = \exp(\mathrm{ad}(x))$ for any $x \in L$. By Hartley [14] lemma 3 p.262 we find that

$$H_{\beta+1}^*/H_\beta^* = \Sigma\ (H_\beta^* + H_\beta^{*\,e(x_1)\dots e(x_n)})/H_\beta^*.$$

Hence there is an ascending series of ideals between H_β^* and $H_{\beta+1}^*$ of which a typical factor is

$$(H_\beta^{*e} + M)/M$$

where $e = e(x_1)\ldots e(x_n)$ is an automorphism of L, $x_i \in H_{\beta+1}$ all i, and $M \lhd H_{\beta+1}{}^*$.

Let $N \lhd H_{\beta+1}{}^*$. By induction there is an ascending $\mathfrak{A}$-series from 0 to $H_\beta{}^*$. Consider the series obtained from this by adding N to each term. A typical factor is of the form $(Y+N)/(X+N)$ where $X < Y \leq H_\beta{}^*$ and $Y/X \in \mathfrak{A}$. Therefore $(Y+N)/(X+N) \in \mathfrak{A}$, and there is an ascending $\mathfrak{A}$-series from N to $H_\beta{}^*+N$. Let $N = M^{e^{-1}}$ and transform by e to get an ascending $\mathfrak{A}$-series from M to $H_\beta{}^{*e}+M$. This establishes the assertion about $H_{\beta+1}{}^*/H_\beta{}^*$. Fitting all these 'subseries' together gives us an ascending $\mathfrak{A}$-series from 0 to $\langle H^L \rangle \lhd L$. Either the quotient $L/\langle H^L \rangle = 0$ or it has nontrivial Gruenberg radical and we can continue the process. Eventually we obtain an ascending $\mathfrak{A}$-series for L.

Thus (2) implies (1). That (1) implies (2) is manifest.

Since $\mathfrak{Gr}$ is Q-closed and contained in $L\mathfrak{N}$ the particular case follows.

Chapter Eight

The existence or otherwise of infinite-dimensional abelian subalgebras

An old problem in group theory is:

Does every infinite group possess an infinite abelian subgroup?

Novikov and Adyan, in their recent work on the Burnside problem, have shown that the answer is in the negative ([27] p.1190 theorem 3); but Hall and Kulatilaka [13] have produced an affirmative answer for locally finite groups. Kulatilaka [22] has also obtained results when certain restrictions are placed on the nature of the required abelian subgroup (e.g that it be subnormal).

In this chapter we consider the analogous problem for Lie algebras:

Does every infinite-dimensional Lie algebra have an infinite-dimensional abelian subalgebra?

First we show that the answer is in general 'no'. Next we obtain analogues of Kulatilaka's results for certain 'generalised soluble' classes of Lie algebras.

Finally we prove the analogue of the Hall-Kulatilaka theorem for $L\mathfrak{F}$ Lie algebras, and deduce a few corollaries.

8.1 A negative result

It is convenient to turn the problem upside-down. Suppose Δ is any of the relations $\leq$, $\triangleleft$, $\triangleleft^{\alpha}$, si , asc. We will say L satisfies Fin-$\Delta\mathfrak{A}$ if and only if $A \,\Delta\, L$ and $A \in \mathfrak{A}$ implies $A \in \mathfrak{F}$. (Instead of Fin-$\leq\mathfrak{A}$ we write Fin-$\mathfrak{A}$). We use the same notation for the class of Lie algebras satisfying the condition.

Clearly if $\mathfrak{X}$ is a class of Lie algebras then the following assertions are equivalent:

1) Every infinite-dimensional $\mathfrak{X}$-algebra L has an infinite dimensional abelian subalgebra $A \,\Delta\, L$.

2) $\mathfrak{X} \cap \text{Fin-}\Delta\mathfrak{A} \leq \mathfrak{F}$.

It is in the second form that we shall state our results.

Theorem 8.1.1

$$\text{Fin-}\mathfrak{A} \not\leq \mathfrak{F} .$$

Proof:

Let L be a free Lie algebra on more than 1 generator. By Witt [43] any subalgebra of L is free. But the only abelian free Lie algebras are of dimension ≤ 1. Thus $L \in \text{Fin-}\mathfrak{A}$. Clearly $L \notin \mathfrak{F}$.

8.2 Generalised Soluble Classes

Let Δ be any of the relations above. We define the class $\acute{E}(\Delta)\mathfrak{A}$ to consist of all Lie algebras L having an ascending $\mathfrak{A}$-series $(L_\alpha)_{\alpha\leq\sigma}$ such that $L_\alpha \; \Delta \; L$ for all $\alpha \leq \sigma$.

