Variants of Gambling in Contests

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To my parents and my grandmother

Declarations

This thesis is submitted to the University of Warwick in support of my application for the degree of Doctor of Philosophy. It has been composed by myself and has not been submitted for a degree at another university.

The work presented in this thesis was carried out by myself, except where stated. Parts of this thesis have been accepted or submitted for publication as follows.

The work in Chapter 2 has been accepted for publication in *Decisions in Economics and Finance* under the title "Gambling in contests modelled with diffusions" with my supervisor Prof. David Hobson as a co-author.

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The work in Chapter 4 has been submitted for publication as a sole author paper.

Abstract

In the Seel-Strack contest (Seel and Strack [2013]), n agents each privately observe an independent copy of a drifting Brownian motion which starts above zero. Each agent chooses when to stop the process she observes, and the winner of the contest is the agent who stops her Brownian motion at the highest value amongst the set of agents. The objective of each agent is to maximise her probability of winning the contest. We will give a new derivation of the results of Seel and Strack [2013] based on a Lagrangian approach. This approach facilitates our analysis of the variants of the Seel-Strack problem.

We will consider a generalisation of the Seel-Strack contest in which the observed processes are independent copies of some time-homogeneous diffusion. We will use a change of scale to reduce this contest to a contest in which the observed processes are diffusions in natural scale. It turns out that, unlike in the Seel-Strack problem, the way of breaking ties becomes important.

Moreover, we will discuss an extension of the Seel-Strack contest to one in which an agent is penalised when her strategy is suboptimal, in the sense that her chosen strategy does not win the contest, but there existed an alternative strategy which would have resulted in victory. We will see that different types of penalty have different effects.

Seel and Strack [2013] studied the asymmetric 2-player contest in which the observed processes start from different constants. We will redrive their results using the Lagrangian method and then study a general asymmetric n-player contest. We will find that some results in the 2-player contest do not hold for the general n-player contest.

In a symmetric 2-player contest, the Seel-Strack model assumes that the observed processes start from the same positive constant. We will extend the results to the case where the starting values of the processes are independent non-negative random variables that have the same distribution.

Chapter 1

Introduction

Seel and Strack [2013] introduces a model of a gambling contest between agents in which each agent privately observes a stochastic process and chooses a stopping time to produce a stopped value. The agent wins the contest if her stopped value is greater than the stopped values of the other agents, and the other agents get nothing. The objective of each agent is not to maximise the expected value of the process, but rather to maximise the probability that her stopped value is the highest amongst the set of agents. Moreover, an agent has to stop if she goes bankrupt, that is when her process hits zero.

The Seel-Strack model investigates the contests in which contestants are unable to observe their rivals and make decisions based only on their own progress. It provides a stylised model for a competition between fund managers. In the competition, each manager wants to outperform the others, only the most successful of them will be given funds to invest over the next time period. In many cases, the manager cannot infer the decisions of other managers. Another distinct strand of the literature on modelling a competition between fund managers is represented by Basak and Makarov [2014].

Although the problem described in Seel and Strack [2013] is very simple, the solution is remarkably rich and subtle. Firstly, in equilibrium, agents must use randomised strategies, so that the level at which the agent should stop is stochastic. Secondly, the set of values at which the agent should stop forms an interval which is bounded above. Several variants are discussed in Seel and Strack [2013], including the extension to the asymmetric case where the starting values of the processes observed by the agents are different.

Besides Seel and Strack [2013], there are many other articles which have discussed contest models. Compared with the Seel-Strack model, one class of contest models assumes that each contestant can observe the performance of all contestants at all points in time. This contains the war of attrition models (see *e.g.* Hendricks et al. [1988]; Bulow and Klemperer [1999]). In a war of attrition, each contestant must choose a time at which she concedes in the event that other contestants have not already conceded. The return to conceding decreases with time, but, at any time, a contestant earns a higher return if other contestants concede first. Unlike the war of attrition models, the Seel-Strack model has no cost over time, but there is a bankruptcy constraint and the probability of bankruptcy is assumed to be strictly positive.

Another class of contest models assumes that the optimal strategy does not depend on information which arrives after the start of the contest, which is different from the Seel-Strack model. This class includes the all-pay auction models (see *e.g.* Hillman and Samet [1987]; Baye et al. [1996]). In an all-pay auction, each bidder submits a non-negative sealed bid simultaneously, all bidders pay their bids, and a prize is awarded to the highest bidder. We will see that the symmetric Nash equilibrium of a Seel-Strack contest is quite similar to that of a common-value all-pay auction with complete information and the stopped value of the process corresponds to the auction bid. The comparison between a Seel-Strack contest and an all-pay auction will be discussed in Section 1.3.

1.1 The Seel-Strack model

Now we introduce the mathematical model of the contest introduced in Seel and Strack [2013].

There are n players with labels $i \in I = \{1, 2, \dots, n\}$ who take part in the contest. Player i privately observes the continuous-time realisation of a stochastic process $X^i = (X^i_t)_{t\in\mathbb{R}^+}$ absorbed at zero with $X^i_0 = x_0$, where x_0 is a positive constant which is the same for all players. We assume for simplicity that X^i is a Brownian motion in this chapter. In fact, Seel and Strack considered the case where X^i is a Brownian motion with drift and scaling, and in Chapter 2 we will see that this can be reduced to the case where X^i is a Brownian motion.

Let $\mathcal{F}^i_t = \sigma(\{X^i_s : s \leq t\})$ and set $\mathbb{F}^i = (\mathcal{F}^i_t)_{t\geq 0}$. The space of strategies for agent i is the space of $\mathbb{F}^i$-stopping times τ^i. Since zero is absorbing for X^i, without loss of generality we may restrict attention to $\tau^i \leq H^i_0$, where H^i_0 is the first time at which X^i hits 0 and is defined via

$$H^i_x = \inf\{t \geq 0 : X^i_t = x\} \text{ for any } x \in \mathbb{R}.$$

Player i observes her own process X^i, but not X^j for $j \neq i$; nor does she observe the stopping times chosen by the other players. Moreover, the processes X^i are assumed to be independent.

The player who stops at the highest value wins a prize. We normalize the prize to one without loss of generality. So $\forall i \in I$, player i wins 1 if she stops at time τ^i such that $X^i_{\tau^i} > X^j_{\tau^j}$ $\forall j \neq i$. If there are k players who stop at the equal highest value, then these players each win $\frac{1}{k}$. Therefore, player i with stopping value $X^i_{\tau^i}$ receives pay-off

$$\frac{1}{k}\mathbf{1}_{\{X^i_{\tau^i}=\max_{j\in I} X^j_{\tau^j}\}},$$

where $k = \left|\left\{i \in I : X^i_{\tau^i} = \max_{j\in I} X^j_{\tau^j}\right\}\right|$. Here $\mathbf{1}_E$ denotes the indicator function of the event E, that is $\mathbf{1}_E$ is a random variable that takes value 1 when E happens and value 0 when it does not happen.

The key insight of Seel and Strack [2013] is to observe that the problem of choosing the optimal stopping time can be reduced to a problem of finding the optimal law for $X^i_{\tau^i}$ or equivalently an optimal target distribution. The pay-offs to the agents only depend upon τ^i via the distribution of $X^i_{\tau^i}$. Hence, the problem can be considered in two stages, firstly find an optimal target distribution F^i, and then verify that there is a choice of τ^i such that $X^i_{\tau^i}$ has law F^i. We focus on the first stage.

The problem of finding τ such that X_τ has law F is a classical problem in probability theory, and is known as the Skorokhod embedding problem (Skorokhod [1965]). Since X is a Brownian motion started at x_0 and absorbed at 0, any distribution on $\mathbb{R}^+$ with mean less than or equal to x_0 can be embedded with a finite stopping time τ (and conversely, for any τ the law of X_τ has mean less than or equal to x_0). Note that there are multiple solutions to the Skorokhod embedding problem for F. All the solutions can be used to construct an optimal stopping strategy, and these strategies will bring equal probability of success to an agent.

Notice that the distribution F of $X^i_{\tau^i}$ should satisfy that F is a distribution function such that $\mathbb{P}(X^i_{\tau^i} < 0) = 0$ and $\mathbb{E}(X^i_{\tau^i}) \leq x_0$, which can be shown by the fact that X^i is a non-negative supermartingale and using Fatou's lemma. We say that such a distribution function F is feasible.

Our aim is to find Nash equilibria for the problem. By the above remarks, a Nash equilibrium can be identified with a family of feasible distribution functions $(F^i)_{i\in I}$. Throughout the thesis, we will say that $(F^i)_{i\in I}$ is a Nash equilibrium if F^i is feasible $\forall i \in I$ and if, for each $i \in I$, if the other agents use stopping rules τ^j such that $X^j_{\tau^j} \sim F^j$, then the optimal target distribution for agent i is F^i, and she may use any stopping rule τ^i such that $X^i_{\tau^i} \sim F^i$. Throughout the thesis, we will say a Nash equilibrium is symmetric if F^i does not depend on i, and we will say that a Nash equilibrium is atom-free if each F^i is atom-free.

Given the symmetry of the situation in the sense that each agent observes a martingale process started from the same level x_0, it seems natural that a Nash equilibrium is symmetric. Moreover, arguments over rearranging mass can be used to show that it is never optimal for agents to put mass at the same positive point.

Theorem 1.1.1. *Suppose a Nash equilibrium is symmetric, then it is atom-free.*

Proof. Since the Nash equilibrium is symmetric, it is identified with a feasible distribution function $F(x)$. Introduce $\theta = 1/n$, and this helps us to extend the proof to more general cases. Notice that $\theta \in [0,1)$ and θ represents the way of breaking ties.

(i) Assume that $F(x)$ places an atom of size $p > 0$ at $z > 0$. Let

$$G(x) = \begin{cases} F(x), & \text{if } x \in [0, z-\epsilon_1) \cup [z+\epsilon_2, \infty), \\ F(x) + q, & \text{if } x \in [z-\epsilon_1, z), \\ F(x) - (p-q), & \text{if } x \in [z, z+\epsilon_2), \end{cases}$$

where $\epsilon_2 \in \left(0, \frac{(1-\theta)p^{n-1}z}{1+\theta p^{n-1}}\right)$, $\epsilon_1 \in \left(\frac{(1+\theta p^{n-1})\epsilon_2}{(1-\theta)p^{n-1}}, z\right)$ and $q = \frac{\epsilon_2 p}{\epsilon_1+\epsilon_2} \in (0,p)$. Observe that $(z-\epsilon_1)q + (z-\epsilon_2)(p-q) = zp$, which means that F and G have the same mean. And this implies that G is a feasible distribution function.

Suppose that $X^i_{\tau^i} \sim F$ for any $i \neq 1$. Let

$$\varphi(x) = \mathbb{P}\left(\max_{i\neq 1} X^i_{\tau^i} < x\right).$$

Let V_F and V_G denote the expected pay-off of player 1 if player 1 chooses F and G, respectively, as her target distribution of $X^1_{\tau^1}$. Then

$$\begin{aligned} V_G - V_F &= \varphi(z-\epsilon_1)q + \varphi(z+\epsilon_2)(p-q) - \varphi(z)p - \theta p^n \\ &\qquad + \theta q \mathbb{P}\left(\max_{i\neq 1} X^i_{\tau^i} = z-\epsilon_1\right) + \theta(p-q)\mathbb{P}\left(\max_{i\neq 1} X^i_{\tau^i} = z+\epsilon_2\right) \\ &\geq \varphi(z-\epsilon_1)q + \varphi(z+\epsilon_2)(p-q) - \varphi(z)p - \theta p^n \\ &= p\left[\varphi(z-\epsilon_1)\frac{\epsilon_2}{\epsilon_1+\epsilon_2} + \varphi(z+\epsilon_2)\frac{\epsilon_1}{\epsilon_1+\epsilon_2} - \varphi(z) - \theta p^{n-1}\right] \\ &= p\left\{[\varphi(z+\epsilon_2) - \varphi(z)]\frac{\epsilon_1}{\epsilon_1+\epsilon_2} - [\varphi(z) - \varphi(z-\epsilon_1)]\frac{\epsilon_2}{\epsilon_1+\epsilon_2} - \theta p^{n-1}\right\}. \end{aligned}$$

Since $\varphi(z+\epsilon_2)-\varphi(z)\ge p^{n-1}$ and $0\le\varphi(z)-\varphi(z-\epsilon_1)\le 1$,

$$\begin{aligned} V_G - V_F &\ge p\left[\frac{\epsilon_1}{\epsilon_1+\epsilon_2}p^{n-1} - \frac{\epsilon_2}{\epsilon_1+\epsilon_2} - \theta p^{n-1}\right] \\ &= p\,\frac{\epsilon_1(1-\theta)p^{n-1} - \left(1+\theta p^{n-1}\right)\epsilon_2}{\epsilon_1+\epsilon_2} > 0. \end{aligned}$$

This means that $V_G > V_F$, which is a contradiction to the definition of Nash equilibrium. Thus, $F(x)$ is continuous on $[0,\infty)$.

(ii) Assume that $F(x)$ places an atom of size $p\in(0,1)$ at 0. Fix any q such that $0<q<\min\left\{p\sqrt[n]{2-\theta}-p, 1-p\right\}$. Since F is continuous on $[0,\infty)$, there exists ϵ such that $F(\epsilon)-p=q$. Let G be given by

$$G(x) = \begin{cases} 0, & \text{if } x\in[0,\delta), \\ p+q, & \text{if } x\in[\delta,\epsilon), \\ F(x), & \text{if } x\in[\epsilon,\infty), \end{cases}$$

where $\delta=\int_0^\epsilon yF(dy)/(p+q)$. Note that $\delta\in(0,\epsilon)$, since $\int_0^\epsilon yF(dy)>0$ and $\int_0^\epsilon yF(dy)=\epsilon F(\epsilon)-\int_0^\epsilon F(y)dy<\epsilon F(\epsilon)=\epsilon(p+q)$. Moreover, G is a distribution function with the same mean as F. Thus, G is feasible.

Suppose that $X^i_{\tau^i}\sim F$ for any $i\ne 1$. Let V_F and V_G denote the expected pay-off of player 1 if player 1 chooses F and G, respectively, as her target distribution of $X^1_{\tau^1}$. Then

$$\begin{aligned} V_G - V_F &= (p+q)F(\delta)^{n-1} - \theta p^n - \int_0^\epsilon F(y)^{n-1}F(dy) \\ &\ge (p+q)p^{n-1} - \theta p^n - (p+q)^{n-1}(p+q-p) \\ &> (2-\theta)p^n - (p+q)^n > 0. \end{aligned}$$

Hence, player 1 would prefer strategy G to F, which is a contradiction to the definition of Nash equilibrium. Thus, $F(0)=0$.

In conclusion, $F(x)$ is atom-free and thus the symmetric Nash equilibrium is atom-free.

□

Remark 1.1.1. The fact that the Nash equilibrium is atom-free relies on the fact that the situation is symmetric in the sense that all agents stop Brownian motions started from a common value x_0. If the agents observe processes with different starting points, then the Nash equilibrium may have masses at zero for some agents. In that case, for a Nash equilibrium, no agent places mass at a positive point, and at least one agent has an atom-free distribution.

1.1.1 Equilibrium distribution

In this section, we explain how Seel and Strack solved the problem in Seel and Strack [2013]. To solve the problem, Seel and Strack started with deriving a candidate Nash equilibrium that is symmetric and atom-free.

A symmetric and atom-free Nash equilibrium is identified with a continuous distribution function F such that $F(0) = 0$. Suppose that the other agents all choose F as their target distribution. If agent i chooses to stop at x then her probability of winning $u_i(x)$ is given by

$$u_i(x) = \mathbb{P}(\max_{j \neq i} X^j_{\tau^j} < x) = F(x)^{n-1}, \tag{1.1}$$

since F is atom-free. Let $b = \sup\{x : F(x) < 1\}$. Let $\tau^i_{(0,b)} = \inf\{t : X^i_t \notin (0,b)\}$, that is $\tau^i_{(0,b)}$ is the first time at which X^i leaves the interval $(0,b)$. Seel and Strack derived an equilibrium in which for any point $x \in (0,b)$, it is indifferent for agent i to stop at x or to play the continuation strategy $\tau^i_{(0,b)}$. Thus,

$$\begin{aligned} u_i(x) &= \mathbb{P}\left(\max_{j \neq i} X^j_{\tau^j} < X^i_{\tau^i_{(0,b)}}\right) = 1 \cdot \mathbb{P}\left(X^i_{\tau^i_{(0,b)}} = b\right) + 0 \cdot \mathbb{P}\left(X^i_{\tau^i_{(0,b)}} = 0\right) \\ &= \mathbb{P}\left(H^i_b < H^i_0\right) = \frac{x}{b}, \end{aligned} \tag{1.2}$$

for any $x \in (0,b)$. Seel and Strack also argued that $u_i(x_0) = 1/n$ by symmetry and optimality of stopping at x_0, which then implies $b = nx_0$ by (1.2). Then using (1.1) and (1.2), they obtained the candidate equilibrium distribution

$$F(x) = \sqrt[n-1]{u_i(x)} = \sqrt[n-1]{\frac{x}{nx_0}},$$

for any $x \in [0, nx_0]$.

Now suppose that agent i chooses a stopping rule τ^i such that $X^i_{\tau^i}$ has distribution F, then her probability of winning is

$$\mathbb{E}\left[u_i(X^i_{\tau^i})\right] = \int_0^\infty F(x)^{n-1} F(dx) = \int_0^{nx_0} \frac{x}{nx_0} d\sqrt[n-1]{\frac{x}{nx_0}} = \frac{1}{n}.$$

Seel and Strack proved that F is indeed an equilibrium distribution by showing that no other stopping rule gives agent i a winning probability greater than $1/n$.

Specifically, consider any stopping time $\tilde{\tau}^i \le H_0^i$. Because $u_i(X^i)$ is non-negative and X^i is a martingale, $u_i(X^i)$ is a non-negative local martingale and thus a non-negative supermartingale. Observe that here H_0^i is finite almost surely, which means $\tilde{\tau}^i$ is also finite almost surely. So we have

$$\mathbb{E}\left[u_i(X^i_{\tilde{\tau}^i})\right] \le \mathbb{E}\left[u_i(X^i_0)\right] = u_i(x_0) = \frac{1}{n}.$$

Thus, the candidate equilibrium distribution

$$F(x) = \min\left\{\sqrt[n-1]{\frac{x}{nx_0}}, 1\right\}$$

is indeed an equilibrium distribution for the contest.

1.2 The Lagrangian method

We rederive the Nash equilibrium using a different approach based on a Lagrangian method. This brings new insights and yields a simpler proof for the Seel-Strack problem and facilitates our analysis of the variants of the Seel-Strack problem.

Suppose that the other agents all choose $F(x)$ as their target distribution, where F is continuous and $F(0) = 0$, then the expected pay-off of agent i with stopping time τ^i is given by

$$\mathbb{E}\left[F(X^i_{\tau^i})^{n-1}\right].$$

Observe that the optimal τ^i must satisfy that $\mathbb{E}(X^i_{\tau^i}) = x_0$. Recall that $\mathbb{E}(X^i_{\tau^i}) \le x_0$. Assume now that $X^i_{\tilde{\tau}^i} \sim \tilde{G}$ and $\tilde{G}$ has mean $x_1 < x_0$, then there exists (G, τ^i) such that G has mean x_0, $G \le \tilde{G}$ and $X^i_{\tau^i} \sim G$. Clearly τ^i dominates $\tilde{\tau}^i$ as a strategy since $F(x)^{n-1}$ is non-decreasing. Hence we may restrict attention to stopping times τ^i such that the distribution of $X^i_{\tau^i}$ has mean x_0.

Let $\mathcal{A}$ be the set of non-decreasing right-continuous functions $f : [0,\infty) \mapsto [0,\infty)$. Then the problem facing the agent i is to choose G to solve

$$\max_{G\in\mathcal{A}} \int_0^\infty F(x)^{n-1} G(dx) \tag{1.3}$$

subject to $\int_0^\infty xG(dx) = x_0$ and $\int_0^\infty G(dx) = 1$. Introducing multipliers λ and γ for the two constraints, the Lagrangian for the optimisation problem (1.3) is then

$$\mathcal{L}_F(G; \lambda, \gamma) = \int_0^\infty \left[F(x)^{n-1} - \lambda x - \gamma\right] G(dx) + \lambda x_0 + \gamma. \tag{1.4}$$

Let $\mathcal{A}_D(x_0)$ be the subset of $\mathcal{A}$ corresponding to distribution functions of random variables with mean x_0, then

$$\mathcal{A}_D(x_0) = \left\{ f \in \mathcal{A} : \lim_{x \uparrow \infty} f(x) = 1 \text{ and } \int_0^\infty x f(dx) = x_0 \right\}.$$

Now we state a variant of the Lagrangian sufficiency theorem for our problem.

Proposition 1.2.1. *If G^*, λ^* and γ^* exist such that $G^* \in \mathcal{A}_D(x_0)$, G^* is continuous, $G^*(0) = 0$ and*

$$\mathcal{L}_{G^*}(G^*; \lambda^*, \gamma^*) \geq \mathcal{L}_{G^*}(G; \lambda^*, \gamma^*) \qquad \textit{for all } G \in \mathcal{A}, \tag{1.5}$$

then G^* is a symmetric, atom-free Nash equilibrium.

Proof. We seek a symmetric atom-free Nash equilibrium. Since there are no atoms, we do not need to consider how to break ties and a symmetric atom-free Nash equilibrium is identified with a continuous distribution function $G^* \in \mathcal{A}_D(x_0)$ with $G^*(0) = 0$ and the property that

$$\int_0^\infty G^*(x)^{n-1} G^*(dx) \geq \int_0^\infty G^*(x)^{n-1} G(dx) \qquad \forall G \in \mathcal{A}_D(x_0).$$

Thus, if all other agents follow a strategy yielding a stopped value with distribution G^*, (and then the maximum of the stopped values of the other agents has a distribution $(G^*)^{n-1}$) then the agent has a higher probability of winning by following a strategy yielding a stopped value also with distribution G^* than with a strategy yielding any other distribution G.

Now suppose that $G \in \mathcal{A}_D(x_0)$, then

$$\int_0^\infty G^*(x)^{n-1} G(dx) = \mathcal{L}_{G^*}(G; \lambda^*, \gamma^*).$$

Then, under the hypotheses of the proposition,

$$\int_0^\infty G^*(x)^{n-1} G^*(dx) = \mathcal{L}_{G^*}(G^*; \lambda^*, \gamma^*) \geq \mathcal{L}_{G^*}(G; \lambda^*, \gamma^*) = \int_0^\infty G^*(x)^{n-1} G(dx).$$

Thus, G^* is a symmetric, atom-free Nash equilibrium.

$\square$

With Proposition 1.2.1, it is easy to verify the candidate Nash equilibrium.

Theorem 1.2.1. *There exists a symmetric, atom-free Nash equilibrium for the problem for which $X^i_{\tau^i}$ has law $F(x)$, where for $x \geq 0$,*

$$F(x) = \min\left\{ \sqrt[n-1]{\frac{x}{nx_0}}, 1 \right\}.$$

Proof. On $[0, \infty)$ let $G^*(x) = \min\left\{ \sqrt[n-1]{x/(nx_0)}, 1 \right\}$, $\lambda^* = 1/(nx_0)$ and $\gamma^* = 0$. From the explicit form of G^* it follows immediately that G^* is continuous, $G^*(0) = 0$ and $G^* \in \mathcal{A}_D(x_0)$. We then verify that for these multipliers (1.5) holds:

$$\begin{aligned}
\mathcal{L}_{G^*}(G; \lambda^*, \gamma^*) &= \int_0^\infty \left[G^*(x)^{n-1} - \lambda^* x - \gamma^* \right] G(dx) + \lambda^* x_0 + \gamma^* \\
&= \int_{nx_0}^\infty \left[1 - \frac{x}{nx_0} \right] G(dx) + \frac{1}{n} \leq \frac{1}{n} = \mathcal{L}_{G^*}(G^*; \lambda^*, \gamma^*).
\end{aligned}$$

Thus, there exists a symmetric, atom-free Nash equilibrium of the given form.

□

1.2.1 Derivation of the equilibrium distribution

This section is intended to illustrate how we derived the optimal multipliers and the candidate Nash equilibrium. The Lagrangian approach gives a general method for finding the optimal solution, which is distinct from the ideas in Seel and Strack [2013], and can be generalised to other settings.

Recall the definition of the Lagrangian $\mathcal{L}_F(G; \lambda, \gamma)$ for the optimisation problem (1.3). Denote by $L_F(x)$ the integrand in $\mathcal{L}_F$, that is

$$L_F(x) = F(x)^{n-1} - \lambda x - \gamma.$$

Then $\mathcal{L}_F(G; \lambda, \gamma) = \int_0^\infty L_F(x) G(dx) + \lambda x_0 + \gamma$.

In order to have a finite optimal solution, we require $L_F(x) \leq 0$ on $[0, \infty)$. Let $\mathcal{D}_F$ be the set of (λ, γ) such that $\mathcal{L}_F(\cdot; \lambda, \gamma)$ has a finite maximum. Then $\mathcal{D}_F$ is defined by

$$\mathcal{D}_F = \{(\lambda, \gamma) : L_F(x) \leq 0 \text{ on } [0, \infty)\}.$$

In order to reach the maximum value, we require $G(dx) = 0$ when $L_F(x) < 0$. This means that for $(\lambda, \gamma) \in \mathcal{D}_F$ the maximum of $\mathcal{L}_F(\cdot; \lambda, \gamma)$ occurs at G^* such that $G^*(dx) = 0$ when $L_F(x) < 0$. Suppose the Nash equilibrium is symmetric then we must have $G^*(x) = F(x)$. And then $L_{G^*}(x) \leq 0$, $G^*(dx) = 0$ when $L_{G^*}(x) < 0$ and $L_{G^*}(x) = 0$ when $G^*(dx) > 0$.

Introduce $a = \inf\{x : G^*(x) > 0\}$ and $b = \sup\{x : G^*(x) < 1\}$. Since

$L_{G^*}(x) = G^*(x)^{n-1} - \lambda x - \gamma$, we must have $G^*(x) = \sqrt[n-1]{\lambda x + \gamma}$ when $G^*(dx) > 0$. Since G^* is non-decreasing and not constant we must have $\lambda > 0$, which means $\sqrt[n-1]{\lambda x + \gamma}$ is strictly increasing. Then, since we are searching for atom-free solutions, we must have $G^*(x) = \sqrt[n-1]{\lambda x + \gamma}$ on the whole of the interval $[a, b]$.

Observe that $0 \leq F(0)^{n-1}$ so that if $(\lambda, \gamma) \in \mathcal{D}_F$ then γ is non-negative. Since G^* is atom-free, $G^*(a) = 0$ and hence $\lambda a + \gamma = 0$. Then, by the non-negativity of a and γ and the positivity of λ, it follows that $\gamma = 0 = a$. Thus $G^*(x) = \sqrt[n-1]{\lambda x}$ on $[0, b]$ for some λ and b which we must find.

For a feasible solution, $\int_0^\infty G^*(dx) = 1$ and $\int_0^\infty xG^*(dx) = x_0$, so that

$$1 = \int_0^b d\left(\sqrt[n-1]{\lambda x}\right) = \sqrt[n-1]{\lambda b}; \quad x_0 = \int_0^b xd\left(\sqrt[n-1]{\lambda x}\right) = \frac{\sqrt[n-1]{\lambda}}{n} b^{\frac{n}{n-1}} = \sqrt[n-1]{\lambda b}\frac{b}{n}.$$

Hence $b = nx_0$ and $\lambda = 1/(nx_0)$. This gives us that G^* is the distribution function given in Theorem 1.2.1.

1.3 Comparison with all-pay auctions

In this section, we study an n-player all-pay auction with complete information. We will use the Lagrangian method to derive the Nash equilibria for the all-pay auction, and we will compare the all-pay auction with the Seel-Strack contest.

Suppose that there are n bidders with labels $i \in I = \{1, 2, \ldots, n\}$ in an all-pay auction. Each bidder i submits a non-negative bid b_i and a prize is awarded to the highest bidder. All bidders pay their bids, and their valuations of the prize are the same, denoted by v. If there are k bidders who have the same highest bid, then these agents each win v/k. Then the pay-off for bidder i is

$$\frac{v}{k} \cdot \mathbf{1}_{\{b_i = \max_{j \in I} b_j\}} - b_i,$$

where $k = |\{i \in I : b_i = \max_{j \in I} b_j\}|$.

It is clear that the expected pay-offs to the bidders only depend on the distributions of $(b_i)_{i \in I}$. Thus, a Nash equilibrium for the auction can be identified with a family of distribution functions $(F_i)_{i \in I}$. We say that $(F_i)_{i \in I}$ is a Nash equilibrium if, for each $i \in I$, if the other bidders chooses b_j such that $b_j \sim F_j$, then the optimal bid b_i for agent i should satisfy that $b_i \sim F_i$.

Baye et al. [1990] found that there exists a unique symmetric equilibrium and a continuum of asymmetric equilibria for the auction. The algebraic forms of the families of equilibrium distributions are given in Theorem 1.3.1. They also show that those equilibria are all the possible equilibria by proving the following lemma.

Lemma 1.3.1. *[Baye et al. [1990]] There are only two types of equilibria: either all players use the same continuous mixed strategy with support* $[0, \bar{x}]$*; or at least two players randomise continuously over* $[0, \bar{x}]$ *with no mass at 0 and each other player* i *randomising continuously over* $(\alpha_i, \bar{x}]$*,* $\alpha_i > 0$*, and having a masspoint at 0 equal to* $F_i(\alpha_i)$*. Here* $\bar{x}$ *is a strictly positive constant (in fact,* $\bar{x} = v$*) and if* $\alpha_i > \bar{x}$ *then player* i *places all mass at 0.*

We will not present the proof of Lemma 1.3.1 here, see Baye et al. [1990] for details. This lemma allows us to construct all the possible equilibrium distributions explicitly, and then we can use the Lagrangian method to verify these Nash equilibria.

Theorem 1.3.1. *[Baye et al. [1990]] Any Nash equilibrium* $(F_i)_{i\in I}$ *has the following form:*

$$F_1(x) = \cdots = F_h(x) = \begin{cases} \sqrt[h-1]{\frac{x}{v\Pi_{k>h}p_k}}, & \text{if } x \in [0, \alpha_{h+1}), \\ \sqrt[k-1]{\frac{x}{v\Pi_{j>k}p_j}}, & \text{if } x \in [\alpha_k, \alpha_{k+1}), k = h+1, \ldots, n-1, \\ \sqrt[n-1]{\frac{x}{v}}, & \text{if } x \in [\alpha_n, v], \end{cases}$$

and, for any $i = h+1, \ldots, n$*,*

$$F_i(x) = \begin{cases} p_i, & \text{if } x \in [0, \alpha_i), \\ F_1(x), & \text{if } x \in [\alpha_i, v], \end{cases}$$

where $h \in \{2, 3, \ldots, n\}$ *is an arbitrary constant,* α_i*'s are also arbitrary constants and satisfy that* $0 < \alpha_{h+1} \leq \alpha_{h+2} \leq \cdots \leq \alpha_n \leq v$ *and* p_i*'s are given by*

$$p_n = \sqrt[n-1]{\frac{\alpha_n}{v}}, \; p_k = \sqrt[k-1]{\frac{\alpha_k}{v\Pi_{j>k}p_j}}, \quad k = h+1, h+2, \ldots, n-1. \tag{1.6}$$

Remark 1.3.1. Equation (1.6) ensures that all the given distribution functions $(F_i)_{i\in I}$ are continuous. In fact, all Nash equilibria are atom-free on $(0, \infty)$.

Proof. Let $\mathcal{A}$ be the set of pairs (G, q) where $q \in \mathbb{R}^+$ and $G : [0, \infty) \mapsto [0, \infty)$ is a non-decreasing right-continuous function with $G(0) = q$. An element of $\mathcal{A}$ is identified with a measure ν on $[0, \infty)$ such that $G(x) = \nu([0, x])$ and $q = \nu(\{0\})$.

Fix any player i. Suppose that any other player j chooses (F_j, p_j) as her strategy with F_j continuous $\forall j \neq i$. Then the objective of agent i is to choose a law of b_i, which corresponds to a pair (F_i, p_i), to solve

$$\max_{b_i} \mathbb{E}\left[v \prod_{j\neq i} F_j(b_i)\cdot \mathbf{1}_{\{b_i>0\}} + \frac{v}{n}\prod_{j\neq i} F_j(0)\cdot \mathbf{1}_{\{b_i=0\}} - b_i\right]$$
$$= \max_{(F_i,p_i)\in\mathcal{A}} \left\{\int_{(0,\infty)} \left(v\prod_{j\neq i} F_j(x) - x\right) F_i(dx) + \frac{v}{n}\left(\prod_{j\neq i} p_j\right) p_i\right\} \tag{1.7}$$

subject to $\int_{(0,\infty)} F_i(dx) + p_i = 1$. Introducing multiplier γ_i for the constraint, the Lagrangian for the optimisation problem (1.7) is then

$$\mathcal{L}_i^{\text{apa}}(F_i, p_i; \gamma_i) = \int_{(0,\infty)} \left(v\prod_{j\neq i} F_j(x) - x - \gamma_i\right) F_i(dx) + \left(\frac{v}{n}\prod_{j\neq i} p_j - \gamma_i\right) p_i + \gamma_i. \tag{1.8}$$

We first derive the candidate Nash equilibria. Suppose that $(F_i, p_i)_{i\in I}$ is a Nash equilibrium. Then, by a similar argument in Section 1.2.1, we get that $v\prod_{j\neq i} F_j(x) - x - \gamma_i = 0$ when $F_i(dx) > 0$ and $\frac{v}{n}\prod_{j\neq i} p_j - \gamma_i = 0$ if $p_i > 0$. By Lemma 1.3.1, F_i are all continuous. Let

$$\operatorname{supp}(F_i) = \{x : F_i(z_1) < F_i(z_2) \text{ for all } z_1 < x < z_2\}.$$

Then, $\operatorname{supp}(F_i) = [\alpha_i, \bar{x}]$, where $\bar{x}$ and α_i are constants such that $0 \le \alpha_1 \le \alpha_2 \le \cdots \le \alpha_n \le \bar{x}$ and $p_i = F_i(\alpha_i)$. Moreover, suppose that $\alpha_1 = \alpha_2 = \cdots = \alpha_h = 0 < \alpha_{h+1}$, where $h \in \{2, 3, \ldots, n\}$ is an arbitrary constant, then $p_1 = p_2 = \cdots = p_h = 0$. Thus, for any $i \in I$,

$$v\prod_{j\neq i} F_j(x) = x + \gamma_i \text{ for any } x \in [\alpha_i, \bar{x}]. \tag{1.9}$$

Since $\alpha_1 = 0$ and $F_2(0) = 0$, we get $\gamma_1 = v\prod_{j\neq 1} F_j(0) = 0$. Then, since $F_i(\bar{x}) = 1$ for any $i \in I$, $\gamma_i = \gamma_1 = 0$ for all $i \in I$ and $\bar{x} = v - \gamma_i = v$. Fix any $k \in \{h, h+1, \ldots, n-1\}$. Then, by (1.9), we have that on $[\alpha_k, \alpha_{k+1}]$, $F_i(x) = p_i \ \forall i \ge k+1$ and

$$F_i(x) = F_1(x) = \sqrt[k-1]{\frac{x}{v\prod_{j>k} p_j}} \quad \forall 1 < i \le k,$$

and on $[\alpha_n, v]$,

$$F_i(x) = F_1(x) = \sqrt[n-1]{\frac{x}{v}} \quad \forall 1 < i \le n.$$

This gives us the family of functions $(F_i)_{i\in I}$ described in the theorem. Moreover, since $F_1(x)$ is continuous, we get (1.6) holds. Notice that $p_{h+1} \le p_{h+2} \le \cdots \le p_n$.

We next verify that $(F_i, p_i)_{i\in I}$ is indeed a Nash equilibrium. Fix any $i \in I$. It is obvious that F_i is continuous and $\int_{(0,\infty)} F_i(dx) + p_i = 1$. Moreover, $v\prod_{j\neq i} F_j(x) - x = 0$ on $[\alpha_i, v]$ and $v\prod_{j\neq i} F_j(x) - x = v - x < 0$ on (v, ∞). Now suppose $i \geq h+1$ and fix any $k \in \{h, h+1, \ldots, i-1\}$. Observe that on $[\alpha_k, \alpha_{k+1}]$

$$\begin{aligned} v\prod_{j\neq i} F_j(x) - x &= v\frac{x}{v\prod_{j>k} p_j}\sqrt[k-1]{\frac{x}{v\prod_{j>k} p_j}}\prod_{j>k, j\neq i} p_j - x \\ &= \frac{x}{p_i}\sqrt[k-1]{\frac{x}{v\prod_{j>k} p_j}} - x \leq \frac{x}{p_i}\sqrt[k-1]{\frac{\alpha_{k+1}}{v\prod_{j>k} p_j}} - x = \frac{p_{k+1}}{p_i}x - x \leq 0. \end{aligned}$$

Thus, $v\prod_{j\neq i} F_j(x) - x \leq 0$ on $(0, \infty)$, and for any $(G_i, q_i) \in \mathcal{A}$

$$\begin{aligned} &\int_{(0,\infty)} \left(v\prod_{j\neq i} F_j(x) - x\right) G_i(dx) + \frac{v}{n}\left(\prod_{j\neq i} p_j\right) q_i \\ &\qquad \leq 0 = \int_{(0,\infty)} \left(v\prod_{j\neq i} F_j(x) - x\right) F_i(dx) + \frac{v}{n}\left(\prod_{j\neq i} p_j\right) p_i. \end{aligned}$$

Then by the definition we get that $(F_i, p_i)_{i\in I}$ is a Nash equilibrium.

$\square$

Remark 1.3.2. The unique symmetric equilibrium is given by $h = n$. In fact, if $h = n$, then by Theorem 1.3.1 the Nash equilibrium $(F_i)_{i\in I}$ is given by $F_i(x) = \min\left\{\sqrt[n-1]{\frac{x}{v}}, 1\right\}$ for any $i \in I$.

We now compare the all-pay auction with the Seel-Strack contest. In the all-pay auction, the cost from making a bid enters directly into the objective function and the agent chooses any probability distribution on $\mathbb{R}^+$. In contrast, in the context of gambling in contests there is no cost associated with the stopped value of the process in the objective function, but the constraint that the target probability distribution has mean x_0 introduces an extra term into the Lagrangian.

Theorem 1.3.1 shows that there exists a continuum of asymmetric equilibria in the all-pay auction. Recall that in the Seel-Strack contest, we focused only on the class of symmetric Nash equilibria. However, it is easy to see that because of the mean constraint in the Seel-Strack contest, the asymmetric candidate Nash equilibria proposed by Lemma 1.3.1 can not be Nash equilibria for the Seel-Strack contest.

On the other hand, in the class of symmetric Nash equilibria, there are strong parallels between the Seel-Strack contest and the all-pay auction in which the stopped value of the process corresponds to the auction bid. Suppose that the

Nash equilibrium is symmetric. Also suppose that other players all choose F as their target distribution and player i chooses G as her target distribution. Then the Lagrangian (1.8) for the all-pay auction becomes

$$\mathcal{L}_F^{\mathrm{apa}}(G, 0; \gamma) = \int_0^\infty \Big(vF(x)^{n-1} - x - \gamma\Big)\, G(dx) + \gamma. \tag{1.10}$$

Comparing with the Lagrangian (1.4) for the Seel-Strack contest, we see that modulo a factor of v representing the size of the winnings and a relabelling of parameters, the main difference is that in (1.10) the multiplier λ on the bid level is set to $1/v$. Moreover, with $v = nx_0$, the equilibrium distribution in the all-pay auction is exactly the same as the equilibrium distribution in the Seel-Strack contest.

1.4 Overview of thesis

Seel and Strack [2013] considers the case where X^i is a Brownian motion with drift and scaling and derives a Nash equilibrium under a joint feasibility condition on the drift parameter, scale parameter and the number of players. In Chapter 2, we consider a generalisation of the Seel-Strack contest in which the observed processes are independent copies of some time-homogeneous diffusion. This naturally leads us to consider the problem in cases where the analogue of the feasibility condition is violated. We solve the problem via a change of scale and a Lagrangian method. Unlike in the Seel-Strack problem it turns out that the optimal strategy may involve a target distribution which has an atom, and the rule used for breaking ties becomes important.

Chapter 3 considers the impact of adding a penalty associated with failure to follow a winning strategy. Again the objective of the agent is to maximise her chances of winning the contest, but now she is penalised if she has not won the contest, but there was an alternative strategy which would have led to her winning the contest. Three variants of the problem will be considered: the agent is penalised for stopping too soon, for stopping too late, or for stopping too soon or too late. We will find that in the first problem, the effect of the penalty is equivalent to an increase in the number of participants, and the second problem is both more difficult and more interesting. Moreover, we will find that the third problem is exactly equivalent to the original Seel-Strack contest.

Several variants of the original Seel-Strack contest have been discussed in Seel and Strack [2013], including an extension to the asymmetric two-player case where the starting values of the processes observed by the players are different constants. In Chapter 4, we start with rederiving the Nash equilibrium obtained by Seel and Strack in the asymmetric 2-player case and then extend the results to the asymmetric n-player case. We will show that there exists a Nash equilibrium $(F_i)_{i\in I}$ that has no atoms in $(0,\infty)$ and such that $\sup(\mathrm{supp}(F_i))$ are the same. Moreover, we will see that in equilibrium, when $n>2$, the agents with lower starting values may choose not to stop at small values but to wait for high values.

In a symmetric 2-player contest, Seel and Strack assumed that the observed processes start from the same positive constant, that is the starting value is common knowledge. In contrast, we assume that the starting values of the processes are independent non-negative random variables that have the same distribution in Chapter 5. So we consider a symmetric 2-player contest in which the starting value of the observed process is private information and only its distribution is commonly known. We will prove the existence and uniqueness of a symmetric Nash equilibrium for the problem.

Chapter 2

Contests modelled with diffusions

In Section 1.1, we described the Seel-Strack model and discussed the case where the processes $(Y^i)_{i\in I}$ privately observed by the contestants are continuous-time realisations of independent Brownian motions. However, Seel and Strack [2013] actually considered the case where Y^i is a Brownian motion with drift and scaling and it starts above zero and is absorbed at zero. And they derived a Nash equilibrium under a joint feasibility condition on the drift parameter, scale parameter and the number of contestants.

Specifically, in the Seel and Strack paper, each contestant privately observes an independent copy of a continuous stochastic process $Y = (Y_t)_{t\geq 0}$, where Y is given by

$$Y_t = y_0 + \mu t + \sigma W_t \tag{2.1}$$

with constant initial value $y_0 > 0$, constant drift coefficient μ and constant diffusion coefficient $\sigma > 0$. Here W is a Brownian motion. It is assumed that if Y hits zero then it is absorbed there, and an agent has to stop if her process hits zero. Each agent chooses a stopping time τ, and the stopped value Y_τ forms her entry into the contest. The agent who stops at the highest value wins the contest, and the objective of each agent is to maximise her probability of winning.

In a contest between n agents, Seel and Strack imposed a feasibility condition

$$\mu < \log\left(1 + \frac{1}{n-1}\right)\frac{\sigma^2}{2y_0}. \tag{2.2}$$

With this condition in force, they showed that there exists a symmetric Nash equilibrium within the class of bounded stopping rules. A feature of the Nash equilibrium is that it involves a randomised strategy, and the aim of each agent is to choose a stopping time τ such that the final value of the stopped process Y_τ has distribution F. The Nash equilibrium is unique in the sense that the target distribution F is unique, but in general there are many stopping times τ such that Y_τ has law F. (The optimal stopping times can be identified as solutions of the Skorokhod embedding problem (Skorokhod [1965]).) It turns out that F is the distribution function of a continuous random variable (in particular it is atom-free) and has a density which is strictly positive on a bounded interval, and is zero elsewhere.

We have explained how Seel and Strack solved the problem in Section 1.1.1. We adopt a different method of proof based on a Lagrangian sufficiency theorem. Moreover, our first step is to transform the problem into natural scale. This allows us to consider more general models for the observed process, beyond drifting Brownian motion, such that Y is a time-homogeneous diffusion. The change of scale method explains the origin of the condition (2.2) and motivates us to study the problem when (2.2) fails.

The effect of using the scale function is to transform the original contest into a simpler contest in which the observed processes are continuous martingales. To illustrate this procedure, consider the model used in Seel and Strack [2013], that is the observed process Y is given by (2.1). Then the scale function of Y is given by

$$s(y) = \frac{\sigma^2}{2\mu} - \frac{\sigma^2}{2\mu} e^{-2\mu y/\sigma^2}$$

(we have chosen a normalisation such that $s(0) = 0$). Define X by

$$X = s(Y),$$

then X is a diffusion in natural scale with starting value $x_0 = s(y_0)$ which solves $dX_t = \sigma \left(1 - \frac{2\mu X_t}{\sigma^2}\right) dW_t$, at least until it first hits zero, which is an absorbing point. Observe that the state space of X is $[0, s(\infty))$. Then it is easy to check that the condition (2.2) imposed by Seel and Strack is equivalent to $s(\infty) > nx_0$. We are interested in the cases where $s(\infty) \leq nx_0$, which are not covered in Seel and Strack [2013].

We introduce the mathematical model of this contest in Section 2.1. We start with a contest in which the observed process is any non-negative time-homogeneous diffusion. Then we explain how this contest is equivalent to a contest in which the observed process is a diffusion in natural scale. In Section 2.2, we derive the Nash equilibrium using a technique based on the Lagrangian sufficiency theorem. We will see that the choice of method used to break ties matters. We solve the problem for two canonical ways of breaking ties. In Section 2.3, we discuss two examples of Y and give explicit expressions for the Nash equilibrium and an optimal stopping rule in each case. Finally, in Section 2.4, we explain the origin of the optimal multipliers and the candidate Nash equilibrium distributions.

