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Controlled Continuous Time Random Walks and their position dependent extensions

by

Maria Veretennikova

Thesis

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Declarations

I submit this thesis to the University of Warwick in partial fulfillment for the requirements for admission to the degree of Doctor of Philosophy. Here I declare that all the work presented here is my own, except where it is indicated as otherwise and referenced. I expect the work presented in this thesis to appear in the following publications:

1. "Fractional Hamilton Jacobi Bellman equations for scaled limits of controlled Continuous Time Random Walks" - accepted by SIMAI CAIM, to appear in the Special issue for BCAM FCNLO 2013 workshop proceedings; to be published in 2014.

2. "Well-posedness and regularity of the Cauchy problem for nonlinear fractional in time and space equations" - accepted by the Fractional Differential Calculus (FDC) Journal, 2014.

3. "Scaled limits of controlled Continuous Time Random Walks and their position dependent extensions", in preparation, to be submitted, 2014

Abstract

Continuous Time Random Walks (CTRWs) are used widely for modelling anomalous diffusion. This thesis is the first research which focuses on optimal control of CTRWs, their modifications and their position dependent extensions. We derive the equation which may be called a fractional Hamilton Jacobi Bellman equation (FHJB), as it is similar to the HJB equation for controlled Markov processes. We present our original analysis of the FHJB equation, firstly working with its simpler linear version and obtaining useful regularity properties, and secondly, deriving the mild form of the FHJB, exploring its regularity properties and well-posedness. We present our novel theorems proving rigorous convergence for optimal payoffs of the scaled stochastic processes of our interest and give an interpretation of the solution of the FHJB equation as a solution to an optimization problem.

Controlled Continuous Time Random Walks and their Position Dependent Extensions

Thesis

Maria Veretennikova

December 18, 2014

Contents

Chapter 1

Introduction

1.1 Outline of the thesis

This thesis focuses on stochastic optimal control models such that the dynamics of the controlled system is a Continuous Time Random Walk (CTRW) with waiting times that have non-exponential distributions. CTRWs and their scaling limits model anomalous diffusion. Our work is the first research development on the new field of controlled anomalous diffusion systems. The main novel results presented here are the derivation of the new Fractional Hamilton Jacobi Bellman (FHJB) equation from a macroscopic model of a controlled CTRW; the derivation of an appropriate mild form for a FHJB equation; existence and uniqueness of a solution to a mild form of the FHJB equation and to the HJB itself, as well as various interesting regularity properties; the rigorous convergence theorems and an interpretation of the solution of the FHJB equation as a solution to an optimization problem. We present two derivations of the FHJB. The first derivation is carried out heuristically in analogy with the standard heuristic derivation of the Bellman dynamic programming equation. It is heuristic in the sense that we assume existence of all the limits that occur in our scaling limit. The full rigorous justification of the limiting procedure is based on a completely different reasoning.

Here is the explanation of the structure for this thesis. Chapter 1 focuses on the theoretical background and the mathematical description of the process we study, with important preliminary definitions, in particular relating fractional integral and differential operators with probabilistic concepts. Chapter 2 concentrates on scaling and control of the CTRW process, introducing the notions of an embedded Markov process and a corresponding payoff function. Then, in a heuristic manner, it presents our results concerning the limiting payoff function. It also extends the scenario to the setting with time and position dependence, as well as an extension to multiple stability parameter dependence. In Chapter 3, after a brief overview of existing theory, the reader can acquaint themselves with our original results on well-posedness and regularity properties of the solution to the mild form of a linear version for the FHJB equation. In Chapter 4 we present our novel important convergence results and a corresponding verification theorem. Here the limiting scaled CTRW is viewed as a component of a Markov process, unlike in the previous chapters. Such representation allows to see the model within the framework of controlled Markov processes for which the theory is well-developed, see [1]. Following this is a chapter on further developments, highlighting other ideas and potential applications. For example, this includes our novel conjecture about the limiting scaled CTRW process, linking it to stochastic differential equations (SDEs). The appendix presents some of the important theorems used to obtain our results.

1.2 Background and literature review

In this section we present the background theory which allows the reader to see how the results in this thesis fit into the wide mathematical picture as the interface of anomalous diffusion models, control theory and fractional calculus.

Diffusion is one of the key mechanisms for material transport in physics, biology and chemistry. The well-established mathematical model for particle diffusion is Brownian motion. One of its characteristic features is the following mean-squared displacement: $\mathbb{E}[(x(t)-x_0)^2] \propto t$. The distribution $P(x,t)$ of the Brownian motion $W_t^{x_0}$ started at a point $x_0 \in (-\infty, \infty)$ is a Normal (Gaussian) distribution:

$$P(x,t) = \frac{1}{\sqrt{4\pi Dt}} \exp\{-(x-x_0)^2/4Dt\} \tag{1.1}$$

for $x \in (-\infty, \infty)$, $t \geq 0$ and $D > 0$. It can be shown that $P(x,t)$ in (1.1) is the solution to the Cauchy problem

$$u_t - Du_{xx} = 0, \text{ for } t > 0, \tag{1.2}$$

with $u(0, x) = \delta(x)$, where $\delta(x)$ is the Dirac delta function, and $D > 0$ is the diffusion constant. This means that the probability of finding the Brownian particle at x at time $t > 0$ is given by the Gaussian probability density function (1.1) which satisfies (1.2).

Since the original results presented in [2] in 1905, other models for diffusion have been proposed, capturing more general and anomalous particle behaviour patterns in comparison to Brownian motion. One of the models developed for anomalous diffusion is the Continuous Time Random Walk (CTRW) model. CTRWs form a class of processes which are continuous from the right with left hand limits (càdlàg). The book [3] provides a thorough and concise exposition of key convergence results for càdlàg processes which includes families of scaled CTRWs. A CTRW has been introduced for the first time in [4] to study random walks on a lattice. Models of CTRWs are ubiquitous in describing numerous natural phenomena, see [5], [6], [7], [8] and [9]. For example a CTRW approach has been taken to model the dynamics of supercooled liquids and prices of financial assets, see [10], [11] and [12]. We focus on CTRWs with non-exponential waiting times between displacements and with jump distributions exhibiting some stable law properties. Most of the background information for the CTRW that we use in this thesis can be found in [5] and [13].

For the first time a symmetric stable law, with the stability parameter $\alpha = 3/2$, appears in a paper dated 1919 in a stellar model, see [14]. More precisely, it is a model for the distribution of the gravitational force due to an infinite number of stars which are uniformly distributed in $\mathbb{R}^3$. This distribution is now referred to as Holtsmark and has extensive uses in physics, see [13] and references therein. As a class, stable laws were characterised for the first time by P. Lévy in 1920's. Much later, in the 1960's, B. Mandelbrot and his successors wrote a series of papers where stable laws were used in specific economic models. Since then there has been an expansion of areas where stable laws have arisen, see [15]. Similarly to (1.2) for the Gaussian law, stable distributions appear as solutions to integro-differential equations, see [15]. Due to the lack of closed formulas for most stable probability densities it was difficult to use them for mathematical modelling for a long time, however research and recent programming advances now allow to compute properties of those, paving the road for direct applications, see [16]. Stable laws have recently manifested themselves in many different physical and social phenomena. One of the reasons to employ stable distributions is the Generalized Central Limit Theorem (GCLT). It states that the only limits of scaled and centered sums of i.i.d. random variables are stable laws with the stability parameter $\alpha \in (0, 2]$. When $\alpha = 2$ the GCLT is the Central Limit Theorem. This reason is strengthened empirically by large data sets that demonstrate heavy tail behaviour and a skewness property, see [16] and references therein. Although β-stable laws with $\beta \in (0, 1)$ feature an infinite mean and an infinite variance, which is unrealistic for real-life data, they may still be used in modelling. If one can find a reason why a considered quantity of interest has a minimal value, the corresponding distribution may be cut-off at that point. In case a stable law is an appropriate approximation for the distribution in a large time interval before the cut-off point, the sum of the i.i.d. random variables from this distribution for a long time may be modelled by the stable one without cut-offs, see [17]. Due to the reasoning above and that we mostly deal with finite time intervals and finite data, we may use stable laws yielding infinite first and second moments for the random variables modelling the waiting times and the jumps, without being unrealistic.

Stable distributions for jumps exhibit power law behaviour for large distances, apart from the normal distribution for which the stability parameter is $\alpha = 2$. We look at even broader range of distributions than stable laws, namely, assume jump distributions belonging to the domain of attraction of a stable law. When data shows asymmetry and heavy tail behaviour and the considered motion may be modelled by a CTRW, the GCLT may be employed to justify the use of domains of attraction of α-stable laws as appropriate distributions in a realistic complete mathematical model for the observed motion, see [18]. The Pareto distribution with a parameter $\alpha < 2$ is in the DOA of an α-stable law. Also the Mittag-Leffler distribution with the parameter $\beta \in (0, 1)$ is in the DOA of a β-stable law, for example see [19], [20] and references therein. Note that every stable distribution is in the DOA of itself. Any finite variance distribution belongs to the DOA of the Gaussian distribution by the Central Limit Theorem (CLT). The rigorous definitions, the mathematical details for stable laws and DOA of a stable law, as well as the detailed explanation of the relation to CTRWs in this thesis, will follow in section 1.3. Here we mention just one specific example where non-exponential waiting times have appeared. Analysis of high-frequency data for daily price changes in 800 companies at the Japanese market in a 27 year interval showed that the waiting times between daily price changes follow a power law. Further details may be found in [21]. The explanation for such behaviour has been proposed in [22], see also [23].

CTRWs may be scaled appropriately to obtain a microscopic model, which is sometimes more convenient for specific purposes. A scaled CTRW with correlated jumps in the limit yields the microscopic model of fractional Brownian motion, see [24]. Another example of the scaled CTRW limiting process is a stable-Lévy motion with a random time change, see [25]. CTRW limit processes have been studied extensively in [26] and references therein.

In a rich class of financial, biophysical, air and hydro-dynamics phenomena one observes constant time periods, which are referred to as trapping events, [27]. To model this pattern a random time change is introduced. This idea was initiated by Bochner in 1949. In essence, the real time of a process is replaced by some non-decreasing process which has independent increments. In some models this external process is the Brownian motion, which in that case may be looked at as a distribution limit of some CTRW, see [27] and references therein. Time changed Lévy processes, inverse stable processes, subordinators and inverse subordinators are standard processes used for modelling the random time change. The reader is referred to [28] and references therein for specific examples. In this thesis we study CTRWs with a random time change. From the microscopic viewpoint of the CTRW model, in this thesis the processes in consideration will have the β-stable Lévy motion for the time process, where $\beta \in (0, 1)$.

Research results presented in [29], [30], [31] and [32] relate CTRWs to fractional differential equations (FDEs). The strong connection between CTRWs and FDEs is explored for example in [5], [33], [34] and [35]. In essence, FDEs appear in mathematical descriptions of CTRW scaling limits, and the order of the derivatives in a FDE depends on the probability density functions for jump lengths and the waiting times of the CTRW.

Fractional calculus generalises the classical calculus to study integral and differential operators of non-integer order. Fractional differential equations are closely related to modelling processes with memory and spatial non-local properties, see [36], [37], [38], [39] for extensive descriptions of models where history impacts the future of the process. Without being exhaustive we name few fields where fractional calculus has recently been useful - viscoelasticity, hydrology, cosmology, finance as well as fire propagation dynamics and plasma turbulence, see [40], [41], [42], [11], [43] and [44] respectively. For example, porous structure and chemical heterogeneity of media such as in geological formations lead to space and time non-local dependency for solute transport, see [45].

Often it is possible for humans to intervene in a dynamical process in order to regulate the behaviour so as to profit the most from the control. The goal may be represented as an optimisation of a payoff/cost function. The methods for stochastic optimal control problems are the Bellman's dynamical programming and Pontryagin's principle, see [46] and [47] respectively. An example of an optimal stochastic control application is controlling a linearly fuelled satellite following a diffusion, with the aim to keep it above a critical level, see [48]. Theory for stochastic optimal control problems is presented in [1] and references therein. However the existing stochastic control theory does not cover controlled CTRWs and their scaling limits. This thesis extends to the scenario where for the first time CTRW processes can be controlled. We also present generalizations of controlled CTRWs by including time and position dependence.

We establish that control of CTRWs with waiting times in the DOA of a β-stable law gives rise to a new kind of equations in control theory. These are fractional in time Hamilton-Jacobi-Bellman (HJB) equations:

$$D_{0,t}^{*\beta} f(t,y) = H(t,y,D_y f(t,y)), \tag{1.3}$$

where $(t,y) \in Q$, $D_{0,t}^{*\beta}$ is a fractional time derivative, which we will define in section 1.3, and H is a continuous function on $\bar{Q} \times \mathbb{R}^n$. It is called the Hamiltonian and contains all the control related terms. In section 1.3 we will discuss fractional derivatives in more detail. In case $\beta = 1$, $D_{0,t}^{*\beta} = \frac{d}{dt}$. Equations such as (1.3) with $\beta = 1$ have been extensively studied, for example see [1].

A solution $f(t, y)$ to a partial differential equation (PDE) is said to be classical when it is differentiable as many times as necessary for the PDE to make sense. Most HJB equations do not have classical solutions and for such cases the theory of viscosity solutions becomes useful, see [1]. However, under certain restrictions and conditions on the Hamiltonian H in (1.3) a classical solution exists. The case without control, when $\beta = 1$ is studied in [13]. In this thesis the reader will find the literature review for the analysis of FDEs. Further, we work with a mild form of the fractional HJB (FHJB) and present our original well-posedness and regularity results.

A recent paper [49] studies equations with a fractional Laplacian and proves existence of weak solutions in a fractional Sobolev space under specified boundary data and the fractional index parameter conditions. It also applies the results to an optimal control problem. More precisely, it proves the existence of optimal processes in the fractional Sobolev space for the optimal control problem associated to the weak formulation of the following FDE

$$(-\Delta)^{\alpha/2} z(x) = G_z(x, z(x), u(x)) \text{ in } \Omega \tag{1.4}$$

for some non-linearity G_z, with the fixed interior boundary condition $z(x) = v(x)$ in $\mathbb{R}^n \setminus \Omega$ and the integral cost functional

$$J(z, u) = \int_\Omega \Phi(x, z(x), u(x))dx, \tag{1.5}$$

with specific conditions on Φ and the control set U, and $\alpha \in (1, 2)$.

This thesis considers a broader range of FDEs than (1.4) and we look primarily for a mild solution to (1.3). We take a different approach to the optimal control problem, however paper [49] is an interesting development in the direction of fractional optimal control problems.

The theory of controlled Random Walks (RW) is only in the beginning of development. The introduction of control paves the road for applications of CTRW processes in the wide range of problems. For example, a recently proposed project in telecommunications industry focuses in particular on applying CTRW models with non-exponential waiting times to model non-standard information transport, as well as understanding to what extent such processes may be controlled, see [50] for details. It is anticipated that the work done in this thesis will be of high value for this project and its further developments. This thesis presents ideas on how the CTRW theory presented here could be applied to game theory and to limit order book modelling.

Various papers presenting convergence results for CTRWs with random time changes prove convergence of the time and the jump processes separately, and then use the continuous mapping theorem (CMT) to prove convergence of the time changed CTRW to a time changed limiting process, [25], [28]. Independent and identical distribution of jumps is a necessary assumption in such papers. The random variables representing the jumps are non i.i.d. when the jumps become controlled. The presence of control in our stochastic processes makes it impossible to use their methods, and we take a different approach.

Limits of controlled scaled CTRWs that we study are non-Markov processes, however enlarging the state space in certain ways enables to present them as components of pair Markov processes. Theory of controlled Markov processes is well studied, see for example [1]. Such an approach enables to view the CTRW in a more general setting and obtain interesting novel results.

1.3 Preliminaries

In this section we introduce the key definitions for describing and understanding the stochastic model which is central in our research. We also discuss some properties of the defined mathematical objects and cite important references.

Definition 1. *Let (γ_1, ξ_1), (γ_2, ξ_2), ... be a sequence of independent and identically distributed (i.i.d.) pairs of independent random variables (r.v.'s) such that the distribution of each γ_i, $i \in \mathbb{N}$, is given by the probability measure $\nu(dr)$, and the distribution of jump sizes $\xi_i, i \in \mathbb{N}$, is given by the probability measure $\mu(d\xi)$. Let $X(n) = \sum_{i=1}^{n} \gamma_i$ and $Y(n) = \sum_{i=1}^{n} \xi_i$. Define* *the renewal process* $M(t)$*:*

$$\sum_{i=1}^{M(t)} \gamma_i \leq t < \sum_{i=1}^{M(t)+1} \gamma_i, \text{ for } t > 0. \tag{1.6}$$

Note that for $t > 0$*,* $M(t) = \inf_n\{n : X(n) > t\}$ *and* $M(0) = 0$*. We will refer to* $M(t)$ *as an inverse time process. A CTRW* $\eta(t)$ *with jumps* $\xi_i \in \mathbb{R}^d$ *and waiting times* $\gamma_i \in \mathbb{R}_+$ *is defined as*

$$\eta(t) = Y(M(t)) = \sum_{i=1}^{M(t)} \xi_i. \tag{1.7}$$

Definition 2. *A random variable* X *is said to have a stable distribution if for any positive numbers* A *and* B*, there is a positive number* C *and a real number* D *such that*

$$AX_1 + BX_2 =_d CX + D \tag{1.8}$$

where X_1 *and* X_2 *are independent copies of* X*, and where "* $=_d$ *" denotes equality in distribution, [51].*

If a distribution is *stable* then there exists an $\alpha \in (0, 2]$, called the index of stability, such that the following holds for the corresponding characteristic function ϕ:

$$\log \phi_\alpha(y) = i(b, y) + \int_0^\infty \int_{S^{d-1}} \left(e^{i(y,\xi)} - 1 - \frac{i(y, \xi)}{1 + \xi^2} \right) \frac{d|\xi|}{|\xi|^{1+\alpha}} \mu(ds) \tag{1.9}$$

where μ is some finite measure in S^{d-1}, see theorem 1.4.3 in [13] and for the proof see [51]. In case $\alpha = 2$ the distribution is *normal.*

Stable distributions depend on four parameters: $\alpha \in (0, 2], \sigma \geq 0, b \in [-1, 1], m \in \mathbb{R}^d$. These are the stability, the scale, the skewness and the shift parameters respectively. Hence, the notation for a stable distribution is often of the form $S_\alpha(\sigma, b, m)$. We will use this notation where necessary. A distribution S is symmetric α-stable if $b = 0$ and $m = \bar{0}$. For $\alpha \neq 1$ a distribution $S_\alpha(\sigma, b, m)$ is strictly stable iff $b = 0$. In case $\alpha = 1$, $S_1(\sigma, b, m)$ is strictly stable iff $b = 0$. The case $\alpha = 2, \sigma = 1, b = 0, m = 0$ corresponds to the standard Normal distribution, [51].

In our research, when referring to the *domain of attraction (DOA) of a stable law*, strictly speaking we will refer to the *domain of normal attraction of a stable law.* This is done for simplicity of notation. The precise definition of DOA is presented just below and is as in [52].

Definition 3. *Let ξ_k, $k \geq 1$, be i.i.d. random variables with the (cumulative) distribution function $F(x)$. We say that $F(x)$ belongs to the domain of normal attraction of the distribution function $V(x)$, if for some $a > 0$ and some A_n the following relation holds:*

$$\lim_{n\to\infty} P\left(\frac{1}{an^{1/\alpha}}\left(\sum_{k=1}^{n}\xi_k\right) - A_n \leq x\right) = V(x) \tag{1.10}$$

for every $x \in \mathbb{R}^d$. Here α is the characteristic exponent (stability parameter/index) of the law $V(x)$.

We say that $f \sim g$ if

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1. \tag{1.11}$$

Let us assume that for $i \in \mathbb{N}$, the probability distributions of r.v.'s γ_i and ξ_i introduced in definition 1 belong to the domains of attraction (DOA) of *stable* distributions $S_\beta(\sigma, 1, 0)$ and $S_\alpha(\sigma, 0, 0)$ respectively, with $\beta \in (0, 1)$ and $\alpha \in (1, 2]$. More precisely, assume that for $n \to \infty$

$$\int_{|r|>n} \nu(dr) \sim \frac{1}{\Gamma[1-\beta]n^\beta}, \tag{1.12}$$

for $\beta \in (0, 1)$ and for every Borel $\Omega \in S^{d-1}$

$$\int_{|y|>n} 1_{(y/|y|\in\Omega)}\mu(dy) \sim \frac{1}{\Gamma(1-\alpha)n^\alpha}\int_\Omega d\sigma(\xi), \tag{1.13}$$

where $\alpha \in (1, 2)$, and σ is a centrally symmetric finite Borel measure on the sphere S^{d-1}.

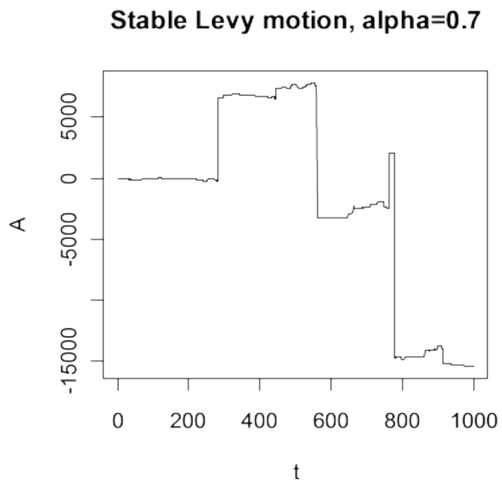
Definition 4. *An α-stable Lévy motion $E(t)$, $0 < \alpha \leq 2$, is a process $\{X(t), t \in \mathbb{R}\}$ with stationary independent increments $E(t) - E(s)$ having a strictly α-stable distribution $S_\alpha((t-s)^{1/\alpha}, b, 0)$, for any $0 \leq s < t < \infty$, where $b \in [-1, 1]$, [51].*

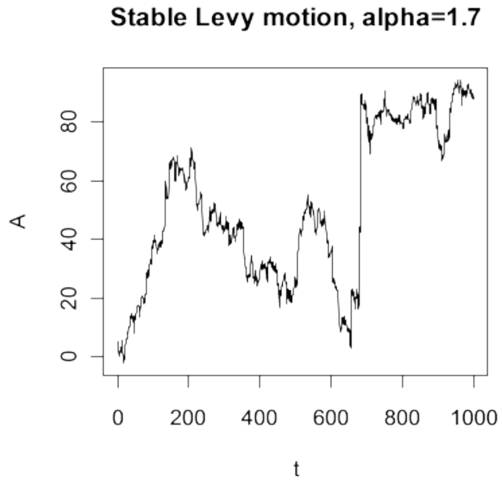
For $\zeta_i \in DOA(Z)$, where Z is stable with the stability index $\alpha \in (1, 2]$, as $n \to \infty$

$$\left\{ n^{-1/\alpha} \sum_{i=1}^{\lfloor nt \rfloor} \zeta_i - A_n \right\}_{t \geq 0} \Rightarrow \{D(t)\}_{t \geq 0}, \tag{1.14}$$

in distribution, for some A_n, where $D(t)$ is a stable Lévy motion with stability parameter $\alpha \in (1, 2]$, [5], [53]. A stable Lévy motion is uniquely described by its infinitesimal generator, which is of a fractional derivative form, [54], [13], which we will discuss after presenting more details about stable laws and DOAs. Stable Lévy motion is used for modelling anomalous diffusion, because of the power law probability tails for the jumps ζ_i, see [53] and references therein.

Below are two graphs of stable Lévy motions with stability parameters $\alpha = 0.7$ and $\alpha = 1.7$. When $\alpha = 2$ the stable Lévy motion is called the Brownian motion. The graphs here show that as α increases, the number of very large jumps decreases and the motion resembles the Brownian motion more, which supports the theory.





Several useful theorems giving necessary and sufficient conditions for a probability distribution to belong to the normal domain of attraction of a stable law with the stability index in the interval $(0, 2]$ are presented, for example, in the books [52] and [51]. In the book [55] there are descriptions of models with random variables in DOA of stable laws. The book [55] also provides a detailed overview of analytic properties of stable laws. Other manuscripts that present necessary and sufficient conditions of random variables belonging to DOA of a stably distributed r.v. include [56] and references therein.

There are only three classes of stable distributions with known explicit expressions for the probability density functions (pdf's). Denote by μ and σ the mean and the variance for the random variable with a specified distribution. Here are the three stable distributions with known pdf's, [51].

- The Gaussian distribution $S_2(\sigma, 0, \mu)$, corresponding to $\alpha = 2$, with the pdf

$$f(x) = (2\sigma\sqrt{\pi})^{-1} e^{-(x-\mu)^2/4\sigma^2}; \tag{1.15}$$

- The Lévy distribution $S_{1/2}(\sigma, 1, \mu)$, corresponding to $\alpha = 1/2$, whose pdf is

$$f(x) = \left(\frac{\sigma}{2\pi}\right)^{1/2} \frac{1}{(x-\mu)^{3/2}} \exp\left\{\frac{-\sigma}{2(x-\mu)}\right\}; \tag{1.16}$$

- The Cauchy distribution $S_1(\sigma, 0, \mu)$, corresponding to $\alpha = 1$, whose pdf is

$$f(x) = \frac{\sigma}{\pi((x-\mu)^2 + \sigma^2)}. \tag{1.17}$$

Despite the lack of stable pdf expressions, for any stability parameter $\alpha \in (0, 2]$ there is an analytic expression for the characteristic function, [15]. In this thesis we concentrate on the case $\alpha \in (1, 2)$. The case $\alpha = 2$ is simpler and is omitted here, apart from the analysis results Chapter.

Now, relating back to (1.14) which relates attracted distributions with fractional operators, we present several definitions and properties of fractional operators. Let us denote the space of functions $f(t, y)$ which are differentiable in t and twice differentiable in y by $C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^d)$. Denote by A^β the infinitesimal generator of a β-stable Lévy motion $\tilde{X}$ with $\beta \in (0, 1)$:

$$A^\beta f(t, y) = -\frac{1}{\Gamma[-\beta]} \int_0^\infty (f(t + r, y) - f(t, y)) \frac{dr}{r^{1+\beta}}, \tag{1.18}$$

for $f \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^d)$ regular enough and with $f(t, y) = 0$ for $t < 0$, $y \in \mathbb{R}$. The dual operator for A is the fractional Caputo derivative $D_{0,t}^{*\beta}$ defined below, [13], [57]. Here *dual* is understood in the sense that for all f, g in the domain of A, $(f, A^* g) = (Af, g)$, where $(\cdot, \cdot)$ is the dual pairing. Let us write $C^1([0, \infty))$ for the space of functions which are continuous and with a continuous first derivative.

Definition 5. *The left-sided Caputo fractional derivative for $\beta \in (0, 1)$ is defined for $f \in C^1([0, \infty))$ regular enough by*

$$D_{0+,x}^{*\beta} f(x) = \frac{1}{\Gamma[1-\beta]} \int_0^x \frac{df(y)}{dy} (x - y)^{-\beta} dy. \tag{1.19}$$

The fractional derivative in (1.19) is named after M. Caputo who used this operator in 1967.

Definition 6. *The fractional Laplacian* $-(-\Delta)^{\alpha/2}$ *with* $\alpha \in (0,2]$ *is defined for* f *regular enough as*

$$-(-\Delta)^{\alpha/2} f(t,y) = p.v.c_{d,s} \int_{\mathbb{R}^d} \frac{f(t,y) - f(t,\xi)}{|y-\xi|^{d+\alpha}} d\xi, \tag{1.20}$$

where p.v. stands for "principal value" and $c_{d,s}$ *is a normalising constant, which ensures that the following Fourier transform in space relation holds*

$$F((-\Delta)^{\alpha/2} f)(t,p) = |p|^{\alpha} F(f)(t,p), \tag{1.21}$$

see [13], [51].