(Thus $\acute{E}(\leq)\mathfrak{A} = \acute{E}(\text{asc})\mathfrak{A} = \acute{E}\mathfrak{A}$; $\acute{E}(\triangleleft)\mathfrak{A}$ and $\acute{E}(\text{si})\mathfrak{A}$ are respectively the classes $\mathfrak{V}$, $\mathfrak{W}$ of chapter 5.

Lemma 8.2.1

Let $0 \neq N \triangleleft L \in \mathfrak{Z}$. Then $N \cap \zeta_1(L) \neq 0$.

Proof:

Let α be the first ordinal such that $N \cap \zeta_\alpha(L) \neq 0$. Then $N \cap \zeta_\alpha(L) \leq N \cap \zeta_1(L)$.

Lemma 8.2.2

If A is a maximal abelian ideal of $L \in \mathfrak{Z}$ then $A = C_L(A)$.

Proof:

Suppose $A < C = C_L(A)$. $L/A \in \mathfrak{Z}$ and $0 \neq C/A \triangleleft L/A$ so by lemma 8.2.1 there exists $x \notin A$, $x+A \in C/A \cap \zeta_1(L/A)$. Then $A + \langle x\rangle \in \mathfrak{A}$, $A + \langle x\rangle \triangleleft L$, contrary to the maximality of A.

Theorem 8.2.3

$\mathfrak{Z} \cap \text{Fin-}\triangleleft\mathfrak{A} \leq \mathfrak{F}$.

Proof:

Let $L \in \mathfrak{Z} \cap \text{Fin-}\triangleleft\mathfrak{A}$. Take a maximal abelian ideal A of L (exists by Zorn). Then $A \in \mathfrak{F}$, and by lemma 8.2.2 $A = C_L(A)$. By lemma 4.2.1 $L/A \in \mathfrak{F}$. Thus $L \in \mathfrak{F}$ as required.

Theorem 8.2.4

$$\acute{E}(\triangleleft)\mathfrak{A} \cap \text{Fin-}\triangleleft^2\mathfrak{A} \leq \mathfrak{F} .$$

Proof:

Let $L \in \acute{E}(\triangleleft)\mathfrak{A} \cap \text{Fin-}\triangleleft^2\mathfrak{A}$ and suppose if possible that $L \notin \mathfrak{F}$. L has an ascending $\mathfrak{A}$-series $(L_\alpha)_{\alpha \leq \sigma}$ with $L_\alpha \triangleleft L$ $(\alpha \leq \sigma)$.

Suppose first that for some finite n $L_n \notin \mathfrak{F}$ but $L_{n-1} \in \mathfrak{F}$. Let $H = C_{L_n}(L_{n-1})$. By lemma 4.2.1 $L_n/H \in \mathfrak{F}$. $H^2 \leq L_{n-1}$ and $[H, L_{n-1}] = 0$ so $H \in \mathfrak{N}_2 \leq \mathfrak{Z}$. $H \triangleleft L \in \text{Fin-}\triangleleft^2\mathfrak{A}$ so $H \in \text{Fin-}\triangleleft\mathfrak{A}$. By theorem 8.2.3 $H \in \mathfrak{F}$. Thus $L_n \in \mathfrak{F}$, a contradiction.

Consequently we may assume that $L_n \in \mathfrak{F}$ for all $n < \omega$, $L_\omega \notin \mathfrak{F}$. Suppose $H_m \in \mathfrak{N} \cap \mathfrak{F}$, $H_m \leq L_\omega$, $H_m \triangleleft L$. Then $C_m = C_{L_\omega}(H_m) \triangleleft L$, and $C_m \notin \mathfrak{F}$. Therefore there exists a first $n = n(m)$ such that $C_m{}^* = L_n \cap C_m \nleq H_m$. $C_m{}^* \triangleleft L$ and $C_m{}^* \in \mathfrak{F}$. Let $H_{m+1} = H_m + C_m{}^*$. Then $C_m{}^{*2} \leq L_{n-1} \cap C_m \leq H_m$ so $C_m{}^* \in \mathfrak{N}_2$. Thus $H_{m+1} \in \mathfrak{N} \cap \mathfrak{F}$, and $H_m < H_{m+1}$. Let $H_1 = L_1$ and set $H = \bigcup_{m=1}^{\infty} H_m \triangleleft L$. $H \notin \mathfrak{F}$. $H = \langle H_{m+1}, C_{m+1}{}^*, C_{m+2}{}^*, \ldots \rangle$