Our results show that the strategy that the agent should use in a Nash equilibrium is determined by both the mechanism for the breaking of ties and the value of the upper bound of the state space of the diffusion in natural scale. Unlike in the Seel-Strack problem, it turns out that the optimal strategy may involve a target distribution which has an atom, and the rule used for breaking ties becomes important. Moreover, there exist multiple Nash equilibria if the way to break the ties has been improperly chosen. There are close links between the problem and an all-pay auction with a bid cap (Che and Gale [1998]).

2.1 The model

In the contest, there are n participants with labels $i \in I = \{1, 2, \ldots, n\}$. Agent i privately observes an independent copy Y^i of a non-negative time-homogeneous diffusion process $Y = (Y_t)_{t\geq 0}$ with constant initial value $Y_0 = y_0 > 0$. Assume the state space of Y is an interval $\mathcal{S}$ with endpoints $\{0, r \in (y_0, \infty]\}$.

If Y can reach an endpoint in finite time, then we assume that the endpoint is absorbing. Further, to exclude degeneracies, we assume that $\lim_{t\uparrow\infty} Y_t$ exists, almost surely (and then $\lim_{t\uparrow\infty} Y_t \in \{0, r\}$) and that $\mathbb{P}(\lim_{t\uparrow\infty} Y_t = 0) > 0$. (We return to this point in Remark 2.1.1 below.) Examples include Brownian motion with drift, absorbed at zero, and exponential Brownian motion, provided that the parameters are such that the process does not diverge to infinity, see Example 2.1.2.

Let $\mathcal{F}_t^{Y^i} = \sigma(\{Y_s^i : s \leq t\})$ and set $\mathbb{F}^{Y^i} = (\mathcal{F}_t^{Y^i})_{t\geq 0}$. The space of strategies for agent i is the space of $\mathbb{F}^{Y^i}$-stopping times τ^i. Without loss of generality, we restrict attention to $\tau^i \leq \inf\{t \geq 0 : Y_t^i = 0 \text{ or } Y_t^i = r\}$. Note that τ^i is not necessarily assumed to be finite, that is agent i may choose to never stop the process Y^i, in which case her entry is taken to be $\lim_{t\uparrow\infty} Y_t$. Note also that agent i observes her own process Y^i, but not Y^j for $j \neq i$; nor does she observe the stopping times chosen by the other agents.

The winner of the contest is the one who stops at the highest value, and she wins unit reward, that is $\forall i \in I$, agent i wins 1 if she stops at a time τ^i such that $Y_{\tau^i}^i > Y_{\tau^j}^j$ $\forall j \neq i$. If there are k agents who stop at the equal highest value, then these agents each win $\theta(k)$, where $\theta(\cdot) : \{1, 2, \ldots, n\} \mapsto [0, 1]$ is some non-increasing (deterministic) function with $\theta(1) = 1$. Therefore agent i with stopping value $Y^i_{\tau^i}$ receives pay-off

$$\theta(k) \cdot \mathbf{1}_{\{Y^i_{\tau^i} = \max_{j \in I} Y^j_{\tau^j}\}},$$

where $k = \left| \left\{ i \in I : Y^i_{\tau^i} = \max_{j \in I} Y^j_{\tau^j} \right\} \right|$.

In general, there are two canonical choices of the ways to break the ties. One choice is to divide the prize evenly, that is to set $\theta(k) = 1/k$. This is equivalent to randomly breaking ties. Another choice is to reward only outright wins, so that no one wins if there is more than one player who stops at the highest value and $\theta(k) = \mathbf{1}_{\{k=1\}}$. (There is a third, less natural policy which is to set $\theta(k) \equiv 1$, or equivalently to reward joint winners with the full prize. In this case, the problem is degenerate, and a Nash equilibrium is obtained by all agents stopping immediately, $\tau^i = 0$.) We will give explicit solutions for the first two cases in Section 2.2.

Suppose that Y is a solution of the stochastic differential equation (SDE)

$$dY_t = a(Y_t)dW_t + b(Y_t)dt$$

where b is continuous and a is continuous and positive on the interior of $\mathcal{S}$. Let $s = s(y)$ be the scale function of Y. Then s is a strictly increasing solution of

$$a(y)^2 s''(y) + 2b(y)s'(y) = 0.$$

In general, $s(\mathcal{S})$ is an interval with endpoints $\{L, U\}$ with $-\infty \leq L < U \leq \infty$ and there are four sub-cases depending on whether either L or U is finite or not. In fact, by the Rogozin trichotomy, our assumption that $\lim_{t\uparrow\infty} Y_t$ exists rules out the case that $s(\mathcal{S}) = \mathbb{R}$, and the assumption that $\mathbb{P}(\lim_{t\uparrow\infty} Y_t = 0) > 0$ rules out the case that $L = -\infty$. Since s is only determined up to affine transformation, we may set $s(0) = 0$, and then $s(\mathcal{S})$ is an interval with endpoints $\{0, U\}$ where $U = s(r)$ may be finite or infinite. We could also insist that $s(y_0) = 1$ but we do not choose to do so.

Define X by

$$X = s(Y).$$

Then X is a diffusion in natural scale on $s(\mathcal{S})$ with starting value $x_0 = s(y_0)$, see Rogers and Williams [2000, Section V.7] or Karatzas and Shreve [1991, Section 5.5]. Set $\mathbb{F}^X$ to be the natural filtration of X (and by extension $\mathbb{F}^{X^i}$ to be the natural filtration of $X^i = s(Y^i)$). Clearly, τ is a $\mathbb{F}^Y$-stopping time if and only if τ is an $\mathbb{F}^X$ stopping time.

Since $s(\cdot)$ is a continuous, strictly increasing function, the pay-off of agent i

with stopping value $Y^i_{\tau^i}$ can be rewritten as

$$\theta(k) \cdot \mathbf{1}_{\{X^i_{\tau^i} = \max_{j \in I} X^j_{\tau^j}\}},$$

where $X^i_{\tau^i} = s(Y^i_{\tau^i})$ and $k = \left|\left\{i \in I : X^i_{\tau^i} = \max_{j \in I} X^j_{\tau^j}\right\}\right|$. This implies the equivalence between the two contests in which players privately observe Y^i and X^i, respectively, and the optimal stopping rule τ^i is the same for both contests. In particular, if we have a Nash equilibrium for which τ^i is optimal for the process X^i, then we also have a Nash equilibrium for the process Y^i. Hence, without loss of generality, we may reduce the problem to the case in which the observed process is a copy of a local martingale diffusion.

One of the insights of Seel and Strack [2013] is that since the pay-offs to the agents are determined by the distribution of $X^i_{\tau^i}$ rather than the stopping time τ^i itself, the problem of choosing the optimal stopping time τ^i can be reduced to a problem of finding the optimal distribution $F^i_X \equiv F^i$ of $X^i_{\tau^i}$. Then, once we have found the optimal target distribution F^i, the remaining work is to verify that there exists τ^i such that $X^i_{\tau^i}$ has law F^i_X. It follows that $Y^i_{\tau^i}$ has law $F^i_Y = F^i_X \circ s$.

Since X is a diffusion in natural scale, by the Dambis-Dubins-Schwarz Theorem (*e.g.* Rogers and Williams [2000], Theorem 34.1, p. 64) X can be expressed as a time change of Brownian motion. Then $X_t = B_{\Gamma_t}$ for some Brownian motion B with $B_0 = x_0$ and an increasing functional $\Gamma_t = [X]_t$. If F is the distribution function of any random variable on $[0, U]$ with mean x_0, there exists a $\mathbb{F}^B$-stopping time ρ such that $\rho \leq \inf\{v \geq 0 : B_v = 0 \text{ or } B_v = U\}$ and B_ρ has distribution F. Such a ρ is known as a solution of the Skorokhod embedding problem. In general there are many such solutions. Then, if we take $\tau = \Gamma^{-1} \circ \rho$, we find $X_\tau = B_\rho \sim F$ and τ is also an embedding of $F_Y = F_X \circ s$ in Y. Moreover τ is a $\mathbb{F}^Y$-stopping time. If $\rho < \inf\{v \geq 0 : B_v = 0 \text{ or } B_v = U\}$, then it follows that τ is finite, and more generally $\tau \leq \inf\{v \geq 0 : X_v = 0 \text{ or } X_v = U\} = \inf\{v \geq 0 : Y_v = 0 \text{ or } Y_v = r\} \leq \infty$.

If $U < \infty$, then $\left(X_{t \wedge \inf\{v \geq 0 : X_v = 0 \text{ or } X_v = U\}}\right)_{t \geq 0}$ is a martingale and every candidate target distribution F_X for X must have mean x_0. However, the stochastic process $\left(X_{t \wedge \inf\{v \geq 0 : X_v = 0 \text{ or } X_v = U\}}\right)_{t \geq 0}$ is a priori only a local martingale if $U = \infty$. Nonetheless, it is a non-negative supermartingale, and hence cannot explode to $U = \infty$, and $\lim_{t \uparrow \infty} X_t = 0 = \lim_{t \uparrow \infty} Y_t$ almost surely. In this case, for any distribution $\tilde{F}$ with mean strictly less than x_0 with associated embedding $\tilde{\tau}$, there exists (F, τ) such that F has mean x_0, $F \leq \tilde{F}$ and $X_\tau \sim F$. Clearly, τ dominates $\tilde{\tau}$ as a strategy, which means such $(\tilde{F}, \tilde{\tau})$ must not be optimal. Thus, we may restrict attention to stopping times τ^i such that $\mathbb{E}\left[X^i_{\tau^i}\right] = x_0$.

Remark 2.1.1. For the duration of this remark, relax the assumptions that $\lim_{t\uparrow\infty} Y_t$ exists and that $\mathbb{P}(\lim_{t\uparrow\infty} Y_t = 0) > 0$. If $\lim_{t\uparrow\infty} Y_t$ does not exist then $s(\mathcal{S}) = \mathbb{R}$. If $\lim_{t\uparrow\infty} Y_t$ exists and is equal to r almost surely, then $s(0) = -\infty$. In either case, $\limsup_{t\uparrow\infty} Y_t = r$ with probability 1, and then any stopping rule that involves stopping at $Y_t = \hat{y}$ for $\hat{y} < r$ can be improved upon by waiting until Y hits $(\hat{y} + r)/2$. Hence, either the optimal strategy is to wait until Y hits r (either in finite time, or in the limit) or there is no optimal strategy.

In terms of the process in natural scale $\limsup_{t\uparrow\infty} Y_t = r$ is equivalent to $\limsup_{t\uparrow\infty} X_t = U$ which is the case if and only if $L = -\infty$. This is why we have excluded the case.

Example 2.1.1. (Drifting Brownian motion) Let $\tilde{Y}$ be drifting Brownian motion so that $\tilde{Y}_t = y_0 + \mu t + \sigma W_t$, where $y_0 > 0$, $\mu \neq 0$ and $\sigma > 0$ are all constants. Let $\tilde{H}_0 = \inf\{u \geq 0 : \tilde{Y}_u = 0\}$ and let $Y_t = \tilde{Y}_{t\wedge\tilde{H}_0}$. Then $Y = (Y_t)_{t\geq 0}$ is drifting Brownian motion absorbed at zero and has state space $\mathcal{S} = [0, \infty)$. The scale function of both $\tilde{Y}$ and Y is $s(y) = \frac{\sigma^2}{2\mu} - \frac{\sigma^2}{2\mu} e^{-2\mu y/\sigma^2}$. Since $s(0) = 0$ and $s(\infty) \leq \infty$, $\lim_{t\uparrow\infty} Y_t$ exists and $\mathbb{P}(\lim_{t\uparrow\infty} Y_t = 0) > 0$. In particular, if $\mu > 0$ then $s(\infty) = \frac{\sigma^2}{2\mu} < \infty$ and $\mathbb{P}(\lim_{t\uparrow\infty} Y_t = 0) = 1 - s(y_0)/s(\infty) = e^{-2\mu y_0/\sigma^2}$, whereas if $\mu < 0$ then $s(\infty) = \infty$ and thus $\mathbb{P}(\lim_{t\uparrow\infty} Y_t = 0) = 1$.

Drifting Brownian motion absorbed at zero is the process considered in the original Seel and Strack paper, although they do not map it into natural scale.

Example 2.1.2. (Exponential Brownian motion) Now suppose Y is exponential Brownian motion, so that Y solves $dY_t = \mu Y_t dt + \sigma Y_t dW_t$, subject to $Y_0 = y_0 > 0$, where y_0, μ and $\sigma \neq 0$ are all constants. Y has state space $(0, \infty)$ and scale function $s(y) = y^\kappa/\kappa$ for $\kappa \neq 0$ and $s(y) = \ln y$ for $\kappa = 0$, where $\kappa = 1 - 2\mu/\sigma^2$. We assume $\kappa > 0$ to ensure that $s(0)$ is finite so that $\lim_{t\uparrow\infty} Y_t$ exists and $\mathbb{P}(\lim_{t\uparrow\infty} Y_t = 0) > 0$. If $\kappa \leq 0$ then $\mathbb{P}(\limsup_{t\uparrow\infty} Y_t = \infty) = 1$ and this example is degenerate.

As in Section 1.1, we say that a distribution function F is feasible if it satisfies that $F(0-) = 0$ and $\int_0^\infty xF(dx) \leq x_0$. We aim to find a Nash equilibrium for the problem in the sense of a family of the optimal feasible target distributions $(F^i)_{i\in I}$ of $(X^i_{\tau^i})_{i\in I}$. We will say that a Nash equilibrium has no atoms in $[0, U)$ if each F^i has no atoms in $[0, U)$.

Given that the contest is symmetric in the sense that each agent observes a local martingale process started from the same level x_0, it seems natural to search for Nash equilibria which are symmetric. Further, a simple argument over rearranging mass shows that provided $\theta(k) < 1$ for some $k \geq 2$, it is never optimal for k agents to put mass at a same point $x \in (0, U)$—any of them could benefit by modifying the target distribution to put a proportion $N/(N+1)$ of this mass at $(x+N^{-2})$ and a proportion $1/(N+1)$ at $(x-N^{-1})$ (where N is a sufficiently large number)—and thus it is possible to deduce that any optimal solution puts no mass at any point which belongs to $(0,U)$. A more detailed proof is provided by Theorem 1.1.1. Notice that this argument does not apply to the case where the mass point is U. In fact, in Section 2.2, we will see that in some cases it is indeed optimal to put mass at the upper bound U.

Remark 2.1.2. Theorem 1.1.1 shows that a symmetric Nash equilibrium has no atom at zero. Notice that the fact that the Nash equilibrium has no atom at 0 relies on the fact that the situation is symmetric. If the observed processes have different starting points, then the Nash equilibrium may have masses at zero for some agents. In that case, for a Nash equilibrium, at least one agent must put no mass at zero. We will only consider the symmetric case in this chapter.

Remark 2.1.3. When $U \geq nx_0$ (for example when $U=\infty$), the solution to our problem can be identified with the solution provided in Seel and Strack [2013]. In this case, we expect that there exists a unique symmetric Nash equilibrium.

The novel part of our solution, beyond the fact that we consider general diffusion processes, is that in the case $U < nx_0$, we identify a Nash equilibrium. This equilibrium may depend on the method used to break ties, but provided this method has been chosen sensibly then there is a symmetric Nash equilibrium, which may involve an atom at U. We expect this to be the unique symmetric Nash equilibrium, but our focus is on proving that such equilibria exist.

2.2 Equilibrium distribution

From Section 2.1, we know that the original contest can be reduced to a new contest in which each agent i privately observes an independent copy X^i of a continuous local martingale process $X=(X_t)_{t\geq 0}$ where $X_0 = x_0 > 0$ is a constant. The process X takes values in $[0,U]$, where $x_0 < U \leq \infty$. Agent i who stops at $X^i_{\tau^i}$ receives pay-off

$$\theta(k)\cdot \mathbf{1}_{\{X^i_{\tau^i}=\max_{j\in I} X^j_{\tau^j}\}},$$

where $k = \left|\left\{i\in I : X^i_{\tau^i} = \max_{j\in I} X^j_{\tau^j}\right\}\right|$.

In this section, we explicitly discuss the two canonical choices of $\theta(\cdot)$, $\theta(k) = 1/k$ and $\theta(k) = \mathbf{1}_{\{k=1\}}$ (and briefly, $\theta(k)=1$ in Remark 2.2.6), for the general n-player contest. These correspond to the cases where ties are broken randomly, and only outright wins earn the prize, respectively (and thirdly the case where all joint winners are rewarded with the full prize). We give the candidate Nash equilibrium solution and then verify the candidate Nash equilibrium using the Lagrangian sufficiency theorem. We will see that the two different choices of $\theta(\cdot)$ give us different results.

Theorem 2.2.1. *(i) Suppose $U \geq nx_0$, and recall $\theta(k) \leq 1$ for all k. Then there exists a symmetric Nash equilibrium for the problem that is atom-free and $X^i_{\tau^i}$ has law $F(x)$, where for $x \geq 0$*

$$F(x) = \min\left\{ \sqrt[n-1]{\frac{x}{nx_0}}, 1 \right\}.$$

(ii) Suppose $x_0 < U < nx_0$.

(a) If $\theta(k) = \mathbf{1}_{\{k=1\}}$, then there exists a symmetric Nash equilibrium for the problem that has no atoms in $[0, U)$ but an atom at U of size $\frac{nx_0 - U}{(n-1)U}$, and $X^i_{\tau^i}$ has law F such that for $0 \leq x < U$

$$F(x) = \frac{n(U - x_0)}{(n-1)U} \sqrt[n-1]{\frac{x}{U}}.$$

(b) If $\theta(k) = 1/k$, then there exists a symmetric Nash equilibrium for the problem that has no atoms in $[0, U)$ but an atom at U of size $p = 1 - \phi$, where $\phi \in \left(0, \sqrt[n-1]{U/(nx_0)}\right)$ solves $\Phi(x) = 0$. Here $\Phi(x) = x_0\,(x^n - 1) - U(x-1)$. Further, $X^i_{\tau^i}$ has law F such that for $0 \leq x < U$

$$F(x) = \min\left\{ \sqrt[n-1]{\frac{x}{nx_0}}, \phi \right\}.$$

Proof. Let $\mathcal{A}$ be the set of pairs (H, h) where $H : [0, U) \mapsto [0, \infty)$ is a non-decreasing right-continuous function and $h \in \mathbb{R}^+$. An element of $\mathcal{A}$ is identified with a measure ν on $[0, U]$ such that $H(x) = \nu([0, x])$ and $h = \nu(\{U\})$.

Fix agent $i \in I$. Suppose that the other players all choose (F, p) as their target measure with F continuous and $F(0) = 0$. Then agent i aims to choose a feasible law of $X^i_{\tau^i}$, which corresponds to a pair (G, q), to solve

$$\max_{(G,q)\in\mathcal{A}} \left\{ \int_{[0,U)} F(x)^{n-1} G(dx) + \left[\sum_{k=1}^{n} \theta(k) C_{n-1}^{k-1} p^{k-1} (1-p)^{n-k} \right] q \right\} \tag{2.3}$$

subject to $\int_{[0,U)} xG(dx) + Uq = x_0$ and $\int_{[0,U)} G(dx) + q = 1$. Introducing multipliers λ and γ for the two constraints, the Lagrangian for the optimisation problem (2.3) is then

$$\mathcal{L}_{F,p}(G,q;\lambda,\gamma) = \int_{[0,U)} \left[F(x)^{n-1} - \lambda x - \gamma\right] G(dx) \\ + \left[\sum_{k=1}^{n} \theta(k) C_{n-1}^{k-1} p^{k-1}(1-p)^{n-k} - \lambda U - \gamma\right] q + \lambda x_0 + \gamma. \tag{2.4}$$

Let $\mathcal{A}_D(x_0)$ be the subset of $\mathcal{A}$ identified with probability measures with mean x_0. Then $\mathcal{A}_D(x_0)$ is given by

$$\mathcal{A}_D(x_0) = \left\{(H,h) \in \mathcal{A} : \lim_{x\uparrow U} H(x) + h = 1 \text{ and } \int_{[0,U)} xH(dx) + Uh = x_0\right\}.$$

Now we state a variant of the Lagrangian sufficiency theorem for our problem.

Proposition 2.2.1. *If there exist* G^**,* q^**,* λ^* *and* γ^* *such that* $(G^*, q^*) \in \mathcal{A}_D(x_0)$*,* $G^*(0) = 0$*,* G^* *is continuous on* $[0, U)$ *and*

$$\mathcal{L}_{G^*,q^*}(G^*, q^*; \lambda^*, \gamma^*) \geq \mathcal{L}_{G^*,q^*}(G, q; \lambda^*, \gamma^*) \qquad \text{for all } (G, q) \in \mathcal{A}, \tag{2.5}$$

then (G^*, q^*) *is a symmetric Nash equilibrium that has no atoms in* $[0, U)$*.*

Proof. We seek a symmetric Nash equilibrium that has no atoms in $[0, U)$. Since there are no atoms in $[0, U)$, a Nash equilibrium is identified with a pair $(G^*, q^*) \in \mathcal{A}_D(x_0)$ such that $G^*(0) = 0$, G^* is continuous on $[0, U)$ and for all $(G, q) \in \mathcal{A}_D(x_0)$

$$\int_{[0,U)} G^*(x)^{n-1} G^*(dx) + \left[\sum_{k=1}^{n} \theta(k) C_{n-1}^{k-1} (q^*)^{k-1}(1-q^*)^{n-k}\right] q^* \\ \geq \int_{[0,U)} G^*(x)^{n-1} G(dx) + \left[\sum_{k=1}^{n} \theta(k) C_{n-1}^{k-1} (q^*)^{k-1}(1-q^*)^{n-k}\right] q. \tag{2.6}$$

If $(G, q) \in \mathcal{A}_D(x_0)$, then using the definition of the Lagrangian,

$$\int_{[0,U)} G^*(x)^{n-1} G(dx) + \left[\sum_{k=1}^{n} \theta(k) C_{n-1}^{k-1} (q^*)^{k-1}(1-q^*)^{n-k}\right] q = \mathcal{L}_{G^*,q^*}(G, q; \lambda^*, \gamma^*).$$

Then, since $\mathcal{L}_{G^*,q^*}(G^*, q^*; \lambda^*, \gamma^*) \geq \mathcal{L}_{G^*,q^*}(G, q; \lambda^*, \gamma^*)$, we have that (2.6) holds. Thus, (G^*, q^*) is a symmetric Nash equilibrium that has no atoms in $[0, U)$.

□

Return to the proof of Theorem 2.2.1.

(i) Suppose $U \geq nx_0$. On $[0,\infty)$ let $G^*(x) = \min\left\{ \sqrt[n-1]{x/(nx_0)}, 1 \right\}$, $q^* = 0$, $\lambda^* = 1/(nx_0)$ and $\gamma^* = 0$. It is immediate that (G^*, q^*) correspond to a distribution with mean x_0, $G^*(0) = 0$ and G^* is continuous on $[0, U)$, and so it remains to verify (2.5) for the given multipliers.

Since $\theta(k) \leq 1$ for all k we have that

$$\begin{aligned}\sum_{k=1}^{n} \theta(k) C_{n-1}^{k-1} (q^*)^{k-1} (1-q^*)^{n-k} &\leq \sum_{k=1}^{n} C_{n-1}^{k-1} (q^*)^{k-1} (1-q^*)^{n-k} \\ &= (q^* + (1-q^*))^{n-1} = 1,\end{aligned}$$

and then

$$\begin{aligned}&\mathcal{L}_{G^*,q^*}(G, q; \lambda^*, \gamma^*) \\ &\leq \int_{[0,U)} \left[G^*(x)^{n-1} - \lambda^* x - \gamma^* \right] G(dx) + (1 - \lambda^* U - \gamma^*) q + \lambda^* x_0 + \gamma^* \\ &= \int_{(nx_0,U)} \left(1 - \frac{x}{nx_0}\right) G(dx) + \left(1 - \frac{U}{nx_0}\right) q + \frac{1}{n} \\ &\leq \frac{1}{n} = \mathcal{L}_{G^*,q^*}(G^*, q^*; \lambda^*, \gamma^*).\end{aligned}$$

Thus, by Proposition 2.2.1, (G^*, q^*) is a symmetric, atom-free Nash equilibrium.

(ii) Suppose $x_0 < U < nx_0$.

(a) Set $\theta(k) = \mathbf{1}_{\{k=1\}}$. On $[0,\infty)$ let $G^*(x) = \left[\frac{n(U-x_0)}{(n-1)U} \sqrt[n-1]{\frac{x}{U}} \right] \cdot \mathbf{1}_{\{x<U\}} + \mathbf{1}_{\{x \geq U\}}$, $q^* = \frac{nx_0 - U}{(n-1)U}$, $\lambda^* = \left[\frac{n(U-x_0)}{(n-1)U} \right]^{n-1} U^{-1}$ and $\gamma^* = 0$. From the explicit form of G^* and q^*, it is clear that $G^*(0) = 0$, G^* is continuous on $[0, U)$ and $(G^*, q^*) \in \mathcal{A}_D(x_0)$. Now we verify that for these multipliers (2.5) holds. Since $\theta(k) = \mathbf{1}_{\{k=1\}}$,

$$\begin{aligned}&\mathcal{L}_{G^*,q^*}(G, q; \lambda^*, \gamma^*) \\ &= \int_{[0,U)} \left[G^*(x)^{n-1} - \lambda^* x - \gamma^* \right] G(dx) \\ &\qquad + \left[(1-q^*)^{n-1} - \lambda^* U - \gamma^* \right] q + \lambda^* x_0 + \gamma^* \\ &= \left[\frac{n(U-x_0)}{(n-1)U} \right]^{n-1} \frac{x_0}{U} = \mathcal{L}_{G^*,q^*}(G^*, q^*; \lambda^*, \gamma^*).\end{aligned}$$

(b) Set $\theta(k) = 1/k$. On $[0,\infty)$ let $G^*(x) = \min\left\{ \sqrt[n-1]{x/(nx_0)}, \phi \right\} \cdot \mathbf{1}_{\{x<U\}} + \mathbf{1}_{\{x \geq U\}}$, $q^* = 1 - \phi$, $\lambda^* = 1/(nx_0)$ and $\gamma^* = 0$, where $\phi \in (0, 1)$ solves

$\Phi(x) = 0$. Because $\Phi(0) = U - x_0 > 0$, $\Phi(1) = 0$, $\Phi'(0) = -U < 0$, $\Phi'(1) = nx_0 - U > 0$ and Φ is convex on $(0, \infty)$, there exists a unique solution ϕ to $\Phi\,(x) = 0$ such that $\phi \in (0,1)$. Moreover, since $\Phi(\phi) = x_0\phi^n - U\phi + U - x_0 = 0$ and $\phi \in (0,1)$, $\frac{U}{nx_0} = \frac{1-\phi^n}{n(1-\phi)} = \frac{1}{n}\left(1 + \phi + \phi^2 + \cdots + \phi^{n-1}\right) > \phi^{n-1}$ and thus $\phi < \sqrt[n-1]{U/(nx_0)}$.

Again by he explicit form of G^* and q^*, we get that $G^*(0) = 0$, G^* is continuous on $[0, U)$ and $(G^*, q^*) \in \mathcal{A}_D(x_0)$. Next we verify that for these multipliers (2.5) holds. Since $q^* \neq 0$ and $\theta(k) = 1/k$, and since $C_{n-1}^{k-1}/k = C_n^k/n$,

$$\begin{aligned}
&\sum_{k=1}^{n} \theta(k) C_{n-1}^{k-1} (q^*)^{k-1} (1-q^*)^{n-k} \\
&\qquad = \frac{1}{nq^*} \sum_{k=1}^{n} C_n^k (q^*)^k (1-q^*)^{n-k} = \frac{1}{nq^*}\left[1 - (1-q^*)^n\right] \\
&\qquad = \frac{1}{n(1-\phi)}\,(1-\phi^n) = \frac{U}{nx_0},
\end{aligned}$$

and we have

$$\begin{aligned}
&\mathcal{L}_{G^*,q^*}(G, q; \lambda^*, \gamma^*) \\
&= \int_{[0,U)} \left[G^*(x)^{n-1} - \lambda^* x - \gamma^*\right] G(dx) + \left[\frac{U}{nx_0} - \lambda^* U - \gamma^*\right] q + \lambda^* x_0 + \gamma^* \\
&= \int_{(nx_0\phi^{n-1}, U)} \left(\phi^{n-1} - \frac{x}{nx_0}\right) G(dx) + \frac{1}{n} \le \frac{1}{n} = \mathcal{L}_{G^*,q^*}(G^*, q^*; \lambda^*, \gamma^*).
\end{aligned}$$

Thus, (G^*, q^*) is a symmetric Nash equilibrium that has no atoms in $[0, U)$ by Proposition 2.2.1.

□

Remark 2.2.1. In the case where U cannot be reached in finite time, if the optimal target law places mass on U then this corresponds to the optimal stopping rule $\tau = \infty$ for that part of the sample space where $X_\tau = U$.

Remark 2.2.2. Similar to Section 1.3, in the class of symmetric Nash equilibria, there are strong parallels between our model and an all-pay auction in which the stopped value of the process corresponds to the auction bid, so that the choice over distributions for the stopped value of the process corresponds to the choice over distributions for the bid size. The upper bound on the state space of the stochastic process in natural scale corresponds to the bid cap.

It follows that the same Lagrangian methods can be applied to the all-pay auction. Consider a symmetric all-pay auction with a cap m on bids, where $m \in (0, \infty]$ is the maximum allowable bid. We assume that there are n bidders in the auction, and all bidders have the same valuation v of the prize. The Lagrangian for this problem is

$$\mathcal{L}^{\text{apa}}_{F,p}(G, q; \gamma) = \int_{[0,m)} \left[vF(x)^{n-1} - x - \gamma \right] G(dx) \\ + \left[v \sum_{k=1}^{n} \theta(k) C_{n-1}^{k-1} p^{k-1} (1-p)^{n-k} - m - \gamma \right] q + \gamma. \tag{2.7}$$

Comparing with (2.4) we see that modulo a factor of v representing the size of the winnings and a relabelling of parameters, the main difference is that in (2.7) the multiplier λ on the bid level is set to $1/v$.

In the case where $m \geq v$ or the case where $m \in (v/n, v)$ and $\theta(k) = 1/k$, the equilibrium distribution in this all-pay auction is exactly the same as the equilibrium distribution in our model with $m = U$ and $v = nx_0$. In the case where $m \in (0, v)$ and $\theta(k) = \mathbf{1}_{\{k=1\}}$, these two equilibrium distributions are also the same with $m = U$ and $v = U \left[\frac{(n-1)U}{n(U-x_0)} \right]^{n-1}$.

Remark 2.2.3. Expanding on the previous remark, for the most standard tie-breaking rule, *i.e.* $\theta(k) = 1/k$, Theorem 2.2.1 shows that if $x_0 < U < nx_0$, then there is a "hole" in the support of the equilibrium distribution. Specifically, in equilibrium, players stop with positive probability on $\left[0, nx_0\phi^{n-1}\right]$, players stop with zero probability on $\left(nx_0\phi^{n-1}, U\right)$, and players stop at U with probability $1 - \phi$. Similar equilibrium distributions with holes have been found in all-pay auctions with bid caps, *e.g.*, Che and Gale [1998], Dechenaux et al. [2006] or Szech [2011] and also in wars of attrition, *e.g.*, Hendricks et al. [1988] and Damiano et al. [2012].

Remark 2.2.4. If $x_0 < U < nx_0$, $\theta(k) = \mathbf{1}_{\{k=1\}}$ (so that only outright wins are rewarded) and all other agents follow strategies which yield the optimal target distribution stated in Case (ii.a) of Theorem 2.2.1, then whatever stopping rule agent i chooses her expected pay-off is equal to $\frac{x_0}{U} \left[\frac{n(U-x_0)}{(n-1)U} \right]^{n-1}$. In the other two cases, Cases (i) and (ii.b) of the theorem, if other agents use the Nash equilibrium strategy, then the agent achieves the same expected pay-off as the optimal strategy, provided she puts no mass in $(nx_0, U]$ or $(nx_0\phi^{n-1}, U)$, respectively.

Remark 2.2.5. The choice of the tie-breaking rule is crucial in determining the Nash equilibrium, at least in cases where the upper bound is sufficiently small. This phenomena is also a feature of some variants of the all-pay auction in which optimal bid distributions include an atom, see for example Che and Gale [1998], Dechenaux et al. [2006], Cohen and Sela [2007], and Szech [2011].

Szech [2011] studies a two-player all-pay auction in which each player is restricted to choose her bid from the interval $[0, m]$. She introduces an asymmetry whereby it is assumed that if both bidders submit the same bid, bidder 1 wins with probability $\alpha \in [0, 1]$, otherwise bidder 2 wins. In this auction, Szech shows that the Nash equilibrium depends on the choice of tie-breaking rule via (α, m), and takes one of three distinct forms.

Remark 2.2.6. If the way of breaking ties has not been chosen appropriately, then there might exist multiple Nash equilibria. (In the context of all-pay auctions, Cohen and Sela [2007] show that there may be multiple symmetric equilibria even in the standard case $\theta(k) = 1/k$, but there the phenomena arises from the discreteness of the set of possible bids.)

Take $\theta(k) \equiv 1$ (in which tied winners all win the full prize) as an example in the context of this section. It is clear that the stopping rule such that every agent stops immediately is a symmetric Nash equilibrium and that the associated target distribution consists of unit mass at $x_0 \in [0, U)$. Moreover, there exists a symmetric Nash equilibrium that has no atoms in $[0, U)$. This can be proved similarly to the proof of Theorem 2.2.1. In fact, if $U \geq nx_0$, then there exists a symmetric Nash equilibrium for the problem that has no atoms and $X^i_{\tau^i}$ has law $F(x)$, where $F(x) = \min\left\{ \sqrt[n-1]{x/(nx_0)}, 1 \right\}$ for $x \geq 0$; if $x_0 < U < nx_0$ then there exists a symmetric Nash equilibrium for the problem that has no atoms in $[0, U)$ but an atom at U of size $(1 - \hat{\phi})$, and $X^i_{\tau^i}$ has law F such that $F(x) = \min\left\{ \sqrt[n-1]{x/(nx_0)}, \hat{\phi} \right\}$ for $0 \leq x < U$. Here, $\hat{\phi} \in (0, 1)$ solves $\hat{\Phi}(x) = 0$, where $\hat{\Phi}(x) = Ux^n - nUx + n(U - x_0)$.

2.3 Examples

In this section, we give explicit expressions for the optimal target distribution and associated stopping time. The optimal stopping time is based on the Azéma-Yor solution of the Skorokhod embedding problem (Azéma and Yor [1979]). Note that any other solution for the Skorokhod embedding problem (see Hobson [2011]; Obłój [2004] for a survey) can also be used to construct an optimal strategy, but the Azéma-Yor solution is both relatively simple and quite concrete.

2.3.1 Drifting Brownian motion

Suppose the diffusion process Y is a drifting Brownian motion absorbed at zero and solves $dY_t = \mu dt + \sigma dW_t$ where $Y_0 = y_0 > 0$, $\mu \neq 0$ and $\sigma > 0$ are all constants. Set

$\gamma = 2\mu/\sigma^2$. The scale function of Y is $s(y) = \frac{1}{\gamma} - \frac{1}{\gamma}e^{-\gamma y}$. Let $X = s(Y)$; then, X is a diffusion in natural scale on $[0, U)$ with starting value $x_0 = s(y_0)$, and $U = s(\infty)$. If $\mu > 0$ then $U = s(\infty) = 1/\gamma$, else if $\mu < 0$ then $U = s(\infty) = \infty$.

Seel and Strack [2013] discussed the case where $U > nx_0$, which is equivalent to the condition (2.2), *i.e.* $\gamma < \frac{1}{y_0} \log \frac{n}{n-1}$. Here we discuss the general case.

(i) Suppose $U \geq nx_0$, that is suppose $\gamma \leq \frac{1}{y_0} \log \frac{n}{n-1}$. Recall that $\theta(k) \leq 1$ for all k, then by Case (i) of Theorem 2.2.1, the optimal distribution of X_τ is $F(x) = \min\left\{ \sqrt[n-1]{x/(nx_0)}, 1\right\}$ for $x \geq 0$. Define

$$\psi(x) = \frac{1}{1 - F(x)} \int_{[x,\infty)} yF(dy) \tag{2.8}$$

for $x \leq \inf\{x : F(x) = 1\}$ and $\psi(x) = x$ otherwise. Then ψ is the barycentre function. Thus, $\psi(x) = x$ for $x \geq nx_0$, and for $0 \leq x < nx_0$,

$$\psi(x) = x_0 \frac{1 - \left(\frac{x}{nx_0}\right)^{n/(n-1)}}{1 - \left(\frac{x}{nx_0}\right)^{1/(n-1)}}. \tag{2.9}$$

The Azéma-Yor embedding of F in Brownian motion started at $x_0 = (1 - e^{-\gamma y_0})/\gamma$ is $\tau = \inf\left\{t \geq 0 : \psi(X_t) \leq \bar{X}_t\right\}$, where $\bar{X}_t = \sup_{s \leq t} X_s$.

We want to reinterpret this solution in terms of the drifting Brownian motion Y. Set $F_Y = F \circ s$ so that $F_Y(y) = 0$ for $y \leq 0$,

$$F_Y(y) = \sqrt[n-1]{\frac{1 - e^{-\gamma y}}{n(1 - e^{-\gamma y_0})}} \quad \text{for } y \in \left(0, -\frac{1}{\gamma} \log\left[1 - n(1 - e^{-\gamma y_0})\right]\right),$$

and $F_Y(y) = 1$ for $y \geq -\frac{1}{\gamma} \log\left[1 - n(1 - e^{-\gamma y_0})\right]$.

Then, since s is strictly increasing, we rewrite τ as $\tau = \inf\left\{t \geq 0 : \Psi(Y_t) \leq \bar{Y}_t\right\}$ where $\Psi(y) := s^{-1}(\psi(s(y)))$ is given by

$$\Psi(y) = -\frac{1}{\gamma} \log\left(1 - (1 - e^{-\gamma y_0}) \frac{1 - \left(\frac{(1-e^{-\gamma y})}{n(1-e^{-\gamma y_0})}\right)^{n/(n-1)}}{1 - \left(\frac{(1-e^{-\gamma y})}{n(1-e^{-\gamma y_0})}\right)^{1/(n-1)}}\right) \tag{2.10}$$

if $0 \leq y < -\frac{1}{\gamma} \log\left[1 - n(1 - e^{-\gamma y_0})\right]$ and $\Psi(y) = y$ otherwise.

(ii) Suppose $x_0 < U < nx_0$, that is suppose $\gamma > \frac{1}{y_0} \log \frac{n}{n-1}$. Assume agents receive no reward if they are a tied winner so that $\theta(k) = \mathbf{1}_{\{k=1\}}$. By Case (ii.a) of Theorem 2.2.1, the optimal distribution of X_τ has law F such that

$F(x)=\frac{n(U-x_0)}{(n-1)U}\sqrt[n-1]{\frac{x}{U}}$ for $0\le x<U$ and $F(x)=1$ for $x\ge U$. Then, by the definition of the barycentre function ψ in (2.8), $\psi(x)=x$ for $x>U$ and

$$\psi(x)=\frac{x_0(n-1)-x\left(1-\frac{x_0}{U}\right)\sqrt[n-1]{\frac{x}{U}}}{n-1-n(1-\frac{x_0}{U})\sqrt[n-1]{\frac{x}{U}}} \text{ for } x\in[0,U]. \tag{2.11}$$

Recalling that $U=\frac{1}{\gamma}$ and substituting $s(y)=\frac{1-e^{-\gamma y}}{\gamma}$, $s^{-1}(x)=-\frac{\log(1-\gamma x)}{\gamma}$ and (2.11) into the expressions $F_Y=F\circ s$ and $\Psi(y)=s^{-1}(\psi(s(y)))$ yields

$$\begin{aligned} F_Y(y) &= \frac{n}{n-1}e^{-\gamma y_0}\sqrt[n-1]{1-e^{-\gamma y}}, \\ \Psi(y) &= y_0-\frac{1}{\gamma}\log\left(\frac{n-1-(n-1+e^{-\gamma y})\sqrt[n-1]{1-e^{-\gamma y}}}{n-1-ne^{-\gamma y_0}\sqrt[n-1]{1-e^{-\gamma y}}}\right), \end{aligned}$$

for $0\le y<\infty$. In particular, if $n=2$ then

$$\Psi(y)=y+\frac{1}{\gamma}\log\left(e^{\gamma(y_0+y)}-2e^{\gamma y}+2\right).$$

Note that $\gamma y_0>\log 2$ by hypothesis, so that the term inside the logarithm is positive.

Then $\tau=\inf\left\{t\ge 0:\Psi(Y_t)\le\bar{Y}_t\right\}$ is the Azéma-Yor optimal stopping rule for the original contest. We have that $\lim_{y\uparrow\infty}F_Y(y)<1$, so there is a nonzero probability that $\tau=\infty$ and that the agent achieves an infinite entry into the contest.

(iii) Again suppose $\gamma>\frac{1}{y_0}\log\frac{n}{n-1}$ but assume $\theta(k)=1/k$. The optimal distribution F of X_τ is given by Case (ii.b) of Theorem 2.2.1 as $F(x)=\min\left\{\sqrt[n-1]{\frac{x}{nx_0}},\phi\right\}$ for $0\le x<U$ and $F(x)=1$ for $x\ge U$, where $U=1/\gamma$ and $\phi\in\left(0,\sqrt[n-1]{\frac{U}{nx_0}}\right)$ solves $\Phi(x)=0$ with $\Phi(x)=x_0\left(x^n-1\right)-U(x-1)$. Then, by the definition (2.8) of the barycentre function, for $0\le x\le nx_0\phi^{n-1}$, $\psi(x)$ is given by (2.9), for $nx_0\phi^{n-1}<x\le U$ we have $\psi(x)=U$, and for $x>U$, $\psi(x)=x$.

Let $F_Y=F\circ s$. Then, for $y>0$, $F_Y(y)=\min\left\{\sqrt[n-1]{\frac{1-e^{-\gamma y}}{n\left(1-e^{-\gamma y_0}\right)}},\phi\right\}$.

Let $\Psi(y)=s^{-1}(\psi(s(y)))$. Then, $\Psi(y)$ is given by (2.10) if $y\in\left[0,s^{-1}(nx_0\phi^{n-1})\right]$ with $s^{-1}\left(nx_0\phi^{n-1}\right)=-\frac{1}{\gamma}\log\left(1-n\phi^{n-1}(1-e^{-\gamma y_0})\right)$ and $\Psi(y)=\infty$ otherwise. Moreover, the Azéma-Yor optimal stopping rule for the original contest is $\tau=\inf\left\{t\ge 0:\Psi(Y_t)\le\bar{Y}_t\right\}$. Again there is a nonzero probability that $\tau=\infty$ and that the agent achieves an infinite entry into the contest.

2.3.2 Exponential Brownian motion

The above methods extend easily to any non-negative time-homogeneous diffusion with state space an interval with endpoints $\{0, r \in (0,\infty]\}$ provided the scale function s satisfies $L = s(0) > -\infty$. Then we can normalise s so that $s(0) = 0$. Depending on the value of $U = s(r)$ we are in one of the cases of Theorem 2.2.1. In each case, for the diffusion in natural scale, the optimal target law is given as in the theorem, and by formula (2.8) for the barycentre ψ, we can construct an optimal stopping rule. The barycentre and the stopping time are exactly as in Section 2.3.1. Finally, it remains to interpret these stopping times as stopping times for the original process, and only at this stage do the calculations look different to the drifting Brownian motion case.

As a further example, now suppose agents privately observe independent copies of an exponential Brownian motion Y. Suppose Y is a solution of $dY_t = \mu Y_t dt + \sigma Y_t dW_t$, where $Y_0 = y_0 > 0$, and y_0, μ and $\sigma \neq 0$ are all constants. In light of the discussion in Example 2.1.2, we assume $\mu \in (-\infty, \sigma^2/2)$. The scale function of Y is $s(y) = y^\kappa/\kappa$, where $\kappa = 1 - 2\mu/\sigma^2$ and $\kappa > 0$. Let $X = s(Y)$; then X is a diffusion in natural scale on $(0,\infty)$ with starting value $x_0 = s(y_0)$.

In this case $U = s(\infty) = \infty$ and trivially $U \geq nx_0$. Provided that $\theta(k) \leq 1$ for all k, then by Case (i) of Theorem 2.2.1, the optimal distribution of X_τ is $F(x) = \min\left\{ \sqrt[n-1]{x/(nx_0)}, 1\right\}$ for $x \geq 0$ and the barycentre function ψ is the same as shown in (2.9). Then the optimal law of Y is $F_Y = F \circ s$ where for $y \geq 0$

$$F_Y(y) = \min\left\{ \sqrt[n-1]{\frac{y^\kappa}{ny_0^\kappa}}, 1\right\}$$

Using the Azéma-Yor embedding, one solution is $\tau = \inf\{t \geq 0 : \Psi(Y_t) \leq \bar{Y}_t\}$, where

$$\Psi(y) = y_0 \left(1 - \left(\frac{y^\kappa}{ny_0^\kappa}\right)^{n/(n-1)}\right)^{1/\kappa} \left(1 - \left(\frac{y^\kappa}{ny_0^\kappa}\right)^{1/(n-1)}\right)^{-1/\kappa}$$

if $0 \leq y < y_0 n^{1/\kappa}$ and $\Psi(y) = y$ otherwise. Note that $\lim_{y\uparrow y_0 n^{1/\kappa}} \Psi(y) = y_0 n^{1/\kappa}$ and hence $\tau \leq \inf\left\{u : Y_u = y_0 n^{1/\kappa}\right\}$.