The symmetric α-stable Lévy motion has the fractional Laplacian $-(-\Delta)^{\alpha/2}$ as its infinitesimal generator [13].

Definition 7. *The left-sided Riemann-Liouville fractional derivative for* $\beta \in (0,1)$ *is defined for* $f \in C^1([0,\infty))$ *regular enough as follows:*

$$D^{\beta}_{0+,x} f(x) = \frac{1}{\Gamma[1-\beta]} \frac{d}{dx} \int_0^x f(y)(x-y)^{-\beta} dy. \tag{1.22}$$

Both the Riemann-Liouville and the Caputo fractional derivatives are special cases of the more general Dzhrbashyan-Nersesyan fractional derivative, which first appeared in publications in 1968, see [58].

Definition 8. *The antiderivative for* $D^{\beta}_{0+,x}$ *is the Riemann-Liouville integral*

$$I^{\beta}_{0,x} f(x) = \frac{1}{\Gamma(\beta)} \int_0^x f(t)(x-t)^{\beta-1} dt. \tag{1.23}$$

The following relation between $D^{\beta}_{0+,x}$ and $I^{1-\beta}_{0+,x}$ holds for $\beta \in (0,1)$:

$$D^{\beta}_{0+,x} f(x) = \frac{d}{dx} I^{1-\beta}_{0+,x} f(x). \tag{1.24}$$

We shall only deal with left-sided fractional Caputo and Riemann-Liouville derivatives; so we drop the terminology *left-sided* from now on.

Definition 9. *The generator form of the Caputo fractional differential operator for every f from the Schwartz space $S(\mathbb{R})$ is defined as:*

$$\frac{d^\beta f(x)}{dx^\beta} = \frac{1}{\Gamma[-\beta]} \int_0^\infty (f(x-y) - f(x)) \frac{dy}{y^{1+\beta}}, \qquad \beta \in (0,1). \tag{1.25}$$

In case $f(x) = 0$ for $x < 0$, the generator form is equivalent to the Caputo, but not to the Riemann-Liouville one. This is discussed in more detail for example in [5]. The Riemann-Liouville (1.22) and the Caputo derivatives (1.19) are related by the formula [57]:

$$D_{0,t}^{\beta} f(t) = D_{0,t}^{*\beta} f(t) + \frac{t^{-\beta} f(0)}{\Gamma[1-\beta]}. \tag{1.26}$$

The physical interpretations of fractional Caputo and Riemann-Liouville derivatives may be found in [36] and in [59].

Chapter 2

Control and Scaling of the Continuous Time Random Walk

2.1 Controlled CTRW and the payoff function.

Let us assume that an agent in charge of the stochastic system described in section 1.3 can control the distribution of jumps $\xi_i, i \in \mathbb{N}$. Let us denote the set of all possible controls during any jump by U. It is assumed to be a closed convex subset in $\mathbb{R}^d$. By $\tilde{U}$ we denote the set of sequences:

$$\tilde{U} = \{\tilde{u} = (u_1, u_2, \ldots)\}, u_i \in U, i \in \mathbb{N}. \tag{2.1}$$

Since ξ_i become dependent on control, we shall write $\xi_i(u_i)$. Control brings about dependence of the probability measure μ on $u \in U$, hence the notation $\mu_u(d\xi)$ appearing later. Now

$$Y(M(t), \tilde{u}) = \sum_{i=1}^{M(t)} \xi_i(u_i) \tag{2.2}$$

is a *controlled CTRW*. For simplicity of notation we write

$$\eta(t, \tilde{u}) = Y(M(t), \tilde{u}). \tag{2.3}$$

We assume that the effect the control has on the jumps may be represented as follows

$$\int_{|y|>n} 1_{(y/|y|\in\Omega)}\mu(dy) \sim \frac{\omega(u)}{\Gamma(1-\alpha)n^{\alpha}} \int_{\Omega} d\sigma(\xi), \tag{2.4}$$

with some continuous and bounded from above and from below function $\omega(u)$.

We are not going to control the distribution of waiting times, so $M(t)$ remains independent of $u \in U$. Let us study the process $Y_u(M(t))$ only at times when jumps take place. This is an embedded Markov process. Let the terminal payoff $S(t=0,y)$ at any $y \in \mathbb{R}^d$ be given by a known bounded function $S(t=0,y) = S_0(y)$.

Assume that there is a cost for jumping as well as for arriving at a new state. Let us represent the running cost due to jumping by the function $f(u,y,\xi)$. Assume also that there is payoff due to waiting which is $g(u,y)$ per time unit. Then the optimal payoff function (the value function) $S(t,y)$ for the process $Y_u(M(t))$ considered at jump times only is defined as follows:

$$S(t,y) = \inf_{\tilde{u}\in\tilde{U}} \mathbb{E}\left[S_0(y+\eta(t,\tilde{u}))+\sum_{i=1}^{M(t)} \gamma_i g(u_i, y+Y(i-1))+\sum_{i=1}^{M(t)} f(u_i, y+Y(i-1), \xi_i(u_i))\right] \tag{2.5}$$

for $t \geq 0$. It is convenient and physically appropriate to assume that for all $t < 0$, $S(t,y) = 0$. The object defined in (2.5) represents the minimal cost that the agent pays when the time until the end of the process is $t \geq 0$ and the agent is at position $y \in \mathbb{R}^d$. Due to the law of total expectation

$$\begin{aligned} &\inf_{\tilde{u}\in\tilde{U}} \mathbb{E}S_0(y+\eta(t,\tilde{u})) \\ &= \inf_{\tilde{u}\in\tilde{U}} \Big(\mathbb{E}(S(t,y)\big|\text{no jump})P(\text{no jump}) \\ &+\mathbb{E}(S(t,y)\big|\text{jump occurred})P(\text{jump occured})\Big). \end{aligned}$$

Given that a jump ξ takes place at a time $r \in [0,t]$, the conditional expectation of the payoff S at time $t-r \in \mathbb{R}_+$ is given by the integral $\int_{\mathbb{R}^d} S(t-r, y+\xi)\mu_u(d\xi)$ where $0 < r < t$,

$r = t - \gamma$. Consequently, $S(t, y)$ satisfies the following dynamic programming equation:

$$S(t,y) = \inf_{u\in U}\left[S_0(y)P[\gamma > t] + \int_0^t \mathbb{E}S(t-r, y+\xi(u)|\gamma = r)\nu(dr)\right]$$

$$= \inf_{u\in U}\left[S_0(y)\int_t^\infty \nu(dr) + \int_0^t\int_{\mathbb{R}^d} S(t-r, y+\xi)\mu_u(d\xi)\nu(dr)\right.$$

$$\left. + \int_0^t\int_{\mathbb{R}^d} g(u,y)r\mu_u(d\xi)\nu(dr) + t\int_t^\infty g(u,y)\nu(dr) + \int_0^t\int_{\mathbb{R}^d} f(u,y,\xi)\mu_u(d\xi)\nu(dr)\right]. \quad (2.6)$$

2.2 Scaling.

Let us make the following scaling which is standard in CTRW theory: $\gamma_i \mapsto \gamma_i\tau^{1/\beta}$ and $\xi_i \mapsto \xi_i\tau^{1/\alpha}$, $i \in \mathbb{N}$, where $\tau > 0$, see [25], [19]. Denote by X^τ the scaled X:

$$X^\tau(n) = \sum_{i=1}^n \tau^{1/\beta}\gamma_i = X^\tau(n-1) + \tau^{1/\beta}\gamma_n. \quad (2.7)$$

Also, by Y^τ denote the scaled Y:

$$Y^\tau(n) = \sum_{i=1}^n \tau^{1/\alpha}\xi_i(u_i) = Y^\tau(n-1) + \tau^{1/\alpha}\xi_n(u_n). \quad (2.8)$$

Define $M^\tau(t) = M(t\tau^{-1/\beta})$ similarly to (1.6) by

$$\tau^{1/\beta}\sum_{i=1}^{M^\tau(t)} \gamma_i \le t < \tau^{1/\beta}\sum_{i=1}^{M^\tau(t)+1} \gamma_i. \quad (2.9)$$

And similarly to (1.7)

$$Y^\tau(M^\tau(t), \tilde{u}) = \sum_{i=1}^{M^\tau(t)} \tau^{1/\alpha}\xi_i(u_i) \quad (2.10)$$

with n replaced by $M^\tau(t)$.

Analogously to (2.5) we will denote the optimal payoff at time t, where $t \ge 0$, and at position $y \in \mathbb{R}^d$ by $S^\tau(t, y)$:

$$S^\tau(t,y) = \inf_{\tilde{u}\in\tilde{U}} \mathbb{E}\left[S_0(y+\eta^\tau(t,\tilde{u})) + \sum_{i=1}^{M^\tau(t)} \gamma_i g(u_i, y+Y(i-1)) + \sum_{i=1}^{M^\tau(t)} f(u_i, y+Y(i-1), \xi_i(u_i))\right].$$

The function $S(t,y)$ defined in (2.5) is $S^1(t,y)$. Now the scaled analogue of the dynamic programming equation (2.6) for $S^\tau(t,y)$ becomes

$$S^\tau(t,y) = \inf_{u\in U}\left[S_0(y)\int_t^\infty \nu(dr/\tau^{1/\beta})\right.$$

$$+\int_0^t\int_{\mathbb{R}^d}\left[S^\tau(t-r,y+\xi)+f(u,y,\xi)\right]\mu_u(d\xi/\tau^{1/\alpha})\nu(dr/\tau^{1/\beta})$$

$$\left.+\,t\int_t^\infty g(u,y)\nu(dr/\tau^{1/\beta})+\int_0^t\int_{\mathbb{R}^d} g(u,y)r\mu_u(d\xi/\tau^{1/\alpha})\nu(dr/\tau^{1/\beta})\right]. \tag{2.11}$$

The scaled measures $\mu(d\xi/\tau^{1/\alpha})$ and $\nu(dr/\tau^{1/\beta})$ can be defined by their integrals with continuous functions

$$\int_{\mathbb{R}^d} h_1(\xi)\mu(d\xi/\tau^{1/\alpha}) = \int_{\mathbb{R}^d} h_1(\tau^{1/\alpha}\xi)\mu(d\xi), \tag{2.12}$$

$$\int_0^\infty h_2(r)\nu(dr/\tau^{1/\beta}) = \int_0^\infty h_2(\tau^{1/\beta}r)\nu(dr), \tag{2.13}$$

for any continuous functions h_1 and h_2 on $\mathbb{R}^d$ and $\mathbb{R}_+$ respectively.

2.3 Heuristic results.

Assume further that as $\tau\to 0$, for $t\geq 0$ and $y\in\mathbb{R}^d$:

$$S^\tau(t,y)\to\tilde{S}(t,y) \tag{2.14}$$

in an appropriate sense, the limit $\tilde{S}(t,y)$ belongs to $C^{1,2}(\mathbb{R}_+\times\mathbb{R}^d)$ and decays at infinity in the spatial variable.

For $f\in C^{1,2}(\mathbb{R}_+\times\mathbb{R}^d)$ which decays at infinity in the spatial variable, let L_u^α be defined as

$$L_u^\alpha f(t,y) = p.v.c_{d,\alpha}\int_{\mathbb{R}^d}\frac{f(t,y)-f(t,\xi)}{|y-\xi|^{d+\alpha}}w(u)d\xi, \tag{2.15}$$

and let $A^{*\beta}$ be the dual for the generator A^β defined in (1.18). Here duality should be understood in the sense that $(f,A^{*\beta}g)=(A^\beta f,g)$ for any f,g in the domain of A^β and $(\cdot,\cdot)$ is the dual pairing. Note that A^β and $A^{*\beta}$ are defined for $f\in C^{1,2}(\mathbb{R}_+\times\mathbb{R}^d)$ and regular enough. Assume that for any $\rho \in \mathbb{R}_+$

$$f(u, y, \rho\xi) = \rho^\alpha f(u, y, \xi), \tag{2.16}$$

i.e., f is homogeneous of order α. We will use this assumption in the case $\rho = \tau^{1/\alpha}$. Let us show that under the above assumptions the function $\tilde{S}(t, y)$ satisfies an equation of the form

$$\frac{t^{-\beta}}{\Gamma[1-\beta]} S_0(y) + A_t^{*\beta} \tilde{S}(t, y)$$
$$+ \inf_{u \in U} \left[L_u^\alpha \tilde{S}(t, y) + F(u, y) + 2g(u, y) \left(\frac{t^{1-\beta}}{\Gamma[1-\beta]} \right) \right] = 0. \tag{2.17}$$

Remark 1. *In the case without control this equation appears in different forms for example in [25], [39], [60].*

We write the dynamic programming equation (2.11) in the form

$$S^\tau(t, y) = \inf_{u \in U} \left[S_0(y) \int_{t\tau^{-1/\beta}}^\infty \nu(dr) + \int_0^{t\tau^{-1/\beta}} \int_{\mathbb{R}^d} f(u, y, \xi\tau^{1/\alpha}) \mu_u(d\xi) \nu(dr) \right.$$
$$+ \int_0^{t\tau^{-1/\beta}} \int_{\mathbb{R}^d} \left[S^\tau(t - r\tau^{1/\beta}, y + \xi\tau^{1/\alpha}) \right] \mu_u(d\xi) \nu(dr)$$
$$\left. + t \int_{t\tau^{-1/\beta}}^\infty g(u, y) \nu(dr) + \int_0^{t\tau^{-1/\beta}} \int_{\mathbb{R}^d} g(u, y) r\tau^{1/\beta} \mu_u(d\xi) \nu(dr) \right] \tag{2.18}$$

or equivalently

$$S^\tau(t, y) = \inf_{u \in U} \left[S_0(y) \int_{t\tau^{-1/\beta}}^\infty \nu(dr) \right.$$
$$+ \tau \int_0^{t\tau^{-1/\beta}} \int_{\mathbb{R}^d} \left[\frac{S^\tau(t - r\tau^{1/\beta}, y + \xi\tau^{1/\alpha}) - S^\tau(t - r\tau^{1/\beta}, y)}{\tau} \right] \mu_u(d\xi) \nu(dr)$$
$$+ \tau \int_0^{t\tau^{-1/\beta}} \int_{\mathbb{R}^d} \left[\frac{S^\tau(t - r\tau^{1/\beta}, y) - S^\tau(t, y)}{\tau} \right] \mu_u(d\xi) \nu(dr)$$
$$+ \int_0^{t\tau^{-1/\beta}} \int_{\mathbb{R}^d} S^\tau(t, y) \mu_u(d\xi) \nu(dr) + t \int_{t\tau^{-1/\beta}}^\infty g(u, y) \nu(dr)$$
$$\left. + \int_0^{t\tau^{-1/\beta}} \int_{\mathbb{R}^d} g(u, y) r\tau^{1/\beta} \mu_u(d\xi) \nu(dr) + \int_0^{t\tau^{-1/\beta}} \int_{\mathbb{R}^d} f(u, y, \xi\tau^{1/\alpha}) \mu_u(d\xi) \nu(dr) \right] \tag{2.19}$$

Let us represent one of the summands as follows:

$$\int_0^{t\tau^{-1/\beta}} \int_{\mathbb{R}^d} \left[\frac{S^\tau(t - r\tau^{1/\beta}, y) - S^\tau(t, y)}{\tau} \right] \mu_u(d\xi)\nu(dr)$$

$$= \int_0^{\infty} \int_{\mathbb{R}^d} \left[\frac{S^\tau(t - r\tau^{1/\beta}, y) - S^\tau(t, y)}{\tau} \right] \mu_u(d\xi)\nu(dr)$$

$$- \int_{t\tau^{-1/\beta}}^{\infty} \int_{\mathbb{R}^d} \left[\frac{S^\tau(t - r\tau^{1/\beta}, y) - S^\tau(t, y)}{\tau} \right] \mu_u(d\xi)\nu(dr). \tag{2.20}$$

Note that

$$\int_{t\tau^{-1/\beta}}^{\infty} \int_{\mathbb{R}^d} S^\tau(t - r\tau^{1/\beta}, y)\mu_u(d\xi)\nu(dr) = 0 \tag{2.21}$$

because $t - r\tau^{1/\beta} \in (-\infty, t - t\tau^{-1/\beta}]$, and $S^\tau(s, y) = 0$ for $s < 0$. Next, by the well-known property of $A^{*\beta}$

$$\int_0^{\infty} \left[\frac{h(t - r\tau^{1/\beta}) - h(t)}{\tau} \right] \nu(dr) \to A^*_\beta h(t), \tag{2.22}$$

for $h \in C^1(\mathbb{R}_+)$. Hence under appropriate assumptions on the convergence of S^τ to $\tilde{S}$ as $\tau \to 0$

$$\int_0^{\infty} \left[\frac{S^\tau(t - r\tau^{1/\beta}, y) - S^\tau(t, y)}{\tau} \right] \nu(dr) \to A^*_\beta \tilde{S}(t, y). \tag{2.23}$$

This is the main part that makes our deduction heuristic. Let us also re-write

$$\int_0^{t\tau^{-1/\beta}} \int_{\mathbb{R}^d} S^\tau(t, y)\mu_u(d\xi)\nu(dr)$$

$$= \int_0^{\infty} \int_{\mathbb{R}^d} S^\tau(t, y)\mu_u(d\xi)\nu(dr) - \int_{t\tau^{-1/\beta}}^{\infty} \int_{\mathbb{R}^d} S^\tau(t, y)\mu_u(d\xi)\nu(dr). \tag{2.24}$$

The first term in (2.24) cancels out with $S^\tau(t, y)$ on the left hand side (LHS) and the second term in (2.24) cancels out with the last term in (2.20). Also we write

$$\int_0^{t\tau^{-1/\beta}} \int_{\mathbb{R}^d} \left[\frac{S^\tau(t - r\tau^{1/\beta}, y + \xi\tau^{1/\alpha}) - S^\tau(t - r\tau^{1/\beta}, y)}{\tau} \right] \mu_u(d\xi)\nu(dr)$$

$$= \int_0^{\infty} \int_{\mathbb{R}^d} \left[\frac{S^\tau(t - r\tau^{1/\beta}, y + \xi\tau^{1/\alpha}) - S^\tau(t - r\tau^{1/\beta}, y)}{\tau} \right] \mu_u(d\xi)\nu(dr)$$

$$-\int_{t\tau^{-1/\beta}}^{\infty}\int_{\mathbb{R}^d}\left[\frac{S^\tau(t-r\tau^{1/\beta},y+\xi\tau^{1/\alpha})-S^\tau(t-r\tau^{1/\beta},y)}{\tau}\right]\mu_u(d\xi)\nu(dr). \tag{2.25}$$

And similarly to (2.23) it follows that

$$\int_0^{\infty}\int_{\mathbb{R}^d}\left[\frac{S^\tau(t-r\tau^{1/\beta},y+\xi\tau^{1/\alpha})-S^\tau(t-r\tau^{1/\beta},y)}{\tau}\right]\mu_u(d\xi)\nu(dr)\to L_\alpha\tilde{S}(t,y), \tag{2.26}$$

Meanwhile

$$\tau\int_{t\tau^{-1/\beta}}^{\infty}\int_{\mathbb{R}^d}\left[\frac{S^\tau(t-r\tau^{1/\beta},y+\xi\tau^{1/\alpha})-S^\tau(t-r\tau^{1/\beta},y)}{\tau}\right]\mu_u(d\xi)\nu(dr)=0, \tag{2.27}$$

since we assume that $S^\tau(t,y)=0$ for $t<0$. For convenience, denote

$$F(u,y)=\int_{\mathbb{R}^d}f(u,y,\xi)\mu_u(d\xi). \tag{2.28}$$

Now using (2.16) and the notation above we write

$$\lim_{\tau\to 0}\frac{1}{\tau}\int_0^{t\tau^{-1/\beta}}\int_{\mathbb{R}^d}f(u,y,\tau^{1/\alpha}\xi)\mu(d\xi)\nu(dr)$$

$$=\lim_{\tau\to 0}\int_0^{t\tau^{-1/\beta}}\int_{\mathbb{R}^d}f(u,y,\xi)\mu(d\xi)\nu(dr)$$

$$=\lim_{\tau\to 0}F(u,y)\left(1-\frac{(t\tau^{-1/\beta})^{-\beta}}{\Gamma[1-\beta]}\right)=F(u,y). \tag{2.29}$$

Terms with g yield in the limit:

$$\lim_{\tau\to 0}\frac{1}{\tau}t\int_t^{\infty}g(u,y)\nu(dr/\tau^{1/\beta})=\lim_{\tau\to 0}\frac{1}{\tau}t\int_{t\tau^{-1/\beta}}^{\infty}g(u,y)\nu(dr)$$

$$=\lim_{\tau\to 0}\frac{t}{\tau}g(u,y)\int_{t\tau^{-1/\beta}}^{\infty}\nu(dr)=\frac{t^{1-\beta}}{\Gamma[1-\beta]}g(u,y), \tag{2.30}$$

and

$$\frac{1}{\tau}\int_0^t\int_{\mathbb{R}^d}g(u,y)r\mu_u(d\xi/\tau^{1/\alpha})\nu(dr/\tau^{1/\beta})$$

$$=\frac{1}{\tau}\int_0^{t\tau^{-1/\beta}}g(u,y)\tau^{1/\beta}r\mu_u(d\xi)\nu(dr)=\frac{t^{1-\beta}}{\Gamma[1-\beta]}g(u,y). \tag{2.31}$$

Here in (2.31) we have used that in the limit $n \to \infty$ the measure $\nu(dr)$ behaves as $\frac{r^{-\beta-1}}{\Gamma[1-\beta]}dr$, since $\int_{|r|>n}^{\infty} \nu(dr)$ behaves as $\frac{n^{-\beta}}{\Gamma[1-\beta]}$ in the limit $n \to \infty$. Here we let $n = t\tau^{-1/\beta}$ and take $\tau \to 0$. Together (2.18), (2.23), (2.26), (2.29), (2.30) and (2.31) yield the theorem result (2.17).

Now we describe a modification of the scaled CTRW process. We present related optimal payoff function equation for the embedded Markov processes and present the equation for $\tilde{S}(t,y)$ in cases with and without running costs due to jumping and waiting. Let us keep all the previous assumptions about the CTRW and also assume there is a stochastic motion during waiting times. This motion may be an arbitrary Feller process $K_t^u(y)$. We assume that its generator is B^u and the control is of Markov type. We will refer to the additional deterministic part of the CTRW as the *inner motion*. In [61] such a process with inner motion is referred to as *a process with a semi-Markov chance interference*. In case without running costs our dynamic programming equation for the optimal payoff function obtains the following form:

$$S^\tau(t,y) = \inf_{u\in U} \Big[(\mathbb{E}[S^\tau(K_t^u(y))\, P(\tau > t)]$$
$$+ \int_0^t \int_{\mathbb{R}^d} \mathbb{E}\,(S^\tau(t-r, K_t^u(y)+\xi)\,|\gamma = r)\mu_u(d\xi/\tau^{1/\alpha})\nu(dr/\tau^{1/\beta})\Big]. \tag{2.32}$$

We scale the waiting times for this inner motion by $\tau^{1/\beta}$, and use the following convenient notation

$$\mathbb{E}[S^\tau(K_r^u(y)+\xi, t-r)] = \int_{\mathbb{R}^d} S^\tau(z+\xi, t-r)P_r(y,dz). \tag{2.33}$$

So after the scaling (2.32) may be written as:

$$S^\tau(t,y) = \inf_{u\in U}\Bigg[\int_t^\infty \int_{\mathbb{R}^d} S^\tau(z)P_t(y,dz)\nu(dr/\tau^{1/\beta})$$
$$+\int_0^t \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} S^\tau(z+\xi,t-r)P_r(y,dz)\mu_u(d\xi/\tau^{1/\alpha})\nu(dr/\tau^{1/\beta})\Bigg]$$
$$= \inf_{u\in U}\Bigg[\int_{t\tau^{-1/\beta}}^\infty \int_{\mathbb{R}^d} S^\tau(z)P_{t\tau^{1/\beta}}(y,dz)\nu(dr)$$
$$+\int_0^{t\tau^{-1/\beta}} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} S^\tau(z+\xi\tau^{1/\alpha},t-r\tau^{1/\beta})P_{r\tau^{1/\beta}}(y,dz)\mu_u(d\xi)\nu(dr)\Bigg]$$
$$= \inf_{u\in U}\Bigg[\int_{t\tau^{-1/\beta}}^\infty \int_{\mathbb{R}^d} S^\tau(z)P_{t\tau^{1/\beta}}(y,dz)\nu(dr)$$
$$+\int_0^{t\tau^{-1/\beta}} \tau\int_{\mathbb{R}^d} \frac{\int_{\mathbb{R}^d} S^\tau(z+\xi\tau^{1/\alpha},t-r\tau^{1/\beta})P_{r\tau^{1/\beta}}(y,dz) - S^\tau(y,t-r\tau^{1/\beta})}{\tau}\mu_u(d\xi)\nu(dr)$$
$$+\int_0^{t\tau^{-1/\beta}} \left(\tau\frac{S^\tau(y,t-r\tau^{1/\beta})-S^\tau(t,y)}{\tau}+S^\tau(t,y)\right)\nu(dr)\Bigg]. \tag{2.34}$$

Hence,

$$\frac{1}{\tau}\Bigg[\int_{t\tau^{-1/\beta}}^\infty S^\tau(t,y)\nu(dr) - \int_{t\tau^{-1/\beta}}^\infty \int_{\mathbb{R}^d} S^\tau(z)P_{t\tau^{1/\beta}}(y,dz)\nu(dr)\Bigg]$$
$$= \inf_{u\in U}\Bigg[\int_0^{t\tau^{-1/\beta}} \frac{S^\tau(t-r\tau^{1/\beta},y)-S^\tau(t,y)}{\tau}\nu(dr)$$
$$+\int_0^{t\tau^{-1/\beta}} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{S^\tau(t-r\tau^{1/\beta},z+\xi\tau^{1/\alpha})-S^\tau(t-r\tau^{1/\beta},z)}{\tau}P_{r\tau^{1/\beta}}(y,dz)\mu_u(d\xi)\nu(dr)$$
$$+\int_0^{t\tau^{-1/\beta}} \int_{\mathbb{R}^d} \frac{S^\tau(t-r\tau^{1/\beta},z)-S^\tau(t-r\tau^{1/\beta},y)}{\tau}P_{r\tau^{1/\beta}}(y,dz)\nu(dr)\Bigg]. \tag{2.35}$$

The last summand can be represented in the form:

$$\int_0^{t\tau^{-1/\beta}} \int_{\mathbb{R}^d} \frac{S^\tau(t-r\tau^{1/\beta},z)-S^\tau(t-r\tau^{1/\beta},y)}{\tau}P_{r\tau^{1/\beta}}(y,dz)\nu(dr)$$
$$= \int_0^\infty \int_{\mathbb{R}^d} \frac{S^\tau(t-r\tau^{1/\beta},z)-S^\tau(t-r\tau^{1/\beta},y)}{\tau}P_{r\tau^{1/\beta}}(y,dz)$$
$$-\int_{t\tau^{-1/\beta}}^\infty \int_{\mathbb{R}^d} \frac{S^\tau(t-r\tau^{1/\beta},z)-S^\tau(t-r\tau^{1/\beta},y)}{\tau}P_{r\tau^{1/\beta}}(y,dz)\nu(dr). \tag{2.36}$$

As $\tau\to 0$, due to the boundary condition the two terms in the second integral are 0's and the first integral satisfies:

$$\int_0^\infty \int_{\mathbb{R}^d} \frac{S^\tau(t-r\tau^{1/\beta},z)-S^\tau(t-r\tau^{1/\beta},y)}{\tau}P_{r\tau^{1/\beta}}(y,dz)\nu(dr) \to B^u\tilde{S}(t,y) \tag{2.37}$$