and C_{m+k}^* centralises H_{m+1} for all $k \geq 1$, so $[H_m, H_{m+1}] \leq H_{m+1}^2$. Since $C_m^{*2} \leq H_m$ and $[H, C_m^*] = 0$ we have $H_{m+1}^2 \leq H_m$. Thus $[H, H_{m+1}] \leq H_m$, and H has an ascending central series. Thus $H \in \mathfrak{Z}$. $H \lhd L \in \text{Fin-}\lhd^2\mathfrak{A}$ so $H \in \text{Fin-}\lhd\mathfrak{A}$. Thus $H \in \mathfrak{F}$, a contradiction.

Corollary

$$\mathfrak{H} \cap \text{Fin-}\lhd^2\mathfrak{A} \leq \mathfrak{F}.$$

Proof:

If $L \in \mathfrak{H}$ then there exists $N \lhd L$, $N \in \mathfrak{N}$. Then $\zeta_1(N) \lhd L$ and lies in $\mathfrak{A}$. The quotient by this also lies in $\mathfrak{H}$ so we may repeat the argument to get $\mathfrak{H} \leq \acute{E}(\lhd)\mathfrak{A}$. Now use theorem 8.2.4.

We shall extend our definition of the class $\mathfrak{B}$ to fields of characteristic $\neq 0$ as follows: $L \in \mathfrak{B}$ if and only if $L \in L\mathfrak{N}$ and $x \in L$ implies $\langle x\rangle$ si L. This clearly does not conflict with earlier usage.

Theorem 8.2.5

$$\mathfrak{B} \cap \text{Fin-si}\mathfrak{A} \leq \mathfrak{F}.$$

Proof:

Let $L \in \mathfrak{B} \cap \text{Fin-si}\mathfrak{A}$. Suppose $0 \neq x \in L$. $L \in \mathfrak{B}$ so $\langle x\rangle$ si L. Let

$$\langle x\rangle = L_0 \lhd L_1 \lhd \ldots \lhd L_n = L$$

be the ideal closure series of $\langle x\rangle$ in L. We show by induction on i that $L_i = \zeta_{m(i)}(L_i) \in \mathfrak{N} \cap \mathfrak{F}$ $(0 \leq i \leq n-1)$.

$\langle x \rangle$ is a minimal ideal of $L_1 \in \mathfrak{B} \leq L\mathfrak{N}$ so by lemma 3.3.3 $\langle x \rangle \leq \zeta_1(L_1)$ ch $L_1 \triangleleft L_2$. By the definition of ideal closure series $L_1 = \zeta_1(L_1)$. L_1 is an abelian subideal of L so $L_1 \in \mathfrak{F}$. Now suppose the assertion true for i-1. Thus $L_{i-1} = \zeta_{m(i-1)}(L_{i-1}) \in \mathfrak{N} \cap \mathfrak{F}$. $L_{i-1} \triangleleft L_i \in L\mathfrak{N}$, so by lemma 3.3.4 $L_{i-1} \leq \zeta_{m(i)}(L_i)$ ch $L_i \triangleleft L_{i+1}$. By the definition of ideal closure series $L_i = \zeta_{m(i)}(L_i)$. L_i si L so $L_i \in \mathfrak{N} \cap$ Fin-si$\mathfrak{A} \leq \mathfrak{F}$ by theorem 8.2.3. Thus the induction step goes through, and $\langle x^L \rangle = L_{n-1} \in \mathfrak{N}$. Thus $L \in \mathfrak{F}t \cap$ Fin-si$\mathfrak{A} \leq \mathfrak{F}$ by the corollary to theorem 8.2.4.

<u>Theorem 8.2.6</u>

$\acute{E}(si)\mathfrak{A} \cap$ Fin-si$\mathfrak{A} \leq \mathfrak{F}$ for fields of characteristic zero.

Proof:

Let $L \in \acute{E}(si)\mathfrak{A} \cap$ Fin-si$\mathfrak{A}$, having an ascending $\mathfrak{A}$-series $(L_\alpha)_{\alpha \leq \sigma}$ with L_α si L $(\alpha \leq \sigma)$. Let $B = \beta(L) \neq 0$. $B \triangleleft L$ (lemma 7.2.1) so $B \in$ Fin-si$\mathfrak{A}$. $B \in \mathfrak{L}$ by definition, so by theorem 8.2.5 $B \in \mathfrak{F}$. Thus $B \in \mathfrak{N}$, so that $Z = \zeta_1(B) \triangleleft L$ and $0 \neq Z \in \mathfrak{A}$.