2.4 Derivation of the equilibrium distribution

This section is devoted to the derivation of the optimal multipliers, and the candidate Nash equilibrium distributions given in Theorem 2.2.1.

Recall the definition of the Lagrangian $\mathcal{L}_{F,p}(G, q; \lambda, \gamma)$ for the optimisation problem (2.3). Let $l(p) = \sum_{k=1}^{n} \theta(k) C_{n-1}^{k-1} p^{k-1} (1-p)^{n-k} - \lambda U - \gamma$ and $L_F(x) = F(x)^{n-1} - \lambda x - \gamma$, then

$$\mathcal{L}_{F,p}(G, q; \lambda, \gamma) = \int_0^{\infty} L_F(x) G(dx) + l(p) q + \lambda x_0 + \gamma.$$

In order to have a finite optimal solution, we require $L_F(x) \leq 0$ on $[0, U)$ and $l(p) \leq 0$. Let $\mathcal{D}_F$ be the set of (λ, γ) such that $\mathcal{L}_{F,p}(\cdot, \cdot; \lambda, \gamma)$ has a finite maximum. Then $\mathcal{D}_F$ is defined by

$$\mathcal{D}_F = \{(\lambda, \gamma) : L_F(x) \leq 0 \text{ on } [0, U) \text{ and } l(p) \leq 0\}.$$

In order to reach the maximum value, we require $G\,(dx) = 0$ when $L_F(x) < 0$ and $q = 0$ if $l(p) < 0$. This means that for $(\lambda, \gamma) \in \mathcal{D}_F$ the maximum of $\mathcal{L}_{F,p}(\cdot, \cdot; \lambda, \gamma)$ occurs at (G^*, q^*) such that $G^*(dx) = 0$ when $L_F(x) < 0$ and $q^* = 0$ when $l(p) < 0$. If the Nash equilibrium is symmetric, then we must have $G^*(x) = F(x)$ and $q^* = p$, which means $L_{G^*}(x) = 0$ when $G^*(dx) > 0$ and $l(q^*) = 0$ when $q^* > 0$.

Because $L_{G^*}(x) = G^*(x)^{n-1} - \lambda x - \gamma$, we have $G^*(x) = \sqrt[n-1]{\lambda x + \gamma}$ when $G^*(dx) > 0$. Since G^* is non-decreasing and not constant, we must have $\lambda > 0$. Set $a = \inf\{x : G^*(x) > 0\}$ and $b = \sup\{x : G^*(x) < (1 - q^*)\}$. Since we are searching for the $G^*(x)$ that has no atom on $[0, U)$ and since $\sqrt[n-1]{\lambda x + \gamma}$ is strictly increasing, we must have $G^*(x) = \sqrt[n-1]{\lambda x + \gamma}$ on the whole of the interval $[a, b)$.

Observe that $0 \leq F^{n-1}(0) \leq \gamma$, so that if $(\lambda, \gamma) \in \mathcal{D}_F$ then γ is non-negative. Since G^* has no atom on $[0, U)$, $G^*(a) = 0$ and hence $\lambda a + \gamma = 0$, and by the non-negativity of a and γ and the positivity of λ, it follows that $\gamma = 0 = a$. Thus $G^*(x) = \sqrt[n-1]{\lambda x}$ on $[0, b)$ for some $\lambda > 0$ and $b \leq U$ which we must find. Further, we must find $q^* \in [0, 1)$ that solves $l(q^*) = 0$ if $q^* \neq 0$.

For a feasible solution, G^* and q^* should satisfy $\int_{[0,U)} x G^*(dx) + U q^* = x_0$ and $\int_{[0,U)} G^*(dx) + q^* = 1$. Thus, to get G^* and q^*, we should solve the following system of equations

$$\begin{cases} x_0 = \int_{[0,b)} x d\left(\sqrt[n-1]{\lambda x}\right) + Uq = b \sqrt[n-1]{\lambda b} - \frac{n-1}{n\lambda} (\lambda b)^{\frac{n}{n-1}} + Uq = \frac{b}{n} \sqrt[n-1]{\lambda b} + Uq, \\ 1 = \int_{[0,b)} d\left(\sqrt[n-1]{\lambda x}\right) + q = \sqrt[n-1]{\lambda b} + q, \\ l(q) = \sum_{k=1}^{n} \theta(k) C_{n-1}^{k-1} q^{k-1} (1-q)^{n-k} - \lambda U = 0, \text{ if } q \neq 0. \end{cases} \tag{2.12}$$

If $q = 0$, then from (2.12) we obtain $b = n x_0$ and $\lambda = 1/(n x_0)$. Thus, $q = 0$ is a feasible solution if $b = n x_0 \leq U$ and is not a feasible solution otherwise. Next we search for nonzero q that is feasible.

(i) Set $\theta(k) = \mathbf{1}_{\{k=1\}}$. Then the third equation in (2.12) can be reduced to $(1-q)^{n-1} - \lambda U = 0$ if $q \neq 0$. Thus, for $q \neq 0$, (2.12) can be reduced to

$$\lambda = \frac{(1-q)^{n-1}}{U}; \qquad b = \frac{(1-q)^{n-1}}{\lambda};$$
$$x_0 = \frac{(1-q)^n}{n\lambda} + Uq = \frac{U(1-q)}{n} + Uq = \frac{U + (n-1)Uq}{n}.$$

This gives us the optimal $q^* = \frac{nx_0 - U}{(n-1)U}$ and then $\lambda^* = \left[\frac{n(U-x_0)}{(n-1)U}\right]^{n-1} U^{-1}$ and $b^* = U$, which then gives us the G^* given in the theorem.

(ii) Set $\theta(k) = 1/k$. Observe that we have shown $\sum_{k=1}^{n} \frac{1}{k} C_{n-1}^{k-1} q^{k-1}(1-q)^{n-k} = \frac{1}{nq}\left[1-(1-q)^n\right]$ if $q \neq 0$ in the proof of Theorem 2.2.1. For $q \neq 0$, (2.12) can be reduced to

$$\lambda = \frac{1-(1-q)^n}{nUq}; \qquad b = \frac{(1-q)^{n-1}}{\lambda}; \tag{2.13}$$
$$x_0 = \frac{(1-q)^n}{n\lambda} + Uq = \frac{Uq(1-q)^n}{1-(1-q)^n} + Uq = \frac{Uq}{1-(1-q)^n}. \tag{2.14}$$

Let $\phi = 1 - q$ then (2.14) can be rewritten as $\Phi(\phi) = 0$, where $\Phi(x) = x_0 x^n - Ux + U - x_0$.

Since Φ is convex on $(0,1)$, we can find that there exists an solution ϕ to $\Phi(x) = 0$ such that $\phi \in (0,1)$ if and only if $U < nx_0$. Moreover, such solution is unique. Denote ϕ^* by this solution if it exists.

Therefore, if $U < nx_0$ then $q^* = 1 - \phi^*$ and then $\lambda^* = 1/(nx_0)$ and $b^* = nx_0(1-q^*)^{n-1}$ using (2.13). And this gives us the G^* given in the theorem.

Chapter 3

Contests with regret

Recall the Seel-Strack contest introduced in Section 1.1. In the contest, each player privately observes a transient diffusion process and chooses when to stop it. The player with the highest stopped value wins the contest, and each player's objective is to maximise her probability of winning the contest. This chapter considers an extension of the contest to one in which a penalty associated with failure to follow a winning strategy is added.

We add a behavioural finance aspect to the contest, in the form of regret theory in the sense of Loomes and Sugden [1982]. Again the objective of the agent is to maximise her chances of winning the contest, but now she is penalised if she has not won the contest, and she has behaved sub-optimally, in the sense that there was an alternative strategy which would have led to her winning the contest. Thus, in a competition between fund managers, a fund manager who has followed a poor strategy is not merely given a new role within the firm, but instead is terminated with disgrace.

Specifically, we will consider the following variant of the problem. The agent's choice of stopping rule determines her stopped value X. But if with hindsight we look at the best possible time she could have chosen, then we get a maximum value M she might have attained. We consider a problem in which the agent receives a reward of 1 if her stopped value is higher than the highest stopped value Y of all other agents, but she is penalised K if her stopped value is not the highest, and yet if she had stopped at the maximum value she might have attained M then she would have been the winner. If she is not the winner, and there is no strategy she might have followed which would led to her being the winner, then her reward is zero. Thus, her objective is to maximise $\mathbb{P}(X \geq Y) - K\mathbb{P}(X < Y \leq M)$.

In fact we consider three variants of the problem, in which an omniscient being (or the agent's supervisor) penalises the agent for stopping too soon, for stopping too late, or for stopping too soon or too late. In the first case, the agent faces regret over stopping too soon, and we consider the maximum value to be the maximum value attained by her process after the moment she chose to stop. In the second case the maximum is taken only over that part of the path which occurs before the chosen stopping time and the agent faces regret over stopping too late. In the third case we take the maximum over the whole path.

Our results are that in the first problem, the effect of the penalty is precisely equivalent to an increase in the number of opponents. An increase in K provides the agent with an incentive to aim for higher values, at the cost of stopping at low values more often. This is the same as the effect of competition from more opposing agents. In the second problem, which is both harder and more interesting, the optimal strategy is modified in a more subtle way. This case is relevant if the agent's process is unobservable from the point at which it is stopped, for instance if it is the gains from trade process arising from a dynamic investment strategy chosen by the agent. Now the agent faces a risk of a penalty whenever she stops below the value of the current maximum. For this reason she is reluctant to do so, although it is also sub-optimal to wait until her process hits zero, as this is a sure losing strategy. An increase in K gives her an incentive to stop more quickly. The third problem might be expected to be a combination of the two previous problems, but in fact there is a natural simplification which leads to the optimum being the same solution as the original Seel-Strack problem.

The remainder of this chapter is constructed as follows. In Section 3.1 we introduce the contest with regrets, which we then solve in the three cases described above in Sections 3.2, 3.3 and 3.4, respectively. In Section 3.5 we explain the origin of the optimal multipliers and the candidate Nash equilibrium distributions. Finally, in Section 3.6 we show how our results for Brownian motion absorbed at zero extend to general time homogeneous diffusions.

3.1 The model

Recall the model introduced in Section 1.1. There are n players with labels $i \in I = \{1, 2, \dots, n\}$ who take part in the contest. Player i privately observes the continuous-time realisation of a Brownian motion $X^i = (X^i_t)_{t \in \mathbb{R}^+}$ absorbed at zero with $X^i_0 = x_0 > 0$, where x_0 is a constant. Let $\mathcal{F}^i_t = \sigma(\{X^i_s : s \leq t\})$ and set $\mathbb{F}^i = (\mathcal{F}^i_t)_{t \geq 0}$. Player i chooses an $\mathbb{F}^i$-stopping time τ^i, and without loss of generality we restrict attention to $\tau^i \leq H^i_0 = \inf\{t \geq 0 : X^i_t = 0\}$. Player i observes her own process X^i, but not X^j for $j \neq i$; nor does she observe the stopping times chosen by the other agents. Moreover, the processes X^i are assumed to be independent.

We now assume that there is an omniscient judge who can observe the path of X^i, and not just the stopped value, and who penalises the agent for the failure to use a winning stopping rule if such a strategy exists. This judge represents the supervisor of the agent, and the agent faces penalties (such as dismissal) in cases where after the fact she is seen to have followed a losing strategy, when a winning strategy existed.

The player with the highest stopping value wins unit reward, that is $\forall i \in I$, player i wins 1 if she stops at time τ^i such that $X^i_{\tau^i} > X^j_{\tau^j}$ $\forall j \neq i$. (We discuss the award of the prize in the case of a tie below.) In addition, the player is penalised $K \geq 0$ if her stopped value is not the highest, and if she had an alternative strategy which would, with the benefit of hindsight, have allowed her to win. (The case $K = 0$ corresponds to the standard problem.) Given that the best strategy for agent i is to stop at the maximum value M^i attained by X^i, this means that player i loses K if she stops at τ^i such that $X^i_{\tau^i} < \max_{j\neq i} X^j_{\tau^j} < M^i$.

There are several different potential definitions for the quantity M^i which represents the maximum the agent could have achieved. Depending on the interpretation, this could be the maximum over the entire path $M^i = \max\{X^i_t; 0 \leq t \leq H^i_0\}$, or it could be that only that part of the path before the agent's chosen stopping time is considered, $M^i = \max\{X^i_t; 0 \leq t \leq \tau^i\}$, or only that part of the path after the agent's chosen stopping time, $M^i = \max\{X^i_t; \tau^i \leq t \leq H^i_0\}$. These different interpretations will lead to different Nash equilibria. We consider the three cases separately in the next three sections.

As before, ties are broken randomly. If there are k players who stop at the highest value then these players each wins $\frac{1}{k}$. Further, player i loses K_2 if she stops at τ^i such that $X^i_{\tau^i} < \max_{j\neq i} X^j_{\tau^j} = M^i$, where $0 \leq K_2 \leq K$. Hence player i who stops at $X^i_{\tau^i}$ with maximum value M^i has pay-off

$$\frac{1}{k}\mathbf{1}_{\{X^i_{\tau^i}=\max_{j\in I} X^j_{\tau^j}\}} - K\mathbf{1}_{\{X^i_{\tau^i}<\max_{j\neq i} X^j_{\tau^j}<M^i\}} - K_2\mathbf{1}_{\{X^i_{\tau^i}<\max_{j\neq i} X^j_{\tau^j}=M^i\}},$$

where $k = \left|\left\{i \in I : X^i_{\tau^i} = \max_{j\in I} X^j_{\tau^j}\right\}\right|$.

Our objective is to find a Nash equilibrium which is represented by a family of stopping rules $(\tau^i)_{i\in I}$. Since the values $(X^i_{\tau^i}, M^i)_{i\in I}$ are sufficient statistics for the problem, the Nash equilibrium can be characterised by the laws $(\nu^i)_{i\in I}$ of $(X^i_{\tau^i}, M^i)_{i\in I}$. Then, in equilibrium, the agent can use any stopping rule for which $(X^i_{\tau^i}, M^i)$ has law ν^i.

In our generalised setting we will limit our search to symmetric atom-free Nash equilibria. Then, since there are no atoms, the probability of a tie is zero. Thus, neither the method of breaking ties nor the value of K_2 will affect our results.

Suppose that the other players all choose $F(x)$ as their target distribution of X_τ, where F is continuous and such that $F(0) = 0$. Let $Y = \max_{j \neq i} X^j_{\tau^j}$. Then Y has cumulative distribution function F_Y given by $F_Y(y) = F(x)^{n-1}$ and conditional on $(X^i_{\tau_i} = x, M^i = m)$, the expected pay-off to agent i is

$$\begin{aligned}\mathbb{P}(Y \leq x) - K\mathbb{P}(x < Y \leq m) &= F(x)^{n-1} - K\left[F(m)^{n-1} - F(x)^{n-1}\right] \\ &= (1+K)F(x)^{n-1} - KF(m)^{n-1}.\end{aligned}$$

We say $\nu = \nu(dx, dm)$ is a feasible measure if ν is a possible joint law of $(X^i_{\tau^i}, M^i)$. Then the aim of agent i is to choose a feasible measure to maximise

$$(1+K)\mathbb{E}[F(X^i_{\tau^i})^{n-1}] - K\mathbb{E}[F(M^i)^{n-1}]. \tag{3.1}$$

Note that the set of feasible measures depends on the definition of M. In particular, we must have $\mathbb{E}(X^i_{\tau^i}) \leq x_0$, since X^i is a non-negative supermartingale and τ^i is finite almost surely.

3.2 Contest with regret over future failure to stop

In this section we consider the contest in which the agent is penalised for stopping too soon. We consider the maximum value M^i to be defined by

$$M^i := M^i_{[\tau^i, H^i_0]} = \sup_{\tau^i \leq t \leq H^i_0} X^i_t.$$

Theorem 3.2.1. *There exists a symmetric, atom-free Nash equilibrium for the problem for which $X^i_{\tau^i}$ has law $F(x)$, where for $x \geq 0$*

$$F(x) = \min\left\{ \sqrt[N-1]{\frac{x}{Nx_0}}, 1 \right\}$$

with $N = n + K(n-1)$.

Remark 3.2.1. The agent follows exactly the same optimal strategy as an agent in a different setup, where there is no penalty, but the total number of contestants is increased to $N = n + K(n-1)$.

Proof. Fix any $i \in I$. Denote by ν the joint distribution of $X^i_{\tau^i}$ and $M^i_{[\tau^i, H^i_0]}$ and denote by $G(x)$ the marginal distribution of $X^i_{\tau^i}$. Then, using the strong Markov property and the martingale property of X^i,

$$
\begin{aligned}
\nu([0,x]\times[0,y]) &= \mathbb{P}\left(X^i_{\tau^i}\le x, M^i_{[\tau^i,H^i_0]}\le y\right)\\
&= \int_0^x \mathbb{P}\left(M^i_{[\tau^i,H^i_0]}\le y|X^i_{\tau^i}=z\right)G(dz)\\
&= \int_0^x \mathbb{P}\left(\tilde{H}^i_0<\tilde{H}^i_{y\vee z}|X^i_{\tau^i}=z\right)G(dz)\\
&= \int_0^x \mathbb{P}\left(H^i_0<H^i_{y\vee z}|X^i_0=z\right)G(dz) = \int_0^x \frac{y\vee z - z}{y}G(dz), \quad (3.2)
\end{aligned}
$$

where $H^i_w=\inf\{t\ge 0: X^i_t=w\}$ and $\tilde{H}^i_w=\inf\{t\ge\tau^i: X^i_t=w\}$ for any $w\ge 0$.

Let $\mathcal{A}$ be the set of non-decreasing right-continuous functions $f:[0,\infty)\mapsto[0,\infty)$. Suppose that the other players all choose $F(x)$ as their target distribution of X_τ, where $F(x)$ is continuous and satisfies $F(0)=0$. Substituting (3.2) into (3.1), the expected pay-off of player i becomes

$$
\begin{aligned}
&\int_0^\infty\int_0^\infty\left[(1+K)F(x)^{n-1}-KF(y)^{n-1}\right]\nu(dx,dy)\\
&\quad = \int_0^\infty (1+K)F(x)^{n-1}G(dx) - \int_0^\infty\int_x^\infty KF(y)^{n-1}\frac{x}{y^2}dyG(dx)\\
&\quad = \int_0^\infty\left[(1+K)F(x)^{n-1} - Kx\int_x^\infty\frac{F(y)^{n-1}}{y^2}dy\right]G(dx).
\end{aligned}
$$

Thus, given other players' choices, player i would like to choose G to solve

$$
\max_{G\in\mathcal{A}}\int_0^\infty\left[(1+K)F(x)^{n-1} - Kx\int_x^\infty\frac{F(y)^{n-1}}{y^2}dy\right]G(dx) \qquad (3.3)
$$

subject to $\int_0^\infty xG(dx)\le x_0$ and $\int_0^\infty G(dx)=1$.

Introducing multipliers λ and γ for the two constraints, the Lagrangian for the optimisation problem (3.3) is then

$$
\begin{aligned}
\mathcal{L}_F(G;\lambda,\gamma) = \int_0^\infty\left[(1+K)F(x)^{n-1} - Kx\int_x^\infty\frac{F(y)^{n-1}}{y^2}dy - \lambda x-\gamma\right]G(dx)\\
+\lambda x_0+\gamma.
\end{aligned}
$$

Let $\mathcal{A}_D(x_0)$ be the subset of $\mathcal{A}$ corresponding to distribution functions of random variables with mean less than or equal to x_0, that is

$$\mathcal{A}_D(x_0) = \left\{ f \in \mathcal{A} : \lim_{x \uparrow \infty} f(x) = 1 \text{ and } \int_0^\infty x f(dx) \le x_0 \right\}.$$

Now we state a variant of the Lagrangian sufficiency theorem for our problem.

Proposition 3.2.1. *If G^*, λ^* and γ^* exist such that $\lambda^* \ge 0$, $G^*(0) = 0$, G^* is continuous, $G^* \in \mathcal{A}_D(x_0)$, $\int_0^\infty x G^*(dx) = x_0$ and*

$$\mathcal{L}_{G^*}(G^*; \lambda^*, \gamma^*) \ge \mathcal{L}_{G^*}(G; \lambda^*, \gamma^*) \qquad \textit{for all } G \in \mathcal{A}, \tag{3.4}$$

then G^* is a symmetric, atom-free Nash equilibrium.

Proof. We seek a symmetric atom-free Nash equilibrium. A symmetric atom-free Nash equilibrium is identified with a continuous distribution function $G^* \in \mathcal{A}_D(x_0)$ with $G^*(0) = 0$ and the property that for any $G \in \mathcal{A}_D(x_0)$

$$\begin{aligned} &\int_0^\infty \left[(1+K) G^*(x)^{n-1} - Kx \int_x^\infty \frac{G^*(y)^{n-1}}{y^2} dy \right] G^*(dx) \\ &\qquad \ge \int_0^\infty \left[(1+K) G^*(x)^{n-1} - Kx \int_x^\infty \frac{G^*(y)^{n-1}}{y^2} dy \right] G(dx). \end{aligned}$$

If $G \in \mathcal{A}_D(x_0)$ then

$$\begin{aligned} &\int_0^\infty \left[(1+K) G^*(x)^{n-1} - Kx \int_x^\infty \frac{G^*(y)^{n-1}}{y^2} dy \right] G(dx) \\ &\qquad = \mathcal{L}_{G^*}(G; \lambda^*, \gamma^*) - \lambda^* \left[x_0 - \int_0^\infty x G(dx) \right] \le \mathcal{L}_{G^*}(G; \lambda^*, \gamma^*), \end{aligned}$$

since $\lambda^* \ge 0$ and $\int_0^\infty x G(dx) \le x_0$. Then, under the hypotheses of the proposition,

$$\begin{aligned} &\int_0^\infty \left[(1+K) G^*(x)^{n-1} - Kx \int_x^\infty \frac{G^*(y)^{n-1}}{y^2} dy \right] G^*(dx) \\ &\qquad = \mathcal{L}_{G^*}(G^*; \lambda^*, \gamma^*) \ge \mathcal{L}_{G^*}(G; \lambda^*, \gamma^*) \\ &\qquad \ge \int_0^\infty \left[(1+K) G^*(x)^{n-1} - Kx \int_x^\infty \frac{G^*(y)^{n-1}}{y^2} dy \right] G(dx). \end{aligned}$$

Thus, G^* is a symmetric, atom-free Nash equilibrium.

□

Return to the proof of Theorem 3.2.1. Let $G^*(x) = \min\left\{1, \sqrt[N-1]{x/(Nx_0)}\right\}$

on $[0,\infty)$, $\lambda^* = 1/(Nx_0)$ and $\gamma^* = 0$, where $N = n + K(n-1)$. It is easy to check that $\lambda^* \geq 0$, $G^*(0) = 0$, G^* is continuous, $G^* \in \mathcal{A}_D(x_0)$ and $\int_0^\infty xG^*(dx) = x_0$. Moreover,

$$\mathcal{L}_{G^*}(G;\lambda^*,\gamma^*) = \int_0^\infty \left[(1+K)G^*(x)^{n-1} - Kx\int_x^\infty \frac{G^*(y)^{n-1}}{y^2}dy - \lambda^* x - \gamma^*\right] G(dx) + \lambda^* x_0 + \gamma^*$$

$$= \int_{Nx_0}^\infty \left[1 - \frac{x}{Nx_0}\right] G(dx) + \frac{1}{N} \leq \frac{1}{N} = \mathcal{L}_{G^*}(G^*;\lambda^*,\gamma^*).$$

Hence, by the Lagrangian sufficiency theorem (Proposition 3.2.1), G^* is a symmetric, atom-free Nash equilibrium.

□

Remark 3.2.2. In this version of the problem, the stopping decision depends on the current value of X alone, and not on the current maximum. This is because the penalty depends on the future maximum, which conditional on the current value of the process is independent of the past maximum.

3.3 Contest with regret over past failure to stop

This section discusses the contest with regret over past failure to stop, that is player is penalised when she could have won if she had stopped sooner. This case is relevant when the omniscient being can only observe the realisation of X^i up to the stopping time chosen by the agent. In this case the maximum value M^i is defined by

$$M^i := M^i_{\tau^i} = \sup_{0\leq t\leq \tau^i} X^i_t.$$

Consider the problem facing a single agent under the assumption that the strategies of the other agents in the contest are fixed. Temporarily we drop the subscript denoting the label of the agent. Recall that the pay-off to the agent is $(1+K)F(X_\tau)^{n-1} - KF(M_\tau)^{n-1}$. For a continuous martingale Kertz and Rösler [1990] characterises all possible joint laws of (X_τ, M_τ) and hence the problem is reduced to a search over measures with these characteristics. However, an alternative is to split the optimisation problem into a two-stage procedure: first for any feasible distribution of X_τ (a non-negative random variable with mean less than or equal to x_0) find the joint law of (X_τ, M_τ) for which M_τ is as small as possible in distribution (in the sense of first order stochastic dominance)—such a joint law exists by results of Perkins [1986]—and then minimise a modified objective function over feasible laws of X_τ.

For a given atom-free law of X_τ (recall that we are seeking a symmetric, atom-free distribution, so we focus on this case), the joint law of (X_τ, M_τ) for which M_τ is minimised is such that mass is placed only on the set $A = \{(x,x) : x \geq x_0\} \cup \{(x, \Phi(x)) : x < x_0\}$, where $\Phi : (0, x_0) \mapsto (x_0, \infty)$ is a strictly decreasing function. Let ϕ be inverse to Φ. Then, if G denotes the marginal law of X_τ and $\mathbb{E}(X_\tau) = x_0$, we can conclude from Doob's submartingale inequality, in conjunction with the set identity $(M_\tau \geq m) = (X_\tau \geq m) \cup (X_\tau \leq \phi(m))$, that for $m \geq x_0$

$$\begin{aligned} 0 &= \mathbb{E}[m - X_\tau; X_\tau \geq m] + \mathbb{E}[m - X_\tau; X_\tau \leq \phi(m)] \\ &= \int_m^\infty (m-y)G(dy) + \int_0^{\phi(m)} (m-y)G(dy) \end{aligned} \tag{3.5}$$

which, since X_τ has mean x_0, is equivalent to

$$0 = m - x_0 + (m - \phi(m))G(\phi(m)) - \int_{\phi(m)}^m G(y)dy. \tag{3.6}$$

In differential form, assuming G and ϕ are differentiable, this becomes

$$0 = \phi'(m)(m - \phi(m))G'(\phi(m)) + 1 + G(\phi(m)) - G(m). \tag{3.7}$$

It follows from the results of Perkins [1986] and Hobson and Pedersen [2002], that if G is the law of an atom-free non-negative random variable, then there exists a decreasing function ϕ solving (3.5). Further, if ξ is a random variable such that

$$\mathbb{P}(\xi \geq s) = \exp\left(- \int_{(x_0,s)} \frac{G(du)}{1 - G(u) + G(\phi(u))} \right) \qquad \text{for any } s \geq x_0$$

and if $\tau = \tau_\xi \wedge \tau_\phi$ where $\tau_\xi = \inf\{t > 0 | M_t \geq \xi\}$ and $\tau_\phi = \inf\{t > 0 | X_t \leq \phi(M_t)\}$, then X_τ has law G and (X_τ, M_τ) places no mass outside of A. Moreover, amongst the class of joint laws for (X_τ, M_τ) such that X_τ has law G, M_τ is as small as possible in distribution.

Theorem 3.3.1. *Suppose there exists a finite real number $r > x_0$, a once differentiable strictly decreasing function $\phi : [x_0, r] \mapsto [0, x_0]$, a thrice differentiable strictly increasing and strictly convex function $\psi : [x_0, r] \mapsto [0, 1]$ and a once differentiable strictly decreasing function $\theta : [x_0, r] \mapsto [0, 1]$ such that ϕ, ψ and θ solve the following* *system of equations*

$$
(\dagger)\quad \begin{cases} \phi'(y)\psi'(y) = (1+K)\theta'(y), & (3.8)\\ K\psi'(y) = (y-\phi(y))\psi''(y), & (3.9)\\ \dfrac{y-\phi(y)}{n-1}\theta'(y) = \left(\psi(y)^{\frac{1}{n-1}} - 1\right)\theta(y)^{\frac{n-2}{n-1}} - \theta(y), & (3.10)\end{cases}
$$

and satisfy that $\phi(x_0) = x_0$, $\psi(r) = 1$, $\psi'(r-) = \frac{K+1}{r}$, $\psi''(r-) = \frac{K(K+1)}{r^2}$ *and* $\theta(x_0) = \psi(x_0)$.

(i) Then

$$\theta(y) = \psi(y) - \frac{K}{K+1}\frac{\psi'(y)^2}{\psi''(y)}. \tag{3.11}$$

(ii) Moreover, there exists a symmetric, atom-free Nash equilibrium for the problem for which $X^i_{\tau^i}$ *and* $M^i_{\tau^i}$ *have joint law* ν^* *that is determined by the marginal distribution* G^* *of* $X^i_{\tau^i}$, *given by* $G^*(x) = 0$ *for* $x \le 0$, $G^*(x) = 1$ *for* $x \ge r$ *and*

$$G^*(x) = \begin{cases} \theta(\phi^{-1}(x))^{\frac{1}{n-1}} & , \text{ if } 0 < x < x_0,\\ \psi(x)^{\frac{1}{n-1}} & , \text{ if } x_0 \le x < r,\end{cases} \tag{3.12}$$

otherwise, and the conditional distribution of $M^i_{\tau^i}$ *given* $X^i_{\tau^i}$ *such that*

$$M^i_{\tau^i} = \begin{cases} X^i_{\tau^i} & , \text{ if } X^i_{\tau^i} \ge x_0,\\ \phi^{-1}(X^i_{\tau^i}) & , \text{ if } 0 \le X^i_{\tau^i} < x_0.\end{cases}$$

Proof. The conditions in the theorem imply some properties of function ϕ: let $y = r-$ in (3.9), then since $\psi'(r-) = \frac{K+1}{r}$ and $\psi''(r-) = \frac{K(K+1)}{r^2}$ we have $\phi(r) = 0$; since (3.9) holds and by the positivity of ψ' and ψ'', we have $\phi(y) < y$ on (x_0, r).

(i) Integrating (3.8) with respect to y,

$$
\begin{aligned}
&(K+1)\theta(z) - (K+1)\theta(x_0)\\
&= \int_{x_0}^{z} \phi'(y)\psi'(y)dy = \phi(z)\psi'(z) - \phi(x_0)\psi'(x_0+) - \int_{x_0}^{z}\phi(y)\psi''(y)dy. \qquad (3.13)
\end{aligned}
$$

Rearranging (3.9) and integrating it,

$$
\begin{aligned}
\int_{x_0}^{z}\phi(y)\psi''(y)dy &= \int_{x_0}^{z} y\psi''(y)dy - \int_{x_0}^{z} K\psi'(y)dy\\
&= z\psi'(z) - x_0\psi'(x_0+) - (1+K)\psi(z) + (1+K)\psi(x_0).
\end{aligned}
$$

Combining this equation with (3.13), we find

$$\theta(y) = \frac{1}{K+1}(\phi(y) - y)\psi'(y) + \psi(y). \tag{3.14}$$

Then, substituting (3.9) into (3.14), (3.11) follows.

(ii) Let $\mathcal{E}(x_0)$ be the set of measures $\nu(dx, dm)$ on $[0, \infty) \times [0, \infty)$ such that $\nu(dx, dm)$ has no mass on $\{(x, m) : m < x \text{ or } m < x_0\}$, and let $\mathcal{E}_D(x_0)$ be the subset of $\mathcal{E}(x_0)$ corresponding to probability measures of a pair of random variables $X \leq M$ such that X is a random variable with mean less than or equal to x_0 and $\mathbb{E}[X - z; M \geq z] \leq 0$ for all $z \geq x_0$. Note that the last equation comes from the Doob's submartingale inequality, applied in the continuous supermartingale case.

Fix player $i \in I$. Suppose that the other players all choose $F(x)$ as their target distribution of X_τ and suppose that F is continuous with $F(0) = 0$. Then the aim of player i is to choose ν to solve

$$\max_{\nu \in \mathcal{E}(x_0)} \left\{ \int_0^\infty \int_0^\infty \left[(1+K)F(x)^{n-1} - KF(m)^{n-1} \right] \nu(dx, dm) \right\} \tag{3.15}$$

subject to $\int_0^\infty \int_0^\infty x\nu(dx, dm) \leq x_0$, $\int_0^\infty \int_0^\infty \nu(dx, dm) = 1$ and $\int_{x=0}^\infty \int_{m=z}^\infty (x - z)\nu(dx, dm) \leq 0 \ \forall z \geq x_0$.

Introduce multipliers λ and γ for the first two constraints, and for each $z \geq x_0$ introduce a Lagrange multiplier $\eta(z)$ for the last constraint: the constraint becomes $\int_0^\infty \int_0^\infty \int_{z=x_0}^m \{\eta(z)(x - z)dz\} \nu(dx, dm) = 0$. Then the Lagrangian for the optimisation problem (3.15) is

$$\begin{aligned} \mathcal{L}_F(\nu; \lambda, \gamma, \eta) = \int_0^\infty \int_0^\infty \Big[& (1+K)F(x)^{n-1} - KF(m)^{n-1} - \lambda x - \gamma \\ & - \int_{x_0}^m \eta(z)(x - z)dz \Big] \nu(dx, dm) + \lambda x_0 + \gamma. \end{aligned} \tag{3.16}$$

Now we state a variant of the Lagrangian sufficiency theorem for problem (3.15).

Proposition 3.3.1. *If ν^*, λ^*, γ^* and η^* exist such that $\lambda^* \geq 0$, $\eta^* \geq 0$, $\nu^* \in \mathcal{E}_D(x_0)$, $\int_0^\infty \int_0^\infty x\nu^*(dx, dm) = x_0$, $\int_{x=0}^\infty \int_{m=z}^\infty (x - z)\nu^*(dx, dm) = 0$ for all $z \geq x_0$, G^* is continuous, $G^*(0) = 0$ and*

$$\mathcal{L}_{G^*}(\nu^*; \lambda^*, \gamma^*, \eta^*) \geq \mathcal{L}_{G^*}(\nu; \lambda^*, \gamma^*, \eta^*) \qquad \text{for all } \nu \in \mathcal{E}(x_0), \tag{3.17}$$

where $G^*(x) = \nu^*([0, x] \times [0, \infty))$, then ν^* is a symmetric, atom-free Nash equilibrium.

Proof. A symmetric atom-free Nash equilibrium is identified with a measure $\nu^* \in \mathcal{E}_D(x_0)$ with the property that G^* is continuous, $G^*(0) = 0$ and for any measure $\nu \in \mathcal{E}_D(x_0)$

$$\begin{aligned}\int_0^\infty \int_0^\infty &\left[(1+K)G^*(x)^{n-1} - KG^*(m)^{n-1}\right] \nu^*(dx, dm)\\ &\geq \int_0^\infty \int_0^\infty \left[(1+K)G^*(x)^{n-1} - KG^*(m)^{n-1}\right] \nu(dx, dm),\end{aligned}$$

where $G^*(x) = \nu^*([0,x] \times [0,\infty))$ for all $x \geq 0$.

If $\nu \in \mathcal{E}_D(x_0)$ then

$$\begin{aligned}\int_0^\infty \int_0^\infty &\left[(1+K)G^*(x)^{n-1} - KG^*(m)^{n-1}\right] \nu(dx, dm)\\ &= \mathcal{L}_{G^*}(\nu; \lambda^*, \gamma^*, \eta^*) - \lambda^* \left[x_0 - \int_0^\infty \int_0^\infty x\nu(dx, dm)\right]\\ &\qquad - \int_{z=x_0}^\infty \eta^*(z) \int_{x=0}^\infty \int_{m=z}^\infty (x-z)\nu(dx, dm)dz\\ &\leq \mathcal{L}_{G^*}(\nu; \lambda^*, \gamma^*, \eta^*).\end{aligned}$$

Then, under the hypotheses of the proposition,

$$\begin{aligned}\int_0^\infty \int_0^\infty &\left[(1+K)G^*(x)^{n-1} - KG^*(m)^{n-1}\right] \nu^*(dx, dm)\\ &= \mathcal{L}_{G^*}(\nu^*; \lambda^*, \gamma^*, \eta^*) \geq \mathcal{L}_{G^*}(\nu; \lambda^*, \gamma^*, \eta^*)\\ &\geq \int_0^\infty \int_0^\infty \left[(1+K)G^*(x)^{n-1} - KG^*(m)^{n-1}\right] \nu(dx, dm).\end{aligned}$$

Thus, ν^* is a symmetric, atom-free Nash equilibrium.

□

Now return to the proof of Theorem 3.3.1. On $[0,\infty) \times [0,\infty)$ let ν^* be the joint law given in the theorem and G^* be its marginal distribution with respect to X_τ. In particular, G^* is given by (3.12). Let $\lambda^* = \psi'(x_0+)$, $\gamma^* = \psi(x_0) - x_0\psi'(x_0+)$, $\eta^*(y) = \psi''(y)$ for $x_0 < y < r$, and $\eta^*(y) = 0$ for $y \geq r$. It is clear that $\lambda^* \geq 0$ and $\eta^* \geq 0$.

We first show that $G^*(0+) = 0$, $\lim_{y\uparrow\infty} G^*(y) = 1$, $G^*(y)$ is continuous and non-decreasing, $\int_0^\infty uG^*(du) = x_0$ and $\int_{x=0}^\infty \int_{y=z}^\infty (x-z)\nu^*(dx, dy) = 0$ for all $z \geq x_0$. This implies that $\nu^* \in \mathcal{E}_D(x_0)$.

Letting $y = r-$ in (3.11), we find $\theta(r) = 0$. Then since $\phi(r) = 0$, we have $G^*(0+) = \theta(\phi^{-1}(0))^{1/(n-1)} = \theta(r)^{1/(n-1)} = 0$. Moreover, $\lim_{y\uparrow\infty} G^*(y) = 1$

follows from the finiteness of r. We have $\phi^{-1}(x_0) = x_0$ and hence G^* is continuous at x_0. Since both ϕ and θ are decreasing and continuous on $[x_0, r]$, $G^*(x) = \theta(\phi^{-1}(x))^{1/(n-1)}$ is increasing and continuous on $[0, x_0]$. Then, since $\psi(y)$ is increasing and continuous on $[x_0, r]$, G^* is continuous and non-decreasing on the whole interval of $[0, r]$. Note that this implies $r = \sup\{x \geq 0 : G^*(x) < 1\}$.

For $y > x_0$, we have $G^*(\phi(y)) = \theta(y)^{1/(n-1)}$ and so $\phi'(y)(G^*)'(\phi(y)) = \theta(y)^{1/(n-1)-1}\theta'(y)/(n-1)$. Then, using (3.10),

$$\begin{aligned}\phi'(y)(y-\phi(y))(G^*)'(\phi(y)) &= \theta(y)^{\frac{2-n}{n-1}}\frac{y-\phi(y)}{n-1}\theta'(y)\\ &= \psi(y)^{\frac{1}{n-1}} - 1 - \theta(y)^{\frac{1}{n-1}} = G^*(y) - 1 - G^*(\phi(y)).\end{aligned}$$

Hence,

$$\phi'(y)(y-\phi(y))(G^*)'(\phi(y)) + (1-\phi'(y))G^*(\phi(y)) = G^*(y) - 1 - \phi'(y)G^*(\phi(y)),$$

and integrating from x to r

$$-(x-\phi(x))G^*(\phi(x)) = -(r-x) + \int_x^r G^*(y)dy + \int_0^{\phi(x)} G^*(y)dy. \tag{3.18}$$

Then, setting $x = x_0$, we recover $x_0 = \int_0^r (1 - G^*(y))dy$ so that a random variable with distribution function G^* has mean x_0.

Finally, from its construction we have that ν^* only puts mass on A. Hence, from (3.5),

$$\int_{x=0}^{\infty}\int_{y=z}^{\infty}(x-z)\nu^*(dx,dy) = \int_0^{\phi(z)}(x-z)G^*(dx) + \int_z^r (x-z)G^*(dx) = 0.$$

Now we prove that (3.17) holds. Let

$$L^*(x,y) = (1+K)G^*(x)^{n-1} - KG^*(y)^{n-1} - \lambda^* x - \gamma^* - \int_{x_0}^{y}\eta^*(z)(x-z)dz$$

and then $\mathcal{L}_{G^*}(\nu; \lambda^*, \gamma^*, \eta^*) = \int_0^\infty \int_0^\infty L^*(x,y)\nu(dx,dy) + \lambda^* x_0 + \gamma^*$.

For notational convenience, extend the domain of ψ to $[0, r]$ by defining $\psi(x) = \theta(\phi^{-1}(x))$ for $x \in [0, x_0)$. Then, for $x < x_0$, $\psi'(x) = \frac{\theta'(\phi^{-1}(x))}{\phi'(\phi^{-1}(x))} = \frac{\psi'(\phi^{-1}(x))}{K+1} > 0$, where the last equality comes from (3.8). Moreover $\psi''(x) = \frac{\psi''(\phi^{-1}(x))}{(1+K)\phi'(\phi^{-1}(x))} < 0$. Thus, ψ is increasing on $[0, r]$, $\psi''(x) < 0$ if $x \in (0, x_0)$ and $\psi''(x) > 0$ if $x \in (x_0, r)$.

Fix $y \in [x_0, r]$. For any $0 \le x \le y$,

$$\begin{aligned}
L^*(x,y) &= (1+K)\psi(x) - K\psi(y) - \psi'(x_0+)x \\
&\qquad\qquad - \psi(x_0) + x_0\psi'(x_0+) - \int_{x_0}^{y} \psi''(z)(x-z)dz \\
&= (1+K)(\psi(x) - \psi(y)) + (y-x)\psi'(y). \qquad (3.19)
\end{aligned}$$

We have $L^*(\phi(y), y) = (1+K)(\theta(y) - \psi(y)) + (y - \phi(y))\psi'(y)$, and then by (3.11) and (3.9), $L^*(\phi(y), y) = 0$. It is also clear that $L^*(y,y) = 0$. Differentiating (3.19) with respect to x,

$$\frac{\partial L^*}{\partial x}(x,y) = (1+K)\psi'(x) - \psi'(y); \qquad \frac{\partial^2 L^*}{\partial x^2}(x,y) = (1+K)\psi''(x).$$

Then, $\frac{\partial^2 L^*}{\partial x^2}(x,y) < 0$ on $(0, x_0)$ and $\frac{\partial^2 L^*}{\partial x^2}(x,y) > 0$ on (x_0, y). Since $\frac{\partial L^*}{\partial x}(\phi(y), y) = (1+K)\psi'(\phi(y)) - \psi'(y) = (1+K)\frac{\psi'(y)}{K+1} - \psi'(y) = 0$ and $\frac{\partial L^*}{\partial x}(y,y) = K\psi'(y) > 0$, it follows that $\frac{\partial L^*}{\partial x}(x,y) > 0$ if $x \in (0, \phi(y))$, $\frac{\partial L^*}{\partial x}(x,y) < 0$ if $x \in (\phi(y), \tilde{x})$ and $\frac{\partial L^*}{\partial x}(x,y) > 0$ if $x \in (\tilde{x}, y)$, where $\tilde{x} \in (x_0, y)$ is such that $\frac{\partial L^*}{\partial x}(x,y)|_{x=\tilde{x}} = 0$. It follows that $L^*(x,y) < 0$ for $x \in [0, \phi(y)) \cup (\phi(y), y)$.

Now fix $y > r$. For any $0 \le x \le y$, since $\psi'(r-) = (K+1)/r$ and writing $\tilde{\psi}(x) = \psi(x) - x/r$,

$$\begin{aligned}
L^*(x,y) &= (1+K)\psi(x) - K - \psi'(x_0+)x \\
&\qquad\qquad - \psi(x_0) + x_0\psi'(x_0+) - \int_{x_0}^{r} \psi''(z)(x-z)dz \\
&= (1+K)(\psi(x) - 1) + (r-x)\psi'(r-) = (1+K)\tilde{\psi}(x).
\end{aligned}$$

If $x \in (r, y]$ then $L^*(x,y) = (1+K)\frac{1}{r}[r-x] < 0$. Now suppose $x \in (0, r)$. Since $\phi(r) = 0$, we have $\psi'(0+) = \frac{1}{r}$ and thus $\tilde{\psi}'(0+) = 0$. Further $\tilde{\psi}'(r-) = \frac{K}{r} > 0$. Then, by the sign of $\psi''(x)$, we get $\tilde{\psi}'(x)$ is negative and then positive on $(0, r)$. Since $\tilde{\psi}(0) = \tilde{\psi}(r) = 0$, we deduce that $\tilde{\psi}(x) < 0$ on $(0, r)$. Thus $L^*(x,y) < 0$ for $x \in (0, r)$.

From above analysis, we know $L^*(x,y) \le 0$ for any (x,y) such that $0 \le x \le y$ and $y \ge x_0$. This means that, for any $\nu \in \mathcal{E}(x_0)$, since $\lambda^* x_0 + \gamma^* = \psi(x_0)$,

$$\mathcal{L}_{G^*}(\nu; \lambda^*, \gamma^*, \eta^*) \le \psi(x_0) = \mathcal{L}_{G^*}(\nu^*; \lambda^*, \gamma^*, \eta^*).$$

Thus, ν^* is a symmetric, atom-free Nash equilibrium from Proposition 3.3.1.