As for the term

$$\frac{1}{\tau}\int_{t\tau^{-1/\beta}}^{\infty}\int_{\mathbb{R}^d} S^{\tau}(z)P_{t\tau^{1/\beta}}(y,dz)\nu(dr), \tag{2.38}$$

we add and subtract

$$\frac{1}{\tau}\int_{t\tau^{-1/\beta}}^{\infty} S_0(y)\nu(dr) \tag{2.39}$$

and this gives us:

$$\begin{aligned} &\frac{1}{\tau}\int_{t\tau^{-1/\beta}}^{\infty}\int_{\mathbb{R}^d} S^{\tau}(z)P_{t\tau^{1/\beta}}(y,dz)\nu(dr) \\ &= \int_{t\tau^{-1/\beta}}^{\infty} \frac{\left[\int_{\mathbb{R}^d} S^{\tau}(z)P_{t\tau^{1/\beta}}(y,dz) - S_0(y)\right]}{\tau}\nu(dr) \\ &\quad + \frac{1}{\tau}\int_{t\tau^{-1/\beta}}^{\infty} S_0(y)\nu(dr). \end{aligned} \tag{2.40}$$

Now,

$$\begin{aligned} &\int_{t\tau^{-1/\beta}}^{\infty} t\tau^{1/\beta-1}\int_{\mathbb{R}^d}\frac{S^{\tau}(z)-S_0(y)}{t\tau^{1/\beta}}P_{t\tau^{1/\beta}}(y,dz)\nu(dr) \\ &\quad \to \int_{t\tau^{-1/\beta}}^{\infty} t\tau^{1/\beta-1}B^u S_0(y)\nu(dr) \to 0, \end{aligned} \tag{2.41}$$

and

$$\frac{1}{\tau}\int_{t\tau^{-1/\beta}}^{\infty} S_0(y)\nu(dr) \sim \frac{t^{-\beta}}{\Gamma(1-\beta)}S_0(y), \tag{2.42}$$

Assuming that the limit $\tilde{S}(t,y) := \lim_{\tau\to 0} S^{\tau}(t,y)$ exists in an appropriate sense, it follows

$$\begin{aligned} &\frac{1}{\Gamma(1-\beta)}t^{-\beta}S_0(y) \\ &+ \inf_{u\in U}\left[L_{\alpha}^u\tilde{S}(t,z) + B^u\tilde{S}(t,y)\right] + A_{\beta}^*\tilde{S}(t,y) = 0. \end{aligned} \tag{2.43}$$

If we also add the costs for jumping and costs for waiting, then the extra terms with $g_{u,t}(y)$ and $F(u,y)$ appear as in the previous version of the process, yielding the following equation for $\tilde{S}(t,y)$:

$$\frac{1}{\Gamma(1-\beta)}t^{-\beta}S_0(y)+A^*_\beta\tilde{S}(t,y)$$
$$+\inf_{u\in U}\left[L^u_\alpha\tilde{S}(t,z)+B^u\tilde{S}(t,y)+2g_{u,t}(y)t^{1-\beta}+F(u,y)\right]=0. \tag{2.44}$$

2.4 Process description forward in time.

Sometimes it is appropriate to describe the motion time-forwards, rather than time-backwards, i.e., $t>0$ represents time elapsed since the beginning of the walk, rather than until its termination. In particular, this is convenient when jump or waiting time distributions are position dependent. Let us denote by T the time we finish observing the process. Then what used to be denoted by t becomes $T-t$ in the new notation. The terminal payoff is now $S(T,y)=S_T(y)$. Define $M^\tau(T-t)$ by

$$\tau^{1/\beta}\sum_{i=1}^{M^\tau(T-t)}\gamma_i\le T-t<\tau^{1/\beta}\sum_{i=1}^{M^\tau(T-t)+1}\gamma_i. \tag{2.45}$$

Then

$$\eta^\tau(T-t,\tilde{u})=Y^\tau(M^\tau(T-t,\tilde{u}))=\tau^{1/\alpha}\sum_{i=1}^{M^\tau(T-t)}\xi_i(u_i). \tag{2.46}$$

Then the new definition of the optimal payoff $S^\tau(t,y)$ is

$$S^\tau(t,y)=\inf_{\tilde{u}\in\tilde{U}}\mathbb{E}\Bigg[S_T(y+\eta^\tau(T-t,\tilde{u}))+\sum_{i=1}^{M^\tau(T-t)}\gamma_i g(u_i,y+Y(i-1))$$
$$+\sum_{i=1}^{M^\tau(T-t)}f(u,y+Y(i-1),\xi_i(u_i))\Bigg], \tag{2.47}$$

We now assume that $S^\tau(t,y)=0$ for $t>T$, replacing the assumption in the previous process formulation: $S^\tau(t,y)=0$ for $t<0$.

Assume again that $S^\tau(\cdot)\to\tilde{S}(\cdot)$ in the appropriate sense and L^α_u is as before. Let us show that as $\tau\to\infty$ the function $\tilde{S}(t,y)$ should satisfy the equation

$$\frac{(T-t)^{-\beta}S_T(y)}{\Gamma[1-\beta]}+A^\beta\tilde{S}(t,y)$$

$$+\inf_{u\in U}\left[L_u^\alpha \tilde{S}(t,y) + 2g(u,y)\frac{(T-t)^{1-\beta}}{\Gamma[1-\beta]} + F(u,y)\right] = 0. \tag{2.48}$$

Analogously to the previous derivation, we derive from the law of total expectation that $S^\tau(t,y)$ defined in (2.47) satisfies the following:

$$S^\tau(t,y) = \inf_{u\in U}\left[S_T(y)\int_{(T-t)\tau^{-1/\beta}}^{\infty}\nu(dr) + \int_0^{(T-t)\tau^{-1/\beta}}\int_{\mathbb{R}^d} S^\tau(t,y)\mu_u(d\xi)\nu(dr)\right.$$

$$+\tau\int_0^{(T-t)\tau^{-1/\beta}}\int_{\mathbb{R}^d}\frac{S^\tau(t+r\tau^{1/\beta}, y+\xi\tau^{1/\alpha}) - S^\tau(t+r\tau^{1/\beta}, y)}{\tau}\mu_u(d\xi)\nu(dr)$$

$$\left. +\tau\int_0^{(T-t)\tau^{-1/\beta}}\int_{\mathbb{R}^d}\frac{S^\tau(t+r\tau^{1/\beta}, y) - S^\tau(t,y)}{\tau}\mu_u(d\xi)\nu(dr)\right]. \tag{2.49}$$

Next we carry out the same procedures as for the previous derivation and obtain equation (2.4). Note that (2.4) may be written in the form

$$A^\beta \tilde{S}(t,y) + \inf_{u\in U}(L_u^\alpha \tilde{S}(t,y) + h(t,u,y)) = 0 \tag{2.50}$$

or

$$A^\beta \tilde{S}(t,y) + \inf_{u\in U} H\big(\tilde{S}(t,y), t, u, y\big) = 0. \tag{2.51}$$

The next section will concentrate on the scenario where jump sizes depend on the current position of the agent, and for every $i \geq 1$ waiting time γ_i depends on $\sum_{j=1}^{i-1}\gamma_j$. There we use the forwards-in-time description that we have just introduced.

2.5 Time and position dependent extensions of CTRWs.

In this section we present position dependent extensions of controlled CTRW models studied in the previous sections of this Chapter, and corresponding limiting payoff function theorems. These position dependent CTRWs are Markov Chains with nonexponential waiting times. Here we generalize the fractional operators to generators of Feller processes. These include the

Brownian motion, all Lévy processes as well as solutions to Stochastic Differential Equations (SDEs) with Lipschitz continuous coefficients. The definitions, properties and a detailed exposition of Feller processes is presented for example in [54].

Let $\mathscr{L}_u$ be an infinitesimal generator of a Feller process, and $\mathscr{A}$ be an infinitesimal generator of an increasing Feller process. Let the dynamics of the process $Y^\tau(M^\tau(t))$ be determined by

$$\begin{cases} Y^\tau(n) = Y^\tau(n-1) + \xi_n^\tau(Y^\tau(n-1)), \\ X^\tau(n) = X^\tau(n-1) + \gamma_n^\tau(X^\tau(n-1)), \\ M^\tau(t) = \inf_n\{n : X^\tau(n) > t\} - 1, \end{cases}$$

for every $n \in \mathbb{N}$; and for $i \in \mathbb{N}$ the r.v.'s ξ_i, γ_i obey probability laws $\mu_{u,\tau}(d\xi)$ and $\nu_\tau(dr)$ respectively, such that as $\tau \to 0$ the following convergence holds:

$$\int_0^\infty \frac{f(t+r,y) - f(t,y)}{\tau}\nu_\tau(dr) \to \mathscr{A} f(t,y) \tag{2.52}$$

for any $f(t,y)$ in the domain of $\mathscr{A}$, and

$$\int_{\mathbb{R}^d} \frac{f(t,y+\xi) - f(t,y)}{\tau}\mu_{u,\tau}(d\xi) \to \mathscr{L}_u f(t,y), \tag{2.53}$$

for any $f(t,y)$ in the domain of $\mathscr{L}_u$.

Let us consider $Y^\tau(M^\tau(t))$ at jump times only. Define the payoff for this dynamics:

$$S^\tau(t,y) = \inf_{\tilde{u}\in\tilde{U}} \mathbb{E}\Bigg[S_T(y + Y^\tau_{M^\tau(T-t)}) + \sum_{i=1}^{M^\tau(T-t)} \gamma_i^\tau g(u_i, y + Y_{i-1}^\tau) + \sum_{i=1}^{M^\tau(T-t)} f(u_i, y + Y_{i-1}^\tau, \xi_i)\Bigg], \tag{2.54}$$

where $S_T(y) = S(T,y)$ is a known bounded function, $S^\tau(t,y) = 0$ for all $t > T$; $g(u,y)$ and $f(u,y,\xi)$ are the running cost functions corresponding to waiting and jumping costs respectively. Assume also that as $\tau \to 0$, $S^\tau(\cdot) \to \tilde{S}(\cdot)$ in an appropriate sense, where $\tilde{S}(t,y)$ belongs to domains of the operators $\mathscr{L}_u$ and $\mathscr{A}$. Denote the dual operator for $\mathscr{A}$ by $\mathscr{A}^*$. Let us show that the equation for $\tilde{S}(t,y)$ becomes

$$\mathscr{A}\tilde{S}(t,y) - \mathscr{A}^*\mathbf{1}_{(T-t>0)}S_T(y) + \inf_{u\in U}\Big[\mathscr{L}_u\tilde{S}(t,y)$$

$$+g(u,y)\mathscr{A}^*((T-t)\mathbf{1}_{(\mathbf{T-t>0})}) - F(u,y)\mathscr{A}^*\mathbf{1}_{(T-t>0)}\Big] = 0. \tag{2.55}$$

Remark 2. *It is well-known that if there is no control, then the Markov chains $X^{\tau}_{[t/\tau]}, Y^{\tau}_{[t/\tau]}$ converge to Feller processes with generators $\mathscr{A}$ and $\mathscr{L}$ respectively as $\tau \to 0$, see e.g. [54]. In the case without control the equation (2.55) is a well-known result for the limits of functions of scaled CTRW, see e.g. [13], [62].*

Due to the law of total expectation, analogously to (2.47), the optimal payoff defined in (2.54) satisfies:

$$S^{\tau}(t,y) = \inf_{u\in U}\Bigg[[S_T(y) + g(u,y)(T-t)]\int_{T-t}^{\infty}\nu_{\tau}(dr)$$

$$+\int_0^{T-t}\int_{\mathbb{R}^d}[S^{\tau}(t+r,y+\xi) + g(u,y)r + f(y+\xi,u)]\,\mu_{u,\tau}(d\xi)\nu_{\tau}(dr)\Bigg]. \tag{2.56}$$

Let us re-write (2.56) as

$$S^{\tau}(t,y) = \inf_{u\in U}\Bigg[S_T(y)\int_t^{\infty}\nu_{\tau}(dr) + \tau\int_0^{T-t}\int_{\mathbb{R}^d}\frac{S^{\tau}(t+r,y+\xi) - S(t+r,y)}{\tau}\mu_{u,\tau}(d\xi)\nu_{\tau}(dr)$$

$$+\tau\int_0^{T-t}\int_{\mathbb{R}^d}\frac{S^{\tau}(t+r,y) - S^{\tau}(t,y)}{\tau}\mu_{u,\tau}(d\xi)\nu_{\tau}(dr) + \int_0^{T-t}\int_{\mathbb{R}^d}S^{\tau}(t,y)\mu_{u,\tau}(d\xi)\nu_{\tau}(dr)$$

$$g(u,y)(T-t)\int_{T-t}^{\infty}\nu_{\tau}(dr) + \int_0^{T-t}\int_{\mathbb{R}^d}r\mu_{u,\tau}(d\xi)\nu_{\tau}(dr) + \int_0^{T-t}\int_{\mathbb{R}^d}f(y+\xi,u)\mu_{u,\tau}(d\xi)\nu_{\tau}(dr)\Bigg]. \tag{2.57}$$

Now, to show (2.55) we follow the same method as in proving (2.17). Let us re-write

$$\tau\int_0^{T-t}\int_{\mathbb{R}^d}\frac{S^{\tau}(t+r,y+\xi) - S^{\tau}(t+r,y)}{\tau}\mu_{u,\tau}(d\xi)\nu_{\tau}(dr)$$

$$= \tau\int_0^{\infty}\int_{\mathbb{R}^d}\frac{S^{\tau}(t+r,y+\xi) - S^{\tau}(t+r,y)}{\tau}\mu_{u,\tau}(d\xi)\nu_{\tau}(dr)$$

$$-\tau\int_{T-t}^{\infty}\int_{\mathbb{R}^d}\frac{S^{\tau}(t+r,y+\xi) - S^{\tau}(t+r,y)}{\tau}\mu_{u,\tau}(d\xi)\nu_{\tau}(dr). \tag{2.58}$$

The double integral that we are subtracting is equal to 0 because we have assumed $S^\tau(t,y) = 0$ for $t > T$. Due to (2.53), our assumption that $S^\tau \to \tilde{S}$ in an appropriate sense, and due to the properties of the generators $\mathscr{A}$ and $\mathscr{L}_u$, it follows that

$$\lim_{\tau\to 0}\int_0^\infty \frac{\mathscr{L}\tilde{S}(t+r,y) - \mathscr{L}S(t,y)}{\tau}\nu_\tau(dr) = \mathscr{A}(\mathscr{L}\tilde{S}(t,y)), \tag{2.59}$$

so

$$\begin{aligned}\lim_{\tau\to 0}\int_0^\infty \mathscr{L}\tilde{S}(t+r,y)\nu_\tau(dr) &= \lim_{\tau\to 0}\left(\int_0^\infty \mathscr{L}\tilde{S}(t,y)\nu_\tau(dr) + \tau\mathscr{A}(\mathscr{L}\tilde{S}(t,y))\right)\\ &= \lim_{\tau\to 0}\int_0^\infty \mathscr{L}\tilde{S}(t,y)\nu_\tau(dr) = \mathscr{L}\tilde{S}(t,y),\end{aligned} \tag{2.60}$$

similarly to (2.59) it follows that as $\tau \to 0$

$$\int_{\mathbb{R}^d} \frac{S^\tau(t+r,y+\xi) - S^\tau(t+r,y)}{\tau}\mu_{u,\tau}(d\xi) \to \mathscr{L}_u\tilde{S}(t,y). \tag{2.61}$$

Now express

$$\begin{aligned}&\tau\int_0^{T-t}\int_{\mathbb{R}^d} \frac{S^\tau(t+r,y) - S^\tau(t,y)}{\tau}\mu_{u,\tau}(d\xi)\nu_\tau(dr)\\ &= \tau\int_0^\infty\int_{\mathbb{R}^d} \frac{S^\tau(t+r,y) - S^\tau(t,y)}{\tau}\mu_{u,\tau}(d\xi)\nu_\tau(dr)\\ &-\tau\int_{T-t}^\infty\int_{\mathbb{R}^d} \frac{S^\tau(t+r,y) - S^\tau(t,y)}{\tau}\mu_{u,\tau}(d\xi)\nu_\tau(dr).\end{aligned} \tag{2.62}$$

Due to the definition of $\mathscr{A}$ and (2.52) and the assumption $S^\tau \to \tilde{S}$, as $\tau \to 0$, similarly to how (2.59) and (2.61) have arisen earlier, it follows that

$$\int_0^\infty \frac{S^\tau(t+r,y) - S^\tau(t,y)}{\tau}\nu_\tau(dr) \to \mathscr{A}\tilde{S}(t,y). \tag{2.63}$$

The first term in the double integral that we subtract is equal to 0, as we have assumed $S^\tau(t,y) = 0$ for $t > T$. Now,

$$\begin{aligned}&\int_0^{T-t}\int_{\mathbb{R}^d} S^\tau(t,y)\mu_{u,\tau}(d\xi)\nu_\tau(dr)\\ &= \int_0^\infty\int_{\mathbb{R}^d} S^\tau(t,y)\mu_{u,\tau}(d\xi)\nu_\tau(dr) - \int_{T-t}^\infty\int_{\mathbb{R}^d} S^\tau(t,y)\mu_{u,\tau}(d\xi)\nu_\tau(dr).\end{aligned} \tag{2.64}$$

The double integral we subtract here cancels out with the last term in (2.62), and the first cancels out with $S^\tau(t,y)$ on LHS. As for the running cost terms the following calculations prove to be useful:

$$\mathscr{A}^*\mathbf{1}_{(T-t>0)} = -\lim_{\tau\to 0}\int_0^\infty \left(\mathbf{1}_{(T-t-r>0)} - \mathbf{1}_{(T-t>0)}\right)\nu_\tau(dr)$$

$$= \lim_{\tau\to 0}\int_{T-t}^\infty \mathbf{1}_{(T-t>0)}\nu_\tau(dr) = \lim_{\tau\to 0}\int_{T-t}^\infty \nu_\tau(dr), \tag{2.65}$$

as we assume $T-t\geq 0$. Also,

$$\mathscr{A}^*\left((T-t)\mathbf{1}_{(T-t>0)}\right) = -\lim_{\tau\to 0}\int_0^\infty \left((T-t-r)\mathbf{1}_{(T-t-r>0)} - (T-t)\mathbf{1}_{(T-t>0)}\right)\nu_\tau(dr)$$

$$= \lim_{\tau\to 0}\int_0^\infty \left(-(T-t)(\mathbf{1}_{(T-t-r>0)} - \mathbf{1}_{(T-t>0)}) + r\mathbf{1}_{(T-t-r>0)}\right)\nu_\tau(dr)$$

$$= \lim_{\tau\to 0}\int_0^{T-t} r\nu_\tau(dr) + (T-t)\mathscr{A}^*(\mathbf{1}_{(T-t>0)}). \tag{2.66}$$

Hence terms with $g(u,y)$ yield:

$$\lim_{\tau\to 0}\left(g(u,y)(T-t)\int_{T-t}^\infty \nu_\tau(dr) + g(u,y)\int_0^{T-t} r\nu_\tau(dr)\right)$$

$$= g(u,y)\mathscr{A}^*\left((T-t)\mathbf{1}_{(T-t>0)}\right). \tag{2.67}$$

The term with f becomes in the limit

$$\lim_{\tau\to 0}\int_{T-t}^\infty\int_{\mathbb{R}^d} f(y+\xi,u)\mu_{u,\tau}(d\xi)\nu_\tau(dr) = -F(y,u)\mathscr{A}^*(\mathbf{1}_{(T-t>0)}), \tag{2.68}$$

in analogy to the more specific case (2.29). The result (2.55) follows. In the particular case when we assume $\mathscr{L} = \sum_{j=1}^n L_u^{\alpha_j}$ for $\alpha_j \in (0,2)$ and assume $\mathscr{A} = \sum_{j=1}^n A^{\beta_j}$ for $\beta_j \in (0,1)$, $j \in [1,\cdots,n]$, the limiting equation has the form

$$\sum_{j=1}^n A^{\beta_j}\tilde{S}(t,y) - \sum_{j=1}^n A^{*\beta_j}(1_{\{T-t>0\}})S_T(y)$$
$$+\inf_{u\in U}\left[\sum_{j=1}^n L_u^{\alpha_j}\tilde{S}(t,y) + g(u,y)\sum_{j=1}^n A^{*\beta_j}((T-t)1_{T-t>0}) - F(u,y)\sum_{j=1}^n A^{*\beta_j}1_{T-t>0}\right] = 0, \tag{2.69}$$

where $L_u^{\alpha_j}$ is defined as in (2.15) for each $j\in[1,\cdots,n]$. Physically this corresponds to the situation where each of the jumps ξ_i depends on n different stable laws. I.e. each jump size belongs to the normal domain of attraction on n different stable r.v.'s, representing n factors affecting the sizes on jumps $\xi_i, i \in \mathbb{N}$. Similarly, the second assumption models the scenario in which the waiting times γ_i depend on n different stable laws. For example, if $n = 2$, and $\alpha_2 = 1$, the measure $\mu_{\alpha_1}(d\xi)$ is independent of control, and β is fixed for $j \in [1, 2]$, then $L^{\alpha_2} = -w(u)\frac{d}{dy}$, the drift operator. In case when there are no running costs, the equation for $\tilde{S}(t, y)$, written backwards in time, has the form

$$A^{*\beta}\tilde{S}(t, y) - A^{*\beta}S_0(y) + \inf_{u\in U}\left[\left(L^{\alpha_1} + w(u)\frac{d}{dy}\right)\tilde{S}(t, y)\right] = 0. \tag{2.70}$$

Using the definition of the Caputo derivative $D^{*\beta}_{0,t}$ in (1.19), equation (2.70) can be presented in the form

$$D^{*,\beta}_{0^+,t}\tilde{S}(t, y) = L^{\alpha_1}\tilde{S}(t, y) + H\left(t, y, D_y\tilde{S}(t, y)\right), \tag{2.71}$$

where H contains all the control related terms and the initial value term. Analysis of well-posedness for a mild form of (2.71) in case $L^{\alpha_1} = -(-\Delta)^{\alpha/2}$ with $\alpha \in (1, 2]$ is presented in the next Chapter, together with various regularity properties. The rigorous proof for $S^\tau(t, y) \to \tilde{S}(t, y)$ as $\tau \to 0$ is presented in Chapter 6.

Chapter 3

Wellposedness

The purpose of this Chapter is to present our results concerning well-posedness of the Cauchy problem for the non-linear fractional in time and space differential equation

$$D_{0,t}^{*\beta} f(t,y) = -a(-\Delta)^{\alpha/2} f(t,y) + H(t,y,\nabla f(t,y)) \tag{3.1}$$

where $y \in \mathbb{R}^d, t \geq 0,\ \beta \in (0,1), \alpha \in (1,2],\ H(t,y,p)$ is a Lipschitz function in all of its variables, $f(0,y) = f_0(y)$ is known and bounded, and a is a constant, $a > 0$. Here ∇ denotes the gradient with respect to the spatial variable. For a function dependent on several spatial variables, say x, y, we may occasionally indicate the variable with respect to which the gradient is taken, by a subscript, ∇_x. For the readers' convenience, below we write the definitions of the Caputo and Laplacian fractional operators. The Caputo derivative $D_{0,t}^{*\beta}$ is defined for $f \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^d)$ regular enough as

$$D_{0,t}^{*\beta} f(t,y) = \frac{1}{\Gamma(1-\beta)} \int_0^t \frac{df(s,y)}{ds} (t-s)^{-\beta} ds, \tag{3.2}$$

whilst $-(-\Delta)^{\alpha/2}$ is the fractional Laplacian

$$-(-\Delta)^{\alpha/2} f(t,y) = p.v. C_{d,\alpha} \int_{\mathbb{R}^d} \frac{f(t,y) - f(t,x)}{|y-x|^{d+\alpha}} dx, \tag{3.3}$$

defined for $f \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^d)$ and decaying at infinity in y, and where $C_{d,\alpha}$ is a normalizing constant and $p.v.$ stands for the "principal value". The extension of our results for (3.1) to the case where $H = H(t, y, f(t, y), \nabla f(t, y))$ is straightforward and we omit it here.

As a preliminary analysis we establish the regularity properties for solutions of the linear equations such as

$$D_{0,t}^{*\beta} f(t, y) = -a(-\Delta)^{\alpha/2} f(t, y) + h(t, y), \tag{3.4}$$

with a given function h, an initial condition $f(0, y) = f_0(y)$, $\beta \in (0, 1)$, $\alpha \in (1, 2]$, and a constant $a > 0$. This allows one to reduce the analysis of (3.1) to a fixed point problem. Firstly we study the linear problem (3.4) and secondly we formulate and prove our main results for equation (3.1).

Our regularity estimates for G_β and $S_{\beta,1}$ have already been applied to analysis of SIR systems described with Caputo time derivatives and fractional Laplacians. The paper proves existence of a global solution and establishes solution time bounds, see [63].

Before the fractional Cauchy problem analysis we present a literature review for recent developments in analysis of fractional differential equations (FDEs). Among researchers who studied solutions to FDEs are [64], [36], [65], [66], [67], [57], [68], [69], [37], [70], [71]. More results and reviews can be found in references therein. FDEs appear for example in modelling processes with memory, see [39], [38], [72], [73].

Several authors solve FDEs using Laplace transforms in time, see [66], [37] and [67] for example.

The book [57] covers analysis for Caputo time-fractional differential equations with the parameter $\beta > 0$, for example

$$D_{0,t}^{*\beta} y(x) = -\mu y(x) + q(x), \tag{3.5}$$

with $y(0) = y_0^{(0)}$, $Dy(0) = y_0^{(1)}$, $\beta \in (1, 2)$, $\mu > 0$.

In [65] the theory for FDEs in L^p spaces is developed. Well-posedness of (3.4) in L^p may be deduced from there.

In [64] the authors consider classical solutions for fractional Cauchy problems in bounded domains $D \subset \mathbb{R}^d$ with Dirichlet boundary conditions.

In [74] one may find the analysis for the non-local Cauchy problem in a Banach space, where instead of $-(-\Delta)^{\alpha/2}$ there is a general infinitesimal generator of a strongly continuous semigroup of bounded linear operators. The authors present conditions that need to hold to ensure existence of mild forms of the FDEs.

The paper [75] establishes asymptotic estimates of solutions to the following FDE and its similar versions:

$$D_{0,t}^{*\alpha} u(x,t) = a^2 \frac{d^2 u(x,t)}{dx^2}, \tag{3.6}$$

for $t > 0, x \in \mathbb{R}$, $\alpha \in (0,1)$, $u(x,0) = \phi(x)$, $\lim_{|x|\to+\infty} u(x,t) = 0$, however the case of the spatial fractional Laplacian is not included and there is no $h(x,t)$ term on the right hand side (RHS).

In [76] the author studies the uniqueness of a solution to

$$D_{0,t}^{*\alpha} u(t) = Au(t), \tag{3.7}$$

where $t > 0, u(0) = u_0$, and A is an unbounded closed operator in a Banach space, $\alpha \in (0,1)$. However there is no non-homogeneity term $h(t)$ on the RHS. For solvability of linear FDEs in Banach spaces one may see [77], where

$$D_{0,t}^{*\alpha} x(t) = Ax(t), \text{ for } m-1 < \alpha \leq m \in \mathbb{N}, \tag{3.8}$$

and $\frac{d^k}{dt^k} x(t)|_{t=0} = \xi_k$, for $k = 0, \ldots, m-1$. The authors give sufficient conditions under which the set of initial data ξ_k for $k = 0, \ldots, m-1$ provides a solution to (3.8) of the form $\sum_{k=0}^{m-1} t^k E_{\alpha,k+1}(At^\alpha)\xi_k$. In particular, these conditions depend on Roumieu, Gevrey and Beurling spaces related to the operator A.