$L/Z \in \acute{E}(si)\mathfrak{A}$. Suppose A/Z si L/Z, $A/Z \in \mathfrak{A}$. Then A si L and $A \in \mathfrak{A}^2 \cap$ Fin-si$\mathfrak{A} \leq \mathfrak{F}$ by theorem 8.2.4. Thus $L/Z \in$ Fin-si$\mathfrak{A}$. We may therefore repeat the argument, until either we show $L \in \mathfrak{F}$ or we find an infinite-dimensional $\acute{E}(\triangleleft)\mathfrak{A}$-subalgebra $W \triangleleft L$. Then $W \in \acute{E}(\triangleleft)\mathfrak{A} \cap$ Fin-si $\mathfrak{A}$ so by theorem 8.2.5 $W \in \mathfrak{F}$ contradiction. Hence $L \in \mathfrak{F}$.

The obvious theorem to complete the hierarchy:

Theorem 8.2.7

Over fields of characteristic zero,
$\acute{E}\mathfrak{A} \cap$ Fin-asc $\mathfrak{A} \leq \mathfrak{F}$.

Proof:

Let $L \in \acute{E}\mathfrak{A} \cap$ Fin-asc $\mathfrak{A}$. Let $(L_\alpha)_{\alpha \leq \sigma}$ be an ascending $\mathfrak{A}$-series of L. If $L_n \notin \mathfrak{F}$ for some $n < \omega$ then $L_n \in E\mathfrak{A} \cap$ Fin-asc $\mathfrak{A} \leq \mathfrak{F}$ by theorem 8.2.4, a contradiction. Thus we may assume $L_n \in \mathfrak{F}$ if $n < \omega$, and $L = L_\omega \notin \mathfrak{F}$.

Let $F_n = \nu(L_n) \in \mathfrak{H}$. F_n asc L so $F_n \in$ Fin-asc $\mathfrak{A}$ so by corollary to theorem 8.2.4 $F_n \in \mathfrak{F}$. Therefore $F_n \in \mathfrak{N} \cap \mathfrak{F}$. F_n ch L_n by lemma 7.2.1 and $L_n \triangleleft L_{n+1}$ so $F_n = L_n \cap F_{n+1} \triangleleft F_{n+1}$. Let $F = \bigcup_{n=1}^{\infty} F_n \triangleleft L$ (since each element of L_n idealises F_{n+k} for all $k \geq 0$.)

Suppose if possible $F \in \mathfrak{F}$. Then $C = C_L(F) \notin \mathfrak{F}$ since $L \notin \mathfrak{F}$, so for some n $C_{L_n}(F) = C \cap L_n \nleq F_n$. $L_n \in E\mathfrak{A}$ and $C_{L_n}(F_n) \nleq F_n$ which contradicts lemma 6.1.1.2.

Hence $F \notin \mathfrak{F}$. Clearly $F \in L\mathfrak{N}$, so without loss of generality $L \in L\mathfrak{N}$. $\zeta_r(L_n) = L_n$ for some

$r = r(n)$. Let $Z = \langle \zeta_r(L_n) : n = 1,2,\ldots \rangle$. $Z_r \leq Z_{r+1}$ and $L = \bigcup_{r=1}^{\infty} Z_r$. Let $x \in \zeta_r(L_n)$, $y \in \zeta_r(L_m)$ where $m \leq n$. Then $[x,y]$ lies in $\zeta_{r-1}(L_n)$ so $Z_r^2 \leq Z_{r-1}$. Thus (Z_r) forms an ascending $\mathfrak{A}$-series for L. Z_1 asc L and $Z_1 \in \mathfrak{A}$ so $Z_1 \in \mathfrak{F}$. Consequently $Z_1 \leq L_k$ for some k so that $\zeta_1(L_n) \leq L_k$ for all n. Thus $0 \neq \zeta_1(L_k) \geq \zeta_1(L_{k+1}) \geq \ldots$ so that $Y = \bigcap_{r=1}^{\infty} \zeta_1(L_{k+r}) \neq 0$. Clearly $Y = \zeta_1(L)$.