□

It remains to show that there exists a constant r and functions (ϕ, θ, ψ) which satisfy the hypotheses of Theorem 3.3.1 and hence that a symmetric atom-free Nash equilibrium always exists. The following lemma is key in defining the appropriate entities.

Lemma 3.3.1. *Let $J(u)$ solve the ordinary differential equation*

$$J'(u) = \frac{J(u) + 1 - (1-u)^{1/(n-1)}}{(K+1)\left[1 - u - J(u)^{n-1}\right]} \tag{3.20}$$

subject to $J(0) = 0$ and $u \geq 0$. Let $u^* = \sup\left\{u : J(u) < (1-u)^{1/(n-1)}\right\}$.

(i) Let $z^* = 1 - u^*$ and define

$$H(z) = \frac{K}{(K+1)\left[z - J(1-z)^{n-1}\right]} \tag{3.21}$$

on $[z^*, 1]$. Then, $z^* > 0$, H is positive on $(z^*, 1)$ and $\int_{z^*}^1 \exp\left(\int_w^1 H(v)dv\right) dw < (K+1)$.

(ii) Define

$$r = \frac{x_0(K+1)}{(K+1) - \int_{z^*}^1 \exp\left(\int_w^1 H(v)dv\right) dw}$$

and

$$\Psi(z) = \frac{r}{K+1}\left[(K+1) - \int_z^1 \exp\left(\int_w^1 H(v)dv\right) dw\right]$$

on $[z^*, 1]$. Let $\psi = \Psi^{-1}$ be the inverse function of Ψ. Then, $x_0 < r < \infty$ and $\psi : [x_0, r] \mapsto [0, 1]$ is a thrice differentiable strictly increasing and strictly convex function that satisfies $\psi(r) = 1$, $\psi'(r-) = \frac{K+1}{r}$ and $\psi''(r-) = \frac{K(K+1)}{r^2}$.

(iii) Define

$$\phi(y) = y - \frac{K\psi'(y)}{\psi''(y)}. \tag{3.22}$$

Then $\phi : [x_0, r] \mapsto [0, x_0]$ is a once differentiable strictly decreasing function with $\phi(x_0) = x_0$.

(iv) Define

$$\theta(y) = \psi(x_0) + \frac{1}{K+1}\int_{x_0}^y \phi'(z)\psi'(z)dz$$

Then $\theta : [x_0, r] \mapsto [0, 1]$ is a once differentiable strictly decreasing function with $\theta(x_0) = \psi(x_0)$. Moreover, $\theta(y) = \psi(y) - (y - \phi(y))\psi'(y)/(K+1)$.

Proof. It is easily seen that $J(u)$ is a strictly increasing function at least until $J(u) = \sqrt[n-1]{1-u}$, and that $u^* < 1$.

(i) Since $u^* < 1$, $z^* = 1 - u^* > 0$. Since J is increasing, for any $u \in (0, u^*)$, $J(u)^{n-1} \leq J(u^*)^{n-1}$ and thus $1 - u - J(u)^{n-1} \geq 1 - u - J(u^*)^{n-1} = u^* - u$. This means that, for any $z \in (z^*, 1)$, we have $z - J(1-z)^{n-1} \geq z - z^* > 0$ and then $0 < H(z) \leq \frac{K}{(K+1)(z-z^*)}$. Moreover,

$$\begin{aligned}
&\int_{z^*}^{1} \exp\left(\int_w^1 \frac{K}{(K+1)(v-z^*)} dv\right) dw \\
&\quad = \int_{z^*}^{1} \exp\left(\frac{K}{K+1} \ln \frac{1-z^*}{w-z^*}\right) dw = \int_{z^*}^{1} \left(\frac{1-z^*}{w-z^*}\right)^{\frac{K}{K+1}} dw \\
&\quad = (1-z^*)^{\frac{K}{K+1}} (1-z^*)^{\frac{1}{K+1}} (K+1) = (1-z^*)(K+1) < (K+1),
\end{aligned}$$

and it follows that $\int_{z^*}^1 \exp\left(\int_w^1 H(v)dv\right) dw < (K+1)$.

(ii) Since $0 < (K+1) - \int_{z^*}^1 \exp\left(\int_w^1 H(v)dv\right) dw < (K+1)$, we have that $x_0 < r < \infty$. By the differentiability of J and the form of Ψ, Ψ is thrice differentiable. Taking derivatives of Ψ on $(z^*, 1)$, we find $\Psi'(z) = \frac{r}{K+1} \exp\left(\int_z^1 H(v)dv\right) > 0$ and $\Psi''(z) = -H(z)\Psi'(z) < 0$. Then, since $\Psi(z^*) = x_0$ and $\Psi(1) = r$, Ψ is a strictly increasing and strictly concave function from $[z^*, 1]$ to $[x_0, r]$. Thus, $\psi = \Psi^{-1} : [x_0, r] \mapsto [0, 1]$ is a thrice differentiable strictly increasing and strictly convex function satisfying $\psi(r) = 1$.

Moreover, $\psi'(y) = \frac{1}{\Psi'(\psi(y))}$ and thus $\psi''(y) = -\frac{\Psi''(\psi(y))\psi'(y)}{\Psi'(\psi(y))^2}$. Then, since $\Psi(1) = r$, $\Psi'(1-) = \frac{r}{K+1}$ and $\Psi''(1-) = -\frac{rK}{(K+1)^2}$, we get $\psi(r) = 1$, $\psi'(r-) = \frac{1}{\Psi'(1-)} = \frac{K+1}{r}$ and $\psi''(r-) = -\frac{\Psi''(1-)\psi'(r-)}{\Psi'(1-)^2} = \frac{K(K+1)}{r^2}$.

(iii) Letting $u = 1 - z$ in (3.20), we get

$$J'(1-z) = \frac{J(1-z) + 1 - z^{1/(n-1)}}{(K+1)\left[z - J(1-z)^{n-1}\right]} = \frac{H(z)}{K}\left(J(1-z) + 1 - z^{\frac{1}{n-1}}\right). \quad (3.23)$$

Letting $x = z$ in (3.21) and differentiating it with respect to z,

$$H'(z) = -\frac{K\left[1 + (n-1)J(1-z)^{n-2}J'(1-z)\right]}{(K+1)\left[z - J(1-z)^{n-1}\right]^2}.$$

Substituting (3.23) into previous equation,

$$H'(z) = -\frac{K+1}{K} H(z)^2 \left[1 + (n-1)J(1-z)^{n-2} \frac{H(z)}{K}\left(J(1-z) + 1 - z^{\frac{1}{n-1}}\right)\right],$$

which implies that $\frac{H'(z)}{H(z)^2} < -\frac{K+1}{K}$ and is equivalent to

$$\begin{aligned} H'(z) = &-\frac{(K+1)(n-1)}{K^2} H(z)^3 \left[J(1-z)^{n-2} \left(1 - z^{\frac{1}{n-1}}\right) + J(1-z)^{n-1} \right] \\ &- \frac{K+1}{K} H(z)^2. \end{aligned} \tag{3.24}$$

Note that ϕ is once differentiable because ψ is thrice differentiable and $\phi'(y) = 1 - K\left(1 - \frac{\psi'(y)\psi'''(y)}{\psi''(y)^2}\right)$. Since $H(z) = -\frac{\Psi''(z)}{\Psi'(z)} = \frac{\psi''(\Psi(z))}{\psi'(\Psi(z))^2}$, we have that

$$H(\psi(y)) = \frac{\psi''(y)}{\psi'(y)^2} \tag{3.25}$$

and thus $H'(\psi(y))\psi'(y) = \frac{\psi'''(y)}{\psi'(y)^2} - \frac{2\psi''(y)^2}{\psi'(y)^3}$. This implies that $1 - \frac{\psi'(y)\psi'''(y)}{\psi''(y)^2} = -H'(\psi(y))\frac{\psi'(y)^4}{\psi''(y)^2} - 1$ and therefore

$$\phi'(y) = K\frac{H'(\psi(y))}{H(\psi(y))^2} + K + 1. \tag{3.26}$$

It follows that $\phi'(y) < 0$.

Since $\psi'(r-) = \frac{K+1}{r}$, $\psi'(y) > 0$ and $\psi''(y) > 0$ on (x_0, r), we know $\psi'(x_0+)$ is bounded. Then, since $\frac{\psi''(x_0+)}{\psi'(x_0+)^2} = H(\psi(x_0)) = H(z^*) = +\infty$, we get $\psi''(x_0+) = +\infty$. Substituting these values into (3.22), we obtain $\phi(x_0) = x_0$.

(iv) The statements about θ are either trivial, or follow as in the derivation of (3.14).

□

Theorem 3.3.2. *Let r, ψ, ϕ, θ be as defined in Lemma 3.3.1.*

Then there exists a symmetric, atom-free Nash equilibrium for the problem for which $X^i_{\tau^i}$ has distribution F where $F(x) = 0$ for $x \le 0$, $F(x) = 1$ for $x \ge r$ and otherwise

$$F(x) = \begin{cases} \theta(\phi^{-1}(x))^{\frac{1}{n-1}} & \text{if } 0 < x < x_0, \\ \psi(x)^{\frac{1}{n-1}} & \text{if } x_0 \le x < r. \end{cases}$$

Proof. By (3.22) and (3.25), $\frac{1}{H(\psi(y))} = \frac{\psi'(y)^2}{\psi''(y)} = \frac{y-\phi(y)}{K}\psi'(y)$. Hence $\theta(y) = \psi(y) - \frac{K}{(K+1)H(\psi(y))} = J(1-\psi(y))^{n-1}$. Thus, letting $z = \psi(y)$ in (3.24),

$$\begin{aligned} H'(\psi(y)) = -\frac{(K+1)(n-1)}{K^2} H(\psi(y))^3 \left[\left(1 - \psi(y)^{\frac{1}{n-1}}\right)\theta(y)^{\frac{n-2}{n-1}} + \theta(y)\right] \\ - \frac{K+1}{K} H(\psi(y))^2, \end{aligned}$$

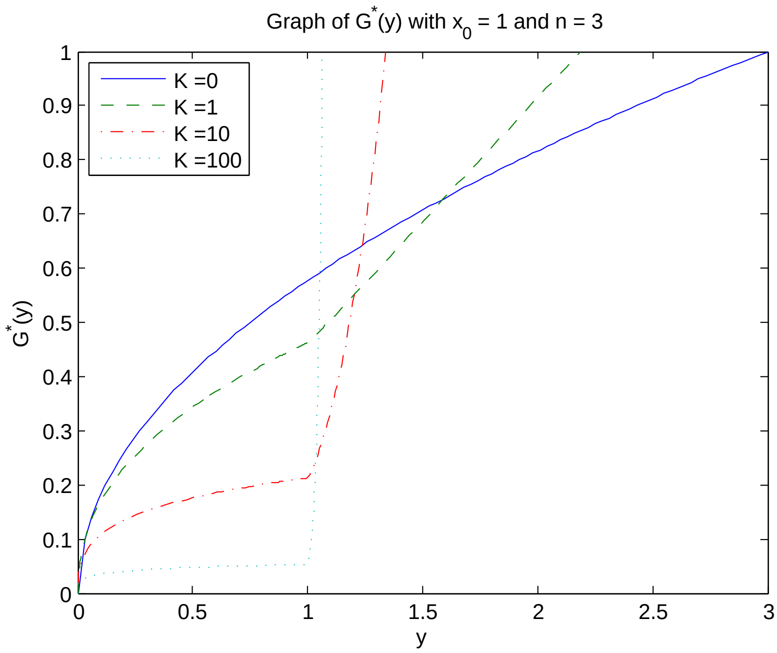

Figure 3.1: Graph of $G^*(x)$ for different K with $x_0 = 1$ and $n = 3$.

and using (3.26) and rearranging above equation,

$$\frac{y - \phi(y)}{n-1} \frac{\phi'(y)\psi'(y)}{K+1} = -\left[\theta(y) + \left(1 - \psi(y)^{\frac{1}{(n-1)}}\right)\theta(y)^{\frac{n-2}{n-1}}\right].$$

Then, substituting (3.8) into above equation, (3.10) follows. Therefore, (†) holds. Then, using Theorem 3.3.1, we obtain the symmetric, atom-free Nash equilibrium given in Theorem 3.3.2.

□

Example 3.3.1. As an example we consider a 3-player contest. Set $x_0 = 1$ and $n = 3$. In Figures 3.1 and 3.2 we give graphs of the optimal distribution $G^*(x)$ and its density function $g^*(x)$ for various values of K.

As we can see in Figure 3.1, the right endpoint $r = r(K)$ of $G^*(x)$ decreases as K increases. Moreover, $r(K)$ tends to $n = 3$ as K decreases to 0 and tends to x_0 as K increases to $+\infty$. We also find that $G^*(x)$ tends to the equilibrium distribution of the original contest as K decreases to 0 and $G^*(x)$ tends to the Heaviside function $\mathcal{H}_{\{x \geq x_0\}}$ as K increases to $+\infty$. From Figure 3.2, we find $g^*(x)$ jumps at x_0 if $K > 0$ and $g^*(x)$ tends to $+\infty$ as y tends to 0.

Intuitively, if K is very large, then the player does not aim for large values of the stopped process, for then she risks a moderate value of the maximum together

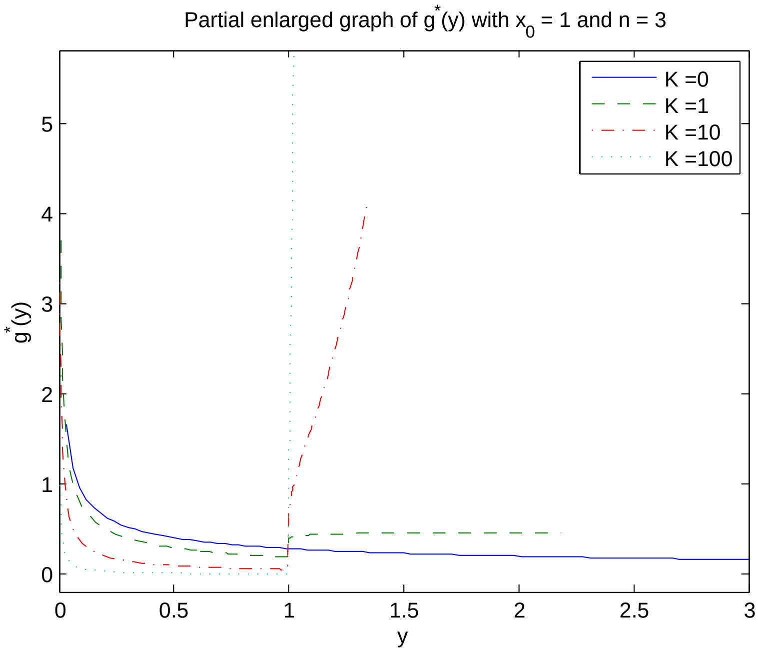

Figure 3.2: Enlarged graph of $g^*(x)$ for different K with $x_0 = 1$ and $n = 3$.

with a small and losing value for the stopped process. Because of the large penalty she wishes to avoid such outcomes.

Example 3.3.2. In the 2-player contest we can give explicit expressions for several of the quantities of interest.

Set $n = 2$. Substituting (3.8), (3.9) and (3.11) into (3.10), we get

$$(y - \phi(y))\frac{1}{1+K}\phi'(y)\psi'(y) = (\psi(y) - 1) - \psi(y) + \frac{K}{K+1}\frac{y - \phi(y)}{K}\psi'(y).$$

Defining $\varphi(y) = y - \phi(y)$, the above equation simplifies to $\varphi(y)\varphi'(y) = \frac{K+1}{\psi'(y)}$. Differentiating this expression and using (3.9), we have

$$[\varphi(y)\varphi'(y)]' = -(K+1)\frac{\psi''(y)}{\psi'(y)^2} = -(K+1)\frac{K}{\varphi(y)\psi'(y)} = -K\varphi'(y),$$

and then

$$\varphi(y)\varphi'(y) = -K\varphi(y) + K\varphi(r) + \varphi(r)\varphi'(r-). \tag{3.27}$$

Since $\varphi(r) = r$ and $\psi'(r-) = \frac{K+1}{r}$, we have $\varphi'(r-) = 1$. Then (3.27) becomes $\varphi(y)\varphi''(y) = -K\varphi(y) + r(K+1)$, which using the boundary condition $\varphi(r) = r$ has solution

$$r - y = \frac{r(K+1)}{K^2}\ln\left((K+1) - \frac{K\varphi(y)}{r}\right) - \frac{r - \varphi(y)}{K}$$

Using $\varphi(x_0) = x_0 - \phi(x_0) = 0$,

$$r = x_0 \frac{K^2}{(K+1)\left[K - \ln(1+K)\right]}$$

and therefore the implicit form of $\varphi(y)$ for $y \in [x_0, r]$ is

$$y = x_0 - \frac{\varphi(y)}{K} - \frac{x_0}{K - \ln(1+K)} \ln\left[1 - \varphi(y)\frac{K - \ln(1+K)}{Kx_0}\right]$$

and $\phi(y) = y - \varphi(y)$. It is possible to express ψ and θ in terms of φ, and thence the optimal distribution G^* of $X^i_{\tau^i}$ and the optimal conditional distribution of $M^i_{\tau^i}$ given $X^i_{\tau^i}$, but these expressions are not so compact.

3.4 Contest with regret over failure to stop at the best time

In this section, we discuss the contest with regret over failure to stop at a winning time, when alternative times both before and after the chosen time are permitted. A player experiences regret if she could have won if she had stopped at the maximum value over the whole path. The maximum value M^i is given by

$$M^i := M^i_{H^i_0} = \sup_{0 \le t \le H^i_0} X^i_t.$$

Theorem 3.4.1. *There exists a symmetric, atom-free Nash equilibrium for the problem for which $X^i_{\tau^i}$ has law $F(x)$, where for $x \geq 0$*

$$F(x) = \min\left\{\sqrt[n-1]{\frac{x}{nx_0}}, 1\right\}.$$

Proof. The agent's expected pay-off is

$$(1+K)\mathbb{E}[F(X^i_{\tau^i})^{n-1}] - K\mathbb{E}[F(M^i_{H^i_0})^{n-1}]$$

But the latter term is independent of the stopping rule used by the agent. Hence, in determining her optimal strategy the agent need only consider $(1+K)\mathbb{E}[F(X^i_{\tau^i})^{n-1}]$. Modulo the factor of $(1+K)$, this is the same objective function as in the standard case.

□

Remark 3.4.1. The agent follows exactly the same Nash equilibrium strategy as an agent in the original contest, in which there is no penalty. The intuition behind is that the regret is determined by $M^i_{H^i_0}$ but player cannot change the distribution of $M^i_{H^i_0}$ by changing the choices of stopping time τ^i.

3.5 Derivation of the equilibrium distribution

This section is intended to illustrate how we derived the optimal multipliers and the candidate Nash equilibria in Sections 3.2 and 3.3 and also the boundary conditions in Section 3.3. The Lagrangian approach gives a general method for finding the optimal solution, which is distinct from the ideas in Seel and Strack [2013], and can be generalised to other settings.

3.5.1 Contest with regret over stopping too soon

Recall the definition of the Lagrangian $\mathcal{L}_F(G;\lambda,\gamma)$ for the optimisation problems (3.3). Denote by $L_F(x)$ the integrand in $\mathcal{L}_F$, that is

$$L_F(x) = (1+K)F(x)^{n-1} - Kx\int_x^\infty \frac{F(y)^{n-1}}{y^2}dy - \lambda x - \gamma,$$

so $\mathcal{L}_F(G;\lambda,\gamma) = \int_0^\infty L_F(x)G(dx) + \lambda x_0 + \gamma$. In order to have a finite optimal solution, we require $L_F(x) \leq 0$ on $[0,\infty)$. Let $\mathcal{D}_F$ be the set of (λ,γ) such that $\mathcal{L}_F(\cdot;\lambda,\gamma)$ has a finite maximum. Then $\mathcal{D}_F$ is defined by

$$\mathcal{D}_F = \{(\lambda,\gamma) : L_F(x) \leq 0 \text{ on } [0,\infty)\}.$$

Observe that $L_F(0) = (1+K)F(0)^{n-1} - \gamma$, which implies that if $(\lambda,\gamma) \in \mathcal{D}_F$ then γ is non-negative. In order to reach the maximum value, we require $G(dx) = 0$ when $L_F(x) < 0$. This means that for $(\lambda,\gamma) \in \mathcal{D}_F$ the maximum of $\mathcal{L}_F(\cdot;\lambda,\gamma)$ occurs at G^* such that $G^*(dx) = 0$ when $L_F(x) < 0$. If the Nash equilibrium is symmetric, then we must have $G^*(x) = F(x)$, and then $L_{G^*}(x) \leq 0$, and $L_{G^*}(x) = 0$ when $G^*(dx) > 0$. Introduce $a = \inf\{x : G^*(x) > 0\}$ and $b = \sup\{x : G^*(x) < 1\}$ which are the limits on the support of G^*.

Let $\psi(x) = G^*(x)^{n-1}$, then $L_{G^*}(x)$ becomes

$$L_{G^*}(x) = (1+K)\psi(x) - Kx\int_x^\infty \frac{\psi(y)}{y^2}dy - \lambda x - \gamma. \tag{3.28}$$

Thus, we expect $\psi(x)$ is the solution to $L_{G^*}(x) = 0$ at least when $\psi(dx) > 0$.

Setting $L_{G^*}(x) = 0$ and differentiating (3.28) twice with respect to x, we find

$$(1+K)\psi''(x)x + K\psi'(x) = 0.$$

Thus, $\psi(x) = C_1 x^{\frac{1}{K+1}} + C_2$, where C_1 and C_2 are some constants, and then $G^*(x) = \sqrt[n-1]{C_1 x^{\frac{1}{K+1}} + C_2}$ when $G^*(dx) > 0$. Since we are seeking an atom-free non-constant solution, we must have $G^*(x) = \sqrt[n-1]{C_1 x^{\frac{1}{K+1}} + C_2}$ on the whole interval of $[a,b]$, where $C_1 > 0$.

Substituting $\psi(x) = G^*(x)^{n-1} = (C_1 x^{\frac{1}{K+1}} + C_2) \wedge 1$ into (3.28), and setting $L_{G^*}(x) = 0$, we have that, $\forall x \in [a,b]$,

$$\begin{aligned} 0 &= (1+K)\left(C_1 x^{\frac{1}{K+1}} + C_2\right) - Kx\int_x^b \frac{C_1 y^{\frac{1}{K+1}} + C_2}{y^2}dy - Kx\int_b^\infty \frac{1}{y^2}dy - \lambda x - \gamma \\ &= \left[(1+K)C_1 b^{\frac{1}{K+1}} + KC_2 - K - \lambda b\right]\frac{x}{b} + C_2 - \gamma. \end{aligned}$$

This gives us optimal multipliers $\lambda^* = \frac{1}{b}\left[(1+K)C_1 b^{\frac{1}{K+1}} + KC_2 - K\right]$ and $\gamma^* = C_2$.

Since G^* is atom-free, $G^*(a) = 0$ and hence $C_1 a^{\frac{1}{K+1}} + C_2 = 0$. Then, from the non-negativity of a and $\gamma^* = C_2$ and the positivity of C_1, it follows that $C_2 = a = 0$. Thus, $G^*(x) = \sqrt[n-1]{C_1 x^{\frac{1}{K+1}}}$ on $[0,b]$ for some C_1 and b which can be identified using the fact that G^* corresponds to a probability distribution with mean x_0. In particular, setting $N = 1 + (K+1)(n-1)$, for a feasible solution,

$$\begin{cases} 1 = \int_0^b d\left(\sqrt[n-1]{C_1 x^{\frac{1}{K+1}}}\right) = \sqrt[n-1]{C_1 b^{1/(K+1)}}, \\ x_0 = \int_0^b x d\left(\sqrt[n-1]{C_1 x^{\frac{1}{K+1}}}\right) = \frac{\sqrt[n-1]{C_1}}{(K+1)(n-1)+1} b^{\frac{(K+1)(n-1)+1}{(K+1)(n-1)}} = \sqrt[N-1]{C_1^{K+1} b}\,\frac{b}{N}. \end{cases}$$

Hence, $C_1 = b^{-1/(K+1)}$ and then $b = Nx_0$ and $C_1 = \sqrt[K+1]{\frac{1}{Nx_0}}$. Thus $G^*(x) = \sqrt[N-1]{x/Nx_0}$ on $[0, Nx_0]$.

3.5.2 Contest with regret over past failure

Recall the definition of the Lagrangian $\mathcal{L}_F(\nu;\lambda,\gamma,\eta)$ for the optimisation problem of Section 3.3. Let $L_F(x,y)$ be the integrand in the definition of $\mathcal{L}_F$ as given in (3.16). In order to have a finite optimal solution, we require $L_F(x,y) \le 0$ on $[0,\infty)\times[x_0,\infty)$. Let $\mathcal{D}_F$ be the set of (λ,γ,η) such that $\mathcal{L}_F(\cdot;\lambda,\gamma,\eta)$ has a finite maximum. Then $\mathcal{D}_F$ is defined by

$$\mathcal{D}_F = \{(\lambda,\gamma,\eta) : L_F(x,y) \le 0; x \ge 0, y \ge x_0\}.$$

For $(\lambda, \gamma, \eta) \in \mathcal{D}_F$, the maximum of $\mathcal{L}_F(\cdot; \lambda, \gamma, \eta)$ occurs at a measure ν^* such that $\nu^*(dx, dy) = 0$ when $L_F(x, y) < 0$.

Let $G^*(x) = \nu^*(\{(u, y) : u \leq x, x_0 \leq y < \infty\})$ be the marginal of ν^*. If the Nash equilibrium is symmetric, then we must have $G^*(x) = F(x)$ and $L_{G^*}(x, y) = 0$ when $\nu^*(dx, dy) > 0$. Motivated by the results of previous sections, we expect G^* to place mass on an interval $[a, b]$ where $0 = a < x_0 < b$. In this section we write $b = r$ for the upper limit.

It follows from the discussion before Theorem 3.3.1 that for an optimal solution either $X^i_{\tau^i} = M^i_{\tau^i}$ or $X^i_{\tau^i} = \phi(M^i_{\tau^i})$ for some decreasing function ϕ. Hence, for $x_0 \leq y \leq r$, we expect $\nu^*(dx, dy) > 0$ if and only if either $x = y$ or $x = \phi(y)$. Let $\psi(x) = G^*(x)^{n-1}$. Then $L_{G^*}(x, y)$ becomes

$$L(x, y) := L_{G^*}(x, y) = (1 + K)\psi(x) - K\psi(y) - \lambda x - \gamma - \int_{x_0}^{y} \eta(z)(x - z)dz.$$

Fixing $y \in (x_0, r)$, and using $L(x, y) \leq 0$ for any $0 \leq x \leq y$, together with $L(\phi(y), y) = 0$, we expect $\frac{\partial L}{\partial x}(\phi(y), y) = 0$.

Thus, $\forall y \in (x_0, r)$, ψ and ϕ must solve

$$\begin{cases} L(y, y) = \psi(y) - \lambda y - \gamma - \int_{x_0}^{y} \eta(z)(y - z)dz = 0, & (3.29) \\ L(\phi(y), y) = (1 + K)\psi(\phi(y)) - K\psi(y) - \lambda\phi(y) & \\ \qquad\qquad -\gamma - \int_{x_0}^{y} \eta(z)(\phi(y) - z)dz = 0, & (3.30) \\ \frac{\partial L}{\partial x}(\phi(y), y) = (1 + K)\psi'(\phi(y)) - \lambda - \int_{x_0}^{y} \eta(z)dz = 0. & (3.31) \end{cases}$$

Differentiating (3.29) with respect to y yields

$$\psi'(y) - \lambda - \int_{x_0}^{y} \eta(z)dz = 0. \tag{3.32}$$

Comparing (3.31) with (3.32), we find $\psi'(y) = (1 + K)\psi'(\phi(y))$. If we now set $\theta(y) = \psi(\phi(y))$, then (3.8) follows. From (3.32) we find

$$\psi''(y) - \eta(y) = 0. \tag{3.33}$$

Then, differentiating (3.30) with respect to y, and using (3.31), we obtain $-K\psi'(y) - \eta(y)(\phi(y) - y) = 0$ and (3.9). Finally, (3.10) comes directly from (3.7) on noting that $G(\phi(m)) = \theta(m)^{1/(n-1)}$.

Next we deduce the boundary conditions. First note that from (3.5) we can infer that $\phi(x_0) = x_0$ and $\phi(r) = 0$. Hence $\theta(x_0) = \psi(x_0)$ and $\theta(r) = 0$. Given that (3.8) and (3.9) hold, as in the proof of Theorem 3.3.1, we have that (3.14) holds. Letting $y = r$ and using $\psi(r) = 1$, $\phi(r) = 0$ and $\theta(r) = 0$, we find $0 = \frac{-r}{K+1}\psi'(r-) + \psi(r)$ and hence $\psi''(r-) = \frac{K+1}{r}$. Further, letting $y = r$ in (3.9), we get $\psi''(r-) = K(K+1)/r^2$ as required.

Lastly, we derive the optimal multipliers which we write as η^*, λ^* and γ^*. From (3.33), $\eta^*(y) = \psi''(y)$ for $y \in (x_0, r)$. Then, from (3.32), $\lambda^* = \psi'(y) - \int_{x_0}^{y} \eta^*(z)dz = \psi'(y) - \int_{x_0}^{y} \psi''(z)dz = \psi'(x_0+)$. Finally, (3.29) yields

$$\begin{aligned}\gamma^* &= \psi(y) - \lambda y - \int_{x_0}^{y} \eta^*(z)(y-z)dz \\ &= \psi(y) - y\psi'(x_0+) - \int_{x_0}^{y} \psi''(z)(y-z)dz = \psi(x_0) - x_0\psi'(x_0+).\end{aligned}$$

3.6 Extension to the case of time homogeneous diffusions

Our results in this chapter can be extended to the case where the processes observed by the agents are independent copies of some time-homogeneous diffusion process Y which converges almost surely to the lower bound on its state space. The idea has been specifically explained in Chapter 2. Briefly speaking, the idea is to use a change of scale to transform the problem into natural scale.

Denote by $s(\cdot)$ the scale function of Y and denote by $\{l_1, l_2\}$ the endpoints of the state space of Y with $-\infty \le l_1 < Y_0 = y_0 < l_2 \le \infty$. We assume that Y converges to the lower boundary, which implies that $s(l_1)$ is finite whereas $s(l_2) = \infty$. Without loss of generality, we may set $s(l_1) = 0$. Then, $X = s(Y)$ is a continuous local martingale that converges to zero almost surely (and if zero can be reached in finite time, then zero is absorbing).

As explained in Section 2.1, the contest in which players privately observe independent copies of Y is equivalent to the contest in which players privately observe independent copies of X, and the choice of the optimal τ^i is the same for both contests.

The problem is then to find a Nash equilibrium $(G^i)_{i\in I}$ for $Y^i_{\tau^i}$ and then verify that there exists τ^i such that $Y^i_{\tau^i}$ has law G^i. Under our transformation, this is the same as finding a Nash equilibrium $(F^i)_{i\in I}$ for $X^i_{\tau^i}$ where $F^i = G^i \circ s^{-1}$, where s^{-1} is the inverse of s. To solve the problem for X, then either we argue that the only properties of X that we use are the strong Markov property, the local martingale property, and the fact that X converges to zero, so that the theory of this chapter applies to the local martingale diffusion X, or we argue that since X is a non-negative martingale diffusion, X is a time-change of Brownian motion and $X_t = B_{\Gamma_t}$ for some increasing functional Γ_t. Then, if F is any distribution with mean less than or equal to x_0, and σ is a stopping time such that $B_\sigma \sim F$, then we may take $\tau = \Gamma^{-1} \circ \sigma$ and then $X_\tau = B_\sigma \sim F$ and τ is an embedding of G in Y.

Chapter 4

Contests: Asymmetric case

Several variants of the original Seel-Strack contest have been discussed in Seel and Strack [2013], including an extension to the asymmetric two-player case where the starting values of the processes observed by the players are different constants. In this chapter, we start with rederiving the Nash equilibrium obtained by Seel and Strack in an asymmetric 2-player contest and then extend the results to an asymmetric n-player contest. We assume that the starting values of the observed processes are strictly positive constants and all of these constants may be different.

By analogy with the symmetric case studied in Chapter 1, we expect that in equilibrium no agent places mass at a positive point and not all agents can have a mass point at zero. However, the Nash equilibrium may have masses at zero for some agents. We will show that there exists a Nash equilibrium such that it has no atoms in $(0, \infty)$ and satisfies that the highest levels at which the agents should stop are the same. Furthermore, we will see that in the equilibrium, when $n > 2$, the agents with lower starting values may choose not to stop at small values but to wait for high values.

The remainder of this chapter is constructed as follows. In Section 4.1, we introduce the mathematical model of the asymmetric 2-player contest used in Seel and Strack [2013] and rederive the equilibrium distributions using the Lagrangian method. Then we study an asymmetric n-player contest in Section 4.2. In Section 4.2.1 we provide and prove the sufficient conditions for the Nash equilibrium distributions. Then, based on the conditions, we give a more detailed form of the equilibrium distributions. We will analytically prove the existence of the equilibrium in two special cases. For more complicated cases, we may need to solve the problem numerically. We will take the 3-player case as an example. And in Section 4.2.2 we explain the origin of the candidate Nash equilibrium distributions.

4.1 2-player contest

In an asymmetric 2-player contest, Seel and Strack [2013] assumed that the processes privately observed by the two players are given by

$$Y_t^i = y_i + \mu_i t + \sigma_i W_t^i, \tag{4.1}$$

where $y_i > 0$, $\mu_i \neq 0$ and $\sigma_i > 0$ are all constants, $i \in \{1, 2\}$. It is assumed that Y^1 and Y^2 are absorbed at zero and W^1 and W^2 are independent Brownian motions. Seel and Strack also imposed an upper bound on the drifts μ_i, that is they assumed that $\mu_1 < 0$ and $\mu_2 < 0$.

Under these assumptions, they have proved the following theorem. Note that the scale function of Y^i is given by $s_i(y) = \frac{\sigma_i^2}{2\mu_i} - \frac{\sigma_i^2}{2\mu_i}\exp(-2\mu_i y/\sigma_i^2)$ (we set $s_i(0) = 0$).

Theorem 4.1.1. *[Seel and Strack [2013]] There exists unique $b_i > y_i$ which satisfies*

$$\int_0^{b_i} [s_i(y_i) - s_i(z)]\, s_j(dz) = 0 \tag{4.2}$$

with $j \neq i$, for $i \in \{1, 2\}$. Suppose $b_2 \geq b_1$. Let $b^* = b_2$ and $p_2 = 0$.

(i) There exists $p_1 \in [0, 1]$ that satisfies

$$p_1 = 1 - \frac{s_1(y_1)s_2(b^*)}{\int_0^{b^*} s_1(z)s_2(dz)}. \tag{4.3}$$

(ii) There exists a Nash equilibrium in which the equilibrium distributions G_1 and G_2 satisfy that

$$G_i(y) = \min\left\{p_i + (1 - p_i)\frac{s_j(y)}{s_j(b^*)}, 1\right\}, \tag{4.4}$$

where $j \neq i$ and $s_i(\cdot)$ is the scale function of Y^i with $s_i(0) = 0$, $i = 1, 2$.

Seel and Strack solved the problem using a similar approach as described in Section 1.1.1. Briefly speaking, they solved the problem by writing down candidate value functions for the problem, and then verifying that the candidate value functions are martingales up to the optimal stopping times for each player.

We solve the problem using the Lagrangian approach, and it is easy to see that Theorem 4.1.1 holds for a more general model rather than (4.1).

4.1.1 Equilibrium distribution

In this section, we use a more general model instead of (4.1). We assume that Y^1 and Y^2 are two independent non-negative time-homogeneous diffusion processes with constant starting values $Y_0^1 = y_1 > 0$ and $Y_0^2 = y_2 > 0$. In addition, assume that Y^1 and Y^2 have the same state space which is an interval with endpoints $\{0, r\}$, where $\max\{y_1, y_2\} < r \leq \infty$, and assume that Y^1 and Y^2 both converge almost surely to zero.

Let $s_i(\cdot)$ be the scale function of Y^i, then $s_i(0)$ is finite and $s_i(r) = \infty$ by the convergence assumption. Without loss of generality we set $s_i(0) = 0$.

Fix player $i \in \{1, 2\}$. Given that player $j \neq i$ chooses G_j as her target distribution, where G_j is continuous, then the expected pay-off of agent i with stopping time τ^i is given by

$$\mathbb{E}\left[G_j(Y^i_{\tau^i}) \cdot \mathbf{1}_{\{Y^i_{\tau^i} > 0\}} + \frac{1}{2} G_j(0) \cdot \mathbf{1}_{\{Y^i_{\tau^i} = 0\}}\right]. \tag{4.5}$$

Let $X_t^i = s_i(Y_t^i)$. Note that s_i is strictly increasing. Denote by s_i^{-1} the inverse function of s_i. Then, s_i^{-1} is also strictly increasing and $s_i^{-1}(0) = 0$. Moreover, the expected pay-off becomes

$$\mathbb{E}\left[G_j \circ s_i^{-1}(X^i_{\tau^i}) \cdot \mathbf{1}_{\{X^i_{\tau^i} > 0\}} + \frac{1}{2} G_j(0) \cdot \mathbf{1}_{\{X^i_{\tau^i} = 0\}}\right].$$

Since $\left[G_j \circ s_i^{-1}(x) \cdot \mathbf{1}_{\{x>0\}} + \frac{1}{2} G_j(0) \cdot \mathbf{1}_{\{x=0\}}\right]$ is non-decreasing and X^i is a non-negative super-martingale and using a similar argument as described in Section 1.2, we get that every candidate stopping rule τ^i should satisfy that $\mathbb{E}\left[X^i_{\tau^i}\right] = X^i_0$, which is equivalent to $\mathbb{E}\left[s_i(Y^i_{\tau^i})\right] = s_i(Y^i_0) = s_i(y_i)$.

Let $\mathcal{A}$ be the set of pairs (G, p), where $p \in \mathbb{R}^+$ and $G : [0, \infty) \mapsto [0, \infty)$ is a non-decreasing right-continuous function with $G(0) = p$. An element of $\mathcal{A}$ is identified with a measure ν on $[0, \infty]$ such that $G(y) = \nu([0, y])$ and $p = \nu(\{0\})$.

Suppose that the other player $j \neq i$ chooses (G_j, p_j) as her target measure with G_j continuous. Then by (4.5), the problem facing agent i is to choose a law of $Y^i_{\tau^i}$, which corresponds to a pair (G_i, p_i), to solve

$$\max_{(G_i, p_i) \in \mathcal{A}} \left\{ \int_{(0,\infty)} G_j(z) G_i(dz) + \frac{1}{2} p_j p_i \right\} \tag{4.6}$$

subject to $\int_{(0,\infty)} s_i(z) G_i(dz) = s_i(y_i)$ and $\int_{(0,\infty)} G_i(dz) + p_i = 1$. Introducing multipliers λ_i and γ_i for the two constraints, the Lagrangian for the optimisation problem

(4.6) is then

$$\mathcal{L}_i(G_i, p_i; \lambda_i, \gamma_i) = \int_{(0,\infty)} [G_j(z) - \lambda_i s_i(z) - \gamma_i]\, G_i(dz) + \left[\frac{1}{2}p_j - \gamma_i\right] p_i + \lambda_i s_i(y_i) + \gamma_i.$$

Let $\mathcal{A}_D^i(y_i)$ be the subset of $\mathcal{A}$ that consists of probability measures of a random variable Y such that $\mathbb{E}[s_i(Y)] = s_i(y_i)$, that is $\mathcal{A}_D^i(y_i)$ is given by

$$\mathcal{A}_D^i(y_i) = \left\{(G, p) \in \mathcal{A} : \lim_{y\uparrow\infty} G(y) = 1 \text{ and } \int_{(0,\infty)} s_i(z)G(dz) = s_i(y_i)\right\}.$$

Now we state a variant of the Lagrangian sufficiency theorem for this problem.

Proposition 4.1.1. *Suppose that, for all $i \in \{1, 2\}$, there exists $(G_i^*, p_i^*; \lambda_i^*, \gamma_i^*)$ such that $(G_i^*, p_i^*) \in \mathcal{A}_D^i(y_i)$, G_i^* is continuous and*

$$\mathcal{L}_i(G_i^*, p_i^*; \lambda_i^*, \gamma_i^*) \geq \mathcal{L}_i(G_i, p_i; \lambda_i^*, \gamma_i^*) \qquad \text{for all } (G_i, p_i) \in \mathcal{A}. \tag{4.7}$$

Then the family $(G_i^*, p_i^*)_{i\in\{1,2\}}$ is a Nash equilibrium that has no atoms in $(0, \infty)$.

Proof. Our aim is to find a Nash equilibrium that has no atoms in $(0, \infty)$. Such a Nash equilibrium is identified with a pair of probability measures $(G_i^*, p_i^*)_{i\in\{1,2\}}$ such that, for any $i \in \{1, 2\}$, $(G_i^*, p_i^*) \in \mathcal{A}_D^i(y_i)$, G_i^* is continuous and with $j \neq i$

$$\int_{(0,\infty)} G_j^*(x)G_i^*(dx) + \frac{1}{2}p_j^* p_i^* \geq \int_{(0,\infty)} G_j^*(x)G_i(dx) + \frac{1}{2}p_j^* p_i \qquad \forall (G_i, p_i) \in \mathcal{A}_D^i(y_i).$$

Fix any $i \in \{1, 2\}$. If $(G_i, p_i) \in \mathcal{A}_D^i(y_i)$ then

$$\int_{(0,\infty)} G_j^*(x)G_i(dx) + \frac{1}{2}p_j^* p_i = \mathcal{L}_i(G_i, p_i; \lambda_i^*, \gamma_i^*).$$

Then, under the hypotheses of the proposition,

$$\begin{aligned}\int_{(0,\infty)} G_j^*(x)G_i^*(dx) + \frac{1}{2}p_j^* p_i^* &= \mathcal{L}_i(G_i^*, p_i^*; \lambda_i^*, \gamma_i^*) \\ &\geq \mathcal{L}_i(G_i, p_i; \lambda_i^*, \gamma_i^*) = \int_{(0,\infty)} G_j^*(x)G_i(dx) + \frac{1}{2}p_j^* p_i.\end{aligned}$$

Thus, $(G_i^*, p_i^*)_{i\in\{1,2\}}$ is a Nash equilibrium that has no atoms in $(0, \infty)$.

□

Now we prove Theorem 4.1.1 using Proposition 4.1.1.

Proof of Theorem 4.1.1. Fix any $i \in \{1,2\}$ and let $j \neq i$. Define $B_i(x) = \int_0^x [s_i(y_i) - s_i(z)]\, s_j(dz)$. Since s_i and s_j are strictly increasing, B_i is strictly increasing on $(0, y_i)$ and strictly decreasing on (y_i, ∞). It is easy to get that $B_i(\infty) = -\infty$, since s_i and s_j are strictly increasing, $s_i(\infty) = \infty$ and $s_j(\infty) = \infty$. Then, because $B_i(0) = 0$, there exists unique $b_i > y_i$ that satisfies $B_i(b_i) = 0$.

Suppose $b_2 \geq b_1$ and let $b^* = b_2$. Since $b^* \geq b_1$ and by the shape of B_1, $B_1(b^*) \leq 0$, which is equivalent to $\int_0^{b^*} s_1(z)s_2(dz) \geq \int_0^{b^*} s_1(y_1)s_2(dz) = s_1(y_1)s_2(b^*)$. This implies that p_1 given by (4.3) is non-negative. And it is obvious that $p_1 \leq 1$. Thus, $p_1 \in [0,1]$.

Suppose G_i^* are given by (4.4). Let $p_1^* = \gamma_2^* = p_1$, $p_2^* = \gamma_1^* = 0$, $\lambda_1^* = 1/s_1(b^*)$ and $\lambda_2^* = (1-p_1)/s_2(b^*)$. It is easy to get that G_i^* is continuous on $(0,\infty)$ and $(G_i^*, p_i^*) \in \mathcal{A}_D^i(y_i)$, for $i \in \{1,2\}$. We then verify that for these multipliers (4.1.1) holds. For any $i \in \{1,2\}$ and with $j \neq i$, since $\lambda_i^* s_i(b^*) + \gamma_i^* = 1$ and s_i is strictly increasing,

$$
\begin{aligned}
\mathcal{L}_i(G_i, p_i; \lambda_i^*, \gamma_i^*) &= \int_{(0,\infty)} \left[G_j^*(z) - \lambda_i^* s_i(z) - \gamma_i^*\right] G_i(dz) \\
&\qquad\qquad + \left[\frac{1}{2}p_j^* - \gamma_i^*\right] p_i + \lambda_i^* s_i(y_i) + \gamma_i^* \\
&= \int_{b^*}^{\infty} [1 - \lambda_i^* s_i(z) - \gamma_i^*]\, G_i(dz) + \left[\frac{1}{2}p_j^* - \gamma_i^*\right] p_i + \lambda_i^* s_i(y_i) + \gamma_i^* \\
&\leq \lambda_i^* s_i(y_i) + \gamma_i^* = \mathcal{L}_i(G_i^*, p_i^*; \lambda_i^*, \gamma_i^*).
\end{aligned}
$$

Thus, there exists a Nash equilibrium of the given form by Proposition 4.1.1.

□

In the rest of this section we explain how to derive the candidate Nash equilibrium and the optimal multipliers. Recall the definition of the Lagrangian $\mathcal{L}_i(G_i, p_i; \lambda_i, \gamma_i)$ for the optimisation problem (4.6), and let $L_i(z) = G_j(z) - \lambda_i s_i(z) - \gamma_i$.