In [78] the authors use fixed point theorems to prove existence and uniqueness of a positive solution for the problem

$$D^{\alpha}_{0,t}x(t) = f(t, x(t), -D^{\beta}_{0,t}x(t)), t \in (0, 1), \tag{3.9}$$

with non-local Riemann-Stieltjes integral condition

$$D^{\beta}_{0,t}x(0) = D^{\beta+1}_{0,t}x(0) = 0, \tag{3.10}$$

and $D^{\beta}_{0,t}x(1) = \int_0^1 D^{\beta}_{0,t}x(s)dA(s)$, where A is a function of bounded variation, $\alpha \in (2, 3], \beta \in (0, 1), \alpha-\beta > 2$. In [78] there are references to papers where FDEs are inspected with the help of various fixed point theorems. Our analysis also includes a fixed point theorem, however its use and the problem itself are different from the one in [78].

In [69] there is a construction and investigation of a fundamental solution for the Cauchy problem with a regularised fractional derivative $D^{\alpha}_{0,t,reg}$, and $\alpha \in (0, 1)$ defined by

$$D^{\alpha}_{0,t,reg}u(t, x) = \frac{1}{\Gamma(1-\alpha)} \left[\frac{\partial}{\partial t} \int_0^t (t-\tau)^{-\alpha} u(\tau, x) d\tau - t^{-\alpha} u(0, x) \right]. \tag{3.11}$$

Note that

$$D^{\alpha}_{0,t}u(t, x) = \frac{1}{\Gamma(1-\alpha)} \frac{\partial}{\partial t} \int_0^t (t-\tau)^{-\alpha} u(\tau, x) d\tau \tag{3.12}$$

is the definition of the Riemann-Liouville fractional derivative. Since $D^{*\alpha}_{0,t}f(t, x) = D^{\alpha}_{0,t}f(t, x) - \frac{t^{-\alpha}}{\Gamma(1-\alpha)}f(0, x)$, the regularised derivative in (3.11) is in fact identical to our definition of the Caputo derivative in (3.2).

The problem studied there is

$$D^{\alpha}_{0,t,reg}u(t, x) - Bu(t, x) = f(t, x), \tag{3.13}$$

$t \in (0, T], x \in \mathbb{R}^n$, where

$$B = \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{j=1}^{n} b_j(x) \frac{\partial}{\partial x_j} + c(x) \tag{3.14}$$

with bounded real-valued coefficients. Our analysis goes beyond to include $B = -a(-\Delta)^{\alpha/2}$, with $a > 0$. Theorems 3, 6 and 7 concerning the case $\alpha = 2$ are obtainable from the results in [69] by slightly different arguments.

The paper [79] presents in particular the fundamental solution to the multi-time FDE

$$\sum_{k=1}^{m} \lambda_k D^{*\beta_k} u(t,y) - \Delta_x u(t,y) = f(t,y), \tag{3.15}$$

for $t = (t_1, \ldots, t_n) \in \mathbb{R}^n, y = (y_1, \ldots, y_m) \in \mathbb{R}^m$, and $\lambda = (\lambda_1, \ldots, \lambda_m) \in \mathbb{R}^m$, whilst Δ_x is the standard Laplacian operator and $\beta_k \in (0,1)$ for all $1 \geq k \leq m$. There is also the proof of that the fundamental solution for (3.15) is unique. The uniqueness result covers a more broad range of FDEs involving Dzhrbashyan-Nersesyan fractional in time differential equations. In our case there are fractional operators with respect to both spatial and temporal variables.

Denote a bounded domain by D. Taking $\alpha \in (0,2)$, $\beta \in (0,1)$ the paper [80] explores strong solutions to the equation

$$D_{0,t}^{*\beta} u(t,x) = \Delta_x^{\alpha/2} u(t,x), \tag{3.16}$$

for $x \in D$, $t > 0$, $u(0,x) = f(x)$ for $x \in D$ and $u(t,x) = 0$ for $x \in D^c$, $t > 0$.

Our approach to the non-linear FDE is different and includes the fractional Laplacian $-(-\Delta)^{\alpha/2}$ instead of the standard one Δ_y. We extend to the scenario with the RHS term including $H(t,y,\nabla f(t,y))$. We concentrate on the case with only one fractional time derivative $D_{0,t}^{*\beta}$.

In [81] the authors present conditions under which a Caputo FDE has a classical solution. However, their equation considers a fractional derivative in time only, whilst the spatial operator is the Laplacian. Also, apart from the fractional time operator, their equation of interest involves a Laplacian in time.

3.1 The mild form for the linear fractional dynamics

Our analysis of equation (3.4) is based on the Fourier transform in space, where for a function $g(y)$ its Fourier transform will be defined in the following way

$$\hat{g}(p) = \int_{R^d} e^{-i\langle p,y\rangle} g(y)dy, \tag{3.17}$$

where $\langle \cdot, \cdot \rangle$ is the scalar product. Applying the Fourier transform in y to (3.4) yields

$$D^{*\beta}_{0,t}\hat{f}(t,p) = -a|p|^{\alpha}\hat{f}(t,p) + \hat{h}(t,p). \tag{3.18}$$

This is a standard linear equation with the Caputo fractional derivative. For continuous h its solution is given by

$$\hat{f}(t,p) = \hat{f}_0(p)E_{\beta,1}(-a|p|^{\alpha}t^{\beta}) + \int_0^t (t-s)^{\beta-1}E_{\beta,\beta}(-a(t-s)^{\beta}|p|^{\alpha}))\hat{h}(s,p)ds, \tag{3.19}$$

where $E_{\beta,1}$ and $E_{\beta,\beta}$ are Mittag-Leffler functions, see formulas (7.3) – (7.4) in [57]. Let us recall that the Mittag-Leffler functions are defined for $Re(\beta) > 0$, and $\gamma, z \in \mathbb{C}$:

$$E_{\beta,\gamma}(z) = \sum_{k=1}^{\infty} \frac{z^k}{\Gamma(\beta k + \gamma)}. \tag{3.20}$$

We will use the following connection between $E_{\beta,\beta}$ and $E_{\beta,1}$:

$$x^{\beta-1}E_{\beta,\beta}(-a|p|^{\alpha}x^{\beta}) = -\frac{1}{a|p|^{\alpha}}\frac{d}{dx}E_{\beta,1}(-a|p|^{\alpha}x^{\beta}). \tag{3.21}$$

To prove (3.21) one may use the representation of $E_{\beta,1}(-a|p|^{\alpha}x^{\beta})$ in (3.20) and differentiate with respect to x term by term. Now we present two convenient notations for further analysis.

Let us denote

$$S_{\beta,1}(t,y) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{i\langle p,y\rangle} E_{\beta,1}(-a|p|^\alpha t^\beta)dp \tag{3.22}$$

and

$$G_\beta(t,y) := \frac{t^{\beta-1}}{(2\pi)^d}\int_{\mathbb{R}^d} e^{i\langle p,y\rangle} E_{\beta,\beta}(-a|p|^\alpha t^\beta)dp. \tag{3.23}$$

Using (3.21) we can re-write (3.23) as

$$G_\beta(t,y) := -\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{i\langle p,y\rangle} \frac{1}{a|p|^\alpha}\frac{d}{dt} E_{\beta,1}(-a|p|^\alpha t^\beta)dp. \tag{3.24}$$

Applying the inverse Fourier transform to (3.19) we obtain:

$$f(t,y) = \int_{\mathbb{R}^d} S_{\beta,1}(t,y-x)f_0(x)dx + \int_0^t\int_{\mathbb{R}^d} G_\beta(t-s,y-x)h(s,x)dxds. \tag{3.25}$$

It is natural to call this integral equation the mild form of the fractional linear equation (3.4). In particular we see that the function $S_{\beta,1}(t,y-y_0)$ is the solution of equation (3.4) with $f_0(y) = \delta(y-y_0)$ and $h(t,y) = 0$. The function $G_\beta(t-t_0,y-y_0)$ is the solution of (3.4) with $f_0(y) = 0$ and $h(t,y) = \delta(t-t_0,y-y_0)$. Thus the functions $S_{\beta,1}$ and G_β may be called Green functions of the corresponding Cauchy problems. Notice the crucial difference with the usual evolution corresponding to $\beta = 1$ where G_β and $S_{\beta,1}$ coincide.

In order to clarify the properties of f in (3.25) we are now going to carefully analyse properties of the integral kernels $S_{\beta,1}(t,y)$ and $G_\beta(t,y)$.

3.2 Regularity properties of $S_{\beta,1}$ and G_β

For $d \geq 1$ let us define the symmetric stable density g in $\mathbb{R}^d$ as the Fourier transform of the corresponding characteristic function

$$g(y;\alpha,\sigma,\gamma=0)=\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\exp\{-i\langle p,y\rangle-a\sigma|p|^{\alpha}\}dp, \tag{3.26}$$

where α is the stability parameter, σ is the scaling parameter and γ is the skewness parameter which is $\gamma=0$ for symmetric stable densities. In $d=1$ and $\alpha\neq 1$ we define the fully skewed density with $\gamma=1$ and without scaling:

$$w(x;\alpha,1)=\frac{1}{2\pi}Re\int_{-\infty}^{\infty}\exp\left\{-ipx-|p|^{\alpha}\exp\left\{-i\frac{\pi}{2}K(\alpha)\right\}\right\}dp, \tag{3.27}$$

where $K(\alpha)=\alpha-1+\text{sign}(1-\alpha)$. The function $w(x;\alpha,1)$ is infinitely differentiable and vanishes identically for $x<0$, see [15], theorem C.3 and §2.2, equation (2.2.1a).

The starting point of the analysis of $S_{\beta,1},G_{\beta}$ is the following representation of the Mittag-Leffler function due to [15], see chapter 2.10, Theorem 2.10.2, equations (2.10.8 − 2.10.9). For $\beta\in(0,1)$

$$E_{\beta,1}(-a\lambda)=\frac{1}{\beta}\int_0^{\infty}\exp(-a\lambda x)x^{-1-1/\beta}w(x^{-1/\beta},\beta,1)dx. \tag{3.28}$$

Substitute $\lambda=|p|^{\alpha}t^{\beta}$:

$$E_{\beta,1}(-a|p|^{\alpha}t^{\beta})=\frac{1}{\beta}\int_0^{\infty}\exp(-a|p|^{\alpha}t^{\beta}x)x^{-1-1/\beta}w(x^{-1/\beta},\beta,1)dx. \tag{3.29}$$

So then

$$t^{\beta-1}E_{\beta,\beta}(-a|p|^{\alpha}t^{\beta})=\frac{-1}{a|p|^{\alpha}}\frac{d}{dt}E_{\beta,1}(-a|p|^{\alpha}t^{\beta})$$
$$=t^{\beta-1}\int_0^{\infty}x^{-1/\beta}\exp(-a|p|^{\alpha}t^{\beta}x)w(x^{-1/\beta},\beta,1)dx, \tag{3.30}$$

implying

$$G_\beta(t,y)=\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{i\langle p,y\rangle}E_{\beta,\beta}(-a|p|^\alpha t^\beta)t^{\beta-1}dp$$
$$=\frac{t^{\beta-1}}{(2\pi)^d}\int_0^\infty\int_{\mathbb{R}^d} e^{i\langle p,y\rangle}\exp\{-a|p|^\alpha t^\beta x\}x^{-1/\beta}w(x^{-1/\beta},\beta,1)dpdx$$
$$=t^{\beta-1}\int_0^\infty x^{-1/\beta}w(x^{-1/\beta},\beta,1)g(-y,\alpha,t^\beta x)dx, \tag{3.31}$$

where g is as in (3.26) and w is as in (3.27).

$$S_{\beta,1}(t,y)=\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{i\langle p,y\rangle}E_{\beta,1}(-a|p|^\alpha t^\beta)dp$$
$$=\frac{1}{\beta(2\pi)^d}\int_0^\infty\int_{\mathbb{R}^d} e^{i\langle p,y\rangle}e^{-a|p|^\alpha t^\beta x}x^{-1-1/\beta}w(x^{-1/\beta},\beta,1)dpdx$$
$$=\frac{1}{\beta(2\pi)^d}\int_0^\infty x^{-1-1/\beta}w(x^{-1/\beta},\beta,1)g(-y,\alpha,t^\beta x)dx. \tag{3.32}$$

Throughout this paper we shall denote by C various constants that may be different from formula to formula and line to line.

Theorem 1. *For $\beta\in(0,1)$, $\alpha\in(1,2)$ and $G_\beta(t,y)$ defined in (3.23)*

$$\int_{\mathbb{R}^d}|G_\beta(t,y)|dy\le Ct^{\beta-1}, \tag{3.33}$$

where $C>0$ is a constant.

Proof. Let us write the integral representing $G_\beta(t,y)$ as the sum of two, so that

$$G_\beta(t,y)=I_A+I_B, \tag{3.34}$$

where

$$I_A = \frac{t^{\beta-1}}{(2\pi)^d} \int_{|y|^\alpha t^{-\beta}}^{\infty} \int_{\mathbb{R}^d} e^{i\langle p,y\rangle} \exp\{-a|p|^\alpha t^\beta x\} x^{-1/\beta} w(x^{-1/\beta}, \beta, 1) dp dx$$
$$= t^{\beta-1} \int_{|y|^\alpha t^{-\beta}}^{\infty} x^{-1/\beta} w(x^{-1/\beta}, \beta, 1) g(-y, \alpha, t^\beta x) dx \qquad (3.35)$$

and

$$I_B = \frac{t^{\beta-1}}{(2\pi)^d} \int_0^{|y|^\alpha t^{-\beta}} \int_{\mathbb{R}^d} e^{i\langle p,y\rangle} \exp\{-a|p|^\alpha t^\beta x\} x^{-1/\beta} w(x^{-1/\beta}, \beta, 1) dp dx$$
$$= t^{\beta-1} \int_0^{|y|^\alpha t^{-\beta}} x^{-1/\beta} w(x^{-1/\beta}, \beta, 1) g(-y, \alpha, t^\beta x) dx. \qquad (3.36)$$

To estimate $|I_A|$ and $|I_B|$, let us examine cases $|y| > t^{\beta/\alpha}$ and $|y| \leq t^{\beta/\alpha}$ and start with $|y| > t^{\beta/\alpha}$. Note that the asymptotic expansions for $g(y, \alpha, \sigma)$ and $g(-y, \alpha, \sigma)$, namely, (7.2) and (7.6) appearing in the Appendix, are the same, by inspection. Since $x > |y|^\alpha t^{-\beta}$ in I_A we may use the asymptotic for $|y|/x^{1/\alpha} t^{\beta/\alpha} \to 0$, see (7.2). We also use that for $x \to \infty$, $x^{-1/\beta} \to 0$, so for $x \to \infty$ we have $w(x^{-1/\beta}, \beta, 1) \sim C$, where $C \geq 0$ is a constant. Thus we have

$$|I_A| \leq \left| \int_{|y|^\alpha t^{-\beta}}^{\infty} x^{-1/\beta - d/\alpha} w(x^{-1/\beta}, \beta, 1) A_0 t^{\beta-1-d\beta/\alpha} dx \right|$$
$$\leq C t^{\beta-1-d\beta/\alpha} |A_0| \frac{(|y|^\alpha t^{-\beta})^{1-1/\beta-d/\alpha}}{|1 - 1/\beta - d/\alpha|}$$
$$\leq C t^{\beta-1-d\beta/\alpha} \frac{|A_0|}{|1 - 1/\beta - d/\alpha|} (|y|^\alpha t^{-\beta})^{1-1/\beta-d/\alpha}$$
$$\leq C t^{\beta-1+1-\beta} |y|^{\alpha-\alpha/\beta-d}. \qquad (3.37)$$

Now, let us study I_B in case $|y| > t^{\beta/\alpha}$. Here we use the asymptotic expansion for $|y|/x^{1/\alpha} t^{\beta/\alpha} \to \infty$ as it appears in (7.6) in the Appendix and take the first term only. Here we use the change of variables $z = x^{-1/\beta}$.

$$|I_B| \leq \left| A_1 t^{\beta-1} \int_0^{|y|^\alpha t^{-\beta}} x^{-1/\beta} w(x^{-1/\beta}, \beta, 1) |y|^{-d-\alpha} t^\beta x dx \right|$$
$$\leq C t^{2\beta-1} |y|^{-d-\alpha} \int_{|y|^{-\alpha/\beta} t}^{\infty} z^{-2\beta} w(z, \beta, 1) dz. \tag{3.38}$$

We split this integral into two parts: $z \in [1, \infty)$ and $z \in (|y|^{-\alpha/\beta} t, 1)$. Firstly,

$$t^{2\beta-1} |y|^{-d-\alpha} \int_1^\infty z^{-2\beta} w(z, \beta, 1) dz$$
$$\leq t^{2\beta-1} |y|^{-d-\alpha} \int_0^\infty w(z, \beta, 1) dz \leq C t^{2\beta-1} |y|^{-d-\alpha}, \tag{3.39}$$

In case $z \in (|y|^{-\alpha/\beta} t, 1)$ we may use that z is small and so $z^{-2\beta} w(z, \beta, 1) < C z^{-2\beta+q-3}$, for any $q > 1$. So

$$t^{2\beta-1} |y|^{-d-\alpha} \int_{|y|^{-\alpha/\beta} t}^{1} z^{-2\beta} w(z, \beta, 1) dz \leq C t^{2\beta-1} |y|^{-d-\alpha} \int_{|y|^{-\alpha/\beta}}^{1} z^{-2\beta+q-3} dz$$
$$= C t^{2\beta-1} |y|^{-d-\alpha} \left(1 - (|y|^{-\alpha/\beta} t)^{-2\beta+q-2}\right). \tag{3.40}$$

Now let us study the case $|y| \leq t^{\beta/\alpha}$. For I_A we use that x is large, so $x^{-1/\beta}$ is small, and that for $q \geq 4$ we have $x^{-d/\alpha-1/\beta} w(x^{-1/\beta}) < C x^{-d/\alpha-(\frac{q-2}{\beta})}$. Here $|y|^\alpha \leq t^\beta$ and we obtain

$$|I_A| \leq C t^{\beta-1-d\beta/\alpha} \left| \int_{|y|^\alpha t^{-\beta}}^{\infty} A_0 x^{-d/\alpha - \frac{q-2}{\beta}} dx \right|$$
$$\leq t^{\beta-1-\beta d/\alpha} \left(y^\alpha t^{-\beta}\right)^{-d/\alpha - \frac{q-2}{\beta}+1} C$$
$$\leq t^{\beta-1-d\beta/\alpha} t^{-d\beta/\alpha-(q-2)+\beta+d\beta/\alpha+(q-2)-\beta} C$$
$$\leq t^{\beta-1-d\beta/\alpha} C. \tag{3.41}$$

As for I_B in case $|y| \leq t^{\beta/\alpha}$,

$$|I_B| \le C \int_0^{|y|^\alpha t^{-\beta}} x^{1-1/\beta} w(x^{-1/\beta}, \beta, 1) t^{2\beta-1} |y|^{-d-\alpha} dx$$
$$\le C|y|^{-d-\alpha} t^{2\beta-1} \int_0^{|y|^\alpha t^{-\beta}} x^{1-1/\beta} (x^{-1/\beta})^{-1-\beta} dx \le C|y|^{2\alpha-d} t^{-\beta-1}. \tag{3.42}$$

Integrating (3.37) in polar coordinates gives

$$\int_{|y|>t^{\beta/\alpha}} |I_A| dy \le C \int_{|r|>t^{\beta/\alpha}} |r|^{\alpha-\alpha/\beta-d+d-1} d|r| \le (t^{\beta/\alpha})^{\alpha-\alpha/\beta} C = t^{\beta-1} C, \tag{3.43}$$

Integration of (3.40) in polar coordinates gives

$$\int_{|y|>t^{\beta/\alpha}} |I_B| dy \le C t^{2\beta-1} \int_{|r|>t^{\beta/\alpha}} |r|^{-d-\alpha+d-1} dr$$
$$+ C t^{2\beta-1} \int_{|r|>t^{\beta/\alpha}} |r|^{d-1-d-\alpha} |r|^{2\alpha} t^{-2\beta} dr$$
$$= C t^{2\beta-1} (t^{\beta/\alpha})^{-\alpha} + C t^{2\beta-1-2\beta} (t^{\beta/\alpha})^{\alpha} = C t^{\beta-1}. \tag{3.44}$$

Integration of (3.41) gives

$$\int_{|y|\le t^{\beta/\alpha}} |I_A| dy \le C t^{\beta-1-d\beta/\alpha} \int_{|r|\le t^{\beta/\alpha}} |r|^{d-1} d|r|$$
$$\le t^{d\beta/\alpha-d\beta/\alpha+\beta-1} \frac{C|A_0|}{|d|} \le t^{\beta-1} \frac{|A_0|C}{d}. \tag{3.45}$$

Integration of (3.42) yields

$$\int_{|y|\le t^{\beta/\alpha}} |I_B| dy \le C \int_{|r|\le t^{\beta/\alpha}} t^{-\beta-1} |r|^{-d+2\alpha+d-1} dr \le C t^{-\beta-1} (t^{\beta/\alpha})^{2\alpha} = C t^{\beta-1}. \tag{3.46}$$

Combining (3.43)–(3.46) yields (3.33).

□

Theorem 2. *For $\beta \in (0,1)$, $\alpha \in (1,2)$ and $G_\beta(t,y)$ defined in (3.23)*

$$\int_{\mathbb{R}^d} |\nabla G_\beta(t,y)| dy \leq t^{\beta-1-\beta/\alpha} C. \tag{3.47}$$

Proof. In case $|y| > t^{\beta/\alpha}$, we have $|y|^{-1} < t^{-\beta/\alpha}$ and so differentiation with respect to y yields

$$|\nabla I_A| \leq C t^{-\beta/\alpha} |I_A| \tag{3.48}$$

and

$$|\nabla I_B| \leq C t^{-\beta/\alpha} |I_B|. \tag{3.49}$$

In case $|y| \leq t^{\beta/\alpha}$ we need to take into account the second term of the asymptotic expansion, since the first term is independent of $|y|$. Consequently,

$$\begin{aligned} |\nabla I_A| &\leq C \int_{|y|^\alpha t^{-\beta}}^{\infty} x^{-d/\alpha} t^{-d\beta/\alpha} |y| (x t^\beta)^{-2/\alpha} x^{-1/\beta} t^{\beta-1} dx \\ &\leq C \int_{|y|^\alpha t^{-\beta}}^{\infty} x^{-d/\alpha-2/\alpha-1/\beta} t^{-d\beta/\alpha-2\beta/\alpha+\beta-1} dx \\ &\leq C t^{-d\beta/\alpha-2\beta/\alpha+\beta-1} |y| - C(|y|^\alpha t^{-\beta})^{-d/\alpha-2/\alpha-1/\beta+1} t^{\beta/\alpha} \\ &= C t^{-d\beta/\alpha-2\beta/\alpha+\beta-1} |y| - C t^{\beta/\alpha} |y|^{-d-2-\alpha/\beta+\alpha}. \end{aligned} \tag{3.50}$$

Integration of the first term in (3.50) yields

$$\begin{aligned} &C \int_{|r| \leq t^{\beta/\alpha}} t^{-d\beta/\alpha-2\beta/\alpha+\beta-1} |r|^{d-1+1} dr \\ &\leq C t^{-d\beta/\alpha-\beta/\alpha+\beta-1+d\beta/\alpha} \leq C t^{\beta-1-\beta/\alpha}. \end{aligned} \tag{3.51}$$

Integration of the second term in (3.50) gives

$$\int_{|y|\le t^{\beta/\alpha}} t^{\beta/\alpha}|y|^{-d+d-3-\alpha/\beta+\alpha}dy$$
$$\le t^{\beta/\alpha}(t^{\beta/\alpha})^{-2-\alpha/\beta+\alpha} \le t^{\beta-1-\beta/\alpha}. \tag{3.52}$$

Combining (3.51) and (3.52)

$$\int_{|y|\le t^{\beta/\alpha}} |\nabla I_A|dy \le Ct^{\beta-1-\beta/\alpha}. \tag{3.53}$$

As for $|\nabla I_B|$ for $|y| \le t^{\beta/\alpha}$

$$|\nabla I_B| \le Ct^{2\beta-1}|y|^{-d-\alpha-1}\int_0^{|y|^\alpha t^{-\beta}} \xi^{1-1/\beta}w(\xi^{-1/\beta},\beta,1)d\xi$$
$$\le Ct^{2\beta-1}|y|^{-d-\alpha-1}\int_0^{|y|^\alpha t^{-\beta}} \xi^{1-1/\beta}(\xi^{-1/\beta})^{-1-\beta}d\xi$$
$$\le Ct^{2\beta-1}|y|^{-d-\alpha-1}(|y|^\alpha t^{-\beta})^3 \le Ct^{-\beta-1}|y|^{-d+2\alpha-1}. \tag{3.54}$$

Integration gives

$$\int_{|y|\le t^{\beta/\alpha}} |\nabla I_B|dy \le C\int_{|y|\le t^{\beta/\alpha}} t^{-\beta-1}|y|^{-d+d+2\alpha-2}dy \le Ct^{-\beta-1-\beta/\alpha}. \tag{3.55}$$

So

$$\int_{|y|\le t^{\beta/\alpha}} |\nabla I_B|dy \le Ct^{\beta-1-\beta/\alpha}. \tag{3.56}$$

Since

$$\int_{\mathbb{R}^d} |\nabla G_\beta(t,y)|dy \le \int_{\mathbb{R}^d} |\nabla I_A|dy + \int_{\mathbb{R}^d} |\nabla I_B|dy \tag{3.57}$$

combining results (3.48), (3.49), (3.53) and (3.56) we obtain

$$\int_{\mathbb{R}^d} |\nabla G_\beta(t,y)|dy \le Ct^{\beta-1-\beta/\alpha}. \tag{3.58}$$

which proves (3.47).

□

Now let us consider the case $\alpha = 2$.

Theorem 3. *Let $G_\beta(t,y)$ be as in (3.23) and (3.31). For $\alpha = 2$ and any $\beta \in (0,1)$:*

- $\int_0^t \int_{\mathbb{R}^d} |G_\beta(t,y)|dyds \le Ct^\beta$,
- $\int_0^t \int_{\mathbb{R}^d} |\nabla G_\beta(t,y)|dyds \le Ct^{\beta/2}$.