Let $H = \bigcup_{\alpha} \zeta_{\alpha}(L) \in \mathfrak{Z}$. From theorem 8.2.3 $H \in \mathfrak{F}$, so $H \in \mathfrak{N}$. Suppose A/H asc L/H, $A/H \in \mathfrak{A}$. Then $A \in \mathfrak{N}$ and satisfies Fin-asc$\mathfrak{A}$ so by theorem 8.2.3 $A \in \mathfrak{F}$. Thus $L/H \in$ Fin-asc$\mathfrak{A}$. By the above reasoning, L/H has non-trivial centre, contrary to the definition of H. This contradiction establishes the theorem.

Corollary

For fields of characteristic zero,

$$\mathfrak{Gr} \cap \text{Fin-asc}\mathfrak{A} \leq \mathfrak{F}.$$

Proof:

Use lemma 7.6.1.

8.3 Locally finite algebras

In this section we prove the Lie-theoretic version of the theorem of Hall and Kulatilaka:

Theorem 8.3.1

Over fields of characteristic zero

$$L\mathfrak{F} \cap \text{Fin-}\mathfrak{A} \leq \mathfrak{F}.$$

The proof begins by following Hall and Kulatilaka, but parts company as soon as things start to get interesting.

Let $\mathfrak{Q}$ denote the class of all Lie algebras L such that either $L \in \mathfrak{F}$ or L has an infinite-dimensional abelian subalgebra. Let $\mathfrak{R}$ denote the class of Lie algebras L such that $L \in \mathfrak{F}$ or there exists $x \in L$ with $C_L(x) \notin \mathfrak{F}$ and $x \neq 0$.

Lemma 8.3.2

Suppose $\mathfrak{X} = QS\mathfrak{X}$ is a class of Lie algebras. Then $\mathfrak{X} \leq \mathfrak{Q}$ if and only if $\mathfrak{X} \leq \mathfrak{R}$.

Proof:

$\mathfrak{Q} \leq \mathfrak{R}$ so one implication is clear. Suppose now that $L \in \mathfrak{X} \leq \mathfrak{R}$, $L \notin \mathfrak{F}$. Consider the set $\mathcal{B}$ of all finite-dimensional abelian subalgebras A of L for which $C_L(A) \notin \mathfrak{F}$. $\mathcal{B} \neq \emptyset$ since $0 \in \mathcal{B}$.

Suppose $A \in \mathcal{B}$. Then $A \lhd C = C_L(A)$ and $C/A \notin \mathfrak{F}$. $C/A \in QS\mathfrak{X} = \mathfrak{X} \leq \mathfrak{R}$, so there exists $x \in C \setminus A$ such that $D/A = C_{C/A}(A+x) \notin \mathfrak{F}$. For all $d \in D$ $[d,x] \equiv 0 \pmod{A}$ so $[D,x] \leq A$. Let $A_1 = A+\langle x\rangle \in \mathfrak{A} \cap \mathfrak{F}$ since $A \in \mathfrak{A}$, $A \lhd C$. $A_1 > A$. $C_1 = C_L(A_1) = C_D(x)$.

Let $V = [D,x]$ qua vector space, and consider the linear map $\lambda: D \rightarrow V$ defined by $d\lambda = [d,x]$ $(d \in D)$. $\ker(\lambda) = C_1$, $\operatorname{im}(\lambda) = V \leq A \in \mathfrak{F}$. Thus $\dim(D/C_1) < \infty$

so $C_1 \notin \mathfrak{F}$. Thus $A_1 \in \mathcal{B}$.

We have shown that $\mathcal{B}$, ordered by inclusion, has no maximal element. Take a properly ascending chain $A_1 < A_2 < \ldots$ of elements of $\mathcal{B}$. The union is infinite-dimensional and abelian. Thus $L \in \mathfrak{Q}$ as required.

<u>Lemma 8.3.3</u>

Suppose $L \in (L\mathfrak{F} \cap LE\mathfrak{A}) \setminus \mathfrak{R}$. Then there exists $H \leq L$, $H \in \mathfrak{F}$, such that $C_L(H^2) = 0$.

Proof:

We show that if $F \in \mathfrak{F}$, $F \leq L$, then there exists $F^* \leq L$, $F^* \in \mathfrak{F}$, such that $C_L(F^{*2}) < C_L(F^2)$.

Suppose $L^2 \in \mathfrak{F}$. Since $L \in LE\mathfrak{A}$ $L^2 \in E\mathfrak{A}$ so $L \in E\mathfrak{A}$. By theorem 8.2.4 $L \in \mathfrak{Q}$. $E\mathfrak{A}$ is QS-closed so by lemma 8.3.2 $L \in \mathfrak{R}$, a contradiction. Consequently L^2 is infinite-dimensional.