In order to have a finite optimal solution, we require $L_i(z) \leq 0$ on $(0,\infty)$ and $\frac{1}{2}p_j - \gamma_i \leq 0$. Let $\mathcal{D}_i$ be the set of (λ_i, γ_i) such that $\mathcal{L}_i(\cdot,\cdot;\lambda_i,\gamma_i)$ has a finite maximum. Then $\mathcal{D}_i$ is defined by

$$
\mathcal{D}_i = \left\{ (\lambda_i, \gamma_i) : L_i(z) \leq 0\ \forall z > 0 \text{ and } \frac{1}{2}p_j - \gamma_i \leq 0 \right\}.
$$

For $(\lambda_i, \gamma_i) \in \mathcal{D}_i$, the maximum of $\mathcal{L}_i(\cdot,\cdot;\lambda_i,\gamma_i)$ occurs at (G_i, p_i) such that $G_i(dz) = 0$ when $L_i(z) < 0$ and $p_i = 0$ if $\frac{1}{2}p_j - \gamma_i < 0$. Thus, if $G_i(dz) > 0$ then $L_i(z) = 0$ and if $p_i > 0$ then $\frac{1}{2}p_j - \gamma_i = 0$.

Since $L_i(z) = G_j(z) - \lambda_i s_i(z) - \gamma_i$, we must have $G_j(z) = \lambda_i s_i(z) + \gamma_i$ when $G_i(dz) > 0$. Since G_j is non-decreasing and not constant, we must have $\lambda_i > 0$, which means $\lambda_i s_i(z) + \gamma_i$ is strictly increasing. This implies that $G_j(dz) > 0$ if and only if $G_i(dz) > 0$, $\inf\{x : G_i(x) > 0\} = \inf\{x : G_j(x) > 0\} \triangleq a$ and $\sup\{x : G_i(x) < 1\} = \sup\{x : G_j(x) < 1\} \triangleq b$. Then since we are searching for solutions that have no positive atoms, we must have $G_j(z) = \lambda_i s_i(z) + \gamma_i$ on the whole of the interval $[a, b]$.

Observe that $p_j \geq 0$, so that if $(\lambda_i, \gamma_i) \in \mathcal{D}_i$ then γ_i is non-negative. Since either $G_i(a) = 0$ or $G_j(a) = 0$ and by the non-negativity of a, γ_i and γ_j and the positivity of λ_i and λ_j, it follows that $a = 0$. Furthermore, $p_i = G_i(0) = \gamma_j$ and $p_j = G_j(0) = \gamma_i$.

For a feasible solution, $\int_{(0,\infty)} s_1(z)G_1(dz) = s_1(y_1)$, $\int_{(0,\infty)} G_1(dz) + p_1 = 1$, $\int_{(0,\infty)} s_2(z)G_2(dz) = s_2(y_2)$ and $\int_{(0,\infty)} G_2(dz) + p_2 = 1$, so that

$$\int_0^b s_1(z)\lambda_2 s_2(dz) = s_1(y_1); \qquad \int_0^b \lambda_2 s_2(dz) + \gamma_1 = 1; \tag{4.8}$$

$$\int_0^b s_2(z)\lambda_1 s_1(dz) = s_2(y_2); \qquad \int_0^b \lambda_1 s_1(dz) + \gamma_2 = 1. \tag{4.9}$$

Recall that we expect that in equilibrium not all agents can have a mass point at zero. This means either $p_2 = 0$ or $p_1 = 0$, and thus either $\gamma_1 = 0$ or $\gamma_2 = 0$. Assume that $\gamma_2 = 0$. Then, using (4.9) and (4.2), we get $b = b_2$. Further, by (4.8), we get $\gamma_1 = 1 - \frac{s_1(y_1)\int_0^{b_2} s_2(dz)}{\int_0^{b_2} s_1(z)s_2(dz)}$. Note that we require $\gamma_1 \geq 0$, which means that $\gamma_2 = 0$ is feasible only if $\int_0^{b_2} [s_1(y_1) - s_1(z)]\, s_2(dz) \leq 0$. However, this inequality holds if and only if $b_1 \leq b_2$. Thus, if $b_2 \geq b_1$ then $\gamma_2 = 0$; if $b_1 \geq b_2$ then $\gamma_1 = 0$. Now suppose $b_2 \geq b_1$. Then, $\gamma_2 = 0$, $b = b_2$, $\gamma_1 = 1 - \frac{s_1(y_1)s_2(b_2)}{\int_0^{b_2} s_1(z)s_2(dz)}$ and we get the Nash equilibrium described in Theorem 4.1.1.

4.2 n-player contest

We study an asymmetric n-player contest in this section and focus on the case where heterogeneity only exists in the starting values of the observed processes. The problem is simplified by homogeneity of the scale functions of the observed processes.

Recall the model introduced in Section 1.1. There are n agents with labels $i \in I = \{1, 2, \ldots, n\}$ in the contest, where $n \geq 2$. Agent i privately observes the continuous-time realisation of a Brownian motion $X^i = (X^i_t)_{t\geq 0}$ absorbed at zero with $X^i_0 = x_i$ where x_i is a constant. We assume that $0 < x_1 \leq x_2 \leq \cdots \leq x_n$.

We also assume that there exists $i \in I$ such that $x_i < x_n$, that is we only consider asymmetric cases.

Each agent decides when to stop her own process, and the agent who stops at the highest value wins unit reward. Agent i observes her own process X^i, but not X^j for $j \neq i$; nor does she observe the stopping times chosen by the other agents. Moreover, the processes X^i are assumed to be independent.

Fix any agent $i \in I$. Given that agent j chooses F_j as her target measure with F_j continuous $\forall j \neq i$, then the expected pay-off of agent i with stopping time τ^i is given by

$$\mathbb{E}\left[\prod_{j\neq i} F_j(X^i_{\tau^i}) \cdot \mathbf{1}_{\{X^i_{\tau^i}>0\}} + \frac{1}{n}\prod_{j\neq i} F_j(0) \cdot \mathbf{1}_{\{X^i_{\tau^i}=0\}}\right].$$

Then, by a similar argument as described in Section 1.2 and in Section 4.1.1, we get that every candidate stopping rule τ^i should satisfy that $\mathbb{E}\left[X^i_{\tau^i}\right] = X^i_0 = x_i$. Thus, we restrict attention to stopping times τ^i such that $\mathbb{E}\left[X^i_{\tau^i}\right] = x_i$.

Remark 4.2.1. The results in this chapter can be extended to the case where the observed processes are independent realisations of some time-homogeneous diffusion process Y which converges almost surely to the lower bound on its state space. The idea has been explained in Chapter 2 and in Section 3.6.

4.2.1 Equilibrium distribution

In this section, we first use the following Lemma 4.2.1 to describe a Nash equilibrium for the contest, and a more explicit form of such a Nash equilibrium will be given in Theorem 4.2.1. Lemma 4.2.1 provides the sufficient conditions for a family of equilibrium distributions that has no positive atoms. The candidate Nash equilibrium is verified using the Lagrangian sufficiency theorem. Recall that the support of a distribution function F is defined by

$$\text{supp}(F) = \{x > 0 : F(z_1) < F(z_2) \text{ for all } z_1 < x < z_2\}.$$

Lemma 4.2.1. *Suppose a family of functions $(F_i)_{i\in I}$ satisfies the following conditions:*

(i) F_i is a distribution function with mean x_i, F_i is continuous and $F_n(0) = 0$, for any $i \in I$;

(ii) $\text{supp}(F_i) = [a_i, b_i]$, where $0 \leq a_i < b_i < \infty$ are some constants, for any $i \in I$;

(iii) $b_i = a_0$ *for any* $i \in I$ *and* $0 = a_n = a_{n-1} \le a_{n-2} \le \cdots \le a_1 < a_0$*, where* a_0 *is some constant;*

(iv) For any $i < n$*,*

$$\prod_{j \neq i} F_j(x) = \frac{x}{a_0}$$

on $[a_i, a_0]$*, and*

$$\prod_{j \neq n} F_j(x) = \left(1 - \prod_{j \neq n} F_j(0)\right) \frac{x}{a_0} + \prod_{j \neq n} F_j(0)$$

on $[0, a_0]$*.*

Then the family $(F_i)_{i \in I}$ *is a Nash equilibrium for the problem.*

Proof. Let $\mathcal{A}$ be the set of pairs (F, p) where $p \in \mathbb{R}^+$ and $F : [0, \infty) \mapsto [0, \infty)$ is a non-decreasing right-continuous function with $F(0) = p$. A pair $(F, p) \in \mathcal{A}$ is identified with a measure ν on $[0, \infty)$ such that $F(x) = \nu([0, x])$ and $p = \nu(\{0\})$.

Fix agent $i \in I$. Suppose that, for any $j \neq i$, agent j chooses (F_j, p_j) as her target measure with F_j continuous, then the aim of agent i is to choose a law of $X^i_{\tau^i}$, which corresponds to a pair (F_i, p_i), to solve

$$\max_{(F_i, p_i) \in \mathcal{A}} \left\{ \int_{(0,\infty)} \prod_{j \neq i} F_j(x) F_i(dx) + \frac{1}{n} \prod_{j \neq i} p_j p_i \right\} \tag{4.10}$$

subject to $\int_0^\infty x F_i(dx) = x_i$ and $\int_0^\infty F_i(dx) + p_i = 1$. Introducing multipliers λ_i and γ_i for the two constraints, the Lagrangian for the optimisation problem (4.10) is then

$$\mathcal{L}_i(F_i, p_i; \lambda_i, \gamma_i) = \int_{(0,\infty)} \left[\prod_{j \neq i} F_j(x) - \lambda_i x - \gamma_i\right] F_i(dx) + \left[\frac{1}{n} \prod_{j \neq i} p_j - \gamma_i\right] p_i + \lambda_i x_i + \gamma_i.$$

Let $\mathcal{A}_D(x)$ be the subset of $\mathcal{A}$ identified with probability measures with mean x, then $\mathcal{A}_D(x)$ is given by

$$\mathcal{A}_D(x) = \left\{(F, p) \in \mathcal{A} : \int_{(0,\infty)} F(dz) + p = 1 \text{ and } \int_{(0,\infty)} z F(dz) = x\right\}.$$

Now we state a variant of the Lagrangian sufficiency theorem for our problem.

Proposition 4.2.1. *Suppose that, for all* $i \in I$*, there exists* $(F_i^*, p_i^*; \lambda_i^*, \gamma_i^*)$ *such* *that* $(F_i^*, p_i^*) \in \mathcal{A}_D(x_i)$*,* F_i^* *is continuous and*

$$\mathcal{L}_i(F_i^*, p_i^*; \lambda_i^*, \gamma_i^*) \geq \mathcal{L}_i(F_i, p_i; \lambda_i^*, \gamma_i^*) \qquad \text{for all } (F_i, p_i) \in \mathcal{A}. \tag{4.11}$$

Then the family $(F_i^*, p_i^*)_{i\in I}$ *is a Nash equilibrium that has no atoms in* $(0, \infty)$*.*

Proof. We seek a Nash equilibrium that has no atoms in $(0, \infty)$. Such a Nash equilibrium is identified with a family of probability measures $(F_i^*, p_i^*)_{i\in I}$ such that, for any $i \in I$, $(F_i^*, p_i^*) \in \mathcal{A}_D(x_i)$, F_i^* is continuous and

$$\int_{(0,\infty)} \prod_{j\neq i} F_j^*(x) F_i^*(dx) + \frac{1}{n} \prod_{j\neq i} p_j^* p_i^* \geq \int_{(0,\infty)} \prod_{j\neq i} F_j^*(x) F_i(dx) + \frac{1}{n} \prod_{j\neq i} p_j^* p_i \tag{4.12}$$

for all $(F_i, p_i) \in \mathcal{A}_D(x_i)$.

Fix any $i \in I$. If $(F_i, p_i) \in \mathcal{A}_D(x_i)$, then, using the definition of the Lagrangian,

$$\int_{(0,\infty)} \prod_{j\neq i} F_j^*(x) F_i(dx) + \frac{1}{n} \prod_{j\neq i} p_j^* p_i = \mathcal{L}_i(F_i, p_i; \lambda_i^*, \gamma_i^*).$$

Then, since $\mathcal{L}_i(F_i^*, p_i^*; \lambda_i^*, \gamma_i^*) \geq \mathcal{L}_i(F_i, p_i; \lambda_i^*, \gamma_i^*)$, we have (4.12) holds. Hence, $(F_i^*, p_i^*)_{i\in I}$ is a Nash equilibrium for the contest.

□

Return to the proof of Lemma 4.2.1. Suppose $(F_i^*)_{i\in I}$ satisfies all the conditions described in the lemma. Let $p_i^* = F_i^*(0)$. Let $\gamma_i^* = 0$ and $\lambda_i^* = 1/a_0$ for any $i < n$, and let $\gamma_n^* = \prod_{j<n} p_j^*$ and $\lambda_n^* = (1 - \gamma_n^*)/a_0$. It is clear that $(F_i^*, p_i^*) \in \mathcal{A}_D(x_i)$ and F_i^* is continuous. We only need to verify that (4.11) holds for all $i \in I$.

Fix any $i \in \{1, 2, \ldots, n-2\}$. Observe that $F_{n-1}^*(x) \leq F_{n-1}^*(a_i) = p_i^* = F_i^*(x)$ and $\prod_{j\neq(n-1)} F_j^*(x) = \frac{x}{a_0}$ on $[0, a_i]$. Also observe that $\prod_{j\neq i} F_j^*(x) = \frac{x}{a_0}$ on $[a_i, a_0]$. Then, since $p_n^* = F_n^*(0) = 0$,

$$\begin{aligned}
&\mathcal{L}_i(F_i, p_i; \lambda_i^*, \gamma_i^*) \\
&\quad = \int_{(0,\infty)} \left[\prod_{j\neq i} F_j^*(x) - \lambda_i^* x - \gamma_i^*\right] F_i(dx) + \left[\frac{1}{n} \prod_{j\neq i} p_j^* - \gamma_i^*\right] p_i + \lambda_i^* x_i + \gamma_i^* \\
&\quad = \int_{(0,a_i)} \left[\frac{x}{a_0} \frac{F_{n-1}^*(x)}{F_i^*(x)} - \frac{x}{a_0}\right] F_i(dx) + \int_{a_0}^{\infty} \left(1 - \frac{x}{a_0}\right) F_i(dx) + \frac{x_i}{a_0} \\
&\quad \leq \frac{x_i}{a_0} = \mathcal{L}_i(F_i^*, p_i^*; \lambda_i^*, \gamma_i^*).
\end{aligned}$$

Moreover, since $\prod_{j\neq(n-1)} F_j^*(x) = \frac{x}{a_0}$ on $[0, a_0]$ and $p_n^* = 0$,

$$\begin{aligned}\mathcal{L}_{n-1}(F_{n-1}, p_{n-1}; \lambda_{n-1}^*, \gamma_{n-1}^*) &= \int_{a_0}^{\infty} \left(1 - \frac{x}{a_0}\right) F_i(dx) + \frac{x_{n-1}}{a_0} \\ &\leq \frac{x_{n-1}}{a_0} = \mathcal{L}_{n-1}(F_{n-1}^*, p_{n-1}^*; \lambda_{n-1}^*, \gamma_{n-1}^*),\end{aligned}$$

and since $\prod_{j\neq n} F_j^*(x) = \left(1 - \prod_{j\neq n} p_j^*\right) \frac{x}{a_0} + \prod_{j\neq n} p_j^*$ on $[0, a_0]$,

$$\begin{aligned}&\mathcal{L}_n(F_n, p_n; \lambda_n^*, \gamma_n^*) \\ &= \left(1 - \prod_{j\neq n} p_j^*\right) \int_{a_0}^{\infty} \left(1 - \frac{x}{a_0}\right) F_n(dx) + \frac{(1-n)p_n}{n} \prod_{j\neq n} p_j^* + \lambda_n^* x_n + \gamma_n^* \\ &\leq \lambda_n^* x_n + \gamma_n^* = \mathcal{L}_n(F_n^*, p_n^*; \lambda_n^*, \gamma_n^*).\end{aligned}$$

Thus, $(F_i^*)_{i\in I}$ is a Nash equilibrium for the problem by Proposition 4.2.1.

□

Now we explicitly characterize the algebraic form of the Nash equilibrium described in Lemma 4.2.1.

Theorem 4.2.1. *Let $a_{n-1} = a_n = 0$, $x_0 = 0$, $p_0 = 1$ and $\prod_{j<0} p_j = p_0 = 1$. Suppose that there exists $\{p_1, p_2, \ldots, p_{n-1}, a_0, a_1, \ldots a_{n-2}, \gamma_n, \lambda_n\}$ such that $p_i \geq 0$ for any $i = 1, 2, \ldots, n-1$, $0 \leq a_{n-2} \leq \cdots \leq a_1 < a_0$ and $\lambda_n > 0$ and it solves the following system of equations*

$$(\star)\quad \begin{cases} \lambda_n a_i + \gamma_n = p_i^{n-i} \prod_{j<i} p_j, & \forall i = 0, 1, \ldots, n-1, \quad (4.13) \\ x_i - x_{i-1} = \dfrac{a_{i-1}p_{i-1} - a_i p_i}{n-i+1} + \dfrac{(n-i)\gamma_n}{(n-i+1)\lambda_n}(p_i - p_{i-1}), & \\ & \forall i = 1, 2, \ldots, n-1, \quad (4.14) \\ x_n = \dfrac{1}{a_0} \displaystyle\sum_{j=1}^{n-1} \Bigg[\dfrac{(n-j)(a_j p_j - a_{j-1}p_{j-1})}{(n-j+1)\lambda_n} + \dfrac{a_{j-1}^2 p_{j-1}}{\lambda_n a_{j-1} + \gamma_n} & \\ \qquad - \dfrac{a_j^2 p_j}{\lambda_n a_j + \gamma_n} + \dfrac{(n-j)^2 \gamma_n}{(n-j+1)\lambda_n^2}(p_{j-1} - p_j) \Bigg], & \\ & \text{if } x_{n-1} < x_n, \quad (4.15) \\ \gamma_n = 0 \text{ and } p_{n-1} = 0, & \text{if } x_{n-1} = x_n. \quad (4.16) \end{cases}$$

(i) If $x_{n-1} < x_n$, then $0 < p_{n-1} \leq p_{n-2} \leq \cdots \leq p_1 < 1$ and $\gamma_n > 0$. If $x_m < x_{m+1} = \cdots = x_n$, for some $m \in \{1, 2, \ldots, n-2\}$, then $0 = p_{n-1} = \cdots = p_{m+1} < p_m \leq \cdots \leq p_1 < 1$, $\gamma_n = 0$ and $0 = a_{n-1} = \cdots = a_{m+1} < a_m$.

(ii) For any $i \in \{2, 3, \ldots, n\}$, if $x_i = x_{i-1}$, then $p_i = p_{i-1}$ and $a_i = a_{i-1}$.

(iii) Define function $F_{n-1}(\cdot)$ by

$$F_{n-1}(x) = \begin{cases} \sqrt[n-k]{\frac{\lambda_n x + \gamma_n}{\prod_{j<k} p_j}}, & \text{if } x \in [a_k, a_{k-1}),\ k = 1, 2, \ldots, n-1, \\ 1, & \text{if } x \in [a_0, \infty), \end{cases}$$

and define function $F_n(\cdot)$ by

$$F_n(x) = \frac{x}{(\lambda_n x + \gamma_n) a_0} F_{n-1}(x).$$

For any $i < n-1$, define

$$F_i(x) = \max\{p_i, F_{n-1}(x)\}.$$

Then, the family $(F_i)_{i \in I}$ satisfies all the conditions in Lemma 4.2.1 and thus is a Nash equilibrium for the asymmetric contest.

Moreover, for any $i \in \{2, 3, \ldots, n\}$, if $x_i = x_{i-1}$, then $F_i(x) = F_{i-1}(x)$ for all $x \in [0, \infty)$.

Proof. (i) Suppose that $x_{n-1} < x_n$. Assume that there exists $k \in \{1, 2, \ldots, n-1\}$ such that $p_k = 0$. Then $\gamma_n = \prod_{j<n} p_j = 0$. By (4.14),

$$x_i - x_{i-1} = \frac{a_{i-1} p_{i-1} - a_i p_i}{n - i + 1}, \quad \forall i = 1, 2, \ldots, n-1, \tag{4.17}$$

which means that $x_{n-1} = \sum_{j=1}^{n-1} \frac{a_{j-1} p_{j-1} - a_j p_j}{n-j+1}$. However, since $\gamma_n = 0$ and $\lambda_n a_0 = 1$ and by (4.15), $x_n = \sum_{j=1}^{n-1} \frac{a_{j-1} p_{j-1} - a_j p_j}{n-j+1} = x_{n-1}$, which is a contradiction. Thus, $p_i > 0$ for all $i = 1, 2, \ldots, n-1$, and then $\gamma_n = \prod_{j<n} p_j > 0$. By (4.13),

$$\lambda_n (a_i - a_{i-1}) = \left(p_i^{n-i} - p_{i-1}^{n-i}\right) \prod_{j<i} p_j, \quad \forall i = 1, 2, \ldots, n-1. \tag{4.18}$$

Then, since $\lambda_n > 0$ and $0 = a_{n-1} \le a_{n-2} \le \cdots \le a_1 < a_0$, we get that $0 < p_{n-1} \le p_{n-2} \le \cdots \le p_1 < p_0 = 1$.

Suppose that $x_m < x_{m+1} = \cdots = x_n$, for some $m \in \{1, 2, \ldots, n-2\}$. Then, by (4.16), $\gamma_n = 0$ and $p_{n-1} = 0$. By (4.17) and since $a_{n-1} = 0$, we get

$$a_{i-1} p_{i-1} = \sum_{j=i}^{n-1} (n-j+1)(x_j - x_{j-1}) = \sum_{j=i}^{n} x_j - (n-i+1) x_{i-1}, \quad \forall i = 1, 2, \ldots, n-1. \tag{4.19}$$

Then, since $x_m < x_{m+1} = \cdots = x_n$, $a_i p_i = 0$ if $m+1 \le i \le n-1$ and $a_i p_i > 0$ if $0 \le i \le m$. Thus, $a_i > 0$ and $p_i > 0$ $\forall 0 \le i \le m$. Further, by (4.13) and $\gamma_n = 0$, we get that $a_i = p_i = 0$ $\forall m+1 \le i \le n-1$. Then, by (4.18) and since $\lambda_n > 0$ and $0 < a_m \le \cdots \le a_1 < a_0$, we get that $0 = p_{n-1} = \cdots = p_{m+1} < p_m \le \cdots \le p_1 < p_0 = 1$.

(ii) Fix any $i \in \{2, 3, \ldots, n-1\}$ and suppose that $x_{i-1} = x_i$. Substituting (4.13) into (4.14) and since $\gamma_n = \prod_{j<n} p_j$, we get that

$$\begin{aligned} x_i - x_{i-1} &= \frac{\left(p_{i-1}^{n-i+1} - p_i^{n-i+1}\right)\prod_{j<i} p_j + (n-i+1)\gamma_n(p_i - p_{i-1})}{(n-i+1)\lambda_n} \\ &= \frac{(p_{i-1} - p_i)\left(\sum_{j=0}^{n-i} p_{i-1}^j p_i^{n-i-j}\right)\prod_{j<i} p_j + (n-i+1)(p_i - p_{i-1})\prod_{j<n} p_j}{(n-i+1)\lambda_n} \\ &= (p_{i-1} - p_i)\frac{\sum_{j=0}^{n-i} p_{i-1}^j p_i^{n-i-j} - (n-i+1)\prod_{j=i}^{n-1} p_j}{(n-i+1)\lambda_n}\prod_{j<i} p_j. \end{aligned}$$

Observe that by (i) we know $0 \le p_{n-1} \le p_{n-2} \le \cdots \le p_1 < 1$. Now assume that $p_{i-1} > p_i$. Then, $\sum_{j=0}^{n-i} p_{i-1}^j p_i^{n-i-j} > (n-i+1)p_i^{n-i} \ge (n-i+1)\prod_{j=i}^{n-1} p_j$ and $\prod_{j<i} p_j \ge p_{i-1}^{i-1} > 0$. This means that $x_i > x_{i-1}$, which is a contradiction. Thus, $p_{i-1} = p_i$. Then, by (4.18), we get $a_{i-1} = a_i$. Now if $x_{n-1} = x_n$, then, by (4.16), it is clear that $p_{n-1} = p_n = 0$ and $a_{n-1} = a_n = 0$.

(iii) By the definitions of F_i, it is clear that the second and third condition given in Lemma 4.2.1 hold and $F_n(0) = 0$.

We next show that $F_{n-1}(x)$ is continuous at $\{a_0, a_1, a_2, \ldots, a_{n-2}\}$. Fix any $k \in \{1, 2, \ldots, n-2\}$. By the definition of $F_{n-1}(x)$,

$$F_{n-1}(a_k-) = \sqrt[n-k-1]{\frac{\lambda_n a_k + \gamma_n}{\prod_{j<(k+1)} p_j}} \quad \text{and} \quad F_{n-1}(a_k) = \sqrt[n-k]{\frac{\lambda_n a_k + \gamma_n}{\prod_{j<k} p_j}}.$$

Then, using (4.13), we get $F_{n-1}(a_k-) = p_k = F_{n-1}(a_k)$. Further, $F_{n-1}(a_0-) = \sqrt[n-1]{\frac{\lambda_n a_0 + \gamma_n}{\prod_{j<1} p_j}} = 1 = F_{n-1}(a_0)$. Hence, F_{n-1} is continuous, which implies that all F_i are continuous.

Since $\lambda_n > 0$ and by the continuity of F_i, it is clear that F_i is non-decreasing. Then, since $F_i(a_0) = 1$, F_i is a distribution function.

Now we show that the mean of F_i is equal to x_i. Fix any $i \in \{1, 2, \ldots, n-1\}$. By the definition of F_i and since F_i is continuous,

$$\int_0^\infty x F_i(dx) = \sum_{k=1}^{i} \int_{a_k}^{a_{k-1}} x d \sqrt[n-k]{\frac{\lambda_n x + \gamma_n}{\prod_{j<k} p_j}}.$$

Using the method of integration by parts

$$\int_0^\infty xF_i(dx) = \sum_{k=1}^{i}\Bigg[a_{k-1}\sqrt[n-k]{\frac{\lambda_n a_{k-1}+\gamma_n}{\prod_{j<k}p_j}} - a_k\sqrt[n-k]{\frac{\lambda_n a_k+\gamma_n}{\prod_{j<k}p_j}} \\ - \frac{n-k}{n-k+1}\frac{\prod_{j<k}p_j}{\lambda_n}\int_{a_k}^{a_{k-1}} d\left(\frac{\lambda_n x+\gamma_n}{\prod_{j<k}p_j}\right)^{\frac{n-k+1}{n-k}}\Bigg].$$

Then, using $(\star)$, we get

$$\begin{aligned}\int_0^\infty xF_i(dx) &= \sum_{k=1}^{i}\left[a_{k-1}p_{k-1} - a_k p_k - \frac{n-k}{n-k+1}\frac{\prod_{j<k}p_j}{\lambda_n}\left(p_{k-1}^{n-k+1} - p_k^{n-k+1}\right)\right] \\ &= \sum_{k=1}^{i}\left[\frac{a_{k-1}p_{k-1}-a_k p_k}{n-k+1} + \frac{(n-k)\gamma_n}{(n-k+1)\lambda_n}(p_k - p_{k-1})\right] \qquad (4.20) \\ &= \sum_{k=1}^{i}(x_k - x_{k-1}) = x_i,\end{aligned}$$

that is F_i has mean x_i. Next calculate the mean of F_n. Suppose that $x_{n-1} = x_n$. Then, since $\gamma_n = 0$ and $\lambda_n a_0 = 1$, it is clear that $\int_0^\infty xF_n(dx) = \int_0^\infty xF_{n-1}(dx) = x_{n-1} = x_n$. Suppose that $x_{n-1} < x_n$. Then, by the form of F_n,

$$\int_0^\infty xF_n(dx) = \frac{1}{a_0}\sum_{k=1}^{n-1}\Bigg\{\frac{n-k}{\lambda_n}\int_{a_k}^{a_{k-1}} xd\sqrt[n-k]{\frac{\lambda_n x+\gamma_n}{\prod_{j<k}p_j}} \\ + \int_{a_k}^{a_{k-1}} x^2 d\frac{(\lambda_n x+\gamma_n)^{\frac{1}{n-k}-1}}{\sqrt[n-k]{\prod_{j<k}p_j}}\Bigg\}.$$

Because

$$\int_{a_k}^{a_{k-1}} x^2 d\frac{(\lambda_n x+\gamma_n)^{\frac{1}{n-k}-1}}{\sqrt[n-k]{\prod_{j<k}p_j}} = \frac{a_{k-1}^2 p_{k-1}}{\lambda_n a_{k-1}+\gamma_n} - \frac{a_k^2 p_k}{\lambda_n a_k+\gamma_n} \\ - \frac{2(n-k)}{\lambda_n}\int_{a_k}^{a_{k-1}} xd\sqrt[n-k]{\frac{\lambda_n x+\gamma_n}{\prod_{j<k}p_j}}$$

and using (4.20) and (4.15), we get

$$\int_0^\infty xF_n(dx) = \frac{1}{a_0}\sum_{k=1}^{n-1}\Bigg\{\frac{(n-k)(a_k p_k - a_{k-1}p_{k-1})}{(n-k+1)\lambda_n} + \frac{a_{k-1}^2 p_{k-1}}{\lambda_n a_{k-1}+\gamma_n} \\ - \frac{a_k^2 p_k}{\lambda_n a_k+\gamma_n} + \frac{(n-k)^2\gamma_n}{(n-k+1)\lambda_n^2}(p_{k-1}-p_k)\Bigg\} = x_n.$$

Thus, the mean of F_n is x_n.

Fix any $i < n$ and fix any $k \le i$. For any $x \in [a_k, a_{k-1}]$,

$$\prod_{j\neq i} F_j(x) = \left(\prod_{j<k} p_j\right)\left(\prod_{j\ge k, j\neq i} F_j(x)\right) = \left(\prod_{j<k} p_j\right)\left(\frac{\lambda_n x + \gamma_n}{\prod_{j<k} p_j}\right)\frac{x/a_0}{\lambda_n x + \gamma_n} = \frac{x}{a_0}.$$

Now consider $i = n$. Fix any $k \le n-1$. For any $x \in [a_k, a_{k-1}]$,

$$\prod_{j\neq n} F_j(x) = \left(\prod_{j<k} p_j\right)\left(\prod_{j\ge k, j\neq n} F_j(x)\right) = \left(\prod_{j<k} p_j\right)\left(\frac{\lambda_n x + \gamma_n}{\prod_{j<k} p_j}\right) = \lambda_n x + \gamma_n.$$

Further, because $a_{n-1} = 0$ and by (4.13), we get $\gamma_n = \prod_{j<n} p_j$ and $\lambda_n = (1 - \gamma_n)/a_0 = \left(1 - \prod_{j<n} p_j\right)/a_0$. Thus, $(F_i)_{i\in I}$ satisfies the last condition in Lemma 4.2.1.

Using (ii) and (4.16), it is clear that, for any $i \in \{2, 3, \ldots, n\}$, if $x_i = x_{i-1}$, then $F_i(x) = F_{i-1}(x)$ for all $x \in [0, \infty)$.

$\square$

Remark 4.2.2. It is clear that the optimal distribution functions $(F_i)_{i\in I}$ in Theorem 4.2.1 satisfy that $F_1(x) \ge \cdots \ge F_{n-1}(x) \ge F_n(x)$.

In general, it is difficult to prove the existence of a solution to the system of equations ($\star$) and also is difficult to analytically solve ($\star$). However, there are two special cases where ($\star$) can be significantly simplified and the existence of a Nash equilibrium can be analytically proved. One case corresponds to the scenario where there are multiple highest starting values of the observed processes, and another case corresponds to the scenario where the highest starting value is unique and the rest of the starting values are the same.

Theorem 4.2.2. *Recall the definitions* $a_{n-1} = a_n = 0$, $x_0 = 0$, $p_0 = 1$ *and* $\prod_{j<0} p_j = p_0 = 1$.

(i) Suppose $0 < x_1 \le \cdots \le x_m < x_{m+1} = \cdots = x_n$ *for some* $m \in \{1, 2, \ldots, n-2\}$. *Let* $\gamma_n = 0$, $a_0 = \sum_{j=1}^n x_j$, $\lambda_n = 1/a_0$, $p_{n-1} = p_{n-2} = \cdots = p_{m+1} = 0$, $a_{n-2} = a_{n-3} = \cdots = a_{m+1} = 0$,

$$p_i = \prod_{k=1}^{i} \sqrt[n-k+1]{\frac{\sum_{j=k+1}^n x_j - (n-k)x_k}{\sum_{j=k}^n x_j - (n-k+1)x_{k-1}}},\ i = 1, 2, \ldots, m,$$

and

$$a_i = \frac{\sum_{j=i+1}^n x_j - (n-i)x_i}{p_i},\ i = 1, 2, \ldots, m.$$

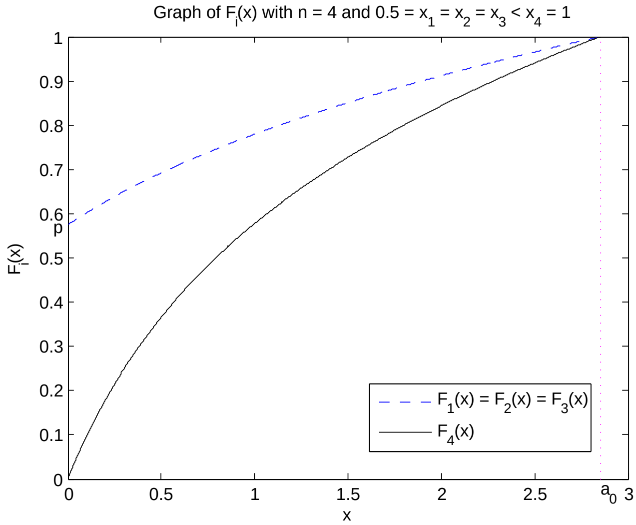

Figure 4.1: Graph of $(F_i)_{i\in I}$ with $n = 4$ and $0.5 = x_1 = x_2 = x_3 < x_4 = 1$.

Then, $\{p_1, p_2, \ldots, p_{n-1}, a_0, a_1, \ldots a_{n-2}, \gamma_n, \lambda_n\}$ *is the unique solution to* $(\star)$ *such that* $p_i \geq 0$ *for any* $i = 1, 2, \ldots, n-1$, $0 \leq a_{n-2} \leq \cdots \leq a_1 < a_0$ *and* $\lambda_n > 0$. *Moreover,* $(F_i)_{i\in I}$ *defined by Theorem 4.2.1 is a Nash equilibrium for the asymmetric contest.*

(ii) Suppose $0 < x_1 = \cdots = x_{n-2} = x_{n-1} < x_n$. *Let* $p \in (0,1)$ *be the unique solution to* $P(z) = x_1/x_n$, *where*

$$P(z) = \left(1 - z^{n-1}\right) \frac{(n-1)z^n - nz^{n-1} + 1}{nz^{2n-2} - (n-1)^2 z^n + n\,(n-3)\,z^{n-1} + 1}.$$

And let $\gamma_n = p^{n-1}$, $p_1 = p_2 = \cdots = p_{n-2} = p$, $a_1 = a_2 = \cdots = a_{n-2} = 0$,

$$a_0 = \frac{nx_1\left(1 - p^{n-1}\right)}{(1 - p^{n-1}) + (n-1)(p-1)p^{n-1}} \quad \textit{and} \quad \lambda_n = \frac{1 - p^{n-1}}{a_0}.$$

Then, $\{p_1, p_2, \ldots, p_{n-1}, a_0, a_1, \ldots a_{n-2}, \gamma_n, \lambda_n\}$ *is the unique solution to* $(\star)$ *such that* $p_i \geq 0$ *for any* $i = 1, 2, \ldots, n-1$, $0 \leq a_{n-2} \leq \cdots \leq a_1 < a_0$ *and* $\lambda_n > 0$. *Furthermore,* $(F_i)_{i\in I}$ *defined by Theorem 4.2.1 is a Nash equilibrium for the asymmetric contest.*

Example 4.2.1. Before proving Theorem 4.2.2, we give two examples of these two special cases to illustrate the general shape for graphs of the distribution functions $(F_i)_{i\in I}$.

In Figure 4.1, we study the special case where the highest starting value is unique and the other starting values are the same. It can be seen that the player

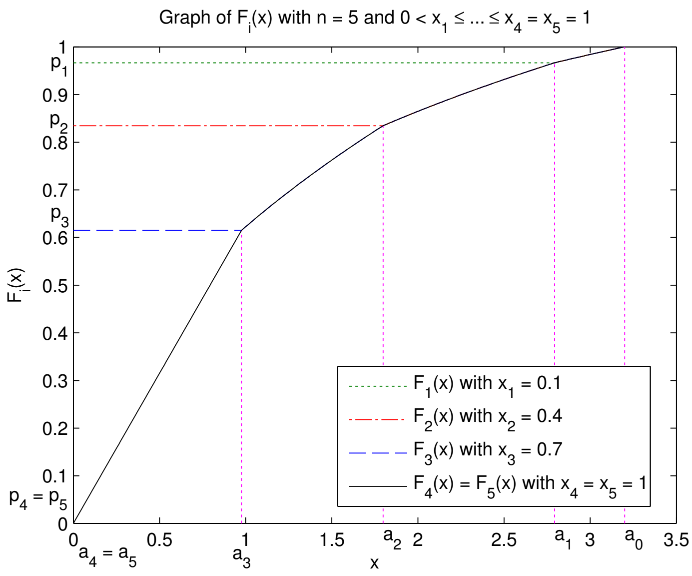

Figure 4.2: Graph of $(F_i)_{i\in I}$ with $n = 5$ and $0.1 = x_1 < x_2 = 0.4 < x_3 = 0.7 < x_4 = x_5 = 1$.

with the higher starting value puts no mass on zero while the other players with the lower starting value put a mass of size p on zero, but the support of the distribution functions are all $[0, a_0]$.

The other special case is shown in Figure 4.2. As is shown in the figure, similar to the previous case, the player with a lower starting value has a larger probability of stopping at zero. But in this case, the support of the distribution functions are no longer the same. In particular, the largest values at which the players should stop are all a_0, while the smallest nonzero value at which the player should stop increases as the starting value of the player decreases.

Proof of Theorem 4.2.2. (i) Suppose that $x_m < x_{m+1} = \cdots = x_n$ for some $m \in \{1, 2, \ldots, n-2\}$. Then, by (i) in Theorem 4.2.1, $\gamma_n = 0$, $0 = p_{n-1} = \cdots = p_{m+1} < p_m \le \cdots \le p_1 < 1$ and $0 = a_{n-1} = \cdots = a_{m+1} < a_m$. Further, by (4.19),

$$a_i = \frac{\sum_{j=i+1}^n x_j - (n-i)x_i}{p_i}, \ i = 0, 1, \ldots, m.$$

Because $p_0 = 1$ and $x_0 = 0$, $a_0 = \sum_{j=1}^n x_j$. Now observe that $\lambda_n a_i p_i = p_i^{n-i+1} \prod_{j<i} p_j$, which leads to $\lambda_n \left(\sum_{j=i+1}^n x_j - (n-i)x_i \right) = p_i^{n-i+1} \prod_{j<i} p_j$. Thus,

$$\frac{p_k}{p_{k-1}} = \sqrt[n-k+1]{\frac{p_k^{n-k+1} \prod_{j<k} p_j}{p_{k-1}^{n-k+2} \prod_{j<k-1} p_j}} = \sqrt[n-k+1]{\frac{\sum_{j=k+1}^n x_j - (n-k)x_k}{\sum_{j=k}^n x_j - (n-k+1)x_{k-1}}},$$

for any $k = 1, 2, \ldots, m$. Then, by $p_i = \prod_{k=1}^{i} \frac{p_k}{p_{k-1}}$, we get the p_i stated in the theorem.

(ii) Suppose $x_1 = \cdots = x_{n-2} = x_{n-1} < x_n$. First we show that there exists a unique solution p to $P(z) = x_1/x_n$ such that $p \in (0, 1)$. Observe that $P(0) = 1$ and $P(1) = 0$. Next we show that P is strictly decreasing on $(0, 1)$. Let

$$P_1(p) = \frac{(n-1)p^n - np^{n-1} + 1}{1 - p^{n-1}}$$

and

$$P_2(p) = \frac{np^{2n-2} - (n-1)^2 p^n + n\,(n-3)\,p^{n-1} + 1}{(1 - p^{n-1})^2},$$

then $P(p) = P_1(p)/P_2(p)$. Because

$$P_1'(p) = \frac{(n-1)p^{n-2}}{(1 - p^{n-1})^2} \left[-p^n + np - (n-1)\right] < 0$$

and

$$P_2'(p) = \frac{(n-1)^2 p^n}{(1 - p^{n-1})^3} \Big[-(n-2)p^n + np^{n-1} - np + n - 2\Big] > 0$$

on $(0, 1)$, $P(p) = P_1(p)/P_2(p)$ is strictly decreasing on $(0, 1)$. Then, since $0 < x_1/x_n < 1$, there exists a unique solution p to $P(z) = x_1/x_n$ such that $p \in (0, 1)$.

Because $x_1 = \cdots = x_{n-2} = x_{n-1}$ and by Theorem 4.2.1, $p_1 = \cdots = p_{n-2} = p_{n-1}$ and $a_1 = \cdots = a_{n-2} = a_{n-1} = 0$. Then $(\star)$ can be reduced to

$$\begin{cases} \lambda_n a_0 + \gamma_n = 1, \\ \gamma_n = p_1^{n-1}, \\ x_1 = \frac{a_0}{n} + \frac{(n-1)\gamma_n}{n\lambda_n}(p_1 - 1), \\ x_n = a_0 \left[1 - \frac{(n-1)}{n(1-\gamma_n)} + \frac{(n-1)^2\gamma_n}{n(1-\gamma_n)^2}(1 - p_1)\right]. \end{cases}$$

Thus, $\gamma_n = p_1^{n-1}$, $\lambda_n = \frac{1-\gamma_n}{a_0}$, $a_0 = \frac{nx_1(1-\gamma_n)}{(1-\gamma_n)+(n-1)\gamma_n(p-1)}$ and

$$\frac{x_n}{x_1} = \frac{1}{1 - p_1^{n-1}} \frac{np_1^{2n-2} - (n-1)^2 p_1^n + n(n-3)p_1^{n-1} + 1}{(n-1)p_1^n - np_1^{n-1} + 1} = \frac{1}{P(p_1)}.$$

The last equation shows that p_1 is a solution to $P(z) = x_1/x_n$.

$\square$

Remark 4.2.3. Recall that the optimal distribution functions $(F_i)_{i \in I}$ for a 2-player asymmetric contest is given by Theorem 4.1.1. Suppose $0 < x_1 \leq x_2$. Since now

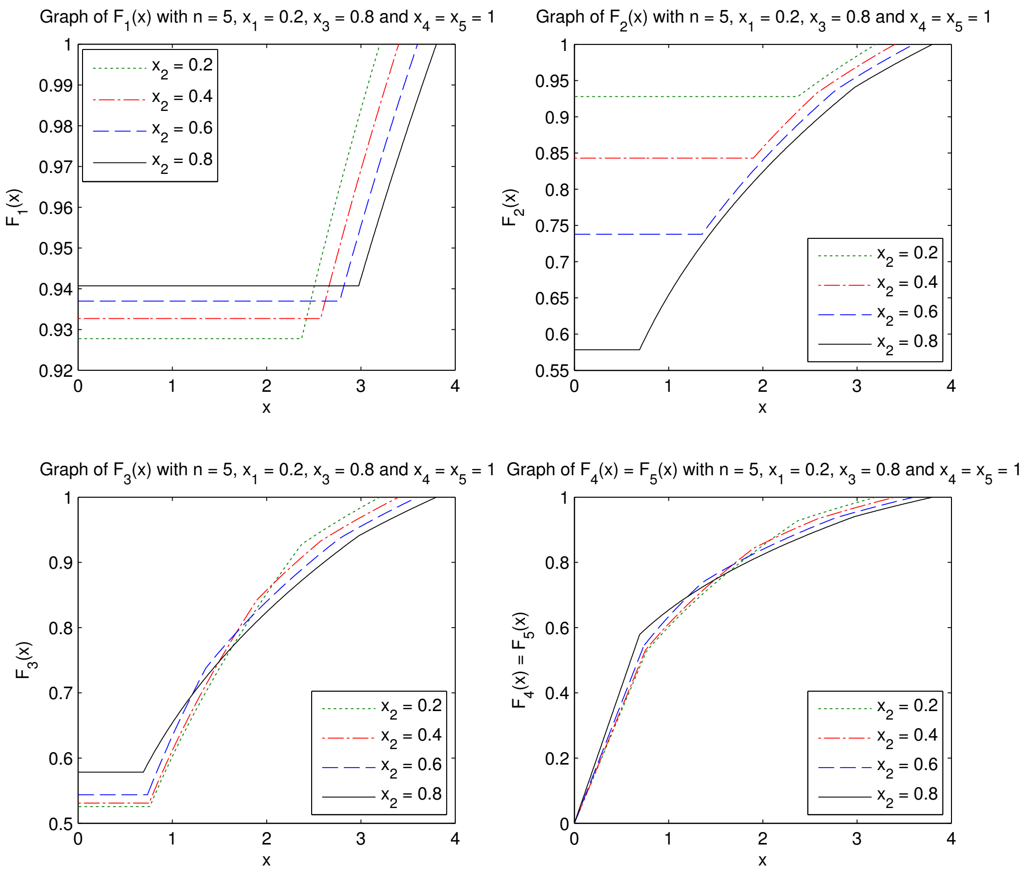

Figure 4.3: Graph of $(F_i)_{i \in I}$ for different x_2 with $n = 5$, $x_1 = 0.2$, $x_3 = 0.8$ and $x_4 = x_5 = 1$.

the processes that observed by the players are Brownian motions, in a 2-player asymmetric contest $(F_i)_{i \in I}$ are given by

$$F_1(x) = \min\left\{\left(1 - \frac{x_1}{x_2}\right) + \frac{x_1}{x_2}\frac{x}{2x_2}, 1\right\}, \qquad F_2(x) = \min\left\{\frac{x}{2x_2}, 1\right\}.$$

Since $\mathbb{P}\left(H^1_{x_2} < H^1_0\right) = \frac{x_1}{x_2}$, $X^2_{\tau^2} \cdot \mathbf{1}_{\{H^1_{x_2} < H^1_0\}}$ has the same distribution as $X^1_{\tau^1}$. This implies that, in the 2-player asymmetric contest, an optimal strategy for player 1 is to wait until $X^1_t \in \{0, x_2\}$ and then use the same strategy as player 2 if X^1 hits x_2. But this result does not hold for a general asymmetric n-player contest.