Proof. Note that

$$\int_{\mathbb{R}^d} \exp\{-a\sigma p^2 - i\langle y,p\rangle\}dp = \left(\frac{\sqrt{\pi}}{\sqrt{\sigma}}\right)^d \exp\left\{-\frac{y^2}{4a\sigma}\right\}, \tag{3.59}$$

where in our case $\sigma = xt^\beta$. Substitute this into (3.31) to obtain

$$G_\beta(t,y) = \frac{t^{\beta-1}}{(2\pi)^d}\int_0^\infty x^{-1/\beta} w(x^{-1/\beta},\beta,1)\left(\frac{\sqrt{\pi}}{\sqrt{t^\beta x}}\right)^d \exp\left\{\frac{-y^2}{4at^\beta x}\right\}dx \tag{3.60}$$

where $y^2 = y_1^2 + y_2^2 + \ldots + y_d^2$. We are interested in $\int_0^t \int_{\mathbb{R}^d} |G_\beta(t,y)|dyds$. Integrating y-dependent terms in G_β with respect to y gives

$$\int_{\mathbb{R}^d} \exp\left\{-|y|^2/4axt^\beta\right\}dy = (4\pi xt^\beta)^{d/2} = Cx^{d/2}t^{\beta d/2}. \tag{3.61}$$

The term $x^{d/2}t^{\beta d/2}$ cancels out with $\left(\frac{1}{\sqrt{t^\beta x}}\right)^d$ and we obtain

$$\int_{\mathbb{R}^d} |G_\beta(t,y)|dy = I(t) = C\int_0^\infty x^{-1/\beta} w(x^{-1/\beta},\beta,1)t^{\beta-1}dx. \tag{3.62}$$

Now we split the integral $I(t)$ into 2 parts: $I_a(t)$ for $x > 1$ and $I_b(t)$ for $0 \le x \le 1$. In $I_a(t)$, $x > 1$ and so $x^{-1/\beta} < 1$ and $w(x^{-1/\beta}, \beta, 1) \sim C$, so we have

$$I_a(t) = \int_1^\infty C t^{\beta-1} x^{-1/\beta} w(x^{-1/\beta}, \beta, 1) dx$$
$$\le t^{\beta-1} \int_1^\infty x^{-1/\beta} C dx \le C 1^{1-1/\beta} t^{\beta-1} = C t^{\beta-1}. \tag{3.63}$$

Integrating with respect to s gives

$$\int_0^t |I_a(t-s)| ds \le \int_0^t C(t-s)^{\beta-1} ds = C t^\beta. \tag{3.64}$$

For $I_b(t)$, $x \le 1$, so $x^{-1/\beta} \ge 1$ and $w(x^{-1/\beta}, \beta, 1) \sim (x^{-1/\beta})^{-1-\beta} = x^{1+1/\beta}$ and

$$I_b(t) = \int_0^1 C x^{-1/\beta} w(x^{-1/\beta}, \beta, 1) t^{\beta-1} dx$$
$$\le C t^{\beta-1} \int_0^1 x^{-1/\beta+1/\beta+1} dx \le C. \tag{3.65}$$

with a constant $C_2 > 0$. Now we integrate with respect to s

$$\int_0^t |I_b(t-s)| ds = \int_0^t C(t-s)^{\beta-1} ds = C t^\beta. \tag{3.66}$$

Together with (3.63) and (3.64) this yields the first statement of the theorem.

Differentiating G_β with respect to y gives us

$$I_1(t) = \int_{\mathbb{R}^d} |\nabla G_\beta(t, y)| dy$$
$$= \int_0^\infty \int_{\mathbb{R}^d} t^{\beta-\beta-1-\beta d/2} x^{-1-1/\beta-d/2} |y| \exp\{-|y|^2/4axt^\beta\} w(x^{-1/\beta}, \beta, 1) dy dx. \tag{3.67}$$

Since

$$\int_{\mathbb{R}^d} |y| \exp\{-|y|^2/4axt^\beta\} dy = Cxt^\beta (\sqrt{xt^\beta})^{d-1} = Cx^{\frac{d+1}{2}} t^{\frac{\beta(d+1)}{2}}, \tag{3.68}$$

we have

$$I_1(t) = \int_0^\infty \int_{\mathbb{R}^d} t^{\beta-\beta-1-\beta d/2} x^{-1-1/\beta-d/2} |y| \exp\{-|y|^2/4axt^\beta\} w(x^{-1/\beta}, \beta, 1) dy dx$$
$$= C \int_0^\infty t^{-1+\beta/2} x^{-1/2-1/\beta} w(x^{-1/\beta}, \beta, 1) dx. \tag{3.69}$$

Now we split the integral $I_1(t)$ into parts corresponding to $x \in (0,1)$ and $x \in [1, \infty)$:

$$I_2(t) = \int_0^1 t^{-1+\beta/2} x^{-1/2-1/\beta} w(x^{-1/\beta}, \beta, 1) dx \tag{3.70}$$

and

$$I_3(t) = \int_1^\infty t^{-1+\beta/2} x^{-1/2-1/\beta} w(x^{-1/\beta}, \beta, 1) dx. \tag{3.71}$$

Let us examine $I_2(t)$. Since $x \in (0,1)$, we have $w(x^{-1/\beta}, \beta, 1) \sim (x^{-1/\beta})^{-1-\beta}$, so

$$I_2(t) = \int_0^1 t^{-1+\beta/2} x^{-1/2-1/\beta} w(x^{-1/\beta}, \beta, 1) dx$$
$$= \int_0^1 t^{-1+\beta/2} x^{-1/2-1/\beta+1+1/\beta} dx = 2t^{-1+\beta/2}/3. \tag{3.72}$$

Integrating

$$\int_0^t |I_2(t-s)| ds \le \int_0^t (t-s)^{-1+\beta/2} ds = t^{\beta/2}. \tag{3.73}$$

Now, for $I_3(t)$ we use that $x^{-1/\beta} \le 1$ and so $w(x^{-1/\beta}, \beta, 1) \sim C$.

$$
\begin{aligned}
|I_3(t)| &\leq \left| \int_1^{\infty} t^{-1+\beta/2} x^{-1/2-1/\beta} w(x^{-1/\beta}, \beta, 1) dx \right| \\
&\leq Ct^{-1+\beta/2} \left| \int_1^{\infty} x^{-1/2-1/\beta} dx \right| = Ct^{-1+\beta/2}.
\end{aligned}
\tag{3.74}
$$

Integrating with respect to s

$$
\int_0^t |I_3(t-s)| ds \leq \int_0^t (t-s)^{\beta/2-1} ds = Ct^{\beta/2}. \tag{3.75}
$$

Note that $\beta/2 = \beta - \beta/\alpha$ for $\alpha = 2$. So for $\alpha = 2$ the form of the estimate is the same as for $\alpha \in (1,2)$.

□

The following corollary is a consequence of the previous theorem.

Corollary 1. *For $\alpha = 2$ and $\beta \in (0,1)$*

$$
\int_0^t \int_{\mathbb{R}^d} \left(|\nabla G_\beta(t,y)| + |G_\beta(t,y)| \right) dy ds \leq Ct^{\beta/2}. \tag{3.76}
$$

Proof. Since $\beta/2 < \beta$, we take the minimum power, $\beta/2$, to write the common estimate of the terms $\int_{\mathbb{R}^d} |\nabla G_\beta(t,y)| dy$ and $\int_{\mathbb{R}^d} |G_\beta(t,y)| dy$, obtaining

$$
\int_{\mathbb{R}^d} \left(|\nabla G_\beta(t,y)| + |G_\beta(t,y)| \right) dy \leq Ct^{\beta/2-1}, \tag{3.77}
$$

substitute t by $t-s$ and we use that

$$
\int_0^t (t-s)^{\beta/2-1} ds = Ct^{\beta/2}, \tag{3.78}
$$

which yields the result (3.76). □

Here we present several theorems regarding $S_{\beta,1}(t,y)$ which are particularly useful for the well-posedness analysis of (3.4) and (3.1).

Theorem 4. *For $\alpha \in (1,2)$ and $\beta \in (0,1)$ the first term from the RHS of (3.4) satisfies*

$$\left| \int_{\mathbb{R}^d} S_{\beta,1}(t, y-x) f_0(x) dx \right| \leq C t^0. \tag{3.79}$$

Proof. Using (3.22) and (3.29) we represent $S_{\beta,1}(t,y)$ as

$$I = \frac{1}{\beta(2\pi)^d} \int_{\mathbb{R}^d} \int_0^\infty e^{i\langle p,y\rangle} \exp\{-a|p|^\alpha t^\beta \xi\} \xi^{-1-1/\beta} w(\xi^{-1/\beta}, \beta, 1) d\xi dp \tag{3.80}$$

and use the assumption $|f_0(y)| < C$. We split the integral I into two parts: I_A for $\xi \in [|y|^\alpha t^{-\beta}, \infty)$ and I_B for $\xi \in (0, |y|^\alpha t^\beta)$. There are 2 cases for each of the integrals: $|y| \leq t^{\beta/\alpha}$ and $|y| > t^{\beta/\alpha}$. Let us study $|I_B|$ in the case $|y| \leq t^{\beta/\alpha}$.

$$\begin{aligned} |I_B| &\leq C \int_0^{|y|^\alpha t^{-\beta}} \xi^{-1-1/\beta} w(\xi^{-1/\beta}, \beta, 1) |y|^{-d-\alpha} t^\beta \xi d\xi \\ &\leq C \int_0^{|y|^\alpha t^{-\beta}} \xi^{-1/\beta} (\xi^{-1/\beta})^{-1-\beta} t^\beta |y|^{-d-\alpha} d\xi \\ &\leq C \int_0^{|y|^\alpha t^{-\beta}} \xi t^\beta |y|^{-d-\alpha} d\xi \\ &= C(|y|^\alpha t^{-\beta})^2 |y|^{-d-\alpha} t^\beta = C t^{-\beta} |y|^{-d+\alpha}. \end{aligned} \tag{3.81}$$

Now, integrating gives

$$\int_{|y|\leq t^{\beta/\alpha}} |I_B| dy \leq C \int_{|y|\leq t^{\beta/\alpha}} t^{-\beta} |y|^{-d+\alpha+d-1} dy = C t^{-\beta} (t^{\beta/\alpha})^\alpha \leq C t^0. \tag{3.82}$$

Let us study $|I_B|$ in case $|y| > t^{\beta/\alpha}$. Here we split the integral I_B into 2 parts: when $\xi \in (0,1]$ and when $\xi \in (1, |y|^\alpha t^{-\beta})$.

$$|I_B| \leq C \int_0^{|y|^\alpha t^{-\beta}} \xi^{-1/\beta} w(\xi^{-1/\beta}, \beta, 1) t^\beta |y|^{-d-\alpha} d\xi, \tag{3.83}$$

so since for $\xi \leq 1$, $w(\xi^{-1/\beta}, \beta, 1) \sim (\xi^{-1/\beta})^{-1-\beta}$, we have

$$\int_0^1 \xi^{-1/\beta} w(\xi^{-1/\beta}, \beta, 1) t^\beta |y|^{-d-\alpha} d\xi \leq C|y|^{-d-\alpha} t^\beta. \tag{3.84}$$

Integration yields

$$\int_{|y|>t^{\beta/\alpha}} t^\beta |y|^{-d-\alpha+d-1} dy = t^\beta (t^{\beta/\alpha})^{-\alpha} \leq Ct^0. \tag{3.85}$$

When $\xi \in (1, |y|^\alpha t^{-\beta})$

$$\begin{aligned}\int_1^{|y|^\alpha t^{-\beta}} \xi^{-1/\beta} w(\xi^{-1/\beta}, \beta, 1) t^\beta |y|^{-d-\alpha} d\xi &\leq C_q \int_1^{|y|^\alpha t^{-\beta}} \xi^{-1/\beta} \xi^{-q/\beta} t^\beta |y|^{-d-\alpha} d\xi \\ &\leq C_q t^\beta |y|^{-d-\alpha} \left((|y|^\alpha t^{-\beta})^{-1/\beta - q/\beta + 1} - 1 \right) \\ &\leq C_q \left(t^{1+q+\beta-\beta} |y|^{-d-\alpha-\alpha/\beta - q\alpha/\beta + \alpha} - t^\beta |y|^{-d-\alpha} \right). \end{aligned} \tag{3.86}$$

Integration gives

$$\int_{|y|>t^{\beta/\alpha}} t^{1+q} |y|^{-\alpha/\beta - q\alpha/\beta - 1} dy = t^{1+q} (t^{\beta/\alpha})^{-\alpha/\beta - q\alpha/\beta} = t^0, \tag{3.87}$$

and

$$\int_{|y|>t^{\beta/\alpha}} |y|^{-d-\alpha+d-1} t^\beta dy = t^\beta (t^{\beta/\alpha})^{-\alpha} \leq Ct^0. \tag{3.88}$$

Combining (3.85), (3.87) and (3.88) gives

$$\int_{\mathbb{R}^d} |I_B| dy \leq Ct^0. \tag{3.89}$$

Let us study $|I_A|$ case $|y| > t^{\beta/\alpha}$. Here $\xi^{-1/\beta}$ is small, so $w(\xi^{-1/\beta}, \beta, 1) \sim C$, where C is a constant.

$$|I_A| \le C \int_{|y|^\alpha t^{-\beta}}^{\infty} \xi^{-1-1/\beta} w(\xi^{-1/\beta}, \beta, 1) t^{-d\beta/\alpha} \xi^{-d/\alpha} d\xi$$
$$= C \int_{|y|^\alpha t^{-\beta}}^{\infty} \xi^{-1-1/\beta} t^{-\beta d/\alpha} \xi^{-d/\alpha} d\xi$$
$$\le C t^{-\beta d/\alpha} (|y|^\alpha t^{-\beta})^{-1/\beta - d/\alpha} \le C|y|^{-\alpha/\beta - d} t, \tag{3.90}$$

Integrating gives

$$\int_{|y|>t^{\beta/\alpha}} |I_A| dy \le C \int_{|y|>t^{\beta/\alpha}} |y|^{-d-\alpha/\beta+d-1} t dy = Ct(t^{\beta/\alpha})^{-\alpha/\beta} \le Ct^0. \tag{3.91}$$

Let us study $|I_A|$, case $|y| \le t^{\beta/\alpha}$. Here we need to split the integral I_A into 2 parts. The first one is

$$\int_1^\infty \xi^{-d/\alpha} \xi^{-1-1/\beta} t^{-\beta d/\alpha} w(\xi^{-1/\beta}, \beta, 1) d\xi. \tag{3.92}$$

Here ξ is large, so $\xi^{-1/\beta}$ is small, so $w(\xi^{-1/\beta}, \beta, 1) \le C_q (\xi^{-1/\beta})^q$, for all $q > 1$, which enables us to estimate (3.92) by

$$C_q \int_1^\infty \xi^{-d/\alpha} t^{-\beta d/\alpha} \xi^{-1-1/\beta} \xi^{-q/\beta} d\xi = C_q t^{-\beta d/\alpha} \int_1^\infty \xi^{-d/\alpha - 1 - 1/\beta - q/\beta} d\xi$$
$$\le Ct^{-\beta d/\alpha} \left(\lim_{K \to \infty} K^{-d/\alpha - 1/\beta - q/\beta} - 1 \right) = Ct^{-\beta d/\alpha}. \tag{3.93}$$

Integrating gives

$$\int_{|y| \le t^{\beta/\alpha}} t^{-\beta d/\alpha} |y|^{d-1} dy = t^{-\beta d/\alpha} (t^{\beta/\alpha})^d \le Ct^0. \tag{3.94}$$

The second part of I_A is

$$\int_{|y|^\alpha t^{-\beta}}^1 \xi^{-1-1/\beta} w(\xi^{-1/\beta}, \beta, 1) \xi^{-d/\alpha} t^{-\beta d/\alpha} d\xi. \tag{3.95}$$

Since $\xi < 1$, $\xi^{-1/\beta} > 1$, so $w(\xi^{-1/\beta}, \beta, 1) \sim (\xi^{-1/\beta})^{-1-\beta}$, so we re-write (3.95) and estimate it as follows

$$\int_{|y|^\alpha t^{-\beta}}^1 \xi^{-d/\alpha} t^{-\beta d/\alpha} \xi^{-1-1/\beta+1+1/\beta} d\xi$$
$$\leq \int_{|y|^\alpha t^{-\beta}}^1 \xi^{-d/\alpha} t^{-\beta d/\alpha} d\xi = Ct^{-\beta d/\alpha}(1 - |y|^\alpha t^{-\beta}). \tag{3.96}$$

Integrating (3.96) in polar coordinates

$$C \int_{|y|\leq t^{\beta/\alpha}} |y|^{d-1} \left(t^{-\beta d/\alpha} - |y|^\alpha t^{-\beta} t^{-\beta d/\alpha} \right) dy$$
$$\leq Ct^{-\beta d/\alpha} t^{\beta d/\alpha} - Ct^{\beta + d\beta/\alpha - \beta - \beta d/\alpha} = Ct^0. \tag{3.97}$$

Combining (3.97) and (3.94) gives that for $|y| \leq t^{\beta/\alpha}$

$$\int_{\mathbb{R}^d} |I_A| dy \leq Ct^0. \tag{3.98}$$

Using the assumption $|f_0(y)| < C$ and putting together estimates (3.89) and (3.98) yields the theorem statement (3.79). □

Theorem 5. *For $\alpha \in (1,2)$, $\beta \in (0,1)$, $S_{\beta,1}(t,y)$ defined in (3.22) and a bounded function $f_0(y)$*

$$\int_{\mathbb{R}^d} \nabla S_{\beta,1}(t,y) f_0(x-y) dy \leq Ct^{-\beta/\alpha}. \tag{3.99}$$

Proof. We differentiate $S_{\beta,1}(t,y)$ with respect to y

$$|\nabla S_{\beta,1}(t,y)| = \left| \frac{1}{\beta(2\pi)^d} \nabla \int_{\mathbb{R}^d} \int_0^\infty e^{i\langle p,y\rangle} \exp\{-a|p|^\alpha t^\beta x\} x^{-1-1/\beta} w(x^{-1/\beta}, \beta, 1) dx dp \right|$$
$$= \frac{1}{\beta(2\pi)^d} \int_{\mathbb{R}^d} \int_0^\infty |ip| e^{i\langle p,y\rangle} \exp\{-a|p|^\alpha t^\beta x\} x^{-1-1/\beta} w(x^{-1/\beta}, \beta, 1) dx dp$$
$$= \frac{1}{\beta(2\pi)^d} \int_{\mathbb{R}^d} \int_0^\infty |p| e^{i\langle p,y\rangle} \exp\{-a|p|^\alpha t^\beta x\} x^{-1-1/\beta} w(x^{-1/\beta}, \beta, 1) dx dp. \tag{3.100}$$

Here we use the asymptotic expansions from Theorems 7.2.1 and 7.2.2 and Theorem 7.3.2, which are in the appendix as equations (7.2) and (7.6), and we use the inequality (7.40) in

[82], which also appears in the appendix for reader's convenience, as (7.11) and (7.12). For I_A in case $|y| > t^{\beta/\alpha}$ we use that for $\xi > 1$, $\xi^{-1/\beta} < 1$ and $w(\xi^{-1/\beta}, \beta, 1) < C_q(\xi^{-1/\beta})^q$, for any $q > 1$. Then

$$|\nabla I_A| \leq C \int_{|y|^\alpha t^{-\beta}}^{\infty} \xi^{-1/\alpha-1-1/\beta-d/\alpha} w(\xi^{-1/\beta}, \beta, 1) t^{-\beta/\alpha-d\beta/\alpha} d\xi$$
$$\leq Ct^{-\beta/\alpha-d\beta/\alpha}(|y|^\alpha t^{-\beta})^{-1/\alpha-d/\alpha-1/\beta-q/\beta} \leq Ct^{1+q}|y|^{-1-q\alpha/\beta-\alpha/\beta-d}. \tag{3.101}$$

Integrating gives

$$\int_{|y|>t^{\beta/\alpha}} |\nabla I_A| \, dy \leq C \int_{|y|>t^{\beta/\alpha}} t^{1+q}|y|^{-d+d-1-1-q\alpha/\beta-\alpha/\beta} dy$$
$$= Ct^{1+q-\beta/\alpha-q-1} = Ct^{-\beta/\alpha}. \tag{3.102}$$

Now, let us look at I_B in case $|y| > t^{\beta/\alpha}$. Proposition 1 in the Appendix and the change of variables $\xi^{-1/\beta} = z$ yield

$$|\nabla I_B| \leq C \int_0^{|y|^\alpha t^{-\beta}} \xi^{-1-1/\beta} w(\xi^{-1/\beta}, \beta, 1) t^{-\beta} \xi^{-1} |y|^{-\alpha-1} |y|^{-d-\alpha} t^\beta \xi d\xi$$
$$\leq C|y|^{-d-1} \int_{|y|^{-\alpha/\beta}t}^{\infty} w(z, \beta, 1) dz \leq C|y|^{-d-1}. \tag{3.103}$$

Integration gives

$$\int_{|y|>t^{\beta/\alpha}} |\nabla I_B| \, dy \leq C \int_{|y|>t^{\beta/\alpha}} |y|^{-d+d-1-1} dy \leq Ct^{-\beta/\alpha}. \tag{3.104}$$

Now, let us look at I_A in case $|y| < t^{\beta/\alpha}$.

$$|\nabla I_A| \leq C \int_{|y|^\alpha t^{-\beta}}^{\infty} \xi^{-1-1/\beta} w(\xi^{-1/\beta}, \beta, 1) t^{-\beta/\alpha} \xi^{-1/\alpha} t^{-\beta d/\alpha} \xi^{-d/\alpha} d\xi. \tag{3.105}$$

We split this integral into cases $\xi \in (|y|^\alpha t^{-\beta}, 1)$ and $\xi \in [1, \infty)$. For $\xi \in (|y|^\alpha t^{-\beta}, 1)$, $\xi^{-1/\beta} > 1$

and we may use that $w(\xi^{-1/\beta}, \beta, 1) \sim (x^{-1/\beta})^{-1-\beta}$. So

$$
\begin{aligned}
|\nabla I_A| &\leq C \int_{|y|^\alpha t^{-\beta}}^{1} \xi^{-1-1/\beta} w(\xi^{-1/\beta}, \beta, 1) t^{-\beta/\alpha} \xi^{-1/\alpha} t^{-\beta d/\alpha} \xi^{-d/\alpha} d\xi \\
&\leq C \int_{|y|^\alpha t^{-\beta}}^{\infty} t^{-\beta/\alpha - \beta d/\alpha} \xi^{-1/\alpha - d/\alpha} d\xi \\
&= Ct^{-\beta/\alpha - \beta d/\alpha} - Ct^{-\beta} |y|^{-1-d+\alpha}. \qquad (3.106)
\end{aligned}
$$

Integration yields

$$
\begin{aligned}
\int_{|y| \leq t^{\beta/\alpha}} |\nabla I_A| \, dy &\leq C \int_{|y| \leq t^{\beta/\alpha}} y^{d-1} t^{-\beta/\alpha - \beta d/\alpha} dy \\
&+ C \int_{|y| \leq t^{\alpha/\beta}} t^{-\beta} |y|^{-1-\alpha-1-d+d} dy \\
&\leq Ct^{-\beta/\alpha} + Ct^{-\beta} (t^{\beta/\alpha})^{-1+\alpha} \leq Ct^{-\beta/\alpha}. \qquad (3.107)
\end{aligned}
$$

As for $\xi \in [1, \infty)$, then $\xi^{-1/\beta} < 1$ and so $w(\xi^{-1/\beta}, \beta, 1) \sim C$ and

$$
\begin{aligned}
&\int_1^\infty \xi^{-2-1/\beta} C \xi^{-d/\alpha} t^{-\beta/\alpha - \beta d/\alpha} d\xi \\
&\leq t^{-\beta/\alpha - \beta d/\alpha} C \int_1^\infty \xi^{-2-1/\beta - d/\alpha} d\xi \\
&\leq t^{-\beta/\alpha - \beta d/\alpha} C \left(1 - \lim_{K \to \infty} \frac{1}{K} \right) = Ct^{-\beta/\alpha - \beta d/\alpha}. \qquad (3.108)
\end{aligned}
$$

Integration yields

$$
\int_{|y| < t^{\beta/\alpha}} t^{-\beta/\alpha - \beta d/\alpha} |y|^{d-1} dy \leq Ct^{-\beta/\alpha - \beta d/\alpha} t^{\beta d/\alpha} = Ct^{-\beta/\alpha}. \qquad (3.109)
$$

Finally, the estimate for $|\nabla I_B|$ in case $|y| \le t^{\beta/\alpha}$ is

$$\begin{aligned}
|\nabla I_B| &\le C \int_0^{|y|^\alpha t^{-\beta}} \xi^{-1-1/\beta} w(\xi^{-1/\beta}, \beta, 1) \xi^{-1} t^{-\beta} |y|^{\alpha-1} |y|^{-d-\alpha} t^\beta \xi d\xi \\
&\le C \int_0^{|y|^\alpha t^{-\beta}} \xi^{-1-1/\beta} w(\xi^{-1/\beta}, \beta, 1) |y|^{-1-d} d\xi \\
&\le C \int_0^{|y|^\alpha t^{-\beta}} \xi^{-1-1/\beta} (\xi^{-1/\beta})^{-1-\beta} |y|^{-1-d} d\xi \\
&\le C |y|^{-1-d-\alpha} t^{-\beta}.
\end{aligned} \tag{3.110}$$

Integration yields

$$\begin{aligned}
\int_{|y|\le t^{\beta/\alpha}} |\nabla I_B|\, dy &\le C \int_{|y|\le t^{\beta/\alpha}} |y|^{\alpha-1-1-d+d} t^{-\beta} dy \\
&= C t^{-\beta} (t^{\beta/\alpha})^{\alpha-1} = C t^{-\beta/\alpha}.
\end{aligned} \tag{3.111}$$

Hence (3.102), (3.104), (3.107), (3.109) and (3.111) together with the assumption $|f_0(y)| < C$ yield (3.98). □

Theorem 6. *For $\alpha = 2$, $S_{\beta,1}$ defined as in (3.22) and assuming $|f_0(y)| < C$*

$$\int_{\mathbb{R}^d} S_{\beta,1}(t, y - x) f_0(x) dx \le C t^0. \tag{3.112}$$

Proof. Using (3.61)

$$\begin{aligned}
\int_{\mathbb{R}^d} S_{\beta,1}(t,y) dy &= \int_0^\infty \int_{\mathbb{R}^d} (xt^\beta)^{-d/2} \exp\{-y^2/(4axt^\beta)\} x^{-1-1/\beta} w(x^{-1/\beta}, \beta, 1) dy dx \\
&= C \int_0^\infty x^{-1-1/\beta} w(x^{-1/\beta}, \beta, 1) dx.
\end{aligned} \tag{3.113}$$

We split this integral into two parts: $x \in [0,1]$ and $x \in (1, \infty)$. In the first case $x \le 1$ and $x^{-1/\beta} > 1$ so we may use $w(x^{-1/\beta}, \beta, 1) \sim (x^{-1/\beta})^{-1-\beta}$. In case $x > 1$ we may use that $w(x^{-1/\beta}, \beta, 1) \sim C$. So we obtain

$$\int_0^1 x^{-1-1/\beta} w(x^{-1/\beta}, \beta, 1) dx = C \int_0^1 dx = C, \tag{3.114}$$

and

$$\int_1^\infty x^{-1-1/\beta} w(x^{-1/\beta},\beta,1)dx = \int_1^\infty x^{-1-1/\beta} C dx$$
$$= C\left(\lim_{K\to\infty} K^{-1/\beta} - 1^{-1/\beta}\right) = C. \quad (3.115)$$

Together with the assumption $|f_0(y)| < C$, the result (3.112) follows. □

Theorem 7. *For $\alpha = 2$, $\beta \in (0,1)$, $S_{\beta,1}(t,y)$ defined in (3.22) and assuming $|f_0(y)| < C$*

$$\int_{\mathbb{R}^d} \nabla S_{\beta,1}(t,y) f_0(x-y)dy \le Ct^{-\beta/2}. \quad (3.116)$$

Proof. We use the representation of $S_{\beta,1}(t,y)$ in (3.80) and write

$$\int_{\mathbb{R}^d} \nabla S_{\beta,1}(t,y)dy$$
$$= \int_0^\infty x^{-3/2-1/\beta} t^{-\beta/2} w(x^{-1/\beta},\beta,1)dx. \quad (3.117)$$

We split the above integral into two: for $x \in [0,1]$ and for $x > 1$. In case $x \in [0,1]$ we use that $w(x^{-1/\beta},\beta,1) \sim (x^{-1/\beta})^{-1-\beta}$. In case $x > 1$ we use that $w(x^{-1/\beta},\beta,1) \sim C$. So we get

$$t^{-\beta/2}\int_0^1 x^{-3/2-1/\beta} w(x^{-1/\beta},\beta,1)dx = t^{-\beta/2}\int_0^1 x^{-1/2}dx = t^{-\beta/2}/2 \quad (3.118)$$

and

$$t^{-\beta/2}\int_1^\infty x^{-3/2-1/\beta} w(x^{-1/\beta},\beta,1)dx = t^{-\beta/2}. \quad (3.119)$$

So from (3.118) and (3.119) and that $|f_0(y)| < C$ and we obtain (3.116). □

3.3 Smoothing properties for the linear equation

Let us denote by $C^p(\mathbb{R}^d)$ the space of p times continuously differentiable in y functions $f(t,y)$. By $C^1_\infty(\mathbb{R}^d)$ we shall denote functions f in $C^1(\mathbb{R}^d)$ such that $f(t,y)$ and $\nabla f(t,y)$ are rapidly decreasing continuous functions on $\mathbb{R}^d$, with the sum of sup-norms of the function and all of its derivatives up to and including the order p as the corresponding norm. Let us denote by H_1^p the Sobolev space of functions $f(t,y)$ with generalised spatial derivative up to and including p, being in $L^1(\mathbb{R}^d)$. For $f \in H_1^p(\mathbb{R}^d)$ the sup-norm is $\|f\| = \sup_{t\in[0,T]} \|f(t,y)\|_{H_1^p}$ and for $f \in C^p(\mathbb{R}^d)$ the sup-norm is $\|f\| = \sup_{t\in[0,T]} \|f(t,y)\|_{C^p}$. Here and in what follows we often identify the function $f_0(y)$ with the function $g(t,y) := f_0(y)$, $\forall t \geq 0$, $\forall y \in \mathbb{R}^d$.