Let $c \in C_L(F^2) \in \mathfrak{F}$ since $L \notin \mathfrak{R}$. Then $C_L(c) \in \mathfrak{F}$ so there exists $x \in L^2 \setminus C_L(c)$. For some $x_i, y_i \in L$ $x = [x_1, y_1] + \ldots + [x_m, y_m]$. Let

$$F^* = \langle F, x_1, \ldots, x_m, y_1, \ldots, y_m \rangle$$

which is in $\mathfrak{F}$ by local finiteness of L. Now

$$\begin{aligned} C_L(F^{*2}) &\leq C_L(F^2) \cap C_L([x_1,y_1] + \ldots + [x_m,y_m]) \\ &\leq C_L(F^2) \setminus \langle c \rangle \\ &< C_L(F^2) \end{aligned}$$

as claimed. The conclusion of the lemma follows.

Corollary 1

$L(\mathfrak{N}\mathfrak{A}) \leq \mathfrak{Q}$. In particular $L\mathfrak{N} \leq \mathfrak{Q}$.

Proof:

Let $L \in L(\mathfrak{N}\mathfrak{A})$. If $L \notin L\mathfrak{F}$ then there exists an infinite-dimensional $\mathfrak{N}\mathfrak{A}$-subalgebra of L. $\mathfrak{N}\mathfrak{A} \leq E\mathfrak{A}$ so by theorem 8.2.4 L has an infinite-dimensional abelian subalgebra. Now suppose $L \in L\mathfrak{F} \setminus \mathfrak{Q}$. By lemma 8.3.2 $L \notin \mathfrak{R}$. By lemma 8.3.3 There exists $H \leq L$, $H \in \mathfrak{F}$, with $C_L(H^2) = 0$. $H \in \mathfrak{N}\mathfrak{A}$ so $H^2 \in \mathfrak{N}$. $H^2 \neq 0$ (or else $C_L(H^2) = L$) so $\zeta_1(H^2) \neq 0$ and $C_L(H^2) \neq 0$ contradiction.

Corollary 2

Over fields of characteristic zero, $LE\mathfrak{A} \leq \mathfrak{Q}$.

Proof:

If $L \in LE\mathfrak{A}$ is not in $L\mathfrak{F}$ proceed as above. If $L \in L\mathfrak{F}$ then $L \in L(\mathfrak{F} \cap E\mathfrak{A}) \leq L(\mathfrak{N}\mathfrak{A})$ by Jacobson [17] p.51.

We note that the Mal'cev correspondence now enables us to assert

Theorem 8.3.4

Let G be a complete locally nilpotent torsion-free group of infinite rank. Then

1) G has an abelian subgroup of infinite rank.

2) If G is a Gruenberg group it has an infinite-rank abelian ascendant subgroup.

3) If G is a Baer group it has an abelian subnormal subgroup of infinite rank.

4) If G is a Fitting group it has an abelian subnormal subgroup of defect ≤ 2 of infinite rank.

5) If G is a ZA-group it has an abelian normal subgroup of infinite rank.

To prove theorem 8.3.1 we need a lemma about Cartan subalgebras, which is given as an exercise in Jacobson [17] p.149 ex.3. The lemma (for which we have provided a proof) is as follows:

<u>Lemma 8.3.5</u>

Let L, L* be semisimple Lie algebras over a field of characteristic zero, and suppose $L \leq L^*$. Let H be a Cartan subalgebra of L. Then there exists a Cartan subalgebra H* of L* with $H \leq H^*$.

Proof:

(For unexplained terminology see Jacobson [17] or Carter [4]).

L* is an L-module in the natural fashion. L is semisimple, and the theorem of complete reducibility (Jacobson [17] p.79 theorem 8) implies that L* is a direct sum of irreducible L-modules. Each of these is also an H-module. By Carter [4] p.70 theorem 24 every irreducible L-module is a direct sum of 1-dimensional H-submodules. Thus

$$L^* = V_1 \oplus \dots \oplus V_t$$

where each V_i is a 1-dimensional H-module. Thus if $v \in V_i$, $h \in H$, we must have $[v,h] = \lambda_i(h)v$ where $\lambda_i(h)$ lies in the field k. We collect together those V_i for which λ_i = a given λ, and let their sum be W_λ. Thus

$$L^* = W_0 \oplus W_{\lambda_1} \oplus \dots \oplus W_{\lambda_r}.$$

Clearly W_λ is the weight-space for H with weight λ. It is shown in Jacobson [17] p.64 that

$$\begin{aligned}[W_\lambda, W_\mu] &\leq W_{\lambda+\mu} \quad \text{if } \lambda+\mu \text{ is a weight,} \\ &= 0 \quad \text{otherwise.}\end{aligned}$$

Thus W_0 is a subalgebra of L^*. H is abelian ([17] p.110) and $H \leq W_0$. If $h \in H$, $w \in W_0$ then by definition of W_0 $[w,h] = 0.w = 0$. Thus $H \leq \zeta_1(W_0)$. Let H^*/H be a Cartan subalgebra for W_0/H. We claim that H^* is a Cartan subalgebra for L^*.