Example 4.2.2. Consider an example of the first special case where there are multiple highest starting values. Set $n = 5$, $x_1 = 0.2$, $x_3 = 0.8$ and $x_4 = x_5 = 1$. We give the graphs of the optimal distribution functions $(F_i)_{i \in I}$ for various values of x_2. Figure 4.3 shows that when x_2 is larger, player 2 has a smaller probability of

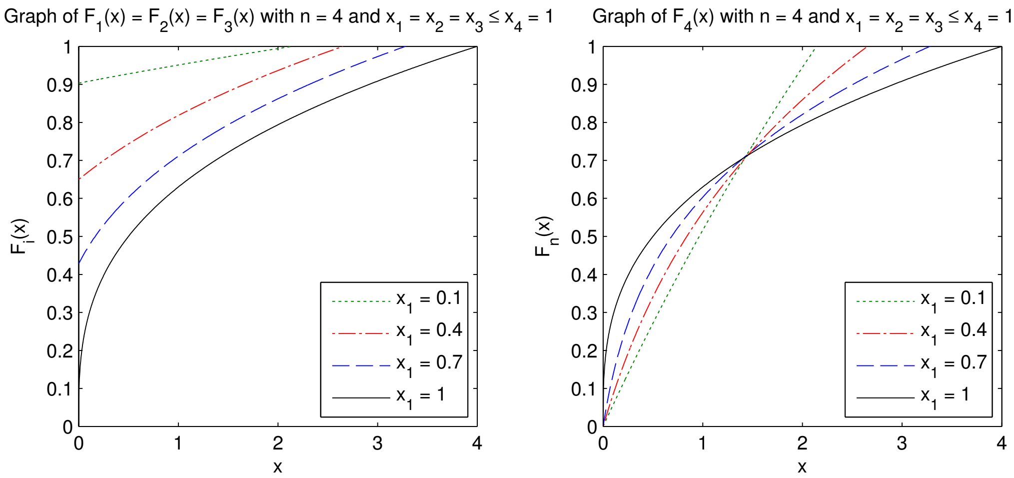

Figure 4.4: Graph of $(F_i)_{i\in I}$ for different x_1 with $n = 4$ and $x_1 = x_2 = x_3 \leq x_4 = 1$.

stopping at zero while the other players $j \in \{i \in I : x_i < x_n\} \setminus \{2\}$ have a larger probability of stopping at zero. It also shows that when x_2 increases, the smallest nonzero value a_j at which player $j \in \{i \in I : x_i < x_2\}$ should stop increases, while the smallest nonzero value a_j at which player $j \in \{i \in I : x_2 \leq x_i < x_n\}$ should stop decreases. And when x_2 increases, the largest value a_0 at which a player should stop increases. Moreover, as x_2 increases, the probability that player 2 will stop at small values decreases while the probability that player $j \in \{i \in I : x_2 < x_i\}$ will stop at small values increases.

Example 4.2.3. We consider an example of the second special case where the highest starting value is unique. Set $n = 4$ and $x_4 = 1$ and let $x_1 = x_2 = x_3$. We show the graphs of $(F_i)_{i\in I}$ for various values of x_1. As shown in Figure 4.4, when x_1 increases, the largest value a_0 at which a player should stop increases and the probability p that the player with a lower starting value (player 1, 2 or 3) will stop at zero decreases. Further, as x_1 increases, the probability that player 1 will stop at small values decreases while the probability that player 4 will stop at small values increases. And Figure 4.5 shows that $F_1(x)$ tends to $F_4(x)$ as x_1 tends to x_4.

Example 4.2.4. Now we consider another example of the second special case. Set $0.5 = x_1 = \cdots = x_{n-1} < x_n = 1$. We give the graphs of $(F_i)_{i\in I}$ for various values of n. It can be seen from Figure 4.6 that when the number of participants n increases, the largest value a_0 at which a player should stop increases and the probability that a player will stop at small values increases (the same trends exist in the symmetric case where $x_1 = \cdots = x_{n-1} = x_n$). In particular, Figure 4.7 shows that when n is very large, the probability p that player 1 will stop at zero is close to 1 and the

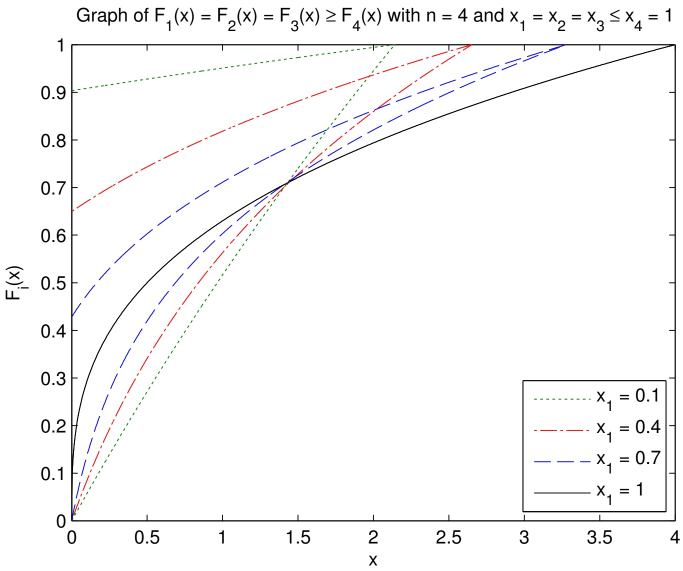

Figure 4.5: Graph of $(F_i)_{i \in I}$ for different x_1 with $n = 4$ and $x_1 = x_2 = x_3 \leq x_4 = 1$.

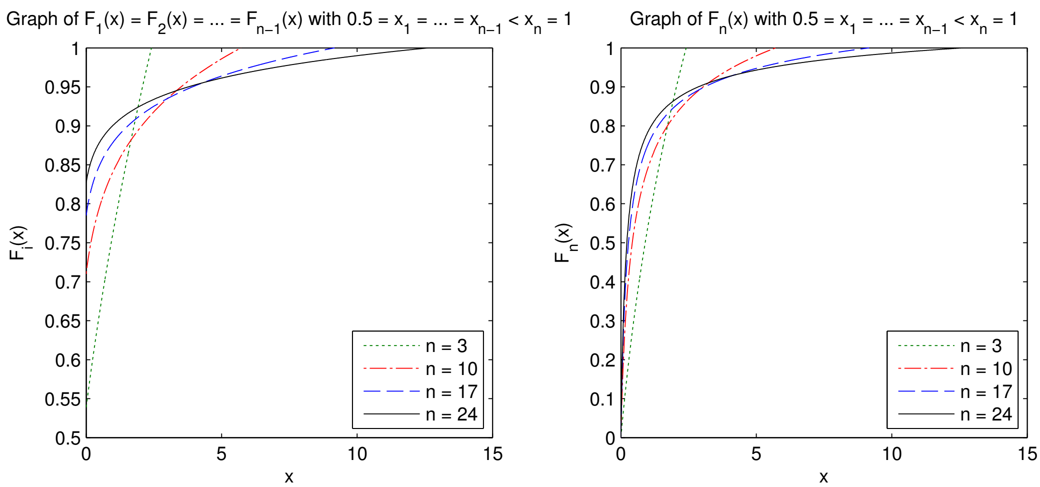

Figure 4.6: Graph of $(F_i)_{i \in I}$ for different n with $0.5 = x_1 = \cdots = x_{n-1} < x_n = 1$.

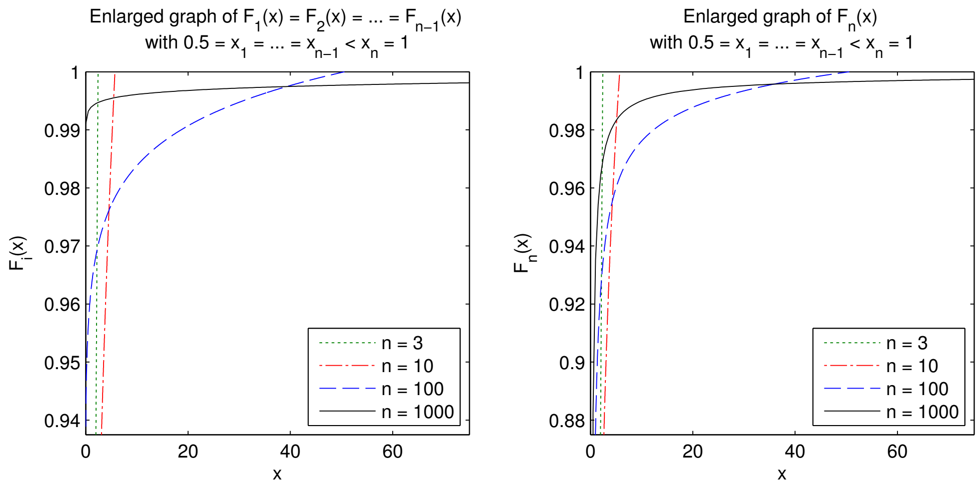

Figure 4.7: Enlarged graph of $(F_i)_{i\in I}$ for different n with $0.5 = x_1 = \cdots = x_{n-1} < x_n = 1$.

probability that player n will stop at very small values is also close to 1.

Fix any $i \in I$. Define $C_i(x) = \int_x^\infty (y-x)F_i(dy)$ and observe that $C_i^{'}(x) = F_i(x) - 1$. Thus, Figure 4.6 implies that as n increases, the function $C_i(x)$ increases for any $x \geq 0$, which can also be seen from Figure 4.8. Suppose that τ^i is a stopping time such that $X^i_{\tau^i} \sim F_i$. Then, by Chacon and Walsh [1976] and the fact that $\mathbb{E}\left[X^i_{\tau^i}\right] = x_i$ for any n, we get that as n increases τ^i increases (the same trend exists in the symmetric case where $x_1 = \cdots = x_{n-1} = x_n$), that is as the number of participants increases a player should stop later.

For more complicated cases, it might be difficult to analytically solve $(\star)$ which gives us a Nash equilibrium for our problem, but it may be possible to find solve $(\star)$ numerically. We take a 3-player game as an example.

Example 4.2.5. Set $n = 3$ and suppose that $0 < x_1 < x_2 < x_3$. Then, $a_2 = a_3 = 0$, $x_0 = 0$, $p_0 = 1$, and $(\star)$ can be reduced to

$$\begin{cases} p_2 = p_1 - \frac{2(x_2-x_1)}{a_1}, \\ \gamma_3 = p_1 p_2 = p_1^2 - \frac{2(x_2-x_1)}{a_1} p_1, \\ \lambda_3 = \frac{p_1^2 - p_1 p_2}{a_1} = \frac{2(x_2-x_1)p_1}{a_1^2}, \\ a_0 = \frac{1-\gamma_3}{\lambda_3} = a_1 + \frac{1-p_1^2}{2(x_2-x_1)p_1} a_1^2, \\ x_1 = (1-p_1)a_1 + \frac{2p_1^3 - 3p_1^2 + 1}{6(x_2-x_1)p_1} a_1^2, \\ x_3 = \frac{6(2p_1^2-p_1)(x_2-x_1)^2 a_1 + 12(x_2-x_1)(1-p_1)p_1^2 a_1^2 + (3p_1^4 - 4p_1^3 + 1)a_1^3}{6p_1(x_2-x_1)(a_1 - 2x_1 p_1 + 2x_2 p_1 - a_1 p_1^2)}. \end{cases}$$

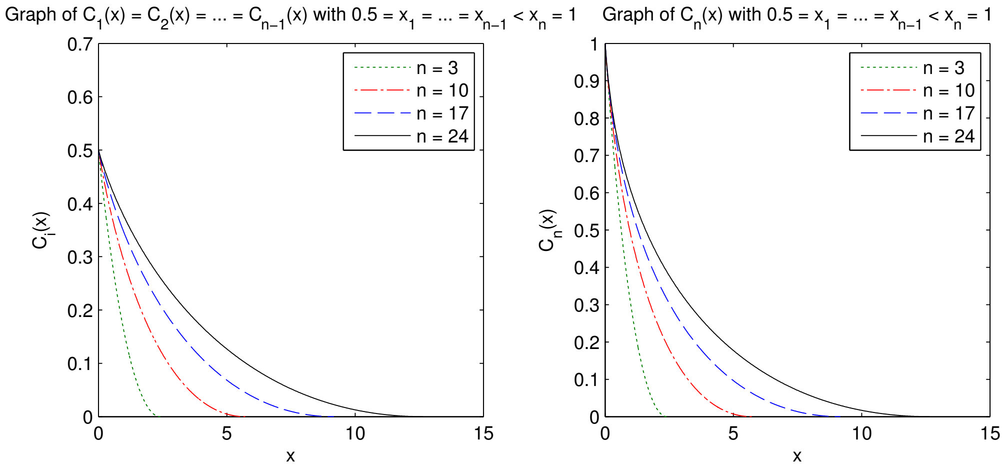

Figure 4.8: Graph of $(C_i)_{i\in I}$ for different n with $0.5 = x_1 = \cdots = x_{n-1} < x_n = 1$.

Using the last two equations, we get that $a_1 = A_1(p_1)/A_2(p_1)$ and $p_1 \in (0,1)$ solves $Q(p) = 0$, where $A_1(p) \triangleq 2(x_2 - x_1)p\left[4x_3p^2 + (3x_1 + 4x_3)p + (3x_1 + x_3)\right]$, $A_2(p) \triangleq 4x_3p^4 + (4x_1 + 2x_2 + 4x_3)p_1^3 + (3x_1 + 4x_2 - 3x_3)p^2 + 4(x_2 - x_3)p + 2x_1 - x_2 - x_3$ and

$$
\begin{aligned}
Q(p) \triangleq\ & 4(x_1 + 2x_2)x_3^2p^6 + 2x_3(x_1 + 2x_2)(3x_1 + 3x_2 + 2x_3)p^5 \\
& + \left[3x_1^3 + 12(x_2 + x_3)x_1^2 + (12x_2^2 + 18x_2x_3 + x_3^2)x_1 + 12x_2^2x_3 - 22x_2x_3^2\right] p^4 \\
& + \left[6x_1^3 + (12x_2 - 6x_3)x_1^2 + (18x_2^2 - 6x_2x_3 - 16x_3^2)x_1 - 8x_2x_3^2\right] p^3 \\
& + \left[(18x_2 - 6x_3)x_1^2 + (12x_2^2 - 4x_3^2)x_1 - 30x_2^2x_3 + 10x_2x_3^2\right] p^2 \\
& + \left[6x_1^3 + (24x_2 + 6x_3)x_1^2 + (-18x_2^2 - 36x_2x_3 + 8x_3^2)x_1 + 6x_2^2x_3 + 4x_2x_3^2\right] p \\
& + 3x_1(x_2 - 2x_1 + x_3)^2.
\end{aligned}
$$

It is difficult to find an explicit solution to $Q(p) = 0$, but we can numerically solve it. Moreover, it can be shown that there is one and only one solution p_1 to $Q(p) = 0$ such that $p_1 \in (0,1)$. The numerical result is given in Figure 4.9. In this figure, we also present the graphs of the Nash equilibrium distributions $(F_i)_{i\in I}$ for the other three cases.

4.2.2 Derivation of the equilibrium distribution

This section illustrates how we obtained the sufficient conditions for the Nash equilibrium given in Lemma 4.2.1. Recall that $0 \le x_1 \le x_2 \le \cdots \le x_n$ and recall the definition of the Lagrangian $\mathcal{L}_i(F_i, p_i; \lambda_i, \gamma_i)$ for the optimisation problem (4.10).

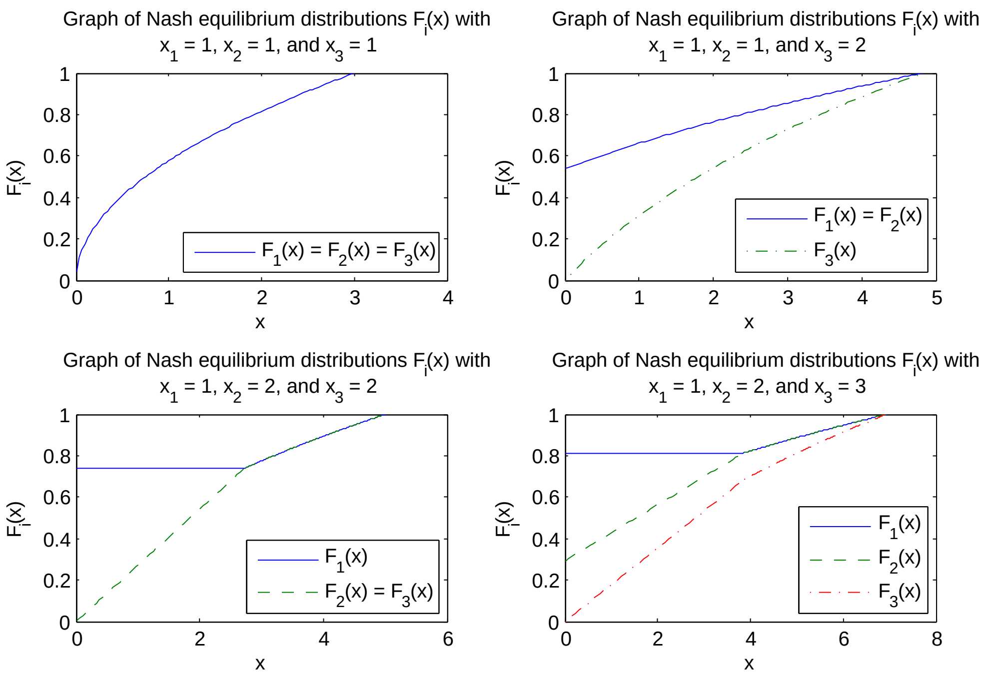

Figure 4.9: Graph of Nash equilibria $(F_i)_{i\in I}$ with $n = 3$.

Let $L_i(x) = \prod_{j\neq i} F_j(x) - \lambda_i x - \gamma_i$, then the Lagrangian is given by

$$\mathcal{L}_i(F_i, p_i; \lambda_i, \gamma_i) = \int_{(0,\infty)} L_i(x) F_i(dx) + \left[\frac{1}{n} \prod_{j\neq i} p_j - \gamma_i \right] p_i + \lambda_i x_i + \gamma_i.$$

In order to have a finite optimal solution, we require $L_i(x) \leq 0$ on $(0, \infty)$ and $\frac{1}{n}\prod_{j\neq i} p_j - \gamma_i \leq 0$. Let $\mathcal{D}_i$ be the set of (λ_i, γ_i) such that $\mathcal{L}_i(\cdot, \cdot; \lambda_i, \gamma_i)$ has a finite maximum. Then $\mathcal{D}_i$ is defined by

$$\mathcal{D}_i = \left\{ (\lambda_i, \gamma_i) : L_i(x) \leq 0 \ \forall x > 0 \text{ and } \frac{1}{n} \prod_{j\neq i} p_j - \gamma_i \leq 0 \right\}.$$

For $(\lambda_i, \gamma_i) \in \mathcal{D}_i$, the maximum value of $\mathcal{L}_i(\cdot, \cdot; \lambda_i, \gamma_i)$ occurs at (F_i, p_i) such that $F_i(dx) = 0$ when $L_i(x) < 0$ and $p_i = 0$ when $\frac{1}{n}\prod_{j\neq i} p_j - \gamma_i < 0$. Thus, if $F_i(dx) > 0$ then $L_i(x) = \prod_{j\neq i} F_j(x) - \lambda_i x - \gamma_i = 0$ and if $p_i > 0$ then $\frac{1}{n}\prod_{j\neq i} p_j - \gamma_i = 0$.

By analogy with the symmetric case, we expect that the support of the distribution functions F_i are all intervals. Denote by $\{a_i, b_i\}$ the endpoints of supp(F_i) with $a_i < b_i$. We also expect that the right-endpoints are the same, that is $b_i = a_0$ for any $i \in I$, where a_0 is some positive constant. Thus, $\prod_{j\neq i} F_j(x) - \lambda_i x - \gamma_i = 0$

on $[a_i, a_0]$. Further, the left-endpoints are expected to satisfy that $0 = a_n = a_{n-1} \leq a_{n-2} \leq \cdots \leq a_1 < a_0$.

Recall that we expect that there is at least one agent who puts no mass at zero and all agents put no mass at any positive point, which implies F_i is continuous and there exists $k \in I$ such that $F_k(0) = 0$. We now argue that $F_n(0) = 0$. Assume that $F_n(0) \neq 0$, that is $k \neq n$. Then, since $p_n > 0$ and $p_k = 0$, $\gamma_n = \frac{1}{n} \prod_{j \neq n} p_j = 0$. Further, since $F_k(a_k) = 0$ and $a_k \geq 0 = a_n$, $\lambda_n a_k = \prod_{j \neq n} F_j(a_k) - \gamma_n = 0$. Observe that $\lambda_n a_0 = \prod_{j \neq n} F_j(a_0) - \gamma_n = 1 > 0$, which means $\lambda_n > 0$, $a_0 > 0$ and $a_k = 0$. Since $\prod_{j \neq k} F_j(a_k) - \lambda_k a_k - \gamma_k = 0$, $\gamma_k = \prod_{j \neq k} F_j(0) \geq 0$. Then, by $\lambda_n a_0 = 1 = \lambda_k a_0 + \gamma_k$, $\lambda_n \geq \lambda_k$. Thus, on $(0, a_0)$, since $\prod_{j \neq n} F_j(x) = \lambda_n x$ and $\prod_{j \neq k} F_j(x) = \lambda_k x + \gamma_k$, we get $\frac{F_n(x)}{F_k(x)} - 1 = \frac{\lambda_k x + \gamma_k}{\lambda_n x} - 1 = \frac{(\lambda_n - \lambda_k)(a_0 - x)}{\lambda_n x} \geq 0$. This means that $F_n(x) \geq F_k(x)$ on $[0, a_0]$. But this is possible only if $x_n = x_k$, since we require F_n has mean x_n and F_k has mean x_k. Now suppose $x_n = x_k$, then $F_n(x) = F_k(x)$ on $[0, a_0]$, which contradicts $F_n(0) \neq 0$. Thus, $F_n(0) = 0$.

Fix any $i < n$. Suppose that $x_i = x_n$. We claim that $F_i(x) = F_n(x)$, $\gamma_i = 0$ and $\lambda_i = 1/a_0$. Assume that $F_i(0) \neq 0$. Then, since $F_n(0) = 0$ and by previous arguments, we get that $F_i(x) \geq F_n(x)$ on $[0, a_0]$. Further, since $x_i = x_n$, $F_i(x) = F_n(x)$ on $[0, a_0]$, which contradicts $F_i(0) \neq 0$. Thus, $F_i(0) = 0$. Then, we get $\gamma_n = \prod_{j \neq n} F_j(0) = 0$, $F_i(a_i) = 0$ and $\lambda_n a_i = \prod_{j \neq n} F_j(a_i) - \gamma_n = 0$. Because $\lambda_n a_0 = 1 > 0$, we have $a_0 > 0$, $\lambda_n > 0$ and $a_i = 0$. Then since $F_n(0) = 0$, $\gamma_i = \prod_{j \neq i} F_j(0) = 0$ and thus $\lambda_i a_0 = 1 = \lambda_n a_0$. This means that $\lambda_i = \lambda_n = 1/a_0$ and $F_i(x) = F_n(x)$.

Fix any $i < n$. Suppose that $x_i < x_n$. We claim that $F_i(0) > 0$, $\gamma_i = 0$ and $\lambda_i = 1/a_0$. Assume that $F_i(0) = 0$. Then, by previous arguments, we get that $F_i(x) = F_n(x)$, which contradicts $\int_0^\infty x F_i(dx) = x_i < x_n = \int_0^\infty x F_n(dx)$. Thus, $F_i(0) > 0$. Moreover, since $p_i > 0$ and $p_n = 0$, $\gamma_i = \frac{1}{n} \prod_{j \neq i} p_j = 0$. Further, since $\lambda_i a_0 + \gamma_i = \prod_{j \neq i} F_j(a_0) = 1$, $\lambda_i = 1/a_0$.

It is now clear that $\gamma_i = 0$ and $\lambda_i = 1/a_0$ for all $i < n$. Then, since $F_i(dx) > 0$ on (a_i, a_0), $\prod_{j \neq i} F_j(x) = \lambda_i x + \gamma_i = x/a_0$ on $[a_i, a_0]$.

Because $F_n(dx) > 0$ on $(0, a_0)$, $\prod_{j \neq n} F_j(x) - \lambda_n x - \gamma_n = 0$ on $[0, a_0]$. Letting $x = 0$ and $x = a_0$, we get $\gamma_n = \prod_{j \neq n} F_j(0)$ and $\lambda_n = (1 - \gamma_n)/a_0 = \left(1 - \prod_{j \neq n} F_j(0)\right)/a_0$. Thus, $\prod_{j \neq n} F_j(x) = \left(1 - \prod_{j \neq n} F_j(0)\right) \frac{x}{a_0} + \prod_{j \neq n} F_j(0)$ on $[0, a_0]$.

Chapter 5

Contests with random initial law

Recall the Seel-Strack contest introduced in Section 1.1. In the contest, each player chooses a stopping rule to stop a privately observed stochastic process. The player who stops her process at the highest value wins a prize. And the objective of each player is not to maximise the expected stopping value, but rather to maximise the probability that her stopping value is the highest amongst the set of all players.

Seel and Strack studied the symmetric case where all the processes start from the same strictly positive real number. In contrast, we will discuss the symmetric case where all the starting values are independently drawn from the (commonly known) distribution μ with $\mu\left((-\infty, 0)\right) = 0$ and finite mean (*i.e.* $\int_0^\infty x\mu(dx) < \infty$). In this case, we will also see that in equilibrium, players use randomised strategies, so that the level at which the player should stop is stochastic. Moreover, the set of values at which the agent should stop forms an interval, but now the interval may not be bounded from above. We will restrict attention to the 2-player case. This allows us to describe an explicit way of constructing a Nash equilibrium.

The difference between the Seel-Strack model and our setting is that we assume the initial starting value is also private information and only its distribution is commonly known. This assumption is relevant if the ability of each contestant is private information or if a random or luck component is considered.

The remainder of this chapter is constructed as follows. In Section 5.1, we introduce the mathematical model of the contest, and we will see that a Nash equilibrium is identified by a pair of probability measures. Then in Section 5.2, we explain that if a measure satisfies certain conditions then this measure gives us a symmetric Nash equilibrium. Some preliminary properties are proved in Section 5.3. In Section

5.4.1, we show an explicit way of constructing a symmetric Nash equilibrium when the starting random variable takes only a finite number of distinct values. Then in Section 5.4.2, we discuss the existence of a symmetric Nash equilibrium in the general case where the starting value is some non-negative random variable. And the uniqueness of a symmetric Nash equilibrium will be proved in Section 5.5. Finally in Section 5.6, we explain how to derive the sufficient conditions that gives us the unique symmetric Nash equilibrium.

5.1 The model

Suppose that there are only two agents in the contest. Agent i privately observes the continuous-time realisation of a Brownian motion $X^i = (X^i_t)_{t\in\mathbb{R}^+}$ absorbed at zero, where $i \in \{1,2\}$. Assume X^i_0 has law μ, which is the same for both players, and assume that μ is the law of a non-negative random variable with finite mean $x_0 \in \mathbb{R}^+$. Moreover, we assume the processes X^i are independent.

Let $\mathcal{F}^i_t = \sigma(\{X^i_s : s \le t\})$ and set $\mathbb{F}^i = (\mathcal{F}^i_t)_{t\ge 0}$. A strategy of agent i is a $\mathbb{F}^i$-stopping times τ^i. Since zero is absorbing for X^i, without loss of generality we may restrict attention to $\tau^i \le H^i_0 = \inf\{t \ge 0 : X^i_t = 0\}$. Both the process X^i and the stopping time τ^i are private information to agent i. That is, X^i and τ^i cannot be observed by the other agent.

The agent who stops at the highest value wins a prize, which we normalize to one without loss of generality. In the case of a tie in which both agents stop at the equal highest value, we assume that each of them wins θ, where $\theta \in [0,1)$. Therefore player i with stopping value $X^i_{\tau^i}$ receives payoff

$$\mathbf{1}_{\{X^i_{\tau^i} < X^j_{\tau^j}\}} + \theta \mathbf{1}_{\{X^i_{\tau^i} = X^j_{\tau^j}\}},$$

where $j \ne i$, $i \in \{1,2\}$.

Again since the payoffs to the agents only depend upon τ^i via the distribution of $X^i_{\tau^i}$, a Nash equilibrium can be characterised by the laws $(\nu^i)_{i\in\{1,2\}}$ of $(X^i_{\tau^i})_{i\in\{1,2\}}$. Then, in equilibrium, the agent can use any stopping time τ^i such that $X^i_{\tau^i} \sim \nu^i$.

We now introduce some notations. Throughout the chapter, for any measure ϖ, define $F_\varpi(x) = \varpi((-\infty, x])$,

$$C_\varpi(x) = \int_x^\infty (y-x)\varpi(dy) \text{ and } P_\varpi(x) = \int_0^x (x-y)\varpi(dy),$$

for any $x \ge 0$. Note that $C_\varpi(x) - P_\varpi(x) = \int_0^\infty y\varpi(dy) - x\int_0^\infty \varpi(dy)$.

Suppose that ν^i has the same mean as μ and $C_{\nu^i}(x) \geq C_\mu(x)$ for any $x \geq 0$, then by Chacon and Walsh [1976] there exists τ^i such that $X^i_{\tau^i} \sim \nu^i$. Conversely, suppose that there exists τ^i such that $X^i_{\tau^i} \sim \nu^i$, then a simple application of Jensen's inequality shows that ν^i must satisfy $C_{\nu^i}(x) \geq C_\mu(x)$ for all $x \geq 0$, and since X^i is a non-negative supermartingale, the mean of ν^i must be less than or equal to the mean of μ, that is $\int_0^\infty x\nu^i(dx) \leq x_0$. Further, $C_{\nu^i} \geq C_\mu$ implies that $F_{\nu^i}(0) \geq F_\mu(0)$. We will say that a measure ν^i is admissible if there exists a stopping time τ^i such that $X^i_{\tau^i} \sim \nu^i$.

Our aim is to find a pair of admissible measures (ν^1, ν^2) such that, for each $i \in \{1, 2\}$, given that the other agent $j \neq i$ uses a stopping rule τ^j such that $X^j_{\tau^j} \sim \nu^j$, then the optimal target law for agent i is ν^i, and she may use any stopping rule τ^i such that $X^i_{\tau^i} \sim \nu^i$. And we say that this pair of admissible measures (ν^1, ν^2) is a Nash equilibrium.

We will say a Nash equilibrium is symmetric if $\nu^1((-\infty, x]) = \nu^2((-\infty, x])$ for any $x \in \mathbb{R}$. This chapter investigates the existence and uniqueness of a symmetric Nash equilibrium. It seems natural that a Nash equilibrium is symmetric, since the contest is symmetric in the sense that each agent observes a martingale process started from the same law μ. Then simple arguments over rearranging mass can be used to show that it is never optimal for two agents to put mass at the same positive point x.

Theorem 5.1.1. *Suppose (ν, ν) is a Nash equilibrium, then $F_\nu(x)$ is continuous on $[0, \infty)$ and $F_\nu(0) = F_\mu(0)$.*

Proof. (i) Assume that $F_\nu(x)$ places an atom of size $p > 0$ at $z > 0$. Let measure σ be given by

$$F_\sigma(x) = \begin{cases} F_\nu(x), & \text{if } x \in [0, z - \epsilon_1) \cup [z + \epsilon_2, \infty), \\ F_\nu(x) + q, & \text{if } x \in [z - \epsilon_1, z), \\ F_\nu(x) - (p - q), & \text{if } x \in [z, z + \epsilon_2), \end{cases}$$

where $\epsilon_2 \in \left(0, \frac{(1-\theta)zp}{1+\theta p}\right)$, $\epsilon_1 \in \left(\frac{(1+\theta p)\epsilon_2}{(1-\theta)p}, z\right)$ and $q = \frac{\epsilon_2 p}{\epsilon_1 + \epsilon_2} \in (0, p)$. Observe that $(z - \epsilon_1)q + (z + \epsilon_2)(p - q) = zp$, which means that F_ν and F_σ have the same mean. Moreover, observe that $C_\sigma(x) = C_\nu(x)$ if $x \in [0, z - \epsilon_1) \cup [z + \epsilon_2, \infty)$, $C_\sigma(x) = C_\nu(x) + [x - (z - \epsilon_1)]q$ if $x \in [z - \epsilon_1, z)$, and $C_\sigma(x) = C_\nu(x) + (z + \epsilon_2 - x)(p - q)$ if $x \in [z, z + \epsilon_2)$. This implies that $C_\sigma(x) \geq C_\nu(x) \geq C_\mu(x)$. Thus, σ is admissible.

Suppose that $X^2_{\tau^2} \sim F_\nu$. Let $\varphi(x) = \mathbb{P}\left(X^2_{\tau^2} < x\right)$. Let V_ν and V_σ denote the expected payoff of player 1 if player 1 chooses ν and σ, respectively, as her target law of $X^1_{\tau^1}$. Then

$$\begin{aligned}
V_\sigma - V_\nu &= \varphi(z-\epsilon_1)q + \varphi(z+\epsilon_2)(p-q) - \varphi(z)p - \theta p^2 \\
&\quad + \theta\mathbb{P}\left(X^2_{\tau^2} = z-\epsilon_1\right)q + \theta\mathbb{P}\left(X^2_{\tau^2} = z+\epsilon_2\right)(p-q) \\
&\geq \varphi(z-\epsilon_1)q + \varphi(z+\epsilon_2)(p-q) - \varphi(z)p - \theta p^2 \\
&= p\left[\varphi(z-\epsilon_1)\frac{\epsilon_2}{\epsilon_1+\epsilon_2} + \varphi(z+\epsilon_2)\frac{\epsilon_1}{\epsilon_1+\epsilon_2} - \varphi(z) - \theta p\right] \\
&= p\left\{[\varphi(z+\epsilon_2)-\varphi(z)]\frac{\epsilon_1}{\epsilon_1+\epsilon_2} - [\varphi(z)-\varphi(z-\epsilon_1)]\frac{\epsilon_2}{\epsilon_1+\epsilon_2} - \theta p\right\}.
\end{aligned}$$

Since $\varphi(z+\epsilon_2)-\varphi(z) \geq p$ and $0 \leq \varphi(z)-\varphi(z-\epsilon_1) \leq 1$,

$$V_\sigma - V_\nu \geq p\left[\frac{\epsilon_1}{\epsilon_1+\epsilon_2}p - \frac{\epsilon_2}{\epsilon_1+\epsilon_2} - \theta p\right] = p\,\frac{\epsilon_1(1-\theta)p-(1+\theta p)\epsilon_2}{\epsilon_1+\epsilon_2} > 0,$$

which contradicts the definition of Nash equilibrium. Thus, $F_\nu(x)$ is continuous on $[0,\infty)$.

(ii) Let $p = F_\nu(0)$ and $p_\mu = F_\mu(0)$. Since $\mu \preceq_{cx} \nu$, $p \geq p_\mu$. Assume that $p > p_\mu$. Fix any q such that $0 < q < \min\left\{p\sqrt{1-\theta}, 1-p\right\}$. Since F_ν is continuous on $[0,\infty)$, there exists $\epsilon > 0$ such that $\nu((0,\epsilon)) = q$. Then $F_\nu(\epsilon) = p+q$. For any $\phi \in (0,1)$, let measure σ_ϕ be given by

$$F_{\sigma_\phi}(x) = \begin{cases} (1-\phi)F_\nu(x), & \text{if } x \in [0,\delta), \\ \phi(p+q)+(1-\phi)F_\nu(x), & \text{if } x \in [\delta,\epsilon), \\ F_\nu(x), & \text{if } x \in [\epsilon,\infty), \end{cases}$$

where $\delta = \int_0^\epsilon y\nu(dy)/(p+q)$. Then σ_ϕ is a probability measure with the same mean as ν.

Suppose that $X^2_{\tau^2} \sim F_\nu$. Let V_ν and V_{σ_ϕ} denote the expected payoff of player 1 if player 1 chooses ν and σ_ϕ, respectively, as her target law of $X^1_{\tau^1}$. Then

$$\begin{aligned}
V_{\sigma_\phi} - V_\nu &= \phi\left\{(p+q)F_\nu(\delta) - \theta p^2 - \int_0^\epsilon F_\nu(y)\nu(dy)\right\} \\
&\geq \phi\left\{(p+q)p - \theta p^2 - (p+q)q\right\} = \phi\left\{(1-\theta)p^2 - q^2\right\} > 0.
\end{aligned}$$

Hence, if σ_ϕ is admissible then player 1 would refer strategy σ_ϕ to ν.

Making q and ϵ smaller if necessary, and using the fact that $C'_\nu(0+) = p - 1 > p_\mu - 1 = C'_\mu(0+)$, we can insist that $C_\nu(x) - C_\mu(x) > (p-p_\mu)x/2$ for $x \in (0,\epsilon)$. Observe that $C_\nu(x) - C_{\sigma_\phi}(x) = 0$ for $x \geq \epsilon$. Moreover, for $x \in [0,\epsilon)$, since

$F_\nu(x) - F_{\sigma_\phi}(x) \le \phi F_\nu(\delta)$, $C_\nu(x) - C_{\sigma_\phi}(x) \le \phi F_\nu(\delta)x$. Then, if $\phi \le \frac{p-p_\mu}{2F_\nu(\delta)}$, we have

$$\begin{aligned} C_{\sigma_\phi}(x) - C_\mu(x) &= (C_\nu(x) - C_\mu(x)) - \Big(C_\nu(x) - C_{\sigma_\phi}(x)\Big) \\ &> \frac{1}{2}(p - p_\mu)x - \phi F_\nu(\delta)x \ge 0 \end{aligned}$$

for all $x \in (0, \epsilon)$, and thus $\mu \preceq_{cx} \sigma_\phi$. This contradicts the definition of Nash equilibrium. Hence, $F_\nu(0) = F_\mu(0)$.

□

Remark 5.1.1. Our results in this chapter can be extended to the case where the processes observed by the agents are independent copies of a time-homogeneous diffusion process Y under certain assumptions. The idea has been explained in Chapter 2 and in Section 3.6. Again we use a change of scale to transform the problem into natural scale.

Denote by $\{l_1, l_2\}$ the endpoints of the state space of Y with $-\infty \le l_1 < l_2 \le \infty$, and denote by $s(\cdot)$ the scale function of Y. Assume that $s(l_1) > -\infty$, $s(l_2) = \infty$ and $\mathbb{E}[s(Y_0)] < \infty$. Set $s(l_1) = 0$. Then, $X = s(Y)$ converges to zero almost surely (and if zero can be reached in finite time, then zero is absorbing), $X_0 \ge 0$ almost surely since $X_0 = s(Y_0) \ge s(l_1) = 0$, and $\mathbb{E}[X_0] < \infty$.

Again the contest in which players privately observe independent copies of Y is equivalent to the contest in which players privately observe independent copies of X, and the choice of optimal stopping rule is the same for both contests. In particular, once we have found a Nash equilibrium (F_1, F_2) for $\big(X^1_{\tau^1}, X^2_{\tau^2}\big)$, where F_i is the distribution function of $X^i_{\tau^i}$, then letting $G_i = F_i \circ s$, we see that (G_1, G_2) is a Nash equilibrium for $\big(Y^1_{\tau^1}, Y^2_{\tau^2}\big)$.

5.2 Nash equilibrium

In this section, we first provide the sufficient conditions for a symmetric Nash equilibrium for the contest.

Theorem 5.2.1. *Let $\mathcal{A}^*_\mu$ be the set of all measures ν satisfying all the following conditions:*

(i) $\nu((-\infty, 0)) = 0$, $\nu((-\infty, \infty)) = 1$, F_ν is continuous on $(0, \infty)$, $F_\nu(0) = F_\mu(0)$, $\int_0^\infty x\nu(dx) = x_0$, and $C_\nu(x) \ge C_\mu(x)$ for all $x \ge 0$;

(ii) $F_\nu(x)$ is strictly increasing and concave on $[0, r]$;

(iii) if $C_\nu(x) > C_\mu(x)$ on some interval $\mathcal{J} \subset [0, r]$ then $C_\nu(x)$ is quadratic on $\mathcal{J}$,

where $r = \sup\{x : F_\nu(x) < 1\}$ and r may be infinity. If $\nu^* \in \mathcal{A}_\mu^*$, then (ν^*, ν^*) is a symmetric Nash equilibrium for the problem.

Remark 5.2.1. We explain the necessity of the first condition in this remark. The necessity of the other two conditions can be seen in Section 5.6. Recall that $X_0^i \sim \mu$ and μ has mean x_0. It is clear that the optimal law ν of $X_{\tau^i}^i$ should be a probability measure such that $\nu((-\infty, 0)) = 0$, $\int_0^\infty x\nu(dx) \leq x_0$ and $C_\nu(x) \geq C_\mu(x)$ for all $x \geq 0$. Recall that if (ν, ν) is a Nash equilibrium then F_ν is continuous on $[0, \infty)$ and $F_\nu(0) = F_\mu(0)$ by Theorem 5.1.1. Now observe that if $X_{\tilde{\tau}} \sim \tilde{\nu}$ and $\tilde{\nu}$ has mean strictly less than x_0, then there exists (ν, τ) such that ν has mean x_0, $F_\nu(x) \leq F_{\tilde{\nu}}(x)$ for all x, and $X_\tau \sim \nu$. Clearly τ dominates $\tilde{\tau}$ as a strategy. Hence the optimal law ν must have mean x_0.

We will show that the number of members of set $\mathcal{A}_\mu^*$ is less than or equal to 1 in Section 5.5. In Section 5.4.1 and Section 5.4.2, we will see that $\mathcal{A}_\mu^*$ is non-empty, and thus $\mathcal{A}_\mu^*$ is actually a singleton.

Theorem 5.2.2. $\left|\mathcal{A}_\mu^*\right| = 1$.

Before proving Theorem 5.2.1 we first present some examples. And Theorem 5.2.2 will be proved in later sections.

Example 5.2.1. Let $\upsilon = \mathcal{U}[0, 2x_0]$, where $\mathcal{U}$ stands for the continuous uniform distribution. Suppose μ satisfies that $C_\mu \leq C_\upsilon$. Then, it is easy to see that $\upsilon \in \mathcal{A}_\mu^*$, and thus (υ, υ) is the unique symmetric Nash equilibrium for the problem.

Example 5.2.2. Suppose μ satisfies that $F_\mu(x)$ is continuous on $[0, \infty)$ and strictly increasing on $[0, r_\mu]$, where $r_\mu = \sup\{x : F_\mu(x) < 1\}$. If F_μ is concave on $[0, r_\mu]$, then $\mu \in \mathcal{A}_\mu^*$ and (μ, μ) is the unique symmetric Nash equilibrium for the problem. If F_μ is convex on $[0, r_\mu]$ and $F_\mu(0) = 0$, then it can be verified that $C_\mu \leq C_\upsilon$ (see Proposition 5.3.1 for a detailed proof), where $\upsilon = \mathcal{U}[0, 2x_0]$, and thus (υ, υ) is the unique symmetric Nash equilibrium for the problem.

Example 5.2.3. (Beta distribution) Suppose μ is a beta distribution with shape parameters $\alpha = 2$ and $\beta = 3$, that is $\mu = \text{Beta}(2, 3)$. Then, the mean of μ is $2/5$, $C_\mu(x) = \left(\frac{3}{5}x^5 - 2x^4 + 2x^3 - x + \frac{2}{5}\right)\mathbf{1}_{\{x \leq 1\}}$, and $F_\mu(x) = \min\{3x^4 - 8x^3 + 6x^2, 1\}$. Since $F_\mu(x)$ is convex on $(0, \frac{1}{3})$ and concave on $(\frac{1}{3}, 1)$, the equilibrium measure ν must satisfy that $C_\nu(x) = c_1x^2 - x + \frac{2}{5}$ on $[0, c_2]$, where $c_1 > 0$ and $c_2 > 0$ are constants. Moreover, c_1 and c_2 satisfy that $C_\mu(c_2) = c_1c_2^2 - c_2 + \frac{2}{5}$ and $C_\mu'(c_2) = 2c_1c_2 - 1$. Solving the system of equations, we get $c_1 = \frac{4\sqrt{10}+140}{243}$ and $c_2 = \frac{10-\sqrt{10}}{9}$. Define function $C(x)$ by

$$C(x) = \left(c_1 x^2 - x + \frac{2}{5}\right) \cdot \mathbf{1}_{x\in[0,c_2)} + C_\mu(x) \cdot \mathbf{1}_{x\in[c_2,\infty)},$$

and let ν be given by

$$F_\nu(x) = 2c_1 x \cdot \mathbf{1}_{x\in[0,c_2)} + F_\mu(x) \cdot \mathbf{1}_{x\in[c_2,\infty)}.$$

Then $C_\nu(x) = C(x)$ for any $x \geq 0$. Note that $c_2 \in (\frac{1}{3}, 1)$ and $F_\nu^{'}(c_2-) = 2c_1 > F_\mu^{'}(c_2)$, which means that $F_\nu(x)$ is concave. And by the exact form of $F_\nu(x)$, it can be seen that $\nu \in \mathcal{A}_\mu^*$. Thus, (ν, ν) is the unique symmetric Nash equilibrium for the problem.