Theorem 8 (Solution regularity)**.** *For $\alpha \in (1,2]$ and $\beta \in (0,1)$ the resolving operator*

$$\Psi_t(f_0) = \int_{\mathbb{R}^d} S_{\beta,1}(t, y-x) f_0(x) dx + \int_0^t \int_{\mathbb{R}^d} G_\beta(t-s, y-x) h(s,x) dx ds \tag{3.120}$$

where $S_{\beta,1}(t,y)$ and $G_\beta(t,y)$ are as in (3.22) and (3.23), satisfies the following properties

- $\Psi_t : C^p(\mathbb{R}^d) \mapsto C^p(\mathbb{R}^d)$, *and* $\|\Psi_t\|_{C^p} < C(t)$,
- $\Psi_t : H_1^p(\mathbb{R}^d) \mapsto H_1^p(\mathbb{R}^d)$ *and* $\|\Psi_t\|_{H_1^p} < C(t)$.

Proof. We look at the C^p norm of $\Psi_t f_0$ and use Theorems 1 and 4

$$\begin{aligned} \|\Psi_t(f_0)\|_{C^p(\mathbb{R}^d)} &\leq \int_{\mathbb{R}^d} S_{\beta,1}(t,y) |f_0^{(p)}| dy + Ct^\beta \sup_{s\in[0,t]} \|h(s,\cdot)\|_{C^p(\mathbb{R}^d)} \\ &\leq C\|f_0\|_{C^p(\mathbb{R}^d)} + Ct^\beta \sup_{t\in[0,t]} \|h(s,\cdot)\|_{C^p(\mathbb{R}^d)}, \end{aligned} \tag{3.121}$$

for some constant $C > 0$. Analogously,

$$\begin{aligned} \|\Psi_t f_0\|_{H_1^p(\mathbb{R}^d)} &\leq \int_{\mathbb{R}^d} S_{\beta,1}(t,y) |f_0^{(p)}| dy + Ct^\beta \sup_{s\in[0,t]} \|h(s,\cdot)\|_{H_1^p(\mathbb{R}^d)} \\ &\leq C\|f_0\|_{H_1^p(\mathbb{R}^d)} + Ct^\beta \sup_{s\in[0,t]} \|h\|_{H_1^p(\mathbb{R}^d)}. \end{aligned} \tag{3.122}$$

□

Theorem 9 (Solution smoothing)**.** *For $\alpha \in (1,2]$ and $\beta \in (0,1)$ the resolving operator (3.120) satisfies the following smoothing properties*

- *If $f_0, h \in C^p(\mathbb{R}^d)$ uniformly in time, then $f \in C^{p+1}(\mathbb{R}^d)$ and for any $t \in (0,T]$*

$$\|\Psi_t(f_0)\|_{C^{p+1}(\mathbb{R}^d)} \leq Ct^{-\beta/\alpha} \|f_0\|_{C^p(\mathbb{R}^d)} + Ct^{\beta-\beta/\alpha} \|h\|_{C^p(\mathbb{R}^d)} \tag{3.123}$$

- *If $f_0, h \in H_1^p(\mathbb{R}^d)$ uniformly in time, then $f \in H_1^{p+1}(\mathbb{R}^d)$ and for any $t \in (0,T]$*

$$\|\Psi_t(f_0)\|_{H_1^{p+1}(\mathbb{R}^d)} \leq Ct^{-\beta/\alpha}\|f_0\|_{H_1^p(\mathbb{R}^d)} + Ct^{\beta-\beta/\alpha}\|h\|_{H_1^p(\mathbb{R}^d)}. \tag{3.124}$$

In particular we may choose $p = 0$, when $H_1^0(\mathbb{R}^d) = L^1(\mathbb{R}^d)$.

Proof. We study the $C^{p+1}(\mathbb{R}^d)$ norm of $\Psi_t(f_0)$ and use theorems 1, 2, 4 and 5

$$\begin{aligned}\|\Psi_t(f_0)\|_{C^{p+1}} &\leq \sup_{x\in\mathbb{R}^d}\int_{\mathbb{R}^d}\left|\nabla_x S_{\beta,1}(t,x-y)f_0^{(p)}(y)\right|dy \\ &+ \sup_{x\in\mathbb{R}^d}\int_0^t\int_{\mathbb{R}^d}\left|\nabla_x G_\beta(t-s,x-y)h_y^{(p)}(s,y)\right|dyds \\ \leq Ct^{-\beta/\alpha}\sup_{x\in\mathbb{R}^d}|f_0^{(p)}(x)| &+ C\sup_{x\in\mathbb{R}^d}|h^{(p)}(s,x)|\int_0^t(t-s)^{\beta-\beta/\alpha-1}ds \\ &\leq Ct^{-\beta/\alpha}\|f_0\|_{C^p} + Ct^{\beta-\beta/\alpha}\|h\|_{C^p}.\end{aligned} \tag{3.125}$$

The proof for (3.124) is analogous. □

3.4 Well-posedness

Now we study well-posedness of the full non-linear equation (3.1):

$$D_{0,t}^{*\beta}f(t,y) = -a(-\Delta)^{-\alpha/2}f(t,y) + H(t,y,\nabla f(t,y)), \tag{3.126}$$

with the initial condition $f(0,y) = f_0(y)$, and $a > 0$ is a constant. This FDE has the following mild form:

$$f(t,y) = \int_{\mathbb{R}^d} f_0(x)S_{\beta,1}(t,y-x)dx + \int_0^t\int_{\mathbb{R}^d} G_\beta(t-s,y-x)H(s,x,\nabla f(s,x))dxds, \tag{3.127}$$

which follows from (3.19).

Lemma 1. *Let us define by $C([0,T], C_\infty^1(\mathbb{R}^d))$ the space of functions $f(t,y), t \in [0,T], y \in \mathbb{R}^d$ such that $f(t,y)$ is continuous in t and $f(t,\cdot) \in C_\infty^1(\mathbb{R}^d)$ for all t. Denote by $B_{f_0}^T$ the closed convex subset of $C([0,T], C_\infty^1(\mathbb{R}^d))$ consisting of functions with $f(0,\cdot)) = f_0(\cdot) = S_0(\cdot)$ for* *some given function* S_0*. Let us define a non-linear mapping* $f \to \{\Psi_t(f)\}$ *defined for* $f \in B^T_{f_0}$*:*

$$\Psi_t(f)(y) = \int_{\mathbb{R}^d} f_0(x) S_{\beta,1}(t, y-x)dx$$
$$+ \int_0^t \int_{\mathbb{R}^d} G_\beta(t-s, y-x) H(s, x, \nabla f(s,x)) dx ds. \tag{3.128}$$

Suppose $H(s, y, p)$ *is Lipschitz in* p *with the Lipschitz constant* L*. Let us take* $f_1, f_2 \in B^T_{f_0}$*. Then for* $K = \frac{1}{\beta - \beta/\alpha}$ *and for any* $t \in [0, T]$*:*

$$\|\Psi^n_t(f_1) - \Psi^n_t(f_2)\|_{C^1} \le \frac{(\beta - \beta/\alpha) L^n (K t^{(\beta-\beta/\alpha)})^n}{n^{n\beta - n\beta/\alpha + 1}} \sup_{s \in [0,t]} \|f_1(s) - f_2(s)\|_{C^1}. \tag{3.129}$$

Proof. Due to regularity estimates for $S_{\beta,1}$ and G_β:

$$\|\Psi_t(f_1) - \Psi_t(f_2)\|_{C^1} \le C L t^{\beta - \beta/\alpha} \sup_{s \in [0,t]} \|f_1(s) - f_2(s)\|_{C^1}. \tag{3.130}$$

and

$$\|\Psi^2_t(f_1) - \Psi^2_t(f_2)\|_{C^1} \le C^2 L^2 \sup_{s \in [0,t]} \|f_1(s) - f_2(s)\|_{C^1} \int_0^t (t-s)^{\beta - \beta/\alpha - 1} s^{\beta - \beta/\alpha} ds. \tag{3.131}$$

We calculate the integral above using the change of variables $z = s/t$:

$$\int_0^t (t-s)^{\beta-\beta/\alpha-1} s^{\beta-\beta/\alpha} ds$$
$$= \int_0^1 t^{\beta-\beta/\alpha-1} (1-z)^{\beta-\beta/\alpha-1} z^{\beta-\beta/\alpha} t^{\beta-\beta/\alpha+1} dz$$
$$= t^{2\beta - 2\beta/\alpha} B(\beta - \beta/\alpha + 1, \beta - \beta/\alpha). \tag{3.132}$$

Now, when we estimate $\|\Psi^3_t(f_1) - \Psi^3_t(f_2)\|_{C^1}$ we calculate

$$\int_0^t s^{2\beta-2\beta/\alpha} (t-s)^{\beta-\beta/\alpha-1} ds$$
$$= t^{\beta-\beta/\alpha-1} \int_0^1 t^{2\beta-2\beta/\alpha+1} z^{2\beta-2\beta/\alpha} (1-z)^{\beta-\beta/\alpha-1} dz$$
$$= t^{3(\beta-\beta/\alpha)} B(2\beta - 2\beta/\alpha + 1, \beta - \beta/\alpha). \tag{3.133}$$

This yields

$$\|\Psi_t^3(f_1) - \Psi_t^3(f_2)\|_{C^1} \le C^3 L^3 t^{3\beta - 3\beta/\alpha} B(2\beta - 2\beta/\alpha + 1, \beta - \beta/\alpha) \sup_{s\in[0,t]} \|f_1(s) - f_2(s)\|_{C^1}. \tag{3.134}$$

As the inductive step, assume that the following is true for some $n \in \mathbb{N}$:

$$\|\Psi_t^n(f_1) - \Psi_t^n(f_2)\|_{C^1} \le C^n L^n t^{n\beta - n\beta/\alpha} \frac{(\Gamma(\beta - \beta/\alpha))^{n-1}\Gamma(\beta - \beta/\alpha + 1)}{\Gamma(n\beta - n\beta/\alpha + 1)} \sup_{s\in[0,t]} \|f_1(s) - f_2(s)\|_{C^1}. \tag{3.135}$$

Let us check that then (3.135) holds for $k = n + 1$.

$$\begin{aligned}
&\|\Psi_t^{n+1}(f_1) - \Psi_t^{n+1}(f_2)\|_{C^1} \\
&= \left\| \int_0^t \int_{\mathbb{R}^d} G_\beta(t-s, x-y) \left(H(s, y, \nabla\Psi_t^n(f_1)) - H(s, y, \nabla\Psi_t^n(f_2))\right) \right\|_{C^1} \\
&\le CL \int_0^t (t-s)^{\beta - \beta/\alpha - 1} ds \|\Psi_t^n(f_1) - \Psi_t^n(f_2)\|_{C^1} \\
&\le C^{n+1} L^{n+1} t^{n\beta - n\beta/\alpha} M_n \int_0^t (t-s)^{\beta-\beta/\alpha-1} s^{n\beta - n\beta/\alpha} ds \sup_{s\in[0,t]} \|f_1(s) - f_2(s)\|_{C^1} \\
&\le C^{n+1} L^{n+1} t^{n\beta - n\beta/\alpha} M_n B_n \sup_{s\in[0,T]} \|f_1(s) - f_2(s)\|_{C^1} \\
&\le C^{n+1} L^{n+1} t^{n\beta - n\beta/\alpha} M_{n+1} \sup_{s\in[0,t]} \|f_1(s) - f_2(s)\|_{C^1},
\end{aligned} \tag{3.136}$$

where

$$M_n = \frac{(\Gamma(\beta - \beta/\alpha))^{n-1}\Gamma(\beta - \beta/\alpha + 1)}{\Gamma(n\beta - n\beta/\alpha + 1)}, \tag{3.137}$$

M_{n+1} is as in (3.137) with n replaced by $n + 1$, and B_n is the Beta function

$$B_n = B(n\beta - n\beta/\alpha + 1, \beta - \beta/\alpha). \tag{3.138}$$

The inequality (3.136) is (3.135) with $k = n$ replaced by $k = n + 1$. We have shown (3.135) is true for $k = 1$ and $k = 2$. So by induction on k we obtain (3.135) for any $k \in \mathbb{N}$. Using that $g(x) = x^n$ is a convex function for $n \in \mathbb{N}$ and substituting $x = \Gamma(\beta - \beta/\alpha)$, it may be shown by Jensen's inequality that

$$(\Gamma(\beta - \beta/\alpha))^n \leq \frac{\Gamma(n\beta - n\beta/\alpha - n + 1)}{n^{n\beta - n\beta/\alpha - n + 1}}, \tag{3.139}$$

and using Stirling's formula we obtain the quotient approximation

$$\frac{\Gamma(n(\beta - \beta/\alpha) + B)}{\Gamma(n(\beta - \beta/\alpha) + A)} \approx (n(\beta - \beta/\alpha))^{B-A}. \tag{3.140}$$

Let us substitute $A = 1$ and $B = -n + 1$. Then

$$\begin{aligned} &\|\Psi_t^n(f_1) - \Psi_t^n(f_2)\|_{C^1} \\ \leq \frac{\Gamma(1 + \beta - \beta/\alpha)t^{n\beta - n\beta/\alpha}}{n^{n\beta - n\beta/\alpha + 1}(\beta - \beta/\alpha)^n \Gamma(\beta - \beta/\alpha)} &\sup_{s \in [0,t]} \|f_1(s) - f_2(s)\|_{C^1} \\ \leq \frac{t^{n\beta - n\beta/\alpha}}{n^{n\beta - n\beta/\alpha + 1}(\beta - \beta/\alpha)^{n-1}} &\sup_{s \in [0,t]} \|f_1(s) - f_2(s)\|_{C^1}, \end{aligned} \tag{3.141}$$

so (3.129) holds. □

Theorem 10. *Assume that*

- *$H(s, y, p)$ is Lipschitz in p with the Lipschitz constant L independent of y.*
- *$|H(s, y, 0)| \leq h$, for a constant h independent of y.*
- *$f_0(y) \in C^1_\infty(\mathbb{R}^d)$.*

Then the equation (3.127) *has a unique solution $S(t, y) \in C^1(\mathbb{R}^d)$.*

Proof. Let us denote by $C([0, T], C^1_\infty(\mathbb{R}^d))$ and $B^T_{f_0}$ as in Lemma 1. Let $\Psi_t(f)$ be defined as in (3.128). Take $f_1(s, x), f_2(s, x) \in B^T_{f_0}$. Note that due to our choice of f_1, f_2,

$$\int_{\mathbb{R}^d} f_1(0, x) S_{\beta,1}(t, y - x) dx = \int_{\mathbb{R}^d} f_2(0, x) S_{\beta,1}(t, y - x) dx. \tag{3.142}$$

We would like to prove the existence and uniqueness result for all $t \leq T$ and any $T \geq 0$. For this we use (3.129) in Lemma 1. As $n \to \infty$, n^n grows faster than m^n for any fixed $m > 0$. Hence for any $t \geq 0$

$$\|\Psi_t^n(f_1) - \Psi_t^n(f_2)\|_{C^1} \leq \frac{L^n(t^{\beta-\beta/\alpha}(\beta-\beta/\alpha)^{-1})^n(\beta-\beta/\alpha)}{n^{n\beta-n\beta/\alpha+1}} \sup_{s\in[0,t]} \|f_1(s) - f_2(s)\|_{C^1}. \tag{3.143}$$

The sum $\sum_{n=1}^{\infty} \frac{(t^{\beta-\beta/\alpha}(\beta-\beta/\alpha)^{-1})^n}{n^{n(\beta-\beta/\alpha)+1}}$ is convergent by the ratio test. By Weissinger's fixed point theorem, see [57] Theorem D.7, Ψ_t has a unique fixed point f^* such that for any $f_1 \in B_{f_0}^T$

$$\|\Psi_t^n(f_1) - f^*\|_{C^1} \leq \sum_{k=n}^{\infty} \frac{(t^{\beta-\beta/\alpha}(\beta-\beta/\alpha)^{-1})^n(\beta-\beta/\alpha)}{n^{n\beta-n\beta/\alpha+1}} \|\Psi_t(f_1) - f_1\|_{C^1}. \tag{3.144}$$

So $S(t,y) = f^*$ is the solution of (3.127) of class $C^1(\mathbb{R}^d)$. □

Theorem 11. *Assume that*

- *$H(s,y,p)$ is Lipschitz in p with the Lipschitz constant L_1 independent of y.*
- *H is Lipschitz in y independently of p, with a Lipschitz constant L_2*

$$|H(s,y_1,p) - H(s,y_2,p)| \leq L_2|y_1 - y_2|(1+|p|) \tag{3.145}$$

- *$|H(s,y,0)| \leq h$, for a constant h independent of y.*
- *$f_0(y) \in C_\infty^2(\mathbb{R}^d)$.*

Then there exists a unique solution $f^*(t,y)$ of the FDE equation (3.126) for $\beta \in (0,1)$ and $\alpha \in (1,2]$, and f^* satisfies

$$ess \sup_y |\nabla^2(f^*(t,y))| < C. \tag{3.146}$$

Proof. Let us work with the mild form of the equation (3.126). Let $B_{f_0}^{T,2}$ denote the subset of $B_{f_0}^T$ which is twice continuously differentiable in y and with $f_0(y) = f_0(y)$, for all $y \in \mathbb{R}^d$. Let the mapping Ψ_t on $B_{f_0}^{T,2}$ be defined as in (3.128). Take $f_0 \in B_{f_0}^{T,2}$, which continues

$f_0(y) = S_0(y)$ to all $t \geq 0$. Then

$$\|\Psi_t(f_0)\|_{C^2} \leq t^{\beta-\beta/\alpha} \sup_{s\in[0,t]} \|H(s,x,\nabla f_0(x))\|_{C^1}$$
$$+\| \int_{\mathbb{R}^d} S_{\beta,1}(t,y-x) f_0(x)dx\|_{C^2}$$
$$\leq t^{\beta-\beta/\alpha} L_1 \sup_{s\in[0,t]} \|f_0\|_{C^2} + t^{\beta-\beta/\alpha} L_2 \sup_{s\in[0,t]} \|f_0\|_{C^1} + Ct^{\beta-\beta/\alpha}\|\nabla f_0(x)\|_{C^0} + C_3$$
$$\leq Lt^{\beta-\beta/\alpha} \sup_{s\in[0,t]} \|f_0\|_{C^2} + Ct^{\beta-\beta/\alpha} \sup_{s\in[0,t]} \|f_0(x)\|_{C^1} + C_3$$
$$\leq Ct^{\beta-\beta/\alpha} \left(\sup_{s\in[0,t]} \|f_0\|_{C^2} + 1 \right) + C_3. \tag{3.147}$$

Iterations and induction yield

$$\|\Psi_t^n(f_0)\|_{C^2} \leq C_3 \sum_{m=1}^{n} t^{m\beta-m\beta/\alpha} K_m + \sum_{m=1}^{n} t^{m(\beta-\beta/\alpha)} C_m \left(1 + \sup_{s\in[0,t]} \|f_0\|_{C^2} \right), \tag{3.148}$$

for constants $K_m = B_2 \times \cdots \times B_{m-1}$ and $C_m = B_2 \times \cdots \times B_m$, where $B_k = B(k\beta - k\beta/\alpha + 1, \beta - \beta/\alpha)$, for any $k \in \mathbb{N}$. We use that for x large and y fixed $B(x,y) \sim \Gamma(y)x^{-y}$ to obtain that $B_{m+1} < B_m$, for all $m \in \mathbb{N}$ which yields that the sums $\sum_{m=1}^{n} t^{m\beta-m\beta/\alpha} K_m$ and $\sum_{m=1}^{n} t^{m\beta-m\beta/\alpha} C_m$ are convergent as $n \to \infty$. So for some constants $A_1, A_2, C_{f_0} > 0$,

$$\|\Psi_t^n f_0\|_{C^2} < A_1 + A_2 \sup_{s\in[0,t]} \|f_0\|_{C^2} < C_{f_0}. \tag{3.149}$$

Hence, $\forall n \in \mathbb{N}$

$$\|\nabla(\Psi_t^n f_0)\|_{Lip} < C_{f_0}. \tag{3.150}$$

Hence we obtain

$$\| \lim_{n\to\infty} \nabla(\Psi_t^n f_0)\|_{Lip} < 2C_{f_0}. \tag{3.151}$$

By Rademacher's theorem it follows that $\lim_{n\to\infty}(\nabla^2(\Psi_t^n(f_0))$ exists almost everywhere. We invite the reader to see [83] for the Rademacher's theorem and its proof. From the previous theorem $\lim_{n\to\infty} \Psi_t^n(f_0) = f^*$. The limit is understood in the sense of convergence in $C^1(\mathbb{R}^d)$.

Therefore f^* satisfies (3.146).

□

Theorem 12. *Assume that*

- $H(s,y,p)$ *is Lipschitz in* p *with the Lipschitz constant* L *independent of* y.
- H *is Lipschitz in* y *independently of* p, *with a Lipschitz constant* L_2

$$|H(s,y_1,p)-H(s,y_2,p)|\leq L_2|y_1-y_2|(1+|p|) \tag{3.152}$$

- $|H(s,y,0)|\leq h$, *for a constant* h *independent of* y.
- $f_0(y)\in C^2_\infty(\mathbb{R}^d)$.

Then a solution to the mild form

$$f(t,y)=\int_{\mathbb{R}^d}S_{\beta,1}(t,x-y)f_0(y)dy+\int_0^t\int_{\mathbb{R}^d}G_\beta(t-s,x-y)H(s,y,\nabla f(s,y))dsdy \tag{3.153}$$

which satisfies (3.146), is a classical solution to

$$D^{*\beta}_{0,t}f(t,y)=-(-\Delta)^{\alpha/2}f(t,y)+H(t,y,\nabla f(t,y)). \tag{3.154}$$

Proof. Let us define $\Psi_t(f)$ as in (3.128). By [57]

$$\hat{f}(t,p)=\hat{f}_0(p)E_{\beta,1}(-a|p|^\alpha t^\beta)+\int_0^t(t-s)^{\beta-1}E_{\beta,\beta}(-a(t-s)^\beta|p|^\alpha))\hat{H}(s,y,p)ds, \tag{3.155}$$

is equivalent to

$$D^{*\beta}_{0,t}\hat{f}(t,p)=-a|p|^\alpha\hat{f}(t,p)+\hat{H}(t,y,\nabla f(t,y)), \tag{3.156}$$

which in turn is equivalent to (3.154) as its Fourier transform. Also, (3.153) is equivalent to (3.19) as its inverse Fourier transform. Therefore (3.153) is equivalent to (3.154). We may carry out these equivalence procedures when $D^{*\beta}_{0,t}\Psi_t(f)$ and $-(-\Delta)^{\alpha/2}f$ are defined for f satisfying (3.146). Due to theorem assumptions:

$$|H(s,y,\nabla f(s,\cdot))| \leq h + L|\nabla f(s,\cdot)| < \infty. \tag{3.157}$$

So

$$\begin{aligned} & D^{*\beta}_{0,t}\left(\int_0^t \int_{\mathbb{R}^d} G_\beta(t,y) H(s,y,\nabla f(s,y)) dy ds\right) \\ \leq & \frac{C}{\Gamma[1-\beta]} \int_0^t (t-s)^{-\beta} s^\beta ds \leq C_1 \int_0^1 (t-tz)^{-\beta} \beta (tz)^{\beta-1} t dz \\ & \leq C_1 \beta \int_0^1 (1-z)^{1-\beta-1} z^{\beta-1} dz \leq C_1 \beta B(1-\beta,\beta) < \infty. \end{aligned} \tag{3.158}$$

Similarly

$$D^{*\beta}_{0,t} \int_{\mathbb{R}^d} S_{\beta,1}(t,x-y) f_0(y) dy \tag{3.159}$$

exists when $f_0(y)$ gives dependence of $\int_{\mathbb{R}^d} S_{\beta,1}(t,x-y)f_0(y)dy$ on t such as t^k, where $k > -1$. This is because

$$\begin{aligned} & \int_0^t (t-s)^{-\beta} \left(\frac{d}{ds} s^k\right) ds \\ = t^{k+1-\beta} & \int_0^1 (1-z)^{-\beta} z^{k-1} dz = t^{k+1-\beta} B(1-\beta, k+1), \end{aligned} \tag{3.160}$$

where for any $\beta \in (0,1)$ the Beta function $B(1-\beta,k+1)$ is defined for $k+1 > 0$. Hence, due to (3.157), (3.158) and (3.159), $D^{*\beta}_{0,t}\Psi_t(f)$ is defined for the solution f for (3.154). For f satisfying (3.146), when $\alpha \in (1,2]$, $-(-\Delta)^{\alpha/2}f$ is defined. Now, let us study the solution $f^*(t,y)$

$$\begin{aligned} f^*(t,y) = & \int_0^t \int_{\mathbb{R}^d} G_\beta(t-s,y-x) H(s,x,\nabla f^*(t,x)) dx ds \\ & + \int_{\mathbb{R}^d} S_{\beta,1}(t,y-x) f_0(x) dx. \end{aligned} \tag{3.161}$$

Differentiating (3.161) twice w.r.t. y gives:

$$\nabla^2 \int_0^t \int_{\mathbb{R}^d} G_\beta(t-s, y-x) H(s, x, \nabla f^*(s, x)) dx ds$$
$$= \int_0^t \int_{\mathbb{R}^d} \nabla_y G_\beta(t-s, y-x) \nabla_x H(s, x, \nabla f^*(s, x)) dx ds. \tag{3.162}$$

From the representations of $G_\beta(t, y)$ and $\nabla G_\beta(t, y)$ used in theorems (1), (2) it is clear that $\nabla G_\beta(t, y)$ exists and is continuous in t and in y. From theorem 10 we know ∇f^* exists and is Lipschitz continuous. Since we assumed H to be Lipschitz, it follows from Rademacher's theorem that $\nabla_x H(s, x, \nabla f^*(s, x))$ is almost everywhere defined and bounded. Hence (3.162) represents a continuous function in y and in t. Since $f_0 \in C^2_\infty(\mathbb{R}^d)$ and due to theorem 4

$$\nabla^2 \int_{\mathbb{R}^d} S_{\beta,1}(t, y-x) f_0(x) dx$$
$$= \int_{\mathbb{R}^d} S_{\beta,1}(t, x) \nabla^2 f_0(y-x) dx < \infty. \tag{3.163}$$

Thus, $\nabla^2 f^*(t, y)$ exists and so $f^*(t, y) \in C^2(\mathbb{R}^d)$. This completes the necessary requirements for the solution of the mild form (3.153) to be the solution of (3.154) of class $C^2(\mathbb{R}^d)$. For f^* to be a classical solution to the original FDE, we also need to prove that f^* decays at infinity in y. This follows from the representations of $S_{\beta,1}$ and G_β in (3.80) and (3.31). Since they are written in terms of a stable density g which decays spatially at infinity, so do $S_{\beta,1}$ and G_β decay and hence the solution f^* also decays rapidly enough as a function of y. Therefore f^* is of class $C^2_\infty(\mathbb{R}^d)$ and so is a solution to the original FDE (3.1) in the classical sense.