H^* is nilpotent: $H^*/H \in \mathfrak{N}$ by definition, and H is central in H^*, so $H^* \in \mathfrak{N}$.

H^* is self-idealising: suppose $x \in I_{L^*}(H^*)$. Then $x = x_0 + x_{\lambda_1} + \dots + x_{\lambda_r}$ where $x_\lambda \in W_\lambda$. Let $h \in H$. Then $[x,h] \in H^* \leq W_0$. But $[x,h] = \lambda_1(h)x_{\lambda_1} + \dots + \lambda_r(h)x_{\lambda_r}$ which lies in W_0 if and only if $x_{\lambda_1} = \dots = x_{\lambda_r} = 0$

since the deomposition into weight spaces W_λ is a direct sum. Thus $x \in W_0$. Now $[x,H^*] \leq H^*$ so the coset $x+H$ idealises H^*/H, which is a Cartan subalgebra of W_0/H. Thus $x \in H^*$. Consequently H^* is self-idealising.

Thus H^* is a Cartan subalgebra of L^* as required.

We may now prove theorem 8.3.1 in the form $L\mathfrak{F} \leq \mathfrak{Q}$. The proof utilises most of the major results of the classical theory of finite-dimensional Lie algebras!

Let $L \in L\mathfrak{F}$ (over a field k of characteristic zero). Without loss of generality $L = \bigcup_{n=1}^{\infty} L_i$ where $L_i < L_{i+1} \in \mathfrak{F}$ for all i. Let R_i be the soluble radical of L_i. Then $R_i \triangleleft L_i$. $R = \sum_i R_i \in LE\mathfrak{A}$. If $R \notin \mathfrak{F}$ then R (and so L) has an infinite-dimensional abelian subalgebra by lemma 8.3.3 corollary 2. Thus we may assume $R \in \mathfrak{F}$, so $\dim(R_i)$ is bounded. By Jacobson [17] p.91 and p.93 cor 1 there exist semisimple <u>Levi factors</u> S_i such that

1) $L_i = R_i \oplus S_i$, $R_i \cap S_i = 0$,

2) $S_i \leq S_{i+1}$.

Since $\dim(R_i)$ is bounded but $L \notin \mathfrak{F}$,

$$\dim(S_i) \text{ is unbounded.} \qquad (*)$$

Thus without loss of generality $L = \bigcup_{i=1}^{\infty} S_i$.

Let C_i be a Cartan subalgebra of S_i. Using lemma 8.3.5 we may arrange matters so that $C_i \leq C_{i+1}$ for all i. $C_i \in \mathfrak{A}$ ([17] p.110) so that $C = \bigcup_{i=1}^{\infty} C_i \in \mathfrak{A}$. If $C \notin \mathfrak{F}$ then the theorem follows. Thus we may assume (for a contradiction) that

$$\dim(C_i) \leq c \quad \text{for all } i.$$

Suppose now that S is a semisimple Lie algebra over a field k of characteristic zero, H a Cartan subalgebra of S. Let dim(S) = s, sim(H) = h. Let k* be the algebraic closure of k, and denote the algebras over k* corresponding to H, S by H*, S* (formed by taking tensor products with k*). S* is semisimple ([17] p.70) and H* is a Cartan subalgebra of S* ([17] p.61). Clearly $\dim_{k^*}(S^*) = s$ and $\dim_{k^*}(H^*) = h$.

By [17] p.71

$$S^* = J_1 \oplus \ldots \oplus J_m$$

where each J_i is a classical simple Lie algebra. If H_i is a Cartan subalgebra of J_i then clearly $H_1 \oplus \ldots \oplus H_m$ is a Cartan subalgebra of S*. All Cartan subalgebras of S* are conjugate via an automorphism of S* ([17] p.273) so they have the same dimension, and

$$h = h_1 + \ldots + h_m$$

where $h_i = \dim(H_i) > 0$.