Example 5.2.4. (Atomic measure) Suppose that $\mu = \frac{1}{2}\delta_{1-\epsilon} + \frac{1}{2}\delta_{1+\epsilon}$, where $\epsilon \in (0,1)$, and δ_x is a Dirac measure on set $\{x\}$. Then the mean of μ is 1 and $C_\mu(x) = (1-x)\cdot \mathbf{1}_{x\in[0,1-\epsilon)} + \frac{1}{2}(1+\epsilon-x)\cdot \mathbf{1}_{x\in[1-\epsilon,1+\epsilon)}$. By calculations, if $\epsilon \in (0, 1/2]$ then $C_\mu \leq C_\upsilon$, where $\upsilon = \mathcal{U}[0,2]$. Thus, if $\epsilon \in (0,1/2]$ then (υ, υ) is the unique symmetric Nash equilibrium for the problem. Next suppose $\epsilon \in (1/2, 1)$. Define function $C(x)$ by

$$C(x) = \frac{x^2 - 8(1-\epsilon)(x-1)}{8(1-\epsilon)} \cdot \mathbf{1}_{x\in[0,2(1-\epsilon))} + \frac{x^2 - 8\epsilon x + 16\epsilon^2}{8(3\epsilon-1)} \cdot \mathbf{1}_{x\in[2(1-\epsilon),4\epsilon)},$$

and let ν be given by

$$F_\nu(x) = \frac{x}{4(1-\epsilon)} \cdot \mathbf{1}_{x\in[0,2(1-\epsilon))} + \frac{x+4(2\epsilon-1)}{4(3\epsilon-1)} \cdot \mathbf{1}_{x\in[2(1-\epsilon),4\epsilon)} + \mathbf{1}_{x\in[4\epsilon,\infty)}.$$

Then $C_\nu(x) = C(x)$ for any $x \geq 0$. It can be seen that $\nu \in \mathcal{A}_\mu^*$. Hence, (ν, ν) is the unique symmetric Nash equilibrium for the problem if $\epsilon \in (1/2, 1)$.

In the rest of this section we present the proof of Theorem 5.2.1.

Proof of Theorem 5.2.1. The proof is based on a Lagrangian method. Let $\mathcal{A}_\mu$ be the set of all measures ν satisfying that $\nu((-\infty,0)) = 0$, $\nu((-\infty,\infty)) = 1$, F_ν is continuous on $(0,\infty)$, $F_\nu(0) \geq F_\mu(0)$, $\int_0^\infty x\nu(dx) = x_0$, and $C_\nu(x) \geq C_\mu(x)$ for all $x \geq 0$. A symmetric Nash equilibrium is identified with a measure $\nu^* \in \mathcal{A}_\mu$ with the property that, for any $\nu \in \mathcal{A}_\mu$,

$$\int_{(0,\infty)} F_{\nu^*}(x)\nu^*(dx) + \theta F_{\nu^*}(0)F_{\nu^*}(0) \geq \int_{(0,\infty)} F_{\nu^*}(x)\nu(dx) + \theta F_{\nu^*}(0)F_\nu(0). \quad (5.1)$$

Fix any $\nu^* \in \mathcal{A}^*_\mu$ and fix any $\nu \in \mathcal{A}_\mu$. Suppose that λ, γ and $\zeta \geq 0$ are some finite constants and $\eta(\cdot)$ satisfies that $\eta(dz) \geq 0$ $\forall z \geq 0$. Define

$$\begin{aligned}
\mathcal{L}_{\nu^*}(\nu; \lambda, \gamma, \zeta, \eta) = &\int_{(0,\infty)} F_{\nu^*}(x)\nu(dx) + \theta F_{\nu^*}(0)F_\nu(0) + \lambda \left(x_0 - \int_{(0,\infty)} x\nu(dx) \right) \\
&+ \gamma \left(1 - \int_{(0,\infty)} \nu(dx) - F_\nu(0) \right) + \zeta \left(F_\nu(0) - F_\mu(0) \right) \\
&+ \int_{(0,\infty)} \left(C_\nu(z) - C_\mu(z) \right) \eta(dz) \qquad (5.2) \\
= &\int_{(0,\infty)} \left(F_{\nu^*}(x) - \lambda x - \gamma + \int_{(0,x)} (x-z)\eta(dz) \right) \nu(dx) + \lambda x_0 + \gamma \\
&+ \left(\theta F_{\nu^*}(0) - \gamma + \zeta \right) F_\nu(0) - \int_0^\infty C_\mu(z)\eta(dz) - \zeta F_\mu(0). \qquad (5.3)
\end{aligned}$$

Rearranging (5.2), and since $\zeta \geq 0$, $F_\nu(0) \geq F_\mu(0)$, $\eta(dz) \geq 0$ and $C_\nu(z) \geq C_\mu(z)$ $\forall z \geq 0$,

$$\begin{aligned}
&\int_{(0,\infty)} F_{\nu^*}(x)\nu(dx) + \theta F_{\nu^*}(0)F_\nu(0) \\
&\quad = \mathcal{L}_{\nu^*}(\nu; \lambda, \gamma, \zeta, \eta) - \zeta \left(F_\nu(0) - F_\mu(0) \right) - \int_{(0,\infty)} \left(C_\nu(z) - C_\mu(z) \right) \eta(dz) \\
&\quad \leq \mathcal{L}_{\nu^*}(\nu; \lambda, \gamma, \zeta, \eta). \qquad (5.4)
\end{aligned}$$

Furthermore, if η satisfies that $C_{\nu^*}(z) = C_\mu(z)$ for any z where $\eta(dz) > 0$, then since $F_{\nu^*}(0) = F_\mu(0)$,

$$\int_{(0,\infty)} F_{\nu^*}(x)\nu^*(dx) + \theta F_{\nu^*}(0)F_{\nu^*}(0) = \mathcal{L}_{\nu^*}(\nu^*; \lambda, \gamma, \zeta, \eta). \qquad (5.5)$$

Define $r^* = \sup\{x : F_{\nu^*}(x) < 1\}$ and $r^* \leq \infty$. Because F_{ν^*} is continuous and concave, it is absolutely continuous, which means that there exists a function f_{ν^*} such that $F_{\nu^*}(x) = \int_0^x f_{\nu^*}(y)dy + F_{\nu^*}(0)$. Moreover, $f_{\nu^*}(x) = \int_x^{r^*} \psi(dz)$, where ψ is some measure given by $\psi((z_1, z_2]) = f_{\nu^*}(z_1+) - f_{\nu^*}(z_2+)$ for any $z_1 < z_2$. Since f_{ν^*} is non-increasing, $\psi(dz) \geq 0$. And since $\nu^* \in \mathcal{A}^*_\mu$, if $C_{\nu^*}(z) > C_\mu(z)$ then f_{ν^*} is constant near z, which implies that $\psi(dz) = 0$. Also observe that, for any $x \in (0, r^*)$, since $\int_0^x z\psi(dz) = \int_0^x \int_0^z dy\psi(dz) = \int_0^x \int_y^x \psi(dz)dy$,

$$\int_0^x z\psi(dz) = \int_0^x \left(f_{\nu^*}(y) - f_{\nu^*}(x) \right) dy = F_{\nu^*}(x) - F_{\nu^*}(0) - x f_{\nu^*}(x). \qquad (5.6)$$

(i) Suppose $f_{\nu^*}(0) < \infty$. Let $\lambda^* = f_{\nu^*}(0)$, $\gamma^* = F_{\nu^*}(0)$, $\zeta^* = (1-\theta)F_{\nu^*}(0)$

and $\eta^*(dz) = \psi(dz)$ for any $z \in (0, r^*)$ with $\eta^*(dz) = 0$ elsewhere. Notice that $\lambda^* < \infty$, $\gamma^* < \infty$, $0 \leq \zeta^* < \infty$ and η^* satisfies that $\eta^*(dz) \geq 0 \; \forall z \geq 0$ and $C_{\nu^*}(z) = C_\mu(z)$ for any z where $\eta^*(dz) > 0$.

Define $\Gamma(x) = \lambda^* x + \gamma^* - \int_0^x (x-z)\eta^*(dz)$ for all $x \geq 0$. Then, for any $x \in (0, r^*)$, $\Gamma(x) = f_{\nu^*}(0)x + F_{\nu^*}(0) - \int_0^x (x-z)\psi(dz)$. By (5.6), $-\int_0^x (x-z)\psi(dz) = -x\left(f_{\nu^*}(0) - f_{\nu^*}(x)\right) + F_{\nu^*}(x) - F_{\nu^*}(0) - x f_{\nu^*}(x)$. Thus, $\Gamma(x) = F_{\nu^*}(x)$ for any $x \in (0, r^*)$.

Observe that Γ is continuous and non-decreasing. This implies that $F_{\nu^*}(x) \leq \Gamma(x)$ for any $x \geq 0$. Also observe that $\theta F_{\nu^*}(0) - \gamma^* + \zeta^* = 0$. Thus, by (5.4) and (5.3),

$$\int_0^\infty F_{\nu^*}(x)\nu(dx) + \theta F_{\nu^*}(0)F_\nu(0) \leq \lambda^* x_0 + \gamma^* - \int_0^\infty C_\mu(z)\eta^*(dz) - \zeta^* F_\mu(0), \quad (5.7)$$

and by (5.5) and (5.3),

$$\int_0^\infty F_{\nu^*}(x)\nu^*(dx) + \theta F_{\nu^*}(0)F_{\nu^*}(0) = \lambda^* x_0 + \gamma^* - \int_0^\infty C_\mu(z)\eta^*(dz) - \zeta^* F_\mu(0). \quad (5.8)$$

Note $\int_0^\infty C_\mu(z)\eta^*(dz) = \int_0^{r^*} C_\mu(z)\psi(dz) \leq x_0 f_{\nu^*}(0) < \infty$ so that the right-hand side of (5.7) and (5.8) is well defined and positive. Thus, for any $\nu \in \mathcal{A}_\mu$,

$$\int_0^\infty F_{\nu^*}(x)\nu(dx) + \theta F_{\nu^*}(0)F_\nu(0) \leq \int_0^\infty F_{\nu^*}(x)\nu^*(dx) + \theta F_{\nu^*}(0)F_{\nu^*}(0).$$

Hence, from the definition (5.1) of a Nash equilibrium, (ν^*, ν^*) is a symmetric Nash equilibrium for the problem.

(ii) Suppose $f_{\nu^*}(0) = \infty$. Fix any $\epsilon > 0$. Let $\lambda_\epsilon = f_{\nu^*}(\epsilon)$, $\gamma_\epsilon = F_{\nu^*}(0) + \int_0^\epsilon z\psi(dz)$, $\zeta_\epsilon = \gamma_\epsilon - \theta F_{\nu^*}(0)$ and $\eta_\epsilon(dz) = \psi(dz)$ for any $z \in (\epsilon, r^*)$ with $\eta_\epsilon(dz) = 0$ elsewhere. Notice that $\lambda_\epsilon < \infty$, $\gamma_\epsilon = F_{\nu^*}(\epsilon) - \epsilon f_{\nu^*}(\epsilon) < \infty$, $0 \leq (1-\theta)F_{\nu^*}(0) + \int_0^\epsilon z\psi(dz) = \zeta_\epsilon < \infty$ and η_ϵ satisfies that $\eta_\epsilon(dz) \geq 0 \; \forall z \geq 0$ and $C_{\nu^*}(z) = C_\mu(z)$ for any z where $\eta_\epsilon(dz) > 0$.

Define $\Gamma_\epsilon(x) = \lambda_\epsilon x + \gamma_\epsilon - \int_0^x (x-z)\eta_\epsilon(dz)$ for any $x \geq 0$. Then, for any $x \in (\epsilon, r^*)$, $\Gamma_\epsilon(x) = f_{\nu^*}(\epsilon)x + F_{\nu^*}(0) + \int_0^x z\psi(dz) - x\int_\epsilon^x \psi(dz)$. Using (5.6), $\int_0^x z\psi(dz) - x\int_\epsilon^x \psi(dz) = F_{\nu^*}(x) - F_{\nu^*}(0) - x f_{\nu^*}(x) - x\left(f_{\nu^*}(\epsilon) - f_{\nu^*}(x)\right)$. Thus, $\Gamma_\epsilon(x) = F_{\nu^*}(x)$ for any $x \in (\epsilon, r^*)$. Further, $\Gamma_\epsilon(x) = f_{\nu^*}(\epsilon)(x-\epsilon) + F_{\nu^*}(\epsilon)$ on $[0, \epsilon]$, and if $r^* < \infty$ then $\Gamma_\epsilon(x) = 1$ on $[r^*, \infty)$.

On $(0, \epsilon)$, $\Gamma_\epsilon(x)$ is linear and $F_{\nu^*}(x)$ is concave. Then, since $\Gamma_\epsilon'(\epsilon) = f_{\nu^*}(\epsilon)$ and $\Gamma_\epsilon(\epsilon) = F_{\nu^*}(\epsilon)$, we have $F_{\nu^*}(x) \leq \Gamma_\epsilon(x)$ and $F_{\nu^*}(x) - \Gamma_\epsilon(x) \geq F_{\nu^*}(0) - \Gamma_\epsilon(0) = -\int_0^\epsilon z\psi(dz)$, for any $x \in (0, \epsilon)$. Thus, $F_{\nu^*}(x) \leq \Gamma_\epsilon(x)$ for any $x \geq 0$. Then, since

$\theta F_{\nu^*}(0) - \gamma_\epsilon + \zeta_\epsilon = 0$ and by (5.4) and (5.3),

$$\int_{(0,\infty)} F_{\nu^*}(x)\nu(dx) + \theta F_{\nu^*}(0)F_\nu(0) \leq \lambda_\epsilon x_0 + \gamma_\epsilon - \int_0^\infty C_\mu(z)\eta_\epsilon(dz) - \zeta_\epsilon F_\mu(0),$$

and since $\int_0^\epsilon \left(F_{\nu^*}(x) - \Gamma_\epsilon(x)\right)\nu^*(dx) \geq -\left(F_{\nu^*}(\epsilon) - F_{\nu^*}(0)\right)\int_0^\epsilon z\psi(dz)$ and using (5.5) and (5.3),

$$\begin{aligned}
&\int_{(0,\infty)} F_{\nu^*}(x)\nu^*(dx) + \theta F_{\nu^*}(0)F_{\nu^*}(0) \\
&\quad \geq -\left(F_{\nu^*}(\epsilon) - F_{\nu^*}(0)\right)\int_0^\epsilon z\psi(dz) + \lambda_\epsilon x_0 + \gamma_\epsilon - \int_0^\infty C_\mu(z)\eta_\epsilon(dz) - \zeta_\epsilon F_\mu(0).
\end{aligned}$$

Observe that, since $\lim_{\epsilon\downarrow 0}\int_0^\epsilon z\psi(dz) = 0$,

$$\begin{aligned}
&\lim_{\epsilon\downarrow 0}\left\{\lambda_\epsilon x_0 + \gamma_\epsilon - \int_0^\infty C_\mu(z)\eta_\epsilon(dz) - \zeta_\epsilon F_\mu(0)\right\} \\
&\qquad = \lim_{\epsilon\downarrow 0}\left\{f_{\nu^*}(\epsilon)x_0 + F_{\nu^*}(0) - \int_\epsilon^{r^*} C_\mu(z)\psi(dz) - (1-\theta)F_{\nu^*}(0)F_\mu(0)\right\} \\
&\qquad = \lim_{\epsilon\downarrow 0}\left\{F_{\nu^*}(0) + \int_\epsilon^{r^*}\left(x_0 - C_\mu(z)\right)\psi(dz) - (1-\theta)F_{\nu^*}(0)F_\mu(0)\right\} \\
&\qquad = F_{\nu^*}(0) + \int_0^{r^*}\left(x_0 - C_\mu(z)\right)\psi(dz) - (1-\theta)F_{\nu^*}(0)F_\mu(0).
\end{aligned}$$

Since $x_0 - C_\mu(z) \leq z \wedge x_0$ for any $z \geq 0$ and using (5.6),

$$\int_0^{r^*}\left(x_0 - C_\mu(z)\right)\psi(dz) \leq \int_0^{r^*}(z\wedge x_0)\psi(dz) = \int_0^{x_0} z\psi(dz) + x_0 f_{\nu^*}(x_0) < \infty,$$

which means that $\lim_{\epsilon\downarrow 0}\left\{\lambda_\epsilon x_0 + \gamma_\epsilon - \int_0^\infty C_\mu(z)\eta_\epsilon(dz) - \zeta_\epsilon F_\mu(0)\right\} < \infty$. Then, since $\lim_{\epsilon\downarrow 0}\left\{-\left(F_{\nu^*}(\epsilon) - F_{\nu^*}(0)\right)\int_0^\epsilon z\psi(dz)\right\} = 0$,

$$\begin{aligned}
&\int_{(0,\infty)} F_{\nu^*}(x)\nu^*(dx) + \theta F_{\nu^*}(0)F_{\nu^*}(0) \\
&\qquad\qquad \geq \lim_{\epsilon\downarrow 0}\left\{\lambda_\epsilon x_0 + \gamma_\epsilon - \int_0^\infty C_\mu(z)\eta_\epsilon(dz) - \zeta_\epsilon F_\mu(0)\right\} \\
&\qquad\qquad \geq \int_{(0,\infty)} F_{\nu^*}(x)\nu(dx) + \theta F_{\nu^*}(0)F_\nu(0).
\end{aligned}$$

Hence, from the definition (5.1) of a Nash equilibrium, (ν^*, ν^*) is a symmetric Nash equilibrium for the problem.

□

5.3 Preliminaries

In this section we state and prove some technique results which will be required later.

Proposition 5.3.1. *Let $\mathcal{Y}$ be the set of non-negative random variables Y with mean $\bar{y} \in (0,\infty)$. Denote by F^Y the distribution function of Y. Let $s_1^Y = \inf\{u : F^Y(u) > 0\} \geq 0$ and $s_2^Y = \sup\{u : F^Y(u) < 1\} \leq \infty$.*

(i) Let $\mathcal{Y}_{cx} = \left\{Y \in \mathcal{Y} : F^Y \text{ is convex on } \left(-\infty, s_2^Y\right]\right\}$. Then, for $Y \in \mathcal{Y}_{cx}$, $\mathbb{P}(Y = 0) = 0$.

(a) Suppose H is convex. Then, $\sup_{Y\in\mathcal{Y}_{cx}} \mathbb{E}[H(Y)] = \int_0^{2\bar{y}} \frac{1}{2\bar{y}} H(y)dy$.

(b) Suppose H is concave. Then, $\sup_{Y\in\mathcal{Y}_{cx}} \mathbb{E}[H(Y)] = H(\bar{y})$.

(ii) Let $\mathcal{Y}_{cv} = \left\{Y \in \mathcal{Y} : F^Y \text{ is concave on } [0,\infty)\right\}$. Then, for $Y \in \mathcal{Y}_{cv}$, $s_1^Y = 0$.

(a) Suppose H is convex. Then, $\sup_{Y\in\mathcal{Y}_{cv}} \mathbb{E}[H(Y)] = H(0) + \bar{y}\lim_{x\uparrow\infty} \frac{H(x)}{x}$.

(b) Suppose H is concave. Then, $\sup_{Y\in\mathcal{Y}_{cv}} \mathbb{E}[H(Y)] = \int_0^{2\bar{y}} \frac{1}{2\bar{y}} H(y)dy$.

The bounds are all best possible. The bounds in (i.a) and (ii.b) are attained by $Y \sim \mathcal{U}[0, 2\bar{y}]$. The bound in (i.b) is attained by $Y \sim \delta_{\bar{y}}$. The bounds in (i.b) and (ii.a) are valid for all distributions on $\mathbb{R}^+$ with mean $\bar{y}$ and not just those with convex (or concave) distribution functions.

Remark 5.3.1. This result is stated for completeness; the result we will use is (i.a).

Proof. Let U be a $\mathcal{U}[0,1]$ random variable.

(i.a) Suppose $Y \in \mathcal{Y}_{cx}$. It is obvious that the distribution function of Y, $F = F^Y$, is strictly increasing on (s_1^Y, s_2^Y), and $F(s_1^Y) = 0$. So the inverse function $G(y) \triangleq F^{-1}(y)$ of F exists on $[0,1]$. Since $G(U)$ is distributed as Y, $\mathbb{E}[H(Y)] = \mathbb{E}[H(G(U))] = \int_0^1 H(G(u))du$ and $\int_0^1 G(u)du = \mathbb{E}[G(U)] = \mathbb{E}[Y] = \bar{y}$.

It is clear that G is concave on $[0,1]$ and $G(0) = s_1^Y \geq 0$. Then, since $\int_0^1 2\bar{y}udu = \int_0^1 G(u)du$, there exists a unique $u^* \in (0,1)$ such that $G(u^*) = 2\bar{y}u^*$ (see the left graph in Figure 5.1). Moreover, $2\bar{y}u \leq G(u) \leq 2\bar{y}u^*$ for $u \in [0,u^*]$ and $2\bar{y}u^* \leq G(u) \leq 2\bar{y}u$ for $u \in [u^*,1]$. Since $\mathbb{E}[H(2\bar{y}U)] = \int_0^1 H(2\bar{y}u)du$,

$$\mathbb{E}[H(Y)] - \mathbb{E}[H(2\bar{y}U)] = \int_0^{u^*} (H(G(u)) - H(2\bar{y}u))\,du + \int_{u^*}^1 (H(G(u)) - H(2\bar{y}u))\,du.$$

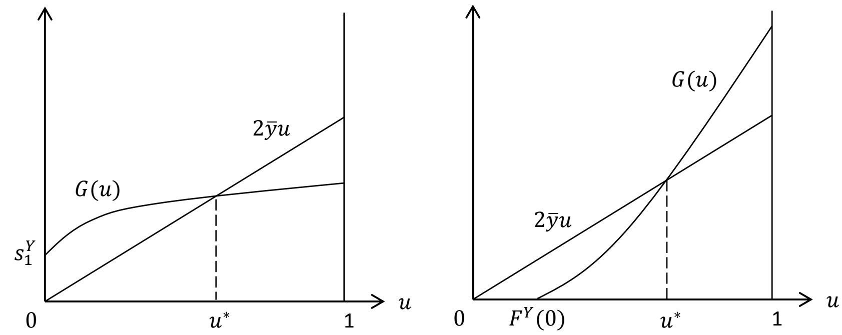

Figure 5.1: Comparison of $G(u)$ and $2\bar{y}u$. Since G is the inverse of the CDF of a mean $\bar{y}$ random variable, the area under G is $\bar{y}$. Then the areas under G and the line $2\bar{y}u$ are the same. Hence, if G is either convex or concave, there is a unique crossing point u^* of G and the line $2\bar{y}u$.

Let $H_-^{'}$ denote the left derivative of the convex function H, then $H(u_2) - H(u_1) \leq (u_2 - u_1)H_-^{'}(u_2)$ for any u_1 and u_2. This means that

$$\begin{aligned}
&\mathbb{E}[H(Y)] - \mathbb{E}[H(2\bar{y}U)] \\
&\qquad \leq \int_0^{u^*} (G(u) - 2\bar{y}u)H_-^{'}(2\bar{y}u^*)du + \int_{u^*}^1 (G(u) - 2\bar{y}u)\, H_-^{'}(2\bar{y}u^*)du \\
&\qquad = H_-^{'}(2\bar{y}u^*) \int_0^1 (G(u) - 2\bar{y}u)du = 0. \qquad (5.9)
\end{aligned}$$

Thus, $\mathbb{E}[H(Y)] \leq \mathbb{E}[H(2\bar{y}U)] = \int_0^{2\bar{y}} \frac{1}{2\bar{y}} H(y)dy$. Moreover, it is obvious that the bound is attained by $Y \sim \mathcal{U}[0, 2\bar{y}]$.

(i.b) Since H is concave for any $Y \in \mathcal{Y}$, Jensen's inequality gives $\mathbb{E}[H(Y)] \leq H(\mathbb{E}[Y]) = H(\bar{y})$. Also if $Y \sim \delta_{\bar{y}}$ then $Y \in \mathcal{Y}_{cx}$.

(ii.a) Fix any $z > 0$. Since H is convex, $H(z) \leq H(0) + \frac{z}{x}(H(x) - H(0))$ for any $x \geq z$. Letting x tends to ∞, $H(z) \leq H(0) + z \lim_{x\uparrow\infty} H(x)/x$. This implies that for $Y \in \mathcal{Y}$, $\mathbb{E}[H(Y)] \leq \mathbb{E}\left[H(0) + Y \lim_{x\uparrow\infty} H(x)/x\right] = H(0) + \bar{y} \lim_{x\uparrow\infty} H(x)/x$.

It remains to show that the bound is best possible. Let $Y_n \sim \left(1 - \frac{1}{n}\right)\delta_0 + \frac{1}{n}\mathcal{U}[0, 2\bar{y}n]$. Then $Y_n \in \mathcal{Y}_{cv}$. Also

$$\begin{aligned}
\lim_{n\to\infty} \mathbb{E}[H(Y_n)] &= \lim_{n\to\infty} \left[\left(1 - \frac{1}{n}\right) H(0) + \frac{1}{n}\int_0^{2\bar{y}n} \frac{H(z)}{2\bar{y}n}dz\right] \\
&= \lim_{n\to\infty} \left[\left(1 - \frac{1}{n}\right) H(0) + 2\bar{y}\int_0^1 u\frac{H(2\bar{y}nu)}{2\bar{y}nu}du\right] \\
&= H(0) + 2\bar{y} \lim_{x\uparrow\infty} \frac{H(x)}{x} \int_0^1 udu = H(0) + \bar{y} \lim_{x\uparrow\infty} \frac{H(x)}{x}.
\end{aligned}$$

(ii.b) Suppose $Y \in \mathcal{Y}_{cv}$. It is clear that $s_1^Y = 0$ and $F = F^Y$ is strictly increasing on $(0, s_2^Y)$. This means that its inverse function $F^{-1}(\cdot)$ exists on $[F(0), 1]$. Now define $G(y) = F^{-1}(y)$ for any $y \in [F(0), 1]$ and $G(y) = 0$ for any $y < F(0)$. Similar to (i.a), we have that $G(U)$ is distributed as Y, $\mathbb{E}[H(Y)] = \int_0^1 H(G(u))du$, $\int_0^1 G(u)du = \bar{y}$ and $\int_0^1 2\bar{y}udu = \bar{y}$.

It is clear that $G(y)$ is convex on $[F(0), 1]$ and $G(y) = 0$ on $[0, F(0)]$. Then, since $\int_0^1 2\bar{y}udu = \int_0^1 G(u)du$, there exists a unique $u^* \in (0, 1)$ such that $G(u^*) = 2\bar{y}u^*$ (see the right graph in Figure 5.1). Moreover, $G(u) \leq 2\bar{y}u \leq 2\bar{y}u^*$ for $u \in [0, u^*]$ and $2\bar{y}u^* \leq 2\bar{y}u \leq G(u)$ for $u \in [u^*, 1]$.

Then, since H is concave, $H(u_2) - H(u_1) \leq (u_2 - u_1)H'_-(u_1)$, for any u_1 and u_2. Again we have (5.9) holds. Thus, $\mathbb{E}[H(Y)] \leq \mathbb{E}[H(2\bar{y}U)] = \int_0^{2\bar{y}} \frac{1}{2\bar{y}} H(y)dy$, and it is obvious that the bound is attained by $Y \sim \mathcal{U}[0, 2\bar{y}]$.

□

Proposition 5.3.2. *Let H be twice differentiable, and suppose that $h = H'$ is concave. Suppose that $H(0) = 0$, $h(0) > 0$, $h'(0) \leq 0$ and h is not constant. Then, for any $\hat{w} > 0$ such that $H(\hat{w}) = 0$, we have $h(\hat{w}) + h(0) \leq 0$, i.e. $|h(\hat{w})| \geq h(0)$.*

Proof. Since h is not constant and h is concave, there is a solution $\tilde{w}$ to $h(w) = -h(0)$. Let $\delta = -2h(0)/\tilde{w}$ be the slope of the line joining $(0, h(0))$ to $(\tilde{w}, -h(0))$. Then, on $(0, \tilde{w})$, $h(w) \geq h(0) + \delta w$ and $H(w) = \int_0^w h(x)dx \geq h(0)w + \delta w^2/2$. So that $H(\tilde{w}) \geq 0$. Then, by concavity of H and since $H(\hat{w}) = 0$, $\hat{w} \geq \tilde{w}$. Thus, $h(\hat{w}) \leq h(\tilde{w}) = -h(0)$ and the result follows.

□

Proposition 5.3.3. *For any measure $\nu \in \mathcal{A}_\mu^*$, if $C_\nu(x) = C_\mu(x)$ for some $x > 0$, then $C_\mu(\cdot)$ is differentiable at x and $C'_\nu(x) = C'_\mu(x)$.*

Proof. Suppose that $\nu \in \mathcal{A}_\mu^*$. Then, C_ν is continuously differentiable and $C_\nu \geq C_\mu$. Because $C_\nu(y) - C_\mu(y) \geq 0 = C_\nu(x) - C_\mu(x)$ for any $y < x$ and by the definition of left derivatives, we get $C'_\nu(x-) - C'_\mu(x-) \leq 0$. Similarly, we have $C'_\nu(x+) - C'_\mu(x+) \geq 0$. Thus, $C'_\mu(x+) \leq C'_\nu(x+) = C'_\nu(x-) \leq C'_\mu(x-)$. Observe that $C'_\mu(x-) \leq C'_\mu(x+)$ since C_μ is convex. Hence, $C'_\mu(x-) = C'_\mu(x+)$, which means that $C'_\mu(x)$ exists and $C'_\nu(x) = C'_\mu(x)$.

□

Proposition 5.3.4. *Fix any measure $\nu \in \mathcal{A}_\mu^*$. Suppose that $C_\nu(x) = \phi(x)$ on some closed interval $\mathcal{J} = [j_1, j_2]$, where $0 \leq j_1 < j_2$ and $\phi(\cdot)$ is a quadratic function defined on $(-\infty, \infty)$ with $\phi'' > 0$. Then $j_2 < \infty$ and $C_\nu(x) \leq \phi(x)$ on (j_2, ∞).*

Proof. Assume that $j_2 = \infty$. Then C_ν is quadratic on (j_2, ∞) with strictly positive quadratic coefficient. This means that C_ν is not ultimately decreasing, which is a contradiction. Thus, $j_2 < \infty$. Since $C_\nu^{'}$ is continuous and concave and since $\phi^{'}$ is linear, it is clear that $C_\nu(x) \leq \phi(x)$ on $[j_2, \infty)$.

□

5.4 Existence of a Nash equilibrium

This section shows how to construct a measure that belongs to $\mathcal{A}_\mu^*$.

5.4.1 Atomic Initial Measure

We start with the case where the initial law μ is an atomic probability measure. We will construct a function $P(x)$ that satisfies certain conditions, and then define a measure ν via $\nu((-\infty, x]) = P^{'}(x)$. Using the conditions that P satisfies, it will be seen that this measure ν belongs to $\mathcal{A}_\mu^*$. Then, by Theorem 5.2.1 and Theorem 5.2.2, (ν, ν) is the unique symmetric Nash equilibrium for the problem.

Theorem 5.4.1. *Suppose μ is a measure with finitely many atoms, that is $\mu = \sum_{j=1}^N p_j \delta_{\xi_j}$, where $0 \leq \xi_1 < \xi_2 < \cdots < \xi_N$ and $p_j \in (0,1)$ for all $1 \leq j \leq N$. Let $m = \sum_{j=1}^N p_j \xi_j$ and suppose that $m \in [0, \infty)$. Then, there exists a unique measure ν such that $\nu((-\infty, 0)) = 0$, $\nu(\mathbb{R}) = \mu(\mathbb{R})$, F_ν is continuous on $(0, \infty)$, $F_\nu(0) = F_\mu(0)$, $\int_0^\infty x\nu(dx) = m$, $C_\nu(x) \geq C_\mu(x)$ $\forall x \geq 0$, and ν has a piecewise constant density ρ and ρ only decreases when $C_\nu(x) = C_\mu(x)$.*

Proof. Let $\xi_0 = 0$, $\xi_{N+1} = \infty$ and $p_0 = 0$. Then

$$P_\mu(y) = \sum_{j=0}^{i} p_j(y - \xi_j) \text{ if } y \in [\xi_i, \xi_{i+1}),\ i = 0, 1, \ldots, N.$$

If μ is a point mass at zero, then set $\nu(\{0\}) = \mu(\{0\})$ and the construction is complete. Otherwise, let $Q_1(r, y) = yF_\mu(0) + ry^2/2$. Then there exists a unique value of r (r_1 say) such that

$$Q_1(r_1, y) \geq P_\mu(y)\ \forall y \geq 0 \text{ and } Q_1(r_1, y) = P_\mu(y) \text{ for some } y > 0.$$

Let $y_1 = \max\{y > 0 : Q_1(r_1, y) = P_\mu(y)\}$. Then necessarily $P_\mu^{'}(y_1)$ exists and $\frac{\partial}{\partial y} Q_1(r_1, y_1) = P_\mu^{'}(y_1)$. Note that $y_1 \notin \{\xi_1, \xi_2, \cdots, \xi_N\}$ since $P_\mu^{'}$ has a kink at these points. Let n_1 be such that $\xi_{n_1} < y_1 < \xi_{n_1+1}$. If $n_1 = N$ (equivalently $P_\mu^{'}(y_1) = \sum_{j=0}^N p_j = \mu(\mathbb{R})$) then stop. Otherwise we proceed inductively.

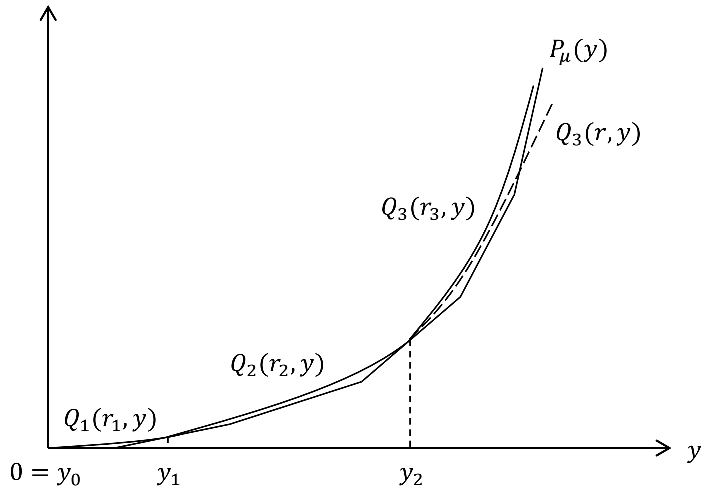

Figure 5.2: Construction of $Q(y)$. The dashed curve is $Q_3(r,y)$ with $r<r_3$. $Q_i(r,y)$ are quadratic functions of y such that $Q_i(r,y_{i-1})=P_\mu(y_{i-1})$ and $\frac{\partial}{\partial y}Q_i(r,y_{i-1})=P_\mu^{'}(y_{i-1})$. And r_i is the unique value of r such that $Q_i(r_i,y)\geq P_\mu(y)$ for all $y\geq y_{i-1}$ and $Q_i(r_i,y)=P_\mu(y)$ for some $y>y_{i-1}$.

Let $y_0=0$. Suppose we have found $0<y_1<y_2<\cdots<y_k<\xi_N$ ($y_i\notin\{\xi_1,\xi_2,\cdots,\xi_N\}$ $\forall 1\leq i\leq k$) and $Q(\cdot)$ on $[0,y_k]$ such that Q is continuously differentiable, $Q(y_i)=P_\mu(y_i)$ and $Q^{'}(y_i)=P_\mu^{'}(y_i)$ for any $0\leq i\leq k$, Q is piecewise quadratic, in particular Q is quadratic on $\{(y_{i-1},y_i)\}_{1\leq i\leq k}$ with representation

$$Q(y)=Q_i(r_i,y)\triangleq P_\mu(y_{i-1})+(y-y_{i-1})P_\mu^{'}(y_{i-1})+\frac{1}{2}r_i(y-y_{i-1})^2$$

on $[y_{i-1},y_i]$. We will see that $(r_i)_{1\leq i\leq k}$ is a decreasing sequence. Let $Q_{k+1}(r,y)=P_\mu(y_k)+(y-y_k)P_\mu^{'}(y_k)+\frac{1}{2}r(y-y_k)^2$, then there exists a unique r (r_{k+1} say) such that

$$Q_{k+1}(r_{k+1},y)\geq P_\mu(y)\ \forall y\geq y_k \text{ and } Q_{k+1}(r_{k+1},y)=P_\mu(y) \text{ for some } y>y_k.$$

Since $Q_k(r_k,y)>P_\mu(y)$ for all $y>y_k$, it is clear that $0<r_{k+1}<r_k$. Set $y_{k+1}=\max\{y>y_k:Q_{k+1}(r_{k+1},y)=P_\mu(y)\}$. Then, $P_\mu^{'}(y_{k+1})$ exists, $P_\mu^{'}(y_{k+1})=\frac{\partial}{\partial y}Q_{k+1}(r_{k+1},y_{k+1})$, and $y_{k+1}\notin\{\xi_1,\xi_2,\cdots,\xi_N\}$ since P_μ has changes in slope at these points. Set $Q(y)=Q_{k+1}(r_{k+1},y)$ on $[y_k,y_{k+1}]$. Repeat up to and including the index $T-1$ for which $y_{k+1}>\xi_N$. Then $y_{T-1}<\xi_N<y_T$ and then set $Q(y)=P_\mu(y)$ for $y\geq y_T$.

Now set $\rho(y)=Q^{''}(y)$. Then $\rho(y)=r_i$ on $(y_{i-1},y_i)_{1\leq i\leq T}$ and $\rho(y)=0$ on (y_T,∞). Furthermore, ρ is decreasing and ρ only decreases at points where $P_\mu(y)=$

$Q(y)$. Let ν be the measure with density ρ and an atom at 0 of size $F_\mu(0)$. Recall that $y_T > \xi_N$. Then, $P_\nu(y) = Q(y) \geq P_\mu(y)$, $F_\nu(0) = F_\mu(0)$, $F_\nu(y) = P_\nu^{'}(y_T) = P_\mu^{'}(y_T) = \mu(\mathbb{R})$ for any $y \geq y_T$ and $\int_0^\infty y\nu(dy) = y_T F_\nu(y_T) - P_\nu(y_T) = y_T\mu(\mathbb{R}) - P_\mu(y_T) = m$. Furthermore, $C_\nu(y) = P_\nu(y) + m - y\mu(\mathbb{R}) \geq P_\mu(y) + m - y\mu(\mathbb{R}) = C_\mu(y)$. It then follows that ν satisfies all the conditions in the theorem.

$\square$

Remark 5.4.1. Fix any $k \in \{1, 2, \dots, T\}$. Let n_k be such that $\xi_{n_k} < y_k < \xi_{n_k+1}$. Then, in the mapping $\mu \mapsto \nu$, the atoms $(\xi_1, \xi_2, \cdots, \xi_{n_k})$ of μ are mapped to $[0, y_k]$, and $\nu([0, y_k]) = P_\mu^{'}(y_k) = \sum_{j=1}^{n_k} p_j$. Moreover, $\int_0^{y_k} y\nu(dy) = y_k F_\nu(y_k) - P_\nu(y_k) = y_k \sum_{j=1}^{n_k} p_j - P_\mu(y_k) = y_k \sum_{j=1}^{n_k} p_j - \sum_{j=1}^{n_k} p_j(y_k - \xi_j) = \sum_{j=1}^{n_k} p_j \xi_j$.

Example 5.4.1. Suppose that $\mu = p\delta_\xi$ with $\xi > 0$. Then $P_\mu(y) = p(y - \xi)^+$. Let $Q_1(r, y) = ry^2/2$. Then, $Q_1(r, y) \geq P_\mu(y)$ if and only if $r \geq p/(2\xi) = r_1$, and for $r = r_1$ we have $Q_1(r_1, y) \geq P_\mu(y)$ with equality at $y = 0$ and $y = y_1 = 2\xi$. Then $y_1 > \xi = \xi_N$ so that the construction ends and $\nu = p\mathcal{U}[0, 2\xi]$.

The rest of this section proves a useful lemma which will be used in the next section to find the optimal target law for a general initial measure.

Lemma 5.4.1. *Suppose μ is a probability measure with finitely many atoms. Suppose that $\mu((-\infty, 0)) = 0$ and μ has mean $x_0 \in \mathbb{R}^+$. Denote by ν the probability measure that satisfies all the conditions in Theorem 5.4.1. Suppose ω is any probability measure such that ω has mean x_0 and $\mu \preceq_{cx} \omega$, where $\preceq_{cx}$ denotes "less than or equal to in convex order."*

(i) Let X_ω be a random variable has law ω. Define $\bar{X}_\omega$ as a random variable that has conditional distribution $\bar{X}_\omega \sim \mathcal{U}[0, 2x]$ given $X_\omega = x$. Let $\bar{\omega}$ be the law of $\bar{X}_\omega$. Then $\nu \preceq_{cx} \bar{\omega}$. In particular, $C_\nu(x) \leq C_{\bar{\omega}}(x)$.

(ii) Define $D_\nu(x) = -C_\nu^{'}(x)$ for any $x \geq 0$. Then,

$$D_\nu(x) \leq \inf_{y<x} \frac{C_\nu(y)}{x - y} \leq \inf_{y<x} \frac{C_{\bar{\omega}}(y)}{x - y} \quad \textit{for any } x \geq 0.$$

Further, $\lim_{x\uparrow\infty} x \inf_{y<x} \frac{C_{\bar{\omega}}(y)}{x-y} = 0$.

Proof. Suppose $\mu = \sum_{j=1}^N p_j \delta_{\xi_j}$, where $0 \leq \xi_1 < \xi_2 < \cdots < \xi_N$, $p_i \in (0, 1)$, $\sum_{j=1}^N p_j = 1$ and $\sum_{j=1}^N p_j \xi_j = x_0$.

(i) Let $\bar{\mu} = \sum_{j=1}^N p_j \mathcal{U}[0, 2\xi_j]$ where $\mathcal{U}[0, 0]$ denotes δ_0. We first show that $\nu \preceq_{cx} \bar{\mu}$. For any $m \in \{1, 2, \dots, N\}$, define $\mu^m = \sum_{j=1}^m p_j \delta_{\xi_j}$ and suppose ν^m is the corresponding measure derived using the algorithm in Theorem 5.4.1.

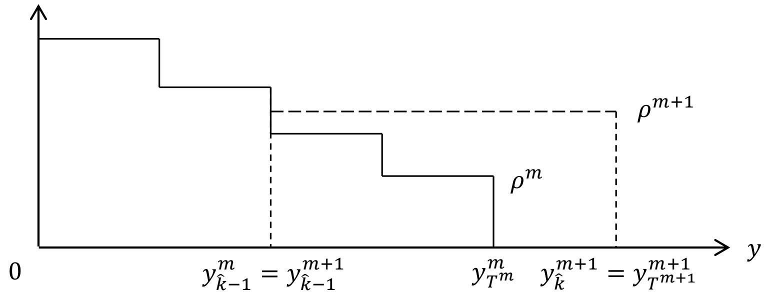

Figure 5.3: Graph of constant piecewise functions $\rho^m(y)$ and $\rho^{m+1}(y)$. Claim that $\rho^m \leq \rho^{m+1}$. In particular, $(\rho^{m+1} - \rho^m)$ is non-decreasing on $(0, y^{m+1}_{T^{m+1}})$.

If $N = 1$, then $\mu = p_1\delta_{\xi_1}$ and $\nu = p_1\mathcal{U}[0, 2\xi_1]$ and the result holds. Now suppose that $N \geq 2$, then $\nu = \nu^N = \sum_{m=1}^{N-1}(\nu^{m+1} - \nu^m) + \nu^1$. Provided we can show that $(\nu^m - \nu^{m-1})$ has increasing density, it then follows from Proposition 5.3.1 that $(\nu^j - \nu^{j-1}) \preceq_{cx} p_j\mathcal{U}[0, 2\xi_j]$. Then, since convex order is preserved under addition of measures, $\nu \preceq_{cx} \sum_{j=1}^N p_j\mathcal{U}[0, 2\xi_j] = \bar{\mu}$.

We use a suffix m to label quantities constructed in Theorem 5.4.1, to show that they are constructed from measure μ^m. Fix any $1 \leq k \leq T^m$. By construction $Q^m_k(r^m_k, y) \geq P_{\mu^m}(y)$ on (y^m_{k-1}, ∞). If also $Q^m_k(r^m_k, y) \geq P_{\mu^{m+1}}(y)$ on (y^m_{k-1}, ∞), then $\left(Q^m_j(r^m_j, y), y^m_j\right)_{j\geq 1}$ and $\left(Q^{m+1}_j(r^{m+1}_j, y), y^{m+1}_j\right)_{j\geq 1}$ will be the same up to $j = k$.