□

Here is an example of $H(u, p)$ which is Lipschitz in p and the control space U is compact:

$$H(u, p) = \inf_{u \in U} [u^2 - u + 2sin(p) - p], \tag{3.164}$$

with $U = (0, 1)$ and $p = D_y f(t, y)$.

Chapter 4

Convergence theorems

4.0.1 Process description

In this Chapter we present our new convergence results for the position dependent controlled CTRW. In this section we describe the general stochastic process these convergence results apply to. For $i \in \mathbb{N}$ let γ_i be i.i.d. waiting times, whose distribution belongs to the domain of attraction of a β-stable law with $\beta \in (0,1)$. For $n \geq 1$ let ξ_n denote the jump occurring after the waiting time γ_n. Since the jumps depend on control, they are not identically distributed. Assume that the jump distributions are independent from the waiting time distribution. Assume that the jumps occur instantaneously. For $n \geq 1$ let

$$\begin{cases} X_n = t + \sum_{j=1}^{n} \gamma_i, \\ Y_n = Y_{n-1} + \xi(Y_{n-1}, u(n-1, X_{n-1}, Y_{n-1})), \end{cases}$$

where $(X_0, Y_0) = (t, y)$ and $u(\cdot)$ is a given control function taking values in the set U. For $n \geq 1$, at the n-th jump the function $u(\cdot)$ is dependent on $n-1$, on the previous position $y \in \mathbb{R}^d$ of the process Y_{n-1} and on the state $x \in \mathbb{R}_+$ of X_{n-1}. Denote $\tilde{U} := \{u((n-1), x, y) : \mathbb{R}_+ \times [0,T] \times \mathbb{R}^d \to U\}$. The sets U and $\tilde{U}$ will be specified in appropriate theorems. Assume that

$$\frac{\mathbb{E} f\left(x + \tau^{1/\beta}, y + \xi(u,x,y)\right) - f(x,y)}{\tau} \to \left(L_x^1 + L_y^2\right) f(x,y). \tag{4.1}$$

Then the limiting process $(\lim_{\tau\to 0} X^\tau(t), \lim_{\tau\to 0} Y^\tau_{[t/\tau]}$ exists. Let $X(t) := \lim_{\tau\to 0} X^\tau(t)$ and $Y^{u(\cdot)}(t) := \lim_{\tau\to 0} Y^\tau_{[t/\tau]}$. The limiting pair $(X(t), Y^{u(\cdot)}(t))$ is a Markov process and its generating family is $L_{x,y,u(t,x,y)} = L^1_X + L^2_{Y,u(t,x,y)} = L^1_x + L^2_{x,y,u(t,x,y)}$. Denote for any $\tau > 0$ the inverse process for $X^\tau(t)$ by $M_{X^\tau}(t)$ and similarly denote by $M_X(t)$ the inverse process for the limiting process $X(t)$. The controlled processes we are interested in are $Y^{\tau,u(\cdot)}(M_{X^\tau}(t))$ and $Y^{u(\cdot)}(M_X(t)) := \lim_{\tau\to 0} Y^{\tau,u(\cdot)}(M_{X^\tau}(t))$. For ease of notation from now on for any $\tau > 0$ the process $M_{X^\tau}(t)$ will be denoted by $M^\tau(t)$. The definition of the optimal payoff (value) function $S^\tau(t,y)$ without running costs is:

$$S^\tau(t,y) = \inf_{u\in\tilde{U}} \mathbb{E}_t S_0 \left(y + Y^{\tau,u(\cdot)}_{[M^\tau(t)/\tau]} \right) \tag{4.2}$$

and with running costs it is

$$S^\tau(t,y) = \inf_{u\in\tilde{U}} \mathbb{E}_t S_0 \left(y + Y^{\tau,u(\cdot)}_{[M^\tau(t)/\tau]} + \int_0^t R\left(Y^{\tau,u(\cdot)}_{[M^\tau(t)/\tau]} \right) ds \right). \tag{4.3}$$

For simplicity of notation we will write $Y^{\tau,u(\cdot)}(M^\tau(t))$ instead of $Y^{\tau,u(\cdot)}_{[M^\tau(t)/\tau]}$. Corollaries 1, 2 and the theorem 16 will concern the definition 4.3.

4.0.2 Main results

Before we present the main convergence results, we state preliminary results, definitions and a theorem that we use. Let μ, ν be two arbitrary elements in the space of probability measures $P(\Omega)$ for some Polish space Ω.

Definition 10. *The bounded Lipschitz distance between two measures μ and ν is:*

$$\rho_{b\ Lip}(\mu,\nu) = \sup\{|(\mu,f) - (\nu,f)| : ||f||_\infty + ||f||_{Lip} \le 1\}. \tag{4.4}$$

There are several ways to metrize weak convergence, and one of them is using the above definition of the bounded Lipschitz distance, see [84] for details. The next theorem is the bounded Lipschitz distance convergence theorem, [84].

Theorem 13. *Let (Ω, m) be a Polish space with a metric m. Let $\{\mu^\tau\}_{\tau\ge 0}$ and μ be measures* *in* $P(\Omega)$*. Convergence* $\rho_{b\ Lip}(\mu^\tau, \mu) \to 0$ *as* $\tau \to 0$ *is equivalent to the weak convergence of measures* $\mu^\tau \to \mu$*.*

Lemma 2. *Let* $\bar{s} = (s_1, \cdots, s_n) \in \mathbb{R}_+^n$ *and* $\bar{y} = (y_1, \cdots, y_n) \in \mathbb{R}^{nd}$ *with each* $y_i \in \mathbb{R}^d$ *for* $i \in [1, \cdots, n]$*. Let the probability kernels* $\{P^\tau\}_{\tau \geq 0}, P$ *in* $\mathbf{P}(\mathbb{R}^{nd})$ *and* $\{\nu^\tau\}_{\tau \geq 0}, \nu$ *in* $\mathbf{P}(\mathbb{R}_+^n)$ *be such that as* $\tau \to 0$ *for any fixed* $\bar{s}$*,* $P^\tau(\bar{s}, d\bar{y}) \to P(\bar{s}, d\bar{y})$ *and* $\nu^\tau(d\bar{s}) \to \nu(d\bar{s})$ *weakly. Then the following holds uniformly for* f *which is both bounded and Lipschitz,* $||f||_\infty \leq C_1$ *and* $||f||_{Lip} \leq C_2$*:*

$$\int_{\mathbb{R}_+^n} \int_{\mathbb{R}^{nd}} f(\bar{s}, \bar{y}) P^\tau(\bar{s}, \bar{x}, d\bar{y}) \nu^\tau(d\bar{s}) \to \int_{\mathbb{R}_+^n} \int_{\mathbb{R}^{nd}} f(\bar{s}, \bar{y}) P(\bar{s}, \bar{x}, d\bar{y}) \nu(d\bar{s}). \tag{4.5}$$

Proof.

$$\int_{\mathbb{R}^{nd}} \int_{\mathbb{R}_+^n} f(\bar{s}, \bar{y}) P^\tau(\bar{s}, \bar{x}, dy_1 \cdots dy_n) \nu^\tau(ds_1 \cdots ds_n)$$

$$- \int_{\mathbb{R}^{nd}} \int_{\mathbb{R}_+^n} f(\bar{s}, \bar{y}) P(\bar{s}, \bar{x}, dy_1 \cdots dy_n) \nu(ds_1 \cdots ds_n)$$

$$= \int_{\mathbb{R}^{nd}} \bigg(\int_{\mathbb{R}_+^n} f(\bar{s}, \bar{y}) \nu^\tau(ds_1, \cdots, ds_n) \tag{4.6}$$

$$- \int_{\mathbb{R}_+^n} f(\bar{s}, \bar{y}) \nu(ds_1, \cdots, ds_n) \bigg) P^\tau(\bar{s}, \bar{x}, dy_1, \cdots, dy_n)$$

$$+ \int_{\mathbb{R}_+^n} \bigg(\int_{\mathbb{R}^{nd}} f(\bar{s}, \bar{y}) P^\tau(\bar{s}, \bar{x}, dy_1, \cdots, dy_n)$$

$$- \int_{\mathbb{R}^{nd}} f(\bar{s}, \bar{y}) P(\bar{s}, \bar{x}, dy_1, \cdots dy_n) \bigg) \nu(ds_1, \cdots, ds_n).$$

By theorem 13 it follows that as $\tau \to 0$

$$\rho_{\text{b Lip}}(P^\tau, P) \to 0 \text{ and } \rho_{\text{b Lip}}(\nu^\tau, \nu) \to 0. \tag{4.7}$$

Now, due to the definition of $\rho_{\text{b Lip}}$ and the assumption that f is bounded and Lipschitz, (4.7) implies that uniformly for all such f

$$\int_{\mathbb{R}^n_+} f(\bar{s}, \bar{y})\nu^\tau(ds_1, \cdots, ds_n) - \int_{\mathbb{R}^n_+} f(\bar{s}, \bar{y})\nu(ds_1, \cdots, ds_n) \to 0 \tag{4.8}$$

and

$$\int_{\mathbb{R}^{nd}} f(\bar{s}, \bar{y})P^\tau(\bar{s}, \bar{x}, dy_1, \cdots, dy_n) - \int_{\mathbb{R}^{nd}} f(\bar{s}, \bar{y})P(\bar{s}, \bar{x}, dy_1, \cdots dy_n) \to 0. \tag{4.9}$$

Since P^τ and ν are probability measures, it follows that both summands in (4.6) converge to 0 uniformly in f. Therefore

$$\begin{aligned} &\int_{\mathbb{R}^n_+} \int_{\mathbb{R}^{nd}} f(\bar{s}, \bar{y})P^\tau(\bar{s}, \bar{x}, dy_1 \cdots dy_n)\nu^\tau(ds_1 \cdots ds_n) \\ &- \int_{\mathbb{R}^n_+} \int_{\mathbb{R}^{nd}} f(\bar{s}, \bar{y})P(\bar{s}, \bar{x}, dy_1 \cdots dy_n)\nu(ds_1 \cdots ds_n) \to 0, \end{aligned} \tag{4.10}$$

uniformly for all f satisfying $||f||_{\text{Lip}} \leq C_2$ and $||f||_\infty \leq C_1$. □

The Skorokhod space D of càdlàg functions is a Polish space with respect to the standard metric d. We invite the reader to see [85] for the definition of d and more about the Skorokhod space. This makes the definition 10, theorem 13 and lemma 2 applicable to study convergence of càdlàg function sequences. Since CTRWs are càdlàg, the theory applies for scaled CTRWs too. Here we present the novel application to CTRW processes discussed in the subsection 4.0.1.

Theorem 14. *Assume the processes $X^\tau(r), X(r), Y^\tau(r)$, $Y(r)$, $M^\tau(r)$ and $M(r)$ are defined as before in subsection 4.0.1, but there is no control. Let us take any $t_1, \cdots, t_n \in [0, T]$ such that $t_i \neq t_j$ for $i \neq j$. Then for any initial positions $y_i \in \mathbb{R}^d$, $i \in [1, \cdots, n]$, and any f Lipschitz in the spatial variable, with the Lipschitz constant $L \in \mathbb{R}_+$, as $\tau \to 0$ the following* *convergence holds uniformly in f:*

$$\mathbb{E}_{t_1,\cdots,t_n} f\big(Y^\tau(M^\tau(t_1)), M^\tau(t_1), t_1, \cdots, Y^\tau(M^\tau(t_n)), M^\tau(t_n), t_n\big)$$
$$\to \mathbb{E}_{t_1,\cdots,t_n} f\big(Y(M(t_1)), M(t_1), t_1, \cdots, Y(M(t_n)), M(t_n), t_n\big). \tag{4.11}$$

Proof. Denote

$$P^\tau(\bar{s}, \bar{x}, d\bar{y}) = P\Big(Y^\tau(M^\tau(s_1)) \in x_1 + dy_1, \cdots, Y^\tau(M^\tau(s_n)) \in x_n + dy_n\Big) \tag{4.12}$$

and

$$\nu^\tau(s_1, \cdots, s_n) = P(M^\tau(s_1) < a_1, \cdots, M^\tau(s_n) < a_n)$$
$$= P(X^\tau_{a_1} > s_1, \cdots, X^\tau_{a_n} > s_n). \tag{4.13}$$

By our assumption on the distribution of γ_i it follows that ν does not have any atoms. Now, since

$$P(X^\tau_{a_1} > s_1, \cdots, X^\tau_{a_n} > s_n) \to P(X_{a_1} > s_1, \cdots, X_{a_n} > s_n) \tag{4.14}$$

pointwise, it follows that $\nu^\tau(s_1, \cdots, s_n) \to \nu(s_1, \cdots, s_n)$. Since for every $\tau > 0$

$$\mathbb{E}_{t_1,\cdots,t_n} f\big(Y^\tau(M^\tau(t_1)), M^\tau(t_1), t_1, \cdots, Y^\tau(M^\tau(t_n)), M^\tau(t_n), t_n\big)$$
$$= \int_{\mathbb{R}^{nd}} \int_{\mathbb{R}^n_+} f(\bar{y}, \bar{s}) P^\tau(\bar{s}, \bar{x}, d\bar{y}) \nu^\tau(d\bar{s}), \tag{4.15}$$

the proof follows from the application of Lemma 2. □

Theorem 15. *Let the processes X^τ, X, Y^τ, Y, M^τ and M be defined as before and with control. Take any $t_1, \cdots, t_n \in [0, T]$. Let the position dependent control function*

$$u\Big(t, M(t), Y(M(t))\Big) \tag{4.16}$$

be uniformly Lipschitz in all variables. Denote by $U \subset \mathbb{R}_+ \times \mathbb{R}^d$ a compact set where the *control function u takes its values, and denote by $\tilde{U}_{\mathscr{L}} \subset \tilde{U}$ the compact set of all Lipschitz functions $u \in \tilde{U}$ which have the same Lipschitz constant $\mathscr{L} \in \mathbb{R}_+$. Assume $f(Y, M, u)$ is Lipschitz in Y, M and u. Denote the Lipschitz constant for f in u by L. Then the following convergence holds:*

$$\begin{aligned}
&\inf_{u \in \tilde{U}_{\mathscr{L}}} \mathbb{E}_{t_1, \cdots, t_n, 0} f\Big(t_1, Y^\tau(M^\tau(t_1)), M^\tau(t_1), u\left(t_1, M^\tau(t_1), Y^\tau(M^\tau(t_1))\right), \cdots, \\
&\qquad t_n, Y^\tau(M^\tau(t_n)), M^\tau(t_n), u\left(t_n, M^\tau(t_n), Y^\tau(M^\tau(t_n))\right)\Big) \\
&\to \inf_{u \in \tilde{U}_{\mathscr{L}}} \mathbb{E}_{t_1, \cdots, t_n, 0} f\Big(t_1, Y(M(t_1)), M(t_1), u\left(t_1, M(t_1), Y(M(t_1))\right), \cdots, \\
&\qquad t_n, Y(M(t_n)), M(t_n), u\left(t_n, M(t_n), Y(M(t_n))\right)\Big).
\end{aligned} \tag{4.17}$$

Proof. To make some of our formulas shorter let us set $\bar{x} = (\bar{Y}, \bar{M}, \bar{t})$ and

$$\begin{aligned}
F\left(\bar{x}, u(\bar{t}, \bar{x})\right) := f\big(&t_1, Y^\tau\left(M^\tau(t_1)\right), M^\tau(t_1), u(t_1, Y^\tau(M^\tau(t_1)), M^\tau(t_1)), \cdots, \\
&t_n, Y^\tau\left(M^\tau(t_n)\right), M^\tau(t_n), u(t_n, Y^\tau(M^\tau(t_n)), M^\tau(t_n))\big).
\end{aligned} \tag{4.18}$$

Then due to Lipschitz assumptions on f and u:

$$\begin{aligned}
&|F\big(\bar{x_1}, u(\bar{t_1}, \bar{x_1})\big) - F\big(\bar{x_2}, u(\bar{t_2}, \bar{x_2})\big)| \\
&\qquad \le L(1 + \mathscr{L})\Big(|\bar{x}_1 - \bar{x}_2|\Big),
\end{aligned} \tag{4.19}$$

where $|\bar{x}_1 - \bar{x}_2| = |\bar{Y_1} - \bar{Y_2}| + |M_1 - M_2| + |\bar{t}_1 - \bar{t}_2|$. Hence F is uniformly Lipschitz. Note that for any real-valued functions $a(u), b(u)$:

$$|\inf_{u \in \tilde{U}_{\mathscr{L}}} a(u) - \inf_{u \in \tilde{U}_{\mathscr{L}}} b(u)| \le \sup_{u \in \tilde{U}_{\mathscr{L}}} |a(u) - b(u)|. \tag{4.20}$$

Let

$$\begin{aligned}
a(u) = \mathbb{E}_{y_1, \cdots, y_n, 0} f\Big(&t_1, M^\tau(t_1), Y^\tau(M^\tau(t_1)), u\left(t_1, M^\tau(t_1), Y^\tau(M^\tau(t_1))\right), \cdots, \\
&t_n, M^\tau(t_n), Y^\tau(M^\tau(t_n)), u\left(M^\tau(t_n), Y^\tau(M^\tau(t_n))\right)\Big)
\end{aligned} \tag{4.21}$$

and

$$b(u) = \mathbb{E}_{y_1,\cdots,y_n,0} f\Bigg(t_1, Y\big(M(t_1)), M(t_1), u\,(t_1, M(t_1), Y(M(t_1)))\big), \cdots,$$

$$t_n, Y\big(M(t_n), M(t_n), u\,(t_n, M(t_n), Y(M(t_n)))\big)\Bigg). \tag{4.22}$$

Substituting these into (4.20) enables us to write

$$\begin{aligned}
&\Bigg|\inf_{u\in\tilde{U}_{\mathscr{L}}} \mathbb{E}_{t_1,\cdots,t_n} f\Bigg(\bar{t}, \bar{M}^\tau(t), \bar{Y}^\tau(M^\tau(t)), u\left(\bar{t}, \bar{M}^\tau(t), \bar{Y}^\tau(M^\tau(t))\right)\Bigg) \\
&\qquad - \inf_{u\in\tilde{U}_{\mathscr{L}}} \mathbb{E}_{t_1,\cdots,t_n} f\Bigg(\bar{t}, \bar{M}(t), \bar{Y}(M(t)), u\left(t, M(t), \bar{Y}(M(t))\right)\Bigg)\Bigg| \\
&\le \sup_{u\in\tilde{U}_{\mathscr{L}}} \Bigg|\mathbb{E}_{x_1,\cdots,x_n} f\Bigg(\bar{t}, \bar{M}^\tau(t), \bar{Y}^\tau(M^\tau(t)), u\left(\bar{t}, \bar{M}^\tau(t), \bar{Y}^\tau(M^\tau(t))\right)\Bigg) \\
&\qquad - \mathbb{E}_{t_1,\cdots,t_n} F\Bigg(\bar{t}, \bar{M}(t), \bar{Y}(M(t)), u\left(\bar{t}, \bar{M}(t), \bar{Y}(M(t))\right)\Bigg)\Bigg|.
\end{aligned} \tag{4.23}$$

Since F is uniformly Lipschitz and bounded, we may use Lemma 2 to deduce that

$$\begin{aligned}
&\sup_{u\in\tilde{U}_{\mathscr{L}}} \Bigg|\mathbb{E}_{t_1,\cdots,t_n} f\Bigg(\bar{t}, \bar{M}^\tau(t), \bar{Y}^\tau(M^\tau(t)), u\left(\bar{t}, \bar{M}^\tau(t), \bar{Y}^\tau(M^\tau(t))\right)\Bigg) \\
&\qquad - \mathbb{E}_{t_1,\cdots,t_n} f\Bigg(\bar{t}, \bar{M}(t), \bar{Y}(M(t)), u\left(\bar{t}, \bar{M}(t), \bar{Y}(M(t))\right)\Bigg)\Bigg| \to 0
\end{aligned} \tag{4.24}$$

and hence that

$$\begin{aligned}
&\Bigg|\inf_{u\in\tilde{U}_{\mathscr{L}}} \mathbb{E}_{t_1,\cdots,t_n} f\Bigg(\bar{t}, \bar{M}^\tau(t), \bar{Y}^\tau(M^\tau(t)), u\left(\bar{t}, \bar{M}^\tau(t), \bar{Y}^\tau(M^\tau(t))\right)\Bigg) \\
&\quad - \inf_{u\in\tilde{U}_{\mathscr{L}}} \mathbb{E}_{t_1,\cdots,t_n} f\Bigg(\bar{t}, \bar{M}(t), \bar{Y}(M(t)), u\left(\bar{t}, \bar{M}(t), \bar{Y}(M(t))\right)\Bigg)\Bigg| \to 0.
\end{aligned} \tag{4.25}$$

□

Corollary 1. *Take arbitrary $t_1, \cdots, t_n \in [0, T]$ such that $t_i \neq t_j$ for $i \neq j$. Let the assumptions of the previous theorem hold. Then as $\tau \to 0$ the following convergence result is* *true:*

$$\begin{aligned}&\inf_{u\in\tilde{U}_{\mathscr{L}}} \mathbb{E}_{t_1,\cdots,t_n} S_0\left(y_1+Y^{\tau,u(\cdot)}(M^\tau(t_1)),M^\tau(t_1),t_1,\cdots,y_n+Y^{\tau,u(\cdot)}(M^\tau(t_n)),M^\tau(t_n),t_n\right)\\&\quad\to\inf_{u\in\tilde{U}_{\mathscr{L}}} \mathbb{E}_{t_1\cdots,t_n} S_0\left(y_1+Y^{u(\cdot)}(M(t_1)),M^\tau(t_1),t_1\cdots,y_n+Y^{u(\cdot)}(M(t_n)),M(t_n),t_n\right).\end{aligned} \tag{4.26}$$

Proof. The proof is analogous to the proof for the previous theorem, with $S_0(y+\cdot)$ taking the role of $f(\cdot)$. □

The next theorems deal with a more general payoff which includes running costs.

Theorem 16. *Let us keep assumptions of theorem 15 and also assume that the running cost function is of the following form*

$$\sum_{i=1}^{n} R\left(Y^{\tau,u(\cdot)}(M^\tau(s_i))\right), \tag{4.27}$$

where $R(\cdot)$ is a Lipschitz function with the Lipschitz constant L and $n\in\mathbb{N}$. Then as $\tau\to 0$ the following convergence result holds:

$$\inf_{u\in\tilde{U}_{\mathscr{L}}} \mathbb{E}\sum_{i=1}^{n} R\left(Y^{\tau,u(\cdot)}(M^\tau(s_i))\right)\to\inf_{u\in\tilde{U}_{\mathscr{L}}} \mathbb{E}\sum_{i=1}^{n} R\left(Y^{u(\cdot)}(M(s_i))\right). \tag{4.28}$$

Proof. Note that the sum in equation (4.27) is of the form

$$f\big(Y^{\tau,u(\cdot)}(s_1),Y^{\tau,u(\cdot)}(s_2),\cdots,Y^{\tau,u(\cdot)}(s_m)\big) \tag{4.29}$$

where f is Lipschitz in all variables. Therefore, we may apply theorem 15 and result follows. □

Corollary 2. *Let us keep assumptions of theorem 16. Let us assume that the function $R\left(Y^{\tau,u(\cdot)}(M^\tau(s))\right)$ models the running cost associated to the process. Assume the optimal* *payoff function (the value function) is*

$$S^\tau(t,y) = \inf_{u\in\tilde{U}_{\mathscr{L}}} \mathbb{E}_t\Bigg[\sum_{i=1}^{n} R(Y^{\tau,u(\cdot)}(M^\tau(s_i))) + S_0\Big(y + Y^{\tau,u(\cdot)}(M^\tau(t))\Big)\Bigg], \tag{4.30}$$

where y is the initial position of the process at time $t = 0$. Then as $\tau \to 0$ the following convergence result holds:

$$S^\tau(t,y) \to \inf_{u\in\tilde{U}_{\mathscr{L}}} \mathbb{E}_t\Bigg[\sum_{i=1}^{n} R\left(Y^{u(\cdot)}(M(s_i))\right) + S_0\Big(y + Y^{u(\cdot)}(M(t))\Big)\Bigg]. \tag{4.31}$$

Proof. The proof combines the one for theorem 15 with the one for theorem 16, as in this case $f\left(Y^{\tau,u(\cdot)}(M^\tau(t))\right) = S_0(y + Y^{\tau,u(\cdot)}(M^\tau(t))) + \sum_{i=1}^n R(Y^{\tau,u(\cdot)}(M^\tau(s_i)))$, and the mentioned theorems deal with the two summands separately. □

Theorem 17. *Let the controlled stochastic processes be as in the previous theorems. Let us assume that the cost function is of the form*

$$S^\tau(t,y) = \inf_{u\in U_{\mathscr{L}}} \mathbb{E}\Bigg[\int_0^t R\left(Y^{\tau,u(\cdot)}(M^\tau(s)), M^\tau(s), s\right) ds + S_0\left(y + Y^{\tau,u(\cdot)}(M^\tau(t))\right)\Bigg]. \tag{4.32}$$

Then the following convergence holds:

$$S^\tau(t,y) \to \inf_{u\in U_{\mathscr{L}}} \mathbb{E}\Bigg[\int_0^t R\left(Y^{u(\cdot)}(M(s)), M(s), s\right) ds + S_0\left(y + Y^{u(\cdot)}(M(t))\right)\Bigg]. \tag{4.33}$$

Proof. The proof is based on the same ideas as in the previous theorems and on the interchangeability of the integral with respect to time and the expectation.

$$
\begin{aligned}
&\left|\inf_{u\in U_{\mathscr{L}}} \mathbb{E}\left[\int_0^t R(Y^{\tau,u(\cdot)}(M^\tau(s)), M^\tau(s), s)ds + S_0(y+Y^{\tau,u(\cdot)}(M^\tau(t))\right]\right.\\
&\qquad\left. - \inf_{u\in U_{\mathscr{L}}} \mathbb{E}\left[\int_0^t R(Y^{u(\cdot)}(M(s)), M(s), s)ds + S_0(y+Y^{u(\cdot)}(M(t))\right]\right|\\
&\le \sup_{u\in U_{\mathscr{L}}}\left|\mathbb{E}\int_0^t R(Y^{\tau,u(\cdot)}(M^\tau(s)), M^\tau(s), s)ds - \mathbb{E}\int_0^t R(Y^{u(\cdot)}(M(s)), M(s), s)ds\right.\\
&\qquad\left. + \mathbb{E}S_0(y+Y^{\tau,u(\cdot)}(M^\tau(t)) - \mathbb{E}S_0(y+Y^{u(\cdot)}(M(t))\right|\\
&\le \sup_{u\in U_{\mathscr{L}}}\int_0^t\left|\mathbb{E}\Big(Y^{\tau,u(\cdot)}(M^\tau(s)), M^\tau(s), s\Big) - \mathbb{E}\Big(Y^{u(\cdot)}(M(s)), M(s), s\Big)\right|ds\\
&\qquad + \sup_{u\in U_{\mathscr{L}}}\left|S_0(y+Y^{\tau,u(\cdot)}(M^\tau(t)) - S_0(y+Y^{u(\cdot)}(M(t))\right|\\
&\le t\sup_{u\in U_{\mathscr{L}}}\sup_{s\in[0,t]}\left|\mathbb{E}\Big(Y^{\tau,u(\cdot)}(M^\tau(s)), M^\tau(s), s\Big) - \mathbb{E}\Big(Y^{u(\cdot)}(M(s)), M(s), s\Big)\right|\\
&\qquad + \sup_{u\in U_{\mathscr{L}}}\left|\mathbb{E}S_0(y+Y^{\tau,u(\cdot)}(M^\tau(t)) - \mathbb{E}S_0(y+Y^{u(\cdot)}(M(t))\right| \to 0, \qquad (4.34)
\end{aligned}
$$

since $\left|\mathbb{E}\Big(Y^{\tau,u(\cdot)}(M^\tau(s)), M^\tau(s), s\Big) - \mathbb{E}\Big(Y^{u(\cdot)}(M(s)), M(s), s\Big)\right| \to 0$ and by the previous theorems $\left|\mathbb{E}S_0(y+Y^{\tau,u(\cdot)}(M^\tau(t)) - \mathbb{E}S_0(y+Y^{u(\cdot)}(M(t))\right| \to 0.$ □

The following theorem is an application of a verification theorem in [1] to the limiting process in our framework.