Therefore $m \leq h$.

Now the classical simple algebras comprise the following list:

A_ℓ	of dimension	$\ell(\ell+2)$	$(\ell \geq 1)$
B_ℓ	of dimension	$\ell(2\ell+1)$	$(\ell \geq 2)$
C_ℓ	of dimension	$\ell(2\ell+1)$	$(\ell \geq 3)$
D_ℓ	of dimension	$\ell(2\ell-1)$	$(\ell \geq 4)$
G_2	of dimension	14	
F_4	of dimension	52	
E_6	of dimension	78	
E_7	of dimension	133	
E_8	of dimension	248	

where the subscript denotes the dimension of any Cartan subalgebra.

Thus, if $\dim(J_i) = j_i$, by inspection of this list we see that $j_i \leq 4h_i^2 \leq 4h^2$. Therefore $s \leq 4h^3$.

In the original situation, therefore, we deduce that $s \leq 4c^3$ and $\dim(S_i)$ is bounded, contrary to (*).

This completes the proof of theorem 8.3.1.

We may summarise our results about $\mathfrak{Q}$ by stating

Theorem 8.3.6

$\mathfrak{Q}$ is $\{L, \acute{E}\}$-closed, for fields of characteristic zero.

Proof:

Let $L \in L\mathfrak{Q}$. Either L has an infinite-dimensional $\mathfrak{Q}$-subalgebra or $L \in L\mathfrak{F}$. Either way $L \in \mathfrak{Q}$.

Now let $L \in \acute{E}\mathfrak{Q}$. L has an ascending $\mathfrak{Q}$-series $(L_\alpha)_{\alpha \leq \sigma}$. Without loss of generality $L_n < L_{n+1}$ for all finite n and $L = L_\omega \notin \mathfrak{F}$. If $L_{n+1}/L_n \in \mathfrak{F}$ for all n then $L_\omega \in L\mathfrak{F} \leq \mathfrak{Q}$. Otherwise for some first integer n L_{n+1}/L_n contains an infinite-dimensional abelian subalgebra A/L_n, then $A \in \mathfrak{F}\mathfrak{A}$ which is easily seen to lie inside $L\mathfrak{F}$. Thus A has an infinite-dimensional abelian subalgebra and again $L \in \mathfrak{Q}$.

Corollary

$\{L, \acute{E}\}\mathfrak{F} \leq \mathfrak{Q}$. (characteristic zero).

Proof:

$\mathfrak{F} \leq \mathfrak{Q}$ by definition.

Remarks

This is genuinely stronger than theorem 8.3.1 since, unlike group theory, for Lie algebras $L\mathfrak{F}$ is not even E-closed, let alone É-closed. To see this consider the Lie algebra $L = P \oplus Q$ described just before lemma 4.1.1. $P \in \mathfrak{A} \leq L\mathfrak{F}$, and $Q \in \mathfrak{F} \leq L\mathfrak{F}$. But $L \in \mathfrak{G} \setminus \mathfrak{F}$ so $L \notin L\mathfrak{F}$.

Since $\mathfrak{A} \leq \acute{E}\mathfrak{F}$ this result also implies $\{L, \acute{E}\}\mathfrak{A} \leq \mathfrak{Q}$ superseding lemma 8.3.3 corollary 2.

Finally, two deductions from theorem 8.3.1 which are of a rather different nature.

Theorem 8.3.7

Let A be a locally finite associative algebra of infinite dimension over a field k of characteristic zero. Then A has an infinite-dimensional commutative subalgebra. (A is said to be locally finite if every finite subset of A is contained in a finite-dimensional associative subalgebra.)

Proof:

Let L be the associated Lie algebra. Then $L \in L\mathfrak{F}$ and is infinite-dimensional so by theorem 8.3.1 L has an infinite-dimensional abelian subalgebra B. If $b,c \in B$ then $bc-cb = 0$ so $bc = cb$. Thus B generates a commutative subalgebra of A, which contains B so is of infinite dimension.

(This theorem applies in particular to the group algebra kG of a locally finite group G).

Theorem 8.3.8

A locally finite Lie algebra over a field of characteristic zero satisfies the minimal condition for subalgebras if and only if it is finite-dimensional.

Proof:

The implication is easy in one direction. If $L \in L\mathfrak{F} \setminus \mathfrak{F}$ then L has an infinite-dimensional abelian subalgebra by theorem 8.3.1, and clearly this does not satisfy the minimal condition for subalgebras. This contradiction completes the proof.

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