Suppose that $Q^m_k(r^m_k, y) \geq P_{\mu^{m+1}}(y)$ on (y^m_{k-1}, ∞) for all $1 \leq k \leq T^m$. Then, it is clear that $T^{m+1} = T^m + 1$, $y^m_{T^m} < y^{m+1}_{T^{m+1}}$ and densities ρ^{m+1} and ρ^m satisfy that $\rho^{m+1} = \rho^m$ on interval $(0, y^m_{T^m} = y^{m+1}_{T^m})$, ρ^{m+1} is constant on $(y^{m+1}_{T^m}, y^{m+1}_{T^{m+1}})$ and ρ^m is zero on $(y^{m+1}_{T^m}, y^{m+1}_{T^{m+1}})$. In particular, $(\rho^{m+1} - \rho^m)$ is non-decreasing $(0, y^{m+1}_{T^{m+1}})$.

Suppose that there exists

$$\hat{k} \triangleq \inf\left\{1 \leq k \leq T^m : Q^m_k(r^m_k, y) \not\geq P_{\mu^{m+1}}(y) \text{ on } (y^m_{k-1}, \infty)\right\}.$$

Then, it must be that in the construction we have $\rho^{m+1} = \rho^m$ on interval $(0, y^m_{\hat{k}-1} = y^{m+1}_{\hat{k}-1})$, ρ^{m+1} is constant (denoted as $r^{m+1}_{T^{m+1}}$) on $(y^{m+1}_{\hat{k}-1}, y^{m+1}_{T^{m+1}})$, ρ^m is decreasing and strictly less than $r^{m+1}_{T^{m+1}}$ on $(y^{m+1}_{\hat{k}-1}, y^m_{T^m})$, ρ^m is zero on $(y^m_{T^m}, \infty)$ and that $T^{m+1} = \hat{k}$. We want to argue that $y^m_{T^m} < y^{m+1}_{T^{m+1}}$, which then implies that $(\rho^{m+1} - \rho^m)$ is non-decreasing on $(0, y^{m+1}_{T^{m+1}})$ (see *e.g.* Figure 5.3).

We first construct a new measure $\tilde{\nu}^{m+1}$. Let $P_{m+1}(y) = \sum_{j=1}^{m+1} p_j(y - \xi_j)$ and define

$$\tilde{Q}^{m+1}_{T^m}(r, y) \triangleq P_{\mu^m}(y^m_{T^m-1}) + (y - y^m_{T^m-1})P'_{\mu^m}(y^m_{T^m-1}) + \frac{1}{2}r(y - y^m_{T^m-1})^2.$$

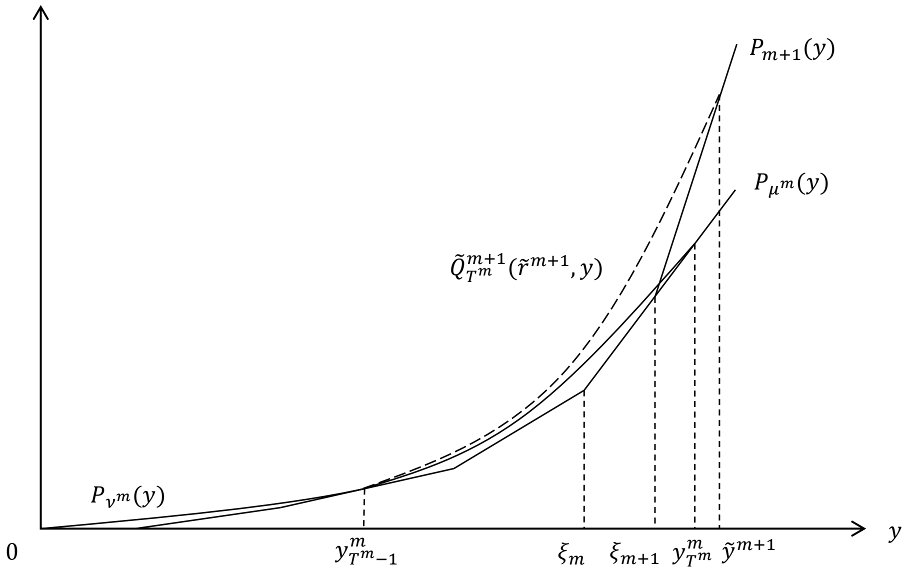

Figure 5.4: Graph of $\tilde{Q}^{m+1}(y)$. The dashed curve $\tilde{Q}_{T^m}^{m+1}(\tilde{r}^{m+1},y)$ is a quadratic function of y. Let $\tilde{\nu}^{m+1}$ be the measure with density $\tilde{\rho}^{m+1} = (\tilde{Q}^{m+1})''$ and an atom at 0 of size $F_\mu(0)$. Then, $\tilde{\rho}^{m+1} = \rho^m$ on $(0, y^m_{T^m-1})$ and $\tilde{\rho}^{m+1}$ is constant on $(y^m_{T^m-1}, \tilde{y}^{m+1})$. Note that ρ^m is constant on $(y^m_{T^m-1}, y^m_{T^m})$. Further, $\tilde{\nu}^{m+1}((y^m_{T^m-1}, \tilde{y}^{m+1})) = \mu([y^m_{T^m-1}, \xi_{m+1}])$ and $\nu^m((y^m_{T^m-1}, y^m_{T^m})) = \mu([y^m_{T^m-1}, \xi_m])$.

Then, there exists a unique r ($\tilde{r}^{m+1}$ say) such that

$$\tilde{Q}_{T^m}^{m+1}(\tilde{r}^{m+1},y) \geq P_{m+1}(y)\ \forall y \geq 0 \text{ and } \tilde{Q}_{T^m}^{m+1}(\tilde{r}^{m+1},y) = P_{m+1}(y) \text{ for some } y > 0.$$

Let $\tilde{y}^{m+1}$ be the point such that $\tilde{Q}_{T^m}^{m+1}(\tilde{r}^{m+1},\tilde{y}^{m+1}) = P_{m+1}(\tilde{y}^{m+1})$, then $\tilde{y}^{m+1} > y^m_{T^m-1}$ and $\frac{\partial}{\partial y}\tilde{Q}_{T^m}^{m+1}(\tilde{r}^{m+1},\tilde{y}^{m+1}) = P'_{m+1}(\tilde{y}^{m+1})$. Now let $\tilde{Q}^{m+1}(\cdot)$ be given by

$$\tilde{Q}^{m+1}(y) = P_{\nu^m}(y)\cdot \mathbf{1}_{[0,y^m_{T^m-1})} + \tilde{Q}_{T^m}^{m+1}(\tilde{r}^{m+1},y)\cdot \mathbf{1}_{[y^m_{T^m-1},\tilde{y}^{m+1})} + P_{m+1}(y)\cdot \mathbf{1}_{[\tilde{y}^{m+1},\infty)}$$

(see Figure 5.4, which illustrates the shape of $\tilde{Q}^{m+1}$).

Let $\tilde{\rho}^{m+1}(y) = (\tilde{Q}^{m+1})''(y)$, and let $\tilde{\nu}^{m+1}$ be the measure with density $\tilde{\rho}^{m+1}$ and an atom at 0 of size $F_\mu(0)$. Then, $P_{\tilde{\nu}^{m+1}}(y) = \tilde{Q}^{m+1}(y)$, $F_{\tilde{\nu}^{m+1}}(y) = P'_{m+1}(\tilde{y}^{m+1}) = \sum_{j=1}^{m+1} p_j$ if $y \geq \tilde{y}^{m+1}$ and $\int_0^\infty y\tilde{\nu}^{m+1}(dy) = \tilde{y}^{m+1}F_{\tilde{\nu}^{m+1}}(\tilde{y}^{m+1}) - P_{\tilde{\nu}^{m+1}}(\tilde{y}^{m+1}) = \sum_{j=1}^{m+1} p_j\xi_j$.

Note that in the construction of ν^m the masses at points $(\xi_{n^m_{T^m-1}+1},\dots,\xi_m)$ are embedded in the interval $(y^m_{T^m-1}, y^m_{T^m})$, and in the construction of $\tilde{\nu}^{m+1}$ the masses at points $(\xi_{n^m_{T^m-1}+1},\dots,\xi_{m+1})$ are embedded in $(y^m_{T^m-1}, \tilde{y}^{m+1})$. Moreover, ν^m has constant density over $(y^m_{T^m-1}, y^m_{T^m})$ and $\tilde{\nu}^{m+1}$ has constant density over

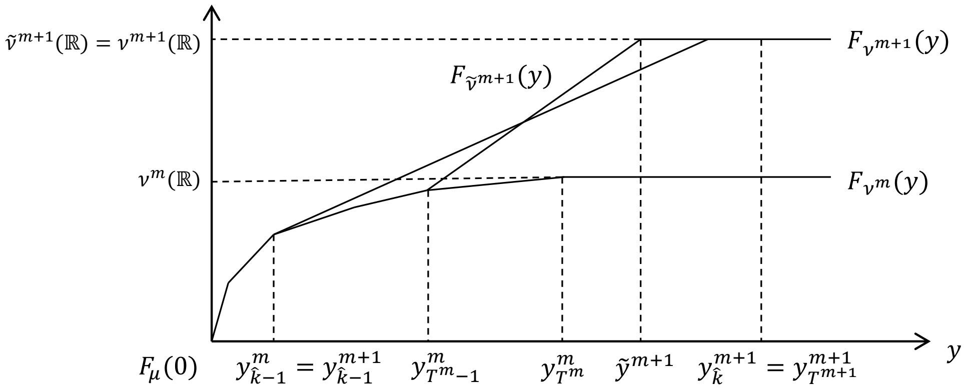

Figure 5.5: Graph of $F_{\nu^{m+1}}$, F_{ν^m} and $F_{\tilde{\nu}^{m+1}}$ with $\hat{k} < T^m$. By the constructions, $F_{\nu^{m+1}}(y) = F_{\nu^m}(y) = F_{\tilde{\nu}^{m+1}}(y)$ on $[0, y^{m+1}_{\hat{k}-1}]$, $F_{\nu^{m+1}}(y) > F_{\nu^m}(y) = F_{\tilde{\nu}^{m+1}}(y)$ on $(y^{m+1}_{\hat{k}-1}, y^m_{T^m-1}]$ and $F_{\tilde{\nu}^{m+1}}$ is linear on $(y^m_{T^m-1}, \tilde{y}^{m+1})$. Since $F_{\nu^{m+1}}$ and $F_{\tilde{\nu}^{m+1}}$ have the same mean, the area between the line $\tilde{\nu}^{m+1}(\mathbb{R})$ and $F_{\nu^{m+1}}$ must be equal to the area between the line $\tilde{\nu}^{m+1}(\mathbb{R})$ and $F_{\tilde{\nu}^{m+1}}$. Hence $\tilde{y}^{m+1} < y^{m+1}_{T^{m+1}}$.

$(y^m_{T^m-1}, \tilde{y}^{m+1})$. Comparing the means of ν^m and $\tilde{\nu}^{m+1}$, we have

$$\frac{1}{2}\left(y^m_{T^m-1} + y^m_{T^m}\right) \sum_{j=n^m_{T^m-1}+1}^{m} p_j = \sum_{j=n^m_{T^m-1}+1}^{m} p_j \xi_j$$

and

$$\frac{1}{2}\left(y^m_{T^m-1} + \tilde{y}^{m+1}\right) \sum_{j=n^m_{T^m-1}+1}^{m+1} p_j = \sum_{j=n^m_{T^m-1}+1}^{m+1} p_j \xi_j.$$

Hence,

$$\begin{aligned} y^m_{T^m} - y^m_{T^m-1} &= \frac{2\sum_{j=n^m_{T^m-1}+1}^{m} p_j \left(\xi_j - y^m_{T^m-1}\right)}{\sum_{j=n^m_{T^m-1}+1}^{m} p_j} \\ &< \frac{2\sum_{j=n^m_{T^m-1}+1}^{m+1} p_j \left(\xi_j - y^m_{T^m-1}\right)}{\sum_{j=n^m_{T^m-1}+1}^{m+1} p_j} = \tilde{y}^{m+1} - y^m_{T^m-1} \end{aligned}$$

and then $y^m_{T^m} < \tilde{y}^{m+1}$.

Next compare $\tilde{\nu}^{m+1}$ with ν^{m+1}. Recall that $\tilde{\nu}^{m+1}(\mathbb{R}) = \nu^{m+1}(\mathbb{R})$ and $\tilde{\nu}^{m+1}$ and ν^{m+1} have the same mean. Moreover, $F_{\nu^{m+1}}(y) = F_{\nu^m}(y) = F_{\tilde{\nu}^{m+1}}(y)$ on $[0, y^{m+1}_{\hat{k}-1}]$, and if $\hat{k} < T^m$ then $F_{\nu^{m+1}}(y) > F_{\nu^m}(y) = F_{\tilde{\nu}^{m+1}}(y)$ on $(y^{m+1}_{\hat{k}-1}, y^m_{T^m-1}]$ (see *e.g.* Figure 5.5). This implies that $\tilde{y}^{m+1} < y^{m+1}_{T^{m+1}}$ if $\hat{k} < T^m$ and $\tilde{y}^{m+1} = y^{m+1}_{T^{m+1}}$ if $\hat{k} = T^m$. Thus, $\tilde{y}^{m+1} \le y^{m+1}_{T^{m+1}}$, and then $y^m_{T^m} < y^{m+1}_{T^{m+1}}$.

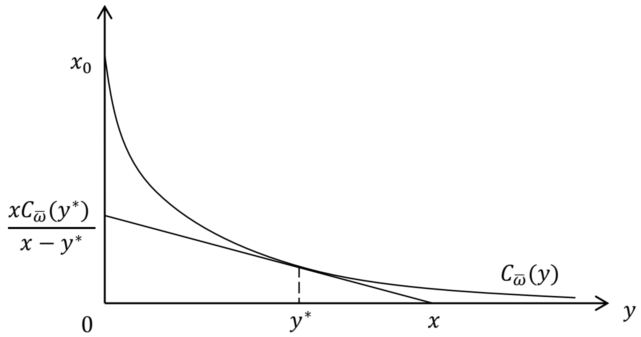

Figure 5.6: The infimum of $\frac{C_{\bar{\omega}}(y)}{x-y}$ over $y \in [0,x]$ is attained at y^*.

Therefore, we find that $(\rho^{m+1} - \rho^m)$ is non-decreasing on $(0, y^{m+1}_{T^{m+1}})$, and $(\nu^{m+1} - \nu^m)$ is a positive measure with increasing density on its support. Hence $\nu \preceq_{cx} \bar{\mu}$.

Next we show that $\bar{\mu} \preceq_{cx} \bar{\omega}$. Observe that $\mathbb{E}\left[\bar{X}_\omega\right] = \int_0^\infty xF_\omega(dx) = x_0$. Let $\bar{X}_\mu \sim \bar{\mu}$, then

$$\begin{aligned}
P_{\bar{\mu}}(x) = \mathbb{E}\left[(x - \bar{X}_\mu)^+\right] &= \int_{(0,\infty)} \mu(du) \int_0^{2u} \frac{(x-z)^+}{2u} dz + xF_\mu(0) \\
&= \int_{(0,\frac{x}{2})} (x-u)\mu(du) + \int_{\frac{x}{2}}^\infty \frac{x^2}{4u}\mu(du) + xF_\mu(0) \\
&= F_\mu(\frac{x}{2})\frac{x}{2} + \int_{(0,\frac{x}{2})} F_\mu(u)du - F_\mu(\frac{x}{2})\frac{x}{2} + \int_{x/2}^\infty F_\mu(u)\frac{x^2}{4u^2}du,
\end{aligned}$$

and then since $P_\mu'(x) = F_\mu(x)$,

$$P_{\bar{\mu}}(x) = P_\mu(\frac{x}{2}) - P_\mu(\frac{x}{2}) + \int_{x/2}^\infty P_\mu(u)\frac{x^2}{2u^3}du = \int_{x/2}^\infty P_\mu(u)\frac{x^2}{2u^3}du.$$

Similarly, we have $P_{\bar{\omega}}(x) = \int_{x/2}^\infty P_\omega(u)\frac{x^2}{2u^3}du$. Since $P_\mu \le P_\omega$, $P_{\bar{\mu}}(x) \le P_{\bar{\omega}}(x)$. Thus, $\bar{\mu} \preceq_{cx} \bar{\omega}$ and then $\nu \preceq_{cx} \bar{\omega}$. In particular, $C_\nu(x) \le C_{\bar{\omega}}(x)$.

(ii) Fix any $x \ge 0$. Since C_ν is convex, for any $y \le x$, $C_\nu(x) - C_\nu(y) \le (x-y)C_\nu'(x)$, and then $C_\nu(y) \ge C_\nu(x) + (x-y)D_\nu(x) \ge (x-y)D_\nu(x)$. Thus,

$$D_\nu(x) \le \inf_{y \le x} \frac{C_\nu(y)}{x-y} \le \inf_{y \le x} \frac{C_{\bar{\omega}}(y)}{x-y}.$$

Observe that $C_{\bar{\omega}}$ is decreasing and convex. Denote by y^* the point where the infimum of $C_{\bar{\omega}}(y)/(x-y)$ over $y \in [0,x]$ is attained (see *e.g.* Figure 5.6). Then, y^* is in fact the point supporting the tangent to $C_{\bar{\omega}}(\cdot)$ which crosses point $(x,0)$, and $xC_{\bar{\omega}}(y^*)/(x-y^*)$ is actually the y-coordinate of the y-intercept of this tangent. Since $\lim_{x\uparrow\infty} C_{\bar{\omega}}(x) = 0$, $\lim_{x\uparrow\infty} x\frac{C_{\bar{\omega}}(y^*)}{x-y^*} = 0$, that is $\lim_{x\uparrow\infty} x \inf_{y<x} \frac{C_{\bar{\omega}}(y)}{x-y} = 0$.

□

5.4.2 General Initial Measure

This section discusses the case where the initial law is a general probability measure with finite mean. We show that there exists a measure ν such that ν belongs to set $\mathcal{A}^*_\mu$. Then by Theorem 5.2.1 and Theorem 5.2.2, (ν, ν) is the unique symmetric Nash equilibrium for the problem.

Theorem 5.4.2. *Let μ be the law of a non-negative random variable with finite mean $x_0 \in \mathbb{R}^+$. Then, there exists a unique probability measure ν such that $\nu((-\infty, 0)) = 0$, F_ν is continuous on $(0, \infty)$, $F_\nu(0) = F_\mu(0)$, ν has mean x_0, $C_\nu(x) \geq C_\mu(x)$ for all $x \geq 0$, ν has a decreasing density ρ, and ρ only decreases at points such that $C_\nu(x) = C_\mu(x)$. Moreover, (ν, ν) is the unique symmetric Nash equilibrium for the problem.*

Proof. Let $\{\mu_n\}_{n\geq 1}$ be a sequence of atomic probability measures with finite support such that $F_{\mu_n}(0) = F_\mu(0)$, μ_n has mean x_0 and $\mu_n \uparrow \mu$ in convex order, *i.e.* $C_{\mu_n} \uparrow C_\mu$. For every μ_n, there exists a probability measure ν_n that satisfies all of the conditions described in Theorem 5.4.1 and thus $\nu_n \in \mathcal{A}^*_{\mu_n}$. Define

$$D_n(x) = -C^{'}_{\nu_n}(x) = 1 - F_{\nu_n}(x) \text{ for all } x \geq 0.$$

Let $r_n = \sup\{x : F_{\nu_n}(x) < 1\}$. By the construction of ν_n in the proof of Theorem 5.4.1 and Proposition 5.3.4, it can be seen that r_n is finite. Thus, D_n is a decreasing, convex function with $D_n(0) = 1 - F_\mu(0)$, $D_n(r_n) = 0$, $D_n \geq 0$ and

$$\int_0^\infty D_n(x)dx = \int_0^{r_n} D_n(x)dx = r_n D_n(r_n) - \int_0^{r_n} xD_n(dx) = \int_0^{r_n} x\nu_n(dx) = x_0. \tag{5.10}$$

Now let us introduce the Helly theorem (see *e.g.* Helly [1912]; Filipów et al. [2012], Theorem 1.3).

Lemma 5.4.2. *(Helly) If $\{f_n\}_{n\geq 1}$ is a uniformly bounded sequence of monotone real-valued functions defined on $\mathbb{R}$ then there is a subsequence $\{f_{n_k}\}_{k\geq 1}$ which is pointwise convergent.*

This theorem means that there exists a convergent subsequence of $\{D_n\}_{n\geq 1}$. Without loss of generality, we assume $\{D_n\}_{n\geq 1}$ is pointwise convergent. Denote by

D_∞ the limit function. Since D_n is decreasing and convex for any $n \geq 1$, D_∞ is decreasing and convex. Moreover, by Fatou's lemma and (5.10),

$$\int_0^\infty D_\infty(x)dx = \int_0^\infty \liminf_{n\to\infty} D_n(x)dx \leq \liminf_{n\to\infty} \int_0^\infty D_n(x)dx = x_0.$$

Because D_∞ is decreasing and $\int_0^\infty D_\infty(x)dx < \infty$, $\lim_{x\uparrow\infty} D_\infty(x) = 0$.

Define a probability measure ν via $\nu((-\infty, x]) = 1 - D_\infty(x)$. It is clear that $F_\nu(0) = F_\mu(0)$, $F_\nu(x)$ is continuous and ν has a non-increasing density ρ. Next we show that ν has mean x_0. Suppose $Y_\infty \sim \nu$ and $Y_n \sim \nu_n$ for any $n \geq 1$. Recall that $F_{\nu_n}(x) = 1 - D_n(x)$. Because $\lim_{n\to\infty} D_n(x) = D_\infty(x)$, Y_n converges in distribution to Y_∞. We next argue that the random variables $\{Y_n\}_{n\geq 1}$ are uniformly integrable. For any $\alpha \geq 0$,

$$\mathbb{E}[Y_n; Y_n \geq \alpha] = \mathbb{E}[Y_n - \alpha; Y_n \geq \alpha] + \alpha\mathbb{P}(Y_n \geq \alpha) = C_{\nu_n}(\alpha) + \alpha D_n(\alpha).$$

Let $X \sim \mu$ and define $\bar{X}$ via $\bar{X}|X \sim \mathcal{U}[0, 2X]$. Let $\bar{C}(x) = \mathbb{E}[(\bar{X} - x)^+]$. Then, since $C_{\mu_n} \leq C_\mu$ and by Lemma 5.4.1, $C_{\nu_n}(\alpha) \leq \bar{C}(\alpha)$ and $\alpha D_n(\alpha) \leq \alpha \inf_{y<\alpha} \frac{\bar{C}(\alpha)}{\alpha - y}$. Thus, since $\lim_{\alpha\to\infty} \bar{C}(\alpha) = 0$ and $\lim_{\alpha\to\infty} \alpha \inf_{y<\alpha} \frac{\bar{C}(\alpha)}{\alpha-y} = 0$,

$$\begin{aligned}
\lim_{\alpha\to\infty} \sup_n \mathbb{E}[Y_n; Y_n \geq \alpha] &= \lim_{\alpha\to\infty} \sup_n \left(C_{\nu_n}(\alpha) + \alpha D_n(\alpha)\right) \\
&\leq \lim_{\alpha\to\infty} \left(\bar{C}(\alpha) + \alpha \inf_{y<\alpha} \frac{\bar{C}(\alpha)}{\alpha - y}\right) = 0.
\end{aligned}$$

From the definition, we get that $\{Y_n\}_{n\geq 1}$ are uniformly integrable. Then, since Y_n converges to Y_∞ in distribution, $\mathbb{E}[Y_\infty] = \lim_{n\uparrow\infty} \mathbb{E}[Y_n] = x_0$, that is ν has mean x_0.

Observe that $P_{\nu_n}(x) = \mathbb{E}[(x - Y_n)^+]$ and $P_\nu(x) = \mathbb{E}[(x - Y_\infty)^+]$. Fix any $x \geq 0$. Since $p(y) = (x - y)^+$ is a bounded, continuous, real-valued function and since Y_n converges in distribution to Y_∞, we have $\lim_{n\uparrow\infty} \mathbb{E}[p(Y_n)] = \mathbb{E}[p(Y_\infty)]$, that is $P_\nu(x) = \lim_{n\to\infty} P_{\nu_n}(x)$. Thus, $C_\nu(x) = \lim_{n\to\infty} C_{\nu_n}(x)$. Then because $C_{\nu_n}(x) \geq C_{\mu_n}(x)$, $C_\nu(x) \geq C_\mu(x)$.

Suppose $C_\nu(x) > C_\mu(x)$ on some interval $\mathcal{J}$, then there exists $N_0 > 0$ such that $C_{\nu_n}(x) > C_\mu(x)$ on $\mathcal{J}$ for all $n \geq N_0$. Then on $\mathcal{J}$, $D_n(x)$ is a linear function. It is easy to see that $D_\infty(x) = \lim_{n\uparrow\infty} D_n(x)$ is also linear on $\mathcal{J}$. Thus, the density of ν only decreases when $C_\nu(x) = C_\mu(x)$.

It can be seen that ν satisfies all the conditions in the theorem, which implies that $\nu \in \mathcal{A}^*_\mu$. The uniqueness of ν then follows from Theorem 5.2.2. Furthermore, by Theorem 5.2.1, (ν, ν) is the unique symmetric Nash equilibrium for the problem. $\square$

5.5 Uniqueness of a Nash equilibrium

Section 5.4 shows that set $\mathcal{A}_\mu^*$ is non-empty. In this section, we prove that $\left|\mathcal{A}_\mu^*\right| \leq 1$, which then proves Theorem 5.2.2 that states $\mathcal{A}_\mu^*$ is a singleton.

Proof of Theorem 5.2.2. Assume that there exists two distinct elements in set $\mathcal{A}_\mu^*$, ν and σ. Recall that if $C_\nu(x) > C_\mu(x)$ then C_ν is locally a quadratic function near x, so it is with C_σ. Moreover, both $C_\nu^{'}$ and $C_\sigma^{'}$ are concave.

Observe that, for any $x \geq 0$, we cannot have $C_\nu(y) > C_\sigma(y)$ for all $y \in (x, \infty)$: if so then $C_\nu(y) > C_\sigma(y) \geq C_\mu(y)$ on (x, ∞) and C_ν is quadratic on (x, ∞), which is impossible by Proposition 5.3.4.

Let $K > 0$ be such that $C_\nu(K) \neq C_\sigma(K)$. Without loss of generality, suppose that $C_\nu(K) > C_\sigma(K)$. Define $x_1 = \inf\{x > K : C_\nu(x) = C_\sigma(x)\}$. By the observation above $x_1 < \infty$. Also note that $C_\nu(x) > C_\sigma(x)$ for all $x \in [K, x_1)$.

Suppose that $C_\nu(x_1) = C_\sigma(x_1) > C_\mu(x_1)$. Then, near x_1,

$$\begin{cases} C_\nu(x) = C_\nu(x_1) + \beta_{\nu,1}(x - x_1) + \gamma_{\nu,1}(x - x_1)^2, \\ C_\sigma(x) = C_\sigma(x_1) + \beta_{\sigma,1}(x - x_1) + \gamma_{\sigma,1}(x - x_1)^2, \end{cases}$$

for some constants $\beta_{\nu,1} < 0$, $\beta_{\sigma,1} < 0$, $\gamma_{\nu,1} > 0$ and $\gamma_{\sigma,1} > 0$. Here, the signs of these constants come from $C_\nu^{'}(x_1) < 0$, $C_\sigma^{'}(x_1) < 0$, $C_\nu^{''}(x_1) > 0$ and $C_\sigma^{''}(x_1) > 0$. Since $C_\nu(x) > C_\sigma(x)$ to the left of x_1, it is clear that $\beta_{\nu,1} \leq \beta_{\sigma,1}$.

Assume that $\beta_{\nu,1} = \beta_{\sigma,1}$. Then since $C_\nu(x) > C_\sigma(x)$ on $[K, x_1)$, we have $\gamma_{\nu,1} > \gamma_{\sigma,1}$. Let $\overline{x}_{\nu,1} = \inf\{x > x_1 : C_\nu(x) = C_\mu(x)\}$, then $C_\nu(x) = C_\nu(x_1) + \beta_{\nu,1}(x - x_1) + \gamma_{\nu,1}(x - x_1)^2$ on $[x_1, \overline{x}_{\nu,1}]$ and $\overline{x}_{\nu,1} < \infty$ by Proposition 5.3.4. Further, by Proposition 5.3.4, $C_\sigma(x) \leq C_\sigma(x_1) + \beta_{\sigma,1}(x - x_1) + \gamma_{\sigma,1}(x - x_1)^2$ on (x_1, ∞). Thus, $C_\sigma(\overline{x}_{\nu,1}) < C_\nu(\overline{x}_{\nu,1}) = C_\mu(\overline{x}_{\nu,1})$, which is a contradiction. Hence $\beta_{\nu,1} < \beta_{\sigma,1} < 0$. Similarly, let $\overline{x}_{\sigma,1} = \inf\{x > x_1 : C_\sigma(x) = C_\mu(x)\} < \infty$ and if $\gamma_{\nu,1} \leq \gamma_{\sigma,1}$ then $C_\nu(\overline{x}_{\sigma,1}) < C_\sigma(\overline{x}_{\sigma,1}) = C_\mu(\overline{x}_{\sigma,1})$, which is a contradiction. So we conclude that $\beta_{\nu,1} < \beta_{\sigma,1} < 0$ and $\gamma_{\nu,1} > \gamma_{\sigma,1} > 0$. Set $\vartheta = \beta_{\sigma,1} - \beta_{\nu,1} > 0$.

Now we introduce a useful lemma.

Lemma 5.5.1. *Suppose x_k is such that $C_\nu(x_k) = C_\sigma(x_k) > C_\mu(x_k)$. Then, $x_k > 0$ and in a neighbourhood of x_k we can write*

$$\begin{cases} C_\nu(x) = C_\nu(x_k) + \beta_{\nu,k}(x - x_k) + \gamma_{\nu,k}(x - x_k)^2, \\ C_\sigma(x) = C_\sigma(x_k) + \beta_{\sigma,k}(x - x_k) + \gamma_{\sigma,k}(x - x_k)^2. \end{cases}$$

Moreover, there is an interval to the left of x_k on which $C_\nu(x) - C_\sigma(x)$ is either

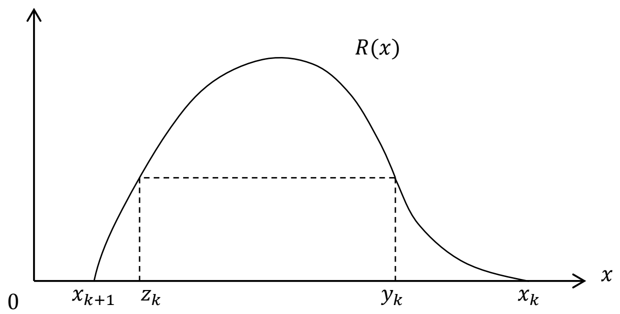

Figure 5.7: Graph of $R(x)$. Since $R^{''}(x) = f_\nu(x) - f_\sigma(x)$ is non-decreasing on (x_{k+1}, x_k) and $R^{''}(y_k) = 0$, R is concave on $[x_{k+1}, y_k]$ and strictly convex on $[y_k, x_k]$. Since $R^{'}(x) = C_\nu^{'}(x) - C_\sigma^{'}(x)$, $R^{'}(x_{k+1}) = \beta_{\nu,k+1} - \beta_{\sigma,k+1} > 0$ and $R^{'}(x_k) = \beta_{\nu,k} - \beta_{\sigma,k} < 0$. Then, $R^{'}(y_k) < R^{'}(x_k) < 0$, and there exists a unique $z_k \in (x_{k+1}, y_k)$ such that $R(z_k) = R(y_k)$. Further, $R^{'}(x_{k+1}) \geq R^{'}(z_k) > 0$.

strictly positive or strictly negative. Suppose that $C_\nu(x) - C_\sigma(x) > 0$ on some interval $(x_k - \epsilon, x_k)$: if not then interchange the roles of ν and σ. Then, $\beta_{\nu,k} < \beta_{\sigma,k} < 0$ and $\gamma_{\nu,k} > \gamma_{\sigma,k} > 0$.

Define $x_{k+1} = \sup\{x < x_k : C_\nu(x) = C_\sigma(x)\}$. Then, $x_{k+1} > 0$, $C_\nu(x_{k+1}) = C_\sigma(x_{k+1}) > C_\mu(x_{k+1})$ and hence in a neighbourhood of x_{k+1} we can write

$$\begin{cases} C_\nu(x) = C_\nu(x_{k+1}) + \beta_{\nu,k+1}(x - x_{k+1}) + \gamma_{\nu,k+1}(x - x_{k+1})^2, \\ C_\sigma(x) = C_\sigma(x_{k+1}) + \beta_{\sigma,k+1}(x - x_{k+1}) + \gamma_{\sigma,k+1}(x - x_{k+1})^2. \end{cases} \tag{5.11}$$

Further, $\beta_{\sigma,k+1} < \beta_{\nu,k+1} < 0$, $\gamma_{\sigma,k+1} > \gamma_{\nu,k+1} > 0$ and $\beta_{\nu,k+1} - \beta_{\sigma,k+1} > \beta_{\sigma,k} - \beta_{\nu,k} > 0$.

Proof. Exactly as in the case $k = 1$ from the proof of Theorem 5.2.2, we conclude that $\beta_{\nu,k} < \beta_{\sigma,k} < 0$ and $\gamma_{\nu,k} > \gamma_{\sigma,k} > 0$.

Assume that $C_\nu(x_{k+1}) = C_\sigma(x_{k+1}) = C_\mu(x_{k+1})$. Then $C_\nu^{'}(x_{k+1}) = C_\sigma^{'}(x_{k+1})$ by Proposition 5.3.3. Since $C_\nu(x) > C_\sigma(x)$ on (x_{k+1}, x_k), $C_\nu^{'}$ is linear on $[x_{k+1}, x_k]$. Then because $C_\sigma^{'}$ is concave, it is clear that $C_\sigma^{'}(x) \leq C_\nu^{'}(x)$ on (x_{k+1}, x_k). This implies that $C_\sigma(x_k) < C_\nu(x_k)$, which is a contradiction. Hence $C_\nu(x_{k+1}) = C_\sigma(x_{k+1}) > C_\mu(x_{k+1})$.

It then follows that $x_{k+1} \in (0, x_k)$ and both C_ν and C_σ are quadratic in a neighbourhood of x_{k+1}. In particular, C_ν is quadratic on (x_{k+1}, x_k) and $\gamma_{\nu,k} = \gamma_{\nu,k+1}$. Using a similar argument described in the case $k = 1$ from the proof of Theorem 5.2.2, we get that $\beta_{\sigma,k+1} < \beta_{\nu,k+1} < 0$ and $\gamma_{\sigma,k+1} > \gamma_{\nu,k+1} > 0$.

Denote by f_ν the density function of measure ν. Then f_ν is constant on

(x_{k+1}, x_k). In contrast, f_σ is non-increasing, $f_\sigma(x_k) = \gamma_{\sigma,k} < \gamma_{\nu,k} = f_\nu(x_k)$ and $f_\sigma(x_{k+1}) = \gamma_{\sigma,k+1} > \gamma_{\nu,k+1} = f_\nu(x_{k+1})$. Set $y_k = \sup\{y < x_k : f_\sigma(y) \geq f_\nu(y)\}$, then $y_k \in (x_{k+1}, x_k)$. Further, $R(x) \triangleq C_\nu(x) - C_\sigma(x)$ defined on $[x_{k+1}, x_k]$ is zero at the endpoints, has increasing second derivative and is concave on $[x_{k+1}, y_k]$ and strictly convex on $[y_k, x_k]$ (see *e.g.* Figure 5.7).

Let $z_k \in (x_{k+1}, y_k)$ be the unique value such that $R(z_k) = R(y_k)$. Then $R^{'}(x_{k+1}) \geq R^{'}(z_k) > 0$ and $R^{'}(y_k) < R^{'}(x_k) \leq 0$. Set $H(x) = R(y_k - x) - R(y_k)$ on $[0, y_k - z_k]$ in Proposition 5.3.2, then we get $R^{'}(z_k) \geq \left|R^{'}(y_k)\right|$. Hence, $\beta_{\nu,k+1} - \beta_{\sigma,k+1} = R^{'}(x_{k+1}) \geq R^{'}(z_k) \geq \left|R^{'}(y_k)\right| > \left|R^{'}(x_k)\right| = \beta_{\sigma,k} - \beta_{\nu,k} > 0$.

□

Return to the proof of Theorem 5.2.2. Using Lemma 5.5.1, we construct a decreasing sequence of points $(x_k)_{k\geq 1}$ at which $C_\nu - C_\sigma$ changes sign. Moreover, $\left|C_\nu^{'}(x_k) - C_\sigma^{'}(x_k)\right| = |\beta_{\nu,k} - \beta_{\sigma,k}| \geq \vartheta$. Let $x_\infty = \lim_{k\uparrow\infty} x_k$ then $x_\infty \geq 0$. Observe that $\lim_{k\uparrow\infty}(\beta_{\nu,k} - \beta_{\sigma,k}) = C_\nu^{'}(x_\infty) - C_\sigma^{'}(x_\infty)$ exists. However, $\limsup_{k\uparrow\infty}(\beta_{\nu,k} - \beta_{\sigma,k}) \geq \vartheta > 0$ and $\liminf_{k\uparrow\infty}(\beta_{\nu,k} - \beta_{\sigma,k}) \leq -\vartheta < 0$, which is a contradiction. Hence, there cannot be distinct elements ν and σ in set $\mathcal{A}_\mu^*$.

The above is predicated on the assumption that $C_\nu(x_1) = C_\sigma(x_1) > C_\mu(x_1)$. Now suppose $C_\nu(x_1) = C_\sigma(x_1) = C_\mu(x_1)$. Recall that $C_\nu > C_\sigma$ on an interval to the left of x_1. Let $x_2 = \sup\{x < x_1 : C_\nu(x) = C_\sigma(x)\}$. Then, $x_2 \in [0, K)$ and $C_\nu(x) > C_\sigma(x)$ on (x_2, x_1).

Assume that $C_\nu(x_2) = C_\sigma(x_2) = C_\mu(x_2)$. Then, by Proposition 5.3.3, $C_\nu^{'}(x_1) = C_\sigma^{'}(x_1)$ and $C_\nu^{'}(x_2) = C_\sigma^{'}(x_2)$ (recall that $C_\nu^{'}(0) = C_\sigma^{'}(0) = F_\mu(0) - 1$). Because $C_\nu^{'}$ is linear and $C_\sigma^{'}$ is concave on (x_2, x_1), $C_\nu^{'}(x) \leq C_\sigma^{'}(x)$ for all $x \in [x_2, x_1]$. This means that $C_\nu(x) \leq C_\sigma(x)$ for all $x \in [x_2, x_1]$, which is a contradiction. Hence $C_\nu(x_2) = C_\sigma(x_2) > C_\mu(x_2)$.

Now starting the construction at x_2, rather than x_1, we are in the same case as discussed previously. In particular, there cannot be distinct elements ν and σ in set $\mathcal{A}_\mu^*$.

□

5.6 Derivation of the sufficient conditions

This section is devoted to the derivation of the last two conditions in Theorem 5.2.1.

Let $\mathcal{A}$ be the set of all measures ν satisfying $\nu((-\infty, 0)) = 0$. Given that the other agent j chooses ν_j as her target law, where F_{ν_j} is continuous, agent i aims to choose a measure ν_i to solve

$$\max_{\nu_i \in \mathcal{A}} \left\{ \int_{(0,\infty)} F_{\nu_j}(x)\nu_i(dx) + \theta F_{\nu_j}(0) F_{\nu_i}(0) \right\} \tag{5.12}$$

subject to $\int_{(0,\infty)} x\nu_i(dx) = x_0$, $\int_{(0,\infty)} \nu_i(dx) + F_{\nu_i}(0) = 1$, $F_{\nu_i}(0) \geq F_\mu(0)$ and $C_{\nu_i}(z) \geq C_\mu(z)$ for all $z > 0$.

Introduce multipliers λ, γ and $\zeta \geq 0$ for the first three constraints, and for each $z > 0$ introduce a multiplier $\eta(z)$ for the last constraint. The Lagrangian for problem (5.12) is

$$\begin{aligned}\mathcal{L}_{\nu_j}(\nu_i; \lambda, \gamma, \zeta, \eta) = &\int_{(0,\infty)} F_{\nu_j}(x)\nu_i(dx) + \theta F_{\nu_j}(0) F_{\nu_i}(0) + \lambda \left(x_0 - \int_{(0,\infty)} x\nu_i(dx) \right) \\ &+ \gamma \left(1 - \int_{(0,\infty)} \nu_i(dx) - F_{\nu_i}(0) \right) + \zeta \left(F_{\nu_i}(0) - F_\mu(0) \right) \\ &+ \int_{(0,\infty)} \left(C_{\nu_i}(z) - C_\mu(z) \right) \eta(dz),\end{aligned}$$

which gives us (5.2). Since $\int_{(0,\infty)} C_{\nu_i}(z)\eta(dz) = \int_{(0,\infty)} \int_{(0,x)} (x - z)\eta(dz)\nu_i(dx)$,

$$\begin{aligned}\mathcal{L}_{\nu_j}(\nu_i; \lambda, \gamma, \zeta, \eta) = &\int_{(0,\infty)} \left(F_{\nu_j}(x) - \lambda x - \gamma + \int_{(0,x)} (x - z)\eta(dz) \right) \nu_i(dx) + \lambda x_0 + \gamma \\ &+ \left(\theta F_{\nu_j}(0) - \gamma + \zeta \right) F_{\nu_i}(0) - \int_{(0,\infty)} C_\mu(z)\eta(dz) - \zeta F_\mu(0),\end{aligned}$$

which gives us (5.3). Here η should satisfy that $\eta(dz) \geq 0$ for all $z \geq 0$.

Define $L_{\nu_j}(x) = F_{\nu_j}(x) - \lambda x - \gamma + \int_{(0,x)} (x - z)\eta(dz)$. Let $\mathcal{D}_{\nu_j}$ be the set of $(\lambda, \gamma, \zeta, \eta)$ such that $\mathcal{L}_{\nu_j}(\cdot; \lambda, \gamma, \zeta, \eta)$ has a finite maximum, then $\mathcal{D}_{\nu_j}$ is defined by

$$\mathcal{D}_{\nu_j} = \{ (\lambda, \gamma, \zeta, \eta) : L_{\nu_j}(x) \leq 0 \ \text{ for all } \ x \geq 0 \ \text{ and } \ \theta F_{\nu_j}(0) - \gamma + \zeta \leq 0 \}.$$

In order to reach the maximum value, we must have $\nu_i(dx) = 0$ when $L_{\nu_j}(x) < 0$ and $F_{\nu_i}(0) = 0$ when $\theta F_{\nu_j}(0) - \gamma + \zeta < 0$. This means that for $(\lambda, \gamma, \zeta, \eta) \in \mathcal{D}_{\nu_j}$, the maximum of $\mathcal{L}_{\nu_j}(\cdot; \lambda, \gamma, \zeta, \eta)$ occurs at ν^* such that $\nu^*(dx) = 0$ when $L_{\nu_j}(x) < 0$ and $F_{\nu^*}(0) = 0$ when $\theta F_{\nu_j}(0) - \gamma + \zeta < 0$. Since we search for a symmetric Nash equilibrium, we must have $F_{\nu_j}(x) = F_{\nu^*}(x)$. Thus, $L_{\nu^*}(x) \leq 0$ for all $x \geq 0$, $L_{\nu^*}(x) = 0$ when $\nu^*(dx) > 0$, and $\theta F_{\nu^*}(0) - \gamma + \zeta = 0$ if $F_{\nu^*}(0) > 0$.

Let $\Gamma(x) = \lambda x + \gamma - \int_0^x (x - z)\eta(dz)$, then $\nu^*(x) \leq \Gamma(x)$ for any $x \geq 0$ and $\nu^*(x) = \Gamma(x)$ when $\nu^*(dx) > 0$. Observe that $\Gamma^{'}(x) = \lambda - \int_0^x \eta(dz)$ and $\Gamma^{''}(x) = -\eta(dx)$. Since $\eta(dx) \geq 0$, Γ is concave on $[0, \infty)$. Let $l = \inf\{x : F_{\nu^*}(x) > 0\}$

and $r = \sup\{x : F_{\nu^*}(x) < 1\}$. By the concavity of Γ and the definition of r, we must have that $\Gamma(x)$ is strictly increasing on $(-\infty, r)$. Then since F_{ν^*} is continuous, $F_{\nu^*}(x) = \Gamma(x)\ \forall x \in [l, r]$. Thus, F_{ν^*} is continuous, strictly increasing and concave on $[l, r]$.

Assume that $l > 0$. Because $\Gamma^{'}(x) > 0$ for $x < r$, $\Gamma(x) < \Gamma(l) = F_{\nu^*}(l)$ for all $x \in [0, l)$. But notice that $F_{\nu^*}(x) = F_{\nu^*}(l)$ for all $x \in [0, l)$, since F_{ν^*} is continuous on $(0, \infty)$ and $l > 0$. This means $F_{\nu^*}(x) > \Gamma(x)$ on $[0, l)$, which is a contradiction. Thus, $l \leq 0$. Then by the non-negativity of l, it follows that $l = 0$.

Now by Kuhn-Tucker condition, ν^* should satisfy that if $C_{\nu^*}(x) > C_\mu(x)$ then $\eta(dx) = 0$. So if $C_{\nu^*}(x) > C_\mu(x)$ on some interval $\mathcal{J} \subset [0, r]$, then $F_{\nu^*}(x) = \Gamma(x)$ is linear on $\mathcal{J}$.

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