4.0.3 A verification theorem

The limiting process $(t+X(r), y+Y^{u(\cdot)}(r))$ in the subsection 4.0.1 is Markov and assume has the generating family

$$
L_G = L_x^1 + L_y^2 = D_{0,r}^{*\beta} + \omega(u,y)D_y - (-\Delta)^{\alpha/2}. \qquad (4.35)
$$

We observe the process until $t+X(r)=T$. At this moment the state of the process is $(T, Y^{u(\cdot)}(M(T-t)))$. The goal of controlling the process is to minimise

$$
J(t,x,y;u) = \mathbb{E}\left\{\int_t^T R(x(s), y(s), u(s, x(s), y(s)))ds + g(T, y+Y(M(T-t)))\right\}, \qquad (4.36)
$$

where the function g represents the terminal payoff function.

Define the value function $V_u := \inf_{u \in \tilde{U}_{\mathscr{L}}} J(x, y; u(t, x, y))$, and denote by $u^*(\cdot)$ an optimal control strategy and by $(x^*(\cdot), y^*(\cdot))$ the corresponding trajectory.

Theorem 18. *Let us study the process $(t+X(r), y+Y^{u(\cdot)}(r))$ defined above. Assume the set of admissible control functions $\tilde{U}_{\mathscr{L}}$ for the process $(t+X(r), y+Y^{u(\cdot)}(r))$ is compact. Assume also that the process starts at a point (x, y) in $\mathbb{R}_+ \times \mathbb{R}^d$. Let R be a running cost function on $\mathbb{R}_+ \times \mathbb{R}^d \times \tilde{U}_{\mathscr{L}}$, which is Lipschitz and bounded. Assume g to be continuous and bounded on $\mathbb{R}_+ \times \mathbb{R}^d$.*

Let $W \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^d)$ be a solution to

$$H(x, y, L_G V) = 0, \tag{4.37}$$

with

$$H(x, y, L_G V) = \inf_{u \in \tilde{U}_{L_G}} \left[R(t, s, x(s), y(s), u(s, x(s), y(s))) + L_G V\right] \tag{4.38}$$

and with the boundary condition $V(T, y) = g(T, y)$ for $y \in \mathbb{R}^d$. Then for every $(x, y) \in (\mathbb{R}_+ \times \mathbb{R}^d)$:

1. *$W(x, y) \leq J(x, y; u)$ for any admissible measurable control process $u(\cdot)$.*
2. *Suppose that there exists $u^*(\cdot) \in \tilde{U}_{\mathscr{L}}$ such that*

$$u^*(t, x, y) \in argmin \left[L_G W(x^*(t), y^*(t)) + R(x^*(t), y^*(t))\right]. \tag{4.39}$$

Then $W(x, y) = J(x, y; u^*)$, where J is as defined in (4.36).

Proof. The proof is analogous to that of the verification theorem for the infinite horizon problem for a controlled Markov diffusion in [1], Chapter IV, Theorem 5.1. □

Theorem 18 verifies that the solution to the fractional HJB equation (4.37) is a solution to the optimisation problem described in the beginning of the subsection 4.0.3. Previous theorems 14, 15, 16 and related corollaries give general convergence of $S^{\tau}(t, y)$ to solutions of optimisation problems as $\tau \to 0$. Note that (4.37), (2.50), (2.51) and (2.69) are of the same form. Together with the verification theorem 18 these convergence results yield convergence of $S^\tau(t, y)$ in (4.32) to the solution of (4.37), which exists and is unique by our well-posedness analysis in Chapter 3.

Chapter 5

Further developments

In this Chapter we present several ideas about alternative approaches to analysing different aspects of the model presented in this thesis. The first one is related to analysis of solutions $S^\tau(t,y)$ to the pre-limiting dynamic programming equations without running costs. We found the approach presented in the earlier Chapters more straightforward, so this line of thought was not continued. However it resulted in a property for $S^\tau(t,y)$ which we will present in the subsection 5.0.4, and it is interesting in its own right. The second idea is a stochastic differential equation (SDE) approach to the limiting controlled processes with random time changes. The third idea is an application of the CTRW to a many particle system modelling and we extend this even further to a model of competing particle systems. The last further development idea is a suggestion to apply a CTRW model to limit order book modelling.

5.0.4 Analysis

Let us study the position dependent model described in the subsection 4.0.1. Assume that $S_0(y)$ is Lipschitz in y with the Lipschitz constant L_0. Then for fixed τ, t, S^τ defined in (4.2) is Lipschitz in y.

The reasoning follows from the definition (4.2)

$$S^\tau(t,y) = \inf_{u\in U} \mathbb{E}[S_0(y + Y^\tau(M^\tau(t)))]. \tag{5.1}$$

Using (4.20) we obtain

$$|S^\tau(t,y_1) - S^\tau(t,y_2)|$$

$$\le \sup_{u\in U}\,[\mathbb{E}(S_0(y_1 + Y^\tau(M^\tau(t))) - S_0(y_2 + Y^\tau(M^\tau(t))))]$$

$$\le L_0 \sup_{u\in U}[Y^\tau(M^\tau(t), y_1) - Y^\tau(M^\tau(t), y_2)] \le L_0 C(t)|y_1 - y_2|. \tag{5.2}$$

Let us study the optimal payoff functions $S^\tau(t,y)$ as they appear earlier in the Chapter. As $\tau \to 0$ the functions $S^\tau(t,y)$ converge to a limit $\tilde{S}(t,y)$ for every $t \ge 0, y \in \mathbb{R}^d$. Here we present the reasoning. Let us recall that $S^\tau(t,y)$ satisfies:

$$S^\tau(t,y) = \int_{t\tau^{-1/\beta}}^{\infty} \nu(dr) S_0(y) + \sup_{u\in U} \int_0^{t\tau^{-1/\beta}} \int_{\mathbb{R}^d} S^\tau(t - r\tau^{1/\beta}, y + \xi\tau^{1/\alpha})\mu_u(d\xi)\nu(dr).$$

Due to the definition of $S^\tau(t,y)$, for $y_1, y_2 \in \mathbb{R}^d$:

$$|S^\tau(t,y_1) - S^\tau(t,y_2)| \le |S_0(y_1) - S_0(y_2)|$$

$$+ \sup_{u\in U} \int_0^{t\tau^{-1/\beta}} \int_{\mathbb{R}^d} \Big| S^\tau(t - r\tau^{1/\beta}, y_1 + \xi\tau^{1/\alpha})$$

$$- S^\tau(t - r\tau^{1/\beta}, y_2 + \xi\tau^{1/\alpha}) \Big| \mu_u(d\xi)\nu(dr)$$

$$\le |S_0(y_1) - S_0(y_2)| + \int_0^{t\tau^{-1/\beta}} \int_{\mathbb{R}^d} L^{\tau, t - r\tau^{1/\beta}} |y_1 - y_2| \mu_u(d\xi)\nu(dr), \tag{5.3}$$

where $L^{\tau,t}$ is the Lipschitz constant for $S^\tau(t,y)$. Dividing through by $|y_1 - y_2|$:

$$L^{\tau,t} \le L_0 + \int_0^{t\tau^{-1/\beta}} L^{\tau, t - t\tau^{1/\beta}} \nu(dr) \le L_0 + \int_0^t L^{\tau, t-r} \nu(dr/\tau^{1/\beta}).$$

Now Gronwall's inequality gives us uniform boundedness of $L^{\tau,t}$.

For fixed $\tau > 0, t \ge 0$, $S^\tau(t,y)$ defined in (4.2) for the model in the subsection 4.0.1 and re-written in (5.1) is equicontinuous in y. Here we present the reasoning.

Let us take $\delta > 0$. Let $|y_1 - y_2| < \delta$. Then

$$|S^\tau(t,y_1) - S^\tau(t,y_2)| \le |S_0(y_1) - S_0(y_2)|$$

$$
\begin{aligned}
&+|y_1 - y_2| \int_0^{t\tau^{-1/\beta}} L^{\tau, t - r\tau^{1/\beta}} \nu(dr) \\
&\leq L_0 \delta + \delta \int_0^{t\tau^{-1/\beta}} L^{\tau, t - r\tau^{1/\beta}} \nu(dr) \\
&\leq \delta \big(L_0 + \int_0^{t\tau^{-1/\beta}} L_0 C(t) \nu(dr) \big) \\
&\leq \delta L_0 (1 + C(t)).
\end{aligned} \tag{5.4}
$$

Here we have used the uniform boundedness of $L^{\tau,t} \leq L_0 C(t)$ discussed earlier.

So for any $\epsilon > 0$ there exists $\delta_\epsilon = \frac{\epsilon}{L_0(1+C(t))}$ s.t. $|y_1 - y_2| < \delta \Rightarrow |S^\tau(t, y_1) - S^\tau(t, y_2)| < \epsilon$. Hence $S^\tau(t, y)$ is equicontinuous.

Therefore by Arzela-Ascoli theorem for any fixed $t \in \mathbb{R}_+$ there exists a sequence $\{\tau_n\}_{n \geq 1}$ such that $S^{\tau_n}(t, y) \to \tilde{S}(t, y)$ for some $\tilde{S}(t, y)$. Then $\tilde{S}$ is Lipschitz too with the Lipschitz constant $L_0 C(t)$ for some $C(t)$, and moreover, by Rademacher's theorem $\tilde{S}(t, y)$ is differentiable in y.

5.0.5 Linear quadratic control

Let us assume that the Hamiltonian H has the following form

$$
\begin{aligned}
H(t, y, p) &= \inf_{u \in U} (fp + g) \\
&= \inf_{u \in U} \left((au + by)\, p - \left(ru^2 + suy + wy^2 \right) \right),
\end{aligned} \tag{5.5}
$$

Minimising over the control parameter u, i.e. differentiating the function inside the infimum with respect to u, gives us optimal control $u^* = \frac{ap - sy}{2r}$, substituting it we have

$$
\begin{aligned}
H^*(t, y, p) &= \left(a \left(\frac{ap - sy}{2r} \right) + by \right) p \\
&- \left(r \left(\frac{ap - sy}{2r} \right)^2 + s \left(\frac{ap - sy}{2r} \right) y + wy^2 \right),
\end{aligned} \tag{5.6}
$$

i.e. the control is linear quadratic. Now assume that $S(t,y) = h(t)y^2$ and $L = \frac{y^2}{2}\frac{d^2}{dy^2}$. Then the FHJB equation reduces to

$$\begin{aligned} y^2 D_{0,t}^{*\beta} h(t) = h(t)y^2 + 2y^2\frac{a^2}{\alpha}h^2(t) + 2bh(t)y^2 \\ -\frac{y^2}{4r}\left(4a^2h^2(t) - 4ash(t) + s^2\right) \\ -\frac{sah(t)y^2}{r} + \frac{s^2y^2}{2r} - wy^2. \end{aligned} \tag{5.7}$$

The term y^2 cancels out and we are left with a fractional Riccati equation for $h(t)$, with constant coefficients. If $L = \frac{1}{2}\frac{d^2}{dy^2}$ instead then the coefficients will depend on y, however the equation is for the function $h(t)$ and so powers of y in the coefficients can still be treated as constants, since they do not depend on t. This leaves us with a fractional Riccati equation for $h(t)$:

$$D_{0,t}^{*\beta} h(t) = Ah^2(t) + Bh(t) + C, \tag{5.8}$$

for some A, B, C independent of the time parameter t.

Different numerical methods for fractional Riccati equations have been proposed recently, however yet there is no proof of well-posedness. The paper [86] presents numerical results for the case $A = 1, B = 0, C = -1$.

5.0.6 Stochastic Differential Equations for the limiting processes

Here we propose an alternative description of the limiting process via Stochastic Differential Equations (SDEs). Stochastic processes are solutions of SDEs. For example, the solution to

$$dX(t) = \mu dt + \sigma dW(t) \tag{5.9}$$

where $W(t)$ denotes the Wiener process, is a Gaussian process with mean μt and variance $\sigma^2 t$. The solution to

$$dX(t) = f(t, X(t))dt + g(t, X(t))dW(t) \tag{5.10}$$

for $t_1 \leq t \leq t_2$ with the initial data $X(t) = x$ is pathwise unique and is a Markov diffusion process, please see the book by [1] and references therein. In case of a controlled Markov diffusion, $X(t)$ is the solution to an SDE of the form

$$dX(t) = f(s, X(s), u(s))ds + g(s, X(s), u(s))dW(s), \tag{5.11}$$

where $u(\cdot)$ is the admissible control function. Under specific conditions on the functions f, g and u there exists a unique solution to (5.11). Recent developments extend to theory for solutions to SDEs for which the term dW_t is replaced by dL_t^α, where L_t^α is an α-stable Lévy motion. We invite the reader to see for example [87] and [88] for more details.

So far there is no research on the SDE approach to controlled stable processes with random time changes. Following ideas in [1], [89], [90] and [91], we make the following conjecture:

Conjecture 1. *An α-stable Lévy motion $L_\alpha(t)$ subordinated to $S_\beta(t)$ an inverse of a β-stable Lévy motion is a solution to the following stochastic differential equation (SDE):*

$$dY_t = F(Y_t)dS_\beta(t) + \sigma(Y_t)dL_\alpha(S_\beta(t)). \tag{5.12}$$

In the case when the process $L_\alpha(S_\beta(t))$ is controlled, we propose that the corresponding SDE is of the form

$$dY_t = F(Y_t, u)dS_\beta(t) + \sigma(Y_t, u)dL_\alpha(S_\beta(t)), \tag{5.13}$$

where u is the control parameter.

5.0.7 Mean field game application

We looked at the application of the CTRW model to interacting particle systems. This work was then handed over to E. Hernández-Hernández and the reader is invited to read [92] for interesting results in this direction. Here we present the model we have looked at. It is based on the work developed in this thesis and on the nonlinear Markov game models presented in [93].

Often to obtain a description for a stochastic system with $|N|$ particles where $|N|$ is large, it is easier to approximate it by the analogous system with an infinite number of particles. Assume that there is a large number $|N|$ of particles, and they are of d types. For example, in industry there could be $|N|$ robots of d different types with the same controller. Let us denote by S the state space in $\mathbb{Z}^d$. It is the sequence of length d with each element $n_i, i \in \{1, \ldots, d\}$ representing the number of particles of the i-th type. We denote this sequence by $N = (n_1, \ldots, n_d)$. Denote by $|N| = n_1 + \ldots + n_d$ the total number of particles. Assume the system evolves as particles jump one at a time and each jump is a transformation of a particle from one type to another. If the system is at a state $\frac{N}{|N|}$ and a particle of type i transforms to type j, where $i \neq j$, we denote the new state of the system by $\frac{N^{ij}}{|N|}$. Assume there are no jumps from type i to type i, i.e. a transformation must be a change in type.

Assume the waiting times $\gamma_i, i \in \mathbb{N}$ between the jumps belong to the DOA of a β-stable law with $\beta \in (0, 1)$. By $Q(u, t, y)$ we denote the rate matrices. These may depend on a control parameter $u \in U$ as we may control the jumps to optimise the system's payoff. Assume the particles can not control their jumps themselves, and that controlling the evolution is only possible to the unique controlling agent. Each matrix element Q_{ij} in $Q(u, t, y)$ is the rate with which a particle can jump/transform from i-th type to the j-th type.

Denote $X(n) := \sum_{i=1}^{n} \gamma_i$. Let $Z_X(t) = \inf_{n \in \mathbb{N}}\{n : X(n) > t\} - 1$ and let $Y(t) = Y(Z_X(t))$ be the process of the system evolution with time. Assume that as $N \to \infty$, $N/|N| \to y$ for $y \in \mathbb{R}^d$. Let a payoff function for the whole system with an infinite number of participants represent the total cost for the system to be at a particular position y at any given time $t \geq 0$. The aim of the system coordinator is to optimise the payoff. By $S\left(\frac{N}{|N|}, t\right)$ we shall denote the value of the payoff function at the state $\frac{N}{|N|}$ and at time t until the end of the process. Assume that there are no costs for jumping and for waiting, only for arriving at a new state and at a terminal state.

Let the scaling parameter be $\tau = \frac{1}{|N|}$. Let us scale the waiting times by $\tau^{1/\beta}$ and scale the configuration N by τ. We propose that as $\tau \to 0$ the limiting equation for S obtains the following form:

$$A^* S(y,t) + \sup_{u \in U} \left[\sum_{i,j=1, i \neq j}^{d} C^{ij} \left(\frac{dS(y,t)}{dy_j} - \frac{dS(y,t)}{dy_i} \right) \right] + S_0(y) \frac{t^{-\beta}}{\Gamma[1-\beta]} = 0, \qquad (5.14)$$

where $C^{ij} = |N| \frac{Q_{ij}}{\sum_k n_k |Q_{kk}|} = \frac{R^{ij}}{|N|}$ $i \neq j$. Here $A^* = -D_{0,t}^{*\beta}$ is the dual of the generator A.

However we did not go ahead with this research direction. A result similar to the one proposed above has recently been derived heuristically by E. Hernández-Hernández for a slightly different system of interacting particles, see [92] for details. The rigorous proof is also ongoing research of V. Kolokoltsov and E. Hernández-Hernández.

We also looked at two competing controlled systems, each behaving as a CTRW such as the one described above. Assume there are two agents each controlling a system of many particles. The goal of each of the two agents is to optimise their own payoff.Let $\frac{N}{|N|} = \frac{1}{|N|}(n_1, n_2, \ldots, n_d)$ denote the state of the system controlled by the first agent and $\frac{M}{|M|} = \frac{1}{|M|}(m_1, m_2, \ldots, m_d)$ denote the state of the system controlled by the second agent. Denote by U and V the control sets for the first and the second agent. The control parameter $u \in U$ depends on the state of the other agent's system $\frac{M}{|M|}$ and the control parameter $v \in V$ depends on the state $\frac{N}{|N|}$ of the opposing agent's particle system. Assume that the only payoff is due to arriving to a new state and the terminal payoff. Let $Q(t, u, x)$ and $P(t, v, y)$ denote the rate matrices which depend on position and control, bringing non-linearity into the game. Let the scaling parameter be $\tau = \frac{1}{|N||M|}$. Then, after the scaling, the value function equation obtains the following form:

$$S\left(\frac{N}{|N|},\frac{M}{|M|},t\right)=S_0\left(\frac{N}{|N|},\frac{M}{|M|},t\right)\int_{t\tau^{-1/\beta}}^{\infty}\nu(dr)$$

$$+\sup_{u\in U}\inf_{v\in V}\left[\sum_{i,j=1,i\neq j}^{d}\int_0^{t\tau^{-1/\beta}}\frac{n_iQ_{ij}}{\sum_{i=1}^{d}|Q_{ii}|n_i}S\left(\frac{N^{ij}}{|N|},\frac{M}{|M|},t-r\tau^{1/\beta}\right)\nu(dr)\right.$$

$$\left.+\sum_{i,j=1,i\neq j}^{d}\int_0^{t\tau^{-1/\beta}}\frac{m_iP_{ij}}{\sum_{i=1}^{d}|P_{ii}|m_i}S\left(\frac{N}{|N|},\frac{M^{ij}}{|M|},t-r\tau^{1/\beta}\right)\nu(dr)\right]. \tag{5.15}$$

We propose that the limiting equation will have a similar form to (5.14).

5.0.8 Application to finance

Together with A. V. Chertok we proposed to apply the CTRW model studied in [94] for the net flow of orders in a limit order book in a liquid market. We invite the reader to see [95] for the background theory on stochastic limit order book modelling. The authors of the paper comment on the potential usefulness of heavy-tailed distributions in limit order book stochastic modelling, and we tried to develop this idea further. This work did not go ahead due to specific properties of the limit order book behaviour which could not be captured by our methods. The key property to allow for in an accurate model of limit orders was the tendency for the CTRW to frequently reach the zero level.

Here we present the model we had in mind. Let $\gamma_i \in$ DOA (β-stable law) denote waiting times between order arrivals, $\beta \in (0,1)$. Let $\xi_i \in DOA$ (α-stable law) be the sizes or orders, $\alpha \in (1,2]$. Define the inverse process $M_X(t) = inf\{n : \sum_{i=1}^{n}\gamma_i > t\} - 1$. Then $Y(t) = \sum_{i=1}^{M_X(t)}\xi_i$ may model the net flow of orders. Let us scale the waiting times by $\tau^{1/\beta}$ and the jumps by $\tau^{1/\alpha}$. It is possible to describe the distribution of the limiting process after scaling and letting $\tau \to 0$ via *fractional* distributions:

$$g(t,x,y)=\int_0^{\infty}G_{\alpha,\theta}(xs^{\beta/\alpha})dG_{\beta,1}(s), \tag{5.16}$$

where $G_{\alpha,\theta}$ is the stable distribution with the characteristic function of the form

$$g(\alpha,\theta) = exp\left\{-|s|^{\alpha} exp\left(-i\frac{\pi\alpha\theta}{2} sign(s)\right)\right\}, \tag{5.17}$$

and $G_{\beta,1}$ is the stable distribution with the characteristic function of the form

$$g(\beta,1) = exp\left\{-|s|^{\beta} exp\left(-i\frac{\pi\beta}{2} sign(s)\right)\right\}. \tag{5.18}$$

In the view of controlled diffusion models in reinsurance studied by [96], we propose that the controlled CTRW model is potentially useful in this field too.

Chapter 6

Conclusions

In this thesis we presented the first research on controlled Continuous Time Random Walks (CTRWs). Using one heuristic and one rigorous approach we derived that for such a dynamics the optimal payoff will satisfy a fractional Hamilton Jacobi Bellman (FHJB) type equation. We have also studied extensions of the CTRW models and derived more general equations for the corresponding optimal payoffs. We focused on the analysis of the FHJB equation, for the first time deriving the mild form of the FHJB and studying its well-posedness and regularity properties. We have also expressed the solution to the FHJB equation as a solution to an optimization problem, which may also be extended to a more general scenario where the operators are generators of Feller processes.

Chapter 7

Appendix

Let us recall the asymptotic properties of stable densities defined in (3.26)

$$g(y, \alpha, \sigma) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \exp\{-\sigma|p|^\alpha\} e^{-ipy} dp, \tag{7.1}$$

see [82] for details. For $|y|/\sigma^{1/\alpha} \to 0$ the following asymptotic expansion for g holds

$$g(y, \alpha, \sigma) \sim \frac{|S^{d-2}|}{(2\pi\sigma^{1/\alpha})^d} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} a_k \left(\frac{|y|}{\sigma^{1/\alpha}} \right)^{2k}, \tag{7.2}$$

where

$$a_k = \alpha^{-1} \Gamma\left(\frac{2k+d}{\alpha} \right) B\left(k + \frac{1}{2}, \frac{d-1}{2} \right), \tag{7.3}$$

where

$$B(q, p) = \int_0^1 x^{p-1}(1-x)^q dx = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} \tag{7.4}$$

is the Beta function, and

$$|S^{d-2}| = 2\frac{\pi^{(d-1)/2}}{\Gamma(\frac{d-1}{2})} \tag{7.5}$$

and $|S^0| = 2$, see [82] for the proof.

For $|y|/\sigma^{1/\alpha} \to \infty$ the following asymptotic expansion holds

$$g(y;\alpha,\sigma) \sim (2\pi)^{-(d+1)/2}\frac{2}{|y|^d}\sum_{k=1}^{\infty}\frac{a_k}{k!}(\sigma|y|^{-\alpha})^k \tag{7.6}$$

where

$$a_k = (-1)^{k+1}\sin\left(\frac{k\pi\alpha}{2}\right)\int_0^{\infty}\xi^{\alpha k+(d-1)/2}W_{0,\frac{d}{2}-1}(2\xi)d\xi \tag{7.7}$$

and $W_{0,n}(z)$ is the Whittaker function

$$W_{0,n}(z) = \frac{e^{-z/2}}{\Gamma(n+1/2)}\int_0^{\infty}[t(1+t/z)]^{n-1/2}e^{-t}dt, \tag{7.8}$$

see [82] for the proof. We refer the interested reader to [97] for background information on asymptotic expansions.

In case $d = 1$ the stable density function $w(x,\beta,1)$ defined in (3.27) is infinitely smooth for $x = 0$ and $w(x,\beta,1) = 0$ for $x < 0$. Hence w grows at 0 slower than any power. This gives rise to the inequalities such as $w(x,\beta,1) < C_q x^{q-1}$ for any $q > 1$, for $x < 1$. The property $w(x) \sim x^{-1-\beta}$ for $x >> 1$, may be found for example in [82]. This may be deduced from the asymptotic expansions in equations 7.7 and 7.9 in [82] with $\gamma = 1$.

The following result is part of the proposition 7.3.2 from [82]:

Proposition 1. *Let*

$$\phi(y,\alpha,\beta,\sigma) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} |p|^\beta exp\{-i(p,y) - \sigma|p|^{-\alpha}\}dp, \tag{7.9}$$

so that

$$\frac{\partial\phi}{\partial\beta}(y,\alpha,\beta,\sigma) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} |p|^\beta log|p| exp\{-i(p,y) - \sigma|p|^{-\alpha}\}dp. \tag{7.10}$$

Then if $\frac{|y|}{\sigma^{1/\alpha}} \le K$

$$|\phi(y,\alpha,\beta,\sigma)| \le c\sigma^{-\beta/\alpha} g(y,\alpha,\sigma) \tag{7.11}$$

and if $\frac{|y|}{\sigma^{1/\alpha}} > K$

$$|\phi(y,\alpha,\beta,\sigma)| \le c\sigma^{-1}|y|^{\alpha-\beta} g(y,\alpha,\sigma), \tag{7.12}$$

where g is as in (7.1) and (3.26).

Definition 11. *For an open set* $\Omega \subset \mathbb{R}^n$ *a fractional Sobolev space* $H^{p,1}(\Omega)$ *is defined as*

$$H^{p,1}(\Omega) := \left\{ u \in L^1(\Omega) : \frac{|u(x) - u(y)|}{|x-y|^{n+p}} \in L^1(\Omega\times\Omega) \right\} \tag{7.13}$$

with the norm

$$||u||_{H^{p,1}(\Omega)} := \int_\Omega |u|dx + \int_\Omega\int_\Omega \frac{|u(x)-u(y)|}{|x-y|^{n+p}}dxdy. \tag{7.14}$$

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